Akio Matsumoto
Ferenc Szidarovszky

Game Theory and Its Applications

Second Edition

Institute of Economic Research Chuo University
IERCU
中央大学経済研究所
1964

Springer

Game Theory and Its Applications

Akio Matsumoto · Ferenc Szidarovszky

Game Theory and Its Applications

Second Edition

Institute of Economic Research, Chuo University
IERCU
中央大学経済研究所
1964

Springer

Akio Matsumoto
Department of Economics
Chuo University
Hachioji, Tokyo, Japan

Ferenc Szidarovszky
Department of Mathematics
Corvinus University
Budapest, Budapest, Hungary

ISBN 978-981-96-0589-7 ISBN 978-981-96-0590-3 (eBook)
https://doi.org/10.1007/978-981-96-0590-3

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

The authors of this book had several decades of research in different areas of game theory until the mid-1990s, when they met in a conference in Odense, Denmark. Since then they work together on oligopolies and different dynamic economic systems and meet at least once every year in Tokyo and either in Tucson, Arizona or in Budapest and Pécs, Hungary.

This book has two origins. First, it is based on game theory short courses presented in several countries including Japan, Hungary, China, and Taiwan among others. The second author introduced and taught for several years a one-semester graduate-level game theory course at the Eötvös University of Sciences and at the University of Arizona for students in engineering and management. The class notes of that course is the second origin of this book. The objective of this book is to introduce the readers into the main concepts, methods, and applications of game theory, the subject, which has continuously increasing importance in applications in many fields of quantitative sciences including economics, social science, engineering, biology, etc. The wide variety of applications are illustrated with the particular examples introduced in the second and third chapters as well as with the case studies of the last chapters.

After the first edition of this book was published, the authors continued their joint research and teaching game theory that resulted in ideas of new interesting applications, examples, and some additional theoretical issues. They are added into this second edition.

We strongly recommend this book to undergraduate and graduate students, researchers and practitioners in all fields of quantitative science where decision problems might arise involving more than one decision makers, stakeholders, or interest groups. As we will see later in the different chapters, the most appropriate solution concept and the corresponding solution methodology for any problem is a function of the behavior of the decision makers and their interrelationships, and the available information. So before applying any method from this book, these conditions have to be examined. Then the most appropriate method has to be selected and applied to get the solution, which has to be then interpreted and applied in practice.

We sincerely hope that this book will help the readers to understand the main concepts and methodology of game theory, it will help to select the most appropriate model, solution concept, and method, and to use the obtained result in applying it in their practical problems.

The authors are thankful to the Department of Economics of Chuo University, Tokyo, as well as the Mathematics Department of the Corvinus University of Budapest for their hospitality during the joint works of the authors. The support of the Applied Mathematics Department of the University of Pécs, Hungary, is also appreciated.

In addition, the authors wish to express their special thanks to Dr. Mark Molnar at ELTE GTK, Budapest, for the assistance in preparing the manuscript and the final edited version of this book.

Kawasaki, Japan — Akio Matsumoto
Budapest, Hungary — Ferenc Szidarovszky

Contents

List of Figures

List of Tables

Chapter 1
Introduction

In our private life and also in our professional life we have to make decisions repeatedly. Some decisions might have very small consequences and there are others, the consequences of which might have significant effects in our life. As such examples we can consider the choice of an item in our lunch and accepting or rejecting a job offer. Decision science is dealing with all kinds of decision problems, concepts and solution methodologies.

In formulating a mathematical model of a decision problem there are two conflicting tendencies. In one hand we would like to include as many variables, constraints and possible consequences as possible in order to get close to reality. However on the other hand we would like to solve the models, so they must not be too complicated. In creating a decision making model we have to identify the person or persons who are in charge, that is, who is or are responsible to decide. There are two major possibilities: one or more decision makers are present. In order to decide in any choice, the set of all possible decision alternatives have to be made clear to the decision makers. If this set is finite or countable, then the decision problem is called discrete, and if it is a connected set (like an interval), then the problem is considered to be continuous. In the first case the alternatives are simply listed in an order, and in the second case the alternatives are characterized by decision variables and the set of the alternatives is defined by certain inequalities and equations containing the decision variables. We usually make decisions to gain or avoid something. The goodness of any decision can be measured by the different levels of attributes such as received profit, economic loss, level of pollution, water supply, etc. We can usually attach a utility value to each possible level of the attributes which represents the goodness of that value. This function is sometimes called the value or the utility function attached to the attribute. We usually assume that higher utility value is better for the decision makers. In a decision making problem the decision makers might face with single utility or with multiple utilities. In the optimization literature they are referred to as single objective or multiple objective problems. Regarding the numbers of the

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_1

decision makers and the objective functions we might divide the decision making problems into several groups. In the presence of a single decision maker and a single objective function we have an optimization problem the type of which depends on the structure of the set of alternatives and the properties of the objective function. Some of the most frequently applied optimum problems are linear, nonlinear, discrete, mixed, dynamic and stochastic programming, and their solution methods are well taylored to the nature of the problems in hand. If a single decision maker is faced with more than one objectives, then the problem is modeled as a multiobjective optimization problem. There are many different solution concepts and methods for their solutions. Assume next that there are multiple decision makers. If their priorities, and therefore their objectives are the same or almost the same, then a common objective can be formulated, and the group of decision makers can be substituted with a single decision maker instead of the group. We face a very different situation, when the decision makers have conflicting interests, each of them wants to get as high as possible objective function value, however, the conflicts in their interests force them to reach some agreement or mutually acceptable solution. The kind of solution to be obtained largely depends on the available information and the attitude of the decision makers toward each other. We have now arrived into the territory of game theory. As every scientific discipline, game theory also has its own language. The decision makers are called *players*, even if the decision problem is not a game and the decision makers are not playing at all. The decision alternatives are called the *strategies*, and the objective functions of the players are called the *payoffs* or *payoff functions*. Game theory can be divided into two major groups. If there is no information sharing, negotiation or mediation between the players, and they select strategies independently from each other, then the game is *noncooperative*, otherwise *cooperative*. A new concept, the idea of partial cooperation is getting more attention recently, which is between no cooperation and total cooperation. The most simple situation occurs if each player knows the set of feasible strategies and payoff functions of all players, in which case we face a game with *complete information*. Otherwise the game is *incomplete*. In the case of *repeated* or *dynamic games with perfect information* the players have complete knowledge at each time period about the complete history of the game with all previous strategy selections and payoff values. Games with *imperfect information* occur if some of the above mentioned information is not available to the players. In most cases the missing information is considered as a random variable and therefore probabilistic methods are involved in the analysis. If the game is played only ones, each player selects a strategy simultaneously with the others and they receive the corresponding payoffs instantly, then the game is *static*. However in many cases the game is repeated and the set of feasible strategies and payoff values of each time period might depend on the previous strategy selections of the players, in which case we face *repeated* or *dynamic* games. The overall strategy of each player consists of his/her decisions at any time period and in any possible situation of the game at that time period.

The aim of this book is to give an introduction to the theory of games and their applications, so both researches and application oriented experts can benefit from it and can use the material of this book in their work. The solution concepts and the associated methodology largely depend on the types of the game under consideration. This book is structured accordingly. Part I of the book is devoted to noncooperative games. In Chap. 2 we start with examples of static two-person discrete games with complete information, and then some examples of their N-person extensions are introduced. Continuous static games are discussed then in Chap. 3 with examples including the well-known Cournot oligopoly, and the first and second-price auctions. In Chap. 4 the relation of the Nash-equilibrum with fixed-point and optimization problems is discussed, which can be used to guarantee the existence of equilibria and to construct computer methods for finding equilibria. Existence results are presented in Chap. 5, bimatrix and matrix games, mixed extensions of finite games, and multiproduct oligopolies are selected as applications of the general results. Chapter 6 introduces the most common computer methods to find equilibria. They are based on the solution of a certain system of (usually) nonlinear equations and inequalities, or on the solution of a (usually) nonlinear programming problem. The general methodology is illustrated with bimatrix and matrix games and single-product N-firm oligopolies. Chapter 7 is devoted to special matrix games and their relation to linear programming. Two special methods for solving matrix games are introduced: method of fictitious play as an iteration process, and the method of von Neumann as a "interior point" method giving the equilibrium as the limit of the trajectories of a nonlinear ordinary differential equation system. Chapter 8 gives conditions for the uniqueness of equilibria based on conditions on the best response mappings as well as on the strict diagonal convexity of the payoff functions. Chapter 9 on dynamic games starts with the most simple case of leader-follower games, where the concept of backward induction is introduced. Dynamic games with simultenous moves are illustrated with dynamic oligopolies, and games with sequential moves are discussed using the case of oligopolies, bargaining, and finite rooted tree games. Games under uncertainty are discussed in Chap. 10, which is divided into two parts; static and dynamic games. Part II of the book discusses the main issues of cooperative games. Chapter 11 introduces solution concepts based on characteristic functions including the core, stable sets, the nucleolus, the Shapley values, the kernel and bargaing sets. Chapter 12 discusses games when the players partially cooperate with each other. Chapter 13 introduces the main concepts of conflict resolution. The symmetric and nonsymmetric Nash bargaining solutions are introduced and some alternative methods are outlined. The fundamentals of multiobjective optimization are discussed in Chap. 14, which methods are important if a mediator is hired to find solution for the dispute among the players. If no quantifiable payoff functions are available and the players only can rank the alternatives, then social choice procedures are the most appropriate methods, which are introduced in Chap. 15. In the previous chapters we already introduced particular games arising in several areas and showed their solutions. In the last two chapters some additional case studies are discussed showing the broad applicability of the material discussed in this book. In the Appendices some mathematical background materials are briefly discussed which are repeatedly used in the book.

Chapter 2
Discrete Static Games

In an optimization problem we have a single decision maker, his feasible decision alternative set and an objective function depending on the selected alternative. In game theoretical models we have several decision makers who are called the *players*, each of them has a feasible alternative set, which is called the player's *strategy set*, and each player has an objective function what is called the player's *payoff function*. The payoff of each player depends on the strategy selections of all players, so the outcome depends on his own decision as well as on the decisions of the other players. Let N be the number of players, S_k the strategy set of player $k\,(k = 1, 2, \ldots, N)$ and it is assumed that the payoff function ϕ_k of player k is defined on $S_1 \times S_2 \times \cdots \times S_N$ and is real valued. That is, $\phi_k : S_1 \times S_2 \times \cdots \times S_N \mapsto \mathbb{R}$. So if $s_1, s_2, \ldots, s_N$ are the strategy selections of the players, $s_k \in S_k (k = 1, 2, \ldots, N)$, then the payoff of player k is $\phi_k(s_1, s_2, \ldots, s_N)$. The game can be denoted as $\Gamma(N; S_1, S_2, \ldots, S_N; \phi_1, \phi_2, \ldots, \phi_N)$ which is usually called the *normal form* representation of the game.

A game is called discrete, if the strategy sets are countable, in most cases only finite. The most simple discrete game has only two players, each of them has only two possible strategies to select from. Therefore there are only four possible outcomes of the game.

2.1 Examples of Two-Person Finite Games

We start with the prisoner's dilemma game, which is the starting example in almost all game theory books and courses.

Example 2.1 (Prisoner's dilemma) Assume two criminals robbed a jewellery store for hire. After doing this job they escaped with a stolen can and delivered the stolen items to a mafia boss who hired them. After getting rid of the clear evidence the police

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_2

Table 2.1 Payoff table of Example 2.1

1 \ 2	C	D
C	$(-2, -2)$	$(-10, -1)$
D	$(-1, -10)$	$(-5, -5)$

stopped them for a traffic violation and arrested them for using a stolen car. However the police had a very strong suspicion that they robbed the jewellery store because the method they used was already known to the authorities, but there was no evidence for the serious crime, only for the minor offense of using a stolen car. In order to have evidence, the two prisoners were placed to separate cells from each other, so they could not communicate, and investigators told to each of them that his partner already admitted the robbery and encouraged him to do the same for a lighter sentence. In this situation the two criminals are the players, each of them has the choice from two alternatives: cooperate (C) with his partner by not confessing or defect (D) from his partner by confessing. So we have four possible states, (C, C), (C, D), (D, C) and (D, D) where the first (second) symbol shows the strategy of the first (second) player. The payoff values are the lengths of the prison sentences given to the two players. They are given in Table 2.1, where the first number is the payoff value of player 1 and the second number is that of player 2. The rows correspond to the strategies of player 1 and the columns to the strategies of player 2.

If both players cooperate, then they get only a light sentence because the police has no evidence for the robbery. If only one player defects, then he gets a very light sentence as the exchange for his testimony against his partner, who will receive a very harsh punishment. If both players confess, then they get stronger punishment then in the case of (C, C) but lighter than the cooperating player in the case when his partner defects.

In this situation the players can think in several different ways. They can look for a stable outcome or they can try to get as good as possible outcome under this condition.

The state (C, C) is not stable, since it is the interest of the first player to change his strategy from C to D, when his 2-years sentence would decrease to only 1 year. By this change the second player would get a very harsh 10 years sentence. The state (C, D) is not stable either, since if the first player would change his strategy to D, then his sentence would decrease in the expense of the second player. The state (D, C) is similar by interchanging the two players. The state (D, D) is stable in the sense that none of the players has the incentive to change strategy, that is, if any of the players changes strategy and the other player keeps his choice, then the strategy change can result in the same or worse payoff values. So the state (D, D) is the only stable state. It is usually called the *Nash-equilibrium*.

Definition 2.1 A Nash-equilibrium gives a strategy choice for all players such that no player can increase his payoff by unilaterally changing strategy.

Table 2.2 Payoff table of Example 2.2

1 \ 2	H	L
H	(40, 40)	(10, 50)
L	(50, 10)	(20, 20)

Another way of leading to the same solution is based on the notion of *best response*, which is the best strategy selection of each player given the strategy selection(s) of the other player(s). We can find the best response function of player 1 as follows. If player 2 selects C, then the payoff of player 1 is either -2 or -1 depending on his choice of C or D. Since -1 is more preferable than -2, player 1 selects D in this case:

$$R_1(C) = D.$$

Similarly, if player 2 selects D, then the payoff of player 1 is either -10 or -5, and again -5 is better with the strategy choice of D,

$$R_1(D) = D.$$

We can see that strategy D is the best response of player 1 regardless of the strategy selection of the other player. Therefore D is called a *dominant strategy*, so it is the players' optimal choice. Player 2 thinks in the same way, so his optimal choice is always D. So the players select the state (D, D). ▼

Example 2.2 (Competition of gas stations) Two gas stations compete in an intersection of a city. They are the players, and for the sake of simplicity assume that they can select only low (L) or high (H) selling price.

The payoff values are given in Table 2.2. If both charge high price, then they share the market and both enjoy high profit. If only one charges high price, then almost all customers select the station with low price, so its profit will be high by the high volume, while the other station will get only small profit by the very low volume. If both select low price, then they share the market with low profits. By using the same argument as in the previous example we can see that the only stable state is (L, L), and L is dominant strategy for both players. The state (L, L) provides 20 units profit to each player. Notice that by cooperating with both selecting high price their profits would be 40 units. However without cooperation such case cannot occur because of the usual lack of trust between the players. The same comment can be made in Example 2.1. as well, however, the illegality of price fixing also prohibits the players to cooperate. ▼

Example 2.3 (Game of privilege) Consider a house with two apartments and several common areas, such as laundry, storage, stairs etc. The two families are supposed to take turns in cleaning and maintaining these common areas. In this situation the two families are the players, their possible strategies are participating (P) in the joint effort or not (N). The payoff table is given in Table 2.3. If both families participate, then the common areas are always nice and clean resulting in the highest payoff for both players. If only one participates, then the common areas are not as clean as in the previous case, and payoff of the participating player is even less than that of the other player because of its efforts.

Table 2.3 Payoff table of Example 2.3

1 \ 2	P	N
P	(3, 3)	(1, 2)
N	(2, 1)	(0, 0)

If none of the players participate, then the common areas are not taken care resulting in the least payoffs. The best responses of the first player are as follows:

$$R_1(P) = P \text{ and } P_1(N) = P.$$

That is, P is dominant strategy. The same holds for player 2 as well, so the only Nash equilibrium is (P, P). ▼

Example 2.4 (Chicken game) Consider a very narrow street in which two teenagers stand against each other on motorbikes. For a signal they start driving toward each other. The one who gives way to the other is called the chicken. In this situation the teenagers want to show to their friends or to a gang that how determined they are. They are the two players with two possible strategies: becoming a chicken (C) or not (N). Table 2.4 shows the payoff values.

If both players are chickens, then their payoffs are higher than the payoff of a single chicken and lower than a non-chicken when the other player is a chicken. The worst possible outcome occurs with the state (N, N), when they collide and might suffer serious injuries. The best responses are the followings:

$$R_k(C) = N \text{ and } R_k(N) = C\,(k = 1, 2).$$

Therefore both states (C, N) and (N, C) are Nash equilibria, since in both cases the strategy choice of each player is its best response against the corresponding strategy of the other player. This result however does not help the players in their choices in a particular situation, since both strategies are equilibibrium strategies and a choice among them requires the knowledge of the selected strategy of the other player. ▼

Table 2.4 Payoff table of Example 2.4

1 \ 2	C	N
C	(3, 3)	(2, 4)
N	(4, 2)	(1, 1)

Example 2.5 (Battle of sexes) A husband (H) and wife (W) want to spend an evening together. There are two possibilities, either they can go to a football game (F) or to a movie (M). The husband would prefer F, while the wife would like to go to M. They do not decide on the common choice in the morning and plan to call each other in the afternoon to finalize the evening program. However they cannot communicate for some reason (unexpected meeting in work or power shortage), so each of them selects F or M independently of the other, travels there hoping to meet his/her spouse. The payoff values are given in Table 2.5.

Table 2.5 Payoff table of Example 2.5

H \ W	F	M
F	(2, 1)	(0, 0)
M	(0, 0)	(1, 2)

If both players go to F, then they spend the evening together with positive payoff values, and since F is the preferred choice of the husband, his payoff is higher than that of his wife. The state (M, M) is similar in which case the wife gets a bit higher payoff. In the cases of (F, M) and (M, F) they cannot meet, no joint event occurs with zero payoff values. Clearly for both players $k = H, W$,

$$R_k(F) = F \text{ and } R_k(M) = M$$

so both states (F, F) and (M, M) are equilibria. Similarly to the previous example this solution does not give a clear choice in particular situations. ▼

Example 2.6 (Good citizens) Assume a robbery takes place in a dark alley and there are two witnesses of this crime. Both of them have a mobile phone, so they have the choice of either calling the police (C) or not (N). If at least one of them makes the call, then the criminal is arrested resulting in a positive payoff to the society including both witnesses. However the caller will be used to testify in the trial against the criminal, which takes time and possible revenge from the criminal's partners. So the possible strategies of the witnesses are C and N, and the corresponding payoff values are given in Table 2.6.

Table 2.6 Payoff table of Example 2.6

1 \ 2	C	N
C	(7, 7)	(7, 10)
N	(10, 7)	(0, 0)

The arrest of the criminal gives a 10 units benefit, however making the call to the police decreases it by 3 units. If no phone call is made, then no benefit is obtained without any cost. In this case

$$R_k(C) = N \text{ and } R_k(N) = C \, (k = 1, 2)$$

resulting in two equilibria (C, N) and (N, C). ▼

The previous examples show that equilibrium can be unique or multiple. In the following example we will show case when no equilibrium exists.

Example 2.7 (Checking tax return) A tax payer (T) has to pay an income tax of 5,000 dollars, however he has the option of not declaring his income and to avoid paying tax. However in this second case he might get into trouble if IRS checks his tax return. In formulating this situation as a two-person game, player 1 is the taxpayer with two possible strategies: cheating (C) or being honest (H) with the tax return; and player 2 is the IRS who can check (C) the tax return or not (N). In determining the payoff values we notice that in the case of cheating the taxpayer has to pay his entire income tax of \$5, 000 and a penalty \$5, 000 as well if his tax return is checked. In checking a tax return the IRS has a cost of \$1,000. Table 2.7 shows the payoff values of the two players.

Table 2.7 Payoff table of Example 2.7

T \ IRS	C	N
C	$(-10, 9)$	$(0, 0)$
H	$(-5, 4)$	$(-5, 5)$

The best responses of the two players are as follows:

$$R_T(C) = H \text{ and } R_T(N) = C,$$
$$R_{IRS}(C) = C \text{ and } R_{IRS}(H) = N.$$

We can easily verify, that there is no equilibrium, that is, no state is stable in the sense that in the cases of all states at least one player can increase its payoff by changing strategy.

In the case of state (C, C) player 1 has the incentive to change its strategy to H. In the case of (C, N) player 2 can increase its payoff by changing strategy to C. In the case of state (H, C) player 2 has again the incentive to change strategy to N, and finally, in the case of state (H, N) player 1 would want to change to C. ▼

Example 2.8 (Waste management) A waste management company plans to place dangerous waste on the border between two counties causing damages D_1 and D_2 units to them. In order to avoid these damages at least one county has to support intensive lobbying against the waste management company, which would cost them C_1 and C_2 units, respectively. Both counties have two possible strategies: supporting (S) the lobbying or not (N). So we have four possible states with payoff values given in Table 2.8.

Table 2.8 Payoff table of Example 2.8

1 \ 2	S	N
S	$(-C_1, -C_2)$	$(-C_1, 0)$
N	$(0, -C_2)$	$(-D_1, -D_2)$

If both support lobbying, then both counties face costs but there is no damage. If only one of them is supporter, then neither county faces damage but only one of them pays for lobbying. If none of them is supporter, then both face damages without any cost.

We can easily check the conditions under which the different states provide equilibrium. State (S, S) is an equilibrium, if S is best response of both players against the strategy choice of S of the other player, which occurs when $-C_1 \geqq 0$ and $-C_2 \geqq 0$. This is impossible, so (S, S) cannot be an equilibrium. State (N, S) is an equilibrium if $0 \geqq -C_1$ and $-C_2 \geqq -D_2$, which can be rewritten as $C_2 \leqq D_2$. State (S, N) is an equilibrium if $-C_1 \geqq -D_1$ and $0 \geqq -C_2$, that is, when $C_1 \leqq D_1$. And finally, (N, N) is an equilibrium if $-D_1 \geqq -C_1$, and $-D_2 \geqq -C_2$, which can be rewritten as $D_1 \leqq C_1$ and $D_2 \leqq C_2$. Figure 2.1. shows these cases. Clearly there is always an equilibrium, and it is not unique if $C_1 \leqq D_1$ and $C_2 \leqq D_2$. ▼

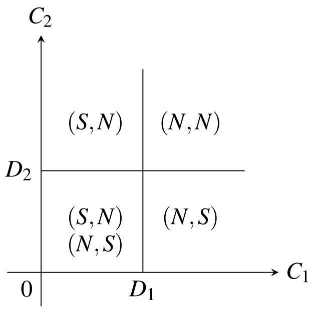

Fig. 2.1 Equilibria in Example 2.8

Example 2.9 (Advertisement game) Consider m markets of potential customers and assume that each of two agencies plans an intensive advertisement campaign on one of the markets. So they select a market and perform intensive advertisement there. If only one agency advertises on a market, then it will get all customers, however, if they select the same market, then they have to share the customers. So the set of strategies of both agencies is $\{1, 2, \ldots, m\}$. Let $a_1 \geqq a_2 \geqq \cdots \geqq a_m$ denote the number of potential customers in the different markets. The payoff values ϕ_1 and ϕ_2 of the two agencies are given in Table 2.9, where $q_k = 1 - p_k$ for $k = 1, 2, \ldots, m$. A strategy pair (i, j) is an equilibrium if strategy i is the best response of player 1 if player 2 selects strategy j, and also strategy j is best response of player 2 if player 1 choses strategy i. That is, the $\phi_1(i, j)$ payoff value in the ϕ_1 table is largest in its column, and $\phi_2(i, j)$ is largest in its row in the ϕ_2 table. Notice first that in the ϕ_1 table the elements of the first row and the value at (2, 1) can be largest in their colums, so only these elements can provide equilibrium. In the ϕ_2 table only the first column and element $\phi_2(1, 2)$ can be largest in their rows. There are only three strategy pairs satisfying both row and column maximum conditions,

$$(2, 1), (1, 1) \text{ and } (1, 2).$$

The state (2, 1) is equilibrium, if $a_2 \geqq p_1 a_1$; the state (1, 1) is equilibrium if $p_1 a_1 \geqq a_2$ and $q_1 a_1 \geqq a_2$, and similarly (1, 2) is an equilibrium if $a_2 \geqq q_1 a_1$. ▼

Table 2.9 Payoff tables of Example 2.9

1 \ 2	1	2	...	m
1	$p_1 a_1$	a_1	...	a_1
2	a_2	$p_2 a_2$	...	a_1
⋮	⋮	⋮		⋮
m	a_m	a_m	...	$p_m a_m$

ϕ_1

1 \ 2	1	2	...	m
1	$q_1 a_1$	a_2	...	a_m
2	a_1	$q_2 a_2$	...	a_m
⋮	⋮	⋮		
m	a_1	a_2	...	$q_m a_m$

ϕ_2

Example 2.10 (A game with coins) Each of two boys has 1 coin in his hand. At the beginning each of them places 0 or 1 coin into his pocket. First, player 1 guesses the total number of coins under the condition that no bluffing is allowed, that means that 0 cannot be said if this player has a coin in his pocket, and 2 cannot be said if there is no coin in his pocket. Then player 2 makes a guess without bluffing and under the additional condition that the guessing of player 1 cannot be repeated. The player with the correct guessing wins and will receive a 1 dollar reward. The loosing player has to pay this reward.

This is a two person zero-sum game (n =2). The strategy of any player has two components: number of coins placed into his pocket (0 or 1) and guessing of the total number of coins (0, 1 or 2). The process of the game with payoffs is illustrated in Fig. 2.1, where it is assumed that Player 1 places 0 or 1 coin into his pocket, and then Player 2 does the same. This is equivalent with the case when they do it simultaneously.

The game starts at point I, where Player 1 can decide the number of coins placed into his pocket (0 or 1) represented by the two arcs originated at this point. The two outcomes are denoted by A_1 and A_2, where Player 2 has similar decisions, 0 or 1 again, which are represented by two arcs originated at points A_1 and A_2. The four endpoints are B_1, B_2 , B_3 and B_4 at which points player 1 has to guess the total number of coins without bluffing. Since he does not have any coin in his pocket at ponts B_1 and B_2 guessing 2 is impossible. Similarly at points B_3, B_4 one coin is in the pocket, so a zero guess is impossible. These impossible guesses are crossed out in the figure. Since at each point B_1, B_2, B_3 or B_4 only two feasible guesses remain, two arcs originate from these nodes. Their endpoints are C_1–C_8, where Player 2 gives his guessing .

At point C_1 guessing 0 would repeat the guessing of Player 1 and guessing 2 is also impossible since there is no coin in his pocket. Therefore only one possibility remains, so at point C_1 Player 2 does not have any choice, his guess has to be 1. At point C_2, guessing 1 would repeat Player 1 and since there is no coin in his pocket, guessing 2 is also impossible, showing that guessing 0 is the only choice. At point C_3, 0 would repeat Player 1, and since his guessing is 0 and bluffing is impossible, Player 1 must have no coin in his pocket. Therefore only one possibility remains, so at point C_3 Player 2 must select 1.

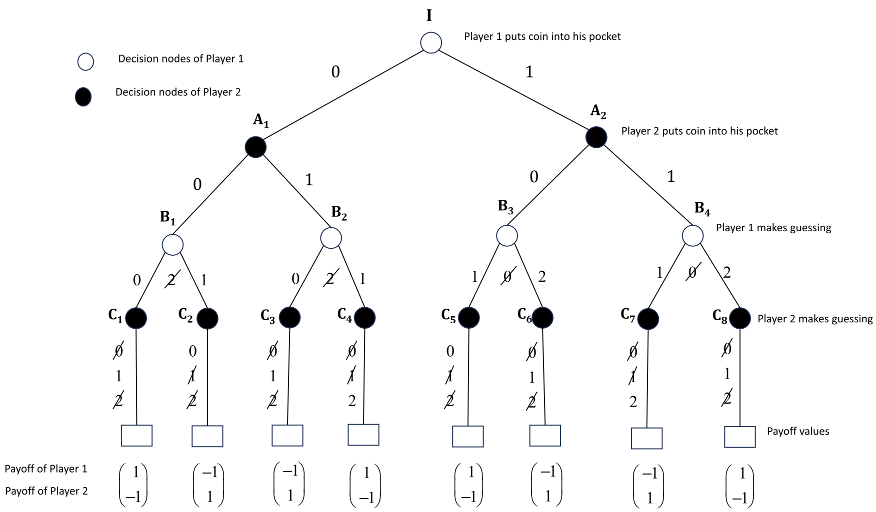

Fig. 2.2 Illustration of Example 2.10

Very similar reasons show that at all points C_4, C_5, C_6, C_7, C_8, the situation is similar, only one feasible choice is possible for Player 2.

The payoff values are shown under the terminal points. For example under C_1 the total number of coins is $0 + 0 = 0$, which is the guess of Player 1, who is therefore the winner.

The above discussion shows that the guessing of Player 2 is predetermined by the numbers of coins in the pockets and the guess of Player 1, therefore Player 2 has only one component in strategy: 0 or 1. Player 1 has two componenets, each strategy can be given as (k, l) where k is the number of coins in his pocket and l represents his guessing.

This game is zero-sum, it is sufficient to give payoffs of Player 1 only in Table 2.10.

Equilibrium payoffs are those payoff values, which are largest in their columns and smallest in their rows. The largest values in both columns are 1, and there is a -1 next to them showing that they cannot be also smallest in their rows. Thus the game has no equilibrium. ▼

Table 2.10 Payoff table of Player 1 in Example 2.10

1 \ 2	(0)	(1)
(0, 0)	1	−1
(0, 1)	−1	1
(1, 1)	1	−1
(1, 2)	−11	1

2.2 General Description of Two-Person Finite Games

Up to this point we have introduced two-person finite games, when the players had only finitely many strategies to select from. Assume that player 1 has m strategies and player 2 has n strategies. Then the strategy sets are $S_1 = \{1, 2, \ldots, m\}$ and $S_2 = \{1, 2, \ldots, n\}$ for the two players. As we did in the examples, the payoff values can be shown in the payoff tables, the general forms of which are given in Table 2.11.

Table 2.11 Payoff tables of two-person finite games

1 \ 2	1	2	…	n
1	a_{11}	a_{12}	…	a_{1n}
2	a_{21}	a_{22}	…	a_{2n}
⋮	⋮	⋮		⋮
m	a_{m1}	a_{m2}	…	a_{mn}

ϕ_1

1 \ 2	1	2	…	n
1	b_{11}	b_{12}	…	b_{1n}
2	b_{21}	b_{22}	…	b_{2n}
⋮	⋮	⋮		⋮
m	b_{m1}	b_{m2}	…	b_{mn}

ϕ_2

A strategy pair (or state) (i, j) is an equilibrium, if the element a_{ij} is largest in its column in the ϕ_1 table, and the element b_{ij} is largest in its row in the ϕ_2 table. As it was illustrated in the previous examples there is no guarantee for the existence of an equilibrium, and even if it exists the uniqueness of the equilibrium is not guaranteed either.

A two-person game is called *zero-sum* if $\phi_1(i, j) + \phi_2(i, j) = 0$ with all strategy pairs (i, j). That is, the gain of a player is the loss of the other. In this case $b_{ij} = -a_{ij}$, so there is no need to give the table for ϕ_2, since its elements are the negatives of the corresponding elements of ϕ_1. A strategy pair (i, j) is an equilibrium, if a_{ij} is the largest among the elements $a_{1j}, a_{2j}, \ldots, a_{mj}$ and $-a_{ij}$ is the largest among the elements $-a_{i1}, -a_{i2}, \ldots, -a_{in}$. The second condition can be rewritten as a_{ij} is the smallest among the numbers $a_{i1}, a_{i2}, \ldots, a_{in}$. That is, a_{ij} is the largest in its column and also smallest in its row. The equilibria of zero sum games are often called the *saddle points* (think of a person sitting on a horse who is observed from the side and from the back of the horse). In general, zero-sum, two-person games do not necessarily have equilibrium, and if equilibrium exists, it is not necessarily unique. However we can easily show that in the case of multiple equilibria the strategies are different but the corresponding payoff values are identical.

Lemma 2.1 *Let (i, j) and (k, l) be two equilibria of a two-person zero-sum game. Then $\phi_1(i, j) = \phi_1(k, l)$.*

Proof Let $\phi_1(i, j) = a_{ij}$ and $\phi_1(k, l) = a_{kl}$. Then

$$a_{ij} \geqq a_{kj} \geqq a_{kl}$$

since a_{ij} is largest in its column and a_{kl} is smallest in its row. Similarly

$$a_{ij} \leqq a_{il} \leqq a_{kl}$$

since a_{ij} is smallest in its row and a_{kl} is largest in its column. These relations imply that $a_{ij} = a_{kl}$. ■

It is an interesting problem to find out the proportion of two-person zero-sum finite games which have at least one equilibrium. As the following theorem shows this ratio is getting smaller by increasing the size of the payoff table. Consider a two-person, zero-sum game in which the players have m and n strategies, respectively. Assume that the payoff values a_{ij} are independent, identically distributed random variables with a continuous cumulative distribution function. Then the following fact can be proved.

Theorem 2.1 *Under the above conditions the probability that the game has an equilibrium is*

$$P_{m,n} = \frac{m!n!}{(m+n-1)!}.$$

Proof Notice first that

(i) the elements of the payoff table are different with probability one;
(ii) all elements a_{ij} have the same probability to be equilibrium;
(iii) the probability that there is an equilibrium is mn times the probability that a_{11} is equilibrium.

Fact (i) follows from the assumption that the distribution function is continuous and the table elements are independent, (ii) is implied by the assumption that the table elements are identically distributed. From (i), the probability that multiple equilibria exists is zero.

The element a_{11} is equilibrium if it is the largest in its column and the smallest in its row. So if we list the elements of the first row and column in increasing order, then all other elements of the first column should be before a_{11} and all other elements of the first row have to be after a_{11}. The $(m-1)$ other elements of the first column can be permutted in $(m-1)!$ different ways and the $(n-1)$ other elements of the first row can be permutted in $(n-1)!$ different ways, therefore there are $(m-1)!(n-1)!$ possible permutations in which a_{11} is in the equilibrium position. Since the $m+n-1$ elements of the first row and column have altogether $(m+n-1)!$ permutations, the probability that the element a_{11} is an equilibrium equals

$$\frac{(m-1)!(n-1)!}{(m+n-1)!}.$$

Hence the probability that equilibrium exists is

$$mn\frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{m!n!}{(m+n-1)!}.$$

■

Notice that the value of P_{mn} does not depend on the distribution type of the elements, it depends on only the size of the payoff table.

In order to gain a feeling about this value let's consider some special cases and relations:

$$P_{2,2} = \frac{2!2!}{3!} = \frac{2}{3}$$
$$P_{2,3} = \frac{2!3!}{4!} = \frac{1}{2}$$
$$P_{2,4} = \frac{2!4!}{5!} = \frac{2}{5},$$

from which we see that the probability value decreases if the size of the table becomes larger. This is true in general, since

$$\frac{P_{m,n+1}}{P_{m,n}} = \frac{m!(n+1)!}{(m+n)!} \cdot \frac{(m+n-1)!}{m!n!} = \frac{n+1}{m+n} < 1.$$

The same result is obtained if m increases, since $P_{m,n} = P_{n,m}$. Notice that

$$P_{2n} = \frac{2!n!}{(2+n-1)!} = \frac{2}{n+1} \to 0$$

as $n \to \infty$, therefore with any $m \geq 2$,

$$P_{mn} \leq P_{2n} \to 0$$

as $n \to \infty$. Therefore P_{mn} is decreasing in m and n, furthermore it converges to zero if either m or n tends to infinity.

We can also show that the theorem does not hold necessarily if the distribution of table elements is discrete.

Example 2.11 Consider therefore the case of $m = n = 2$ and assume that the four elements of the table are randomly generated from a Bernoulli distribution such that

$$P(a_{ij} = 1) = p \text{ and } P(a_{ij} = 0) = q = 1 - p.$$

Since each element can take two possible values 0 and 1 furthermore there are four elements, there are $16(= 2^4)$ possible payoff tables:

$$\begin{pmatrix} \textcircled{0} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \textcircled{0} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \textcircled{0} & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & \textcircled{0} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \textcircled{0} & 1 \end{pmatrix}$$
$$\begin{pmatrix} \textcircled{1} & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \textcircled{0} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \textcircled{0} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & \textcircled{1} \end{pmatrix}$$
$$\begin{pmatrix} 1 & \textcircled{1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & \textcircled{1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \textcircled{1} & 1 \end{pmatrix} \begin{pmatrix} \textcircled{1} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \textcircled{1} \\ 1 & 1 \end{pmatrix}.$$

There are only two of them without an equilibrium. In the other tables an equilibrium element is circled. The probability of each table

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

equals p^2q^2, so the probability that no equilibrium exists is $2p^2q^2$, and the probability that there is at least one equilibrium is $1 - 2p^2q^2$, which is not necessarily equal to $\frac{2}{3}$. ▼

As an example of two-person zero-sum finite games consider the following situation.

Example 2.12 (Anti-terrorism game) A rectangular shaped city is divided into m block-rows and n block-columns by $E - W$ and $S - N$ streets as shown in Fig. 2.2. So there are mn blocks, and their values are listed in the figure (Fig. 2.3).

a_{11}	a_{12}	a_{13}	......	$a_{1,n-2}$	$a_{1,n-1}$	a_{1n}
a_{21}	a_{22}	a_{23}	......	$a_{2,n-2}$	$a_{2,n-1}$	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
a_{m1}	a_{m2}	a_{m3}	......	$a_{m,n-2}$	$a_{m,n-1}$	a_{mn}

Fig. 2.3 Structure of the city

Assume now that a terrorist group placed a bomb in one of the city blocks and demands a large amount of money as well as the release of prisoners from jail. The city administration clearly does not want to negotiate, they try to find the bomb and avoid damages. However they have sufficient resources to check only one complete block-row or a complete block-column, so if the bomb is placed there, then it is certainly found. In this situation the city and the terrorist group are the two players. The city can chose the block row or the block column which will be checked, the terrorists can select any block of the city. The payoff of the city is positive when they can find the bomb and save that block. The corresponding payoff of the terrorist group is the negative of that of the city, since they lose the damage opportunity. Table 2.12 shows the payoff table of the city.

Table 2.12 Payoff table of Example 2.12

1 \ 2	(1, 1)	(1, 2)	…	(1, n)	(2, 1)	(2, 2)	…	(2, n)	…	(m, 1)	(m, 2)	…	(m, n)
1	a_{11}	a_{12}	…	a_{1n}									
2					a_{21}	a_{22}	…	a_{2n}					
⋮													
m										a_{m1}	a_{m2}	…	a_{mn}
1	a_{11}				a_{21}					a_{m1}			
2		a_{12}				a_{22}					a_{m2}		
⋮			⋱				⋱					⋱	
n				a_{1n}				a_{2n}					a_{mn}

If the city checks block-row i, then the bomb is found if it is placed in one of the blocks $(i, 1), (i, 2), \ldots, (i, n)$ and if the city checks block-column j, then the bomb is found if it is in one of the blocks $(1, j), (2, j), \ldots, (m, j)$. This is a zero-sum game, so an element of the table provides equilibrium if it is the largest in its column and smallest in its row. Every column has positive element, so the largest element is always positive. Every row has zero elements, so the smallest element is always zero. Therefore there is no element in the table which satisfies both conditions of an equilibrium. Consequently the game has no equilibrium. ▼

2.3 N-Person Finite Games

Let N denote the number of players and assume that the players have finitely many strategies to select from. Assume that player $k\,(1 \leqq k \leqq N)$ has m_k strategies which can be denoted by $1, 2, \ldots, m_k$. So the set of strategies of player k is the finite set $S_k = \{1, 2, \ldots, m_k\}$. If player 1 selects strategy i_1, player 2 selects i_2, and so on, player N selects i_N, then the N-tuple $\underline{s} = (i_1, i_2, \ldots, i_N)$ is called a *simultaneous strategy* of the players. So $\underline{s} \in S_1 \times S_2 \times \cdots \times S_N$, and the payoff function of each player k is a real valued function defined on $S = S_1 \times S_2 \times \cdots \times S_N$ which can be denoted by $\phi_k(\underline{s})$. Similarly to the two-player case a simultaneous strategy $\underline{s}^* = (i_1^*, i_2^*, \ldots, i_N^*) \in S$ is an equilibrium, if i_k^* is the best response of all players k with given strategies $i_1^*, \ldots, i_{k-1}^*, i_{k+1}^*, \ldots, i_N^*$ of the other players.

Example 2.13 (Selecting a number) There are $n \geq 4$ students in a classroom. The professor gives the following assignment to the students. Independently of each other they have to select one number from the set $\{0, 1, 2, \ldots, m\}$. So this is an n-person game, the strategy set of each player is the set $\{0, 1, 2, \ldots, m\}$. The common payoff of the students is determined by the professor: if the students select the same number then each of them receives 1 dollar, otherwise no reward will be given to anyone.

We can prove that all selections are equilibria, except when n-1 students select the same number and one student selects a different number.

It is easy to see that this is not an equilibrium, since if the student with a different number changes his/her strategy to the choice of the others, then everybody's payoff will increase.

If the students select the same number, then their payoff will decrease if any one of the students changes choice to something else.

If at most n-2 students have identical choice, then we have again an equilibrium: if a student changes strategy, then it is still impossible that all students would have identical choices. So the students will remain loosers, no increase in payoff would occur. ▼

Example 2.14 (Voting game) Consider a city with two candidates for an office, like to become the mayor. Let A and B denote the candidates. The potential voters are divided between the candidates. If N denotes the number of voter eligible individuals, then we can define an N-person game in the following way. The potential voters are the players. Each of them has two possible strategies voting or not. In defining the payoff functions two factors have to be taken into consideration. For any voter the benefit is 1 if his/her candidate is the winner, 0 in the case of a tie, and -1 if the other candidate wins. However voting has some cost (time, car usage, etc.), which is assumed to be less than unity. In finding conditions for the existence of an equilibrium we have to consider the following simple facts.
(i) There is no equilibrium when a candidate wins.
If at least one player votes in the losing group, then by not voting he/she would increase payoff by eliminating voting cost. If nobody votes in the losing group, then we have two subcases. If more than one person votes in the winning group, then one of them could change strategy to not voting and would increase payoff. If only one person is in the winning group, then any person in the losing group could make the election result a tie by going to vote, and in this way increase payoff.
(ii) So the election result has to be a tie in any equilibrium, and everybody has to vote.
Assume that there is a person who dose not vote. By going to vote he/she could make his/her group winner and so the payoff would increase.

In summary, the only possibility for an equilibrium is if N is even, equal number of people support the two candidates and everybody votes. This is really an equilibrium, since if any player changes strategy by not voting, then his/her group becomes the losing group and the payoffs decrease for its members. ▼

If at least one player has infinitely many strategies, then the payoff matrices become infinite. Nash equilibria are defined in the same way as in finite games, however the existence of best responses is not guaranteed in general.

Chapter 3
Continuous Static Games

Let N denote the number of players. It is usually assumed that the set of all feasible strategies of each player has at least two elements. If S_k is the strategy set of player k, then its payoff function ϕ_k is defined on the set of all simultaneous strategies, which is denoted by $S = S_1 \times S_2 \times \cdots \times S_N$, and $\phi_k(\underline{s})$ for all $\underline{s} \in S$ is a real number. The normal form of the game is given as $G = \{N; S_1, S_2, \ldots, S_N; \phi_1, \ldots, \phi_N\}$. The game is *continuous*, if all sets S_k are connected and all payoff functions ϕ_k are piece-wise continuous.

The best responses of the players and the Nash-equilibrium can be defined in the same way as they were introduced for discrete games. The best response function of any player k is the following:

$$R_k(\underline{s}_{-k}) = \{s_k^* | s_k^* \in S_k, \phi_k(s_k^*, \underline{s}_{-k}) = \max_{s_k \in S_k} \phi_k(s_k, \underline{s}_{-k})\} \tag{3.1}$$

which is the set of all strategies s_k^* of player k such that his payoff is maximal given the strategy selections $\underline{s}_{-k} = (s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_N)$ of the other players. The Nash-equilibrium is a simultaneous strategy vector $\underline{s}^* = (s_1^*, \ldots, s_N^*)$ such that the equilibrium strategy s_k^* for all players are their best responses given the strategies s_j^* of all other players j. This property can be reformulated as for all players k and $s_k \in S_k$,

$$\phi_k(s_1^*, ..., s_k^*, ..., s_N^*) \geqq \phi_k(s_1^*, \ldots, s_{k-1}^*, s_k, s_{k+1}^*, \ldots, s_N^*)$$

meaning that no player can increase his payoff from the equilibrium by unilaterally changing strategy.

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_3

3.1 Examples of Two-Person Continuous Games

Example 3.1 (Sharing a pie) The mother of two children bakes a pie for her children who were asked to tell the amount they want to get from the pie under two conditions. First, none of them can know the request of the other child before announcing his request and second, if the total amount they request is more than the pie itself, then they do not get any part of the pie. In this case the players are the two children, their strategies are the real numbers x (for players 1) and y (for player 2) such that $x, y \in [0, 1]$. The payoff functions are

$$\phi_1(x, y) = \begin{cases} x & \text{if } x + y \leqq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

and

$$\phi_2(x, y) = \begin{cases} y & \text{if } x + y \leqq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

With given value of $y \in [0, 1]$ the best response of player 1 is to ask the leftover portion after player 2 gets his requested amount: $R_1(y) = 1 - y$. If $x < 1 - y$, then player 1 could increase his payoff by increasing his strategy to $1 - y$, and if $x > 1 - y$, then his payoff is zero, so it could be increased by decreasing the value of x to $1 - y$. Similarly the best response of player 2 is $R_2(x) = 1 - x$. The two best response functions are illustrated in Fig. 3.1 from which it is clear that there are infinitely many equilibria:

$$\{(x^*, y^*) | x^* \in [0, 1], y^* = 1 - x^*\}.$$

▼

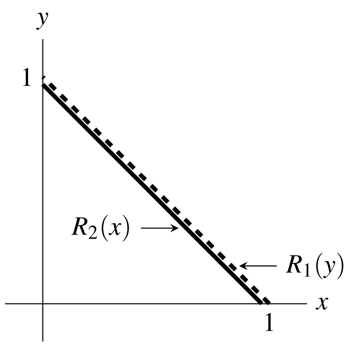

Fig. 3.1 Best responses in Example 3.1

Example 3.2 (Airplane and submarine) This game is a simplified version of the game between British airplanes and German submarines during World War 2 in the British Channel. Assume a submarine is hiding at a certain point x of the unit interval [0, 1] and an airplane drops a bomb into a location y of interval [0, 1]. The damage to the submarine is the payoff of the airplane and its negative is the payoff of the submarine. In this game the submarine is player 1 and the airplane is player 2 with strategy sets $S_1 = S_2 = [0, 1]$. The payoff of player 1 is $\phi_1(x, y) = -d(|x - y|)$ and that of player 2 is $\phi_2(x, y) = d(|x - y|)$, where d strictly decreases in $|x - y|$, that is, the damage is increasing if the bomb is dropped closer to the submarine. With given $y \in [0, 1]$, the submarine wants to find the location which is as far as possible from point y:

$$R_1(y) = \begin{cases} 0 & \text{if } y > \frac{1}{2} \\ 1 & \text{if } y < \frac{1}{2} \\ \{0;\ 1\} & \text{if } y = \frac{1}{2} \end{cases} \tag{3.4}$$

that is, there is a unique best response if $y \neq \frac{1}{2}$ and the two endpoints if $y = \frac{1}{2}$. The best response of player 2 is the exact hit with the bomb if x is known:

$$R_2(x) = x. \tag{3.5}$$

The best responses are illustrated in Fig. 3.2 from which we can conclude that there is no Nash-equilibrium. ▼

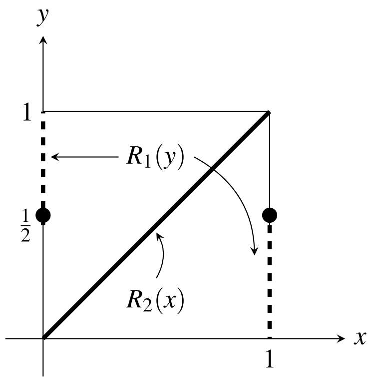

Fig. 3.2 Best responses in Example 3.2

Example 3.3 (Cournot duopoly) Assume that two firms produce identical product or offer the same service to a homogeneous market. Let x and y denote the outputs of the firms, so the total supply to the market is $s = x + y$. Let L_1 and L_2 be the capacity limits of the firms, so $0 \leqq x \leqq L_1$ and $0 \leqq y \leqq L_2$. If $C_1(x)$ and $C_2(y)$ are the cost functions of the firms and $p(s)$ is the inverse demand (or price) function of the market, then the profit functions of the two firms are given as

$$\phi_1(x, y) = xp(x + y) - C_1(x) \tag{3.6}$$

and

$$\phi_2(x, y) = yp(x + y) - C_2(y). \tag{3.7}$$

In this two-person game the two firms are the players with strategy sets $S_1 = [0, L_1]$, $S_2 = [0, L_2]$ and payoff functions ϕ_1 and ϕ_2. ▼

We can illustrate the best response functions and the Nash equilibria in several special cases.

Case 1. Assume linear cost and price functions

$$C_1(x) = x + 1,\ C_2(y) = y + 1,\ p(s) = 10 - s$$

with $L_1 = L_2 = 5$. The payoff function of player 1 is the following:

$$\phi_1(x, y) = x(10 - x - y) - (x + 1) = 9x - x^2 - xy - 1 \tag{3.8}$$

with derivatives

$$\frac{\partial \phi_1}{\partial x} = 9 - 2x - y$$

and

$$\frac{\partial^2 \phi_1}{\partial x^2} = -2 < 0.$$

So ϕ_1 is strictly concave in x, so there is a unique best response of player 1 in interval $[0, 5]$. The first order condition gives the stationary point $x = \frac{9-y}{2}$ which is a feasible strategy for player 1 with all $y \in [0, 5]$, so

$$R_1(y) = \frac{9 - y}{2}.$$

Similarly,

$$R_2(x) = \frac{9 - x}{2}$$

and the Nash equilibrium can be obtained as the unique solution of equations

$$x = \frac{9-y}{2},\ y = \frac{9-x}{2}$$

which is $x^* = y^* = 3$. The best responses are illustrated in Fig. 3.3.
Case 2. Assume linear price and quadratic cost functions:

$$p(s) = 10 - s,\ C_1(x) = x + x^2,\ C_2(y) = y + y^2$$

with $L_1 = L_2 = 5$ as before. The payoff function of player 1 is clearly

$$\phi_1(x, y) = x(10 - x - y) - (x + x^2) = 9x - 2x^2 - xy. \quad (3.9)$$

The stationary point is obtained from equation

$$\frac{\partial \phi_1}{\partial x} = 9 - 4x - y = 0$$

which is $x = \frac{9-y}{4}$ and similarly $y = \frac{9-x}{4}$ resulting again a unique equilibrium: $x^* = y^* = \frac{9}{5} = 1.8$.

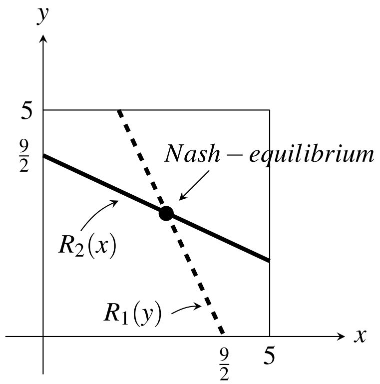

Fig. 3.3 Best responses in Case 1

Case 3. Assume capacity limits $L_1 = L_2 = 1$ and price function $p(s) = 2 - s$ and assume that the cost functions are $C_1(x) = x - \frac{x^2}{4}$ and $C_2(y) = y - \frac{y^2}{4}$. The profit of player 1 is

$$\phi_1(x, y) = x(2 - x - y) - x + \frac{x^2}{4} = x - \frac{3x^2}{4} - xy \quad (3.10)$$

with derivatives

$$\frac{\partial \phi_1}{\partial x} = 1 - \frac{6x}{4} - y$$

and so the stationary point is

$$x = \frac{2(1-y)}{3}$$

which is again feasible strategy for player 1. The best response of player 2 is similarly

$$y = \frac{2(1-x)}{3}$$

leading to a unique equilibrium again: $x^* = y^* = \frac{2}{5} = 0.4$.

Case 4. Keep the same capacity limits and price function but change the cost functions to $C_1(x) = x - x^2$ and $C_2(y) = y - y^2$. The profit of player 1 is now

$$\phi_1(x, y) = x(2 - x - y) - (x - x^2) = x - xy \tag{3.11}$$

with derivative

$$\frac{\partial \phi_1}{\partial x} = 1 - y$$

which is positive as $y < 1$ and zero for $y = 1$. Therefore ϕ_1 strictly increases in x as $y < 1$ and constant for $y = 1$. Therefore

$$R_1(y) = \begin{cases} 1 & \text{if } y < 1 \\ [0, 1] & \text{if } y = 1. \end{cases}$$

The best response of player 2 is similar:

$$R_2(x) = \begin{cases} 1 & \text{if } x < 1 \\ [0, 1] & \text{if } x = 1. \end{cases}$$

The best responses and the Nash equilibria are illustrated in Fig. 3.4.

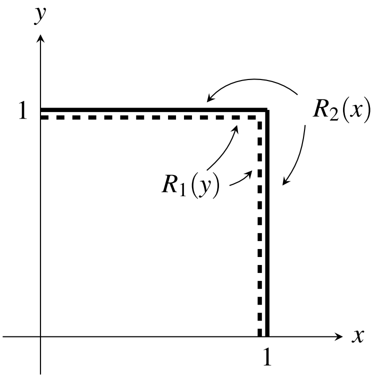

Fig. 3.4 Best responses in Case 4

Clearly there are infinitely many equilibria:

$$\{x^* = 1, \quad 0 \leqq y^* \leqq 1\} \text{ and } \{y^* = 1, \quad 0 \leqq x^* \leqq 1\}.$$

Case 5. Keep the same price function and capacity limits as in the previous case but change the cost functions to $C_1(x) = 2x - 2x^2$ and $C_2(y) = 2y - 2y^2$. The payoff function of player 1 is now

$$\phi_1(x, y) = x(2 - x - y) - 2x + 2x^2 = x^2 - xy \tag{3.12}$$

which is a convex function, so its maximum is obtained either at $x = 0$ or $x = 1$. At $x = 0$, $\phi_1(0, y) = 0$ and at $x = 1$, $\phi_1(1, y) = 1 - y$. So the best response of player 1 is

$$R_1(y) = \begin{cases} 1 & \text{if } y < 1 \\ \{0;\ 1\} & \text{if } y = 1. \end{cases}$$

Similarly,

$$R_2(x) = \begin{cases} 1 & \text{if } x < 1 \\ \{0;\ 1\} & \text{if } x = 1. \end{cases}$$

Figure 3.5 shows the best responses, and the unique equilibrium $x^* = y^* = 1$.

In all previous cases we had examples with unique or infinitely many equilibria. In the next case we will have a duopoly with three equilibria.

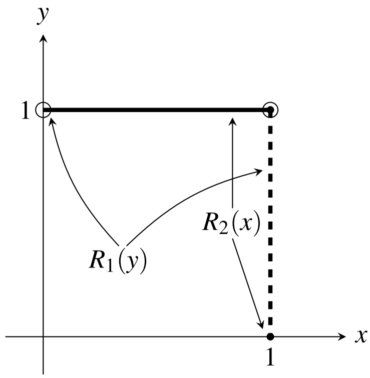

Fig. 3.5 Best responses in Case 5

Case 6. Assume $L_1 = L_2 = 1$, $p(s) = \frac{7}{6} - \frac{s}{2}$, $C_1(x) = x - \frac{x^2}{3}$, and $C_2(y) = y - \frac{y^2}{3}$. In this case

$$\phi_1(x, y) = x(\frac{7}{6} - \frac{x}{2} - \frac{y}{2}) - (x - \frac{x^2}{3}) = \frac{x}{6} - \frac{x^2}{6} - \frac{xy}{2}. \tag{3.13}$$

The stationary point is the solution of the first order condition

$$\frac{1}{6} - \frac{x}{3} - \frac{y}{2} = 0$$

that is,

$$x = \frac{1 - 3y}{2}.$$

So the best response of player 1 is the following :

$$R_1(y) = \begin{cases} \frac{1-3y}{2} & \text{if } y \leqq \frac{1}{3} \\ 0 & \text{if } y > \frac{1}{3}. \end{cases}$$

Similarly,

$$R_2(x) = \begin{cases} \frac{1-3x}{2} & \text{if } x \leqq \frac{1}{3} \\ 0 & \text{if } x > \frac{1}{3}. \end{cases}$$

These functions are shown in Fig. 3.6, and there are three points, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ and $(\frac{1}{5}, \frac{1}{5})$ which are on both best response curves, so they are the equilibria of the duopoly.

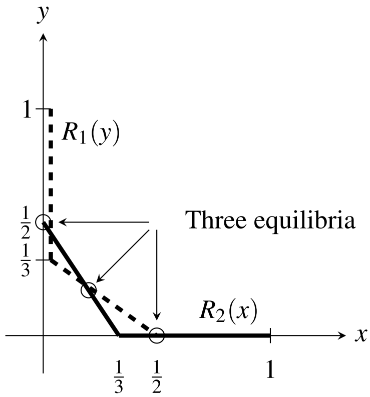

Fig. 3.6 Best responses in Case 6

Example 3.4 (Price war) There are two firms producing and selling the same kind of product. Both decide on the selling prices P_1 and P_2 which are the strategies of the players. Assume that the maximum price they can select is P. Since customers will buy the product from that firm which charges lower price, the demand function is $D = A - \bar{P}$, where $\bar{P} = \min\{P_1, P_2\}$. Assume $A > 2P$. Therefore the strategy set of the players is the interval $[0, P]$ with payoff functions

$$\phi_1(P_1, P_2) = \begin{cases} P_1(A - P_1) & \text{if } P_1 < P_2 \\ \frac{1}{2}P_1(A - P_1) & \text{if } P_1 = P_2 \\ 0 & \text{if } P_1 > P_2 \end{cases}$$

and

$$\phi_2(P_1, P_2) = \begin{cases} P_2(A - P_2) & \text{if } P_2 < P_1 \\ \frac{1}{2}P_2(A - P_2) & \text{if } P_2 = P_1 \\ 0 & \text{if } P_2 > P_1 \end{cases}$$

If $P_1 < P_2$ then all customers will buy the product on the lower price P_1, if $P_2 < P_1$ then they will buy on the lower price P_2, and if $P_1 = P_2$ then the firms will share the market. Figure 3.7. shows the payoff function of Player 1, the payoff function of Player 2 is similar.

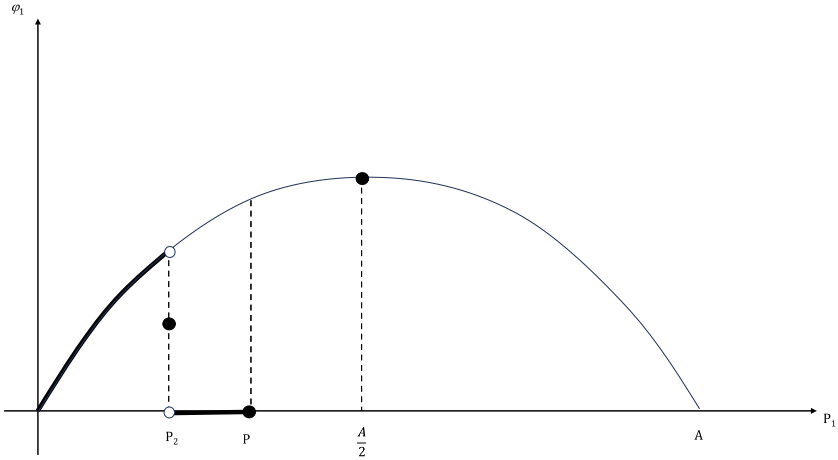

Fig. 3.7 Payoff of Player 1 in Example 3.4

Notice that the graph of function $P_1(A\text{-}P_1)$ is a concave parabola with two zeros, at 0 and A. Therefore the vertex occurs at $P_1 = \frac{A}{2}$ and the function increases in interval $\left[0,\ \frac{A}{2}\right]$ and therefore in intervals $[0,\ P]$ and $\left[P,\ \frac{A}{2}\right]$ as well. Since $P_2 \le P$, function ϕ_1 does not have a maximum in interval $[0,\ P]$ it has a supremum at $P_1 = P_2$, which is not maximum since at this point ϕ_1 jumps down. So no equilibrium exists. ▼

Example 3.5 (Timing game) Two individuals want to get a valuable object, which is valued as v_1 and v_2 by them, respectively. Both of them want to wait with giving an offer hoping that the other will give up, so he can get the object. It is assumed that waiting is costly. Price wars, isolation of a community in a war can be mentioned as particular examples. The individuals are the players, the strategies are their decisions when to quit, t_1 and t_2. This situation can be modeled as a two-person game in which the players are the two competing individuals, their strategy sets are $S_1 = S_2 = [0, \infty)$. In defining the payoff functions we have to consider three cases. If $t_1 < t_2$, then player 1 gives up first, so player 2 gets the item with v_2 benefit, however his waiting time t_1 is equivalent to a loss of t_1. Player 1 does not have any benefit, since he does not get the item, but he also has the same loss as player 2 because of the lost time period of length t_1. If $t_1 = t_2$, then both have the same loss $t_1 (= t_2)$ and each of them has a 50% chance to win the item. If $t_1 > t_2$, then we have the first case with interchanged players. So the payoff function of player 1 is the following:

$$\phi_1(t_1, t_2) = \begin{cases} -t_1 & \text{if } t_1 < t_2 \\ \frac{1}{2}v_1 - t_1 & \text{if } t_1 = t_2 \\ v_1 - t_2 & \text{if } t_1 > t_2. \end{cases} \tag{3.14}$$

The payoff function of player 2 is similar, then player 2 takes over the place of player 1. In order to find the best response of player 1, we illustrate ϕ_1 in Fig. 3.8 in three different cases, when $v_1 > t_2$, $v_1 = t_2$ and $v_1 < t_2$. ▼

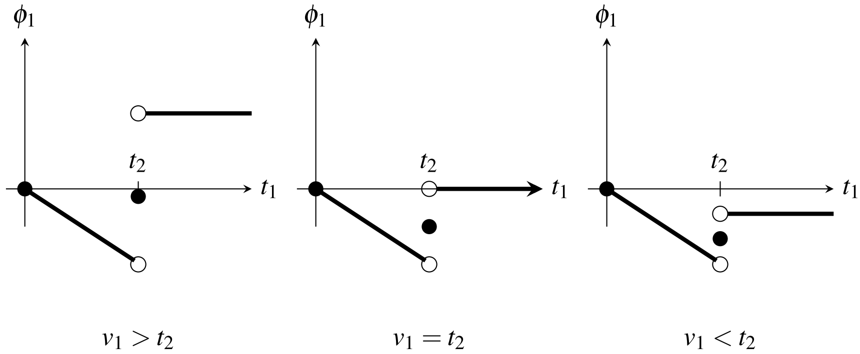

Fig. 3.8 Payoff function of player 1 in Example 3.5

In interval $[0, t_2)$ the function is the -45 degree line starting at the origin. In interval (t_2, ∞) the function is constant, which is positive if $v_1 > t_2$, zero as $v_1 = t_2$ and negative for $v_1 < t_2$. When $t_1 = t_2$, then $\frac{1}{2}v_1 - t_1 = \frac{1}{2}(-t_1 + v_1 - t_2)$, that is, the value of ϕ_1 is the midpoint between the left hand side and right hand side limits. In the first case ϕ_1 is maximal if $t_1 > t_2$, in the second case if $t_1 = 0$ or $t_1 > t_2$, and in the third case only when $t_1 = 0$. That is,

$$R_1(t_2) = \begin{cases} (t_2, \infty) & \text{if } v_1 > t_2 \\ \{0\} \cup (t_2, \infty) & \text{if } v_1 = t_2 \\ 0 & \text{if } v_1 < t_2. \end{cases}$$

The best response of player 2 is similar by interchanging the two players. Figure 3.9 illustrates $R_1(t_2)$ and Fig. 3.10 shows both $R_1(t_2)$ and $R_2(t_1)$, where $R_1(t_2)$ is the lighter area and $R_2(t_1)$ is the darker area, and it is assumed that $v_1 < v_2$.

The points which are in both best responses are the equilibria:

$$\{t_1 \geqq v_2 \text{ and } t_2 = 0\} \cup \{t_2 \geqq v_1 \text{ and } t_1 = 0\}.$$

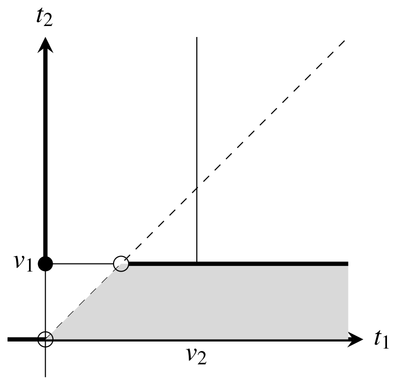

Fig. 3.9 Illustration of $R_1(t_2)$

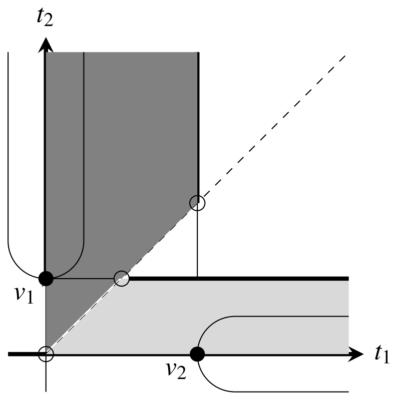

Fig. 3.10 Best responses in Example 3.5

Example 3.6 (Position game) Two manufacturers produce similar products with quality parameters x_1 and x_2. The expectation of the customers is M as the ideal quality indicator. In the competition that manufacturer wins who's quality is closer to the ideal value. So the two players are the manufacturers, their strategies are x_1 and x_2 as their quality indicators with strategy sets $S_1 = S_2 = [0, \infty)$. The payoff function of player 1 is clearly

$$\phi_1(x_1, x_2) = \begin{cases} 1 & \text{if } |x_1 - M| < |x_2 - M| \\ 0 & \text{if } |x_1 - M| = |x_2 - M| \\ -1 & \text{if } |x_1 - M| > |x_2 - M|. \end{cases} \tag{3.15}$$

The payoff of player 2 is similar by interchanging the players. In order to find the best responses we will illustrate again the payoff function of player 1, in which we consider three cases: $x_2 > M$, $x_2 = M$ and $x_2 < M$. In the first case player 1 wins if $|x_1 - M| < x_2 - M$ which can be rewritten as

$$2M - x_2 < x_1 < x_2.$$

Player 1 is the loser if $x_1 < 2M - x_2$ or $x_1 > x_2$. There is a tie if $2M - x_2 = x_1$, or $x_1 = x_2$. If $x_2 = M$, then player 1 cannot win, loses if $x_1 \neq M$ and there is a tie if $x_1 = M$. In the third case player 1 wins if $|x_1 - M| < M - x_2$ which can be rewritten as $x_2 < x_1 < 2M - x_2$. There is a tie if either $x_1 = x_2$ or $x_1 = 2M - x_2$. Otherwise player 1 is the loser. Figure 3.11 shows the payoff of player 1, from which its best response is clearly the following:

$$R_1(x_2) = \begin{cases} (2M - x_2, x_2) & \text{if } x_2 > M \\ M & \text{if } x_2 = M \\ (x_2, 2M - x_2) & \text{if } x_2 < M. \end{cases}$$

▼

The two best responses are shown in Fig. 3.12, where $R_1(x_2)$ is the darker area, and $R_2(x_1)$ is the lighter. There is a unique equilibrium: $x_1 = x_2 = M$, which means that both manufacturers have to produce according to the expectation of the market.

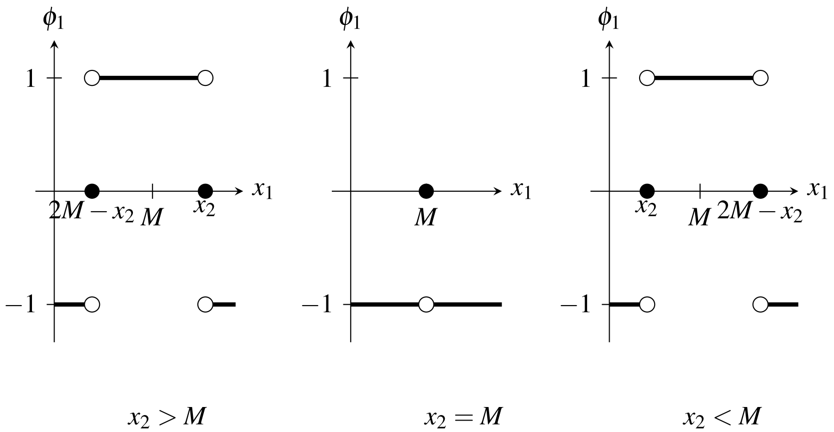

Fig. 3.11 Payoff function of player 1 in Example 3.6

Example 3.7 (Location game) Two icecream sellers compete in a beach, and want to find ideal locations for their shops. It is assumed for the sake of simplicity that the beach is the unit interval [0, 1], so the possible locations are $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$. The possible buyers are uniformly distributed along the beach, and each of them buys icecream from the shop which is closer to his location. The number of customers for each shop is therefore proportional to the length of that part of the unit interval which contains points closer to it than to its competitor. Therefore in this two-person game the icecream sellers are the players with strategy sets $S_1 = S_2 = [0, 1]$, and with payoff function for player 1

$$\phi_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2} & \text{if } x_1 < x_2 \\ \frac{1}{2} & \text{if } x_1 = x_2 \\ 1 - \frac{x_1+x_2}{2} & \text{if } x_1 > x_2 \end{cases} \tag{3.16}$$

and a similar $\phi_2(x_1, x_2)$ for player 2, where the players are interchanged. If $x_1 < x_2$, then the points of interval $[0, \frac{x_1+x_2}{2})$ are closer to x_1, if $x_1 = x_2$ then the players equally share the market, and if $x_1 > x_2$, then the points of interval $(\frac{x_1+x_2}{2}, 1]$ are closer to x_1. The lengths of these intervals in the first and third cases are $\frac{x_1+x_2}{2}$ and $1 - \frac{x_1+x_2}{2}$. Function (3.16) is illustrated in Fig. 3.13, where we distinguish between three cases: $x_2 < \frac{1}{2}$, $x_2 = \frac{1}{2}$ and $x_2 > \frac{1}{2}$. The function values are indicated at the points in the figure.

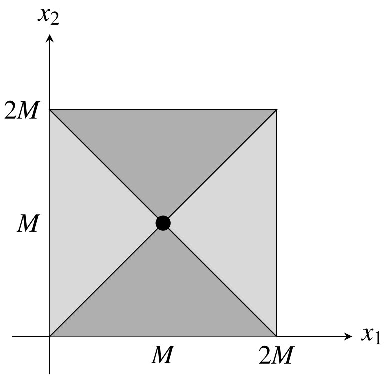

Fig. 3.12 Best responses in Example 3.6

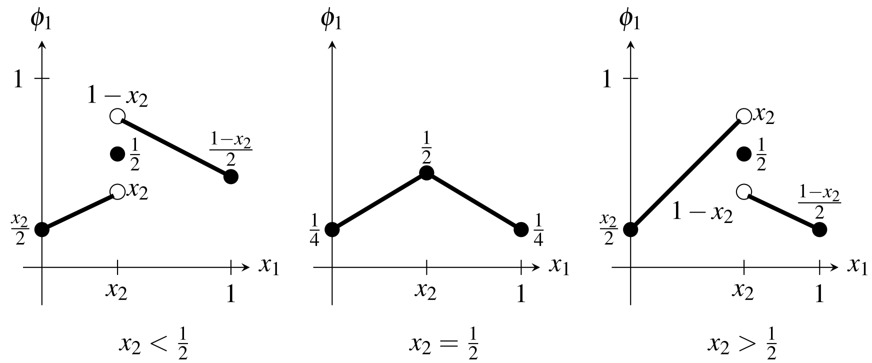

Fig. 3.13 Payoff of player 1 in Example 3.7

In the first and third cases there is no best response, the payoff function has surpremum but no maximum. So best response exists only for $x_2 = \frac{1}{2}$, and then $R_1(x_2) = \frac{1}{2}$. The same holds for the second player, so the game has a unique equilibrium, $x_1 = x_2 = \frac{1}{2}$. ▼

Example 3.8 (Market sharing) Consider two firms competing for a market of unit value. Let x_1 and x_2 denote their efforts to get as large as possible portions of the market. The firms are the players, their strategies are $x_1, x_2 \geqq 0$. So the strategy sets are $S_1 = S_2 = [0, \infty)$. The payoff of player 1 is

$$\phi_1(x_1, x_2) = \frac{x_1}{x_1 + x_2} - x_1 \tag{3.17}$$

since it can get $\frac{x_1}{x_1+x_2}$ portion of the market and player 2 will get the other, $\frac{x_2}{x_1+x_2}$ part of it. In order to determine the best response of player 1, we have to find the maximum of ϕ_1, as a function of x_2. The first order condition

$$\frac{1 \cdot (x_1 + x_2) - x_1 \cdot 1}{(x_1 + x_2)^2} - 1 = 0$$

gives the stationary point

$$x_1 = \sqrt{x_2} - x_2.$$

Since

$$\frac{\partial^2 \phi_1}{\partial x_1^2} = \frac{-x_2 \cdot 2 \cdot (x_1 + x_2)}{(x_1 + x_2)^4} < 0,$$

ϕ_1 is strictly concave in x_1 implying the uniqueness of the optimum. Since the best response has to be non-negative,

$$R_1(x_2) = \begin{cases} \sqrt{x_2} - x_2 & \text{if } x_2 \leqq 1 \\ 0 & \text{if } x_2 > 1. \end{cases}$$

The best response $R_2(x_1)$ is similar by replacing x_2 by x_1. They are illustrated in Fig. 3.14 showing that there are two equilibria: $x_1 = x_2 = 0$ and $x_1 = x_2 = \frac{1}{4}$. However the first equilibrium is only fictitious, since ϕ_1 is not defined there. So only the point $x_1 = x_2 = \frac{1}{4}$ is considered as the equilibrium of the game. ▼

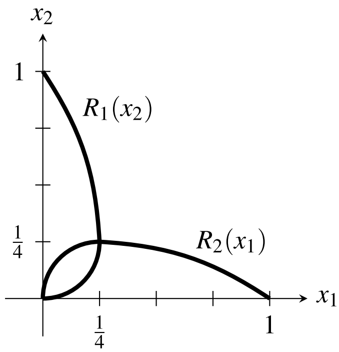

Fig. 3.14 Best responses in Example 3.8

Example 3.9 (Duel without sound) In old times an insult to a gentlemen usually resulted in a duel. The victim of the insult called for a duel, which was performed either with a sword or with a gun. In the second case the insulter and the victim selected a mutually acceptable judge, who gave identical guns to both of them with only one bullet in each. So each of them had one gun with a single bullet in it. Then they had to stand facing each other from a given distance, and at a signal of the judge they had to start walking toward each other. The decision of each participant is the time (or place) when he will shoot at the other, selected at the beginning. The first injury (or death) is the end of the duel, and if there is no injury (when none of them is hurt or killed) then the duel is a tie. There are two possibilities in conducting a duel with respect to the guns. We talk about *duel without sound* if the guns have silencers, in which case the participants do not observe any shot at them if it did not cause damage, that is, they do not observe if the other participant already used up his only bullet. If the guns have no silencers, then we talk about *duel with sound*. In this case a use of the bullet without success results in certain loss (injury or death), since the other participant knows for sure that he cannot shoot at him anymore, so he can walk next to him and shoot at him from a very small distance. In this example we examine duels without sound, the other case will be the subject of the next example.

For the sake of simplicity assume that the initial distance between the two duelists is 2 units and they walk toward each other with equal speed. So without shooting they would meet in the middle. The players are the duelists. The strategy of each player is the place when he will shoot: $x_1 \in [0, 1]$ and $x_2 \in [0, 1]$. Let $P_1(x_1)$ and $P_2(x_2)$ be the hitting probabilities of the players when player 1 shoots at location x_1 and player 2 shoots at x_2. Let the payoff of the winner be 1, that of the loser -1, and 0 in the case of a tie for both. The payoff function of player 1 is the following:

$$\phi_1(x_1, x_2) = \begin{cases} P_1(x_1) \cdot 1 - \big(1 - P_1(x_1)\big) P_2(x_2) & \text{if } x_1 < x_2 \\ P_1(x_1) - P_2(x_2) & \text{if } x_1 = x_2 \\ P_2(x_2) \cdot (-1) + \big(1 - P_2(x_2)\big) P_1(x_1) & \text{if } x_1 > x_2. \end{cases} \tag{3.18}$$

If $x_1 < x_2$, then player 1 is the winner if he injures player 2 and he is the loser if player 2 can injure him later. Otherwise the result is a tie. The third case is similar by interchanging the players and winning by losing. If $x_1 = x_2$, then they shoot together, so player 1 wins if only his shot finds its target, player 2 wins if only he causes damage and the game a tie if either both hurt the other or none of them causes damage to the other. In order to find best responses and possible equilibrium consider the special case of $P_1(x_1) = x_1$ and $P_2(x_2) = x_2$ meaning that the hitting probabilities are equal to zero at the starting locations, equal to 1 in the middle (when the distance between them is zero), and both functions are linear. In this case

$$\phi_1(x_1, x_2) = \begin{cases} x_1 - (1 - x_1)x_2 = x_1 + x_1x_2 - x_2 & \text{if } x_1 < x_2 \\ 0 & \text{if } x_1 = x_2 \\ -x_2 + (1 - x_2)x_1 = x_1 - x_2 - x_1x_2 & \text{if } x_1 > x_2 \end{cases}$$

which is illustrated in Fig. 3.15. There is no maximum if $x_2^2 > 1 - 2x_2$, and if $x_2^2 \leqq 1 - 2x_2$, then ϕ_1 has its maximum at $x_1 = 1$. So best response of player 1 exists only if

$$x_2^2 + 2x_2 - 1 \leqq 0$$

that is, when $x_2 \leqq \sqrt{2} - 1 \approx 0.4142$. The same holds for player 2 as well. The best responses are shown in Fig. 3.16 from which it is clear that no equilibrium exists. ▼

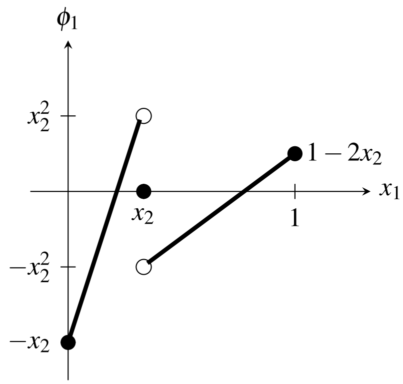

Fig. 3.15 Payoff ϕ_1 in Example 3.9

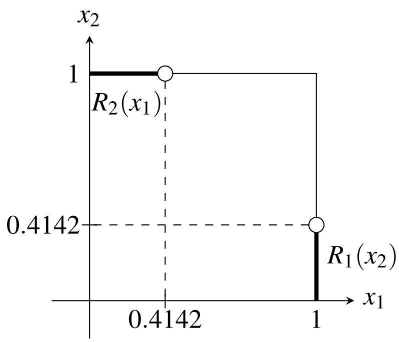

Fig. 3.16 Best responses in Example 3.9

Example 3.10 (Duel with sound) The major difference between this case and the previous example is the fact that shooting without hit results in certain loss. Keeping the same notation as before, the payoff of player 1 is now the following:

$$\phi_1(x_1, x_2) = \begin{cases} P_1(x_1) \cdot 1 - \big(1 - P_1(x_1)\big) & \text{if } x_1 < x_2 \\ P_1(x_1) - P_2(x_2) & \text{if } x_1 = x_2 \\ -P_2(x_2) + \big(1 - P_2(x_2)\big) & \text{if } x_1 > x_2. \end{cases} \tag{3.19}$$

In the special case of $P_1(x_1) = x_1$ and $P_2(x_2) = x_2$ we have

$$\phi_1(x_1, x_2) = \begin{cases} 2x_1 - 1 & \text{if } x_1 < x_2 \\ 0 & \text{if } x_1 = x_2 \\ 1 - 2x_2 & \text{if } x_1 > x_2, \end{cases}$$

which is illustrated in Fig. 3.17 in three cases: $x_2 < \frac{1}{2}$, $x_2 = \frac{1}{2}$ and $x_2 > \frac{1}{2}$.

In the first case all $x_1 \in (x_2, 1]$ give maximum, in the second case all $x_1 \in [\frac{1}{2}, 1]$ provide maximum, but in the third case no maximum exists. Therefore the best response of player 1 can be given as

$$R_1(x_2) = \begin{cases} (x_2, 1] & \text{if } x_2 < \frac{1}{2} \\ [\frac{1}{2}, 1] & \text{if } x_2 = \frac{1}{2} \\ \phi & \text{if } x_2 > \frac{1}{2}. \end{cases}$$

The best response of player 2 is analogous. Figure 3.18 shows the best responses, in which $R_1(x_2)$ is the darker and $R_2(x_1)$ the lighter area. The only equilibrium is $x_1 = x_2 = \frac{1}{2}$, since it is the only point belonging to both $R_1(x_2)$ and $R_2(x_1)$. ▼

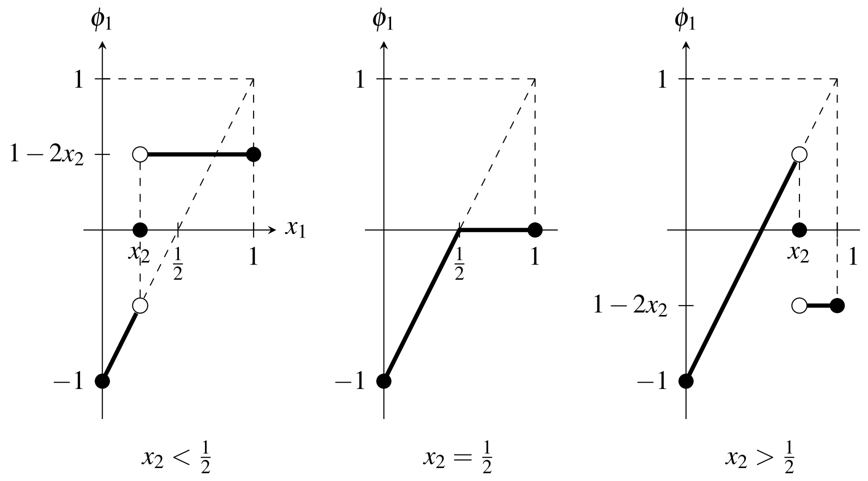

Fig. 3.17 Payoff ϕ_1 in Example 3.10

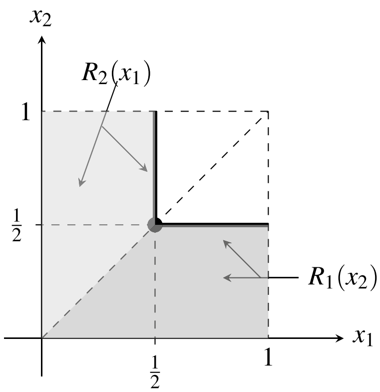

Fig. 3.18 Best responses in Example 3.10

Example 3.11 (Spying game) Assume that a spy goes to an enemy territory to gather important intelligence information. His decision variable x_1 is the level of his effort. He clearly faces the counter intelligence agency, which has its decision variable x_2 representing its effort to catch spies. So the strategy set of both of them is the interval $S_1 = S_2 = [0, \infty)$. For the spy there are two indicators of his activity. First let $P(x_1, x_2)$ denote the probability that his activity will be discovered and therefore he will be arrested. This function is increasing in both variables: larger value of x_1 exposes easier his activity, so he can be caught which is the case with higher value of x_2 as well. If he gets arrested, his organization loses his value, which is denoted by U. If he is not arrested, then let $V(x_1)$ denote the value of the information he obtains and forwards to his organization. The payoff of the spy (player 1) is given as the expectation

$$\phi_1(x_1, x_2) = P(x_1, x_2)(-U) + \big(1 - P(x_1, x_2)\big)V(x_1). \tag{3.20}$$

The gain of the spy is the loss of counter intelligence (player 2), so its payoff is $\phi_2(x_1, x_2) = -\phi_1(x_1, x_2)$. We have now a zero-sum game. In order to find best responses and possible equilibrium consider the special case of

$$U = 4,\ \ P(x_1, x_2) = \frac{1}{8}(x_1 + x_2),\ \ V(x_1) = x_1.$$

To avoid too large probability values assume that the maximum efforts for the players are x_1^* and x_2^* such that $x_1^* + x_2^* \leqq 8$. In this case

$$\phi_1(x_1, x_2) = \frac{1}{8}(x_1 + x_2)(-4) + \big(1 - \frac{1}{8}(x_1 + x_2)\big)x_1 = -\frac{1}{8}x_1^2 - \frac{1}{8}x_1x_2 + \frac{x_1}{2} - \frac{x_2}{2}$$

which is a concave parabola in x_1. Since

$$\frac{\partial \phi_1}{\partial x_1} = -\frac{1}{4}x_1 - \frac{1}{8}x_2 + \frac{1}{2},$$

the stationary point is $x_1 = \frac{4-x_2}{2}$, so the best response of the spy is the following:

$$R_1(x_2) = \begin{cases} 0 & \text{if } x_2 > 4 \\ \frac{4-x_2}{2} & \text{otherwise.} \end{cases}$$

Since

$$\phi_2(x_1, x_2) = -\phi_1(x_1, x_2) = \frac{1}{8}x_1^2 + \frac{1}{8}x_1x_2 - \frac{x_1}{2} + \frac{x_2}{2}$$

strictly increases in x_2, the best response of player 2 is $R_2(x_1) = x_2^*$. ▼

The best responses are illustrated in Fig. 3.19, where we assume that $x_1^* > 2$. If $x_2^* > 4$, then the equilibrium is $x_1 = 0$, $x_2 = x_2^*$ meaning that the best choice of the spy is to go home and do nothing if the counter espionage agency is very strong. Otherwise the equilibrium is $x_1 = \frac{4-x_2^*}{2}$, $x_2 = x_2^*$ meaning that x_1 decreases with increasing strength of the counter espionage agency.

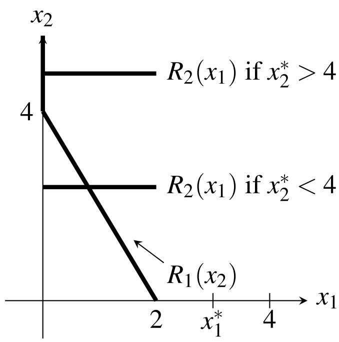

Fig. 3.19 Best responses in Example 3.11

Example 3.12 (First price auction) Two agents are bidding for a valuable item, which is sold in an auction. They have to send their bids to the company who conducts the auction before a given date and the agent with the higher bid can purchase the item. Assume that they have subjective valuations, v_1 and v_2 of the item. The agents are the players and their strategies are their bids, x_1 and x_2. The payoff of player 1 is therefore

$$\phi_1(x_1, x_2) = \begin{cases} v_1 - x_1 & \text{if } x_1 \geqq x_2 \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Here we assume that $v_1 > v_2$ and in the case of a tie the agent with higher valuation gets the item. Similarly

$$\phi_2(x_1, x_2) = \begin{cases} v_2 - x_2 & \text{if } x_2 > x_1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.22}$$

Notice that this game is not symmetric with respect to the players. Payoff ϕ_1 is shown in Fig. 3.20, where three cases are considered. The best response of player 1 is given as follows:

$$R_1(x_2) = \begin{cases} x_2 & \text{if } x_2 < v_1 \\ [0, x_2] & \text{if } x_2 = v_1 \\ [0, x_2) & \text{if } x_2 > v_1. \end{cases}$$

Similarly, payoff ϕ_2 is given in Fig. 3.21, from which the best response of player 2 can be seen as

$$R_2(x_1) = \begin{cases} \phi & \text{if } x_1 < v_2 \\ [0, x_1] & \text{if } x_1 \geq v_2. \end{cases}$$

The best responses are illustrated in Fig. 3.22, from which it is clear that the equilibria set is $\{(x_1, x_2) | v_2 \leqq x_1 = x_2 \leqq v_1\}$. ▼

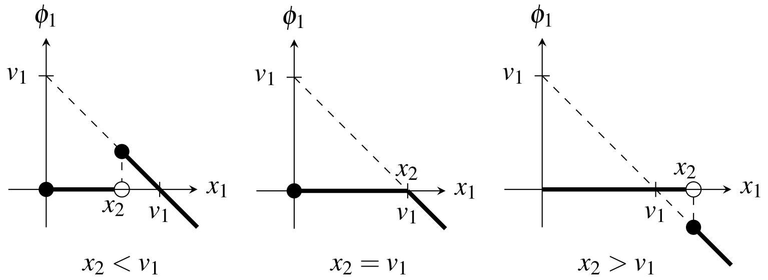

Fig. 3.20 Payoff ϕ_1 in Example 3.12

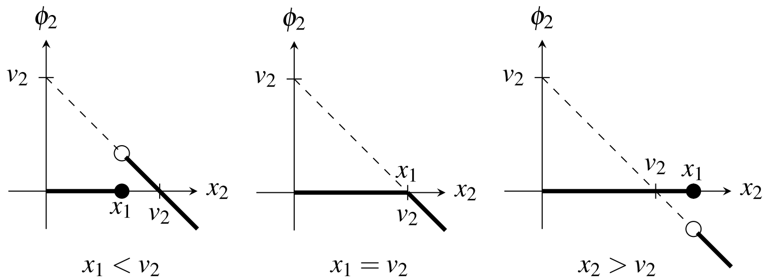

Fig. 3.21 Payoff ϕ_2 in Example 3.12

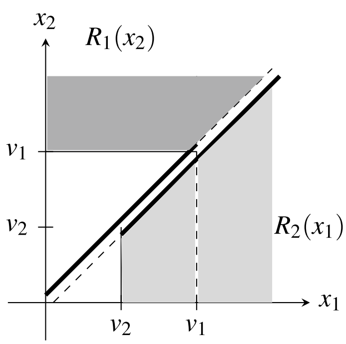

Fig. 3.22 Best responses in Example 3.12

Example 3.13 (Second price auction) In the case of a second price auction the mechanism is the same as in a first price auction with the only difference that higher bidder has to pay the bid of its competitor, that is, the winner does not need to pay his offered price, only the lower bid of the other bidder. So the payoffs are as follows:

$$\phi_1(x_1, x_2) = \begin{cases} v_1 - x_2 & \text{if } x_1 \geq x_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.23}$$

and

$$\phi_2(x_1, x_2) = \begin{cases} v_2 - x_1 & \text{if } x_2 > x_1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.24}$$

Figures 3.23 and 3.24 show these payoffs for three cases similarly to the previous example.

So the best responses are

$$R_1(x_2) = \begin{cases} [x_2, \infty) & \text{if } x_2 < v_1 \\ [0, \infty) & \text{if } x_2 = v_1 \\ [0, x_2) & \text{if } x_2 > v_1 \end{cases}$$

and

$$R_2(x_1) = \begin{cases} (x_1, \infty) & \text{if } x_1 < v_2 \\ [0, \infty) & \text{if } x_1 = v_2 \\ [0, x_1] & \text{if } x_1 > v_2. \end{cases}$$

They are illustrated in Fig. 3.25, from which we see that there are again infinitely many equilibria and they are the points of the darkest region. ▼

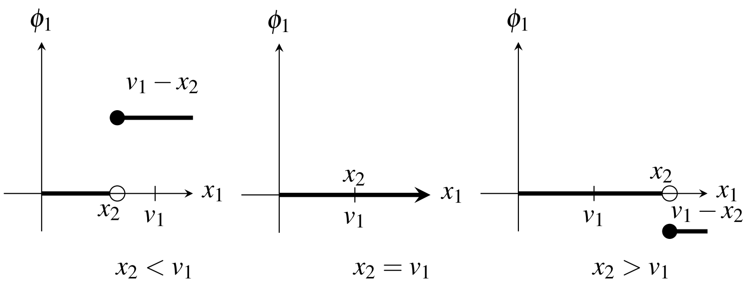

Fig. 3.23 Payoff ϕ_1 in Example 3.13

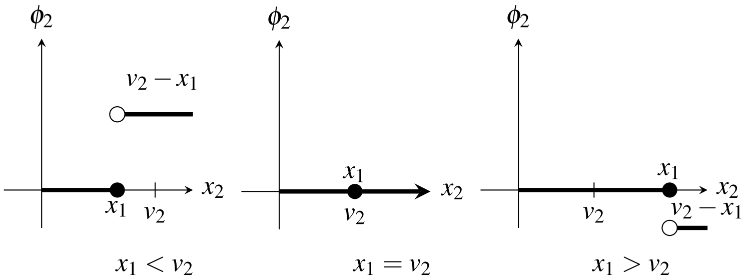

Fig. 3.24 Payoff ϕ_2 in Example 3.13

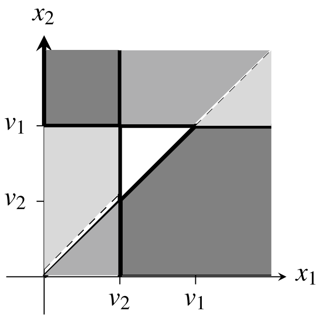

Fig. 3.25 Best responses in Example 3.13

3.2 Examples of N-Person Continuous Games

In this section we will introduce the general N-person extensions of some of the games which were examined in the previous section for the special two-person case.

Example 3.14 (Sharing a pie) Assume now that a group of N children is promised to get a pie to be shared among them. Each of them was asked to present his demand of the pie by telling how big part of the pie he wants. These demands are presented independently when none of the children knows the demands of the others before presenting his request. If the sum of the demands is larger than the entire pie, then none of the children gets anything, and if the total request is feasible, then each child receives the requested amount. The children are the players, their requested amounts $x_k \in [0, 1]$ are the strategies, and the payoff of player k is given as

$$\phi_k(x_1, \ldots, x_N) = \begin{cases} x_k & \text{if } \sum_{i=1}^N x_i \leqq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

With given $x_1, \ldots, x_{k-1}, x_{k+1} \ldots, x_N$ the best choice of player k is to ask for the remaining amount $1 - \sum_{i \neq k} x_i$, which can be proved as follows. If player k selects less than this amount, than he can increase his demand up to this value and increase payoff. If he asks more than this amount, then he will get nothing, so decreasing demand until this value clearly would increase his payoff. So the game has infinitely many equilibria which form the set

$$\Big\{(x_1, \ldots, x_N)|0 \leqq x_k \leqq 1 \text{ for all } k, \sum_{k=1}^{N} x_k = 1\Big\}.$$

▼

Example 3.15 (Cournot oligopoly) Consider now N firms producing the same product or offer the same service to a homogeneous market. Let x_k denote the output (production level) of firm k, then the total industry output is $s = \sum_{k=1}^{N} x_k$. If L_k is the capacity limit of firm k, then $0 \leqq x_k \leqq L_k$. The firms are the players, their strategies are the produced amounts, so the strategy set of player k is $S_k = [0, L_k]$. In addition let $C_k(x_k)$ denote the cost function of firm k, and $p(s)$ the inverse demand (or price) function of the market. The payoff function of firm k is its profit:

$$\phi_k(x_1, \ldots, x_N) = x_k p\Big(\sum_{i=1}^{N} x_i\Big) - C_k(x_k). \tag{3.26}$$

In the earlier Example 3.3 we saw some two-person special cases of this game. In this example we will show that under certain conditions there is always a unique equilibrium. The proof will be constructive giving a computer method to find the equilibrium.

Assume that functions $p(s)$ and $C_k(x_k)(k = 1, 2, \ldots, N)$ are twice continuously differentiable, furthermore

(a) $p'(s) < 0$ for all $s \in \Big[0, \sum_{i=1}^{N} L_i\Big]$;

(b) $x_k p''(s) + p'(s) \leqq 0$ and

(c) $p'(s) - C_k''(x_k) < 0$ for all $s \in \Big[0, \sum_{i=1}^{N} L_i\Big]$ and $x_k \in \Big[0, L_k\Big]$.

Notice first that

$$\frac{\partial \phi_k}{\partial x_k} = p\Big(\sum_{i=1}^{N} x_i\Big) + x_k p'\Big(\sum_{i=1}^{N} x_i\Big) - C_k'(x_k) \tag{3.27}$$

and

$$\frac{\partial^2 \phi_k}{\partial x_k^2} = 2p'\Big(\sum_{i=1}^{N} x_i\Big) + x_k p''\Big(\sum_{i=1}^{N} x_i\Big) - C_k''(x_k).$$

Assumptions (b) and (c) imply that this second order derivative is negative, so ϕ_k is strictly concave in x_k with any given values of $x_1, \ldots, x_{k-1}, x_{k+1} \ldots, x_N$. So the best response of firm k is unique and is given as

$$R_k(s_k) = \begin{cases} 0 & \text{if } p(s_k) - C_k'(0) \leqq 0 \\ L_k & \text{if } p(L_k + s_k) + L_k p'(L_k + s_k) - C_k'(L_k) \geqq 0 \\ z_k^* & \text{otherwise} \end{cases} \tag{3.28}$$

where $s_k = \sum_{i \neq k} x_i$ and z_k^* is the unique solution of equation

$$p(z_k + s_k) + z_k p'(z_k + s_k) - C_k'(z_k) = 0 \tag{3.29}$$

in interval $[0, L_k]$. Let $g_k(z_k)$ denote the left hand side of Eq. (3.29). It is strictly decreasing in z_k because of $\frac{\partial^2 \phi_k}{\partial x_k^2} < 0$, $g_k(0) = p(s_k) - C_k'(0) > 0$ and $g_k(L_k) = p(L_k + s_k) + L_k p'(L_k + s_k) - C_k'(L_k) < 0$ in the third case of (3.28). So Eq. (3.29) has a unique solution. Since functions p, p', C_k' are continuous, $R_k(s_k)$ is also continuous in s_k. The implicit function theorem implies that z_k is a differentiable function of s_k in the third case of (3.28), so (3.29) can be rewritten as

$$p\big(z_k(s_k) + s_k\big) + z_k(s_k) p'\big(z_k(s_k) + s_k\big) - C_k'\big(z_k(s_k)\big) = 0.$$

Implicitly differentiating this equation with respect to s_k we have

$$p' \cdot (z_k' + 1) + z_k' \cdot p' + z_k \cdot p'' \cdot (z_k' + 1) - C_k'' \cdot z_k' = 0$$

implying that

$$z_k' = -\frac{p' + z_k p''}{2p' + z_k p'' - C_k''} \tag{3.30}$$

which is nonpositive, since

$$p' + z_k p'' \leqq 0 \text{ and } 2p' + z_k p'' - C_k'' = (p' + z_k p'') + (p' - C_k'') < 0.$$

So $R_k(s_k)$ has three segments, 0 and L_k, and a strictly decreasing segment in between. From (3.30) we also conclude that

$$-1 < R_k'(s_k) \leqq 0 \tag{3.31}$$

for all s_k.

We can also rewrite the best response of firm k as a function of the industry output s:

$$\overline{R}_k(s) = \begin{cases} 0 & \text{if } p(s) - C_k^{'}(0) \leqq 0 \\ L_k & \text{if } p(s) + L_k p'(s) - C_k^{'}(L_k) \geqq 0 \\ \overline{z}_k^* & \text{otherwise} \end{cases} \tag{3.32}$$

where $\overline{z}_k^*$ is the unique solution of equation

$$p(s) + z_k p'(s) - C_k^{'}(z_k) = 0 \tag{3.33}$$

in interval $[0, L_k]$. Let now $\overline{g}_k(z_k)$ denote the left hand side of Eq. (3.33). Clearly

$$\overline{g}_k^{'}(z_k) = p'(s) - C_k^{''}(z_k) < 0,$$

so $\overline{g}_k(z_k)$ strictly decreases in z_k, furthermore

$$\overline{g}_k(0) = p(s) - C_k^{'}(0) > 0 \text{ and } \overline{g}_k(L_k) = p(s) + L_k p'(s) - C_k^{'}(L_k) < 0$$

in the third case of (3.32). Therefore there is a unique solution which is a differentiable function of s in this case. So the $z_k(s)$ solution satisfies equation

$$p(s) + z_k(s)p'(s) - C_k^{'}\big(z_k(s)\big) = 0.$$

Implicitly differentiating this equation with respect to s we have

$$p' + z_k' p' + z_k \cdot p^{''} - C_k^{''} \cdot z_k' = 0$$

implying that

$$z_k' = -\frac{p' + z_k p^{''}}{p' - C_k{''}} \leqq 0. \tag{3.34}$$

Since in the first two cases of (3.32) the best response is constant we conclude that $\overline{R}_k(s)$ is a continuous nonincreasing function of s. Consider finally equation

$$\sum_{k=1}^{N} R_k(s) - s = 0. \tag{3.35}$$

The left hand side, which is now denoted by $g(s)$, is strictly decreasing in s,

$$g(0) = \sum_{k=1}^{N} R_k(0) \geqq 0 \text{ and } g\Big(\sum_{i=1}^{N} L_i\Big) = \sum_{k=1}^{N} \Big(R_k\Big(\sum_{i=1}^{N} L_i\Big) - L_k\Big) \leqq 0,$$

so there is a unique solution s^* of Eq. (3.35), and then the equilibrium output levels are obtained as $x_k^* = \overline{R}_k(s^*)$. Since $g(s)$ is strictly decreasing, standard methods can be used to solve Eq. (3.35). See for example, Szidarovszky and Yakovitz (1978).

In the case of duopoly the existence of a unique equilibrium follows immediately from the contraction mapping theorem. Notice that the equilibrium is the solution of the following equations:

$$\begin{aligned} x &= R_1(y) \\ y &= R_2(x) \end{aligned} \tag{3.36}$$

where x and y denote the output levels of the firms. Since the feasible sets of x and y are compact and both R_1' and R_2' are piecewise continuous, there is a minimum of both derivatives r_{1min}' and r_{2min}', so from (3.31),

$$-1 < r_{kmin}' \leq R_k' \leq 0.$$

Therefore (3.36) is a fixed point problem of a contraction mapping implying the existence of a unique solution. ▼

Example 3.16 (Timing game) Assume now that N agents want to get a valuable item which is valued as $v_1 > v_2 > \cdots > v_N$ by them. As negotiations are in progress, each of them wants to wait until all competitors give up negotiating and walk away, so he can get the item. Waiting any time period t results in a loss of t units to the agents. The players are the agents, the strategy of player k is $t_k \geqq 0$, and its payoff function is

$$\phi_k(t_1, \dots, t_N) = \begin{cases} v_k - t_k & \text{if } t_k = \max\{t_1, \dots, t_N\} \\ -t_k & \text{otherwise} \end{cases} \tag{3.37}$$

where we assume that the t_k values are different. ▼

Example 3.17 (Position game) Consider now N manufacturers producing similar products with quality parameters $x_1, x_2, \dots, x_N$. The expectation of the market is M as the ideal quality indicator. If the manufacturers are the players and their strategies are $x_1, \dots, x_N$, then the strategy set of player k in $S_k = [0, \infty)$. We assume that the value of the market is 1 and the firms have the market shares with respect to their qualities meaning that the firm who's quality parameter is closest to M gets the largest share, and so on, the firm with quality parameter farthest from M gets the smallest market share. So we may assume that the payoff function of manufacturer k is

$$\phi_k(x_1, \ldots, x_N) = \frac{|x_k - M|^{-1}}{\sum_{i=1}^N |x_i - M|^{-1}}. \tag{3.38}$$

Since this payoff depends on only the quantities $|x_k - M|^{-1}$, we can consider the values $z_k = |x_k - M|^{-1}$ as strategies, $z_k > 0$ with the transformed payoff functions

$$\phi_k(z_1, \ldots, z_N) = \frac{z_k}{\sum_{i=1}^N z_i}. \tag{3.39}$$

Notice that

$$\frac{\partial \phi_k}{\partial z_k} = \frac{\sum_{i=1}^N z_i - z_k}{\left(\sum_{i=1}^N z_i\right)^2} = \frac{\sum_{i \neq k} z_i}{\left(\sum_{i=1}^N z_i\right)^2} > 0,$$

so ϕ_k strictly increases in z_k, so there is no finite best response, only $z_k \to \infty$, in which case $x_k = M$ for all manufacturers. ▼

Example 3.18 (Market sharing) Assume that N agents compete for a market of unit value. Let x_k denote the effort of agent k to get as large as possible portion of the market. The agents are the players and their efforts are their strategies. So the strategy set of player k is $S_k = [0, \infty)$. The payoff of player k is similar to (3.17):

$$\phi_k(x_1, \ldots, x_N) = \frac{x_k}{\sum_{i=1}^N x_i} - x_k \tag{3.40}$$

where the first term shows the proportion of the market that player k will get. Notice that

$$\frac{\partial \phi_k}{\partial x_k} = \frac{1 \cdot \sum_{i=1}^N x_i - x_k \cdot 1}{\left(\sum_{i=1}^N x_i\right)^2} - 1 = \frac{s_k}{(x_k + s_k)^2} - 1$$

with $s_k = \sum_{i \neq k} x_i$, furthermore

$$\frac{\partial^2 \phi_k}{\partial x_k^2} = \frac{-2s_k(x_k + s_k)}{(x_k + s_k)^4} = \frac{-2s_k}{(x_k + s_k)^3} < 0$$

showing that with given s_k, ϕ_k is strictly concave in x_k. Therefore there is a unique best response of player k. The stationary point is the solution of equation

$$\frac{s_k}{(x_k + s_k)^2} - 1 = 0$$

implying that

$$x_k = \sqrt{s_k} - s_k.$$

Therefore the best response function of agent k is as follows:

$$R_k(s_k) = \begin{cases} \sqrt{s_k} - s_k & \text{if } s_k \leqq 1 \\ 0 & \text{if } s_k > 1. \end{cases}$$

The shape of this function is the same as shown in Fig. 3.14. The interior equilibrium is symmetric, $x_1 = \cdots = x_N = x^*$ such that

$$x^* = \sqrt{(N-1)x^*} - (N-1)x^*$$

implying that

$$x^* = \frac{N-1}{N^2}.$$

In this case $s_k = (N-1)x^* = \frac{(N-1)^2}{N^2}$ which is always less than one. ▼

Example 3.19 (First price auction) Assume that N agencies are bidding for a valuable item, and the bids are sent in before a given date. The agents have no information about the bids of the others. The agents can be considered as the players, their bids are their strategies, $x_k \in [0, \infty)$, so the strategy set of each player is $S_k = [0, \infty)$. The agent with the highest bid can purchase the item and in the case of more than one highest bidder the one with the highest valuation can get the item. Assume that the valuations are $v_1 > v_2 > \cdots > v_N$ which always can be guaranteed by renumbering the agents. The payoff of agent k is therefore

$$\phi_k(x_1, \ldots, x_N) = \begin{cases} v_k - x_k & \text{if } x_k = \max\{x_1, \ldots, x_N\} \text{ and either maximum is} \\ & \qquad \text{unique or } k = \min\{l | x_k = x_l\} \\ 0 & \text{otherwise.} \end{cases}$$

In Example 3.11 the two-person version of this game was discussed and it was shown that at any equilibrium the agents have to give identical bids and since player 1 had the higher valuation, he is always the winner. We can easily show that this is the case in the general N-person case as well, that is, always player 1 is the winner at any equilibrium. In order to prove this fact, assume that another player, $i \neq 1$, is the winner. Then $x_i > x_1$. If $x_i > v_2$, then $\phi_i = v_i - x_i \leqq v_2 - x_i < 0$, so player i could increase his payoff to zero by decreasing his bid x_i, so other player could win the item. If $x_i \leqq v_2$, then player 1 could increase his zero payoff to $v_1 - x_i$ by increasing his bid to x_i. ▼

Example 3.20 (Second price auction) The auction proceeds in the same way than in the previous example with the only difference that the winner pays only the second highest bid, not his own bid. In this case the payoff of player k becomes

$$\phi_k(x_1, \ldots, x_N) = \begin{cases} v_k - x_i & \text{if } x_k = \max\{x_1, \ldots, x_N\}, \text{ and either maximum is} \\ & \qquad \text{unique or } k = \min\{l | x_k = x_l\} \\ 0 & \text{otherwise.} \end{cases}$$

where $x_i = \max\{x_l | l \neq k\}$.

We have seen in Example 3.12 that even in the two-person case the set of equilibria is a complicated set, which is the same in the general case. Therefore we leave this issue to the interested readers. ▼

Chapter 4
Relation to Other Mathematical Problems

The equilibrium problem of N-person games have strong relation to other important mathematical problem areas such as optimization, and fixed point problems. In this chapter we will discuss this issue.

4.1 Nonlinear Optimization

Consider a general optimization problem

$$\begin{aligned} &\text{maximize } f(\underline{x}) \\ &\text{subject to } \underline{x} \in X \\ &\underline{g}(\underline{x}) \geqq \underline{0} \end{aligned} \tag{4.1}$$

where $\underline{x} \in \mathbb{R}^n$ is the decision vector, $X \subseteq \mathbb{R}^n$ is any set (which can be even discrete) and $\underline{g}(\underline{x}) \in \mathbb{R}^m$ for all $\underline{x} \in X$. There is no restriction about the objective function

$$f : X \mapsto \mathbb{R}.$$

The Lagrangean of this problem is defined as

$$L(\underline{x}, \underline{u}) = f(\underline{x}) + \underline{u}^T \underline{g}(\underline{x})$$

for all $\underline{x} \in X$ and $\underline{0} \leqq \underline{u} \in \mathbb{R}^m$. We can now define a two-person, zero-sum game with strategy sets $S_1 = X$, $S_2 = \mathbb{R}^m_+ \left(= \{\underline{u} | \underline{0} \leqq \underline{u} \in \mathbb{R}^m\}\right)$ and payoff functions $\phi_1 = L$ and $\phi_2 = -L$.

Theorem 4.1 *If* $(\underline{x}^*, \underline{u}^*)$ *is an equilibrium of the two-person game* $\{2; S_1, S_2; \phi_1, \phi_2\}$, *then* $\underline{x}^*$ *is an optimal solution of problem (4.1)*

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_4

Proof If $(\underline{x}^*, \underline{u}^*)$ is an equilibrium, then for all strategies $\underline{x}$ and $\underline{u}$,

$$L(\underline{x}^*, \underline{u}^*) \geqq L(\underline{x}, \underline{u}^*)$$

and

$$-L(\underline{x}^*, \underline{u}^*) \geqq -L(\underline{x}^*, \underline{u}).$$

These relations can be rewritten as

$$f(\underline{x}^*) + \underline{u}^{*T} \underline{g}(\underline{x}^*) \geqq f(\underline{x}) + \underline{u}^{*T} \underline{g}(\underline{x}) \tag{4.2}$$

and

$$-\left[f(\underline{x}^*) + \underline{u}^{*T} \underline{g}(\underline{x}^*)\right] \geqq -\left[f(\underline{x}^*) + \underline{u}^{T} \underline{g}(\underline{x}^*)\right]. \tag{4.3}$$

From (4.3) with the selection of $\underline{u} = \underline{0}$ we have

$$\underline{u}^{*T} \underline{g}(\underline{x}^*) \leqq 0. \tag{4.4}$$

Next we show that $\underline{g}(\underline{x}^*) \geqq \underline{0}$. Assume that for some i, $g_i(\underline{x}^*) < 0$. Then by selecting sufficiently large value of u_i, (4.3) would be violated. Since $\underline{u}^* \geqq \underline{0}$, clearly $\underline{u}^{*T} \underline{g}(\underline{x}^*) \geqq 0$ and combining this relation with (4.4) we can conclude that

$$\underline{u}^{*T} \underline{g}(\underline{x}^*) = 0. \tag{4.5}$$

And finally we can show the optimality of $\underline{x}^*$. Since we have shown that $\underline{g}(\underline{x}^*) \geqq \underline{0}$, $\underline{x}^*$ is feasible for (4.1). Furthermore from (4.2),

$$f(\underline{x}^*) = \underline{f}(\underline{x}^*) + \underline{u}^{*T} \underline{g}(\underline{x}^*) \geqq f(\underline{x}) + \underline{u}^* \underline{g}(\underline{x}) \geqq f(\underline{x})$$

for all feasible $\underline{x}$, since $\underline{u}^*$ and $\underline{g}(\underline{x})$ are nonnegative in every component.

■

So any optimization problem can be solved by solving for the equilibria of two-person zero-sum games.

4.2 Fixed Point Problems

(A) Consider an N-person game, $\Gamma = \{N; S_1, \ldots, S_N; \phi_1, \ldots, \phi_N\}$ and let $x_k \in S_k$ denote the strategy of player k. Then vector $\underline{x} = (x_1, \ldots, x_N)$ is called a simultaneous strategy vector. Introduce the notation $\underline{x}_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N)$ for the simultaneous strategy vector of all players except player k. The best response set of player k is denoted by $R_k(\underline{x}_{-k})$, where $\underline{x}_{-k}$ does not contain the strategy of player k, however we can consider R_k as the set depending on the entire vector $\underline{x}$ being the same for all x_k with fixed $\underline{x}_{-k}$. So we can use the notation $R_k(\underline{x})$. An equilibrium of the game is a simultaneous strategy vector $\underline{x}^* = (x_1^*, \ldots, x_N^*)$ such that for all players, $x_k^* \in R_k(\underline{x}^*)$. Introduce the set valued mapping $\underline{R}(\underline{x}) = (R_1(\underline{x}), \ldots, R_N(\underline{x}))$.

Theorem 4.2 *A simultaneous strategy vector $\underline{x}^*$ is an equilibrium if and only if*

$$\underline{x}^* \in \underline{R}(\underline{x}^*) \tag{4.6}$$

that is, $\underline{x}^*$ is a fixed point of mapping $\underline{R}$.

(B) For all simultaneous strategy vectors $\underline{x}$ and $\underline{y}$ let

$$\Phi(\underline{x}, \underline{y}) = \sum_{k=1}^{N} \phi_k(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_N). \tag{4.7}$$

This function satisfies the following property:

Lemma 1 *$\underline{x}^*$ is an equilibrium of the N-person game if and only if for all simultaneous strategy vectors $\underline{y}$,*

$$\Phi(\underline{x}^*, \underline{x}^*) \geqq \Phi(\underline{x}^*, \underline{y}). \tag{4.8}$$

Proof Assume first that $\underline{x}^*$ is an equilibrium, then for all k and $y_k \in S_k$,

$$\phi_k(x_1^*, \ldots, x_N^*) \geqq \phi_k(x_1^*, \ldots, x_{k-1}^*, y_k, x_{k+1}^*, \ldots, x_N^*), \tag{4.9}$$

and by summing up this inequality for $k = 1, \ldots, N$ we get (4.8).

Assume next that (4.8) holds. Select $\underline{y} = (x_1^*, \ldots, x_{k-1}^*, y_k, x_{k+1}^*, \ldots, x_N^*)$ then from (4.8),

$$\sum_{l=1}^{N} \phi_l(x_1^*, \ldots, x_N^*) \geqq \sum_{l \neq k} \phi_l(x_1^*, \ldots, x_N^*) + \phi_k(x_1^*, \ldots, x_{k-1}^*, y_k, x_{k+1}^*, \ldots, x_N^*).$$

After cancelation of all terms for $l \neq k$, (4.9) is obtained, so $\underline{x}^*$ is equilibrium.

■

Define now the point-to-set mapping

$$\underline{H}(\underline{x}) = \left\{\underline{z} | \underline{z} \in S, \Phi(\underline{x}, \underline{z}) = \max\left\{\Phi(\underline{x}, \underline{y}) | \underline{y} \in S\right\}\right\} \tag{4.10}$$

where $S = S_1 \times S_2 \times \cdots \times S_N$ is the set of all simultaneous strategy vectors.

Theorem 4.3 *$\underline{x}^* \in S$ is an equilibrium if and only if $\underline{x}^*$ is a fixed point of mapping $\underline{H}(\underline{x})$.*

Proof From (4.10) it is clear that $\underline{x}^* \in \underline{H}(\underline{x}^*)$ if and only if

$$\Phi(\underline{x}^*, \underline{x}^*) = \max \left\{ \Phi(\underline{x}^*, \underline{y}) | \underline{y} \in S \right\}$$

that is, for all $\underline{y} \in S$, (4.8) holds. Then Lemma 1 implies the assertion.

■

Theorems 4.2 and 4.3 reduce the equilibrium problem to fixed point problems of point-to-set or point-to-point mappings. If the best responses of the players are always unique, then mapping $\underline{R}(\underline{x})$ is point-to-point, otherwise point-to-set. Similarly, if the maximum in (4.10) is always unique, then mapping $\underline{H}(\underline{x})$ is point-to-point, otherwise point-to-set. These theorems will be very useful in proving existence of equilibria by applying well-known existence results of fixed points. In addition, they can provide computer methods for finding equilibria by solving the fixed point equations and relations.

Chapter 5
Existence of Equilibria

In Chap. 2 we have seen examples of discrete static games which had no equilibrium (Example 2.7), unique equilibrium (Example 2.1) and multiple equilibria (Example 2.6). We faced similar situation with continuous games in Chap. 3 (Examples 3.2, 3.3 and 3.1). In this chapter we will give sufficient conditions for the existence of equilibria in continuous games. A class of special discrete games will be discussed later in this book.

5.1 General Existence Conditions

We have shown in Chap. 4 the equivalence of equilibrium problems and fixed point problems, therefore any existence result on fixed points can be used to show existence of equilibria. In Appendix D the most frequently applied fixed point theorems are summarized, they will be the basis of the existence theorems presented in this chapter. Consider an N person game with strategy sets $S_1, \ldots, S_N$ and payoff functions $\phi_1, \ldots, \phi_N$. The best response mapping of player k was given in (3.1) as

$$R_k(\underline{s}) = \left\{ s_k^* | s_k^* \in S_k, \phi_k(s_k^*, \underline{s}_{-k}) = \max_{s_k \in S_k} \phi(s_k, \underline{s}_{-k}) \right\}$$

where s_k is the strategy of player k, $\underline{s} = (s_1, \ldots, s_N)$ and $\underline{s}_{-k} = (s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_N)$. A simultaneous strategy vector $\underline{s} = (s_1, s_2, \ldots, s_N)$ is an equilibrium if and only if

$$s_k \in R_k(\underline{s})$$

for all players k. That is, when $\underline{s} \in \underline{R}(\underline{s})$ with

$$\underline{R}(\underline{s}) = \left(R_1(\underline{s}), \ldots, R_N(\underline{s}) \right).$$

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_5

Assume first that for all players k and $\underline{s} \in S_1 \times \cdots \times S_N$, set $R_k(\underline{s})$ has only one element, that is, mapping $\underline{s} \mapsto \underline{R}(\underline{s})$ is point-to-point.

The Brouwer fixed point theorem implies the following simple result.

Theorem 5.1 *Assume that the strategy sets S_k of all players k are nonempty, convex, closed, bounded subsets of finite dimensional Euclidean spaces, and the best response mapping is one-to-one and $\underline{R}(\underline{s})$ is continuous. Then there is at least one equilibrium of the game.*

Proof If all sets S_k are nonempty, closed and bounded, then the same holds for $S = S_1 \times \cdots \times S_N$. Assume next that all sets S_k are convex. We can easily show that S is also convex. Let $\underline{x} = (x_1, \ldots, x_N)$ and $\underline{y} = (y_1, \ldots, y_N)$ be two points in S. Then with $0 \leqq \alpha \leqq 1$,

$$\alpha\underline{x} + (1-\alpha)\underline{y} = \big(\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_N + (1-\alpha)y_N\big)$$

where the convexity of S_k implies that for all k, $\alpha x_k + (1-\alpha)y_k \in S_k$, so $\alpha\underline{x} + (1-\alpha)\underline{y} \in S$. Then the Brouwer fixed point theorem implies the existence of at least one fixed point, which is an equibrium. ■

We can also apply the Banach fixed point theorem as follows.

Theorem 5.2 *Assume that the strategy sets S_k of all players k are nonempty and closed in finite dimensional Euclidean spaces, furthermore mapping $\underline{R}(\underline{s})$ is one-to-one and contraction on S. Then there is a unique equilibrium which can be obtained as the limit of the iteration sequence*

$$\underline{s}^{(k+1)} = \underline{R}(\underline{s}^{(k)}) \tag{5.1}$$

starting with any arbitrary initial approximation $\underline{s}^{(0)} \in S$. ■

Proof It is clear, that if all sets S_k are closed, then $S = S_1 \times \cdots \times S_N$ is also closed and then the Banach fixed point theorem implies the assertion. ■

Theorem 5.1 can be easily extended to the general case when $\underline{R}(\underline{s})$ is a point-to-set mapping. In such cases the Kakutani fixed point theorem provides existence.

Theorem 5.3 *Assume that all strategy sets S_k are nonempty, convex, closed, bounded sets in finite dimensional Euclidean spaces, $\underline{R}(\underline{s})$ for all $\underline{s} \in S$ is nonempty, convex, closed set, furthermore the graph of mapping $\underline{R}$,*

$$G_{\underline{R}} = \big\{(\underline{s}, \underline{s}')|\underline{s}' \in \underline{R}(\underline{s}), \underline{s} \in S\big\} \tag{5.2}$$

is closed, then there is at least one equilibrium. ■

Theorems 5.1, 5.2, 5.3 require the determination and examination of the best response mappings of the players which is a difficult task is many cases. For this reason it would be helpful to check existence without knowing the best response mappings and checking some properties of only the payoff functions. The most commonly used such result is the Nikaido-Isoda theorem, which can be formulated as follows (Nikaido & Isoda, 1955):

Theorem 5.4 *Assume that for all players k,*
(a) S_k is nonempty, convex, closed, bounded in finite dimensional Euclidean space;
(b) $\phi_k(\underline{s})$ is continuous in $\underline{s}$ as an N-variable function;
(c) $\phi_k(s_1, \dots, s_{k-1}, s_k, s_{k+1}, \dots, s_N)$ is concave in s_k with all fixed strategies $s_l \in S_l$ of the other players.
Then there is at least one equilibrium. ■

Proof Let $\underline{R}(\underline{s})$ denote the best response mapping. We will prove that all conditions of the Kakutani fixed point theorem are satisfied.

Notice first that $S = S_1 \times \cdots \times S_N$ is nonempty, convex, closed and bounded in a finite dimensional Euclidean space. Set $R_k(\underline{s})$ is the set of maximum points of the continuous, concave function $\phi_k(s_1, \dots, s_{k-1}, s_k, s_{k+1}, \dots, s_N)$ in S_k with fixed values of $s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_N$, so the best response set is nonempty, convex, closed, furthermore bounded, since S_k is bounded. Therefore $\underline{R}(\underline{s}) = R_1(\underline{s}) \times R_2(\underline{s}) \times \cdots \times R_N(\underline{s})$ is also nonempty, convex, closed and bounded. Consider finally the graph (5.2) of mapping $\underline{R}(\underline{s})$. In order to show that it is a closed set consider a convergent sequence from $G_{\underline{R}}$, $(\underline{s}^{(l)}, \underline{s}^{'(l)}) \to (\underline{s}^*, \underline{s}^{'*})$. We will prove that $(\underline{s}^*, \underline{s}'^*) \in G_{\underline{R}}$ as well. Since $\underline{s}'^{(l)} \in \underline{R}(\underline{s}^{(l)})$, for all players k,

$$\phi_k\left(s_1^{(l)}, \dots, s_{k-1}^{(l)}, s_k'^{(l)}, s_{k+1}^{(l)}, \dots, s_N^{(l)}\right) \geqq \phi_k\left(s_1^{(l)}, \dots, s_{k-1}^{(l)}, s_k', s_{k+1}^{(l)}, \dots, s_N^{(l)}\right) \quad (5.3)$$

for all $s_k' \in S_k$. Let $l \to \infty$, then the continuity of ϕ_k implies that

$$\phi_k(s_1^*, \dots, s_{k-1}^*, s_k'^*, s_{k+1}^*, \dots, s_N^*) \geqq \phi_k(s_1^*, \dots, s_{k-1}^*, s_k', s_{k+1}^*, \dots, s_N^*) \quad (5.4)$$

that is, $s_k'^* \in R_k(\underline{s}^*)$, and so $(\underline{s}^*, \underline{s}'^*) \in G_{\underline{R}}$. ■

We can next show that all conditions of the theorem are needed to guarantee the existence of an equilibrium by presenting particular examples when all by one conditions of the theorem hold and the game has no equilibrium.

Example 5.1 Any discrete game satisfies all but one condition, when the strategy sets are not convex. In Example 2.7 we had no equilibrium. ▼

Example 5.2 Let $N = 2$, $S_1 = S_2 = [0, 1)$ and $\phi_1 = \phi_2 = s_1 + s_2$. The strategy sets are not closed, since the right end point of the interval does not belong to them. All other conditions however hold. The players have no best responses since neither s_1 nor s_2 has maximum in $[0, 1)$, so there is no equilibrium. ▼

Example 5.3 Let $N = 2$, $S_1 = S_2 = [0, \infty)$ and $\phi_1 = \phi_2 = s_1 + s_2$ as before. The strategy sets are unbounded, the players have no best responses, so there is no equilibrium. ▼

Example 5.4 Let $N = 2$, $S_1 = S_2 = [0, 1]$,

$$\phi_1 = \begin{cases} s_1 + s_2 & \text{if } s_1 < 1 \\ s_2 & \text{if } s_1 = 1 \end{cases}$$

and

$$\phi_2 = \begin{cases} s_1 + s_2 & \text{if } s_2 < 1 \\ s_1 & \text{if } s_2 = 1. \end{cases}$$

These payoff functions are shown in Fig. 5.1 from which it is clear that they are not continuous, however all other conditions of the theorem are satisfied. Since no best responses exist, the game has no equilibrium. ▼

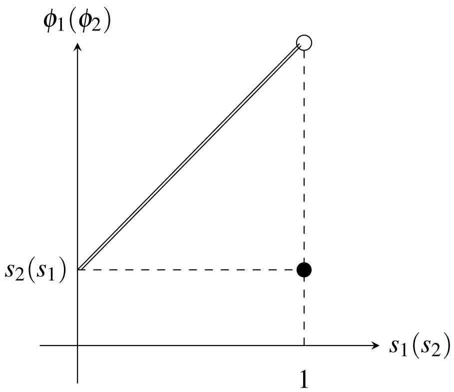

Fig. 5.1 Payoff functions of Example 5.4

Example 5.5 Let $N = 2$, $S_1 = S_2 = [0, 1]$,

$$\phi_1 = (s_1 - s_2)^2 + 1, \phi_2 = -(s_1 - s_2)^2 + 1$$

in which case all conditions of the theorem hold except that ϕ_1 is not concave in s_1. The best responses are as follows:

$$R_1(s_2) = \begin{cases} 1 & \text{if } s_2 < \frac{1}{2} \\ 0 & \text{if } s_2 > \frac{1}{2} \\ \{0, 1\} & \text{if } s_2 = \frac{1}{2} \end{cases}$$

and

$$R_2(s_1) = s_1.$$

These best responses are shown in Fig. 5.2 from which it is clear that no equilibrium exists. ▼

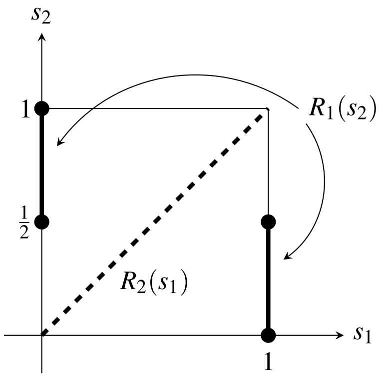

Fig. 5.2 Best responses in Example 5.5

5.2 Bimatrix and Matrix Games

Consider a two-person finite game, when the strategy sets are $S_1 = \{1, 2, \dots, m_1\}$ and $S_2 = \{1, 2, \dots, m_2\}$. Let $a_{ij}^{(1)}$ denote the payoff value $\phi_1(i, j)$, when player 1 choses strategy i and player 2 selects strategy j. Let $a_{ij}^{(2)}$ denote the corresponding payoff value $\phi_2(i, j)$ of player 2.

If there is no equilibrium, then there is no clear strategy selections of the players, so they change their strategies game by game. If there is a deterministic rule of strategy selection of a player, then the other player will successfully predict his strategy choices so appropriate answers can be found in advance. Therefore random strategy selection is the logical solution for this problem. In this case each player defines a discrete probability distribution $\underline{s}_k = \left(x_1^{(k)}, \dots, x_{m_k}^{(k)}\right)^T$ on the strategy set and at each realization of the game a random strategy is selected according to this distribution. By assuming that the game is repeated many times, the average payoff of player k becomes

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} a_{ij}^{(k)} x_i^{(1)} x_j^{(2)} = \underline{s}_1^T \underline{A}^{(k)} \underline{s}_2$$

where

$$\underline{A}^{(k)} = \left(a_{ij}^{(k)}\right)_{i,j}.$$

This extended game with strategy sets

$$\overline{S}_k = \left\{\underline{s}_k = \left(x_1^{(k)}, \dots, x_{m_k}^{(k)}\right) | 0 \leqq x_i^{(k)} \leqq 1, \sum_{i=1}^{m_k} x_i^{(k)} = 1\right\} \tag{5.5}$$

and payoff functions

$$\phi_k(\underline{s}_1, \underline{s}_2) = \underline{s}_1^T \underline{A}^{(k)} \underline{s}_2 \tag{5.6}$$

is called the *mixed extension* of a two-person finite game, or simply *bimatrix game*, since it is characterized by two matrices $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$. The elements of S_k are called *pure strategies* and those of $\overline{S}_k$ are called *mixed strategies*.

It is easy to see that all conditions of the Nikaido-Isoda theorem are satisfied, so bimatrix games always have at least one equilibrium.

Example 5.6 Consider a simple example when the players have only two pure strategies and the matrices are:

$$\underline{A}^{(1)} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \underline{A}^{(2)} = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}.$$

Notice that there is no strategy pair (i, j) such that $a_{ij}^{(1)}$ is the largest element of its column in $\underline{A}^{(1)}$ and $a_{ij}^{(2)}$ is the largest element of its row in $\underline{A}^{(2)}$. So there is no pure strategy equilibrium. By finding the best responses of the players we will be able to find the mixed strategy equilibrium of the game. For the sake of notational simplicity let $\underline{s}_1 = (x, 1-x)^T$ and $\underline{s}_2 = (y, 1-y)^T$ with $0 \leqq x, y \leqq 1$. Then

$$\begin{aligned} \phi_1 = (x, 1-x) \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} &= \big(x + 2(1-x), 2x\big) \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ = (2-x, 2x) \begin{pmatrix} y \\ 1-y \end{pmatrix} = (2-x)y + 2x(1-y) &= 2x + 2y - 3xy \\ &= x(2-3y) + 2y. \end{aligned}$$

The best response of player 1 maximizes this linear function in x,

$$R_1(y) = \begin{cases} 1 & \text{if } y < \frac{2}{3} \\ 0 & \text{if } y > \frac{2}{3} \\ [0, 1] & \text{if } y = \frac{2}{3} \end{cases}$$

since if the multiplier of x is positive, then the largest x value is the maximizer, if the multiplier is negative, then the smallest x value is selected, and if the multiplier is zero, then ϕ_1 does not depend on x, so the entire interval $[0, 1]$ for x has to be considered as the best response. Similarly

$$\begin{aligned} \phi_2 = (x, 1-x) \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} &= \big(2x + 4(1-x), x + 5(1-x)\big) \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ = (4-2x, 5-4x) \begin{pmatrix} y \\ 1-y \end{pmatrix} = (4-2x)y + (5-4x)(1-y) &= -4x - y + 2xy + 5 \\ &= y(2x-1) - 4x + 5 \end{aligned}$$

so the best response of player 2 is the following:

$$R_2(x) = \begin{cases} 1 & \text{if } x > \frac{1}{2} \\ 0 & \text{if } x < \frac{1}{2} \\ [0, 1] & \text{if } x = \frac{1}{2}. \end{cases}$$

The best responses are shown in Fig. 5.3, the only equilibrium is $x = \frac{1}{2}$, $y = \frac{2}{3}$. This result shows that the probability distributions of the players are

$$\underline{s}_1 = (\tfrac{1}{2}, \tfrac{1}{2})^T \text{ and } \underline{s}_2 = (\tfrac{2}{3}, \tfrac{1}{3})^T.$$

So at the equilibrium player 1 selects the strategies with equal, $\frac{1}{2} - \frac{1}{2}$ probability while player 2 selects strategy 1 twice as often than strategy 2. ▼

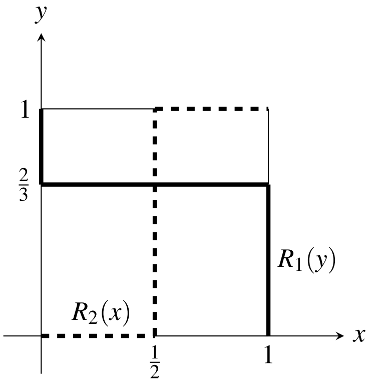

Fig. 5.3 Best responses in Example 5.6

If the two-person finite game is zero sum, then its extension is called *matrix game*, since $\underline{A}^{(2)} = -\underline{A}^{(1)}$, so the game is characterized with a single matrix $\underline{A}^{(1)}$. The payoff functions are

$$\phi_1 = \underline{s}_1^T \underline{A}^{(1)} \underline{s}_2 \text{ and } \phi_2 = -\phi_1.$$

All matrix games have at least one equilibrium.

Example 5.7 Consider again a simple example with two pure strategies of the players, when

$$\underline{A}^{(1)} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Notice that there is no matrix element which is largest in its column and smallest in its row. So there is no pure strategy equilibrium. As before let the random strategies of the players be $\underline{s}_1 = (x, 1 - x)^T$ and $\underline{s}_2 = (y, 1 - y)^T$ with $0 \leqq x, y \leqq 1$. Then

$$\phi_1 = (x, 1-x)\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \big(2x, x+2(1-x)\big)\begin{pmatrix} y \\ 1-y \end{pmatrix} = (2x, 2-x)\begin{pmatrix} y \\ 1-y \end{pmatrix}$$
$$= 2xy + (2-x)(1-y) = -x - 2y + 3xy + 2 =$$
$$= x(3y-1) - 2y + 2,$$

so the best response of player 1 is the following:

$$R_1(y) = \begin{cases} 1 & \text{if } y > \frac{1}{3} \\ 0 & \text{if } y < \frac{1}{3} \\ [0, 1] & \text{if } y = \frac{1}{3}. \end{cases}$$

Since the payoff of player 2 is

$$\phi_2 = -\phi_1 = x + 2y - 3xy - 2 = y(2-3x) + x - 2,$$

his best response is as follows:

$$R_2(x) = \begin{cases} 1 & \text{if } x < \frac{2}{3} \\ 0 & \text{if } x > \frac{2}{3} \\ [0, 1] & \text{if } x = \frac{2}{3}. \end{cases}$$

Figure 5.4 shows these best responses, and clearly $x = \frac{2}{3}$ and $y = \frac{1}{3}$ is the only equilibrium. So the mixed strategies of the players are $\underline{s}_1 = (\frac{2}{3}, \frac{1}{3})^T$ and $\underline{s}_2 = (\frac{1}{3}, \frac{2}{3})^T$, which means that at the mixed equilibrium player 1 selects strategy 1 twice as often than strategy 2, and player 2 selects strategy 2 twice as often than strategy 1. ▼

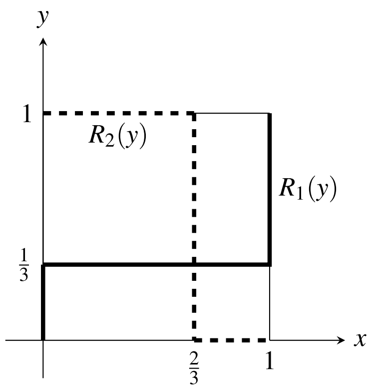

Fig. 5.4 Best responses in Example 5.7

In the above examples the equilibrium was unique. This is not necessarily the case in general. It is sufficient to consider matrices $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ with identical elements. In this case with the notation of $\underline{A}^{(1)} = (a^{(1)})_{ij}$, $\underline{A}^{(2)} = (a^{(2)})_{i,j}$ the payoff functions are constants:

$$\phi_k = \underline{s}_1^T \underline{A}^{(k)} \underline{s}_2 = a^{(k)} \underline{s}_1^T \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix} \underline{s}_2 = a^{(k)} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} x_i^{(1)} x_j^{(2)}$$

$$= a^{(k)} \Big(\sum_{i=1}^{m_1} x_i^{(1)} \Big) \Big(\sum_{j=1}^{m_2} x_j^{(2)} \Big) = a^{(k)}. \tag{5.7}$$

So all strategy pairs $(\underline{s}_1, \underline{s}_2)$ provide equilibria.

5.3 Mixed Extensions of N-Person Finite Games

Consider now an N-person finite game, where the strategy set of player $k (k = 1, 2, \dots, N)$ is $S_k = \{1, 2, \dots, m_k\}$, and his payoff function is denoted as

$$\phi_k(i_1, i_2, \dots, i_N) = a^{(k)}_{i_1 i_2 \dots i_N}.$$

The mixed extension of this game can be defined similarly to bimatrix and matrix games. The mixed strategies of each player k are the probability vectors

$$\underline{s}_k = \left(x_1^{(k)}, x_2^{(k)}, \dots, x_{m_k}^{(k)} \right)^T$$

such that $0 \leqq x_i^{(k)} \leqq 1$ for all i and $\sum_{i=1}^{m_k} x_i^{(k)} = 1$. So the strategy sets are defined again as it was given in (5.5) but the payoff of player k is more complicated:

$$\phi_k = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \dots \sum_{i_N=1}^{m_N} a^{(k)}_{i_1 i_2 \dots i_N} x_{i_1}^{(1)} x_{i_2}^{(2)} \dots x_{i_N}^{(N)}. \tag{5.8}$$

Notice that the strategy sets are nonempty, convex, closed and bounded in the m_k-dimensional space, ϕ_k is continuous in the strategy vectors and linear in $\underline{s}_k$, so concave as well. So all conditions of the Nikaido-Isoda theorem hold implying the existence of at least one equilibrium. The uniqueness of the equilibrium is not guaranteed in general, since we have constant payoff functions if the elements $a^{(k)}_{i_1 i_2 \dots i_N}$ depend on only k, and independent of the strategy selections $i_1, i_2, \dots, i_N$. In this case we have

$$\phi_k = a^{(k)} \Big(\sum_{i_1=1}^{m_1} x_{i_1}^{(1)} \Big) \Big(\sum_{i_2=1}^{m_2} x_{i_2}^{(2)} \Big) \dots \Big(\sum_{i_N=1}^{m_N} x_{i_N}^{(N)} \Big) = a^{(k)},$$

so any strategy N-tuple $\left(\underline{s}_1, \underline{s}_2, \dots, \underline{s}_N \right)$ provides equilibrium.

5.4 Multiproduct Oligopolies

Consider an industry of N firms, each of them produces M items or offers M different services to a homogeneous market. The output of firm k is a production vector $\underline{x}_k = \left(x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(M)}\right)^T$ where $x_k^{(m)}$ denotes the production level of product m by firm k. The total production of the industry is $\underline{s} = \sum_{k=1}^{N} \underline{x}_k$ as an M-element vector. Let $p^{(m)}(\underline{s})$ be the unit price of item m, and let $\underline{p} = \left(p^{(1)}, \ldots, p^{(M)}\right)^T$ be the price vector. If $C_k(\underline{x}_k)$ is the cost function of firm k, then its profit is given as follows:

$$\phi_k(\underline{x}_1, \ldots, \underline{x}_N) = \sum_{m=1}^{M} x_k^{(m)} p^{(m)}(\underline{s}) - C_k(\underline{x}_k) = \underline{x}_k^T \underline{p}(\underline{s}) - C_k(\underline{x}_k). \tag{5.9}$$

Assume that the following conditions hold;

(a) The strategy set S_k of each firm is a nonempty, convex, closed and bounded set in $\mathbb{R}_+^M = \{\underline{x} | \underline{x} \geqq \underline{0}, \underline{x} \in \mathbb{R}^M\}$;
(b) Functions $\underline{p}$ and C_k for all k are continuous;
(c) $\underline{x}_k^T \underline{p}\left(\sum_{l=1}^{N} \underline{x}_l\right)$ is concave in $\underline{x}_k$;
(d) $C_k(\underline{x}_k)$ is convex in $\underline{x}_k$.

Under these assumptions the conditions of the Nikaido-Isoda theorem are satisfied, so the multiproduct oligopoly has at least one equilibrium.

Conditions (a), (b) and (d) can be verified easily, however assumption (c) needs further investigation. A simple sufficient condition is given by the following result (Okuguchi & Szidarovszky, okuszi99):

Lemma 5.1 *Let $\underline{f} : D \mapsto \mathbb{R}^M$ with $D \subseteq \mathbb{R}_+^M$ being a convex set, $-\underline{f}$ is monotonic and all components of $\underline{f}$ are concave. Then $g(\underline{x}) = \underline{x}^T \underline{f}(\underline{x})$ is concave on D.*

Proof Let $\alpha, \beta \geqq 0$ such that $\alpha + \beta = 1$, and $\underline{x}, \underline{y} \in D$. Then

$$(\underline{x} - \underline{y})^T \left(\underline{f}(\underline{x}) - \underline{f}(\underline{y})\right) \leqq 0.$$

By multiplying both sides by $\alpha\beta$ we have

$$\alpha\beta\underline{x}^T \underline{f}(\underline{x}) + \alpha\beta\underline{y}^T \underline{f}(\underline{y}) \leqq \alpha\beta\underline{y}^T \underline{f}(\underline{x}) + \alpha\beta\underline{x}^T \underline{f}(\underline{y}).$$

Since $\alpha\beta = \alpha(1 - \alpha) = \alpha - \alpha^2$ and $\alpha\beta = (1 - \beta)\beta = \beta - \beta^2$ we get

$$(\alpha\underline{x} + \beta\underline{y})^T \left(\alpha\underline{f}(\underline{x}) + \beta\underline{f}(\underline{y})\right) \geqq \alpha\underline{x}^T \underline{f}(\underline{x}) + \beta\underline{y}^T \underline{f}(\underline{y})$$

implying that

$$g(\alpha\underline{x}+\beta\underline{y})=(\alpha\underline{x}+\beta\underline{y})^T\underline{f}(\alpha\underline{x}+\beta\underline{y})\geqq(\alpha\underline{x}+\beta\underline{y})^T\left(\alpha\underline{f}(\underline{x})+\beta\underline{f}(\underline{y})\right)\geqq$$
$$\alpha\underline{x}^T\underline{f}(\underline{x})+\beta\underline{y}^T\underline{f}(\underline{y})=\alpha g(\underline{x})+\beta g(\underline{y}),$$

where we used the concavity of the components of $\underline{f}$. This last inequality shows that $g(\underline{x})$ is concave. ■

Corollary *Assume that in a multiproduct oligopoly-$\underline{p}$ is monotonic and all components of $\underline{p}$ are concave, furthermore conditions (a), (b) and (d) hold, then ϕ_k is concave in $\underline{x}_k$, so there is at least one equilibrium.*

Consider finally the single product case, when $\underline{x}_k = x_k$, a scalar, $s=\sum_{k=1}^N x_k$, $\underline{P}(\underline{s}) = p(s)$ where p is a single-variable real-valued function. The strategy set of each player is a closed interval $[0, L_k]$, where L_k is the capacity limit of firm k. The profit of firm k is given as

$$\phi_k = x_k p(s) - C_k(x_k).$$

We can simplify conditions (a)-(d) in this special case. Observe first that $S_k = [0, L_k]$ satisfies condition (a). By assuming the continuity of functions p and C_k, condition (b) is satisfied. If we assume that the cost functions are convex, then (d) also holds. Condition (c) can be examined by assuming that function p is twice differentiable. In this case

$$\frac{\partial\left(x_k p\left(\sum_{l=1}^N x_l\right)\right)}{\partial x_k} = x_k p'\left(\sum_{l=1}^N x_l\right) + p\left(\sum_{l=1}^N x_l\right)$$

and

$$\frac{\partial^2\left(x_k p\left(\sum_{l=1}^N x_l\right)\right)}{\partial^2 x_k} = 2p'\left(\sum_{l=1}^N x_l\right) + x_k p''\left(\sum_{l=1}^N x_l\right).$$

So by assuming that $2p'(s) + x_k p''(s) \leqq 0$, condition (c) also holds.

Making the natural assumption $p'(s) < 0$ (since price has to be a decreasing function of the supply), the above existence conditions

$$\text{(A) } C_k''(x_k) \geqq 0$$
$$\text{(B) } 2p'(s) + x_k p''(s) \leqq 0$$

are slightly different than conditions (a)–(c) assumed in Example 3.14, since (a) and (b) imply (B), (a) and (A) imply (c).

Chapter 6
Computation of Equilibria

In the previous chapter conditions were given for the existence of equilibria in N-person games. The Nikaido-Isoda theorem was based on the Kakutani fixed point theorem, which guarantees the existence of at least one fixed point without providing computational method for finding the fixed points. In this chapter a general method will be introduced and applied to some special game classes.

6.1 Application of the Kuhn-Tucker Conditions

Assume that the strategy set of each player is defined by certain inequalities:

$$S_k = \{\underline{x}_k | \underline{g}_k(\underline{x}_k) \geqq \underline{0}\} \tag{6.1}$$

where functions $\underline{g}_k : D_k \mapsto \mathbb{R}^{m_k}$ are continuously differentiable where D_k is an open set in finite dimensional Euclidean space. Assume in addition that all payoff functions ϕ_k are also continuously differentiable in $\underline{x}_k$ with any selection of the strategies $\underline{x}_l (l \neq k)$ of the other players. If $(\underline{x}_1^*, \ldots, \underline{x}_N^*)$ is an equilibrium, then $\underline{x}_k^*$ is a solution of the optimization problem

$$\begin{aligned} \text{maximize }_{\underline{x}_k} \phi_k(\underline{x}_1^*, \ldots, \underline{x}_{k-1}^*, \underline{x}_k, \underline{x}_{k+1}^*, \ldots, \underline{x}_N^*) \\ \text{subject to } \underline{g}_k(\underline{x}_k) \geqq \underline{0}. \end{aligned} \tag{6.2}$$

Assume in addition that the Kuhn-Tucker regularity condition holds, then the Kuhn-Tucker necessary conditions imply the existence of an m_k-dimensional vector $\underline{u}_k$ such that

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_6

$$
\begin{aligned}
\underline{u}_k &\geqq \underline{0} \\
\underline{g}_k(\underline{x}_k) &\geqq \underline{0} \\
\nabla_k \phi_k(\underline{x}_1, \ldots, \underline{x}_N) + \underline{u}_k^T \ \nabla_k \ \underline{g}_k(\underline{x}_k) &= \underline{0}^T \\
\underline{u}_k^T \underline{g}_k(\underline{x}_k) &= 0
\end{aligned} \tag{6.3}
$$

where $\nabla_k \phi_k$ is the gradient vector of ϕ_k with respect to $\underline{x}_k$ as a row vector, and $\nabla_k \underline{g}_k$ is the Jacobian matrix of $\underline{g}_k(\underline{x}_k)$. If for all k, ϕ_k and $\underline{g}_k$ are both concave in $\underline{x}_k$, then conditions (6.3) are also sufficient for optimality.

By constructing conditions (6.3) for all players a system of (usually nonlinear) equations and inequalities is obtained, the equilibria have to be among the solutions of the system.

Example 6.1 Consider a duopoly ($N = 2$) with strategy sets $S_1 = S_2 = [0, 10]$ and price function $p(x + y) = 20 - (x + y)$ where x and y are the production levels of the firms. Then

$$\phi_1(x, y) = x(20 - x - y) - x,$$

and

$$\phi_2(x, y) = y(20 - x - y) - y$$

where we assume that the cost functions are $C_1(x) = x$ and $C_2(y) = y$. In this case the optimum problem (6.2) for the two players have the form

$$
\begin{aligned}
&\text{maximize}_x x(20 - x - y) - x = 19x - x^2 - xy \\
&\text{subject to } 0 \leqq x \leqq 10
\end{aligned} \tag{6.4}
$$

and

$$
\begin{aligned}
&\text{maximize}_y y(20 - x - y) - y = 19y - y^2 - xy \\
&\text{subject to } 0 \leqq y \leqq 10.
\end{aligned} \tag{6.5}
$$

In order to rewrite these problems into the form (C3)(see Appendix C), the constraints have to be reformulated as

$$
\begin{aligned}
x &\geqq 0 \\
10 - x &\geqq 0
\end{aligned}
$$

and

$$
\begin{aligned}
y &\geqq 0 \\
10 - y &\geqq 0.
\end{aligned}
$$

Therefore for both players the $\underline{u}_k$ vectors are two dimensional:

$$\underline{u}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \text{ and } \underline{u}_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}.$$

For player 1, conditions (6.3) have the following form:

$$\begin{aligned} u_1, v_1 &\geqq 0 \\ x \geqq 0,\ 10 - x &\geqq 0 \\ 19 - 2x - y + (u_1, v_1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= 0 \\ u_1 x + v_1(10 - x) &= 0. \end{aligned}$$

Notice that both terms of the last equation are nonnegative, so their sum is zero if and only if both terms are equal to zero. Hence the Kuhn-Tucker conditions for player 1 can be reformulated as follows:

$$\begin{aligned} u_1, v_1 &\geqq 0 \\ x \geqq 0,\ 10 - x &\geqq 0 \\ 19 - 2x - y + u_1 - v_1 &= 0 \\ u_1 x = v_1(10 - x) &= 0. \end{aligned} \tag{6.6}$$

For player 2 we get similarly the conditions

$$\begin{aligned} u_2, v_2 &\geqq 0 \\ y \geqq 0,\ 10 - y &\geqq 0 \\ 19 - 2y - x + u_2 - v_2 &= 0 \\ u_2 y = v_2(10 - y) &= 0. \end{aligned} \tag{6.7}$$

System (6.6)–(6.7) has six unknowns, x, y, u_1, v_1, u_2, v_2.

In solving the system we consider three cases: $x = 0$, $x = 10$ and $0 < x < 10$.

If $x = 0$, then the last condition of (6.6) implies that $v_1 = 0$, so from the third condition of (6.6) we see that $19 - y + u_1 = 0$ implying that $y = 19 + u_1$, which contradicts the assumption that $y \leqq 10$.

If $x = 10$, then $u_1 = 0$ implying that $-1 - y - v_1 = 0$, so $y = -1 - v_1 < 0$ contradicting the assumption that $y \geqq 0$.

If $0 < x < 10$, then $u_1 = v_1 = 0$, so $19 - 2x - y = 0$.

By repeating the same argument for system (6.7) we get that $19 - 2y - x = 0$. Hence we have to solve equations

$$19 - 2x - y = 0$$
$$19 - x - 2y = 0$$

giving the solution $x = y = \frac{19}{3}$. ▼

6.2 Reduction to an Optimization Problem

The Kuhn-Tucker conditions (6.3) for all players can be reformulated as the following optimization problem:

$$\begin{aligned} &\text{minimize } \sum_{k=1}^{N} \underline{u}_k^T \underline{g}_k(\underline{x}_k) \\ &\text{subject to } \left.\begin{aligned} \underline{u}_k &\geqq \underline{0}, \\ \underline{g}_k(\underline{x}_k) &\geqq \underline{0}, \text{ and} \\ \nabla_k \phi_k(\underline{x}_1, \ldots, \underline{x}_N) + \underline{u}_k^T \nabla_k \underline{g}_k(\underline{x}_k) &= \underline{0}^T. \end{aligned}\right\} (k = 1, 2, \ldots, N) \end{aligned} \tag{6.8}$$

The last conditions of (6.3) are satisfied if and only if the objective function of (6.8) is zero. The first two constraints of (6.8) imply that the objective function is always nonnegative, so we have two possibilities. If the optimal objective function value is zero, then all optimal solutions satisfy all conditions of (6.3), so they solve the Kuhn-Tucker conditions, and the equilibria are among the solutions. If the objective function at the optimum is positive, then the Kuhn-Tucker conditions do not have a solution implying that no equilibrium exists.

Example 6.2 In the case of the previous example problem (6.8) has the following form:

$$\begin{aligned} \text{minimize } \{u_1 x + v_1(10 - x) + u_2 y + v_2(10 - y)\} \\ \text{subject to } u_1, v_1, u_2, v_2 \geqq 0 \\ 0 \leqq x, y \leqq 10 \\ 19 - 2x - y + u_1 - v_1 = 0 \\ 19 - x - 2y + u_2 - v_2 = 0. \end{aligned} \tag{6.9}$$

It is easy to see that $u_1 = v_1 = u_2 = v_2 = 0$ and $x = y = \frac{19}{3}$ is feasible and provides zero objective function value, so it is optimal. ▼

As an additional illustration consider the following example.

Example 6.3 Let $N = 2$, $S_1 = S_2 = [0, \infty)$ be the number of players and the strategy sets. Assume that the payoff functions are

$$\phi_1(x, y) = x + y - (x + y)^2 = x + y - x^2 - 2xy - y^2$$

and

$$\phi_2(x, y) = x + y - 2(x + y)^2 = x + y - 2x^2 - 4xy - 2y^2.$$

Since the strategy sets are characterized by single inequalities $x \geqq 0$ and $y \geqq 0$, the vectors $\underline{u}_1$ and $\underline{u}_2$ are one-dimensional. The Kuhn-Tucker conditions (6.3) for the two players are as follows:

$$\begin{aligned} x, u_1 &\geqq 0 \\ 1 - 2x - 2y + u_1 &= 0 \\ u_1 x &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} y, u_2 &\geqq 0 \\ 1 - 4x - 4y + u_2 &= 0 \\ u_2 y &= 0. \end{aligned}$$

In solving these equations and inequalities we will consider four cases: $x = 0,\ y = 0$; $x = 0,\ y > 0$; $x > 0,\ y = 0$ and $x > 0,\ y > 0$. If $x = y = 0$, then from the second conditions $1 + u_1 = 1 + u_2 = 0$, which cannot occur, since u_1 and u_2 are nonnegative. If $x = 0$ and $y > 0$, then $u_2 = 0$ and $1 - 4y = 0$. So $y = \frac{1}{4}$ and from the second condition for player 1 we conclude that $1 - \frac{1}{2} + u_1 = 0$, which is again a contradiction. If $x > 0$ and $y = 0$, then similarly $u_1 = 0$ and $1 - 2x = 0$, so $x = \frac{1}{2}$ and then the second condition for player 2 implies that $1 - 2 + u_2 = 0$. Therefore $u_2 = 1$, and the solution is $u_1 = 0$, $u_2 = 1$, $x = \frac{1}{2}$ and $y = 0$. If x and y are positive, then $u_1 = u_2 = 0$ implying that $1 - 2x - 2y = 1 - 4x - 4y = 0$, which is again a contradiction.

The optimum problem (6.8) has now the following form:

$$\begin{aligned} &\text{minimize } u_1 x + u_2 y \\ &\text{subject to } u_1, u_2, x, y \geqq 0 \\ &\qquad\qquad 1 - 2x - 2y + u_1 = 0 \\ &\qquad\qquad 1 - 4x - 4y + u_2 = 0. \end{aligned}$$

From the last two constraints we have

$$u_1 = 2x + 2y - 1 \geqq 0 \quad \text{and} \quad u_2 = 4x + 4y - 1 \geqq 0$$

implying that

$$x + y \geqq \frac{1}{2}.$$

Substituting the above expressions for u_1 and u_2 into the objective function, the problem becomes

$$\begin{aligned} &\text{minimize } x(2x+2y-1)+y(4x+4y-1)=2x^2+4y^2+6xy-x-y \\ &\text{subject to } x, y \geqq 0 \\ &\qquad\qquad x+y \geqq \frac{1}{2}. \end{aligned}$$

At any equilibrium the objective function has to be zero, so

$$0 = 2x^2+4y^2+6xy-x-y=(x+y)(2x+4y-1),$$

that is, $2x+4y-1=0$ or $y=-\frac{x}{2}+\frac{1}{4}$. Then

$$x+y=x-\frac{x}{2}+\frac{1}{4}=\frac{x}{2}+\frac{1}{4}\geq\frac{1}{2},$$

so $x \geqq \frac{1}{2}$. Furthermore $y=-\frac{x}{2}+\frac{1}{4}\geq 0$, implying that $x \leqq \frac{1}{2}$. Hence $x=\frac{1}{2}$, $y=0$ is the solution. ▼

6.3 Solution of Bimatrix Games

Bimatrix games are characterized by two matrices $\underline{A}^{(1)}$ and $\underline{A}^{(2)}$ of the common size $m_1 \times m_2$. The strategy sets of the two players are

$$\overline{S}_1 = \{\underline{s}_1 | \underline{s}_1 \in \mathbb{R}^{m_1}, \underline{s}_1 \geqq \underline{0}, \underline{1}_1^T \underline{s}_1 = 1\}$$

and

$$\overline{S}_2 = \{\underline{s}_2 | \underline{s}_2 \in \mathbb{R}^{m_2}, \underline{s}_2 \geqq \underline{0}, \underline{1}_2^T \underline{s}_2 = 1\}$$

where $\underline{1}_k^T$ is the row vector with m_k unit elements. Notice that with any vector $\underline{s}$, $\underline{1}^T\underline{s}$ is the sum of its elements. The payoff functions are as follows:

$$\phi_1 = \underline{s}_1^T \underline{A}^{(1)} \underline{s}_2 \text{ and } \phi_2 = \underline{s}_1^T \underline{A}^{(2)} \underline{s}_2.$$

In order to construct the Kuhn-Tucker conditions we have to rewrite the constraints of $\overline{S}_1$ and $\overline{S}_2$ in the following way:

$$\underline{s}_k \geqq 0$$
$$\underline{1}_k^T \underline{s}_k - 1 \geqq 0$$
$$-\underline{1}_k^T \underline{s}_k + 1 \geqq 0$$

for $k = 1, 2$. So in the notation of (6.1),

$$\underline{g}_k(\underline{s}_k) = \begin{pmatrix} \underline{s}_k \\ \underline{1}_k^T \underline{s}_k - 1 \\ -\underline{1}_k^T \underline{s}_k + 1 \end{pmatrix},$$

with Jacobian matrix

$$\nabla_k \underline{g}_k(\underline{s}_k) = \begin{pmatrix} \underline{I}_k \\ \underline{1}_k^T \\ -\underline{1}_k^T \end{pmatrix},$$

where $\underline{I}_k$ is the $m_k \times m_k$ identity matrix, and $\underline{1}_k$ is the m_k-element vector with unit components. The $\underline{u}_k$ vector also can be decomposed accordingly as

$$\underline{u}_k = \begin{pmatrix} \underline{v}_k \\ v_k^{(m_k+1)} \\ v_k^{(m_k+2)} \end{pmatrix}.$$

We can now find the special form of the optimization problem (6.8). The objective function can be written as

$$\sum_{k=1}^{2} \Big(\underline{v}_k^T \underline{s}_k + v_k^{(m_k+1)} (\underline{1}_k^T \underline{s}_k - 1) - v_k^{(m_k+2)} (\underline{1}_k^T \underline{s}_k - 1) \Big) = \sum_{k=1}^{2} \Big(\underline{v}_k^T \underline{s}_k - \alpha_k (\underline{1}_k^T \underline{s}_k - 1) \Big) \tag{6.10}$$

with $\alpha_k = v_k^{(m_k+2)} - v_k^{(m_k+1)}$ without any sign constraint. The constraints of (6.8) imply that

$$\underline{v}_1 \geqq \underline{0}, \quad \underline{v}_2 \geqq \underline{0}$$
$$\underline{s}_1 \geqq \underline{0}, \quad \underline{s}_2 \geqq \underline{0}$$

$$\underline{s}_2^T \underline{A}^{(1)T} + \underline{v}_1^T + \big(\underline{v}_1^{(m_1+1)} - \underline{v}_1^{(m_1+2)} \big) \underline{1}_1^T = \underline{0}^T$$

and

$$\underline{s}_1^T \underline{A}^{(2)} + \underline{v}_2^T + \big(\underline{v}_2^{(m_2+1)} - \underline{v}_2^{(m_2+2)} \big) \underline{1}_2^T = \underline{0}^T.$$

From the last two equations we have

$$\underline{v}_1^T = \alpha_1 \underline{1}_1^T - \underline{s}_2^T \underline{A}^{(1)T}, \; \underline{v}_2^T = \alpha_2 \underline{1}_2^T - \underline{s}_1^T \underline{A}^{(2)}$$

so the objective function becomes

$$\begin{aligned}\left(\alpha_1 \underline{1}_1^T - \underline{s}_2^T \underline{A}^{(1)T}\right)\underline{s}_1 + \left(\alpha_2 \underline{1}_2^T - \underline{s}_1^T \underline{A}^{(2)}\right)\underline{s}_2 - \alpha_1(\underline{1}_1^T \underline{s}_1 - 1) - \alpha_2(\underline{1}_2^T \underline{s}_2 - 1) \\ = -\underline{s}_1^T \underline{A}^{(1)} \underline{s}_2 - \underline{s}_1^T \underline{A}^{(2)} \underline{s}_2 + \alpha_1 + \alpha_2\end{aligned}$$

since $\underline{1}_k^T \underline{s}_k = 1$ for $k = 1, 2$. By using the nonnegativity of $\underline{v}_1^T$ and $\underline{v}_2^T$, the optimum problem can be rewritten in a quadratic programming form

$$\begin{aligned}\text{maximize } \underline{s}_1^T \left(\underline{A}^{(1)} + \underline{A}^{(2)}\right)\underline{s}_2 - \alpha_1 - \alpha_2 \\ \text{subject to } \underline{s}_1 \geqq \underline{0},\; \underline{s}_2 \geqq \underline{0} \\ \underline{1}_1^T \underline{s}_1 = 1,\; \underline{1}_2^T \underline{s}_2 = 1 \\ \underline{A}^{(1)} \underline{s}_2 \leqq \alpha_1 \underline{1}_1 \\ \underline{A}^{(2)T} \underline{s}_1 \leqq \alpha_2 \underline{1}_2.\end{aligned} \tag{6.11}$$

A strategy pair $(\underline{s}_1^*, \underline{s}_2^*)$ is an equilibrium if and only if it is an optimal solution of problem (6.11) with some values of (α_1, α_2). Equivalently, a strategy pair (s_1^*, s_2^*) is an equilibrium if with some α_1, α_2 it satisfies all constraints of (6.11) and the corresponding objective function value is zero (Mangasarian and Stone, 1964).

Example 6.4 Consider the bimatrix game with matrices

$$\underline{A}^{(1)} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } \underline{A}^{(2)} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Let $\underline{s}_1 = \left(x_1^{(1)}, x_2^{(1)}\right)^T,\; \underline{s}_2 = \left(x_1^{(2)},\; x_2^{(2)}\right)^T$. Since

$$\underline{A}^{(1)} + \underline{A}^{(2)} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$

problem (6.11) has the special form

$$\begin{aligned}\text{maximize } \left(x_1^{(1)}, x_2^{(1)}\right) \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} - \alpha_1 - \alpha_2 \\ \text{subject to } x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)} \geqq 0 \\ x_1^{(1)} + x_2^{(1)} = x_1^{(2)} + x_2^{(2)} = 1 \\ \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} \leqq \begin{pmatrix} \alpha_1 \\ \alpha_1 \end{pmatrix}\end{aligned}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} \leqq \begin{pmatrix} \alpha_2 \\ \alpha_2 \end{pmatrix}.$$

A simple computer program shows that there are three optimal solutions:

$$\underline{s}_1 = (1, 0)^T,\ \underline{s}_2 = (1, 0)^T,\ \alpha_1 = 2,\ \alpha_2 = 1,$$
$$\underline{s}_1 = (0, 1)^T,\ \underline{s}_2 = (0, 1)^T,\ \alpha_1 = 1,\ \alpha_2 = 2,$$

and

$$\underline{s}_1 = (\frac{3}{5}, \frac{2}{5})^T,\ \underline{s}_2 = (\frac{2}{5}, \frac{3}{5})^T,\ \alpha_1 = \frac{1}{5},\ \alpha_2 = \frac{1}{5}.$$

We can easily obtain the same solutions by using the best responses of the players. For the simple notation let

$$\underline{s}_1 = \begin{pmatrix} x \\ 1-x \end{pmatrix} \text{ and } \underline{s}_2 = \begin{pmatrix} y \\ 1-y \end{pmatrix},$$

then the payoff functions are

$$\phi_1 = (x, 1-x) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = 2xy - x(1-y) - y(1-x) + (1-x)(1-y) =$$
$$= 5xy - 2x - 2y + 1 = x(5y-2) - 2y + 1$$

and

$$\phi_2 = (x, 1-x) \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = xy - (1-x)y - x(1-y) + 2(1-x)(1-y) =$$
$$= 5xy - 3y - 3x + 2 = y(5x-3) - 3x + 2$$

with best responses

$$R_1(y) = \begin{cases} 1 & \text{if } y > \frac{2}{5} \\ 0 & \text{if } y < \frac{2}{5} \\ [0,1] & \text{if } y = \frac{2}{5} \end{cases}$$

and

$$R_2(x) = \begin{cases} 1 & \text{if } x > \frac{3}{5} \\ 0 & \text{if } x < \frac{3}{5} \\ [0,1] & \text{if } x = \frac{3}{5}. \end{cases}$$

These best responses are shown in Figure 6.1, from which we see the existence of three equilibria: $x = y = 0$, $x = y = 1$ and $x = \frac{3}{5}$, $y = \frac{2}{5}$. So the equilibrium strategies are

$$\underline{s}_1 = (0, 1)^T, \underline{s}_2 = (0, 1)^T;$$
$$\underline{s}_1 = (1, 0)^T, \underline{s}_2 = (1, 0)^T$$
$$\underline{s}_1 = (\frac{3}{5}, \frac{2}{5})^T, \underline{s}_2 = (\frac{2}{5}, \frac{3}{5})^T.$$

which is the same result as the one obtained by the optimum model (Fig. 6.1). ▼

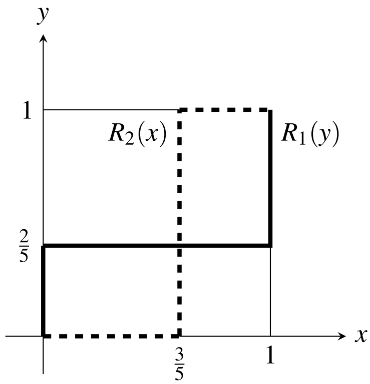

Fig. 6.1 Best responses in Example 6.4

6.4 Solution of Matrix Games

Matrix games are the mixed extensions of zero-sum two-person finite games. By using the notation of the previous section, $\underline{A}^{(1)} + \underline{A}^{(2)} = \underline{0}$, the quadratic programming problem becomes linear:

$$\begin{aligned} &\text{minimize } \alpha_1 + \alpha_2 \\ &\text{subject to } \underline{s}_1 \geqq \underline{0}, \underline{s}_2 \geqq \underline{0} \\ &\underline{1}_1^T \underline{s}_1 = 1, \underline{1}_2^T \underline{s}_2 = 1 \\ &\underline{A}^{(1)} \underline{s}_2 \leqq \alpha_1 \underline{1}_1 \\ &-\underline{A}^{(1)T} \underline{s}_1 \leqq \alpha_2 \underline{1}_2. \end{aligned} \tag{6.12}$$

Notice that variable pairs $\alpha_1, \underline{s}_2$ and $\alpha_2, \underline{s}_1$ are independent, so we can rewrite problem (6.12) as a pair of linear programming problems of much lower sizes than (6.12):

$$\begin{aligned} &\text{minimize } \alpha_1 \\ &\text{subject to } \underline{1}_2^T \underline{s}_2 = 1 \\ &\underline{A}^{(1)} \underline{s}_2 \leqq \alpha_1 \underline{1}_1 \\ &\underline{s}_2 \geqq \underline{0} \end{aligned} \quad \text{and} \quad \begin{aligned} &\text{minimize } \alpha_2 \\ &\text{subject to } \underline{1}_1^T \underline{s}_1 = 1 \\ &\underline{A}^{(1)T} \underline{s}_1 \geqq -\alpha_2 \underline{1}_2 \\ &\underline{s}_1 \geqq \underline{0}. \end{aligned} \tag{6.13}$$

A strategy pair $(\underline{s}_1^*, \underline{s}_2^*)$ is an equilibrium of the matrix game if and only if
(a) They are optimal solutions of problems (6.13) with some values of α_1 and α_2;
or
(b) They are feasible solutions of problems (6.13) with some values of α_1 and α_2 such that $\alpha_1 + \alpha_2 = 0$.
or
(c) They are feasible solutions of the system

$$\begin{aligned} \underline{A}^{(1)} \underline{s}_2 &\leqq \alpha \underline{1}_1, \\ \underline{1}_2^T \underline{s}_2 &= 1 \\ \underline{s}_2 &\geqq 0 \end{aligned} \qquad \begin{aligned} \underline{A}^{(1)T} \underline{s}_1 &\geqq \alpha \underline{1}_2 \\ \underline{1}_1^T \underline{s}_1 &= 1 \\ \underline{s}_1 &\geqq \underline{0} \end{aligned} \tag{6.14}$$

with some value of α.

The independence of the two problems of (6.13) implies that if $(\underline{s}_1^*, \underline{s}_2^*)$ and $(\underline{s}_1^{**}, \underline{s}_2^{**})$ are equilibria, then $(\underline{s}_1^*, \underline{s}_2^{**})$ and $(\underline{s}_1^{**}, \underline{s}_2^*)$ are also equilibria.

Example 6.5 Consider a matrix game with matrix

$$\underline{A}^{(1)} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 3 \\ -1 & 3 & 3 \end{pmatrix}.$$

By using the convenient notation

$$\underline{s}_1 = \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right)^T \text{ and } \underline{s}_2 = \left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right)^T$$

problem (6.13) can be rewritten as follows:

$$\begin{aligned} &\text{minimize } \alpha_1 \\ \text{subject to } &x_1^{(2)} + x_2^{(2)} + x_3^{(2)} = 1 \\ &2x_1^{(2)} + x_2^{(2)} - \alpha_1 \leqq 0 \\ &2x_1^{(2)} + 3x_3^{(2)} - \alpha_1 \leqq 0 \\ &-x_1^{(2)} + 3x_2^{(2)} + 3x_3^{(2)} - \alpha_1 \leqq 0 \\ &x_1^{(2)}, x_2^{(2)}, x_3^{(2)} \geqq 0 \end{aligned} \quad \text{and} \quad \begin{aligned} &\text{minimize } \alpha_2 \\ \text{subject to } &x_1^{(1)} + x_2^{(1)} + x_3^{(1)} = 1 \\ &2x_1^{(1)} + 2x_2^{(1)} - x_3^{(1)} + \alpha_2 \geqq 0 \\ &x_1^{(1)} + 3x_3^{(1)} + \alpha_2 \geqq 0 \\ &3x_2^{(1)} + 3x_3^{(1)} + \alpha_2 \geqq 0 \\ &x_1^{(1)}, x_2^{(1)}, x_3^{(1)} \geqq 0. \end{aligned}$$

A simple linear program solver gives the solution:

$$\underline{s}_1 = \left(\frac{4}{7}, \frac{4}{21}, \frac{5}{21}\right)^T, \alpha_2 = -\frac{9}{7}$$
$$\underline{s}_2 = \left(\frac{3}{7}, \frac{3}{7}, \frac{1}{7}\right)^T, \alpha_1 = \frac{9}{7}.$$

Notice that $\alpha_1 + \alpha_2 = 0$, as it should. ▼

In small dimensional cases the linear programming problems can be solved very easily as it is illustrated in the following example.

Example 6.6 Consider now a simple matrix game with matrix

$$\underline{A}^{(1)} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

which was the subject of the earlier Example 5.7. By introducing the notation

$$\underline{s}_1 = \begin{pmatrix} x \\ 1-x \end{pmatrix} \text{ and } \underline{s}_2 = \begin{pmatrix} y \\ 1-y \end{pmatrix}$$

the linear programming problem pair (6.13) has now the special form:

$$\begin{aligned} &\text{minimize } \alpha_1 \\ &\text{subject to } 0 \leqq y \leqq 1 \\ &2y + (1-y) \leqq \alpha_1 \\ &2(1-y) \leqq \alpha_1 \end{aligned} \quad \text{and} \quad \begin{aligned} &\text{minimize } \alpha_2 \\ &\text{subject to } 0 \leqq x \leqq 1 \\ &2x \geqq -\alpha_2 \\ &x + 2(1-x) \geqq -\alpha_2. \end{aligned} \tag{6.15}$$

The constraints can be rewritten as

$$\begin{aligned} -y + \alpha_1 &\geqq 1 \\ 2y + \alpha_1 &\geqq 2 \\ 0 \leqq y &\leqq 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x + \alpha_2 &\geqq 0 \\ -x + \alpha_2 &\geqq -2 \\ 0 \leqq x &\leqq 1. \end{aligned}$$

The feasible sets are shown in Figs. 6.2A, B.

In Fig. 6.2A the smallest α_1 value occurs at the intersection of the lines $\alpha_1 = y + 1$ and $\alpha_1 = 2 - 2y$ giving the solution $y = \frac{1}{3}, \alpha_1 = \frac{4}{3}$. In Fig. 6.2B the solution is the intersection of lines $\alpha_2 = x - 2$ and $\alpha_2 = -2x$ with the solution $x = \frac{2}{3}, \alpha_2 = -\frac{4}{3}$. So the mixed strategy equilibrium is

$$\underline{s}_1 = \left(\frac{2}{3}, \frac{1}{3}\right)^T \text{ and } \underline{s}_2 = \left(\frac{1}{3}, \frac{2}{3}\right)^T.$$

▼

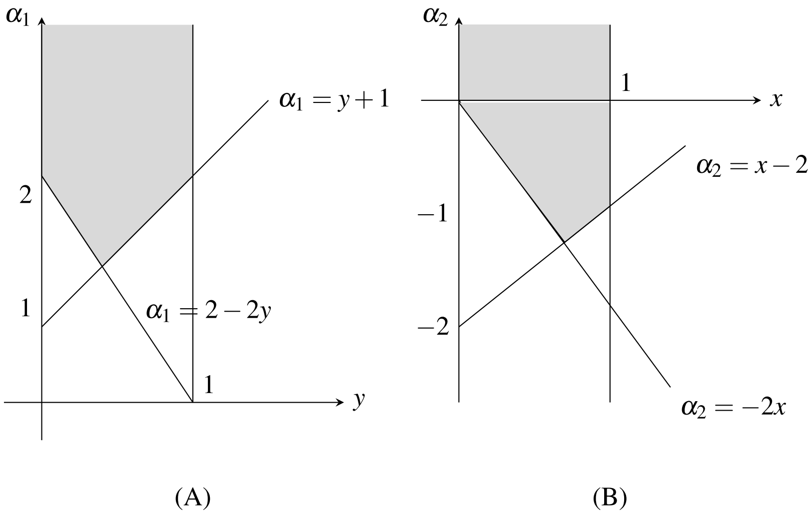

Fig. 6.2 Feasible sets in problems (6.15)

Finally we mention that the optimal value of α_1 in (6.13) has a special meaning. Notice that at any equilibrium $(\underline{s}_1^*, \underline{s}_2^*)$,

$$\underline{s}_1^{*T} \underline{A}^{(1)} \underline{s}_2^* \leqq \underline{s}_1^{*T} (\alpha_1 \underline{1}_1) = \alpha_1 (\underline{s}_1^{*T} \underline{1}_1) = \alpha_1$$

and

$$\begin{aligned} \underline{s}_1^{*T} \underline{A}^{(1)} \underline{s}_2^* = \underline{s}_2^{*T} \underline{A}^{(1)T} \underline{s}_1^* \geqq \underline{s}_2^{*T} (-\alpha_2 \underline{1}_2) = \underline{s}_2^{*T} (\alpha_1 \underline{1}_2) \\ = \alpha_1 (\underline{s}_2^{*T} \underline{1}_2) = \alpha_1 \end{aligned}$$

implying that at any equilibrium the payoff of player 1 equals α_1. This value is sometimes called the *value* of the matrix game.

6.5 Solution of Oligopolies

Consider now an industry of N firms producing identical product or offering the same service to a homogeneous market. Let $x_k \in [0, L_k]$ be the production level of firm k, where L_k is its capacity limit. The profit of firm k is given as

$$\phi_k(x_1, \ldots, x_N) = x_k p\Big(\sum_{l=1}^{N} x_l\Big) - C_k(x_k) \tag{6.16}$$

where p is the price function and C_k is the cost function of firm k. The strategy set of each firm is characterized by two inequalities

$$\begin{aligned} x_k &\geqq 0 \\ L_k - x_k &\geqq 0, \end{aligned}$$

so each firm has a two-dimensional vector $\underline{u}_k = (u_k^{(1)}, u_k^{(2)})^T$, by keeping the notation of (6.3). In addition

$$\underline{g}_k(x_k) = \begin{pmatrix} x_k \\ L_k - x_k \end{pmatrix}$$

with Jacobian matrix

$$\nabla_k \underline{g}_k(x_k) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Notice also that

$$\nabla_k \phi_k(\underline{x}_1, \ldots, \underline{x}_N) = p\Big(\sum_{l=1}^{N} x_l\Big) + x_k p'\Big(\sum_{l=1}^{N} x_l\Big) - C_k'(x_k),$$

so the optimization problem (6.8) can be specialized as follows:

$$\text{minimize} \ \sum_{k=1}^{N} \Big(u_k^{(1)} x_k + u_k^{(2)} (L_k - x_k)\Big)$$

$$\left.\begin{aligned} \text{subject to} \ u_k^{(1)}, u_k^{(2)} \geq 0 \\ 0 \leq x_k \leq L_k \\ p\left(\sum_{l=1}^{N} x_l\right) + x_k p'\left(\sum_{l=1}^{N} x_l\right) - C_k'(x_k) + \left(u_k^{(1)}, u_k^{2}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \end{aligned}\right\} (1 \leq k \leq N) \tag{6.17}$$

By introducing the new variables $\alpha_k = u_k^{(1)} - u_k^{(2)}$ and $\beta_k = u_k^{(2)}$ and by noticing that from the last constraint we have

$$\alpha_k = -\Big[p\Big(\sum_{l=1}^{N} x_l\Big) + x_k p'\Big(\sum_{l=1}^{N} x_l\Big) - C_k'(x_k)\Big],$$

problem (6.17) can be simplified in the following way:

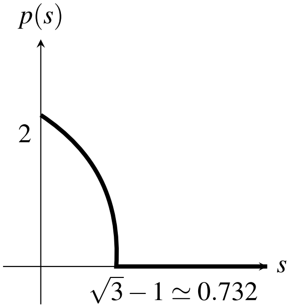

Fig. 6.3 Price function in Example 6.7

$$\text{minimize} \sum_{k=1}^{N} \left(-x_k \left[p\left(\sum_{l=1}^{N} x_l\right) + x_k p'\left(\sum_{l=1}^{N} x_l\right) - C_k'(x_k) \right] + \beta_k L_k \right) \quad (6.18)$$

$$\begin{aligned} &\text{subject to } \beta_k \geqq 0 \qquad\qquad [(1 \leqq k \leqq N).] \\ &\beta_k - \left[p\left(\textstyle\sum_{l=1}^{N} x_l\right) + x_k p'\left(\textstyle\sum_{l=1}^{N} x_l\right) - C_k'(x_k) \right] \geqq 0 \\ &0 \leqq x_k \leqq L_k. \end{aligned}$$

Hence the solutions of this nonlinear optimization problem provide the equilibria.

Example 6.7 Consider now a three-firm oligopoly with $N = 3$, capacity limits $L_1 = L_2 = L_3 = 1$, price function $p(s) = 2 - 2s - s^2 (s = x_1 + x_2 + x_3)$ and cost functions $C_k(x_k) = kx_k^3 + x_k$. The price function is shown in Fig. 6.3. In order to avoid negative price we have to define the price as zero for $s > \sqrt{3} - 1$. At any equilibrium $s \leq \sqrt{3} - 1$, otherwise at least one firm could decrease its output level so that s would still be greater than $\sqrt{3} - 1$ with decreased cost. So its payoff would increase.

In this case problem (6.18) becomes the following:

$$\text{minimize} \sum_{k=1}^{3} \left(-x_k(2 - 2s - s^2 - 2x_k - 2x_k s - 3kx_k^2 - 1) + \beta_k \right)$$

$$\left.\begin{aligned} \text{subject to } 0 \leqq x_k \leqq 1 \\ \beta_k \geqq 0 \\ \beta_k - [2 - 2s - s^2 - 2x_k - 2x_k s - 3kx_k^2 - 1] \geqq 0 \end{aligned}\right\} (1 \leqq k \leqq 3). \quad (6.19)$$

$$x_1 + x_2 + x_3 = s.$$

A computer program gave the optimal solution:

$$x_1^* = 0.1077,\ x_2^* = 0.0986 \text{ and } x_3^* = 0.0919.$$

▼

Chapter 7
Special Matrix Games

In this chapter some special classes of matrix games will be discussed.

7.1 Matrix with Identical Elements

Assume that the matrix game is defined with an $m_1 \times m_2$ matrix such that all elements of $\underline{A}^{(1)}$ are equal to a given constant α. That is,

$$\underline{A}^{(1)} = \begin{pmatrix} \alpha & \cdots & \alpha \\ \vdots & \ddots & \vdots \\ \alpha & \cdots & \alpha \end{pmatrix} = \alpha \underline{1}_1 \underline{1}_2^T,$$

where $\underline{1}_1 \in \mathbb{R}^{m_1}$, $\underline{1}_2 \in \mathbb{R}^{m_2}$ and all elements of these vectors equal unity, so the payoff of player 1 equals

$$\phi_1 = \underline{s}_1^T \alpha \underline{1}_1 \underline{1}_2^T \underline{s}_2 = \alpha(\underline{s}_1^T \underline{1}_1)(\underline{1}_2^T \underline{s}_2) = \alpha.$$

Then $\phi_2 = -\alpha$, that is, the payoff functions do not depend on the strategy selection, therefore any strategy pair gives equilibrium.

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_7

7.2 The Case of Diagonal Matrix

Assume now that the matrix game is defined by the diagonal matrix

$$\underline{A}^{(1)} = \begin{pmatrix} a_1 & & \bigcirc \\ & a_2 & \\ & & \ddots \\ \bigcirc & & a_n \end{pmatrix}.$$

Let $\underline{s}_k = \left(x_1^{(k)}, \ldots, x_n^{(k)}\right)^T$ for $k = 1, 2$, then conditions (6.14) can be written as follows:

$$\begin{aligned} a_k x_k^{(2)} &\leqq \alpha \\ a_k x_k^{(1)} &\geqq \alpha \\ x_k^{(1)}, x_k^{(2)} &\geqq 0 \\ \sum_{k=1}^{n} x_k^{(1)} = \sum_{k=1}^{n} x_k^{(2)} &= 1. \end{aligned} \tag{7.1}$$

We will now consider three cases.

If $a_k > 0$ for all k, then for at least one k, $a_k x_k^{(2)} > 0$, so $\alpha > 0$. Therefore

$$1 = \sum_{k=1}^{n} x_k^{(2)} \leqq \sum_{k=1}^{n} \frac{\alpha}{a_k} \leqq \sum_{k=1}^{n} x_k^{(1)} = 1$$

where the inequality also holds for all terms k. This relation holds if and only if $x_k^{(1)} = x_k^{(2)} = \frac{\alpha}{a_k}$, however

$$1 = \sum_{l=1}^{n} x_l^{(1)} = \alpha \sum_{l=1}^{n} \frac{1}{a_l},$$

so the only equilibrium is

$$x_k^{(1)} = x_k^{(2)} = \frac{\alpha}{a_k} \quad \text{with} \quad \alpha = \frac{1}{\sum_{l=1}^{n} \frac{1}{a_l}}.$$

If $a_k < 0$ for all k, then for at least one k, $a_k x_k^{(1)} < 0$, so $\alpha < 0$. Then from (7.1),

$$\begin{aligned} (-a_k) x_k^{(2)} &\geqq -\alpha \\ (-a_k) x_k^{(1)} &\leqq -\alpha, \end{aligned}$$

furthermore

$$1 = \sum_{k=1}^{n} x_k^{(2)} \geqq \sum_{k=1}^{n} \frac{-\alpha}{-a_k} \geqq \sum_{k=1}^{n} x_k^{(1)} = 1,$$

so for all k, $x_k^{(1)} = x_k^{(2)} = \frac{\alpha}{a_k}$. However

$$1 = \sum_{l=1}^{n} x_l^{(1)} = \alpha \sum_{l=1}^{n} \frac{1}{a_l},$$

the only equilibrium is

$$x_k^{(1)} = x_k^{(2)} = \frac{\alpha}{a_k} \quad \text{with} \quad \alpha = \frac{1}{\sum_{l=1}^{n} \frac{1}{a_l}}.$$

If $a_i \geqq 0$ and $a_j \leqq 0$ with some i and j, then

$$\alpha \geqq a_i x_i^{(2)} \geqq 0 \geqq a_j x_j^{(1)} \geqq \alpha$$

implying that $\alpha = 0$, so for all k conditions (7.1) can be rewritten as

$$a_k x_k^{(2)} \leqq 0 \leqq a_k x_k^{(1)}.$$

Define

$$u_k \begin{cases} = 0 & \text{if } a_k > 0 \\ \geqq 0 & \text{if } a_k \leqq 0 \end{cases}$$

and

$$v_k \begin{cases} = 0 & \text{if } a_k < 0 \\ \geqq 0 & \text{if } a_k \geqq 0 \end{cases}$$

arbitrarily and let

$$u = \sum_{k=1}^{n} u_k \quad \text{and} \quad v = \sum_{k=1}^{n} v_k.$$

Then any strategy pair of the form

$$\underline{s}_1 = \frac{1}{v}(v_1, \ldots, v_n)^T \quad \text{and} \quad \underline{s}_2 = \frac{1}{u}(u_1, \ldots, u_n)^T$$

provides equilibrium.

7.3 Symmetric Matrix Games

Here we assume that $\underline{A}^{(1)}$ is an $n \times n$ skew-symmetric matrix, that is,

$$\underline{A}^{(1)T} = -\underline{A}^{(1)}.$$

In this case problems (6.13) have the following form:

$$\begin{aligned} &\text{minimize } \alpha_1 \\ &\text{subject to } \underline{1}_2^T \underline{s}_2 = 1 \\ &\underline{A}^{(1)} \underline{s}_2 \leqq \alpha_1 \underline{1}_1 \\ &\underline{s}_2 \geqq \underline{0} \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\text{minimize } \alpha_2 \\ &\text{subject to } \underline{1}_1^T \underline{s}_1 = 1 \\ &-\underline{A}_1^{(1)} \underline{s}_1 \geqq -\alpha_2 \underline{1}_2 \\ &\underline{s}_1 \geqq \underline{0} \end{aligned}$$

showing that the two linear programming problems are identical. So the optimal solutions and the optimal objective function values are equal. Since $\alpha_1 + \alpha_2 = 0$ at the optimum, $\alpha_1 = \alpha_2 = 0$. Let X^* denote the set of optimal solutions, then a strategy pair (s_1^*, s_2^*) is an equilibrium if and only if $s_1^*, s_2^* \in X^*$. It is clear, that

$$X^* = \{\underline{s} \in \mathbb{R}^n | \underline{s} \geqq \underline{0}, \underline{1}^T \underline{s} = 1, \underline{A}^{(1)} \underline{s} \leqq \underline{0}\}. \tag{7.2}$$

Example 7.1 Assume that

$$\underline{A}^{(1)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 1 & -2 & 1 \end{pmatrix},$$

then X^* is the set of all three-element vectors such that

$$\begin{aligned} s_1, s_2, s_3 &\geqq 0 \\ s_1 + s_2 + s_3 &= 1 \\ s_1 - s_3 &\leqq 0 \\ 2s_2 + 2s_3 &\leqq 0 \\ s_1 - 2s_2 + s_3 &\leqq 0. \end{aligned}$$

▼

7.4 Relation Between Matrix Games and Linear Programming

Consider a primal-dual pair of a linear programming problem:

$$\text{(P)} \quad \begin{aligned} \text{maximize } & \underline{c}^T \underline{x} \\ \text{subject to } & \underline{A}\underline{x} \leqq \underline{b} \\ & \underline{x} \geqq \underline{0} \end{aligned} \qquad \text{(D)} \quad \begin{aligned} \text{minimize } & \underline{b}^T \underline{y} \\ \text{subject to } & \underline{A}^T \underline{y} \geqq \underline{c} \\ & \underline{y} \geqq \underline{0}. \end{aligned}$$

Construct a skew-symmetric matrix

$$\underline{P} = \begin{pmatrix} \underline{0} & \underline{A} & -\underline{b} \\ -\underline{A}^T & \underline{0} & \underline{c} \\ \underline{b}^T & -\underline{c}^T & 0 \end{pmatrix}$$

which defines a symmetric matrix game with $\underline{A}^{(1)} = \underline{P}$, which will be called the P-game.

The relation between the linear programming primal-dual pair and the P-game is given by the following result.

Theorem 7.1 *Let $\underline{z} = (\underline{u}, \underline{v}, \lambda)$ be an equilibrum strategy of the P-game with $\lambda > 0$, then*

$$\underline{x} = \frac{1}{\lambda}\underline{v} \ \text{ and } \ \underline{y} = \frac{1}{\lambda}\underline{u}$$

are optimal solutions of the primal and dual problems, respectively.

Proof If $\underline{z}$ is an equilibrium, then $\underline{P}\ \underline{z} \leqq \underline{0}$, that is,

$$\underline{A}\underline{v} - \underline{b}\lambda \leqq \underline{0} \tag{7.3}$$

$$-\underline{A}^T\underline{u} + \underline{c}\lambda \leqq \underline{0} \tag{7.4}$$

$$\underline{b}^T\underline{u} - \underline{c}^T\underline{v} \leqq 0. \tag{7.5}$$

Since $\lambda > 0$ and $\underline{z} \geqq \underline{0}$,

$$\underline{x} = \frac{1}{\lambda}\underline{v} \geqq \underline{0} \ \text{ and } \ \underline{y} = \frac{1}{\lambda}\underline{u} \geqq \underline{0},$$

furthermore from (7.3),

$$\underline{A}\underline{x} \leqq \underline{b}$$

and from (7.4),

$$\underline{A}^T\underline{y} \geqq \underline{c}$$

implying that $\underline{x}$ and $\underline{y}$ are feasible solutions of the primal and dual problems, respectively. Furthermore (7.5) implies that

$$\underline{b}^T \underline{y} \leqq \underline{c}^T \underline{x},$$

however from the weak duality property (see Appendix F) we know that

$$\underline{b}^T \underline{y} \geqq \underline{c}^T \underline{x},$$

so $\underline{b}^T \underline{y} = \underline{c}^T \underline{x}$, and then the strong duality theorem (see Appendix F) implies that $\underline{x}$ is an optimal solution of the primal and $\underline{y}$ is an optimal solution of the dual problem.

■

Example 7.2 Consider now the following linear programming problem:

$$\begin{aligned} &\text{maximize } x_1 + 2x_2 \\ &\text{subject to } x_1 \geqq 0 \\ &\qquad -x_1 + x_2 \geqq 1 \\ &\qquad 5x_1 + 7x_2 \leqq 25. \end{aligned}$$

First we have to rewrite this problem into a primal form. Since there is no sign restriction on x_2, we have to rewrite it as the difference of two nonnegative variables,

$$x_2 = x_2^+ - x_2^-$$

where $x_2^+, x_2^- \geqq 0$. Then the objective function and the sign constraints become

$$\begin{aligned} &\text{maximize } x_1 + 2x_2^+ - 2x_2^- \\ &\text{subject to } x_1, x_2^+, x_2^- \geqq 0. \end{aligned}$$

The other two conditions become the following:

$$-x_1 + x_2^+ - x_2^- \geqq 1,$$

which can be rewritten as

$$x_1 - x_2^+ + x_2^- \leqq -1$$

and the last constraint becomes

$$5x_1 + 7x_2^+ - 7x_2^- \leqq 25.$$

Therefore in the primal problem

$$\underline{A} = \begin{pmatrix} 1 & -1 & 1 \\ 5 & 7 & -7 \end{pmatrix}, \; \underline{b} = \begin{pmatrix} -1 \\ 25 \end{pmatrix} \text{ and } \underline{c}^T = (1, 2, -2).$$

The corresponding P-game is defined by the matrix

$$\underline{P} = \left(\begin{array}{cc|ccc|c} 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 5 & 7 & -7 & -25 \\ \hline -1 & -5 & 0 & 0 & 0 & 1 \\ 1 & -7 & 0 & 0 & 0 & 2 \\ -1 & 7 & 0 & 0 & 0 & -2 \\ \hline -1 & 25 & -1 & -2 & 2 & 0 \end{array}\right).$$

The equilibria of this game can provide the primal and dual optimal solutions of the linear programming problem.

▼

Consider next a matrix game with $\underline{A}^{(1)} > \underline{0}$. This condition does not restrict generality, since adding the same constant to all elements of $\underline{A}^{(1)}$ does not change the equilibria. Construct then the symmetric matrix game with matrix

$$\underline{P} = \begin{pmatrix} \underline{0} & \underline{A}^{(1)} & -\underline{1} \\ -\underline{A}^{(1)T} & \underline{0} & \underline{1} \\ \underline{1}^T & -\underline{1}^T & 0 \end{pmatrix}.$$

The following theorem shows the equivalence of matrix game with $\underline{A}^{(1)}$ and the symmetric matrix game with $\underline{P}$.

Theorem 7.2 *The following relations are valid with $\underline{A}^{(1)} > 0$:*
(a) If $\underline{z} = (\underline{u}, \underline{v}, \lambda)$ is an equilibrium of the P-game, then with $a = \frac{1-\lambda}{2}$,

$$\underline{x} = \frac{1}{a}\underline{u} \text{ and } \underline{y} = \frac{1}{a}\underline{v}$$

the strategy pair $(\underline{x}, \underline{y})$ is an equilibrium of the matrix game with matrix $\underline{A}^{(1)}$ and $\frac{\lambda}{a}$ is the value of the game;
(b) If $(\underline{x}, \underline{y})$ is an equilibrium of the matrix game with $\underline{A}^{(1)}$ and v is the value of the game, then vector

$$\underline{z} = \frac{1}{2+v}(\underline{x}, \underline{y}, v)^T$$

is an equilibrium strategy of the P-game.

Proof (a) Assume that $\underline{z} = (\underline{u}, \underline{v}, \lambda)$ is an equilibrium of the P-game, then $\underline{P}\underline{z} \leqq \underline{0}$, that is,

$$\underline{A}^{(1)}\underline{v} - \underline{1}\lambda \leqq \underline{0} \tag{7.6}$$

$$-\underline{A}^{(1)T}\underline{u} + \underline{1}\lambda \leqq \underline{0} \tag{7.7}$$

$$\underline{1}^T\underline{u} - \underline{1}^T\underline{v} \leqq 0. \tag{7.8}$$

First we show that $0 < \lambda < 1$. If $\lambda = 1$, then (since $\underline{z}$ is a probability vector) $\underline{u} = \underline{0}$ and $\underline{v} = \underline{0}$ contradicting to (7.7). If $\lambda = 0$, then $\underline{1}^T\underline{u} + \underline{1}^T\underline{v} = 1$ and by (7.8), $\underline{v}$ must have at least one positive component which contradicts (7.6).

Next we show that $\underline{1}^T\underline{u} = \underline{1}^T\underline{v}$. From (7.6) and (7.7),

$$\underline{u}^T\underline{A}^{(1)}\underline{v} - \underline{u}^T\underline{1}\lambda \leqq 0$$

$$-\underline{v}^T\underline{A}^{(1)T}\underline{u} + \underline{v}^T\underline{1}\lambda \leqq 0$$

and by adding these inequalities we have

$$\lambda(\underline{v}^T\underline{1} - \underline{u}^T\underline{1}) \leqq 0$$

implying that

$$\underline{1}^T\underline{v} \leqq \underline{1}^T\underline{u}.$$

This relation and (7.8) imply that $\underline{1}^T\underline{u} = \underline{1}^T\underline{v}$.

Select $a = \frac{1-\lambda}{2} (> 0)$ then $\underline{1}^T\underline{v} = \underline{1}^T\underline{u} = a$, so both vectors

$$\underline{x} = \frac{1}{a}\underline{u} \quad \text{and} \quad \underline{y} = \frac{1}{a}\underline{v}$$

are probability vectors, furthermore from (7.7)

$$\underline{A}^{(1)T}\underline{x} = \frac{1}{a}\underline{A}^{(1)T}\underline{u} \geqq \frac{\lambda}{a}\underline{1}$$

and from (7.6),

$$\underline{A}^{(1)}\underline{y} = \frac{1}{a}\underline{A}^{(1)}\underline{v} \leqq \frac{\lambda}{a}\underline{1}.$$

Select finally $\alpha_1 = \frac{\lambda}{a}$ and $\alpha_2 = -\frac{\lambda}{a}$, then $(\underline{x}, \alpha_2)$ and $(\underline{y}, \alpha_1)$ are feasible solutions of the linear programming pair (6.12) with $\alpha_1 + \alpha_2 = 0$ implying that they are optimal solutions, so the strategy pair $(\underline{x}, \underline{y})$ is an equilibrium of the matrix game with matrix $\underline{A}^{(1)}$.

(b) The proof of the other part can be made similarly to that of part (a), so it is left as an exercise to the interested reader.

■

Example 7.3 Consider now the matrix game with matrix

$$\underline{A} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 3 \\ -1 & 3 & 3 \end{pmatrix}.$$

Since $\underline{A}$ has a negative element, we can add 2 to all of its elements without changing the equilibria. So we can now consider the matrix game with matrix

$$\underline{A}^{(1)} = \begin{pmatrix} 4 & 3 & 2 \\ 4 & 2 & 5 \\ 1 & 5 & 5 \end{pmatrix}.$$

Then the equivalent symmetric matrix game is defined with matrix

$$\underline{P} = \left(\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 4 & 3 & 2 & -1 \\ 0 & 0 & 0 & 4 & 2 & 5 & -1 \\ 0 & 0 & 0 & 1 & 5 & 5 & -1 \\ \hline -4 & -4 & -1 & 0 & 0 & 0 & 1 \\ -3 & -2 & -5 & 0 & 0 & 0 & 1 \\ -2 & -5 & -5 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & -1 & -1 & -1 & 0 \end{array}\right).$$

▼

7.5 Method of Fictitious Play

In this section an iteration method will be introduced to solve matrix games. The idea is that at each iteration step the players select best responses against the last selected strategies of the competitors. This idea without any modification has the problem that the best choices are usually basis vectors depending on which component of $\underline{A}^{(1)}\underline{s}_2$ or $-\underline{s}_1^T\underline{A}^{(1)}$ is maximal. Sequence of different basis vectors never converge, so mixed strategies have to be introduced into the procedure. The method can be described in the following way:

At the initial step, $k = 1$, player 1 selects an initial strategy, $\underline{s}_1^{(1)}$, and then player 2 finds his best response by finding the basis vector $\underline{e}_{j_1}$ such that

$$\underline{s}_1^{(1)T}\underline{A}^{(1)}\underline{e}_{j_1} = \min_j \left\{\underline{s}_1^{(1)T}\underline{A}^{(1)}\underline{e}_j\right\} \tag{7.9}$$

and let $\underline{s}_2^{(1)} = \underline{e}_{j_1}$. Notice that j_1 is the index of the smallest element of $\underline{s}_1^{(1)T}\underline{A}^{(1)}$.

At each further step $k \geqq 2$ let

$$\underline{\bar{s}}_2^{(k-1)} = \frac{1}{k-1} \sum_{i=1}^{k-1} \underline{s}_2^{(i)} \tag{7.10}$$

be the average of all previous basis vector choices of player 2 and select $\underline{s}_1^{(k)} = \underline{e}_{i_k}$ such that

$$\underline{e}_{i_k}^T \underline{A}^{(1)} \underline{\bar{s}}_2^{(k-1)} = \max_i \left\{ \underline{e}_i^T \underline{A}^{(1)} \underline{\bar{s}}_2^{(k-1)} \right\}. \tag{7.11}$$

Notice that element i_k is the largest in vector $\underline{A}^{(1)} \underline{\bar{s}}_2^{(k-1)}$. Let

$$\underline{\bar{s}}_1^{(k)} = \frac{1}{k} \sum_{i=1}^{k} \underline{s}_1^{(i)} \tag{7.12}$$

be the average of all previous basis vector choices of player 1, and select $\underline{s}_2^{(k)} = \underline{e}_{j_k}$ such that

$$\underline{\bar{s}}_1^{(k)T} \underline{A}^{(1)} \underline{e}_{j_k} = \min_j \left\{ \underline{\bar{s}}_1^{(k)T} \underline{A}^{(1)} \underline{e}_j \right\}, \tag{7.13}$$

where component j_k is the smallest in vector $\underline{\bar{s}}_1^{(k)T} \underline{A}^{(1)}$.

Then go back with the next value of k.

We present the convergence theorem of this algorithm (Shapiro, 1958) without giving the lengthy and complicated proof.

Theorem 7.3 *Any limit point of sequences* $\left\{ \underline{\bar{s}}_1^{(k)} \right\}$ *and* $\left\{ \underline{\bar{s}}_2^{(k)} \right\}$ *gives equilibrium strategies.*

Example 7.4 Consider the matrix game with $\underline{A}^{(1)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, which has no pure strategy equilibrium. The method of fictitious play consists of the following steps.
$k = 1$: Select

$$\underline{s}_1^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then

$$\underline{s}_1^{(1)T} \underline{A}^{(1)} = (1, 0) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (1, 2)$$

where the minimal component is the first. So

$$\underline{s}_2^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$k = 2$:

$$\underline{\bar{s}}_2^{(1)} = \underline{s}_2^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\underline{A}^{(1)}\overline{\underline{s}}_2^{(1)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where the largest component is the second, so

$$\underline{s}_1^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and } \overline{\underline{s}}_1^{(2)} = \frac{1}{2}\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Since

$$\overline{\underline{s}}_1^{(2)T}\underline{A}^{(1)} = (\frac{1}{2}, \frac{1}{2})\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (\frac{3}{2}, \frac{3}{2}),$$

either component can be selected as minimal. For example let $\underline{s}_2^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$k = 3$:

$$\overline{\underline{s}}_2^{(2)} = \frac{1}{2}\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\underline{A}^{(1)}\overline{\underline{s}}_2^{(2)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where the maximum component is the second, so

$$\underline{s}_1^{(3)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \overline{\underline{s}}_1^{(3)} = \frac{1}{3}\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Since

$$\overline{\underline{s}}_1^{(3)T}\underline{A}^{(1)} = (\frac{1}{3}, \frac{2}{3})\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (\frac{5}{3}, \frac{4}{3})$$

with the minimal component being the second, we have

$$\underline{s}_2^{(3)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$k = 4$:

$$\overline{\underline{s}}_2^{(3)} = \frac{1}{3}\Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big] = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

and

$$\underline{A}^{(1)}\overline{\underline{s}}_2^{(3)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$$

where the second component is the largest, so

$$\underline{s}_1^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \overline{\underline{s}}_1^{(4)} = \frac{1}{4}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}.$$

Since

$$\overline{\underline{s}}_1^{(4)T} \underline{A}^{(1)} = (\frac{1}{4}, \frac{3}{4}) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (\frac{7}{4}, \frac{5}{4})$$

with the second component being the smaller, we have

$$\underline{s}_2^{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so on for $k = 5, 6, \ldots$

▼

Since linear programming problems can be reformulated as matrix games, the fictitious play method can be also used to solve linear programming problems. The convergence of this method is usually much slower than the application of the simplex method. Therefore it has only theoretical importance.

7.6 Method of Von Neumann

Consider a symmetric matrix game with matrix $\underline{P}$. Since any matrix game can be reduced to an equivalent symmetric matrix game by Theorem 7.2, this assumption does not restrict the general application of the method. So $\underline{P}$ is an $n \times n$ skew-symmetric matrix.

Introduce the following functions:

$$u_i : \mathbb{R}^n \mapsto \mathbb{R} \text{ such that } u_i(\underline{s}) = \underline{e}_i^T \underline{P}\underline{s} \, (i = 1, 2, \ldots, n)$$

which is the ith element of vector $\underline{P}\underline{s}$;

$$\phi : \mathbb{R} \mapsto \mathbb{R} \text{ such that } \phi(u) = \max\{0; u\} \geqq 0$$

$$\Phi : \mathbb{R}^n \mapsto \mathbb{R} \text{ such that } \Phi(\underline{s}) = \sum_{i=1}^{n} \phi\big(u_i(\underline{s})\big) \geqq 0$$

$$\psi : \mathbb{R}^n \mapsto \mathbb{R} \text{ such that } \psi(\underline{s}) = \sum_{i=1}^{n} \phi^2\big(u_i(\underline{s})\big) \geqq 0,$$

and consider the system of ordinary differential equations:

$$\dot{s}_j(t) = \phi\Big(u_j\big(\underline{s}(t)\big)\Big) - \Phi\big(\underline{s}(t)\big)s_j(t)\,(j = 1, 2, \ldots, n) \tag{7.14}$$

with a probability vector initial condition,

$$s_j(0) = s_j^{(0)}\,(j = 1, 2, \ldots, n)$$

such that $0 \leqq s_j^{(0)} \leqq 1$ and $\sum_{j=1}^{n} s_j^{(0)} = 1$.

The idea behind Eq. (7.14) is the following. The value of the symmetric matrix game is zero. If at a strategy $\underline{s}$, $\phi\big(u_j(\underline{s})\big) > 0$, then $\underline{e}_j^T \underline{P}\underline{s} > 0$ however $\underline{e}_j^T \underline{P}\underline{e}_j = 0$, so player 2 needs to increase s_j to unity (since his payoff is zero at $\underline{e}_j$ and negative at $\underline{s}$). The increase of $\underline{s}_j$ in this case is represented by the first term of the right hand side. Similarly, if $\phi\big(u_j(\underline{s})\big) < 0$, then player 2 wants to decrease s_j. The second term of the right hand side guarantees that vector $\underline{s}(t)$ remains a probability vector for all $t \geqq 0$. The convergence of vectors $\underline{s}(t)$ as $t \to \infty$ is stated in the following theorem.

Theorem 7.4 *Let t_k $(k = 1, 2, \ldots)$ be a positive, strictly increasing sequence which converges to infinity. Then any limit point of the sequence $\big\{\underline{s}(t_k)\big\}$ is an equilibrium strategy and there exists a constant $c > 0$ such that*

$$\underline{e}_j^T \underline{P}\underline{s}(t_k) \leqq \frac{\sqrt{n}}{c + t_k}. \tag{7.15}$$

Proof The proof consists of several steps.
(a) We first show that $\underline{s}(t)$ is a probability vector for all $t \geqq 0$. Assume that with some j and $t_1 > 0$, $s_j(t_1) < 0$. Let

$$t_0 = \sup\big\{t|0 < t < t_1, s_j(t) \geqq 0\big\}.$$

By continuity, $s_j(t_0) = 0$ and for all $\tau \in (t_0, t_1]$, $s_j(\tau) < 0$. Since $\phi(u_j)$ and $\Phi\big(\underline{s}(t)\big)$ are nonnegative,

$$\dot{s}_j(\tau) = \phi\Big(u_j\big(\underline{s}(\tau)\big)\Big) - \Phi\big(\underline{s}(\tau)\big)s_j(\tau) \geqq 0$$

and by the Lagrange mean-value theorem

$$s_j(t_1) = s_j(t_0) + \dot{s}_j(\tau)(t_1 - t_0) \geqq 0$$

contradicting the assumption that $s_j(t_1) < 0$.

Next we show that for all $t \geqq 0$, $\sum_{j=1}^{n} s_j(t) = 1$. Notice that

$$\Big(1 - \sum_{j=1}^{n} s_j(t)\Big)' = -\sum_{j=1}^{n} \dot{s}_j(t) = -\sum_{j=1}^{n} \phi\Big(u_j\big(\underline{s}(t)\big)\Big) + \Phi\big(\underline{s}(t)\big)\sum_{j=1}^{n} s_j(t)$$

$$= -\underline{\Phi}\big(\underline{s}(t)\big)\Big[1 - \sum_{j=1}^{n} s_j(t)\Big].$$

So function

$$f(t) = 1 - \sum_{j=1}^{n} s_j(t)$$

satisfies the homogeneous differential equation

$$\dot{f}(t) = -\Phi\big(\underline{s}(t)\big) f(t) \tag{7.16}$$

with the initial condition $f(0) = 0$, therefore for all $t \geqq 0$, $f(t) = 0$.
(b) Assume that for some $t \geqq 0$, $\phi\Big(u_i\big(\underline{s}(t)\big)\Big) > 0$, then

$$\frac{d}{dt}\phi\Big(u_i\big(\underline{s}(t)\big)\Big) = \frac{d}{dt}u_i\big(\underline{s}(t)\big) = \underline{e}_i^T \underline{P}\dot{\underline{s}}(t) = \sum_{j=1}^{n} p_{ij}\dot{s}_j(t)$$

where p_{ij} is the (i, j) element of matrix $\underline{P}$. Therefore

$$\frac{d}{dt}\phi\Big(u_i\big(\underline{s}(t)\big)\Big) = \sum_{j=1}^{n} p_{ij}\phi\Big(u_j\big(\underline{s}(t)\big)\Big) - \sum_{j=1}^{n} p_{ij}\Phi\big(\underline{s}(t)\big)s_j(t).$$

Notice that the second term equals

$$\Phi\big(\underline{s}(t)\big)\sum_{j=1}^{n} p_{ij}s_j(t) = \Phi\big(\underline{s}(t)\big)\phi\big(u_i(t)\big). \tag{7.17}$$

Multiplying both sides of (7.17) by $\phi\big(u_i(t)\big)$ and adding up the resulting equations for $i = 1, 2, \ldots, n$ we have

$$\sum_{i=1}^{n}\phi(u_i)\frac{d}{dt}\phi(u_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}\phi(u_i)\phi(u_j) - \Phi\big(\underline{s}(t)\big)\psi\big(\underline{s}(t)\big).$$

Since $\underline{P}$ is skew-symmetric, the first term equals zero, so

$$\frac{1}{2}\frac{d}{dt}\psi\big(\underline{s}(t)\big) = -\underline{\Phi}\big(\underline{s}(t)\big)\psi\big(\underline{s}(t)\big). \tag{7.18}$$

Notice that if $\phi\Big(u_i\big(\underline{s}(t)\big)\Big) = 0$, then this relation still holds, since zero terms are added to both sides.

(c) Assume next that with some t_0, $\psi\big(\underline{s}(t_0)\big) = 0$. Then from the differential equation $\psi\big(\underline{s}(t)\big) = 0$ for all $t \geqq t_0$, so for all i, $\phi_i\big(\underline{s}(t)\big) = 0$ implying that $\underline{P}\underline{s}(t)) \leqq \underline{0}$, so $\underline{s}(t)$ is equilibrium.
(d) If $\psi\big(\underline{s}(t)\big) > 0$ for all t, then

$$\frac{1}{2}\frac{d}{dt}\psi\big(\underline{s}(t)\big) \leqq -\psi\big(\underline{s}(t)\big)^{\frac{3}{2}} \tag{7.19}$$

since clearly

$$\Phi\big(\underline{s}(t)\big)^2 \geqq \psi\big(\underline{s}(t)\big).$$

Relation (7.19) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\psi\big(\underline{s}(t)\big)\psi\big(\underline{s}(t)\big)^{-\frac{3}{2}} \leqq -1$$

and by integrating both sides between 0 and t we have

$$-\psi\big(\underline{s}(t)\big)^{-\frac{1}{2}} + c \leqq -t$$

with $c = \psi\big(\underline{s}(0)\big)^{-\frac{1}{2}}$. Therefore

$$\psi\big(\underline{s}(t)\big)^{\frac{1}{2}} \leqq \frac{1}{c+t},$$

and so

$$\underline{e}_i^T \underline{P}\underline{s}(t) \leqq \phi(u_i\big(\underline{s}(t)\big) \leqq \Phi\big(\underline{s}(t)\big) \leqq \sqrt{n\psi\big(\underline{s}(t)\big)} \leqq \frac{\sqrt{n}}{c+t}$$

since the Cauchy-Schwarz inequality implies that

$$\Phi(\underline{s}) = \sum_{i=1}^{n} 1 \cdot \phi\big(u_i(\underline{s})\big) \leqq \sqrt{\sum_{i=1}^{n} 1}\sqrt{\sum_{i=1}^{n} \phi^2\big(u_i(\underline{s})\big)}.$$

Taking any increasing sequence $\{t_k\}$ such that $t_k \to \infty$, for any limit point $\underline{s}^*$ and all i,

$$\underline{e}_i^T \underline{P}\underline{s}^* \leqq 0$$

so $\underline{P}\underline{s}^* \leqq \underline{0}$ implying that $\underline{s}^*$ is an equilibrium.

■

Example 7.5 Consider again the matrix game of the previous example with matrix

$$\underline{A}^{(1)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Since this matrix game is not symmetric, we have to use Theorem 7.2 to construct the equivalent symmetric matrix game with matrix

$$\underline{P} = \left(\begin{array}{cc|cc|c} 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 1 & -1 \\ \hline -1 & -2 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 & 1 \\ \hline 1 & 1 & -1 & -1 & 0 \end{array}\right).$$

Notice

$$\begin{aligned} u_1(\underline{s}) &= s_3 + 2s_4 - s_5 \\ u_2(\underline{s}) &= 2s_3 + s_4 - s_5 \\ u_3(\underline{s}) &= -s_1 - 2s_2 + s_5 \\ u_4(\underline{s}) &= -2s_1 - s_2 + s_5 \\ u_5(\underline{s}) &= s_1 + s_2 - s_3 - s_4, \end{aligned}$$

so

$$\begin{aligned} \phi(u_1) &= \max\{0; s_3 + 2s_4 - s_5) \\ \phi(u_2) &= \max\{0; 2s_3 + s_4 - s_5) \\ \phi(u_3) &= \max\{0; -s_1 - 2s_2 + s_5) \\ \phi(u_4) &= \max\{0; -2s_1 - s_2 + s_5) \\ \phi(u_5) &= \max\{0; s_1 + s_2 - s_3 - s_4). \end{aligned}$$

and

$$\Phi\big(\underline{s}(t)\big) = \sum_{i=1}^{5} \phi(u_i).$$

So the 5-dimensional differential equation (7.14) can be easily formulated and solved by any computer method.

▼

Chapter 8
Uniqueness of Equilibria

In examining the existence conditions for equilibria in N-person games either the Banach or the Kakutani fixed point theorem was used. The Banach fixed point theorem guaranteed the existence of the unique equilibrium and an iteration algorithm was also suggested to compute the equilibrium. However the existence theorems based on the Kakutani fixed point theorem (Theorems 5.3 and 5.4) do not guarantee uniqueness. For example, by selecting constant payoff functions all strategies provide equilibria, and constant functions are continuous as well as concave. So the conditions of the Nikaido-Isoda theorem are satisfied if the strategy sets are nonempty, convex, closed and bounded. It is well known from optimization theory that strictly concave functions cannot have multiple maximum points. Unfortunately for N-person games this result cannot be extended, since as the following example shows, there is the possibility of even infinitely many equilibria if all payoff functions are strictly concave.

Example 8.1 Consider a two-person game ($N = 2$) with strategy sets $S_1 = S_2 = [0, 1]$ and payoff functions

$$\phi_1(x, y) = \phi_2(x, y) = 2x + 2y - (x + y)^2$$

for $x \in S_1$ and $y \in S_2$.

Notice that all conditions of the Nikaido-Isoda Theorem are satisfied and ϕ_1 is strictly concave in x and ϕ_2 strictly concave in y.

Since

$$\frac{\partial \phi_1}{\partial x} = 2 - 2(x + y),$$

the best response of player 1 is given as

$$R_1(y) = 1 - y,$$

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_8

and similarly that of player 2 is the following:

$$R_2(x) = 1 - x.$$

These best responses are illustrated in Fig. 8.1, showing that there are infinitely many equilibria:

$$\{(x, y)|0 \leqq x \leqq 1, y = 1 - x\}.$$

▼

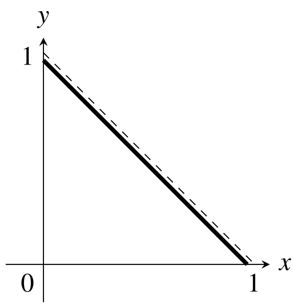

Fig. 8.1 Best responses in Example 8.1

8.1 Criteria Based on Best Responses

Since equilibria are the fixed points of the best response mappings, we will first give sufficient conditions for the uniqueness of equilibria in terms of the best response mappings.

Let $\underline{s}$ denote the simultaneous strategy vector of the players, $\underline{s} = (s_1, \ldots, s_N)$ with $s_k \in S_k$ being the strategy of player k. The best response mapping is defined as

$$\underline{R}(\underline{s}) = (R_1(\underline{s}), \ldots, R_N(\underline{s})),$$

where for $k = 1, 2, \ldots, N$, $R_k(\underline{s})$ is defined in (3.1).

Theorem 8.1 *Assume that* $\underline{R}(\underline{s})$ *is point-to-point and either*

(a) $||\underline{R}(\underline{s}) - \underline{R}(\underline{s}')|| < ||\underline{s} - \underline{s}'||$ *for all* $\underline{s}, \underline{s}' \in S_1 \times \cdots \times S_N$, $\underline{s} \neq \underline{s}'$ *or*

(b) $||\underline{R}(\underline{s}) - \underline{R}(\underline{s}')|| > ||\underline{s} - \underline{s}'||$ *for all* $\underline{s}, \underline{s}' \in S_1 \times \cdots \times S_N$, $\underline{s} \neq \underline{s}'$.

Then the game cannot have multiple equilibria.

Proof Assume $\underline{s}^*$ and $\underline{s}^{**}$ are both equilibria, then $\underline{R}(\underline{s}^*) = \underline{s}^*$ and $\underline{R}(\underline{s}^{**}) = \underline{s}^{**}$, so

$$||\underline{R}(\underline{s}^*) - \underline{R}(\underline{s}^{**})|| = ||\underline{s}^* - \underline{s}^{**}||$$

contradicting the assumptions. ■

Notice that condition (a) is slightly weaker than the contraction condition, which guaranteed the existence of a fixed point. As the next example shows, condition (a) guarantees only uniqueness, no existence is implied by this condition.

Example 8.2 Consider a single-dimensional mapping

$$R(s) = s + \frac{1}{s+1}$$

on the interval $[0, \infty]$, which clearly has no fixed point.

However

$$R'(s) = 1 - \frac{1}{(s+1)^2} \in [0, 1)$$

and with some c between s and s'

$$|R(s) - R(s')| = |R'(c)| \cdot |s - s'| < |s - s'|$$

showing that condition (a) is satisfied. ▼

Condition (b) does not guarantee existence either, as shown in the following example.

Example 8.3 Consider function $R(s) = 2s$ on the interval $[1, \infty)$. Clearly no fixed point exists (since $0 \notin [1, \infty)$), but

$$|R(s) - R(s')| = |2\,s - 2\,s'| = 2\,s - s'| > |s - s'|$$

showing that condition (b) is satisfied. ▼

In Appendix E we show that if $\underline{R}(\underline{s})$ is point-to-point and $-\underline{R}(\underline{s})$ is monotonic, then $\underline{R}(\underline{s})$ cannot have multiple fixed points implying the following result.

Theorem 8.2 *Assume that $\underline{R}(\underline{s})$ is point-to-point and $-\underline{R}(\underline{s})$ is monotonic. Then the game cannot have multiple equilibria.*

The conditions of this theorem do not guarantee existence of a fixed point, as it is demonstrated in the following example.

Example 8.4 Let

$$R(s) = \begin{cases} 1 - \frac{s}{2} & \text{if} s < \frac{1}{2} \\ \frac{1}{2} - \frac{s}{2} & \text{if} s \geqq \frac{1}{2} \end{cases}$$

on interval [0, 1], which has no fixed point. ▼

8.2 Criteria Based on Payoff Functions

The application of Theorems 8.1 and 8.2 in establishing the uniqueness of the equilibrium faces the difficulty of determining and examining the best response mapping. The following result of Rosen guarantees uniqueness based on only certain properties of the strategy sets and the payoff functions, so it can be easily applied in practical cases.

Assume that the N-person game $G = \{N; S_1, \ldots, S_N; \phi_1, \ldots, \phi_N\}$ satisfies the following conditions for all $k = 1, 2, \ldots, N$:
(a) There is a function $\underline{g}_k : D_k \mapsto \mathbb{R}^{p_k}$ such that

$$S_k = \left\{\underline{s}_k | \underline{g}_k(\underline{s}_k) \geqq \underline{0}\right\} \tag{8.1}$$

where $D_k \subset \mathbb{R}^{m_k}$ is an open set, $\underline{g}_k$ is continuously differentiable on D_k and its components are all concave in $\underline{s}_k$;
(b) S_k satisfies the Kuhn-Tucker regularity condition;
(c) ϕ_k is twice continuously differentiable on $D_1 \times D_2 \times \cdots \times D_N$.

Introduce function

$$\underline{h}(\underline{s}, \underline{r}) = \begin{pmatrix} r_1 \, \nabla_1 \, \phi_1(\underline{s}) \\ r_2 \, \nabla_2 \, \phi_2(\underline{s}) \\ \vdots \\ r_N \, \nabla_N \, \phi_N(\underline{s}) \end{pmatrix} \tag{8.2}$$

where $\underline{s}$ is the simultaneous strategy vector, $\underline{r} \in \mathbb{R}^N$ is a nonnegative vector, and for all k, $\nabla_k \phi_k(\underline{s})$ is the gradient vector of ϕ_k with respect to $\underline{s}_k$ as a column vector. Therefore the dimension of vector $\underline{h}(\underline{s}, \underline{r})$ is $M = \sum_{k=1}^{N} m_k$. The game G is called *strictly diagonally concave* if there is an $\underline{r} \geqq \underline{0}$ such that for all $\underline{s} \neq \underline{s}'$, $\underline{s}, \underline{s}' \in S_1 \times \cdots \times S_N$,

$$(\underline{s} - \underline{s}')^T \left(\underline{h}(\underline{s}, \underline{r}) - \underline{h}(\underline{s}', \underline{r})\right) < 0. \tag{8.3}$$

Notice that G is strictly diagonally concave if and only if $-\underline{h}(\underline{s}, \underline{r})$ is strictly monotonic.

Theorem 8.3 (Rosen, 1965) *Assume that conditions (a), (b) and (c) hold, furthermore the game is strictly diagonally concave. Then the game cannot have multiple equilibria.*

Proof Assume $\underline{s}^{(1)} = \left(\underline{s}_1^{(1)}, \ldots, \underline{s}_N^{(1)}\right)$ and $\underline{s}^{(2)} = \left(\underline{s}_1^{(2)}, \ldots, \underline{s}_N^{(2)}\right)$ are both equilibria. Then the Kuhn-Tucker conditions (6.3) imply the existence of vectors $\underline{u}_k^{(1)}$ and $\underline{u}_k^{(2)}$ such that for $l = 1, 2$,

$$\underline{u}_k^{(l)T} \underline{g}_k\left(\underline{s}_k^{(l)}\right) = 0 \tag{8.4}$$

and

$$\nabla_k \phi_k\left(\underline{s}^{(l)}\right) + \underline{u}_k^{(l)T} \nabla_k \, g_k(\underline{s}_k) = \underline{0}^T.$$

That is,

$$\nabla_k \, \phi_k\left(\underline{s}^{(l)}\right) + \sum_{j=1}^{p_k} u_{kj}^{(l)} \nabla \, g_{kj}\left(\underline{s}_k^{(l)}\right) = \underline{0} \tag{8.5}$$

where $\nabla_k \phi_k\left(\underline{s}^{(l)}\right)$ is now a column vector, $u_{kj}^{(l)}$ is the jth element of $\underline{u}_k^{(l)}$ and $\nabla_{kj} g_{kj}\left(\underline{s}_k^{(l)}\right)$ is the gradient vector of the jth component of $\underline{g}_k\left(\underline{s}_k^{(l)}\right)$ as a column vector. For $l = 1$, multiply Eq. (8.5) by $r_k\left(\underline{s}_k^{(2)} - \underline{s}_k^{(1)}\right)^T$ and for $l = 2$ multiply by $r_k\left(\underline{s}_k^{(1)} - \underline{s}_k^{(2)}\right)^T$ from the left hand side and add the resulting equations for $k = 1, 2, \ldots, N$ to have

$$0 = \left[\left(\underline{s}^{(2)} - \underline{s}^{(1)}\right)^T \underline{h}\left(\underline{s}^{(1)}, \underline{r}\right) + \left(\underline{s}^{(1)} - \underline{s}^{(2)}\right)^T h\left(\underline{s}^{(2)}, r\right)\right] +$$
$$\sum_{k=1}^{N} \sum_{j=1}^{p_k} \left\{ r_k \left[u_{kj}^{(1)} \left(\underline{s}_k^{(2)} - \underline{s}_k^{(1)}\right)^T \nabla \, g_{kj}\left(\underline{s}_k^{(1)}\right) + u_{kj}^{(2)} \left(\underline{s}_k^{(1)} - \underline{s}_k^{(2)}\right)^T \nabla \, g_{kj}\left(\underline{s}_k^{(2)}\right) \right] \right\}. \tag{8.6}$$

Since all components g_{kj} of $\underline{g}_k$ are concave, (B3) implies that

$$\left(\underline{s}_k^{(2)} - \underline{s}_k^{(1)}\right)^T \nabla \, g_{kj}\left(\underline{s}_k^{(1)}\right) \geqq g_{kj}\left(\underline{s}_k^{(2)}\right) - g_{kj}\left(\underline{s}_k^{(1)}\right)$$

and

$$\left(\underline{s}_k^{(1)} - \underline{s}_k^{(2)}\right)^T \nabla \, g_{kj}\left(\underline{s}_k^{(2)}\right) \geqq g_{kj}\left(\underline{s}_k^{(1)}\right) - g_{kj}\left(\underline{s}_k^{(2)}\right),$$

furthermore the first term on the right hand side of (8.6) is positive by (8.3). Therefore the right hand side of (8.6) is greater than

$$\sum_{k=1}^{N} \sum_{j=1}^{p_k} \left\{ r_k \left[u_{kj}^{(1)} g_{kj}\left(\underline{s}_k^{(2)}\right) + u_{kj}^{(2)} g_{kj}\left(s_k^{(1)}\right) \right] \right\} \geqq 0,$$

where we also used (8.4). This is a clear contradiction. Hence the equilibrium cannot be multiple. ■

In practical cases it is usually difficult to check if condition (8.3) holds or not, however the following simple result gives a sufficient condition.

Theorem 8.4 *Let $\underline{J}(\underline{s}, \underline{r})$ denote the Jacobian matrix of $\underline{h}(\underline{s}, \underline{r})$ with respect to $\underline{s}$. If $\underline{J}(\underline{s}, \underline{r}) + \underline{J}^T(\underline{s}, \underline{r})$ is negative definite, then condition (8.3) holds.*

This result follows immediately from the sufficient condition for a function being strictly monotonic (Appendix E).

Example 8.5 Consider now a duopoly ($N = 2$) with strategy sets $S_1 = S_2 = [0, 1]$, cost functions $C_k(s_k) = s_k$ and price function $p(s) = 2 - s$ with $s = s_1 + s_2$.

The payoff functions are as follows:

$$\phi_1(s_1, s_2) = s_1(2 - s_1 - s_2) - s_1 = -s_1^2 + s_1(1 - s_2)$$
$$\phi_2(s_1, s_2) = s_2(2 - s_1 - s_2) - s_2 = -s_2^2 + s_2(1 - s_1)$$

with derivatives (which are the gradients in the single-dimensional case)

$$\nabla_1\phi_1(s_1, s_2) = -2s_1 + 1 - s_2$$
$$\nabla_2\phi_2(s_1, s_2) = -2s_2 + 1 - s_1,$$

so

$$\underline{h}(\underline{s}, \underline{r}) = \begin{pmatrix} r_1(-2s_1 + 1 - s_2) \\ r_2(-2s_2 + 1 - s_1) \end{pmatrix}.$$

The Jacobian of $\underline{h}(\underline{s}, \underline{r})$ with respect to $\underline{s}$ has the following form

$$\underline{J}(\underline{s}, \underline{r}) = \begin{pmatrix} -2r_1 & -r_1 \\ -r_2 & -2r_2 \end{pmatrix},$$

so

$$\underline{J}(\underline{s}, \underline{r}) + \underline{J}^T(\underline{s}, \underline{r}) = \begin{pmatrix} -4r_1 & -(r_1 + r_2) \\ -(r_1 + r_2) & -4r_2 \end{pmatrix}.$$

This matrix is negative definite, if the eigenvalues are negative. The characteristic polynomial of this matrix is

$$\det \begin{pmatrix} -4r_1 - \lambda & -(r_1 + r_2) \\ -(r_1 + r_2) & -4r_2 - \lambda \end{pmatrix} = \lambda^2 + \lambda(4r_1 + 4r_2) + \left(16r_1r_2 - (r_1 + r_2)^2\right).$$

Selecting $r_1 = r_2 = 1$ for example, this polynomial becomes

$$\lambda^2 + 8\lambda + 12 = 0$$

with roots $\lambda_1 = -2$, $\lambda_2 = -6$. Since both are negative, $\underline{J}(\underline{s}, \underline{r}) + \underline{J}^T(\underline{s}, \underline{r})$ is negative definite implying that $\underline{h}(\underline{s}, \underline{r})$ is strictly diagonally concave. Therefore there is at most one equilibrium. Notice that this duopoly satisfies all conditions of the Nikaido-Isoda theorem, so there is at least one equilibrium. Thus there is exactly one equilibrium, which can be easily determined based on the best responses of the players. The first order conditions imply that at the stationary points

$$\frac{\partial \phi_1}{\partial s_1} = -2s_1 + 1 - s_2 = 0 \text{ and } \frac{\partial \phi_2}{\partial s_2} = -2s_2 + 1 - s_1 = 0$$

implying that the best responses are

$$R_1(s_2) = \frac{1 - s_2}{2} \text{ and } R_2(s_1) = \frac{1 - s_1}{2}.$$

Since both are feasible with all $s_1, s_2 \in [0, 1]$, the equilibrium is the solution of equations

$$s_1 = \frac{1 - s_2}{2} \text{ and } s_2 = \frac{1 - s_1}{2},$$

which is

$$s_1^* = s_2^* = \frac{1}{3}.$$

▼

Chapter 9
Repeated and Dynamic Games

In this chapter repeated and dynamic games will be discussed in which the players know the strategy sets and payoff functions of all players, that is, the game has *complete information*. It is also assumed that at each time period each player knows the complete history of the game which consists of the past strategy selections and corresponding payoff values of all players. It means that the game also has *perfect information*.

The most simple case occurs when at time periods $t = 0, 1, 2, \ldots,$ a game is played (which can be identical or not), and the strategy selections and payoffs at time t are independent of these occurred in the previous time periods. The games at different time periods are completely independent of each other. The payoff of this repeated game of each player is the (maybe discounted) sum of his payoffs at the different time periods. It is very easy to see that equilibrium of the repeated game occurs when the players select equilibrium strategies at all time periods $t = 0, 1, 2, \ldots$

9.1 Leader-Follower Games

One of the most simple dynamic game is known as *Leader-follower* or *Stackelberg game*. Assume that there are two players, player 1 is the leader and player 2 is the follower. The game is played in two stages. In stage 1, the leader selects a strategy and let the follower know it. In stage 2 the follower selects his strategy and then both players receive the payoffs. This situation can be mathematically modeled as follows. Let S_1 and S_2 be the strategy sets and ϕ_1 and ϕ_2 the payoff functions. In stage 1 the leader selects strategy $s_1 \in S_1$, then the follower (assuming he is a rational player) selects his best response against s_1, which is denoted by $R_2(s_1)$. So the leader knows that his payoff will become $\phi_1(s_1, R_2(s_1))$, which depends on only his strategy choice. So the leader solves the optimum problem

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_9

$$\begin{aligned} &\text{maximize } \phi_1\big(s_1, R_2(s_1)\big) \\ &\text{subject to } s_1 \in S_1. \end{aligned} \tag{9.1}$$

Let s_1^* be an optimum solution, then $\big(s_1^*, R_2(s_1^*)\big)$ is the solution of the game, which is usually called the *Stackelberg equilibrium* (von Stackelberg, 1934). In the procedure first the best response of player 2 is determined and then the payoff of player 1 is maximized subject to the best response selection of player 2. This process is called *backward induction*, which is illustrated by the following examples.

Example 9.1 (Duopoly Stackelberg game) Consider a duopoly of a home-firm and a foreign firm. The price function is linear: $p(s) = a - b(s_1 + s_2)$, where s_1 and s_2 are the production levels of the firms. The marginal cost c is the same for the firms, however the home-firm receives government subsidy α to all units of its product. The strategy sets are clearly $S_1 = S_2 = [0, \infty)$ and the payoff functions are as follows:

$$\begin{aligned} \phi_1(s_1, s_2) &= s_1(a - bs_1 - bs_2) - (c - \alpha)s_1 \\ \phi_2(s_1, s_2) &= s_2(a - bs_1 - bs_2) - cs_2. \end{aligned}$$

Considering the home-firm as the leader and the foreign firm as the follower, first we have to determine the best response of player 2. The first order conditions imply that

$$\frac{\partial \phi_2}{\partial s_2} = a - bs_1 - 2bs_2 - c = 0,$$

so by assuming interior optimum,

$$R_2(s_1) = \frac{a - c - bs_1}{2b}. \tag{9.2}$$

The home-firm knows that the foreign firm is rational, so it will chose $R_2(s_1)$, so the payoff of the home-firm will become

$$\begin{aligned} \phi_1\big(s_1, R_2(s_1)\big) &= s_1(a - bs_1 - \frac{a - c - bs_1}{2} - c + \alpha) \\ &= \frac{s_1}{2}(a - c - bs_1 + 2\alpha), \end{aligned}$$

and the first order conditions for maximizing this function gives equation

$$a - c - 2bs_1 + 2\alpha = 0,$$

so

$$s_1^* = \frac{a - c + 2\alpha}{2b} \tag{9.3}$$

and therefore

$$s_2^* = R_2(s_1^*) = \frac{a - c - 2\alpha}{4b}. \tag{9.4}$$

It will be interesting to compare the Stackelberg equilibrium and the Nash equilibrium of the game. The best response of player 1 is obtained from the first order condition of maximizing ϕ_1,

$$\frac{\partial \phi_1}{\partial s_1} = a - 2bs_1 - bs_2 - c + \alpha = 0$$

implying that

$$R_1(s_2) = \frac{a - c - bs_2 + \alpha}{2b} \tag{9.5}$$

by assuming interior optimum. The Nash equilibrium is then the solution of equations

$$s_1 = \frac{a - c - bs_2 + \alpha}{2b}$$
$$s_2 = \frac{a - c - bs_1}{2b},$$

the solutions of which are

$$s_1^{**} = \frac{a - c + 2\alpha}{3b} \text{ and } s_2^{**} = \frac{a - c - \alpha}{3b}. \tag{9.6}$$

Notice that $s_1^* = s_1^{**} \cdot \frac{3}{2}$ and $s_2^* = s_2^{**} \cdot \frac{3}{4} - \frac{\alpha}{4b}$, so being the leader gives an output increase to the home-firm and being the follower decreases output of the foreign firm. ▼

There is an alternative method to find solutions for leader-follower games. The follower optimizes his payoff function $\phi_2(s_1, s_2)$ with given s_1 and then the leader maximizes $\phi_1(s_1, s_2)$ assuming optimality of s_2. This can be described by a constrained optimization problem where $\phi_1(s_1, s_2)$ is maximized and the constraints are the Kuhn-Tucker optimality conditions for maximizing $\phi_2(s_1, s_2)$ with respect to s_2.

Example 9.2 The previous example is reconsidered now. The optimum problem of player 2 is the following:

$$\begin{aligned} \text{maximize } & s_2(a - bs_1 - bs_2) - cs_2 \\ \text{subject to } & s_2 \geqq 0. \end{aligned}$$

So the Kuhn-Tucker conditions are given as

$$\begin{aligned} u \geq 0, s_2 \geqq 0 \\ a - bs_1 - 2bs_2 - c + u = 0 \\ us_2 = 0 \end{aligned}$$

since we have only one constraint on s_2, so only one Lagrange multiplier is needed and the gradient of ϕ_2 with respect to s_2 is the usual derivative. So the leader-follower solution can be obtained by solving the following maximum problem:

$$\begin{aligned} \text{maximize } & s_1(a - bs_1 - bs_2) - (c - \alpha)s \\ \text{subject to } & s_1, s_2, u \geqq 0 \\ & a - bs_1 - 2bs_2 - c + u = 0 \\ & us_2 = 0. \end{aligned}$$

The last constraint implies that either $u = 0$ or $s_2 = 0$. In the case of interior solution $s_2 > 0$, so $u = 0$. Then the second constraint can be solved for s_2, which is the interior best response of player 2 and by substituting it into the objective function the same problem is obtained as in the previous example. ▼

Example 9.3 (Wages and Employment) Consider a firm which has complete control on its employment and faces a monopoly union which has exclusive control on the wages. The union is considered the leader and presents its wage request w to the firm, and then the firm decides on its employment L based on the wage requirements (Leontief, 1946). The strategy sets of the players are $S_1 = [0, W]$ meaning that W is the largest wage the union can request, and $S_2 = [0, \infty)$. The payoff function of the firm is

$$\phi_2(w, L) = R(L) - wL \tag{9.7}$$

where $R(L)$ is the amount of revenue produced by the work force of size L, and wL is the amount of wages the firm has to pay to the workers. Function $R(L)$ is strictly increasing in L. The payoff of the union is $\phi_1(w, L)$ which strictly increases in both w and L.

As a numerical example select $W = 2$,

$$R(L) = \sqrt{L} \text{ and } \phi_1(w, L) = w^2(-w^2 + 2w + 20)L. \tag{9.8}$$

Notice that

$$-w^2 + 2w + 20 = -(w - 1)^2 + 21 > 0$$

so ϕ_1 strictly increases in L, furthermore

$$\frac{\partial \phi_1}{\partial w} = \big(- 4w^3 + 6w^2 + 40w \big) L = (2w^2(3 - 2w) + 40w) L > 0$$

so ϕ_1 strictly increases in w as well.

The best response of the firm is obtained by maximizing

$$\phi_2(w, L) = \sqrt{L} - wL. \tag{9.9}$$

By differentiation

$$\frac{\partial \phi_2}{\partial L} = \frac{1}{2\sqrt{L}} - w = 0,$$

so

$$L = R_2(w) = \frac{1}{4w^2},$$

and therefore the corresponding payoff of the union becomes

$$\phi_1\big(w, R_2(w)\big) = w^2(-w^2 + 2w + 20)\frac{1}{4w^2} = \frac{1}{4}(-w^2 + 2w + 20)$$

which has maximum at $w^* = 1$, so the best response of the firm becomes $R_2(w^*) = \frac{1}{4}$. The Stackelberg equilibrium is therefore

$$w^* = 1 \text{ and } L^* = \frac{1}{4}.$$

▼

The existence of a Stackelberg equilibrium is not guaranteed in general. However if the set S_1 of the strategies of the leader is compact, its payoff $\phi_1(s_1, s_2)$ is continuous as a bi-variable function and the best response $R_2(s_1)$ is continuous on S_1, then $\phi_1\big(s_1, R_2(s_1)\big)$ is a continuous function on a compact set, which has maximum on S_1, so there is at least one Stackelberg equilibrium.

For N-player games for $N > 2$ the concept of Stackelberg equilibrium can be generalized in several different ways.

A possibility is that the players are ordered in importance order. Assume that player 1 is the least important, player 2 is the second least important, and so on. In this general case the backward induction proceeds as follows. First the best response of the least important player is determined: $R_1(s_2, \ldots, s_N)$ against the strategy selections of all other players. In the second stage the second least important player 2 selects its best strategy against the strategy selections of the other players knowing the choice of player 1. So player 2 maximizes $\phi_2\big(R_1(s_2, \ldots, s_N), s_2, \ldots, s_N\big)$, so the payoff of player 2 depends on only strategies $s_2, \ldots, s_N$, and his best response $R_2(s_3, \ldots, s_N)$ depends on only $s_3, \ldots, s_N$. Next player 3 selects strategy by maximizing his payoff $\phi_3\Big(R_1\big(R_2(s_3, \ldots, s_N), s_3, \ldots, s_N\big), R_2(s_3, \ldots, s_N), s_3, \ldots, s_N\Big)$

and his best response depends on only $s_4, \ldots, s_N$. And finally, player N, the absolute leader considers his payoff which depends on only his own strategy, what he maximizes.

Another possibility is the assumption of a single leader, say player i_1, and the others are all followers who select a Nash equilibrium among them as an $(N-1)$-person game with given choice s_{i_1} of the leader. If $s_k = E_k(s_{i_1})(k \neq i_1)$ denotes the equilibrium strategy of player k given s_{i_1}, then the leader will maximize its payoff

$$\phi_{i_1}\big(E_1(s_{i_1}), \ldots, E_{i_1-1}(s_{i_1}), s_{i_1}, E_{i_1+1}(s_{i_1}), \ldots, E_N(s_{i_1})\big) \tag{9.10}$$

which depends on only s_{i_1}. Let $s_{i_1}^*$ be an optimal solution, then

$$s_k^* = \begin{cases} E_k(s_{i_1}^*) & \text{if } k \neq i_1 \\ s_{i_1}^* & \text{if } k = i_1 \end{cases} \tag{9.11}$$

is the general Stackelberg equilibrium.

The two above concepts can be combined by dividing the players into groups $G_1, G_2, \ldots, G_M$ so that $G_i \cap G_j = \emptyset$ for $i \neq j$ and $G_1 \cup G_2 \cup \ldots \cup G_M = \{1, 2, \ldots, N\}$, and the groups are ordered in importance order, and inside each group a Nash-equilibrium is formed among the members. The only difference between this concept and the above ideas is that at each step the actual group computes its equilibrium based on the unknown equilibrium strategies of the more important groups and predicted equilibrium strategies of the less important groups.

9.2 Dynamic Games with Simultaneous Moves

Let $t = 0, 1, 2, \ldots$ denote again discrete time scales and assume that the decision of each player at each time period depends on the past history of the game. As an illustration of such a game we will now describe the case of "greedy" oligopolists.

Example 9.4 (Dynamic Oligopoly) Consider an N-firm oligopoly with linear price and cost functions, $p(s) = a - bs$ with $s = \sum_{k=1}^{N} x_k$ and $C_k(x_k) = c_k x_k + d_k (k = 1, 2, \ldots, N)$. At the initial time period $t = 0$, each firm selects an initial output level, $x_k(0)$, and at each later time period $t \geqq 1$ the game proceeds as follows. Each firm forms a prediction (expectation) about the current output of the rest of the industry, which can be denoted by $s_k^E(t)$, and then the firm selects its best response $R_k\big(s_k^E(t)\big)$ as its new output level for time period t. When firm k selects its output level $x_k(t)$, it has no knowledge about the simultaneous output choices of the competitors, however based on past observations on the behavior and choices of the other players it is able to form a realistic prediction. In the case of *static expectations* the firms select the last observed data, $s_k(t-1) = \sum_{l \neq k} x_l(t-1)$ as their predictions for the new time period t. The payoff of firm k is given as

$$\phi_k(x_1, \dots, x_N) = x_k(a - bx_k - bs_k) - (c_k x_k + d_k) \tag{9.12}$$

so its best response is obtained by differentiation,

$$a - 2bx_k - bs_k - c_k = 0$$

implying that

$$R_k(s_k) = \frac{a - c_k - bs_k}{2b}. \tag{9.13}$$

So the output level of this firm at any time period t is obtained as $x_k(t) = R_k\big(s_k(t - 1)\big)$, which can be written as

$$x_k(t) = -\frac{1}{2} \sum_{l \neq k} x_l(t-1) + \frac{a - c_k}{2b} \tag{9.14}$$

for $k = 1, 2, \dots, N$. Thus an N-dimensional linear system is obtained with coefficient matrix

$$\underline{A} = \begin{pmatrix} 0 & -\frac{1}{2} & \dots & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \dots & -\frac{1}{2} \\ \vdots & \vdots & & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \dots & 0 \end{pmatrix} = -\frac{1}{2}\underline{1} + \frac{1}{2}\underline{I}$$

where $\underline{1}$ is the $N \times N$ matrix, all elements of which equal unity and $\underline{I}$ is the $N \times N$ identity matrix.

In examining the long-term properties of the oligopoly, stability analysis is the appropriate approach. Since the system is linear, the eigenvalues of matrix $\underline{A}$ decide the stability or instability of the system. It is well known (see Appendix H) that this system is asymprotically stable if and only if all eigenvalues are inside the unit circle. The eigenvalues of matrix $\underline{1}$ have to be determined first. The eigenvalue equation of matrix $\underline{1}$ has the form

$$\sum_{l=1}^{N} u_l = \lambda u_k \; (k = 1, 2, \dots, N) \tag{9.15}$$

where $u_1, u_2, \dots, u_N$ are the components of the eigenvector associated with λ. If $\lambda = 0$, then $\sum_{k=1}^{N} u_k = 0$, so we have $N - 1$ linearly independent eigenvectors. If $\lambda \neq 0$, then $u_1 = u_2 = \dots = u_N$, and (9.15) implies that $\lambda = N$. Therefore the eigenvalues of $\underline{1}$ are 0 and N, and so the eigenvalues of $\underline{A}$ are

$$-\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2} \text{ and } -\frac{1}{2} \cdot N + \frac{1}{2} \cdot 1 = \frac{-N + 1}{2}.$$

Since $-\frac{1}{2}$ is inside the unit circle, the system is asymptotically stable if and only if

$$|\frac{N-1}{2}| < 1,$$

that is, $N < 3$. Consequently an oligopoly is asymptotically stable if and only if it is a duopoly.

Better stability conditions can be guaranteed by assuming *adaptive expectations*, when the firms adaptively correct the errors of their previous expectations :

$$s_k^E(t) = s_k^E(t-1) + \alpha_k\big(s_k(t-1) - s_k^E(t-1)\big) \tag{9.16}$$

where $\alpha_k \in (0, 1]$ is the speed of adjustment of firm k, and $s_k(t-1)$ is the actual output of the rest of the industry in the perspective of firm k. The difference $s_k(t-1) - s_k^E(t-1)$ shows the expectation error in the previous time period $t-1$. If $s_k(t-1) < s_k^E(t-1)$, then firm k overestimated $s_k(t-1)$ and (9.16) shows that the firm decreases its expectation in the next period. If $s_k(t-1) > s_k^E(t-1)$, then firm k underestimated $s_k(t-1)$, so for the next time period it wants to increase its expectation, and if $s_k(t-1) = s_k^E(t-1)$ then the firm believes that the previous expectation was correct, so there is no need to change expectation. As a special case assume that the firms select identical speed of adjustment, that is, $\alpha_1 = \alpha_2 = \cdots = \alpha_N$. If α denotes this common value, then (9.16) for $k = 1, 2, \ldots, N$ provides a discrete linear dynamic system

$$x_k(t) = R_k\big(s_k^E(t)\big) = -\frac{1}{2}\Big(\alpha\sum_{l\neq k} x_l(t-1) + (1-\alpha)s_k^E(t-1)\Big) + \frac{a-c_k}{2b} \tag{9.17}$$

$$s_k^E(t) = \alpha\sum_{l\neq k} x_l(t-1) + (1-\alpha)s_k^E(t-1) \tag{9.18}$$

with $2N$ state variables x_k and s_k^E for $k = 1, 2, \ldots, N$. The coefficient matrix of this system has the special block form

$$\underline{A}_a = \begin{pmatrix} \underline{A}_1 & \underline{A}_2 \\ \underline{A}_3 & \underline{A}_4 \end{pmatrix}$$

with

$$\underline{A}_1 = \begin{pmatrix} 0 & -\frac{\alpha}{2} & \cdots & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & 0 & \cdots & -\frac{\alpha}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\alpha}{2} & -\frac{\alpha}{2} & \cdots & 0 \end{pmatrix}, \underline{A}_2 = \begin{pmatrix} -\frac{1-\alpha}{2} & & & \\ & -\frac{1-\alpha}{2} & & \\ & & \ddots & \\ & & & -\frac{1-\alpha}{2} \end{pmatrix}$$

$$\underline{A}_3 = \begin{pmatrix} 0 & \alpha & \dots & \alpha \\ \alpha & 0 & \dots & \alpha \\ \vdots & \vdots & & \vdots \\ \alpha & \alpha & \dots & 0 \end{pmatrix} \text{ and } \underline{A}_4 = \begin{pmatrix} 1-\alpha & & & \\ & 1-\alpha & & \\ & & \ddots & \\ & & & 1-\alpha \end{pmatrix}.$$

The eigenvalue equation also has a special block form:

$$\underline{A}_1\underline{u} + \underline{A}_2\underline{v} = \lambda\underline{u} \tag{9.19}$$

$$\underline{A}_3\underline{u} + \underline{A}_4\underline{v} = \lambda\underline{v} \tag{9.20}$$

where both $\underline{u}$ and $\underline{v}$ are N-element vectors. Multiply (9.19) by 2 and add the resulting equation to (9.20) to get

$$\lambda(2\underline{u} + \underline{v}) = \underline{0}$$

showing that $\lambda = 0$ is an eigenvalue. If $\lambda \neq 0$, then $\underline{v} = -2\underline{u}$ and by substituting it into (9.19) yields

$$(\underline{A}_1 - 2\underline{A}_2)\underline{u} = \lambda\underline{u},$$

so the nonzero eigenvalues are eigenvalues of matrix

$$\underline{A}_1 - 2\underline{A}_2 = -\frac{\alpha}{2}(\underline{1} - \underline{I}) + (1-\alpha)\underline{I} = -\frac{\alpha}{2}\underline{1} + (1 - \frac{\alpha}{2})\underline{I}.$$

Since the eigenvalues of $\underline{I}$ are all equal unity and the eigenvalues of $\underline{1}$ are 0 and N, the eigenvalues of $\underline{A}_1 - 2\underline{A}_2$ are

$$\lambda = 1 - \frac{\alpha}{2} + \begin{cases} 0 \\ -\frac{\alpha N}{2} \end{cases} = \begin{cases} 1 - \frac{\alpha}{2} \\ 1 - \frac{(N+1)\alpha}{2}. \end{cases}$$

The system is asymptotically stable if and only if these eigenvalues are inside the unit circle, that is, when

$$|1 - \frac{(N+1)\alpha}{2}| < 1$$

which can be rewritten as

$$\alpha < \frac{4}{N+1}. \tag{9.21}$$

This relation shows that dynamic oligopolies can become asymptotically stable with any number of firms if the common speed of adjustment is sufficiently small.

Notice that the case of $\alpha = 1$ reduces adaptive expectations to static expectation, in which case (9.21) holds for only $N = 2$. This is the same result what was obtained earlier.

In the previous discussions we considered discrete time scales. Those models can be easily modified for continuous time scales. In the discrete time scales after time period $t - 1$ there was the next time period t, and the new output levels of the firms were determined based on best response dynamics and given types of expectations. In the case of continuous time scales there is no "next" time period after $t - 1$, so instead of determining the next output level, the "directions of the output changes" can be specified.

In the case of *best response dynamics* each firm adjusts its output in the direction toward its best response. Based on (9.13) this concept results in the dynamic system

$$\dot{x}_k(t) = \alpha_k\Big(R_k\big(s_k(t)\big) - x_k(t)\Big) = \alpha_k\Big(-\frac{1}{2}\sum_{l\neq k} x_l(t) - x_k(t) + \frac{a - c_k}{2b}\Big) \quad (9.22)$$

for $k = 1, 2, \ldots, N$.

In the case of *gradient adjustments* the firms adjust their outputs proportionally to their marginal profits. If the marginal profit of a firm is positive, then it is the interest of the firm to increase output level, and if the marginal profit in negative, then the interest of the firm is to decrease production level. If the marginal profit is zero, then the firm believes that its output level is optimum, so there is no need to change it. This idea can be written as the dynamic system

$$\begin{aligned}\dot{x}_k(t) = \alpha_k \frac{\partial \phi_k(x_1, \ldots, x_N)}{\partial x_k} &= \alpha_k\Big(a - 2bx_k(t) - b\sum_{l\neq k} x_l(t) - c_k\Big) \\ &= 2b\alpha_k\Big(-\frac{1}{2}\sum_{l\neq k} x_l(t) - x_k(t) + \frac{a - c_k}{2b}\Big) \qquad (9.23)\end{aligned}$$

for $k = 1, 2, \ldots, N$. Notice that systems (9.22) and (9.23) are basically identical, instead of α_k, $2b\alpha_k$ is selected in (9.23). So it is sufficient to discuss model (9.22). It is a continuous linear system with coefficient matrix

$$\underline{A}_c = \begin{pmatrix} -\alpha_1 & -\frac{\alpha_1}{2} & \ldots & -\frac{\alpha_1}{2} \\ -\frac{\alpha_2}{2} & -\alpha_2 & \ldots & -\frac{\alpha_2}{2} \\ \vdots & \vdots & & \vdots \\ -\frac{\alpha_N}{2} & -\frac{\alpha_N}{2} & \ldots & -\alpha_N \end{pmatrix}.$$

In the special case of equal speeds of adjustments

$$\underline{A}_c = -\frac{\alpha}{2}\underline{1} - \frac{\alpha}{2}\underline{I}$$

with eigenvalues

$$\lambda = -\frac{\alpha}{2} - \begin{cases} \frac{\alpha N}{2} \\ 0. \end{cases}$$

And since all eigenvalues are negative, the system is always asymptotically stable (see Appendix H) regardless of the number of firms. ▼

9.3 Dynamic Games with Sequential Moves

We consider discrete time scales, $t = 0, 1, 2, \ldots$. At time period $t = 0$ each player selects an initial strategy $s_k^{(0)}$ independently of each other. If this is an equilibrium, then no player has incentive to change strategy. Otherwise a dynamic process develops when at any later time period $t \geqq 1$, one player $k(t)$ selects new strategy. At $t = 0$ the initial strategies are announced to all players, and at each time period $t \geqq 0$, player $k(t)$ makes his strategy $s_{k(t)}^{(t)}$ known to all other players. So the complete history of the game is known to all players at all times.

Assuming greedy and rational players, at time period $t \geqq 1$ player $k(t)$ maximizes his payoff where the strategy of each other player l is assumed to be the last strategy selection of this player. Let $\underline{s}^{(t)}$ denote the simultaneous strategy vector of the players at time period t, then $\underline{s}^{(0)} = \left(s_1^{(0)}, s_2^{(0)}, \ldots, s_N^{(0)}\right)$ and for $t \geqq 1$, $\underline{s}^{(t)} = \left(s_1^{(t)}, s_2^{(t)}, \ldots, s_N^{(t)}\right)$, where

$$s_l^{(t)} = \begin{cases} s_l^{(t-1)} & \text{if } l \neq k(t) \\ s_{k(t)}^{(t)} & \text{if } l = k(t). \end{cases} \tag{9.24}$$

Then at time period $t \geqq 1$, player $k(t)$ solves the optimization problem

$$\begin{aligned} &\text{maximize } \phi_{k(t)}\left(\underline{s}^{(t)}\right) \\ &\text{subject to } s_{k(t)}^{(t)} \in S_{k(t)} \end{aligned} \tag{9.25}$$

where $S_{k(t)}$ is the strategy set of player $k(t)$ where the strategies of the other players are assumed to be given. If $R_{k(t)}(\underline{s})$ denotes the best response mapping of player $k(t)$, then the solution of the optimization problem is

$$s_{k(t)}^{(t)} \in R_{k(t)}\left(\underline{s}^{(t)}\right). \tag{9.26}$$

If for all $T > 0$, every player selects strategy at least ones in time periods $T + 1, T + 2, \ldots,$ then the steady states of the dynamic system (9.26) and the Nash equilibria of the corresponding static game with strategy sets S_l and payoff functions ϕ_l are the same.

Example 9.5 (Oligopoly with Sequential Moves) Consider again the linear oligopoly of Example 9.4, where we assumed that at each time period $t \geqq 1$ the firms select strategies simultaneously and independently of each other. If sequential strategy selection is assumed, then the dynamic equation (9.14) is modified as follows:

$$x_{k(t)}^{(t)} = -\frac{1}{2} \sum_{l \neq k(t)} x_l^{(t-1)} + \frac{a - c_{k(t)}}{2b} \tag{9.27}$$

where the simultaneous strategy vector satisfies the recursive equation

$$x_l^{(t)} = \begin{cases} x_l^{(t-1)} & \text{if } l \neq k(t) \\ x_l^{(t)} & \text{if } l = k(t). \end{cases} \tag{9.28}$$

In the special case of cyclic firm selection, that is, when $k(1) = 1, k(2) = 2, \ldots, k(N) = N, k(N+1) = 1, k(N+2) = 2, \ldots$ etc., system (9.27) is equivalent with the Gauss-Seidel iteraction process (Szidarovszky & Yakowitz, 1978), for solving the linear equations

$$x_k = -\frac{1}{2} \sum_{l \neq k} x_l + \frac{a - c_k}{2b}. (k = 1, 2, \ldots, N).$$

▼

Example 9.6 (Sequential bargaining) Assume that two agents want to get a reasonable business deal, where the set of all possible simultaneous payoff values form a certain set H in the two-dimensional space, where ϕ_1 and ϕ_2 are the coordinate lines. The boundaries of H are the nonnegative segments of the coordinate lines and the graph of a strictly decreasing concave function $\phi_2 = g(\phi_1)$ as shown in Fig. 9.1.

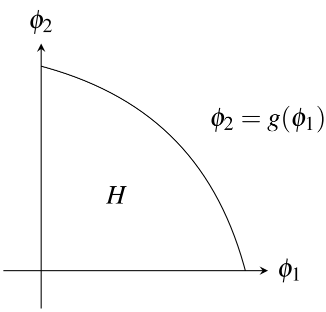

Fig. 9.1 Illustration of the bargaining set in Example 9.6

At the initial time period $t = 0$, each player announces his payoff request $\phi_1^{(0)}$ and $\phi_2^{(0)}$. Assuming rationality of the players, the initial choices of the players are the points $\left(\phi_1^{(0)}, g\left(\phi_1^{(0)}\right)\right)$ and $\left(g^{-1}\left(\phi_2^{(0)}\right), \phi_2^{(0)}\right)$ on the boundary curve, which is also called the *Pareto frontier*. Assume that $\phi_1^{(0)} > g^{-1}\left(\phi_2^{(0)}\right)$ and $\phi_2^{(0)} > g\left(\phi_1^{(0)}\right)$. The value $g\left(\phi_1^{(0)}\right)$ can be imagined that it is the initial offer of player 1 to player 2, and the value $g^{-1}\left(\phi_2^{(0)}\right)$ is the initial offer of player 2 to player 1. Since the initial offers to both players are lower than their payoff requests, they should negotiate in order to reach a mutually acceptable solution. One way of modeling the negotiation process is the following (Szidarovszky, 1998). At each time period $t = 1, 2, \ldots,$ a player makes concession by lowering his demand and therefore increasing the amount of his offer to the other player. This new offer maybe rejected, then negotiation is over, both players get zero payoff, since no business is made. If the offer is considered as a basis for further negotiation, then the other player presents a counteroffer and the game continues.

It is assumed that at each odd time period player 1 presents his offer and at each even time period player 2 makes concession. Consider a time period t when player 1 gives offer. Let $\left(\phi_1^{(t-2)}, g\left(\phi_1^{(t-2)}\right)\right)$ and $\left(g^{-1}\left(\phi_2^{(t-1)}\right), \phi_2^{(t-1)}\right)$ be the last offers of the two players. If player 1 presents his new offer $\left(\phi_1, g(\phi_1)\right)$ to player 2, then the relative gain of player 2 would be

$$\frac{g(\phi_1) - g\left(\phi_1^{(t-2)}\right)}{\phi_2^{(t-1)} - g\left(\phi_1^{(t-2)}\right)}. \tag{9.29}$$

If this relative gain is small, the offer is rejected with high probability, and if it is large, the negotiation will continue with high probability. Therefore assume that probability of continuation is given as

$$\left(\frac{g(\phi_1) - g\left(\phi_1^{(t-2)}\right)}{\phi_2^{(t-1)} - g\left(\phi_1^{(t-2)}\right)}\right)^{P_2}. \tag{9.30}$$

where $P_2 > 0$ is the negotiation power of player 2. Giving the offer player 1 faces a random outcome, since rejection and acceptance would happen with probabilities and not with certainly. The expected payoff of player 1 is therefore

$$E_1 = \phi_1\left(\frac{g(\phi_1) - g\left(\phi_1^{(t-2)}\right)}{\phi_2^{(t-1)} - g\left(\phi_1^{(t-2)}\right)}\right)^{P_2}. \tag{9.31}$$

what player 1 maximizes in order to get his next offer. Since E_1 and $\ln E_1$ are maximal at the same ϕ_1 value, we consider

$$\ln E_1 = \ln \phi_1 + P_2\Big(\ln \Big(g(\phi_1) - g\big(\phi_1^{(t-2)}\big)\Big)\Big) - P_2\Big(\ln \Big(\phi_2^{(t-1)} - g\big(\phi_1^{(t-2)}\big)\Big)\Big) \tag{9.32}$$

and by differentiation

$$\frac{\partial \ln E_1}{\partial \phi_1} = \frac{1}{\phi_1} + \frac{P_2 g'(\phi_1)}{g(\phi_1) - g\big(\phi_1^{(t-2)}\big)} \tag{9.33}$$

and

$$\frac{\partial^2 \ln E_1}{\partial \phi_1^2} = -\frac{1}{\phi_1^2} + P_2 \cdot \frac{g''(\phi_1)\big(g(\phi_1) - g(\phi_1^{(t-2)})\big) - g'(\phi_1)^2}{\Big(g(\phi_1) - g\big(\phi_1^{(t-2)}\big)\Big)^2} < 0$$

implying that E_1 is strictly concave in ϕ_1, so there is a unique optimum. The first order condition gives equation

$$h_1(\phi_1) = g(\phi_1) - g\big(\phi_1^{(t-2)}\big) + P_2\phi_1 g'(\phi_1) = 0. \tag{9.34}$$

Notice that $h_1(\phi_1)$ is strictly decreasing in ϕ_1, furthermore $h_1\big(\phi_1^{(t-2)}\big) < 0$, therefore the optimum value of ϕ_1 can be obtained as follows. If $h_1\Big(g^{-1}\big(\phi_2^{(t-1)}\big)\Big) \leqq 0$, then $\phi_1 = g^{-1}\big(\phi_2^{(t-1)}\big)$ is the optimal choice meaning that player 1 agrees with the previous offer of player 2, so negotiation is terminated, agreement is reached. Otherwise equation (9.34) has a unique solution, and this is the new offer of player 1. Then player 2 presents his optimal offer to player 1, which is obtained similarly to the case of player 1 shown above. Then player 1 accepts or rejects this offer or makes the next offer and the game continues with a new offer of player 2, and so on. It can be proved that the game terminates in finitely many steps when either one of the players rejects an offer or accepts the previous offer of the other player. ▼

9.4 Finite Tree Games

In the previous two examples the order in which the players selected strategies was "predetermined", that is, it did not depend on the history and the current states of the game. As an illustration of other case we will introduce a class of finite dynamic games which can be represented by *finite trees*. A tree is a graph which has no circle. Assume there are N players, who move along the arcs of the tree, in the following way. There is a unique initial node, called the *root*, and a predetermined player moves along a selected arc originating in the root. When the player arrives to the endpoint of the arc, then another player continues along an arc originating at this node. At the endpoint of the arc another player moves forward along an arc, and so on, until the game ends at a terminal node of the tree. Assume that each player knows the entire tree, at each node of the tree a player is assigned who selects the arc originating from that node and makes the next move to the endpoint of that arc. These assignments are known to all players and at any stage of the game all players know the entire history of the game until that stage since there is a unique path between the root and this point. At each endpoint of the game all players have assigned payoff values what will be received by the players if the game terminates at that point. The strategy of each player is a set of decisions at all nodes where that player has to decide about the continuation of the game. This game is a finite game with complete and perfect information.

Example 9.7 A special three-person game is shown in Fig. 9.2, where at each node we show (circled) the player who has to continue the game at that node. The payoff values of the players are also indicated at each endpoint, the first number is the payoff of player 1, the second number is that of player 2 and the third number is the payoff of player 3. ▼

We will next prove the following existence theorem (Kuhn, 1953):

Theorem 9.1 *Every game with complete and perfect information played on a finite, rooted tree always has at least one equilibrium.*

Proof There are several patterns through adjacent arcs from the root to the endpoints of the tree, however from the root to any given endpoint there is only one path, since otherwise the tree would have a circle. The length of each path is the number of arcs contained in the path. Consider the lengths of the paths from the root to any of the endpoints. The length of the tree is defined as the length of the longest path from the root to an endpoint, which is denoted by L. The proof is based on finite induction with respect to L.

If $L = 0$, then the game has only one node, the root. And clearly this is an equilibrium. Assume next that all tree games with lengths less than L have at least one equilibrium, and consider a game with length L. Assume that player k_0 is assigned to the root, and he can select from M arcs with endpoints $I_1, I_2, \ldots, I_M$. Then M subgames are defined, the roots of them are these nodes and the arcs of each subgame are those arcs which can be reached starting from its root. Let $G_1, G_2, \ldots, G_M$ denote these subgames. Since the length of each of them is less than L, all of them have at least one equilibrium. If $\phi_{k_0^{(1)}}, \ldots, \phi_{k_0^{(M)}}$ denote the payoffs of player k_0 at these equilibria, then the equilibrium of the original game can be obtained as follows. For any player $k \neq k_0$, the equilibrium strategies are those of games $G_1, G_2, \ldots, G_M$. For player k_0 the equilibrium strategies are those in games $G_1, G_2, \ldots, G_M$ and arc l starting from the root which gives the largest of the payoff values $\phi_{k_0^{(1)}}, \phi_{k_0^{(2)}}, \ldots, \phi_{k_0^{(M)}}$, that is, l is selected as

$$\phi_{k_0^{(l)}} = \max\left\{\phi_{k_0^{(1)}}, \phi_{k_0^{(2)}}, \ldots, \phi_{k_0^{(M)}}\right\}.$$

■

Before presenting a particular example, some comments are in order. First, the uniqueness of the equilibrium is not guaranteed, since with identical payoff values all strategies are equilibrium strategies. Second, the proof suggests a simple computational method to find the equilibrium, which is known also as *backward induction* similarly to the leader-follower games. Third, the theorem might fail if the graph describing the game is not finite. Assume that in the last stage of the game each player has infinitely many arcs to get to endpoints and the payoff values are there $0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-1}{n}, \dots$. Then the players have no maximal payoff choices at these endpoints. So there is no equilibrium. Fourth, it is also possible that at certain nodes the continuation of the game is random according to discrete probability distributions defined on the sets of arcs originating at these nodes. Such moves are called "chance moves". The theorem remains true in this more general case as well, when the expected values of the payoffs replace the deterministic payoff values.

Example 9.8 Consider again the game introduced in the previous example, which is shown again in Fig. 9.3 We indicated at each node the player assigned to it, and also at each endpoint we have the payoff values of the three players. First we determine the optimum last moves of the game. They start at nodes B_1, B_2, B_3, B_4, B_5, and B_6. At B_1, player three has two choices, moving up and down. If he moves up, then gets 2 and by moving down he would get 3. Since $3 > 2$, from point B_1 the game would terminate on the arc moving down. The thicker arc shows this choice. At point B_2, player 3 would move up, since $2 > 1$. At point B_3, player 2 has the choice between two arcs, where his payoff is 3 or 5, and since $5 > 3$, the arc moving down is selected. Similarly from point B_4 player 2 would move up, at points B_5 and B_6 player 3 would move up. By going back with one stage we get to points A_1, A_2 and A_3. At point A_1 player 2 selects between two arcs, up and down. By moving up, he would get to point B_1, from where the game would continue on the arc pointing down resulting in a payoff value 2. If player 2 would select moving down at point A_1, then he would reach point B_2, where player 3 would move up resulting in a payoff value 1 for player 2. Since $2 > 1$, at point A_1 player 2 selects to move up. It is easy to show in a similar way, that at point A_2 player 3 moves down and at point A_3 player 2 moves up. By going back with one stage again, we reach the root R, where player 1 has 3 choices. By moving up, he would arrive at point A_1, where player 2 would choose to move up to point B_1, from where player 3 would move down to the endpoint giving payoff value 2 to player 1. If he would move to point A_2, then the game would continue to point B_4 and then up to the endpoint giving payoff value 1 to player 1. Similarly, if player 1 would move down to point A_3, then the game would continue to point B_5 and then up to the endpoint giving unit payoff value to player 1. Since the three choices would result in payoffs 2, 1, 1 to player 1, he definitely would select to move up to point A_1. So the equilibrium path is $R \to A_1 \to B_1 \to E$. The thick arcs show the equilibrium choices of the players in all nodes of the game.

▼

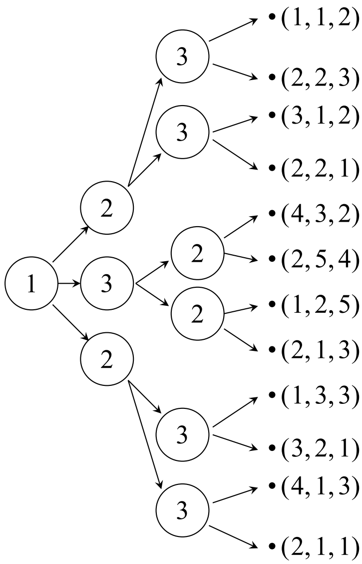

Fig. 9.2 A finite tree game with three players

Table 9.1 Payoff table of Example 9.8

E \ C	H	S
I	(a, b)	(c, d)
O	(α, β)	(α, β)

Applying backward induction we always get an equilibrium, but there is no guarantee that all equilibria can be obtained by this way, as it is illustrated by the following example.

Example 9.9 (Chain store and an enterpreneur) Assume that an enterpreneur is planning to open a small business (e.g. selling hamburgers) next to a chain store selling similar items. After the enterpreneur starts advertising his business and becomes ready to open, the chain store can become hard on the enterpreneur by lowering its prices, giving special coupons in order to drive him out of the business, or can be soft and let the enterpreneur do business without any difficulties from the chain store. The response of the enterpreneur is to go into business or to get out of it by not opening. A two-person game is defined, the enterpreneur (E) and the chain store (C) are the players with strategy sets $S_1 = \{I = in, O = out\}$ for the enterpreneur and $S_2 = \{H = hard, S = soft\}$ for the chain store. The payoff functions are given in Table 9.1., where we assume that $a < c$, $b < d$ (selection of H by C hurts both players), $\beta > \max\{b, d\}$ (the best payoff for C occurs if E is out), furthermore $a < \alpha < c$ (getting out gives higher payoff to E than fighting with C, and gives lower payoff than staying in business without the interference of C).

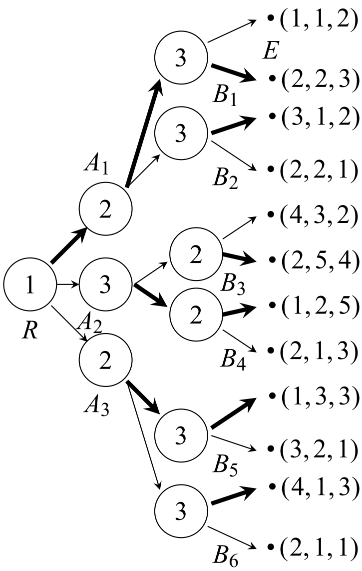

Fig. 9.3 Illustration of the backward induction

We can illustrate this game by a small tree shown in Fig. 9.4. At point A player C can select between two arcs resulting in payoff values b and d. Since $b < d$, the player will select to move down. At the root E has to choose between two arcs. Moving up he would get α but by moving down to point A, the game will continue on arc S giving payoff c to player E. Since $c > \alpha$, player E should select arc I, so the obtained equilibrium is (I, S). ▼

We can also determine the equilibria from Table 9.1. The best responses of the players are as follows:

$$R_E(H) = 0, \quad R_E(S) = I$$

and

$$R_C(I) = S, \quad R_C(0) = \{H;\ S\}$$

so we have two equilibria $(0, H)$ and (I, S). The first equilibrium cannot be obtained by using backward induction. ▼

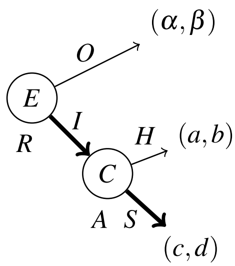

Fig. 9.4 Game tree of Example 9.9

Example 9.10 (Chess-game) The chess-game also satisfies the conditions of Theorem 9.1. The root of the game is the initial setting of the board with all 32 figures in their original positions. The white player has the first choice from the possible 20 moves. Then the black player moves from the 20 possibilities, and after this move the white player comes, and so on. This game is finite, since after repeating the same standing on the board a certain number of time, the game terminates with a tie, and there are only finitely many standing possibilities. So the chess-game has at least one equilibrium. However the huge size of the graph describing all possible situations and moves in a chess-game is so large, that finding an equilibrium is impossible even by using modern high-speed computers. ▼

9.5 Extensive-Forms of Dynamic Games

In the previous section we examined dynamic games in which the dynamics and the evolutions of the games were characterized by graphs. In the case of static games the normal-form representation giving the players, sets of strategies and the payoff function was sufficient to characterize and solve the games. If the game is dynamic, then in general the dynamics of the game can be characterized by its *extensive-form*, which specifies the followings:

1. The players of the game;
2. The time periods when each player has to move;
3. The possible choices of each player when he has the opportunity to move;
4. The payoff values received by each player for all possible combinations of the moves the players can select during the game.

Without saying we have already used extensive-forms in the previous section.

Example 9.11 Consider the tree-game of Fig. 9.5. with two players, and at each time when any of the players has to move, he has two choices, up (U) and down (D). At time 1, player 1 has the choice and then at time 2, player 2 can select the next direction of move. Then the game terminates, and at each of the endpoints both players receive payoff values. ▼

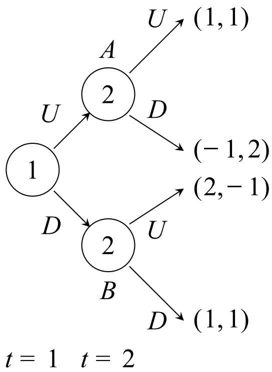

Fig. 9.5 Extensive form of Example 9.11

Example 9.12 In Fig. 9.3. we face a slightly different situation, where the players are assigned to the different states of the game and not to different time periods.

With a small modification this problem can be solved as shown in Fig. 9.6, where at $t = 1$ player 1 moves, at $t = 2$ player 2 moves, at $t = 3$ player 3 has the choice and at $t = 4$ player 2 moves again. And at $t = 5$ the terminal nodes are reached. ▼

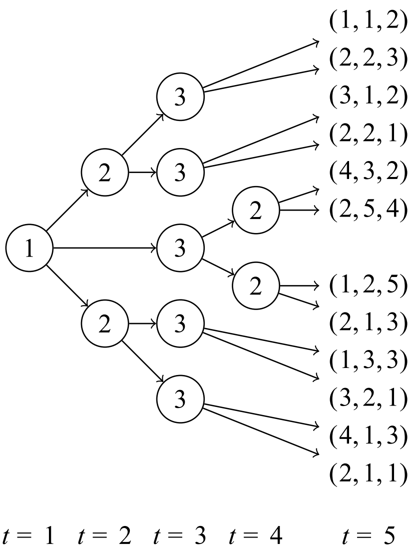

Fig. 9.6 Modified graph of Example 9.3

The strategy of a player is a decision at each time period and each possible past history of the game. In the graph representation it means that the player has to choose direction of continuation at each node the player is assigned to. There is a strong relation between the normal and extensive forms of dynamic games.

Example 9.13 We can illustrate this in the case of the game shown in Fig. 9.5. Player 1 has only one node where he can choose between U and D. Player 2 however has two nodes, A and B. At each node he has two choices, so he has altogether four strategies (U, U), (U, D), (D, U) and (D, D) where the first component gives his choice at point A and the second component is his choice at point B. The payoff matrix is given in Table 9.2. ▼

Table 9.2 Payoff matrix of game of Fig. 9.5

1	2			
	(U, U)	(U, D)	(D, U)	(D, D)
U	(1, 1)	(1, 1)	(−1, 2)	(−1, 2)
D	(2, −1)	(1, 1)	(2, −1)	(1, 1)

Static games also can be represented in extensive form. For example, consider the prisoner's dilemma game introduced in Example 2.1, where the payoff matrix was given in Table 2.1. The players move simultaneously, so when a player selects strategy then he does not know the selected strategy of the other player, and if we consider the game as a dynamic game when player 1 moves first and player 2 moves next, then we get the extensive form shown in Fig. 9.7. When player 2 has to move, he does not know the choice of player 1, so he does not know which is the point (A or B) from which he has to continue moving forward. We can represent this kind of uncertainty about previous moves in an extensive-form game by introducing the notion of a player's *information set*, which contains decision nodes such that the player has the move at every node of the information set, but the player does not know which node of the information set has been reached. The usual way of representating information sets is by connecting its nodes by dotted lines. ▼

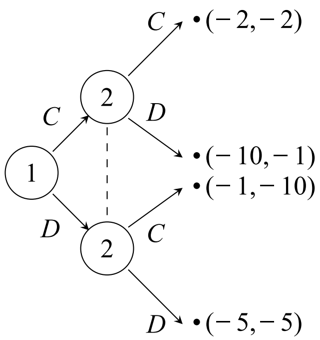

Fig. 9.7 Extensive form in the prisoner's dilemma game

9.6 Subgames and Subgame-Perfect Nash Equilibria

Consider an extensive form game, which might contain information sets. A *subgame* in this extensive-form game is defined as follows. It starts at a decision node, DN, which is not the game's first decision node and is a singleton information set. It consists of all decision and terminal nodes which follow DN and does not cut any information set, that is, if any node of an information set belongs to a subgame, then all nodes of that information set also must belong to the subgame. Notice first that the prisoner's dilemma game has no subgame, since the only singleton information set is the root of the graph, which is the game's first decision node. In the case of the game shown in Fig. 9.3. all points A_1, A_2, A_3, B_1, B_2, B_3, B_4, B_5, B_6 can be the starting points of subgames. For instance the subgame starting at point A_2 has the decision points A_2, B_3 and B_4 and four terminal nodes. The motivations of the above described conditions for a subgraph being a subgame are the requirements that it should be analyzed on its own and the analysis has to be relevant to the original game. If nodes of an information set with multiple nodes would be selected as the initial decision nodes of a subgame, then there would be an uncertainty about the starting move and so about the entire dynamism of the game. Every possible continuation from the initial decision node of the subgame has to be contained in its subgame and nodes which are not reachable from the initial point of the subgame should not be contained in it.

Definition 9.1 A Nash equilibrium is called *subgame perfect*, if the equilibrium strategies of the players form Nash equilibrium in every subgame.

We should not be confused between the outcome obtained by backward induction and the notion of subgame perfect Nash equilibria. Consider first a leader-follower game, in which backward induction gives the outcome $\left(s_1^*, R_2(s_1^*)\right)$ where R_2 is the best response function of player 2 and s_1^* maximizes $\phi_1\left(s_1, R_2(s_1)\right)$ with ϕ_1 being the payoff function of player 1. If this game is considered as a dynamic game, then player 2 has to select strategy in every node where he has to decide. There are infinitely many nodes, since s_1 can usually have infinitely many values. So player 2 has to specify his choice for all possible values of s_1, so his strategy is $s_2 = R_2(s_1)$. So the subgame perfect Nash equilibrium is $\left(s_1^*, R_2(s_1)\right)$.

Chapter 10
Games Under Uncertainty

In the previous chapters we were dealing with games with complete information and in the case of dynamic games perfect information was also assumed. Complete information refers to games when every player knows the strategy sets and payoff functions of all players, that is, all players have complete information about the game. Perfect information refers to dynamic games when at each time period each player knows the previous strategy selections of all players and the previous chance moves if any. In short, complete information refers to the amount of information the players have about the game and perfect information refers to the amount of information the players have about the other players' and their own previous moves (and about the possible chance moves).

In the case of a static game we talk about incomplete information, if there is a lack of information on the strategy set and/or the payoff function of at least one player. Without losing generality we may assume uncertainty in only the payoff functions. If the strategy set of any player is uncertain, then we can consider the union of all possible strategy sets as the strategy set of the player and if at any realization of the game a strategy of the union becomes infeasible, then we can define the associated payoff value as $-\infty$.

Before introducing the theory of Bayesian games some simple examples are introduced.

Example 10.1 Consider a duopoly with price function $p(s_1 + s_2) = a - (s_1 + s_2)$, and assume that the cost function of firm 1 is $C_1(s_1) = c_1 s_1$, and the cost function of firm is 2 uncertain in the sense that it is $C_2(s_2) = c_L s_2$ with probability p and $C_2(s_2) = c_H s_2$ with probability $1 - p$. Firm 1 knows his own cost function and the two possibilities for $C_2(s_2)$ and the occurring probabilities. Firm 2 knows the cost function of firm 1 and his actual choice between the two marginal costs c_L and c_H. Both firms know the price function. This information structure is asymmetric, and only player 1 faces uncertainty in the cost function of his opponent. This structure can

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_10

be used in the case when player 2 is new in the market, or invented a new technology, and firm 1 is not sure that firm 2 selects the technology with low or high marginal cost.

Let's next see how the two players think in this situation. Firm 2's best strategy choice is obtained by maximizing his profit

$$\phi_2 = s_2(a - s_1 - s_2) - c_2 s_2 \tag{10.1}$$

where c_2 is equal to either c_L or c_H. Assuming interior optimum,

$$s_2^*(c_L) = \frac{a - s_1^* - c_L}{2} \tag{10.2}$$

and

$$s_2^*(c_H) = \frac{a - s_1^* - c_H}{2} \tag{10.3}$$

where s_1^* is the optimum choice of firm 1, who's profit can be given as the expectation

$$\phi_1 = \Big[\big(a - s_1 - s_2^*(c_L) - c_1\big)s_1\Big]p + \Big[\big(a - s_1 - s_2^*(c_H) - c_1\big)s_1\Big](1 - p). \tag{10.4}$$

Assuming again interior optimum,

$$s_1^* = \frac{\big[a - s_2^*(c_L) - c_1\big]p + \big[a - s_2^*(c_H) - c_1\big](1 - p)}{2}. \tag{10.5}$$

Notice that equations (10.2), (10.3) and (10.5) give three equations for the three unknowns, s_1^*, $s_2^*(c_L)$ and $s_2^*(c_H)$ and the solutions are as follows:

$$s_1^* = \frac{a - 2c_1 + pc_L + (1 - p)c_H}{3}, \tag{10.6}$$

$$s_2^*(c_L) = \frac{a + c_1}{3} - \frac{c_L(p + 3) + (1 - p)c_H}{6} \text{and} s_2^*(c_H) = \frac{a + c_1}{3} - \frac{pc_L + (4 - p)c_H}{6} \tag{10.7}$$

▼

Example 10.2 Consider again the earlier Example 2.1 of the prisoner's dilemma, where each player had two strategies; cooperate with his partner (C) or defect (D). The payoff matrix of Table 2.1 is now modified as shown in Table 10.1.

Table 10.1 Modified payoff matrix of Example 2.1

1 \ 2	C	D
C	(0, −2)	(−10, −1)
D	(−1, −10)	(−5, −5)

The only modification in the payoffs is in the case, when both players cooperate. In the modified table we assume that in the lack of hard evidence player 1 can go free, while player 2 gets the same sentence as in the original game. This can be the situation if player 1 is an informer and his earlier services to the police are honored by this favor. Player 1 knows that there is the possibility that his partner will have certain emotional damage (like being afraid that the friends of his partner will take a revenge on him) when defecting, which is equivalent to 6 additional years spent in prison. However player 1 does not know for sure that it will happen or not. Here is the uncertainty of the game, since there are two game types: Type I game has the payoff table given in Table 10.1, and Type II game has the payoff table shown in Table 10.2 with asymmetric information structure.

Table 10.2 Payoff matrix of the Type II game

1 \ 2	C	D
C	(0, −2)	(−10, −7)
D	(−1, −10)	(−5, −11)

In the realization of the game player 2 knows the type of the game (his own emotional condition), but it is unknown to player 1. He thinks that the probability that the game is Type I is p, and the probability of the Type II game is $1 - p$. Let's see how the players think in this situation. First player 2 is considered. If the game is Type I, then D is his dominant strategy, and if the game is Type II, when C is his dominant strategy. Therefore his strategy choice would be D or C in Type I or Type II game:

$$s_2^* \text{ (Type I) } = D$$
$$s_2^* \text{ (Type II)} = C.$$

Then Player 1 can determine his expected payoffs as

$$E\big(\phi_1(C)\big) = (-10)p + 0(1-p) = -10p \text{and} \qquad E\big(\phi_1(D)\big) = (-5)p + (-1)(1-p) = -1 - 4p,$$

so his choice will be C if

$$-10p > -1 - 4p,$$

that is, when $p < \frac{1}{6}$; his choice will be D if

$$-1 - 4p > -10p,$$

that is, when $p > \frac{1}{6}$; and the two strategies are equivalent if

$$-10p = -1 - 4p,$$

that is, when $p = \frac{1}{6}$. ▼

Example 10.3 Consider now a two-person game where both players have two possible strategies, s_1, s_2 and t_1, t_2. The payoff values depend on whether the players are in weak or strong position. Let a_1 and a_2 denote the weak and strong positions of player 1, and b_1 and b_2 those of player 2. Therefore there are four different game types: (a_1, b_1), (a_1, b_2), (a_2, b_1) and (a_2, b_2). Their occuring probabilities are given in Table 10.3. That is, $p(a_1, b_1) = 0.4$, $p(a_1, b_2) = 0.1$, $p(a_2, b_1) = 0.2$, and $p(a_2, b_2) = 0.3$.

Table 10.3 Occurance probability values

	b_1	b_2
a_1	0.4	0.1
a_2	0.2	0.3

There are different payoff matrices in the cases of the four game types, the payoff matrices of player 1 are summarized in Table 10.4. Assuming that all game types are zero sum we know that $\phi_2 = -\phi_1$.

Table 10.4 Payoff matrices of player 1 in Example 10.3

$1 \setminus 2$	t_1	t_2
s_1	2	5
s_2	−1	20

(a_1, b_2)

$1 \setminus 2$	t_1	t_2
s_1	−24	−36
s_2	0	24

(a_2, b_1)

$1 \setminus 2$	t_1	t_2
s_1	28	15
s_2	40	4

(a_2, b_2)

$1 \setminus 2$	t_1	t_2
s_1	12	20
s_2	2	13

As in the previous examples, each player has to decide his strategy choices in the case when he is weak and also in the case when he is strong. So each strategy of each player consists of two numbers, (i, j) where i shows his strategy choice when he is weak and j is the same when he is strong. Since both i and j can be either 1 or 2, each player has 4 such strategies. The associated 4×4 payoff matrix is presented in Table 10.5.

Table 10.5 Final payoff matrix of player 1 in Example 10.3

1 \ 2	(1, 1)	(1, 2)	(2, 1)	(2, 2)
(1, 1)	7.6	8.8	6.2	7.4
(1, 2)	7.0	9.1	1.0	3.1
(2, 1)	8.8	13.6	14.6	19.4
(2, 2)	8.2	13.9	9.4	15.1

To illustrate how these payoff values are obtained, consider first the strategy selection(1, 2) and (1, 1). There are four game types. In the game (a_1, b_1) both players play strategy 1 (s_1 and t_1), so the payoff of player 1 is 2. In game (a_1, b_2) both players play again strategy 1 with payoff value -24 of player 1. If the game is (a_2, b_1), then player 1 selects strategy s_2 and player 2 chooses again strategy t_1 giving 40 payoff value to player 1, and finally in game (a_2, b_2) player 1 selects s_2 and player 2 chooses again t_1 so the resulting payoff value is 2 for player 1. Since from Table 10.3 we know that the occuring probabilities of the four game types are 0.4, 0.1, 0.2 and 0.3, respectively, the expected payoff of player 1 becomes

$$0.4(2) + 0.1(-24) + 0.2(40) + 0.3(2) = 7.0.$$

In the case of strategy selection (1, 2) and (2, 1) the strategies of the players in the four game types are (s_1, t_2), (s_1, t_1), (s_2, t_2) and (s_2, t_1), respectively, therefore the associated payoff values for player 1 are 5, -24, 4 and 2, respectively, so the expected payoff for player 1 is the following:

$$0.4(5) + 0.1(-24) + 0.2(4) + 0.3(2) = 1.0.$$

Notice that Table 10.5 is the payoff matrix of a zero-sum game, so the pure strategy equilibria are obtained by matrix elements which are largest in their columns and also smallest in their rows. The largest elements of the four columns are 8.8, 13.9, 14.6 and 19.4, and only 8.8 is the smallest in its row, therefore this matrix element gives the unique pure strategy equilibrium: (2, 1) and (1, 1) as corresponding strategies.▼

10.1 Static Bayesian Games

In the previous examples the uncertainty was in the type of at least one player which was determined by the choice of marginal cost, or the payoff values were based on emotional state, or being weak or strong. The uncertainty was characterized by probability distributions defined on the random possibilities. This idea is generalized in defining the normal forms of games with incomplete information also called *Static Bayesian games*. Let N denote the number of players and assume that each player $k (1 \leqq k \leqq N)$ can have a type $t_k \in T_k$, where T_k is his type space. Let S_k denote the strategy set of player k. The information structure is the following. Each player k knows his own type, and his payoff values $\phi_k(s_1, \ldots, s_N; t_k)$ where $s_1, \ldots, s_N$ are the strategy selections of the players, $s_l \in S_l$ for $l = 1, 2, \ldots, N$, and t_k is his type. He also has belief, $p_k(t_{-k}|t_k)$, which is the conditional distribution (density or mass function) of the joint types of the other players given his own type. In the Bayesian game introduced by Harsanyi (1967) it is assumed that first the nature selects the type vector $\underline{t} = (t_1, \ldots, t_N)$ where $t_i \in T_i$ for all players i, and reveals t_i to player i but not to the other players. Then the players select strategies simultaneously and each player k receives the corresponding payoff $\phi_k(s_1, \ldots, s_N; t_k)$. In the case when a player has private information about the types of some other players, then his belief is modified as follows. Assume that player k knows the types of players $i_1, i_2, \ldots, i_l$ (including himself) then his belief is given by the conditional joint density function of the types of all other players who's types are unknown to player k : $p_k(\underline{t}_{-\{i_1,i_2,\ldots,i_l\}}|t_{i_1}, t_{i_2}, \ldots, t_{i_l})$ where $\underline{t}_{-\{i_1,i_2,\ldots,i_l\}}$ is the vector with components $t_i (i \neq i_1, i_2, \ldots, i_l)$. The conditional densities $p_k(\underline{t}_{-k}|t_k)$ have to be consistent with each other, therefore it is assumed that the joint distribution of the types of all players is a public information, that is, it is known by all players, and the conditional distributions are obtained by using the theorem of Bayes. In the discrete case the values of the conditional mass functions are obtained as

$$p_k(\underline{t}_{-k}|t_k) = \frac{p(t_1, \ldots, t_N)}{\sum_{t_k} p(t_1, \ldots, t_N)} \tag{10.8}$$

and in the continuous case the conditional probability density functions are given as

$$p_k(\underline{t}_{-k}|t_k) = \frac{p(t_1, \ldots, t_N)}{\int p(t_1, \ldots, t_N) dt_k}. \tag{10.9}$$

In the description of a static Bayesian game we have to specify the sets of the feasible strategies, the set of all possible types, the beliefs and the payoff functions of all players, so the normal form representation is usually written as

$$\{N; S_1, \ldots, S_N; T_1, \ldots, T_N; p_1, \ldots, p_N; \phi_1, \ldots, \phi_N\}.$$

Example 10.4 In the case of Example 10.1, firm 1 had only one type and firm 2 had two possible types by selecting low or high marginal cost. Therefore $T_1 = \{1\}$ and $T_2 = \{c_L, c_H\}$, and the beliefs are given as

$$p_1(c_L|1) = p \text{ and } p_1(c_H|1) = 1 - p,$$

furthermore

$$p_2(1|c_L) = p_2(1|c_H) = 1.$$

In Example 10.2 we have a similar situation, since player 1 has only one type and player 2 has two possible types, so the beliefs of the players are as follows:

$$p_1(\text{Type I}|1) = p, \; p_1(\text{Type II}|1) = 1 - p$$

and

$$p_2(1|\text{Type I}) = p_2(1|\text{Type II}) = 1.$$

▼

Example 10.5 In Example 10.3 both players have two types, weak or strong. The joint probability mass function is given in Table 10.3, and therefore the conditional beliefs of the players are computed by using Eq. (10.8):

$$p_1(b_1|a_1) = \frac{4}{5} \qquad p_1(b_1|a_2) = \frac{2}{5}$$

$$p_1(b_2|a_1) = \frac{1}{5} \qquad p_1(b_2|a_2) = \frac{3}{5}$$

and

$$p_2(a_1|b_1) = \frac{4}{6} \qquad p_2(a_1|b_2) = \frac{1}{4}$$

$$p_2(a_2|b_1) = \frac{2}{6} \qquad p_2(a_2|b_2) = \frac{3}{4}.$$

▼

In order to introduce a general definition of the strategies of the players in a static Bayesian game notice that it was a unique choice of players who had only one type, and for players with multiple types it was an instruction of strategy selection in the cases of all possible types. Assume now that for player k the set of all possible types is T_k, then a strategy of this player is a function $s_k(t_k)$ defined on T_k with range in S_k meaning that if the type of the player is t_k, then he selects strategy $s_k(t_k)$. In the case of Example 10.1 the equilibrium strategy for player 1 is unique but for player 2 it depends on his selection between c_L and c_H (Eqs. (10.6)–(10.7)). In the case of Example 10.2, the strategy selection of player 1 is unique, but depends on the probability value p:

$$s_1^*(p) = \begin{cases} C & \text{if } p < \frac{1}{6} \\ D & \text{if } p > \frac{1}{6} \\ \{C, D\} & \text{if } p = \frac{1}{6}. \end{cases}$$

The strategy choice of player 2 depends on the type of the game (which is actually his type):

$$s_2^*(\text{Type I}) = D \;\text{ and }\; s_2^*(\text{Type II}) = C.$$

In the case of Example 10.3 the type-dependent strategies with the associated payoff values are presented in Table 10.5.

The Bayesian payoff function of each player is defined as his expected payoff value given his type, where expectation is taken with respect to the types of the other players given their conditional distributions. In the discrete case it is given as

$$\sum_{\underline{t}_{-k}} \phi_k\Big(s_1(t_1), \ldots, s_N(t_N); t_k\Big) p_k(\underline{t}_{-k}|t_k). \tag{10.10}$$

Similarly, in the continuous case this expression is modified:

$$\int \phi_k\Big(s_1(t_1), \ldots, s_N(t_N); t_k\Big) p_k(\underline{t}_{-k}|t_k) d\underline{t}_{-k} \tag{10.11}$$

so instead of adding the products of the payoff values and their believed occuring probabilities, the product of the payoff function and the associated probability density function is integrated. So we know the players, their strategies and the Bayesian payoff functions are defined, so we are ready to introduce *Bayesian Nash equilibria*. Notice first that expectations (10.11) and (10.12) depend on the strategy selections of all players and only on the type of player k. For the sake of simple notation let $\prod_k \big(s_1, \ldots, s_{k-1}, s_k(t_k), s_{k+1}, \ldots, s_N; t_k\big)$ denote expression (10.11) in the discrete case and (10.12) in the continuous case.

Definition 10.1 A Bayesian Nash equilibrium is an N-tuple of strategies $(s_1^*, ..., s_N^*)$ such that for all players k and types $t_k \in T_k$, $s_k^*(t_k)$ maximizes function $\prod_k \big(s_1^*, \ldots, s_{k-1}^*, a_k, s_{k+1}^*, \ldots, s_N^*; t_k\big)$ where a_k runs through S_k.

10.2 Dynamic Bayesian Games

In the previous parts of this book we have already introduced several equilibrium concepts. In the case of static games with complete and perfect information the Nash equilibrium served as the solution of the game. If a static game has incomplete information, then the uncertainty in the types of the players were modeled by Bayesian methods and the resulted solution concept was Bayesian Nash equilibrium. For dynamic games with complete information we introduced the concept of subgame-perfect equilibria. If a dynamic game has incomplete information, then Bayesian methodology has to be included in the solution concept, which is known as *perfect Bayesian equilibrium*. In order to understand this concept completely, some simple examples will be first introduced.

Example 10.6 Consider again the battle of the sexes game (Example 2.5). Its extensive form representation is given in Fig. 10.1.

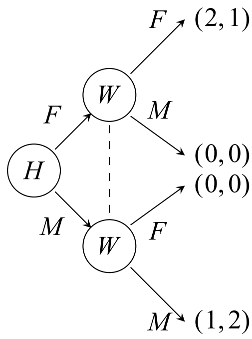

Fig. 10.1 Extensive form of the battle of sexes game

The game has one information set consisting of the two possible choices of the husband. We have two Nash equilibria, (F, F) and (M, M). Notice that both equilibria are subgame perfect, since the game has no subgame (nodes on information set cannot be starting points of subgames). When the wife selects strategy she does not know which equilibrium to be selected. However if she has a probabilistic feeling about the husband's choice, then her choice will be much easier. Assume the wife believes that the husband will select F with 60% probability and M with 40%. If the husband's choice is F, then the wife's best response is F giving her unit payoff value. If the husband selects M, then her best response is M with 2 payoff value. Under the uncertainty of the husband's choice her expected payoff is $0.6(1) + 0.4(0) = 0.6$ by selecting F and $0.6(0) + 0.4(2) = 0.8$ by selecting M. So the best move is M, since it gives the larger expected payoff value. ▼

In this simple game we had and used two assumptions, which can be generalized as follows:

(A) At each information set, the player who has to move must have beliefs about the nodes in the information set which is reached by the play of the game. This belief is a probability distribution defined on the nodes of the information set. In Example 10.6 the distribution was $p(F) = 0.6$ and $p(M) = 0.4$. The probability values are between 0 and 1 and their sum equals unity.

(B) Based on the beliefs of the players, their strategies must be sequentially rational. That is, at each information set the move of the player must be optimal given his belief and the other players' subsequent strategies.
Recall that the strategies of the players are complete plans of actions in every possible situation that might occur during the realizations of the game. In the above example the best response of the wife realized the optimality.
In computing the expected payoff for the wife in the previous example, not only the outcome at the equilibrium was taken into account, it included outcomes that do not occur in the case of an equilibrium. For example consider the (F, F) equilibrium in which case the outcomes (M, F), (F, M) and (M, M) can never occur. Therefore we need the following requirement:

(C) At any information set on the equilibrium path, beliefs are determined by the players' equilibrium strategies and Bayes' rule.
In the case of equilibrium (F, F) the wife's beliefs should be therefore $p(F) = 1$ and $p(M) = 0$ by the players' equilibrium strategies. Assume next that the husband has a third option, staying in office and finish a project, and there is a mixed equilibrium strategy of the husband; selecting F with probability p_1, M with p_2 and stay in office with probability $1 - p_1 - p_2$. In this case the wife's belief should be $p(F) = \frac{p_1}{p_1+p_2}$ and $p(M) = \frac{p_2}{p_1+p_2}$.
The strategy of each player consists of his moves at any circumstances when he has to move and also his beliefs at all information sets where he has to move. It means that the players must have beliefs at all information sets where they have to move and not only at information sets which belong to the equilibrium. Therefore we require condition (D) as follows:

(D) Requirement (C) is assumed at all information sets where the player has to move regardless whether the information set belongs to the equilibrium or not.

Definition 10.2 A *perfect Bayesian equilibrium* consists of strategies and beliefs satisfying conditions (A)–(D).

Example 10.7 (Signaling Games) Consider a dynamic game with two players, a sender (S) and a receiver (R). The game is played in the following steps:
(i) Nature selects the type of S from a finite set of possible types, $T = \{t_1, t_2, \ldots, t_N\}$, according to a discrete probability distribution, which is known by both players. If $p(t_n)$ denotes the probability of selecting t_n, then $0 \leqq p(t_n) \leqq 1$ for all n, and $\sum_{n=1}^{N} p\big(t_n\big) = 1$.
(ii) Player S observed his type t_n, which is unknown to R, and then sends a message m_p to R from a set of feasible messages, $M = \{m_1, m_2, \ldots, m_P\}$.
(iii) The message is observed by R and then he selects an action a_q from a set of feasible actions, $A = \{a_1, a_2, \ldots, a_Q\}$.
(iv) The payoffs received by the two players are

$$\phi_S(t_n, m_p, a_q) \text{ and } \phi_R(t_n, m_p, a_q).$$

This game is clearly a dynamic game with incomplete information, since the type of S is uncertain to R. In certain applications the sets T, M or A may be infinite, for example an interval on the real line.

The strategy of S is a decision on the selected message as function of his type, $m(t_n)$ $(n = 1, 2, \ldots, N)$, since he has to choose message in all possible types, that the nature selects for him. The strategy of R is his action depending on the message he receives from S, $a(m_p)$ $(p = 1, 2, \ldots, P)$. The total number of the feasible strategies of S is P^N since at any fixed t_n he has P possible messages to choose from. Similarly R has Q^P strategies, since at each message m_p he has Q possible actions. In order to see how the concept of perfect Bayesian equilibrium can be applied, we have to revisit requirements (A)–(D) given before, and interpret them in this case.

(A) After receiving a message m_p, R must have a belief about the type of S, which is a conditional probability

$$P(\text{type of } S \text{ is } t_n \,|\, \text{received message is } m_p) \tag{10.12}$$

which can be denoted by $\mu(t_n|m_p)$.

(B) Since both players have to be sequentially rational, we need to separate requirements for R and S as follows. For R, his action must maximize his expected payoff given his beliefs:

$$a^*(m_p) = \arg\max_{a_q \in A} \sum_{t_n \in T} \mu\Big(t_n|m_p\Big)\phi_R\Big(t_n, m_p, a_q\Big). \tag{10.13}$$

Here each possible outcome $\phi_R(t_n, m_p, a_q)$ is multiplied by its believed occurring probability, and these products are added up for all possible types of S. For S, notice that he knows his type before sending the message, so for him the game has complete information, and his choice is given as

$$m(t_n) = \arg\max_{m_p \in M} \phi_S\big(t_n, m_p, a^*(m_p)\big) \tag{10.14}$$

where $a^*(m_p)$ is the strategy of R.

(C) Since only R has beliefs, this requirement has to be posed on R only. Let m_p be a given message received by S, and let T_p denote all possible types t_n of S such that $m(t_n) = m_p$, then the Bayes rule requires that

$$\mu(t_n|m_p) = \frac{p(t_n)}{\sum_{t_r \in T_p} p\big(t_r\big)}. \tag{10.15}$$

In the denominator the probabilities of all types are added at which R could send the message m_p as his choice.

A pure strategy perfect Bayesian equilibrium is a pair of strategies $\big(m^*(t_n), a^*(m_p)\big)$ and beliefs $\mu(t_n|m_p)$ that satisfy the above requirements (A)–(C).

As an illustration of the requirements consider the game shown in Fig. 10.2.

The initial node (I) is a chance-node, where nature decides on the continuation of the game with $p(t_1) = p(t_2) = 0.5$. Depending on the initial chance-move S continues the game either from node A_1 or from node A_2. Then the game reaches one of the nodes B_1, B_2, B_3, B_4. Player R makes the final move, either a_1 or a_2 and the game terminates, giving the corresponding payoff values to the players. The payoff values are indicated in the figure next to the endpoints. The first component gives the payoff of S and the second component is the payoff of R. There are two information sets of R containing nodes B_1, B_3 and B_2, B_4. Let the beliefs of R be denoted by

$$p(B_1) = \mu(t_1|m_1) = \mu, \qquad p(B_3) = \mu(t_2|m_1) = 1 - \mu,$$
$$p(B_2) = \mu(t_1|m_2) = \overline{\mu}, \qquad p(B_4) = \mu(t_2|m_2) = 1 - \overline{\mu}.$$

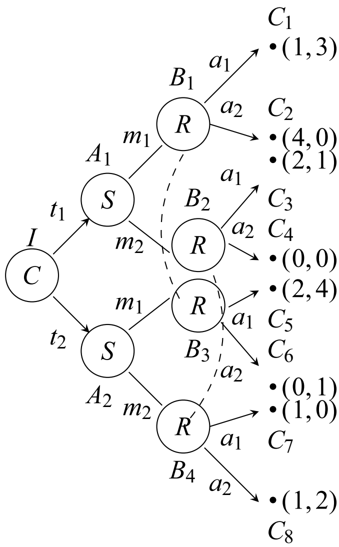

Fig. 10.2 Extensive form of a signaling game

There are four possible types of equilibrium strategies of S:

(i) $m(t_1) = m(t_2) = m_1$ (pooling on m_1)
(ii) $m(t_1) = m(t_2) = m_2$ (pooling on m_2)
(iii) $m(t_1) = m_1, m(t_2) = m_2$ (separation with t_1 playing m_1 and with t_2 playing m_2)
(iv) $m(t_1) = m_2, m(t_1) = m_1$ (separation with t_1 playing m_2 and with t_2 playing m_1).

As an example we show that

$$m(t_1) = m(t_2) = m_1,\ a(m_1) = a_1,\ a(m_2) = a_2,\ \mu = 0.5,\ \overline{\mu}$$

is a perfect Bayesian equilibrium for $\overline{\mu} \leqq \frac{2}{3}$.

Notice first that requirement (C) implies that $\mu = 0.5$.

If message is m_1, then the expected payoff of R with selecting a_1 and a_2 are as follows:

$$E(\phi_R) = 3\mu + 4(1-\mu) = 4 - \mu \text{ by selecting } a_1$$
$$E(\phi_R) = 0\mu + 1(1-\mu) = 1 - \mu \text{ by selecting } a_2,$$

so the choice of R has to be a_1.

If the message is m_2, then the expected payoff of R is similarly,

$$E(\phi_R) = 1\overline{\mu} + 0(1-\overline{\mu}) = \qquad \overline{\mu} \text{ by selecting } a_1$$
$$E(\phi_R) = 0\overline{\mu} + 2(1-\overline{\mu}) = \qquad 2-2\overline{\mu} \text{ by selecting } a_2.$$

Then R will choose a_2, if $2-2\overline{\mu} \geqq \overline{\mu}$, that is, when $\overline{\mu} \leq \frac{2}{3}$. We can now examine the payoff of S. If the message is m_1, then his payoff is 1 or 2 in type t_1 and type t_2, respectively. Assume next the case of message m_2, Player R selects a_2, so the payoff of S is 0 and 1 with types t_1 and t_2, respectively. Since $(0, 1) < (1, 2)$, the sender will not change his message from m_1 to m_2.

By a similar reasoning it is easy to show that there are no perfect Bayesian equilibrium of the kinds (ii) and (iii). Finally we verify that

$$m(t_1) = m_2,\ m(t_2) = m_1,\ a(m_1) = a(m_2) = a_1,\ \mu = 0,\ \overline{\mu} = 1$$

is also a perfect Bayesian equilibrium.

Requirement (C) implies that $\mu = 0$ and $\overline{\mu} = 1$, since in information set (B_1, B_3) only node B_3 is on the equilibrium path and in information set (B_2, B_4) only node B_2 is on the equilibrium path. If the message is m_1, then the payoff of R is as follows:

$$E(\phi_R) = 3\mu + 4(1-\mu) = 4-\mu = 4 \text{ if } a_1 \text{ is chosen}$$
$$E(\phi_R) = 0\mu + 1(1-\mu) = 1-\mu = 1 \text{ if } a_2 \text{ is chosen,}$$

so R will select a_1 and the sender gets payoffs 1 and 2 in cases t_1 and t_2.

If the message is m_2, then

$$E(\phi_R) = 1\overline{\mu} + 0(1-\overline{\mu}) = \overline{\mu} \qquad = 1 \text{ by choosing } a_1$$
$$E(\phi_R) = 0\overline{\mu} + 2(1-\overline{\mu}) = 2-2\overline{\mu} \qquad = 0 \text{ by choosing } a_2,$$

so the choice of R is a_1 giving payoff values 2 and 1 in types t_1 and t_2, respectively, to S.

If sender changes m_2 to m_1 in t_1, then receiver's choice is a_1 giving unit payoff to sender, which is less than 2 what he would get by playing m_2. If sender changes m_1 to m_2 in type t_2, then receiver would play a_1 again which would result in unit payoff for sender, which is less than 2 he would get by keeping m_1 in type t_2. So the sender does not have incentive to change his strategy. ▼

A nice detailed description of signaling games is given in Chapter 4 of Gibbons (1992). We also mention that signaling games have many applications in economy, including models in job market (Spence, 1973) investment and capital structure (Myers & Majluf, 1984), monetary policy (Vickers, 1986) among others.

Chapter 11
Solutions Based on Characteristic Functions

In the cases of noncooperative games the players cannot or do not want to make binding agreements, so they select strategies independently of each other, and receive the corresponding payoffs. The Nash equilibrium does not need agreement between the players, since at an equilibrium situation the interest of each player is to keep the equilibrium strategy, otherwise his payoff decreases. Each player considers his own selfish interest without any consideration to the other players. As the next simple example illustrates, the players can be able to increase their payoffs by cooperation.

Consider a duopoly with price function $10 - (x + y)$, where x and y are the outputs of the firms with $0 \leqq x, y \leqq 5$ and cost functions $C_1(x) = x$ and $C_2(y) = y$. The profit of firm 1 is clearly

$$\phi_1 = x(10 - x - y) - x$$

with best response

$$R_1(y) = \frac{9 - y}{2}.$$

Similarly the best response of firm 2 is

$$R_2(x) = \frac{9 - x}{2},$$

so the Nash equilibrium is the solution of equations

$$x = \frac{9 - y}{2}$$

$$y = \frac{9 - x}{2}$$

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_11

which is $\overline{x} = \overline{y} = 3$. The corresponding profit of both players is $\phi_1(\overline{x}, \overline{y}) = \phi_2(\overline{x}, \overline{y}) = 9$.

If the players cooperate, then they are able to maximize their overall profit in order to gain extra profit in comparison to the case of the Nash equilibrium. Since

$$\phi_1 + \phi_2 = (x + y)(10 - x - y) - (x + y) = u(10 - u) - u = 9u - u^2$$

with $0 \leqq u = x + y \leqq 10$, the overall profit is maximal if $u = \frac{9}{2}$, and the corresponding profit becomes $\frac{81}{4} = 20.25$. So by cooperating the players can increase their overall profit from 18 to 20.25.

In discussing cooperation we have to make it clear what cooperation means and what is the mechanism of cooperation. If cooperation consists of only information sharing when the players inform the others about their strategy sets, payoff functions, and all previous moves, then the game is not cooperative, it is still a noncooperative game with complete and perfect information. Cooperation means a certain way of coordinating their actions. Depending on the nature of the problem and the payoff functions, several possibilities can be considered.

If the payoffs are given in the same unit (e.g. in dollars), then the most logical way is for the players to form a grand coalition and maximize their overall payoff and then distribute the obtained amount among each other. The central issue in such cases is to find a fair and mutually acceptable distribution of this maximal amount. It has to be acceptable by all players and also by all possible coalitions in order to avoid some players to gang up against the others for even higher share. The strengths of the players and all possible coalitions are usually characterized by the amount they can receive without the help of the others. The concept of the *characteristic function* realizes this idea. There are several ways of distributing the total payoff among the players, we will introduce some of them later.

Consider an N-person game and let $\mathbb{N} = \{1, 2, \ldots, N\}$ be the set of players. Any subset of $\mathbb{N}$ is called a coalition. The characteristic function of the game is a real-valued function defined for all possible coalitions of the players. That is, $v : 2^{\mathbb{N}} \mapsto \mathbb{R}$, where $2^{\mathbb{N}}$ denotes the set of all (2^N) possible subsets of $\mathbb{N}$. If S is a coalition then $v(S)$ can be interpreted as the amount of payoff coalition S can get without the cooperation of the rest of the players. Clearly $v(\phi) = 0$ and $v(\mathbb{N})$ is the maximum of the sum of the payoffs of all players. The most common general definition of $v(S)$ is given as follows. Let s_i denote the strategy of player i, $s_i \in S_i$. The payoff of coalition S is clearly $\sum_{i \in S} \phi_i\big(s_1, \ldots, s_N\big)$. If the rest of the players want to punish the members of the coalition then their strategy selection is to minimize the overall payoff of the coalition. And then the members of the coalition want to maximize their overall payoff in this worst case scenario:

$$v(S) = \max_{\substack{s_i \\ i \in S}} \min_{\substack{s_j \\ j \notin S}} \sum_{i \in S} \phi_i\Big(s_1, \ldots, s_N\Big). \tag{11.1}$$

As an illustration consider the following simple example.

Example 11.1 Consider an oligopoly with three players, and let s_1, s_2, s_3 denote the outputs of the players. Assume that the capacity limit of each firm is 3, the price function is $10 - s$, where $s = s_1 + s_2 + s_3$. It is also assumed that the firms have identical cost functions, $C_i(s_i) = s_i + 1$. In the case of three players we have $2^3 = 8$ coalitions:

$$\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\} \text{ and } \{1,2,3\}.$$

Because of symmetry,

$$v(\{1\}) = v(\{2\}) = v(\{3\})$$

and

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}).$$

By using (11.1),

$$v(\{1\}) = \max_{s_1} \min_{s_2, s_3} s_1(10 - s_1 - s_2 - s_3) - (s_1 + 1).$$

The payoff of player 1 is minimal, if the other players can reduce the price as much as possible which is reached if they produce maximum output levels: $s_2 = s_3 = 3$. Then the minimal value of ϕ_1 becomes

$$s_1(4 - s_1) - (s_1 + 1) = -s_1^2 + 3s_1 - 1.$$

This is a concave parabola in s_1, so the first order condition gives maximum:

$$-2s_1 + 3 = 0$$

implying that $s_1 = \frac{3}{2}$. Then the maximal value of ϕ_1 under the minimality condition with respect to s_2 and s_3 becomes

$$-s_1^2 + 3s_1 - 1 = -\frac{9}{4} + \frac{9}{2} - 1 = \frac{5}{4} = 1.25$$

so we have

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = \frac{5}{4}.$$

Similarly,

$$\begin{aligned} v(\{1,2\}) &= \max_{s_1,s_2} \min_{s_3} (s_1+s_2)(10-s_1-s_2-s_3)-(s_1+s_2+2) \\ &= \max_{u} \min_{s_3} u(10-u-s_3)-(u+2) \end{aligned}$$

where we introduce the new variable $u = s_1 + s_2 \in [0, 6]$. The largest possible s_3 value provides the minimum, so players 1 and 2 maximize

$$u(7-u)-(u+2) = -u^2+6u-2.$$

Maximum occurs at $u = 3$ and the corresponding function value is $-9 + 18 - 2 = 7$. So

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 7.$$

And finally we compute the characteristic function value at the grand coalition:

$$v(\{1,2,3\}) = \max_{s_1,s_2,s_3} (s_1+s_2+s_3)(10-s_1-s_2-s_3)-(s_1+s_2+s_3+3).$$

By introducing the new variable $\overline{u} = s_1 + s_2 + s_3 \in [0, 9]$, the total payoff of the players becomes

$$\overline{u}(10-\overline{u})-\overline{u}-3 = -\overline{u}^2+9\overline{u}-3.$$

Its maximum occurs at $\overline{u} = \frac{9}{2}$ with maximum function value

$$v(\{1,2,3\}) = -\frac{81}{4}+\frac{81}{2}-3 = \frac{69}{4} = 17.25.$$

▼

There are other alternative definitions of the characteristic function, and as the different solution concepts are concerned it really does not matter how the characteristic function is obtained. So an N-person cooperative game with characteristic function v can be denoted as $G = \{N, v\}$. The following example shows a case when the characteristic function is obtained by using the nature of the game without a formal definition.

Example 11.2 Assume three children want to buy as much candy as possible. There are three boxes available to purchase, their sizes are 500, 750 and 1000 g with prices \$7, \$9 and \$11. The three children have 6, 4 and 3 dollar budgets, respectively. Since none of the children has enough money to buy any of the boxes alone, they have to form coalitions to do so. Clearly

$$v(\phi) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0.$$

Notice that $v(\{1,2\}) = 750$, since children 1 and 2 have altogether 10 dollars and they can buy the 9 dollar box. Similarly $v(\{1,3\}) = 750$, $v(\{2,3\}) = 500$ and $v(\{1,2,3\}) = 1000$.

▼

A game $\Gamma = \{N, v\}$ is *superadditive*, if for all mutually exclusive $S, T \subseteq \mathbb{N}$,

$$v(S \cup T) \geqq v(S) + v(T) \tag{11.2}$$

meaning that the merging of mutually exclusive coalitions cannot make their situation worse. By induction one can easily show that (11.2) can be generalized: with pairwise mutually exclusive coalitions S_k $(k = 1, 2, ..., K)$,

$$v(S_1 \cup S_2 \cdots \cup S_K) \geqq \sum_{k=1}^{K} v(S_k). \tag{11.3}$$

A game $\Gamma = \{N, v\}$ is *monotonic* if $S \supseteq T$ implies

$$v(S) \geqq v(T)$$

which means that larger coalition cannot have worse situation than the smaller one.

The convexity of games is defined similarly to real functions by requiring the monotonicity of the first differences. The first difference is now the contribution of a player to a coalition,

$$d_i(S) = \begin{cases} v\big(S \cup \{i\}\big) - v(S) & \text{if } i \notin S \\ v(S) - v\big(S - \{i\}\big) & \text{if } i \in S \end{cases} \tag{11.4}$$

where $S - \{i\}$ contains all elements of S except player i. A game $\Gamma = \{N, v\}$ is *convex*, if $S \subseteq T$ implies that for all $i \in \mathbb{N}$,

$$d_i(S) \leqq d_i(T) \tag{11.5}$$

meaning that the values of coalitions increase more rapidly as the coalitions become bigger.

A game $\Gamma\{N, v\}$ is *constant-sum*, if for any coalition $S \subset \mathbb{N}$,

$$v(S) + v(\mathbb{N} - S) = v(\mathbb{N}). \tag{11.6}$$

As a special case the game is *zero-sum* if (11.6) holds with $v(\mathbb{N}) = 0$.

Game $\Gamma\{N, v\}$ is called *rational* if the grand coalition achieves at least as much as the players would get together without cooperation:

$$v(\mathbb{N}) \geqq \sum_{i=1}^{N} v\big(\{i\}\big). \tag{11.7}$$

If this inequality is strict, then the game is *essential*, otherwise the game is called *inessential*. Notice that in inessential games the players have no incentives to cooperate. Therefore in the rest of this chapter we assume that the game under consideration is essential.

In some applications the superadditive property is relaxed by assuming that for all $S \subseteq \mathbb{N}$,

$$v(\mathbb{N}) \geqq v(S) + \sum_{i \notin S} v\big(\{i\}\big). \tag{11.8}$$

In this case the game is *weakly superadditive*.

Sometimes it is useful to consider *strategic equivalence* of cooperative games, since by proving an important property for one of the equivalent games it becomes immediately valid for all of them. We say that games $\{N, v\}$ and $\{N, \overline{v}\}$ are strategically equivalent if there is a positive α and real $\beta_1, \beta_2, \ldots, \beta_N$ such that for all coalitions S,

$$v(S) = \alpha\overline{v}(S) + \sum_{i \in S} \beta_i. \tag{11.9}$$

Strategically equivalent games form equivalence classes, that is, equivalence relation is reflective, symmetric and transitive. The equivalence of games can be also useful to reduce complicated games to games of very simple structure such as *(0, 1)—normalized* games which satisfy the following properties:

$$\begin{aligned} v(\{i\}) &= 0 \ \text{ for all } \ i \in \mathbb{N} \\ v(\mathbb{N}) &= 1. \end{aligned} \tag{11.10}$$

Clearly any (0, 1) normalized game is essential.

Theorem 11.1 *Let $\Gamma = (N, v)$ be an essential game, then any game equivalent to Γ is also equivalent to a unique (0,1)—normalized game.*

Proof If game Γ is equivalent to a (0, 1)—normalized game, then from (11.9),

$$\alpha v\big(\{i\}\big) + \beta_i = 0 \ \text{ and } \ \alpha v(\mathbb{N}) + \sum_{i=1}^{N} \beta_i = 1$$

which is a system of linear equations for unknowns α and β_i $(i = 1, 2, \ldots, N)$. The unique solution is

$$\alpha = \frac{1}{v(\mathbb{N}) - \sum_{i=1}^{N} v(\{i\})} \quad \text{and} \quad \beta_i = -\alpha v(\{i\}), \tag{11.11}$$

where $\alpha > 0$, since Γ is essential.

■

A superadditive (0, 1)—normalized game $G = \{N, v\}$ is called *simple*, if for all coalitions S, either $v(S) = 0$ or $v(S) = 1$. The coalitions with $v(S) = 0$ are called *losers* and coalitions with $v(S) = 1$ are called *winners*.

We mentioned earlier that there are several definitions of the characteristic function depending on how we interpret the term "without cooperation of the rest of the players". In definition (11.1) we assume that the rest of the players form a counter-coalition and punish the members of the coalition as much as they can. This concept is called the *maximin construction*. We selected this construction because of two main reasons. First, it is very often used in applications, and second, any superadditive characteristic function can be obtained as a maximin construction of an N-person cooperative game. So mathematically all superadditive characteristic functions can be considered as maximin constructions (von Neumann and Morgenstern, 1994).

The main question is how the players divide the commonly earned payoff among each other. We know that the maximum amount they can get and therefore can divide among each other is $v(\mathbb{N})$, so the set of all feasible payoff vectors is given as

$$\left\{ \underline{x} = (x_1, \ldots, x_N) | \sum_{i=1}^{N} x_i \leqq v(\mathbb{N}) \right\} \tag{11.12}$$

and the set of all *efficient* payoff vectors is

$$\left\{ \underline{x} = (x_1, \ldots, x_N) | \sum_{i=1}^{N} x_i = v(\mathbb{N}) \right\} \tag{11.13}$$

where x_i is the payoff given in the distribution to player i. The elements of (11.13) are called *preimputations*. A payoff vector $\underline{x}$ is called *individually rational* if for all players i, $x_i \geqq v(\{i\})$. If this condition is violated for any one of the players, then this player will not agree with this obtained payoff, since he can get higher payoff $v(\{i\})$ without the help of the others. Therefore any payoff vector has to be individually rational in order to be accepted by each individual player. If a preimputation is individually rational, then it is called an *imputation*. For a particular game there are usually many (sometimes infinitely many) imputations. Therefore in order to find a mutually acceptable payoff vector solution, additional conditions should be assumed. One of them is based on dominance of imputations.

We say that imputation $\underline{x}$ *dominates* imputation $\overline{\underline{x}}$ on coalition $S \subseteq \mathbb{N}$ if

$$x_i > \overline{x_i} \quad \text{for all} \quad i \in S$$

and

$$\sum_{i \in S} x_i \leqq v(S).$$

This relation is denoted by $\underline{x} >^S \overline{\underline{x}}$. The first condition requires that all members of coalition S get higher payoff in $\underline{x}$ than in $\overline{\underline{x}}$. The second condition makes it sure that coalition S is able to get the higher payoffs to its members. We also say that imputation $\underline{x}$ dominates impution $\overline{\underline{x}}$ if there is a coalition S such that $\underline{x} >^S \overline{\underline{x}}$.

11.1 The Core

We have already explained that any reasonable payoff vector has to be individually rational in order to be acceptable by all individual members. What about acceptance by the possible coalitions? If there is a group S of players who would get together less than $v(S)$, then they would revolt against this agreement, since by ganging up against the others by forming a coalition, they could ensure at least $v(S)$ for themselves. Therefore it is an important extension of individual rationality that the payoff vector $\underline{x}$ has to be rational to all possible coalitions, that is,

$$\sum_{i \in S} x_i \geqq v(S) \quad \text{for all} \quad S \subseteq \mathbb{N},\, S \neq \mathbb{N} \tag{11.14}$$

and

$$\sum_{i=1}^{N} x_i = v(\mathbb{N}). \tag{11.15}$$

The payoff vectors satisfying these conditions form the *core* of the game. The first condition requires that $\underline{x}$ has to be rational to all possible coalitions including the grand coalition $\mathbb{N}$, however the players cannot distribute among each other more than the maximum overall payoff they can get. This is represented in the second condition. Notice that the core is always a convex set.

Example 11.3 We have seen in Example 11.1 that in a three-firm oligopoly

$$v(\phi) = 0,\; v\big(\{1\}\big) = v\big(\{2\}\big) = v\big(\{3\}\big) = \frac{5}{4},$$
$$v\big(\{1,2\}\big) = v\big(\{1,3\}\big) = v\big(\{2,3\}\big) = 7 \;\text{ and }\; v\big(\{1,2,3\}\big) = \frac{69}{4}.$$

Therefore the core consists of all payoff vectors $\underline{x} = (x_1, x_2, x_3)$ such that

$$\begin{aligned} x_1, x_2, x_3 &\geqq \frac{5}{4} \\ x_1 + x_2, x_1 + x_3, x_2 + x_3 &\geqq 7 \\ x_1 + x_2 + x_3 &= \frac{69}{4}. \end{aligned}$$

▼

Example 11.4 In the case of the game introduced in Example 11.2 we have seen that

$$v(\phi) = v\big(\{1\}\big) = v\big(\{2\}\big) = v\big(\{3\}\big) = 0$$
$$v\big(\{1, 2\}\big) = v\big(\{1, 3\}\big) = 750,\ v\big(\{2, 3\}\big) = 500 \text{ and } v\big(\{1, 2, 3\}\big) = 1000.$$

So the core is formed by all payoff vectors such that

$$\begin{aligned} x_1, x_2, x_3 &\geqq 0 \\ x_1 + x_2, x_1 + x_3 \geqq 750,\ x_2 + x_3 &\geqq 500 \\ x_1 + x_2 + x_3 &= 1000. \end{aligned}$$

▼

The core of weakly superadditive games can be characterized by the following result.

Theorem 11.2 *Let $G = \{N, v\}$ be a weakly superadditive game. Then its core consists of all non-dominated imputations.*

Proof It is sufficient to prove that for any coalition S, the following assumptions are equivalent:

(A) $\sum_{i \in S} x_i < v(S)$

(B) there is an imputation $\overline{\underline{x}}$ such that $\overline{\underline{x}} >^S \underline{x}$.

Assume first (A). Clearly $S \neq \phi$ and $S \neq \mathbb{N}$. Introduce the notation

$$\triangle = v(S) - \sum_{i \in S} x_i > 0$$

and

$$\overline{\triangle} = v(\mathbb{N}) - v(S) - \sum_{i \notin S} v\big(\{i\}\big).$$

Since G is superadditive, $\overline{\triangle} \geqq 0$, and let

$$\overline{x_i} = \begin{cases} x_i + \frac{\triangle}{|S|} & \text{if } i \in S \\ v\big(\{i\}\big) + \frac{\overline{\triangle}}{|\mathbb{N}-S|} & \text{if } i \notin S \end{cases}$$

where $|S|$ is the number of players in coalition S, and $|\mathbb{N} - S|$ is the number of players not belonging to S. Notice that

$$\sum_{i \in S} \overline{x_i} = \sum_{i \in S} x_i + v(S) - \sum_{i \in S} x_i = v(S)$$

and clearly $\overline{x_i} > x_i$ for all $i \in S$. Therefore $\underline{\overline{x}} >^S \underline{x}$. Assume next that (B) holds. Then $\underline{\overline{x}} >^S \underline{x}$ implies that

$$v(S) \geqq \sum_{i \in S} \overline{x}_i > \sum_{i \in S} x_i,$$

so (A) holds.

■

The definition of the core implies that if $\underline{x}$ is an imputation from the core, then there is no coalition which would want to change the outcome of the game, otherwise the other players might "punish" them to get lower overall payoff $v(S)$. Unfortunately the core might have infinitely many imputations as in Examples 11.3 and 11.4, and the core might be empty as it is shown in the following result.

Theorem 11.3 *Let $G = \{N, v\}$ be an essential constant-sum game. Then its core is empty.*

Proof Assume an imputation $\underline{x}$ belongs to the core of the game. Then for all players i,

$$\sum_{j \neq i} x_j \geq v\big(\mathbb{N} - \{i\}\big)$$

and since the game is constant sum,

$$V\big(\mathbb{N} - \{i\}\big) = v(\mathbb{N}) - v\big(\{i\}\big).$$

Notice that for all players i,

$$\sum_{j \neq i} x_j + v(\{i\}) \geqq v(\mathbb{N} - \{i\}) + v(\{i\}) = v(\mathbb{N}) = \sum_{j \neq i} x_j + x_i$$

therefore $x_i \leqq v(\{i\})$, so

$$\sum_{i=1}^{N} x_i \leqq \sum_{i=1}^{N} v(\{i\}),$$

and since G is essential,

$$\sum_{i=1}^{N} v(\{i\}) < v(\mathbb{N}).$$

Hence

$$\sum_{i=1}^{N} x_i < v(\mathbb{N}),$$

which is impossible since $\underline{x}$ is an imputation.

■

Example 11.5 Consider a simple game $G = \{N, v\}$. A player $i \in \mathbb{N}$ has veto power if

$$v(\mathbb{N} - \{i\}) = 0.$$

In a simple game $v(\mathbb{N}) = 1$, so without player i the others get zero overall profit.

Assume first that there is no veto player. Then for all $i \in \mathbb{N}$, $v(\mathbb{N} - \{i\}) = 1$, and any imputation from the core would satisfy

$$x_i \geqq v(\{i\}) = 0$$
$$\sum_{j \neq i} x_j \geqq v(\mathbb{N} - \{i\}) = 1$$

for all i, and

$$\sum_{i=1}^{N} x_i = v(\mathbb{N}) = 1.$$

This is an obvious contradiction, so the core is empty.

Assume next that there is at least one veto player. Let S denote their set. Define imputation $\underline{x}$ as

$$x_i \geqq 0 \text{ for } i \in S$$
$$x_i = 0 \text{ for } i \notin S$$
$$\sum_{i \in S} x_i = 1.$$

If T is a losing coalition, then (11.14) clearly holds for T. Let T be next a winning coalition, then $S \subset T$ by the superadditivity of G, since if a veto player $j \notin T$, then $T \subseteq \mathbb{N} - \{j\}$ and

$$0 = v\big(\mathbb{N} - \{j\}\big) \geqq v(T) + v\big(\mathbb{N} - \{j\} - T\big) \geqq 1$$

being a contradiction. Then

$$\sum_{i \in T} x_i \geqq \sum_{i \in S} x_i = 1 = v(T)$$

implying that (11.14) also holds for winning coalitions. Thus $\underline{x}$ belongs to the core of the game.

▼

In Theorem 11.2 the points of the core were characterized, in Theorem 11.3 we presented a class of games with empty core, and in Example 11.5 the class of simple games was examined giving conditions for empty and nonempty cores. The next result shows that there is a large class of games, the core of which is necessarily nonempty.

Theorem 11.4 *Assume game $G = \{N, v\}$ is convex. Then its core is nonempty.*

Proof Let Π be a permutation of the players and for each player i define

$$P_i^{\Pi} = \big\{j \in \mathbb{N} | \Pi(j) < \Pi(i)\big\}$$

as the set of all players who precede player i in the ordering Π. The *marginal worth* of player i is given as

$$d_i^{\Pi} = v\big(P_i^{\Pi} \cup \{i\}\big) - v(P_i^{\Pi}),$$

which is the marginal contribution of player i to the coalition of his predecessors with respect to ordering Π. The *marginal worth vector* is defined as

$$\underline{d}^{\Pi} = (d_1^{\Pi}, d_2^{\Pi}, \dots, d_N^{\Pi}).$$

We will show that vector $\underline{d}^{\Pi}$ is an element of the core of game G.

For each player i define $S_i = \{\Pi^{-1}(j) | j = 1, 2, \ldots, i\}$ with $S_0 = \phi$. Then

$$P^{\Pi}_{\Pi^{-1}(i)} = \left\{ j \in \mathbb{N} | \Pi(j) < i \right\} = S_{i-1}$$

and so

$$d^{\Pi}_{\Pi^{-1}(i)} = v\left(S_{i-1} \cup \left\{\Pi^{-1}(i)\right\}\right) - v(S_{i-1}) = v(S_i) - v(S_{i-1})$$

and for all $i \in \mathbb{N}$

$$\sum_{j \in S_i} d^{\Pi}_j = \sum_{j=1}^{i} d^{\Pi}_{\Pi^{-1}(j)} = \sum_{j=1}^{i} \left(v(S_j) - v(S_{j-1})\right) = v(S_i) - v(S_0) = v(S_i).$$

Since $S_N = \mathbb{N}$, we conclude that the sum of the d^{Π}_i values equals $v(\mathbb{N})$.

Next we prove that $\underline{d}^{\Pi}$ is rational to all possible coalitions. Let now $S \neq \phi$ be a coalition with s players, who are ordered in such a way that $S = \{i_1, \ldots, i_s\}$ and $\Pi(i_1) < \cdots < \Pi(i_s)$. Define $\sum_0 = \phi$ and $\sum_j = \{i_1, \ldots, i_j\}$ for $j = 1, 2, \ldots, s$. The definition of d^{Π}_i implies that $\sum_{j-1} \subset P^{\Pi}_{i_j}$, and from the convexity of game G we have

$$v\Big(P^{\Pi}_{i_j} \cup \sum_j \Big) - v(P^{\Pi}_{i_j}) \geqq v\Big(\sum_j\Big) - v\Big(\sum_{j-1}\Big)$$

and consequently

$$\sum_{j \in S} d^{\Pi}_j = \sum_{j=1}^{s} d^{\Pi}_{i_j} = \sum_{j=1}^{s} \left(v\big(P^{\Pi}_{i_j} \cup \{i_j\}\big) - v(P^{\Pi}_{i_j})\right)$$

$$\geq \sum_{j=1}^{s} \left(v\Big(\sum_j\Big) - v\Big(\sum_{j-1}\Big)\right) = v\Big(\sum_s\Big) - v\Big(\sum_0\Big) = v(S).$$

Hence $\underline{d}^{\Pi}$ is rational to all possible coalitions S.

■

Notice that the definition of the core of an N-person game, relations (11.14)–(11.15) consists of $2^N - 1$ linear inequalities, so the existence of solutions can be checked by the first phase of the simple method. In the case of multiple solutions the choice of a special payoff vector can be determined by optimizing a mutually accepted objective function (such as social welfare, environmental characteristics) subject to constraints (11.14)–(11.15). Since these constraints define a compact set in the N-dimensional vector space, any continuous objective function has optimal solution.

11.2 Stable Sets

We have seen that the core of an N-person game might be empty or might have many imputations, and in many cases the selection of a unique imputation from the core is problematic. An alternative solution concept is the *von Neumann-Morgenstern stable* set, which can be defined as follows.

Let $G = \{N, v\}$ be an N-person cooperative game, and let I denote the set of all imputations of G. A subset $J \subset I$ is called a *stable set* if
(A) there are no $\underline{x}, \overline{x} \in J$ such that $\underline{x}$ dominates $\overline{x}$ (internal stability);
(B) if $\underline{x} \notin J$, then there is an $\overline{x} \in J$ such that $\overline{x}$ dominates $\underline{x}$ (external stability).

The existence of stable sets in general is very difficult mathematical problem, since there are infinitely many imputations for an N-person game and conditions (A) and (B) require the satisfaction of infinitely many relations. An interesting existence theorem is the following.

Theorem 11.5 *Let $G = \{N, v\}$ be a simple game and let S be a minimal winning coalition (such that $v(S) = 1$ and $v(T) = 0$ for all $T \subset S, T \neq S$). Let J be the set of all imputations $\underline{x}$ such that $x_i = 0$ for all $i \notin S$. Then J is a stable set.*

Proof Notice first that if $S = \mathbb{N}$, then J is clearly a stable set, so we may assume that $S \neq \mathbb{N}$.

We prove first internal stability. Assume that $\underline{x}, \overline{x} \in J$ and $\underline{x} >^C \overline{x}$ for a coalition C. Then $x_i > 0$ for $i \in C$ and $\sum_{i \in C} x_i \leqq v(C)$. Since C has to be a subset of S and S is a minimal winning coalition, $C = S$. Observe that

$$\sum_{j \in S} x_j = \sum_{j \in S} \overline{x}_j = 1,$$

since both $\underline{x}$ and $\overline{x}$ are imputations and for $j \notin S$, $x_j = \overline{x}_j = 0$. Therefore dominance is impossible: $x_j > \overline{x}_j$ for all $j \in S$ would imply that $\sum_{j \in S} x_j > \sum_{j \in S} \overline{x}_j$.

We next prove external stability. Select an $\overline{x} \notin J$, then $\overline{x}_j > 0$ for some $j \notin S$ and therefore

$$\triangle = 1 - \sum_{i \in S} \overline{x}_i > 0.$$

Define imputation $\underline{x}$ as follows:

$$x_i = \begin{cases} \overline{x}_i + \frac{\triangle}{|\mathbb{N} - S|} & \text{if } i \in S \\ 0 & \text{if } i \notin S, \end{cases}$$

then clearly $x_i > \overline{x}_i$ for $i \in S$ and

$$\sum_{i \in S} x_i = \sum_{i \in S} \overline{x}_i + \Big(1 - \sum_{i \in S} \overline{x}_i\Big) = 1 = v(S).$$

Hence $\underline{x}$ dominates $\underline{\overline{x}}$ on S.

■

We have seen in Theorem 11.1 that any essential game is equivalent with a (0, 1)—normalized game, and if this is also a simple game, then Theorem 11.5 can be used to show existence of stable sets and also to provide a stable set.

11.3 The Nucleolus

Consider a rational game $G = \{N, v\}$ and let $\underline{x}$ be a payoff vector. For each coalition $S \subseteq \mathbb{N}$ define *excess of* S *on* $\underline{x}$ as

$$e(S, \underline{x}) = v(S) - \sum_{j \in S} x_j \tag{11.16}$$

which can be interpreted as the measure of the dissatisfaction of coalition S if payoff vector $\underline{x}$ is offered to the players. Clearly every coalition wants to minimize dissatisfaction, so a multiobjective optimization problem is obtained:

$$\begin{aligned} \text{minimize } & v(S) - \sum_{j \in S} x_j \quad (S \subseteq \mathbb{N}) \\ \text{subject to } & x_i \geqq v\big(\{i\}\big) \quad (i = 1, 2, \ldots, N) \\ & \sum_{i=1}^{N} x_i = v(\mathbb{N}) \end{aligned} \tag{11.17}$$

where we have $2^N - 1$ objective functions, since the number of nonempty coalitions is $2^N - 1$.

Based on different solution concepts of solving multiobjective optimization problems several alternative versions of solving game G can be offered.

One way is a lexicographic approach in which the maximal complaint, the largest excess, is minimized. If the solution is unique, then it is the final solution. Otherwise minimize the second largest complaint with keeping the largest complaint on its minimal level. If the solution is unique, then it is the final solution. Otherwise minimize the third largest complaint keeping the largest and second largest complaints on their minimal levels and so on. The process terminates if either a unique optimum is found in a step, or all complaints are already minimized. So in the first step we solve problem

$$\begin{aligned}&\text{minimize } m_1\\&\text{subject to } v(S) - \sum_{j\in S} x_j \leqq m_1\,(S \subseteq \mathbb{N})\\&\qquad x_i \geq v\big(\{i\}\big)\,(i = 1, 2, \ldots, \mathbb{N})\\&\qquad \sum_{i=1}^{N} x_i = v(\mathbb{N}).\end{aligned} \tag{11.18}$$

Let m_1^* be the optimum value of m_1, which is obtained with a coalition S_1. If solution is unique, then stop. Otherwise in the second step solve the following problem:

$$\begin{aligned}&\text{minimize } m_2\\&\text{subject to } v(S) - \sum_{j\in S} x_j \leqq m_2\,(S \subseteq \mathbb{N},\, S \neq S_1)\\&\qquad v(S_1) - \sum_{j\in S_1} x_j = m_1^*\\&\qquad x_i \geqq v\big(\{i\}\big)\,(i = 1, 2, \ldots, \mathbb{N})\\&\qquad \sum_{i=1}^{N} x_i = v(\mathbb{N}).\end{aligned} \tag{11.19}$$

In general, in step k, the optimum problem is as follows:

$$\begin{aligned}&\text{minimize } m_k\\&\text{subject to } v(S) - \sum_{j\in S} x_j \leqq m_k\,(S \subseteq \mathbb{N},\, S \neq S_1, S_2, \ldots, S_{k-1})\\&\qquad v(S_l) - \sum_{j\in S_l} x_j = m_l^*\,(l = 1, 2, \ldots, k-1)\\&\qquad x_i \geq v\big(\{i\}\big)\,(i = 1, 2, \ldots, \mathbb{N})\\&\qquad \sum_{i=1}^{N} x_i = v(\mathbb{N}).\end{aligned} \tag{11.20}$$

It can be proved (Schmeidler, 1968) that this procedure produces a unique solution, which is usually called the *lexicographic nucleolus*. Its computation can be very laborious, since in the worst case scenario $2^N - 1$ optimum problems have to be solved. However in practical cases much less number of steps are needed. Notice that all optimum problems are linear, and at each step the objective function changes and an inequality constraint becomes equality. This fact can be used when we start from the final simplex table of the previous problem in solving the new optimum problem.

The solution of problem (11.18) shows if the core is empty or not. If $m_1^* > 0$ then it is impossible to find payoff vector such that for all coalitions,

$$v(S) - \sum_{j \in S} x_j \leqq 0,$$

that is, the core is empty. Otherwise $m_1^* \leqq 0$, then with any solution of (11.18) and all coalitions,

$$v(S) - \sum_{j \in S} x_j \leqq m_1^* \leqq 0,$$

so payoff vector $\underline{x}$ belongs to the core. In other words, the lexicographic nucleolus always belongs to the core as $m_1^* \leqq 0$. In this case it can be considered as a *core allocation rule*.

Another way of solving problem (11.17) is to minimize the variance of the excesses of the coalitions instead of putting down the largest access in order to flatten the excess vector, as it was done in the previous model. So now we solve the following problem.

$$\begin{aligned} \text{minimize } & \sum_{S \subset \mathbb{N}} \big(e(S, \underline{x}) - \overline{e}(S, \underline{x})\big)^2 \\ \text{subject to } & x_i \geqq v\big(\{i\}\big) \, (i = 1, 2, \ldots, N) \\ & \sum_{i=1}^{N} x_i = v(\mathbb{N}) \end{aligned} \tag{11.21}$$

where

$$\overline{e}(S, \underline{x}) = \frac{1}{(2^N - 1)} \sum_{S \subseteqq \mathbb{N}} e(S, \underline{x})$$

is the average excess at $\underline{x}$ not counting the zero excess of the empty coalition. It can be proved that

$$\overline{e}(S, \underline{x}) = \frac{1}{2^N - 1} \sum_{S \subset \mathbb{N}} \Big(v(S) - \sum_{j \in S} x_j\Big) = \frac{1}{2^N - 1} \Big(\sum_{S \subset \mathbb{N}} v(S) - 2^{N-1} v(\mathbb{N})\Big)$$

since each player i can be a member of 2^{N-1} different coalitions, and $\sum_{i=1}^{N} x_i = v(\mathbb{N})$. It is easy to show that the objective function of (11.3) is quadratic with a positive definite Hessian matrix which implies the uniqueness of the solution. The existence of the optimum is implied by the facts that the feasible set is compact in $\mathbb{R}^N$ and the objective function is continuous. The unique solution is usually called the *least-square nucleolus* (Ruiz et al., 1996).

Both solution concepts can be modified by replacing the excesses of the coalitions with their relative excesses,

$$re(S, \underline{x}) = \frac{v(S) - \sum_{j \in S} x_j}{v(S)} \tag{11.22}$$

in the optimization problems.

Assume finally that there are measures showing the relative importances of the different coalitions in the game. Let $\alpha(S)$ be the importance measure for coalition S. Then the objective function of problem (10.21) can be replaced by either

$$\sum_{S \subset \mathbb{N}} \alpha(S) \big(e(S, \underline{x}) - \overline{e}(S, \underline{x}) \big)^2 \tag{11.23}$$

$$\tag{11.24}$$

or by

$$\sum_{S \subset \mathbb{N}} \alpha(S) \big(re(S, \underline{x}) - \overline{re}(S, \underline{x}) \big)^2$$

depending on whether we wish to consider excesses or relative excesses.

11.4 The Shapley Values

The core and stable set gave only a set of solutions and the choice of the most appropriate point of these sets raises further questions. The different kinds of nucleolus gave one-point solutions, which might be different for different kinds of the nucleolus. In this section another one-point solution is introduced based on the fair distribution of the maximum achievable total payoff of the players. The concept is simple, each player has to receive his expected marginal contribution to the different coalitions. So if a player contributes less, then he has to get less from the "common basket", and if he contributes more, then he has to get more. In order to define expected contribution we need a probability model. Assume that the players join each coalition one by one until the coalition fills up. A given player's contribution to the coalition occurs when he is the last who joins the coalition. The question is now to find the probability that this happens. Assume that the players are in random order. Let S be a coaliton with s players. Player k is the last who joins S if he is in position s and all other members of S are before him. The coalition S and the position of player k does not change, if the orders of the first $s-1$ players and the $N-s$ players behind player k change. So the probability that player k is the last who joins S equals

$$\frac{(s-1)!(N-s)!}{N!}, \tag{11.25}$$

since the N players have altogether $N!$ permutations. The marginal contribution of player k to coalition S is given by (11.4), so if this player belongs to S, then

$$d_k(S) = v(S) - v(S - \{k\}).$$

Therefore the expected marginal contribution of player k is given as

$$\phi_k = \sum_{k \in S \subseteq \mathbb{N}} \frac{(s-1)!(N-s)!}{N!} d_k(S), \tag{11.26}$$

which is called the *Shapley-value* of player k (Shapley, 1953). The summation has to be done only for those coalitions S which contain player k, since otherwise $d_k(S) = 0$.

Example 11.6 In the case of $N = 2$ we have four possible coalitions, ϕ, $\{1\}$, $\{2\}$, and $\{1, 2\}$. In determining ϕ_1 we have to consider coalitions $\{1\}$ and $\{1, 2\}$ only, since the others do not contain player 1. Since these coalitions have 1 and 2 members, we have

$$\begin{aligned} \phi_1 &= \frac{(1-1)!(2-1)!}{2!}\Big(v\big(\{1\}\big) - v(\phi)\Big) + \frac{(2-1)!(2-2)!}{2!}\Big(v\big(\{1,2\}\big) - v\big(\{2\}\big)\Big) \\ &= \frac{1}{2}\Big(v\big(\{1\}\big) + v\big(\{1,2\}\big) - v\big(\{2\}\big)\Big). \end{aligned}$$

Similarly,

$$\phi_2 = \frac{1}{2} v\big(\{2\}\big) + v\big(\{1,2\}\big) - v\big(\{1\}\big).$$

Notice that $\phi_1 + \phi_2 = v\big(\{1, 2\}\big)$.

▼

Example 11.7 In the case of $N = 3$, we have eight possible coalitions, ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$. In computing ϕ_1 we need to consider only coalitions $\{1\}$, $\{1, 2\}$, $\{1, 3\}$ and $\{1, 2, 3\}$ with 1, 2, 2 and 3 members, respectively. Therefore

$$\phi_1 = \frac{(1-1)!(3-1)!}{3!}\Big(v(\{1\}) - v(\phi)\Big) + \frac{(2-1)!(3-2)!}{3!}\Big(v(\{1,2\}) - v(\{2\})\Big)$$
$$+ \frac{(2-1)!(3-2)!}{3!}\Big(v(\{1,3\}) - v(\{3\})\Big) + \frac{(3-1)!(3-3)!}{3!}\Big(v(\{1,2,3\}) - v(\{2,3\})\Big)$$
$$= \frac{1}{6}\Big(2v(\{1\}) + v(\{1,2\}) - v(\{2\}) + v(\{1,3\}) - v(\{3\}) + 2v(\{1,2,3\}) - 2v(\{2,3\})\Big).$$

Similarly

$$\phi_2 = \frac{1}{6}\Big(2v(\{2\}) + v(\{1,2\}) - v(\{1\}) + v(\{2,3\}) - v(\{3\}) + 2v(\{1,2,3\}) - 2v(\{1,3\})\Big)$$

and

$$\phi_3 = \frac{1}{6}\Big(2v(\{3\}) + v(\{2,3\}) - v(\{2\}) + v(\{1,3\}) - v(\{1\}) + 2v(\{1,2,3\}) - 2v(\{1,2\})\Big).$$

Notice again that $\phi_1 + \phi_2 + \phi_3 = v\big(\{1,2,3\}\big)$.

▼

Shapley values (11.26) have several favorable properties.

Theorem 11.6 *For any game $G = \{N, v\}$,*
(a) $\sum_{k=1}^{N} \phi_k = v(\mathbb{N})$;
and
(b) If G is superadditive, then $\phi_k \geq v\big(\{k\}\big)$ for all players.

Proof (a) In any ordering of the players, $(i_1, i_2, \ldots, i_N)$ the total contribution of the players to the consecutive coalitions is

$$\Big[v\big(\{i_1\}\big) - v(\phi)\Big] + \Big[v\big(\{i_1, i_2\}\big) - v\big(\{i_1\}\big)\Big] + \cdots + \Big[v\big(\{i_1, \ldots, i_N\}\big) - v\big(\{i_1, \ldots, i_{N-1}\}\big)\Big]$$
$$= v\big(\{i_1, \ldots, i_N\}\big) = v(\mathbb{N}),$$

so its expectation also equals $v(\mathbb{N})$.
(b) By the superadditivity of G we have

$$d_k(S) = v(S) - v\big(S - \{k\}\big) \geqq v\big(\{k\}\big),$$

for all coalitions S including player k. Since ϕ_k is the expectation of $d_k(S)$, it has to be greater than or equal to $v(\{k\})$.

■

This theorem implies that for any superadditive game the Shapley-values represent an imputation.

Theorem 11.7 *The Shapley values are invariant under strategic equivalence and permutations of the players.*

Proof Consider first games $G = \{N, v\}$ and $\overline{G} = \{N, \overline{v}\}$ such that for all $S \subseteq \mathbb{N}$,

$$v(S) = \alpha\overline{v}(S) + \sum_{i \in S} \beta_i .$$

Then

$$\begin{aligned}\phi_k &= \sum_{S \subseteq \mathbb{N}} \frac{(s-1)!(N-s)!}{N!} \Big(\alpha\overline{v}(S) - \alpha\overline{v}\big(S - \{k\}\big) + \beta_k\Big) \\ &= \alpha\overline{\phi}_k + \beta_k \sum_{k \in S \subseteq \mathbb{N}} \frac{(s-1)!(N-s)!}{N!} = \alpha\overline{\phi}_k + \beta_k \sum_{s=1}^{N} \frac{(s-1)!(N-s)!}{N!} \binom{N-1}{s-1} \\ &= \alpha\overline{\phi}_k + \beta_k ,\end{aligned}$$

since

$$\frac{(s-1)!(N-s)!}{N!} \binom{N-1}{s-1} = \frac{(s-1)!(N-s)!(N-1)!}{N!(s-1)!(N-s)!} = \frac{1}{N} .$$

Notice that all $N!$ permutations of the players were accounted for when the expectation of the marginal contribution was determinated, which implies the invariance of the Shapley-values under strategic equivalence and permutations of the players.

■

The first property implies that in determining the Shapley values the answers do not depend an the selected units of the payoffs. The second property guarantees that there is no discrimination among the players in which order the Shapley values are computed.

Theorem 11.8 *Let* $G = \{N, v\}$ *and* $\overline{G} = \{N, \overline{v}\}$ *be two games defined on the same set of players, and let* $G^* = \{N, v + \overline{v}\}$. *Then for all players,* $\phi_k^* = \phi_k + \overline{\phi_k}$, *that is the Shapley-values are additive.*

Proof Since expectation is an additive operation, the Shapley-values are also additive.

■

The Shapley-values have the disadvantage that in many cases it produces an unstable solution, since it does not belong necessarily to the core of the game. However in a large class of games it cannot happen.

Theorem 11.9 *Let $G = \{N, v\}$ be a convex game. Then the payoff vector $(\phi_1, \ldots, \phi_N)$ given by the Shapley-values belongs to the core of the game.*

Proof From Theorem 11.4 we know that the core of game G is nonempty, and in its proof we also verified that the marginal worth vector belongs to the core with every permutation Π of the players. Notice first that (11.26) can be rewritten as

$$\phi_k = \sum_{S \subset \mathbb{N}-\{k\}} \frac{s!(N-s-1)!}{N!} \Big(v\big(S \cup \{k\}\big) - v(S) \Big), \tag{11.27}$$

since for each $S \subset \mathbb{N} - \{k\}$ the number of permutations for which $P_k^\Pi = S$ is $s!(N - s - 1)!$. Here we use the notation of the proof of Theorem 11.4. In addition, the marginal worth of player k,

$$d_k^\Pi = v\big(P_k^\Pi \cup \{k\}\big) - v(P_k^\Pi)$$

is of the form

$$v\big(S \cup \{k\}\big) - v(S),$$

so we conclude that

$$\phi_k = \frac{1}{N!} \sum_\Pi d_k^\Pi \tag{11.28}$$

where the summation is made for all permutations of the N players, that is, with vector $\underline{\phi} = (\phi_1, \ldots, \phi_N)$,

$$\underline{\phi} = \frac{1}{N!} \sum_\Pi \underline{d}^\Pi \tag{11.29}$$

showing that the payoff vector given by the Shapley-values is a convex linear combination of vectors $\underline{d}^\Pi$. Since all vectors $\underline{d}^\Pi$ belong to the core and the core is convex, all convex linear combinations of its elements also belong to it. Hence vector $\underline{\phi}$ is an element of the core.

■

This theorem can be interpreted as the Shapley-values can serve as an alternative *core allocation rule* in addition to the nucleolus.

11.5 The Kernel and the Bargaining Set

In the previous solution concepts it was always assumed that the players form a grand coalition, the maximal obtained overall payoff goes into a common basket which is then distributed among the players. The difference between the different solution concepts is the way how this payoff distribution is conducted. So there was no consideration to possible coalition formations among the players and any actual bargaining process. In this section two new solution concepts will be introduced which are based on coalition structures.

Consider a $G = \{N, v\}$ rational game, and let

$$C = (C_1, \dots, C_M)$$

be a partition of the players into nonempty, disjoint coalitions, that is,

$$C_i \cap C_j \neq \phi \text{ as } i \neq j, \text{ and } C_1 \cup C_2 \cup \dots \cup C_M = \mathbb{N}.$$

The concept of imputations can be generalized by assuming that for a payoff vector $\underline{x}$,

$$\sum_{i \in C_k} x_i = v(C_k)\,(k = 1, 2, \dots, M) \tag{11.30}$$

and for all players

$$x_i \geqq v\big(\{i\}\big)\,(i = 1, 2, \dots, N). \tag{11.31}$$

Then $(\underline{x}, C)$ is called an *individually rational payoff configuration (IRPC).* Notice that if C has only one coalition $\mathbb{N}$, then this concept reduces to that of an imputation.

The excess of a coalition S with respect to payoff vector $\underline{x}$ was defined in (11.16), and define next the *surplus* of player i against player j in the following way:

$$s_{ij}(\underline{x}) = \max\big\{e(S, \underline{x}\}|S \subseteq \mathbb{N}, i \in S, j \notin S\big\}. \tag{11.32}$$

Notice that it represents the largest amount player i can gain (or smallest amount of loss) by withdrawing from agreement $\underline{x}$ and joining a coalition without the consent of player j.

Consider now an IRPC $(\underline{x}, \underline{C})$ and assume players i and j are in a coalition C_k. We say that player i *outweights* player j if

$$s_{ij}(\underline{x}) > s_{ji}(\underline{x}) \text{ and } x_j > v\big(\{j\}\big). \tag{11.33}$$

The first condition shows that player i has the larger incentive to depart from the agreement than player j, so player i can gain more by the threat of excluding player

j than vice versa. The second condition requires that the threat is realistic, since player j could get lower payoff $v(\{j\})$ then in the case of agreement $\underline{x}$.

The *kernel* of a rational game with respect to a coalition structure $C = (C_1, \ldots, C_M)$ is the set of all IRPCs $(\underline{x}, C)$ such that there is no coalition C_k and players $i, j \in C_k$ with $s_{ij}(\underline{x}) > s_{ji}(\underline{x})$. That is, in every coalition for each pair (i, j) of players, $s_{ij}(\underline{x}) = s_{ji}(\underline{x})$. In other words, all pairs of players in every coalition are in equilibrium (Davis & Maschler, 1965).

Example 11.8 Consider a simple coalition structure

$$C = \big(\{1\}, \{2\}, \ldots, \{N\}\big)$$

where each player forms his own coalition. In this case we have a unique IRPC$(\underline{x}, C)$ such that $x_k = v(\{k\})$ for each player. This payoff vector belongs to the kernel, since (11.31) clearly holds, and (11.30) is also true for all single member coalitions.

▼

This example shows the nonemptiness of the kernel in this very special coalition structure. For more complicated coalition structures to establish the nonemptiness of the kernel is a much more difficult task. Instead of going into generalities we consider the other extreme case when $C = (\mathbb{N})$, that is, when the grand coalition is the only coalition.

Theorem 11.10 *Let $G = \{N, v\}$ be a rational game with lexicographic nucleolus $\underline{x}$. Then IRPC $(\underline{x}, \mathbb{N})$ is in the kernel of the game.*

Proof Assume in contrary to the assertion that $(\underline{x}, \mathbb{N})$ is not in the kernel of the game. Then there are players i and j such that

$$s_{ij}(\underline{x}) > s_{ji}(\underline{x}) \quad \text{and} \quad x_j > v\big(\{j\}\big).$$

Define

$$\triangle = \min\left\{\frac{1}{2}\big(s_{ij}(\underline{x}) - s_{ji}(\underline{x})\big), x_j - v\big(\{j\}\big)\right\} \tag{11.34}$$

and let payoff vector $\underline{x}'$ be defined as

$$x_k' = \begin{cases} x_i + \triangle & \text{if } k = i \\ x_j - \triangle & \text{if } k = j \\ x_k & \text{if } k \neq i, j. \end{cases}$$

Then clearly $\underline{x}'$ is an imputation, $\triangle > 0$ and $\underline{x} \neq \underline{x}'$. For the sake of simplicity let λ_{ij} be set of all coalitions containing i but not j and define

$$\Lambda = \big\{S | S \subseteq \mathbb{N},\ e(S, \underline{x}) \geq s_{ij}(\underline{x}) \ \text{and} \ S \notin \lambda_{ij}\big\}. \tag{11.35}$$

We will next prove that

$$e(S,\underline{x}')\begin{cases}= e(S,\underline{x}) \geqq s_{ij}(\underline{x}) & \text{for } S \in \Lambda\\ < s_{ij}(\underline{x}) & \text{for } S \notin \Lambda.\end{cases} \tag{11.36}$$

Notice first that from (11.35),

$$e(S,\underline{x}') = \begin{cases} e(S,\underline{x}) + \triangle & \text{if } S \in \lambda_{ji}\\ e(S,\underline{x}) - \triangle & \text{if } S \in \lambda_{ij}\\ e(S,\underline{x}) & \text{otherwise.}\end{cases}$$

Assume that $S \in \Lambda$, then the definition of Λ implies that $S \notin \lambda_{ij}$, furthermore

$$e(S,\underline{x}) \geqq s_{ij}(\underline{x}) > s_{ji}(\underline{x}) = \max_{T\in\lambda_{ji}} e(T,\underline{x})$$

which would lead to contradiction if $S \in \lambda_{ji}$, so we know that if $S \in \Lambda$, then $S \notin \lambda_{ij}$ and $S \notin \lambda_{ji}$. Therefore the first case of (11.36) is shown.

Notice next that $e(S,\underline{x}')$ satisfies the following conditions:

(a) If $S \in \lambda_{ij}$, then

$$e(S,\underline{x}') = e(S,\underline{x}) - \triangle \leqq s_{ij}(\underline{x}) - \triangle < s_{ij}(x);$$

(b) If $S \in \lambda_{ji}$, then

$$e(S,\underline{x}') = e(S,\underline{x}) + \triangle \leqq s_{ji}(\underline{x}) + \triangle < s_{ij}(x);$$

(c) And if $S \notin \Lambda$, $S \notin \lambda_{ij}$, $S \notin \lambda_{ji}$, then

$$e(S,\underline{x}') = e(S,\underline{x}) < s_{ij}(\underline{x}).$$

Select a coalition $T \in \lambda_{ij}$ such that $e(T,\underline{x}) = s_{ij}(\underline{x})$, then $T \notin \Lambda$ and

$$\max_{S\notin\Lambda} e(S,\underline{x}) = e(T,\underline{x}) = s_{ij}(\underline{x}),$$

since if $S \notin \Lambda$ then either $e(S,\underline{x}) < s_{ij}(\underline{x})$ or $S \in \lambda_{ij}$. Assume finally that the 2^N components of the excess vector are ordered in a nonincreasing order, and let $\underline{e}^{(1)} \geq \underline{e}^{(2)} \geq \cdots \geq \underline{e}^{(2^N)}$ denote these components. Then there is an $l \geqq 1$ such that

$$e^{(i)}(\underline{x}') = e^{(i)}(\underline{x}) \quad \text{for} \quad i = 1, 2, \ldots, l$$

and

$$e^{(l+1)}(\underline{x}') < s_{ij}(\underline{x}) = e^{(l+1)}(\underline{x})$$

implying that in the lexicographic order $\underline{x}'$ is less preferred than $\underline{x}$, so $\underline{x}$ cannot be the lexicographic nucleolus.

■

The Kernel requires that in every coalition each pair of players are in equilibrium, that is, no one outweights the other. The other concept to be introduced in this section is also based on a notion of stability, which is reached if any objection to a suggested payoff vector can be met by a credible and powerful counterobjection.

Let $(\underline{x}, C)$ be an IRPC with $C = (C_1, \ldots, C_M)$ and assume that players i and j belong to coalition C_k. In the bargaining process player i challenges $(\underline{x}, C)$ by presenting an objection against player j as a pair $(\underline{x}', S)$ such that

(a) $S \subseteq \mathbb{N}$, $i \in S$ but $j \notin S$;
(b) $\underline{x}' \in \mathbb{R}^s$ a payoff vector for coalition S and $\sum_{i \in S} x'_i = v(S)$, where S has s players;
(c) $x'_k > x_k$ for all $k \in S$.

These conditions mean that player i offers to form a coalition without player j such that all members of this coalition would receive higher payoff than in the case of agreement payoff vector $\underline{x}$.

If such a challenge is put forth, then player j might come up with a credible counteroffer with a pair $(\underline{\overline{x}}, T)$ satisfying

(a) $T \subseteq \mathbb{N}$, $j \in T$ but $i \notin T$;
(b) $\underline{\overline{x}} \in \mathbb{R}^t$ a payoff vector for coalition T and $\sum_{i \in T} \overline{x}_i = v(T)$ where T has t players;
(c) $\overline{x}_k \geqq x'_k$ for all $k \in S \cap T$;
and
(d) $\overline{x}_l \geqq x_l$ for all $l \in T - S$.

In the counteroffer player j suggests to form a coalition T without player i such that the players in $S \cap T$ receive at least as high payoffs as in the offer of player i and players of $T - S$ (who are in the proposed coalition of player j but not in that of player i will get at least as good payoffs than in the case of the payoff vector $\underline{x}$.

The *bargaining set* for the coalition structure C is the set of all IRPCs such that every challenge against it can be counteroffered.

Theorem 11.11 *Let $G = (N, v)$ be a rational game. Then its kernel is contained in the bargaining set.*

Proof Assume that $(\underline{x}, C)$ is not in the bargaining set, then we will prove that it is not in the kernel either. If $(\underline{x}, C)$ is not in the bargaining set, then player i can have a challenge $(\underline{x}', S)$ against player j which cannot be counteroffered. Notice first that $x_j > v(\{j\})$, since otherwise player j always can have a counteroffer $(\underline{\overline{x}}, T)$ with

$\overline{x}_j = v(\{j\})$ and $j \in T$. Assume that $U \subseteq S$, then $j \notin U$, since $j \notin S$ because $(\underline{x}', S)$ is a challenge against player j. Clearly

$$s_{ij}(\underline{x}) \geq e(U, \underline{x}).$$

Assume also that $s_{ji}(\underline{x}) \geq s_{ij}(\underline{x})$, then there is a coalition U' such that

$$e(U', \underline{x}) = s_{ji}(\underline{x}) \geq s_{ij}(\underline{x}),$$

furthermore

$$\sum_{l \in U \cap U'} x'_l = \sum_{l \in U} x'_l - \sum_{l \in U \setminus U'} x'_l = v(U) - \sum_{l \in U \setminus U'} x'_l \leq v(U) - \sum_{l \in U \setminus U'} x_l$$

so

$$\sum_{l \in U \cap U'} (x'_l - x_l) \leqq v(U) - \sum_{l \in U} x_l \leqq s_{ij}(\underline{x}).$$

Therefore

$$\begin{aligned} \sum_{l \in U \cap U'} x'_l + \sum_{l \in U' \setminus U} x_l &\leqq s_{ij}(\underline{x}) + \sum_{l \in U'} x_l = s_{ij}(\underline{x}) + v(U') - e(U', \underline{x}) \\ &= s_{ij}(\underline{x}) - s_{ji}(\underline{x}) + v(U') \leqq v(U'). \end{aligned}$$

So player j has enough room to give the members of $U' \cap U$ and $U' \setminus U$ as much as needed for a valid counteroffer on U', which contradicts the assumption. Thus $s_{ji}(\underline{x}) < s_{ij}(\underline{x})$ showing that player i outweights j, consequently $(\underline{x}, C)$ is not in the kernel.

■

Chapter 12
Partial Cooperation

In most concepts of solving cooperative games the players first form a grand coalition maximizing their total payoff and they distribute this amount among themselves in a way they feel fair.

The idea of partial cooperation was first introduced and explored by Cyert and DeGroot (1973), in which instead of forming a grand coalition each player takes the interest of the others into account. One way of doing so is the following. Consider N players and let $\alpha_{kl} \leq 1$ be the degree of cooperation of Player k towards Player $l (l \neq k)$. Then each Player k replaces its profit ϕ_k with the linear combination

$$\Psi_k = \phi_k + \sum_{l \neq k} \alpha_{kl} \phi_l \tag{12.1}$$

and considers a new noncooperative game with unchanged strategy sets and payoff function $\Psi_1, \Psi_2, \ldots, \Psi_N$.

Notice that if all $\alpha_{kl} = 0$ then the original noncooperative game is obtained, and if $\alpha_{kl} = 1$ for all k and l then the classical cooperation is derived when players have the total profit as their common new payoff.

The concept of partial cooperation can be also derived from different models of shareholding interlocks.

Joint ventures are modeled with payoff functions

$$\Psi_k = \left(1 - \sum_{l \neq k} \delta_{lk} \right) \phi_k + \sum_{l \neq k} \delta_{kl} \phi_l \tag{12.2}$$

where ϕ_k is the profit function of player k ($k = 1, 2, \ldots, N$) and δ_{kl} is the ownership interest of Player k in Player l (Reynolds and Snapp, 1982). Clearly $0 < \delta_{lk} < 1$ and $\sum_l \delta_{lk} < 1$.

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_12

Since the multiplier of ϕ_k is positive, maximizing Ψ_k is equivalent with maximizing (12.1) with

$$\alpha_{kl} = \frac{\delta_{kl}}{1 - \sum_{l \neq k} \delta_{lk}}.$$

In the case of indirect shareholding (Flath, 1991, 1992) the payoff of player k satisfies relation

$$\Psi_k = \phi_k + \sum_{l \neq k} \delta_{kl} \Psi_l \tag{12.3}$$

in which player k maximizes the sum of its operating earnings ϕ_k and the returns on equity holdings in the other players. Introducing vectors $\underline{\phi} = (\phi_k)$ and $\underline{\psi} = (\psi_k)$, furthermore matrix $\underline{D} = (\delta_{kl})$ with $\delta_{kk} = 0$, relation (12.3) can be rewritten as

$$\underline{\psi} = \underline{\phi} + \underline{D}\underline{\psi} \tag{12.4}$$

or

$$\underline{\psi} = \left(\underline{I} - \underline{D}\right)^{-1} \underline{\phi},$$

where $\underline{I}$ is the $N \times N$ identity matrix. Since $\sum_{l \neq k} \delta_{lk} < 1$, matrix $\left(\underline{I} - \underline{D}\right)$ is diagonally row-dominant with unit diagonal elements, therefore it is an M-matrix (Szidarovszky et al., 2002). In addition,

$$\left(\underline{I} - \underline{D}\right)^{-1} = \underline{I} + \underline{D} + \underline{D}^2 + \ldots.$$

where the right hand side is convergent. Let b_{kl} denote the (k, l) element of $\left(\underline{I} - \underline{D}\right)^{-1}$, then clearly $b_{kl} \geq 0$, $b_{kk} \geq 1$ and

$$\Psi_k = \sum_{l=1}^{N} b_{kl} \phi_l \tag{12.5}$$

Maximizing this function is equivalent with maximizing (12.1) where for $l \neq k$, $\alpha_{kl} = \frac{b_{kl}}{b_{kk}}$

Net indirect shareholding (Merlone, 2001) considers the payoff functions

$$\Psi_k = \left(1 - \sum_{l \neq k} \delta_{lk}\right) \left(\phi_k + \sum_{l \neq k} \delta_{kl} \bar{\Psi}_l\right) \tag{12.6}$$

where the gross profits $\bar{\Psi}_k$ of the players are defined implicitly as

$$\bar{\Psi}_k = \phi_k + \sum_{l \neq k} \delta_{kl} \bar{\Psi}_l \tag{12.7}$$

This equation is identical to (12.3) so with the notation $\underline{\bar{\psi}} = \left(\bar{\Psi}_k\right)$,

$$\underline{\bar{\psi}} = \left(\underline{\mathbf{I}} - \underline{\mathbf{D}}\right)^{-1} \underline{\phi}$$

By introducing matrix

$$\underline{\Delta} = \text{diag}\left(1 - \sum_{l \neq 1} \delta_{l1},\, 1 - \sum_{l \neq 2} \delta_{l2},\, ...,\, 1 - \sum_{l \neq n} \delta_{ln}\right)$$

with (12.6) we have

$$\underline{\Psi} = \underline{\Delta}\left[\underline{\phi} + \underline{D}\underline{\bar{\psi}}\right] = \underline{\Delta}\left[\underline{\phi} + \underline{D}\left(\underline{I} - \underline{D}\right)^{-1}\underline{\phi}\right]$$

Notice that the multiplier of $\underline{\Delta}$ can be rewritten as

$$\left[\left(\underline{I} - \underline{D}\right)\left(\underline{I} - \underline{D}\right)^{-1} + \underline{D}\left(\underline{I} - \underline{D}\right)^{-1}\right]\underline{\phi} = \left(\underline{I} - \underline{D} + \underline{D}\right)\left(\underline{I} - \underline{D}\right)^{-1}\underline{\phi}$$

implying that

$$\underline{\Psi} = \underline{\Delta}\left(\underline{I} - \underline{D}\right)^{-1}\underline{\phi}$$

so for $k = 1, 2, \ldots, n$,

$$\psi_k = \left(1 - \sum_{l \neq k} \delta_{lk}\right) \sum_{l=1}^{n} b_{kl}\phi_l \tag{12.8}$$

which has the same form as (12.5) where b_{kl} is replaced with $\left(1 - \sum_{l \neq k} \delta_{lk}\right) b_{kl}$.

Consider first a two-person zero-sum game with payoff functions ϕ_1 and $\phi_2 = -\phi_1$. Then from (12.1) we see that

$$\psi_1 = \phi_1 + \alpha_{12}\phi_2 = \phi_1 + \alpha_{12}\left(-\phi_1\right) = \left(1 - \alpha_{12}\right)\phi_1$$

and

$$\psi_2 = \phi_2 + \alpha_{21}\phi_1 = \phi_2 + \alpha_{21}\left(-\phi_2\right) = \left(1 - \alpha_{21}\right)\phi_2.$$

Maximizing $\psi_1$and $\psi_2$are equivalent with maximizing $\phi_1$and ϕ_2. That is, in this special case partial cooperation is equivalent with non-cooperation.

12.1 Partial Cooperation in Oligopolies

Cournot oligopolies were introduced in Example 3.15. There are N firms, the players. The strategy of player k is a finite closed interval $[0, L_k]$ where L_k can be interpreted as a capacity limit. Let x_k be the strategy (production level) of player k and $s = \sum_{l=1}^{N} x_l$ the industry output. If $p(s)$ is the unit price and $C_k(x_k)$ the production cost of player k, then its profit is the difference of its revenue and cost:

$$\phi_k\left(x_1, x_2, ..., x_N\right) = x_k p\left(s\right) - C_k\left(x_k\right) \tag{12.9}$$

Assume that the players form partial cooperation, then player k wants to maximize its payoff:

$$\psi_k\left(x_1, ..., x_N\right) = \left(x_k + Q_k\right) p\left(x_k + s_k\right) - C_k\left(x_k\right) - \sum_{l \neq k} \alpha_{kl} C_l\left(x_l\right) \tag{12.10}$$

where $s_k = \sum_{l \neq k} x_l$ and $Q_k = \sum_{l \neq k} \alpha_{kl} x_l$.

Assume that functions $p(s)$ and $C_k(x_k)$ $(k = 1, 2, \ldots, N)$ are twice continuously differentiable, furthermore

1. $p'\left(s\right) < 0$
2. $p'\left(s\right) + \left(x_k + Q_k\right) p''\left(s\right) \leq 0$
3. $p'\left(s\right) - C_k''\left(x_k\right) < 0$

for all k and feasible values of the relevant variables. Notice that these assumptions are the same as those presented in Example 3.15 when $\alpha_{kl} \equiv 0$.

Under these assumptions,

$$\frac{\partial \Psi_k}{\partial x_k} = p\left(x_k + s_k\right) + \left(x_k + Q_k\right) p'\left(x_k + s_k\right) - C_k'\left(x_k\right) \tag{12.11}$$

and

$$\frac{\partial^2 \Psi_k}{\partial x_k^2} = 2p'\left(x_k + s_k\right) + \left(x_k + Q_k\right) p''\left(x_k + s_k\right) - C_k''\left(x_k\right) < 0 \tag{12.12}$$

showing that $\Psi_k$is strictly concave in x_k. Since $x_k \in [0, L_k]$ and Ψ_k is continuous in $x_k$with any fixed values of s_k and Q_k, there is a unique best response of player k, which can be written as

$$R_k\left(s_k, Q_k\right) = \begin{cases} 0 \text{ if } p\left(s_k\right) + Q_k p'\left(s_k\right) - C_k'\left(0\right) \leq 0 \\ L_k \text{ if } \mathrm{p}\left(L_k + s_k\right) + \left(L_k + Q_k\right) p'\left(L_k + s_k\right) - C_k'\left(L_k\right) \geq 0 \\ \tilde{x}_k \text{ otherwise} \end{cases} \tag{12.13}$$

where $\tilde{x}_k$ is the unique solution of the following monotonic equation:

$$p\left(x_k + s_k\right) + \left(x_k + Q_k\right) p'\left(x_k + s_k\right) - C_k'\left(x_k\right) = 0 \tag{12.14}$$

inside interval $(0,\ L_k)$. If the best reponse is interior, then by implicit differentiation with respect to s_k and Q_k, with assuming that $\bar{x}_k = x_k\left(s_k\right)$ and $\tilde{x}_k = x_k\left(Q_k\right)$respectively, we have

$$p'(s)\left(\bar{x}_k' + 1\right) + \bar{x}_k' p'(s) + \left(\bar{x}_k + Q_k\right) p''(s)\left(\bar{x}_k' + 1\right) - C_k''\left(\bar{x}_k\right)\bar{x}_k' = 0$$

and

$$\tilde{x}_k' p'(s) + \left(\tilde{x'}_k + 1\right) p'\left(\tilde{x}_k + s_k\right) + \left(\tilde{x}_k + Q_k\right) p''\left(\tilde{x}_k + s_k\right)\tilde{x}_k' - C_k''\left(\tilde{x}_k\right)\tilde{x}_k' = 0$$

implying that

$$\frac{\partial}{\partial s_k} R_k\left(s_k, Q_k\right) = -\frac{p' + \left(x_k + Q_k\right) p''}{2p' + \left(x_k + Q_k\right) p'' - C_k''} \in (-1, 0]$$

and

$$\frac{\partial}{\partial Q_k} R_k\left(s_k, Q_k\right) = -\frac{p'}{2p' + \left(x_k + Q_k\right) p'' - C_k''} < 0.$$

A strategy vector (x_k^*) is an equilibrium if and only if

$$0 \le x_k^* \le L_k$$

and

$$x_k^* = R_k\left(\sum_{l \neq k} x_l^*; \sum_{l \neq k} \alpha_{kl} x_l^*\right).$$

The existence of an equilibrium is guaranteed by Theorem 5.4. (Nikaido-Isoda theorem). The uniqueness of equilibrium is not guaranteed in general, as the following example illustrates.

Example 12.1 (Multiple equilibria) Consider a duopoly, $N = 2$, with price function $p\left(x_1 + x_2\right) = A - B\left(x_1 + x_2\right), (A, B > 0)$ and cost functions $c_k\left(x_k\right) = c_k x_k + d_k$. The profit of player k towards its competitor is given as

$$\phi_k = x_k\left(A - B\left(x_1 + x_2\right)\right) - C_k(x_k).$$

Let α_k denote the degree of cooperation of player k. Then from (12.1) player k uses the payoff

$$\psi_k = x_k\left(A - Bx_k - Bx_l\right) - \left(c_k x_k + d_k\right) + \alpha_k\left[x_l\left(A - Bx_k - Bx_l\right) - \left(c_l x_l + d_l\right)\right]$$

where $l \neq k$. Clearly

$$\frac{\partial \psi_k}{\partial x_k} = A - 2Bx_k - Bx_l - c_k - \alpha_k Bx_l = A - 2Bx_k - B\left(1 + \alpha_k\right) x_l - c_k$$

showing that the best response of player k is the following:

$$R_k\left(x_l, \alpha_k\right) = \begin{cases} 0 & \text{if} A - B\left(1 + \alpha_k\right) x_l - c_k \leq 0 \\ -\frac{B(1+\alpha_k)x_l}{2B} + \frac{A-c_k}{2B} & \text{otherwise} \end{cases} \tag{12.15}$$

and similarly for Player l,

$$R_l\left(x_k, \alpha_l\right) = \begin{cases} 0 & \text{if} A - B\left(1 + \alpha_l\right) x_k - c_l \leq 0 \\ -\frac{B(1+\alpha_l)x_k}{2B} + \frac{A-c_l}{2B} & \text{otherwise} \end{cases} \tag{12.16}$$

when we ignore capacity limits. Since $\frac{\partial \Psi_k}{\partial x_k}$ strictly decreases, Ψ_k is strictly concave, furthermore Ψ_k tends to $-\infty$ as $x_k \to \infty$, the maximum has to be interior. The best responses of the non-cooperative players can be obtained by selecting $\alpha_1 = \alpha_2 = 0$. The best responses under partial cooperation are illustrated in Fig. 12.1.

Consider now the symmetric case, when $c_1 = c_2 = c$ and $\alpha_1 = \alpha_2 = 1$. Then the players have the common payoff function

$$\psi_k = \left(x_1 + x_2\right)\left(A - B\left(x_1 + x_2\right)\right) - c\left(x_1 + x_2\right) - d_1 - d_2 \tag{12.17}$$

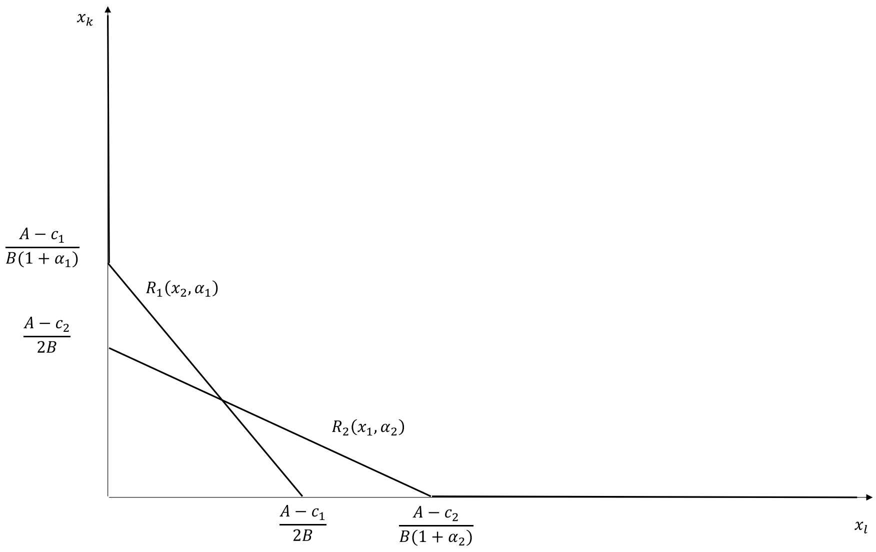

Fig. 12.1 Best responses (12.15) and (12.16)

which is strictly concave in $z = x_1 + x_2$ with a unique optimum

$$x_1 + x_2 = \begin{cases} 0 \ \text{if} A - c \leq 0 \\ \frac{A-c}{2B} \text{otherwise} \end{cases}$$

So we have infinitely many (x_1, x_2) pairs:

$$\left\{ x_1 + x_2 \,\middle|\, 0 \leq x_1 \leq \frac{A-c}{2B}, x_2 = \frac{A-c}{2B} - x_1 \right\}$$

if $A > c$, otherwise $x_1 = x_2 = 0$ is the only equilibrium.

Similarly to the noncooperative case we can rewrite the best responses in the N-player case as functions of s and Q_k as

$$\bar{R}_k (s, Q_k) = \begin{cases} 0 \ \text{if} \ \ p(s) + Q_k p'(s) - C_k'(0) \leq 0 \\ L_k \ \text{if} \ p(s) + (L_k + Q_k) p'(s) - C_k'(L_k) \geq 0 \\ \bar{x}_k \ \ \text{otherwise} \end{cases} \tag{12.18}$$

where $\bar{x}_k$ is the unique solution of equation

$$p(s) + (x_k + Q_k) p'(s) - C_k'(x_k) = 0 \tag{12.19}$$

inside interval $(0, L_k)$. Notice first that $\bar{R}_k$ is constant in the first two segments with zero partial derivatives with respect to s and Q_k. Otherwise we consider $\bar{x}_k$ as a function of s and $\tilde{x}_k$ as a function of Q_k. Implicitly differentiating equation (12.19) with respect to s and Q_k yields

$$p' + \bar{x}_k' p' + (\bar{x}_k + Q_k) p'' - C_k''(\bar{x}_k) \bar{x}_k' = 0 \qquad (\bar{x}_k = x_k(s))$$

and

$$\left(\tilde{x}_k' + 1\right) p' - C_k''(\tilde{x}_k) \tilde{x}_k' = 0 \qquad\qquad (\tilde{x}_k = x_k(Q_k))$$

showing that

$$\frac{\partial \bar{R}_k}{\partial s} = -\frac{p' + (\bar{x}_k + Q_k) p''}{p' - C_k''(\bar{x}_k)} \leq 0 \tag{12.20}$$

and

$$\frac{\partial \bar{R}_k}{\partial Q_k} = -\frac{p'}{p' - C_k''(\tilde{x}_k)} < 0 \tag{12.21}$$

respectively.

Hence both $\frac{\partial \bar{R}_k}{\partial s}$ and $\frac{\partial \bar{R}_k}{\partial Q_k}$ are nonpositive everywhere, so $\bar{R}_k$ decreases in both variables s and Q_k.

In order to compute the equilibrium we need to solve a system of (usually) nonlinear algebraic equations:

$$\begin{aligned} &\sum_{l \neq k} \alpha_{kl} \bar{R}_l (s, Q_l) = Q_k \ (k = 1,\ 2,\ \dots, N) \\ &\sum_{k=1}^{n} \bar{R}_k (s, Q_k) = s \end{aligned} \tag{12.22}$$

for unknowns $s, Q_1, Q_2, \dots, Q_n$.

12.2 Dependence on Model Parameters

First we show that by selecting partial cooperation the industry output decreases in comparison to the noncooperative case. Let $s^{(1)}$ and $s^{(2)}$ be the industry outputs in the partial cooperation and in the noncooperative cases, respectively. We can easily prove that $s^{(1)} \leq s^{(2)}$. Assume on the contrary that $s^{(1)} > s^{(2)}$. Then

$$s^{(1)} = \sum_{k=1}^{N} \bar{R}_k \left(s^{(1)}, Q_k^*\right) \leq \sum_{k=1}^{N} \bar{R}_k \left(s^{(1)}, 0\right) \leq \sum_{k=1}^{N} \bar{R}_k \left(s^{(2)}, 0\right) = s^{(2)}$$

which is an obvious contradiction.

Next we can compare industry outputs in the cases of different degrees of cooperation between the players. For mathematical convenience assume that for all k, $\alpha_{kl} = \alpha_k$ meaning that each player treats all competitors equally.

In this case (12.18) can be rewritten as follows:

$$\bar{R}_k (s, \alpha_k) = \begin{cases} 0 \ \text{if} \ \ p(s) + \alpha_k s p'(s) - C_k'(0) \leq 0 \\ L_k \ \ \text{if} \ \ p(s) + (\alpha_k (s - L_k) + L_k) p'(s) - C_k'(L_k) \geq 0 \\ \bar{x}_k^* \ \ \text{otherwise} \end{cases} \tag{12.23}$$

where $\bar{x}_k^*$ is the unique solution of equation

$$p(s) + (\alpha_k s + (1 - \alpha_k) x_k) p'(s) - C_k'(x_k) = 0 \tag{12.24}$$

in interval $(0, L_k)$. The left hand side is positive at $x_k = 0$, negative at $x_k = L_k$, furthermore strictly decreases if we assume a slightly stronger version of Assumption 3:

3'. $(1 - \alpha_k) p'(s) - C_k''(x_k) < 0$.

So there is a unique solution for x_k. $\bar{R}_k$ is constant in the first two segments of (12.23) with zero partial derivatives, otherwise it can be obtained by using implicit differentiation with respect to s and α_k.

$$\frac{\partial \bar{R}_k}{\partial s} = -\frac{(1 + \alpha_k) p' + (\alpha_k s + (1 - \alpha_k) x_k) p''}{(1 - \alpha_k) p' - C_k''} \leq 0$$

if in addition to assumption 3' we assume a weaker version of Assumption 2:

2'. $(1+\alpha_k)\,p' + (\alpha_k s + (1-\alpha_k)\,x_k)\,p'' \leq 0.$
Similarly,

$$\frac{\partial \bar{R}_k}{\partial \alpha_k} = -\frac{(s - x_k)\,p'}{(1-\alpha_k)\,p' - C_k''} \leq 0.$$

Hence under assumptions 1, 2' and 3' we have the following result:

Theorem 12.1 *Assume* $\alpha_k^{(1)} \leq \alpha_k^{(2)}$ *for all k, and let* $s^{(1)}$ *and* $s^{(2)}$ *denote industry outputs in these cases. If conditions 1, 2' and 3' hold in both cases then* $s^{(1)} \geq s^{(2)}$.

Proof Assume on the contrary that $s^{(1)} < s^{(2)}$. Then

$$s^{(2)} = \sum_{k=1}^{N} \bar{R}_k\left(s^{(2)}, \alpha_k^{(2)}\right) \leq \sum_{k=1}^{N} \bar{R}_k\left(s^{(2)}, \alpha_k^{(1)}\right) \leq \sum_{k=1}^{N} \bar{R}_k\left(s^{(1)}, \alpha_k^{(1)}\right) = s^{(1)}$$

which is an obvious contradiction.

Chapter 13
Conflict Resolution

In Chap. 11 we assumed that the players were able to collect their payoff values to a common basket, and this common payoff was distributed among the players. There are however many cases when this is impossible. First, if the payoffs of the different players are not transferable, second, if the players are unable to agree on the mechanism of side payments. In this chapter solution concepts are introduced in which each player earns his own payoff and receives it directly. In finding any such solution a negotiation process has to take place before reaching an agreement. In order to give sufficient incentive to the players to negotiate and to reach an agreement, very unfavorable payoff values are assigned to the players in case if no agreement is reached. Instead of negotiating on specific strategies, the subject of the negotiation is the received payoff value of each player. For mathematical simplicity we will discuss only two-person conflicts, and the general N-person case will be briefly discussed later.

Let S_1 and S_2 be the strategy sets of players 1 and 2 respectively, and ϕ_1 and ϕ_2 their payoff functions. Then the payoff space is defined as

$$F = \left\{(f_1, f_2) | f_1 = \phi_1(s_1, s_2), f_2 = \phi_2(s_1, s_2), s_1 \in S_1, s_2 = S_2\right\}, \tag{13.1}$$

which consists of all pairs of the simultaneous payoff values of the players. If $\underline{f}_* = (f_{1*}, f_{2*})$ denotes the disagreement point then it is assumed that there is a point $(f_1, f_2) \in F$ such that $f_1 > f_{1*}$ and $f_2 > f_{2*}$ otherwise at least one player has no incentive to negotiate.

A two-person conflict is defined by the pair $(F, \underline{f}_*)$. The players can negotiate on the simultaneous payoff values from the set

$$F^* = \left\{(f_1, f_2) | (f_1, f_2) \in F, f_1 \geqq f_{1*}, f_2 \geqq f_{2*}\right\}, \tag{13.2}$$

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_13

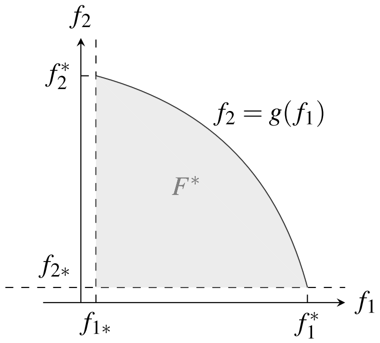

Fig. 13.1 Illustration of a conflict

since no player will accept an agreement which would give him less than the disagreement payoff what he can get anyway without negotiation. It is also assumed that set F is comprehensive, that is, if $(f_1, f_2) \in F$ and $(\overline{f_1}, \overline{f_2}) \leqq (f_1, f_2)$, then $(\overline{f_1}, \overline{f_2}) \in F$. This property means that any payoff vector which is below a feasible payoff vector is also feasible. The players can "waste" payoffs.

The *Pareto frontier* of set F^* consists of all points $(f_1, f_2) \in F^*$ such that there is no point $(\overline{f_1}, \overline{f_2}) \in F^*$ with $\overline{f_1} \geq f_1$ and $\overline{f_2} \geq f_2$ with strict inequality for at least one of these relations. The Pareto frontier therefore consists of all points from which none of the payoffs can be increased without worthening the other payoff. The following assumptions are usually made:

(i) F^* is compact and convex in $\mathbb{R}^2$;
(ii) there is at least one $\underline{f} \in F^*$ such that $\underline{f} > \underline{f}_*$.
The payoffs are in conflict meaning that
(iii) if (f_1, f_2) and $(\overline{f_1}, \overline{f_2})$ are both in the Pareto frontier and $f_1 < \overline{f_1}$, then $f_2 > \overline{f_2}$.

Notice that if F^* is not convex, then it can be made convex by allowing mixed (probabilistic) strategies. The above conditions imply that F^* is bounded by the $f_1 = f_{1*}$ vertical line, the $f_2 = f_{2*}$ horizontal line and the Pareto frontier which is the graph of a strictly decreasing, concave function $f_2 = g(f_1)$. Figure 13.1 shows set F^*.

Introduce the notation $f_1^* = g^{-1}(f_{2*})$ and $f_2^* = g(f_{1*})$, then player 1 can select any payoff value from interval $[f_{1*}, f_1^*]$ and similarly the choices of player 2 are the points of interval $[f_{2*}, f_2^*]$.

We can model this situation as a *noncooperative game* similarly to Example 3.1 when a fair division of a pie was discussed. The strategy sets of the players are $S_1 = [f_{1*}, f_1^*]$ and $S_2 = [f_{2*}, f_2^*]$ with payoff functions

$$\phi_1(f_1, f_2) = \begin{cases} f_1 & \text{if } (f_1, f_2) \in F^* \\ f_{1*} & \text{otherwise} \end{cases} \tag{13.3}$$

and

$$\phi_2(f_1, f_2) = \begin{cases} f_2 & \text{if } (f_1, f_2) \in F^* \\ f_{2*} & \text{otherwise.} \end{cases} \tag{13.4}$$

If the simultaneous payoff vector is feasible, then both players get their requests. If it is not feasible, then the players get their disagreement payoffs.

Clearly any equilibrium has to be on the Pareto frontier, since otherwise at least one of the players can increase his payoff without changing the payoff of the other player. It is also clear, that all points of the Pareto frontier provide equilibrium. So the number of equilibria equals the number of the points in the Pareto frontier, which is infinity, and consequently non-cooperative game theory gives no instruction in selecting a unique solution.

We can also model this problem as a *single-person decision problem*. Consider now player 1, who believes that player 2 selects his strategy randomly from interval $[f_{2*}, f_2^*]$ accordingly to uniform distribution. This is a decision problem under uncertainty, and player 1 wants to find his best choice by maximizing his expected payoff. His payoff is f_1, if $(f_1, f_2) \in F^*$ which is the case if and only if $f_2 \leq g(f_1)$. Otherwise his payoff is f_{1*}, the disagreement payoff. Therefore the expected payoff of player 1 can be given as

$$\begin{aligned} & f_1 \frac{g(f_1) - f_{2*}}{f_2^* - f_{2*}} + f_{1*}\left(1 - \frac{g(f_1) - f_{2*}}{f_2^* - f_{2*}}\right) \\ &= \frac{(f_1 - f_{1*})\big(g(f_1) - f_{2*}\big)}{f_2^* - f_{2*}} + f_{1*}. \end{aligned} \tag{13.5}$$

Since $f_2^* - f_{2*}$ is a positive constant and f_{1*} is a constant, this function is maximal if and only if the product

$$(f_1 - f_{1*})(f_2 - f_{2*}) \tag{13.6}$$

is maximal on the Pareto frontier, which is the same as being maximal on F^*. This product is known as the *Nash-product*. Interchanging the two players we get the same optimum point for player 2, since (13.6) is symmetric in the players. So the point $(f_1, f_2) \in F^*$ which maximizes (13.6) is a common optimum for the players, so it has to be accepted as the solution of the problem. This solution depends on the assumption that both players consider the payoff choice problem as a decision problem under uncertainty with uniform distributions.

In the rest of this chapter several solution concepts will be introduced which do not rely on such heavy assumption than the previous concept.

13.1 The Nash Bargaining Solution

Since any conflict is identified with the pair $(F^*, \underline{f}_*)$, any solution concept has to be defined on the set of all feasible pairs $(F^*, \underline{f}_*)$ and should provide a simultaneous payoff vector. So any solution $\underline{\psi}$ maps all feasible pairs $(F^*, \underline{f}_*)$ into two-dimensional real vectors which belong to F^*. Nash was looking for a solution mapping $\underline{\psi}$ which satisfies certain fairness and reasonable conditions, which are called the *Nash axioms*. They can be explained as follows :

1. $\underline{\psi}(F^*, \underline{f}_*) \in F^*$ (feasibility)
2. $\underline{\psi}(F^*, \underline{f}_*) \geqq \underline{f}_*$ (rationality)
3. $\underline{\psi}(F^*, \underline{f}_*)$ has to be Pareto optimal (Pareto optimality)
4. If $\overline{F}^* \subset F^*$ and $\underline{\psi}(F^*, \underline{f}_*) \in \overline{F}^*$, then $\underline{\psi}(\overline{F}^*, \underline{f}_*) = \underline{\psi}(F^*, \underline{f}_*)$ (independence from unfavorable alternatives)
5. Assume $T(f_1, f_2) = (\alpha_1 f_1 + \beta_1, \alpha_2 f_2 + \beta_2)$ is a linear mapping such that $\alpha_1, \alpha_2 > 0$, and let $T(F^*) = \{T(f_1, f_2) | (f_1, f_2) \in F^*\}$ be the image of F^* with respect to mapping T. Then
$$\underline{\psi}\big(T(F^*), T(\underline{f}_*)\big) = T\big(\underline{\psi}(F^*, \underline{f}_*)\big)$$
(independence from increasing linear transformations)
6. If a conflict is symmetric, that is, $(f_1, f_2) \in F^*$ if and only if $(f_2, f_1) \in F^*$, and $f_{1*} = f_{2*}$, then the two components of the solution vector $\underline{\psi}(F^*, \underline{f}_*)$ are equal (symmetry).

Axiom 1 requires that the solution is feasible and axiom 2 ensures that none of the players get less than in the case of disagreement. Pareto optimality is also a reasonable assumption, otherwise at least one player could increase his payoff without hurting the other player. Assumption 4 is a reformulation of a well known property of optimum problems that the optimum does not change if the feasible set is reduced and the optimum remains feasible. Assumption 5 can be interpreted as changing the units in which the payoffs are computed should not alter the solution. The last axiom 6 requires equal payoff to the players if there is no difference between them in the conflict formulation (in set F^* and disagreement payoff vector $\underline{f}_*$).

The main result of Nash (1953) is the following.

Theorem 13.1 *There is a unique solution function $\underline{\psi}$ satisfying the Nash axioms and it can be obtained as the unique optimal solution of problem*

$$\begin{aligned} &\textit{maximize } (f_1 - f_{1*})(f_2 - f_{2*}) \\ &\textit{subject to } (f_1, f_2) \in F^*. \end{aligned} \tag{13.7}$$

Before a proof is presented we discuss a simple geometric fact, which will be used in the proof. Consider the graph of function $y = \frac{1}{x^\alpha}$ in the first quadrant, and let (x_0, y_0) be a point on the curve. Consider next the tangent line to the curve at this

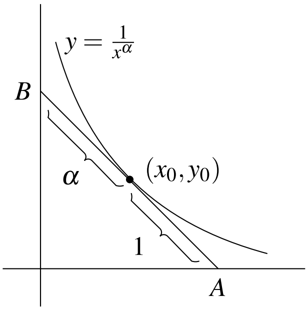

Fig. 13.2 A simple geometric fact

point. Then point (x_0, y_0) divides the segment of the tangent line in the first quadrant into two parts, the ratio of their lengths is $\frac{1}{\alpha}$.

Figure 13.2 shows this situation. The slope of the tangent line is obtained by differentiation: $-\alpha x_0^{-(\alpha+1)}$, so its equation is

$$y - y_0 = \frac{-\alpha}{x_0^{\alpha+1}}\big(x - x_0\big).$$

Its x-intersection is obtained by substituting $y = 0$, so

$$-y_0 = \frac{-\alpha}{x_0^{\alpha+1}}\big(x - x_0\big)$$

resulting in

$$x_A = \frac{y_0 x_0^{\alpha+1}}{\alpha} + x_0 = \frac{x_0}{\alpha} + x_0 = x_0 \frac{1+\alpha}{\alpha}.$$

The y-intersection is obtained by substituting $x = 0$ into the equation of the tangent line,

$$y_B = y_0 + \frac{\alpha x_0}{x_0^{\alpha+1}} = y_0 + \alpha y_0 = (1+\alpha)y_0.$$

So the coordinates of the x-intersection of the tangent line are $A\big(x_0 \frac{1+\alpha}{\alpha}, 0\big)$ and these of the y-intersection are $B\big(0, (1+\alpha)y_0\big)$. Since

$$\frac{\alpha}{1+\alpha}\Big(x_0 \frac{1+\alpha}{\alpha}\Big) + \frac{1}{1+\alpha}(0) = x_0$$

and

$$\frac{\alpha}{1+\alpha}(0) + \frac{1}{1+\alpha}\big((1+\alpha)y_0\big) = y_0,$$

point (x_0, y_0) divides the segment into two parts, the ratio of their lengths is $\frac{1}{\alpha}$ as shown in the figure. In the special case, when $\alpha = 1$, point (x_0, y_0) is in the middle of the segment.

Now we are ready for the proof of Theorem 13.1.

Proof (a) Notice first that the solution of the optimum problem is unique since optimizing its objective function is equivalent with maximizing its logarithm, $\log(f_1 - f_{1*}) + \log(f_2 - f_{2*})$ which is strictly concave implying the uniqueness of the optimal solution. It is also obvious that the optimal solution satisfies the Nash axioms.

(b) We will next prove, that the solution satisfying the axioms is necessarily the optimal solution of problem (13.7). We proceed in several steps as shown in Fig. 13.3. We first assume that $\underline{f}_* = (0, 0)$.

Consider the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Based on axioms 3 and 6, the solution has to be the midpoint between vertices $(0, 1)$ and $(1, 0)$ which is the optimal solution of (12.7). Because of axiom 5, for any triangle of the first quadrant with the horizontal axis and the vertical axis beeing its two sides the same holds, that is, the solution is the midpoint of the third side. Assume next that F^* is a convex closed set. The optimal solution T of (12.7) is obtained graphically by finding a hyperbola $f_2 = \frac{C}{f_1}$ which is the tangent to the Pareto frontier of F^* at point T. The theorem of separating hyperplans guarantees that at this point they have a common tangent line which creates a triangle with vertices A, B and $(0, 0)$. Since segment $\overline{AB}$ is the tangent to the hyperbola, point T is the midpoint of the segment, so it is also the axiomatic solution. Since F^* is a subset of the triangle, axiom 4 guarantees, that the axiomatic solution of problem $(F^*, \underline{0})$ is also point T, which is also the optimal solution of problem (12.7). Axiom 5 implies that the statement remains valid if point $\underline{f}_*$ is not zero, since shifting is a special linear transformation with $\alpha_1 = \alpha_2 = 1$. ∎

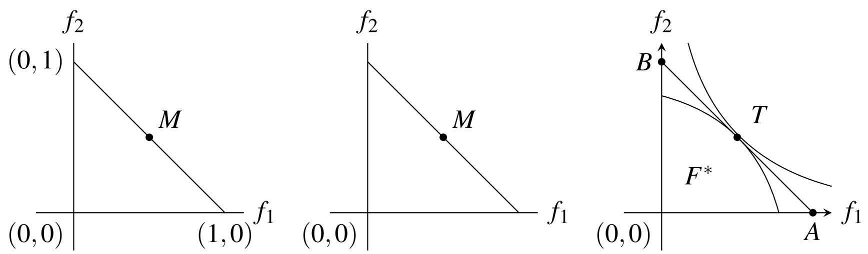

Fig. 13.3 Steps of the proof of Theorem 13.1

Example 13.1 In the case of Example 3.1, when we were looking for a fair share of a pie, set F^* were the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. The Nash bargaining solution is clearly the midpoint between $(1, 0)$ and $(0, 1)$, which is $(\frac{1}{2}, \frac{1}{2})$ meaning that equal shares is the solution. ▼

Example 13.2 Assume that F^* is the convex set between the two axes and the curve $f_2 = 1 - f_1^2$, and the disagreement point is $(0, 0)$. Then problem (12.7) can be written as

$$\begin{aligned} &\text{maximize } f_1 \cdot f_2 \\ &\text{subject to } 0 \leqq f_1, f_2 \leqq 1 \\ &\qquad f_2 \leqq 1 - f_1^2. \end{aligned}$$

At the optimum clearly $f_2 = 1 - f_1^2$, so this problem is equivalent with a single-dimensional problem

$$\begin{aligned} &\text{maximize } f_1(1 - f_1^2) \\ &\text{subject to } 0 \leqq f_1 \leqq 1. \end{aligned}$$

Notice that at both $f_1 = 0$ and $f_1 = 1$ the objective function is zero and for all $f_1 \in (0, 1)$ it is positive. So the optimum is interior. By differentiation,

$$\frac{d}{df_1}\big(f_1(1 - f_1^2)\big) = 1 - 3f_1^2 = 0$$

implying that

$$f_1 = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3} \text{ and } f_2 = 1 - f_1^2 = \frac{2}{3}.$$

▼

13.2 Alternative Solution Concepts

Several authors were criticizing the fairness and reality of the Nash axioms. The symmetry assumption (Axiom 6) was one of them, since the negotiating partners not always have the same negotiating power, so their relative power has to be incorporated into the solution concept. The *Non-symmetric Nash bargaining solution* is based on this idea (Harsanyi & Selten, 1972). Select a positive vector $\underline{\alpha} = (\alpha_1, \alpha_2)$ such that $\alpha_1 + \alpha_2 = 1$, and let $\triangle$ denote the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. In the case of the triangle $\triangle$ we require that the solution is the point on the segment connecting (0, 1) and (1, 0) which divides it to two parts, the ratio of their lengths is $\frac{\alpha_1}{\alpha_2}$.

Theorem 13.2 *For any given positive vector* $\underline{\alpha}$ *with* $\alpha_1 + \alpha_2 = 1$ *there is a unique solution function* $\psi(F^*, \underline{f}_*)$ *satisfying axioms* $1 - 5$ *and it is the unique solution of the optimum problem*

$$\begin{aligned} &\textit{maximize } (f_1 - f_{1*})^{\alpha_1}(f_2 - f_{2*})^{\alpha_2} \\ &\textit{subject to } (f_1, f_2) \in F^*. \end{aligned} \tag{13.8}$$

The proof of this theorem can be made along the lines of the proof of Theorem 13.1 with minor differences. Instead of a hyperbola the curve of function $f_2 = \frac{1}{f_1^{\alpha_1/\alpha_2}}$ has to be considered.

Example 13.3 Consider again the conflict examined in our previous example, in which case problem (13.8) has the form

$$\begin{aligned} &\text{maximize } f_1^{\alpha_1} f_2^{1-\alpha_1} \\ &\text{subject to } 0 \leqq f_1, f_2 \leqq 1 \\ &\qquad f_2 \leqq 1 - f_1^2 \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\text{maximize } f_1^{\alpha_1}(1 - f_1^2)^{1-\alpha_1}, \\ &\text{subject to } 0 \leqq f_1 \leqq 1. \end{aligned}$$

By differentiation,

$$\alpha_1 f_1^{\alpha_1 - 1}(1 - f_1^2)^{1-\alpha_1} + f_1^{\alpha_1}(1 - \alpha_1)(1 - f_1^2)^{-\alpha_1}(-2f_1) = 0,$$

that is,

$$\alpha_1(1 - f_1^2) - 2f_1^2(1 - \alpha_1) = 0$$

implying that

$$f_1 = \sqrt{\frac{\alpha_1}{2 - \alpha_1}} \text{ and } f_2 = 1 - f_1^2 = \frac{2 - 2\alpha_1}{2 - \alpha_1}.$$

In the special symmetric case $\alpha_1 = \frac{1}{2}$, so $f_1 = \sqrt{\frac{1}{3}}$ and $f_2 = \frac{2}{3}$ which is the same result we obtained earlier. Notice also that if $\alpha_1 = 0$, then no power is given to player 1 and $f_1 = 0$, $f_2 = 1$ is the solution giving nothing to player 1. In the other extreme

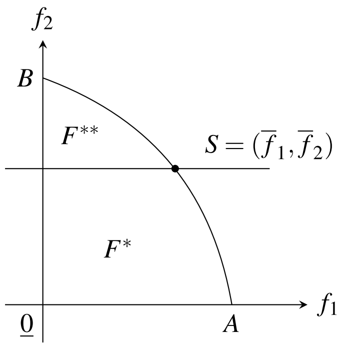

Fig. 13.4 Illustration of Axiom 4

case of $\alpha_1 = 1$, $f_1 = 1$ and $f_2 = 0$ giving nothing to player 2. It is also clear that f_1 increases with increasing value of α_1 and f_2 decreases in the same time. ▼

By dropping Pareto optimality from the requirements we get the following interesting result.

Theorem 13.3 *There are exactly two solution functions satisfying axioms* 1, 2, 4, 5 *and* 6. *One is the Nash bargaining solution and the other is the disagreement payoff vector.*

The theorem can be proved similarly to Theorem 13.1, which can be left as an exercise for interested readers.

Axiom 4 also has some problems. We can illustrate them in two simple examples.

Example 13.4 Consider Fig. 13.4, where F^* is the region with disagreement vector $\underline{0}$, and the conflict is symmetric with vertices, $\underline{0}$, A and B and Nash bargaining solution $S = (\overline{f_1}, \overline{f_2})$. Assume that in a new conflict

$$F^{**} = \left\{(f_1, f_2) | (f_1, f_2) \in F^*, f_2 \geq \overline{f_2}\right\},$$

and since $F^{**} \subseteq F^*$ and solution $(\overline{f_1}, \overline{f_2})$ remains feasible in reducing F^* to F^{**}, the solution of the reduced problem is also $(\overline{f_1}, \overline{f_2})$. This is not fair to player 2, since player 1 receives the highest possible payoff in F^{**} and player 2 gets the worst possible outcome from F^{**}. ▼

Example 13.5 Consider the conflict shown in Fig. 13.5, where the disagreement point is $\underline{0}$ and F^* is the convex hull of $(0, 0)$, $(1, 0)$, $\left(1, \frac{7}{10}\right)$ and $(0, 1)$. Define F^{**} as the convex hull of $(0, 0)$, $(1, 0)$, $\left(\frac{3}{4}, \frac{3}{4}\right)$ and $(0, 1)$, as shown in Fig. 13.5. So we have two conflicts $(F^*, \underline{0})$ and $(F^{**}, \underline{0})$. Notice first that with any value $f_1 \in (0, 1)$ player 2 gets higher payoff in conflict $(F^*, \underline{0})$ than in conflict $(F^{**}, \underline{0})$, so he should expect higher payoff in the solution of conflict $(F^*, \underline{0})$ than in $(F^{**}, \underline{0})$. However $\underline{\psi}(F^*, \underline{0}) = \left(1, \frac{7}{10}\right)$ and $\underline{\psi}(F^{**}, \underline{0}) = \left(\frac{3}{4}, \frac{3}{4}\right)$ conflicting the expectation of player 2. ▼

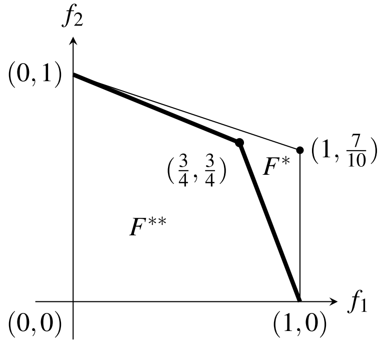

Fig. 13.5 Other illustration of Axiom 4

Kalai and Smorodinsky (1975) replaced Axiom 4 with the requirement of individual monotonicity, which can be defined as follows. Consider a conflict $(F, \underline{f}_*)$ and define

$$f_i^* = \max\left\{f_i|(f_1, f_2) \in F^*\right\}$$

and with fixed f_j, let

$$f_{iF}^*(f_j) = \begin{cases} \max\{f_i|(f_1, f_2) \in F^*\} & \text{if this set is nonempty} \\ f_i^* & \text{otherwise} \end{cases}$$

for $i = 1, 2$ and $j = 3 - i$ (the other player than i).

Monotonicity axiom: Consider two conflicts $(F_1, \underline{f}_*)$ and $(F_2, \underline{f}_*)$ and assume that with a player j and all $f_j \in [f_{j*}, f_j^*]$ for the other player i, $f_{iF_1}^*(f_j) \geqq f_{iF_2}^*(f_j)$. Then at the solution $\psi_i(F_1, \underline{f}_*) \geqq \psi_i(F_2, \underline{f}_*)$.

Notice that $f_{iF_l}^*(f_j)$ is the maximum payoff that player i can obtain in conflict $(F_l, \underline{f}_*)$ with given payoff f_j of player j. This axiom requires that if for every payoff level of player j the maximum feasible payoff that the other player can get increases, then his payoff at the solution also must increase.

Theorem 13.4 *There is a unique solution satisfying axioms 1–3, 5, 6 and monotonicity, which is the intersection of the Pareto frontier and the linear segment connecting points $\underline{f}_* = (f_{1*}, f_{2*})$ and $\underline{f}^* = (f_1^*, f_2^*)$.*

The proof of this theorem is very simple, so left as an exercise to the interested readers. The solution is illustrated in Fig. 13.6, and is called the *Kalai-Smorodinsky solution*. It is easy to see that it can be obtained by solving the following single-dimensional equation

$$\frac{g(f_1) - f_{2*}}{f_1 - f_{1*}} = \frac{f_2^* - f_{2*}}{f_1^* - f_{1*}}. \tag{13.9}$$

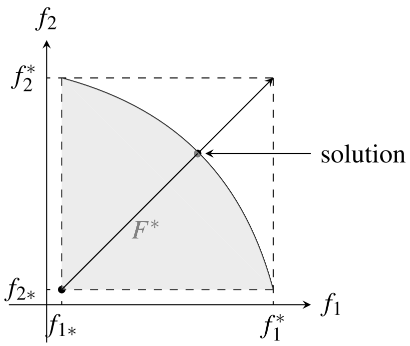

Fig. 13.6 Illustration of the Kalai-Smorodinsky solution

The left hand side strictly decreases in f_1. At $f_1 = f_1^*$ it is zero and as $f_1 \to f_{1*}$ it converges to infinity. Therefore there is a unique solution and it can be computed by simple algorithms.

Example 13.6 Consider again the conflict examined earlier in Examples 13.2 and 13.3. In this case $f_{1*} = f_{2*} = 0$, $f_1^* = f_2^* = 1$, and $g(f_1) = 1 - f_1^2$, so Eq. (13.9) simplifies as

$$\frac{1 - f_1^2}{f_1} = \frac{1}{1}$$

that is,

$$f_1^2 + f_1 - 1 = 0$$

implying that $f_1 = \frac{\sqrt{5}-1}{2} \approx 0.62$ and $f_2 = 1 - \left(\frac{\sqrt{5}-1}{2}\right)^2 = f_1$. ▼

The *Equal loss principle* of Chun (1988) equalizes the losses of the players from the ideal point (f_1^*, f_2^*) by assuming that the players decrease their demands with equal speed until a feasible solution is obtained. If the ideal point is feasible, then clearly it is the solution. Otherwise the solution is obtained by solving the following problem:

$$\begin{aligned} &\text{minimize } v \\ &\text{subject to } (f_1^* - v, f_2^* - v) \in F^* \end{aligned}$$

which can be rewritten as the single-variable nonlinear equation

$$f_2^* - g(f_1) - f_1^* + f_1 = 0. \tag{13.10}$$

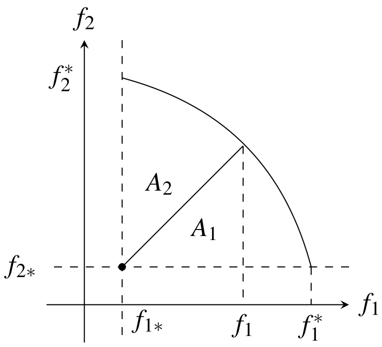

Fig. 13.7 Illustration of the area monotonic solution

Notice that the left hand side strictly increases in f_1. At $f_1 = f_1^*$ its value is $f_2^* - f_{2*} > 0$ and at $f_1 = f_{1*}$ its value equals $-f_1^* + f_{1*} < 0$. Therefore there is a unique solution which can be obtained by simple methods. The players get the payoff vector $\big(f_1, g(f_1)\big)$.

Example 13.7 In the case of the previous example Eq. (13.10) has the special form

$$1 - \left(1 - f_1^2\right) - 1 + f_1 = 0$$

that is,

$$f_1^2 + f_1 - 1 = 0$$

leading to the same solution obtained in the previous example. ▼

We mention here that Chun (1988) has presented an axiomatic development of this method and proved that the equal loss solution is the only solution satisfying his axioms.

The *Area monotonic solution* of Anbarci (1993) is another alternative solution concept for solving conflicts. Consider Fig. 13.7 where we added an arc starting at the disagreement point. This arc represents simultaneous payoff vectors (f_1, f_2) such that the ratio of the increases of the players from the disagreement payoffs is the same.

Any point under this arc gives player 1 a higher ratio and any point above the arc gives higher ratio to player 2. In order to be fair, we are looking for an arc which breaks up F^* into two regions of equal area and the intersection of this arc and the Pareto frontier is accepted as the solution for the conflict.

The area under the arc is

$$A_1 = \frac{1}{2}(f_1 - f_{1*})\big(f_{2*} + g(f_1)\big) + \int_{f_1}^{f_1^*} g(f)df - (f_1^* - f_{1*})f_{2*} \qquad (13.11)$$

and the area above the arc is given as

$$A_2 = \int_{f_{1*}}^{f_1} g(f)df - \frac{1}{2}(f_1 - f_{1*})\big(f_{2*} + g(f_1)\big). \tag{13.12}$$

Notice that A_1 strictly decreases in f_1 and A_2 strictly increases, furthermore $A_1(f_{1*}) = A$, $A_1(f_1^*) = 0$, $A_2(f_{1*}) = 0$ and $A_2(f_1^*) = A$ where A is the total area of F^*. Therefore there is a unique solution.

Example 13.8 In the case of the previous example

$$\begin{aligned} A_1 &= \frac{1}{2}(f_1 - 0)\big(0 + 1 - f_1^2\big) + \int_{f_1}^{1} \Big(1 - f^2\Big)df + (1-0)0 \\ &= \frac{1}{2} f_1\big(1 - f_1^2\big) + \frac{2}{3} - f_1 + \frac{f_1^3}{3} = -\frac{f_1^3}{6} - \frac{f_1}{2} + \frac{2}{3}. \end{aligned}$$

The total area of F^* is clearly

$$\int_0^1 \Big(1 - f^2\Big)df = \Big[f - \frac{f^3}{3}\Big]_0^1 = \frac{2}{3}$$

so we have to solve equation

$$-\frac{f_1^3}{6} - \frac{f_1}{2} + \frac{2}{3} = \frac{1}{3}$$

that is,

$$f_1^3 + 3f_1 - 2 = 0$$

giving the solution

$$f_1 \approx 0.60 \text{ and } f_2 \approx 1 - (0.60)^2 = 0.64$$

where the value of f_1 is computed by using the Newton method. ▼

The noncooperative foundations of the area monotonic solution is discussed in Anbarci (1993); and an axiomatic development is given in Anbarci and Bigelow (1994).

The methods discussed earlier in this section assumed equal players except the non-symmetric Nash method. We can easily extend them to cases of unequal players in the following way. Let α_1 and α_2 be the powers of the two players.

In the case of the *Kalai-Smorodinsky solution* we can replace the ideal point by $\big(\alpha_1 f_1^*, \alpha_2 f_2^*\big)$ and use the method without any further changes. The *equal loss principle* can be modified to the principle of proportional losses in which the stronger player has to decrease his payoff slower than the weaker player resulting in the optimum problem

$$\begin{aligned}&\text{minimize } v\\&\text{subject to } \Big(f_1^* - \frac{1}{\alpha_1}v,\ f_2^* - \frac{1}{\alpha_2}v\Big) \in F^*\end{aligned}$$

which can be rewritten as the single-variable nonlinear equation

$$\alpha_2\big(f_2^* - g(f_1)\big) - \alpha_1(f_1^* - f_1) = 0.$$

Notice that in the special case of $\alpha_1 = \alpha_2$, this problem reduces to (13.10). In the case of the *area monotonic solution* A_1 is replaced by $\alpha_1 A_1$ and A_2 is replaced by $\alpha_2 A_2$ so we have equation $\alpha_1 A_1 - \alpha_2 A_2 = 0$.

13.3 *N*-Person Conflicts

In generalizing the solution concepts and methods the feasible sets F, F^* and the assumptions are extended naturally. In the general case

$$F = \big\{(f_1, \ldots, f_N) | f_k = \phi_k(s_1, \ldots, s_N) \text{ for all } k \text{ with } s_i \in S_i \text{ for all } i\big\}$$

and

$$F^* = \big\{(f_1, \ldots, f_N) | (f_1, \ldots, f_N) \in F \text{ and } f_k \geq f_{k*} \text{ for all } k\big\}$$

where $\underline{f}_* = (f_{1*}, \ldots, f_{N*})$ is the disagreement point. It is also assumed that F^* is compact and convex, there is an $\underline{f} \in F^*$ such that $\underline{f} > \underline{f}_*$. In addition, F is comprehensive and if $(f_1, \ldots, f_N)$ and $(\overline{f}_1, \ldots, \overline{f}_N)$ are both on the Pareto frontier and if $f_i > \overline{f}_i$ for an i, then there is a j with $f_j < \overline{f}_j$.

The Nash axioms can be easily extended to the N-person case and problem (13.7) can be generalized as

$$\begin{aligned}&\text{maximize } \Pi_{i=1}^N (f_i - f_{i*})\\&\text{subject to } (f_1, \ldots, f_N) \in F^*.\end{aligned}$$

The Nonsymmetric Nash solution is obtained by solving the following problem:

$$\begin{aligned}&\text{maximize } \Pi_{i=1}^N (f_i - f_{i*})^{\alpha_i}\\&\text{subject to } (f_1, \ldots, f_N) \in F^*\end{aligned} \tag{13.13}$$

where $\alpha_1, \ldots, \alpha_N$ are the powers of the players.

The ideal point $\underline{f}^*$ is given by coordinates,

$$f_k^* = \max \left\{ f_k | (f_1, \ldots, f_N) \in F^* \right\}.$$

The intersection of the Pareto frontier with the linear segment connecting points $\underline{f}_*$ and $\underline{f}^*$ gives the Kalai-Smorodinsky solution.

The Equal loss principle for the N-person case requires the solution of the simple optimization problem

$$\begin{aligned} &\text{minimize } v \\ &\text{subject to } \left(f_1^* - v, \ldots, f_N^* - v \right) \in F^* \end{aligned} \tag{13.14}$$

which is a clear extension of the 2-player case. The payoffs of the players are then $f_1^* - v, f_2^* - v, \ldots, f_N^* - v$.

Another way to extend conflict resolution to the N-person case is to reduce the problem to a sequence of 2-player conflicts. Let $i_1, \ldots, i_N$ be a permutation of the players as an agenda. First players i_1 and i_2 resolve their conflict with all possible strategies of the other players. After they reach an agreement, then they (as one player) negotiate with player i_3. After they get an agreement, then these three players as one player negotiate with player i_4, and so on. Clearly the outcome depends on the selected agenda, what all players should accept.

There is also a large literature discussing different negotiation processes. Such an example was earlier introduced in Example 9.6.

Chapter 14
Multiobjective Optimization

In the previous chapters we considered three types of cooperation of the players. In the first case they formed a grand coalition, obtained the largest possible overall payoff which is then divided among the players based on certain concepts of fairness. In the second case each player individually took the interests of the other players into account by maximizing a linear combination of the payoffs of all players including itself. In the third case the players negotiate to reach a fair settlement. In many cases a fourth way exists. The players agree in selecting a special person, the mediator, who has the trust of the players and the right and duty of finding a final solution to be given to them. After the mediator reaches his decision, all players are obliged to accept it. In this case we have a single decision maker, who takes the interests of all players into account simultaneously. If $\phi_1, \ldots, \phi_N$ are the payoff functions of the players and higher value is better for all, then the role of the mediator is to find a satisfactory solution for the multiobjective optimization problem

$$\begin{aligned} &\text{maximize } \ (\phi_1(s_1, \ldots, s_N), \ldots, \phi_N(s_1, \ldots, s_N)) \\ &\text{subject to } s_k \in S_k \, (k = 1, 2, \ldots, N) \end{aligned} \tag{14.1}$$

where s_k is the strategy of player k selected from his strategy set S_k for $k = 1, 2, \ldots, N$. The major differences between single-objective and multiobjective optimization problems are given in Appendix G, where the main solution concept, the meaning of nondominated (or Pareto optimal) solutions is outlined. Assuming that the players as well as the mediator are rational, the solution has to be Pareto-optimal, which is called nondominated in the multiobjective programming literature. Since there are usually many (and often infinitely many) nondominated solutions, the choice of a fair solution requires a precise notion of fairness from the mediator. Depending on how fairness is defined several methods can be selected.

In the formulation 14.1 the feasible set is the Chartesian product $S_1 \times S_2 \times \cdots \times S_N$ assuming that the players select strategies independently from each other. In many applications this is not the case, think about limited resources in an industry as an

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_14

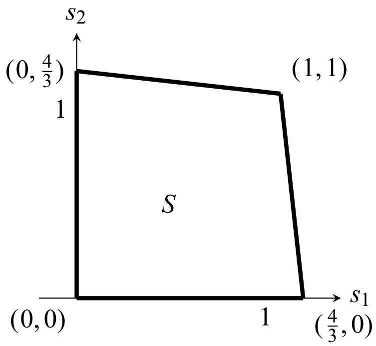

Fig. 14.1 Decision space in Example 13.1

example. In this case the set of simultaneous strategies is a subset of $S_1 \times \cdots \times S_N$ and in this case we can assume that the decision vector is $\underline{s} = (s_1, \ldots, s_N)$ and the constraint is $\underline{s} \in S$.

Set S is the *decision space* containing the possible actions. The elements of S show what the players can do. Introduce the *payoff space*

$$F = \left\{ \left(\phi_1(\underline{s}), \ldots, \phi_N(\underline{s}) \right) | \underline{s} \in S \right\}$$

which contains all feasible simultaneous payoffs. The elements of F show what the players can get.

Example 14.1 Consider the following simple problem with two objectives

$$\begin{aligned} \text{maximize } & s_1 + s_2, s_1 - s_2 \\ \text{subject to } & s_1, s_2 \geqq 0 \\ & 3s_1 + s_2 \leqq 4 \\ & s_1 + 3s_2 \leqq 4. \end{aligned} \tag{14.2}$$

The decision space is shown in Fig. 14.1. In order to find the payoff space introduce the payoff functions as new variables,

$$\phi_1 = s_1 + s_2 \qquad \text{and} \qquad \phi_2 = s_1 - s_2$$

from which we have

$$s_1 = \frac{\phi_1 + \phi_2}{2} \qquad \text{and} \qquad s_2 = \frac{\phi_1 - \phi_2}{2}.$$

By substituting them into the constraints of (14.2) we get the constraints of the payoff space F:

$$
\begin{aligned}
s_1 \geqq 0 \quad &\Longleftrightarrow \quad \frac{\phi_1+\phi_2}{2} \geqq 0 \quad \Longleftrightarrow \quad \phi_2 \geqq -\phi_1 \\
s_2 \geqq 0 \quad &\Longleftrightarrow \quad \frac{\phi_1-\phi_2}{2} \geqq 0 \quad \Longleftrightarrow \quad \phi_2 \leqq \phi_1 \\
3s_1+s_2 \leqq 4 \quad &\Longleftrightarrow \quad \frac{3(\phi_1+\phi_2)}{2}+\frac{\phi_1-\phi_2}{2} \leqq 4 \quad \Longleftrightarrow \quad 2\phi_1+\phi_2 \leqq 4 \\
s_1+3s_2 \leqq 4 \quad &\Longleftrightarrow \quad \frac{\phi_1+\phi_2}{2}+\frac{3(\phi_1-\phi_2)}{2} \leqq 4 \quad \Longleftrightarrow \quad 2\phi_1-\phi_2 \leqq 4.
\end{aligned}
$$

Figure 14.2 shows the feasible set of these constraints. The set of Pareto solutions is the linear segment between points $(2, 0)$ and $\left(\frac{4}{3}, \frac{4}{3}\right)$. If all constraints are linear and S is bounded, then there is an easier way to construct set F. Compute first the images of the vertices of S with respect to the payoff functions, and then F is the convex hull of the images. In our case

$$
\begin{aligned}
(0,0) \quad &\mapsto \quad (0+0, 0-0) \quad = \quad (0,0) \\
\left(\frac{4}{3}, 0\right) \quad &\mapsto \quad \left(\frac{4}{3}+0, \frac{4}{3}-0\right) \quad = \quad \left(\frac{4}{3}, \frac{4}{3}\right) \\
\left(0, \frac{4}{3}\right) \quad &\mapsto \quad \left(0+\frac{4}{3}, 0-\frac{4}{3}\right) \quad = \quad \left(\frac{4}{3}, -\frac{4}{3}\right) \\
(1,1) \quad &\mapsto \quad (1+1, 1-1) \quad = \quad (2,0).
\end{aligned}
$$

As Fig. 14.2 shows these points are the vertices of F. ▼

14.1 Lexicographic Method

Assume that the mediator has ordinal preferences of the players meaning that player i_1 is the most important, i_2 is the second most important, and so on, i_N is the least important player to him. So he wants to satisfy player i_1 first as much as possible without any consideration to the other players. If there is a unique optimum, then it is the final solution. Otherwise the second most important player, i_2 is satisfied as much as possible keeping the payoff of player i_1 at its optimal level. If a unique optimum is obtained, then the process terminates, otherwise player i_3 is satisfied as much as possible, keeping the payoffs of players i_1 and i_2 at their optimal levels, and so on. The process terminates if at a stage unique optimum is found or all payoffs were already optimized. Mathematically this process can be formulated as follows:
Step 1:

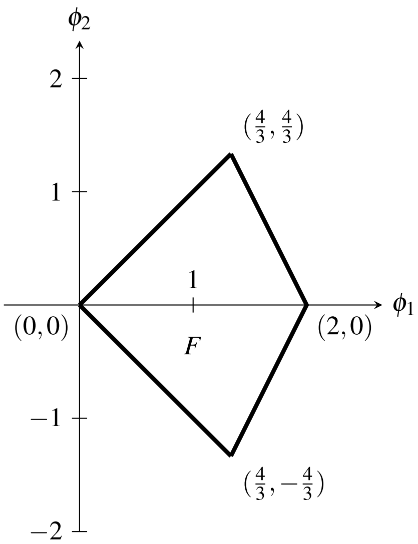

Fig. 14.2 Payoff space in Example 13.1

$$\begin{aligned} &\text{maximize } \phi_{i_1}(\underline{s}) \\ &\text{subject to } \underline{s} \in S \end{aligned} \tag{14.3}$$

and let $\phi^*_{i_1}$ denote the optimal payoff value. If the optimal decision $\underline{s}$ is unique, then process terminates and $\underline{s}$ is the final solution. Otherwise continue with next step.
Step 2:

$$\begin{aligned} &\text{maximize } \phi_{i_2}(\underline{s}) \\ &\text{subject to } \underline{s} \in S \\ &\qquad \phi_{i_1}(\underline{s}) = \phi^*_{i_1} \end{aligned} \tag{14.4}$$

and let $\phi^*_{i_2}$ denote the optimal payoff value. If the optimal decision $\underline{s}$ is unique, then stop, otherwise continue with next step. In general, *Step k* is the following:

$$\begin{aligned} &\text{maximize } \phi_{i_k}(\underline{s}) \\ &\text{subject to } \underline{s} \in S \\ &\qquad \phi_{i_l}(\underline{s}) = \phi^*_{i_l} \; (l = 1, 2, \ldots, k-1). \end{aligned} \tag{14.5}$$

If there is a unique decision then it is the solution, otherwise increase the value of k by 1 and go to the next step. Notice that the computation of the lexicographic nucleolus was also based on the above concept.

Example 14.2 Consider the problem of the previous example. If ϕ_1 is more important than ϕ_2, then ϕ_1 is maximized first. From Figure 14.2 we see that ϕ_1 is maximal at the point (2, 0) with payoff values $\phi_1 = 2$ and $\phi_2 = 0$, so the corresponding strategies are

$$s_1 = \frac{\phi_1 + \phi_2}{2} = 1 \ \text{ and } \ s_2 = \frac{\phi_1 - \phi_2}{2} = 1.$$

▼

It is easy to prove that the lexicographic method always provides a weakly Pareto optimal solution, but has a huge disadvantage. If at step k a unique optimum is obtained, then the process terminates and payoffs of players $i_{k+1}, i_{k+2}, \ldots, i_N$ are not considered at all in the process, so they can have very unfavorable, and therefore unacceptable values. One way to overcome this difficulty is to relax the optimality requirements for payoffs $\phi_{i_l}(\underline{s})$, $l = 1, 2, \ldots, k-1$, in step k, so it is modified as follows:

$$\begin{aligned} &\text{maximize } \phi_{i_k}(\underline{s}) \\ &\text{subject to } \underline{s} \in S \\ &\qquad \phi_{i_l}(\underline{s}) \geqq \phi_{i_l}^* - \epsilon_{i_l} \ (l = 1, 2, \ldots, k-1) \end{aligned} \tag{14.6}$$

where ϵ_{i_l} is a small relaxing constant.

Example 14.3 In the case of the previous example we already showed Step 1 resulting in $\phi_1^* = 2$. If $\epsilon_1 = 0.5$ is chosen, then in the second step we solve problem

$$\begin{aligned} &\text{maximize } s_1 - s_2 \\ &\text{subject to } s_1, s_2 \geqq 0 \\ &\qquad 3s_1 + s_2 \leqq 4 \\ &\qquad s_1 + 3s_2 \leqq 4 \\ &\qquad s_1 + s_2 \geqq 1.5. \end{aligned}$$

The optimal solution is $s_1 = 1.25$ and $s_2 = 0.25$ with payoff values $\phi_1 = 1.5$ and $\phi_2 = 1$. ▼

While the original version of the lexicographic method always provides Pareto optimal solution, this modified version supplies only *weakly Pareto optimal* solutions meaning that starting from the solution it is impossible to increase all payoffs simultaneously.

Example 14.4 Fig. 14.3 shows such an example, where F is the unit square and ϕ_1 is the more important payoff. The maximal value of ϕ_1 is clearly 1, and if the optimality constraints for ϕ_1 is relaxed by ϵ_1, then the feasible set for Step 2 is the shaded region. If ϕ_2 is maximized there, then there are infinitely many optimal solutions which form the linear segment connecting points $(1 - \epsilon_1, 1)$ and $(1, 1)$. Only point $(1, 1)$ is Pareto optimal, all others are only weakly Pareto optimal. ▼

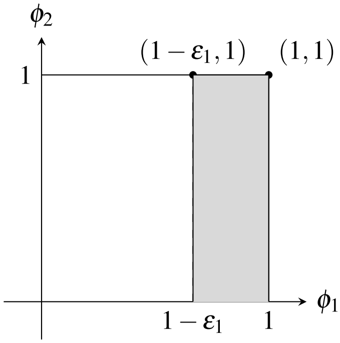

Fig. 14.3 Illustration of Example 14.4

It is also a disadvantage of the lexicographic method that Pareto optimal solutions can be lost by the method selection, like in Example 14.1 only two Pareto optimal solutions can be obtained.

14.2 The ε-Constraint Method

In this method the decision maker does not need to specify ordinal preferences of the players, he needs only to specify the most important player. Then in order to avoid very low payoffs to the other players, he requires all other payoff functions to satisfy certain lower bound constraints. Mathematically this idea is modeled by the following optimization problem:

$$\begin{aligned} &\text{maximize } \phi_k(\underline{s}) \\ &\text{subject to } \underline{s} \in S \\ &\qquad\qquad \phi_l(\underline{s}) \geqq \epsilon_l \; (l \neq k) \end{aligned} \tag{14.7}$$

when player k is the most important to the decision maker.

It is easy to show that the solution is always weakly Pareto optimal.

Example 14.5 Consider again problem 14.2 and assume that ϕ_1 is the more important payoff, and $\epsilon_2 = 1$ is the lower bound for ϕ_2. Then (14.7) is specialized as

$$\begin{aligned} &\text{maximize } s_1 + s_2 \\ &\text{subject to } s_1, s_2 \geqq 0 \\ &\qquad\qquad 3s_1 + s_2 \leqq 4 \\ &\qquad\qquad s_1 + 3s_2 \leqq 4 \\ &\qquad\qquad s_1 - s_2 \geqq 1. \end{aligned}$$

The optimal solution can be obtained easily by the graphical approach or by using the simplex method: $s_1 = 1.25$ and $s_2 = 0.25$ with corresponding payoff $\phi_1 = 1.5$ and $\phi_2 = 1$. ▼

It is easy to prove that if $\underline{s}^*$ is any Pareto optimal solution, then by the appropriate choice of the lower bounds, $\underline{s}^*$ can be obtained as an optimal solution of problem (14.7). A such choice is given by selecting any k as most important player and $\epsilon_l = \phi_l(\underline{s}^*)$ for $l \neq k$. So no Pareto optimal solution is lost by selecting this method.

14.3 The Weighting Method

This method assumes that the decision maker is able to define the relative importances of the players by assigning importance weights $c_1, c_2, \ldots, c_N$ such that all $c_i > 0$ and $\sum_{i=1}^{N} c_i = 1$. These weights can also be interpreted as what proportion of the 100% overall attention to the players is given to each player. This idea can be modeled by the single-objective optimization problem

$$\begin{aligned} &\text{maximize } \sum_{i=1}^{N} c_i \phi_i(\underline{s}) \\ &\text{subject to } \underline{s} \in S. \end{aligned} \tag{14.8}$$

It is clear that any optimum solution is also Pareto optimal.

Example 14.6 Assume that in problem (14.2) the weights are $c_1 = c_2 = \frac{1}{2}$. Then model (14.8) has the form

$$\begin{aligned} \text{maximize } & \frac{1}{2}\left(s_1 + s_2\right) + \frac{1}{2}\left(s_1 - s_2\right) = s_1 \\ \text{subject to } & s_1, s_2 \geqq 0 \\ & 3s_1 + s_2 \leqq 4 \\ & s_1 + 3s_2 \leqq 4. \end{aligned}$$

From Fig. 14.1 we can see that the largest value of s_1 occurs at point $\left(\frac{4}{3}, 0\right)$ so the optimal solution is $s_1 = \frac{4}{3}$ and $s_2 = 0$. ▼

In the cases of lexicographic and ϵ-constraints methods the payoff functions do not need to be transferable since different payoff functions are not compared. However the objective function of (14.8) can be interpreted only if the payoffs are transferable. If the original payoffs are not transferable, then we have to replace them by transformed payoff functions which become transferable. There are several ways to do so. One way is to find the normalized payoff functions;

$$\overline{\phi}_k(s) = \frac{\phi_k(s) - \phi_{k*}}{\phi_k^* - \phi_{k*}} \tag{14.9}$$

with

$$\phi_k^* = \max\left\{\phi_k | (\phi_1, \ldots, \phi_N) \in F\right\}$$

and

$$\phi_{k*} = \min\left\{\phi_k | (\phi_1, \ldots, \phi_N) \in F\right\}.$$

Example 14.7 Consider again problem (14.2). From Fig. 14.2 we can see that

$$\phi_{1*} = 0,\ \phi_1^* = 2,\ \phi_{2*} = -\frac{4}{3} \text{ and } \phi_2^* = \frac{4}{3},$$

so the normalized payoff functions are as follows:

$$\overline{\phi}_1 = \frac{s_1 + s_2 - 0}{2 - 0} = \frac{1}{2}\big(s_1 + s_2\big) \text{ and } \overline{\phi}_2 = \frac{s_1 - s_2 + \frac{4}{3}}{\frac{4}{3} + \frac{4}{3}} = \frac{3}{8}\big(s_1 - s_2\big) + \frac{1}{2}.$$

By selecting equal weights, the objective function of (14.8) becomes

$$\frac{1}{2}\Big(\frac{1}{2}(s_1 + s_2)\Big) + \frac{1}{2}\Big(\frac{3}{8}(s_1 - s_2) + \frac{1}{2}\Big)$$

which is equivalent with maximizing function $7s_1 + s_2$. The optimal solution on set S is: $s_1 = \frac{4}{3}$, $s_2 = 0$. ▼

Another way is when the players can define satisfaction (or utility) functions indicating their satisfaction with the different payoff values: $u_i\big(\phi_i(\underline{s})\big)$. Then the objective function of (14.8) becomes

$$\sum_{i=1}^{N} c_i u_i\big(\phi_i(\underline{s})\big). \tag{14.10}$$

The normalized payoff functions can also be interpreted as the application of a special satisfaction function, which gives 0 satisfaction at the worst payoff value ϕ_{i*}, and 100% satisfaction at the best payoff value ϕ_i^*, and is linear between these extreme values.

Example 14.8 By selecting the weighting method we might lose Pareto optimal solutions which is illustrated in Fig. 14.4, where point A is Pareto optimal and cannot be obtained as an optimal solution with positive weights. ▼

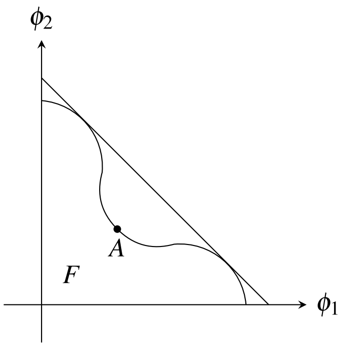

Fig. 14.4 Illustration of Example 13.8

Notice that F is nonconvex in the above example, and as the following theorem states, this problem cannot occur with convex payoff sets.

Theorem 14.1 *Assume F is convex and $\underline{s}^* \in S$ is Pareto optimal. Then there are nonnegative weights such that $\underline{s}^*$ is an optimal solution of problem (13.8).*

Proof Let

$$\overline{F} = \{\underline{\psi} \in \mathbb{R}^N | \text{ there exists } \underline{\phi} \in F \text{ such that } \underline{\psi} \leqq \underline{\phi}\}$$

which is also a convex set with the same Pareto optimal solutions as F. Define $\underline{\phi}^* = \big(\phi_1(\underline{s}^*), \ldots, \phi_N(\underline{s}^*)\big)$ which is clearly a boundary point. So the theorem of separating hyperplanes guarantees the existence of an N-element vector $\underline{c}$ such that

$$\underline{c}^T(\underline{\phi}^* - \underline{\phi}) \geqq 0 \text{ for all } \underline{\phi} \in \overline{F}. \tag{14.11}$$

We will next show that $\underline{c} \geqq \underline{0}$. Assume in contrary that $c_i < 0$ for some i. Then for arbitrary $\epsilon > 0$ and vector $\underline{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$,

$$\underline{\phi} = \underline{\phi}^* - \epsilon \underline{e}_i \in \overline{F}$$

where the ith component of e_i is unity, all others are zeros.

Furthermore

$$\underline{c}^T(\underline{\phi}^* - \underline{\phi}) = \underline{c}^T \epsilon \underline{e}_i = \epsilon c_i < 0$$

which is a clear contradiction. So $\underline{c} \geqq \underline{0}$ and (14.11) implies that

$$\underline{c}^T \underline{\phi}^* \geq \underline{c}^T \underline{\phi} \text{ for all } \underline{\phi} \in F$$

since $F \subseteq \overline{F}$. ∎

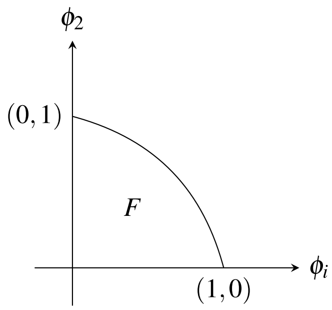

Fig. 14.5 Illustration of Example 14.9

Example 14.9 In Fig. 14.5 set F is given as

$$F = \left\{(\phi_1, \phi_2) | \phi_1 \geqq 0,\ \phi_2 \geqq 0,\ \phi_1^2 + \phi_2^2 \leqq 1\right\},$$

where point $(1, 0)$ is Pareto optimal. It can be obtained as an optimal solution of problem (14.8) with only $c_2 = 0$. ▼

Theorem 14.1 guarantees the existence of only nonnegative weights, as Example 14.9 shows, the positivity of the weights cannot be ensured in general. However if F is a polyhedron and all payoff functions are linear, that is, if the constraints defining the strategy sets and all payoffs are linear, then the positivity of the weights is guaranteed (Szidarovszky et al., 1986).

14.4 Distance-Based Methods

The ideally best solution would give maximum payoffs to all players which is the payoff vector $\underline{\phi}^* = (\phi_1^*, \ldots, \phi_N^*)$ with

$$\phi_k^* = \max\left\{\phi_k | (\phi_1, \ldots, \phi_N) \in F\right\}$$

as before. However this ideal payoff vector is usually infeasible. A logical choice is to find the payoff vector which is as close as possible to this ideal payoff vector. If $\rho(\underline{\phi}, \underline{\psi})$ is a distance function of N-element vectors, then this idea can be modeled as

$$\begin{aligned} &\text{minimize } \rho\left(\underline{\phi}(\underline{s}), \underline{\phi}^*\right) \\ &\text{subject to } \underline{s} \in S. \end{aligned} \tag{14.12}$$

In applications usually one of the Minkowski distances is used:

$$\rho_p(\underline{\phi}, \underline{\psi}) = \left\{ \sum_{i=1}^{N} c_i |\phi_i - \psi_i|^p \right\}^{1/p}$$

with positive weights c_i. If $p = 1$, then

$$\rho_1(\underline{\phi}, \underline{\psi}) = \sum_{i=1}^{N} c_i |\phi_i - \psi_i|,$$

if $p = 2$, then

$$\rho_2(\underline{\phi}, \underline{\psi}) = \left\{ \sum_{i=1}^{N} c_i |\phi_i - \psi_i|^2 \right\}^{1/2}$$

is the weighted Euclidean distance, and if $p = \infty$, then

$$\rho_\infty(\underline{\phi}, \underline{\psi}) = \max_i \left\{ c_i |\phi_i - \psi_i| \right\}$$

is the maximum distance.

Example 14.10 In the case of problem (14.2) the ideal point is $\underline{\phi}^* = \left(2, \frac{4}{3}\right)$, so with different choices of the distance function the objective function of (14.12) becomes:

$$\rho_1(\underline{\phi}, \underline{\phi}^*) = c_1|s_1 + s_2 - 2| + c_2\left|s_1 - s_2 - \frac{4}{3}\right|$$

with $p = 1$,

$$\rho_2(\underline{\phi}, \underline{\phi}^*) = \left\{ c_1|s_1 + s_2 - 2|^2 + c_2\left|s_1 - s_2 - \frac{4}{3}\right|^2 \right\}^{1/2}$$

with $p = 2$, and with $p = \infty$,

$$\rho_\infty(\underline{\phi}, \underline{\phi}^*) = \max \left\{ c_1|s_1 + s_2 - 2|; \; c_2\left|s_1 - s_2 - \frac{4}{3}\right| \right\}.$$

With $c_1 = c_2 = \frac{1}{2}$ the optimal solutions are

$$s_1 = \frac{4}{3}, \qquad s_2 = 0 \qquad \text{with} \quad p = 1;$$

$$s_1 = \frac{19}{15}, \qquad s_2 = \frac{3}{15} \qquad \text{with} \quad p = 2$$

and

$$s_1 = \frac{11}{9}, \qquad s_2 = \frac{3}{9} \qquad \text{with } p = \infty.$$

▼

The geometric distance

$$\rho_G(\underline{\phi}, \underline{\psi}) = \Pi_{i=1}^{N} \left|\phi_i - \psi_i\right|^{c_i}$$

is also used in rare cases. This definition does not satisfy the usual requirements any distance should satisfy. The distance of different vectors can be zero, such as with $c_1 = c_2 = 1$,

$$\rho\big((1,0),(1,2)\big) = |1-1| \cdot |0-2| = 0.$$

The triangle inequality can also be violated, as

$$\rho\big((1,0),(1,2)\big) + \rho\big((1,2),(2,2)\big) = 0 + 0 = 0 < \rho\big((1,0),(2,2)\big) = 1 \cdot 2 = 2$$

with $c_1 = c_2 = 1$. Notice that maximizing geometric distance is the same as finding the symmetric or the nonsymmetric Nash bargaining solution.

Another variant of distance based methods is to find the point with largest distance from the ideally worst point $\underline{\phi}_* = (\phi_{1*}, \ldots, \phi_{N*})$ with

$$\phi_{k*} = \min \left\{\phi_k | (\phi_1, \ldots, \phi_N) \in F\right\}.$$

In this case model (14.12) is modified as

$$\begin{aligned} &\text{maximize } \rho\big(\underline{\phi}(\underline{s}), \underline{\phi}_*\big) \\ &\text{subject to } \underline{s} \in S. \end{aligned} \tag{14.13}$$

Example 14.11 In the case of problem (14.2), $\underline{\phi}_* = \left(0, -\frac{4}{3}\right)$, since $\underline{\phi}_{1*} = 0$ and $\underline{\phi}_{2*} = -\frac{4}{3}$ as seen in Fig. 14.2. In all distances ρ_1, ρ_2 and ρ_∞ the same solution is obtained:

$$s_1 = \frac{4}{3}, \; s_2 = 0.$$

▼

14.5 Direction-Based Methods

The worst possible payoff vector $\underline{\phi}_*$ is unacceptable to the players, so the payoff values have to be improved in comparison to $\underline{\phi}_*$. So the decision maker selects a positive vector $\underline{v}$, as the direction of improvement and wants to increase the simultaneous payoff vector in the direction $\underline{v}$ as much as possible. Mathematically this idea can be rewritten as model:

$$\begin{aligned} &\text{maximize } t \\ &\text{subject to } \underline{\phi}_* + t\underline{v} - \underline{\phi}(\underline{s}) = \underline{0} \\ &\underline{s} \in S. \end{aligned} \tag{14.14}$$

Example 14.12 In the case of problem (14.2) we know that $\underline{\phi}_* = \left(0, -\frac{4}{3}\right)$. By selecting $\underline{v} = (1, 1)$, which improves the payoffs in equal speed, problem (14.14) can have the form

$$\begin{aligned} &\text{maximize } t \\ &\text{subject to } s_1, s_2 \geqq 0 \\ &3s_1 + s_2 \leqq 4 \\ &s_1 + 3s_2 \leqq 4 \\ &0 + t - s_1 - s_2 = 0 \\ &-\frac{4}{3} + t - s_1 + s_2 = 0 \end{aligned}$$

with optimal solution $s_1 = \frac{10}{9}$, $s_2 = \frac{6}{9}$ and $t = \frac{10}{9}$. ▼

There is no guarantee that the obtained solution is Pareto optimal.

Example 14.13 Consider Fig. 14.6, where the arc starting at point $\underline{\phi}_*$ can improve both payoffs in the direction $\underline{v}$ until point A, which is not Pareto optimal. ▼

Another variant of the method is the following. The ideal point $\underline{\phi}^*$ is usually infeasible, so the decision maker has to decrease the payoffs in order to get feasible solution. One way is to decrease the payoffs in a given direction $\underline{v} > \underline{0}$ starting at $\underline{\phi}^*$ until a feasible solution is obtained:

$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } \underline{\phi}^* - t\underline{v} - \underline{\phi}(\underline{s}) = \underline{0} \\ &\underline{s} \in S. \end{aligned} \tag{14.15}$$

Example 14.14 In the case of problem (14.2), $\underline{\phi}^* = \left(2, \frac{4}{3}\right)$, so with $\underline{v} = (1, 1)$, (14.15) can be rewritten as

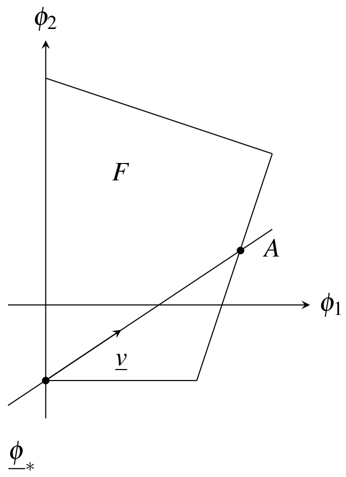

Fig. 14.6 Non-Pareto optimal solution

$$\begin{aligned}
&\text{minimize } t\\
&\text{subject to } s_1,\ s_2 \geqq 0\\
&\qquad 3s_1 + s_2 \leqq 4\\
&\qquad s_1 + 3s_2 \leqq 4\\
&\qquad 2 - t - s_1 - s_2 = 0\\
&\qquad \frac{4}{3} - t - s_1 + s_2 = 0.
\end{aligned}$$

The optimal solution is $s_1 = \frac{11}{9},\ s_2 = \frac{1}{3}$ and $t = \frac{4}{9}$. ▼

Notice that problem (14.14) coincides with the Kalai-Smorodinsky solution if the worst possible payoff vector is the disagreement point and the direction of improvement is $\underline{\phi}^* - \underline{\phi}_*$. Problem (14.15) is equivalent with the equal loss method, if $\underline{v}$ has identical components.

14.6 Pareto Games

Up to this point we assumed that each player had only one payoff function, and the players wanted to get for their payoff functions as high values as possible. However it is often the case, when the players themselves face multiple objectives. In such cases we have to combine game theoretical concepts with multiobjective optimization.

Let N denote the number of players. For Player k let S_k denote its strategy set and let $f_{k_1}, f_{k_2}, \ldots, f_{k_{i_k}}$ be its payoff functions. As we can see, this player usually has to consider the interior conflict between these payoff functions and in addition the exterior conflict with the other players.

The most common way of resolving the interior conflict is the application of multiobjective ideas when the multiple payoffs are comprised into a single payoff. After all players resolve their interior conflicts, then each of them will have a single payoff function and the Pareto game will become a usual N-person game.

Since the different payoff functions have different units and meanings, their comparisons need their transformations into a common measure such as satisfaction or utility functions as shown in Sect. 14.3. The existence of equilibria of Pareto games can be guaranteed in the following way. Assume for all players the strategy sets S_k are nonempty, closed, convex and bounded subsets of finite dimensional Euclidean spaces. Let $\phi_k$denote the single payoff function of Player k, as a result of resolving the conflict between its own payoffs. Let $x_1, x_2, \ldots, x_N$ denote the strategies of the players, $x_k \in S_k$, for all k. Assume that $\phi_k\,(x_1, x_2, \ldots, x_N)$ is continuous as an N-variable function and with fixed values of $x_j\ (j \neq k)$, ϕ_k is concave in x_k. Then the Nikaido-Isoda theorem (Theorem 5.4.) implies the existence of at least one Nash-equilibrium.

These conditions hold if for all players k,

(a) $\phi_k = \left(f_{k1}, f_{k2}, \ldots, f_{ki_k}\right)$ is continuous as an i_k variable function;
(b) all objective functions $f_{kj}\,(x_1, x_2, \ldots, x_N)$ are continuous as N-variable functions;
(c) $\phi_k = \left(f_{k1}\,(x_1, \ldots, x_N), \ldots, f_{ki_k}\,(x_1, \ldots, x_N)\right)$ is concave in x_k in addition to
(d) S_k is a nonempty, closed convex and bounded subset of a finite dimensional Euclidean space.

The most common way of constructing the single payoffs ϕ_k is the weighing method, when after normalizing the original objectives to $\bar{f}_{kj}$, the players consider the following function as their only payoff:

$$\phi_k = \sum_{j=1}^{i_k} c_{kj}\,\bar{f}_{kj} \tag{14.16}$$

where the coefficients $c_{kj} > 0$ show the relative importance of the objectives. If all $f_{kj}$are continuous and concave in x_k then the same holds for ϕ_k as well, so (a), (b) and (c) are satisfied.

The following example will illustrate these ideas.

Example 14.15 (Duopoly) Consider a single product duopoly. The two firms are the players who produce the same product and sell their outputs on a common market. Let x_1 and x_2 be the produced amounts and assume that the entire outputs of the firms are sold in the market.

Let $p\,(x_1 + x_2) = A - B\,(x_1 + x_2)$ denote the price function and $C_k\,(x_k) = c_k x_k + d_k$ the production cost of firm k. The production process also produces pollution, which is proportional to the production level, $\alpha_k x_k\quad(\alpha_k > 0)$.

The firms consider their pollution emission in addition to profit because they care about the public image influencing how the potential and actual customers view them.

Therefore they abate a certain proportion of the pollution $y_k \alpha_k x_k \quad (0 \le y_k \le 1)$, what they want to maximize. However abating pollution is costly, say $y_k \alpha_k x_k$ abated pollution costs an amount of $\beta_k (y_k \alpha_k x_k)$. So this additional cost has to be added to the production cost. The strategy of Player k is $(x_k, y_k)$where $x_k \in [0, L_k]$, $y_k \in [0, 1]$. Here L_k is the production capacity limit of firm k, $y_k = 0$ refers to no abatement and $y_k = 1$ means that all pollution is abated. The profit of this firm is

$$f_{k1}(x_1, y_1, x_2, y_2) = x_k (A - B(x_1 + x_2)) - (c_k x_k + d_k) - \beta_k (y_k \alpha_k x_k) \quad (14.17)$$

and the environmental objective is

$$f_{k2}(x_k, y_k) = \alpha_k x_k y_k - \alpha_l x_l y_l \; (l \neq k) \quad (14.18)$$

showing how better is this firm than the competitor in controlling the environment.

Let us first examine f_{k1}. Its maximum occurs at x_l =0 and y_k =0, and minimum occurs at $x_l = L_l$ and y_k =1. If x_l =0 and y_k =0 then (14.17) is reduced to the following:

$$x_k (A - Bx_k) - (c_k x_k + d_k).$$

Assuming interior optimum, the first order condition shows that

$$A - 2Bx_k - c_k = 0$$

or

$$x_k = \frac{A - c_k}{2B},$$

when

$$f_{kl}^{\max} = \frac{(A - c_k)^2}{4B} - d_k.$$

Minimum of f_{k1} occurs at $x_l = L_l$, and y_k =1, when f_{k1} becomes

$$x_k (A - Bx_k - BL_l) - (c_k x_k + d_k) - \beta_k \alpha_k x_k.$$

Since this is a concave function, its minimum occurs at an endpoint of the domain of x_k. At x_k =0 we have -d_k, and at $x_k = L_k$, we have

$$L_k (A - BL_k - BL_l) - (c_k L_k + d_k) - \beta_k \alpha_k L_k$$

so

$$f_{k1}^{\min} = \min \{-d_k; L_k (A - BL_k - BL_l - c_k - \beta_k \alpha_k) - d_k\}.$$

So the normalised objective f_{k1} becomes (see (14.9))

$$\bar{f}_{k1} = \frac{f_{k1} - f_{k1}^{\min}}{f_{k1}^{\max} - f_{k1}^{\min}}.$$

It is easy to see that $f_{k2}^{\max} = \alpha_k L_k$ and $f_{k2}^{\min} = -\alpha_l L_l$, so

$$\bar{f}_{k2} = \frac{f_{k2} + \alpha_l L_l}{\alpha_k L_k + \alpha_l L_l}.$$

Using the weighting method the resulting single payoff of Player k becomes

$$\phi_k = c_{k1}\bar{f}_{k1} + c_{k2}\bar{f}_{k2} = c_{k1}\frac{f_{k1} - f_{k1}^{\min}}{f_{k1}^{\max} - f_{k1}^{\min}} + c_{k2}\frac{f_{k2} + \alpha_l L_l}{\alpha_k L_k + \alpha_l L_l} = \bar{C}_{k1} f_{k1} + \bar{C}_{k2} f_{k2} + \bar{D}_k$$

with

$$\bar{C}_{k1} = \frac{c_{k1}}{f_{k1}^{\max} - f_{k1}^{\min}};\ \bar{C}_{k2} = \frac{c_{k2}}{\alpha_k L_k + \alpha_l L_l}$$

and

$$\bar{D}_k = -\frac{c_{k1} f_{k1}^{\min}}{f_{k1}^{\max} - f_{k1}^{\min}} + \frac{c_{k2}\alpha_l L_l}{\alpha_k L_k + \alpha_l L_l}$$

Similarly for Player $l (l \neq k)$,

$$\phi_l = \bar{C}_{l1} f_{l1} + \bar{C}_{l2} f_{l2} + \bar{D}_l$$

where $\bar{C}_{l1}$,$\bar{C}_{l2}$ and $\bar{D}_l$ are analogous to $\bar{C}_{k1}$,$\bar{C}_{k2}$ and $\bar{D}_k$.

So the problem is reduced to a 2-person game with strategy sets $S_1 = \{(x_1, y_1)\,|0 \leq x_1 \leq L_1, 0 \leq y_1 \leq 1\}$, $S_2 = \{(x_2, y_2)\,|0 \leq x_2 \leq L_2, 0 \leq y_2 \leq 1\}$ and payoff functions ϕ_1 and ϕ_2.

Chapter 15
Social Choice

There are decision problems where the consequences of the choices of the different alternatives cannot be quantified. This is the case, for example, in considering environmental issues like esthetics. In such cases no numerical payoff functions are given, only the rankings of the alternatives are possible.

Let M denote the number of alternatives which are ranked by the N players by rankings $1, 2, \ldots, M$, where 1 is given to the most preferred alternative, 2 is given to the second most preferred alternative, and so on, and finally M is given to the least preferred alternative. So the data are given in an $N \times M$ matrix, where the rows correspond to the players and the columns to the alternatives. Each row is a permutation of the numbers $1, 2, \ldots, M$.

Example 15.1 Assume $N = 5$ and $M = 4$. A possible data set is given in Table 15.1. ▼

There is no alternative which is best for all players, so instead of looking for an overall best solution a mutually acceptable solution has to be found. Depending on the meaning of "mutual acceptance" several methods can be offered (Taylor, 1995; Bonner, 1986). In this chapter some of the most frequently used methods are introduced and used.

15.1 Methods with Symmetric Players

Notice that in Table 15.1 the players are considered equal, no ordering or importance is given among the players.

The most simple method is *Plurality voting*, in which each alternative gets as many votes as the number of players who gave best ranking 1 to it. The alternative with the most votes is the solution. In the case of Example 15.1, alternatives 1, 3 and

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_15

Table 15.1 Data of Example 15.1

	Alternatives			
	1	2	3	4
Player 1	2	1	3	4
Player 2	1	3	2	4
Player 3	4	1	3	2
Player 4	2	4	1	3
Player 5	4	3	2	1

4 received 1 vote, while alternative 2 got 2 votes, so it is the solution. The formulation of this method is as follows.

Let r_{ij} denote the (i, j) element of the data matrix, when r_{ij} denotes the ranking of alternative j by player i and define

$$\triangle_{ij} = \begin{cases} 1 & \text{if } r_{ij} = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{15.1}$$

then the number of votes to alternative j is given as

$$n_j = \sum_{i=1}^{N} \triangle_{ij} \tag{15.2}$$

and alternative j_0 is the solution if

$$n_{j_0} = \max\{n_1, \ldots, n_M\}. \tag{15.3}$$

The large disadvantage of this method is the fact that it considers only best rankings and lower rankings are not taken into account at all.

This disadvantage is eliminated by *Borda counts*. For each alternative calculate the Borda counts as

$$B_j = \sum_{i=1}^{N} r_{ij}. \tag{15.4}$$

The alternative with the smallest Borda count is the choice. In Example 15.1,

$$B_1 = 13,\ B_2 = 12,\ B_3 = 11 \text{ and } B_4 = 14.$$

Since B_3 is the smallest, alternative 3 is the choice.

The *Hare system* is based on the successive deletions of less favorable alternatives. This method consists of the following steps.

Table 15.2 Reduced data set by eliminating alternative 1

	Alternatives		
	2	3	4
Player 1	1	2	3
Player 2	2	1	3
Player 3	1	3	2
Player 4	3	1	2
Player 5	3	2	1

Table 15.3 Second reduced table by eliminating alternative 4

	Alternatives	
	3	4
Player 1	1	2
Player 2	2	1
Player 3	1	2
Player 4	2	1
Player 5	2	1

Calculate the number of votes n_j for each alternative. If an alternative has more than half of the votes, then it is the choice and procedure terminates. Otherwise delete an alternative with the least number of votes from the table, which has to be adjusted accordingly. If alternative j^* is deleted, then

$$r_{ij}^{new} = \begin{cases} r_{ij} - 1 & \text{if } r_{ij} > r_{ij*} \\ r_{ij} & \text{if } r_{ij} < r_{ij*}, \end{cases}$$

and go back to the beginning of the procedure, which continues until an alternative gets more than half of the votes.

In the case of Example 15.1 we can delete only one from alternatives 1, 3 and 4. If alternative 1 is deleted, then the new table is given in Table 15.2, from which we have $n_2 = n_3 = 2$ and $n_4 = 1$, so alternative 4 has to be deleted next. The resulted table is given in Table 15.3, from which we see that $n_2 = 2$ and $n_3 = 3$. Since $3 > 2.5$, alternative 3 is the choice.

If alternative 3 is deleted instead of alternative 1, then the reduced table is shown in Table 15.4, where $n_1 = n_2 = 2$ and $n_4 = 1$, so alternative 4 is deleted resulting in Table 15.5, from which we have $n_1 = 2$ and $n_2 = 3$, so alternative 2 is the choice.

If alternative 4 is eliminated in the first step, then the resulting table is the one shown in Table 15.6. Since $n_1 = 1$ and $n_2 = n_3 = 2$, alternative 1 is deleted in the next step. The second reduced table is shown in Table 15.7, in which $n_2 = 2$ and $n_3 = 3$, so alternative 3 is the choice.

Table 15.4 Reduced data set by eliminating alternative 3

	Alternatives		
	1	2	4
Player 1	2	1	3
Player 2	1	2	3
Player 3	3	1	2
Player 4	1	3	2
Player 5	3	2	1

Table 15.5 Second reduced table by eliminating alternative 4

	Alternatives	
	1	2
Player 1	2	1
Player 2	1	2
Player 3	2	1
Player 4	1	2
Player 5	2	1

Table 15.6 Reduced data set by eliminating alternative 4

	Alternatives		
	1	2	3
Player 1	2	1	3
Player 2	1	3	2
Player 3	3	1	2
Player 4	2	3	1
Player 5	3	2	1

The example clearly shows that the final result may depend on the choice of the eliminated alternative if more than one have the least number of votes. This is a huge disadvantage of this method.

Pairwise comparisons is a very popular approach. It has two major variants. For each pair (j_1, j_2) of alternatives define

$$N(j_1, j_2) = \text{number of players } i \text{ such that } r_{ij_1} < r_{ij_2}$$

which gives the number of players who consider alternative j_1 better than j_2. If $N(j_1, j_2) > \frac{N}{2}$, then alternative j_1 is considered better than j_2. If $N(j_1, j_2) = \frac{N}{2}$, then the two alternatives are equally preferred.

Table 15.7 Second reduced table by eliminating alternative 1

	Alternatives	
	2	3
Player 1	1	2
Player 2	2	1
Player 3	1	2
Player 4	2	1
Player 5	2	1

In the first version the players agree in an agenda, which is an order of the alternatives, $j^{(1)}, j^{(2)}, \ldots, j^{(M)}$. First alternatives $j^{(1)}$ and $j^{(2)}$ are compared. The better alternative is then compared to $j^{(3)}$, the worse is eliminated. The winner is then compared to $j^{(4)}$, and so on. The final winner is the choice.

Assume in Example 15.1 that the agenda is $1, 2, 3, 4$. First alternatives 1 and 2 are compared. Since $N(1, 2) = 2$ and $N(2, 1) = 3$, alternative 2 is the winner. In comparing alternatives 2 and 3 we have $N(2, 3) = 2$ and $N(3, 2) = 3$, so alternative 3 is the winner. The final comparison is that of alternatives 3 and 4. Here $N(3, 4) = 3$ and $N(4, 3) = 2$, so alternative 3 is the final winner.

The other version is to compare all alternative pairs and draw conclusions from a comparison graph. In our case

$$
\begin{aligned}
N(1, 2) &= 2, & N(2, 1) &= 3 \\
N(1, 3) &= 2, & N(3, 1) &= 3 \\
N(1, 4) &= 3, & N(4, 1) &= 2 \\
N(2, 3) &= 2, & N(3, 2) &= 3 \\
N(2, 4) &= 3, & N(4, 2) &= 2 \\
N(3, 4) &= 3, & N(4, 3) &= 2
\end{aligned}
$$

resulting in the preference graph of Fig. 15.1. An arrow is given from alternative j_1 to j_2 if j_2 is better than j_1, that is, we want to move from j_1 to j_2. Clearly alternative 3 is the overall best, since it is better than all other alternatives.

The most trivial solution is *Dictatorship*, when one player is dictator, and his best choice is accepted as the social choice. In the case when player 1 is dictator in Table 14.1, then alternative 2 is the choice. If player 2 is the dictator, then alternative 1 is the choice, and so on.

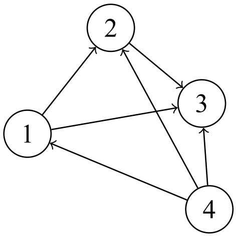

Fig. 15.1 Preference graph

Table 15.8 Data of Example 15.2

	Alternatives				Powersof the players
	1	2	3	4	
Player 1	2	1	3	4	1
Player 2	1	3	2	4	1
Player 3	4	1	3	2	2
Player 4	2	4	1	3	3
Player 5	4	3	2	1	2

15.2 Methods with Powers of Players

In the previous section equal players were assumed, which is not always the case.

Example 15.2 Table 15.8 is almost the same as Table 15.1, the only difference is the added last column containing the powers of the players. ▼

All methods introduced in the previous section can be used with minor modifications.

In the case of *Plurality voting* Eq. 15.2 is modified as

$$n_j = \sum_{i=1}^{N} \triangle_{ij} w_i \tag{15.5}$$

where w_i is the power of player i. That is, best rankings are counted with multiplicities according to the powers of the players. The choice is selected again from (15.3) . In our case $n_1 = 1, n_2 = n_3 = 3, n_4 = 2$, so alternatives 2 and 3 are equally best.

In computing the *Borda counts* equation (15.4) is changed to

$$B_j = \sum_{i=1}^{N} r_{ij} w_i \tag{15.6}$$

where rankings are multiplied by the powers of the players. In the case of Table 15.8,

Table 15.9 Reduced table by eliminating alternative 1

	Alternatives			Powers of the players
	2	3	4	
Player 1	1	2	3	1
Player 2	2	1	3	1
Player 3	1	3	2	2
Player 4	3	1	2	3
Player 5	3	2	1	2

Table 15.10 Second reduced table by eliminating alternative 4

	Alternatives		Powers of the players
	2	3	
Player 1	1	2	1
Player 2	2	1	1
Player 3	1	2	2
Player 4	2	1	3
Player 5	2	1	2

$$B_1 = 25,\ B_2 = 24,\ B_3 = 18 \ \text{ and } \ B_4 = 23$$

and since B_3 is the smallest, alternative 3 is the choice.

In applying *Hare-system* notice first that n_1 is the smallest from the alternatives, so it has to be deleted first. Table 15.9 shows the reduced table. Notice that from this table we have $n_2 = 3$, $n_3 = 4$ and $n_4 = 2$, so alternative 4 has to be then deleted resulting in Table 15.10, where $n_2 = 3$ and $n_3 = 6$ showing that alternative 3 is the solution.

In applying *Pairwise comparisons* the only change is a slightly different definition of $N(j_1, j_2)$:

$N(j_1, j_2) =$ sum of powers of players i such that $r_{ij_1} < r_{ij_2}$

and both versions can be used without further modification.

If the agenda is again 1, 2, 3, 4 then we first compare alternatives 1 and 2. Since $N(1, 2) = 4$ and $N(2, 1) = 5$, alternative 2 is the winner, which is then compared to alternative 3. Then we have $N(2, 3) = 3$ and $N(3, 2) = 6$ showing that 3 is the winner. And finally, comparing alternatives 3 and 4 we see that $N(3, 4) = 5$, $N(4, 3) = 4$ implying that alternative 3 becomes again the overall winner.

If all pairs of alternatives are compared, then

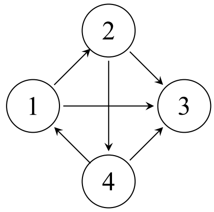

Fig. 15.2 Preference graph with weighted players

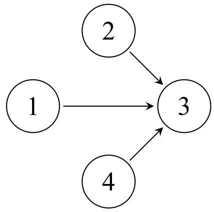

Fig. 15.3 Reduced preference graph

$$\begin{aligned} N(1,2) &= 4 \\ N(1,3) &= 2 \\ N(1,4) &= 5 \\ N(2,3) &= 3 \\ N(2,4) &= 4 \\ N(3,4) &= 5. \end{aligned}$$

Note that $N(j_2, j_1) = \sum_{i=1}^{N} w_i - N(j_1, j_2)$, and we consider alternative j_1 as overall better than j_2, if $N(j_1, j_2) > \frac{1}{2}\sum_{i=1}^{N} w_i$ which is denoted as $j_1 \succ j_2$. If $N(j_1, j_2) = \frac{1}{2}\sum_{i=1}^{N} w_i$ then the two alternatives are considered equal: $j_1 \sim j_2$. In our case $2 \succ 1$, $3 \succ 1$, $1 \succ 4$, $3 \succ 2$, $4 \succ 2$ and $3 \succ 4$.

The preference graph is given in Fig. 15.2.

Again alternative 3 becomes the winner since it is better than any other alternative. There is however a problem with this graph. Consider alternatives 1, 2 and 4. Their preferences are contradictory, since $2 \succ 1$ and $1 \succ 4$ so by transitivity it should follow that $2 \succ 4$. However $4 \succ 2$ which is a clear contradiction. If such cycle appears in the graph then the best is to delete the entire cycle to get the reduced graph shown in Fig. 15.3.

Clearly, alternative 3 remains the best.

Dictatorship can be applied without any modification, the player with largest power can be considered as the dictator. In our case it is player 4, his choice, alternative 3 is therefore the solution.

Before closing this chapter some comments are in order. Notice first that the methods of Section 15.2 are reduced to those of Section 15.1 if we select $w_1 = w_2 =$

$\cdots = w_N = 1$. The methods introduced in this chapter assume that the number of alternatives in finite. In the case of infinitely many alternatives let S denote the set of alternatives and assume that each player k defines a real-valued function $r_k(\underline{x}), \underline{x} \in S$, as the ranking of alternative $\underline{x}$. The best choice of player k is clearly arg $\min_{\underline{x} \in S} r_k(\underline{x})$. In the case of infinitely many alternatives there is a slight chance that more than one players select the same best choice. Otherwise plurality voting is the same as dictatorship, when the player with the largest power is the dictator. Assume that functions are normalized in a way that their common minimal value is unity. Assume in addition that each player selects an $r_k^* > 1$ value such that alternatives with $r_k(\underline{x}) < r_k^*$ are still considered "close to best". Then for each player we can define his "close to best set" as

$$C_k = \left\{\underline{x} | r_k(\underline{x}) < r_k^*\right\} \tag{15.7}$$

which might overlap for different players. Then the number of votes alternative $\underline{x}$ gets is defined as

$$n_{\underline{x}} = \text{ number of players } k \text{ such that } \underline{x} \in C_k,$$

and the solution is alternative $\underline{x}$ with the largest $n_{\underline{x}}$ value.

The Borda count is the same as the weighting method of multiobjective optimization. Let

$$B(\underline{x}) = \sum_{i=1}^{N} r_i(\underline{x}) w_i, \tag{15.8}$$

and the social choice is

$$\underline{x}^* = \arg\min_{\underline{x} \in S} B(\underline{x}). \tag{15.9}$$

The Hare systems and Pairwise comparisons heavily depend on the finiteness of the alternative set, so it is very complicated to extend them to infinite alternative sets. We can always use a multiobjective optimization method with objective functions $r_k(\underline{x})$ and weights w_k when all objectives are minimized.

Chapter 16
Case Studies and Applications of Static Games

In Chaps. 2 and 3 we have already introduced simple examples of games modeling situations in competition, social issues, tax evasion, waste management, advertisement, homeland security, elections, military, economics, location for a business, market sharing, duel, espionage and auctions among others. In this chapter some additional applications of game theory are outlined.

16.1 A Salesman's Dilemma

Consider a salesman who is selling an equipment to a customer. The equipment has three components, each of them can be defective or in working order. After delivering the equipment the customer has to pay α dollars to the salesman, unless at least one of the components turns out to be defective. In this case instead of receiving the price, the salesman has to pay β dollars penalty to the customer. In order to decrease the chance of losing money, the salesman is able to check one or more components before delivery, which costs him γ dollars for each component being checked. So the salesman has 4 strategies depending on the number of checked components, and therefore his strategy set is $S_1 = \{0, 1, 2, 3\}$. If he decides to check more than one components and in the process he discovers a defective component, then he does not need to continue to check more.

This situation can be modeled as a "game against nature", when the nature is the equipment which also has 4 strategies, the possible number of defective components. So its strategy set is $S_2 = \{0, 1, 2, 3\}$. The pessimistic salesman considers this two-person game as a zero-sum game. By assuming that the components may fail with equal probability and the salesman selects the components to be checked with equal probability, the payoff matrix of the salesman (who is player 1) is given in Table 16.1.

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_16

Table 16.1 Payoff matrix of player 1

S \ E	0	1	2	3
0	α	$-\beta$	$-\beta$	$-\beta$
1	$\alpha-\gamma$	$-\frac{2}{3}\beta-\gamma$	$-\frac{1}{3}\beta-\gamma$	$-\gamma$
2	$\alpha-2\gamma$	$-\frac{1}{3}\beta-\frac{5}{3}\gamma$	$-\frac{4}{3}\gamma$	$-\gamma$
3	$\alpha-3\gamma$	-2γ	$-\frac{4}{3}\gamma$	$-\gamma$

Let f_{ij} denote the (i, j) element of the payoff matrix with $0 \leqq i, j \leqq 3$. If the equipment has no defective component (when $j = 0$), then the salesman certainly will receive the price of the equipment and his cost is γ times the number of checked equipments. So

$$f_{i0} = \alpha - i\gamma \quad i = 0, 1, 2, 3.$$

If all components are defective, then the salesman pays the penalty only if he does not check any component, otherwise at the first checking he realizes that the equipment is defective.

Therefore

$$f_{i3} = \begin{cases} -\beta & \text{if } i = 0 \\ -\gamma & \text{if } i \geqq 1. \end{cases}$$

Similarly, if $i = 0$, that is, when the salesman does not check any of the components, then he gets the price if $j = 0$ and pays the penalty otherwise:

$$f_{oj} = \begin{cases} \alpha & \text{if } j = 0 \\ -\beta & \text{if } j \geqq 1. \end{cases}$$

The other matrix elements can be obtained by simple probabilistic considerations.

In the case of f_{11} only one component is defective, which can be discovered with probability $\frac{1}{3}$, so with probability $\frac{2}{3}$ the salesman will deliver the equipment and pay the penalty. In addition to this, checking the component costs him γ dollars. In the case of f_{12}, the salesman checks 1 component, so he has $\frac{1}{3}$ probability to miss one of the two defective components implying that $f_{12} = -\frac{1}{3}\beta - \gamma$. In the case of f_{21} we can reason as follows. One component is defective, so there are three possibilities: (a) the defective component is not discovered with probability $\frac{1}{3}$, (b) the defective component is discovered during the first checking with probability $\frac{1}{3}$; (c) it is discovered during the second checking with probability $\frac{1}{3}$. Therefore

$$f_{21} = \frac{1}{3}\big(-\beta - 2\gamma\big) + \frac{1}{3}\big(-\gamma\big) + \frac{1}{3}\big(-2\gamma\big) = -\frac{1}{3}\beta - \frac{5}{3}\gamma.$$

In the case of f_{22} two components are defective and since the salesman checks 2 components, there is no way he will believe that the equipment is good. He can discover a defective component in either the first or in the second checking. The corresponding probabilities are $\frac{2}{3}$ and $\frac{1}{3}$ with associated costs $-\gamma$ and -2γ, so

$$f_{22} = \frac{2}{3}(-\gamma) + \frac{1}{3}(-2\gamma) = -\frac{4}{3}\gamma.$$

If $i = 3$, then the salesman always finds detective component unless $j = 0$. If only one component is detective, then he can find it either in the first, or in second, or in third checking. So

$$f_{31} = \frac{1}{3}(-\gamma) + \frac{2}{3}\left[\frac{1}{2}(-2\gamma) + \frac{1}{2}(-3\gamma)\right] = -2\gamma.$$

If two components are detective, then it can be discovered either in the first or in the second checking, so

$$f_{32} = \frac{2}{3}(-\gamma) + \frac{1}{3}(-2\gamma) = -\frac{4}{3}\gamma.$$

The Nash equilibrium is a strategy pair (i, j) such that the f_{ij} element is largest in its column and smallest in its row.

The first row has three smallest elements: f_{01}, f_{02} and f_{03}. f_{01} is largest in its column if

$$\begin{aligned}
-\beta &\geqq -\frac{2}{3}\beta - \gamma \\
-\beta &\geq -\frac{1}{3}\beta - \frac{5}{3}\gamma \\
-\beta &\geq -2\gamma.
\end{aligned}$$

Simple algebra shows that these relations hold if and only if $\beta \leqq 2\gamma$. f_{02} is largest in its column, if

$$\begin{aligned}
2 - \beta &\geq -\frac{1}{3}\beta - \gamma \\
-\beta &\geq -\frac{4}{3}\gamma
\end{aligned}$$

which is the case if and only if $\beta \leq \frac{4}{3}\gamma$.

Element f_{03} is largest in its column, if

$$-\beta \geq -\gamma$$

that is, when $\beta \leqq \gamma$.

The smallest element in the second row is f_{11} which is largest in its column if

$$-\frac{2}{3}\beta - \gamma \geqq -\beta$$
$$-\frac{2}{3}\beta - \gamma \geq -\frac{1}{3}\beta - \frac{5}{3}\gamma$$
$$-\frac{2}{3}\beta - \gamma \geq -2\gamma$$

which cannot occur since these inequalities are contradictory.

In the third row we can have f_{20} and f_{21} as smallest elements, however f_{20} cannot be the largest in its column, so only f_{21} can provide equilibrium. It is the case if

$$-\frac{1}{3}\beta - \frac{5}{3}\gamma \leqq \alpha - 2\gamma$$
$$-\frac{1}{3}\beta - \frac{5}{3}\gamma \geq -\beta$$
$$-\frac{1}{3}\beta - \frac{5}{3}\gamma \geq -\frac{2}{3}\beta - \gamma$$
$$-\frac{1}{3}\beta - \frac{5}{3}\gamma \geq -2\gamma$$

which relations are also contradictory.

In the last row both f_{30} and f_{31} can be the smallest. Notice that f_{30} cannot be the largest in its column, so only f_{31} is a potential equilibrium, which is the case if

$$-2\gamma \leqq \alpha - 3\gamma$$
$$-2\gamma \geq -\beta$$
$$-2\gamma \geq -\frac{2}{3}\beta - \gamma$$
$$-2\gamma \geq -\frac{1}{3}\beta - \frac{5}{3}\gamma$$

and these relations hold if $\beta \geq 2\gamma$ and $\gamma \leqq \alpha$.

Figure 16.1 illustrates the different cases and the equilibria.

In region A the unique equilibrium is $(3, 0)$ if $\gamma \leqq \alpha$, and if $\gamma > \alpha$, then no equilibrium exists. In region B there is a unique equilibrium $(0, 1)$, in region C we have two equilibria $(0, 1)$ and $(0, 2)$, and finally in region D there are three equilibria, $(0, 1)$, $(0, 2)$ and $(0, 3)$. It is interesting to notice that the number of equilibria and the equilibria themselves depend on model parameters. We had a similar continuous case in Example 3.3, when the different possibilities of Cournot oligopoly were discussed.

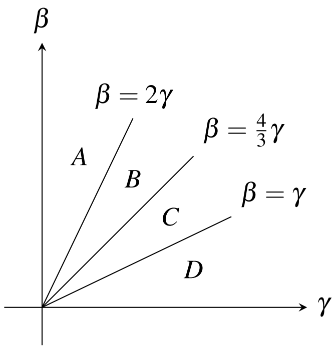

Fig. 16.1 Different cases in equilibrium analysis

16.2 Oligopoly in Water Management

The oligopoly game was introduced as a real economic situation. In this section we will show how the oligopoly game can be applied in water resources management.
1. Consider a region where local companies (industry, agriculture, etc.) produce waste water which has to be cleaned in a common plant. Let N denote the number of companies, and s_k the amount of waste water produced by company k, so the total treated water amount in the treatment plant is $s = \sum_{k=1}^{N} s_k$. The annual benefit for each company (for example, by avoiding penalty) depends on the amount of its waste water to be treated, $B_k(s_k)$, and the total annual cost of the entire plant depends on only s, $A(s)$. It is assumed that the different firms contribute to the total cost in proportion to the amounts of their waste water being treated in the plant. So the net benefit for company k is given as

$$\Pi_k = B_k(s_k) - \frac{s_k}{s} A(s). \tag{16.1}$$

If we introduce the notation

$$p(s) = \frac{-1}{s} A(s) \text{ and } C_k(s_k) = -B_k(s_k)$$

then the payoff function becomes the same as given in (3.26).
2. Consider next N farms who want to construct a joint irrigation system and share costs in proportion to their water usage. This situation can be modeled similarly to the previous case. Let s_k denote the water usage of farm k, then its benefit from irrigation is $B_k(s_k)$ and its contribution to the joint operation is $\frac{s_k}{s} A(s)$, where $s = \sum_{k=1}^{N} s_k$ and $A(s)$ is the operating cost of the system. Then the net benefit of farm k is given in (16.1) leading to the same model as before.
3. In river basin planning several demands have to be met, water supply, flood protection, irrigation, water quality control, and recreation. The development cost of the

system should be distributed among the various beneficiaries. The storage volume of the reservoir system is $\sum s_i$, where s_i is the storage volume utilized by user i. The development cost is given as $C\big(\sum s_i\big)$ which is divided up by the users in proportion to their water usage. The benefit of user i is $B_i(s_i)$ resulting from utilizing a volume s_i of stored water. It is easy to see that the net benefit of the users have the same form as (16.1), so this situation is also equivalent to an oligopoly.

The interested reader may find more details of these models in Bogardi and Szidarovszky (1976).

16.3 A Forestry Management Problem

In the Northern part of the state of Arizona there was a need to find an appropriate forest treatment strategy. Four alternative methods were considered, which were tried out in four different watersheds to assess the consequences of the different treatment strategies.

Alternative 1. On a completely *clear cut* watershed of 184 ha all merchantable poles and saw timber were removed, the remaining noncommercial wood felled. The slash and debris were machine winddrowed to trap and retain snow. In addition, it reduced evapotranspiration losses and increased surface drainage efficiency. The trees then were allowed to sprout and grow.

Alternative 2. *Uniform thinning* was used in a 121 ha watershed where 75% of the initial $\frac{30m^2}{ha}$ of basal was removed, even-aged groups of trees were left with an average basal area of $\frac{7m^2}{ha}$. All slash was also winddrowed.

Alternative 3. In the third watershed of 546 ha an *irregular strip cut* was applied, where all merchantable wood was removed within irregular 20m wide strips and the remaining non-merchantable trees felled. The forest overstory in the intervening leaf strips averaging 35m width were reduced to 25% of the basal area. This combined treatment resulted in almost 60% reduction in the basal area. Slash was piled and burned in the cleared strips.

Alternative 4. The fourth watershed of 351 ha was chosen as the *control*, where minimal managenal input was used for custodial management. This watershed was the basis, to which the other treatment strategies were compared.

Six interest groups were evaluating the consequences of the different treatment strategies and as the result, all groups provided the rankings of the alternatives. The data are given in Table 16.2. In order to find a mutually acceptable alternative, social choice methods were used.

In applying *Plurality voting* notice that the number of votes for the four alternatives are

$$n_1 = 2, n_2 = 0, n_3 = 1, n_4 = 3$$

so A_4 (control) is the solution.

Table 16.2 Rankings of the alternatives

		A_1 Clear Cut	A_2 Uniform thinning	A_3 Irregular strip cut	A_4 Control
P_1	Water users	1	2	3	4
P_2	Wildlife advocates	4	2	1	3
P_3	Livestock producers	1	2	3	4
P_4	Wood producers	3	4	2	1
P_5	Environmentalists	4	3	2	1
P_6	Managers	4	2	3	1

Table 16.3 Reduced table for Hare system

	A_1	A_3	A_4
P_1	1	2	3
P_2	3	1	2
P_3	1	2	3
P_4	3	2	1
P_5	3	2	1
P_6	3	2	1

The *Borda counts* of the alternatives are as follows:

$$B_1 = 17,\ B_2 = 15,\ B_3 = 14,\ B_4 = 14.$$

So alternative A_3(irregular strip cut) and A_4(control) are equally the best.

Since alternative 2 has the least number of votes, in applying the *Hare system* it has to be first deleted. The reduced table is given as Table 16.3, where the new numbers of votes are

$$n_1 = 2,\ n_3 = 1 \text{ and } n_4 = 3.$$

Since alternative A_3 has the least number of votes, it has to be next eliminated resulting in Table 16.4.

In this new table $n_1 = 2$ and $n_4 = 4$, so alternative A_4(control) is the final choice.

In applying *Pair-wise comparisons*, we have the following preferences: so we have only two preference orders: $3 \succ 1$ and $4 \succ 1$. So alternative 1 cannot be considered as the solution.

Table 16.4 Further reduced table for Hare system

	A_1	A_4
P_1	1	2
P_2	2	1
P_3	1	2
P_4	2	1
P_5	2	1
P_6	2	1

Table 16.5 Reduced table in pair-wise comparisons

	A_2	A_3	A_4
P_1	1	2	3
P_2	2	1	3
P_3	1	2	3
P_4	3	2	1
P_5	3	2	1
P_6	2	3	1

$$N(1,2) = 3$$
$$N(1,3) = 2$$
$$N(1,4) = 2$$
$$N(2,3) = 3$$
$$N(2,4) = 3$$
$$N(3,4) = 3$$

After A_1 is eliminated, the reduced table is given in Table 16.5. In this table we have

$$N(1,2) = 3$$
$$N(1,3) = 3$$
$$N(2,3) = 3$$

showing that there is no difference between these alternatives based on pair-wise comparisons.

Finally we note that the problem outlined above is discussed in more details in Eskandari et al. (1995).

16.4 International Fishing

Consider a common fishing ground where N countries are engaged in commercial fishing. Let X be the fish stock in the fishing ground. In each country k there are n_k firms harvesting fish, the fishing effort of firm j from country k is denoted by e_{kj}. Thus the total fishing effort of country k is given as $E_k = \sum_{j=1}^{n_k} e_{kj}$. The profit of firm j from country k can be obtained as the difference of its revenue and cost. The harvest rate of country k is assumed to be $S_k = E_k q_k X$, so the profit of firm j is

$$\pi_{kj} = f_k(S_k) q_k e_{kj} X - c_{kj} e_{kj} \tag{16.2}$$

where f_k is the inverse demand function in country k for the harvested fish, q_k is the common catchability coefficient of the firms in country k, and c_{kj} is the unit cost of fishing effort of firm j of country k.

Assume first that the firms of each country k form an n_k-firm oligopoly and select Nash-equilibrium inside the country. With given fish stock X, the countries are independent of each other. Similarly to the assumptions given in Example 3.14 for Cournot oligopolies we assume that for all k and j,

$$\begin{aligned} &\text{(a)} \quad f_k'(S_k) < 0 \\ &\text{(b)} \quad f_k''(S_k) q_k e_{kj} X + f_k'(S_k) \leqq 0. \end{aligned}$$

Notice that (a) implies that the price decreases if the fish supply increases. Since

$$\frac{\partial \pi_{kj}}{\partial e_{kj}} = f_k'(S_k) q_k^2 e_{kj} X^2 + f_k(S_k) q_k X - c_{kj} \tag{16.3}$$

and

$$\frac{\partial^2 \pi_{kj}}{\partial e_{kj}^2} = f_k''(S_k) q_k^3 e_{kj} X^3 + 2 f_k'(S_k) q_k^2 X^2, \tag{16.4}$$

assumption (b) guarantees that π_{kj} is strictly concave in e_{kj}. It is a natural assumption that the fishing effort of each firm is limited, so we have $0 \leqq e_{kj} \leqq L_{kj}$ for all k and j. Notice that the revenue of firm j in county k can be rewritten as

$$\left[f_k(E_k q_k X) q_k X \right] e_{kj},$$

so the bracketed expression can be imagined as the E_k-dependent price function, since the other quantities q_k and X are given. As a consequence of Example 3.14 there is a unique Nash equilibrium among the firms of country k.

For the sake of simplicity assume that equilibrium is interior, then for all j,

$$f'_k(S_k)q_k^2 e_{kj} X^2 + f_k(S_k)q_k X - c_{kj} = 0. \tag{16.5}$$

By adding up this relation for all values of j,

$$f'_k(E_k q_k X)q_k^2 E_k X^2 + f_k(E_k q_k X)q_k X n_k - \sum_{j=1}^{n_k} c_{kj} = 0, \tag{16.6}$$

from which E_k can be determined, and individual fishing efforts of the firms are obtained from (16.5).

Assume next that the firms of each country form a grand coalition and want to maximize their total profit. In this case the profit of the coalition becomes

$$\pi_k^c = f_k(E_k q_k X)q_k E_k X - c_k E_k \tag{16.7}$$

where we assume for the sake of simplicity that the firms of country k have identical unit cost, $c_{kj} \equiv c_k$. Notice that

$$\frac{\partial \pi_k^c}{\partial E_k} = f_k^{'}(E_k q_k X)q_k^2 E_k X^2 + f_k(E_k q_k X)q_k X - c_k$$

and

$$\frac{\partial^2 \pi_k^c}{\partial E_k^2} = f_k^{''}(E_k q_k X)q_k^3 E_k X^3 + 2 f_k^{'}(E_k q_k X)q_k^2 X^2 < 0$$

by assumption (b) with $e_{kj} = E_k$. So π_k^c is strictly concave in E_k. Since $0 \leqq E_k \leqq \sum_{j=1}^{n_k} L_{kj}$, the domain for E_k is a compact set and function π_k^c is continuous, there is a unique maximal value for E_k. If it is interior, then it satisfies equation

$$f_k^{'}(E_k q_k X)q_k^2 E_k X^2 + f_k(E_k q_k X)q_k X - c_k = 0. \tag{16.8}$$

In order to determine a cooperative solution, the characteristic function of the cooperative game has to be determined. Assume that some firms from country k form a coalition C_k. Then the coalition's profit is given as

$$\pi_k^{c_k} = f_k\big((E_k^{c_k} + E_k^{\bar{c}_k})q_k X\big)q_k E_k^{c_k} X - c_k E_k^{c_k} \tag{16.9}$$

where $E_k^{c_k}$ is the total effort of the members of C_k and $E_k^{\bar{c}_k}$ is the total effort of the other firms. Assuming the characteristic function form (11.1), first this function is minimized by the firms not belonging to the coalition. This can be done by minimizing the price function, which can occur if they have maximal fishing effort, $E_k^{\bar{c}_k} = \sum_{j \notin C_k} L_{kj}$. And then the coalition maximizes its overall profit under this worst case scenario:

$$\text{maximize}_{E_k^{c_k}} f_k\left(\left(E_k^{c_k} + \sum_{j \notin C_k} L_{kj}\right) q_k X\right) q_k E_k^{c_k} X - c_k E_k^{c_k}. \tag{16.10}$$

This objective function is strictly concave and continuous, therefore there is a unique maximal solution. The derivative of the objective function with respect to $E_k^{c_k}$ can be written as follows:

$$f_k'\left(\left(E_k^{c_k} + \sum_{j \notin C_k} L_{kj}\right) q_k X\right) q_k^2 E_k^{c_k} X^2 + f_k\left(\left(E_k^{c_k} + \sum_{j \notin C_k} L_{kj}\right) q_k X\right) q_k X - c_k.$$

If

$$f_k\left(\left(\sum_{j \notin C_k} L_{kj}\right) q_k X\right) q_k X - c_k \leqq 0,$$

then $E_k^{c_k} = 0$ is the optimal solution. Otherwise the optimal solution is positive.

After the characteristic function is determined, the application of the cooperative solution concepts, such as the core, the stable sets, the nucleolus, the Shapley values, the kernel and bargaining sets can be easily applied. We note that Szidarovszky et al. (2005) discusses dynamic extension of the model presented in this section.

16.5 A Water Distribution Problem

Mexico City is one of the most populated cities of the world with a severe water shortage. The main water users are agriculture, industry and domestic users. The question is how to divide the limited water amount between the users. This situation can be considered as a three-person game, where the three water users are the players and their payoff functions are the amounts of water they receive. There are five sources of water: surface water from local source, imported surface water, groundwater from local source, imported ground water, and treated water. So the decision variables (or strategies) of player k ($k = 1, 2, 3$) are as follows:

$$\begin{aligned}
s_k &= \text{surface water usage from local source} \\
s_k^* &= \text{imported surface water usage} \\
g_k &= \text{groundwater usage from local source} \\
g_k^* &= \text{imported groundwater usage} \\
t_k &= \text{treated water usage.}
\end{aligned}$$

Because of water availability and demands, the players have to satisfy the following constraints. First, for each player k,

$$s_k + s_k^* + g_k + g_k^* + t_k \leqq D_k \tag{16.11}$$

where D_k is the total water demand of player k in order to avoid wasting water. A minimum amount has to be given to the players to survive:

$$s_k + s_k^* + g_k + g_k^* + t_k \geqq D_k^{min}. \tag{16.12}$$

The players cannot use more water from the different resources than the available amounts:

$$s_1 + s_2 + s_3 = S_s \tag{16.13}$$
$$s_1^* + s_2^* + s_3^* \leqq S_s^* \tag{16.14}$$
$$g_1 + g_2 + g_3 = S_g \tag{16.15}$$
$$g_1^* + g_2^* + g_3^* \leqq S_g^* \tag{16.16}$$

where S_s, S_s^*, S_g and S_g^* are available water amounts of local and imported surface water, local and imported ground water, respectively. In (16.13) and (16.15) we require equality, so all local resources have to be used before water is imported.

In addition to these requirements each player has its individual constraints.

The *agricultural* users ($k = 1$) must satisfy two major conditions. The following notation is introduced to formulate them:

$$\begin{aligned} G &= \text{set of crops which can use only groundwater} \\ a_i &= \text{area of crop } i \text{ in agricultural area} \\ w_i &= \text{water need of crop } i \text{ per ha} \\ T &= \text{set of crops which can use treated water} \\ W &= \sum a_i w_i = \text{total water need for irrigation.} \end{aligned}$$

Groundwater has the best and treated water has the worst quality, so the most sensitive crops can use only groundwater and the least sensitive crops can be irrigated with treated water. Therefore we have to require that the ratio of available groundwater cannot be less than the relative water need of crops that can use only groundwater:

$$\frac{g_1 + g_1^*}{s_1 + s_1^* + g_1 + g_1^* + t_1} \geqq \frac{\sum_{i \in G} a_i w_i}{W}.$$

Notice that this is equivalent to a linear constraint:

$$\alpha_1 s_1 + \alpha_1 s_1^* + (\alpha_1 - 1) g_1 + (\alpha_1 - 1) g_1^* + \alpha_1 t_1 \leqq 0, \tag{16.17}$$

where α_1 is the right hand since of the above inequality.

Similarly, the ratio of treated water availability cannot be larger than the ratio of water need of crops that can be irrigated with treated water:

$$\frac{t_1}{s_1 + s_1^* + g_1 + g_1^* + t_1} \leqq \frac{\sum_{i \in T} a_i w_i}{W}$$

which is again equivalent to a linear constraint:

$$-\beta_1 s - \beta_1 s_1^* - \beta_1 g_1 - \beta_1 g_1^* + (1 - \beta_1) t_1 \leqq 0 \tag{16.18}$$

where β_1 is the right hand side of the above inequality.

The *industrial* users ($k = 2$) have two conditions concerning water quality. Let

B_g = minimum proportion of groundwater the industry has to receive

B_t = maximum proportion of treated water the industry can use.

Since groundwater has the best quality, in order to ensure a minimum overall water quality for industry, the proportion of groundwater usage of the industry cannot be smaller than B_g:

$$\frac{g_2 + g_2^*}{s_2 + s_2^* + g_2 + g_2^* + t_2} \geqq B_g$$

and since treated water has the worst quality, the proportion of treated water usage cannot exceed B_t:

$$\frac{t_2}{s_2 + s_2^* + g_2 + g_2^* + t_2} \leqq B_t.$$

Both constraints are equivalent with linear inequalities:

$$B_g s_2 + B_g s_2^* + (B_g - 1) g_2 + (B_g - 1) g_2^* + B_g t_2 \leqq 0 \tag{16.19}$$

and

$$-B_t s_2 - B_t s_2^* - B_t g_2 - B_t g_2^* + (1 - B_t) t_2 \leqq 0. \tag{16.20}$$

Domestic water users ($k = 3$) have only one constraint concerning the treated water usage, since its usage is very limited such as irrigation in parks, etc. Let

Table 16.6 Model data

	k = 1	k = 2	k = 3
D_k^{min}	594	177	1092.81
D_k	966	230	2123
α_1	0.41		
β_1	0.33		
B_g		0.066	
B_t		0.20	
B_d			0.06

B_d = maximum allowed treated water proportion for domestic users, then the requirement is as follows:

$$\frac{t_3}{s_3 + s_3^* + g_3 + g_3^* + t_3} \leqq B_d,$$

which can be rewritten as

$$-B_d s_3 - B_d s_3^* - B_d g_3 - B_d g_3^* + (1 - B_d)t_3 \leqq 0. \tag{16.21}$$

The payoff of each user is the total amount of water it receives:

$$\phi_k = s_k + s_k^* + g_k + g_k^* + t_k \, (k = 1, 2, 3). \tag{16.22}$$

In summary, we have a three-person game with payoff functions (16.22). The strategy set of player 1 (agriculture) is defined by constraints (16.11), (16.12) with $k = 1$, (16.17) and (16.18). That of player 2 (industry) is given by (16.11), (16.12) with $k = 2$, (16.19), (16.20) while the strategy set of player 3 (domestic users) is given by (16.11), (16.12) with $k = 3$ and (16.21). However the individual strategy sets are not independent by the additional four constraints (16.13)–(16.16). The actual data are given in Table 16.6 which are real values in the Mexico City region (Ahmadi & Salazar Moreno, 2013).

This problem is considered first as a three-person noncooperative game. The solution algorithm is almost the same as it was explained earlier in Sections 6.1 and 6.2 with the only difference that the "joint" constraints (16.13)–(16.16) also had to be taken into account in formulating the Kuhn-Tucker conditions for the three players. The numerical results are given in Table 16.7. Since the objective function value becomes zero, the results give equilibrium. Notice that all constraints and payoff functions are linear, therefore the Kuhn - Tucker conditions are sufficient and necessary. The results show that agricultural and industrial demands are completely satisfied, however the demand of the domestic users is satisfied in only 59.43%.

We can also consider this problem with having a mediator, the government agency and solving the problem as a multiobjective optimization problem with three objec-

Table 16.7 Nash-equilibrium results

	k = 1	k = 2	k = 3	Total
s_k	0	0	58	58
g_k	966	205.3	530.647	1702
t_k	0	0	75.702	75.702
s_k^*	0	24.64	428.353	453
g_k^*	0	0	169.000	169
Total	966	230	1261.70	

Table 16.8 Weighting method results

	k = 1	k = 2	k = 3	Total
s_k	0	58	0	58
g_k	647.22	83.60	971.180	1702
t_k	318.78	35.40	101.692	455.87
s_k^*	0	0	453	453
g_k^*	0	0	169	169
Total	966	177	1694.87	

tives. The weighting method was used where slightly higher weight was given to domestic users, since Mexico City has one of the largest population among the cities of the World: $w_1 = w_2 = 0.3$ and $w_3 = 0.4$ were chosen. The results are shown in Table 16.8 showing that in this case only the agricultural demands can be completely satisfied while industrial and domestic demands can be satisfied in only 76.96% and 79.83% levels.

From both sets of results we can see that domestic water demands cannot be satisfied. We also recomputed the weighting method results with $w_1 = w_2 = 0$ and $w_3 = 1$, where the maximum supply for domestic users became only 1960, only 92.32% of the demand. That is, there is no way to satisfy the growing water demand of the city in the current system and infrastructure.

Chapter 17
Case Studies and Applications of Repeated and Dynamic Games

In Chap. 3 we introduced static duopolies and N-person oligopolies, furthermore in Sect. 9.2 their dynamic extensions were examined. In this chapter we first further investigate dynamic oligopolies. In addition models on environment friendly companies, competing species and love affairs will be introduced and analysed in this chapter. And finally, the possibility of governmental control of oligopolistic firms will be examined.

17.1 Oligopolies with Pollution Control

In Sect. 3.2. We introduced N-firm Cournot oligopolies, and proved the existence of a unique equilibrium under realistic conditions. For the sake of mathematical simplicity we assume now that the price function and all cost functions are linear:

$$p(s) = a - bs \quad \left(s \leq \frac{a}{b}\right) \text{ and } C_k(x_k) = c_k x_k + d_k \quad (k = 1, 2, \ldots, N)$$

where x_k is the output (production level) of firm k, and $s = \sum_{k=1}^{n} x_k$ is the industry output. The profit of firm k was given in Eq. 3.26, which is simplified as

$$\phi_k(x_1, x_2, \ldots, x_N) = x_k \left(a - b \sum_{l=1}^{N} x_l\right) - (c_k x_k + d_k) \tag{17.1}$$

It is easy to see that conditions (a), (b) and (c) introduced in Example 3.14 hold in this special case. The marginal profit of firm k is clearly

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3_17

$$\frac{\partial \phi_k}{\partial x_k} = a - 2bx_k - b \sum_{i \neq k} x_i - c_k$$

Assuming interior optimum, the best response of this firm is

$$R(s_k) = \frac{a - c_k - bs_k}{2b} \tag{17.2}$$

where $s_k = \sum_{i \neq k} x_i$. At the equilibrium

$$x_k = \frac{a - c_k - bs_k}{2b}$$

for all k, which can be rewritten as

$$2bx_k = a - c_k - b(s - x_k)$$

or

$$x_k = \frac{a - c_k - bs}{b}$$

Adding this relation for all values of k we have

$$bs = Na - \sum_{i=1}^{N} c_i - Nbs$$

implying that the equilibrium industry output level is

$$s* = \frac{Na - \sum_{i=1}^{N} c_i}{(N+1)\,b} \tag{17.3}$$

and the equilibrium output of firm k is therefore

$$x_k^* = \frac{a - c_k}{b} - \frac{Na - \sum_{i=1}^{N} c_i}{(N+1)\,b} = \frac{a + \sum_{i=1}^{N} c_i - (N+1)\,c_k}{(N+1)\,b} \tag{17.4}$$

In Sect. 9.2. we also introduced the dynamic extensions of this model in both discrete and continous time scales. In this section continous time scales are assumed.

Two types of dynamics were introduced: best response dynamics and gradient adjustments, which have the common formulation

$$\dot{x}_k(t) = K_k\left(\alpha_k - \beta\sum_{i\neq k} x_i(t) - x_k(t)\right)(k = 1, 2, \ldots, N) \tag{17.5}$$

where dot indicates derivative with respect to time, furthermore $\alpha_k = \frac{a-c_k}{2b}$, $\beta = \frac{1}{2}$ and K_k is a positive adjustment coefficient. We also demonstrated that system (17.5) is asymptotically stable meaning that with any $x_k(0)$, $(k = 1, 2, \ldots, N)$ initial output levels, $x_k(t)$ for all k converges to the corresponding equilibrium level x_k^* as $t \to \infty$.

In recent decades increasing attention has been given to environmental policies to control environmental degradation. Main sources of degradation are greenhouse gas and pollution emissions by industries using nonrenewable energy sources. One of the many tasks of governments is to find policies that can control pollution emissions. There are two major types of polluters. In the case of point source polluters the government knows the individual emission levels of the firms, so it can make punishments or subsidies individually. In the case of non-point source polluters the government is able to monitor only the total emission level produced by the industry without knowing the individual level of each firm. Therefore collective punishments or subsidies are the only tools for the govermnent to achieve its objectives (Segerson, 1988). In developing a mathematical model describing this situation we will use the N-firm oligopoly introduced before. So N is the number of firms, with cost functions

$$C_k(x_k) = c_k x_k + d_k, (k = 1,\ 2,\ \ldots, N)$$

where x_k is the production level (output) of firm k. Let $s = \sum_{i=1}^{N} x_i$ denote the industry output, then the price function is assumed to be

$$p(s) = a - bs.$$

The production process of the firms creates pollution emissions. Selecting appropriate units we can assume that production of one output unit produces one unit of pollution. In order to avoid or reduce environmental penalties the firms try to abate a part of the emitted pollution. Let 1-e_k denote the pollution reduction coefficient of firm k, so this firm will have the emission level $e_k x_k$, so the total emission of the industry is $\sum_{i=1}^{N} e_i x_i$.The government defines an environmental standard, $\bar{E}$. If the emission of the industry is higher than $\bar{E}$, then each firm has to pay a uniform penalty:

$$\Theta\left(\sum_{i=1}^{N} e_i x_i - \bar{E}\right),$$

and if it is less than the environmental standard, then each firm receives a uniform subsidy

$$\Theta\left(\bar{E}-\sum_{i=1}^{N}e_i x_i\right).$$

The profit of firm k is its profit from production modified with the penalty paid or the subsidy received:

$$\begin{aligned}\phi_k\left(x_1,x_2,...,x_N\right)&=x_k p\left(s\right)-C_k\left(x_k\right)-\Theta\left(\sum_{i=1}^{N}e_i x_i-\bar{E}\right)=\\&=x_k\left(a-b\sum_{i=1}^{N}x_i\right)-\left(c_k x_k+d_k\right)-\Theta\left(\sum_{i=1}^{N}e_i x_i-\bar{E}\right)\end{aligned} \tag{17.6}$$

The marginal profit of this firm is

$$\frac{\partial\phi_k}{\partial x_k}=a-bs-bx_k-c_k-\Theta e_k.$$

Notice that ϕ_k is strictly concave in x_k. Assuming interior optimum, the best response of firm k is the following:

$$x_k=\frac{a-bs-c_k-\Theta e_k}{b} \tag{17.7}$$

which depends on the total industry output. Adding this equation for all firms

$$s=\frac{1}{b}\left(Na-Nbs-\sum_{i=1}^{N}c_i-\Theta\sum_{i=1}^{N}e_i\right)$$

implying that the equilibrium industry output level is

$$s^*=\frac{Na-\sum_{i=1}^{N}c_i-\Theta\sum_{i=1}^{N}e_i}{\left(N+1\right)b} \tag{17.8}$$

Notice that in the case when $e_i=0\ \ (i=1,\ 2,\ldots,N)$, that is, all pollution is abated by the firms, this equation reduces to (17.3). Substituting this relation into (17.7) we have the equilibrium output level of firm k for k =1,2,…, n:

$$x_k^*=\frac{a-c_k-\Theta e_k}{b}-\frac{Na-\sum_{i=1}^{N}c_i-\Theta\sum_{i=1}^{N}e_i}{(N+1)b}= \tag{17.9}$$

$$= \frac{a + \left(\sum_{i=1}^{N} c_i - (N+1)\, c_k \right) + \Theta \left(\sum_{i=1}^{N} e_i - (N+1)\, e_k \right)}{(N+1)\, b}$$

The assumption that the best response is positive is understandable, since if a firm has zero optimal output level, then it ususally leaves the business.

Assuming continous time scales and gradient adjustment the dynamic equation becomes

$$\begin{aligned} \dot{x}_k(t) &= K_k \left(a - b \sum_{i \neq k} x_i(t) - 2b x_k(t) - c_k - \Theta e_k \right) = \\ &= \bar{K}_k \left(\alpha_k - \beta \sum_{i \neq k} x_i(t) - x_k(t) \right) \end{aligned} \tag{17.10}$$

where

$$\alpha_k = \frac{a - c_k - \Theta e_k}{2b}, \ \beta = \frac{1}{2} \text{ and} \bar{K}_k = 2bK_k.$$

This model is identical with (9.24) so the dynamic properties and stability conditions are the same as those discussed in Sect. 9.2.

17.2 Competition of Two Species

In this section the interaction of two competitive species is examined. The mathematical model is based on the works of Volterra (1931), Lotka (1925). At time t let $x(t)$ and $y(t)$ denote the population of the species which are subject to intrinsic growth, inter-, and intra-competition within or between the species. This interaction is usually modeled by the so called Lotka-Volterra equations:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[\varepsilon_1 - a_{11} x(t) - a_{12} y(t)\right] \\ \dot{y}(t) &= y(t)\left[\varepsilon_2 - a_{21} x(t) - a_{22} y(t)\right] \end{aligned} \tag{17.11}$$

Here ε_1 and ε_2 denote the intrinsic growth rates, a_{ii} is the crowding coefficients measuring the strength of the intra-competition within species i, and a_{ij} is the competition coefficient measuring the strength of species i against species j. All model parameters are assumed to be positive. The steady state or equilibrium $(\bar{x}, \bar{y})$ represents the state of the system that will not change anymore, that is, when both $\dot{x}(t)$ and $\dot{y}(t)$ are equal to zero. Several cases should be considered

(a) $\bar{x} = \bar{y} = 0$
(b) $\bar{x} = 0, \bar{y} = \frac{\varepsilon_2}{a_{22}}$
(c) $\bar{y} = 0, \bar{x} = \frac{\varepsilon_1}{a_{11}}$

(d) $\begin{matrix} \varepsilon_1 - a_{11}\bar{x} - a_{12}\bar{y} = 0 \\ \varepsilon_2 - a_{21}\bar{x} - a_{22}\bar{y} = 0 \end{matrix}$ (17.40)

In cases (a), (b) and (c) at least one species has zero population, so we cannot speak about interaction. Therefore we deal only with case (d).

Simple calculation shows that

$$\bar{x} = \frac{\varepsilon_1 a_{22} - \varepsilon_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \ \bar{y} = \frac{\varepsilon_2 a_{11} - \varepsilon_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{17.12}$$

The two species can coexist if $\bar{x} > 0$ and $\bar{y} > 0$ which is the case if

$$\frac{a_{11}}{a_{21}} > \frac{\varepsilon_1}{\varepsilon_2} > \frac{a_{12}}{a_{22}} \tag{17.13}$$

Let g_1 and g_2 denote the right hand sides of equations (17.11), then their partial derivatives at the equilibrium are

$$\frac{\partial g_1}{\partial x} = [\varepsilon_1 - a_{11}\bar{x} - a_{12}\bar{y}] - a_{11}\bar{x} = -a_{11}\bar{x}$$

$$\frac{\partial g_1}{\partial y} = -a_{12}\bar{x}$$

$$\frac{\partial g_2}{\partial x} = -a_{21}\bar{y}$$

$$\frac{\partial g_2}{\partial y} = [\varepsilon_2 - a_{21}\bar{x} - a_{22}\bar{y}] - a_{22}\bar{y} = -a_{22}\bar{y}$$

so the linearised equations have the form

$$\begin{aligned} \dot{x}(t) &= -a_x x(t) - b_x y(t) \\ \dot{y}(t) &= -b_y x(t) - a_y y(t) \end{aligned} \tag{17.14}$$

where $a_x = a_{11}\bar{x}, \ b_x = a_{12}\bar{x}, \ a_y = a_{22}\bar{y}$ and $b_y = a_{21}\bar{y}$.

In order to find the characteristic polynomial assume exponential solutions $x(t) = e^{\lambda t}u$ and $y(t) = e^{\lambda t}v$ and substitute them into (17.14) to have

$$\begin{aligned} \lambda e^{\lambda t}u &= -a_x e^{\lambda t}u - b_x e^{\lambda t}v \\ \lambda e^{\lambda t}v &= -b_y e^{\lambda t}u - a_y e^{\lambda t}v \end{aligned}.$$

Nonzero solutions for u and v exist if and only if the determinant of this system with respect to u and v equals zero. After simplifying by $e^{\lambda t}$ we see that

$$\det\begin{pmatrix} \lambda + a_x & b_x \\ b_y & \lambda + a_y \end{pmatrix} = \lambda^2 + \lambda\left(a_x + a_y\right) + \left(a_x a_y - b_x b_y\right) = 0$$

The linear coefficient is positive, the constant term is also positive by (17.13). Since both are positive (see Appendix H), the equilibrium is asymptotically stable under the linearized dynamics and locally asymptotically stable under dynamics (17.11).

17.3 Love Affair with Cautious Partners

Consider two partners being involved with romantic relationships with each other. Each of them has romantic feelings about the other, which might change in time. There are three major elements influencing their feelings. Oblivion (O_1 or O_2) gives rise to a loss of interest in the partner describing the self-reaction process. Clearly it depends on his/her own level of feeling and the disappeareance of the partner. Return (R_1 or R_2) is a source of interest reacting to the partner's love. And finally instinct (I_1 or I_2) is also a source of interest reacting to the partner's appeal based on physical, intellectual, educational and financia properties.

Based on the works of Strogatz (1988) and Rinaldi (1998a, b) the following functional relations are assumed. Let $x(t)$ and $y(t)$ denote the two individuals' feelings toward each other. Then it is assumed that

$$O(x) = -\alpha_x x, \ O(y) = -\alpha_y y \quad \left(\alpha_x, \alpha_y > 0\right) \tag{17.15}$$

$$R_x(y) = \beta_x \tanh(y), \ R_y(x) = \beta_y \tanh(x) \tag{17.16}$$

and

$$I_x = \gamma_x A_y, \quad I_y = \gamma_y A_x \quad \left(A_x, A_y \geq 0\right) \tag{17.17}$$

and the dynamic evolution of the feelings of the partners are driven by the following two-dimensional system of ordinary differential equations:

$$\dot{x}(t) = O_x(x(t)) + R_x(y(t)) + I_x \tag{17.18}$$

$$\dot{y}(t) = O_y(y(t)) + R_y(x(t)) + I_y \tag{17.19}$$

where dot represents derivative with respect to time t.

Assumption (17.15) gives attention to exponentially vanishing memory. In (17.16) the hyperbolic functions are positive, bounded from above and concave for positive values of y or x. They are negative, bounded from below and convex for negative values of y or x. If β_x (β_y) is positive, then the feeling of the individual is encouraged by his/her partner. Such individuals are called secure. However if β_x (β_y) is negative, then the feeling of the individual is discouraged, and such individuals are called non-secure. Assumption (17.17) shows that individuals have time-invariant positive appeal. Coefficients α_x (α_y) are called forgetting parameters, β_x, β_y, γ_x and γ_y are reaction coefficients of love and appeal. With these terms the model in (17.18) and (17.19) has the form:

$$\dot{x}(t) = -\alpha_x x(t) + \beta_x \tanh(y(t)) + \gamma_x A_y \tag{17.20}$$

$$\dot{y}(t) = \beta_y \tanh(x(t)) - \alpha_y y(t) + \gamma_y A_x \tag{17.21}$$

For the sake of mathematical simplicity assume that $A_x = A_y = 0$, so the right hand sides have only two terms. The steady state of this system has to satisfy $\dot{x}(t) = \dot{y}(t) = 0$ so we get two nonlinear equations

$$\begin{aligned} \alpha_x x &= \beta_x \tanh(y) \\ \beta_y \tanh(x) &= \alpha_y y \end{aligned} \tag{17.22}$$

or

$$y = \tanh^{-1}\left(\frac{\alpha_x}{\beta_x} x\right) = \frac{\beta_y}{\alpha_y} \tanh(x) \tag{17.23}$$

Clearly $x^* = y^* = 0$ is a solution.

Theorem 17.1 *(a) If $\alpha_x \alpha_y \geq \beta_x \beta_y$ then $x^* = y^* = 0$ is the unique steady state.*

(b) If $\alpha_x \alpha_y < \beta_x \beta_y$ then there are three steady states: zero, a positive and a negative.

Proof Let $u(x)$ and $v(x)$ denote the middle and last expressions of (17.23). Clearly x^* is a steady state if it satisfies $u(x) = v(x)$ and y^* is this common value at $x = x^*$. Clearly $x^* = 0$ is a solution so there is at least one steady state.

Simple differentiation shows that

$$u'(x) = \frac{\frac{\alpha_x}{\beta_x}}{1 - \left(\frac{\alpha_x}{\beta_x} x\right)^2}, \; u''(x) = \frac{2x\left(\frac{\alpha_x}{\beta_x}\right)^3}{\left[1 - \left(\frac{\alpha_x}{\beta_x} x\right)^2\right]^2}$$

and

$$v'(x) = \frac{\beta_y}{\alpha_y}\left(\frac{2}{e^x + e^{-x}}\right)^2, \; v''(x) = -\frac{8\beta_y}{\alpha_y} \frac{e^x - e^{-x}}{(e^x + e^{-x})^3}$$

By assumption α_x and α_y are positive, however β_x and β_y do not have definite signs. So we have to consider three cases.

(a) β_x and β_y have different signs. The range of the hyperbolic tangent function is the open interval $(-1, 1)$ which is also the domain of its inverse. Therefore from the first equation of (17.22) we see that $\left|\frac{\alpha_x}{\beta_x} x\right| < 1$ implying that the denominator of $u'(x)$ is positive. Since β_x and β_y have different signs, same holds for $u'(x)$ and $v'(x)$. So one of them is strictly increasing and the other is strictly decreasing. Thus, there is a unique solution $x^* = y^* = 0$.

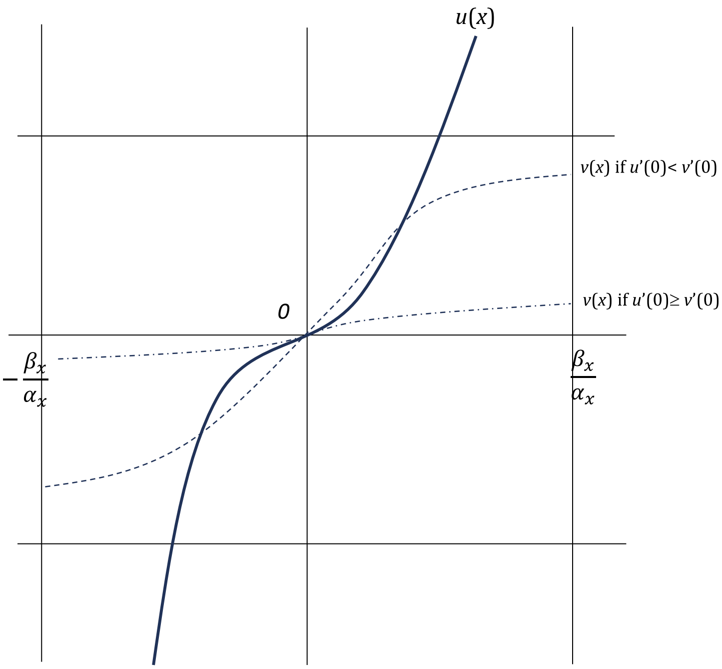

Fig. 17.1 Graphs of $u(x)$ and $v(x)$

(b) β_x and β_y are positive. Notice that $u(0) = 0$, $u\left(\frac{\beta_x}{\alpha_x}\right) = \infty$ with left hand side limit, $u\left(-\frac{\beta_x}{\alpha_x}\right) = -\infty$ with right hand side limit, and

$$u'(x) > 0, \quad u''(x) \begin{cases} > 0 \text{ if } x > 0 \\ < 0 \text{ if } x < 0 \end{cases}.$$

Similarly $v(0) = 0$, $v(\infty) = \frac{\beta_y}{\alpha_y}$, $v(-\infty) = -\frac{\beta_y}{\alpha_y}$,

$$v'(x) > 0, v''(x) \begin{cases} < 0 \text{ if } x > 0 \\ > 0 \text{ if } x < 0 \end{cases}.$$

In addition $u'(0) = \frac{\alpha_x}{\beta_x}$ and $v'(0) = \frac{\beta_y}{\alpha_y}$. Figure 17.1 shows how the shapes of $u(x)$ and $v(x)$ look like. Both $u(x)$ and $v(x)$ are strictly increasing, $u(x)$ is convex for $x > 0$ and concave for $x < 0$, while $v(x)$ is concave for $x > 0$ and convex for $x < 0$. Therefore $x = 0$ is the only solution if $u'(0) \geq v'(0)$ or $\alpha_x \alpha_y \geq \beta_x \beta_y$, otherwise in addition to $x = 0$, there is a positive and a negative solution.

(c) β_x and β_y are negative. This case is identical with case (b) when β_x and β_y are replaced with $-\beta_x$ and $-\beta_y$.

The asymptotic behaviour of the steady states is examined by linearization. Let

$$d_x = \frac{d}{dx}\tanh(x)\Big|_{x=x*} \text{ and } d_y = \frac{d}{dy}\tanh(y)\Big|_{y=y*} \quad \left(d_x, d_y > 0\right)$$

where (x^*, y^*) is a steady state.

Then the linearized equations can be written as

$$\begin{aligned} \dot{x}(t) &= -\alpha_x x(t) + \beta_x d_y y(t) \\ \dot{y}(t) &= \beta_y d_x x(t) - \alpha_y y(t) \end{aligned} \tag{17.24}$$

Notice that at the zero steady state $d_x = d_y = 1$ and $d_x, d_y < 1$ at the nonzero steady states. In analyzing asymptotical behaviour we find the Jacobian of the system (17.24) and obtain the characteristic equation:

$$\begin{aligned} &\det(J - \lambda I) = \det\begin{pmatrix} -\alpha_x - \lambda & \beta_x d_y \\ \beta_y d_x & -\alpha_y - \lambda \end{pmatrix} \\ &= \lambda^2 + \left(\alpha_x + \alpha_y\right)\lambda + \left(\alpha_x\alpha_y - \beta_x\beta_y d_x d_y\right) = 0 \end{aligned} \tag{17.25}$$

Since the linear coefficient is positive, we need to check the sign of the costant term. At the zero steady state $d_x = d_y = 1$, so the constant term is $\alpha_x\alpha_y - \beta_x\beta_y$. For the nonzero steady states it can be proven that $\alpha_x\alpha_y > \beta_x\beta_y d_x d_y$(see Matsumoto, 2017). Thus we have the following result:

Theorem 17.2 *(a) The zero steady state is locally asymptotically stable if* $\alpha_x\alpha_y > \beta_x\beta_y$*, and unstable if* $\alpha_x\alpha_y < \beta_x\beta_y$*. (b) The nonzero steady states are always locally asymptotically stable.*

17.4 Control in Oligopolies

In this section the possibility of government control of oligopolies will be examined. For the sake of simplicity only N-firm single-product oligopolies will be considered without product differentiation. Let x_k denote the output of firm $k (1 \leqq k \leqq N)$, then $s = \sum_{k=1}^{N} x_k$ is the industry output. Assume that the price function and all cost functions are linear, $p(s) = a - bs$ and $C_k(x_k) = c_k x_k + d_k$ for all k. The profit of firm k without governmental control is the difference of its revenue and cost,

$$\phi_k = x_k(a - bs) - (c_k x_k + d_k).$$

Assume that the market is controlled with the cost functions of the firms, for example, by tax rates, subsidies, etc. So the profit of firm k with control variable $u > 0$ can be given as

$$\phi_k = x_k\Big(a - bx_k - b\sum_{l\neq k} x_l\Big) - u(c_k x_k + d_k) \tag{17.26}$$

where we assume that the same control applies to all firms. Consider first the case of discrete time scales and static expectations. The best response of firm k is obtained by differentiation. Assuming interior maximum,

$$a - 2bx_k - b\sum_{l\neq k} x_l - c_k u = 0$$

implying that

$$x_k = -\frac{1}{2}\sum_{l\neq k} x_l + \frac{a - c_k u}{2b}.$$

The dynamic process with static expectation is therefore

$$x_k(t+1) = -\frac{1}{2}\sum_{l\neq k} x_l(t) + \frac{a - c_k u(t)}{2b}. \tag{17.27}$$

By introducing the new variable

$$z_k(t) = x_k(t) - \frac{a}{(N+1)b},$$

this system can be simplified:

$$z_k(t+1) = -\frac{1}{2}\sum_{l\neq k} z_l(t) - \frac{c_k}{2b}u(t). \tag{17.28}$$

This is a discrete system of the form (H9), where

$$\underline{x}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \cdots \\ z_N(t) \end{pmatrix}, \ \underline{A} = \begin{pmatrix} 0 & -\frac{1}{2} & \ldots & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \ldots & -\frac{1}{2} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{2} & -\frac{1}{2} & \ldots & 0 \end{pmatrix}, \ \text{and } \underline{B} = \begin{pmatrix} -\frac{c_1}{2b} \\ -\frac{c_2}{2b} \\ \cdots \\ -\frac{c_N}{2b} \end{pmatrix}.$$

Consider first the case of duopoly, when $N = 2$. The Kalman matrix (H11) has now the form

$$\underline{K} = (\underline{B}, \underline{A}\underline{B}) = \begin{pmatrix} -\frac{c_1}{2b} & \frac{c_2}{4b} \\ -\frac{c_2}{2b} & \frac{c_1}{4b} \end{pmatrix}.$$

System (17.28) with $N = 2$ is controllable if rank $(\underline{K}) = 2$, which is the case if determinant of $\underline{K}$ is nonzero:

$$-\frac{c_1^2}{8b^2} + \frac{c_2^2}{8b^2} \neq 0$$

that is, if $c_1 \neq c_2$.

Assume next that $N \geqq 3$. Observe first that $\underline{A} = \frac{1}{2}(\underline{I} - \underline{1})$, where $\underline{I}$ is the $N \times N$ identify matrix and all elements of $\underline{1}$ are equal to unity. Notice that $\underline{1}^2 = N\underline{1}$ and so

$$\underline{A}^2 = \frac{1}{4}\big(\underline{I} - 2\underline{1} + \underline{1}^2\big) = \frac{1}{4}\big(\underline{I} + (N-2)\underline{1}\big) = \frac{N-1}{4}\underline{I} + \frac{2-N}{2}\underline{A}$$

implying that

$$\underline{A}^2\underline{B} = \frac{N-1}{4}\underline{B} + \frac{2-N}{2}\underline{AB}.$$

That is, the columns of $\underline{K} = (\underline{B}, \underline{AB}, \ldots, \underline{A}^{N-1}\underline{B})$ are linearly dependent, so the system is not controllable.

If continuous time scales are assumed, then the difference equation (17.27) is replaced by the differential equation

$$\dot{x}_k(t) = K_k\Big(-\frac{1}{2}\sum_{l\neq k} x_l(t) + \frac{a - c_k u(t)}{2b} - x_k(t)\Big) \tag{17.29}$$

where $K_k > 0$ is the speed of adjustment of firm k. This can be rewritten as system (H10) with

$$\underline{A} = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ldots & \\ & & & K_N \end{pmatrix} \begin{pmatrix} -1 & -\frac{1}{2} & \ldots & -\frac{1}{2} \\ -\frac{1}{2} & -1 & \ldots & -\frac{1}{2} \\ \ldots & \ldots & \ldots & \ldots \\ -\frac{1}{2} & -\frac{1}{2} & \ldots & -1 \end{pmatrix},$$

$$\underline{B} = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ldots & \\ & & & K_N \end{pmatrix} \begin{pmatrix} -\frac{c_1}{2b} \\ -\frac{c_2}{2b} \\ \ldots \\ -\frac{c_N}{2b} \end{pmatrix}$$

and state variables $z_k(t)$ as in the discrete case. In the duopoly case

$$\underline{A} = \begin{pmatrix} K_1 & \\ & K_2 \end{pmatrix} \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix} \qquad = \begin{pmatrix} -K_1 & -\frac{K_1}{2} \\ -\frac{K_2}{2} & -K_2 \end{pmatrix}$$

and so

$$\underline{AB} = \begin{pmatrix} -K_1 & -\frac{K_1}{2} \\ -\frac{K_2}{2} & -K_2 \end{pmatrix} \begin{pmatrix} -\frac{c_1 K_1}{2b} \\ -\frac{c_2 K_2}{2b} \end{pmatrix} \qquad = \begin{pmatrix} \frac{c_1 K_1^2}{2b} + \frac{K_1 K_2 c_2}{4b} \\ \frac{c_1 K_1 K_2}{4b} + \frac{c_2 K_2^2}{2b} \end{pmatrix}.$$

Therefore the Kalman matrix has the form

$$\underline{K} = \begin{pmatrix} -\frac{c_1 K_1}{2b} & \frac{2c_1 K_1^2 + K_1 K_2 c_2}{4b} \\ -\frac{c_2 K_2}{2b} & \frac{K_1 K_2 c_1 + 2c_2 K_2^2}{4b} \end{pmatrix}.$$

The system is controllable if its determinant is nonzero:

$$\begin{aligned} 0 &\neq \frac{-c_1 K_1 (K_1 K_2 c_1 + 2c_2 K_2^2) + c_2 K_2 (2c_1 K_1^2 + K_1 K_2 c_2)}{8b^2} \\ &= -\frac{K_1 K_2}{8b^2} \left[K_1 c_1^2 + 2(K_2 - K_1) c_1 c_2 - K_2 c_2^2 \right] \end{aligned}$$

which is the case if

$$\frac{c_1}{c_2} \neq \frac{K_1 - K_2 + \sqrt{(K_2 - K_1)^2 + K_1 K_2}}{K_1}. \tag{17.30}$$

In the special case of $K_1 = K_2$, this relation is simplified to $c_1 \neq c_2$. If $N \geqq 3$, the controllability conditions are much more complicated, so we consider here only the special case when $K_1 = K_2 = \cdots = K_N \equiv K$. Matrix $\underline{A}$ has the special form:

$$\underline{A} = -\frac{K}{2} \left(\underline{I} + \underline{1} \right)$$

and therefore

$$\underline{A}^2 = \frac{K^2}{4} \left(\underline{I} + 2\underline{1} + N\underline{1} \right) = -\frac{K^2(N+1)}{4} \underline{I} - \frac{K(N+2)}{2} \underline{A}$$

since $\underline{1} = -\underline{I} - \frac{2}{K}\underline{A}$. Therefore the columns of the Kalman matrix $(\underline{B}, \underline{AB}, \ldots, \underline{A}^{N-1}\underline{B})$ are linearly dependent implying that the system is not controllable.

Okuguchi and Szidarovszky (1999) offer more details of this model and other applications.

17.5 Effect of Information Lag in Oligopoly

Consider an N-firm single product oligopoly without product differentiation with linear price and cost functions. Let x_k denote the output of firm k, then $s = \sum_{k=1}^{N} x_k$

is the industry output. Assume that the price function is $p(s) = a - bs$ and the cost function of firm k is $c_k(x_k) = c_k x_k + d_k$. The profit of firm k is given as

$$\phi_k = x_k(a - bx_k - bs_k) - (c_k x_k + d_k) \tag{17.31}$$

where s_k is the output of the rest of the industry, $s_k = \sum_{l \neq k} x_l$. Assuming interior best responses of the firms, they can be obtained by simple differentiation

$$a - 2bx_k - bs_k - c_k = 0$$

implying that

$$x_k = -\frac{s_k}{2} + \frac{a - c_k}{2b}. \tag{17.32}$$

In many situations the output data of the competitors are not available, however the firms are able to determine the industry output from price information, since from the linear form of the price function,

$$s = \frac{a - p}{b}. \tag{17.33}$$

Assume that at each time period only delayed price information is available to the firms, where the length of the delay is either unknown to them or they simply believe that the price information is instantaneous. So at time period t from the delayed price information firm k believes that the industry output currently is $s(t - \theta)$, where θ is the delay, so believes that

$$s_k(t) = s(t - \theta) - x_k(t),$$

and based on this belief its believed best response is

$$-\frac{s(t - \theta) - x_k(t)}{2} + \frac{a - c_k}{2b}.$$

Assuming continuous time scales and output adjustments toward best responses, the dynamic evolution of the oligopoly is described by the following system of differential-difference equations;

$$\dot{x}_k(t) = K_k \Big(-\frac{\sum_{l=1}^{N} x_l(t - \theta) - x_k(t)}{2} + \frac{a - c_k}{2b} - x_k(t) \Big) \tag{17.34}$$

where K_k is the speed of adjustment of firm k.

For the sake of mathematical simplicity assume symmetric firms: $K_k \equiv K$, $c_k \equiv c$ and identical initial output levels. Then the entire trajectories of the firms are also the same. Let $x(t)$ denote this common trajectory, then (17.34) is reduced to a one-dimensional system:

$$\dot{x}(t) = K\Big(- \frac{Nx(t-\theta)}{2} - \frac{x(t)}{2} + \frac{a-c}{2b}\Big).$$

The eigenvalues of the system can be determined by substituting the exponential form, $x(t) = e^{\lambda t}u$, into the corresponding homogeneous equation:

$$\lambda e^{\lambda t}u = K\Big(- \frac{Ne^{\lambda(t-\theta)}u}{2} - \frac{e^{\lambda t}u}{2}\Big)$$

resulting in the characteristic equation

$$\lambda + \frac{KN}{2}e^{-\lambda\theta} + \frac{K}{2} = 0. \tag{17.35}$$

Without delay ($\theta = 0$) the only eigenvalue is $\lambda = -\frac{K(N+1)}{2} < 0$ implying the asymptotical stability of the system. If the value of θ increases, then stability maybe lost. At the stability switches λ must have zero real part, $\lambda = iw$. We can assume that $w > 0$, since the conjugate of any eigenvalue is also an eigenvalue. Substituting $\lambda = iw$ into (17.35) we have

$$iw + \frac{KN}{2}\big(\cos\theta w - i\sin\theta w\big) + \frac{K}{2} = 0.$$

Separating the real and imaginary parts two equations are obtained for two unknowns, w and θ, as follows:

$$\frac{KN}{2}\cos\theta w + \frac{K}{2} = 0 \tag{17.36}$$

$$w - \frac{KN}{2}\sin\theta w = 0. \tag{17.37}$$

Since $\sin^2\theta w + \cos^2\theta w = 1$, from these equations we have

$$\frac{K^2}{4} + w^2 = \frac{K^2N^2}{4}$$

so

$$w^2 = \frac{K^2(N^2-1)}{4}$$

and

$$w = \frac{K\sqrt{N^2-1}}{2}. \tag{17.38}$$

From (17.36) we know that $\cos\theta w < 0$, so from (17.37) we see that

$$\begin{aligned}\theta_n &= \frac{1}{w}\big(\pi - \sin^{-1}\frac{2w}{KN} + 2n\pi\big) \\ &= \frac{2}{K\sqrt{N^2-1}}\big(\pi - \sin^{-1}\frac{\sqrt{N^2-1}}{N} + 2n\pi\big)\,(n = 0, 1, 2, \ldots).\end{aligned}$$

The direction of the stability switch can be determined by assuming that the eigenvalue is a function of θ and with increasing values of θ at the stability switches stability is lost if $Re\dot{\lambda}(\theta) > 0$ and stability might be regained if $Re\dot{\lambda}(\theta) < 0$. Substituting $\lambda = \lambda(\theta)$ into the characteristic equation (17.35) and implicitly differentiating with respect to θ,

$$\dot{\lambda} + \frac{KN}{2}e^{-\lambda\theta}(-\dot{\lambda}\theta - \lambda) = 0$$

implying that

$$\dot{\lambda} = \frac{\frac{KN}{2}\lambda e^{-\lambda\theta}}{1 - \frac{KN}{2}\theta e^{-\lambda\theta}} = \frac{\lambda(-\lambda - \frac{K}{2})}{1 - \theta(-\lambda - \frac{K}{2})} = \frac{-\lambda^2 - \lambda\frac{K}{2}}{\lambda\theta + (1 + \frac{\theta K}{2})}.$$

With $\lambda = iw$,

$$\dot{\lambda} = \frac{w^2 - iw\frac{K}{2}}{iw\theta + (1 + \frac{\theta K}{2})} = \frac{(w^2 - iw\frac{K}{2})(1 + \theta\frac{K}{2} - iw\theta)}{w^2\theta^2 + (1 + \frac{\theta K}{2})^2}$$

and so

$$Re\dot{\lambda} = \frac{w^2(1 + \frac{\theta K}{2}) - w\frac{K}{2}w\theta}{w^2\theta^2 + (1 + \frac{\theta K}{2})^2} = \frac{w^2}{w^2\theta^2 + (1 + \frac{\theta K}{2})^2} > 0.$$

This relation implies that at all critical values θ_n at least one pair of eigenvalues changes the sign of their real parts from negative to positive, that is, stability is lost at the smallest stability switch:

$$\theta = \theta_0 = \frac{2}{K\sqrt{N^2-1}}\big(\pi - \sin^{-1}\frac{\sqrt{N^2-1}}{N}\big),$$

and the stability cannot be regained with further increase of the value of θ.

The interested reader can find further details of systems with one or two delays in Matsumoto and Szidarovszky (2012, 2013) or in any book on delay differential equations such as Bellman and Cooke (1963).

Appendix A
Vector and Matrix Norms

The lengths of n-element vectors can be characterized by the introduction of their norms. A vector-variable, real valued function $\underline{x} \mapsto ||\underline{x}||$ is called a *norm* of vector $\underline{x}$, if it satisfies the following properties:

(a) $||\underline{x}|| \geqq 0$ and $||\underline{x}|| = 0$ if and only if $\underline{x}$ is the zero vector;
(b) $||\alpha \underline{x}|| = |\alpha| \cdot ||\underline{x}||$ for all vectors $\underline{x}$ and real numbers α;
(c) $||\underline{x} + \underline{y}|| \leqq ||\underline{x}|| + ||\underline{y}||$ for all vectors $\underline{x}$ and $\underline{y}$.

The last property is known as the triangle inequality.

The most frequently used vector norms are given as follows:

$$||\underline{x}||_1 = \sum_{i=1}^{n} |x_i|,$$

where the ith element of vector $\underline{x}$ is x_i;

$$||\underline{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

which is called the *Euclidean* norm; and

$$||\underline{x}||_\infty = \max_i |x_i|$$

which is called the *maximum* norm.

The topology of n-element vector spaces can be easily developed based on vector norms. The distance of vectors $\underline{x}$ and $\underline{y}$ is the norm of their difference, $||\underline{x} - \underline{y}||$.

A sequence of vectors $\underline{x}_k$ converges to a vector $\underline{x}^*$ if with some vector norm

$$||\underline{x}_k - \underline{x}^*|| \to 0$$

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

as $k \to \infty$. This definition does not depend on the choice of a particular norm, since the norms of n-element vectors are equivalent to each other, that is, if $||\cdot||_a$ and $||\cdot||_b$ are two vector norms then there are positive constants K_1 and K_2 such that for all n-element vectors $\underline{x}$,

$$K_1||\underline{x}||_a \leqq ||\underline{x}||_b \leqq K_2||\underline{x}||_a.$$

Similarly to the usual notation $\mathbb{R}$ of the set of real numbers, we use the notation $\mathbb{R}^n$ for the set of all n-element real vectors. An open ball with center $\underline{x} \in \mathbb{R}^n$ and radius r is defined as

$$B(\underline{x}, r) = \left\{\underline{y}|\underline{y} \in \mathbb{R}^n, ||\underline{x} - \underline{y}|| < r\right\},$$

and the corresponding closed ball is given similarly as

$$\overline{B}(\underline{x}, r) = \left\{\underline{y}|\underline{y} \in \mathbb{R}^n, ||\underline{x} - \underline{y}|| \leqq r\right\}.$$

Let $D \subseteq \mathbb{R}^n$ be any set of n-element vectors. A point $\underline{x} \in D$ is called *interior*, if there is an $r > 0$ such that

$$B(\underline{x}; r) \subseteq D$$

that is, set D contains point $\underline{x}$ and an open ball centered at $\underline{x}$. A point $\underline{x} \in \mathbb{R}^n$ is a *boundary* point of D if every ball $B(\underline{x}, r)$ with $r > 0$ contains infinitely many points of D and also infinitely many points which do not belong to D. Set D is called *open* if its every point is interior. Set D is called *closed* if it contains its all boundary points. Clearly the complement of an open set is closed, and the complement of a closed set is open.

A set $D \subseteq \mathbb{R}^n$ is called *bounded*, if there is a constant K such that $||\underline{x}|| \leqq K$ for all $\underline{x} \in D$. Since the norms are equivalent, this definition does not depend on the norm selection. The closed and bounded subsets of $\mathbb{R}^n$ are called *compact*.

An important property of finite dimensional vector spaces is the following, which is known as the Bolzano-Weierstrass theorem:

Let $\{\underline{x}_k\}$ be a bounded infinite sequence of n-element vectors. Then it has a convergent subsequence. In one-dimension for sequences of real numbers this statement can be proved easily. Its n-dimensional extension can be shown by selecting a subsequence where the first component is convergent. Then taking a subsequence of this where the second component converges, then do the same with the third component, and so on. This very important property of finite dimensional vector spaces is the basis for proving many other results.

Assume next that D is compact in $\mathbb{R}^n$ and let $f : D \mapsto \mathbb{R}$ be a vector-variable, real-valued continuous function. Then there is a point $\underline{x}^* \in D$ such that

$$f(\underline{x}^*) = \max\left\{f(\underline{x})|\underline{x} \in D\right\}.$$

In other words, a continuous function on a compact set D reaches its maximal value on D. Since $-f$ is also continuous, function f also reaches its minimal value on D. This property is known as the Weierstrass theorem.

Assume again that D is compact in $\mathbb{R}^n$, and a vector $\underline{x} \in \mathbb{R}^n$ does not belong to D. Then there is a point $\underline{y} \in D$ such that

$$||\underline{x} - \underline{y}|| = \min \left\{ ||\underline{x} - \underline{z}|| \mid \underline{z} \in D \right\},$$

that is, there is a point $\underline{y} \in D$ with minimum distance from $\underline{x}$.

In examining the structure of vector spaces linear mappings $\underline{x} \mapsto \underline{Ax}$ are often examined where $\underline{x} \in \mathbb{R}^n$, and $\underline{A}$ is an nth order square matrix. We are often interested in how large the image of a given vector can be. For this reason we introduce matrix norms in the following way. Let $|| \cdot ||$ be a given vector norm in $\mathbb{R}^n$, and compute the quantity

$$\max \left\{ ||\underline{Ax}|| \mid ||\underline{x}|| = 1 \right\}$$

which shows the largest norm of the images of the points from the unit ball with respect to the selected vector norm.

This quantity is considered as the norm of matrix $\underline{A}$, $||\underline{A}||$, generated from (or associated to) the vector norm $|| \cdot ||$. It can be shown that any such matrix norm satisfies the following properties;

(a) $||\underline{A}|| \geqq 0$ and $||\underline{A}|| = 0$ if and only if $\underline{A} = \underline{0}$;
(b) $||\alpha \underline{A}|| = |\alpha| \cdot ||\underline{A}||$ for all $n \times n$ matrices and real numbers α;
(c) $||\underline{A} + \underline{B}|| \leqq ||\underline{A}|| + ||\underline{B}||$ for all $n \times n$ matrices $\underline{A}$ and $\underline{B}$;
(d) $||\underline{AB}|| \leqq ||\underline{A}|| \cdot ||\underline{B}||$ for all $n \times n$ matrices $\underline{A}$ and $\underline{B}$;
(e) $||\underline{Ax}|| \leqq ||\underline{A}|| \cdot ||\underline{x}||$ for all $n \times n$ matrices $\underline{A}$ and n-element vectors $\underline{x}$, where the matrix norm is generated from the vector norm being in both sides of the inequality.

Let a_{ij} denote the (i, j) element of matrix $\underline{A}$, then the matrix norms generated by vector norms $|| \cdot ||_1$, $|| \cdot ||_2$ and $|| \cdot ||_\infty$ are as follows:

$$||\underline{A}||_1 = \max \sum_{i=1}^{n} |a_{ij}| \quad \text{(column norm)}$$

$$||\underline{A}||_2 = \max \sqrt{\lambda_{A^T A}} \quad \text{(Euclidean norm)}$$

where $\lambda_{A^T A}$ denotes the eigenvalues of matrix $\underline{A}^T \underline{A}$. Notice that this matrix is positive semidefinite with nonnegative eigenvalues. Furthermore

$$||\underline{A}||_\infty = \max \sum_{j=1}^{n} |a_{ij}| \text{ (row norm).}$$

In addition to these matrix norms, the Frobenius matrix norm has certain importance in applications:

$$||\underline{A}||_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}.$$

It is easy to show that it satisfies properties a) - d) of matrix norms and satisfies property e) with the vector norm $||.||_2$, that is, for all $n \times n$ matrices $\underline{A}$ and n-element vectors $\underline{x}$,

$$||\underline{Ax}||_2 \leqq ||\underline{A}||_F \cdot ||\underline{x}||_2.$$

We mention here, that the Frobenius norm cannot be generated from any vector norm, since $||\underline{I}||_F = \sqrt{n}$ and with any matrix norm generated from a vector norm,

$$||\underline{I}|| = \max\left\{||\underline{Ix}|| \mid ||\underline{x}|| = 1\right\} = 1.$$

Let $\underline{A}$ be an $n \times n$ real matrix and λ one of its eigenvalues. Then $|\lambda| \leqq ||\ \underline{\mathrm{A}}\ ||$ with any matrix norm. In examining the stability of discrete dynamic systems the order of magnitude of the eigenvalues plays an important role. Matrix norms can provide a simple bound. More details and proofs of the facts discussed above can be found for example in Szidarovszky and Molnàr (2002).

Appendix B
Convexity, Concavity

Let D $\subseteq \mathbb{R}^n$ be an arbitrary set. We say that D is *convex*, if for all $\underline{x}, \underline{y} \in D$ and $0 \leqq \alpha \leqq 1$, the point $\alpha\underline{x} + (1-\alpha)\underline{y}$ also belongs to D. That is, with any two points a convex set also contains the linear segment between the two points. Clearly, any intersection of convex sets is also convex, but the union of convex sets is not necessarily convex. Assume next that $D \subseteq \mathbb{R}^n$ is a convex set. A real-valued function $f : D \mapsto \mathbb{R}$ defined on D is called *convex*, if for all $\underline{x}, \underline{y} \in D$ and $0 \leqq \alpha \leqq 1$ (Fig. B.1),

$$f\big(\alpha\underline{x} + (1-\alpha)\underline{y}\big) \leqq \alpha f(\underline{x}) + (1-\alpha) f(\underline{y}). \tag{B.1}$$

Function $f : D \mapsto \mathbb{R}$ with D being a convex set is called *strictly convex*, if for all $\underline{x}, \underline{y} \in D,\ \underline{x} \neq \underline{y}$ and $0 < \alpha < 1$,

$$f\big(\alpha\underline{x} + (1-\alpha)\underline{y}\big) < \alpha f(\underline{x}) + (1-\alpha) f(\underline{y}). \tag{B.2}$$

A function is called (strictly) concave if $-f$ is (strictly) convex, that is in (B.1) or (B.2) the opposite inequality direction holds. Assume next, that $f : D \mapsto \mathbb{R}$ is convex and differentiable. Then from (B.1),

$$\frac{f\big(\underline{y} + \alpha(\underline{x} - \underline{y})\big) - f(\underline{y})}{\alpha} \leqq f(\underline{x}) - f(\underline{y})$$

and if $\alpha \to 0$, then the left hand side converges to the derivative of $f\big(\underline{y} + \alpha(\underline{x} - \underline{y})\big)$ with respect to α, therefore

$$\bigtriangledown f(\underline{y})(\underline{x} - \underline{y}) \leqq f(\underline{x}) - f(\underline{y}) \tag{B.3}$$

and by interchanging $\underline{x}$ and $\underline{y}$,

$$\bigtriangledown f(\underline{x})(\underline{y} - \underline{x}) \leqq f(\underline{y}) - f(\underline{x}) \tag{B.4}$$

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

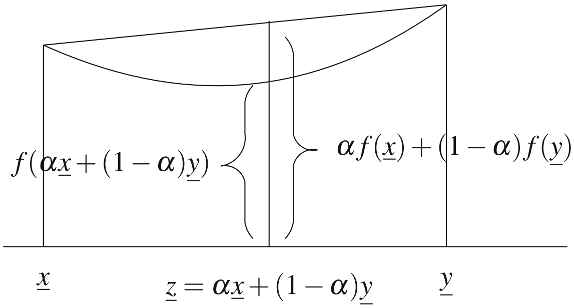

Fig. B.1 Convex function

where $\bigtriangledown f$ is the gradient vector of f the component of which are the partial derivatives of f with respect to its variables. Note, that for concave functions (B.3) and (B.4) hold with opposite inequality directions.

In minimizing convex functions an important fact is the following. Let $D \subseteq \mathbb{R}^n$ be a closed convex set and function $f : D \mapsto \mathbb{R}$ a continuous convex function. Define the minimum set of f as follows:

$$f_{min} = \left\{\underline{x} | \underline{x} \in D,\ f(\underline{x}) \leqq f(\underline{y}) \text{ for all } \underline{y} \in D\right),$$

then f_{min} is either empty or a closed, convex set.

The uniqueness of minimizer is not true in general, it is sufficient to consider a constant function which is convex and concave.

However, if f is strictly convex (concave) then f_{min} (f_{max}) is either empty or has only one point. That is, the minimizer (maximizer) of strictly convex (concave) functions cannot be multiple.

The convexity and concavity of single variable real valued functions can be verified by checking the monotonicity of its derivative. Assume that f is convex and differentiable on a finite or infinite interval.

Since $\bigtriangledown f = f'$, if $x < y$ then relations (B.3) and (B.4) imply that

$$f'(y) \geqq \frac{f(x) - f(y)}{x - y} \geqq f'(x), \tag{B.5}$$

so f' is an increasing function. If f'' exists, then it implies that $f''(x) \geqq 0$ for all x from the interior of the interval. This condition is also sufficient for the convexity of function f.

If $f''(x) > 0$ for all x from the interior of the interval, then f is strictly convex. If f is concave, then (B.5) holds with opposite inequality directions, so f' is decreasing and $f''(x) \leqq 0$ for all x. If $f''(x) < 0$ for all x, then function f is strictly concave.

In the case of vector variable functions the Hessian matrix is used, which is an $n \times n$ matrix, when n is the number of variables of the function. Its (i, j) element is the second order partial derivative

$$\frac{\partial^2 f}{\partial x_i \partial x_j}$$

and so the Hessian matrix, usually denoted by $\underline{H}(\underline{x})$, is symmetric with real eigenvalues. Function f defined on a convex set is convex if and only if $\underline{H}(\underline{x})$ is positive semidefinite for all $\underline{x}$, and if it is positive definite, then f is strictly convex. Similarly, if f is concave, then $\underline{H}(\underline{x})$ is negative semidefinite for all $\underline{x}$, and if it is negative definite, then f is strictly concave.

In order to check if $\underline{H}(\underline{x})$ is definite or semidefinite, its eigenvalues are usually determined. For positive (negative) semidefinite matrices the eigenvalues are nonnegative (nonpositive) and for positive (negative) definite matrices the eigenvalues are all positive (negative). If there are positive and negative eigenvalues than the function is neither convex nor concave.

A comprehensive summary of the properties of convex sets and functions can be found for example in Nikaido (1968).

Appendix C
Optimum Conditions

Consider first a differentiable real-valued function f defined on a finite or infinite interval. If it has its minimum or maximum in an interior point, then the derivative of f equals zero at the optimum point. However if the interval is bounded, then the optimum might occur either in an interior point or at an endpoint of the interval. So we have to find all stationary points and compare the function values at the endpoints and all stationary points and select the largest or smallest value(s). If a function f is differentiable and concave in a closed interval $[a, b]$, then $x = a$ gives maximum if $f'(a) \leqq 0$, value $x = b$ gives maximum if $f'(b) \geqq 0$. If $f'(a) > 0$ and $f'(b) < 0$, then the optimum is interior, where $f'(x) = 0$. If f is a multivariable differentiable function defined on a set $D \subseteq \mathbb{R}^n$ and has its maximum or minimum in an interior point of D, then all first order partial derivatives of f are equal to zero. That is, the gradient vector is zero at the optimum. If optimum occurs in the boundary of D, then the situation is much more complicated than in the one-dimensional case. We need to find equations describing the points on the boundary and add these equations as constraints to the optimization problem to get the optimum at the boundary.

Consider now a constrained optimum problem

$$\begin{aligned} &\text{maximize } f(\underline{x}) \\ &\text{subject to } \underline{g}(\underline{x}) = \underline{0} \end{aligned} \tag{C.1}$$

where f and $\underline{g}$ are continuously differentiable. This problem can be reduced to an unconstrained optimum problem by introducing the *Lagrange multipliers* as a vector $\underline{u}$ with the same dimension as $\underline{g}$, and considering the unconstrained problem

$$\text{maximize } f(\underline{x}) + \underline{u}^T \underline{g}(\underline{x}). \tag{C.2}$$

If $\underline{x}^*$ is an optimal solution of (C.1), then there is an $\underline{u}^*$ such that $(\underline{x}^*, \underline{u}^*)$ is a stationary point of (C.2). We note here, that not all stationary points of (C.2) give optimal solution of (C.1), so the Lagrange - multipliers are only necessary and not always sufficient conditions.

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

In the cases of inequality constraints the *Kuhn-Tucker conditions* give necessary optimum conditions. Consider therefore the optimum problem

$$\begin{aligned} &\text{maximize } f(\underline{x}) \\ &\text{subject to } \underline{g}(\underline{x}) \geqq \underline{0}. \end{aligned} \tag{C.3}$$

It can be rewritten as

$$\begin{aligned} &\text{maximize } f(\underline{x}) \\ &\text{subject to } g_i(\underline{x}) - v_i^2 = 0 \, (i = 1, 2, \ldots, m) \end{aligned}$$

where $\underline{g}$ has m components, and g_i is its ith component.

By applying Lagrange multipliers to this problem after a lengthy derivation the following conditions can be obtained: If $\underline{x}^*$ is an optimal solution of (C.3), then there exists an $\underline{u}^* \in \mathbb{R}^m$ such that

$$\underline{u}^* \geqq \underline{0} \tag{C.4}$$

$$\underline{g}(\underline{x}^*) \geqq \underline{0} \tag{C.5}$$

$$\nabla f(\underline{x}^*) + \underline{u}^{*T} \nabla \underline{g}(\underline{x}^*) = \underline{0}^T \tag{C.6}$$

$$\underline{u}^{*T} \underline{g}(\underline{x}^*) = 0. \tag{C.7}$$

The meanings of these conditions can be explained as follows. The nonnegativity of the Kuhn-Tucker multipliers is required, and (C.5) shows that the original constraints of problem (C.3) have to be satisfied. In (C.6), ∇f is the gradient vector of f as a row vector, $\nabla \underline{g}(\underline{x}^*)$ is the Jacobian matrix of $\underline{g}$, and $\underline{0}^T$ is a zero row vector with the same dimension as $\underline{x}$. Condition (C.6) shows that the gradient of the objective function f is a linear combination of the gradients of the constraints g_i. The last condition is known as the complementarity condition, which can be rewritten as

$$u_i^* g_i(\underline{x}^*) = 0, i = 1, 2, \ldots, m,$$

since all components of $\underline{u}^*$ and $\underline{g}(\underline{x}^*)$ are nonnegative, and the sum of nonnegative numbers is zero if and only if all terms are equal to zero.

The Kuhn-Tucker conditions are only necessary similarly to the Lagrange multipliers. One can easily derive the Lagrange multiplier method from the Kuhn-Tucker conditions by rewriting the equality constraint $\underline{g}(\underline{x}) = \underline{0}$ as two inequalities

$$\begin{aligned} \underline{g}(\underline{x}) &\geqq \underline{0} \\ -\underline{g}(\underline{x}) &\geqq \underline{0} \end{aligned}$$

and applying the Kuhn-Tucker conditions.

The interested reader can find more details in any reference on nonlinear optimization, for example in Forgó (1988) or in Bazara et al. (2006).

Appendix D
Fixed Point Theorems

Let D be an arbitrary set and $f : D \mapsto D$ a single-valued mapping defined on D mapping D into itself. A point $x \in D$ is called a *fixed point* of f, if $x = f(x)$. Clearly we need special conditions on both set D and mapping f in order to guarantee the existence and maybe even the uniqueness of fixed points. One of the most often applied fixed point result is the well known Brouwer fixed point theorem (Brouwer, 1912) :

Let $D \subseteq \mathbb{R}^n$ be a nonempty, closed, convex, bounded set and $f : D \mapsto D$ a continuous function. Then function f has at least one fixed point on D.

In one dimension it is very easy to prove this result, since $D = [a, b]$ is a closed bounded interval (Fig. D.1).

If $f(a) = a$, then a is a fixed point, so we may assume that $f(a) > a$. If $f(b) = b$, then b is a fixed point, so we may assume that $f(b) < b$. So at $x = a$, the curve of f is above the 45 degree line and at $x = b$ it is under it. So between a and b the curve must intercept the 45 degree line giving a fixed point.

Instead of trying to show the main ideas of the proof we can easily show that all conditions of the theorem are really needed by giving examples. In each of them one condition is violated and all others still hold and the mapping has no fixed point:

(a) D is not convex: $D = [-2, -1] \cup [1, 2]$, $f(x) = -x$
(b) D is not closed: $D = (0, 1]$, $f(x) = \frac{x}{2}$
(c) D is not bounded: $D = [0, \infty)$, $f(x) = x + 1$
(d) f is not continuous: $D = [0, 1]$,

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

The constant mapping shows that the uniqueness of the fixed point is not guaranteed in general.

A nice generalization of the Brouwer fixed point theorem was introduced and proved by Kakutani (1941). Let now D be any set and $f : D \mapsto 2^D$ a point to set

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

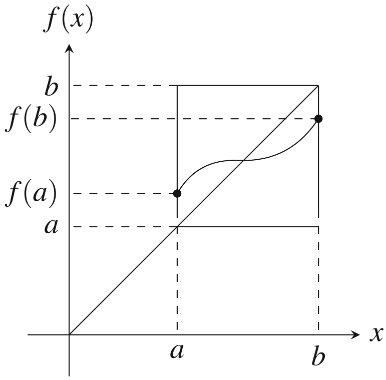

Fig. D.1 Illustration of Brouwer's fixed point theorem

mapping defined on D which maps all points of D into subsets of D. A point $x \in D$ is called a *fixed point* of f if $x \in f(x)$. The existence of at least one fixed point is guaranteed under the following conditions. If $D \subseteq \mathbb{R}^n$ is a nonempty, convex, closed, bounded set, $f(x)$ for all $x \in D$ is nonempty, convex, closed set and the graph of mapping f,

$$G_f = \{(x, y) | x \in D,\ y \in f(x)\}$$

is closed, then there is at least one $x \in D$ such that $x \in f(x)$.

Notice that if f is point-to-point (that is, $f(x)$ has only one point), then the Kakutani fixed point theorem contains Brouwer's theorem as special case.

If these fixed point theorems are used to prove the existence of Nash equilibrium, then uniqueness is not guaranteed resulting in the equilibrium selection problem, which might become a difficult problem to solve.

By relaxing the conditions on set D and by assuming more restrictive conditions on mapping f not only existence but even the uniqueness of fixed points can be guaranteed.

In order to formulate Banach's fixed point theorem we have to introduce the *contraction* property. Let $D \in \mathbb{R}^n$ be a set and $\underline{f} : D \mapsto D$ a point-to-point mapping. It is called a contraction, if there is a constant $q \in [0, 1)$ such that for all $\underline{x}, \underline{y} \in D$,

$$||\underline{f}(\underline{x}) - \underline{f}(\underline{y})|| \leq q \cdot ||\underline{x} - \underline{y}||. \tag{D.1}$$

Notice that this condition depends on the selected norm. So it is often the case that a mapping is contraction in one norm but not in another. Notice also, that any contraction mapping is continuous, since if sequence $\underline{x}_k \to \underline{x}^*$, then

$$0 \leqq ||\underline{f}(\underline{x}_k) - \underline{f}(\underline{x}^*)|| \leqq q||\underline{x}_k - \underline{x}^*|| \to 0 \text{ as } k \to \infty.$$

The existence and uniqueness result is the following (Banach, 1922):

Assume $D \in \mathbb{R}^n$ is a nonempty, closed set and mapping $\underline{f}$ is a contraction on D. Then $\underline{f}$ has exactly one fixed point on D, which can be computed by the following iteration process:

$\underline{x}_0 \in D$ is arbitrary,

$$\underline{x}_{k+1} = \underline{f}(\underline{x}_k).$$

Sequence $\{\underline{x}_k\}$ always converges to the unique fixed point regardless of the initial point $\underline{x}_0$.

Notice that example b) after Brouwer's theorem shows that D has to be closed and example c) shows that we cannot drop the contraction assumption, since in that case $q = 1$.

In comparing the Brouwer and Banach fixed point theorems we can observe the following: The conditions on set D are more restrictive in Brouwer's theorem, since in the case of Banach's result D does not need to be convex and bounded. However, Brouwer's result is less restrictive with mapping f, since contraction implies continuity but not otherwise. For example, $f(x) = x^2$ on set $D = [0, 2]$ is continuous but not a contraction, since

$$x^2 - y^2 = (x + y)(x - y)$$

and $x + y$ can have values larger than unity.

In checking the conditions of the Brouwer fixed point theorem we need to verify the continuity of mapping $\underline{f}$ in addition to simple properties of set D. However to verify the contraction property of mapping $\underline{f}$ needs further elaboration.

If $f : D \mapsto \mathbb{R}$ is a differentiable real function defined on an interval D, then it is contraction if there is a scalar $q \in [0, 1)$ such that $|f'(x)| \leqq q$ for all $x \in D$. This observation follows immediately from the fact that for all $x, y \in D$,

$$|f(x) - f(y)| = |\int_x^y f'(z)dz| \leqq |\int_x^y |f'(z)|dz| \leqq |\int_x^y qdz| = q \cdot |x - y|.$$

If $\underline{f} : D \mapsto \mathbb{R}^n$ is a differentiable vector valued function defined on a convex set $D \in \mathbb{R}^n$, then $\underline{f}$ is contraction in a vector norm, if with the associated matrix norm and $q \in [0, 1)$,

$$|| \bigtriangledown \underline{f}(\underline{z})|| \leq q \text{ for all } \underline{z} \in D$$

where $\bigtriangledown \underline{f}$ is the Jacobian matrix of $\underline{f}$ with (i, j) element $\frac{\partial f_i}{\partial z_j}$.

This fact is a simple consequence of the relation

$$||\underline{f}(\underline{x}) - \underline{f}(\underline{y})|| = ||\int_0^1 \bigtriangledown \underline{f}\big(\underline{y} + t(\underline{x} - \underline{y})\big)(\underline{x} - \underline{y})dt||$$

$$\leqq \int_0^1 || \bigtriangledown \underline{f}\big(\underline{y} + t(\underline{x} - \underline{y})\big)|| \cdot ||\underline{x} - \underline{y}||dt \leqq \int_0^1 q \cdot ||\underline{x} - \underline{y}||dt = q \cdot ||\underline{x} - \underline{y}||.$$

The above conditions require that $f'(z)$ has to be small in the single-dimensional case and all partial derivatives of all components of $\underline{f}$ have to be sufficiently small if $\underline{f}$ is vector-valued. We mention here as well, that function $\underline{f}$ can be contraction in one norm and not in another as the following examples show: Consider Jacobians

$$\underline{A}_1 = \begin{pmatrix} 0.8 & 0.8 \\ 0 & 0 \end{pmatrix}, \; \underline{A}_2 = \begin{pmatrix} 0.8 & 0 \\ 0.8 & 0 \end{pmatrix} \text{ and } \underline{A}_3 = \begin{pmatrix} 0.51 & 0.51 \\ 0.51 & 0 \end{pmatrix}$$

then

$$||\underline{A}_1||_1 = 0.8 < 1, \; ||\underline{A}_1||_\infty = 1.6 > 1, \; ||\underline{A}_1||_2 = \sqrt{1.28} > 1$$
$$||\underline{A}_2||_1 = 1.6 > 1, \; ||\underline{A}_2||_\infty = 0.8 < 1, \; ||\underline{A}_2||_2 = \sqrt{1.28} > 1$$
$$||\underline{A}_3||_1 = 1.02 > 1, \; ||\underline{A}_3||_\infty = 1.02 > 1, \; ||\underline{A}_3||_2 = \sqrt{\frac{3+\sqrt{5}}{2}} \simeq 0.825 < 1.$$

We mention in addition that a slightly relaxed form of the contraction property guerantees uniqueness of fixed points but does not guarantee existence. Instead of (D.1) assume that

$$||\underline{f}(\underline{x}) - \underline{f}(\underline{y})|| < ||\underline{x} - \underline{y}||. \tag{D.2}$$

Then mapping $\underline{f}$ cannot have multiple fixed points, since if both $\underline{x}^{(1)}$ and $\underline{x}^{(2)}$ would be fixed points, then we would have

$$||\underline{f}\big(\underline{x}^{(1)}\big) - \underline{f}\big(\underline{x}^{(2)}\big)|| = ||\underline{x}^{(1)} - \underline{x}^{(2)}||$$

contradicting to (D.2).

Appendix E
Monotonic Mappings

In the case of single-variable, single-valued functions their monotonic properties are strongly related to the uniqueness of solutions of equations $f(x) = 0$ and $x = f(x)$. It is well known that function f is *increasing* if $x < y$ implies $f(x) \leq f(y)$, and is *strictly increasing* if $x < y$ implies $f(x) < f(y)$. Similarly, f is *decreasing* if $x < y$ implies $f(x) \geq f(y)$ and is *strictly decreasing* if $x < y$ implies $f(x) > f(y)$.

Consider first equation $f(x) = 0$. If f is either strictly increasing or strictly decreasing, then this equation cannot have multiple solutions. Otherwise if there are two solutions $x^* < y^*$, then either $f(x^*) < f(y^*)$ or $f(x^*) > f(y^*)$, so it is impossible that both function values are equal to zero.

In the case of equation $x = f(x)$ we have a similar observation: if f is decreasing, then the equation cannot have multiple solutions. Otherwise, if $x^* < y^*$ are two different solutions, then

$$x^* = f(x^*) \geqq f(y^*) = y^*$$

giving an obvious contradiction.

The monotonicity of vector valued and vector variable functions can be defined as the n-dimensional generalization of the single-dimensional concept. One way is to assume that the function is monotonic in each variable. That is, if $\underline{f}(x_1, \ldots, x_n)$ increases if any x_i value increases, then $\underline{f}$ is considered as an increasing function. If $\underline{f}(x_1, \ldots, x_n)$ strictly increases in all variables, then $\underline{f}$ is called strictly increasing. Decreasing and strictly decreasing functions are defined similarly.

Unfortunately this kind of monotonicity cannot guarantee the uniqueness of the solutions of equations $\underline{f}(\underline{x}) = \underline{0}$ and $\underline{x} = \underline{f}(\underline{x})$ as the following simple examples illustrate.

Consider first equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} -x_1 - x_2 &= 0 \\ -2x_1 - 2x_2 &= 0 \end{aligned}$$

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

where the left hand sides are strictly increasing in the first case and strictly decreasing in the second. However both systems have infinitely many solutions, $x_2 =$ arbitrary and $x_1 = -x_2$.

Consider next the fixed point problem

$$x_1 = -x_1 - 2x_2$$
$$x_2 = -2x_1 - x_2$$

where the right hand sides are strictly decreasing, but there are again infinitely many solutions: $x_2 =$ arbitrary and $x_1 = -x_2$.

Another way of defining n-dimensional monotonicity is the following. Let $\underline{f} : D \mapsto \mathbb{R}^n$ be a function where $D \subseteq \mathbb{R}^n$.

It is called *monotonic*, if for all $\underline{x},\ \underline{y} \in D$,

$$(\underline{x} - \underline{y})^T \left(\underline{f}(\underline{x}) - \underline{f}(\underline{y})\right) \geqq 0. \tag{E.1}$$

This clearly generalizes the concept of increasing single-dimensional functions, since if $x < y$, (E.1) can hold if and only if $f(x) \leqq f(y)$.

We say similarly that function $\underline{f}$ is *strictly monotonic*, if for all $\underline{x}, \underline{y} \in D$ and $\underline{x} \neq \underline{y}$,

$$(\underline{x} - \underline{y})^T \left(\underline{f}(\underline{x}) - \underline{f}(\underline{y})\right) > 0. \tag{E.2}$$

This condition clearly generalizes the concept of strictly increasing real functions, since if $x < y$, then (E.2) holds if and only if $f(x) < f(y)$.

Assume now that D is a convex set and $\underline{f}$ is differentiable. Let $\underline{J}(\underline{x})$ denote the Jacobian matrix of $\underline{f}$, the (i, j) element of which is $\frac{\partial f_i(\underline{x})}{\partial x_j}$, where f_i denotes the ith component of $\underline{f}$ and x_j is the jth component of $\underline{x}$. The monotonicity of the function $\underline{f}$ can be characterized with the definiteness of its Jacobian. The two fundamental facts are as follows. Function $\underline{f}$ is monotonic if and only if $\underline{J}(\underline{x}) + \underline{J}(\underline{x})^T$ is positive semidefinite for all $\underline{x} \in D$, and if $\underline{J}(\underline{x}) + \underline{J}(\underline{x})^T$ is positive definite for all $\underline{x} \in D$, then $\underline{f}$ is strictly monotonic. Notice that in the one-dimensional case $\underline{J}(\underline{x}) = f'(x)$, which is positive semidefinite as a 1×1 matrix if $f'(x) \geqq 0$, and it is positive definite if $f'(x) > 0$.

So the n-dimensional definiteness conditions reduce to the usual single-dimensional conditions for $n = 1$ concerning the derivatives of the functions. This kind of monotonicity can also be used to guarantee uniqueness of solutions of equations of the types $\underline{f}(\underline{x}) = \underline{0}$ and $\underline{x} = \underline{f}(\underline{x})$.

Assume first that either $\underline{f}$ or $-\underline{f}$ is strictly monotonic. Then equation $\underline{f}(\underline{x}) = \underline{0}$ cannot have multiple solutions. If it does, then we have $\underline{x}^{(1)} \neq \underline{x}^{(2)}$ such that $\underline{f}(\underline{x}^{(1)}) = \underline{f}(\underline{x}^{(2)}) = 0$ implying that

$$\underline{0} = (\underline{x}^{(1)} - \underline{x}^{(2)})^T \left(\underline{f}(\underline{x}^{(1)}) - \underline{f}(\underline{x}^{(2)})\right)$$

where the right hand side is either positive or negative giving a contradiction. Assume next that $-\underline{f}$ is monotonic. Then equation $\underline{x} = \underline{f}(\underline{x})$ cannot have multiple solutions. Assume it does, then we have $\underline{x}^{(1)} \neq \underline{x}^{(2)}$ such that $\underline{f}(\underline{x}^{(1)}) = \underline{x}^{(1)}$ and $\underline{f}(\underline{x}^{(2)}) = \underline{x}^{(2)}$.

Then clearly

$$0 \geqq (\underline{x}^{(1)} - \underline{x}^{(2)})^T \big(\underline{f}(\underline{x}^{(1)}) - \underline{f}(\underline{x}^{(2)})\big) = (\underline{x}^{(1)} - \underline{x}^{(2)})^T (\underline{x}^{(1)} - \underline{x}^{(2)}) = ||\underline{x}^{(1)} - \underline{x}^{(2)}||_2^2 > 0$$

giving a contradiction.

A good survey of generalized monotonicity is given in Schaible (1994), it can be recommended as further reading.

Appendix F
Duality in Linear Programming

A linear programming (LP) problem is defined as a constrained optimization problem, where the unknown is an n-element vector $\underline{x}$, the objective function and all constraints are linear and each constraint is either $\geqq$, $=$ or $\leqq$ type. The primal form of an LP problem can be obtained in several steps. If there is no sign restriction on a variable x_i, then it can be represented as $x_i = x_i^+ - x_i^-$, where both new variables x_i^+ and x_i^- are assumed to be nonnegative.

If a constraint has the form $\underline{a}_i^T \underline{x} \geqq b_i$, then it can be rewritten as a $\leqq$ type constraint by multiplying both sides by (-1) to get : $(-\underline{a}_i^T)\underline{x} \leqq -b_i$. If a constraint is of equation type, $\underline{a}_i^T \underline{x} = b_i$, then it can be replaced by two inequalities:

$$
\begin{aligned}
\underline{a}_i^T \underline{x} &\geqq b_i \\
\underline{a}_i^T \underline{x} &\leqq b_i
\end{aligned}
$$

where we can multiply the first condition by (-1) to get

$$
\begin{aligned}
(-\underline{a}_i^T)\underline{x} &\leqq (-b_i) \\
\underline{a}_i^T \underline{x} &\leqq b_i .
\end{aligned}
$$

After these modifications are done, the LP problem will have the following *primal form*:

$$
\begin{aligned}
\text{maximize } & \underline{c}^T \underline{x} \\
\text{subject to } & \underline{x} \geqq \underline{0} \\
& \underline{Ax} \leqq \underline{b}.
\end{aligned}
\tag{F.1}
$$

If the original problem minimizes the objective function then by multiplying the objective function by (-1) transforms it into a maximum problem.

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

To each primal form we can introduce its *dual form*:

$$\begin{aligned} \text{minimize } & \underline{b}^T \underline{y} \\ \text{subject to } & \underline{y} \geqq \underline{0} \\ & \underline{A}^T \underline{y} \geqq \underline{c}. \end{aligned} \tag{F.2}$$

It is clear that the dual of the dual problem (F.2) is the primal problem (F.1).

The relation of the primal and dual of an LP problem can be expressed by the following duality theorems.

The *weak duality property* states that if $\underline{x}$ and $\underline{y}$ are feasible solutions of the primal and dual problems, respectively, then

$$\underline{c}^T \underline{x} \leqq \underline{b}^T \underline{y}. \tag{F.3}$$

This inequality can be proved easily, since

$$\underline{c}^T \underline{x} \leqq (\underline{A}^T \underline{y})^T \underline{x} = \underline{y}^T \underline{Ax} \leqq \underline{y}^T \underline{b} = \underline{b}^T \underline{y}.$$

An important corollary of this theorem is that if the primal has unbounded objective function, then the dual has no feasible solution. Similar argument can be concluded in an analogous manner for the dual problem. It is also obvious that if both the primal and the dual problems have feasible solutions, then both objectives are bounded by (F.3). If in addition $\underline{x}$ and $\underline{y}$ are feasible solutions of the primal and dual problems, respectively, such that $\underline{c}^T \underline{x} = \underline{b}^T \underline{y}$, then both are optimal for their respective problems.

The *strong duality* theorem goes one step further asserting that if the primal (dual) problem has a finite optimal solution, then so does the dual (primal) and the two optimal objective values are equal. Padberg (1999) offers a comprehensive summary of duality and related subjects.

Appendix G
Multiobjective Optimization

In a single objective optimization problem

$$\begin{aligned} &\text{maximize } f(\underline{x}) \\ &\text{subject to } \underline{x} \in X, \end{aligned} \tag{G.1}$$

where X is the set of feasible solutions, the values of $f(\underline{x})$ when $\underline{x}$ runs through set X is a subset of the real line. Therefore any optimal solution satisfies the following properties:

(a) optimal solution is at least as good as any other solution;
(b) there is no better solution;
(c) all optimal solutions have the same objective function value.

The last property tells us that it does not matter which optimal solution is selected, since all of them provide the same outcome as the objective function is concerned.

A multiobjective optimization problem is usually formulated as

$$\begin{aligned} &\text{maximize } \underline{f}(\underline{x}) = \big(f_1(\underline{x}), f_2(\underline{x}), \ldots, f_n(\underline{x})\big) \\ &\text{subject to } \underline{x} \in X. \end{aligned} \tag{G.2}$$

We do not restrict generality by assuming that all objectives are maximum types, since in the case of a minimum type objective it can be transformed into maximum type by multiplying it by (-1). Maximizing each objective function individually on the feasible set X we can get n optimal solutions, which are usually different, so there is no optimal solution of problem (G.2) which would satisfy properties (a), (b) and (c). Instead of looking for a usually non-existing optimal solution, we relax the above conditions by looking for nondominated solutions. A feasible $\underline{x}$ is called *weakly nondominated* if there is no $\underline{y} \in X$ such that $f_i(\underline{y}) > f_i(\underline{x})$ for all $i = 1, 2, \ldots, n$. That is, we cannot improve all objectives simultaneously on the feasible set. A feasible $\underline{x}$ is called *strongly nondominated* if no objective can be

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

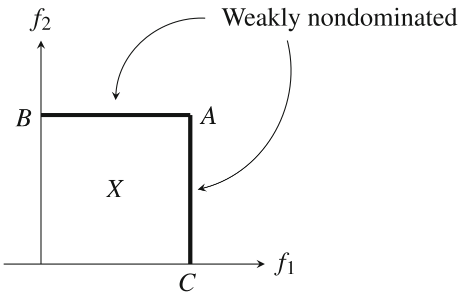

Fig. G.1 Weakly and strongly nondominated solutions

improved without worthening at least one other objective. That is, there is no $\underline{y} \in X$ such that $f_i(\underline{y}) \geqq f_i(\underline{x})$ for all $i = 1, \ldots, n$ and $\underline{f}(\underline{x}) \neq \underline{f}(\underline{y})$. Figure G.1 shows the difference between weakly and strongly nondominated solutions, where $f_1 = x_1$, $f_2 = x_2$. Notice that point A is the only strongly nondominated solution, but there are infinitely many weakly nondominated solutions: the linear segments $\overline{BA}$ and $\overline{AC}$. Clearly every strongly nondominated solution is also weakly nondominated, but the weakly nondominated solutions are not necessarily strongly nondominated (Fig. G.1).

The feasible set X shows the possible decision alternatives giving us all possibilities what we can do. In addition to X, the *objective space* is usually considered:

$$F = \{\underline{f} = (f_1, \ldots, f_n) \mid \text{there exists } \underline{x} \in X \text{ such that } f_i = f_i(\underline{x}), i = 1, 2, \ldots, n\}.$$

This set gives all possible simultaneous objective values, that is, it represents what we can get. Similarly to set X, a vector $\underline{f} \in F$ is weakly nondominated, if there is no other point $\overline{\underline{f}} \in F$ such that $\overline{\underline{f}} > \underline{f}$ in all components. A point $\underline{f} \in F$ is strongly nondominated if there is no other point $\overline{\underline{f}} \neq \underline{f}$ in F such that $\overline{\underline{f}} \geqq \underline{f}$ in every component. Clearly all weakly or strongly nondominated points of F are boundary points. In the economic literature nondominated solutions are often called Pareto optimal.

There is no guarantee in general that a problem (G.2) has nondominated solution, and even if it has, the solution is not necessarily unique. For example the problem

$$\begin{aligned} &\text{maximize } (x_1, x_2) \\ &\text{subject to } x_1^2 + x_2^2 < 1 \end{aligned}$$

has no nondominated solution, and problem

$$\begin{aligned} &\text{maximize } (x_1, x_2) \\ &\text{subject to } x_1 + x_2 \leqq 1 \end{aligned}$$

has infinitely many nondominated solutions: x_2 = arbitrary, $x_1 = 1 - x_2$.

The existence of nondominated solutions is guaranteed by the following general result. Assume F is nonempty, closed and for each $i = 1, 2, \ldots, n$ there exists a value f_i^* such that $f_i(\underline{x}) \leqq f_i^*$ for all $\underline{x} \in F$. Then problem (G.2) has at least one strongly nondominated solution.

In the case of multiple nondominated solutions the choice of the most appropriate solution needs additional preference information since they are not satisfying properties (a) and (c) of the optima of single-objective optimum problems. Different nondominated solutions usually give different objective function values, and in order to compare them we need additional preference information about n-element vectors.

Chapter 14 shows some of the most popular methods for solving multiobjective optimization problems. A comprehensive summary and discussion on the most commonly used methods can be found in many textbooks or monographs, for example in Szidarovszky et al. (1986).

Appendix H
Stability and Controllability

A time-invariant nonlinear system is given by the difference equation

$$\underline{x}(t+1) = \underline{g}\big(\underline{x}(t)\big) \tag{H.1}$$

with discrete time scales and as

$$\dot{\underline{x}}(t) = \underline{g}\big(\underline{x}(t)\big) \tag{H.2}$$

in continuous time scales where $\underline{g} : D \mapsto \mathbb{R}^n$ with $D \subseteq \mathbb{R}^n$.

It is usually assumed that $\underline{g}$ is continuous on D and starting from arbitrary initial value $\underline{x}(0)$, Eqs. (H.1) and (H.2) have unique trajectories in D. Vector $\underline{x}(t)$ is called the state of the system at time t.

The *equilibrium* or *steady state* of system (H.1) is an $\overline{\underline{x}} \in D$ such that $\overline{\underline{x}} = \underline{g}(\overline{\underline{x}})$, and that of system (H.2) is an $\overline{\underline{x}} \in D$ such that $\underline{g}(\overline{\underline{x}}) = \underline{0}$. That is, if the state becomes $\overline{\underline{x}}$ at any time, then the state will remain $\overline{\underline{x}}$ for all future times. The stability theory of dynamic systems tries to answer the question that what is the asymptotic behavior of the state trajectory if $\underline{x}(0)$ differs from the equilibrium; and under what condition $\underline{x}(t)$ approaches the equilibrium in the long run.

The equilibrium $\overline{\underline{x}}$ is called *locally asymptotically stable*, if there is an $\epsilon > 0$ such that $\underline{x}(t) \to \overline{\underline{x}}$ as $t \to \infty$ if $||\underline{x}(0) - \overline{\underline{x}}|| < \epsilon$. That is, if the initial state is selected sufficiently close to the equilibrium, then the state trajectory converges back to the equilibrium as $t \to \infty$.

An equilibrium $\overline{\underline{x}}$ is called *globally asymptotically stable* if $\underline{x}(t) \to \overline{\underline{x}}$ as $t \to \infty$ regardless of the selection of the initial state.

We can call a system asymptotically stable if its equilibrium is asymptotically stable.

One of the most frequently applied method of checking stability of nonlinear systems is the local linearization, when the right hand sides of Eqs. (H.1) and (H.2) are replaced by their linear Taylor polynomials centered at $\overline{\underline{x}}$:

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

$$\underline{g}(\underline{x}(t)) \approx \underline{g}(\overline{\underline{x}}) + \underline{J}(\overline{\underline{x}})(\underline{x}(t) - \overline{\underline{x}}) \tag{H.3}$$

where $\underline{J}(\overline{\underline{x}})$ denotes the Jacobian matrix of $\underline{g}(\underline{x})$ at $\underline{x} = \overline{\underline{x}}$.

So the linearized version of the Eq. (H.1) has the form

$$\underline{x}(t+1) = \overline{\underline{x}} + \underline{J}(\overline{\underline{x}})(\underline{x}(t) - \overline{\underline{x}})$$

and by introducing the notation $\underline{x}_\delta(t) = \underline{x}(t) - \overline{\underline{x}}$, it can be rewritten as

$$\underline{x}_\delta(t+1) = \underline{J}(\overline{\underline{x}})\underline{x}_\delta(t). \tag{H.4}$$

The linearized version of Eq. (H.2) can be written as

$$\dot{\underline{x}}(t) = \underline{J}(\overline{\underline{x}})(\underline{x}(t) - \overline{\underline{x}}),$$

that is,

$$\dot{\underline{x}}_\delta(t) = \underline{J}(\overline{\underline{x}})\underline{x}_\delta(t) \tag{H.5}$$

since $\overline{\underline{x}}$ is a constant with zero derivative. In both cases the linearized equation becomes a homogeneous linear equation and the asymptotic stability of $\overline{\underline{x}}$ in the linearized equations is equivalent to the asymptotic stability of the zero equilibrium of Eqs. (H.4) and (H.5).

It is easy to prove that in the case of linear systems local asymptotic stability implies global asymptotic stability; however this is not true for nonlinear systems. However the following fact gives a practical method to check stability of nonlinear systems. Assume that the homogeneous system (H.4) (or (H.5)) is asymptotically stable, then the nonlinear system (H.1) (or (H.2)) is locally asymptotically stable. So the asymptotic stability of the linearized system implies the local asymptotic stability of the nonlinear system. The reverse of this fact is not true, there are asymptotically stable nonlinear systems with linearized system which are not asymptotically stable. Such example in the discrete case is system

$$x(t+1) = x(t)e^{-x(t)^2}$$

with the unique equilibrium $\overline{x} = 0$, and in the continuous case

$$\dot{x}(t) = -x(t)^3$$

with $\overline{x} = 0$.

The asymptotical stability of the linear systems (H.4) and (H.5) can be decided based on the eigenvalues of the Jacobian matrix $\underline{J}(\overline{\underline{x}})$ as follows. System (H.4) is asymptotically stable if and only if $|\lambda| < 1$ for all eigenvalues of $\underline{J}(\overline{\underline{x}})$, and system (H.5) is asymptotically stable if and only if $\text{Re}\,\lambda < 0$, that is, the real parts of all eigenvalues are negative.

The eigenvalues of an $n \times n$ matrix are the roots of the nth degree characteristic polynomial, so there are n eigenvalues where multiple eigenvalues are counted with their multiplicities.

In the cases of low degree polynomials the stability conditions can be verified without solving the polynomial equations. If $n = 2$, the characteristic polynomial is quadratic :

$$\lambda^2 + a_1\lambda + a_0 = 0.$$

Then both roots have negative real parts if and only if both coefficients a_0 and a_1 are positive. It can be also proved that both roots are inside the unit circle if and only if

$$\begin{aligned} a_0 &< 1 \\ 1 + a_0 + a_1 &> 0 \\ 1 + a_0 - a_1 &> 0. \end{aligned} \tag{H.6}$$

If $n = 3$, then the characteristic polynomial is cubic:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

All roots have negative real parts if and only if all coefficients a_0, a_1, a_2 are positive and

$$a_1 \cdot a_2 > a_0. \tag{H.7}$$

In the discrete case, all roots are inside the unit circle if and only if

$$\begin{aligned} 1 + a_2 + a_1 + a_0 &> 0 \\ 1 - a_2 + a_1 - a_0 &> 0 \\ 1 - a_1 + a_2 a_0 - a_0^2 &> 0 \\ a_1 &< 3. \end{aligned} \tag{H.8}$$

A time invariant discrete linear control system can be written as

$$\underline{x}(t+1) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t),\ \underline{x}(0) = \underline{x}_0 \tag{H.9}$$

where $\underline{A}$ is $n \times n$, $\underline{B}$ is $n \times m$ constant matrix, $\underline{x}(t)$ is n-element, $\underline{u}(t)$ is an m-element vector. Here, as before, $\underline{x}(t)$ is the state of the system and $\underline{u}(t)$ is the input, or control of the system at time t. In the continuous case the system equation has the form:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t),\ \underline{x}(0) = \underline{x}_0. \tag{H.10}$$

Let $T > 0$ be a given future time period, and $\underline{x}^*$ a selected given state vector. We say that system (H.9) or (H.10) is controlable to $\underline{x}^*$ at time T if there is an input function $\underline{u}(t)$ such that the state of the system becomes $\underline{x}^*$ at $t = T$. This type of control is called the *final state control*, since we do not care about the system for $t > T$. We say that a system is completely controllable at time T, if it is controllable to any final state at time T.

The controllability of systems (H.9) and (H.10) can be easily verified by the Kalman controllablility conditions. Define the Kalman controllability matrix

$$\underline{K} = (\underline{B}, \underline{A}\underline{B}, \underline{A}^2\underline{B}, \ldots, \underline{A}^{n-1}\underline{B}) \tag{H.11}$$

which is an $n \times (mn)$ constant matrix. Then system (H.9) is completely controllable at any time $T \geqq n$, if and only if matrix $\underline{K}$ has full rank, that is, $rank(\underline{K}) = n$. System (H.10) is completely controllable at any time $T > 0$ if and only if $rank(\underline{K}) = n$.

Since matrix $\underline{K}$ has n rows, its rank cannot exceed n, so the maximum possible rank of matrix $\underline{K}$ is n.

For more details the interested reader may consult with any textbook on linear systems, for example with Szidarovszky and Bahill (1992).

References

Ahmadi, A., & Salazar Moreno, R. (2013). Game theory applications in a water distribution problem. *Journal of Water Resources and Protection, 2013*(5), 91–96.

Anbarci, N. (1993). Noncooperative foundations of the area monotonic solution. *Quarterly Journal of Economics, 108*, 245–258.

Anbarci, N., & Bigelow, J. (1994). The area monotonic solution to the cooperative bargaining problem. *Mathematical Social Sciences, 28*(2), 133–142.

Banach, S. (1922). Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales. *Fundamenta Mathematicae, 3*, 133–181.

Bazara, M. S., Sherali, H. D., & Shetty, C. M. (2006). *Nonlinear Programming*. Theory and AlgorithmsHoboken, NJ: Wiley.

Bellman, R., & Cooke, K. L. (1963). *Differential-Difference Equations*. New York, NY: Academic Press.

Bischi, G.-I., Chiarella, C., Kopel, M., & Szidarovszky, F. (2010). *Nonlinear Oligopolies: Stability and Bifurcations*. Berlin, Heidelberg: Springer.

Bogardi, I., & Szidarovszky, F. (1976). Application of game theory in water manegement. *Applied Mathematical Modeling, 1*, 16–20.

Bonner, J. (1986). *Introduction to the Theory of Social Choice*. Baltimore, MD: The Johns Hopkins University Press.

Brouwer, L. E. J. (1912). Über Abbildung von Mannigfaltigkeiten. *Mathematische Annalen, 71*, 97–115.

Chun, Y. (1988). The equal-loss principle for bargaining problems. *Economics Letters, 26*, 103–106.

Cyert, R. M., & DeGroot, M. H. (1973). An analysis of cooperation and learning in a duopoly context. *American Economic Review, 63*(1), 24–37.

Davis, M., & Maschler, M. (1965). The Kernel of a Cooperative game. *Naval Research Logistics Quarterly, 12*, 223–259.

Davis, M., & Maschler, M. (1967). Existence of stable payoff configurations for cooperative games. In M. Shubik (Ed.), *Essays in Mathematical Economics in Honor of Oskar Morgenstern* (pp. 39–52). Princeton, NJ: Princeton University Press.

Eskandari, A., Pfolliott, P., Szidarovszky, F. (1995). Uncertainty in multicriterion watershed management problems. *Technology: Journal of the Franklin Institute, 332A*, 199–207.

Flath, D. (1991). When is it rational for firms to acquire silent interests in rivals? *International Journal of Industrial Organisation, 9*, 573–583.

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3

Flath, D. (1992). Horizontal shareholding interlocks. *Managerial and Decision Economics, 13*, 75–77.

Forgó, F. (1988). *Nonconvex Programming*. Budapest, Hungary: Akadémiai Kiadó.

Gibbons, R. (1992). *Game Theory for Applied Economists*. Princeton, NJ: Princeton University Press.

Goldberg, K., Goldman, A. J., & Newman, M. (1968). The Probability of an Equilibrium Point. *Journal of Research of National Institute of Standards, 72B*, 93–101.

Harsanyi, J. (1967). Games with incomplete information played by bayesian players, Parts I, II and III. *Manegement Science, 14*, 159–182, 320–334, 486–502.

Harsanyi, J., & Selten, R. (1972). A generalized nash solution for two-person bargaining games with incomplete information. *Management Science, 18*, 80–106.

Kakutani, S. (1941). A generalization of brouwer's fixed point theorem. *Duke Journal of Mathematics, 8*, 457–459.

Kalai, E., & Smorodinsky, M. (1975). Other solutions to nash's bargaining problem. *Econometrica, 43*, 513–518.

Kuhn, H. W. (1953). Extensive Games and the Problem of Information. In H. W. Kuhn & A. W. Tucker (Eds.), *Contributions to the Theory of Games II* (pp. 193–216). Princeton, NJ: Princeton University Press.

Leontief, W. (1946). The pure theory of the guaranteed annual wage contract. *Journal of Political Economy, 54*, 76–79.

Lotka, A. J. (1910). Contributions to the theory of periodic reaction. *Journal of Physical Chemistry, 14*(3), 271–274.

Lotka, A. J. (1925). *Elements of Physical Biology*. Baltimore: Williams and Wilkins.

Mangasarian, O. L., & Stone, H. (1964). Two-person nonzero-sum games and quadratic programming. *Journal of Mathematical Analysis and Applications, 9*, 348–355.

Matsumoto, A., & Szidarovszky, F. (2012). An elementary study of a class of dynamic systems with two time delays. *CUBO, A Mathematical Journal, 14*(3), 103–113.

Matsumoto, A., & Szidarovszky, F. (2013). An Elementary Study of a Class of Dynamic Systems with Single Time Delay. *CUBO, A Mathematical Journal, 15*(3), 1–7.

Merlone, U. (2001). Cartelizing effects of horizontal shareholding interlocks. *Managerial and Decision Economics, 22*, 333–337.

Myers, S., & Majluf, N. (1984). Corporate financing and investment decisions when firms have information that investors do not have. *Journal of Financial Economics, 13*, 187–221.

Nash, J. F., Jr. (1950). Equilibrium points in N-person Games. *Proceedings of the National Academy of Sciences USA, 36*, 48–49.

Nash, J. F., Jr. (1953). Two-person cooperative games. *Econometrica, 21*, 128–140.

Nikaido, H. (1968). *Convex Structures and Economic Theory*. New York, NY: Academic Press.

Nikaido, H., & Isoda, K. (1955). Note on noncooperative convex games. *Pacific Journal of Mathematics, 5*, 807–815.

Okuguchi, K., & Szidarovszky, F. (1999). *The Theory of Oligopoly with Multi-product Firms*. Berlin, Heidelberg, New York: Springer.

Padberg, M. (1999). *Linear Optimization and Extensions* (2nd ed.). Berlin, Heidelberg, New York: Springer.

Reynolds, R. J., & Snapp, B. R. (1982). *The Economic Effects of Partial Equity Interests and Joint Ventures* (p. 3). US Department of Justice Antitrust Division, Economic Policy Office Discussion Paper.

Rinaldi, S. (1998). Love dynamics: The case of linear couples. *Applied Mathematics and Computation, 95*, 181–192.

Rinaldi, S. (1998). Laura and Petrach: An intriguing case of cyclical love dynamics. *SIAM Journal on Applied Mathematics, 58*, 1205–1221.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica, 33*, 520–534.

Ruiz, L. M., Valenciano, F., & Zarzuelo, J. M. (1996). The least square prenucleolus and the least square nucleolus. two values for TU games based on the excess vector. *International Journal of Game Theory, 25*, 113–134.

Schaible, S. (1994). Generalized monotonicity-a survey. In S. Komlosi, T. Rapcsak, & S. Schaible (Eds.), *Generalized Convexity* (pp. 229–249). Heidelberg, New York: Springer.

Schmeidler, D. (1968). The nucleolus of a characteristic function game. *SIAM Journal of Applied Mathematics, 17*, 1163–1170.

Segerson, K. (1988). Uncertainty and incentives for non-point pollution control. *Journal of Environmental Economics and Management, 15*, 87–98.

Selten, R. (1965). Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetragheit. *Zeitschrift f* ür *Gesamte Staatswissenschaft, 121*, 301–324.

Shapiro, H. N. (1958). Note on a computation method in the theory of games. *Communications on Pure and Applied Mathematics, 11*, 587–593.

Shapley, L. S. (1953). A Value for N-person Games. In H. W. Kuhn & A. W. Tucker (Eds.), *Contributions to the Theory of Games II* (pp. 307–317). Princeton, NJ: Princeton University Press.

Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory, 1*, 11–26.

Spence, A. M. (1973). Job market signaling. *Quaterly Journal of Economics, 87*, 355–374.

von Stackelberg, H. (1934). *Marktform und Gleichgewicht*. Vienna, Austria: Springer.

Strogatz, S. (1988). Love affairs and differential equations. *Mathematics Magazine, 61*, 35.

Szidarovszky, F. (1998). Bargaining with offer dependent break-down probabilities. *Applied Mathematics and Computation, 90*, 117–127.

Szidarovszky, F., & Bahill, A. T. (1992). *Linear Systems Theory*. Boca Raton, London: CRC Press.

Szidarovszky, F., Gershon, M. E., & Duckstein, L. (1986). *Techniques for Multiobjective Decision Making in Systems Management*. Amsterdam, The Netherlands: Elsevier Science Publishers.

Szidarovszky, F., Okuguchi, K., & Kopel, M. (2005). International fishing with several countries. *Pure Mathematics and Applications, 16*(4), 493–514.

Szidarovszky, F., & Molnàr, S. (2002). *Introduction to Matrix Theory*. Singapore: World Scientific.

Szidarovszky, F., Molnàr, S., & Molnàr, M. (2022). *Introduction to Matrix Theory* (2nd ed.). Singapore: World Scientific.

Szidarovszky, F., & Yakovitz, S. (1978). *Principles and Procedures of Numerical Analysis*. New York/London: Plenum.

Taylor, A. D. (1995). *Mathematics and Politics*. New York, NY: Springer.

Vickers, J. (1986). Signalling in a model of monetary policy with incomplete information. *Oxford Economic Papers* (Vol. 38, pp. 443–455).

Volterra, V. (1928). Variations and fluctuations of the number of individuals in animal species living together. *ICES Journal of Marine Science, 3*(1), 3–51.

von Neumann, J., & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton, NJ: Princeton University Press.

Index

A. Matsumoto and F. Szidarovszky, *Game Theory and Its Applications*,
https://doi.org/10.1007/978-981-96-0590-3