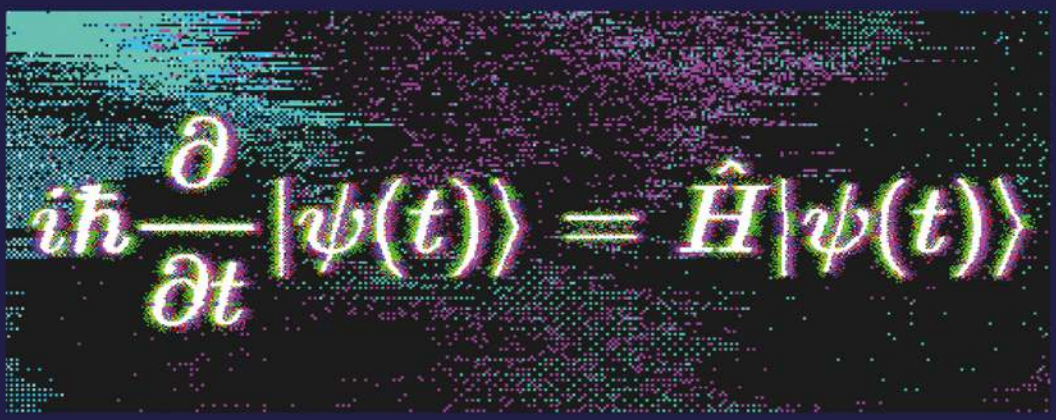
MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Abstract Calculus

A Categorical Approach

SECOND EDITION



Francisco Javier Garcia-Pacheco

A **Chapman & Hall** Book

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Abstract Calculus

Abstract Calculus: A Categorical Approach provides an abstract approach to calculus. Any calculus text for undergraduate students majoring in engineering, mathematics, or physics deals with the classical concepts of limits, continuity, differentiability, optimization, integrability, summability, and approximation. This book covers the exact same topics, but from a categorical perspective, making the classification of topological modules the main category.

This book is primarily intended for graduate students pursuing PhDs in pure mathematics, but junior and senior researchers in basically any field of mathematics and theoretical physics will also find the material of interest.

New to the Second Edition:

- The first section of Chapter One (Set Theory section) has been moved to the Appendix and has been endowed with more content related to the Zermelo-Fraenkel Axiomatic System.
- The first chapter of the Appendix (Category Theory chapter) has been restructured to reinforce universal objects, universal properties, and representations.
- The order topology in partially ordered sets has been reviewed in depth and applied to the study of ordered universal algebras, such as ordered groups/rings/modules.
- Convergence topologies, as well as convergence uniformities, are fully developed and have been relocated in the second chapter (Limits).
- Unit segments in rings have been completely covered and related to ring orderings, highlighting applications to convexity in both modules and abstract cones.

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A Categorical Approach

Second Edition

Francisco Javier Garcia-Pacheco

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To the Memory of my Beloved Mother.
It is hard to keep going.
To my Father.
And, to ω-Tet.

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Preface

This is the second edition of the book "Abstract Calculus: a categorical approach". It is a profound refinement and update of the first edition. However, it follows the same philosophy, which was to review all the classical notions of any Calculus textbook from the perspective of Category Theory. Special emphasis is paid to universal algebras, and to the most known and extended classes of Arithmetics present in the human brain:

- *Natural Arithmetics*: The essence of counting, adding, and multiplying.
- *Modular Arithmetics*: Basic Abstract Algebra.
- *Arithmetics of Infinity*: Mathematical Analysis and Calculus.
- *Boolean Arithmetics*: Set Theory, Measure Theory, and Topology.
- *Quantum Arithmetics*: Advanced Abstract Algebra.

Of course, universal algebras allow many more kind of Arithmetics. As we know, the enhancement of Natural Arithmetics through more complex and abstract ways of thinking fostered the development of the different areas and fields of Pure Mathematics and Theoretical Physics. The philosophy and aim of this book is to explore this kind of different Arithmetics, providing a pool of tools for the working Physicist and Mathematician.

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Symbol Descriptions

$\mathcal{P}(X)$	power set of X
Δ_A	diagonal of $A \times A$
I_A	identity on A
ι_E	inclusion of E
$\mathcal{P}_{\mathtt{f}}(X)$	finite subsets of X
$[a]_{\mathcal{R}}$	equivalence class of a by $\mathcal{R}$
${}^{A}\!/_{\mathcal{R}}$	quotient set of A by $\mathcal{R}$
$\pi_{\mathcal{R}}$	projection of A onto ${}^{A}\!/_{\mathcal{R}}$
$\mathcal{J}(B)$	upward closure of B
$\uparrow B$	upward closure of B
$\mathcal{I}(B)$	downward closure of B
$\downarrow B$	downward closure of B
$\uparrow a$	elements $\geq a$
$\downarrow b$	elements $\leq b$
$\max(A)$	maximum of A
$\min(A)$	minimum of A
$\sup(B)$	supremum of B
$\bigvee B$	supremum of B
$\inf(B)$	infimum of B
$\bigwedge B$	infimum of B
$1 \setminus a$	complement of a
$\mathscr{I}_{\mathcal{F}}$	ideal base induced by $\mathcal{F}$
$\mathcal{F}_{\mathcal{I}}$	filter base induced by $\mathcal{I}$
$\mathcal{F}_X$	Fréchet filter in X
$\mathcal{B}_D$	filter base in directed set D
$\mathrm{fix}(H)$	fixed points of H
$\mathcal{H}_Y$	hull operator induced by Y
$\mathrm{span}(A)$	spanned subalgebra by A
$A^{\times}$	symmetrics of elements of A
A'	commutant of A
$\mathsf{Z}(G)$	center of G
$\ker(f)$	kernel of f
$\mathrm{char}(G)$	characteristic of G
$[G : H]$	index of H in G
Gm	orbit of m
G_m	stabilizer of m
$\mathcal{U}(R)$	invertibles of R
$\mathcal{U}_1(R)$	absolutely invertibles of R
$\mathtt{ld}(s)$	left divisors of s
$\mathtt{rd}(s)$	right divisors of s
$\mathrm{aff}(A)$	affine hull of A
$\mathcal{L}(M, N)$	linear operators
$\mathrm{ext}_2(A)$	2-extreme points of A
$\mathrm{ext}(A)$	extreme points of A
$\mathrm{extl}(A)$	extremal points of A
a^{c}	complement of a
$p^{\perp}$	orthocomplement of p
$\tau(\mathcal{A})$	topology generated by $\mathcal{A}$
$\mathrm{int}(A)$	interior of A
$\mathrm{cl}(A)$	closure of A
$\mathrm{bd}(A)$	boundary of A
∂A	boundary of A
$\mathrm{ac}(A)$	accumulation of A
$\mathrm{iso}(A)$	isolated points of A
$\mathcal{N}_x(X)$	neighborhoods of x
$\mathcal{N}_x^{\times}(X)$	reduced neighborhoods of x
$U[x]$	entourage of x
$\mathsf{B}_X(x, \delta)$	closed ball
$\mathsf{U}_X(x, \delta)$	open ball
$\mathsf{S}_X(x, \delta)$	sphere
U_{δ}	δ-entourage
$d(x, y)$	pseudometric between x, y
$\mathrm{d}(X)$	diameter of X
$\mathcal{B}_{\mathtt{d}}(X)$	bounded subsets of X
$\mathrm{r}_x(X)$	x-radius of X
$\mathrm{r}(X)$	radius of X
X_{c}	center of X
X_{d}	diametrality of X
$\mathcal{A}_X$	admissible subsets of X
$\|m\|$	seminorm of m
$\|r\|$	absolute semivalue of r
$\mathcal{U}_1(R)$	absolute invertibles
B_G	closed unit ball
U_G	open unit ball
S_G	unit sphere
$\lim$	limit
agl	agglomeration
$\limsup$	limit superior
$\liminf$	limit inferior
$\mathcal{CS}$	Cauchy sequences
$\mathcal{U}(G, U)$	convergence neighborhood
$\sigma(X^I, \mathcal{G})$	convergence topology

$\mathcal{V}(G,V)$ convergence entourage
$\upsilon(X^I,\mathcal{G})$ convergence uniformity
$\mathrm{cont}(f)$ continuity points of f
$\mathcal{C}(X,Y)$ continuous functions
$\sigma(X,F)$ initial topology
$\widehat{x}$ induced function by x
$\tau(Y,F)$ final topology
$\mathtt{tld}(0)$ topological left zero-divisor
$\mathtt{trd}(0)$ topological right zero-divisor
B^+ unit segment
B^- negative unit segment
$\overline{\mathrm{span}}(A)$ closed linear span of A
$\mathcal{CL}(M,N)$ cont. linear operators
M^* dual of M
$A^\perp$ annihilator of A
$C^\top$ preannihilator of C
$\mathrm{inter}(A)$ internal points of A
$\mathrm{inn}(A)$ inner points of A
$\mathcal{B}_0(M)$ bounded subsets of M
$\ell_\infty(\Gamma,M)$ bounded functions
$\Delta(\dots)$ geometric simplex
$\mathrm{co}(A)$ convex hull of A
$\sigma(a)$ spectrum of a
$D_m(r)$ inner derivation of r by m
$\int m$ indefinite integral of m
$\Delta_l f(m)(r)$ rate of change of f
$\mathrm{D}_l\, f(m)$ derivative of f
$f_l'(m)$ derivative of f
$\frac{\partial f}{\partial l}(m)$ derivative of f
$\mathcal{D}(A,N)$ differentiable functions
$\mathcal{CD}(A,N)$ cont. differentiable functions
$\mathcal{T}_x(X)$ tangent space
fea feasible solutions
sol optimal solutions
Par Pareto optimal solutions

$\mathrm{epi}(f)$ epigraph of f
$\mathrm{slc}(C,f,r)$ slice of C by f
$\exp(C)$ exposed points of C
$\mathcal{B}(M,N)$ bounded linear operators
$\mathrm{suppv}(T)$ supporting vectors of T
$\mathrm{suppv}^{w^*}(m)$ w^*-supporting functionals
$\Pi_{M,N}$ supporting pairs
$\mathrm{smo}(M)$ smooth points of M
$\mathrm{smo}^{w^*}(M^*)$ w^*-smooth points of M^*
$\mathsf{J}(m)$ unit supporting functional of m
$\mathcal{G}_M$ surjective linear isometries on M
$c_{00}(M)$ eventually null sequences
$c_0(M)$ null sequences
$c(M)$ convergent sequences
$\ell_\infty(M)$ bounded sequences
δ_{ij} Kronecker delta
$\Pi(A)$ permutations of A
$\mathcal{HB}(\lim)$ extensions of lim
$\mathrm{ba}(L,M)$ bounded variation measures
$\mu(\ |a)$ conditional probability
χ_A characteristic function
$\mathsf{L}^0(\Omega,R)$ measurable functions
$\mathsf{E}(\Omega,R)$ measurable simple functions
$\mathrm{ob}(\mathcal{C})$ objects of $\mathcal{C}$
$\mathrm{mor}(\mathcal{C})$ morphisms of $\mathcal{C}$
$\hom_{\mathcal{C}}$ homomorphisms
I_A identity of object A
$\mathrm{mono}_{\mathcal{C}}$ monomorphisms
$\mathrm{epi}_{\mathcal{C}}$ epimorphisms
$\mathrm{iso}_{\mathcal{C}}$ isomorphisms
$\mathrm{end}_{\mathcal{C}}$ endomorphisms
$\mathrm{aut}_{\mathcal{C}}$ automorphisms
$\prod_{i\in I} A_i$ product
π_i projection morphism
$\coprod_{i\in I} A_i$ coproduct
ι_i injection morphism

Author Biography

Francisco Javier Garcia-Pacheco graduated in Mathematics from the University of Cadiz (Spain, EU) in 2000. Later, he became an Adjunct Professor at the University of Cadiz and did a doctorate in Pure Mathematics, under the tutelage of Prof. Dr. Antonio Aizpuru, defending his dissertation on May 27, 2005. One year before defending his dissertation, in August 2004, he was awarded a Graduate Teaching Assistantship at Kent State University (Ohio, USA) to pursue a second doctorate in Nonlinear Analysis, under the tutelage of Prof. Dr. Richard Aron. He defended this second dissertation on January 19, 2007, obtaining the Kent State Outstanding Dissertation Award. During the first two years at Kent State University, he was hired as a Graduate Teaching Assistant. He was then awarded a Teaching Fellowship during his last year at Kent State. In August 2007, he was hired by Texas A&M University (Texas, USA) as a Visiting Assistant Professor until August 2010, under the tutelage of Prof. Dr. Bill Johnson. Since September 2010, he has been at the University of Cadiz, first as an Assistant Professor (September 2010 through May 2012), then as an Associate Professor (June 2012 through February 2019), and finally as a Full Professor (March 2019 to present). During the Summer semester of 2011, he did a research and teaching stay at Texas A&M University-Central Texas as Visiting Associate Professor. He also has experience in Academic Management. From December 2017 through July 2019, he held the academic position of Director of Secretariat for International Projects at the Vice-rectorate for Transference and Technological Innovation of the University of Cadiz. From March 2020 through October 2021, he was Director of the Departmental Section of Mathematics at the College of Engineering of the University of Cadiz. From October 2021 through February 2024, he was the Head of the Mathematics Department at the University of Cadiz. Finally, from February 2024, he is currently Director of Secretariat for Academic Affairs.

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Part I

Functional Calculus

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Chapter 1

Functions

Evariste Galois

1.1 Relations

Relations are fundamental concepts in the mathematical language. They are precisely what gives mathematical structure to sets, conforming well-known entities such as topological spaces, ordered spaces, vector spaces, etc.

1.1.1 Structures

The concept of universal structure simply pretends to englobe, in just one notion, many different mathematical structures that have something in common: a set of multiary relations. However, we are not yet in the right position to formally introduce the notion of universal structure, so we will, for now, unveil a simpler version.

1.1.1.1 Multiary relations

The notion of a multiary relation strongly relies on the product of a finite amount of sets. The Zermelo-Fraenkel Axiomatic System allows the direct construction of the product of a finite amount of sets, such as $A_1 \times \cdots \times A_n$ for $A_1, \ldots, A_n$ sets. The elements of $A_1 \times \cdots \times A_n$ are often called n-tuples and denoted by $(a_1, \ldots, a_n)$ for $a_i \in A_i$ when $i \in \{1, \ldots, n\}$. If $n = 2$, they are also called ordered pairs or, simply, pairs. If all the A_i's are equal to a given set A, then $A \overset{n}{\times \cdots \times} A$ is often denoted as A^n.

Definition 1 (Relation) *An n-ary relation of sets $A_1, \ldots, A_n$ is simply a subset $\mathcal{R} \subseteq A_1 \times \cdots \times A_n$. Also:*

- *If $A_i = A_j$ for all $i, j \in \{1, 2, \ldots, n\}$, then we will say that $\mathcal{R}$ is an internal relation.*
- *If $n = 2$, then $\mathcal{R}$ is called binary.*
- *If $n = 1$, then $\mathcal{R}$ is called unary.*

Given a set A, an n-ary external relation on A is an n-ary relation of sets $A_1, \ldots, A_n$ such that there exists $\varnothing \neq M \subsetneq \{1, \ldots, n\}$ with $A_m = A$ if $m \in M$.

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There are three main examples of binary relations: maps, equivalence relations, and order relations. The following definition describes how to operate with binary relations.

Definition 2 *Let A, B, C be sets. Let $\mathcal{R}, \mathcal{S}$ be binary relations from A to B and from B to C, respectively. Then:*

- *The composition of $\mathcal{R}, \mathcal{S}$ is defined as the binary relation from A to C given by $\mathcal{S} \circ \mathcal{R} := \{(a, c) \in A \times C : \exists b \in B \ (a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{S}\}$.*
- *The inverse of $\mathcal{R}$ is defined as the binary relation from B to A given by $\mathcal{R}^{-1} := \{(b, a) \in B \times A : (a, b) \in \mathcal{R}\}$.*
- *A subrelation of $\mathcal{R}$ is any binary relation $\mathcal{T}$ from A to B such that $\mathcal{T} \subseteq \mathcal{R}$.*
- *The identity relation on A is the binary internal relation on A given by $I_A := A \times A$.*
- *If $A' \subseteq A$, then the restricted relation of $\mathcal{R}$ to A' is defined as $\mathcal{R}|_{A'} := (A' \times B) \cap \mathcal{R}$.*
- *The classes of related elements to given elements $a \in A$ and $b \in B$ are defined as $[a]_{\mathcal{R}} := \{b' \in B : (a, b') \in \mathcal{R}\}$ and ${}_{\mathcal{R}}[b] := \{a' \in A : (a', b) \in \mathcal{R}\}$, respectively.*

Notice that under the settings of the previous definition, any restricted relation $\mathcal{R}|_{A'}$ is also a subrelation of $\mathcal{R}$ since obviously $\mathcal{R}|_{A'} \subseteq A \times B$, so it is a binary relation from A to B, and $\mathcal{R}|_{A'} \subseteq \mathcal{R}$. Next remark shows that the composition of binary relations is associative.

Remark 1 *Let A, B, C, D be sets. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be binary relations from A to B, from B to C, and from C to D, respectively. Note that $(\mathcal{T} \circ \mathcal{S}) \circ \mathcal{R} = \mathcal{T} \circ (\mathcal{S} \circ \mathcal{R}) = \{(a, d) \in A \times D : \exists b \in B, \exists c \in C \ (a, b) \in \mathcal{R}, (b, c) \in \mathcal{S}, (c, d) \in \mathcal{T}\}$.*

1.1.1.2 Binary internal relations

Probably, the most important and relevant examples of multiary relations are the binary internal relations.

Definition 3 *A binary internal relation $\mathcal{R} \subseteq A \times A$ on a set A is said to be:*

- *Reflexive provided that $\forall a \in A \ (a, a) \in \mathcal{R}$, that is, $\Delta_A \subseteq \mathcal{R}$, where $\Delta_A := \{(a, a) : a \in A\}$ stands for the diagonal of $A \times A$.*
- *Symmetric provided that $\forall a, b \in A \ (a, b) \in \mathcal{R} \Rightarrow (b, a) \in \mathcal{R}$, that is, $\mathcal{R} = \mathcal{R}^{-1}$.*
- *Antisymmetric provided that $\forall a, b \in A \ \{(a, b), (b, a)\} \subseteq \mathcal{R} \Rightarrow a = b$.*
- *Transitive provided that $\forall a, b, c \in A \ \{(a, b), (b, c)\} \subseteq \mathcal{R} \Rightarrow (a, c) \in \mathcal{R}$, that is, $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.*
- *Upward directed provided that $\forall a, b \in A \ \exists c \in A \ \{(a, c), (b, c)\} \subseteq \mathcal{R}$.*
- *Downward directed provided that $\forall a, b \in A \ \exists c \in A \ \{(c, a), (c, b)\} \subseteq \mathcal{R}$.*
- *Total provided that $\forall a, b \in A \ (a, b) \in \mathcal{R} \vee (b, a) \in \mathcal{R}$.*

Under the settings of Definition 3, notable subsets of A are $\uparrow a := [a]_{\mathcal{R}} = \{x \in A : (a, x) \in \mathcal{R}\}$ and $\downarrow b := {}_{\mathcal{R}}[b] = \{x \in A : (x, b) \in \mathcal{R}\}$, where $a, b \in A$.

Definition 4 (Upward/downward closure) *Let A be a set endowed with a binary internal relation $\mathcal{R}$. Consider a subset $B \subseteq A$. Then:*

- *The upward closed hull or upward closure of B is defined as*

$$\mathcal{J}(B) := \uparrow B = \bigcup_{b \in B} \uparrow b.$$

- *The downward closed hull or downward closure of B is defined as*

$$\mathcal{I}(B) := \downarrow B = \bigcup_{b \in B} \downarrow b.$$

The previous definition motivates the following concepts, which apply more often to order relations.

Definition 5 (Coinitiality/cofinality) *Let A be a set endowed with a binary internal relation $\mathcal{R}$. Consider a subset $B \subseteq A$. Then:*

- *B is called upward closed whenever $B = \mathcal{J}(B)$.*
- *B is called downward closed provided that $B = \mathcal{I}(B)$.*
- *B is said to be coinitial in A provided that $A = \mathcal{J}(B)$.*
- *B is said to be cofinal in A whenever $A = \mathcal{I}(B)$.*

Since the composition of binary relations is associative in accordance with Remark 1, the following notation will be employed to simplify writing.

Notation 1 *If A is a set and $\mathcal{R}$ is a binary internal relation on A, then $\mathcal{R}^n := \mathcal{R} \circ \cdots \circ \mathcal{R}$ for every $n \in \mathbb{N}$.*

After reviewing the most basic aspects of binary internal relations, it is time now to introduce the notion of structure.

Definition 6 (Structure) *A structure is a pair $(A, \mathcal{R})$, where A is a non-empty set and $\mathcal{R}$ is a binary internal relation on A.*

Maps have not been formally introduced yet, although it is convenient to use the concept of mapping in the next definition. However, they will be properly overviewed with more precision in the upcoming subsection.

Definition 7 (Relator) *Let $(A, \mathcal{R}), (B, \mathcal{S})$ be structured sets. A map $f : A \to B$ is said to be a relator if it is relation-preserving, meaning that $f(\mathcal{R}) \subseteq \mathcal{S}$, or, in other words, $\forall (a_1, a_2) \in \mathcal{R}\ (f(a_1), f(a_2)) \in \mathcal{S}$.*

In general, a subset of a set endowed with a binary internal relation can be endowed with such internal relation. Indeed, if A is a set endowed with a binary internal relation $\mathcal{R}$, then a subset B of A inherits the relation provided that $\mathcal{R} \cap (B \times B) \neq \varnothing$.

1.1.2 Maps

Maps are among the most relevant binary relations. In fact, in this book, we will go through maps before dealing with equivalence or order relations.

1.1.2.1 Composition

Composition of maps is a particular case of composition of binary relations. In fact, composition is a binary external operation, which will be described later on with much more detail. However, we need now to introduce the notion of composition in order to keep constructing more structures.

Definition 8 (Map) *A map or function from a set A to a set B is a binary relation f from A to B, $f \subseteq A \times B$, satisfying that $\forall a \in A\ \exists! b \in B\ (a,b) \in f$.*

The set of maps from A to B is often denoted as B^A. The set A is often called the domain of f, and B is called the range or image of f. Maps are informally denoted by

$$\begin{array}{rccl} f: & A & \to & B \\ & a & \mapsto & b = f(a). \end{array}$$

Definition 9 *Let $f : A \to B$ be a map between sets A, B. Consider subsets $C \subseteq A$ and $D \subseteq B$. Then:*

1. *$f(C) := \{b \in B : \exists c \in C\ \ f(c) = b\}$, called image of C under f.*
2. *$f^{-1}(D) := \{a \in A : f(a) \in D\}$, called preimage of D under f.*
3. *$f|_C$ stands for the map f restricted to C, that is,*

$$\begin{array}{rccl} f|_C: & C & \to & B \\ & c & \mapsto & f|_C(c) := f(c). \end{array}$$

4. *f is called injective if $\forall a, b \in A\ f(a) = f(b) \Rightarrow a = b$.*
5. *f is called surjective if $f(A) = B$.*
6. *f is called bijective if f is injective and surjective.*
7. *If $A = B$, then C is called invariant provided that $f(C) \subseteq C$.*

Under the settings of the previous definition, we will informally write $f(C) := \{f(c) : c \in C\}$.

Remark 2 *Let $f : A \to B$ be a map between sets A, B. Let $C, C_1, C_2 \subseteq A$ and $D, D_1, D_2 \subseteq B$. Then:*

1. *If $C_1 \subseteq C_2$, then $f(C_1) \subseteq f(C_2)$.*
2. *If $D_1 \subseteq D_2$, then $f^{-1}(D_1) \subseteq f^{-1}(D_2)$.*
3. *$f(C_1) \setminus f(C_2) \subseteq f(C_1 \setminus C_2)$.*
4. *$f^{-1}(D_1 \setminus D_2) = f^{-1}(D_1) \setminus f^{-1}(D_2)$.*
5. *$f\left(f^{-1}(D)\right) \subseteq D$.*
6. *$C \subseteq f^{-1}(f(C))$.*

The Axiom of Powers allows the construction of the power set $\mathcal{P}(X)$ of a given set X.

Remark 3 *Every function $f : X \to Y$ between sets X, Y induces two maps:*

$$\begin{array}{rccc} f: & \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ & A & \mapsto & f(A), \end{array} \qquad \begin{array}{rccc} f^{-1}: & \mathcal{P}(Y) & \to & \mathcal{P}(X) \\ & B & \mapsto & f^{-1}(B). \end{array} \tag{1.1}$$

To avoid future confusion, we will strongly rely on the following notation, which is consistent with the previous remarks.

Notation 2 *Let $f : X \to Y$ be a function between sets X, Y. Then:*

- *If $\mathcal{A} \subseteq \mathcal{P}(X)$, then $f(\mathcal{A}) := \{f(A) : A \in \mathcal{A}\}$.*
- *If $\mathcal{B} \subseteq \mathcal{P}(Y)$, then $f^{-1}(\mathcal{B}) := \{A \in \mathcal{P}(X) : f(A) \in \mathcal{B}\}$.*

Under the settings of the previous notation, if we let $\mathcal{P}_{\mathtt{f}}(X)$ denote the set of all finite subsets of X, then $f\left(\mathcal{P}_{\mathtt{f}}(X)\right) \subseteq \mathcal{P}_{\mathtt{f}}(Y)$.

Definition 10 (Composition) *Let A, B, C be sets. Let $f : A \to B$, $g : B \to C$ be maps. Then:*

- *The composition map of f and g is defined as*

$$\begin{array}{rccc} g \circ f: & A & \to & C \\ & a & \mapsto & (g \circ f)(a) := g(f(a)). \end{array}$$

- *The identity map on A is defined as*

$$\begin{array}{rccc} I_A: & A & \to & A \\ & a & \mapsto & I_A(a) := a. \end{array}$$

- *The inclusion map of $E \subseteq A$ is defined as*

$$\begin{array}{rccc} \iota_E: & E & \hookrightarrow & A \\ & e & \mapsto & \iota_E(e) := e. \end{array}$$

- *$h : B \to A$ is said to be a left inverse for f if $h \circ f = I_A$.*
- *$h : B \to A$ is said to be a right inverse for f if $f \circ h = I_B$.*
- *$h : B \to A$ is said to be an inverse for f if it is a left and right inverse for f.*

Under the settings of the above definition, $f|_E = f \circ \iota_E$. Next, the power of the Axiom of Choice is displayed with majesty.

Proposition 1 *Let A, B, C, D be nonempty sets. Let $f : A \to B$, $g : B \to C$, $h : C \to D$ be maps. Then:*

1. *$h \circ (g \circ f) = (h \circ g) \circ f$.*
2. *If $p : B \to Y$ is such that $p \circ f$ is injective, then f is injective.*
3. *If $q : X \to A$ is such that $f \circ q$ is surjective, then f is surjective.*
4. *The following conditions are equivalent:*
   (a) *f is injective.*
   (b) *f has a left inverse.*

(c) $C = f^{-1}(f(C))$ *for every* $C \subseteq A$.

(d) *For every set* X *and every functions* $p, q : X \to A$, *the equality* $f \circ p = f \circ q$ *implies* $p = q$.

5. *The following conditions are equivalent:*

(a) f *is surjective.*

(b) f *has a right inverse.*

(c) $f\left(f^{-1}(D)\right) = D$ *for every* $D \subseteq B$.

(d) *For every set* Y *and every functions* $p, q : B \to Y$, *the equality* $p \circ f = q \circ f$ *implies* $p = q$.

6. f *is bijective if and only if* f *has an inverse.*

Proof 1 *We will only prove some implications of the fourth and fifth items. The rest is left as exercises. For instance, suppose that* f *is injective. Observe that* $B = (B \setminus f(A)) \cup f(A)$. *Fix* $a_0 \in A$. *Define*

$$\begin{array}{rccl} m: & B & \mapsto & A \\ & b & \mapsto & m(b) := \left\{ \begin{array}{ll} a_0 & b \in B \setminus f(A), \\ a & b = f(a). \end{array} \right. \end{array}$$

Note that $m \circ f = I_A$. *As a consequence,* f *has a left inverse. Conversely, assume that for every set* X *and every functions* $p, q : X \to A$, *the equality* $f \circ p = f \circ q$ *implies* $p = q$. *Let us show that* f *is injective. Indeed, fix arbitrary elements* $a_1, a_2 \in A$ *such that* $f(a_1) = f(a_2)$. *Fix a nonempty set* X *and define*

$$\begin{array}{rccl} p: & X & \mapsto & A \\ & x & \mapsto & p(x) := a_1 \end{array} \quad \textit{and} \quad \begin{array}{rccl} q: & X & \mapsto & A \\ & x & \mapsto & q(x) := a_2. \end{array}$$

Note that $f \circ p = f \circ q$. *By hypothesis,* $p = q$, *which implies that* $a_1 = a_2$. *Thus,* f *is injective. Now, let us suppose that* f *is surjective. Let* $T := \left\{E \in \mathcal{P}(A) : \exists b \in B \;\; E = f^{-1}(b)\right\}$. *Since* f *is surjective,* $\varnothing \notin T$. *Also, notice that* $\bigcup T = A$. *By the Axiom of Choice, there exists a choice function* $c : T \to A$ *such that if* $t \in T$, *then* $c(t) \in t$. *Next, define*

$$\begin{array}{rccl} l: & B & \mapsto & A \\ & b & \mapsto & l(b) := c\left(f^{-1}(b)\right). \end{array}$$

Then $f \circ l = I_B$. *In other words,* f *has a right inverse. Finally, assume that for every set* Y *and every functions* $p, q : B \to Y$, *the equality* $p \circ f = q \circ f$ *implies* $p = q$. *We will show that* f *is surjective. Suppose to the contrary that* $f(A) \subsetneq B$. *Let* $m : f(A) \to A$ *be a right inverse for* $f : A \to f(A)$. *Fix elements* $b_0 \in B \setminus f(A)$ *and* $a_0 \in A$. *Define*

$$\begin{array}{rccl} p: & B & \mapsto & A \cup B \\ & b & \mapsto & p(b) := \left\{ \begin{array}{ll} b_0 & b \in B \setminus f(A), \\ m(b) & b \in f(A), \end{array} \right. \end{array}$$

and

$$\begin{array}{rccl} q: & B & \mapsto & A \cup B \\ & b & \mapsto & q(b) := \left\{ \begin{array}{ll} f(a_0) & b \in B \setminus f(A), \\ m(b) & b \in f(A). \end{array} \right. \end{array}$$

Note that $p \circ f = q \circ f$. *However,* $p \neq q$.

1.1.2.2 Families and collections

The direct product of a family of sets is a fundamental concept that we need to introduce as soon as possible, simply because we will rely on products of sets to keep constructing more structures.

Definition 11 (Indexed family) *Let I be a nonempty set. An I-indexed family of sets is simply a function*

$$\begin{array}{rccc} f: & I & \to & \mathcal{A} \\ & i & \mapsto & f(i) := A_i \end{array}$$

where I is called the index set. The family is usually represented by $(A_i)_{i\in I}$.

Under the settings of the above definition, $\bigcup_{i\in I} A_i := \bigcup f(I)$ and $\bigcap_{i\in I} A_i := \bigcap f(I)$, by relying on the Axiom of Union.

Remark 4 *Let $f : A \to B$ be a map between sets A, B. Consider families $(A_i)_{i\in I}, (B_i)_{i\in I}$ of subsets of A, B, respectively. Then:*

- $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$ *and* $f\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} f(A_i)$.
- $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$ *and* $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$.

The direct product and the disjoint union of a family of sets are very important mathematical objects present in every area of Pure and Applied Mathematics.

Definition 12 (Direct product & disjoint union) *Let $(A_i)_{i\in I}$ be an indexed family of sets. Then:*

- *The direct product of $(A_i)_{i\in I}$ is defined as the set*

$$\prod_{i\in I} A_i := \left\{ g \in \left(\bigcup_{i\in I} A_i\right)^I : \forall i \in I \;\; g(i) \in A_i \right\}.$$

  The elements of $\prod_{i\in I} A_i$ are called I-tuples and denoted as $(a_i)_{i\in I}$, where $a_i \in A_i$ for all $i \in I$. The constant I-tuple is a^I whenever there exists a such that $a_i = a$ for all $i \in I$. For every $i_0 \in I$, the map

$$\begin{array}{rccc} \pi_{i_0}: & \prod_{i\in I} A_i & \to & A_{i_0} \\ & (a_i)_{i\in I} & \mapsto & \pi_{i_0}\left((a_i)_{i\in I}\right) := a_{i_0} \end{array}$$

  is called the canonical coordinate projection, or simply projection, on the i_0^{th}-coordinate.

- *The disjoint union of $(A_i)_{i\in I}$ is defined as*

$$\bigsqcup_{i\in I} A_i := \bigcup_{i\in I} \left\{ (x, i) \in \left(\bigcup_{i\in I} A_i\right) \times I : x \in A_i \right\}.$$

  For every $i_0 \in I$, the map

$$\begin{array}{rccc} \iota_{i_0}: & A_{i_0} & \to & \bigsqcup_{i\in I} A_i \\ & a_{i_0} & \mapsto & \iota_{i_0}(a_{i_0}) := (a_{i_0}, i_0) \end{array}$$

  is called the canonical injection of the i_0^{th}-coordinate.

Under the settings of the previous definition, whenever $I := \{1, 2, \ldots, n\}$, $\prod_{i\in I} A_i$ is usually identified with $A_1 \times \cdots \times A_n$ in the sense that there exists a canonical bijection between them. If $A_i = A$ for all $i \in I$, then $\prod_{i\in I} A_i = A^I$.

Definition 13 (Coordinate & marginal maps) *Let $(A_i)_{i\in I}$ be a family of sets. Let B be another set. Then:*

- *Consider a map $f : \prod_{i\in I} A_i \to B$. For every $i_0 \in I$ and every $(a_i)_{i\in I\setminus\{i_0\}} \in \prod_{i\in I\setminus\{i_0\}} A_i$, the following map*

$$\begin{array}{rccl} f_{(a_i)_{i\in I\setminus\{i_0\}}}: & A_{i_0} & \to & B \\ & a_{i_0} & \mapsto & f_{(a_i)_{i\in I\setminus\{i_0\}}}(a_{i_0}) := f\left((a_i)_{i\in I}\right) \end{array}$$

  is called a marginal map of f.

- *Consider a map $f : B \to \prod_{i\in I} A_i$. For every $i_0 \in I$, the map $\pi_{i_0} \circ f$ is called a coordinate map of f.*

In virtue of the classical literature, we will be indistinctively using the names marginal and coordinate for both notions. Under the settings of the above definition, if $I = \{1, 2\}$, then the marginal maps are usually denoted as $f(a_1, \ ) := f_{a_1} : A_2 \to B$ and $f(\ , a_2) := f_{a_2} : A_1 \to B$ for every $a_1 \in A_1$ and every $a_2 \in A_2$.

Notation 3 *Let $(A_i)_{i\in I}, (B)_{i\in I}$ be families of sets. Let $(f_i)_{i\in I} \in \prod_{i\in I} B_i^{A_i}$. We denote the function:*

$$\begin{array}{rccl} \prod_{i\in I} f_i: & \prod_{i\in I} A_i & \mapsto & \prod_{i\in I} B_i \\ & (a_i)_{i\in I} & \mapsto & \left(\prod_{i\in I} f_i\right)\left((a_i)_{i\in I}\right) := (f_i(a_i))_{i\in I}. \end{array}$$

$$\begin{array}{ccc} \prod_{i\in I} A_i & \xrightarrow{\prod_{i\in I} f_i} & \prod_{i\in I} B_i \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i} \\ A_i & \xrightarrow{\quad f_i \quad} & B_i \end{array}$$

If $A_i = A$ for all $i \in I$, then we also have the notation

$$\begin{array}{rccl} \prod_{i\in I} f_i: & A & \mapsto & \prod_{i\in I} B_i \\ & a & \mapsto & \left(\prod_{i\in I} f_i\right)(a) := (f_i(a))_{i\in I}, \end{array}$$

which is the previous function restricted to the constant tuples.

Marginal maps induce a bijection with multiple applications in different areas of Applied Sciences.

Remark 5 (Bimaps) *Let A, B, C be sets. There is a bijection between $C^{A\times B}$ and $\left(C^A\right)^B$ via marginal maps. Indeed:*

$$\begin{array}{rcccl} C^{A\times B} & \to & \left(C^A\right)^B & & \\ & & B & \to & C^A \\ T & \mapsto & b & \mapsto & T(\ , b) \end{array} \tag{1.2}$$

with inverse given by

$$\begin{array}{rcccl} \left(C^A\right)^B & \to & C^{A\times B} & & \\ & & A\times B & \to & C \\ S & \mapsto & (a, b) & \mapsto & S(b)(a). \end{array} \tag{1.3}$$

The power and influence of the Axiom of Choice go beyond imagination, as shown in the next corollary of Proposition 1.

Corollary 1 *Let X, Y be nonempty sets. Let $f : X \to Y$ be a surjective function. If $(y_i)_{i\in I} \subseteq Y$ is a family of elements of Y, then there exists a family $(x_i)_{i\in I} \subseteq X$ of elements of X such that $f(x_i) = y_i$ for every $i \in I$.*

Proof 2 *Let us denote by $y : I \to Y$ the family $(y_i)_{i\in I}$. According to Proposition 1(5), there exists a right inverse $g : Y \to X$. Then simply define the family $x := g \circ y : I \to X$. Indeed, $\forall i \in I\ f(x_i) = f(x(i)) = f(g(y(i))) = y(i) = y_i$.*

The end of this subsection is reserved for a very useful construction associated with duality.

Remark 6 (Duality) *Let X, Y be sets. Let $\mathcal{F} \subseteq Y^X$. Then X can be identified with a subset of $Y^{\mathcal{F}}$. Indeed, consider the embedding*

$$\begin{array}{ccccc} X & \to & Y^{\mathcal{F}} & & \\ x & \mapsto & \mathcal{F} & \to & Y \\ & & f & \mapsto & f(x). \end{array} \tag{1.4}$$

Under the settings of the previous remark, the image of $x \in X$ under the mapping (1.4) is often denoted by $\widehat{x}$. Notice that (1.4) is not necessarily an injective map.

Definition 14 (Separating family) *Let X, Y be sets. A family $\mathcal{F} \subseteq Y^X$ is said to separate points of X if whenever $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$.*

Under the settings of Remark 6, $\mathcal{F}$ separates points of X if and only if (1.4) is injective.

1.1.3 Equivalence relations

Equivalence relations are a particular case of binary internal relations, with multiple applications in Topology, Algebra, and Analysis. Equivalence relations are closely related to partitions of a set.

1.1.3.1 Equivalence classes

An equivalence relation divides a set into equivalence classes, forming a partition of the set. This partition is called the quotient set. These notions are very much used in Geometry as part of a topological process called identification.

Definition 15 (Equivalence relation) *Let A be a set. Then:*

- *An equivalence relation on A is a reflexive, symmetric, and transitive binary internal relation on A.*
- *If $\mathcal{R}$ is an equivalence relation on A, then the equivalence classes of $\mathcal{R}$ are defined as the classes of related elements to a given $a \in A$, that is, $[a]_{\mathcal{R}} := \{b \in A : (a, b) \in \mathcal{R}\}$.*
- *The quotient set of A by $\mathcal{R}$ is defined by ${}^{A}/_{\mathcal{R}} := \{[a]_{\mathcal{R}} : a \in A\}$.*
- *The quotient projection of A onto $A/\mathcal{R}$ is defined by*

$$\begin{array}{rccc} \pi_{\mathcal{R}} : & A & \to & {}^{A}/_{\mathcal{R}} \\ & a & \mapsto & \pi_{\mathcal{R}}(a) := [a]_{\mathcal{R}}. \end{array}$$

The following is a very representative example of an equivalence relation with multiple applications in Topology and Optimization Theory.

Example 1 (Kernel) *Let X, Y be sets. Consider a family $\mathcal{F} \subseteq Y^X$ of functions from X to Y. The following is an equivalence relation on X:*

$$\ker(\mathcal{F}) := \left\{(x,y) \in X^2 : \forall f \in \mathcal{F} \;\; f(x) = f(y)\right\}.$$

We will prove now that the quotient set by any equivalence relation is always a partition, and, conversely, every partition induces an equivalence relation whose quotient set is precisely the partition.

Definition 16 (Partition) *Let X be a set. A subset $\mathcal{P} \subseteq \mathcal{P}(X)$ is called a partition of X if $\bigcup \mathcal{P} = X$ and, for every $A, B \in \mathcal{P}(X)$ with $A \neq B$, then $A \cap B = \varnothing$.*

The notion of partition will be properly generalized to the scope of distributive complemented lattices further on.

Proposition 2 *Let A be a nonempty set. If $\mathcal{R}$ is an equivalence relation on A, then the quotient set ${}^{A}/_{\mathcal{R}}$ is a partition of A. Conversely, if $\mathcal{P}$ is a partition of A, then there exists an equivalence relation $\mathcal{R}$ on A such that ${}^{A}/_{\mathcal{R}} = \mathcal{P}$.*

Proof 3 *The proof will be itemized in accordance with the statement.*

- $\Rightarrow$ *If $c \in [a]_{\mathcal{R}} \cap [b]_{\mathcal{R}}$, then every $d \in [a]_{\mathcal{R}}$ verifies that $(d,b) \in \mathcal{R}$ because $(d,a), (a,c), (c,b) \in \mathcal{R}$. In a similar way, every $d \in [b]_{\mathcal{R}}$ verifies that $(d,a) \in \mathcal{R}$. As a consequence, $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$. Finally, it is obvious that every $a \in A$ is in $[a]_{\mathcal{R}}$.*
- $\Leftarrow$ *It suffices to define $\mathcal{R} := \{(a,b) \in A \times A : \exists T \in \mathcal{P} \;\; a, b \in T\}$.*

1.1.3.2 First Isomorphism Theorem

The concept of the kernel of a function is typical in Linear Algebra. However, it can be extended to general functions as an application of equivalence relations.

Remark 7 (Kernel equivalence relation) *Let A, B be sets. Let $f \in B^A$. The following is an equivalence relation*

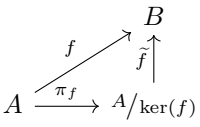
$$\ker(f) := \{(a_1, a_2) \in A \times A : f(a_1) = f(a_2)\}.$$

The following theorem is one of the most important results in Discrete Mathematics and settles the bases for the development of Category Theory.

Theorem 1 (First Isomorphism Theorem) *Let A, B be sets. Let $f : A \to B$ be a map. Then*

$$\begin{array}{rccl} \widetilde{f}: & {}^{A}/_{\ker(f)} & \to & f(A) \\ & [a]_{\ker(f)} & \mapsto & \widetilde{f}\left([a]_{\ker(f)}\right) := f(a) \end{array}$$

is bijective and unique satisfying that $\widetilde{f} \circ \pi_f = f$, where $\pi_f : A \to {}^{A}/_{\ker(f)}$ stands for the canonical projection.



Proof 4 *Let us show first that $\widetilde{f}$ is injective. Indeed, suppose that $a_1, a_2 \in A$ are so that $\widetilde{f}\left([a_1]_{\ker(f)}\right) = \widetilde{f}\left([a_2]_{\ker(f)}\right)$, then $f(a_1) = f(a_2)$; hence, $(a_1, a_2) \in \ker(f)$, so $[a_1]_{\ker(f)} = [a_2]_{\ker(f)}$. Finally, $\widetilde{f}$ is clearly surjective because for each $a \in A$, $\widetilde{f}\left([a]_{\ker(f)}\right) := f(a)$.*

1.1.4 Order relations

Order relations are also a particular case of binary internal relations, with multiple applications in different areas of Pure Mathematics. For instance, order relations are crucial in Abstract Calculus to define the concepts of net and limit.

1.1.4.1 Preorders and orders

By means of preorders and directed sets, we will be able to define nets, which are crucial structures in General Topology. Totally ordered sets will also play a fundamental role in Differential Calculus and Integral Calculus.

Definition 17 (Preorder) *Let A be a set. Let $\leq$ be a binary internal relation on A. Then:*

- *$\leq$ is called a preorder in A if it is reflexive and transitive. The tuple $(A, \leq)$ is called a preordered set.*
- *$\leq$ is called a direction if it is an upward directed preorder. The tuple $(A, \leq)$ is called a directed set.*
- *$\leq$ is called a partial order if it is an antisymmetric preorder. The tuple $(A, \leq)$ is called a partially ordered set (poset).*
- *$\leq$ is called a total order if it is a partial order and for every $a, b \in A$, either $a \leq b$ or $b \leq a$. The tuple $(A, \leq)$ is called a totally ordered set (toset).*

Maps between preordered sets which preserve the preorders are usually called increasing maps, that is, $f(a) \leq f(b)$ whenever $a \leq b$. To avoid confusion, the symbol $<$ means $\leq$ and $\neq$, that is, $a < b \Leftrightarrow (a \leq b) \wedge (a \neq b)$.

Remark 8 *If A is a preordered set, then the binary internal relation given by $\{(a, b) \in A \times A : a \leq b \text{ and } b \leq a\}$ is an equivalence relation on A whose quotient set can be endowed with structure of poset.*

The product of preordered sets can be endowed with a preorder, which turns out to be an ordering if the sets are ordered.

Example 2 (Product preorder) *Let $(X_i)_{i \in I}$ be a family of preordered sets. The product preorder is the preorder defined on $\prod_{i \in I} X_i$ given by*

$$(x_i)_{i \in I} \leq (y_i)_{i \in I} \Leftrightarrow \forall i \in I \;\; x_i \leq y_i \tag{1.5}$$

Another typical example is the pointwise ordering, which is a generalization of the previous example.

Example 3 (Pointwise preorder) *Let X be a set. Let Y be a preorder set. The set of functions Y^X can be endowed with the pointwise preorder given by $f \leq g \Leftrightarrow \forall x \in X \; f(x) \leq g(x)$.*

Under the settings of the previous example, if X_i is a poset for all $i \in I$, then (1.5) is a partial ordering. Notable subsets of a preordered set are the rays and intervals.

Notation 4 *Let A be a preordered set. Let $a, b \in A$. We denote:*

- *Closed rays: $[a, \infty) := \{x \in A : a \leq x\} = \uparrow a$ and $(-\infty, b] := \{x \in A : x \leq b\} = \downarrow b$.*

- *Open rays:* $(a,\infty) := \{x \in A : a < x\} = \uparrow_{\times} a$ *and* $(-\infty, b) := \{x \in A : x < b\} = \downarrow^{\times} b$.
- *Closed bounded interval:* $[a,b] := \{x \in A : a \leq x \leq b\} = \uparrow a \cap \downarrow b$.
- *Open bounded interval:* $(a,b) := \{x \in A : a < x < b\} = \uparrow_{\times} a \cap \downarrow^{\times} b$.
- *Semiopen bounded intervals:* $[a,b) := \{x \in A : a \leq x < b\} = \uparrow a \cap \downarrow^{\times} b$ *and* $(a,b] := \{x \in A : a < x \leq b\} = \uparrow_{\times} a \cap \downarrow b$.

Every subset of a preordered set, poset, or toset inherits the preorder, partial order, or total order, respectively. Cofinal subsets play important roles towards convergence and limit properties. Recall that a sequence in a set D is simply a map $\mathbb{N} \to D$.

Proposition 3 *Let D be a directed set. Let $E \subseteq D$ be cofinal. Let $d \in D$. Then:*

1. *E is also directed.*
2. *$\uparrow d$ is cofinal.*
3. *If E is countable, then there exists a sequence $(j_n)_{n\in\mathbb{N}} \subseteq D$ which is cofinal and increasing.*
4. *If $(A_d)_{d\in D}$ is a decreasing family of sets, then $\bigcap_{d\in D} A_d = \bigcap_{e\in E} A_e$.*
5. *If $(A_d)_{d\in D}$ is an increasing family of sets, then $\bigcup_{d\in D} A_d = \bigcup_{e\in E} A_e$.*

Proof 5 *The proof will be itemized in accordance with the statement.*

1. *Fix arbitrary elements $e_1, e_2 \in E$. There exists $d \in D$ with $d \geq e_1$ and $d \geq e_2$. Since E is cofinal, we can find $e \in E$ with $e \geq d$. Hence, $e \geq e_1$ and $e \geq e_2$.*
2. *Take any $d_1 \in D$. There exists $d_2 \in D$ with $d_2 \geq d_1$ and $d_2 \geq d$. Notice that $d_2 \in \uparrow d$.*
3. *We can write $E := \{e_n : n \in \mathbb{N}\}$. We will follow an inductive process. For $n = 1$, we choose $j_1 := e_1$. Suppose that we have constructed $j_1, j_2, \ldots, j_k$ for $k \geq 1$. There exists $j_{k+1} \in D$ such that $j_{k+1} \geq j_k$ and $j_{k+1} \geq e_{k+1}$. This way we obtain an increasing sequence $(j_n)_{n\in\mathbb{N}} \subseteq D$ which is cofinal in D because $j_n \geq e_n$ for all $n \in \mathbb{N}$.*
4. *Note that $\bigcap_{d\in D} A_d \subseteq \bigcap_{e\in E} A_e$. If $a \in \bigcap_{e\in E} A_e$, then for every $d \in D$ there exists $e \in E$ with $e \geq d$, so $A_e \subseteq A_d$; hence, $a \in A_e \subseteq A_d$.*
5. *Note that $\bigcup_{d\in D} A_d \supseteq \bigcup_{e\in E} A_e$. If $a \in \bigcup_{d\in D} A_d$, then there exists $d \in D$ with $a \in A_d$. We can find $e \in E$ with $e \geq d$, so $A_e \supseteq A_d$; hence, $a \in A_d \subseteq A_e$.*

The following definition carries out the most important notions in Order Theory: maximum, minimum, maximal, minimal, supremum, and infimum elements.

Definition 18 *Let A be a poset. Let $B \subseteq A$ and $a, b \in A$. Then:*

- *a is called the maximum of A provided that $A = \downarrow a$ and we denote it by* $\max(A)$.
- *a is called the minimum of A provided that $A = \uparrow a$ and we denote it by* $\min(A)$.
- *a is called maximal in A provided that $\{b \in A : a \in \downarrow b\} = \{a\}$.*
- *a is called minimal in A provided that $\{b \in A : a \in \uparrow b\} = \{a\}$.*
- *a is said to be an upper bound for B provided that $B \subseteq \downarrow a$.*

- *a is said to be a lower bound for B provided that $B \subseteq \uparrow a$.*
- *a is said to be the supremum of B provided that $\uparrow a = \{c \in A : B \subseteq \downarrow c\}$ and is denoted by $\sup(B)$ or $\bigvee B$.*
- *a is said to be the infimum of B provided that $\downarrow a = \{c \in A : B \subseteq \uparrow c\}$ and is denoted by $\inf(B)$ or $\bigwedge B$.*
- *B is said to be bounded above (below) if it has an upper (lower) bound, and it is said to be bounded if it is bounded above and below.*
- *B is said to be a generator of A if, for every $a \in A$, there exists an increasing sequence $(b_n)_{n\in\mathbb{N}} \subseteq B$ such that $a = \bigvee_{n\in\mathbb{N}} b_n$.*
- *B is said to be a chain of A if it is a totally ordered subset.*
- *a, b are said to form a hole provided that $a < b$ and $(a, b) = \varnothing$.*

Under the settings of the previous definition, observe that if any one of $\max(A), \min(A), \sup(B), \inf(B)$ exists, then it has to be unique. On the other hand, $\sup(B) = \min(\{c \in A : B \subseteq \downarrow c\})$ and $\inf(B) = \max(\{c \in A : B \subseteq \uparrow c\})$.

Definition 19 *Let $f : A \to B$ be a function between a set A and a poset B. We define the sets $\arg\max f(A) := \{a \in A : f(a) = \max f(A)\}$ and $\arg\min f(A) := \{a \in A : f(a) = \min f(A)\}$.*

In the previous definition, $\max_f(A) := \max f(A)$, $\min_f(A) := \min f(A)$, $\arg\max_f(A) := \arg\max f(A)$, $\arg\min_f(A) := \arg\min f(A)$.

Remark 9 *Let $f : A \to B$ be a function between a set A and a poset B. Consider $R, S \subseteq A$. Notice that $\arg\min_f(R \cap S) \supseteq \arg\min_f(R) \cap S$. If $\arg\min_f(R) \cap S \neq \varnothing$, then $\arg\min_f(R \cap S) = \arg\min_f(R) \cap S$. A dual reasoning applies for $\arg\max$.*

The following proposition generalizes Proposition 3 for sups and infs.

Proposition 4 *Let D be a directed set. Let $E \subseteq D$ be cofinal. Let L be a poset. Consider a family $(a_d)_{d\in D} \subseteq L$. Then:*

1. *If $(a_d)_{d\in D}$ is decreasing and $\bigwedge_{d\in D} a_d$ exists in L, then $\bigwedge_{d\in D} a_d = \bigwedge_{e\in E} a_e$.*
2. *If $(a_d)_{d\in D}$ is increasing and $\bigvee_{d\in D} a_d$ exists in L, then $\bigvee_{d\in D} a_d = \bigvee_{e\in E} a_e$.*

Proof 6 *The proof will be itemized in accordance with the statement.*

1. *Note that $\bigwedge_{d\in D} a_d \leq a_e$ for all $e \in E$. Let $b \in L$ such that $b \leq a_e$ for all $e \in E$. For every $d \in D$ there exists $e \in E$ with $e \geq d$, so $a_e \leq a_d$; hence, $b \leq a_e \leq a_d$. Thus, $b \leq \bigwedge_{d\in D} a_d$. This shows that $\bigwedge_{e\in E} a_e = \bigwedge_{d\in D} a_d$.*
2. *Note that $\bigvee_{d\in D} a_d \geq a_e$ for all $e \in E$. Let $b \in L$ such that $b \geq a_e$ for all $e \in E$. For every $d \in D$ there exists $e \in E$ with $e \geq d$, so $a_e \geq a_d$; hence, $b \geq a_e \geq a_d$. Thus, $b \geq \bigvee_{d\in D} a_d$. This shows that $\bigvee_{e\in E} a_e = \bigvee_{d\in D} a_d$.*

The following lemma (Zorn's Lemma) is a crucial result in the development of Modern Mathematics. We will use it throughout this text in order to construct important structures. It can be proved that Zorn's Lemma is equivalent to the Axiom of Choice under the rest of Zermelo-Fraenkel axioms.

Lemma 1 (Zorn's Lemma) *If P is a poset all of whose chains have upper (lower) bounds, then P has maximal (minimal) elements.*

The proof of Zorn's Lemma will not be included here, since it requires certain knowledge of Set Theory and Cardinality Theory that we have not covered in this book. We refer the reader to [1, 2] for a very rigorous proof. A direct consequence of Zorn's Lemma is the following result.

Lemma 2 *If P is a poset all of whose chains have upper (lower) bounds, then every element of P is less (greater) than or equal to a maximal (minimal) element of P.*

Proof 7 *Fix an arbitrary $p \in P$. If C is a chain of $\uparrow p$, then C is a chain of P. By hypothesis, C has an upper bound in P. This upper bound clearly belongs to $\uparrow p$, so C has an upper bound in $\uparrow p$. Finally, Zorn's Lemma assures the existence of a maximal element in $\uparrow p$. It is trivial to realize that this maximal element is also maximal in P.*

The existence of supremums and infimums for bounded subsets is a very helpful condition when working with posets. It is worth mentioning that finite subsets of a poset are not necessarily bounded unless, for instance, the poset is totally ordered.

Definition 20 (Dedekind completeness) *Let X be a poset. Then X is said to*

- *satisfy the (finite/countable) sup property if, for every bounded above (finite/countable) subset $Y \subseteq X$, there exists $\sup(Y)$;*
- *satisfy the (finite/countable) inf property if, for every bounded below (finite/countable) subset $Y \subseteq X$, there exists $\inf(Y)$;*
- *be Dedekind (finitely/countably) complete if it satisfies both the (finite/countable) sup property and the (finite/countable) inf property.*

Dedekind countable completeness is often referred to as Dedekind σ-completeness. Next theorem shows that Dedekind completeness, sup property, and inf property are equivalent properties.

Theorem 2 *A poset X satisfies the sup property if and only if it verifies the inf property. If, in addition, X is totally ordered, then X is Dedekind complete if and only if every bounded subset has and infimum and a supremum.*

Proof 8 *Suppose first that X is a poset satisfying the sup property. Fix any arbitrary bounded below subset Y of X. Consider the set of all lower bounds of Y, $Z := \{x \in X : Y \subseteq \uparrow x\}$. Note that every element of Y is an upper bound of Z. Since X satisfies the sup property, there exists $\sup(Z)$. We will show that $\sup(Z) = \inf(Y)$. Indeed, fix an arbitrary $y \in Y$. As mentioned before, y is an upper bound of Z, so by definition of supremum, $\sup(Z) \leq y$, meaning that $\sup(Z)$ is a lower bound for Y. Finally, assume that $x \in X$ is any lower bound for Y. Then $x \in Z$, meaning that $\sup(Z) \geq x$ again by definition of sup. As a consequence, $\sup(Z) = \inf(Y)$. This shows that X satisfies the inf property. The converse follows a dual proof. Next, suppose that X is totally ordered and that every bounded subset has and infimum and a supremum. Let us show that X satisfies the sup property. Indeed, take Y an arbitrary bounded above subset of X. Fix any $y_0 \in Y$. Then $Z := Y \cap \uparrow y_0$ is bounded, so by assumption there exists $\sup(Z)$. We will show that $\sup(Z) = \sup(Y)$. Indeed, let us prove first that $\sup(Z)$ is an upper bound for Y. If $y \in Y$, then there are two options: either $y \geq y_0$ (in which case $y \in Z$ and $y \leq \sup(Z)$) or $y < y_0$ (in which case $\sup(Z) \geq y_0 > y$). Finally, if p is any other upper bound for Y, then p is an upper bound for Z as well, meaning that $\sup(Z) \leq p$.*

The previous notions motivate the following definition, crucial for constructing hull operators.

Definition 21 (Dedekind closure) *Let X be a poset. Let Y be a subset of X. Then:*

1. *Suppose that X satisfies the (sup) inf property. We say that Y is (sup-)inf-closed provided that every bounded (above) below subset of Y has its (supremum) infimum in Y.*
2. *Suppose that X satisfies the countable (sup) inf property. We say that Y is countably (sup-) inf closed provided that every bounded (above) below countable subset of Y has its (supremum) infimum in Y.*
3. *Suppose that X satisfies the finite (sup) inf property. We say that Y is finitely (sup-) inf-closed provided that every bounded (above) below finite subset of Y has its (supremum) infimum in Y.*

The following example reveals the existence of subsets which are not sup-closed but satisfy the sup-property.

Example 4 *Let Ω be a set with at least three distinct elements α, β, γ. Let $X := \mathcal{P}(\Omega)$, which satisfies the sup property. Let $Y := \{\{\alpha\}, \{\beta\}, \{\alpha, \beta, \gamma\}\}$. Note that Y is not sup-closed in X because $\{\alpha, \beta\} \notin Y$. However, Y satisfies the sup property.*

By relying on the previous concepts, we present the following definition, which generalizes the notion of topology to the scope of order relations. We will explore topologies in the upcoming sections.

Definition 22 (Topology structure) *Let L be a Dedekind complete bounded poset. A subset $\mathcal{T}$ of L is called a topology structure provided that the following conditions are satisfied:*

- *Totality:* $0, 1 \in \mathcal{T}$.
- *Sup closedness: If $O \subseteq \mathcal{T}$, then $\bigvee O \in \mathcal{T}$.*
- *Finite inf closedness: If $F \subseteq \mathcal{T}$ is finite, then $\bigwedge F \in \mathcal{T}$.*

Later on we will see that T-posets are special cases of universal algebras satisfying particular conditions.

Definition 23 (Compactness/openness structure) *Let L be a Dedekind finitely complete bounded poset. Let $K, T \subseteq L$. We say that:*

- *K is a compactness structure provided that $0 \in K$ and K is finitely inf-closed.*
- *T is an openness structure provided that $1 \in T$ and T is finitely sup-closed.*

In the above definition, the elements of K are called compact elements, and the elements of T are called open elements. Compactness and openness structures are generalizations of the collections of compact subsets and open subsets, respectively, in a Hausdorff topological space.

Definition 24 (Finite union/intersection property) *Let X be a bounded poset satisfying the finite (sup) inf property. A subset Y of X is said to verify the finite (union) intersection property if every finite subset Z of Y satisfies that* $(\sup Z \neq 1)$ $\inf Z \neq 0$.

1.1.4.2 Lattices

Lattices are very particular cases of posets. They generalize the poset $(\mathcal{P}(X), \subseteq)$.

Definition 25 (Lattice) *A lattice is a poset A in which, for every $\{a, b\} \subseteq A$, there exist both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$.*

Notice that, under the settings of the previous definition, $\wedge$ and $\vee$ are, in fact, binary internal operations on A, notion that will be studied later.

Example 5 (Lattice of sets) *Let Ω be a set. Then $(\mathcal{P}(\Omega), \subseteq)$ is a lattice, where, for every $A, B \in \mathcal{P}(\Omega)$, $A \vee B := A \cup B$ and $A \wedge B := A \cap B$.*

Basic properties of lattices follow in the next proposition.

Proposition 5 *Let A be a lattice. Let $a, b, c \in A$. Let $B \subseteq A$. Then:*

1. *Idempotence: $a \vee a = a = a \wedge a$.*
2. *Commutativity: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.*
3. *Associativity: $a \vee (b \vee c) = (a \vee b) \vee c = a \vee b \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge b \wedge c$.*
4. *Absorption: $a \vee (a \wedge b) = a = a \wedge (a \vee b)$.*
5. *$a \vee b = b$ if and only if $a \leq b$.*
6. *$a \wedge b = a$ if and only if $a \leq b$.*
7. *$a \wedge b = a$ if and only if $a \vee b = b$.*
8. *$a \leq c$ and $b \leq c$ if and only if $a \vee b \leq c$.*
9. *$c \leq a$ and $c \leq b$ if and only if $c \leq a \wedge b$.*
10. *Compatibility: If $a \leq b$ and $c \leq d$, then $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$.*
11. *Semidistributivity: $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ and $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$. More generally, $\bigvee_{b \in B}(a \wedge b) \leq a \wedge \bigvee_{b \in B} b$ and $a \vee \bigwedge_{b \in B} b \leq \bigwedge_{b \in B}(a \vee b)$, provided that the previous sups and infs exist.*

Proof 9 *We will only prove the semidistributivity. Indeed, $a \wedge b \leq a$ and $a \wedge b \leq b \leq b \vee c$, therefore $a \wedge b \leq a \wedge (b \vee c)$. A similar reasoning shows that $a \wedge c \leq a \wedge (b \vee c)$. As a consequence, $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$. On the other hand, $a \leq a \vee b$ and $a \leq a \vee c$, meaning that $a \leq (a \vee b) \wedge (a \vee c)$. Then $b \wedge c \leq b \leq a \vee b$ and $b \wedge c \leq c \leq a \vee c$, meaning that $b \wedge c \leq (a \vee b) \wedge (a \vee c)$. As a consequence, $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$. Finally, let us prove that $a \wedge \bigvee_{b \in B} b \geq \bigvee_{b \in B}(a \wedge b)$. The other inequality follows by a dual proof. Fix an arbitrary $b \in B$. Then $a \wedge b \leq a$ and $a \wedge b \leq b \leq \bigvee_{b \in B} b$. Thus, $a \wedge b \leq a \wedge \bigvee_{b \in B} b$. As a consequence, $a \wedge \bigvee_{b \in B} b$ is an upper bound for $\{a \wedge b : b \in B\}$, meaning that $\bigvee_{b \in B}(a \wedge b) \leq a \wedge \bigvee_{b \in B} b$. We will prove this inequality in a different way. Again, a is an upper bound for $a \wedge b$ for all $b \in B$, meaning that $\bigvee_{b \in B}(a \wedge b) \leq a$. In a similar way, $\bigvee_{b \in B}(a \wedge b) \leq \bigvee_{b \in B} b$. Therefore, $\bigvee_{b \in B}(a \wedge b)$ is a lower bound for $\left\{a, \bigvee_{b \in B} b\right\}$, which implies that $\bigvee_{b \in B}(a \wedge b) \leq a \wedge \bigvee_{b \in B} b$.*

A direct consequence of the associativity of the binary internal operations $\wedge$ and $\vee$ is that for any lattice L and for each $a_1, \ldots, a_k \in L$, there exist $a_1 \wedge \cdots \wedge a_k$ and $a_1 \vee \cdots \vee a_k$.

Proposition 6 *Let X be a poset. Then X is a lattice if and only if all finite subsets of X are bounded, and X is Dedekind finitely complete.*

Proof 10 *Suppose first that X is a lattice. If $a_1, \ldots, a_k \in X$, then the associativity of the binary internal operations $\wedge$ and $\vee$ allow the existence of $a_1 \wedge \cdots \wedge a_k$ and $a_1 \vee \cdots \vee a_k$. As a consequence, X satisfies the finite inf property and the finite sup property. Conversely, assume that all finite subsets of X are bounded and X is Dedekind finitely complete. If x, y are arbitrary elements of the poset, then $\{x, y\}$ is bounded below and bounded above; therefore, there exists $x \wedge y$ because the poset satisfies the finite inf property, and there exists $x \vee y$ because the poset also satisfies the finite sup property.*

On the other hand, it is not hard to check that a map between lattices is increasing if and only if it is supadditive.

Remark 10 *Let X, Y be lattices and let $T : X \to Y$ be a map. If T is increasing and $x_1, x_2 \in X$, then $x_1 \leq x_1 \vee x_2$ and $x_2 \leq x_1 \vee x_2$, thus $T(x_1) \leq T(x_1 \vee x_2)$ and $T(x_2) \leq T(x_1 \vee x_2)$, therefore $T(x_1) \vee T(x_2) \leq T(x_1 \vee x_2)$, resulting in T being supadditive. Conversely, if $x_1 \leq x_2$, then $T(x_2) = T(x_1 \vee x_2) \geq T(x_1) \vee T(x_2) \geq T(x_1)$, meaning that T is increasing.*

Due to the associativity of the binary internal operations $\wedge$ and $\vee$, in a lattice X it occurs that a subset $Y \subseteq X$ is finitely inf-closed if and only if $y_1 \wedge y_2 \in Y$ for every $y_1, y_2 \in Y$. A dual characterization applies to finitely sup-closed subsets.

Definition 26 (Sublattice) *If L is a lattice and $M \subseteq L$, then M is said to be a sublattice of L if M is closed under $\wedge$ and $\vee$, that is, $a \wedge b \in M$ and $a \vee b \in M$ for all $a, b \in M$.*

As mentioned before and under the settings of the previous definition, for each $a_1, \ldots, a_k \in M$, $a_1 \wedge \cdots \wedge a_k, a_1 \vee \cdots \vee a_k \in M$. In other words, sublattices are exactly the finitely inf and sup-closed subsets. Classic examples of sublattices are the closed rays.

Definition 27 (Distributive lattice) *A lattice A is called distributive provided that it verifies the distributive property: $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in A$.*

In a bounded poset L, we denote $0 := \min(L)$ and $1 = \max(L)$. Two elements $a, b \in L$ are said to be disjoint if $a \wedge b = 0$.

Definition 28 (Partition) *Let A be a bounded poset. A subset $P \subseteq A$ is said to be a partition if the elements of P are pairwise disjoint, that is, $a \wedge b = 0$ for all $a, b \in P$ with $a \neq b$, and $\bigvee P = 1$.*

The notion of partition motivates that of complementation.

Definition 29 (Complement) *Let A be a bounded lattice. An element $b \in A$ is said to be a complement of another element $a \in A$ provided that $\{a, b\}$ is a partition, that is, $a \wedge b = 0$ and $a \vee b = 1$. The set of complements of a is denoted by $1 \setminus a$.*

Under the settings of the previous definition, notice that $a \in 1 \setminus b$ if and only if $b \in 1 \setminus a$. On the other hand, if $a \leq b$, then we say that c is a complement of a in b if $a \wedge c = 0$ and $a \vee c = b$. The set of complements of a in b is denoted by $b \setminus a$. Notice that c is, in fact, the complement of a in the sublattice $\downarrow b$ since $0 = \min(\downarrow b)$ and $b = \max(\downarrow b)$.

Lemma 3 *Let A be a distributive bounded lattice. Let $a, b, c, a', b', c' \in A$. Then:*

1. *If $a = b \vee c$, $a' = b' \vee c'$, $a \leq a'$, $b \leq b'$, and $b' \wedge c = 0$, then $c \leq c'$.*
2. *$1 \setminus a$ is either empty or a singleton.*
3. *If $1 \setminus a$ exists, then so does $b \setminus (a \wedge b)$, and $b \setminus (a \wedge b) = (1 \setminus a) \wedge b$.*

Proof 11 *Let us itemize the proof as the statement is.*

1. *Note that $c \leq a \leq a'$, thus $c = c \wedge a' = c \wedge (b' \vee c') = (c \wedge b') \vee (c \wedge c') = 0 \vee (c \wedge c') = c \wedge c' \leq c'$.*
2. *If $b, c \in 1 \setminus a$, then $c = 1 \wedge c = (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = 0 \vee (b \wedge c) = b \wedge c$, which implies that $c \leq b$. In a similar way, it can be proved that $b \leq c$.*
3. *On the one hand, $b \wedge (1 \setminus a) \leq b$ and $a \wedge b \leq b$, so $(b \wedge (1 \setminus a)) \vee (a \wedge b) \leq b$. On the other hand, $(b \wedge (1 \setminus a)) \vee (a \wedge b) = (b \vee (a \wedge b)) \wedge ((1 \setminus a) \vee (a \wedge b)) = b \wedge [((1 \setminus a) \vee a) \wedge ((1 \setminus a) \vee b)] = b \wedge [1 \wedge ((1 \setminus a) \vee b)] = b \wedge ((1 \setminus a) \vee b) = (b \wedge (1 \setminus a)) \vee b \geq b$. Finally, $(b \wedge (1 \setminus a)) \wedge (a \wedge b) = b \wedge ((1 \setminus a) \wedge a) = b \wedge 0 = 0$.*

Under the settings of the previous lemma, if $1 \setminus a$ is a singleton, then $1 \setminus (1 \setminus a) = a$. Lemma 3 motivates the following definition.

Definition 30 (Complemented lattice) *A bounded lattice is said to be (uniquely) complemented when every element has a (unique) complement.*

A direct consequence of Lemma 3 is the fact that every distributive complemented lattice is, in fact, uniquely complemented.

Theorem 3 *Let A be a distributive complemented lattice. Let $B \subseteq A$. Let $(a_n)_{n \in \mathbb{N}} \subseteq A$. Let $a, b, c, a', b', c' \in A$. Then:*

1. *De Morgan laws: $1 \setminus (a \vee b) = (1 \setminus a) \wedge (1 \setminus b)$ and $1 \setminus (a \wedge b) = (1 \setminus a) \vee (1 \setminus b)$.*
2. *$a \wedge b = a \wedge [1 \setminus (a \wedge (1 \setminus b))]$.*
3. *$a \wedge b = 0$ if and only if $a \leq 1 \setminus b$, and $a \vee b = 1$ if and only if $a \geq 1 \setminus b$.*
4. *$a \leq b$ if and only if $1 \setminus b \leq 1 \setminus a$. In particular, if $a = b \vee c = b' \vee c'$ with $b \wedge c = 0$ and $b' \wedge c' = 0$, and $b \leq b'$ and $c \leq c'$, then $b = b'$ and $c = c'$.*
5. *If $a \leq a'$, then $(a' \wedge b) \setminus (a \wedge b) \leq a' \setminus a$ and $(a' \vee b) \setminus (a \vee b) \leq a' \setminus a$. In particular, if $a \leq a'$ and $b \leq b'$, then $(a \wedge b') \setminus (a \wedge b) \leq (a' \wedge b') \setminus (a' \wedge b)$.*
6. *$\bigwedge_{b \in B} b$ exists if and only if $\bigvee_{b \in B} 1 \setminus b$ exists. In this situation, $\bigvee_{b \in B} 1 \setminus b = 1 \setminus \bigwedge_{b \in B} b$.*
7. *$\downarrow b$ is a distributive complemented lattice and if $a \leq b$, then $b \setminus a = (1 \setminus a) \wedge b$.*
8. *There are $(b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}} \subseteq A$ such that $(b_n)_{n \in \mathbb{N}}$ is increasing, $(c_n)_{n \in \mathbb{N}}$ is pairwise disjoint, and $a_1 \vee \cdots \vee a_n = b_1 \vee \cdots \vee b_n = c_1 \vee \cdots \vee c_n$ for all $n \in \mathbb{N}$. If, in addition, $\bigvee_{n \in \mathbb{N}} a_n$ exists, then so do $\bigvee_{n \in \mathbb{N}} b_n, \bigvee_{n \in \mathbb{N}} c_n$ and $\bigvee_{n \in \mathbb{N}} a_n = \bigvee_{n \in \mathbb{N}} b_n = \bigvee_{n \in \mathbb{N}} c_n$.*
9. *σ-Distributivity: $a \wedge \bigvee_{n \in \mathbb{N}} a_n = \bigvee_{n \in \mathbb{N}} (a \wedge a_n)$ and $a \vee \bigwedge_{n \in \mathbb{N}} a_n = \bigwedge_{n \in \mathbb{N}} (a \vee a_n)$, provided that the previous sups and infs exist.*

Proof 12 *The proof follows, itemized in the same way as the statement of the theorem.*

1. *Notice that* $((1 \setminus a) \wedge (1 \setminus b)) \wedge (a \vee b) = [(1 \setminus a) \wedge (1 \setminus b) \wedge a] \vee [(1 \setminus a) \wedge (1 \setminus b) \wedge b] = 0 \vee 0 = 0$, *and that* $((1 \setminus a) \wedge (1 \setminus b)) \vee (a \vee b) = [(1 \setminus a) \vee a \vee b] \wedge [(1 \setminus b) \vee a \vee b] = 1 \wedge 1 = 1$. *The other equality follows by a dual proof.*

2. *Observe that* $a \wedge [1 \setminus (a \wedge (1 \setminus b))] = a \wedge [(1 \setminus a) \vee (1 \setminus (1 \setminus b))] = a \wedge [(1 \setminus a) \vee b] = (a \wedge (1 \setminus a)) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b$.

3. *In the first place, if* $a \leq 1 \setminus b$, *then* $a \wedge b \leq (1 \setminus b) \wedge b = 0$. *Suppose now that* $a \wedge b = 0$. *On the one hand,* $1 = (a \vee b) \vee (1 \setminus (a \vee b)) = b \vee [a \vee (1 \setminus (a \vee b))]$, *and on the other hand,* $b \wedge [a \vee (1 \setminus (a \vee b))] = (b \wedge a) \vee [b \wedge (1 \setminus (a \vee b))] = (b \wedge a) \vee [b \wedge (1 \setminus a) \wedge (1 \setminus b)] = (b \wedge a) \vee 0 = b \wedge a = 0$. *The uniqueness of the complement allows to deduce that* $a \vee (1 \setminus (a \vee b)) = 1 \setminus b$, *which automatically implies that* $a \leq 1 \setminus b$. *The other equivalence follows by a dual proof.*

4. *It only suffices to prove that, if* $a \leq b$, *then* $1 \setminus b \leq 1 \setminus a$. *Observe that* $a \leq b = 1 \setminus (1 \setminus b)$, *therefore,* $a \wedge (1 \setminus b) = 0$, *which implies that* $1 \setminus b \leq 1 \setminus a$. *Suppose now that* $a = b \vee c = b' \vee c'$ *with* $b \wedge c = 0$ *and* $b' \wedge c' = 0$, *and* $b \leq b'$ *and* $c \leq c'$. *Then* $1 \setminus b' \leq 1 \setminus b$, *so* $c' = a \setminus b' = a \wedge (1 \setminus b') \leq a \wedge (1 \setminus b) = a \setminus b = c$. *In a similar way,* $b' \leq b$.

5. *Let us show first that* $(a' \wedge b) \setminus (a \wedge b) \leq a' \setminus a$. *Observe that* $a' \wedge b = (a \wedge b) \vee ((a' \wedge b) \setminus (a \wedge b))$ *and* $a' = a \vee (a' \setminus a)$. *According to Lemma 3, it only suffices to prove that* $a \wedge ((a' \wedge b) \setminus (a \wedge b)) = 0$. *If* $l \leq a$ *and* $l \leq (a' \wedge b) \setminus (a \wedge b)$, *then* $l \leq (a' \wedge b) \setminus (a \wedge b) \leq a' \wedge b \leq b$, *meaning that* $l \leq a \wedge b$, *so* $l \leq (a \wedge b) \wedge ((a' \wedge b) \setminus (a \wedge b)) = 0$. *As a consequence,* $a \wedge ((a' \wedge b) \setminus (a \wedge b)) = 0$. *Let us finally show that* $(a' \vee b) \setminus (a \vee b) \leq a' \setminus a$. *Take any* $l \leq (a' \vee b) \setminus (a \vee b)$. *Note that* $0 = l \wedge (a \vee b) = (l \wedge a) \vee (l \wedge b)$, *meaning that* $l \wedge a = 0 = l \wedge b$. *Next,* $l \leq 1 \setminus b$, *so* $l \leq (1 \setminus b) \wedge (a' \vee b) = ((1 \setminus b) \wedge a') \vee ((1 \setminus b) \wedge b) = ((1 \setminus b) \wedge a') \vee 0 = (1 \setminus b) \wedge a' \leq a'$. *Next,* $l = l \wedge a' = (l \wedge a) \vee (l \wedge (a' \setminus a)) = 0 \vee (l \wedge (a' \setminus a)) = l \wedge (a' \setminus a) \leq a' \setminus a$.

6. *Suppose, for instance, that* $\bigwedge_{b \in B} b$ *exists. We will show first that* $1 \setminus \bigwedge_{b \in B} b$ *is an upper bound of* $\{1 \setminus b : b \in B\}$. *Indeed, for every* $b_0 \in B$, $\bigwedge_{b \in B} b \leq b_0$, *so* $1 \setminus \bigwedge_{b \in B} b \geq 1 \setminus b_0$. *Next, let* $a \in A$ *be any other upper bound of* $\{1 \setminus b : b \in B\}$. *Then* $1 \setminus a$ *is a lower bound of* B. *By definition of inf,* $1 \setminus a \leq \bigwedge_{b \in B} b$, *so* $1 \setminus \bigwedge_{b \in B} b \leq a$. *This shows that* $\bigvee_{b \in B} 1 \setminus b = 1 \setminus \bigwedge_{b \in B} b$. *The rest follows by a dual proof.*

7. *It is straightforward by bearing in mind Lemma 3(3).*

8. *Let* $b_1 := a_1$ *and* $c_1 := a_1$, *and, for every* $n > 1$, *let us inductively define* $b_n := a_1 \vee \cdots \vee a_n$ *and* $c_n := b_n \setminus b_{n-1} = b_n \wedge (1 \setminus b_{n-1})$. *The desired properties are naturally satisfied for* $(b_n)_{n \in \mathbb{N}}$. *Suppose that* $b_n = c_1 \vee \cdots \vee c_n$ *up to a certain* $n \geq 1$. *Then* $1 = b_n \vee (1 \setminus b_n)$, *so* $b_{n+1} = (b_{n+1} \wedge b_n) \vee (b_{n+1} \wedge (1 \setminus b_n)) = b_n \vee c_{n+1} = c_1 \vee \cdots \vee c_n \vee c_{n+1}$.

9. *Let us prove that* $a \wedge \bigvee_{n \in \mathbb{N}} a_n = \bigvee_{n \in \mathbb{N}} (a \wedge a_n)$. *The other equality follows by a dual proof. On the one hand,*

$$\bigvee_{n \in \mathbb{N}} a_n = \left(a \wedge \bigvee_{n \in \mathbb{N}} a_n \right) \vee \left((1 \setminus a) \wedge \bigvee_{n \in \mathbb{N}} a_n \right). \tag{1.6}$$

On the other hand, for every $n \in \mathbb{N}$,

$$\bigvee_{k=1}^{n} a_k = \left(a \wedge \bigvee_{k=1}^{n} a_k \right) \vee \left((1 \setminus a) \wedge \bigvee_{k=1}^{n} a_k \right). \tag{1.7}$$

Then

$$\bigvee_{n\in\mathbb{N}} a_n = \bigvee_{n\in\mathbb{N}} \left(a \wedge \bigvee_{k=1}^{n} a_k \right) \vee \bigvee_{n\in\mathbb{N}} \left((1 \setminus a) \wedge \bigvee_{k=1}^{n} a_k \right). \tag{1.8}$$

If we compare (1.6) *with* (1.8) *and notice that* $\bigvee_{n\in\mathbb{N}} (a \wedge \bigvee_{k=1}^{n} a_k) \leq a \wedge \bigvee_{n\in\mathbb{N}} a_n$ *and* $\bigvee_{n\in\mathbb{N}} ((1 \setminus a) \wedge \bigvee_{k=1}^{n} a_k) \leq (1 \setminus a) \wedge \bigvee_{n\in\mathbb{N}} a_n$*, then* $\bigvee_{n\in\mathbb{N}} (a \wedge \bigvee_{k=1}^{n} a_k) = a \wedge \bigvee_{n\in\mathbb{N}} a_n$ *and* $\bigvee_{n\in\mathbb{N}} ((1 \setminus a) \wedge \bigvee_{k=1}^{n} a_k) = (1 \setminus a) \wedge \bigvee_{n\in\mathbb{N}} a_n$*. In particular,*

$$a \wedge \bigvee_{n\in\mathbb{N}} a_n = \bigvee_{n\in\mathbb{N}} \left(a \wedge \bigvee_{k=1}^{n} a_k \right) = \bigvee_{n\in\mathbb{N}} \bigvee_{k=1}^{n} (a_k \wedge b) = \bigvee_{n\in\mathbb{N}} (a_n \wedge b).$$

Although symmetrization operations have not been introduced yet, the next remark makes use of them in a very intuitive way. Symmetrization operations will be studied in the upcoming sections.

Remark 11 *If P is a distributive complemented lattice, then there is a symmetrization operation called complementation:*

$$\begin{array}{rccc} {}^{\mathsf{c}}: & P & \to & P \\ & p & \mapsto & p^{\mathsf{c}} := 1 \setminus p. \end{array} \tag{1.9}$$

This symmetrization operation is an involution. If $Q \subseteq P$, then $Q^{\mathsf{c}} := \{q^{\mathsf{c}} : q \in Q\}$.

If $f : P \to Q$ is an increasing map between lattices P, Q, then

$$f(a \wedge b) \leq f(a) \wedge f(b) \leq f(a) \vee f(b) \leq f(a \vee b)$$

for all $a, b \in P$.

Definition 31 (Projection) *Let P be a bounded lattice. For every $p \in P$, the projection of P over p is defined as the increasing map*

$$\begin{array}{ccc} P & \to & P \\ a & \mapsto & a \wedge p \end{array} \tag{1.10}$$

Under the settings of the previous definition, the range of (1.10) is $\downarrow p$. The preimage of $(\downarrow p) \setminus \{0\}$ under (1.10) is denoted as $p^{\circ} := \{a \in P : p \wedge a > 0\}$. Observe that $p^{\circ} \cap 1 \setminus p = \varnothing$.

Definition 32 *Let P, Q be bounded lattices. Let $f : P \to Q$ be an increasing function. Then f is called:*

- *Initial provided that $\{0\} = f^{-1}(\{0\})$.*
- *Regular provided that, for every $q \in Q$, there exists $p \in f^{-1}(\downarrow q)$ such that $f^{-1}(q^{\circ}) \subseteq p^{\circ}$.*
- *Strongly regular provided that, for every $q \in Q$, there exists $p \in f^{-1}(\{q\})$ such that $f^{-1}(q^{\circ}) \subseteq p^{\circ}$.*

The final result in this subsubsection displays an example of initial and regular maps.

Proposition 7 *If $f : X \to Y$ is a (surjective) function between sets X, Y, then the induced map $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is initial and (strongly) regular.*

Proof 13 *First off, note that $\{\varnothing\} = f^{-1}(\{\varnothing\})$. Fix an arbitrary $Q \in \mathcal{P}(Y)$ and let $P := f^{-1}(Q)$. Notice that $f(P) = f\left(f^{-1}(Q)\right) \subseteq Q$, so $P \in f^{-1}(\downarrow Q)$. Now, consider $A \in \mathcal{P}(X)$ such that $f(A) \cap Q \neq \varnothing$. Then $A \cap P = A \cap f^{-1}(Q) \neq \varnothing$. This shows that $f^{-1}(Q^{\circ}) \subseteq P^{\circ}$. Finally, suppose that $f : X \to Y$ is surjective. Fix again an arbitrary $Q \in \mathcal{P}(Y)$ and let $P := f^{-1}(Q)$. Notice that $f(P) = f\left(f^{-1}(Q)\right) = Q$, so $P \in f^{-1}(\{Q\})$. The rest of the proof follows as in the previous case.*

1.1.4.3 Filters/ideals; bornologies

Filters and ideals are fundamental concepts involved in the definition of limits and the construction of convergence methods.

Definition 33 (Ideal) *Let P be a bounded poset. A subset $I \subseteq P$ is called:*

- *A preideal provided that $I \neq \varnothing$, $1 \notin I$, and I is upward directed.*
- *An ideal provided that I is a downward-closed preideal.*
- *An ideal base provided that its downward closure $\mathcal{I}(I)$ is an ideal.*

Under the settings of the previous definition, I is a preideal if and only if it is an ideal base.

Remark 12 *Let P be a bounded poset. Let $I \subseteq P$. Let $p \in P$. Then:*

- *If I is an ideal, then $0 \in I$ because I is downward closed and nonempty. However, ideal bases do not necessarily contain 0.*
- *If P is a bounded lattice and I is an ideal of P, then $a \vee b \in I$ for all $a, b \in I$ because I is upward directed and downward closed. In particular, $a \vee b < 1$.*
- *I is an ideal base if and only if I is a preideal.*
- *If I is an ideal, then $I = \mathcal{I}(I) = {\downarrow}I = \bigcup_{p \in I} {\downarrow}p$.*
- *If $p < 1$, then ${\downarrow}p$ is an ideal (called a principal ideal).*

Under the settings of the previous remark, if $(I_i)_{i \in I}$ is a family of ideals of P, then $\bigcup_{i \in I} I_i$ is an ideal of P. Bornologies are special classes of ideals.

Definition 34 (Bornology) *A subset B of a bounded poset P is said to be a bornology provided that B is an ideal and $1 = \bigvee B$.*

An ideal base is said to be a bornology base whenever its generated ideal is a bornology.

Example 6 *Let X be an unbounded poset. If X is upward (downward) directed, then the collection $\mathcal{B}_{\mathtt{a}}(X)$ ($\mathcal{B}_{\mathtt{b}}(X)$) of all bounded above (below) subsets of X is a bornology of $\mathcal{P}(X)$.*

Under the settings of the previous example, if X is upward and downward-directed, then the collection $\mathcal{B}(X) := \mathcal{B}_{\mathtt{a}}(X) \cap \mathcal{B}_{\mathtt{b}}(X)$ of all bounded subsets of X is a bornology of $\mathcal{P}(X)$. The notion of a filter is coming up now.

Definition 35 (Filter) *Let P be a bounded poset. A subset $F \subseteq P$ is called:*

- *A prefilter provided that $F \neq \varnothing$, $0 \notin F$, and F is downward directed.*
- *A filter provided that F is an upward-closed prefilter.*
- *A filter base provided that its upward closure $\mathcal{J}(F)$ is a filter.*

Under the settings of the previous definition, F is a filter base if and only if it is a prefilter.

Remark 13 *Let P be a bounded poset. Let $F \subseteq P$. Let $p \in P$. Then:*

- *If F is a filter, then $1 \in I$ because F is upward closed and nonempty. However, filter bases do not necessarily contain 1.*
- *If P is a bounded lattice and F is a filter of P, then $a \wedge b \in F$ for all $a, b \in F$ because F is downward directed and upward closed. In particular, $a \wedge b > 0$.*
- *F is a filter base if and only if F is a prefilter.*
- *If F is a filter, then $F = \mathcal{J}(F) = \uparrow F = \bigcup_{p \in F} \uparrow p$.*
- *If $p > 0$, then $\uparrow p$ is a filter (called a principal filter).*

Under the settings of the previous remark, if $(F_j)_{j \in J}$ is a family of filters of P, then $\bigcup_{j \in J} F_j$ is a filter of P. Free filters are a special class of filters.

Definition 36 (Free filter) *A subset F of a bounded poset P is said to be a free filter provided that F is a filter and $0 = \bigwedge F$.*

Filters and ideals are dual concepts. In this duality, free filters correspond with bornologies. Recall that if P is a distributive complemented lattice, then there is a symmetrization operation called complementation:

$$\begin{array}{rccc} {}^{\mathsf{c}}: & P & \to & P \\ & p & \mapsto & p^{\mathsf{c}} := 1 \setminus p. \end{array} \tag{1.11}$$

This symmetrization operation is an involution. If $Q \subseteq P$, then $Q^{\mathsf{c}} := \{q^{\mathsf{c}} : q \in Q\}$.

Proposition 8 *Let P be a distributive complemented lattice. Then:*

1. *If F is a filter (base) of P, then F^{c} is an ideal (base) of P.*
2. *If F is a filter base of P, then $\mathcal{I}(F^{\mathsf{c}}) = \mathcal{J}(F)^{\mathsf{c}}$.*
3. *If F is a free filter of P, then F^{c} is a bornology.*
4. *If I is an ideal (base) of P, then I^{c} is a filter (base) of P.*
5. *If I is an ideal base of P, then $\mathcal{J}(I^{\mathsf{c}}) = \mathcal{I}(I)^{\mathsf{c}}$.*
6. *If B is a bornology of P, then B^{c} is a free filter.*

Proof 14 *We will only prove the first three items:*

1. *We will only prove the case that F is a filter. Let us check the three properties verified by ideals:*
   - *$F^{\mathsf{c}} \neq \varnothing$ and $1 \notin F^{\mathsf{c}}$. Indeed, $F \neq \varnothing$ so $F^{\mathsf{c}} \neq \varnothing$. If $1 \in F^{\mathsf{c}}$, then there exists $p \in F$ with $1 = 1 \setminus p$, thus $p = 0$, which contradicts the fact that $0 \notin F$.*
   - *F^{c} is upward directed. Indeed, let $p, q \in F$. There is $r \in F$ with $r \leq p$ and $r \leq q$, which implies that $1 \setminus p \leq 1 \setminus r$ and $1 \setminus q \leq 1 \setminus r$.*
   - *F^{c} is downward closed. Indeed, let $p \in F$ and $r \in P$ such that $r \leq 1 \setminus p$. Then $p \leq 1 \setminus r$ so $1 \setminus r \in F$; hence, $r = 1 \setminus (1 \setminus r) \in F^{\mathsf{c}}$.*
2. *Note that $F \subseteq \mathcal{J}(F)$, hence $F^{\mathsf{c}} \subseteq \mathcal{J}(F)^{\mathsf{c}}$. Since $\mathcal{I}$ is a hull operator and $\mathcal{J}(F)^{\mathsf{c}}$ is an ideal, $\mathcal{I}(F^{\mathsf{c}}) \subseteq \mathcal{I}(\mathcal{J}(F)^{\mathsf{c}}) = \mathcal{J}(F)^{\mathsf{c}}$. Conversely, take any $p \in \mathcal{J}(F)^{\mathsf{c}}$. Then $1 \setminus p \in \mathcal{J}(F)$, so there exists $q \in F$ such that $q \leq 1 \setminus p$, meaning that $p \leq 1 \setminus q \in F^{\mathsf{c}}$, that is, $p \in \mathcal{I}(F^{\mathsf{c}})$.*

3. We already know that F^{c} is an ideal. Next, by bearing in mind Proposition 3(7),

$$\bigvee_{p\in F} 1 \setminus p = 1 \setminus \bigwedge_{p\in F} p = 1 \setminus 0 = 1.$$

The most natural examples of filters and ideals can be found in the distributive complemented lattice $(\mathcal{P}(X), \subseteq)$, where X is a nonempty set. Whenever we talk about prefilters, filters, preideals, or ideals in a set, then we will be referring to that of its power set.

Proposition 9 *Let P be a bounded poset. Let F be a filter of P. If F is finite, then it is principal.*

Proof 15 *Write $F = \{p_1, \ldots, p_k\}$. By using that F is downward directed, we have that there exists $j \in \{1, \ldots, k\}$ such that $p_j \leq p_i$ for all $i \in \{1, \ldots, k\}$. Therefore, $F \subseteq \uparrow p_j$. Finally, since F is upward closed, we deduce that $\uparrow p_j \subseteq F$.*

By duality, Proposition 9 encounters its analogue for ideals. On the other hand, let P be a bounded poset and consider $(F_i)_{i\in I}$ a chain of filters (ordered by the inclusion). Then $\bigcup_{i\in I} F_i$ is a filter of P. Therefore, Zorn's Lemma assures the existence of maximal filters in bounded posets.

Definition 37 (Ultrafilter) *Let P be a bounded poset. A filter F of P is called an ultrafilter if it is a maximal filter.*

By duality, the notion of ultrafilter also applies to the scope of ideals as ultraideals.

Proposition 10 *Let P be a bounded poset. Let $p \in P \setminus \{0\}$. Then p is a minimal element in $P \setminus \{0\}$ if and only if $\uparrow p$ is an ultrafilter.*

Proof 16 *The proof is as itemized as the statement of the proposition.*

- $\Rightarrow$ *Let F be a filter containing $\uparrow p$ and suppose to the contrary that we can find $y \in F \setminus \uparrow p$. Notice that $p, y \in F$, therefore, there exists $z \in F$ with $z \leq p$ and $z \leq y$. By hypothesis, $z = p$, thus, $p \leq y$, which implies the contradiction that $y \in \uparrow p$.*
- $\Leftarrow$ *If $\uparrow p$ is an ultrafilter and $y \leq p$, then $\uparrow p \subseteq \uparrow y$, so they must be equal by hypothesis. Thus, $y \in \uparrow p$, which means that $p \leq y$.*

Again, as expected, Proposition 10 finds its analogue for ideals by duality interchanging maximal for minimal and ultraideal for ultrafilter.

Lemma 4 *Let P be a bounded lattice. Let F be an ultrafilter of P. Let $p \in P$. If $F \subseteq p^{\circ}$, then $p \in F$.*

Proof 17 *Notice that $F \cup \{a \wedge p : a \in F\}$ is a filter base, therefore $F \subseteq F \cup \{a \wedge p : a \in F\} \subseteq \mathcal{J}(F \cup \{a \wedge p : a \in F\})$. By maximality, $F = \mathcal{J}(F \cup \{a \wedge p : p \in F\})$ and so $p \in F$.*

Initial, regular, and strongly regular functions have a strong impact on transporting prefilters.

Proposition 11 *Let $f : P \to Q$ be an initial increasing function between bounded lattices P, Q. Then:*

1. If B is a filter base of P, then $f(B)$ is a filter base of Q and $\mathcal{J}(f(\mathcal{J}(B))) = \mathcal{J}(f(B))$.

2. *If f is regular and F is an ultrafilter of P, then $\mathcal{J}(f(F))$ is an ultrafilter of Q.*

3. *If f is strongly regular and F is a filter (ultrafilter) of P, then $f(F)$ is a filter (ultrafilter) of Q.*

Proof 18 *The proof will be itemized in accordance with the statement.*

1. *Observe that $f(B) \neq \varnothing$ because so is B. If $0 \in f(B)$, there exists $b \in B$ with $f(b) = 0$, meaning that $b \in f^{-1}(\{0\}) = \{0\}$, so we get the contradiction that $0 = b \in B$. Finally, if $b, c \in B$, there exists $d \in B$ with $d \leq b$ and $d \leq c$, so $f(d) \leq f(b)$ and $f(d) \leq f(c)$. Thus, $f(B)$ is downward directed.*

2. *Let G be a filter of Q containing $\mathcal{J}(f(F))$. Fix an arbitrary $g \in G$. Then $g \wedge f(a) > 0$ for all $a \in F$ in view of Remark 13, in other words, $F \subseteq f^{-1}(g^{\circ})$. Therefore, by hypothesis, there exists $p \in f^{-1}(\{\downarrow g\})$ such that $f^{-1}(g^{\circ}) \subseteq p^{\circ}$. As a consequence, $F \subseteq p^{\circ}$. In virtue of Lemma 4, we have that $p \in F$. Then $f(p) \in f(F)$ and $g \in \mathcal{J}(f(F))$.*

3. *In accordance with Proposition 11(1), we have that $f(F)$ is a filter base of Q. Let G be a filter of Q containing $f(F)$. Fix an arbitrary $g \in G$. Note that $g \wedge f(a) > 0$ for all $a \in F$ (see Remark 13); therefore, by hypothesis, there exists $p \in P$ such that $f(p) = g$ and $p \wedge a > 0$ for all $a \in F$. By applying Lemma 4, we conclude that $p \in F$, hence $g = f(p) \in f(F)$.*

The following corollary is a direct consequence of Proposition 11 together with Proposition 7.

Corollary 2 *Let X, Y be nonempty sets. Consider a function $f : X \to Y$ together with its induced map $f : \mathcal{P}(X) \to \mathcal{P}(Y)$. Then:*

1. *If $\mathcal{B}$ is a filter base of $\mathcal{P}(X)$, then $f(\mathcal{B})$ is a filter base of $\mathcal{P}(Y)$ and $\mathcal{J}\left(f(\mathcal{J}(\mathcal{B}))\right) = \mathcal{J}(f(\mathcal{B}))$.*

2. *If $\mathcal{F}$ is a filter of $\mathcal{P}(X)$, then $f^{-1}(W) \in \mathcal{F}$ for every $W \in \mathcal{J}(f(\mathcal{F}))$.*

3. *If $\mathcal{F}$ is an ultrafilter of $\mathcal{P}(X)$, then $\mathcal{J}(f(\mathcal{F}))$ is an ultrafilter of $\mathcal{P}(Y)$.*

4. *If $f : X \to Y$ is surjective and $\mathcal{F}$ is a filter (ultrafilter) of $\mathcal{P}(X)$, then $f(\mathcal{F})$ is a filter (ultrafilter) of $\mathcal{P}(Y)$.*

Notice that a filter $\mathcal{F}$ of $\mathcal{P}(X)$, for X a given nonempty set, is a free filter if and only if $\bigcap \mathcal{F} = \varnothing$.

Definition 38 (Fréchet filter) *Let X be an infinite set. The Fréchet filter of $\mathcal{P}(X)$ is defined by $\mathcal{F}_X := \{A \subseteq X : X \setminus A \in \mathcal{P}_{\mathtt{f}}(X)\}$.*

The Fréchet filter is one of the most important filters when dealing with convergence and summability.

Proposition 12 *Let X be a nonempty set. Let $\mathcal{F}$ be a filter of $\mathcal{P}(X)$. Then:*

1. *If $\mathcal{F}$ is principal, then it is not free.*

2. *$\mathcal{F}$ is not free if and only if there is $A \in \mathcal{P}(X) \setminus \{\varnothing\}$ so that $\mathcal{F} \subseteq \uparrow A$.*

3. *$\mathcal{F}$ is a nonfree ultrafilter if and only if there is $x \in X$ so that $\mathcal{F} = \uparrow\{x\}$.*

4. *Assuming that X is infinite, $\mathcal{F}$ is free if and only if it contains the Fréchet filter.*

5. *Assume that X is infinite. If $\mathcal{F}$ is free, then no element of $\mathcal{F}$ is a finite set.*

Proof 19 *Let us itemize the proof via the itemization of the statement.*

1. *If $\mathcal{F}$ is principal, then there exists a nonempty $A \subseteq X$ such that $\mathcal{F} = \uparrow A$. Then $\bigcap \mathcal{F} = A \neq \varnothing$, so $\mathcal{F}$ is not free.*

2. *If $\mathcal{F}$ is not free, then $A := \bigcap \mathcal{F} \neq \varnothing$, therefore $\mathcal{F} \subseteq \uparrow A$. Conversely, if $\mathcal{F} \subseteq \uparrow A$ for some nonempty $A \subseteq X$, then $\varnothing \neq A \subseteq \bigcap \mathcal{F}$, hence $\mathcal{F}$ is not free.*

3. *Suppose that $\mathcal{F}$ is a nonfree ultrafilter. There exists $A \in \mathcal{P}(X) \setminus \{\varnothing\}$ so that $\mathcal{F} \subseteq \uparrow A$. By maximality, $\mathcal{F} = \uparrow A$. By Proposition 10, we conclude that A is minimal in $\mathcal{P}(X) \setminus \{\varnothing\}$, which means that there exists $x \in A$ with $\{x\} = A$.*

4. *Assume that $\mathcal{F}_X \subseteq \mathcal{F}$. For every $x \in X$, we have that $X \setminus \{x\} \in \mathcal{F}_X \subseteq \mathcal{F}$. Now,*

$$\bigcap \mathcal{F} \subseteq \bigcap \mathcal{F}_X \subseteq \bigcap_{x \in X} X \setminus \{x\} = \varnothing.$$

   Conversely, assume that $\mathcal{F}$ is free. Fix an arbitrary $x \in X$. If $X \setminus \{x\} \notin \mathcal{F}$, then, for every $B \in \mathcal{F}$, we have that $B \nsubseteq X \setminus \{x\}$, which implies that $x \in B$. This means that $\uparrow\{x\} \subseteq \mathcal{F}$ and thus $\mathcal{F}$ is not free. As a consequence, $X \setminus \{x\} \in \mathcal{F}$ for all $x \in X$. Finally, if A is finite, then

$$X \setminus A = \bigcap_{a \in A} X \setminus \{a\} \in \mathcal{F}.$$

5. *Let $A \in \mathcal{F}$ and suppose, to the contrary, that A is finite. Then $X \setminus A \in \mathcal{F}_X \subseteq \mathcal{F}$. Thus, $\varnothing = A \cap (X \setminus A) \in \mathcal{F}$, which is impossible.*

1.1.4.4 Nets

Nets are natural generalizations of sequences when the set of natural numbers is endowed with its usual total order, which turns it into a directed set.

Definition 39 (Net) *A net is simply a map $\gamma : D \to X$ from a directed set D into a nonempty set X.*

Nets are typically denoted by $(x_d)_{d \in D}$ (under the settings of the previous definition). If D is a directed set, then $\mathcal{B}_D := \{\uparrow d : d \in D\}$ is a filter base in D (the union of whose elements is the whole of D).

Remark 14 *Let X a nonempty set. Then:*

- *Every net $\gamma : D \to X$ from a directed set D into X induces a filter base in X given by $\mathcal{B}_\gamma := \gamma(\mathcal{B}_D) = \{\gamma(\uparrow d) : d \in D\}$.*

- *Every filter base $\mathcal{B}$ in X induces a net in X given by*

$$\begin{array}{rccl} \gamma_{\mathcal{B}} : & D_{\mathcal{B}} & \to & X \\ & (x, B) & \mapsto & \gamma_{\mathcal{B}}(x, B) := x, \end{array} \tag{1.12}$$

  where $D_{\mathcal{B}} := \{(x, B) \in X \times \mathcal{B} : x \in B\}$ endowed with the preorder given by $(x, B) \leq (y, C) \Leftrightarrow C \subseteq B$. Notice that $\gamma_{\mathcal{B}}(\uparrow(x, B)) = B$ for all $B \in \mathcal{B}$ and all $x \in B$, meaning that $\mathcal{B}_{\gamma_{\mathcal{B}}} = \mathcal{B}$.

Sequences are the most representative example of net.

Example 7 (Sequence) *A sequence is, by definition, any function $f : \mathbb{N} \to X$ from the naturals into a nonempty set X. Observe that, when $\mathbb{N}$ is endowed with its usual total order, $\mathbb{N}$ trivially becomes a directed set, which turns any sequence into a net. Sequences are usually denoted by $(x_n)_{n\in\mathbb{N}}$ for historical reasons.*

Given directed sets E, D, nonempty sets X, Y, nets $\gamma : D \to X$ and $\lambda : E \to Y$, and $d \in D$, we define the following notable subsets:

- $E_d := \{e \in E : \lambda(\uparrow e) \subseteq \gamma(\uparrow d)\}$.
- $D_E := \{d \in D : E_d \neq \varnothing\}$.
- $E_d^0 := \{e \in E_d : \lambda(e) = \gamma(d)\}$.
- $D_E^0 := \{d \in D : E_d^0 \neq \varnothing\}$.

Definition 40 (Subnet) *Let E, D be directed sets. Let X, Y be nonempty sets. A net $\lambda : E \to Y$ is said to be a subnet of another net $\gamma : D \to X$ provided that $D_E = D$.*

Under the settings of Definition 40, $\mathcal{B}_D \subseteq \mathcal{J}(\mathcal{B}_E)$, hence $\mathcal{J}(\mathcal{B}_D) \subseteq \mathcal{J}(\mathcal{B}_E)$. Keep in mind that a subnet of a sequence is not necessarily a subsequence.

Lemma 5 *Let F, E, D be directed sets. Let X, Y, Z be nonempty sets. If $\delta : F \to Z$ is a subnet of $\lambda : E \to Y$, and $\lambda : E \to Y$ is a subnet of $\gamma : D \to X$, then $\delta : F \to Z$ is a subnet of $\gamma : D \to X$.*

Proof 20 *Fix an arbitrary $d \in D$. There exists $e \in E$ such that $\lambda(\uparrow e) \subseteq \gamma(\uparrow d)$. There also exists $f \in F$ such that $\delta(\uparrow f) \subseteq \lambda(\uparrow e) \subseteq \gamma(\uparrow d)$.*

Lemma 5 shows that the subnet relation is a preorder. Next result is aimed at showing that every subnet has a further subnet with the same range as the original net.

Lemma 6 *Let E, D be directed sets. Let X be a nonempty set. Let $\gamma : D \to X$ be a net. Then:*

1. *If $\alpha : E \to D$ is increasing and $\alpha(E)$ is cofinal in D, then $\gamma \circ \alpha : E \to X$ is a subnet of γ.*
2. *If $E \subseteq D$ is cofinal, then $\gamma|_E$ is a subnet of γ.*
3. *If $d \in D$, then $\gamma|_{\uparrow d}$ is a subnet of γ.*
4. *It $\delta : E \to Y$ is a subnet of γ, then there exists $e \in E$ such that the subnet $\delta|_{\uparrow e}$ of δ satisfies that $\delta(\uparrow e) \subseteq X$.*

Proof 21 *The proof will be itemized in accordance with the statement.*

1. *Fix $d \in D$. By hypothesis, there exists $e \in E$ such that $\alpha(e) \geq d$. As a consequence, $\alpha(\uparrow e) \subseteq \uparrow\alpha(e) \subseteq \uparrow d$. Finally, $\gamma(\alpha(\uparrow e)) \subseteq \gamma(\uparrow d)$.*
2. *Since E is cofinal in D, we have that E is directed by Remark 3(1). Also, the inclusion $\iota_E : E \hookrightarrow D$ is trivially increasing and satisfies that $\iota_E(E) = E$ is cofinal in D. Finally, Lemma 6(1) assures that $\gamma|_E = \gamma \circ \iota_E$ is a subnet of γ.*

3. *In view of Remark 3(2), $\uparrow d$ is cofinal in D. As a consequence, Corollary 6 assures that $\gamma|_{\uparrow d}$ is a subnet of γ.*

4. *Fix $d \in D$. There exists $e \in E$ such that $\delta(\uparrow e) \subseteq \gamma(\uparrow d) \subseteq X$.*

Note that if E, D are directed sets, X, Y are nonempty sets, $f : X \to Y$ is a function, $\alpha : D \to X$ is a net, and $\beta : E \to X$ is a subnet of α, then $f \circ \beta$ is a subnet of $f \circ \alpha$.

Lemma 7 *Let E, D be directed sets. Let X, Y be nonempty sets. Let $f : X \to Y$ be a surjective function. Let $\alpha : D \to Y$ be a net. Then:*

1. *There exists a net $\gamma : D \to X$ such that $f \circ \gamma = \alpha$.*

2. *If $\beta : E \to Y$ is a subnet of α, then there exists a subnet $\delta : E \to X$ of γ such that $f \circ \delta = \beta$.*

Proof 22 *First off, in accordance with Proposition 1(5), there exists a right-inverse $g : Y \to X$ for f. Thus, it simply suffices to define $\gamma := g \circ \alpha$. Next, suppose that $\beta : E \to Y$ is a subnet of α. Then take $\delta := g \circ \beta$ which is a subnet of γ. Finally, $f \circ \delta = \beta$.*

Under the settings of Lemma 7, if $\beta : E \to Z$ is a subnet of α, then, bearing in mind Lemma 6(4), there exists a subnet $\beta' : F \to Y$ of β. Then Lemma 7(2) assures the existence of a subnet $\delta : F \to X$ of γ such that $f \circ \delta = \beta'$, which is a subnet of β.

Theorem 4 *Let E, D be directed sets. Let X be a nonempty set. Consider a net $\gamma : D \to X$ and a subnet $\lambda : E \to X$ of γ. Then:*

1. *If D is well-ordered, then D_E^0 is cofinal in D.*

2. *If D and E are well ordered, then there exists a subset E^0 of E and a map $\beta : D_E^0 \to E^0$ such that $\lambda \circ \beta = \gamma$.*

Proof 23 *The proof will be itemized in accordance with the statement.*

1. *Fix $d_0 \in D = E_D$. By hypothesis, we can find*

$$d_1 := \min\{d \in \uparrow d_0 : \exists e \in E_{d_0}, \lambda(e) = \gamma(d)\}.$$

It is obvious that $d_1 \geq d_0$. We will show that $d_1 \in D_E^0$. Indeed, let $e \in E_{d_0}$ with $\lambda(e) = \gamma(d_1)$. Note that $\lambda(\uparrow e) \subseteq \gamma(\uparrow d_0)$. If there exists $e_1 \in \uparrow e$ and $d_2 \in \uparrow d_0 \setminus \uparrow d_1$ with $\lambda(e_1) = \gamma(d_2)$, then, by minimality, $d_1 \leq d_2$, which contradicts the fact that $d_2 \notin \uparrow d_1$. As a consequence, $\lambda(\uparrow e) \subseteq \gamma(\uparrow d_1)$, that is, $e \in E_{d_1}$. This implies that $e \in E_{d_1}^0 \neq \varnothing$, hence $d_1 \in D_E^0$.

2. *For every $d \in D'$, by hypothesis, there exists $e_d := \min(E'_d)$. By the Axiom of Choice, there exists a set E' whose elements are exactly all the e_d's. Now, it is possible to define the map*

$$\begin{array}{rccc} \beta : & D' & \to & E' \\ & d & \mapsto & \beta(d) := e_d. \end{array}$$

Finally, $\lambda \circ \beta = \gamma$. Indeed, $\lambda(\beta(d)) = \lambda(e_d) = \gamma(d)$ because $e_d \in E'_d$.

Double nets are important objects in certain aspects when uniform convergence comes into play.

Definition 41 (Double) *Let I, J be directed sets. Let X be a nonempty set. A double net is a net of the form $c : I \times J \to X$.*

The marginal nets are defined as

$$\begin{array}{rccc} c(\cdot,\ ): & I & \to & X^J \\ & i & \mapsto & c(i,\ ): \ \begin{array}{ccc} J & \to & X \\ j & \mapsto & c(i,j), \end{array} \end{array}$$

and

$$\begin{array}{rccc} c(\ ,\cdot): & J & \to & X^I \\ & j & \mapsto & c(\ ,j): \ \begin{array}{ccc} I & \to & X \\ i & \mapsto & c(i,j). \end{array} \end{array}$$

Definition 42 (Subsequence) *A subsequence of a sequence $f : \mathbb{N} \to X$ is a sequence of the form $i \circ f$, where $i : \mathbb{N} \to \mathbb{N}$ is strictly increasing.*

Notice that, if $i : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then $i(\mathbb{N})$ is clearly cofinal in $\mathbb{N}$, therefore, in view of Lemma 6(1), every subsequence is trivially a subnet. The converse is, in general, not true; that is, there can be easily constructed subnets of sequences which are not sequences. Very often, subsequences are denoted simply by $(x_{n_k})_{k\in\mathbb{N}}$. Notice that every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ satisfies that $n_k \geq k$ for all $k \in \mathbb{N}$.

1.1.4.5 Hull operators and convex structures

Hull operators have their origins in the linear span, the convex hull, and the topological closure. The notion of hull operator [3, 4] is then original from Linear Algebra and Operator Theory, although it can also be generalized to Category Theory. The notion of unary internal operator will be used in the upcoming definitions, but it will be properly introduced in the next section.

Definition 43 (Hull operator) *Let X be a poset. A unary internal operation $H : X \to X$ is called a hull operator if it verifies the following three conditions for all $x, y \in X$:*

- *Extensivity or extensionality: $x \leq H(x)$.*
- *Increasingness or monotonicity: $x \leq y \Rightarrow H(x) \leq H(y)$.*
- *Idempotency: $H(H(x)) = H(x)$.*

There are trivial examples of hull operators. For instance, if X is a bounded poset, then $H(x) = 1$ for all $x \in X$ is trivially a hull operator, which really has no interest. A nontrivial example follows.

Example 8 (Closure operator) *Let X be a bounded lattice. A closure operator is a unary internal operation $C : X \to X$ satisfying the following conditions for all $x, y \in X$:*

- *Nullity: $C(0) = 0$.*
- *Extensivity or extensionality: $x \leq C(x)$.*
- *Additivity: $C(x \vee y) = C(x) \vee C(y)$.*
- *Idempotency: $C(x) = C(C(x))$.*

According to Remark 10, a closure operator is always a hull operator. From now on, we will work with bounded posets satisfying the inf property. Bear in mind that every bounded poset satisfying the inf property is a lattice.

Proposition 13 *Let X be a bounded poset satisfying the inf property. Let $H : X \to X$ be a hull operator. Let $\mathrm{fix}(H) := \{x \in X : H(x) = x\}$ be the set of fixed points of H. Let $Y \subseteq X$. Then:*

1. *$H(1) = 1$.*
2. *$H(0) = \min\{H(x) : x \in X\}$.*
3. *If $H(0) > 0$, then $H(0) = \min\{H(x) : x \in X \setminus \{0\}\}$ and $H(x) = H(0)$ for all $0 < x < H(0)$.*
4. *$\mathrm{fix}(H)$ is inf-closed and contains 1.*

If, in addition, H is a closure operator, then $0 \in \mathrm{fix}(H)$ and $\mathrm{fix}(H)$ is finitely sup closed.

Proof 24 *The proof will be itemized according to the items of the statement of the proposition.*

1. *It directly follows from the extensivity that $1 \leq H(1) \leq 1$. In particular, $1 \in \mathrm{fix}(H)$.*
2. *Observe that $0 \leq x$ for all $x \in X$, meaning that $H(0) \leq H(x)$ for all $x \in X$. As a consequence, $H(0) = \min\{H(x) : x \in X\}$.*
3. *Next, suppose that $H(0) > 0$. Then $H(0) \leq H(x)$ for all $x \in X \setminus \{0\}$ and $H(0) = H(H(0))$, which implies that $H(0) \in \{H(x) : x \in X \setminus \{0\}\}$. This shows that $H(0) = \min\{H(x) : x \in X \setminus \{0\}\}$. If there exists $0 < x < H(0)$, then $H(0) \leq H(x) \leq H(H(0)) = H(0)$, obtaining that $H(x) = H(0)$.*
4. *It only remains to show that $\mathrm{fix}(H)$ is inf-closed. Indeed, let $Z \subseteq \mathrm{fix}(H)$. Note that Z is bounded below because X has a minimum 0. Since X satisfies the inf property, there exists $z_m \in X$, the infimum of Z. We have to show that $z_m \in Y$. In the first place, $z_m \leq H(z_m)$ by extensivity. Conversely, $z_m \leq z$ for all $z \in Z$, meaning that $H(z_m) \leq H(z)$ for all $z \in Z$ by monotonicity. However, $H(z) = z$ for all $z \in Z$ because $Z \subseteq \mathrm{fix}(H)$. Then $H(z_m) \leq z$ for all $z \in Z$, which implies that $H(z_m)$ is a lower bounded for Z. Therefore, $H(z_m) \leq z_m$. As a consequence, we obtain that $H(z_m) = z_m$; hence, $z_m \in \mathrm{fix}(H)$. This shows that $\mathrm{fix}(H)$ is inf closed.*

Finally, assume that H is a closure operator. By nullity, $H(0) = 0$ so then $0 \in \mathrm{fix}(H)$. Let us check that $\mathrm{fix}(H)$ is finitely sup-closed. Indeed, fix arbitrary elements $y_1, y_2 \in \mathrm{fix}(H)$. Since H is a closure operator, then $y_1 \vee y_2 = H(y_1) \vee H(y_2) = H(y_1 \vee y_2) \in \mathrm{fix}(H)$.

Under the settings of Proposition 13, if $H : X \to X$ is a closure operator, then $0, 1 \in \mathrm{fix}(H)^{\mathsf{c}}$ and $\mathrm{fix}(H)^{\mathsf{c}}$ is sup-closed and finitely inf-closed in X; hence, $\mathrm{fix}(H)^{\mathsf{c}}$ is a topology structure in X. Proposition 13 motivates the following result, which is a relevant improvement of the classical way for constructing hull operators on power sets (see, for instance, [5, Proposition 1]).

Theorem 5 (Construction) *Let X be a bounded poset satisfying the inf property. Let $Y \subseteq X$ be inf-closed containing 1. The map*

$$\begin{array}{rccl} \mathcal{H}_Y : & X & \to & X \\ & x & \mapsto & \mathcal{H}_Y(x) := \inf\{y \in Y : x \leq y\}. \end{array} \tag{1.13}$$

is a hull operator verifying that $\mathcal{H}_Y(X) = Y$ and $\mathrm{fix}(\mathcal{H}_Y) = Y$, where $\mathrm{fix}(\mathcal{H}_Y)$ is the set of fixed points of $\mathcal{H}_Y$ as described in Proposition 13. If, in addition, $0 \in Y$ and Y is finitely sup-closed, then $\mathcal{H}_Y$ is a closure operator.

Proof 25 *First of all, $\mathcal{H}_Y$ is well defined because X satisfies the inf property and $\{y \in Y : x \leq y\}$ is never empty (it contains 1) no matter which $x \in X$ is chosen. It is obvious that $\mathcal{H}_Y(y) = y$ for all $y \in Y$, which shows that $Y \subseteq \operatorname{fix}(\mathcal{H}_Y)$. Also, note that $\mathcal{H}_Y(x) \in Y$ for all $x \in X$ because Y is inf-closed. Therefore, if $x \in X$ is a fixed point of H, then $x = \mathcal{H}_Y(x) \in Y$. As a consequence, $\operatorname{fix}(\mathcal{H}_Y) \subseteq Y$. We will check that all three properties that characterize hull operators are verified for all $x, x_1, x_2 \in X$:*

- *Extensivity or extensionality: $x \leq \mathcal{H}_Y(x)$. This holds by definition.*
- *Increasingness or monotonicity: $x_1 \leq x_2 \Rightarrow \mathcal{H}_Y(x_1) \leq \mathcal{H}_Y(x_2)$. Indeed, $x_1 \leq x_2 \leq \mathcal{H}_Y(x_2)$ and $\mathcal{H}_Y(x_2) \in Y$, therefore, $\mathcal{H}_Y(x_1) \leq \mathcal{H}_Y(x_2)$.*
- *Idempotency: $\mathcal{H}_Y(\mathcal{H}_Y(x)) = \mathcal{H}_Y(x)$. By extensivity, $\mathcal{H}_Y(x) \leq \mathcal{H}_Y(\mathcal{H}_Y(x))$. The reverse inequality will be proved by showing that $\mathcal{H}_Y(\mathcal{H}_Y(x))$ is a lower bound for the set $\{y \in Y : x \leq y\}$. Indeed, fix an arbitrary $y \in Y$ with $x \leq y$. By monotonicity, $\mathcal{H}_Y(x) \leq \mathcal{H}_Y(y) = y$. So, again by monotonicity, $\mathcal{H}_Y(\mathcal{H}_Y(x)) \leq \mathcal{H}_Y(y) = y$. As a consequence, $\mathcal{H}_Y(\mathcal{H}_Y(x))$ is a lower bound for the set $\{y \in Y : x \leq y\}$. By definition of infimum, $\mathcal{H}_Y(x) \geq \mathcal{H}_Y(\mathcal{H}_Y(x))$.*

Finally, suppose, in addition, that $0 \in Y$ and Y is finitely sup-closed. We will prove that $\mathcal{H}_Y$ is a closure operator. We already know that $\mathcal{H}_Y$ is subadditive due to the increasingness. So, fix arbitrary elements $x_1, x_2 \in X$. Like we just mentioned, $\mathcal{H}_Y(x_1) \vee \mathcal{H}_Y(x_2) \leq \mathcal{H}_Y(x_1 \vee x_2)$. Next, by assumption, Y is finitely sup-closed, meaning that $\mathcal{H}_Y(x_1) \vee \mathcal{H}_Y(x_2) \in Y$. Notice that $x_1 \leq \mathcal{H}_Y(x_1)$ and $x_2 \leq \mathcal{H}_Y(x_2)$ by extensionality, therefore, $x_1 \vee x_2 \leq \mathcal{H}_Y(x_1) \vee \mathcal{H}_Y(x_2)$. Since $\mathcal{H}_Y(x_1) \vee \mathcal{H}_Y(x_2) \in Y$, by construction of $\mathcal{H}_Y$, we conclude that $\mathcal{H}_Y(x_1 \vee x_2) \leq \mathcal{H}_Y(x_1) \vee \mathcal{H}_Y(x_2) \in Y$, obtaining the reverse inequality.

Under the settings of Theorem 5, if T is a topology structure in X, then $0, 1 \in T^{\mathsf{c}}$ and T^{c} is inf-closed and finitely sup-closed in X; hence, $\mathcal{H}_{T^{\mathsf{c}}}$ is a closure operator.

Theorem 6 (Classification) *Let X be a bounded poset satisfying the inf property. Let $H : X \to X$ be a hull operator. Let $\operatorname{fix}(H) := \{x \in X : H(x) = x\}$ be the set of fixed points of H. Then $\mathcal{H}_{\operatorname{fix}(H)} = H$, where $\mathcal{H}_{\operatorname{fix}(H)}$ is the hull operator induced by $\operatorname{fix}(H)$ in* (1.13).

Proof 26 *Fix an arbitrary element $x \in X$. By following* (1.13), *$\mathcal{H}_{\operatorname{fix}(H)}(x) := \inf\{y \in \operatorname{fix}(H) : x \leq y\}$. We will show that $\mathcal{H}_{\operatorname{fix}(H)}(x) = H(x)$. First off, if $y \in \operatorname{fix}(H)$ and $x \leq y$, then $H(x) \leq H(y) = y$ by monotonicity, meaning that $H(x)$ is a lower bound for the set $\{y \in \operatorname{fix}(H) : x \leq y\}$. Thus, $H(x) \leq \mathcal{H}_{\operatorname{fix}(H)}(x)$. Next, if $y \in \operatorname{fix}(H)$ and $H(x) \leq y$, then $x \leq H(x) \leq y$ by extensionality; therefore $\mathcal{H}_{\operatorname{fix}(H)}(x) \leq y$ by monotonicity of $\mathcal{H}_{\operatorname{fix}(H)}$ together with the fact that $\operatorname{fix}(H)$ is the set of fixed points of $\mathcal{H}_{\operatorname{fix}(H)}$ in virtue of Theorem 5. This shows that $\mathcal{H}_{\operatorname{fix}(H)}(x)$ is a lower bound for the set $\{y \in \operatorname{fix}(H) : H(x) \leq y\}$. As a consequence, $\mathcal{H}_{\operatorname{fix}(H)}(x) \leq \mathcal{H}_{\operatorname{fix}(H)}(H(x))$. However, $H(x) \in \operatorname{fix}(H)$ by idempotency, so Theorem 5 assures that $\operatorname{fix}\left(\mathcal{H}_{\operatorname{fix}(H)}\right) = \operatorname{fix}(H)$, that is, $\mathcal{H}_{\operatorname{fix}(H)}(H(x)) = H(x)$, obtaining the desired inequality $\mathcal{H}_{\operatorname{fix}(H)}(x) \leq \mathcal{H}_{\operatorname{fix}(H)}(H(x)) = H(x)$.*

Theorem 6 unveils that hull operators are uniquely determined by their set of fixed points. The following remark is a note on the composition of hull operators.

Remark 15 (Composition of hull operators) *Let X be a bounded poset satisfying the inf property. Let $H, K : X \to X$ be hull operators. Let $\operatorname{fix}(H) := \{x \in X : H(x) = x\}$ be the set of fixed points of H. Suppose that $\operatorname{fix}(H)$ is invariant under K, that is, $K(\operatorname{fix}(H)) \subseteq \operatorname{fix}(H)$. Then:*

1. *$K \circ H$ is a hull operator. Indeed, the only tricky property to be checked is idempotency, which holds sense because, for every $x \in X$, $H(x) \in \mathrm{fix}(H)$, so $K(H(x)) \in \mathrm{fix}(H)$, thus $H(K(H(x)) = K(H(x))$, that is, $K(H(K(H(x)))) = K(K(H(x))) = K(H(x))$.*

2. *$\mathrm{fix}(K \circ H) = \mathrm{fix}(K) \cap \mathrm{fix}(H)$. Indeed, if $x \in \mathrm{fix}(K \circ H)$, then $x = K(H(x))$, meaning that $x \in \mathrm{fix}(K)$ because x is the image of an element of X under K, and $x \in \mathrm{fix}(H)$ because $H(x) \in \mathrm{fix}(H)$ and $K(\mathrm{fix}(H)) \subseteq \mathrm{fix}(H)$. Conversely, if $x \in \mathrm{fix}(K) \cap \mathrm{fix}(H)$, then $x = K(x)$ and $x = H(x)$, so $x = K(x) = K(H(x))$.*

The upward and downward closures are hull operators. Recall that a preordered set is a set endowed with a reflexive and transitive binary internal relation.

Theorem 7 *Let X be a set endowed with a reflexive and transitive binary internal relation $\mathcal{R}$. Then $\mathcal{J} : \mathcal{P}(X) \to \mathcal{P}(X)$ and $\mathcal{I} : \mathcal{P}(X) \to \mathcal{P}(X)$ are hull operators.*

Proof 27 *We will only prove that $\mathcal{I}$ is a hull operator since the other case requires a similar proof. Indeed, let us check that all necessary conditions are satisfied for all $A, B \in \mathcal{P}(X)$:*

- *Extensivity or extensionality: $A \subseteq \mathcal{I}(A)$. Indeed, this holds since $\mathcal{I}(A) := \bigcup_{a \in A} {\downarrow}a$.*
- *Increasingness or monotonicity: If $A \subseteq B$ and $x \in \mathcal{I}(A)$, then there exists $a \in A$ such that $(x, a) \in \mathcal{R}$. Since $a \in B$, we conclude that $x \in \mathcal{I}(B)$.*
- *Idempotency: $\mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)$. Since $A \subseteq \mathcal{I}(A)$, we have that $\mathcal{I}(A) \subseteq \mathcal{I}(\mathcal{I}(A))$. Take any $x \in \mathcal{I}(\mathcal{I}(A))$. There exists $y \in \mathcal{I}(A)$ such that $(x, y) \in \mathcal{R}$. There also exists $a \in A$ such that $(y, a) \in \mathcal{R}$. By the transitivity of $\mathcal{R}$, we conclude that $(x, a) \in \mathcal{R}$, meaning that $x \in \mathcal{I}(A)$.*

The notion of convex structure [6] plays a fundamental role in different areas such as Fixed-Point Metric Theory.

Definition 44 (Convex structure) *Let $\mathcal{X}$ be a bounded poset. A subset $\mathcal{C}$ of $\mathcal{X}$ is called a convex structure provided that the following conditions are satisfied:*

- *Totality: $0, 1 \in \mathcal{C}$.*
- *Inf closedness: If $D \subseteq \mathcal{C}$ and $\exists \bigwedge D$, then $\bigwedge D \in \mathcal{C}$.*
- *Upward-directed sup closedness: If $D \subseteq \mathcal{C}$ is upward directed and $\exists \bigvee D$, then $\bigvee D \in \mathcal{C}$.*

Under the settings of the previous definition and by bearing in mind Theorem 5, if $\mathcal{X}$ satisfies the inf property and $\mathcal{C}$ is a convex structure in $\mathcal{X}$, then $\mathcal{C}$ is inf-closed and contains 1 by definition; hence, it induces a hull operator $\mathcal{H}_{\mathcal{C}}$ known as the convex hull.

Remark 16 *Let $\mathcal{X}$ be a bounded poset. If $(\mathcal{C}_i)_{i \in I}$ is a family of convex structures in $\mathcal{X}$, then $\bigcap_{i \in I} \mathcal{C}_i$ is a convex structure in X.*

1.2 Operations

Operations are special classes of maps (although any map can be regarded as a unitary operation). Likewise with relations, operations serve to construct rich mathematical structures.

1.2.1 Universal algebras

The concept of universal algebra simply pretends to englobe, in just one notion, many different mathematical structures that have something in common: a set of multiary operations. Examples of universal algebras are the magmas, semigroups, monoids, groups, rings, fields, modules, algebras, effect algebras, and Boolean algebras. We will bring our attention to modules.

1.2.1.1 Multiary operations

In order to define multiary operations, we will rely on products of sets and on multiary relations.

Definition 45 (Operation) *A multiary operation, or simply an operation, is a map*

$$\begin{array}{rccc} \star: & \Sigma \subseteq \prod_{\lambda\in\Lambda} A_\lambda & \to & A_\omega \\ & (a_\lambda)_{\lambda\in\Lambda} & \mapsto & \star(a_\lambda)_{\lambda\in\Lambda}, \end{array}$$

where $(A_\lambda)_{\lambda\in\Lambda}$ is a family of sets, A_ω is also a set, and Σ is a multiary relation on $(A_\lambda)_{\lambda\in\Lambda}$, that is, $\Sigma \subseteq \prod_{\lambda\in\Lambda} A_\lambda$.

- *If $\Sigma = \prod_{\lambda\in\Lambda} A_\lambda$, the operation is said to be totally defined.*
- *If $A_\lambda = A_\omega$ for all $\lambda \in \Lambda$, the operation is called internal.*
- *If $\Lambda = \{1, \dots, n\}$, the operation is called n-ary and we denote*

$$\star(a_1, \dots, a_n) = a_1 \star \cdots \star a_n.$$

- *If $\Lambda = \{1, 2\}$, we call it a binary operation.*
- *If $\Lambda = \{1\}$, we call it a unary operation.*
- *If $\Lambda = \{1\}$, $\Sigma = A_1$ and the operation is a constant function, we call it a nullary operation.*

Given a set A, a multiary external operation on A is a multiary operation such as $\star : \Sigma \subseteq \prod_{\lambda\in\Lambda} A_\lambda \to A_\omega$, where there exists $\varnothing \subsetneq \Lambda_\star \subsetneq \Lambda \cup \{\omega\}$ such that $A_\lambda = A$ for all $\lambda \in \Lambda_\star$.

Under the settings of the previous definition, it is interesting to observe that a multiary external operation on A such as $\star : \Sigma \subseteq \prod_{\lambda\in\Lambda} A_\lambda \to A_\omega$, could verify that $A_\lambda = A$ for all $\lambda \in \Lambda \cup \{\omega\}$ even though $\varnothing \subsetneq \Lambda_\star \subsetneq \Lambda \cup \{\omega\}$. In this situation, $\star$ is also, by definition, a multiary internal operation on A, however, we will still regard it as an external operation. This is the case, for instance, of rings seen as modules over themselves. The multiplication on a ring, which is naturally a binary internal operation, is considered a binary external operation in order to see the ring as a module over itself.

Notation 5 *If $\star : \Sigma \subseteq \prod_{\lambda\in\Lambda} A_\lambda \to A_\omega$ is a multiary operation and $C_\lambda \subseteq A_\lambda$ for all $\lambda \in \Lambda$, then we denote*

$$\star(C_\lambda)_{\lambda\in\Lambda} := \star\left(\Sigma \cap \prod_{\lambda\in\Lambda} C_\lambda\right) = \left\{\star(c_\lambda)_{\lambda\in\Lambda} : (c_\lambda)_{\lambda\in\Lambda} \in \Sigma \cap \prod_{\lambda\in\Lambda} C_\lambda\right\}.$$

If $\Lambda = \{1, \dots, n\}$, then we denote

$$\star(C_\lambda)_{\lambda\in\Lambda} = C_1 \star \cdots \star C_n.$$

Once multiary operations have been properly defined, we are in the right position to define universal algebras.

Definition 46 (Universal algebra) *A universal algebra is a 3-tuple (A, I, E), where:*

- *A is a nonempty set.*
- *I is a set of multiary internal operations on A, such as $\star : \Sigma_\star \subseteq A^\Lambda \to A$.*
- *E is a set of totally defined multiary external operations on A, such as $\diamond : \prod_{\gamma\in\Gamma} T_\gamma \to T_\omega$, where there exists $\varnothing \neq \Gamma_\diamond \subsetneq \Gamma \cup \{\omega\}$ such that $T_\gamma = A$ for all $\gamma \in \Gamma_\diamond$.*

Under the settings of the previous definition, a universal algebra is called intrinsic provided that $\omega \in \Gamma_\diamond$ for every totally defined multiary external operation.

Definition 47 (Compatible universal algebras) *Two universal algebras (A, I, E) and (B, J, F) are said to be compatible if there are bijective functions $\hat{} : I \to J$, $\bar{} : E \to F$ verifying that:*

- *For every multiary internal operation $\star : \Sigma_\star \subseteq A^\Lambda \to A$ in I, $\hat{\star} : \Sigma_{\hat{\star}} \subseteq B^\Lambda \to B$ is a multiary internal operation in J, with the same indexing set Λ.*
- *For every totally defined multiary external operation $\diamond : \prod_{\gamma\in\Gamma} T_\gamma \to T_\omega$ in E, $\bar{\diamond} : \prod_{\gamma\in\Gamma} P_\gamma \to P_\omega$ is a totally defined multiary external operation in F, with same indexing sets $\Gamma, \{\omega\}$, such that $\Gamma_{\bar{\diamond}} = \Gamma_\diamond$ and, for every $\gamma \in (\Gamma \cup \{\omega\}) \setminus \Gamma_{\bar{\diamond}}$, $P_\gamma = T_\gamma$.*

The 2-tuple $(\hat{},\bar{})$ is called a compatible pair.

Notice that, if two universal algebras are compatible, then one is intrinsic if and only if so is the other.

Example 9 (Direct product of intrinsic universal algebras) *Consider a family of compatible intrinsic universal algebras $((A_k, I_k, E_k))_{k\in K}$. Then $\prod_{k\in K} A_k$ can be given structure of intrinsic universal algebra. Indeed:*

1. *Multiary internal operations: If $\star_k : \Sigma_{\star_k} \subseteq A_k^\Lambda \to A_k$ is in I_k, for every $k \in K$, and correspond to each other under the compatibility, then*

$$\begin{array}{rccl} (\star_k)_{k\in K} : & \Sigma_{(\star_k)_{k\in K}} \subseteq \left(\prod_{k\in K} A_k\right)^\Lambda & \to & \prod_{k\in K} A_k \\ & \left(\left(a_k^\lambda\right)_{k\in K}\right)_{\lambda\in\Lambda} & \mapsto & (\star_k)_{k\in K}\left(\left(a_k^\lambda\right)_{k\in K}\right)_{\lambda\in\Lambda} := \\ & & & \left(\star_k\left(a_k^\lambda\right)_{\lambda\in\Lambda}\right)_{k\in K} \end{array}$$

   is a multiary internal operation in $\prod_{k\in K} A_k$, where

$$\Sigma_{(\star_k)_{k\in K}} := \left\{ \left(\left(a_k^\lambda\right)_{k\in K}\right)_{\lambda\in\Lambda} \in \left(\prod_{k\in K} A_k\right)^\Lambda : \forall k \in K \ \left(a_k^\lambda\right)_{\lambda\in\Lambda} \in \Sigma_{\star_k} \right\}.$$

$$\begin{array}{ccc} \Sigma_{(\star_k)_{k\in K}} \subseteq \left(\prod_{k\in K} A_k\right)^\Lambda & & \\ & \searrow^{(\star_k)_{k\in K}} & \\ & & \prod_{k\in K} A_k \\ & \nearrow_{\prod_{k\in K}\star_k} & \\ \prod_{k\in K}\Sigma_{\star_k} \subseteq \prod_{k\in K} A_k^\Lambda & & \end{array}$$

2. *Totally defined multiary external operations: Take* $\diamond_k : \prod_{\gamma\in\Gamma} T_\gamma^k \to A_k$ *in* E_k*, for every* $k \in K$*, corresponding to each other under the compatibility. Note that* $\Gamma_{\diamond_k} = \Gamma_{\diamond_l}$ *for all* $k, l \in K$*. Let* $\Gamma_{\prod_{k\in K}\diamond_k} := \Gamma_{\diamond_k}$ *for any* $k \in K$*. This way,* $T_\gamma^k = A_k$*, for all* $k \in K$*, whenever* $\gamma \in \Gamma_{\prod_{k\in K}\diamond_k}$*. The compatibility also implies that* $T_\gamma^k = T_\gamma^l$ *for all* $k, l \in K$ *and all* $\gamma \in \Gamma \setminus \Gamma_{\prod_{k\in K}\diamond_k}$*. Fix any* $k_0 \in K$ *and let*

$$T_\gamma := \begin{cases} T_\gamma^{k_0} & \gamma \in \Gamma \setminus \Gamma_{\prod_{k\in K}\diamond_k}, \\ \prod_{k\in K} A_k & \gamma \in \Gamma_{\prod_{k\in K}\diamond_k}. \end{cases}$$

An element $(t_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} T_\gamma$ *satisfies that* $t_\gamma = (a_k^\gamma)_{k\in K} \in \prod_{k\in K} A_k$ *for every* $\gamma \in \Gamma_{\prod_{k\in K}\diamond_k}$*. Then*

$$\begin{array}{rccl} (\diamond_k)_{k\in K} : & \prod_{\gamma\in\Gamma} T_\gamma & \to & \prod_{k\in K} A_k \\ & (t_\gamma)_{\gamma\in\Gamma} & \mapsto & (\diamond_k)_{k\in K}(t_\gamma)_{\gamma\in\Gamma} := \left(\diamond_k \left(t_\gamma^k\right)_{\gamma\in\Gamma}\right)_{k\in K} \end{array}$$

is a totally defined multiary external operation in $\prod_{k\in K} A_k$*, where*

$$t_\gamma^k := \begin{cases} t_\gamma & \gamma \in \Gamma \setminus \Gamma_{\prod_{k\in K}\diamond_k}, \\ a_k^\gamma & \gamma \in \Gamma_{\prod_{k\in K}\diamond_k}. \end{cases}$$

1.2.1.2 Operators and multioperators

Operators generalize the concept of linear maps to the scope of universal algebras.

Definition 48 (Operator) *Let* $(A, I, E), (B, J, F)$ *be compatible universal algebras. An operator between universal algebras is a* 3*-tuple* $(f, \hat{}, \bar{})$*, where* $f : A \to B$ *is a function and* $(\hat{}, \bar{})$ *is a compatible pair verifying that:*

- *For every multiary internal operation* $\star : \Sigma_\star \subseteq A^\Lambda \to A$ *in* I *and for every* $(a_\lambda)_{\lambda\in\Lambda} \in \Sigma_\star$*,* $(f(a_\lambda))_{\lambda\in\Lambda} \in \Sigma_{\hat{\star}}$ *and* $f(\star(a_\lambda)_{\lambda\in\Lambda}) = \hat{\star}(f(a_\lambda))_{\lambda\in\Lambda}$*.*

$$\begin{array}{ccc} \Sigma_\star \subseteq A^\Lambda & \xrightarrow{f^\Lambda} & \Sigma_{\hat{\star}} \subseteq B^\Lambda \\ \downarrow{\scriptstyle \star} & & \downarrow{\scriptstyle \hat{\star}} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

- *For every totally defined multiary external operation* $\diamond : \prod_{\gamma\in\Gamma} T_\gamma \to T_\omega$ *in* E *and for every* $(t_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} T_\gamma$*, letting*

$$p_\gamma := \begin{cases} t_\gamma & \gamma \in (\Gamma \cup \{\omega\}) \setminus \Gamma_\diamond, \\ f(t_\gamma) & \gamma \in \Gamma_\diamond, \end{cases}$$

it is verified that:

  - *If* $\omega \in \Gamma_\diamond$*, then* $f(\diamond(t_\gamma)_{\gamma\in\Gamma}) = \bar{\diamond}(p_\gamma)_{\gamma\in\Gamma}$*.*
  - *If* $\omega \notin \Gamma_\diamond$*, then* $\diamond(t_\gamma)_{\gamma\in\Gamma} = \bar{\diamond}(p_\gamma)_{\gamma\in\Gamma}$*.*

The set of operators between two compatible universal algebras A, B is denoted by $\mathcal{L}(A, B)$. When $A = B$, we will simply denote it by $\mathcal{L}(A)$.

Remark 17 *Let* $(f,\hat{},\bar{}) : (A,I,E) \to (B,J,F)$ *and* $(g,\check{},\breve{}) : (B,J,F) \to (C,K,G)$ *be operators between compatible universal algebras. Then:*

1. *The composition of* $(f,\hat{},\bar{})$ *and* $(g,\check{},\breve{})$ *is defined as*

$$(g,\check{},\breve{}) \circ (f,\hat{},\bar{}) := (g \circ f, \check{} \circ \hat{}, \breve{} \circ \bar{}) : (A,I,E) \to (C,K,G)$$

   and is an operator of universal algebras.

2. *The identity operator on* (A,I,E) *is defined as* (I_A, I_I, I_E)*, and is clearly an operator of universal algebras.*

3. *If* f *is bijective and* $f(\Sigma_\star) := \{(f(a_\lambda))_{\lambda\in\Lambda} : (a_\lambda)_{\lambda\in\Lambda} \in \Sigma_\star\} = \Sigma_{\hat{\star}}$ *for every multiary internal operation* $\star : \Sigma_\star \subseteq A^\Lambda \to A$ *in* I*, then* $\left(f^{-1},\hat{}^{-1},\bar{}^{-1}\right) : (B,J,F) \to (A,I,E)$ *is an operator between universal algebras.*

The notion of isomorphism is reviewed in Chapter A as those morphisms which have an inverse. In the different categories of universal algebras, Remark 17(3) simplifies the notion of isomorphism.

Definition 49 (Isomorphism) *Let* $(A,I,E),(B,J,F)$ *be compatible universal algebras. An operator* $(f,\hat{},\bar{}) : (A,I,E) \to (B,J,F)$ *is said to be an isomorphism provided that* f *is bijective. In this situation, we say that* $(A,I,E),(B,J,F)$ *are isomorphic.*

The composition of isomorphisms is clearly an isomorphism. Multioperators generalize the concept of multilinear maps to the scope of universal algebras.

Definition 50 (Multioperator) *Let* $((A_k,I_k,E_k))_{k\in K}$ *be a family of compatible universal algebras. Let* (B,J,F) *be another universal algebra compatible with all of them. A map* $T : \prod_{k\in K} A_k \to B$ *is said to be a multioperator provided that, for every* $k_0 \in K$ *and every* $(a_k)_{k\in K\setminus\{k_0\}} \in \prod_{k\in K\setminus\{k_0\}} A_k$*, the marginal map*

$$\begin{array}{rccl} T_{(a_k)_{k\in K\setminus\{k_0\}}} : & A_{k_0} & \to & B \\ & a_{k_0} & \mapsto & T_{(a_k)_{k\in K\setminus\{k_0\}}}(a_{k_0}) := T((a_k)_{k\in K}) \end{array}$$

is an operator of universal algebras.

Under the settings of the above definition, if $K = \{1,2\}$, then the multioperator T is simply called a bioperator. Multioperators will be studied with precision in the context of particular universal algebras, such as groups and modules, where we will see that commutativity is required for them to exist.

Definition 51 (Symmetric multioperator) *Let* $((A_k,I_k,E_k))_{k\in K}$ *be a family of compatible universal algebras. Let* (B,J,F) *be another universal algebra compatible with all of them. A multioperator* $T : \prod_{k\in K} A_k \to B$ *is called symmetric if* $T((a_k)_{k\in K}) = T\left(\left(a_{\pi(k)}\right)_{k\in K}\right)$ *for every permutation (bijection)* $\pi : K \to K$ *and every* $(a_k)_{k\in K} \in \prod_{k\in K} A_k$*.*

Defining universal subalgebras will help us understand a bit better the concept and behavior of subspaces.

Definition 52 (Universal subalgebra) *Let* (A,I,E) *be a universal algebra. A subset* C *of* A *is called a universal subalgebra of* A *if* C *is closed under all the operations of* I *and* E*, that is:*

- *For every multiary internal operation* $\star : \Sigma \subseteq A^{\Lambda} \to A$ *in* I*,*

$$\Sigma \cap C^{\Lambda} \neq \varnothing \text{ and } \star\left(\Sigma \cap C^{\Lambda}\right) \subseteq C.$$

- *For every totally defined multiary external operation* $\diamond : \prod_{\gamma\in\Gamma} T_\gamma \to T_\omega$ *in* E *such that* $\omega \in \Gamma_\diamond$*,* $\diamond(L_\gamma)_{\gamma\in\Gamma} \subseteq C$*, where*

$$L_\gamma := \begin{cases} T_\gamma & \gamma \in \Gamma \setminus \Gamma_\diamond, \\ C & \gamma \in \Gamma_\diamond. \end{cases}$$

Notice that, under the settings of the previous definition, $(C, I|_C, E|_C)$ is a universal algebra, where $I|_C$ and $E|_C$ consist of all the multiary internal operations of I and all the multiary external operations of E, respectively, restricted to C.

Remark 18 *Let* $(f, \hat{}, \bar{}) : (A, I, E) \to (B, J, F)$ *be an operator between compatible universal algebras. Then:*

1. *If* $C \subseteq A$ *is a universal subalgebra of* A*, then* $f(C)$ *is a universal subalgebra of* B*.*
2. *If* $D \subseteq B$ *is a universal subalgebra of* B *and* $\Sigma \cap f^{-1}(D)^{\Lambda} \neq \varnothing$ *for every multiary internal operation* $\star : \Sigma \subseteq A^{\Lambda} \to A$ *in* I*, then* $f^{-1}(D)$ *is a universal subalgebra of* A*.*

Next example shows that the intersection of universal subalgebras is not necessarily a universal subalgebra.

Example 10 *The universal algebra* $(\{-1, 0, 1\}, \{0, \star\}, \varnothing)$*, where*

$$\begin{array}{rccc} 0: & \{-1,0,1\} & \to & \{-1,0,1\} \\ & a & \mapsto & 0 \end{array}$$

is a nullary internal operation, and

$$\begin{array}{rccc} \star: & \{-1,1\} & \to & \{-1,1\} \\ & a & \mapsto & a \end{array}$$

is a unary internal operation, verifies that $\{0, 1\}$ *and* $\{-1, 0\}$ *are universal subalgebras whose intersection* $\{0\}$ *is not a universal subalgebra.*

The previous example somehow suggests that the nonempty intersection of universal subalgebras is a universal subalgebra if all the multiary internal operations on the universal algebra are totally defined.

Remark 19 *Let* (A, I, E) *be a universal algebra. Let* $(C_\delta)_{\delta\in\Delta}$ *be a family of universal subalgebras of* A*. Then:*

- *If* $\Sigma \cap \left(\bigcap_{\delta\in\Delta} C_\delta\right)^{\Lambda} \neq \varnothing$ *for every multiary internal relation* $\star : \Sigma \subseteq A^{\Lambda} \to A$ *in* I*, then* $\bigcap_{\delta\in\Delta} C_\delta$ *is a universal subalgebra of* A*.*
- *If* $(C_\delta)_{\delta\in\Delta}$ *is an upward directed family for the inclusion and all the internal and external operations are* n*-ary, then* $\bigcup_{\delta\in\Delta} C_\delta$ *is a universal subalgebra of* A*.*
- *If all the multiary internal operations in* I *are totally defined and there exists a nullary internal operation in* I*, then* $\bigcap_{\delta\in\Delta} C_\delta \neq \varnothing$ *and* $\bigcap_{\delta\in\Delta} C_\delta$ *is a universal subalgebra.*

The previous remark motivates the concept of hereditary universal algebra. Although most of the well-known universal algebras are hereditary, we will unveil examples of universal algebras which are not.

Definition 53 (Hereditary universal algebra) *A universal algebra is said to be hereditary if every family of universal subalgebras has a nonempty intersection and its intersection is a universal subalgebra.*

An immediate consequence of the previous remark is that, if all the multiary internal operations in a universal algebra are totally defined and there exists a nullary internal operation, then the universal algebra is hereditary.

Definition 54 (Spanned universal subalgebra) *Let (A, I, E) be a hereditary universal algebra. Let $C \subseteq A$. The universal subalgebra of A spanned by C is defined as the intersection of all universal subalgebras of A containing C and denoted by* $\operatorname{span}(C)$.

Under the settings of the previous definition, $\operatorname{span}(C)$ is the smallest universal subalgebra of A containing C. Notice that span is a hull operator.

Theorem 8 *Let $(A, I, E), (B, J, F)$ be compatible universal algebras. Assume that all multiary internal operations in I, J are totally defined. Suppose that I, J contain at least a nullary internal operation. Let $(T, \hat{}, \bar{}) : (A, I, E) \to (B, J, F)$ be an operator. For every $C \subseteq A$, $T(\operatorname{span}(C)) = \operatorname{span}(T(C))$.*

Proof 28 *By Remark 18(1), $T(\operatorname{span}(C))$ is a universal subalgebra of B containing $T(C)$, meaning that $\operatorname{span}(T(C)) \subseteq T(\operatorname{span}(C))$. Conversely, Remark 18(2), assures that $T^{-1}(\operatorname{span}(T(C)))$ is a universal subalgebra of A containing C, meaning that $\operatorname{span}(C) \subseteq T^{-1}(\operatorname{span}(T(C)))$, that is $T(\operatorname{span}(C)) \subseteq \operatorname{span}(T(C))$.*

It is the turn now for the notion of spanning system, which is a very important concept of Linear Algebra.

Definition 55 (Spanning system) *Let (A, I, E) be a hereditary universal algebra. A subset $S \subseteq A$ is called a spanning system of A provided that $\operatorname{span}(S) = A$.*

A special subalgebra of the product of intrinsic universal algebras is given in the next example.

Example 11 (Direct sum of intrinsic universal algebras) *Consider a family of compatible intrinsic universal algebras $((A_k, I_k, E_k))_{k \in K}$. For each $k \in K$, assume the following:*

- *All multiary internal operations in I_k and all totally defined multiary external operations in E_k are defined on a finite product of sets.*
- *I_k has a nullary internal operation $\boldsymbol{o}_k$ such that, for each multiary internal operation $\star_k : \Sigma_{\star_k} \subseteq A_k^{\Lambda} \to A_k$ in I_k, $\boldsymbol{o}_k^{\Lambda} \in \Sigma_{\star_k} \subseteq A_k^{\Lambda}$ and $\star_k \boldsymbol{o}_k^{\Lambda} = \boldsymbol{o}_k$. Notice that $\boldsymbol{o} := (\boldsymbol{o}_k)_{k \in K} \in \Sigma_{\star}$ for $\star := \prod_{k \in K} \star_k$ is a nullary internal operation in $\prod_{k \in K} A_k$.*

Then

$$\bigoplus_{k \in K} A_k := \left\{ (a_k)_{k \in K} \in \prod_{k \in K} A_k : \exists L \in \mathcal{P}_{\mathtt{f}}(K) \ \ \forall k \in K \setminus L \ \ a_k = \boldsymbol{o}_k \right\}$$

is a universal subalgebra of $\prod_{k \in K} A_k$.

1.2.1.3 Binary internal operations

We will focus on binary internal operations on a given set A, such as

$$\begin{array}{rccc} \star: & A \times A & \to & A \\ & (a,b) & \mapsto & a \star b. \end{array}$$

The most usual notations are: *additive* $+$, *multiplicative* $\cdot$, and *composition* $\circ$.

Definition 56 *Let $\star$ be a binary internal operation on a nonempty set A. Then:*

- *$\star$ is associative if $\forall a,b,c \in A \;\; a \star (b \star c) = (a \star b) \star c$.*
- *$\star$ is commutative if $\forall a,b \in A \;\; a \star b = b \star a$.*
- *$\star$ is strongly commutative if $\forall a,b \in A \;\; (a \star a) \star b = b \star a$.*
- *$\star$ is left (right) cancellative provided that $\forall a,b,c \in A \;\; a \star b = a \star c \Rightarrow b = c$ ($a \star c = b \star c \Rightarrow a = b$).*
- *An element $e \in A$ is said to be an identity if $\forall a \in A \;\; e \star a = a \star e = a$.*
- *If there exists an identity $e \in A$, then:*
  - *$\star$ is strict provided that $\forall a,b \in A \;\; a \star b = e \Rightarrow a = b = e$.*
  - *An element $b \in A$ is said to be a left (right) symmetric of another element $a \in A$ if $b \star a = e$ ($a \star b = e$).*
  - *If b is both a left and a right symmetric of a, then we call it a symmetric of a.*
  - *$A^{\times} := \{a \in A : a \text{ has a symmetric}\}$.*

Typical examples of binary internal operations are $\wedge$ and $\vee$ in a lattice. Basic properties satisfied by binary internal operations are gathered in the following proposition.

Remark 20 *Let $\star$ be a binary internal operation on a nonempty set A with an identity e. Let $a, a', b, b', c, f \in A$. Then:*

1. *If f is another identity, then $e = f$. Indeed, $e = e \star f = f$.*
2. *If b is a left (right) symmetric for a, then a is a right (left) symmetric for b.*
3. *If $\star$ is associative, b is a left symmetric for a, and c is a right symmetric for a, then $b = c$. Indeed, $b = b \star e = b \star (a \star c) = (b \star a) \star c = e \star c = c$.*
4. *If $\star$ is associative, a' is a left (right) symmetric for a, and b' is a left (right) symmetric for b, then $b' \star a'$ ($a' \star b'$) is a left (right) symmetric for $a \star b$ ($b \star a$). Indeed, $(b' \star a')(a \star b) = b' \star [(a' \star a) \star b] = b' \star [e \star b] = b' \star b = e$.*
5. *If $\star$ is associative and every element has a left (right) symmetric, then $\star$ is left (right) cancellative. Indeed, if $a \star b = a \star c$ and a' is a left symmetric of a, then $b = e \star b = (a' \star a) \star b = a' \star (a \star b) = a' \star (a \star c) = (a' \star a) \star c = e \star c = c$.*
6. *If $\star$ is strongly commutative, then it is commutative. Indeed, $a \star a = (a \star a) \star a = (a \star a) \star (a \star a)$, $a \star b = (b \star b) \star a = (a \star a) \star (b \star b)$, and $b \star a = (a \star a) \star b = (b \star b) \star (a \star a) = ((a \star a) \star (a \star a)) \star (b \star b) = (a \star a) \star (b \star b) = a \star b$.*

Let $\star$ be a binary internal operation on a nonempty set A. If there exists an identity $e \in A$, then it defines a nullary internal operation on A,

$$\begin{array}{rccc} e: & A & \to & A \\ & a & \mapsto & e, \end{array}$$

called the identity operation. If, in addition, $\star$ is associative, then Remark 20(3) allows to define a unary internal operation on $A^\times$,

$$\begin{array}{rccc} {}^\star: & A^\times & \to & A^\times \\ & a & \mapsto & a^\star \text{ the symmetric of } a, \end{array}$$

called the symmetry operation or symmetrization.

Notation 6 *Consider a binary internal operation $\star$ on a set A. Then:*

- *Additive notation: $\star = +$, the identity is called the neutral and denoted by 0, and the symmetrics are called opposites and denoted by $-a$.*
- *Multiplicative notation: $\star = \cdot$, the identity is called the unity and denoted by 1, and the symmetrics are called inverses and denoted by a^{-1} or $1/a$.*
- *Composition notation: $\star = \circ$, the identity is still called the identity and denoted by I, and the symmetrics are called inverses and denoted by a^{-1}.*

1.2.2 Groups

Groups are fundamental structures in the mathematical world. They are extremely important in Analysis, Algebra, Geometry, Statistics, Computer Science, Physics, Chemistry, Biology, etc. Groups are among the mathematical structures that have more interdisciplinary applications.

1.2.2.1 Magmas

From now on until the subsection on rings, we will use multiplicative notation, unless we explicitly state otherwise. Additive notation is commonly used under the hypothesis of commutativity.

Definition 57 (Magma) *A magma is a universal algebra $(G, \{\cdot\}, \varnothing)$, where $\cdot : G \times G \to G$ is a binary internal operation on G.*

Magmas are also known as groupoids. As expected, submagmas are universal subalgebras of magmas. The following notation is consistent with Notation 5.

Notation 7 *Let G be a magma. Let $A, B \subseteq G$. With additive notation, $A + B := \{a + b : a \in A, b \in B\}$. With multiplicative notation, $AB = \{ab : a \in A, b \in B\}$.*

With additive notation, operators between magmas are called additive maps. With multiplicative notation, they are called multiplicative maps.

Definition 58 *Let G be a magma. Consider $A \subseteq G$ and $p \in G$. Then:*

- *p is said to be idempotent provided that $pp = p$.*
- *The commutant or centralizer of A is defined as $A' := \{g \in G : \forall a \in A \;\; ga = ag\}$. The commutant of G is often called the center of G and denoted by $\mathsf{Z}(G)$.*

- *The left- and right-multiplication operators by p on G are defined as the unary internal operations*

$$\begin{array}{rccl} \mathsf{L}_p: & G & \to & G \\ & g & \mapsto & \mathsf{L}_p(g) := pg \end{array} \qquad \begin{array}{rccl} \mathsf{R}_p: & G & \to & G \\ & g & \mapsto & \mathsf{R}_p(g) := gp. \end{array} \tag{1.14}$$

The multiplication operators, which are called translations with additive notation, can be seen as unary internal operations, and they are exactly the marginal maps of the magma operation. In general, neither the left-multiplication operator nor the right-multiplication operator is a multiplicative maps; that is, they are not operators of magmas.

Definition 59 (Coset) *Let G be a magma. Let H be a subset of G. The left cosets of H are defined as the sets of the form gH with $g \in G$. The right cosets of H are defined as the sets of the form Hg with $g \in G$. Whenever $Hg = gH$ for all $g \in G$, we will say that H is normal.*

Cosets can be seen as the image of a subset under the multiplication operators. On the other hand, observe that if $f : G \to H$ is a multiplicative operator between magmas, then $f(G)$ is a submagma of H in virtue of Remark 18(1).

Remark 21 *Let G, H be magmas. Let $f : G \to H$ be a multiplicative operator. Consider $A, B \subseteq G$ and $p \in G$. Then:*

1. *If p is idempotent, then $f(p)$ is idempotent in H.*
2. *If G has a unity 1, then $f(1)$ is the unity of $f(G)$ and an idempotent of H.*
3. *If $A \subseteq B$, then $B' \subseteq A'$.*
4. $\operatorname{span}(A)' \subseteq A'$.
5. $f(A') \subseteq f(A)'$.

Let G, H, L be magmas and $f : G \times H \to L$ a map. If f is a bioperator of magmas, then we will call it biadditive or bimultiplicative, depending on whether we are using the additive or multiplicative notion. If only the left marginal maps

$$\begin{array}{rccl} f_h: & G & \to & L \\ & g & \mapsto & f(g,h) \end{array}$$

of f are additive/multiplicative, then we will say that f is left additive/mutiplicative. Similarly, we can define right additivity/multiplicativity for f.

Definition 60 (Structured magma) *Let G be a magma. Let $\mathcal{R}$ be a binary internal relation on G. Then:*

- *We say that $\mathcal{R}$ is left compatible with G if it is left-translation invariant, that is, for all $g_1, g_2, h \in G$, $(g_1, g_2) \in \mathcal{R}$ implies that $(hg_1, hg_2) \in \mathcal{R}$.*
- *We say that $\mathcal{R}$ is right compatible with G if it is right-translation invariant, that is, for all $g_1, g_2, h \in G$, $(g_1, g_2) \in \mathcal{R}$ implies that $(g_1h, g_2h) \in \mathcal{R}$.*
- *We say that $\mathcal{R}$ is compatible with G if it is left and right compatible.*

Under the settings of the previous definition, if $\mathcal{R}$ is transitive and compatible with G and $g_1, g_2, h_1, h_2 \in G$ are so that $(g_1, g_2), (h_1, h_2) \in \mathcal{R}$, then $(g_1h_1, g_2h_2) \in \mathcal{R}$.

Definition 61 (Congruent magma) *Let G be a magma. Let $\mathcal{R}$ be a binary internal relation on G. We say that $\mathcal{R}$ is a left (right) congruence relation on G if $\mathcal{R}$ is a left (right) compatible equivalence relation on G. If $\mathcal{R}$ is a compatible equivalence relation on G, then we call G a congruent magma.*

Congruent relations are also called congruencies. The final definition in this subsubsection is the version of structured magma for order relations.

Definition 62 (Ordered magma) *Let G be a magma. Let $\mathcal{R}$ be a binary internal relation on G. We say that $\mathcal{R}$ is a left (right) ordering on G if $\mathcal{R}$ is a left (right) compatible order relation on G. If $\mathcal{R}$ is a compatible order relation on G, then we call G an ordered magma.*

1.2.2.2 Semigroups

Semigroups are magmas whose binary internal operation enjoys associativity.

Definition 63 (Semigroup) *A semigroup is a universal algebra $(G, \{\cdot\}, \varnothing)$ such that $(G, \{\cdot\}, \varnothing)$ is a magma and $\cdot$ is associative.*

Due to associativity, important notation is introduced next, which will be consistent with Notation 7.

Notation 8 *Let G be a semigroup. Consider $n \in \mathbb{N}$, subsets $A, A_1, \ldots, A_n$ of G, and $g \in G$. With additive notation, $A_1 + \cdots + A_n := \{a_1 + \cdots + a_n : a_i \in A_i,\ i = 1, \ldots, n\}$, $ng := g + \overset{n}{\cdots} + g$, and $nA := A + \overset{n}{\cdots} + A$. With multiplicative notation, $A_1 \cdots A_n := \{a_1 \cdots a_n : a_i \in A_i,\ i = 1, \ldots, n\}$, $g^n := g \overset{n}{\cdots} g$, and $A^n := A \overset{n}{\cdots} A$.*

As expected, subsemigroups are universal subalgebras of semigroups.

Proposition 14 *Let G be a semigroup. Consider $A \subseteq G$ and $p, q \in G$. Then:*

1. *The subsemigroup spanned by A is given by $\operatorname{span}(A) = \bigcup_{n\in\mathbb{N}} A \overset{n}{\cdots} A = \left\{a_1^{k_1} \cdots a_n^{k_n} : n \in \mathbb{N}, a_i \in A, k_i \in \mathbb{N}, i = 1, \ldots, n\right\}$.*
2. *A' is a subsemigroup of G satisfying that $A' = \operatorname{span}(A)'$.*
3. *If p, q commute, then $(pq)^n = p^n q^n$ for every $n \in \mathbb{N}$. Hence if G is commutative, then G^n and $\{g^n : g \in G\}$ are subsemigroups for every $n \in \mathbb{N}$.*
4. *If p is idempotent and $p \in G'$, then L_p and R_p are multiplicative.*
5. *$\mathsf{L}_p \circ \mathsf{L}_q = \mathsf{L}_{pq}$ and $\mathsf{R}_p \circ \mathsf{R}_q = \mathsf{R}_{qp}$.*

Proof 29 *Only the first item will be proved. Note that $\bigcup_{n\in\mathbb{N}} A \overset{n}{\cdots} A$ is a subsemigroup of G containing A, thus $\operatorname{span}(A) \subseteq \bigcup_{n\in\mathbb{N}} A \overset{n}{\cdots} A$. Now, let H be a subsemigroup of G containing A. Observe that $A \overset{n}{\cdots} A \subseteq H \overset{n}{\cdots} H \subseteq H$ for every $n \in \mathbb{N}$, thus, $\bigcup_{n\in\mathbb{N}} A \overset{n}{\cdots} A \subseteq H$. The arbitrariness of H allows that $\bigcup_{n\in\mathbb{N}} A \overset{n}{\cdots} A \subseteq \operatorname{span}(A)$.*

Under the settings of Proposition 14 but with additive notation, $\operatorname{span}(A) = \bigcup_{n\in\mathbb{N}} A + \overset{n}{\cdots} + A = \{k_1 a_1 + \cdots + k_n a_n : n \in \mathbb{N}, a_i \in A, k_i \in \mathbb{N}, i = 1, \ldots, n\}$.

Proposition 15 *Let G, H be semigroups. Let $T, S : G \to H$ be multiplicative operators. If H is commutative, then*

$$\begin{array}{rccl} TS: & G & \to & H \\ & g & \mapsto & (TS)(g) := T(g)S(g) \end{array}$$

is a multiplicative operator.

Proof 30 *Fix arbitrary elements $g_1, g_2 \in G$. Observe that $(TS)(g_1g_2) = T(g_1g_2)S(g_1g_2) = T(g_1)T(g_2)S(g_1)S(g_2) = T(g_1)S(g_1)T(g_2)S(g_2) = (TS)(g_1)(TS)(g_2)$.*

1.2.2.3 Monoids

Like in the previous subsubsection, we will keep on using multiplicative notation by default.

Definition 64 (Monoid) *A monoid is a universal algebra $(G, \{\cdot, 1\}, \varnothing)$, where $(G, \{\cdot\}, \varnothing)$ is a semigroup and $1 : G \to G$ is a unity operation for the multiplication $\cdot$.*

Following and being consistent with Notation 8, the following notation for monoids will be employed.

Notation 9 *Let G be a monoid. Consider $A \subseteq G$, $B \subseteq G^\times$, $g \in G$, $h \in G^\times$, and $n \in \mathbb{N}$. With additive notation, $0g := 0$, $0A := \{0\}$, $(-n)h := n(-h)$, and $(-n)B := n(-B)$, where $-B := \{-b : b \in B\}$. With multiplicative notation, $g^0 := 1$, $A^0 := \{1\}$, $h^{-n} := \left(h^{-1}\right)^n$, and $B^{-n} := \left(B^{-1}\right)^n$, where $B^{-1} := \left\{b^{-1} : b \in B\right\}$.*

Notice that if $(G, \{\cdot, 1\}, \varnothing)$ is a monoid, then $\left(G^\times, \left\{\ ^{-1}\right\}, \varnothing\right)$ is a magma. As expected, submonoids are the universal subalgebras of monoids.

Definition 65 *Let G be a monoid. Consider $A, B \subseteq G^\times$ and $p, q \in G$. Then:*

- *p is said to be involutive provided that $p^2 = 1$.*
- *p is said to be a torsion element if there exists $n \geq 1$ such that $p^n = 1$. The subset of all torsion elements of G is denoted by $\mathrm{tor}(G)$.*
- *p, q are said to be conjugate if there exists $g \in G^\times$ such that $p = g^{-1}qg$.*
- *A is called symmetric if A is a submagma of $\left(G^\times, \left\{\ ^{-1}\right\}, \varnothing\right)$.*
- *$\mathrm{span}(B)$ in the magma $\left(G^\times, \left\{\ ^{-1}\right\}, \varnothing\right)$ is called the symmetric hull of B and denoted by $\mathrm{sym}(B)$.*

Under the settings of the previous definition, $1 \in \mathrm{tor}(G) \subseteq G^\times$ and $\mathrm{tor}(G)$ is symmetric. This way, the torsionfree elements are exactly those in $G \setminus \mathrm{tor}(G)$, and G is said to be torsionfree provided that $\mathrm{tor}(G) = \{1\}$.

Example 12 *Given a nonempty set A, in the monoid $\left(A^A, \{\circ, I_A\}, \varnothing\right)$, the idempotent elements are called projections and the involutive elements are called involutions.*

A typical example of involution is the inversion on a monoid, or more generally, the symmetrization operation in a universal algebra endowed with such a unary internal operation. On the other hand, notice that in a monoid G, $\mathsf{L}_1 = \mathsf{R}_1 = I_G$.

Proposition 16 *Let G be a monoid. Let $p, g, h \in G$. Then:*

1. *If p is an idempotent admitting a left or right inverse, then p is the unity.*
2. *If gh has a left inverse a or a right inverse b, then ag is a left inverse of h or hb is a right inverse of g, respectively.*
3. *If $g, h \in G^{\times}$, then $gh \in G^{\times}$ and $(gh)^{-1} = h^{-1}g^{-1}$. In particular, $G^{\times}$ is a submonoid of G.*
4. *$G^{\times} = G$ if and only if every element of G has a left inverse, if and only if every element of G has a right inverse.*
5. *Conjugation is an equivalence relation.*
6. *If G is commutative, then $\operatorname{tor}(G)$ and $\{g^n : g \in G\}$, $n \in \mathbb{Z}^+$, are submonoids of G.*
7. *If g is a left inverse of h, then L_g is a left inverse of L_h.*
8. *If g is a right inverse of h, then R_h is a right inverse of R_g.*

Proof 31 *We will only show one implication of the fourth item. Indeed, suppose that every element of G has a left inverse. We will show that $G^{\times} = G$ by proving that every element of G has a right inverse and then we will apply Remark 20(3). Indeed, fix an arbitrary $a \in G$. By hypothesis, there exists $b \in G$ such that $ba = 1$. Next, b has also a left inverse, so there exists $c \in G$ with $cb = 1$. Then $c = c1 = c(ba) = (cb)a = 1a = a$. In other words, $ab = cb = 1$, so $b = a^{-1}$. The arbitrariness of $a \in G$ means that $G^{\times} = G$.*

Under the settings of Proposition 16, if $g \in G^{\times}$, then $\mathsf{L}_g^{-1} = \mathsf{L}_{g^{-1}}$ and $\mathsf{R}_g^{-1} = \mathsf{R}_{g^{-1}}$. The following example relies on Remark 17(1).

Example 13 *If A is a universal algebra, then $\mathcal{L}(A)$ is a submonoid of A^A under the composition. Also, the set of isomorphisms on A is itself a submonoid of $\mathcal{L}(A)$.*

In general terms, a subset A of a monoid G is called symmetric if so is $A \cap G^{\times}$. Recall that the set of natural numbers $\mathbb{N}$ does not typically contain 0, hence $\mathbb{Z}^+ := \mathbb{N} \cup \{0\}$.

Proposition 17 *Let G be a monoid. Consider $H, A \subseteq G$ and $g \in G^{\times}$. Then:*

1. *The submonoid spanned by A is $\operatorname{span}(A) = \bigcup_{n\in\mathbb{N}}(A \cup \{1\}) \overset{n}{\cdots} (A \cup \{1\}) = \left\{a_1^{k_1} \cdots a_n^{k_n} : n \in \mathbb{N}, a_i \in A, k_i \in \mathbb{Z}^+, i = 1, \ldots, n\right\}$.*
2. *A' is a symmetric submonoid of G satisfying that $A' = \operatorname{span}(A)'$.*
3. *If $H \subseteq G^{\times}$ is symmetric, then $H^{-1} = H$.*
4. *If H is a submonoid of G, then $HH = H$ and $g^{-1}Hg$ is a submonoid of G.*
5. *If $A \subseteq G^{\times}$, then $\operatorname{sym}(A) = A \cup A^{-1}$, $A \cap A^{-1}$ is the largest symmetric subset of A, $A' = \operatorname{sym}(A)'$, and $\left(g^{-1}Ag\right)^{-1} = g^{-1}A^{-1}g$.*

Proof 32 *All items are easily verifiable. We will only show that $\left(A' \cap G^{\times}\right)^{-1} \subseteq A'$. If $c \in A' \cap G^{\times}$, then $c^{-1}a = \left(c^{-1}a\right)1 = \left(c^{-1}a\right)\left(cc^{-1}\right) = c^{-1}\left((ac)c^{-1}\right) = c^{-1}\left((ca)c^{-1}\right) = \left(c^{-1}c\right)\left(ac^{-1}\right) = 1\left(ac^{-1}\right) = ac^{-1}$ for all $a \in A$.*

Under the settings of Proposition 17 but with additive notation, $\operatorname{span}(A) = \bigcup_{n\in\mathbb{N}}(A \cup \{0\}) + \overset{n}{\cdots} + (A \cup \{0\}) = \{k_1a_1 + \cdots + k_na_n : n \in \mathbb{N}, a_i \in A, k_i \in \mathbb{Z}^+, i = 1, \ldots, n\}$. With additive notation, the operators of monoids are called neutral-additive maps, and with multiplicative notation, they are called unital-multiplicative maps.

Definition 66 (Kernel) *Let G, H be monoids. Let $f : G \to H$ be a unital-multiplicative operator. The kernel of f is defined as* $\ker(f) := f^{-1}(\{1\}) = \{g \in G : f(g) = 1\}$.

Observe that if $f : G \to H$ is a unital-multiplicative operator between monoids G, H, then $f(G)$ is a submonoid of H in virtue of Remark 18(1).

Proposition 18 *Let G, H be monoids. Consider a unital-multiplicative operator $f : G \to H$. Then* $\ker(f), f(G)$ *are symmetric submonoids of G, H, respectively, and f preserves inverses, that is, $f\left(g^{-1}\right) = f(g)^{-1}$ for all $g \in G^{\times}$.*

Proof 33 *It is clear that* $1 \in \ker(f)$ *because by hypothesis $f(1) = 1$. If $g, h \in \ker(f)$, then $f(gh) = f(g)f(h) = 1 \cdot 1 = 1$, so $gh \in \ker(f)$. This shows that* $\ker(f)$ *is a submonoid of G. If $g \in \ker(f) \cap G^{\times}$, then $f\left(g^{-1}\right) = f(g)^{-1} = 1^{-1} = 1$, so $g^{-1} \in \ker(g)$. This shows that* $\left(\ker(f) \cap G^{\times}\right)^{-1} \subseteq \ker(f)$, *that is,* $\ker(f)$ *is a symmetric submonoid of G. Next, if $g \in G$ and $f(g) \in H^{\times}$, then $f(g)^{-1} = f\left(g^{-1}\right) \in f(G)$, that is, $\left(f(G) \cap H^{\times}\right)^{-1} \subseteq f(G)$. Finally, if $g \in G^{\times}$, then $f(g)f\left(g^{-1}\right) = f\left(gg^{-1}\right) = f(1) = 1 = f\left(g^{-1}g\right) = f\left(g^{-1}\right)f(g)$, meaning that f preserves inverses.*

As for now, let us regard $\mathbb{Z}^+$ as the additive monoid of positive integers with 0. Note that the subsets of the form $n\mathbb{Z}^+ := \{np : p \in \mathbb{Z}^+\}$, for $n \in \mathbb{Z}^+$, are submonoids of $\mathbb{Z}^+$. However, they are not the only submonoids since, if $p, q \in \mathbb{Z}^+$ are coprime, then $p\mathbb{Z}^+ + q\mathbb{Z}^+$ is also a submonoid of $\mathbb{Z}^+$. It is not hard to show that if $n\mathbb{Z}^+ = m\mathbb{Z}^+$, then $n = m$.

Lemma 8 *Let G be an additive monoid. If $f : \mathbb{Z}^+ \to G$ is neutral-additive, then* $\ker(f) = n\mathbb{Z}^+$ *for some $n \in \mathbb{Z}^+$.*

Proof 34 *If* $\ker(f) = \{0\}$, *then it has the form $n\mathbb{Z}^+$ for $n = 0$. Assume then that* $\ker(f) \neq \{0\}$ *and let* $n := \min(\ker(f) \setminus \{0\})$. *Suppose that there exists $m \in \ker(f) \setminus n\mathbb{Z}^+$. By the division algorithm, we have that $m = pn + q$, where $0 < q < n$. We will reach the contradiction that $q \in \ker(f)$. Observe that $0 = f(m) = pf(n) + f(q) = f(q)$, so $q \in \ker(f)$, which contradicts that n is the minimum of the nonzero elements of the kernel of f.*

The previous lemma allows the definition of the characteristic of a monoid, which is based upon the notion of torsion.

Definition 67 (Characteristic) *Let G be an additive monoid. For every $g \in G$, $n_g \in \mathbb{Z}^+$ is defined as the only element of $\mathbb{Z}^+$ that satisfies that $n_g\mathbb{Z}^+$ is the kernel of*

$$\begin{array}{rcl} \mathbb{Z}^+ & \to & G \\ n & \mapsto & ng. \end{array} \tag{1.15}$$

The characteristic of G is defined as $\operatorname{char}(G) := \operatorname{lcm}\{n_g : g \in G\}$.

In the previous definition, lcm stands for the least common multiple, and it can perfectly occur that $\operatorname{char}(G) = \infty$. Observe that $\operatorname{tor}(G) = \{g \in G : n_g \geq 1\}$.

Remark 22 *Let G be an additive monoid. Let $g \in G$. Then:*

1. *If g does not have an opposite, then the kernel of (1.15) is $\{0\}$, that is, $n_g = 0$.*
2. *If g has an opposite, then $n_g = n_{-g}$.*
3. *$n_0 = 1$.*
4. *$n_g = \operatorname{card}(\mathbb{Z}^+ g)$. Indeed,* $\operatorname{span}(g)$ *in G is $\mathbb{Z}^+ g = \{0, g, 2g, \ldots, (n_g - 1)g\}$.*

The following result in this subsubsection shows that the existence of biadditive or bimultiplicative maps is only possible under commutativity.

Proposition 19 *Let G, H be magmas. Let L be a monoid. If there exists a surjective bimultiplicative map $T : G \times H \to L$ such that $T(G \times H) \subseteq L^\times$, then L is commutative.*

Proof 35 *Fix arbitrary elements $l_1, l_2 \in L$. There exist $g_1, g_2 \in G$ and $h_1, h_2 \in H$ such that $T(g_1, h_1) = l_1$ and $T(g_2, h_2) = l_2$. On the one hand, $T(g_1g_2, h_2h_1) = T(g_1, h_2h_1)T(g_2, h_2h_1) = T(g_1h_2)T(g_1, h_1)T(g_2, h_2)T(g_2, h_1)$. Nevertheless, on the other hand, $T(g_1g_2, h_2h_1) = T(g_1g_2, h_2)T(g_1g_2, h_1) = T(g_1h_2)T(g_2, h_2)T(g_1, h_1)T(g_2, h_1)$. So, $T(g_1h_2)T(g_1, h_1)T(g_2, h_2)T(g_2, h_1) = T(g_1h_2)T(g_2, h_2)T(g_1, h_1)T(g_2, h_1)$. By hypothesis, $T(G \times H) \subseteq L^\times$, meaning that $l_1l_2 = T(g_1, h_1)T(g_2, h_2) = T(g_2h_2)T(g_1, h_1) = l_2l_1$. As a consequence, L is commutative.*

Consider a magma G endowed with a compatible binary internal relation $\mathcal{R}$. If $\mathcal{R}$ is transitive and $g_1, g_2, h_1, h_2 \in G$ are so that $(g_1, g_2), (h_1, h_2) \in \mathcal{R}$, then $(g_1h_1, g_2h_2) \in \mathcal{R}$. In particular, $[g]_\mathcal{R}[h]_\mathcal{R} \subseteq [gh]_\mathcal{R}$ and ${}_\mathcal{R}[g]_\mathcal{R}[h] \subseteq {}_\mathcal{R}[gh]$ for all $g, h \in G$. If G is a monoid, then $\left(h^{-1}, g^{-1}\right) \in \mathcal{R}$ whenever $g, h \in G^\times$ and $(g, h) \in \mathcal{R}$. In particular, $\left([g]_\mathcal{R} \cap G^\times\right)^{-1} = {}_\mathcal{R}[g^{-1}] \cap G^\times$ and $\left({}_\mathcal{R}[g] \cap G^\times\right)^{-1} = [g^{-1}]_\mathcal{R} \cap G^\times$ for all $g \in G^\times$.

Remark 23 *Let G be a monoid endowed with a binary internal relation $\mathcal{R}$. Suppose that $\mathcal{R}$ is left compatible with G. Consider $g \in G^\times$ and $g_1, g_2 \in G$. Then:*

- *$g_1[g_2]_\mathcal{R} \subseteq [g_1g_2]_\mathcal{R}$ and $g[g_1]_\mathcal{R} = [gg_1]_\mathcal{R}$.*
- *$g_1{}_\mathcal{R}[g_2] \subseteq {}_\mathcal{R}[g_1g_2]$ and $g{}_\mathcal{R}[g_1] = {}_\mathcal{R}[gg_1]$.*
- *$\left([1]_\mathcal{R} \cap G^\times\right)^{-1} = {}_\mathcal{R}[1] \cap G^\times$ and $\left({}_\mathcal{R}[1] \cap G^\times\right)^{-1} = [1]_\mathcal{R} \cap G^\times$.*
- *If $\mathcal{R}$ is reflexive and transitive then ${}_\mathcal{R}[1]$ and $[1]_\mathcal{R}$ are submonoids of G.*

Under the settings of the previous remark, if $\mathcal{R}$ is right compatible, then we have the corresponding dual version of the above.

Remark 24 *Let G be a left-ordered monoid. Consider $g_1, g_2, h_1, h_2 \in G$ and $g, h \in G^\times$. The following statements hold:*

- *$g_1 < g_2$ if and only if $hg_1 < hg_2$.*
- *$g > 1$ if and only if $g^{-1} < 1$.*
- *$\uparrow g = g \uparrow 1$, $\uparrow_\times g = g \uparrow_\times 1$, $\downarrow g = g \downarrow 1$, and $\downarrow^\times g = g \downarrow^\times 1$.*
- *$\uparrow_\times g$ is downward directed if and only if $\uparrow_\times 1$ is downward directed.*
- *$\downarrow^\times g$ is upward directed if and only if $\downarrow^\times 1$ is upward directed.*
- *$\uparrow_\times 1 \cap G^\times$ is coinitial in $\uparrow_\times 1$ if and only if $\uparrow_\times g \cap G^\times$ is coinitial in $\uparrow_\times g$.*
- *$\downarrow^\times 1 \cap G^\times$ is cofinal in $\downarrow^\times 1$ if and only if $\downarrow^\times g \cap G^\times$ is cofinal in $\downarrow^\times g$.*
- *If $\uparrow 1$ is totally ordered, then $\uparrow g$ is totally ordered. Indeed, if $g_1, g_2 \in \uparrow g$, then $g^{-1}g_1, g^{-1}g_2 \in \uparrow 1$, meaning that either $g^{-1}g_1 \leq g^{-1}g_2$ or $g^{-1}g_2 \leq g^{-1}g_1$, resulting in either $g_1 \leq g_2$ or $g_2 \leq g_1$.*

If G is right-ordered, then we have the corresponding dual version of the above. If G is ordered, then:

- *If $g_1 < g_2$ and $h_1 < h_2$ in G, and any one of g_1, g_2, h_1, h_2 is invertible, then $h_1 g_1 < h_2 g_2$.*
- *$g > h$ if and only if $g^{-1} < h^{-1}$.*
- *$\uparrow_\times 1$ is upward directed and $\downarrow^\times 1$ is downward directed.*
- *$\uparrow_\times 1 \cap G^\times$ is downward directed if and only if $\downarrow^\times 1 \cap G^\times$ is upward directed.*

The previous remark was on ordered monoids. The next remark is on structured monoids.

Remark 25 *Let G be a congruent monoid. Consider $g \in G^\times$ and $h \in G$. Then $\left[g^{-1}\right]_{\mathcal{R}} = [g]_{\mathcal{R}}^{-1}$, $[gh]_{\mathcal{R}} = [g]_{\mathcal{R}}[h]_{\mathcal{R}}$, and $[hg]_{\mathcal{R}} = [h]_{\mathcal{R}}[g]_{\mathcal{R}}$.*

The following shows an example of compatible relations from one side but not from both sides.

Example 14 *Let X be a set endowed with a binary internal relation $\mathcal{S}$. Consider the monoid X^X endowed with the composition. Consider the binary internal relation $\mathcal{R} := \left\{(f,g) \in X^X \times X^X : \forall x \in X \ \ (f(x), g(x)) \in \mathcal{S}\right\}$. Observe that if $\mathcal{S}$ is an order relation or an equivalence relation, then so is $\mathcal{R}$, respectively. On the one hand, $\mathcal{R}$ is right compatible with X^X. Indeed, if $f, g, h \in X^X$ and $(f,g) \in \mathcal{R}$, then $(f(x), g(x)) \in \mathcal{S}$ for each $x \in X$, so in particular $(f(h(x)), g(h(x))) \in \mathcal{S}$ for each $x \in X$, meaning that $(f \circ h, g \circ h) \in \mathcal{R}$. On the other hand, if $\mathcal{M}$ denotes the submonoid of X^X of relators on X, then the inherited relation on $\mathcal{M}$ is compatible with $\mathcal{M}$.*

The previous example motivates the following generalization of hull operators. Indeed, let X be a bounded poset satisfying the inf property. Consider the monoid X^X endowed with the composition. Note that X^X acquires the structure of a poset, as in the previous example, induced by the order of X. Let $\mathcal{M}$ denote the submonoid of X^X of increasing functions on X. According to the previous example, $\mathcal{M}$ is an ordered monoid, that is, its order is left and right compatible. Then a hull operator $H : X \to X$ is simply an idempotent element of $\mathcal{M}$ which satisfies that $I \leq H$.

Definition 68 (Hull) *Let M be an ordered monoid. An idempotent element $h \in M$ is said to be a hull provided that $1 \leq h$.*

1.2.2.4 Groups

Groups are monoids in which every element has an inverse.

Definition 69 (Group) *A group is a universal algebra $\left(G, \left\{\cdot, 1, {}^{-1}\right\}, \varnothing\right)$, where $(G, \{\cdot, 1\}, \varnothing)$ is a monoid and the inversion ${}^{-1}$ is defined on the whole of G, in other words, $G^\times = G$.*

If $(G, \{\cdot, 1\}, \varnothing)$ is a monoid, the universal algebra $\left(G^\times, \left\{\cdot, 1, {}^{-1}\right\}, \varnothing\right)$ is a group. As expected, subgroups are the universal subalgebras of groups.

Proposition 20 *Let G be a group. Consider $A, H \subseteq G$ and $g \in G$. Then:*

1. *The subgroup spanned by A is $\operatorname{span}(A) = \bigcup_{n\in\mathbb{N}} \left(A \cup \{1\} \cup A^{-1}\right) \overset{n}{\cdots} \left(A \cup \{1\} \cup A^{-1}\right) = \left\{a_1^{k_1} \cdots a_n^{k_n} : n \in \mathbb{N}, a_i \in A, k_i \in \mathbb{Z}, i = 1, \ldots, n\right\}$.*
2. *A' is a subgroup of G satisfying that $A' = \operatorname{span}(A)'$.*

3. *If G is commutative, then $\{g^n : g \in G\}$ is a subgroup for every $n \in \mathbb{Z}$.*

4. *If H is a subgroup of G, then $g^{-1}Hg$ is a subgroup of G.*

Proof 36 *All items are easily verifiable. We will only prove that $A' = \text{span}(A)'$. Since the group span is a hull operator, $A \subseteq \text{span}(A)$; therefore, $\text{span}(A)' \subseteq A'$. If $g \in A'$, then $g \in \left[\left(A \cup \{1\} \cup A^{-1}\right) \overset{n}{\cdots} \left(A \cup \{1\} \cup A^{-1}\right)\right]'$ for all $n \in \mathbb{N}$.*

Under the settings of Proposition 20 but with additive notation, $\text{span}(A) = \bigcup_{n\in\mathbb{N}}(A \cup \{0\} \cup -A) \overset{n}{\cdots} (A \cup \{0\} \cup -A) = \{k_1a_1 + \cdots + k_na_n : n \in \mathbb{N}, a_i \in A, k_i \in \mathbb{Z}, i = 1, \ldots, n\}$.

Remark 26 (Quotient group) *Let G be a congruence group. The quotient set ${}^G/_{\mathcal{R}}$ acquires the structure of a group under the binary internal operation given by the set product $[g]_{\mathcal{R}}[h]_{\mathcal{R}} = [gh]_{\mathcal{R}}$ for all $g, h \in G$.*

The most extended example of a quotient group is that given by the partition into cosets.

Proposition 21 *Let G be a group. Let H be a subgroup of G. Then:*

1. *The left cosets and the right cosets form partitions of G.*

2. *The map $gH \mapsto Hg^{-1}$ establishes a bijection between the left cosets and the right cosets.*

3. *The following conditions are equivalent:*

   (a) *The partition of left cosets coincides with the partition of right cosets.*
   (b) *$Hg = gH$ for all $g \in G$.*
   (c) *$Hg \subseteq gH$ for all $g \in G$.*
   (d) *$Hg \supseteq gH$ for all $g \in G$.*
   (e) *$(g_1H)(g_2H) = (g_1g_2)H$ for all $g_1, g_2 \in G$.*
   (f) *$(Hg_1)(Hg_2) = H(g_1g_2)$ for all $g_1, g_2 \in G$.*
   (g) *$(gH)^{-1} = g^{-1}H$ for all $g \in G$.*
   (h) *$(Hg)^{-1} = Hg^{-1}$ for all $g \in G$.*

Proof 37 *We will only prove that the map $gH \mapsto Hg^{-1}$ is a well-defined bijection. Indeed, $gH = g'H \Leftrightarrow g^{-1}g' \in H \Leftrightarrow (g')^{-1} g \in H \Leftrightarrow Hg^{-1} = H\,(g')^{-1}$.*

Proposition 21(2) motivates the notion of the index of a subgroup.

Definition 70 (Index of a subgroup) *Let G be a group. Let H be a subgroup of G. The index of H in G is defined as the cardinality of the collection of left cosets, or equivalently, the cardinality of the collection of right cosets, and it is denoted by $[G : H]$.*

Proposition 21(3) historically motivated the notion of a normal subgroup. If a group is commutative, then all of its subgroups are normal. More generally, any subgroup contained in the center is normal. Also, in every congruent group G, $[1]_{\mathcal{R}}$ is always a normal subgroup.

Example 15 (Quotient group) *Let G be a group. Let H be a normal subgroup of G. The partition of cosets of H forms an equivalence relation compatible with G. As a consequence, ${}^G/_H$ acquires the structure of a group with the set product given by $(g_1H)(g_2H) = (g_1g_2)H$ for all $g_1, g_2 \in G$.*

Normal subgroups are the only groups for which the partition of cosets is compatible with the group. As expected, the below theorem has its dual version for right cosets.

Theorem 9 *Let G be a group. Let H be a subgroup. The equivalence relation induced by the partition of left cosets $\{gH : g \in G\}$ is left compatible with G. If, in addition, such an equivalence relation is right compatible with G, then H is normal.*

Proof 38 *Let us prove first that the equivalence relation is left compatible with G. Indeed, fix arbitrary elements $g_1, g_2, g \in G$ such that (g_1, g_2) is a related pair. There exists $g_0 \in G$ for which $g_1, g_2 \in g_0 H$. Then $gg_1, gg_2 \in (gg_0)H$, resulting in the pair (gg_1, gg_2) being related. This shows that the equivalence relation is left compatible with G. Next, assume that the equivalence relation is right compatible with G. We will show that H is a normal subgroup. Fix an arbitrary $g \in G$. It suffices to check that $gH \subseteq Hg$. Indeed, take any $h \in H$. Notice that (g, gh) is a related pair since $g = g1 \in gH$ and $gh \in gH$. By assumption, $\left(1, ghg^{-1}\right)$ is a related pair, meaning that there exists $g' \in G$ with $1 \in g'H$ and $ghg^{-1} \in g'H$ as well. Obviously, $g' \in H$, resulting in $ghg^{-1} \in H$; that is, $gh \in Hg$. This concludes the proof.*

The following result unveils that group quotients by normality are equivalent to group quotients by congruences.

Theorem 10 *Let G be a congruent monoid by a compatible equivalence relation $\mathcal{R}$. Then $[1]_{\mathcal{R}}$ is a normal symmetric submonoid of G. As a consequence, if G is a congruent group, then $[1]_{\mathcal{R}}$ is a normal subgroup of G satisfying that ${}^{G}/_{[1]_{\mathcal{R}}} = {}^{G}/_{\mathcal{R}}$.*

Proof 39 *For every $g, h \in [1]_{\mathcal{R}}$, we know that $1 \in [1]_{\mathcal{R}} = [1 \cdot 1]_{\mathcal{R}} = [1]_{\mathcal{R}}[1]_{\mathcal{R}} = [g]_{\mathcal{R}}[h]_{\mathcal{R}} \subseteq [gh]_{\mathcal{R}}$, meaning that $gh \in [1]_{\mathcal{R}}$. This shows that $[1]_{\mathcal{R}}$ is a submonoid of G. Next, we already know that $g^{-1} \in [1]_{\mathcal{R}}$ whenever $g \in [1]_{\mathcal{R}} \cap G^{\times}$. Also, $g[1]_{\mathcal{R}} = [g]_{\mathcal{R}} = [1]_{\mathcal{R}} g$ for all $g \in G$. All of these imply that $[1]_{\mathcal{R}}$ is a normal symmetric submonoid of G. As a consequence, if G is a congruent group, then $[1]_{\mathcal{R}}$ is a normal subgroup of G. Finally, it only suffices to observe that the cosets $g[1]_{\mathcal{R}}$ are precisely the elements $[g]_{\mathcal{R}}$ of ${}^{G}/_{\mathcal{R}}$.*

If $f : G \to H$ is an operator of monoids (that is, unital-multiplicative), then $f|_{G^{\times}} : G^{\times} \to H^{\times}$ is an operator of groups.

Remark 27 *Let G, H be groups. Let $f : G \to H$ be multiplicative. Then f preserves the unity and the inverses; hence, f is an operator of groups, $\ker(f)$ is a normal subgroup of G, $f(G)$ is a subgroup of H, and f is injective if and only if $\ker(f) = \{1\}$.*

The following theorem is a fundamental one in Abstract Algebra.

Theorem 11 (First Isomorphism Theorem) *Let G, H be groups. Let $f : G \to H$ be multiplicative. Then ${}^{G}/_{\ker(f)}$ is isomorphic to $f(G)$ via the (unique) isomorphism*

$$\begin{array}{rccc} \widetilde{f}: & {}^{G}/_{\ker(f)} & \to & f(G) \\ & g\ker(f) & \mapsto & \widetilde{f}\,(g\ker(f)) := f(g). \end{array}$$

$$\begin{array}{ccc} & & H \\ & {\scriptstyle f}\nearrow & \uparrow{\scriptstyle \widetilde{f}} \\ G & \xrightarrow{\pi_f} & {}^{G}/_{\ker(f)} \end{array}$$

Proof 40 *It is immediate to check that $\widetilde{f}$ is well defined and multiplicative, hence an operator of groups. According to Remark 17(3), it only suffices to show that $\widetilde{f}$ is bijective. Indeed, let us show first that $\widetilde{f}$ is injective by proving that* $\ker\left(\widetilde{f}\right) = \{\ker(f)\}$. *Take any $g \in G$ such that $\widetilde{f}\left(g\ker(f)\right) = 1$. Notice that $\widetilde{f}\left(g\ker(f)\right) = f(g)$, thus $g \in \ker(f)$, hence $g\ker(f) = \ker(f)$. In view of Proposition 15 and Proposition 27(3), $\widetilde{f}$ is injective. Finally, it is easy to see that $\widetilde{f}$ is surjective.*

The following is a very curious example of commutative group for which neither addition notation nor multiplication notation seem to apply naturally.

Example 16 *Let Ω be a set. Then $(\mathcal{P}(\Omega), \triangle)$ is a group, where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ for all $A, B \subseteq \Omega$. Observe that $A \triangle A = \varnothing$ for all $A \subseteq \Omega$. Therefore,* $\operatorname{char}\left((\mathcal{P}(\Omega), \triangle)\right) = 2$.

Recall (see Example 12) that, if G is a group, then a projection $p : G \to G$ is simply an idempotent of the monoid $\left(G^G, \circ\right)$.

Lemma 9 *If $p : G \to G$ is a multiplicative projection on a group G, then $I \cdot \left({}^{-1} \circ p\right)$ is a projection.*

Proof 41 *Fix an arbitrary $g \in G$. Then $\left[I \cdot \left({}^{-1} \circ p\right)\right](g) = gp(g)^{-1}$. Thus,*

$$\begin{aligned} & \left[I \cdot \left({}^{-1} \circ p\right)\right]\left(gp(g)^{-1}\right) = gp(g)^{-1} p\left(gp(g)^{-1}\right)^{-1} \\ = \;& gp(g)^{-1}\left(p(g)p\left(p(g)^{-1}\right)\right)^{-1} = gp(g)^{-1}\left(p(g)p\left(p\left(g^{-1}\right)\right)\right)^{-1} \\ = \;& gp(g)^{-1}\left(p(g)p\left(g^{-1}\right)\right)^{-1} = gp(g)^{-1}\left(p(g)p\left(g\right)^{-1}\right)^{-1} = gp(g)^{-1}. \end{aligned}$$

The above lemma motivates the concept of complemented subgroup.

Definition 71 (Complemented subgroup) *A subgroup of a group is said to be complemented if it is the range of a multiplicative projection.*

1.2.2.5 Actions

From the rest on throughout this subsubsection, we will keep on using multiplicative notation, unless otherwise explicitly stated.

Definition 72 (Action) *A left action of a magma G over a set M is a binary external operation $G \times M \to M$. In this sense, M is called a G-set. In addition:*

- *If G is a semigroup, then the action is asked to be associative, that is, $(gh)m = g(hm)$ for all $g \in G$ and all $m \in M$.*
- *If G is a monoid, then the action is assumed to be unital, that is, $1m = m$ for all $m \in M$.*

Right actions can be dually defined; however, all actions throughout this book will be regarded as left actions by default, unless otherwise explicitly stated. Every monoid acts on itself by left or right multiplication. If M is a G-set, for $(G, \{\cdot\}, \varnothing)$ a magma, and we denote the action by $\cdot$, then $(M, \varnothing, \{\cdot\})$ is a universal algebra. In fact, if G is a semigroup, then the G-subset spanned by $N \subseteq M$ is GN, which is $\operatorname{span}(N)$ in the universal algebra $(M, \varnothing, \{\cdot\})$.

Example 17 *Let G be an (additive or multiplicative) semigroup. Then:*

1. *The multiplicative monoid* $\mathbb{N}$ *acts over* G *as*

$$\begin{array}{rcl} \mathbb{N}\times G & \to & G \\ (n,g) & \mapsto & ng \end{array} \quad or \quad \begin{array}{rcl} G\times\mathbb{N} & \to & G \\ (g,n) & \mapsto & g^n \end{array}$$

depending whether the semigroup G *is additive or multiplicative, respectively.*

2. *If* G *is a monoid, then the multiplicative monoid* $\mathbb{Z}^+$ *acts over* G *as*

$$\begin{array}{rcl} \mathbb{Z}^+\times G & \to & G \\ (n,g) & \mapsto & ng \end{array} \quad or \quad \begin{array}{rcl} G\times\mathbb{Z}^+ & \to & G \\ (g,n) & \mapsto & g^n \end{array}$$

depending whether the monoid G *is additive or multiplicative, respectively.*

3. *If* G *is a group, then the multiplicative monoid* $\mathbb{Z}$ *acts over* G *as*

$$\begin{array}{rcl} \mathbb{Z}\times G & \to & G \\ (n,g) & \mapsto & ng \end{array} \quad or \quad \begin{array}{rcl} G\times\mathbb{Z} & \to & G \\ (g,n) & \mapsto & g^n \end{array}$$

depending whether the group G *is additive or multiplicative, respectively.*

Notice that, if M is a G-set for G a magma, then it makes sense to conceive the left- and right-multiplication operators by $g \in G$ and $m \in M$, respectively:

$$\begin{array}{rcl} \mathsf{L}_g: & M & \to & M \\ & m & \mapsto & \mathsf{L}_g(m) := gm \end{array} \qquad \begin{array}{rcl} \mathsf{R}_m: & G & \to & Gm \subseteq M \\ & g & \mapsto & \mathsf{R}_m(g) := gm. \end{array} \tag{1.16}$$

Observe that the multiplication operators are exactly the marginal maps of the action. In fact, if G is a group, then the action induces a multiplicative map

$$\begin{array}{rcl} G & \to & \mathsf{S}_M \\ g & \mapsto & \mathsf{L}_g. \end{array} \tag{1.17}$$

Conversely, every multiplicative map $\pi : G \to \mathsf{S}_M$ induces an action

$$\begin{array}{rcl} G\times M & \to & M \\ (g,m) & \mapsto & \pi(g)(m). \end{array} \tag{1.18}$$

Definition 73 (Orbit/Stabilizer) *Let* G *be a magma acting on a set* M*. Let* $m \in M$ *and* $N \subseteq M$*. Then:*

- *The orbit of* m *is defined as* $Gm := \{gm : g \in G\}$*.*
- *The stabilizer of* m *is defined as* $G_m := \{g \in G : gm = m\}$*.*

Under the settings of the previous definition, if G is a group, then the orbits form a partition of M. The Axiom of Choice allows to construct a set that will be denoted by $\mathscr{K}(M)$, containing one representative of each orbit. Also, if G is a semigroup or a monoid or a group, then the stabilizer is a subsemigroup or a submonoid or a subgroup, respectively.

Definition 74 (Torsion) *Let* G *be a magma acting on a set* M*. An element* $m \in M$ *is called a torsion element if there are* $g \neq h$ *in* G *such that* $gm = hm$*. The set of torsion elements of* M *is denoted by* $\mathrm{tor}(M)$*.*

Under the settings of the previous definition, the torsionfree elements are exactly those in $M \setminus \mathrm{tor}(M)$. If G is a group acting on a set M, then $M \setminus \mathrm{tor}(M) = \{m \in M : G_m = \{1\}\}$.

Definition 75 (Invariance/Independence) *Let G be a magma acting on a set M. Let $N \subseteq M$. Then:*

- *N is said to be invariant if so it is under the left-multiplication operators.*
- *N is called free or independent provided that $Gn \cap N = \{n\}$ for all $n \in N$.*

Under the settings of the previous definition, the set of invariant (or fixed) elements of M is $\mathrm{fix}(M) := \{m \in M : G_m = G\} = \{m \in M : Gm = \{m\}\}$. Note that $\mathrm{fix}(M) \subseteq \mathscr{K}(M)$.

Remark 28 *Let G be a magma acting on a set M. Let $N \subseteq M$. Then N is invariant if and only if $GN \subseteq N$, that is, N is a universal subalgebra of $(M, \varnothing, \{\cdot\})$. If, in addition, G is a monoid, then:*

- *N is invariant if and only if $GN = N$.*
- *N is free if and only if $\forall g \in G\ \forall h \in G^{\times}\ \forall p, n \in N\ gp = hn \Rightarrow p = n$.*
- *N is invariant and free if only if N is composed only of invariant elements.*

The following definition has crucial applications to Measure Theory, particularly to the case of the Haar measure.

Definition 76 *Let G be a monoid acting on a set S. The action is said to be:*

- *Complete if every element of G is in an estabilizer.*
- *Uniform if for every $s, t \in S$ there exists $g \in G$ such that $G_s = g^{-1}G_t g$.*
- *Separated if for every $s \neq t$ in S, $G_s \cap G_t = \{1\}$.*
- *Transitive if for every $s \in S$, $Gs = S$.*

Transitive actions are well known in Group Theory whereas complete, uniform, and separated actions have never been considered yet, as far as we know.

Lemma 10 *Let G be a group acting on a set S. Let $s, t \in S$. Then:*

1. *If $s \in Gt$, then there exists $g \in G$ such that $g^{-1}G_s g = G_t$.*
2. *If G_s is not trivial and the action is uniform and separated, then $Gs = S$.*

Proof 42 *Let us itemize the proof according to the statememt.*

1. *Take $g \in G$ in such a way that $s = gt$. Fix an arbitrary element $h \in G_s$. Then $\left(g^{-1}hg\right)t = \left(g^{-1}h\right)(gt) = \left(g^{-1}h\right)s = g^{-1}(hs) = g^{-1}s = t$. This shows that $g^{-1}G_s g \subseteq G_t$. The same argument applied to the fact that $t = g^{-1}s$ shows that $gG_t g^{-1} \subseteq G_s$, obtaining the desired equality.*
2. *Fix an arbitrary $t \in S$. Since the action is uniform, there exists $g \in G$ such that $g^{-1}G_s g = G_t$. By hypothesis, there exists $h \in G_s \setminus \{1\}$. Then $g^{-1}hg \in G_t$, that is, $g^{-1}hgt = t$, meaning that $h(gt) = gt$. Therefore, $h \in G_s \cap G_{gt}$. Since the action is separated and $h \neq 1$, the only possibility is that $s = gt$, that is, $t = g^{-1}s \in Gs$.*

A direct consequence of the previous lemma is the fact that if a group acts transitively on a set, then it also acts uniformly on the set. Another direct consequence is that if a group action is uniform and separated and there exists a nontrivial stabilizer, then it is transitive.

Definition 77 (Homogeneous map) *Let G be a semigroup. Let M, N be G-sets. If $f : M \to N$ is a map between G-sets and $n \in \mathbb{N}$, then f is said to be homogeneous of degree n if $f(gm) = g^n f(m)$ for all $g \in G$ and all $m \in M$.*

The homogeneous maps of degree 1 will be simply called homogeneous. These maps are precisely the operators between G-sets regarded as universal algebras. Observe that, if M is a G-set for G a semigroup, then R_m is homogeneous for every $m \in M$. Indeed, $\mathsf{R}_m(gh) = (gh)m = g(hm) = g\mathsf{R}_m(h)$ for all $g, h \in G$. With respect to the left-multiplication operator, it can be asserted that $\mathsf{L}_{gh}(m) = (gh)m = g(hm) = g\mathsf{L}_h(m)$ for all $g, h \in G$ and all $m \in M$.

Example 18 *Let G, H be additive or multiplicative semigroups. Let $f : G \to H$ be additive or multiplicative, respectively. Then:*

1. *f is $\mathbb{N}$-homogeneous.*
2. *If G, H are monoids and f is neutral-additive or unital-multiplicative, then f is $\mathbb{Z}^+$-homogeneous.*
3. *If G, H are groups, then f is $\mathbb{Z}$-homogeneous.*

The following result shows that the existence of bihomogeneous maps is only possible under commutativity.

Proposition 22 *Let G be a magma. Let M, N, P be G-sets. If there exists a bihomogeneous map $T : M \times N \to P$ such that $T(M \times N)$ contains a torsionfree element, then G is commutative.*

Proof 43 *Fix arbitrary elements $g_1, g_2 \in G$. Take $m \in M$ and $n \in N$ such that $T(m, n)$ is torsionfree. On the one hand, $T(g_1 m, g_2 n) = g_1 T(m, g_2 n) = g_1 g_2 T(m, n)$. On the other hand, $T(g_1 m, g_2 n) = g_2 T(g_1 m, n) = g_2 g_1 T(m, n)$. Thus, $g_1 g_2 T(m, n) = g_2 g_1 T(m, n)$. Since $T(m, n)$ is torsionfree, $g_1 g_2 = g_2 g_1$. As a consequence, G is commutative.*

There is another commonly used notation for right actions and orbits: the exponential notation. Indeed, if G is a magma and M is a right G-set, then m^g means g acting on m, for all $g \in G$ and all $m \in M$. More generally, if $H \subseteq G$ and $N \subseteq M$, then N^H is the set of all elements n^h with $n \in N$ and $h \in H$. Exponential notation is more common for right selfactions (right actions of a multiplicative group on itself). Group selfactions will be considered from the right by default. Conjugation is a classic example of group selfaction.

Example 19 (Conjugation) *Consider the right action of a group G on itself given by conjugation*

$$\begin{array}{rcl} G \times G & \to & G \\ (g, h) & \mapsto & g^h := h^{-1} g h. \end{array} \tag{1.19}$$

The orbits of this action are called the conjugation classes $\left\{g^G : g \in G\right\}$.

The final results in this subsection are aimed at characterizing group actions whose partition of orbits conforms a congruency, that is, a compatible equivalence relation.

Lemma 11 *Let G be a group acting on itself. The following conditions are equivalent:*

1. *$(gh)^G \subseteq g\left(h^G\right)$ for every $g, h \in G$.*
2. *$(gh)^G \supseteq g\left(h^G\right)$ for every $g, h \in G$.*

3. $(gh)^G = g\left(h^G\right)$ *for every* $g, h \in G$.

Proof 44 *Suppose first that 1. holds. We will show 3. Fix arbitrary elements* $g, h \in G$. *On the one hand, by hypothesis,*

$$(gh)^G \subseteq g\left(h^G\right) \Rightarrow g^{-1}\left((gh)^G\right) \subseteq h^G. \tag{1.20}$$

On the other hand, again by hypothesis,

$$h^G = \left(g^{-1}gh\right)^G \subseteq g^{-1}\left((gh)^G\right) \tag{1.21}$$

By combining (1.20) *and* (1.21) *together, we obtain that*

$$g^{-1}\left((gh)^G\right) \subseteq h^G = \left(g^{-1}gh\right)^G \subseteq g^{-1}\left((gh)^G\right). \tag{1.22}$$

This shows that

$$h^G = g^{-1}\left((gh)^G\right) \Rightarrow g\left(h^G\right) = (gh)^G. \tag{1.23}$$

In a dual way, it can be shown that 2. also implies 3.

The previous lemma motivates the following definition.

Definition 78 (Intertwining action) *An action of a group* G *on itself is called left intertwining if for every* $g, h \in G$, $(gh)^G = g\left(h^G\right)$. *And the action is called right intertwining if for every* $g, h \in G$, $(hg)^G = \left(h^G\right)g$. *The action is called intertwining if it is both left and right intertwining.*

Notice that Lemma 11 has its dual version for right intertwining.

Theorem 12 *Let* G *be a group acting on itself. Then the action is left (right) intertwining if and only if the equivalence relation* $\mathcal{R}$ *on* G *determined by the collection of orbits is left (right) compatible with* G.

Proof 45 *Suppose first that the action is left intertwining. Let us prove that the equivalence relation is left compatible. Fix arbitrary elements* $g_1, g_2, g \in G$ *such that* $(g_1, g_2) \in \mathcal{R}$. *It only suffices to prove that* $gg_1 \in (gg_2)^G$. *Indeed, by assumption,* $(g_1, g_2) \in \mathcal{R}$, *meaning that* $g_1 \in (g_2)^G$. *Then* $gg_1 \in g(g_2)^G = (gg_2)^G$, *hence* $(gg_1, gg_2) \in \mathcal{R}$. *Conversely, assume that the equivalence relation is left compatible. Let us prove that the action is left intertwining. Fix arbitrary elements* $g, g_2, h \in G$. *Observe that* $\left((g_2)^h, g_2\right) \in \mathcal{R}$, *thus, by assumption,* $\left(g(g_2)^h, gg_2\right) \in \mathcal{R}$, *that is,* $g(g_2)^h \in (gg_2)^G$. *This proves that* $(gg_2)^G \supseteq g(g_2)^G$. *At this stage, Lemma 11 does the rest.*

A direct consequence of the previous theorem establishes that an action of a group on itself is intertwining if and only if the equivalence relation given by the orbits is a congruency.

Definition 79 (Pseudoconjugation) *An action of a group* G *on itself is called a pseudoconjugation provided that the following conditions hold for all* $g, h \in G$*:*

- $g^g = g$.
- $g \in G_h \Leftrightarrow h \in G_g$.
- *The right-multiplication operator* $g \mapsto g^h$ *is a group isomorphism on* G.

Pseudoconjugation actions [7, 8] are complete. In fact, under the settings of the previous definition, since $g^1 = g$ for all $g \in G$, the stabilizer of 1 is G and the orbit of 1 is $\{1\}$. In other words, $\mathrm{fix}(G) = \bigcap_{g \in G} G_g$.

1.2.3 Rings and fields

Here we will be assuming that all rings are associative and unitary. Most of the results of this subsection can be easily readapted to fit nonunitary associative rings. Nevertheless, they can be hardly readapted to fit nonassociative rings.

1.2.3.1 Rings

If a set A is endowed with binary internal operations, say $+$ and $\cdot$, then it is natural to wonder if, for instance, $\cdot$ is biadditive. If it is so, then we will call it distributive. The term of distributivity will be only employed for biadditive or bimultiplicative binary internal operations. All other multioperators will not be called distributive.

Definition 80 (Ring) *A ring is a universal algebra $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$, where $(R, \{+, 0, -\}, \varnothing)$ is a commutative additive group and $(R, \{\cdot, 1\}, \varnothing)$ is a multiplicative monoid such that the multiplication $\cdot$ is distributive.*

The distributivity of $\cdot$ with respect to $+$ means that $r(s+t) = rs+rt$ and $(s+t)r = sr+st$ for all $r, s, t \in R$. The following proposition highlights why the addition on a ring must be asked to be commutative.

Proposition 23 *If $(R, \{+, 0, -\}, \varnothing)$ is an additive group and $(R, \{\cdot, 1\}, \varnothing)$ is a multiplicative monoid such that the multiplication $\cdot$ is distributive, then $+$ is commutative.*

Proof 46 *If $r, s \in R$, then $(1 + 1)(a + b) = (1 + 1)a + (1 + 1)b = a + a + b + b$ and $(1 + 1)(a + b) = 1(a + b) + 1(a + b) = a + b + a + b$. As a consequence, $a + b = b + a$.*

As expected, subrings are just universal subalgebras of rings. On the other hand, following Notation 9, we introduce next a new notation consistent with the previously mentioned one.

Notation 10 *Let R be a ring. Consider $n \in \mathbb{Z}$, $A \subseteq R$, and $r \in R$. Then $rA := \{ra : a \in A\}$. Therefore, since $n1 := 1+ \overset{n}{\cdots} +1$ if n is positive, then $(n1)A := \{na : a \in A\}$ for all $n \in \mathbb{Z}$. Hence $\mathbb{Z}A := \{na : n \in \mathbb{Z}, a \in A\}$.*

If R is a ring, then $R^\times$ will stand for the group of invertibles of the multiplicative monoid of R. The set of invertibles $R^\times$ is also denoted by $\mathcal{U}(R)$.

Definition 81 (Integral domain) *Let R be a ring. Consider $s, a, b \in R$ and $A \subseteq R$. Then:*

- *The set of left divisors of s is defined by $\mathtt{ld}(s) := \{r \in R : \exists t \in R \setminus \{0\} \;\; rt = s\}$. Similarly, $\mathtt{rd}(s)$ stands for the set of right divisors of s.*
- *$R \setminus \mathtt{ld}(0)$ is called the set of left regular elements, $R \setminus \mathtt{rd}(0)$ is the set of right regular elements, and $R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$ is the set of regular elements.*
- *R is called an integral domain if $\mathtt{ld}(0) = \mathtt{rd}(0) = \{0\}$.*

Under the settings of the previous definition, $\mathtt{ld}(1)$ and $\mathtt{rd}(1)$ are precisely the sets of elements admitting a right inverse and a left inverse, respectively. Also, if R is an integral domain, then $R \setminus \{0\}$ is a cancellative multiplicative monoid.

Proposition 24 *Let R be a ring. Consider $r, s \in R$ and $A \subseteq R$. Then:*

1. $r0 = 0r = 0$.
2. $1 = 0$ *if and only if* $R = \{0\}$.
3. $-rs = (-r)s = r(-s)$.
4. $R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$ *is a submonoid of the multiplicative monoid of R. In fact, R is an integral domain if and only if $R \setminus \{0\}$ is a submonoid of the multiplicative monoid of R.*
5. $\mathcal{U}(R) = \mathtt{ld}(1) \cap \mathtt{rd}(1)$, $\mathtt{ld}(0) \cap \mathtt{rd}(1) = \varnothing = \mathtt{rd}(0) \cap \mathtt{ld}(1)$, *and* $(\mathtt{ld}(0) \cup \mathtt{rd}(0)) \cap \mathcal{U}(R) = \varnothing$.
6. *If* $1 - rs \in \mathcal{U}(R)$, *then* $1 - sr \in \mathcal{U}(R)$ *and* $(1 - sr)^{-1} = 1 + s(1 - rs)^{-1}r$.

Proof 47 *We will only show the last item. Simply observe that* $(1-sr)\left(1 + s(1 - rs)^{-1}r\right) = 1 - sr + s(1 - rs)^{-1}r - srs(1 - rs)^{-1}r = 1 - sr + s(1 - rs)(1 - rs)^{-1}r = 1 - sr + sr = 1$. *In a similar way, it can be shown that* $\left(1 + s(1 - rs)^{-1}r\right)(1 - sr) = 1$.

If a ring R is an integral domain, then $0, 1$ are the only idempotents of R. As a consequence, if A is an additive subgroup and a multiplicative subsemigroup of R containing an idempotent $r \in A$, then $r = 1$, hence A is a subring of R.

Proposition 25 *Let R be a ring. Let $A \subseteq R$. Then:*

1. *The subring spanned by A is* $\operatorname{span}(A) = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \Big[(A \cup \{0,1\} \cup -A) \overset{m}{\cdots} (A \cup \{0,1\} \cup -A) + \overset{n}{\cdots} + (A \cup \{0,1\} \cup -A) \overset{m}{\cdots} (A \cup \{0,1\} \cup -A)\Big] = \Big\{ \sum_{i=1}^{n} k_i a_{i1}^{k_{i1}} \cdots a_{im_i}^{k_{im_i}} : n, m_i \in \mathbb{N}, a_{ij} \in A, k_i \in \mathbb{Z}, k_{ij} \in \mathbb{Z}^+, j = 1, \ldots, m_i, i = 1, \ldots, n \Big\}$.
2. *A' is a subring of R satisfying that $A' = \operatorname{span}(A)'$.*
3. *$(n1)R$ is an additive subgroup and a multiplicative subsemigroup of R for all $n \in \mathbb{Z}$.*

Proof 48 *We will only prove the last item. We already know that $(n1)R := \{nr : r \in R\}$ is an additive subgroup of R. Let us show that nR is also a multiplicative subsemigroup of R. Indeed, fix arbitrary elements $r, s \in R$. Then $(nr)(ns) = n^2(rs) = n(n(rs)) \in (n1)R$.*

For the ring of integers $\mathbb{Z}$, we will follow the classical notation $n\mathbb{Z} := \{nz : z \in \mathbb{Z}\}$ for all $n \in \mathbb{Z}$. On the other hand, Lemma 8 assures that if R is a ring, then for every $r \in R$, the kernel of the following additive and unital-multiplicative map

$$\begin{array}{ccc} \mathbb{Z} & \to & R \\ n & \mapsto & nr \end{array}$$

is precisely $n_r\mathbb{Z}$, where n_r is given by Definition 67.

Proposition 26 *Let R be a ring. Then:*

1. *If $n_1 > 0$, then for all $r \in R$, n_r is a divisor of n_1.*
2. *If there exists $r \in R \setminus \{0\}$ such that $n_r > 0$, then $n_r 1 \in \mathtt{ld}(0)$. If, in addition, $n_r 1 = 0$, then $n_r = n_1$.*

3. *If* $n1 \notin \mathtt{ld}(0) \setminus \{0\}$ *for every* $n \in \mathbb{N}$, *then* $n_1 = n_r$ *for all* $r \in R \setminus \{0\}$.

4. $\mathrm{char}(R) = n_1$.

Proof 49 *The proof is itemized in accordance with the statement.*

1. *Observe that* $n_1 r = r + \overset{n_1}{\cdots} + r = (1 + \overset{n_1}{\cdots} + 1)r = 0r = 0$. *So,* $n_1 \in n_r\mathbb{Z}^+$.

2. *Note that* $0 = n_r r = r + \overset{n_r}{\cdots} + r = (1 + \overset{n_r}{\cdots} + 1)r$. *Since* $r \neq 0$, *we conclude that* $n_r 1 \in \mathtt{ld}(0)$. *Suppose that* $1 + \overset{n_r}{\cdots} + 1 = 0$. *Then* $n_r \in n_1\mathbb{Z}^+$. *Now, by Proposition 26(1),* $n_1 \in n_r\mathbb{Z}^+$. *This implies that* $n_1 = n_r$.

3. *We will be assuming that* R *is not zero. We will distinguish between two possibilities:*

   - $n_1 > 0$. *Now, fix any arbitrary* $r \in R \setminus \{0\}$. *Then, by Proposition 26(1),* n_r *and is a divisor of* n_1. *This implies that* $n_r > 0$. *Now, we apply Proposition 26(2) to conclude that* $n_r 1 \in \mathtt{ld}(0)$. *By hypothesis,* $n_r 1 \notin \mathtt{ld}(0) \setminus \{0\}$, *thus* $n_r 1 = 0$. *By using again Proposition 26(2), we deduce that* $n_1 = n_r$.
   - $n_1 = 0$. *Suppose that there exists* $r \in R \setminus \{0\}$ *such that* $n_r > 0$. *By using again Proposition 26(2),* $n_r 1 \in \mathtt{ld}(0)$. *By hypothesis,* $n_r 1 \notin \mathtt{ld}(0) \setminus \{0\}$, *thus* $n_r 1 = 0$. *By using again Proposition 26(2), we deduce that* $n_1 = n_r$. *This contradicts the fact that* $n_r > 0$. *As a consequence,* $n_r = 0$.

4. *If* $n_1 = 0$, *then* $\mathrm{char}(R) = \mathrm{mcm}\{n_r : r \in R\} = 0$. *If* $n_1 > 0$, *then by Proposition 26(1),* n_1 *is a multiple of* n_r *for all* $r \in R$, *so* n_1 *is a multiple of* $\mathrm{char}(R)$. *However, by definition,* $\mathrm{char}(R)$ *is a multiple of* n_1. *As a consequence,* $\mathrm{char}(R) = n_1$.

The fact for a ring to have characteristic 0 does not mean that a nonzero element a cannot be found in such a way that $a + a = 0$. For instance, $\mathbb{Z} \times \mathbb{Z}_2$ is a ring of characteristic 0 in which $(1,1) + (1,1)$ is a zero divisor; that is, $((1,1) + (1,1))(0,1) = (0,0)$.

Lemma 12 *Let* R *be a ring. Consider a regular element* $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$. *Let* $A \subseteq R$ *be such that* $rA = Ar$. *Let* $a_1, \ldots, a_k \in A$ *be such that* $a_1 + \cdots + a_k = 1$. *For every* $i \in \{1, \ldots, k\}$, *consider* $b_i, c_i \in A$ *with* $rb_i = a_i r$ *and* $c_i r = r a_i$. *Then* $b_1 + \cdots + b_k = 1 = c_1 + \cdots + c_k$.

Proof 50 *Notice that*

$$r\left(\sum_{i=1}^{k} b_i\right) = \sum_{i=1}^{k} r b_i = \sum_{i=1}^{k} a_i r = \left(\sum_{i=1}^{k} a_i\right) r = r,$$

thus, $r\left(\sum_{i=1}^{k} b_i - 1\right) = 0$. *Since* r *is not a zero divisor, we conclude that* $\sum_{i=1}^{k} b_i = 1$. *A similar reasoning applies for the* c_i*'s.*

The following notion has plenty of applications in nearly all areas of Mathematics.

Definition 82 (Ideal) *Let* R *be a ring. A subset* I *of* R *is called a left ideal of* R *if* I *is a universal subalgebra of* $(R, \{+, 0, -\}, \{\cdot\})$, *where the multiplication* $\cdot : R \times R \to R$ *is now seen as a left action of the multiplicative monoid of* R *on* R.

Under the settings of the previous definition, a left ideal I is simply an additive subgroup of R which is R-invariant, that is, $RI \subseteq I$. In particular, I is a subsemigroup of the multiplicative monoid of R. Right ideals are defined similarly. A two-sided ideal is a left and right ideal. It is not hard to check that, if an ideal contains an invertible, then the ideal is the whole ring.

Lemma 13 *Let R be a ring. Let I be an additive subgroup of R. Then:*

1. *I is a two-sided ideal of R if and only if $(r+I)(s+I) \subseteq rs+I$ for all $r, s \in R$.*
2. *If I is a two-sided ideal of R and either $r \in \mathtt{ld}(1)$ or $s \in \mathtt{rd}(1)$, then $(r+I)(s+I) = rs+I$.*

Proof 51 *Let us itemize the proof as the statement is.*

1. *Suppose first that I is a two-sided ideal. Take any $a, b \in I$. Notice that $(r+a)(s+b) = rs+(rb+as+ab) \in rs+I$ since I is two-sided. Conversely, fix an arbitrary elements $r, s \in R$. Note that $rI = (r+\{0\})(0+I) \subseteq (r+I)(0+I) \subseteq r0+I = I$. Similarly, $Is = (0+I)(s+\{0\}) \subseteq (0+I)(s+I) \subseteq 0s+I = I$.*
2. *Suppose, for instance, that $r \in \mathtt{ld}(1)$. There exists $r' \in R$ such that $rr' = 1$. Fix an arbitrary $a \in I$. Then $rs+a = (r+0)(s+r'a) \in (r+I)(s+I)$. The same conclusion can be similarly reached if $s \in \mathtt{rd}(1)$.*

Ideals allow the construction of quotient rings.

Proposition 27 (Quotient ring) *Let R be a ring. Let I be a two-sided ideal. Then ${}^{R}\!/_{I}$ acquires structure of ring under the product defined by $(r+I)(s+I) := rs+I$ for all $r, s \in R$.*

Proof 52 *It only suffices to show that the product is well defined. The rest is easy to check. Indeed, suppose that $r', s' \in R$ are so that $r-r', s-s' \in I$. Then $rs-r's' = rs-rs'+rs'-r's' = r(s-s')+(r-r')s' \in rI+Is' \subseteq I$. This shows that $r's'+I = rs+I$. Hence the product is well defined.*

The operators of rings are the additive and unital-multiplicative maps. In nonunitary rings, the operators are simply the additive and multiplicative maps. If $f : R \to S$ is an additive and unital-multiplicative operator between rings, that is, an operator of rings, then $f(R)$ is a subring of S in virtue of Remark 18(1). Notice that, if f is not unital, then Remark 18(1) only assures that $f(R)$ is a nonunitary subring of S.

Remark 29 *Let R, S be rings. Let $f : R \to S$ be additive and multiplicative. Then:*

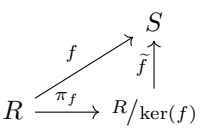
1. *$\ker(f)$ is a two-sided ideal of R.*
2. *$f(1)$ is the unity of $f(R)$ and $f : R \to f(R)$ is unital-multiplicative, so in particular it is an operator of rings.*
3. *If $1 \in \ker(f)$, then $f = 0$.*

The following result is the version of the First Isomorphism Theorem for rings.

Theorem 13 (First Isomorphism Theorem) *Let R, S be rings. Let $f : R \to S$ be additive and multiplicative. Then*

$$\begin{array}{rccl} \widetilde{f}: & {}^{R}\!/_{\ker(f)} & \to & f(R) \\ & r+\ker(f) & \mapsto & \widetilde{f}(r+\ker(f)) := f(r) \end{array}$$

is a (unique) ring isomorphism.



Proof 53 *In accordance with Theorem 11, $\widetilde{f}$ is an isomorphism of additive groups. It is easy to check that $\widetilde{f}$ is also unital-multiplicative, that is, an operator of rings. Finally, Remark 17(3) assures that $\widetilde{f}$ is an isomorphism of rings.*

Notice that, under the settings of Theorem 25, f is not necessarily unital, whereas so is $\widetilde{f}$ over its image.

Definition 83 ($*$-Ring) *A $*$-ring is a universal algebra $(R, \{+, 0, -, \cdot, 1, {}^*\}, \varnothing)$, where $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$ is a ring and*

$$\begin{array}{rccc} {}^*: & R & \to & R \\ & r & \mapsto & r^* \end{array}$$

is a unary internal operation, which is an additive and antimultiplicative involution.

Under the settings of the previous definition, by antimultiplicative is meant that $\forall r, s \in R\ (rs)^* = s^* r^*$. The operators of $*$-rings are the additive and multiplicative maps $f : R \to S$ which also are $*$-preserving: $\forall r \in R\ f(r^*) = f(r)^*$.

Remark 30 *Let R be a $*$-ring. Then:*

1. $1^* = 1$. *Indeed,* $1^* = 1^* 1 = (1^* 1)^* = (1^*)^* = 1$.
2. *If $r \in \mathcal{U}(R)$, then $r^* \in \mathcal{U}(R)$ and $(r^*)^{-1} = \left(r^{-1}\right)^*$. Indeed,* $\left(r^{-1}\right)^* r^* = \left(r r^{-1}\right)^* = 1^* = 1$ *and* $r^* \left(r^{-1}\right)^* = \left(r^{-1} r\right)^* = 1^* = 1$.

A subset A of a $*$-ring R is said to be $*$-invariant provided that $A^* = A$. Note that the $*$-invariant subsets are precisely the universal subalgebras of $(R, \{{}^*\}, \varnothing)$. In this sense, the $*$-hull of a subset A, which is by definition $\operatorname{span}(A)$ in the previous universal algebra, is in fact $A \cup A^*$.

Definition 84 (Unitary element) *An invertible element u of a $*$-ring R is called unitary provided that $u^* = u^{-1}$. The subset of unitary elements is denoted by $\mathcal{U}_{\mathfrak{u}}(R)$.*

Note that $\mathcal{U}_{\mathfrak{u}}(R)$ is an $*$-invariant additively symmetric subgroup of $\mathcal{U}(R)$.

Definition 85 (Extended ring) *An extended ring is a universal algebra of the form $(R, \{+, 0, -, \cdot, 1, \circ, I\}, \varnothing)$, where $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$ is a ring and $(R, \{\circ, I\}, \varnothing)$ is a monoid such that the composition operation $\circ : (R, +) \times (R, +) \to (R, +)$ is left distributive, that is, $(r + s) \circ t = r \circ t + s \circ t$ for all $r, s, t \in R$.*

In an extended ring, the inverse for multiplication is usually denoted by $1/r$, and the inverse for composition is often denoted as r^{-1}.

Remark 31 *Let R be an extended ring. Then:*

1. $0 \circ r = 0$ *for every $r \in R$. Indeed,* $0 \circ r = (0 + 0) \circ r = 0 \circ r + 0 \circ r$, *so $0 \circ r$ is an idempotent of the additive group of R. Thus,* $0 \circ r = 0$.
2. $(-r) \circ s = -(r \circ s)$ *for all $r, s \in R$. Indeed,* $(-r) \circ s + r \circ s = (-r + r) \circ s = 0 \circ s = 0$, *therefore,* $(-r) \circ s = -(r \circ s)$.

1.2.3.2 Ordered rings

Ordered rings are just rings endowed with a partial order compatible with the ring operations.

Remark 32 *Let R be a ring endowed with a partial order $\leq$ compatible with the addition, that is, translation-invariant in the sense that for all $r, s \in R$ with $r \leq s$, $r+t \leq s+t$ for all $t \in R$. Let $R^+ := \uparrow 0 = \{t \in R : t \geq 0\}$. The following left- and right-homothety invariances are equivalent:*

- *For all $r, s \in R$ with $r \leq s$, we have that $tr \leq ts$ for all $t \in R^+$.*
- *For all $r, s \in R$ with $r \leq s$, we have that $rt \leq st$ for all $t \in R^+$.*
- *For all $r, s \in R^+$, $rs \in R^+$.*

The above remark has strong consequences in the definition of an ordered ring since left and right homothety invariances are equivalent.

Definition 86 (Ordered ring) *An ordered ring is a ring R endowed with a partial order $\leq$ compatible with the addition and multiplication operations, that is, for all $r, s \in R$ such that $r \leq s$, then*

- *Translation invariance: $r + t \leq s + t$ for all $t \in R$.*
- *Homothety invariance: $rs \in R^+$ for all $r, s \in R^+$.*

Under the settings of the previous definition, R^+ is a multiplicative subsemigroup of R and $R^- := \downarrow 0 = \{a \in R : a \leq 0\}$. Some basic, and easily verifiable, properties of ordered rings are displayed in the next proposition.

Proposition 28 *Let R be an ordered ring. For all $r, s, t, u \in R$:*

1. *If $r < s$ and $t \leq u$, then $r + t < s + u$.*
2. *If $r < s$ and $t > 0$ is not a left zero-divisor, then $tr < ts$.*
3. *If $r < s$ and $t > 0$ is not a right zero-divisor, then $rt < st$.*
4. *If $r < s$, then $s-r > 0$ and $-s < -r$. In particular, $R^- = -R^+$. Also, $-(\downarrow r) = \uparrow(-r)$, $-(\uparrow r) = \downarrow(-r)$, $\downarrow r = r + R^-$, and $\uparrow r = r + R^+$.*
5. *If $r, 0$ are comparable, then $r^2 \geq 0$.*
6. *If $r, s > 0$ and $rs = 0$, then $[0, r][0, s] = \{0\}$.*
7. *R is totally ordered if and only if $R = R^+ \cup R^-$.*
8. *$R^+ + R^-$ is a subring of R.*
9. *If the ring ordering of R is not trivial, then $\mathrm{char}(R) = 0$.*

Proof 54 *The proof is itemized according to the statement.*

1. *By the translation invariance, we know that $r + t \leq s + t \leq s + u$. If $r + t = s + u$, then $r + t = s + t$, thus, we obtain the contradiction that $r = s$. Hence $r + t < s + u$.*
2. *By the left-homothety invariance, $tr \leq ts$. Suppose on the contrary that $tr = ts$. Then $t(r-s) = 0$, meaning the contradiction that $t \in \mathtt{ld}(0)$ in view of the fact that $r-s \neq 0$.*

3. *By the right-homothety invariance,* $rt \leq st$. *Suppose on the contrary that* $rt = st$. *Then* $(r-s)t = 0$, *meaning the contradiction that* $t \in \mathtt{rd}(0)$ *in view of the fact that* $r - s \neq 0$.

4. *By the translation invariance,* $0 = r - r < s - r$. *If we apply again the translation invariance,* $-s = -s + 0 < -s + (s - r) = -r$.

5. *Suppose first that* $r \geq 0$. *By the left-homothety invariance,* $0 = r0 \leq rr = r^2$. *Suppose next that* $r \leq 0$. *By Proposition 28(4),* $0 \leq -r$. *Then* $0 \leq (-r)^2 = r^2$.

6. *If* $0 \leq a \leq r$ *and* $0 \leq b \leq s$, *then* $0 \leq ab \leq rs = 0$.

7. *If* R *is totally ordered, then every* $r \in R$ *is comparable to* 0, *so either* $r \geq 0$ *or* $r \leq 0$, *hence* $r \in R^+ \cup R^-$. *Conversely, suppose that* $R^+ \cup R^- = R$ *and consider any* $r, s \in R$. *Then* $r - s \in R = R^+ \cup R^-$, *that is, either* $r - s \geq 0$ *or* $r - s \leq 0$, *resulting in either* $r \geq s$ *or* $r \leq s$.

8. *Simply observe that if* $a_1, a_2, b_1, b_2, r_1, r_2, s_1, s_2 \in R^+$, *then* $(r_1 - r_2)(a_1 - a_2) + (s_1 - s_2)(b_1 - b_2) = (r_1a_1 + r_2a_2 + s_1b_1 + s_2b_2) - (r_1a_2 + r_2a_1 + s_1b_2 + s_2b_1) \in R^+ + R^-$.

9. *Suppose on the contrary that* $\mathrm{char}(R) = p > 1$. *If the ring ordering of* R *is not trivial, then there exists* $r > 0$. *Then* $0 = 0 + \overset{n}{\cdots} + 0 < r + \overset{p}{\cdots} + r = \left(1 + \overset{p}{\cdots} + 1\right) r = 0$, *which is a contradiction.*

Under the settings of Proposition 28, if $r < s$ and $t < u$, then $(r, s) + (t, u) \subseteq (r+t, s+u)$, and if $0 \leq r < s$ and $0 \leq t < u$, then $(r, s)(t, u) \subseteq [rt, su]$.

Example 20 *Let* R, S *be totally ordered rings. Then* $T := R \times S$ *satisfies that* $T = T^+ + T^-$ *when endowed with the product order. Indeed, take any* $(r, s) \in T$. *By assumption, either* $r \geq 0$ *or* $r \leq 0$ *and the same happens to* s. *Then* $(r, s) = (r, 0) + (0, s) \in T^+ + T^-$.

Ring orderings for which $0, 1$ are comparable are called unital ring orderings. Notice that if R is a unital ordered ring, then $R^+ + R^-$ is a unitary subring of R.

Proposition 29 *Let* R *be a unital ordered ring. For all* $r, s, t, u \in R$:

1. $0 < 1$ *and* $\mathrm{char}(R) = 0$.

2. $(0, 1)(0, 1) \subseteq [0, 1)$.

3. *If* $r \in \mathcal{U}(R)$, $r > 0$, *and* $r^{-1}, 0$ *are comparable, then* $r^{-1} > 0$.

4. *If* $r, s \in \mathcal{U}(R)$, $0 < r < s$, *and* $r^{-1}, 0$ *and* $s^{-1}, 0$ *are comparable, then* $0 < s^{-1} < r^{-1}$.

5. *If* $r, s \geq 0$, *then* $[-r, r][-s, s] \subseteq [-3rs, 3rs]$.

Proof 55 *Let us itemized the proof following the statement of the proposition.*

1. *Suppose on the contrary that* $1 < 0$. *By Proposition 28(4),* $0 < -1$. *By the left-homothety invariance,* $-1 = (-1)1 < (-1)0 = 0$, *obtaining a contradiction. Next, suppose on the contrary that there exists* $n \in \mathbb{N}$ *for which* $1 + \overset{n}{\cdots} + 1 = 0$. *By Proposition 28(1),* $0 = 0 + \overset{n}{\cdots} + 0 < 1 + \overset{n}{\cdots} + 1 = 0$, *which is a contradiction.*

2. *Fix arbitrary elements* $r, s \in (0, 1)$. *We definition of ring ordering,* $rs \geq 0$. *Finally,* $r < 1$ *and* $s > 0$ *meaning that* $rs \leq s < 1$.

3. *Suppose on the contrary that* $r^{-1} < 0$. *By the left-homothety invariance,* $1 = rr^{-1} < r0 = 0$, *contradicting our hypothesis.*

4. *By Proposition 29(3),* $r^{-1} > 0$ *and* $s^{-1} > 0$. *By the left-homothety invariance,* $0 = r^{-1}0 < r^{-1}r < r^{-1}s$. *The right-homothety invariance assures that* $0 = 0s^{-1} < 1s^{-1} < r^{-1}ss^{-1} = r^{-1}$.

5. *Take arbitrary elements* $a, b \in R$ *with* $-r \leq a \leq r$ *and* $-s \leq b \leq s$. *Note that* $r - a \geq 0$ *and* $s - b \geq 0$, *meaning that* $rs - rb - as + ab = (r - a)(s - b) \geq 0$. *Then* $ab \geq -rs + rb + as$. *Since* $b \geq -s$ *and* $r \geq 0$, *we have that* $rb \geq -rs$. *Also,* $-r \leq a$ *and* $s \geq 0$, *so* $-rs \leq as$. *As a consequence,* $ab \geq -rs + rb + as \geq -rs - rs - rs$. *By repeating the same reasoning for* a *and* $-b$, *we conclude that* $-ab = a(-b) \geq -rs - rs - rs$. *This shows that* $[-r, r][-s, s] \subseteq [-3rs, 3rs]$.

In a unital ordered ring R, the positive cone R^+ is a multiplicative submonoid of R. On the other hand, there are other examples of ordered rings for which $0, 1$ are not comparable than the rings of characteristic different from 0. Indeed, the trivial partial order $r \leq s \Leftrightarrow r = s$ is clearly a ring order for which $0, 1$ are not comparable (unless $1 = 0$).

Proposition 30 *Let* R *be a ring. If* $\leq$ *is a ring order on* R, *then the positive cone* R^+ *satisfies the following properties:*

1. $0 \in R^+$.

2. $R^+ \cap R^- = \{0\}$.

3. $R^+ + R^+ \subseteq R^+$.

4. $R^+R^+ \subseteq R^+$.

Conversely, if $S \subseteq R$ *is a subset of* R *satisfying the four conditions above, then there exists a unique ring order on* R, *given by* $r \leq s \Leftrightarrow s - r \in S$ *for all* $r, s \in R$, *for which* $S = R^+$.

Proof 56 *If* $\leq$ *is a ring order on* R, *then the positive cone* R^+ *trivially satisfies all four conditions above in view of Proposition 28. Conversely, let* $S \subseteq R$ *satisfying all four conditions above. We will check first that* $r \leq s \Leftrightarrow s - r \in S$ *is a partial order in* R. *Indeed, fix arbitrary elements* $r, s, t \in R$:

- *Reflexivity:* $r - r = 0 \in S \Rightarrow r \leq r$.

- *Antisymmetricity: If* $r \leq s$ *and* $s \leq r$, *then* $s - r \in S$ *and* $r - s \in S$, *so* $s - r \in S \cap (-S) = \{0\}$, *meaning that* $s = r$.

- *Transitivity: If* $r \leq s$ *and* $s \leq t$, *then* $s - r \in S$ *and* $t - s \in S$, *so* $t - r = (t - s) + (s - r) \in S + S \subseteq S$, *meaning that* $r \leq t$.

Next, let us check that this is a ring order. Indeed, fix arbitrary elements $r, s, t \in R$:

- *Translation invariance: If* $r \leq s$, *then* $s - r \in S$, *so* $(s + t) - (r + t) = s - r \in S$, *meaning that* $r + t \leq s + t$.

- *Homothety invariance: If* $r, s \geq 0$, *then* $r, s \in S$, *so* $rs \in SS \subseteq S$, *meaning that* $0 \leq rs$.

It is immediate to check that $S = R^+$. *Finally, if* $\preceq$ *is another ring order whose positive cone is* S, *then* $r \leq s \Leftrightarrow s - r \in S \Leftrightarrow r \preceq s$ *for all* $r, s \in R$, *reaching the conclusion that* $\leq$ *and* $\preceq$ *are in fact the same ordering.*

Proposition 30 motivates the following definition.

Definition 87 (Ordering set) *Let R be a ring. A subset $S \subseteq R$ satisfying all four conditions of Proposition 30 will be called an ordering set. If $\mathrm{char}(R) = 0$ and $1 \in S$, then S is called a unital ordering set.*

Note that the intersection of any family of (unital) ordering subsets is again a (unital) ordering subset, and the union of any increasing family of (unital) ordering subsets is again a (unital) ordering subset. By means of Zorn's Lemma, we will demonstrate the existence of ring orderings.

Lemma 14 *Let R be a ring. There exists a maximal ordering subset $S \subseteq R$. If, in addition, $\mathrm{char}(R) = 0$, then there exists a maximal unital ordering subset.*

Proof 57 *Let $\mathcal{L} := \{T \subseteq R : T \text{ is an ordering subset}\}$. Notice that $\mathcal{L} \neq \varnothing$ since $\{0\} \in \mathcal{L}$. If $\mathcal{C} \subseteq \mathcal{L}$ is a chain of $\mathcal{L}$, then $\bigcup \mathcal{C} \in \mathcal{L}$. Finally, Zorn's Lemma guarantees the existence of maximal elements in $\mathcal{L}$. If $\mathrm{char}(R) = 0$, then we can consider $\mathcal{L}_1 := \{T \in \mathcal{L} : 1 \in T\}$. Notice that $\mathcal{L}_1 \neq \varnothing$ since $\{0\} \cup \{1+\overset{n}{\cdots}+1 : n \in \mathbb{N}\} \in \mathcal{L}_1$. Also, any chain $\mathcal{C} \subseteq \mathcal{L}_1$ satisfies that $\bigcup \mathcal{C} \in \mathcal{L}_1$. Zorn's Lemma guarantees again the existence of maximal elements in $\mathcal{L}_1$.*

Notice that $\{0\}$ is the smallest ordering subset and $\{0, 1, 1+1, 1+1+1, \dots\}$ is the smallest unital ordering subset (in characteristic 0). A special kind of ordered rings will serve in further chapters, among other things, to construct regular measures.

Lemma 15 *Let R be a (unital) ordered ring. The following conditions are equivalent:*

1. *R^+ is totally ordered.*
2. *$R^+ \cup R^-$ is a (unitary) subring of R.*
3. *$R^+ + R^- = R^+ \cup R^-$.*
4. *$R^+ \cup R^-$ is totally ordered.*

Proof 58 *Suppose that R^+ is totally ordered. Since $R^+R^+ \subseteq R^+$ and $R^- = -R^+$, we have that $(R^+ \cup R^-)(R^+ \cup R^-) \subseteq (R^+ \cup R^-)$ and $-(R^+ \cup R^-) = R^+ \cup R^-$. So, it only remains to show that $(R^+ \cup R^-) + (R^+ \cup R^-) \subseteq (R^+ \cup R^-)$. Since $R^+ + R^+ \subseteq R^+$, it only suffices to show that $R^+ + R^- \subseteq R^+ \cup R^-$. Fix arbitrary elements $r, s \in R^+$. Since R^+ is totally ordered, we may assume that $r \leq s$, meaning that $r - s \leq 0$, that is, $r - s \in R^-$. Conversely, assume that $R^+ + R^- \subseteq R^+ \cup R^-$. Fix arbitrary elements $r, s \in R^+ \cup R^-$. Then $r - s \in R^+ \cup R^-$, meaning that either $r \geq s$ or $r \leq s$, resulting in $R^+ \cup R^-$ being totally ordered.*

The previous lemma motivates the definition of faithful ordered ring.

Definition 88 (Faithful ordered ring) *Let R be a ring. A ring order on R is called faithful provided that any of the equivalent conditions of Remark 15 is satisfied.*

By means of Zorn's Lemma, maximal faithful ring orderings can be constructed. An ordering subset inducing a faithful ring ordering will be called faithful ordering subset.

Lemma 16 *Let R be a ring. If $\mathrm{char}(R) = 0$, then there exists a maximal unital and faithful ordering subset.*

Proof 59 *Let* $\mathcal{L} := \{T \subseteq R : T \text{ is an ordering subset}\}$ *and* $\mathcal{L}_{\mathtt{F}} := \{T \in \mathcal{L} : 1 \in T \text{ and } T - T \subseteq T \cup -T\}$. *Notice that* $\mathcal{L}_{\mathtt{F}} \neq \varnothing$ *since* $\{0\} \cup \{1+ \overset{n}{\cdots} +1 : n \in \mathbb{N}\} \in \mathcal{L}_{\mathtt{F}}$. *Also, any chain* $\mathcal{C} \subseteq \mathcal{L}_{\mathtt{F}}$ *satisfies that* $\bigcup \mathcal{C} \in \mathcal{L}_{\mathtt{F}}$. *Zorn's Lemma guarantees again the existence of maximal elements in* $\mathcal{L}_{\mathtt{F}}$.

Observe that $\{0\} \cup \{1+ \overset{n}{\cdots} +1 : n \in \mathbb{N}\}$ is the smallest faithful unital ordering subset (in characteristic 0).

Definition 89 (Archimedean ring) *An ordered ring* R *is said to verify the Archimedean property provided that for any positive elements* r, s *in* R, *there exists* $n \in \mathbb{N}$ *such that* $nr > s$.

1.2.3.3 Unit segments

We will rely on this new concept of unit segment [9, Definition 3.2] to define convexity on modules.

Definition 90 (Unit segment) *Let* R *be a ring. A subset* $B^+ \subseteq R$ *is called a unit segment if it satisfies:*

- $0, 1 \in B^+$.
- $B^+ \cap B^- = \{0\}$.
- $1 + B^- \subseteq B^+$.
- $B^+B^+ \subseteq B^+$.

Here, $B^- := -B^+$ *is called negative unit segment.*

Under the settings of the previous definition, it actually holds that $B^+B^+ = B^+$ and $1 + B^- = B^+$.

Example 21 *In* $\mathbb{R}$, *both* $[0, 1]$ *and* $\mathbb{Q} \cap [0, 1]$ *are unit segments.*

The intersection of any family of unit segments is again a unit segment, and the union of any increasing family of unit segments is also a unit segment. The existence of unit segments is directly linked to the characteristic of the ring via Zorn's Lemma.

Lemma 17 *If* R *is a ring, then there exists a unit segment of* R *if and only if* $\operatorname{char}(R) \neq 2$. *In this situation, it can be found a maximal unit segment of* R.

Proof 60 *Suppose that there exists a unit segment* $B^+ \subseteq R$. *If* $\operatorname{char}(R) = 2$, *then* $1+1 = 0$, *so* $1 = -1 \in -B^+ = B^-$ *reaching the contradiction that* $1 \in B^+ \cap B^- = \{0\}$. *Conversely, suppose that* $\operatorname{char}(R) \neq 2$. *Consider the set* $\mathcal{L} := \{A \subseteq R : A \text{ is a unit segment}\}$ *partially ordered by the inclusion. Notice that* $\mathcal{L}$ *is not empty because* $\{0, 1\} \in \mathcal{L}$ *due to the fact that* $\operatorname{char}(R) \neq 2$. *Note also that, if* $\mathcal{C}$ *is a chain of* $\mathcal{L}$, *then it is easy to verify that* $\bigcup \mathcal{C} \in \mathcal{L}$. *By Zorn's Lemma, there exists a maximal element.*

Note that $\{0, 1\}$ is the smallest unit segment (in characteristic different from 2). Basic properties of unit segments are presented in the next lemma.

Lemma 18 *Let* R *be a ring. Let* $B^+ \subseteq R$ *be a unit segment. Then:*

1. $(1 + B^+) \cap B^+ = \{1\}$.

2. *If* $r_1, \ldots, r_k \in B^+$ *with* $r_1 + \cdots + r_k = 1$, *then* $r_1 + \cdots + r_{k-1} \in B^+$.

3. *If* $a, b \in B^+$ *and* $ab = 0$, *then* $a + b \in B^+$.

Proof 61 *The proof will be as itemized as the statement is.*

1. *If* $a, b \in B^+$ *satisfy that* $1 + a = b$, *then* $1 - b = -a \in B^+ \cap B^- = \{0\}$.

2. *Simply notice that* $r_1 + \cdots + r_{k-1} = 1 - r_k \in 1 + B^- = B^+$.

3. *Note that* $1 - a, 1 - b \in B^+$, *therefore* $1 - (a + b) = 1 - (a + b) + ab = (1 - a)(1 - b) \in B^+B^+ = B^+$, *hence* $a + b = 1 - [1 - (a + b)] \in B^+$.

Lemma 18(2) motivates the following definition.

Definition 91 (Full unit segment) *Let* R *be a ring. A unit segment* $B^+ \subseteq R$ *is called full if for every* $r_1, \ldots, r_k \in B^+$ *with* $r_1 + \cdots + r_k \in B^+$, *then* $r_1 + \cdots + r_{k-1} \in B^+$.

Basic properties of full unit segments are displayed in the next technical lemma.

Lemma 19 *Let* R *be a ring. Let* $B^+ \subseteq R$ *be a full unit segment. Then:*

1. *If* $r_1, \ldots, r_k \in B^+$ *with* $r_1 + \cdots + r_k \in B^+$, *then* $\sum_{j \in J} r_j \in B^+$ *for every* $J \subseteq \{1, \ldots, k\}$. *If, in addition,* $r_1 = 1$, *then* $r_2 = \cdots = r_k = 0$.

2. *If* $r_1, \ldots, r_k \in B^+ \setminus \{0\}$ *with* $r_1 + \cdots + r_k \in B^+$, *then* $r_1 + \cdots + r_k \in B^+ \setminus \{0\}$.

3. $\mathrm{char}(R) = 0$.

Proof 62 *By recursively applying the definition of full unit segment, one can see that if* $r_1, \ldots, r_k \in B^+$ *with* $r_1 + \cdots + r_k \in B^+$, *then* $\sum_{j \in J} r_j \in B^+$ *for every* $J \subseteq \{1, \ldots, k\}$. *Next, if* $r_1 = 1$, *then* $1 - (r_1 + r_j) = -r_j \in B^+ \cap B^- = \{0\}$ *for all* $j = 2, \ldots, k$. *Suppose next that* $r_i \neq 0$ *for all* $i \in \{1, \ldots, k\}$. *If* $r_1 + \cdots + r_k = 0$, *then* $r_1 + \cdots + r_{k-1} = -r_k \in B^+ \cap B^- = \{0\}$, *which contradicts that* $r_k \neq 0$. *Finally, let us prove that* $\mathrm{char}(R) = 0$. *Suppose that there exists* $n \in \mathbb{N}$ *such that* $1 + \overset{n}{\cdots} + 1 = 0$. *Notice that* $n > 1$ *since, by default, we are assuming that* $R \neq \{0\}$. *On the one hand,* $1 + \overset{n-1}{\cdots} + 1 = -1$. *On the other hand,* $1 + \overset{n}{\cdots} + 1 = 0 \in B^+$ *so* $-1 = 1 + \overset{n-1}{\cdots} + 1 \in B^+$, *that is,* $-1 \in B^- \cap B^+ = \{0\}$, *reaching the contradiction that* $-1 = 0$.

The intersection of any family of full unit segments is again a full unit segment, and the union of any increasing family of full unit segments is also a full unit segment. Zorn's Lemma assures the existence of maximal full unit segments in rings of characteristic 0.

Lemma 20 *If* R *is a ring with* $\mathrm{char}(R) = 0$, *then there exists a maximal full unit segment of* R.

Proof 63 *Consider the set* $\mathcal{L} := \{A \subseteq R : A \text{ is a full unit segment}\}$ *partially ordered by the inclusion. Notice that* $\mathcal{L}$ *is not empty because* $\{0, 1\} \in \mathcal{L}$ *due to the fact that* $\mathrm{char}(R) = 0$. *Note also that, if* $\mathcal{C}$ *is a chain of* $\mathcal{L}$, *then it is easy to verify that* $\bigcup \mathcal{C} \in \mathcal{L}$. *By the Zorn's Lemma, there exists a maximal element.*

Note that $\{0, 1\}$ is the smallest full unit segment (in characteristic 0). Full unit segments endow rings with a unital ring ordering.

Theorem 14 *Let R be a ring. Let $B^+ \subseteq R$ be a full unit segment. The set $S := \bigcup_{n\in\mathbb{N}} B^+ + \overset{n}{\cdots} + B^+$ is a unital ordering subset; hence, it endows R with a unique ring ordering whose positive cone is S and for which $0, 1$ are comparable. Even more, this ring ordering satisfies that $B^+ = [0, 1]$.*

Proof 64 *Let us verify all four conditions of Proposition 30 one by one:*

1. *$0 \in S$. Indeed, $0 \in B^+ \subseteq S$.*
2. *$S \cap (-S) = \{0\}$. Indeed, take $b_1, \ldots, b_n, c_1, \ldots, c_m \in B^+ \setminus \{0\}$ such that $b_1 + \cdots + b_m = -c_1 - \cdots - c_m$. Then $b_1 + \cdots + b_m + c_1 + \cdots + c_m = 0$, contradicting Lemma 19(2).*
3. *$S + S \subseteq S$. This is automatically satisfied by construction of S.*
4. *$SS \subseteq S$. This is automatically satisfied since $B^+B^+ = B^+$.*

Finally, let us prove that $B^+ = [0, 1]$. Indeed, if $r \in B^+$, then $r \in S$, so $r \geq 0$. Also, $1 - r \in 1 + B^- = B^+ \subseteq S$, hence $1 - r \geq 0$, meaning that $r \leq 1$. As a consequence, $B^+ \subseteq [0, 1]$. Conversely, take $r \in [0, 1]$. Then $r \geq 0$ so $r \in S$. Also, $r \leq 1$, hence $1 - r \in S$. We can then find $a_1, \ldots, a_n \in B^+$ and $b_1, \ldots, b_m \in B^+$ such that $r = a_1 + \cdots + a_n$ and $1 - r = b_1 + \cdots + b_m$. Therefore, $a_1 + \cdots + a_n + b_1 + \cdots + b_m = 1 \in B^+$. By Lemma 19(1), $r = a_1 + \cdots + a_n \in B^+$.

In Theorem 14, the fullness of the unit segment has only been used to show the second condition of Proposition 30. From now on, whenever we have a full unit segment, we will implicitly consider its induced unital ring ordering.

Lemma 21 *Let R be a ring. Let $B^+ \subseteq R$ be a full unit segment. If $a, b \in B^+$ and $a \leq b$, then $b - a \in B^+$.*

Proof 65 *According to Theorem 14, $b - a \in \bigcup_{n\in\mathbb{N}} B^+ + \overset{n}{\cdots} + B^+$, so take $b_1, \ldots, b_n \in B^+$ such that $a + b_1 + \cdots + b_n = b \in B^+$. By Lemma 19(1), $b_1 + \cdots + b_n \in B^+$.*

According to [9, Theorem 3.3], if R is a totally ordered ring, then $B^+ := [0, 1]$ is a unit segment of R. The next result is an improvement of [9, Theorem 3.3] by showing that if R is a partially ordered ring with $0, 1$ comparable, then $B^+ := [0, 1]$ is a full unit segment of R.

Theorem 15 *If R is an ordered ring such that $0, 1$ are comparable, then $B^+ := [0, 1]$ is a full unit segment of R.*

Proof 66 *Let us first check that all four properties of Definition 90 are satisfied:*

1. *By choice of B^+, we clearly have that $0, 1 \in B^+$.*
2. *It suffices to show that $B^+B^+ \subseteq B^+$. Indeed, if $r, s \in B^+$, then $0 \leq r \leq 1$ and $0 \leq s \leq 1$, therefore $0 \leq rs \leq 1$. Thus, $rs \in B^+$.*
3. *$B^+ \cap B^- = \{0\}$. Notice that if $r, s \in [0, 1]$ and $r = -s$, then $-1 \leq -s \leq 0$, so we obtain that $r = -s \leq 0 \leq r$. In other words, $r = 0$.*
4. *$1 + B^- \subseteq B^+$. Indeed, if $r \in [0, 1]$, then $-1 \leq -r \leq 0$, hence $0 \leq 1 - r \leq 1$, that is, $1 - r \in [0, 1] = B^+$.*

Finally, let us prove that $B^+ := [0,1]$ *is full. Indeed, let* $r_1, \ldots, r_k \in B^+$ *with* $r_1 + \cdots + r_k \in B^+$. *Since* $0 \leq r_i \leq 1$ *for all* $i \in \{1, \ldots, k\}$, *we obtain that* $0 \leq r_1 + \cdots + r_{k-1}$. *Next,* $r_1 + \cdots + r_k \leq 1$, *so* $r_1 + \cdots + r_{k-1} \leq 1 - r_k \leq 1$ *since* $1 - r_k \in 1 - B^+ = B^+$. *As a consequence,* $r_1 + \cdots + r_{k-1} \in B^+$.

In order to define convexity on modules, it only suffices to rely on unit segments. However, in order to achieve certain geometrical properties, we will have to consider special cases of unit segments.

Definition 92 *Let* R *be a ring. Let* $B^+ \subseteq R$ *be a unit segment. Then* B^+ *is called:*

- *Completely full if* B^+ *is full,* R^+ *is multiplicatively symmetric, and* $B^+ \setminus \{0\} \subseteq \mathcal{U}(R)$.
- *Totally full if* B^+ *is full and totally ordered.*
- *Strong if for every* $r \in B^+$, *either* $2r \in B^+$ *or* $2(1-r) \in B^+$.
- *Commuting if* $rB^+ = B^+r$ *for all* $r \in R$.
- *Selfcommuting if* $B^+ \subseteq (B^+)'$.
- *Centered if* $2 \in \mathcal{U}(R)$ *and* $2^{-1} \in B^+$.

Recall that a subset A of a multiplicative monoid G is said to be multiplicatively symmetric provided that $(A \cap G^\times)^{-1} \subseteq A$.

Theorem 16 *Let* R *be a ring. Let* $B^+ \subseteq R$ *be a completely full unit segment. If* $a, b \in R$ *satisfy that* $0 < a \leq b$, *then* $b^{-1}a, ab^{-1} \in B^+$.

Proof 67 *By hypothesis,* $(R^+ \cap \mathcal{U}(R))^{-1} \subseteq R^+$, *thus* $b^{-1} > 0$, *hence* $0 = b^{-1}0 < b^{-1}a \leq b^{-1}b = 1$. *As a consequence,* $b^{-1}a \in [0,1]$. *By relying on Theorem 14, we conclude that* $b^{-1}a \in [0,1] = B^+$. *On the other hand,* $b^{-1}a \leq 1$ *and* $a > 0$ *so* $ab^{-1}a \leq a$, *hence* $ab^{-1}aa^{-1} \leq 1$ *because* $a^{-1} > 0$, *meaning that* $ab^{-1} \leq 1$. *Finally, it is clear that* $ab^{-1} \geq 0$, *so it only suffices to apply again Theorem 14 to conclude that* $ab^{-1} \in [0,1] = B^+$.

As an immediate corollary, we obtain that centered, completely and totally full unit segments are strong.

Corollary 3 *Let* R *be a ring such that* $2 \in \mathcal{U}(R)$. *Let* $B^+ \subseteq R$ *be a unit segment. If* B^+ *is centered as well as completely and totally full, then* B^+ *is strong.*

Proof 68 *Fix an arbitrary* $r \in B^+$. *Notice that* $2^{-1}, r, 1-r$ *are all comparable because* B^+ *is totally ordered. If* $2^{-1} < r$ *and* $2^{-1} < 1 - r$, *then* $1 = 2^{-1} + 2^{-1} < r + (1-r) = 1$, *which is a contradiction. As a consequence, either* $r \leq 2^{-1}$ *or* $1 - r \leq 2^{-1}$. *In accordance with Theorem 16, either* $2r \in B^+$ *or* $2(1-r) \in B^+$.

The notion of totally full unit segment is directly linked to the concept of faithfullness for unital ordering sets.

Theorem 17 *Let* R *be a ring. If* $B^+ \subseteq R$ *is a totally full unit segment, then the unital ring ordering induced by* B^+ *on* R *is faithful. Conversely, if* R *is endowed with a faithful unital ring ordering, then* $B^+ := [0,1]$ *is a totally full unit segment.*

Proof 69 *If R is endowed with a faithful unital ring ordering, then R^+ is totally ordered, so in particular, $B^+ := [0,1] \subseteq R^+$ is totally ordered, hence a totally full unit segment. Conversely, suppose that $B^+ \subseteq R$ is a totally full unit segment. The unital ring ordering induced by B^+ on R is given by the unital ordering subset $S := \bigcup_{n\in\mathbb{N}} B^+ + \overset{n}{\cdots} + B^+$ in view of Theorem 14. We will prove that S is faithful by showing that $S - S \subseteq S \cup -S$ (see Remark 15). Indeed, let $s_1, s_2 \in S$. We can find $a_1, \ldots, a_n, b_1, \ldots, b_m \in B^+$ such that $s_1 = a_1 + \cdots + a_n$ and $s_2 = b_1 + \cdots + b_m$. Since $0 \in B^+$, by adding zeros if necessary, we may assume that $m = n$. Then $s_1 - s_2 = (a_1 - b_1) + \cdots + (a_n - b_n)$. Next, for all $i \in \{1, \ldots, n\}$, either $a_i \leq b_i$ or $b_i \leq a_i$, because by hypothesis B^+ is totally full. As a consequence and by relying on Lemma 21, for all $i \in \{1, \ldots, n\}$, either $b_i - a_i \in B^+$ or $a_i - b_i \in B^+$. If $b_i - a_i \in B^+$ for all $i \in \{1, \ldots, n\}$, then $s_1 - s_2 \in B^- + \overset{n}{\cdots} + B^- \subseteq -S$. If $a_i - b_i \in B^+$ for all $i \in \{1, \ldots, n\}$, then $s_1 - s_2 \in B^+ \overset{n}{\cdots} + B^+ \subseteq S$. Otherwise, again we have $s_1 - s_2 = c_1 + \cdots + c_k - d_1 - \cdots - d_l$, where $c_1, \ldots, c_k, d_1, \ldots, d_l \in B^+$ and $k + l < n$. By repeating this process as finitely many times as necessary, we end up having that either $s_1 - s_2 \in S$ or $s_1 - s_2 \in -S$, obtaining the desired result.*

The final lemma in this subsubsection shows the existence of maximal unit segments via Zorn's Lemma. Keep in mind that the intersection of any family of completely full/totally full/strong/communting unit segments is again a completely full/totally full/strong/commuting unit segment, respectively, and the union of any increasing family of completely full/totally full/strong/commuting unit segments is also a completely full/totally full/strong/commuting unit segment, respectively.

Lemma 22 *Let R be a ring with $\operatorname{char}(R) = 0$. There exists a maximal completely full unit segment, a maximal totally full unit segment, a maximal strong unit segment, and a maximal commuting unit segment. There also exists a maximal completely, totally, strong, and commuting full unit segment.*

Proof 70 *Consider the sets*

$$\begin{aligned}
\mathcal{L}_c &:= \{A \subseteq R : A \text{ is a completely full unit segment of } R\},\\
\mathcal{L}_t &:= \{A \subseteq R : A \text{ is a totally full unit segment of } R\},\\
\mathcal{L}_s &:= \{A \subseteq R : A \text{ is a strong unit segment of } R\},\\
\mathcal{L}_{co} &:= \{A \subseteq R : A \text{ is a commuting unit segment of } R\},
\end{aligned}$$

and $\mathcal{L} := \mathcal{L}_c \cap \mathcal{L}_t \cap \mathcal{L}_s \cap \mathcal{L}_{co}$, partially ordered by inclusion. Notice that $\mathcal{L}$ is not empty because $\{0,1\} \in \mathcal{L}$. If $\mathcal{C}$ is a chain in any of the previous sets, then it is easy to verify that $\bigcup \mathcal{C}$ also lies in that set. By Zorn's Lemma, there exists a maximal element.

1.2.3.4 Factorization

Any invertible element u in a commutative ring R satisfies that if $u = vw$ with $v, w \in R$, then $v, w \in \mathcal{U}(R)$. If, in addition, R is an integral domain, then 0 satisfies that, if $0 = ab$ with $a, b \in R$, then either $a = 0$ or $b = 0$. These two facts motivate the next definitions.

Definition 93 (Equivalent elements) *Let R be a commutative ring. Two elements $a, b \in R$ are said to be equivalent if there exists $u \in R^\times$ such that $a = ub$.*

For noncommutative rings, two elements r, s are said to be equivalent if there are invertibles $u, v \in R^\times$ such that $a = ubv$. The equivalence of elements is an equivalence relation.

Definition 94 (Irreducible element) *Let R be a commutative ring. An element $r \in R \setminus (\mathcal{U}(R) \cup \{0\})$ is called irreducible provided that whenever $r = ab$ with $a, b \in R$, then either a or b is equivalent to r.*

Under the settings of the previous definition, if R is an integral domain, then r is irreducible if and only if whenever $r = ab$ with $a, b \in R$, then either $a \in \mathcal{U}(R)$ or $b \in \mathcal{U}(R)$. Irreducible elements play the role of prime numbers in the ring of integers.

Lemma 23 *Let R be an integral domain, commutative ring. Two elements $r, s \in R$ are equivalent if and only if r, s generate the same ideal, that is,* $\text{span}(r) = \text{span}(s)$.

Proof 71 *If r, s are equivalent, then there exists $u \in \mathcal{U}(R)$ with $r = us$, meaning that* $\text{span}(r) = \text{span}(s)$. *Conversely, assume that* $\text{span}(r) = \text{span}(s)$. *If one of r, s is 0, then so is the other. Thus, we may assume that both $r, s \neq 0$. There are $a, b \in R$ such that $r = as$ and $s = br$. Therefore, $r = as = (ab)r$, meaning that $ab = 1$ because $r \neq 0$ and R is an integral domain. As a consequence, $a, b \in \mathcal{U}(R)$ and r, s are equivalent.*

Notice that in a commutative ring R, $\text{span}(r) = rR$ for all $r \in R$. Next notion is the one of prime ideal.

Definition 95 (Prime ideal) *Let R be a commutative ring. A prime ideal P of R is an ideal satisfying that $P \neq R$ and that, if $r, s \in R$ with $rs \in P$, then either r or s belong to P.*

Notice that a commutative ring is an integral domain if and only if $\{0\}$ is a prime ideal.

Proposition 31 *Let R be an integral domain, commutative ring. Let $r \in R \setminus (\mathcal{U}(R) \cup \{0\})$. If the ideal spanned by r,* $\text{span}(r)$, *is a prime ideal or a maximal ideal, then r is irreducible.*

Proof 72 *Let us write $r = ab$ with $a, b \in R$.*

- *We will assume first that* $\text{span}(r)$ *is maximal. Since* $\text{span}(r) \subseteq \text{span}(b)$, *the maximality of* $\text{span}(r)$ *forces two situations. Either* $\text{span}(r) = \text{span}(b)$ *or* $\text{span}(b) = R$. *If* $\text{span}(r) = \text{span}(b)$, *then there exists $s \in R$ with $b = sr = sab$. Notice that $b \neq 0$ because $r \neq 0$. Then the equality $b = sab$ implies that $sa = 1$, so a is invertible. Finally, if* $\text{span}(b) = R$, *then b is invertible. Either way, r is irreducible.*
- *We will assume now that* $\text{span}(r)$ *is prime. Note that $ab \in \text{span}(r)$, so either $a \in \text{span}(r)$ or $b \in \text{span}(r)$. Suppose, for instance, that $a \in \text{span}(r)$. Then $a = cr$ for some $c \in R$, so $r = ab = (cb)r$, meaning that $cb = 1$, so $b \in \mathcal{U}(R)$.*

What follows is one of the most important notions in Commutative Algebra.

Definition 96 (Factorization domain) *Let R be an integral domain, commutative ring. Then R is said to be:*

- *A factorization domain if every $r \in R \setminus (\mathcal{U}(R) \cup \{0\})$ is equivalent to a product of irreducible elements.*
- *A unique factorization domain if every $r \in R \setminus (\mathcal{U}(R) \cup \{0\})$ is equivalent to a unique product of irreducible elements.*

Under the settings of Definition 96, by "$r \in R \setminus (\mathcal{U}(R) \cup \{0\})$ being equivalent to a unique product of irreducible elements" we mean that if $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_l\}$ are subsets of different irreducible elements of R in such a way that r is equivalent to $p_1^{n_1} \cdots p_k^{n_k}$ and to $q_1^{m_1} \cdots q_l^{m_l}$, then $k = l$ and the p_i's and q_i's can be reordered in such a way that $n_i = m_i$ and p_i is equivalent to q_i for all $i \in \{1, \ldots, k\}$.

Definition 97 (Principal ideal domain) *Let R be an integral domain, commutative ring. Then R is said to be a principal ideal domain if every ideal is principal.*

Next result characterizes irreducible elements in principal ideal domains following the philosophy established in Proposition 31.

Proposition 32 *Let R be a principal ideal domain. Let $r \in R \setminus (\mathcal{U}(R) \cup \{0\})$. The following conditions are equivalent:*

1. *r is irreducible.*
2. *$\text{span}(r)$ is a maximal ideal.*
3. *$\text{span}(r)$ is a prime ideal.*

Proof 73 *From Proposition 31 it only remains to show that if r is irreducible, then $\text{span}(r)$ is a maximal ideal. Indeed, in the first place, $\text{span}(r)$ is a proper ideal of R because r is neither 0 nor invertible by hypothesis. Let I be an ideal of R containing $\text{span}(r)$. Since R is a principal ideal domain, there exists $b \in R$ such that $I = \text{span}(b)$. This means that $r = ab$ for some $a \in R$. Since r is irreducible, we conclude that either a is invertible or b is invertible. If a is invertible, then $\text{span}(r) = \text{span}(b)$. If b is invertible, then $\text{span}(b) = R$. Either way, $\text{span}(r)$ is a maximal ideal.*

Next definition is the notion of Euclidean domain. Euclidean domains make possible the Division Algorithm.

Definition 98 (Euclidean domain) *Let R be an integral domain, commutative ring. Then R is said to be an Euclidean domain if there exists a Euclidean function, that is, a function $\phi : R \setminus \{0\} \to \mathbb{Z}^+$ satisfying the following conditions for all $a \in R, b \in R \setminus \{0\}$:*

- *Division algorithm: There exist $q, r \in R$ with $a = bq + r$ and either $r = 0$ or $\phi(r) < \phi(b)$.*
- *If $a \neq 0$, then $\phi(a) \leq \phi(ab)$.*

Classic examples of Euclidean functions are the absolute value and the polynomial degree. Minimal elements play a crucial role in Galois Theory.

Definition 99 (Minimal element) *Let R be an Euclidean domain. Let I be a proper ideal of R. Consider $n_I := \min\{n \in \mathbb{N} : \exists a \in I \setminus \{0\}\ \phi(a) = n\}$. A minimal element of I is an element of $a \in I \setminus \{0\}$ such that $\phi(a) = n_I$.*

Under the settings of the previous definition, if $a \in I \setminus \{0\}$ is a minimal element of I, then $\text{span}(a) = I$. Indeed, take any $x \in I$. By the division algorithm, there are $q, r \in R$ with $x = aq + r$ and either $r = 0$ or $\phi(r) < \phi(a)$. Notice that $r = x - aq \in I$. If $r \neq 0$, then $n_I \leq \phi(r) < \phi(a) = n_I$, which is a contradiction; therefore, $r = 0$ and $x = aq \in \text{span}(a)$.

Lemma 24 *Let R be an Euclidean domain. Let I be a proper ideal of R. An element $b \in I \setminus \{0\}$ is a minimal element of I if and only if $I = \text{span}(b)$.*

Proof 74 *We already know that $I = \text{span}(b)$ if $\phi(b) = n_I$. Conversely, assume that $I = \text{span}(b)$. Take any $a \in I \setminus \{0\}$ with $\phi(a) = n_I$. By the implication right above, we know that $\text{span}(a) = I = \text{span}(b)$. According to Lemma 23, a, b are equivalent. As a consequence, $\phi(a) = \phi(b)$.*

The final result in this subsubsection is a classical key theorem of Factorization Theory.

Theorem 18 *Let R be an integral domain, commutative ring. If R is an Euclidean domain, then R is a principal domain.*

Proof 75 *Let I be an ideal of R. Since $\{0\} = \text{span}(0)$ and $R = \text{span}(1)$ are principal ideals of R, we may assume that I is proper. Take any $b \in I \setminus \{0\}$ with $\deg(b) = n_I$. Then $I = \text{span}(b)$.*

The conclusion of this subsubsection is the generalization of the polynomial degree as a kind of Euclidean function.

Definition 100 (Degree function) *Let R be an integral domain, commutative ring. A degree function $\deg : R \to \mathbb{Z}^+$ is a function satisfying the following conditions for all $p, p' \in R$:*

- *Subadditivity:* $\deg(p + p') \leq \deg(p) + \deg(p')$.
- *Multiplicativity:* $\deg(pp') = \deg(p) + \deg(p')$.
- *Nullity:* $p = 0 \Rightarrow \deg(p) = 0$.
- *Division algorithm: If* $0 < \deg(p') \leq \deg(p)$, *there exist* $q, r \in R$ *with* $\deg(r) < \deg(p')$ *and* $p = p'q + r$.

1.2.3.5 Fields

Fields are a particular case of rings. They are fundamental in Linear Algebra since fields make possible all the nice properties verified by vector spaces.

Definition 101 (Field) *A division ring is a universal algebra $\left(K, \left\{+, 0, -, \cdot, 1, {}^{-1}\right\}, \varnothing\right)$ such that $(K, \{+, 0, -, \cdot, 1\}, \varnothing)$ is a ring and $\left(K \setminus \{0\}, \left\{\cdot, 1, {}^{-1}\right\}, \varnothing\right)$ is a group. If, in addition, $\left(K \setminus \{0\}, \left\{\cdot, 1, {}^{-1}\right\}, \varnothing\right)$ is a commutative group, then we call $\left(K, \left\{+, 0, -, \cdot, 1, {}^{-1}\right\}, \varnothing\right)$ a field.*

Notice that a division ring is simply a ring K in which $K^\times = K \setminus \{0\}$.

Lemma 25 *Let K, L be nonzero division rings. Let $f : K \to L$ be a map. Then:*

1. *If I is an ideal of K, then either $I = \{0\}$ or $I = K$.*
2. *If f is multiplicative, then either $f(1) = 1$ or $f = 0$. If $f(1) = 1$, then $f(K^\times) \subseteq L^\times$.*
3. *If f is additive and multiplicative, then either $\ker(f) = \{0\}$ or $\ker(f) = K$. If $\ker(f) = \{0\}$, then f is an operator of division rings.*

Proof 76 *The proof is itemized by following the statement.*

1. *Assume without loss that I is a left ideal. If there exists $k \in I \setminus \{0\}$, then $1 = k^{-1}k \in k^{-1}I \subseteq I$, therefore, $m = m \cdot 1 = mI \subseteq I$ for all $m \in K$, which means that $I = K$.*
2. *Assume that $f \neq 0$. It suffices to show that $f(K^\times) \subseteq L^\times$ and apply Proposition 27(1). Suppose that there exists $k \in K \setminus \{0\}$ such that $f(k) = 0$. Then $f(1) = f\left(k \cdot k^{-1}\right) = f(k) \cdot f\left(k^{-1}\right) = 0$. Next, $1 \in \ker(f)$, so, $f = 0$ by applying Proposition 27(2). This is a contradiction; therefore, $f(K^\times) \subseteq L^\times$.*
3. *This is a direct consequence of Lemma 25(1) together with Proposition 29(1). If $\ker(f) = \{0\}$, then $f(K^\times) \subseteq L^\times$ in view of Lemma 25(2). By bearing in mind Proposition 27(1), $f : K \to L$ is an operator of additive groups and $f|_{K^\times} : K^\times \to L^\times$ is an operator of multiplicative groups.*

In the upcoming proposition, we will make proper use of the notation $B^{-1} := \{b^{-1} : b \in B \setminus \{0\}\}$, for B any subset of a field.

Proposition 33 *Let K be a division ring. Let $k, l \in K$. Let $A \subseteq K$. Then:*

1. *If $kl = 0$, then either $k = 0$ or $l = 0$. As a consequence, K is an integral domain.*
2. *If K is a field and R is a subring of K, then the subfield spanned by R is* $\mathrm{span}(R) = \{rs^{-1} : r \in R, s \in R \setminus \{0\}\}$.
3. *If K is a field, then the subfield spanned by A is given by* $\mathrm{span}(A) = \left(\bigcup_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\left[(A \cup \{0,1\} \cup -A) \overset{m}{\cdots} (A \cup \{0,1\} \cup -A) + \overset{n}{\cdots} + (A \cup \{0,1\} \cup -A) \overset{m}{\cdots} (A \cup \{0,1\} \cup -A)\right]\right) \cdot \left(\bigcup_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\left[(A \cup \{1\} \cup -A) \overset{m}{\cdots} (A \cup \{1\} \cup -A) + \overset{n}{\cdots} + (A \cup \{1\} \cup -A) \overset{m}{\cdots} (A \cup \{1\} \cup -A)\right]\right)^{-1} = \left\{ \left(\sum_{i=1}^{n} k_i a_{i1}^{k_{i1}} \cdots a_{im_i}^{k_{im_i}}\right) \left(\sum_{h=1}^{p} l_h b_{h1}^{l_{h1}} \cdots b_{hq_h}^{l_{hq_h}}\right)^{-1} : n, m_i, p, q_h \in \mathbb{N}, a_{ij}, b_{hg} \in A, k_i, k_{ij}, l_h, l_{hg} \in \mathbb{Z}, j = 1, \ldots, m_i, i = 1, \ldots, n, g = 1, \ldots, q_h, h = 1, \ldots, p, \sum_{h=1}^{p} l_h b_{h1}^{l_{h1}} \cdots b_{hq_h}^{l_{hq_h}} \neq 0 \right\}$.
4. *A' is a subfield of K.*
5. $\mathrm{span}(A)' \subseteq A'$. *If K is a field, then $A' = \mathrm{span}(A)'$.*

Proof 77 *We will only prove the second item. Observe that every subfield of K containing R clearly contains $\{rs^{-1} : r \in R, s \in R \setminus \{0\}\}$. It only suffices to show that $\{rs^{-1} : r \in R, s \in R \setminus \{0\}\}$ is a subfield of K. Indeed, it is sufficient to take into account that $r_1 s_1^{-1} + r_2 s_2^{-1} = (r_1 s_2 + r_2 s_1)(s_1 s_2)^{-1}$ for all $r_1, r_2 \in R$ and all $s_1, s_2 \in R \setminus \{0\}$.*

The following remark is a direct consequence of Proposition 35(4). The reason why we place it here is that it is necessary for the proof of the next theorem.

Remark 33 *Let R be a ring. Consider $A \subseteq R$. The left ideal spanned by A is* $\mathrm{span}(A) = \{r_1 a_1 + \cdots + r_k a_k : k \in \mathbb{N}, \forall i \in \{1, \ldots, k\}\ r_i \in R, a_i \in A\}$.

Under the settings of the previous remark, $\mathrm{span}(A)$ refers to the smallest left ideal of R containing A, in other words, the universal subalgebra spanned by A in the universal algebra $(R, \{+, 0, -\}, \{\cdot\})$, where $\cdot$ is the multiplication of R seen as a left action of the multiplicative monoid of R on R. A similar result applies for the spanned right ideal.

Proposition 34 (Quotient field) *Let R be a ring. Let $I \subseteq R$ be a two-sided ideal. The following conditions are equivalent:*

1. *R/I is a division ring.*
2. *I is a maximal left ideal.*
3. *I is a maximal right ideal.*

Proof 78 *We will begin by assuming first that R/I is a division ring. We will only prove that I is a maximal left ideal. In a similar way, it can be shown that I is a maximal right ideal. Indeed, take any left ideal $I \subsetneq J \subseteq R$ and consider $r \in J \setminus I$. Notice that $r + I$ is not 0 in R/I, thus, by hypothesis, there exists its inverse $s + I = (r + I)^{-1}$. Notice that $sr + I = (s + I)(r + I) = 1 + I$, meaning that $1 - sr \in I \subseteq J$. On the other hand, $sr \in J$ because J is a left ideal, hence $1 \in J$ meaning that $J = R$. This proves that I is a maximal left ideal. Conversely, assume that I is a maximal left ideal. We will show that every nonzero*

element of R/I *is left invertible and then we will apply Proposition 16(4) to assure that the multiplicative monoid* $(R/I) \setminus \{0\}$ *is a group. Indeed, take any nonzero* $r+I \in R/I$*. In view of Remark 33, let us consider the left ideal* J *spanned by* $I \cup \{r\}$*. Since* $I \subsetneq J$*, it must occur by hypothesis that* $J = R$*. Then* $1 \in J$*, so there exist* $a, b \in R$ *and* $c \in I$ *such that* $1 = ar + bc$*. Observe that* $1 - ar = bc \in I$*, hence* $1 + I = ar + I = (a + I)(r + I)$*. As a consequence,* $r + I$ *is left invertible in* R/I*. According to Proposition 16(4), the multiplicative monoid* $(R/I) \setminus \{0\}$ *is a group; hence,* R/I *is a division ring. Finally, if* I *is a maximal right ideal, a similar proof applies.*

1.2.3.6 Galois Theory

A ring extension is simply an injective (unital) operator $i : R \hookrightarrow S$ of rings R, S. Ring extensions are formally denoted by (R, S, i) and informally denoted by $R \subseteq S$.

Definition 102 (Field extension) *A field extension is a* 3*-tuple* (E, F, i)*, where* E, F *are fields and* $i : E \hookrightarrow F$ *is an injective operator of rings, that is, injective, additive, and multiplicative.*

Notice that if (E, F, i) is a field extension, then $i(E)$ is a subfield of F isomorphic to E; hence, it is common to denote field extensions such as $E \subseteq F$ or F/E. Examples of classical field extensions are $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Definition 103 (Algebraic element) *Let* $R \subseteq S$ *be a ring extension. An element* $s \in S$ *is algebraic over* R *if there exists* $p \in R[x] \setminus \{0\}$ *such that* $p(r) = 0$*.*

Under the settings of the previous definition, it makes sense to consider $n_s := \min \{\deg(p) : p \in R[x] \setminus \{0\},\ p(s) = 0\}$.

Remark 34 *Let* $E \subseteq F$ *be a field extension. Let* $\alpha \in F$ *be algebraic over* E*. Let* $p, q \in E[x] \setminus \{0\}$ *such that* $p(\alpha) = q(\alpha) = 0$ *and* $\deg(p) = \deg(q) = n_\alpha$*. According to the division algorithm, there are* $c(x), r(x) \in E[x]$ *such that* $p(x) = c(x)q(x) + r(x)$ *and* $0 \leq \deg(r(x)) < \deg(q(x)) = n_\alpha$*. Then* $r(\alpha) = p(\alpha) - c(\alpha)q(\alpha) = 0$*. The minimality of* n_α *forces that* $r(x) = 0$*. Next,* $n_\alpha = \deg(p(x)) = \deg(c(x)q(x)) = \deg(c(x)) + \deg(q(x)) = \deg(c(x)) + n_\alpha$*, meaning that* $\deg(c(x)) = 0$*, so there exists* $\lambda \in E$ *such that* $c(x) = \lambda$*, hence* $p(x) = \lambda q(x)$*.*

The previous remark motivates the following definition. Bear in mind that polynomial rings over a field are Euclidean domains.

Definition 104 (Minimal polynomial) *Let* $E \subseteq F$ *be a field extension. Let* $\alpha \in F$ *be algebraic over* E*. We say that* $p_\alpha(x) \in E[x] \setminus \{0\}$ *is the minimal polynomial for* α *if* $p_\alpha(\alpha) = 0$*,* $\deg(p_\alpha(x)) = n_\alpha$*, and* $\operatorname{coef}(p_\alpha(x)) = 1$*.*

The following theorem characterizes algebraic elements and serves to construct intermediate fields.

Theorem 19 *Let* $E \subseteq F$ *be a field extension. An element* $\alpha \in F$ *is algebraic over* E *if and only if* $E[\alpha]$ *is a field. In this situation,* $E \subseteq E[\alpha] \subseteq F$ *are field extensions.*

Proof 79 *The proof follows itemized according to the statement.*

- $\Rightarrow$ *Suppose that* α *is algebraic over* E *and consider the minimal polynomial* $p_\alpha(x) \in E[x]$ *of* α*. Notice that*

$$\begin{array}{rcl} E[x]/\operatorname{span}(p_\alpha(x)) & \to & E[\alpha] \\ q(x) + \operatorname{span}(p_\alpha(x)) & \mapsto & q(\alpha) \end{array}$$

is a ring isomorphism. On the other hand, $\operatorname{span}(p_\alpha(x)) = E[x]p_\alpha(x)$ *is a maximal ideal of* $E[x]$*. Therefore, according to Proposition 34,* $E[x]/\operatorname{span}(p_\alpha(x))$ *is a field, hence* $E[\alpha]$ *is a field as well.*

$\Leftarrow$ *There exists* $q(x) \in E[x]$ *such that* $q(\alpha)\alpha = 1$. *Let* $r(x) := q(x)x - 1$. *Notice that* $r(x) \in E[x]$ *and* $r(\alpha) = 0$.

1.2.4 Modules and algebras

By means of their weaker form of vector spaces, modules constitute the most extended and used algebraic structure in Analysis and Calculus.

1.2.4.1 Modules

Since we are going to consider left actions in this book (unless we otherwise state it explicitly), we are forced to consider left modules. Right modules can be defined in a similar way and all the results for left modules can be easily readapted for right modules.

Definition 105 (Module) *A left module over a ring* $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$ *is a universal algebra* $(M, \{+, 0, -\}, \{\cdot\})$, *where* $(M, \{+, 0, -\}, \varnothing)$ *is a commutative additive group endowed with a distributive (biadditive) left action* $\cdot : R \times M \to M$.

Notice that any commutative additive group is a $\mathbb{Z}$-module with the action given in Example 17. On the other hand, every ring is a left and right module over itself by seeing the multiplication as a left or right action. The submodules of a ring as a left (right) module over itself are its left (right) ideals.

Remark 35 *Let* M *be a left module over a ring* R. *The right action*

$$\begin{array}{rcl} M \times R & \to & M \\ (m, r) & \mapsto & mr := rm \end{array}$$

does not provide structure of right R*-module to* M *unless, for instance,* R *is commutative. Indeed, for all* $r, s \in R$ *and all* $m \in M$, *it must occur that* $(rs)m = m(rs) = (mr)s = (rm)s = s(rm) = (sr)m$, *being possible when* R *is commutative.*

Proposition 35 *Let* M *be a module over a ring* R. *Consider* $r, s \in R$, $m \in M$, *and* $A \subseteq M$. *Then:*

1. $r0 = 0m = 0$.
2. $(-r)m = r(-m) = -rm$.
3. *If* $r \in \mathcal{U}(R)$ *and* $rm = 0$, *then* $r = 0$.
4. *The submodule spanned by* A *is*

$$\operatorname{span}(A) = \bigcup_{n \in \mathbb{N}} RA + \overset{n}{\cdots} + RA = \left\{ \sum_{i=1}^{n} r_i a_i : n \in \mathbb{N}, a_i \in A, i = 1, \ldots, n \right\}.$$

5. $\operatorname{span}(rA) \cup r\operatorname{span}(A) \subseteq \operatorname{span}(A)$.
6. *If* $r \in \mathcal{U}(R)$, *then* $\operatorname{span}(rA) = r\operatorname{span}(A) = \operatorname{span}(A)$.

Proof 80 *We will only show the last two items. If* $a_1, \ldots, a_n \in A$ *and* $r_1, \ldots, r_n \in R$, *then* $\sum_{i=1}^{n} r_i(ra_i) = \sum_{i=1}^{n} (r_i r)a_i \in \operatorname{span}(A)$ *and* $r\sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} (rr_i)a_i \in \operatorname{span}(A)$. *Conversely, assume that* $r \in \mathcal{U}(R)$. *If* $a_1, \ldots, a_n \in A$ *and* $r_1, \ldots, r_n \in R$, *then* $\sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} \left(r_i r^{-1}\right)(ra_i) \in \operatorname{span}(rA)$ *and* $\sum_{i=1}^{n} r_i a_i = r\sum_{i=1}^{n} \left(r^{-1} r_i\right) a_i \in r\operatorname{span}(A)$.

The following remark, although it is trivial, will be extremely useful to perform operations with limits and series.

Remark 36 *Let M be a module over a ring R. Let $A, B, C \subseteq M$. Then:*

1. *If $A + B \subseteq C$ and $C - A \subseteq B$, then $A + B = C$.*
2. *If $C - A \subseteq B$ and $C - B \subseteq A$, then $B = C - A$.*

The notion of equivalence of elements in the scope of rings can also be transported to modules.

Definition 106 (Equivalence of elements) *Let M be a module over a ring R. We say that two elements $m, n \in M$ are equivalent if there exists $u \in \mathcal{U}(R)$ such that $m = un$.*

As expected, the equivalence of elements in a module is an equivalence relation. In fact, under the settings of the previous definition, the equivalence classes are given by $\mathcal{U}(R)m$ for all $m \in M$.

Definition 107 (Equivalence-free set) *Let M be a module over a ring R. A subset $A \subseteq M$ is said to be equivalence free provided that $0 \notin A$ and $\mathcal{U}(R)a \cap \mathcal{U}(R)b = \varnothing$ whenever $a, b \in A$ with $a \neq b$.*

The following lemma establishes the existence of maximal equivalence-free subsets.

Lemma 26 *Let R be a ring. Let M be an R-module. If $A \subseteq M$ satisfies that $A \setminus \{0\} \neq \varnothing$, then there exists a maximal equivalence-free subset $D \subseteq A$. Furthermore,* $\operatorname{span}(D) = \operatorname{span}(A)$.

Proof 81 *Define $\mathcal{L} := \{C \subseteq A : C \text{ is equivalence free}\}$ partially ordered by the inclusion. Notice that $\mathcal{L} \neq \varnothing$ because if $a \in A \setminus \{0\}$, then $C := \{a\} \in \mathcal{L}$. Next, $\mathcal{L}$ is an inductive set in the sense that every chain of $\mathcal{L}$ has an upper bound, which consists of the union of the elements of the chain. Next, Zorn's Lemma assures the existence of a maximal element in $\mathcal{L}$, that is, a maximal equivalence-free subset $D \subseteq A$. Finally, let us show that* $\operatorname{span}(D) = \operatorname{span}(A)$. *Indeed, $D \subseteq A$ so* $\operatorname{span}(D) \subseteq \operatorname{span}(A)$. *In order to show the reverse inclusion, we will prove that $A \subseteq \operatorname{span}(D)$. Assume on the contrary that there exists $a \in A \setminus \operatorname{span}(D)$. Note that $D \cup \{a\}$ is equivalence free, reaching a contradiction with the maximality of D. As a consequence, $A \subseteq \operatorname{span}(D)$ meaning that* $\operatorname{span}(A) \subseteq \operatorname{span}(D)$.

The operators of modules are called linear maps, that is, the additive and homogeneous maps. If M, N are modules over a ring R, then $\mathcal{L}(M, N)$ denotes the set of linear maps from M to N. When $M = N$, we simply write $\mathcal{L}(M)$. The following remark highlights that $\mathcal{L}(M, N)$ acquires structure of commutative additive group.

Remark 37 *Let M, N be modules over a ring R. Then:*

1. *$\mathcal{L}(M, N)$ is a commutative additive group. In fact, it is a $\mathsf{Z}(R)$-module.*
2. *$\mathcal{L}(M)$ is a ring.*
3. *If $f \in \mathcal{L}(M, N)$, then* $\ker(f)$ *is a submodule of M.*

In accordance with Remark 18(1), if $f : M \to N$ is a linear map between modules M, N, then $f(M)$ is a submodule of N. Also, notice that if $A \subseteq M$ and $B \subseteq N$, then $f(\operatorname{span}(A)) = \operatorname{span}(f(A))$ and $f^{-1}(\operatorname{span}(B)) \supseteq \operatorname{span}\left(f^{-1}(B)\right)$.

Lemma 27 *Let M be a module over a division ring R. Let $f, g \in \mathcal{L}(M,R) \setminus \{0\}$. Then $\ker(f)$ is a maximal submodule of M. If $\ker(f) = \ker(g)$, then there exists $r \in R$ such that $g = fr$.*

Proof 82 *Let V be a submodule of M strictly containing $\ker(f)$ and fix an arbitrary $v \in V \setminus \ker(f)$. We will show that $M = V$. Indeed, if $m \in M$, then*

$$m = f(m)f(v)^{-1}v + \left(-f(m)f(v)^{-1}v + m\right) \in RV + \ker(f) \subseteq V.$$

Next, assume that $\ker(f) = \ker(g)$. Fix an arbitrary $v \in M \setminus \ker(f)$. Take $r := f(v)^{-1}g(v)$. Take $m \in M$. Note that $m - f(m)f(v)^{-1}v \in \ker(f) = \ker(g)$, therefore $g(m) = f(m)r$.

Recall that $R_1[x_1, \ldots, x_n]$ stands for the set of polynomials in n-variables of degree 1 with coefficients in R, that is,

$$R_1[x_1, \ldots, x_n] := \left\{ \sum_{i=0}^{n} r_i x_i + \cdots + r_n x_n : r_i \in R_i, i = 1, \ldots, n \right\}.$$

Notice that $R_1[x_1, \ldots, x_n]$ is clearly an R-submodule of $R[x_1, \ldots, x_n]$.

Definition 108 (Linearly independence) *Let M be a module over a ring R. Consider $A \subseteq M$ as well as distinct elements $m_1, \ldots, m_k \in M$. Then:*

- *$\{m_1, \ldots, m_k\}$ is said to be linearly independent or free if the linear map*

$$\begin{array}{rcl} R_1[x_1, \ldots, x_k] & \to & M \\ p(x_1, \ldots, x_k) & \to & p(m_1, \ldots, m_k) \end{array}$$

  has null kernel.

- *A is said to be linearly independent or free if so is every finite subset of A.*

- *A is said to be a basis of M if it is linearly independent and a spanning system of M.*

Next theorem relies on the notion of free object of a category (see Definition 369). By following the classical literature, a module is said to be a free module if it has a basis.

Theorem 20 *Let M be a module over a ring R. The following conditions are equivalent:*

1. *The exists basis for M.*
2. *There exists an index set I such that M is isomorphic to $\bigoplus_{i \in I} R$.*

Proof 83 *If there exists a basis $\mathcal{B}$ of M, then*

$$\begin{array}{rcl} \bigoplus_{b \in \mathcal{B}} R & \to & M \\ (r_b)_{b \in \mathcal{B}} & \mapsto & \sum_{b \in \mathcal{B}} r_b b \end{array}$$

is an isomorphism. Conversely, if there exists an index set I and an isomorphism $T : \bigoplus_{i \in I} R \to M$, then $\{T(e_i) : i \in I\}$ is a basis of M, where $(e_i)_{i \in I}$ is the canonical basis of $\bigoplus_{i \in I} R$.

The following theorem is a classical result of Linear Algebra, which shows that modules over division rings are free modules. Notice that, in a module over a division ring, a subset of two elements is linearly independent if and only if it is equivalence free.

Theorem 21 *Let M be a module over a division ring R. Let $A \subseteq M$. There exists a maximal linearly independent subset $L \subseteq A$. Furthermore,* $\operatorname{span}(L) = \operatorname{span}(A)$. *In particular, if A is a spanning system of A, then A is a basis of M.*

Proof 84 *First off, if $A = \varnothing$ or $A = \{0\}$, then we simply take $L = \varnothing$. Suppose then that $A \setminus \{0\} \neq \varnothing$. Define $\mathcal{L} := \{C \subseteq A : C$ is linearly independent$\}$ partially ordered by the inclusion. Notice that $\mathcal{L} \neq \varnothing$ because if $a \in A \setminus \{0\}$, then $C := \{a\} \in \mathcal{L}$. Next, $\mathcal{L}$ is an inductive set in the sense that every chain of $\mathcal{L}$ has an upper bound, which consists of the union of the elements of the chain. Next, Zorn's Lemma assures the existence of a maximal element in $\mathcal{L}$, that is, a maximal linearly independent subset $L \subseteq A$. Finally, let us show that* $\operatorname{span}(L) = \operatorname{span}(A)$. *Indeed, $L \subseteq A$ so* $\operatorname{span}(L) \subseteq \operatorname{span}(A)$. *In order to show the reverse inclusion, we will prove that $A \subseteq \operatorname{span}(L)$. Assume on the contrary that there exists $a \in A \setminus \operatorname{span}(L)$. Note that $L \cup \{a\}$ is linearly independent, reaching a contradiction with the maximality of L. As a consequence, $A \subseteq \operatorname{span}(L)$ meaning that* $\operatorname{span}(A) \subseteq \operatorname{span}(L)$.

Under the settings of the previous theorem, every linearly independent subset L of M can be enlarged to a basis of M. Indeed, $\mathcal{L} := \{C \subseteq M : L \subseteq C$ and C is linearly independent$\}$ is an inductive set having maximal elements, which are bases of M containing L.

Definition 109 (Torsion) *Let M be a module over a ring R. An element $m \in M$ is said to be a torsion element of M provided that there exists a regular element $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$ such that $rm = 0$. The set of torsion elements of M is denoted as* $\operatorname{tor}(M)$.

For the following remark, it is preceptive to recall that the right-multiplication operator

$$\begin{array}{rccl} \mathsf{R}_m : & R & \to & Rm \subseteq M \\ & r & \mapsto & \mathsf{R}_m(r) := rm \end{array} \tag{1.24}$$

for m in a module M over a ring R, is an R-linear operator.

Remark 38 *Let M be a module over a ring R. Then:*

- $0 \in \operatorname{tor}(M)$ *because $1 \in R$ is regular. This way, the torsionfree elements of M are exactly those in $M \setminus \operatorname{tor}(M)$, and M is called torsionfree provided that* $\operatorname{tor}(M) = \{0\}$.
- *If R is an integral domain, then $M \setminus \operatorname{tor}(M) = \{m \in M : \mathsf{R}_m$ is an isomorphism over its image$\}$.*
- *If R is commutative, then* $\operatorname{tor}(M)$ *is a submodule. Indeed, if $m, n \in M$ and $r, s \in R$, then there are $t_m, t_n \in R$ regular such that $t_m m = t_n n = 0$, so $(t_m t_n)(rm + sn) = 0$ and $t_m t_n$ is regular.*
- *If R is a division ring, then M is torsionfree.*

Modules over a division rings are typically known as vector spaces and their elements are called vectors.

Proposition 36 (Quotient module) *Let M be a module over a ring R. Let N be a submodule of M. Consider $m \in M$ and $r \in R$. Then:*

1. *$r(m + N) \subseteq rm + N$.*
2. *If $r \in \mathcal{U}(R)$, then $r(m + N) = rm + N$.*
3. *M/N becomes an R-module with the action given by $r(m + N) := rm + N$.*

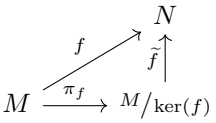
Proof 85 *Fix any arbitrary $n \in N$. Then $r(m+n) = rm + rn \in rm + N$. This shows that $r(m+N) \subseteq rm + N$. Suppose now that $r \in \mathcal{U}(R)$. Fix again an arbitrary $n \in N$. Then $rm + n = r\left(m + r^{-1}n\right) \in r(m+N)$. This proves that $r(m+N) = rm + N$. Finally, in order to show that ${}^{M}\!/_{N}$ becomes an R-module, it only suffices to make sure that the action $r(m+N) := rm + N$ is well defined. Indeed, suppose that $m' \in M$ is so that $m - m' \in N$. Then $rm - rm' = r(m-m') \in N$, meaning that $r(m+N) = rm+N = rm'+N = r(m'+N)$.*

The final result in this subsubsection is the First Isomorphism Theorem in Linear Algebra.

Theorem 22 (First Isomorphism Theorem) *Let M, N be modules over a ring R. Let $f : M \to N$ be a linear operator. Then*

$$\begin{array}{rccl} \widetilde{f}: & {}^{M}\!/_{\ker(f)} & \to & f(N) \\ & m + \ker(f) & \mapsto & \widetilde{f}(m + \ker(f)) := f(m) \end{array}$$

is a (unique) module isomorphism.



Proof 86 *In accordance with Theorem 11, $\widetilde{f}$ is an isomorphism of additive groups. It is easy to check that $\widetilde{f}$ is also linear, that is, an operator of modules. Finally, Remark 17(3), assures that $\widetilde{f}$ is an isomorphism of modules.*

1.2.4.2 Affine manifolds

Affine manifolds play a very important role in the study of the geometry of topological modules.

Definition 110 (Affine manifold) *An affine manifold in a module M over a ring R is a translation of a submodule of M, that is, $m + N$ where $m \in M$ and N is a submodule of M.*

Under the settings of the previous definition, an affine manifold $m+N$ contains 0 if and only if $m + N = N$.

Proposition 37 *Let M be a module over a ring R. If $(m_i + N_i)_{i\in I}$ is a family of affine manifolds of M such that $\bigcap_{i\in I}(m_i + N_i) \neq \varnothing$, then $\bigcap_{i\in I}(m_i + N_i) = m + \operatorname{span}\left(\bigcap_{i\in I}(m_i + N_i) - m\right)$ for every $m \in \bigcap_{i\in I}(m_i + N_i)$. In other words, $\bigcap_{i\in I}(m_i + N_i)$ is an affine manifold of M.*

Proof 87 *It is clear that $\bigcap_{i\in I}(m_i + N_i) \subseteq m + \operatorname{span}\left(\bigcap_{i\in I}(m_i + N_i) - m\right)$ for every $m \in \bigcap_{i\in I}(m_i + N_i)$. A generic element of $m + \operatorname{span}\left(\bigcap_{i\in I}(m_i + N_i) - m\right)$ has the form $m + \sum_{j=1}^{k} r_j\,(b_j - m)$, where $b_j \in \bigcap_{i\in I}(m_i + N_i)$ for all $j \in \{1, \ldots, k\}$. Fix an arbitrary $i_0 \in I$. Then $b_j = m_{i_0} + n_{i_0 j}$, with $n_{i_0 j} \in N_{i_0}$ for all $j \in \{1, \ldots, k\}$. Observe that $m = m_{i_0} + n_{i_0}$ for some $n_{i_0} \in N_{i_0}$. Then*

$$m + \sum_{j=1}^{k} r_j\,(b_j - m) = m_{i_0} + n_{i_0} + \sum_{j=1}^{k} r_j\,(n_{i_0 j} - n_{i_0}) \in m_{i_0} + N_{i_0}.$$

Since i_0 is arbitrary, $m + \operatorname{span}\left(\bigcap_{i\in I}(m_i + N_i) - m\right) \subseteq \bigcap_{i\in I}(m_i + N_i)$.

Definition 111 (Affine hull) *The affine hull of a subset A of a module M over a ring R is defined as the intersection of all affine manifolds of M containing A. The affine hull of A is denoted by* $\mathrm{aff}(A)$.

In accordance with Theorem 5, aff is a hull operator. Under the settings of the above definition, $\mathrm{span}(A)$ is trivially an affine manifold containing A; therefore, $A \subseteq \mathrm{aff}(A) \subseteq \mathrm{span}(A)$. In terms of pointwise ordering, $I \leq \mathrm{aff} \leq \mathrm{span}$.

Lemma 28 *Let M, N be modules over a ring R. Let $T : M \to N$ be a linear operator. Let $A \subseteq M$ and $B \subseteq N$ be nonempty. Let $r \in \mathcal{U}(R)$ and $m \in M$. Then:*

1. $\mathrm{aff}(A) = a + \mathrm{span}(A - a)$ *for every* $a \in A$.
2. $\mathrm{aff}(A) = \mathrm{span}(A)$ *if and only if* $0 \in \mathrm{aff}(A)$.
3. $\mathrm{aff}(m + A) = m + \mathrm{aff}(A)$.
4. $\mathrm{aff}(rA) = r\mathrm{aff}(A)$.
5. $\mathrm{aff}(T(A)) = T(\mathrm{aff}(A))$.
6. $\mathrm{aff}\left(T^{-1}(B)\right) \subseteq T^{-1}(\mathrm{aff}(B))$.

Proof 88 *The proof follows as itemized as the statement of the lemma.*

1. *Fix an arbitrary $a_0 \in A$. By definition,* $\mathrm{aff}(A)$ *is the intersection of all affine manifolds containing A, therefore* $\mathrm{aff}(A) \subseteq a_0 + \mathrm{span}(A - a_0)$. *Now let $m + N$ be an affine manifold of M containing A, where N is a submodule of M and $m \in M$. Take an element $a_0 + r_1(a_1 - a_0) + \cdots + r_k(a_k - a_0) \in a_0 + \mathrm{span}(A - a_0)$, where $a_1, \ldots, a_k \in A$ and $r_1, \ldots, r_k \in R$. For every $i \in \{0, 1, \ldots, k\}$ there exists $n_i \in N$ with $a_i = m + n_i$. Then*
$$a_0 + \sum_{j=1}^{k} r_j(a_j - a_0) = m + n_0 + \sum_{j=1}^{k} r_j(n_j - n_0) \in m + N.$$
This shows that $a_0 + \mathrm{span}(A - a_0) \subseteq m + N$ and hence $a_0 + \mathrm{span}(A - a_0) \subseteq \mathrm{aff}(A)$.
2. *If* $\mathrm{aff}(A) = \mathrm{span}(A)$, *then* $0 \in \mathrm{span}(A) = \mathrm{aff}(A)$. *Conversely, assume that* $0 \in \mathrm{aff}(A)$. *By the previous item, if we choose any $a_0 \in A$, then we have that* $0 \in a_0 + \mathrm{span}(A - a_0)$, *which implies that $-a_0 \in \mathrm{span}(A - a_0)$ and so $a_0 + \mathrm{span}(A - a_0) = \mathrm{span}(A - a_0)$. Then* $\mathrm{aff}(A) = \mathrm{span}(A - a_0)$ *so the affine hull of A is a submodule of M, which then implies that* $\mathrm{span}(A) \subseteq \mathrm{aff}(A)$. *Finally, we know that* $\mathrm{aff}(A) \subseteq \mathrm{span}(A)$.
3. *Fix any $a \in A$. By Lemma 28(1),* $\mathrm{aff}(m + A) = (m + a) + \mathrm{span}((m + A) - (m + a)) = m + (a + \mathrm{span}(A - a)) = m + \mathrm{aff}(A)$.
4. *Fix any $a \in A$. By Lemma 28(1) together with Proposition 35(6),* $\mathrm{aff}(rA) = ra + \mathrm{span}(rA - ra) = ra + \mathrm{span}(r(A - a)) = ra + r\mathrm{span}(A - a) = r\left(a + \mathrm{span}(A - a)\right) = r\mathrm{aff}(A)$.
5. *Fix $x \in A$. Simply observe that* $T(\mathrm{aff}(A)) = T(x + \mathrm{span}(A - x)) = T(x) + T(\mathrm{span}(A - x)) = T(x) + \mathrm{span}(T(A - x)) = T(x) + \mathrm{span}(T(A) - T(x)) = \mathrm{aff}(T(A))$.
6. *Fix $x \in T^{-1}(B)$. Observe that $T^{-1}(B) - x \subseteq T^{-1}(B - T(x))$. Then*
$$\begin{aligned} \mathrm{aff}\left(T^{-1}(B)\right) &= x + \mathrm{span}\left(T^{-1}(B) - x\right) \subseteq x + \mathrm{span}\left(T^{-1}(B - T(x))\right) \\ &\subseteq x + T^{-1}\left(\mathrm{span}(B - T(x))\right) \\ &\subseteq T^{-1}\left(T(x) + \mathrm{span}(B - T(x))\right) = T^{-1}(\mathrm{aff}(B)) \end{aligned}$$

There are several simple examples of affine manifolds. For instance, singletons are trivially affine manifolds since they are translations of the null submodule. The whole module is itself an affine manifold. Other examples follow.

Example 22 (Straight line) *In a module M over a ring R, the line passing through $m, n \in M$ is defined as the affine manifold* $\mathrm{stl}(m,n) := m + R(n-m)$. *Special attention will be paid to a notable subset of* $\mathrm{stl}(m,n)$ *given by* $\mathrm{stl}_{\mathcal{U}}(m,n) := m + (\mathcal{U}(R) \cup \{0\})(n-m)$. *The following properties are satisfied for all $m, n \in M$:*

1. $\mathrm{stl}(m,n) = \mathrm{aff}(\{m,n\})$.
2. $\mathrm{stl}(m,m) = \{m\}$.
3. $\mathrm{stl}(m,n) = \mathrm{stl}(n,m)$, *because* $m + r(n-m) = n + (1-r)(m-n)$ *for all* $r \in R$.
4. *If* $u \in \mathcal{U}(R)$ *and* $m \neq n$, *then* $m \neq m + u(n-m)$.
5. $n \in \mathrm{stl}_{\mathcal{U}}(m,p)$ *for all* $p \in \mathrm{stl}_{\mathcal{U}}(m,n) \setminus \{m\}$ *in view of the fact that if* $p = m + u(n-m)$ *for some* $u \in \mathcal{U}(R)$, *then* $n = m + u^{-1}(p-m) \in \mathrm{stl}_{\mathcal{U}}(m,p)$.

In the vector space ambience, straight lines are the affine manifolds of dimension 1, and hyperplanes are the affine manifolds of codimension 1.

Example 23 (Hyperplane) *In a module M over a ring R, the affine manifold $(m^*)^{-1}(\{s\})$, for $m^* \in M^*$ and $s \in R$ in the range of m^*, represents a hyperplane. Note that $(m^*)^{-1}(\{s\}) = m + \ker(m^*)$ for each $m \in (m^*)^{-1}(\{s\})$.*

In accordance with Example 9, the product of any family of modules is a module. And according to Definition 50, multioperators of modules can also be considered, which will be called multilinear operators. However, in virtue of Proposition 19 and Proposition 22, multilinear operators will be dealt with from the commutativity perspective.

Definition 112 (Multilinear operator) *Let $(M_i)_{i\in I}$ be a family of modules over a commutative ring R. Let P be another module over R. A map $f : \prod_{i\in I} M_i \to P$ is said to be multilinear provided that f is a multioperator of modules, that is, for every $i_0 \in I$ and every $(m_i)_{i\in I\setminus\{i_0\}} \in \prod_{i\in I\setminus\{i_0\}} M_i$:*

- *Multiadditivity: The marginal map*

$$\begin{array}{rccl} f_{(m_i)_{i\in I\setminus\{i_0\}}}: & M_{i_0} & \to & P \\ & m_{i_0} & \mapsto & f_{(m_i)_{i\in I\setminus\{i_0\}}}(m_{i_0}) := f\left((m_i)_{i\in I}\right) \end{array}$$

  is additive.

- *Multihomogeneousity: The marginal map*

$$\begin{array}{rccl} f_{(m_i)_{i\in I\setminus\{i_0\}}}: & M_{i_0} & \to & P \\ & m_{i_0} & \mapsto & f_{(m_i)_{i\in I\setminus\{i_0\}}}(m_{i_0}) := f\left((m_i)_{i\in I}\right) \end{array}$$

  is homogeneous.

The set of all multilinear operators from $\prod_{i\in I} M_i$ to P is usually denoted by $\mathcal{L}^I\left(\prod_{i\in I} M_i, P\right)$.

Under the settings of the previous definition, it is not hard to check that $\mathcal{L}^I\left(\prod_{i\in I} M_i, P\right)$ is an R-module. When $P := R$, we denote $\mathcal{L}^I\left(\prod_{i\in I} M_i\right) := \mathcal{L}^I(\prod_{i\in I} M_i, R)$.

1.2.4.3 Ordered modules

Ordered modules are just modules, over ordered rings, endowed with a partial order compatible with the module operations.

Remark 39 *Let M be a module, over an ordered ring R, endowed with a partial order $\leq$ compatible with the addition, that is, translation-invariant in the sense that for all $m, n \in M$ with $m \leq n$, $m+p \leq n+p$ for all $p \in M$. Let $M^+ := \uparrow 0 = \{m \in M : m \geq 0\}$. The following left and right homothety invariances are equivalent:*

- *For all $m, n \in M$ with $m \leq n$, we have that $tm \leq tn$ for all $t \in R^+$.*
- *For all $r, s \in R$ with $r \leq s$, we have that $rp \leq sp$ for all $p \in M^+$.*

The above remark allows to properly define the notion of ordered module.

Definition 113 (Ordered module) *An ordered module is a module M, over an ordered ring R, endowed with a partial order $\leq$ compatible with the addition and left-multiplication operations, that is, for all $m, n \in M$ with $m \leq n$, then*

- *Translation invariance: $m + p \leq n + p$ for all $p \in M$.*
- *Left homothety invariance: $tm \leq tn$ for all $t \in R^+$.*

Under the settings of the previous definition, $M^- := \downarrow 0 = \{x \in M : x \leq 0\}$. Notice that ordered rings are ordered modules over themselves. Next proposition follows a very similar proof as that of Proposition 28.

Proposition 38 *Let R be a ordered ring and M an ordered R-module. For all $m, n, p, q \in M$ and all $r, s \in R$:*

1. *If $m < n$ and $p \leq q$, then $m + p < n + q$.*
2. *If M is torsionfree, $m < n$, and $r > 0$ is not a zero-divisor, then $rm < rn$.*
3. *If M is torsionfree, $m > 0$, $r < s$, and $s - r$ is not a zero-divisor, then $rm < sm$.*
4. *If $m < n$, then $n - m > 0$ and $-n < -m$. In particular, $M^- = -M^+$. Also, $-(\downarrow m) = \uparrow(-m)$, $-(\uparrow m) = \downarrow(-m)$, $\downarrow m = m + M^-$, and $\uparrow m = m + M^+$.*
5. *$M^+ + M^-$ is an $(R^+ + R^-)$-submodule of M.*
6. *M is totally ordered if and only if $M^+ \cup M^- = M$.*
7. *If $r \in R^+$, $m \in M^+$ and $rm = 0$, then $[0, r][0, m] = \{0\}$.*
8. *If $r \in R^+$ and $m \in M^+$, then $[-r, r][-m, m] \subseteq [-3rm, 3rm]$.*
9. *If R is unital, $1 + 1 \in \mathcal{U}(R)$, and $(1 + 1)^{-1} > 0$, then M is free of holes. If $m, -m$ are comparable, then either $m \geq 0$ or $m \leq 0$.*

Proof 89 *Let us itemize the proof in view of the statement.*

1. *By the translation invariance, we know that $m + p \leq n + p \leq n + q$. If $m + p = n + q$, then $m + p = n + p$, so we obtain the contradiction that $m = n$. Hence $m + p < n + q$.*
2. *By the left-homothety invariance, $rm \leq rn$. Suppose on the contrary that $rm = rn$. Then $r(m - n) = 0$, meaning the contradiction that $r \in \mathtt{ld}(0)$ in view of the fact that $m - n \neq 0$ and M is torsionfree.*

3. *By the left-homothety invariance, $rm \leq sm$. Suppose on the contrary that $rm = sm$. Then $(s-r)m = 0$, meaning the contradiction that $s - r \in \mathtt{ld}(0)$ in view of the fact that $m \neq 0$ and M is torsionfree.*

4. *By the translation invariance, $0 = m - m < n - m$. If we apply again the translation invariance, $-n = -n + 0 < -n + (n - m) = -m$.*

5. *Simply observe that if $m_1, m_2, n_1, n_2 \in M^+$ and $r_1, r_2, s_1, s_2 \in R^+$, then $(r_1 - r_2)(m_1 - m_2) + (s_1 - s_2)(n_1 - n_2) = (r_1m_1 + r_2m_2 + s_1n_1 + s_2n_2) - (r_1m_2 + r_2m_1 + s_1n_2 + s_2n_1) \in M^+ + M^-$.*

6. *If M is totally ordered, then every $m \in M$ is comparable to 0, so either $m \geq 0$ or $m \leq 0$, hence $m \in M^+ \cup M^-$. Conversely, suppose that $M^+ \cup M^- = M$ and consider any $m, n \in M$. Then $m - n \in M^+ \cup M^-$, that is, either $m - n \geq 0$ or $m - n \leq 0$, resulting in either $m \geq n$ or $m \leq n$.*

7. *If $0 \leq a \leq r$ and $0 \leq b \leq m$, then $0 \leq ab \leq rm = 0$.*

8. *Take arbitrary elements $a \in R$ and $b \in M$ with $-r \leq a \leq r$ and $-m \leq b \leq m$. Note that $r - a \geq 0$ and $m - b \geq 0$, meaning that $rm - rb - am + ab = (r - a)(m - b) \geq 0$. Then $ab \geq -rm + rb + am$. Since $b \geq -m$ and $r \geq 0$, we have that $rb \geq -rm$. Also, $-r \leq a$ and $m \geq 0$, so $-rm \leq am$. As a consequence, $ab \geq -rm + rb + am \geq -rm - rm - rm$. By repeating the same reasoning for a and $-b$, we conclude that $-ab = a(-b) \geq -rm - rm - rm$. This shows that $[-r, r][-m, m] \subseteq [-3rm, 3rm]$.*

9. *Take arbitrary elements $p_0, q_0 \in M$ with $p_0 < q_0$. Observe that $p_0 + p_0 < p_0 + q_0 < q_0 + q_0$, hence $p_0 = 2^{-1}(p_0 + p_0) < 2^{-1}(p_0 + q_0) < q_0 = 2^{-1}(q_0 + q_0)$, where here $2 := 1 + 1$ as expected. On the other hand, let us assume without any loss of generality that $-m \leq m$. Then $0 \leq 2m$, meaning that $0 \leq m$ by multiplying by $2^{-1} > 0$.*

Under the settings of Proposition 38, if $m < n$ and $p < q$, then $(m, n) + (p, q) \subseteq (m + p, n + q)$; if $0 \leq r < s$ and $0 \leq m < n$, then $(r, s)(m, n) \subseteq [rm, sn]$; if $m \leq n$ and $r \geq 0$, then $r[m, n] \subseteq [rm, rn]$; if $r \leq s$ and $m \geq 0$, then $[r, s]m \subseteq [rm, sm]$; if $A \subseteq {\downarrow}m$ and A is additively symmetric, then $A \subseteq [-m, m]$; and $[-m, m]$ is additively symmetric whenever it is not empty.

Example 24 *Let R be an ordered ring. Let M, N be totally ordered R-modules. Then $P := M \times N$ satisfies that $P = P^+ + P^-$ when endowed with the product order. Indeed, take any $(m, n) \in P$. By assumption, either $m \geq 0$ or $m \leq 0$ and the same happens to n. Then $(m, n) = (m, 0) + (0, n) \in P^+ + P^-$.*

On the other hand, the trivial ordering $m \leq n \iff m = n$ is clearly a module order on any module over an ordered ring.

Proposition 39 *Let R be an ordered ring. Let M be an R-module. If $\leq$ is a module order on M, then the positive cone M^+ satisfies the following properties:*

1. $0 \in M^+$.
2. $M^+ \cap M^- = \{0\}$.
3. $M^+ + M^+ \subseteq M^+$.
4. $R^+M^+ \subseteq M^+$.

Conversely, if $S \subseteq M$ is a subset of M satisfying the four conditions above, then there exists a unique module order on M, given by $m \leq n \Leftrightarrow n - m \in S$ for all $m, n \in M$, for which $S = M^+$.

Proof 90 *If $\leq$ is a module order on R, then the positive cone M^+ trivially satisfies all four conditions above in view of Proposition 38. Conversely, let $S \subseteq M$ satisfying all four conditions above. We will check first that $m \leq n \Leftrightarrow n - m \in S$ is a partial order in M. Indeed, fix arbitrary elements $m, n, p \in M$:*

- *Reflexivity: $m - m = 0 \in S \Rightarrow m \leq m$.*
- *Antisymmetricity: If $m \leq n$ and $n \leq m$, then $n - m \in S$ and $m - n \in S$, so $n - m \in S \cap (-S) = \{0\}$, meaning that $n = m$.*
- *Transitivity: If $m \leq n$ and $n \leq p$, then $n - m \in S$ and $p - n \in S$, so $p - m = (p - n) + (n - m) \in S + S \subseteq S$, meaning that $m \leq p$.*

Next, let us check that this is a module order. Indeed, fix arbitrary elements $m, n, p \in M$ and $r \in R^+$:

- *Translation invariance: If $m \leq n$, then $m - n \in S$, so $(n + p) - (m + p) = n - m \in S$, meaning that $m + p \leq n + p$.*
- *Left-homothety invariance: If $m \leq n$, then $n - m \in S$, so $t(n - m) \in R^+ S \subseteq S$, meaning that $rm \leq rn$.*

It is immediate to check that $S = M^+$. Finally, if $\preceq$ is another module order whose positive cone is S, then $m \leq n \Leftrightarrow n - m \in S \Leftrightarrow m \preceq n$ for all $m, n \in M$, reaching the conclusion that $\leq$ and $\preceq$ are in fact the same ordering.

Proposition 39 motivates the following definition.

Definition 114 (Ordering set) *Let R be an ordered ring. Let M be an R-module. A subset $S \subseteq R$ satisfying all four conditions of Proposition 39 will be called an ordering set.*

It is trivial that the intersection of any family of ordering subsets is again an ordering subset, and the union of any increasing family of ordering subsets is again an ordering subset. By means of Zorn's Lemma, we will demonstrate the existence of module orderings.

Lemma 29 *Let R be an ordered ring. Let M be an R-module. There exists a maximal ordering subset $S \subseteq M$.*

Proof 91 *Let $\mathcal{L} := \{T \subseteq M : T \text{ is an ordering subset}\}$. Notice that $\mathcal{L} \neq \varnothing$ since $\{0\} \in \mathcal{L}$. If $\mathcal{C} \subseteq \mathcal{L}$ is a chain of $\mathcal{L}$, then $\bigcup \mathcal{C} \in \mathcal{L}$. Finally, Zorn's Lemma guarantees the existence of maximal elements in $\mathcal{L}$.*

Faithful ring orderings motivate the same notion for ordered modules.

Lemma 30 *Let R be a faithful ordered ring. Let M be an ordered module over R. The following conditions are equivalent:*

1. *M^+ is totally ordered.*
2. *$M^+ \cup M^-$ is a $(R^+ \cup R^-)$-submodule of M.*
3. *$M^+ + M^- = M^+ \cup M^-$.*

4. $M^+ \cup M^-$ is totally ordered.

Proof 92 *Suppose that M^+ is totally ordered. Since $R^+M^+ \subseteq M^+$ and $M^- = -M^+$, we have that $(R^+ \cup R^-)(M^+ \cup M^-) \subseteq (M^+ \cup M^-)$ and $-(M^+ \cup M^-) = M^+ \cup M^-$. So, it only remains to show that $(M^+ \cup M^-) + (M^+ \cup M^-) \subseteq (M^+ \cup M^-)$. Since $M^+ + M^+ \subseteq M^+$, it only suffices to show that $M^+ + M^- \subseteq M^+ \cup M^-$. Fix arbitrary elements $m, n \in M^+$. Since M^+ is totally ordered, we may assume that $m \leq n$, meaning that $m - n \leq 0$, that is, $m - n \in M^-$. Conversely, assume that $M^+ + M^- \subseteq M^+ \cup M^-$. Fix arbitrary elements $m, n \in M^+ \cup M^-$. Then $m - n \in M^+ \cup M^-$, meaning that either $m \geq n$ or $m \leq n$, resulting in $M^+ \cup N^-$ being totally ordered.*

The previous remark motivates the definition of faithful ordered module.

Definition 115 (Faithful ordered module) *Let R be a faithful ordered ring. Let M be an R-module. A module order on M is called faithful provided that any of the equivalent conditions of Lemma 30 is satisfied.*

By means of Zorn's Lemma, maximal faithful module orderings can be constructed.

Lemma 31 *Let R be an integral domain faithful ordered ring. If M is a torsionfree R-module, then there exists a maximal faithful ordering subset.*

Proof 93 *Let $\mathcal{L} := \{T \subseteq R : T$ is an ordering subset and $T - T \subseteq T \cup -T\}$. Notice that $\mathcal{L} \neq \varnothing$ since $R^+m \in \mathcal{L}$ for all $m \in M \setminus \{0\}$. Also, any chain $\mathcal{C} \subseteq \mathcal{L}$ satisfies that $\bigcup \mathcal{C} \in \mathcal{L}$. Zorn's Lemma guarantees again the existence of maximal elements in $\mathcal{L}$.*

1.2.4.4 Convexity and extremality

By relying upon unit segments, the concept of convexity for modules can be defined.

Definition 116 (Geometric simplex) *Let M be a module over a ring R. Let B^+ be a unit segment of R. A geometric simplex is defined as the set*

$$\Delta[a_1, \ldots, a_k] := \left\{ \sum_{i=1}^{k} r_i a_i : r_1, \ldots, r_k \in B^+, \sum_{i=1}^{k} r_1 = 1 \right\}$$

for $a_1, \ldots, a_k \in M$. The interior of $\Delta[a_1, \ldots, a_k]$ is defined as

$$\Delta(a_1, \ldots, a_k) := \left\{ \sum_{i=1}^{k} r_i a_i : r_1, \ldots, r_k \in B^+ \setminus \{0, 1\}, \sum_{i=1}^{k} r_1 = 1 \right\}.$$

It is clear that $\Delta(a_1, \ldots, a_k) \subseteq \Delta[a_1, \ldots, a_k]$. If $B^+ = \{0, 1\}$ is the trivial unit segment, then $\Delta[a_1, \ldots, a_k] = \{a_1, \ldots, a_k\}$ and $\Delta(a_1, \ldots, a_k) = \varnothing$ for all $a_1, \ldots, a_k \in M$. On the other hand, notice that $\Delta(a_1, \ldots, a_k)$ is not necessarily the topological interior of $\Delta[a_1, \ldots, a_k]$. If $a \neq b$, the geometric simplex $\Delta[a, b]$ is called a segment. Observe that, since $1 + B^- = B^+$, $\Delta[a, b] = a + B^+(b - a) \subseteq \mathrm{stl}(a, b)$.

Remark 40 *Let M be a module over a ring R. Let B^+ be a unit segment of R. For every $a, b \in M$, $\Delta[a, b] \setminus \{a, b\} \subseteq \Delta(a, b)$. If, in addition, M is a torsionfree module, R is an integral domain, B^+ is nontrivial, and $a \neq b$, then $\Delta[a, b] \setminus \{a, b\} = \Delta(a, b)$. If $a = b$, then $\Delta[a, a] = \{a\} = \Delta(a, a)$, hence $\Delta[a, a] \setminus \{a\} = \varnothing \neq \Delta(a, a)$.*

The hypothesis of fullness is, for now, necessary if repetition of elements in a geometric simplex wants be avoided. For instance, note that $\Delta[a,a,b] \supseteq \Delta[a,b]$, however, in order to assure that $\Delta[a,a,b] \subseteq \Delta[a,b]$, it is needed that B^+ be full, because in such case, if we have an element $ra + sa + tb \in \Delta[a,a,b]$, with $r,s,t \in B^+$ and $r+s+t=1$, then we can obtain $(r+s)a + t \in \Delta[a,b]$ with $r+s \in B^+$.

Definition 117 (Convex set) *Let M be a module over a ring R. Let B^+ be a unit segment of R. A subset A of M is said to be convex if, for every $a_1,\ldots,a_k \in A$, $\Delta[a_1,\ldots,a_k] \subseteq A$.*

Whenever we talk about convexity in the ring R, we will be considering R as a left R-module. Geometric simplices are the easiest examples of convex sets.

Proposition 40 *Let M be a module over a ring R. Let B^+ be a full unit segment of R. Let $a_1,\ldots,a_k \in M$. Then $\Delta[a_1,\ldots,a_k]$ is convex. If, in addition, $(B^+ \setminus \{0\})(B^+ \setminus \{0\}) \subseteq B^+ \setminus \{0\}$, then $\Delta(a_1,\ldots,a_k)$ is convex as well.*

Proof 94 *We will prove the convexity of $\Delta(a_1,\ldots,a_k)$. The other is similar. Let $b_1,\ldots,b_l \in \Delta(a_1,\ldots,a_k)$. Let $s_1,\ldots,s_l \in B^+$ such that $s_1+\cdots+s_l=1$. We may assume without any loss of generality that $s_1,\ldots,s_l \in B^+ \setminus \{0\}$. For every $i=1,\ldots,l$, let $r_{i1},\ldots,r_{ik} \in B^+ \setminus \{0,1\}$ such that $r_{i1}+\cdots+r_{ik}=1$ and $b_i = r_{i1}a_1+\cdots+r_{ik}a_k$. Then $s_1b_1+\cdots+s_lb_l = \sum_{j=1}^{k}\left(\sum_{i=1}^{l} s_i r_{ij}\right)a_j$. Next, $s_i r_{ij} \in (B^+ \setminus \{0\})(B^+ \setminus \{0\}) \subseteq B^+ \setminus \{0\}$ and*

$$\sum_{j=1}^{k}\left(\sum_{i=1}^{l} s_i r_{ij}\right) = \sum_{i=1}^{l} s_i \sum_{j=1}^{k} r_{ij} = \sum_{i=1}^{l} s_i = 1.$$

According to Lemma 19, $\sum_{i=1}^{l} s_i r_{ij} \in B^+ \setminus \{0,1\}$. As a consequence, $s_1b_1+\cdots+s_lb_l = \sum_{j=1}^{k}\left(\sum_{i=1}^{l} s_i r_{ij}\right)a_j \in \Delta(a_1,\ldots,a_k)$.

The following technical lemma shows that geometric simplices behave well towards translations and linearity.

Lemma 32 *Let M,N be modules over a ring R. Let B^+ be a unit segment of R. Let $a_1,\ldots,a_k \in M$. Then:*

1. *If $m \in M$, then $m + \Delta[a_1,\ldots,a_k] = \Delta[m+a_1,\ldots,m+a_k]$.*
2. *If $T : M \to N$ is linear, then $T(\Delta[a_1,\ldots,a_k]) = \Delta[T(a_1),\ldots,T(a_k)]$.*
3. *If $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$ and $B^+r = rB^+$, then $r\Delta[a_1,\ldots,a_k] = \Delta[ra_1,\ldots,ra_k]$.*

Proof 95 *We will show the last item. The others are trivial. Let $r_1,\ldots,r_k \in B^+$ such that $\sum_{i=1}^{k} r_i = 1$. For every $i \in \{1,\ldots,k\}$, there exists $s_i \in B^+$ with $r_i r = r s_i$. According to Lemma 12, we have that $\sum_{i=1}^{k} s_i = 1$. Finally*

$$\sum_{i=1}^{k} r_i(ra_i) = \sum_{i=1}^{k}(r_i r)a_i = \sum_{i=1}^{k}(rs_i)a_i = r\sum_{i=1}^{k} s_i a_i.$$

This shows that $\Delta[ra_1,\ldots,ra_k] \subseteq r\Delta[a_1,\ldots,a_k]$. The reverse inclusion can be proved similarly.

Since geometric simplices behave well towards translations and linearity, so does convexity.

Corollary 4 *Let M, N be modules over a ring R. Let B^+ be a unit segment of R. Let $A \subseteq M$ be convex. Then:*

1. *If $m \in M$, then $m + A$ is convex.*
2. *If $T : M \to N$ is linear, then $T(A)$ is also convex.*
3. *If $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$ and $B^+r = rB^+$, then rA is convex as well.*

The following technical lemma relates geometric simplices with affine intervals.

Lemma 33 *Let M be a module over a ring R. Let B^+ be a unit segment of R. If $a, b \in M$, then $2\Delta[a, b] = (a + b) + (1 + 2B^-)(b - a)$.*

Proof 96 *Fix an arbitrary $s = 1 - 2r \in 1 + 2B^-$ with $r \in B^+$. Then $(a + b) + s(b - a) = (1 - s)a + (1 + s)b = (2r)a + (2 - 2r)b = 2(ra + (1 - r)b) \in 2\Delta[a, b]$. Conversely, let $r \in B^+$. Then $2(ra + (1 - r)b) = (2r)a + (2 - 2r)b = (1 - s)a + (1 + s)b = (a + b) + s(b - a) \in (a + b) + (1 + 2B^-)(b - a)$, where $s = 1 - 2r \in 1 + 2B^-$.*

The following proposition characterizes convexity in the classical way.

Proposition 41 *Let M be a module over a ring R. Let B^+ be a completely full unit segment of R. Let $A \subseteq M$. The following conditions are equivalent:*

1. *A is convex.*
2. *For every $a, b \in A$, $\Delta[a, b] \subseteq A$.*

Proof 97 *By definition, if A is convex, then $\Delta[a, b] \subseteq A$ for every $a, b \in A$. So, assume that for every $a, b \in A$, $\Delta[a, b] \subseteq A$. Consider $a_1, \ldots, a_k \in A$ and $r_1, \ldots, r_k \in B^+$ with $\sum_{i=1}^{k} r_1 = 1$. We have to prove that $\sum_{i=1}^{k} r_i a_i \in A$. We will proceed by induction on $k \in \mathbb{N}$. For $k = 1$, it is trivial and, for $k = 2$, it is our hypothesis. So, assume that the result holds for $k \geq 2$ and let us prove it for $k + 1$. We may assume without any loss of generality that at least one $r_i \neq 1$, say $r_1 \neq 1$ (otherwise, if $r_1 = \cdots = r_{k+1} = 1$, then $\mathrm{char}(R) \neq 0$ contradicting Lemma 19(3)). Since B^+ is completely full, we have that $1 - r_1 = r_2 + \cdots + r_{k+1} \in B^+ \setminus \{0\} \subseteq \mathcal{U}(R)$ and $(1 - r_1)^{-1} r_i \in B^+$ for all $i \in \{2, \ldots, k + 1\}$ in virtue of Theorem 16. On the other hand,*

$$\sum_{i=2}^{k+1} (1 - r_1)^{-1} r_i = (1 - r_1)^{-1} \sum_{i=2}^{k+1} r_i = (1 - r_1)^{-1}(1 - r_1) = 1.$$

This means that $\sum_{i=2}^{k+1} (1 - r_1)^{-1} r_i a_i \in A$ by the induction hypothesis. Finally,

$$\begin{aligned} \sum_{i=1}^{k+1} r_i a_i &= r_1 a_1 + (1 - r_1) \sum_{i=2}^{k+1} (1 - r_1)^{-1} r_i a_i \\ &\in \Delta \left[a_1, \sum_{i=2}^{k+1} (1 - r_1)^{-1} r_i a_i \right] \subseteq A. \end{aligned}$$

The following lemma allows the split of a segment into the union of two segments.

Lemma 34 *Let M be a module over a ring. Let B^+ be a full unit segment of R. Let $a, b \in R$. Then*

1. $\Delta[c_1, c_2] \subseteq \Delta[a, b]$ *for all* $c_1, c_2 \in \Delta[a, b]$.

2. *If* B^+ *is completely and totally full, then* $\Delta[a, b] = \Delta[a, c] \cup \Delta[c, b]$ *for all* $c \in \Delta[a, b]$.

Proof 98 *The proof will follow as itemized as the statement of the lemma.*

1. *Let* $r_i, s_i \in B^+$ *such that* $1 = r_i + s_i$ *and* $c_i = r_i a + s_i b$ *for* $i = 1, 2$. *Let* $r, s \in B^+$ *with* $r + s = 1$. *Then*

$$r(r_1 a + s_1 b) + s(r_2 a + s_2 b) = (rr_1 + sr_2)a + (rs_1 + ss_2)b.$$

Note that

$$rr_1 + sr_2 + rs_1 + ss_2 = 1$$

and $rr_1, sr_2, rs_1, ss_2 \in B^+B^+ = B^+$. *Since B+ is full by hypothesis,* $rr_1 + sr_2, rs_1 + ss_2 \in B^+$ *in virtue of Lemma 19(1). As a consequence,* $rc_1 + sc_2 \in \Delta[a, b]$.

2. *By Lemma 34(1),* $\Delta[a, b] \supseteq \Delta[a, c] \cup \Delta[c, b]$ *for all* $c \in \Delta[a, b]$. *Let us prove the reverse inclusion. Fix an arbitrary* $c \in \Delta[a, b]$ *and write* $c = r_0 a + s_0 b$ *with* $r_0, s_0 \in B^+$ *and* $r_0 + s_0 = 1$. *Now, take any* $x = ra + sb \in \Delta[a, b]$ *with* $r, s \in B^+$ *and* $r + s = 1$. *Since* B^+ *is completely and totally full, according to Theorem 16, we have that either* $ss_0^{-1} \in B^+$ *or* $rs_0^{-1} \in B^+$ *or* $sr_0^{-1} \in B^+$ *or* $rr_0^{-1} \in B^+$. *We may assume without any loss of generality that* $ss_0^{-1} \in B^+$. *Then*

$$\begin{aligned}
x &= ra + sb \\
&= (1 - s)a + \left(ss_0^{-1}s_0\right) b \\
&= \left(1 - ss_0^{-1}s_0\right) a + \left(ss_0^{-1}s_0\right) b \\
&= \left(1 - ss_0^{-1}(1 - r_0)\right) a + \left(ss_0^{-1}s_0\right) b \\
&= \left(1 - ss_0^{-1} + ss_0^{-1}r_0\right) a + \left(ss_0^{-1}s_0\right) b \\
&= \left(1 - ss_0^{-1}\right) a + ss_0^{-1}r_0 a + \left(ss_0^{-1}s_0\right) b \\
&= \left(1 - ss_0^{-1}\right) a + ss_0^{-1}\left(r_0 a + s_0 b\right) \\
&= \left(1 - ss_0^{-1}\right) a + \left(ss_0^{-1}\right) c \\
&\in \Delta[a, c].
\end{aligned}$$

Let M be a module over a ring R. Let B^+ be a unit segment of R. The nonempty intersection of convex sets is convex. This allows to define the convex hull of a subset $A \subseteq M$ as the intersection of all convex subsets containing A. The convex hull of A will be denoted by $\mathrm{co}(A)$. Observe that, in view of Theorem 5, the convex hull is a hull operator. In fact, the family of all convex subsets of M together with the empty set forms a convex structure (see Definition 44).

Theorem 23 *Let* M *be a module over a ring* R. *Let* B^+ *be a unit segment of* R. *If* $A \subseteq M$, *then*

$$\begin{aligned}
\mathrm{co}(A) &= \bigcup \{\Delta[a_1, \ldots, a_k] : a_1, \ldots, a_k \in A, k \in \mathbb{N}\} \\
&= \left\{ \sum_{i=1}^{k} r_i a_i : a_1, \ldots, a_k \in A, r_1, \ldots, r_k \in B^+, \sum_{i=1}^{k} r_i = 1, k \in \mathbb{N} \right\}.
\end{aligned}$$

Proof 99 *Denote* $H := \bigcup \{\Delta[a_1, \ldots, a_k] : a_1, \ldots, a_k \in A, k \in \mathbb{N}\}$. *If* D *is a convex subset of* M *containing* A, *then* $H \subseteq D$ *by definition. In particular,* $H \subseteq \mathrm{co}(A)$. *Notice that* $A \subseteq H$

so if we prove that H is convex, then we can conclude that $H = \mathrm{co}(A)$. Consider $a_{ij} \in A$ and $s_j, r_{ij} \in B^+$ for $i = 1, \ldots, k_j$ and $j = 1, \ldots, l$ such that $\sum_{j=1}^{l} s_j = 1 = \sum_{i=1}^{k_j} r_{ij}$ for all $j = 1, \ldots, l$. Notice that

$$\sum_{j=1}^{l} s_j \sum_{i=1}^{k_j} r_{ij} a_{ij} = \sum_{j=1}^{l} \sum_{i=1}^{k_j} s_j r_{ij} a_{ij} \in \Delta\left[a_{11}, \ldots, a_{k_1 1}, \ldots, a_{1l}, \ldots, a_{k_l l}\right] \subseteq H$$

because $s_j r_{ij} \in B^+ B^+ = B^+$ for all $i = 1, \ldots, k_j$ and all $j = 1, \ldots, l$ and

$$\sum_{j=1}^{l} \sum_{i=1}^{k_j} s_j r_{ij} = 1.$$

An immediate consequence of Theorem 23 together with Lemma 32 is the following corollary.

Corollary 5 *Let M, N be modules over a ring R. Let B^+ be a unit segment of R. Let $A \subseteq M$. Then:*

1. *If $m \in M$, then $m + \mathrm{co}(A) = \mathrm{co}(m + A)$.*
2. *If $T : M \to N$ is linear, then $T(\mathrm{co}(A)) = \mathrm{co}(T(A))$.*
3. *If $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$ and $B^+ r = r B^+$, then $r\,\mathrm{co}(A) = \mathrm{co}(rA)$.*

Next result is also a direct consequence of Theorem 23.

Corollary 6 *Let M be a module over a ring. Let B^+ a full unit segment of R. Then $\Delta[a_1, \ldots, a_k] = \mathrm{co}(\{a_1, \ldots, a_k\})$ for all $a_1, \ldots, a_k \in M$.*

Proof 100 *According to Proposition 40, $\Delta[a_1, \ldots, a_k]$ is convex and contains $\{a_1, \ldots, a_k\}$, hence it contains $\mathrm{co}(\{a_1, \ldots, a_k\})$. Conversely, by bearing in mind Theorem 23, we have that $\Delta[a_1, \ldots, a_k] \subseteq \mathrm{co}(\{a_1, \ldots, a_k\})$.*

The following theorem unveils the expression of the convex hull of a finite union of convex sets.

Theorem 24 *Let M be a module over an integral domain R. Let B^+ be a completely full unit segment of R. If $A_1, \ldots, A_k \subseteq M$ are convex, then*

$$\begin{aligned} \mathrm{co}\left(\bigcup_{i=1}^{k} A_i\right) &= \bigcup \{\Delta[a_1, \ldots, a_k] : a_1 \in A_1, \ldots, a_k \in A_k\} \\ &= \left\{\sum_{i=1}^{k} r_i a_i : \forall i \in \{1, \ldots, k\}\ a_i \in A_i, r_i \in B^+, \sum_{i=1}^{k} r_i = 1\right\}. \end{aligned}$$

Proof 101 *Denote $K := \bigcup \{\Delta[a_1, \ldots, a_k] : a_1 \in A_1, \ldots, a_k \in A_k\}$. By definition, K is contained in any convex subset containing $\bigcup_{i=1}^{k} A_i$. So in particular, $K \subseteq \mathrm{co}\left(\bigcup_{i=1}^{k} A_i\right)$. It is clear that $\bigcup_{i=1}^{k} A_i \subseteq K$, so if we show that K is convex, then we can deduce that $K = \mathrm{co}\left(\bigcup_{i=1}^{k} A_i\right)$. Consider $a_{ij} \in A_i$ and $s_j, r_{ij} \in B^+ \setminus \{0\}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$ such that $\sum_{j=1}^{l} s_j = 1 = \sum_{i=1}^{k} r_{ij}$ for all $j = 1, \ldots, l$. For simplification purposes, denote $c_j := \sum_{i=1}^{k} r_{ij} a_{ij} \in K$ for $j = 1, \ldots, l$ and $t_i := \sum_{j=1}^{l} s_j r_{ij}$ for*

$i = 1, \ldots, k$. We have to prove that $\sum_{j=1}^{l} s_j c_j \in K$. In the first place, B^+ is multiplicatively idempotent so $s_j r_{ij} \in B^+B^+ = B^+$ for all $i = 1, \ldots, k$ and all $j = 1, \ldots, l$. Then $\{s_j r_{ij} : i = 1, \ldots, k, j = 1, \ldots, l\} \subseteq B^+ \setminus \{0\}$ because R is an integral domain. Also

$$\sum_{i=1}^{k} t_i = \sum_{i=1}^{k} \sum_{j=1}^{l} s_j r_{ij} = \sum_{j=1}^{l} s_j \sum_{i=1}^{k} r_{ij} = 1. \tag{1.25}$$

By gluing together Lemma 19 and Theorem 16, we obtain that $t_i = \sum_{j=1}^{l} s_j r_{ij} \in B^+ \setminus \{0\} \subseteq \mathcal{U}(R)$ for every $i \in \{1, \ldots, k\}$ and $t_i^{-1} s_{ij} r_{ij} \in B^+$ for every $j \in \{1, \ldots, l\}$ and every $i \in \{1, \ldots, k\}$. Finally, by bearing in mind Equation (1.25)*, we have that*

$$\begin{aligned} \sum_{j=1}^{l} s_j c_j &= \sum_{j=1}^{l} s_j \sum_{i=1}^{k} r_{ij} a_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{l} s_j r_{ij} a_{ij} = \sum_{i=1}^{k} t_i \sum_{j=1}^{l} \left(t_i^{-1} s_j r_{ij}\right) a_{ij} \\ &= \sum_{i=1}^{k} t_i d_i \in \Delta\left[d_1, \ldots, d_k\right] \subseteq K, \end{aligned}$$

where

$$d_i := \sum_{j=1}^{l} \left(t_i^{-1} s_j r_{ij}\right) a_{ij} \in A_i$$

for all $i \in \{1, \ldots, k\}$, because $t_i^{-1} s_{ij} r_{ij} \in B^+$ for every $j \in \{1, \ldots, l\}$ and

$$\sum_{j=1}^{l} t_i^{-1} s_j r_{ij} = t_i^{-1} t_i = 1.$$

A particular case of convex hull is the one given by the notion of drop.

Definition 118 (Drop) *Let M be a module over a ring R. Let B^+ be a unit segment of R. Let $A \subseteq M$ convex. Let $m \in M \setminus A$. The drop of A and m is defined by* $\operatorname{drop}(m, A) := \operatorname{co}(\{m\} \cup A)$.

Under the settings of the previous definition and in view of Theorem 24, if R is an integral domain and B^+ is completely full, then $\operatorname{drop}(m, A) = \{rm + sa : r, s \in B^+, r + s = 1\}$.

Definition 119 (Convex component) *Let M be a module over a ring R. Let B^+ be a unit segment of R. Let A be a nonempty subset of M. A convex component of A is a maximal convex subset of A.*

The existence of convex components is guaranteed by the Zorn's Lemma.

Proposition 42 *Let M be a module over a ring R. Let B^+ be a unit segment of R. Let A be a nonempty subset of M. Every convex subset of A is contained in a convex component of A.*

Proof 102 *Take a convex subset C of A and consider the set $\mathcal{L} := \{D \subseteq A : C \subseteq D$ and D is convex$\}$ partially ordered by the inclusion. Note that $\mathcal{L} \neq \varnothing$ because $C \in \mathcal{L}$. On the other hand, if $\mathcal{C}$ is a totally ordered subset of $\mathcal{L}$, then $\bigcup \mathcal{C} \in \mathcal{L}$. Indeed, take $a_1, \ldots, a_k \in \bigcup \mathcal{C}$. There are $C_1, \ldots, C_k \in \mathcal{C}$ such that $a_i \in C_i$ for all $i \in \{1, \ldots, k\}$. Since $\mathcal{C}$ is totally ordered, we may assume without any loss of generality that $C_1 \subseteq \cdots \subseteq C_k$. Then $a_1, \ldots, a_k \in C_k$ and hence $\Delta[a_1, \ldots, a_k] \subseteq C_k \subseteq \bigcup \mathcal{C}$ because C_k is convex. This shows that $\bigcup \mathcal{C}$ is convex as well. As a consequence, Zorn's Lemma guarantees that $\mathcal{L}$ has maximal elements. It is immediate that these maximal elements are convex components of A.*

In [10, Definition 3.12] and [11, Definition 4.1], the notion of 2-extreme point for rings and modules was introduced.

Definition 120 (2-Extreme point) *Let M be a module over a ring R. Let $e \in A \subseteq M$. We will say that e is a 2-extreme point of A provided that the condition $2e = a + b$, with $a, b \in A$, implies that $a = b = e$. The set of 2-extreme points of A is denoted by* $\text{ext}_2(A)$.

Next, we will make use of unit segments to introduce the concept of an extreme point [12].

Definition 121 (Extreme point) *Let M be a module over a ring R. Let B^+ be a unit segment of R. Let $e \in A \subseteq M$. We will say that e is an extreme point of A provided that, for all $a, b \in A$, if $e \in \Delta[a, b]$, then either $a = e$ or $b = e$. The set of extreme points of A is denoted by* $\text{ext}(A)$.

The notion of extremal set [12] also relies on unit segments.

Definition 122 (Extremal set) *Let M be a module over a ring R. Let B^+ be a nontrivial unit segment of R. Let $E \subseteq A \subseteq M$. We will say E is extremal in A if, for any $a, b \in A$ such that $E \cap \Delta(a, b) \neq \varnothing$, then $\{a, b\} \subseteq E$.*

Notice that, under the settings of the previous definition, if F is extremal in E and E is extremal in A, then F is extremal in A. Also, the nonempty intersection of extremal subsets is extremal.

Definition 123 (Extremal point) *Let M be a module over a ring R. Let B^+ be a nontrivial unit segment of R. Let $e \in A \subseteq M$. We will say that e is an extremal point of A if $\{e\}$ is extremal in A. The set of extremal points of A is denoted by* $\text{extl}(A)$.

The following proposition reunites all the most basic relationships between the previous concepts. Recall that a unit segment is called centered provided that $2 := 1 + 1 \in \mathcal{U}(R)$ and $2^{-1} \in B^+$.

Proposition 43 *Let M be a module over a ring R. Let B^+ be a nontrivial unit segment of R. Take $A \subseteq M$. Then:*

1. $\text{extl}(A) \subseteq \text{ext}(A)$.
2. *If M is torsionfree and R is an integral domain, then* $\text{extl}(A) = \text{ext}(A)$.
3. *If B^+ is centered, then* $\text{ext}(A) \subseteq \text{ext}_2(A)$.
4. *If M is torsionfree, R is an integral domain, B^+ is strong, and A is convex, then* $\text{ext}_2(A) \subseteq \text{extl}(A)$.

Proof 103 *The proof will follow as itemized as the statement of the proposition.*

1. *Fix any arbitrary $e \in \text{extl}(A)$. Suppose to the contrary that there exist $a, b \in A$ satisfying that $e \in \Delta[a, b] \setminus \{a, b\}$. According to Remark 40, $\Delta[a, b] \setminus \{a, b\} \subseteq \Delta(a, b)$. Since $e \in \text{extl}(A)$, we reach the contradiction that $\{a, b\} \subseteq \{e\}$.*
2. *Take any $e \in \text{ext}(A)$. Let $a, b \in A$ such that $e \in \Delta(a, b)$. We have to prove that $\{a, b\} \subseteq \{e\}$. If $a = b$, then $e \in \Delta(a, a) = \{a\}$, so we are done. Let us assume then that $a \neq b$. There are $r, s \in B^+ \setminus \{0, 1\}$ with $r + s = 1$ such that $e = ra + sb$. By hypothesis, either $a = e$ or $b = e$. Let us assume, without any loss of generality, that $a = e$. Then $a = ra + sb$, hence $s(a - b) = 0$. Since M is torsionfree and $a \neq b$, we conclude that s is a zero divisor. Since R is an integral domain, $s = 0$, which is a contradiction. As a consequence, $a = b = e$.*

3. *Take any* $e \in \text{ext}(A)$. *Suppose that* $2e = a+b$ *with* $a, b \in A$. *Then* $e = 2^{-1}a+2^{-1}b$. *By hypothesis,* $2^{-1} \in B^+$, *therefore,* $e \in \Delta(a,b) \subseteq \Delta[a,b]$. *Since* $e \in \text{ext}(A)$, *either* $a = e$ *or* $b = e$. *Suppose, without any loss of generality, that* $a = e$. *Then* $2e = a+b = e+b$, *hence* $e = b$. *This shows that* $e \in \text{ext}_2(A)$.

4. *Take* $e \in \text{ext}_2(A)$ *and write* $e = ra+sb$ *with* $a, b \in A$, $r, s \in B^+ \setminus \{0,1\}$, *and* $r+s = 1$. *We have to prove that* $\{a,b\} \subseteq \{e\}$. *If* $a = b$, *then* $e = ra + sa = (r+s)a = a = b$. *Thus, let us suppose that* $a \neq b \in A$. *By hypothesis, we may assume, without any loss of generality, that* $2r \in B^+$. *Then* $2e = 2(ra + (1-r)b) = b + ((2r)a + (1-2r)b)$. *Observe that* $(2r)a + (1-2r)b \in \Delta[a,b] \subseteq A$. *Since* $e \in \text{ext}_2(A)$, *we conclude that* $b = (2r)a + (1-2r)b$, *which implies that* $(2r)(a-b) = 0$. *Since* M *is torsionfree and* R *is an integral domain, we have that* $2r = 0$. *In accordance with Proposition 17, the existence of a unit segment* B^+ *in* R *implies that* $\text{char}(R) \neq 2$. *Then* $r = 0$. *This is a contradiction. Therefore,* $a = b = e$.

The following corollary is an immediate consequence of Proposition 43.

Corollary 7 *Let* M *be a torsionfree module over an integral domain ring* R *such that* $2 \in \mathcal{U}(R)$. *Let* B^+ *be a strong and centered unit segment of* R. *For every convex subset* $A \subseteq M$, $\text{ext}_2(A) = \text{extl}(A) = \text{ext}(A)$.

Next lemma characterizes when the neutral element is a 2-extreme point.

Lemma 35 *Let* R *be a ring. Let* M *be an* R*-module. Let* $A \subseteq M$. *Then* $A \cap -A = \{0\}$ *if and only if* $0 \in \text{ext}_2(A)$.

Proof 104 *Suppose first that* $0 \in \text{ext}_2(A)$. *Let* $a_1, a_2 \in A$ *such that* $a_1 = -a_2$. *Then* $2 \cdot 0 = a_1 + a_2$. *Since* $0 \in \text{ext}_2(A)$, *we conclude that* $0 = a_1 = a_2$. *Conversely, suppose that* $A \cap -A = \{0\}$ *and let* $a_1, a_2 \in A$ *such that* $2 \cdot 0 = a_1 + a_2$. *Then* $a_1 = -a_2 \in A \cap -A = \{0\}$, *meaning that* $a_1 = a_2 = 0$.

Next lemma unveils extremal properties of extremal subsets of convex sets.

Lemma 36 *Let* M *be a module over a ring* R. *Let* B^+ *be a completely full nontrivial unit segment of* R. *Let* $E \subseteq A \subseteq M$ *such that* A *is convex and* E *is extremal in* A. *For every* $a_1, \ldots, a_k \in A$, *if* $E \cap \Delta(a_1, \ldots, a_k) \neq \varnothing$, *then* $\{a_1, \ldots, a_k\} \subseteq E$.

Proof 105 *Take* $e \in E$ *and* $r_1, \ldots, r_k \in B^+ \setminus \{0,1\}$ *with* $r_1 + \cdots + r_k = 1$ *and* $e = r_1a_1 + \cdots + r_ka_k$. *In accordance with Lemma 19(1),* $1 - r_1 = r_2 + \cdots + r_k \in B^+ \setminus \{0\}$. *Since* B^+ *is completely full, Theorem 16 assures that* $(1-r_1)^{-1}r_j \in B^+$ *for all* $j = 2, \ldots, k$. *Observe that*

$$\sum_{j=2}^{k}(1-r_1)^{-1}r_j = (1-r_1)^{-1}\sum_{j=2}^{k} r_j = (1-r_1)^{-1}(1-r_1) = 1,$$

meaning that

$$\sum_{j=2}^{k}(1-r_1)^{-1}r_ja_j \in \Delta[a_2, \ldots, a_k] \subseteq A.$$

Then

$$e = r_1a_1 + (1-r_1)\sum_{j=2}^{k}(1-r_1)^{-1}r_ja_j \in \Delta\left(a_1, \sum_{j=2}^{k}(1-r_1)^{-1}r_ja_j\right).$$

Finally, since E is extremal in A,

$$\left\{ a_1, \sum_{j=2}^{k} (1-r_1)^{-1} r_j a_j \right\} \subseteq A.$$

By interchanging the roles of the a_i's, we end up having that $\{a_1, \ldots, a_k\} \subseteq E$.

The final result in this subsubsection is a proposition which reveals how extremality behaves under linearity.

Proposition 44 *Let M, N be modules over a ring R. Let B^+ be a nontrivial unit segment of R. Let $E \subseteq A \subseteq M$. Then:*

1. *If M is torsionfree and $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$ verifies that $rB^+ = B^+r$, then E is extremal in A if and only if rE is extremal in rA. In particular,* $\mathrm{extl}(rA) = r\mathrm{extl}(A)$.
2. *E is extremal in A if and only if $m + E$ is extremal in $m + A$ for all $m \in M$. In particular,* $\mathrm{extl}(m + A) = m + \mathrm{extl}(A)$.
3. *If $T : M \to N$ is injective and linear, then $T(E)$ is extremal in $T(A)$ if and only if E is extremal in A. In particular,* $T(\mathrm{extl}(A)) = \mathrm{extl}(T(A))$.

Proof 106 *We will only prove the first item. The others are trivial.*

- $\Rightarrow$ *Suppose first that E is extremal in A. Let $a, b \in A$ such that $rE \cap \Delta(ra, rb) \neq \varnothing$. Take $e \in E$ and $t, s \in B^+ \setminus \{0, 1\}$ with $t + s = 1$ such that $re = t(ra) + s(rb)$. By hypothesis, $re = r(t'a + s'b)$ for some $t', s' \in B^+$. In view of Lemma 12, $t' + s' = 1$. Since $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$, it can be easily shown that $t', s' \in B^+ \setminus \{0, 1\}$. Then $t'a + s'b \in \Delta(a, b)$. Next, M is torsionfree by hypothesis; therefore, $e = t'a + s'b \in \Delta(a, b)$. Since E is extremal in A, we conclude that $\{a, b\} \subseteq E$. In other words, $\{ra, rb\} \subseteq rE$.*
- $\Leftarrow$ *Conversely, suppose that rE is extremal in rA. Let $a, b \in A$ such that $E \cap \Delta(a, b) \neq \varnothing$. Take $e \in E$ and $t, s \in B^+ \setminus \{0, 1\}$ with $t + s = 1$ such that $e = ta + sb$. Note that $re = r(ta) + r(sb)$. By hypothesis, $re = t'(ra) + s'(rb)$ for some $t', s' \in B^+$. In view of Lemma 12, $t' + s' = 1$. Since $r \in R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$, it can be easily shown that $t', s' \in B^+ \setminus \{0, 1\}$. Then $re = t'(ra) + s'(rb) \in \Delta(ra, rb)$. Since rE is extremal in rA, we conclude that $\{ra, rb\} \subseteq rE$. Next, M is torsionfree by hypothesis; therefore, $\{a, b\} \subseteq E$.*

1.2.4.5 Cones

Abstract cones are, essentially, analogous to modules over ordered rings, except that we take the nonnegative elements of the rings as scalars.

Definition 124 (Cone) *A left cone over a unital ordered ring $((R, \leq), \{+, 0, -, \cdot, 1\}, \varnothing)$ is a universal algebra $(C, \{+, 0\}, \{\cdot\})$, where $(C, \{+, 0\}, \varnothing)$ is a strict, cancellative, commutative, additive monoid endowed with a distributive (biadditive) left action $\cdot : R^+ \times C \to C$.*

Right cones can be similarly defined, although left cones will be considered by default unless otherwise explicitly stated. For simplicity, whenever R is a unital ordered ring, we will be referring to cones over R as R^+-cones.

Example 25 *If R is a unital ordered cone, then R^+ is a cone. In fact, if M is an ordered R-module, then M^+ is also a cone.*

Strict, cancellative, commutative, additive monoids can be extended to commutative additive groups by establishing a simple equivalence relation.

Remark 41 *Let C be a strict, cancellative, commutative, additive monoid. The binary internal relation on $C \times C$ given by $(c,d) \sim (c',d') \Leftrightarrow c+d' = c'+d$ endoses the quotient set ${}^{C\times C}/_{\sim}$ with structure of commutative additive group via $[(c,d)]_\sim + [(c',d')]_\sim := [(c+c', d+d')]_\sim$. Observe that, as subsets of the monoid $C \times C$, $[(c,d)]_\sim + [(c',d')]_\sim = [(c+c', d+d')]_\sim$. The embedding*

$$\begin{array}{ccc} C & \to & {}^{C\times C}/_{\sim} \\ c & \mapsto & [(c,0)]_\sim \end{array} \tag{1.26}$$

is an isomorphism of additive monoids over its image. If, in addition, C is an R^+-cone for R a unital ordered ring, then ${}^{C\times C}/_{\sim}$ can be endowed with structure of R-module via $r[(c,d)]_\sim := [(rc,rd)]_\sim$. As above, as subsets of the R^+-cone $C \times C$, $r[(c,d)]_\sim \subseteq [(rc,rd)]_\sim$. In this situation, (1.26) *is an isomorphism of R^+-cones over its image.*

The notion of convexity considered in the previous subsubsection for modules can be easily extended to cones.

Definition 125 (Convex set) *A subset D of an R^+-cone, for R a unital ordered ring, is said to be convex provided that for all $d, d' \in D$, the geometric simplex $\Delta[d,d'] := \{rd+sd' : 0 \leq r,s \leq 1,\ r+s=1\} \subseteq D$ for all $d,d' \in D$.*

A partial order can be defined on abstract cones, which is going to be translation- and homothety-invariant.

Remark 42 *Let R be a unital ordered ring. Let C be an R^+-cone. The following is a partial order:*

$$c \leq d \Leftrightarrow \exists c' \text{ with } c+c' = d.$$

The following properties are satisfied for each $c,c',d,d' \in C$ and each $r,s \in R^+$:

1. *c' is unique verifying $c+c'=d$.*
2. *$0 = \min(C)$.*
3. *$c+d' \leq d+d'$ if and only if $c \leq d$.*
4. *If $c+d = c'+d'$, $c \leq c'$ and $d \leq d'$, then $c=c'$ and $d=d'$.*
5. *If $c \leq d$, then $rc \leq rd$.*
6. *If $r \leq s$, then $rc \leq sc$.*

Recall that, in a poset X, the downward closure of a subset $Y \subseteq X$ is given by $\downarrow Y := \bigcup_{y\in Y} \downarrow y$, where $\forall y \in Y \downarrow y := \{x \in X : x \leq y\}$. Another notation for $\downarrow Y$ is $\mathcal{I}(Y)$.

Lemma 37 *Let R be a unital ordered ring. Let C be an R^+-cone. If D is a convex subset of C, then its downward closure $\downarrow D$ is convex as well.*

Proof 107 *Let $x,y \in \downarrow D$. Fix arbitrary elements $0 \leq r,s \leq 1$ with $r+s=1$. There are $d,d' \in D$ such that $x \leq d, y \leq d'$. Then $rx+sy \leq rd+sd' \in D$, meaning that $rx+sy \in \downarrow D$.*

Since cones are universal algebras, it makes sense to talk about subcones and about operators of cones. The operators of cones are called linear maps or linear operators, like in the module settings, and they are additive and R^+-homogeneous maps.

Lemma 38 *Let R be a unital ordered ring. Let C,D be R^+-cones. If $T : C \to D$ is R^+-linear, then T is monotone.*

Proof 108 *Let $c \leq c'$ in C. There exists $c_0 \in C$ with $c+c_0 = c'$. Then by additivity, $f(c)+f(c_0) = f(c+c_0) = f(c')$, meaning that $f(c) \leq f(c')$ in D.*

1.2.4.6 Algebras

Every module over a commutative ring is trivially a left and right module over that ring. Also, every ring is trivially a left and right module over itself. These facts motivate the study of bimodules.

Definition 126 (Bimodule) *A bimodule over rings* $(R, \{+, 0, -, \cdot, 1\}, \varnothing), (S, \{+, 0, -, \cdot, 1\}, \varnothing)$ *is a universal algebra* $(M, \{+, 0, -\}, \{\cdot, \cdot\})$*, where* $(M, \{+, 0, -\}, \{\cdot\})$ *is a left* R*-module,* $(M, \{+, 0, -\}, \{\cdot\})$ *is a right* S*-module, and* $(rm)s = r(ms)$ *for all* $r \in R$*, all* $s \in S$*, and all* $m \in M$*.*

Every module over a commutative ring is trivially a bimodule over that ring. Every ring is a bimodule over itself by seeing the multiplication as a left and right action. The submodules of a ring as a bimodule over itself are precisely its two-sided ideals.

Remark 43 *If* M *is a left module over a* $*$*-ring* R*, then* M *can be given structure of right* R*-module as follows:* $mr := r^*m$ *for all* $m \in M$ *and all* $r \in R$*. Indeed,* $m(rs) = (rs)^*m = (s^*r^*)\, m = s^*\,(r^*m) = s^*(mr) = (mr)s$ *for all* $m \in M$ *and all* $r, s \in R$*.*

As a consequence of the previous remark, we will not distinguish between left and right for modules over $*$-rings.

Notation 11 *If* M, N *are* (R, S)*-bimodules and there is no confusion with the rings* R, S*, then* $\mathcal{L}(M, N)$ *will stand for the additive group of* (R, S)*-linear maps. If there is confusion, then we will denote it by* $\mathcal{L}_{(R,S)}(M, N)$*. As expected,* $\mathcal{L}_R(M, N)$ *denotes the additive group of* R*-linear maps, and* $\mathcal{L}_S(M, N)$ *stands for the additive group of* S*-linear maps.*

The bimodule structure endows the additive group of linear maps with a module structure.

Remark 44 *Let* R, S *be rings. Let* M, N *be left* R*-modules. Then:*

1. *If* M *is an* (R, S)*-bimodule, then* $\mathcal{L}_R(M, N)$ *is a left* S*-module by taking the left action* $(sf)(m) := f(ms)$*. Nevertheless,* $\mathcal{L}_R(M)$ *is not, in general, an* $(S, \mathcal{L}_R(M))$*-bimodule. Indeed, fixing arbitrary elements* $f, g \in \mathcal{L}_R(M)$ *and* $s \in S$*, we cannot assure that* $s(f \circ g) = (sf) \circ g$ *unless, for instance,* g *is* S*-linear.*
2. *If* N *is an* (R, S)*-bimodule, then* $\mathcal{L}_R(M, N)$ *is a right* S*-module by taking the right action* $(fs)(m) := f(m)s$*. Nevertheless,* $\mathcal{L}_R(M)$ *is not, in general, an* $(\mathcal{L}_R(M), S)$*-bimodule. Indeed, fixing arbitrary elements* $f, g \in \mathcal{L}_R(M)$ *and* $s \in S$*, we cannot assure that* $(f \circ g)s = f \circ (gs)$ *unless, for instance,* f *is* S*-linear.*
3. $\mathcal{L}_R(M, R)$ *is a right* R*-module.*

If R, S are rings and M is an (R, S)-bimodule, then the left multiplication operator (see Definition 58),

$$\begin{array}{rccc} \mathsf{L}_r: & M & \to & M \\ & m & \mapsto & \mathsf{L}_r(m) := rm, \end{array}$$

is S-linear but not (R, S)-linear unless R is commutative.

Remark 45 *Let* R, S *be rings. Let* M *be an* (R, S)*-bimodule. The following map*

$$\begin{array}{rccc} \mathsf{L}: & R & \to & \mathcal{L}_S(M) \\ & r & \mapsto & \mathsf{L}_r \end{array}$$

is additive and multiplicative when considering the composition in $\mathcal{L}_S(M)$*. It also maps* 1 *to the identity on* M*,* $\mathsf{L}_1 = I_M$*; hence, it is an operator of rings, and, in particular,* $\mathsf{L}_{r^{-1}} = \mathsf{L}_r^{-1}$ *for all* $r \in \mathcal{U}(R)$*.*

The previous remark has its dual version for the right multiplication operator. The notion of balanced operator serves to construct tensor products.

Definition 127 (Balanced operator) *Let R be a ring. Let M, N be a right R-module and a left R-module, respectively. Let P be a commutative additive group. A map $f : M \times N \to P$ is said to be balanced provided that the following is satisfied for all $m, m' \in M$, all $n, n' \in N$, and all $r \in R$:*

- *Biadditivity: $f(m+m', n) = f(m, n)+f(m', n)$ and $f(m, n+n') = f(m, n)+f(m, n')$.*
- *Pairingness: $f(mr, n) = f(m, rn)$.*

The set of all balanced operators from $M \times N$ to P is denoted by $\mathcal{L}^{\mathtt{a}}(M \times N, P)$.

It is not hard to check that $\mathcal{L}^{\mathtt{a}}(M \times N, P)$ is a commutative group with the regular addition of operators. By means of the next definition, bilinear operators can be taken to the noncommutative setting.

Definition 128 (Bilinear operator) *Let R, S, T be rings. Let M, N, P be an (S, R)-bimodule, an (R, T)-bimodule, and an (S, T)-bimodule, respectively. A map $f : M \times N \to P$ is said to be bilinear provided that f is R-balanced and (S, T)-bihomogeneous, that is, $f(sm, n) = sf(m, n)$ and $f(m, nt) = f(m, n)t$ for all $m \in M$, all $n \in N$, all $s \in S$, and all $t \in T$. The set of all bilinear operators from $M \times N$ to P is denoted by $\mathcal{L}^2(M \times N, P)$.*

It is not hard to check that $\mathcal{L}^2(M \times N, P)$ is a commutative group with the regular addition of operators. If $R = S = T$ is a commutative ring, then $\mathcal{L}^2(M \times N, P)$ is also an R-bimodule. In this case, when $P := R$, we denote $\mathcal{L}^2(M \times N) := \mathcal{L}^2(M \times N, R)$.

Definition 129 ($*$-Bilinear operator) *Let M, N, P be modules over a $*$-ring R. A map $f : M \times N \to P$ is said to be $*$-bilinear provided that f is balanced and $*$-bihomogeneous, that is, $f(rm, n) = rf(m, n)$ and $f(m, nr) = f(m, n)r^*$ for all $m \in M$, all $n \in N$, and all $r \in R$. The set of all $*$-bilinear operators from $M \times N$ to P is denoted by $\mathcal{L}^2(M \times N, P)$.*

It is not hard to check that $\mathcal{L}^2(M \times N, P)$ is a commutative group with the regular addition of operators. If R is commutative, then $\mathcal{L}^2(M \times N, P)$ is also an R-bimodule. In this case, when $P := R$, we denote $\mathcal{L}^2(M \times N) := \mathcal{L}^2(M \times N, R)$.

Definition 130 (Hilbert bimodule) *A Hilbert bimodule over a $*$-ring $(R, \{+, 0, -, \cdot, 1, {}^*\}, \varnothing)$, is a universal algebra $(M, \{+, 0, -\}, \{\cdot, \cdot, (\ |\ )\})$ such that $(M, \{+, 0, -\}, \{\cdot\})$ is a left R-module and $(\ |\ ) : M \times M \to R$ is a binary external operation satisfying the following for all $l, m, n \in M$ and all $r \in R$:*

- *Biadditivity: $(l + m|n) = (l|n) + (m|n)$ and $(l|m + n) = (l|m) + (l|n)$.*
- *Left-R-homogeneousity: $(rm|n) = r(m|n)$.*
- *Right-$*$-R-homogeneousity: $(m|nr) = (m|n)r^*$.*
- *Balancedness: $(mr|n) = (m|rn)$.*
- *$*$-Symmetricity: $(m|n) = (n|m)^*$.*

Under the settings of Definition 130, $(\ |\ )$ is a $*$-bilinear operator.

Definition 131 (Algebra) *An algebra over a ring $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$ is a universal algebra $(A, \{+, 0, -, \cdot, 1\}, \{\cdot\})$, where $(A, \{+, 0, -, \cdot, 1\}, \varnothing)$ is a ring and $(A, \{+, 0, -\}, \{\cdot, \cdot\})$ is an (R, A)-bimodule in such a way that $r(ab) = a(rb)$ for all $r \in R$ and all $a, b \in A$.*

Under the settings of the above definition, the condition that $r(ab) = a(rb)$, for all $r \in R$ and all $a, b \in A$, is somehow restrictive on R as, in some cases, could imply that R be commutative.

Proposition 45 *Let A be an algebra over a ring R. For every $a, b \in A$ and every $r, s \in R$, $a[(rs)b] = a[(sr)b]$. In particular, the right action $ar := ra$, defined for all $r \in R$ and all $a \in A$, confers A with structure of right R-module. Furthermore, if R is an integral domain and A is R-torsionfree, then R is commutative.*

Proof 109 *Simply observe that $a[(rs)b] = (rs)(ab) = r[s(ab)] = r[(sa)b] = (sa)(rb) = s[a(rb)] = a[s(rb)] = a[(sr)b]$. If we take $a = b = 1$, then $(rs)1 = (sr)1$. As a consequence, if A is R-torsionfree and R is an integral domain, then $rs = sr$, meaning that R is commutative due to the arbitrariness of $r, s \in R$. Finally, note that $a(rs) = (rs)a = (sr)a = s(ra) = (ra)s = (ar)s$ for all $r, s \in R$ and all $a \in A$, meaning that the right action $ar := ra$, defined for all $r \in R$ and all $a \in A$, confers A with structure of right R-module in view of Remark 35.*

Since every division ring is trivially an integral domain and every module over a division ring is trivially torsionfree, we immediately obtain the following corollary.

Corollary 8 *If A is an algebra over a division ring R, then R is a field.*

Recall that every ring is a bimodule over its self. Also, notice that every ring is an algebra over its center.

Proposition 46 *Let R be a ring. Consider an algebra A over R. Let $B \subseteq A$. Then:*

1. *The subalgebra spanned by B is* $\operatorname{span}(B) = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \Big[R(B \cup \{1\} \cup -B) \overset{m}{\cdots} (B\cup\{1\}\cup -B) + \overset{n}{\cdots} + R(B\cup\{1\}\cup -B) \overset{m}{\cdots} (B\cup\{1\}\cup -B) \Big] = \Big\{ \sum_{i=1}^{n} r_i b_{i1}^{k_{i1}} \cdots b_{im_i}^{k_{im_i}} : n, m_i \in \mathbb{N},\ b_{ij} \in B,\ r_i \in R,\ k_{ij} \in \mathbb{Z}^+,\ j = 1, \ldots, m_i,\ i = 1, \ldots, n \Big\} = \bigcup \{ R[b_1, \ldots, b_n] : n \in \mathbb{N}, b_i \in B, i = 1, \ldots, n \}$.
2. *B' is a subalgebra of A.*
3. *$B' = \operatorname{span}(B)'$.*
4. *If B is a left ideal of A, then B is a subalgebra of A.*

Proof 110 *Only the last item will be shown. Indeed, simply observe that if $r \in R$ and $b \in B$, then $rb = r(1b) = (r1)b \in B$.*

The following example shows how to construct unit segments in algebras from unit segments in the underline ring.

Example 26 *Let A be a torsionfree algebra over an integral domain ring R. If B_R^+ is a unit segment of R, then $B_A^+ := B_R^+ 1$ is a unit segment of A. Indeed, it is straight forward to check that $0, 1 \in B_R^+ 1$ and $\left(B_R^+ 1\right)\left(B_R^+ 1\right) = B_R^+ 1$. Also, $1 + B_R^- 1 = \left(1 + B_R^-\right) 1 = B_R^+ 1$. Finally, $B_R^+ 1 \cap B_R^- 1 = \{0\}$ because A is torsionfree and R is an integral domain.*

As a consequence of the previous example, if A is a real algebra, then $[0, 1]1$ is a unit segment of A.

Definition 132 (Algebraically independent elements) *Let A be an algebra over a ring R. A finite subset $\{a_1, \ldots, a_n\} \subseteq A$ is said to be algebraically independent if the following linear unital-multiplicative map*

$$\begin{array}{rcl} R[x_1, \ldots, x_n] & \to & A \\ p(x_1, \ldots, x_n) & \to & p(a_1, \ldots, a_n) \end{array}$$

has null kernel.

The operators of algebras are precisely the unital-multiplicative linear maps. Notice that if $f : A \to B$ is an operator of algebras between two algebras A, B over a ring R, then $\ker(f)$ is a two-sided ideal of A and $f(A)$ is a subalgebra of B.

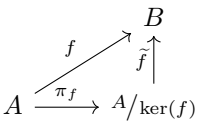
Remark 46 (Quotient algebra) *Let A be an algebra over a ring R. Let I be a two-sided ideal. Then A/I acquires structure of algebra under the internal product defined by $(a+I)(b+I) := ab+I$ and under the external product defined by $r(a+I) := ra+I$ for all $a, b \in A$ and all $r \in R$.*

The final result is the version of the First Isomorphism Theorem for algebras.

Theorem 25 (First Isomorphism Theorem) *Let A, B be algebras over a ring R. Let $f : A \to B$ be additive and multiplicative. Then*

$$\begin{array}{rrcl} \widetilde{f}: & {}^{A}/_{\ker(f)} & \to & f(A) \\ & a + \ker(f) & \mapsto & \widetilde{f}(a + \ker(f)) := f(a) \end{array}$$

is a (unique) algebra isomorphism.



1.2.5 Effect algebras and Boolean algebras

Effect algebras and Boolean algebras are examples of universal algebras which are in between Group Theory and Order Theory. They have diverse origins. For example, Boolean algebras have historically been involved in Measure Theory, Electronics, Computer Science, etc. However, effect algebras were originated from Quantum Mechanics. We refer the reader to [13] for a wide perspective on effect algebras and Boolean algebras.

1.2.5.1 Effect algebras

Effect algebras were introduced for the first time in [14] in the context of quantum mechanics.

Definition 133 (Effect algebra) *An effect algebra is a universal algebra $(L, \{\oplus, 0, 1, {}^{\perp}\}, \varnothing)$, where*

$$\begin{array}{rrcl} \oplus: & \Sigma \subseteq L \times L & \to & L \\ & (p, q) & \mapsto & p \oplus q \end{array}$$

is a partially defined binary internal operation,

$$\begin{array}{rrcl} {}^{\perp}: & L & \to & L \\ & p & \mapsto & p^{\perp} \end{array}$$

is a unary internal operation (called orthocomplementation), and

$$0: \begin{array}{ccc} L & \to & L \\ p & \mapsto & 0 \end{array} \quad \textit{and} \quad 1: \begin{array}{ccc} L & \to & L \\ p & \mapsto & 1 \end{array}$$

are nullary internal operations satisfying the following conditions for all $p, q, r \in L$*:*

- Commutativity: Σ *is a symmetric binary relation on* L *and if* $(p, q) \in \Sigma$*, then* $p \oplus q = q \oplus p$*.*
- Associativity: *If* $(q, r), (p, q \oplus r) \in \Sigma$*, then* $(p, q), (p \oplus q, r) \in \Sigma$ *and* $(p \oplus q) \oplus r = p \oplus (q \oplus r)$*.*
- Orthocomplementation: $p^{\perp}$ *is the only element in* L *such that* $\left(p, p^{\perp}\right) \in \Sigma$ *and* $p \oplus p^{\perp} = 1$*.*
- Zero-One Law: $1 \neq 0$ *and* 0 *is the only element in* L *such that* $(1, 0) \in \Sigma$*.*

Under the settings of the previous definition, two elements p, q of an effect algebra are said to be orthogonal if $p \oplus q$ exists, that is, $(p, q) \in \Sigma$. Whenever we write $p \oplus q$, we will be assuming that they are orthogonal.

Example 27 (Effect algebra of sets) *A effect algebra of sets is a subset* L *of the power set* $\mathcal{P}(X)$ *of a given nonempty set* X *such that* $\{\varnothing, X\} \subseteq L$ *and* $(L, \{\cup|_{\Sigma}, \varnothing, X, X \setminus \}, \varnothing)$ *has effect algebra structure under the partial binary internal operation* $A \oplus B := A \cup B$ *defined on* $\Sigma := \{(A, B) \in L \times L : A \cap B = \varnothing\}$*.*

If L is a uniquely complemented lattice, then $(L, \{\vee|_{\Sigma}, 0, 1, 1 \setminus \}, \varnothing)$ becomes an effect algebra under the partial binary internal operation $p \oplus q := p \vee q$ defined on the set Σ of all $(p, q) \in L \times L$ such that $p \wedge q = 0$.

Lemma 39 *Let* L *be an effect algebra. Let* $p, q, r \in L$*. Then:*

1. $1^{\perp} = 0$*.*
2. $p \oplus 0 = p$*.*
3. $\left(p^{\perp}\right)^{\perp} = p$*.*
4. *If* $p \oplus q = p \oplus r$*, then* $q = r$*.*

Proof 111 *Let us itemize the proof in accordance with the statement of the lemma.*

1. *By the Zero-One Law, we have that* $1^{\perp} = 0$*.*
2. *Fix an arbitrary* $p \in L$*. Observe that* $1 = 1 \oplus 0 = \left(p^{\perp} \oplus p\right) \oplus 0 = p^{\perp} \oplus (p \oplus 0)$*. The orthocomplementation forces that* $p \oplus 0 = p$*.*
3. *Trivial by orthocomplementation.*
4. *By commutativity and associativity, we have that* $1 = q \oplus \left(p \oplus (p \oplus q)^{\perp}\right) = r \oplus \left(p \oplus (p \oplus q)^{\perp}\right)$*. Finally, the orthocomplementation forces that* $q = r$*.*

An immediate consequence of the previous lemma is the fact that every element of an effect algebra is orthogonal with zero, and the only element orthogonal to the unity is zero. Also, the orthocomplementation is an involution.

Proposition 47 *A partial order can be defined in a effect algebra L:*

$$p \leq q \Leftrightarrow \exists r \text{ with } p \oplus r = q. \tag{1.27}$$

The following properties are satisfied for each $p, q, r, s \in L$:

1. *r is unique verifying $p \oplus r = q$ and thus it is denoted by $q \ominus p$.*
2. *$0 = \min(L)$ and $1 = \max(L)$.*
3. *$p^\perp = 1 \ominus p$ and $p \ominus p = 0$.*
4. *$p \leq q$ if and only if $1 \ominus q \leq 1 \ominus p$.*
5. *If $p \oplus r$ and $q \oplus r$ exist, then $p \oplus r \leq q \oplus r$ if and only if $p \leq q$.*
6. *If $r \leq p \leq q$, then $p \ominus r \leq q \ominus r$ and $q \ominus p \leq q \ominus r$.*
7. *If p, q are orthogonal, $r \leq p$ and $s \leq q$, then r, s are orthogonal as well.*
8. *If $p \oplus q = r \oplus s$, $p \leq r$ and $q \leq s$, then $p = r$ and $q = s$.*

Proof 112 *It is not hard to check that (1.27) is a partial order in L. The first three properties are direct consequences of Lemma 39. Let us check the last five.*

4. *Suppose that $p \leq q$ and take $r \in L$ with $p \oplus r = q$. Then $1 = q \oplus q^\perp = (p \oplus r) \oplus q^\perp = p \oplus \left(r \oplus q^\perp\right)$, meaning that $p^\perp = r \oplus q^\perp$, that is, $1 \ominus p = p^\perp \leq q^\perp = 1 \ominus q$.*
5. *Suppose first that $p \leq q$. Let $s \in L$ with $p \oplus s = q$. Then $(p \oplus r) \oplus s = (p \oplus s) \oplus r = q \oplus r$, hence $p \oplus r \leq q \oplus r$. Conversely, if $p \oplus r \leq q \oplus r$, then there exists $t \in L$ with $p \oplus r \oplus t = q \oplus r$. Orthocomplementation provides the existence of $s \in L$ such that $p \oplus r \oplus t \oplus s = q \oplus r \oplus s = 1$. As a consequence, $p \oplus t = q$, meaning that $p \leq q$.*
6. *Let $s, t \in L$ with $r \oplus s = p$ and $p \oplus t = q$. Then $r \oplus (s \oplus t) = (r \oplus s) \oplus t = p \oplus t = q$, so by Proposition 47(1), $q \ominus r = s \oplus t \geq s = p \ominus r$ and $q \ominus r = s \oplus t \geq t = q \ominus p$.*
7. *Let $r', s' \in L$ such that $r \oplus r' = p$ and $s \oplus s' = q$. Since $r \oplus r', s \oplus s'$ are orthogonal, the associativity and commutativity of orthosums allow that $p \oplus q = (r \oplus s) \oplus (r' \oplus s')$, resulting in r, s being orthogonal.*
8. *Let $p', q' \in L$ such that $p \oplus p' = r$ and $q \oplus q' = s$. Then $p \oplus q \oplus p' \oplus q' = r \oplus s = p \oplus q$, meaning that $p' \oplus q' = 0$, so $1 \oplus p' \oplus q' = 1 \oplus 0 = 1$, which implies that $p' = q' = 0$ for being $1, p'$ and $1, q'$ orthogonal pairs.*

If $A = \{a_1, \ldots, a_k\}$ is a finite subset of orthogonal elements of an effect algebra L, then we will denote $\bigoplus A := \bigoplus_{n=1}^{k} a_n := a_1 \oplus \cdots \oplus a_k$.

Definition 134 *Let L be an effect algebra. A subset A of L is said to be orthogonal provided that for every finite subset $B \subseteq A$, $\bigoplus B$ exists. We will denote $\bigoplus A := \sup\{\bigoplus B : B \subseteq A \text{ is finite}\}$, provided that this sup exists.*

If $(a_n)_{n\in\mathbb{N}}$ is an orthogonal sequence in an effect algebra L such that $\bigoplus\{a_n : n \in \mathbb{N}\}$ exists, then $\bigoplus\{a_n : n \in \mathbb{N}\} = \sup\left\{\bigoplus_{n=1}^{k} a_n : k \in \mathbb{N}\right\}$. We will often denote $\bigoplus\{a_n : n \in \mathbb{N}\} := \bigoplus_{n\in\mathbb{N}} a_n = \bigoplus_{n=1}^{\infty} a_n$.

Definition 135 (Effect σ-algebra) *An effect algebra in which every orthogonal sequence $(a_n)_{n\in\mathbb{N}}$ verifies that $\bigoplus_{n\in\mathbb{N}} a_n$ exists is called an effect σ-algebra.*

Next lemma shows, among other things, that Dedekind σ-complete distributive complemented lattices are effect σ-algebras.

Lemma 40 *Let L be an effect algebra. Let $A \subseteq L$. Let $(a_n)_{n\in\mathbb{N}} \subseteq L$. Then:*

1. *$\bigwedge_{a\in A} a$ exists if and only if $\bigvee_{a\in A}(1 \ominus a)$ exists. In this situation, $\bigvee_{a\in A}(1 \ominus a) = 1 \ominus \bigwedge_{a\in A} a$.*
2. *If $(a_n)_{n\in\mathbb{N}}$ is orthogonal and $\bigoplus_{n\in\mathbb{N}} a_n$ exists, then there exists $(b_n)_{n\in\mathbb{N}}$ increasing such that $\bigoplus_{n\in\mathbb{N}} a_n = \bigvee_{n\in\mathbb{N}} b_n$.*
3. *If $(a_n)_{n\in\mathbb{N}}$ is increasing and $\bigvee_{n\in\mathbb{N}} a_n$ exists, then there exists $(c_n)_{n\in\mathbb{N}}$ orthogonal such that $\bigvee_{n\in\mathbb{N}} a_n = \bigoplus_{n\in\mathbb{N}} c_n$.*

Proof 113 *The proof will follow as itemized as the statement is.*

1. *Suppose, for instance, that $\bigwedge_{a\in A} a$ exists. On the one hand, $\bigwedge_{a\in A} a \leq a_0$ for all $a_0 \in A$, thus $1 \ominus a_0 \leq 1 \ominus \bigwedge_{a\in A} a$ for all $a_0 \in A$, that is, $1 \ominus \bigwedge_{a\in A} a$ is an upper bound for $\{1 \ominus a : a \in A\}$. Consider another upper bound $d \in L$ for $\{1 \ominus a : a \in A\}$. Then $1 \ominus a \leq d$ for every $a \in A$, so $1 \ominus d \leq a$ for all $a \in A$, hence $1 \ominus d \leq \bigwedge_{a\in A} a$, meaning that $1 \ominus \bigwedge_{a\in A} a \leq d$. This shows that $\bigvee_{a\in A}(1 \ominus a) = 1 \ominus \bigwedge_{a\in A} a$.*
2. *Simply take $b_n := a_1 \oplus \cdots \oplus a_n$ for all $n \in \mathbb{N}$.*
3. *Take $c_1 := a_1$ and $c_n := a_n \ominus a_{n-1}$ for all $n \geq 2$. For all $n \in \mathbb{N}$, $c_1 \oplus \cdots \oplus c_n = a_n \leq \bigvee_{n\in\mathbb{N}} a_n$. Therefore, $\bigvee_{n\in\mathbb{N}} a_n$ is an upper bound for $\{c_1 \oplus \cdots \oplus c_n : n \in \mathbb{N}\}$. Consider another upper bound $d \in L$ for $\{c_1 \oplus \cdots \oplus c_n : n \in \mathbb{N}\}$, that is, $a_n = c_1 \oplus \cdots \oplus c_n \leq d$ for every $n \in \mathbb{N}$. Then $\bigvee_{n\in\mathbb{N}} a_n \leq d$.*

A nonzero element of an effect algebra is called isotropic when it is orthogonal with itself. An effect algebra is said to be isotropicless if it is free of isotropic elements.

Theorem 26 (Isotropic Orthodisjointness Dichotomy) *Let L be an effect algebra. Let $a, b \in L$ be orthogonal. Then:*

1. *If $t \in L$ is a lower bound for $\{a, b\}$, then either $t = 0$ or t is isotropic.*
2. *If $a \wedge b = 0$ and $a \vee b$ exists, then $a \vee b = a \oplus b$.*

Proof 114 *We will itemize the proof in the same way as the statement is.*

1. *Let $c_1, c_2 \in L$ with $t \oplus c_1 = a$ and $t \oplus c_2 = b$. Then $a \oplus b = t \oplus c_1 \oplus t \oplus c_2 = t \oplus t \oplus c_1 \oplus c_2$, resulting in either $t = 0$ or t is isotropic.*
2. *Note that $a \leq a \oplus b$ and $b \leq a \oplus b$, therefore $a \vee b \leq a \oplus b$. Let $a_1, b_1, c \in L$ such that $a \oplus b_1 = a \vee b = a_1 \oplus b$ and $(a \vee b) \oplus c = a \oplus b$. Then $a \oplus b = (a \vee b) \oplus c = (a \oplus b_1) \oplus c = a \oplus (b_1 \oplus c)$ and $a \oplus b = (a \vee b) \oplus c = (a_1 \oplus b) \oplus c = (a_1 \oplus c) \oplus b$, which implies that $b = b_1 \oplus c$ and $a = a_1 \oplus c$. As a consequence, $c \leq a$ and $c \leq b$, so $c \leq a \wedge b = 0$, hence $c = 0$ and $a \vee b = a \oplus b$.*

A lattice effect algebra is an effect algebra which is a lattice under the partial order induced by the orthoaddition.

Corollary 9 *An effect algebra L is an isotropicless lattice effect algebra if and only if L is a uniquely complemented lattice. In this situation, $a \wedge b = 0$ and $a \vee b = a \oplus b$ for all orthogonal $a, b \in L$.*

Two elements of an effect algebra are called orthodisjoint when they are orthogonal and disjoint.

Corollary 10 *Let L be an effect algebra. The following conditions are equivalent:*

1. *L is isotropicless.*
2. *If $p, q \in L$ are orthogonal, then they are orthodisjoint.*
3. *$(q \ominus p, p)$ is an orthodisjoint pair for every $p, q \in L$ with $p \leq q$.*

Proof 115 *Assume first that L is isotropicless. Suppose on the contrary that either $p \wedge q$ does not exist or $0 < p \wedge q$. In any case, there exists $c \in L \setminus \{0\}$ with $c \leq p$ and $c \leq q$. By the Isotropic Orthodisjointness Dichotomy, $c = 0$, which is a contradiction. Conversely, assume that $(q \ominus p, p)$ is an orthodisjoint pair for every $p, q \in L$ with $p \leq q$. If $p \in L \setminus \{0\}$ is isotropic, then $((p \oplus p) \ominus p, p) = (p, p)$ is not an orthodisjoint pair.*

An effect algebra satisfies the finite inf property if and only if it is a lattice effect algebra. Indeed, if L is an effect algebra satisfying the finite inf property, then $a \vee b$ exists for all $a, b \in L$ and $a \vee b = 1 \ominus [(1 \ominus a) \wedge (1 \ominus b)]$.

Definition 136 (Distributive effect algebra) *Let L be an effect algebra. We say that L is distributive if L satisfies the finite inf property and for all $a, b, c, d \in L$ such that a, b are orthogonal, then $(a \oplus b) \wedge c = (a \wedge c) \oplus (b \wedge c)$.*

Next theorem is a profound result that assures that distributive effect algebras are just distributive complemented lattices.

Theorem 27 *An effect algebra L is a distributive effect algebra if and only if L is a distributive complemented lattice.*

Proof 116 *Assume that L is a distributive effect algebra. By hypothesis, $a \wedge b$ exists for all $a, b \in L$ in view of the fact that L satisfies the finite inf property. Let us check the following properties for all $u, a, b \in L$:*

- *$u \vee a$ exists and $u \vee a = (u \wedge (1 \ominus a)) \oplus a$. First off, $u \wedge (1 \ominus a)$ and a are orthogonal because $u \wedge (1 \ominus a) \leq 1 \ominus a$, $a \leq a$, and $a, 1 \ominus a$ are orthogonal. Also, $u = u \wedge 1 = u \wedge (a \oplus (1 \ominus a)) = (u \wedge a) \oplus (u \wedge (1 \ominus a))$. Therefore, $(u \wedge (1 \ominus a)) \oplus a \geq (u \wedge (1 \ominus a)) \oplus (u \wedge a) = u$. This means that $(u \wedge (1 \ominus a)) \oplus a$ is an upper bound for $\{u, a\}$. Let $t \in L$ be any other upper bound of $\{u, a\}$. Then $t = t \wedge 1 = t \wedge (a \oplus (1 \ominus a)) = (t \wedge a) \oplus (t \wedge (1 \ominus a)) = a \oplus (t \wedge (1 \ominus a)) \geq a \oplus (u \wedge (1 \ominus a))$. This shows that $(u \wedge (1 \ominus a)) \oplus a = u \vee a$.*
- *$a \wedge (1 \ominus a) = 0$. Let $t \in L$ be any lower bound for $\{a, 1 \ominus a\}$. Then $t = t \wedge 1 = t \wedge (a \oplus (1 \ominus a)) = (t \wedge a) \oplus (t \wedge (1 \ominus a)) = t \oplus t$, which means that $t = 0$ by the orthocomplementation.*
- *$b \ominus (a \wedge b) = (1 \ominus a) \wedge b$. Indeed, since $b = b \wedge 1 = b \wedge (a \oplus (1 \ominus a)) = (b \wedge a) \oplus (b \wedge (1 \ominus a))$, we conclude that $b \ominus (a \wedge b) = (1 \ominus a) \wedge b$.*
- *If a, b are orthogonal, then $a \wedge b = 0$. By orthocomplementation, take $t \in L$ with $a \oplus b \oplus t = 1$. Then $0 \leq a \wedge b \leq a \wedge (b \oplus t) = a \wedge (1 \ominus a) = 0$.*
- *If $a \wedge b = 0$, then $a \leq 1 \ominus b$; hence, a, b are orthogonal and $a \vee b = a \oplus b$. Note that $a = a \wedge 1 = a \wedge (b \oplus (1 \ominus b)) = (a \wedge b) \oplus (a \wedge (1 \ominus b)) = 0 \oplus (a \wedge (1 \ominus b)) = a \wedge (1 \ominus b) \leq 1 \ominus b$. There exists $c \in L$ with $a \oplus c = 1 \ominus b$, so $a \oplus c \oplus b = 1$, resulting in a, b being orthogonal. Next, $a \vee b = (a \wedge (1 \ominus b)) \oplus b = a \oplus b$.*

- *If $u \geq b$, then $(a \wedge u) \ominus (a \wedge b) \leq u \ominus b$. Let $c, d \in L$ such that $u = b \oplus d$ and $a \wedge u = (a \wedge b) \oplus c$. Then $a \wedge u = a \wedge (b \oplus d) = (a \wedge b) \oplus (a \wedge d)$. Orthocomplementation in effect algebras forces $c = a \wedge d \leq d$.*

- *If $u \geq b$ and $v \geq a$, then $(a \wedge u) \ominus (a \wedge b) \leq (u \wedge v) \ominus (b \wedge v)$. Note that $u \wedge v \geq b \wedge v$. Therefore, by the previous item, $(u \wedge v) \ominus (b \wedge v) \geq (a \wedge u \wedge v) \ominus (a \wedge b \wedge v) = (a \wedge u) \ominus (a \wedge b)$.*

All of the above shows that L is a uniquely complemented lattice. It only remains to show the distributive property. Fix arbitrary elements $u, a, b \in L$. It always occurs that $(u \vee a) \wedge b \geq (u \wedge b) \vee (a \wedge b)$. On the one hand, $(u \vee a) \wedge b = ((u \wedge (1 \ominus a)) \oplus a) \wedge b = (u \wedge (1 \ominus a) \wedge b) \oplus (a \wedge b)$. On the other hand, $(u \wedge b) \vee (a \wedge b) = (u \wedge b \wedge (1 \ominus (a \wedge b))) \oplus (a \wedge b)$. Next, $a \wedge b \leq a$, so $1 \ominus a \leq 1 \ominus (a \wedge b)$, hence $(u \vee a) \wedge b = (u \wedge (1 \ominus a) \wedge b) \oplus (a \wedge b) \leq (u \wedge b \wedge (1 \ominus (a \wedge b))) \oplus (a \wedge b) = (u \wedge b) \vee (a \wedge b)$. It only remains to show that $(u \wedge a) \vee b = (u \vee b) \wedge (a \vee b)$. However, this is a direct consequence of the fact that the equalities $1 \ominus (a \wedge b) = (1 \ominus a) \vee (1 \ominus b)$ and $1 \ominus (a \vee b) = (1 \ominus a) \wedge (1 \ominus b)$ always hold on any effect algebra.

The following property can be consulted in the excellent text [15].

Definition 137 (Riesz Decomposition Property) *Let L be an effect algebra. Then L is said to enjoy the*

- *Riesz Decomposition Property (RDP) if $c \leq a \oplus b$ implies that $c = c_1 \oplus c_2$ with $c_1 \leq a$ and $c_2 \leq b$;*

- *strong Riesz Decomposition Property (sRDP) if $c \leq \bigoplus_{n \in \mathbb{N}} a_n$ implies that $c = \bigoplus_{n \in \mathbb{N}} c_n$ and $c_n \leq a_n$ for all $n \in \mathbb{N}$.*

Under the settings of the previous definition, notice that the condition $c \leq a \oplus b$ implies that $c = c_1 \oplus c_2$ with $c_1 \leq a$ and $c_2 \leq b$, easily extends to finite orthosums.

Definition 138 (Natural effect algebra) *Let L be an effect algebra. An orthogonal sequence $(b_n)_{n \in \mathbb{N}} \subseteq L$ is called a natural basis of L provided that $1 = \bigoplus_{n \in \mathbb{N}} b_n$ and b_n is minimal in $L \setminus \{0\}$ for every $n \in \mathbb{N}$. We will say that the effect algebra L is natural if it has the sRDP and has a natural basis.*

Observe that $\mathcal{P}(\mathbb{N})$ is a natural effect algebra of sets having $(\{n\})_{n \in \mathbb{N}}$ as a natural basis.

Proposition 48 *Let L be an sRDP effect algebra. Let $(b_n)_{n \in \mathbb{N}}$ be a natural basis of L. For every $a \in L$, there exists a subsequence $(b_{n_k})_{k \in \mathbb{N}}$ of $(b_n)_{n \in \mathbb{N}}$ such that $a = \bigoplus_{k \in \mathbb{N}} b_{n_k}$. In particular, $B := \{b_{i_1} \oplus \cdots \oplus b_{i_k} : i_1, \ldots, i_k \in \mathbb{N}\}$ is a generator of L.*

Proof 117 *Fix an arbitrary $a \in L$. Since L is sRDP and $a \leq 1 = \bigoplus_{n \in \mathbb{N}} b_n$, there exists an orthogonal sequence $(a_n)_{n \in \mathbb{N}}$ such that $a = \bigoplus_{n \in \mathbb{N}} a_n$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$. By the minimality of the b_n's in $L \setminus \{0\}$, we conclude that, for every $n \in \mathbb{N}$, either $a_n = b_n$ or $a_n = 0$. This shows the existence of a subsequence $(b_{n_k})_{k \in \mathbb{N}}$ of $(b_n)_{n \in \mathbb{N}}$ such that $a = \bigoplus_{k \in \mathbb{N}} b_{n_k}$. Finally, since $(b_{n_1} \oplus \cdots \oplus b_{n_k})_{k \in \mathbb{N}}$ is an increasing sequence of B, this also shows that B is a generator of L.*

Since effect algebras are universal algebras, it makes sense to consider effect algebra operators.

Lemma 41 *Let L, G be effect algebras. Let $f : L \to G$ be an effect algebra operator. Then f is increasing.*

Proof 118 *Take $l_1 \leq l_2$ in L. There exists $m \in L$ such that $l_1 \oplus m = l_2$. Then $f(l_2) = f(l_1 \oplus m) = f(l_1) \oplus f(m)$, meaning that $f(l_1) \leq f(l_2)$.*

Effect algebras also have dual spaces. Given two effect algebras E, F, the set of all effect algebra operators between E, F is denoted by $\mathscr{E}(E, F)$.

Definition 139 (Dual of an effect algebra) *Let L be an effect algebra. The dual of L is defined as $L^* := \mathscr{E}(L, \{0, 1\})$.*

Full unit segments in rings also acquire structure of effect algebras.

Theorem 28 *If R is a ring and $B^+ \subseteq R$ is a full unit segment, then the universal algebra $\left(B^+, \left\{+|_\Sigma, 0, 1, {}^\perp\right\}, \varnothing\right)$ becomes an effect algebra, where the ring addition has been restricted to the symmetric binary internal relation on B^+ given by $\Sigma := \{(r, s) \in B^+ \times B^+ : r + s \in B^+\}$ and the orthocomplementation is defined as*

$$\begin{array}{rccc} {}^\perp : & B^+ & \to & B^+ \\ & r & \mapsto & r^c := 1 - r. \end{array}$$

Proof 119 *Notice that $+|_\Sigma$ is clearly associative whenever it is defined. Indeed, if $a, b, c \in B^+$ with $b + c \in B^+$ and $a + (b + c) \in B^+$, then $a + b \in B^+$ due to the fullness of B^+ and $(a + b) + c = a + (b + c) \in B^+$. In a similar way, $+|_\Sigma$ is commutative whenever it is defined. The uniqueness of the orthocomplement is also inferred from the uniqueness of the opposite. Finally, the Zero-One Law is also trivially verified by bearing in mind Lemma 18(1).*

The previous theorem can be generalized to ordered modules.

Theorem 29 *If M is an ordered module over a unital ordered ring R and $m \in M$ is so that $m > 0$, then $[0, m]$ is an effect algebra with the partially defined binary internal operation given by $p \oplus q := p + q \Leftrightarrow p + q \leq m$ for all $p, q \in [0, m]$ and orthocomplementation given by $p^\perp := m - p$ for all $p \in [0, m]$.*

Proof 120 *Notice that $\oplus$ is clearly associative whenever it is defined. Indeed, let $p, q, r \in [0, m]$ with $q + r \in [0, m]$ and $p + (q + r) \in [0, m]$. We have to show that $p + q \in [0, m]$. On the one hand, $p + q \geq 0$ because $p, q \geq 0$. On the other hand, $(p + q) + r = p + (q + r) \in [0, m]$ so $(p + q) + r \leq m$, that is, $m - [(p + q) + r] \geq 0$, so by compatibility with addition $m - (p + q) = m - [(p + q) + r] + r \geq r \geq 0$, in other words, $p + q \in [0, m]$. In a similar way, $\oplus$ is commutative whenever it is defined. The uniqueness of the orthocomplement is also inferred from the uniqueness of the opposite. Finally, the Zero-One Law is also trivially verified.*

1.2.5.2 Boolean algebras

Here, we will make a brief introduction to Boolean algebras. We will define them and check their most basic properties. We will see them as examples of effect algebras. And, finally, we will highlight that Boolean algebras and distributive complemented lattices are, essentially, the same structures. We refer the reader to [16] for a wider perspective on Boolean algebras.

Definition 140 (Boolean algebra) *A Boolean algebra is a universal algebra $(B, \{+, 0, \cdot, 1, {}^c\}, \varnothing)$, where:*

- *Associativity and commutativity: $(B, \{+\}, \varnothing)$ and $(B, \{\cdot\}, \varnothing)$ are commutative semigroups, so, in particular, $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in A$.*

- *Distributivity: the addition* $+ : (B,\cdot) \times (B,\cdot) \to (B,\cdot)$ *and the multiplication* $\cdot : (B,+)\times(B,+) \to (B,+)$ *are both distributive, in other words,* $(a+b)\cdot c = (a\cdot c)+(b\cdot c)$ *and* $(a \cdot b) + c = (a + c) \cdot (b + c)$ *for all* $a, b \in A$.
- *Absorption:* $a + (a \cdot b) = a = a \cdot (a + b)$ *for all* $a, b \in A$.
- *Complementation: The unary internal operation*

$$\begin{array}{rccc} {}^{\mathsf{c}}: & B & \to & B \\ & b & \mapsto & b^{\mathsf{c}} \end{array}$$

and the nullary internal operations

$$0: \begin{array}{ccc} B & \to & B \\ a & \mapsto & 0 \end{array} \quad \textit{and} \quad 1: \begin{array}{ccc} B & \to & B \\ a & \mapsto & 1 \end{array}$$

satisfy that, for all $a \in A$, $a + a^{\mathsf{c}} = 1$ *and* $a \cdot a^{\mathsf{c}} = 0$.

The most representative examples of Boolean algebras are the Boolean algebras of sets. Later on, we will see, by means of the Stone Representation Theorem, that every Boolean algebra is isomorphic to a Boolean algebra of sets.

Example 28 (Boolean algebra of sets) *If* Ω *is a set, then the universal algebra* $(\mathcal{P}(\Omega), \{\cup, \varnothing, \cap, \Omega, \Omega\setminus\}, \varnothing)$ *is a Boolean algebra. Its Boolean subalgebras are known as fields of sets. The rings of sets are, by definition, the universal subalgebras of the universal algebra* $(\mathcal{P}(\Omega), \{\cup, \cap, \Omega\setminus\}, \varnothing)$.

The following lemma unveils basic properties satisfied by Boolean algebras, will become crucial later on to understand the algebraic structure underneath.

Lemma 42 *Let* B *be a Boolean algebra. Let* $a, b, c \in B$. *Then:*

1. $1 \cdot a = a$, $1 + a = 1$, $0 + a = a$, $0 \cdot a = 0$.
2. *If* $a + b = 1 = a + c$ *and* $a \cdot b = 0 = a \cdot c$, *then* $b = c$. *In particular, complements are unique.*
3. $1^{\mathsf{c}} = 0$ *and* $0^{\mathsf{c}} = 1$.
4. $(a^{\mathsf{c}})^{\mathsf{c}} = a$.
5. $(a + b)^{\mathsf{c}} = a^{\mathsf{c}} \cdot b^{\mathsf{c}}$ *and* $(a \cdot b)^{\mathsf{c}} = a^{\mathsf{c}} + b^{\mathsf{c}}$.
6. $a \cdot b = a \cdot (a \cdot b^{\mathsf{c}})^{\mathsf{c}}$ *and* $a + b = a + (a + b^{\mathsf{c}})^{\mathsf{c}}$.
7. $a + a = a = a \cdot a$.
8. $a + b = 0 \Rightarrow a = b = 0$ *and* $a \cdot b = 1 \Rightarrow a = b = 1$.

In particular, $(B, \{+, 0\}, \varnothing)$ *and* $(B, \{\cdot, 1\}, \varnothing)$ *are commutative monoids.*

Proof 121 *Only a few items will be proved and the rest are left as simple exercises.*

1. *By the absorption,* $a = a+(a \cdot a^{\mathsf{c}}) = a+0$, $1 = 1\cdot(1+a) = 1+a$, $a = a\cdot(a + a^{\mathsf{c}}) = a\cdot 1$ *and* $0 = 0 + (0 \cdot a) = 0 \cdot a$.
2. $b = 1 \cdot b = (a + c) \cdot b = (a \cdot b) + (c \cdot b) = 0 + (c \cdot b) = c \cdot b$. *Interchanging the roles of* b, c, *we obtain that* $c = b \cdot c$. *Since* $c \cdot b = b \cdot c$, *we reach the conclusion that* $b = c$.

5. By the distributive property,

$$(a+b)+(a^{\mathsf{c}}\cdot b^{\mathsf{c}})=((a+b)+a^{\mathsf{c}})\cdot((a+b)+b^{\mathsf{c}})=1\cdot 1=1,$$
$$(a+b)\cdot(a^{\mathsf{c}}\cdot b^{\mathsf{c}})=(a\cdot(a^{\mathsf{c}}\cdot b^{\mathsf{c}}))+(b\cdot(a^{\mathsf{c}}\cdot b^{\mathsf{c}}))=0+0=0.$$

6. By the distributive property,

$$a\cdot(a\cdot b^{\mathsf{c}})^{\mathsf{c}}=a\cdot(a^{\mathsf{c}}+(b^{\mathsf{c}})^{\mathsf{c}})=a\cdot(a^{\mathsf{c}}+b)=(a\cdot a^{\mathsf{c}})+(a\cdot b)=0+a\cdot b=a\cdot b.$$

Observe that the complementation is an involution. Lemma 42 also helps to understand how Boolean algebra operators work.

Proposition 49 *Let B,C be Boolean algebras. If $f:B\to C$ is neutral-additive and unital-multiplicative map, then f preserves the complements. As a consequence, f is an operator of Boolean algebras.*

Proof 122 *Take $b\in B$, then $1=f(1)=f(b+b^{\mathsf{c}})=f(b)+f(b^{\mathsf{c}})$ and $0=f(0)=f(b\cdot b^{\mathsf{c}})=f(b)\cdot f(b^{\mathsf{c}})$, which implies that $f(b^{\mathsf{c}})=f(b)^{\mathsf{c}}$.*

Boolean algebras also have dual spaces. Given two Boolean algebras B,C, the set of all Boolean algebra operators between B,C is denoted by $\mathscr{B}(B,C)$.

Definition 141 (Dual of a Boolean algebra) *Let B be a Boolean algebra. The dual of B is defined as $B^*:=\mathscr{B}(B,\{0,1\})$.*

The following remark highlights the trivial fact that Boolean algebras are effect algebras.

Remark 47 *If $(B,\{+,0,\cdot,1,\ ^{\mathsf{c}}\},\varnothing)$ is a Boolean algebra, then the universal algebra $(B,\{\oplus,0,1,\ ^{\mathsf{c}}\},\varnothing)$ is an effect algebra, where $\oplus$ is the partial binary internal operation defined by $a\oplus b:=a+b\Leftrightarrow a\cdot b=0$.*

In line with Remark 47, the following lemma shows that every Boolean algebra operator is an operator of effect algebras.

Lemma 43 *Let B,C be Boolean algebras. Let $f:B\to C$ be a Boolean algebra operator. Then f is an effect algebra operator.*

Proof 123 *Let $a,b\in B$ orthogonal elements. By definition, $a\cdot b=0$. We have to check first that $f(a),f(b)$ are orthogonal. Indeed, by hypothesis, $0=f(0)=f(a\cdot b)=f(a)\cdot f(b)$. As a consequence, $f(a),f(b)$ are orthogonal. Next, $f(a\oplus b)=f(a+b)=f(a)+f(b)=f(a)\oplus f(b)$. This shows that f is additive. Finally, by hypothesis, $f(0)=0$, $f(1)=1$, and $f(a^{\mathsf{c}})=f(a)^{\mathsf{c}}$ for all $a\in B$, meaning that f is an effect algebra operator.*

Since every Boolean algebra is an effect algebra, every Boolean algebra is conferred a partial order, given by Proposition 47.

Remark 48 *Let B be a Boolean algebra. The following is a partial order in B inherited from its effect algebra structure:*

$$a\leq b\Leftrightarrow\exists c\in B\ \ a+c=b,\ a\cdot c=0. \tag{1.28}$$

By bearing in mind Proposition 47(1), c is unique satisfying (1.28)*. In the scope of Boolean algebras, c is denoted by $b-a$.*

According to Lemma 41 together with Lemma 43, every Boolean algebra operator is increasing.

Proposition 50 *Let B be a Boolean algebra. Let $a, b \in B$. Then:*

1. *If $a \leq b$, then $a + b = b$ and $a \cdot b = a$.*
2. *If $a \leq b$, then $b - a = b \cdot a^{\mathsf{c}}$.*
3. *If there exists $d \in B$ with $a + d = b$, then $a \leq b$. In this situation, $d \cdot a^{\mathsf{c}} = b \cdot a^{\mathsf{c}}$.*
4. *If there exists $d \in B$ with $d \cdot b = a$, then $a \leq b$.*
5. *$a \wedge b = a \cdot b$.*
6. *$a \vee b = a + b$.*
7. *$a \leq b$ if and only if $b^{\mathsf{c}} \leq a^{\mathsf{c}}$.*
8. *$a \cdot b = 0$ if and only if $a \leq b^{\mathsf{c}}$.*
9. *$a + b = 1$ if and only if $a^{\mathsf{c}} \leq b$.*
10. *If $a \leq b$ and $d \in B$, then $a + d \leq b + d$ and $a \cdot d \leq b \cdot d$.*

Proof 124 *The proof of the lemma will follow as itemized as its statement.*

1. *Let $c \in B$ such that $a + c = b$ and $a \cdot c = 0$. The absorption yields $a + b = a + (a + c) = (a + a) + c = (a + (a \cdot 1)) + c = a + c = b$ and $a \cdot b = a \cdot (a + c) = a$.*
2. *By the uniqueness provided by Remark 48, it only suffices to show that $a + (b \cdot a^{\mathsf{c}}) = b$ and $a \cdot (b \cdot a^{\mathsf{c}}) = 0$. Indeed, by Proposition 50(1), $a + (b \cdot a^{\mathsf{c}}) = (b \cdot a) + (b \cdot a^{\mathsf{c}}) = b \cdot (a + a^{\mathsf{c}}) = b \cdot 1 = b$. On the other hand, $a \cdot (b \cdot a^{\mathsf{c}}) = b \cdot (a \cdot a^{\mathsf{c}}) = b \cdot 0 = 0$.*
3. *We will show that $a + (d \cdot a^{\mathsf{c}}) = b$ and $a \cdot (d \cdot a^{\mathsf{c}}) = 0$. Indeed, $a \cdot (d \cdot a^{\mathsf{c}}) = d \cdot (a \cdot a^{\mathsf{c}}) = b \cdot 0 = 0$. Next, by hypothesis and by the absorption, $b = a + d = a + (d \cdot 1) = a + (d \cdot (a + a^{\mathsf{c}})) = a + (d \cdot a) + (d \cdot a^{\mathsf{c}}) = a + (d \cdot a^{\mathsf{c}})$. By Remark 48, we conclude that $a \leq b$. Finally, the uniqueness provided by Remark 48 forces that $d \cdot a^{\mathsf{c}} = b \cdot a^{\mathsf{c}}$.*
4. *By the absorption, $a + b = (d \cdot b) + b = b$, hence by Proposition 50(3), $a \leq b$.*
5. *By the absorption, $(a \cdot b) + a = a$ and $(a \cdot b) + b = b$, thus by Proposition 50(3), $a \cdot b \leq a$ and $a \cdot b \leq b$. Let $d \in B$ with $d \leq a$ and $d \leq b$. By Remark 48, there exists $c_a, c_b \in B$ such that $d + c_a = a$, $d + c_b = b$, and $d \cdot c_a = 0 = d \cdot c_b$. Then $a \cdot b = (d + c_a) \cdot (d + c_b) = (d \cdot d) + (c_a \cdot d) + (d \cdot c_b) + (c_a \cdot c_b) = d + (c_a \cdot c_b)$. As a consequence, $d \leq a \cdot b$. This shows that $a \wedge b = a \cdot b$.*
6. *By Proposition 50(3), $a \leq a + b$ and $b \leq a + b$. Take $d \in B$ with $a \leq d$ and $b \leq d$. Take $c_a, c_b \in B$ with $a + c_a = d$ and $b + c_b = d$. Then $d = d + d = (a + c_a) + (b + c_b) = (a + b) + (c_a + c_b)$. According to Proposition 50(3), $a + b \leq d$. This shows that $a \vee b = a + b$.*
7. *Suppose that $a \leq b$ and take $c \in B$ with $a + c = b$. Then $b^{\mathsf{c}} = (a + b)^{\mathsf{c}} = a^{\mathsf{c}} \cdot b^{\mathsf{c}}$. By Proposition 50(4), $b^{\mathsf{c}} \leq a^{\mathsf{c}}$.*
8. *Suppose that $a \cdot b = 0$. Observe that $b^{\mathsf{c}} \cdot a = (b^{\mathsf{c}} \cdot a) + (b \cdot a) = (b^{\mathsf{c}} + b) \cdot a = 1 \cdot a = a$, hence, by Proposition 50(4), $a \leq b^{\mathsf{c}}$. Conversely, suppose that $a \leq b^{\mathsf{c}}$. Take $c \in B$ with $a + c = b^{\mathsf{c}}$. Then $0 = b^{\mathsf{c}} \cdot b = (a + c) \cdot b = (a \cdot b) + (c \cdot b)$. In accordance with Lemma 42(10), $a \cdot b = 0$.*

9. *By relying on Lemma 42(4,5,6) together with Proposition 50(7,8), we have that* $1 = a + b \Leftrightarrow 0 = a^{\mathsf{c}} \cdot b^{\mathsf{c}} \Leftrightarrow a^{\mathsf{c}} \leq b \Leftrightarrow b^{\mathsf{c}} \leq a$.

10. *Take* $c \in B$ *with* $a + c = b$. *Then* $(a+d)+c = (a+c)+d = b+d$ *and* $b \cdot d = (a+c) \cdot d = (a \cdot d) + (c \cdot d)$, *so by Proposition 50(3),* $a + d \leq b + d$ *and* $a \cdot d \leq b \cdot d$.

The following theorem, although extremely important, has an easy proof by bearing in mind Proposition 50.

Theorem 30 *If* L *is a distributive complemented lattice, then the universal algebra* $(L, \{\vee, 0, \wedge, 1, 1\setminus\ \}, \varnothing)$ *is a Boolean algebra. Conversely, if* B *is a Boolean algebra, then* B *is a distributive complemented lattice with the order given by Remark 48.*

Proof 125 *The proof is itemized according to its statement.*

$\Rightarrow$ *It only suffices to define the binary internal operations* $p + q := p \vee q$ *and* $p \cdot q := p \wedge q$ *for all* $p, q \in L$, *and the unary internal operation* $1 \setminus p := p^{\mathsf{c}}$. *Proposition 5 does the rest.*

$\Leftarrow$ *By bearing in mind Proposition 50,* B *is clearly a distributive complemented lattice where* $a \wedge b = a \cdot b$ *and* $a \vee b = a + b$ *for all* $a, b \in B$.

The following lemma is aimed at studying the dual of a Boolean algebra, that is, the set of all Boolean algebra operators from the given Boolean algebra to the Boolean algebra $\{0, 1\}$.

Lemma 44 *Let* B *be a Boolean algebra. Let* $\mathcal{J}$ *be an ultrafilter of* B. *Let* $\chi : B \to \{0, 1\}$ *be a Boolean algebra operator. Let* $b \in B$. *Then:*

1. *Either* $b \in \mathcal{J}$ *and* $b^{\mathsf{c}} \in B \setminus \mathcal{J}$, *or vice versa.*

2. $\chi^{-1}(\{1\})$ *is an ultrafilter of* B *and* $\chi^{-1}(\{0\})$ *is a maximal ideal of* B.

3. $\chi_{\mathcal{J}}$ *is a Boolean algebra operator.*

Proof 126 *We will itemize the proof in accordance with the statement.*

1. *Suppose on the contrary that* $b, b^{\mathsf{c}} \in \mathcal{J}$. *Then* $0 = b \cdot b^{\mathsf{c}} = b \wedge b^{\mathsf{c}} \in \mathcal{J}$, *which is impossible. Suppose next that* $b^{\mathsf{c}} \in B \setminus \mathcal{J}$. *Notice that, for every* $a \in \mathcal{J}$, $a \cdot b \neq 0$, *since otherwise* $a \leq b^{\mathsf{c}}$, *resulting in* $b^{\mathsf{c}} \in \mathcal{J}$ *As a consequence,* $\mathcal{J} \subseteq b^{\mathsf{o}}$. *By applying Lemma 4, we conclude that* $b \in \mathcal{J}$.

2. *In the first place,* $\chi^{-1}(\{1\}) \neq \varnothing$ *since* $1 \in \chi^{-1}(\{1\})$. *Next,* $\chi^{-1}(\{1\})$ *is upward closed since if* $b \in \chi^{-1}(\{1\})$ *and* $c \geq b$, *then* $1 \leq \chi(b) \leq \chi(c) \leq 1$ *due to the fact that* f *is increasing by bearing in mind Lemma 43 and Lemma 41. Now,* $\chi^{-1}(\{1\})$ *is downward directed, because if* $b, c \in \chi^{-1}(\{1\})$, *then* $b \cdot c \in \chi^{-1}(\{1\})$ *and* $b \cdot c \leq b$ *and* $b \cdot c \leq c$. *Finally, let* $\mathcal{J} \subseteq B$ *a filter of* B *containing* $\chi^{-1}(\{1\})$. *Note that either* $\mathcal{J} \subseteq \chi^{-1}(\{1\})$ *or* $\mathcal{J} \subseteq \chi^{-1}(\{0\})$. *Since* $\chi^{-1}(\{1\}) \cap \chi^{-1}(\{0\}) = \varnothing$, *the only possibility left is that* $\mathcal{J} \subseteq \chi^{-1}(\{1\})$, *reaching that* $\mathcal{J} = \chi^{-1}(\{1\})$.

3. *In accordance with Proposition 49, it only suffices to show that* $\chi_{\mathcal{J}}$ *is neutral-additive and unital-multiplicative. Indeed, it is clear that* $\chi_{\mathcal{J}}(0) = 0$ *and* $\chi_{\mathcal{J}}(1) = 1$ *since* $0 \notin \mathcal{J}$ *and* $1 \in \mathcal{J}$ *for being* $\mathcal{J}$ *an ultrafilter of* B. *Fix arbitrary elements* $b, c \in B$. *We have the following options:*

- $\chi_{\mathcal{J}}(b) = \chi_{\mathcal{J}}(c) = 1$. *Then* $b, c \in \mathcal{J}$, *so* $b + c \in \mathcal{J}$ *because* $c \leq b + c$ *and* $\mathcal{J}$ *is upward closed, meaning that* $\chi_{\mathcal{J}}(b + c) = 1 = 1 + 1 = \chi_{\mathcal{J}}(b) + \chi_{\mathcal{J}}(c)$. *On the other hand,* $\mathcal{J}$ *is downward directed, so there exists* $e \in \mathcal{J}$ *such that* $e \leq b$ *and* $e \leq c$, *hence* $e \leq b \cdot c$, *so* $b \cdot c \in \mathcal{J}$ *for being* $\mathcal{J}$ *upward closed, resulting in* $\chi_{\mathcal{J}}(b \cdot c) = 1 = 1 \cdot 1 = \chi_{\mathcal{J}}(b) \cdot \chi_{\mathcal{J}}(c)$.
- $\chi_{\mathcal{J}}(b) = \chi_{\mathcal{J}}(c) = 0$. *Then* $b, c \in B \setminus \mathcal{J}$, *so* $b^{\mathsf{c}}, c^{\mathsf{c}} \in \mathcal{J}$ *in view of Lemma 44, hence* $b + c \notin \mathcal{J}$ *because otherwise we obtain the contradiction that* $0 = b^{\mathsf{c}}(b + c)c^{\mathsf{c}} \in \mathcal{J}$, *meaning that* $\chi_{\mathcal{J}}(b+c) = 0 = 0+0 = \chi_{\mathcal{J}}(b)+\chi_{\mathcal{J}}(c)$. *On the other hand,* $b \cdot c \leq b$, *so* $b \cdot c \notin \mathcal{J}$ *because* $b \notin \mathcal{J}$, *resulting in* $\chi_{\mathcal{J}}(b \cdot c) = 0 = 0 \cdot 0 = \chi_{\mathcal{J}}(b) \cdot \chi_{\mathcal{J}}(c)$.
- $\chi_{\mathcal{J}}(b) = 1$ *and* $\chi_{\mathcal{J}}(c) = 0$. *Then* $b \in \mathcal{J}$ *and* $c \in B \setminus \mathcal{J}$, *so* $b + c \in \mathcal{J}$ *because* $\mathcal{J}$ *is upward closed and* $b \leq b+c$, *meaning that* $\chi_{\mathcal{J}}(b+c) = 1 = 1+0 = \chi_{\mathcal{J}}(b)+\chi_{\mathcal{J}}(c)$. *On the other hand,* $b \cdot c \leq c$, *so* $b \cdot c \notin \mathcal{J}$ *because* $c \notin \mathcal{J}$, *resulting in* $\chi_{\mathcal{J}}(b \cdot c) = 0 = 1 \cdot 0 = \chi_{\mathcal{J}}(b) \cdot \chi_{\mathcal{J}}(c)$.

Let us introduce now the concept of Boolean σ-algebra, which is crucial in the development of Measure Theory.

Definition 142 (Boolean σ-algebra) *A Boolean* σ*-algebra is a universal algebra* $(B, \{+, 0, \cdot, 1, {}^{\mathsf{c}}, \sum\}, \varnothing)$, *where* $(B, \{+, 0, \cdot, 1, {}^{\mathsf{c}}\}, \varnothing)$ *is a Boolean algebra and*

$$\begin{array}{rccc} \sum: & B^{\mathbb{N}} & \to & B \\ & (b_n)_{n\in\mathbb{N}} & \mapsto & \sum_{n=1}^{\infty} b_n \end{array}$$

is an $\mathbb{N}$*-ary internal operation satisfying:*

- *Commutativity: For every* π *permutation of* $\mathbb{N}$, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} b_{\pi(n)}$.
- *Extension: For every* $b_1, b_2 \in B$, $b_1 + b_2 = \sum_{n=1}^{\infty} b_n$, *where* $b_n = 0$ *for* $n \geq 3$.

Trivial examples of Boolean σ-algebras are the Boolean σ-algebras of sets.

Example 29 (Boolean σ-algebra of sets) *If* Ω *is a set, then the universal algebra* $(\mathcal{P}(\Omega), \{\cup, \varnothing, \cap, \Omega, \Omega\setminus, \bigcup\}, \varnothing)$, *where*

$$\begin{array}{rccc} \bigcup: & \mathcal{P}(\Omega)^{\mathbb{N}} & \to & \mathcal{P}(\Omega) \\ & (A_n)_{n\in\mathbb{N}} & \mapsto & \bigcup_{n=1}^{\infty} A_n, \end{array}$$

is a Boolean σ*-algebra. Its Boolean* σ*-subalgebras are known as* σ*-fields of sets. The* σ*-ring of sets are, by definition, the universal subalgebras of the universal algebra* $(\mathcal{P}(\Omega), \{\cup, \cap, \Omega\setminus, \bigcup\}, \varnothing)$.

A more sophisticated example of Boolean σ-algebras is the one given by σ-complete distributive complemented lattices.

Example 30 (σ-Complete distributive complemented lattice) *If* L *is a* σ*-complete distributive complemented lattice, then the universal algebra* $(L, \{\vee, 0, \wedge, 1, 1\setminus, \bigvee\}, \varnothing)$, *where*

$$\begin{array}{rccc} \bigvee: & L^{\mathbb{N}} & \to & L \\ & (a_n)_{n\in\mathbb{N}} & \mapsto & \bigvee_{n=1}^{\infty} a_n, \end{array}$$

is a Boolean σ*-algebra.*

Next remark highlights that inverse-image maps are operators on Boolean σ-algebras of sets in view of Theorem 8.

Remark 49 *Let* $f : X \to Y$ *be a function between sets* X, Y. *The induced inverse-image map* (1.1) *given in Remark 3 is an operator on Boolean* σ*-algebras of sets.*

1.2.6 Topologies and uniformities

In this subsection, we will make a brief introduction to General Topology. We will review the main topological concepts involved in Analysis and Calculus, upon which we will rely to develop this theoretical and categorical approach to Abstract Calculus. We refer the reader to [17] for an excellent text on General Topology.

1.2.6.1 Topological spaces

A topology on a given set is a topology structure on its power set. Defining topologies as universal algebras is not quite an easy task. In the first place, note that, for a given set X, the following are internal operations on X:

$$\begin{array}{rccc} \cap: & \mathcal{P}(X)^2 & \to & \mathcal{P}(X) \\ & (A,B) & \mapsto & A\cap B, \end{array} \qquad \begin{array}{rccc} \varnothing: & \mathcal{P}(X) & \to & \mathcal{P}(X) \\ & A & \mapsto & \varnothing, \end{array}$$

$$\begin{array}{rccc} X: & \mathcal{P}(X) & \to & \mathcal{P}(X) \\ & A & \mapsto & X, \end{array} \qquad \begin{array}{rccc} \bigcup_{i\in I}: & \mathcal{P}(X)^I & \to & \mathcal{P}(X) \\ & (A_i)_{i\in I} & \mapsto & \bigcup_{i\in I} A_i \end{array}$$

for every subset $I \subseteq \mathcal{P}(X)$.

Definition 143 (Topological space) *Let X be a set. We will say that τ is a topology on X if τ is a universal subalgebra of*

$$\left(\mathcal{P}(X), \left\{\bigcup_{i\in I} : I \in \mathcal{P}(\mathcal{P}(X))\right\} \cup \{\cap, \varnothing, X\}, \varnothing\right). \tag{1.29}$$

The pair (X,τ) is called a topological space.

In virtue of Remark 19, since all the internal operations in the universal algebra (1.29) are totally defined and there are nullary internal operations, the intersection of any family of universal subalgebras is a universal subalgebra. As a consequence, if $(\tau_j)_{j\in J}$ is a family of topologies on X, then $\bigcap_{j\in J}\tau_j$ is a topology on X. The topology generated by a subset $\mathcal{A}\subseteq\mathcal{P}(X)$ is defined as the intersection of all topologies on X containing $\mathcal{A}$ and denoted by $\tau(\mathcal{A})$. According to Proposition 5, the map

$$\begin{array}{rcl} \mathcal{P}\left(\mathcal{P}(X)\right) & \to & \mathcal{P}\left(\mathcal{P}(X)\right) \\ \mathcal{A} & \mapsto & \tau(\mathcal{A}) \end{array}$$

is a hull operator on $\mathcal{P}(X)$. Notice that $\tau(\mathcal{A})$ is, precisely, $\operatorname{span}(\mathcal{A})$ in the universal algebra (1.29).

Definition 144 (Coraser/finer topology) *Let X be a set. Let τ_1,τ_2 be two topologies on X such that $\tau_1\subseteq\tau_2$. We say that τ_1 is coarser than τ_2 or τ_2 is finer than τ_1.*

Under the settings of the previous definition, it is clear that $\tau := \{\varnothing, X\}$ is the coarsest topology on X, called the trivial topology, and $\tau := \mathcal{P}(X)$ is the finest topology on X, called the discrete topology.

1.2.6.2 Subsets of a topological space

The following definition describes the most notable subsets of a topological space.

Definition 145 *Let X be a topological space. Let $A\subseteq X$. Let $x\in X$. Then:*

- *A is said to be open provided that A is an element of the topology of X.*

- *A is said to be closed if $X \setminus A$ is open.*
- *The interior of A is defined as* $\operatorname{int}(A) := \bigcup\{U \in \mathcal{P}(X) : U \text{ is open and } U \subseteq A\}$.
- *The closure of A is defined as* $\operatorname{cl}(A) := \bigcap\{F \in \mathcal{P}(X) : F \text{ is closed and } A \subseteq F\}$.
- *The boundary of A is defined as* $\operatorname{bd}(A) := \operatorname{cl}(A) \setminus \operatorname{int}(A)$. *Also denoted by ∂A.*
- *The set of limit (or cluster or accumulation) points of A is defined as* $\operatorname{ac}(A) := \{x \in X : x \in \operatorname{cl}(A \setminus \{x\})\}$.
- *A is said to be a body if A is closed and* $\operatorname{int}(A) \neq \varnothing$.
- *A is said to be dense if* $\operatorname{cl}(A) = X$.
- *A is said to be nowhere dense (or rare) if* $\operatorname{int}(\operatorname{cl}(A)) = \varnothing$.
- *x is said to be an isolated point provided that $\{x\}$ is open. The set of isolated points of X is denoted by* $\operatorname{iso}(X)$.

A topological space X has the discrete topology if and only if $X = \operatorname{iso}(X)$. The notion of neighborhood follows next.

Definition 146 *Let X be topological space X. Let $V \subseteq X$. Let $x \in X$. Then:*

- *We say that V is a neighborhood of x if there exists an open subset $U \subseteq X$ such that $x \in U \subseteq V$. The subset of $\mathcal{P}(X)$ containing all the neighborhoods of x is denoted by $\mathcal{N}_x(X)$.*
- *We say that V is a reduced neighborhood of x if there exists a neighborhood U of x such that $V = U \setminus \{x\}$. The subset of $\mathcal{P}(X)$ containing all the reduced neighborhoods of x is denoted by $\mathcal{N}_x^{\times}(X)$.*

Under the settings of the previous definition, we will use the notation $\mathcal{N}_x, \mathcal{N}_x^{\times}$ instead of $\mathcal{N}_x(X), \mathcal{N}_x^{\times}(X)$, respectively, if there is no confusion with X. If $A \subseteq X$, then a neighborhood of A is a subset $V \subseteq X$ containing an open set U of X such that $A \subseteq U \subseteq V$.

Lemma 45 *Let X be topological space. Let $A, B \subseteq X$. Let $x \in X$. Then:*

1. *The relative or inherited topology of A is $\{U \cap A : U \text{ open in } X\}$, which is, indeed, a topology on A.*
2. $\operatorname{int}(A) = \{x \in X : \exists V \in \mathcal{N}_x(X)\ V \subseteq A\}$.
3. $\operatorname{cl}(A) = \{x \in X : \forall V \in \mathcal{N}_x(X)\ V \cap A \neq \varnothing\}$.
4. $\{\operatorname{int}(A), \operatorname{bd}(A)\}$ *and* $\{\operatorname{ac}(A), \operatorname{iso}(A)\}$ *are partitions of* $\operatorname{cl}(A)$.
5. cl *is a hull operator and* ac *is an increasing operator.*
6. $\operatorname{iso}(A) = A \setminus \operatorname{ac}(A)$.
7. $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.
8. $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$.
9. $\operatorname{ac}(A \cup B) = \operatorname{ac}(A) \cup \operatorname{ac}(B)$.
10. *$\mathcal{N}_x$ is a filter of $\mathcal{P}(X)$.*

11. *If $x \in \mathrm{ac}(X)$, then $\mathcal{N}_x^{\times}$ is a prefilter of $\mathcal{P}(X)$ and a filter of $\mathcal{P}(X \setminus \{x\})$.*

Proof 127 *We will only prove certain items since the rest are essentially elementary.*

6. *Fix an arbitrary $b \in \mathrm{iso}(A)$. Then $\{b\}$ is open in A so there exists U open in X such $U \cap A = \{b\}$. Thus, $U \cap (A \setminus \{b\}) = \varnothing$, meaning that $b \notin \mathrm{cl}(A \setminus \{b\})$, hence $b \notin \mathrm{ac}(A)$. Conversely, fix an arbitrary $b \notin \mathrm{cl}(A \setminus \{b\})$. There exists an open neighborhood U of b such that $U \cap (A \setminus \{b\}) = \varnothing$, that is, $U \cap A = \{b\}$, which implies that $\{b\}$ is open in A and $b \in \mathrm{iso}(A)$.*

9. *Since $A \subseteq A \cup B$, $B \subseteq A \cup B$, and* ac *is an increasing operator, we have that $\mathrm{ac}(A) \subseteq \mathrm{ac}(A \cup B)$ and $\mathrm{ac}(B) \subseteq \mathrm{ac}(A \cup B)$, meaning that $\mathrm{ac}(A) \cup \mathrm{ac}(B) \subseteq \mathrm{ac}(A \cup B)$. Conversely, fix an arbitrary element $x \in \mathrm{ac}(A \cup B)$. Assume on the contrary that $x \notin \mathrm{ac}(A)$ and consider $U_A \in \mathcal{N}_x$ such that $U_A \setminus \{x\} \cap A = \varnothing$. We will show then that $x \in \mathrm{ac}(B)$. Indeed, take any neighborhood $V \in \mathcal{N}_x$. Note that $(U_A \cap V) \setminus \{x\} \cap (A \cup B) \neq \varnothing$, therefore, $V \setminus \{x\} \cap B \neq \varnothing$. This shows that $x \in \mathrm{ac}(B)$.*

Recall that the notation $\bigcap \mathcal{N}_x$ stands for $\bigcap_{V \in \mathcal{N}_x} V$. For simplicity, we will denote $X_{(x)} := \bigcap \mathcal{N}_x$. It is immediate to check that, if $y \in Y \subseteq X$, then $X_{(y)} \cap Y = Y_{(y)}$.

Definition 147 *In a topological space X:*

- *A subset U is called regular open provided that $U = \mathrm{int}(\mathrm{cl}(U))$.*
- *A subset B is called regular closed provided that $B = \mathrm{cl}(\mathrm{int}(B))$.*

The following proposition unveils examples of regular open sets and regular closed sets.

Proposition 51 *In a topological space X, if $U \subseteq X$ is open and $B \subseteq X$ is closed, then $\mathrm{cl}(U)$ and $\mathrm{int}(B)$ are regular closed and regular open, respectively.*

Proof 128 *On the one hand, $U \subseteq \mathrm{cl}(U)$ so $U \subseteq \mathrm{int}(\mathrm{cl}(U))$ and $\mathrm{cl}(U) \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U)))$. On the other hand, $\mathrm{int}(\mathrm{cl}(U)) \subseteq \mathrm{cl}(U)$ so $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) \subseteq \mathrm{cl}(U)$. As a consequence, $\mathrm{cl}(U) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U)))$. Next, observe that $\mathrm{int}(B) \subseteq B$ so $\mathrm{cl}(\mathrm{int}(B)) \subseteq B$ and $\mathrm{int}(\mathrm{cl}(\mathrm{int}(B))) \subseteq \mathrm{int}(B)$. Finally, $\mathrm{int}(B) \subseteq \mathrm{cl}(\mathrm{int}(B)$, so $\mathrm{int}(B) \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))$. Therefore, $\mathrm{int}(B) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(B)))$.*

The following technical lemma shows that sphere-like sets have empty interior.

Lemma 46 *Let X be a topological space. Let U be an open subset of X. Let B be a closed subset of X containing U. Consider $S := B \setminus U$. If either $\mathrm{int}(B) = U$ or $\mathrm{cl}(U) = B$, then $\mathrm{int}(S) = \varnothing$.*

Proof 129 *We will suppose first that $\mathrm{int}(B) = U$. Note that $U = \mathrm{int}(B) = \mathrm{int}(U \cup S) \supseteq \mathrm{int}(U) \cup \mathrm{int}(S) = U \cup \mathrm{int}(S)$. This implies that $\mathrm{int}(S) = \varnothing$ because $U \cap S = \varnothing$. Suppose now that $\mathrm{cl}(U) = B$. If there exists $s \in \mathrm{int}(S)$, then we can find an open subset V of X such that $s \in V \subseteq S$. Note that $s \in B = \mathrm{cl}(U)$, so $V \cap U \neq \varnothing$, which implies that $S \cap U \neq \varnothing$.*

An example follows next displaying that the interior operator composed with the closure operator is not necessarily the identity operator.

Example 31 *Take $X := \mathbb{R}$, $U := (-1, 1)$, and $B := [-1, 1] \cup \{2\}$. Then $\mathrm{int}(B) = U$ but $\mathrm{cl}(U) \neq B$.*

Next example shows that the closure operator composed with the interior operator is not necessarily the identity operator.

Example 32 *Take* $X := \mathbb{R}$, $U := (-1,0) \cup (0,1)$, *and* $B := [-1,1]$. *Then* $\mathrm{cl}(U) = B$ *but* $\mathrm{int}(B) \neq U$.

Given a topological space X, a subset Y of X, and $A \subseteq Y$, the relative interior of A with respect to Y is denoted by $\mathrm{int}_Y(A)$, the relative closure of A with respect to Y is denoted by $\mathrm{cl}_Y(A)$, and the relative boundary of A with respect to Y is denoted by $\mathrm{bd}_Y(A)$. On the other hand, observe that the closed subsets of Y are of the form $F \cap Y$ for F closed in X since $F \cap Y = Y \setminus ((X \setminus F) \cap Y)$.

Lemma 47 *Let* X *be a topological space. Let* $Y, A \subseteq X$. *Then:*

1. $\mathrm{int}(A) \cap Y \subseteq \mathrm{int}_Y(A \cap Y)$.
2. $\mathrm{cl}_Y(A \cap Y) = \mathrm{cl}(A \cap Y) \cap Y$.
3. $\mathrm{bd}_Y(A \cap Y) \subseteq \mathrm{bd}(A \cap Y) \cap \mathrm{bd}(A)$.

If Y *is dense in* X *and* A *is closed in* X, *then* $\mathrm{int}(A) \cap Y = \mathrm{int}_Y(A \cap Y)$, $\mathrm{bd}(A) \cap Y = \mathrm{bd}_Y(A \cap Y)$, *and* $\mathrm{cl}(A) \cap Y = \mathrm{cl}_Y(A \cap Y)$.

Proof 130 *The proof will follow as itemized as the statement of the lemma.*

1. *First off,* $\mathrm{int}(A) \cap Y$ *is open in* Y *by definition and is contained in* $A \cap Y$*; therefore,* $\mathrm{int}(A) \cap Y \subseteq \mathrm{int}_Y(A \cap Y)$.
2. *Notice that* $\mathrm{cl}(A \cap Y) \cap Y$ *is closed in* Y *and contains* $A \cap Y$, *meaning that* $\mathrm{cl}_Y(A \cap Y) \subseteq \mathrm{cl}(A \cap Y) \cap Y$. *Also, if* $y \in \mathrm{cl}(A \cap Y) \cap Y$ *and* V *is open in* X *such that* $y \in V \cap Y$, *then* $V \cap (A \cap Y) \neq \varnothing$, *therefore* $(V \cap Y) \cap (A \cap Y) \neq \varnothing$, *implying that* $y \in \mathrm{cl}_Y(A \cap Y)$. *As a consequence,* $\mathrm{cl}(A \cap Y) \cap Y \subseteq \mathrm{cl}_Y(A \cap Y)$, *hence* $\mathrm{cl}_Y(A \cap Y) = \mathrm{cl}(A \cap Y) \cap Y$.
3. *Take any arbitrary* $y \in \mathrm{bd}_Y(A \cap Y)$. *Then* $y \in \mathrm{bd}_Y(A \cap Y) \subseteq \mathrm{cl}_Y(A \cap Y) = \mathrm{cl}(A \cap Y) \cap Y \subseteq \mathrm{cl}(A)$. *On the one hand, either* $y \in \mathrm{bd}(A \cap Y)$ *or* $y \in \mathrm{int}(A \cap Y)$. *In the latter case,* $y \in \mathrm{int}(A \cap Y) \subseteq \mathrm{int}(A) \cap Y \subseteq \mathrm{int}_Y(A \cap Y)$, *which is impossible since* $y \in \mathrm{bd}_Y(A \cap Y)$ *by assumption. As a consequence,* $y \in \mathrm{bd}(A \cap Y)$. *On the other hand, either* $y \in \mathrm{bd}(A)$ *or* $y \in \mathrm{int}(A)$. *In the latter case,* $y \in \mathrm{int}(A) \cap Y \subseteq \mathrm{int}_Y(A \cap Y)$, *which is impossible since* $y \in \mathrm{bd}_Y(A \cap Y)$ *by assumption. As a consequence,* $y \in \mathrm{bd}(A)$.

Finally, suppose that Y *is dense in* X *and* A *is closed in* X. *We will show first that* $\mathrm{int}(A) \cap Y \supseteq \mathrm{int}_Y(A \cap Y)$. *Indeed, fix an arbitrary* $y \in \mathrm{int}_Y(A \cap Y)$. *There exists an open set* V *in* X *in such a way that* $y \in V \cap Y \subseteq A \cap Y \subseteq A$. *Then* $\mathrm{cl}(V \cap Y) \subseteq A$. *Since* V *is open in* X *and* Y *is dense in* X, *we have that* $V \subseteq \mathrm{cl}(V \cap Y) \subseteq A$. *This shows that* $y \in \mathrm{int}(A)$. *It only remains to show that* $\mathrm{bd}(A) \cap Y \subseteq \mathrm{bd}_Y(A \cap Y)$. *Indeed, take* $y \in \mathrm{bd}(A) \cap Y$. *Since* A *is closed in* X, *we have that* $\mathrm{bd}(A) \subseteq A$ *and* $A \cap Y$ *is closed in* Y. *Therefore,* $y \in \mathrm{bd}(A) \cap Y \subseteq A \cap Y = \mathrm{cl}_Y(A \cap Y)$. *Thus, either* $y \in \mathrm{bd}_Y(A \cap Y)$ *or* $y \in \mathrm{int}_Y(A \cap Y)$. *In the latter case,* $y \in \mathrm{int}_Y(A \cap Y) = \mathrm{int}(A) \cap Y$, *contradicting that* $y \in \mathrm{bd}(A)$. *As a consequence,* $y \in \mathrm{bd}_Y(A \cap Y)$. *It only remains to show that* $\mathrm{cl}(A) \cap Y = \mathrm{cl}_Y(A \cap Y)$. *Indeed,* $\mathrm{cl}(A) \cap Y = (\mathrm{int}(A) \cup \mathrm{bd}(A)) \cap Y = (\mathrm{int}(A) \cap Y) \cup (\mathrm{bd}(A) \cap Y) = \mathrm{int}_Y(A \cap Y) \cup \mathrm{bd}_Y(A \cap Y) = \mathrm{cl}_Y(A \cap Y)$.

The previous lemma allows to obtain regular open/closed subsets in dense subsets.

Corollary 11 *Let* X *be a topological space. Let* Y *be a dense subset of* X. *Let* $U, B \subseteq X$. *Then:*

1. *If* U *is regular open, then* $U \cap Y$ *is regular open in* Y.
2. *If* B *is regular closed, then* $B \cap Y$ *is regular closed in* Y.

Proof 131 *By applying Lemma 47,* $\mathrm{int}_Y\left(\mathrm{cl}_Y(U \cap Y)\right) = \mathrm{int}_Y\left(\mathrm{cl}(U) \cap Y\right) = \mathrm{int}\left(\mathrm{cl}(U)\right) \cap Y = U \cap Y$ *and* $\mathrm{cl}_Y\left(\mathrm{int}_Y(B \cap Y)\right) = \mathrm{cl}_Y\left(\mathrm{int}(B) \cap Y\right) = \mathrm{cl}\left(\mathrm{int}(B)\right) \cap Y = B \cap Y$.

1.2.6.3 Bases and subbases

The notions of base of a topology and base of neighborhoods are crucial in order to construct topologies, such as uniform topologies or linear topologies.

Definition 148 *Let X be a topological space. Let $\mathcal{B}, \mathcal{S} \subseteq \mathcal{P}(X)$. Then:*

- *$\mathcal{B}$ is called a base of X if every element of $\mathcal{B}$ is open and, for every open subset U of X, there exists $\mathcal{C} \subseteq \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{C}} B$.*
- *$\mathcal{S}$ is called a subbase of X if the family of finite intersections of $\mathcal{S}$,*

$$\mathcal{B}(\mathcal{S}) := \left\{ \bigcap_{B \in M} B : M \subseteq \mathcal{S} \text{ finite} \right\},$$

  is a base of X.
- *$\mathcal{B}$ is said to be a base of neighborhoods of an element $x \in X$ if $\mathcal{B}$ is a filter base of $\mathcal{P}(X)$ and $\mathcal{J}(\mathcal{B}) = \mathcal{N}_x(X)$.*

The following remark establishes some trivial, but not less important, connections between bases of a topology and bases of neighborhoods. Recall that an open covering of a topological space X is a family of open subsets $\{U_i\}_{i \in I}$ of X such that $X = \bigcup_{i \in I} U_i$.

Remark 50 *Let X be a topological space. Let $\mathcal{B} \subseteq \mathcal{P}(X)$. Then:*

1. *$\mathcal{B}$ is a base of X if and only if, for every $x \in X$, $\mathcal{B}_x := \{U \in \mathcal{B} : x \in U\}$ is a base of open neighborhoods of x.*
2. *Suppose that $\mathcal{B}$ is a base of X. Then:*
   (a) *If we choose $x_B \in B$ for every nonempty $B \in \mathcal{B}$, then $\{x_B : B \in \mathcal{B}\}$ is clearly dense in X.*
   (b) *Consider an open covering $\{U_i\}_{i \in I}$ of X. For every $B \in \mathcal{B}$ such that $L_B := \{i \in I : B \subseteq U_i\} \neq \varnothing$, we can choose $i_B \in L_B$. Then $\{U_{i_B} : B \in \mathcal{B}, L_B \neq \varnothing\}$ is trivially a subcovering.*

The following theorem characterizes those collections of sets which are bases for a topology.

Theorem 31 *Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$. There exists a topology on X for which $\mathcal{B}$ is a base of if and only if $\mathcal{B}$ satisfies the following two conditions:*

1. *$X = \bigcup \mathcal{B}$.*
2. *For every $A, B \in \mathcal{B}$ and every $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.*

In this situation, there exists only one topology on X for which $\mathcal{B}$ is a base and it is precisely the coarsest topology on X containing $\mathcal{B}$.

Proof 132 *Suppose first that $X = \bigcup \mathcal{B}$. If there exists a topology for which $\mathcal{B}$ is a base, then, by definition, $X = \bigcup \mathcal{B}$. If $A, B \in \mathcal{B}$ and $x \in A \cap B$, then $A \cap B$ is a nonempty open subset of X, therefore, $A \cap B$ is union of elements of $\mathcal{B}$. Conversely, assume that for every $A, B \in \mathcal{B}$ and every $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$. It only suffices to realize that $\tau(\mathcal{B}) = \left\{U \in \mathcal{P}(X) : \exists \mathcal{C} \subseteq \mathcal{B} \;\; U = \bigcup_{C \in \mathcal{C}} C\right\}$ is a topology on X.*

It is not hard to understand that two topologies are the same if they share a common base. More generally, if X is a set and τ_1, τ_2 are topologies on X, then $\tau_1 \subseteq \tau_2$ if there exists a subbase $\mathcal{S}_1$ of τ_1 satisfying that $\mathcal{S}_1 \subseteq \tau_2$.

Example 33 (Product topology) *If $(X_i)_{i\in I}$ is a family of topological spaces, then the product topology on $\prod_{i\in I} X_i$ is defined as the topology generated by the base given by*

$$\left\{\prod_{i\in I} V_i : \forall i \in I \;\; V_i \text{ is open in } X_i \text{ and } \exists J \subseteq I \text{ finite } \;\forall i \in I \setminus J \;\; V_i = X_i\right\}.$$

In the third chapter, the product topology will be constructed by means of continuous functions.

Example 34 (Uniform product topology) *If $(X_i)_{i\in I}$ is a family of topological spaces, then the uniform product topology on $\prod_{i\in I} X_i$ is defined as the topology generated by the base given by*

$$\left\{\prod_{i\in I} V_i : \forall i \in I \;\; V_i \text{ is open in } X_i\right\}.$$

This topology is clearly finer than the product topology.

1.2.6.4 Separation properties

In a topological space X, a covering of X is a collection $\{V_i : i \in I\}$ of subsets of X such that $X = \bigcup_{i\in I} V_i$. If all the V_i's are open, then $\{V_i : i \in I\}$ is called an open covering. An open partition of X is a partition $\{U, V\}$ of X such that U, V are open and nonempty. Two points x, y of X are indistinguishable whenever $\mathcal{N}_x = \mathcal{N}_y$. Two subsets A, B of X are

- separated whenever $A \cap \mathrm{cl}(B) = \mathrm{cl}(A) \cap B = \varnothing$;
- separated by neighborhoods whenever there are disjoint neighborhoods V_A, V_B of A, B, respectively;
- separated by a continuous function if there exists a continuous function $f : X \to \mathbb{R}$ such that $A \subseteq f^{-1}(\{0\}$ and $B \subseteq f^{-1}(\{1\})$.

Continuous functions are mentioned here for the sake of completeness, but they will be covered in the upcoming chapters.

Definition 149 *A topological space X is said to be:*

- $\mathtt{T_0}$ *(Kolmogorov) if every $x \neq y$ in X are distinguishable, that is, $\mathcal{N}_x \neq \mathcal{N}_y$. In other words, the map $x \mapsto \mathcal{N}_x$ is injective.*
- $\mathtt{R_0}$ *(symmetric) if every distinguishable $x \neq y$ in X are separated.*
- $\mathtt{T_1}$ *(accesible or Fréchet) if every $x \neq y$ in X are separated.*
- $\mathtt{R_1}$ *(preregular) if every distinguishable $x \neq y$ in X are separated by neighborhoods.*
- $\mathtt{T_2}$ *(Hausdorff) if every $x \neq y$ in X are separated by neighborhoods.*
- $\mathtt{T_{2^1/_2}}$ *(Urysohn) if every $x \neq y$ in X are separated by closed neighborhoods.*
- *Completely* $\mathtt{T_2}$ *(completely Hausdorff) if every $x \neq y$ in X are separated by a continuous function.*

- *Regular if every proper closed subset $F \subsetneq X$ and every $x \in X \setminus F$ can be separated by neighborhoods.*
- $\mathtt{T}_3$ *(Regular Hausdorff) if X is regular and Hausdorff.*
- *Completely regular if every proper closed subset $F \subsetneq X$ and every $x \in X \setminus F$ can be separated by a continuous function.*
- $\mathtt{T}_{3^1/_2}$ *(Tychonoff) if X is Kolmogorov and completely regular.*
- *Normal if every two disjoint closed subsets of X are separated by neighborhoods.*
- *First countable if, for every $x \in X$, there exists a countable base of neighborhoods of x.*
- *Second countable if there exists a countable base for the topology of X.*
- *Separable if there exists a countable dense subset in X.*
- *Lindelöf if it is possible to extract a countable subcovering $\{V_{i_k} : k \in \mathbb{N}\}$ from every open covering $\{V_i : i \in I\}$ of X.*
- *Meager (or of first category) when X is a countable union of rare sets.*
- *Nonmeager (or of second category) when X is not meager.*
- *Baire if every countable intersection of open dense subsets is dense.*
- *Connected if there does not exist an open partition $\{U, V\}$ of X.*
- *Locally connected if every $x \in X$ has a base of connected neighborhoods.*
- *Compact provided that it is possible to extract a finite subcovering $\{V_{i_k} : k \in \{1, \ldots, n\}\}$ from every open covering $\{V_i : i \in I\}$ of X.*
- *Locally compact if every $x \in X$ has a base of compact neighborhoods.*

We will not go deeply over all these concepts because a whole book only for that would be required. Only some notable and helpful properties will be reviewed.

Lemma 48 *Let X be a topological space. Let $A \subseteq B \subseteq C$ be subsets of X. If A is nowhere dense or of first category in B, then so is A in C, respectively.*

Proof 133 *It only suffices to prove the lemma for nowhere dense sets. If A is nowhere dense in B but not in C, then A is dense in a nonempty set $G \subseteq C$ which is open in C. Since B is dense in G, the set $G \cap B$ is a nonempty relatively open set in B such that $G \cap B \subseteq \mathrm{cl}(A) \cap B = \mathrm{cl}_B(A)$, which implies that A is not nowhere dense in B.*

For every topological space X and each $x \in X$, it is satisfied that

$$X_{(x)} := \bigcap \mathcal{N}_x(X) = \bigcap_{V \in \mathcal{N}_x(X)} V = \{y \in X : x \in \mathrm{cl}(\{y\})\}.$$

On the other hand, for the upcoming lemma it is worth recalling that X is regular if and only if for every $x \in X$ and every $U \in \mathcal{N}_x$ there exists $V \in \mathcal{N}_x$ with $\mathrm{cl}(V) \subseteq U$.

Lemma 49 *Let X be a regular topological space. Let $x, y \in X$. Then:*

1. $\mathrm{cl}(\{x\}) = \bigcap \mathcal{N}_x$.
2. $x \in \bigcap \mathcal{N}_y = \mathrm{cl}(\{y\}) \Leftrightarrow y \in \mathrm{cl}(\{x\}) = \bigcap \mathcal{N}_x$.
3. *If, in addition,* X *is* $\mathtt{T_0}$*, then* X *is* $\mathtt{T_3}$.

Proof 134 *It is trivial that* $y \in \bigcap \mathcal{N}_x$ *if and only if* $x \in \mathrm{cl}(\{y\})$*. By switching* x *and* y*,* $x \in \bigcap \mathcal{N}_y$ *if and only if* $y \in \mathrm{cl}(\{x\})$*. Suppose that* $y \in \bigcap \mathcal{N}_x$ *but* $x \notin \bigcap \mathcal{N}_y$*. There exists* $U \in \mathcal{N}_y$ *with* $x \notin U$*. There exists* $V \in \mathcal{N}_y$ *with* $\mathrm{cl}(V) \subseteq U$*. Then* $x \in X \setminus \mathrm{cl}(V)$ *so* $X \setminus \mathrm{cl}(V) \in \mathcal{N}_x$*. By assumption,* $y \in \bigcap \mathcal{N}_x \subseteq X \setminus \mathrm{cl}(V)$*. This is a contradiction. As a consequence,* $x \in \bigcap \mathcal{N}_y$*. By swapping* x *and* y*, we obtain that* $y \in \bigcap \mathcal{N}_x$ *if* $x \in \bigcap \mathcal{N}_y$*. Next,*

$$X_{(x)} = \bigcap_{V \in \mathcal{N}_x(X)} V = \{y \in X : x \in \mathrm{cl}(\{y\})\} = \{y \in X : y \in \mathrm{cl}(\{x\})\} = \mathrm{cl}(\{x\}).$$

Finally, assume that X *is* $\mathtt{T_0}$*. In order to see that* X *is* $\mathtt{T_3}$*, we only need to show that* X *is* $\mathtt{T_2}$*. Fix arbitrary elements* $x, y \in X$ *with* $x \neq y$*. We may assume without any loss the existence of an open set* U *such that* $x \in U$ *but* $y \notin U$*. There exists an open set* V *such that* $x \in V$ *and* $\mathrm{cl}(V) \subseteq U$*. Then* V *and* $X \setminus \mathrm{cl}(V)$ *are disjoint open neighborhoods of* x *and* y*, respectively.*

Under the settings of the previous lemma, $X_{(x)} = \mathrm{cl}(\{x\})$, hence $X_{(x)}$ is contained in any closed subset of X that contains x.

Lemma 50 *Let* X *be a Hausdorff topological space. Let* $T \subseteq X$*. If* $x, y \in \mathrm{cl}(T)$ *with* $x \neq y$*, then there exists a subset* $S \subseteq T$ *such that* $x \in \mathrm{cl}(S)$ *and* $y \notin \mathrm{cl}(S)$*.*

Proof 135 *Since* X *is Hausdorff, we can take* U, V *neighborhoods of* x, y*, respectively, such that* $U \cap V = \varnothing$*. Notice that* $I := \{W \subseteq U : W$ *is a neighborhood of* $x\}$ *is a directed set when endowed with the partial order* $W_1 \leq W_2 \iff W_2 \subseteq W_1$*. Since* $x \in \mathrm{cl}(T)$*, for every* $W \in I$ *there exists* $t_w \in T \cap W$*. It is clear that the net* $(t_W)_{W \in I}$ *converges to* x*. However, by construction,* $y \notin \mathrm{cl}(\{t_W : W \in I\})$*. Finally, simply take* $S := \{t_W : W \in I\}$*.*

Recall that a family $(A_i)_{i \in I}$ of sets is said to verify the finite intersection property if every finite subfamily $(A_{i_k})_{k=1}^n$ of $(A_i)_{i \in I}$ has nonempty intersection, that is, $\bigcap_{k=1}^n A_{i_k} \neq \varnothing$.

Proposition 52 *Let* X *be a topological space. Then:*

1. *If* X *is Hausdorff, then, for all* $x \in X$*,* $X_{(x)} = \{x\}$ *and* $\{x\}$ *is closed.*
2. X *is compact if and only if every family of closed subsets of* X *satisfying the finite intersection property has nonempty intersection.*
3. *If* X *is compact and* F *is closed in* X*, then* F *is compact.*
4. *If* $F \subseteq X$ *is compact and* X *is Hausdorff, then* F *is closed in* X*.*
5. *If* X *is second countable and* $\mathcal{B}$ *is a countable base of* X*, then:*
   (a) *For every* $x \in X$*,* $\mathcal{B}_x := \{U \in \mathcal{B} : x \in U\}$ *is a countable base of neighborhoods of* x*.*
   (b) *If we choose* $x_B \in B$ *for every nonempty* $B \in \mathcal{B}$*, then* $\{x_B : B \in \mathcal{B}\}$ *is a countable dense subset of* X*.*
   (c) *If* $\{U_i\}_{i \in I}$ *is an open covering of* F*, then* $\{U_{i_B} : B \in \mathcal{B}, L_B \neq \varnothing\}$ *is a countable subcovering, where* i_B *is a chosen element in* L_B *for every* $B \in \mathcal{B}$ *such that* $L_B := \{i \in I : B \subseteq U_i\} \neq \varnothing$*.*

6. If X is compact, then X is Lindelöf.

Proof 136 *Only the fourth item will be proved. The rest are left as exercises. For each $x \in X, y \in F$, there exist neighborhoods $U_{xy} \in \mathcal{N}_x(X), V_y \in \mathcal{N}_y(X)$ with $U_{xy} \cap V_y = \varnothing$. Observe that $\{\text{int}\,(V_y) \cap F : y \in F\}$ is an open covering of F. By hypothesis, we can extract a finite subcovering $\{\text{int}\,(V_{y_k}) \cap F : k = 1, \ldots, p\}$. Then $U := U_{xy_1} \cap \cdots \cap U_{xy_p}$ is a neighborhood of x entirely contained in $X \setminus F$. This shows that $X \setminus F$ is open, hence F is closed.*

Proposition 52(5) assures that every second countable topological space is first countable, separable, and Lindelöf.

Theorem 32 *Let $(X_i)_{i\in I}$ be a family of topological spaces. Then $\prod_{i\in I} X_i$ is Hausdorff if and only if X_i is Hausdorff for each $i \in I$.*

Proof 137 *The proof is itemized via the statement.*

$\Rightarrow$ *Fix an arbitrary $i_0 \in I$ and take any $x_{i_0}, y_{i_0} \in X_{i_0}$ with $x_{i_0} \neq y_{i_0}$. For every $i \in I \setminus \{i_0\}$, fix any $x_i \in X_i$. Now, let $x, y \in \prod_{i\in I} X_i$ defined as*

$$x(i) := \begin{cases} x_i & i \neq i_0, \\ x_{i_0} & i = i_0, \end{cases} \qquad y(i) := \begin{cases} x_i & i \neq i_0, \\ y_{i_0} & i = i_0. \end{cases}$$

Note that $x \neq y$, so, by hypothesis, we can find $V_x \in \mathcal{N}_x\left(\prod_{i\in I} X_i\right), V_y \in \mathcal{N}_y\left(\prod_{i\in I} X_i\right)$ with $V_x \cap V_y = \varnothing$. By bearing in mind Example 33, there are neighborhoods $\prod_{i\in I} A_i \in \mathcal{N}_x\left(\prod_{i\in I} X_i\right), \prod_{i\in I} B_i \in \mathcal{N}_y\left(\prod_{i\in I} X_i\right)$ contained in V_x, V_y, respectively, for which there are $J_A, J_B \subseteq I$ finite such that, for all $i \in I$, A_i, B_i are open in X_i and, for every $i \in I \setminus (J_A \cup J_B)$, $A_i = B_i = X_i$. Note that $\prod_{i\in I} A_i \cap \prod_{i\in I} B_i = \varnothing$ and $x_i \in A_i \cap B_i$ whenever $i \neq i_0$. This forces that $A_{i_0} \cap B_{i_0} = \varnothing$. Finally, notice that $A_{i_0} \in \mathcal{N}_{x_{i_0}}(X_{i_0})$ and $B_{i_0} \in \mathcal{N}_{y_{i_0}}(X_{i_0})$.

$\Leftarrow$ *Take any $x, y \in \prod_{i\in I} X_i$ with $x \neq y$. There exists $i_0 \in I$ such that $x(i_0) \neq y(i_0)$. Since X_{i_0} is Hausdorff, we can find disjoint $V_{x(i_0)} \in \mathcal{N}_{x(i_0)}(X_{i_0}), V_{y(i_0)} \in \mathcal{N}_{y(i_0)}(X_{i_0})$. Finally, it suffices to define $\prod_{i\in I} A_i \in \mathcal{N}_x\left(\prod_{i\in I} X_i\right), \prod_{i\in I} B_i \in \mathcal{N}_y\left(\prod_{i\in I} X_i\right)$ by*

$$A_i := \begin{cases} X_i & i \neq i_0, \\ V_{x(i_0)} & i = i_0, \end{cases} \qquad B_i := \begin{cases} X_i & i \neq i_0, \\ V_{y(i_0)} & i = i_0. \end{cases}$$

Next result assures that the product of a countable family of (second-) first-countable topological spaces is (second) first countable.

Theorem 33 *Let $(X_i)_{i\in I}$ be a countable family of topological spaces. If X_i is (second) first countable for all $i \in I$, then $\prod_{i\in I} X_i$ is (second) first countable as well.*

Proof 138 *We will only prove the first-countable case. Fix an arbitrary $(x_i)_{i\in I}$. For every $i \in I$, let $\mathcal{B}_{x_i} \subseteq \mathcal{N}_{x_i}$ be a countable base of neighborhoods of x_i. Then*

$$\mathcal{B}_{(x_i)_{i\in I}} := \left\{ \prod_{i\in I} V_i : \exists J \subseteq I \text{ finite } \forall j \in J\ V_j \in \mathcal{B}_{x_j} \wedge \forall i \in I \setminus J\ \ V_i = X_i \right\}$$

is a countable base of neighborhoods of $(x_i)_{i\in I}$.

The existence of isolated points does not imply lack of connectedness. Indeed, let X be a set with at least two points. For every $x \in X$, $\tau := \{\varnothing, \{x\}, X\}$ is a topology on X that makes X connected and for which $x \in \text{iso}(X)$.

Proposition 53 *Let X be a Hausdorff topological space with more than two points. If $\mathrm{iso}(X) \neq \varnothing$, then X is not connected.*

Proof 139 *Take any $x \in \mathrm{iso}(X)$. Notice that $\{x\}$ is open in X. Since X is Hausdorff, by Remark 52(1), $\{x\}$ is closed; hence, $X \setminus \{x\}$ is a nonempty open subset of X. Finally, $\{\{x\}, X \setminus \{x\}\}$ is an open partition of X. As a consequence, X is not connected.*

1.2.6.5 Order topology

If X is a set and $\mathcal{S}$ is a nonemtpy subset of $\mathcal{P}(X)$, then the set of finite intersections of $\mathcal{S}$, $\mathcal{B}(\mathcal{S}) := \left\{\bigcap_{T\in\mathcal{T}} T : \mathcal{T} \subseteq \mathcal{S} \text{ finite}\right\}$, is trivially closed under finite intersections, thus $\mathcal{B}(\mathcal{S})$ is a base for a topology on X if and only if $\bigcup_{S\in\mathcal{S}} S = X$. The following theorem provides ground for the definition of the order topology and extends [18, Theorem 2.15] and some other results in [19].

Theorem 34 *Let X be a poset. Then:*

1. *$\{\uparrow_\times x : x \in X\} \cup \{\downarrow^\times x : x \in X\}$ is a subbase for a topology on X if and only if for every $x \in X$ either $\uparrow_\times x \neq \varnothing$ or $\downarrow^\times x \neq \varnothing$.*
2. *$\{\uparrow_\times x : x \in X\} \cup \{\downarrow^\times x : x \in X\} \cup \{X\}$ is a subbase for a topology on X.*

Proof 140 *It is clear that $\{\uparrow_\times x : x \in X\} \cup \{\downarrow^\times x : x \in X\} \cup \{X\}$ is a subbase for a topology on X. So, let us show the first item of the theorem. Observe that $\{\uparrow_\times x : x \in X\} \cup \{\downarrow^\times x : x \in X\}$ is a subbase for a topology on X if and only if $\bigcup_{x\in X} \uparrow_\times x \cup \bigcup_{x\in X} \downarrow^\times x = X$, if and only if for every $x \in X$ there exists $y \in X$ such that either $x \in \uparrow_\times y$ or $x \in \downarrow^\times y$, if and only if for every $x \in X$ there exists $y \in X$ such that either $y \in \downarrow^\times x$ or $y \in \uparrow_\times x$.*

Theorem 34 makes possible the definition of the order topology.

Definition 150 (Order topology) *Let X be a poset. The order topology on X is the topology generated by the subbase $\{\uparrow_\times x : x \in X\} \cup \{\downarrow^\times x : x \in X\} \cup \{X\}$. And we call X a topological poset.*

If X is a topological space endowed with the trivial topology, then the trivial ordering on X given by $x \leq y \Leftrightarrow x = y$ satisfies that its induced order topology is precisely the trivial topology. Something similar occurs with the discrete topology.

Theorem 35 *If X is a nonempty set, then there exists an ordering on X whose induced order topology is the discrete topology.*

Proof 141 *If $X = \{x_1, \ldots, x_n\}$ is finite, then it suffices to establish the total order $x_1 < \cdots < x_n$. Suppose that X is infinite. Let us write $X = \dot{\bigcup}_{i\in I} X_i$ as a disjoint union of infinitely countable subsets. For each $i \in I$, let us write $X_i = \{x_{ij} : j \in \mathbb{N}\}$ endowed with a total order such as $x_{i1} < \cdots < x_{in} < \cdots$. Consider the partial ordering on X given by $x \leq y \Leftrightarrow \exists i \in I$ such that $x, y \in X_i$ and $x \leq y$ in X_i. Finally, it only suffices to observe that, for every $i \in I$, $\{x_{i1}\} = \downarrow^\times x_{i2}$ and $\{x_{ij}\} = \uparrow_\times x_{ij-1} \cap \downarrow^\times x_{ij+1}$ for each $j > 1$.*

Let us next display a counterintuitive, although trivial, example showing that the relative order topology in a subset does not always coincide with the order topology of the subset.

Example 35 *Let $X := \mathbb{R}$ endowed with its usual order topology. Let $Y := \{0\} \cup (2, 3)$. Let τ_1 denote the order topology of Y as a totally ordered set. Let τ_2 denote the relative topology of Y (inherited from the order topology of X). Note that $\tau_1 \subseteq \tau_2$. However, $\tau_1 \neq \tau_2$. Indeed, $\{0\} \in \tau_2$ because $\{0\} = Y \cap (-1, 1)$. However, $\{0\} \notin \tau_1$ since $\{(-\infty, y) : y \in (2, 3)\}$ is a base of neighborhoods of 0 in τ_1.*

The following proposition unveils a sufficient condition to ensure that the relative order topology of a subset coincides with its own order topology. It is worth recalling that if $\mathcal{S}_1, \mathcal{S}_2$ are subbases for topologies τ_1, τ_2 in a set Y and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\mathcal{B}(\mathcal{S}_1) \subseteq \mathcal{B}(\mathcal{S}_2)$, hence $\tau_1 \subseteq \tau_2$.

Proposition 54 (Relative order topology) *Let X be a topological poset. Let Y be a subset of X. Then:*

1. *The order topology of Y is coarser than its inherited topology from X.*
2. *If, for all $y \in Y$, $\uparrow_{\times} y \cap Y$ is coinitial in $\uparrow_{\times} y$ and $\downarrow^{\times} y \cap Y$ is cofinal in $\downarrow^{\times} y$, then the order topology of Y coincides with its inherited topology from X.*

Proof 142 *The proof will follow as itemized as the statement of the proposition*

1. *The subbase of open rays within Y together with Y, which generates its order topology, is clearly contained in $\{\uparrow_{\times} x \cap Y : x \in X\} \cup \{\downarrow^{\times} x \cap Y : x \in X\} \cup \{X \cap Y\}$, which is the subbase of the inherited topology from X. As a consequence, the order topology of Y is coarser than its inherited topology from X.*
2. *Let U be an open subset in the order topology of X such that $U \cap Y \neq \varnothing$ and fix an arbitrary $y \in U \cap Y$. Without loss of generality, there can be found $x_1, \ldots, x_p, x_{p+1}, \ldots, x_q \in X$ such that $y \in (\uparrow_{\times} x_1 \cap \cdots \cap \uparrow_{\times} x_p \cap \downarrow^{\times} x_{p+1} \cap \cdots \cap \downarrow^{\times} x_q) \cap Y \subseteq U \cap Y$. Then $x_i < y < x_j$ for each $i \in \{1, \ldots, p\}$ and each $j \in \{p+1, \ldots, q\}$. By hypothesis, $\uparrow_{\times} y \cap Y$ and $\downarrow^{\times} y \cap Y$ are coinitial and cofinal in $\uparrow_{\times} y$ and $\downarrow^{\times} y$, respectively. Thus, for every $i \in \{1, \ldots, p\}$ and every $j \in \{p+1, \ldots, q\}$, there can be found $y_i, y_j \in Y$ such that $y < y_j \leq x_j$ and $x_i \leq y_i < y$. As a consequence, $y \in (\uparrow_{\times} y_1 \cap Y) \cap \cdots \cap (\uparrow_{\times} y_p \cap Y) \cap (\downarrow^{\times} y_{p+1} \cap Y) \cap \cdots \cap (\downarrow^{\times} y_q \cap Y) \subseteq (\uparrow_{\times} x_1 \cap \cdots \cap \uparrow_{\times} x_p \cap \downarrow^{\times} x_{p+1} \cap \cdots \cap \downarrow^{\times} x_q) \cap Y \subseteq U \cap Y$. This means that $U \cap Y$ is open in the order topology of Y.*

A direct consequence of Proposition 54(1) is the fact that if the order topology of Y is the discrete topology or its inherited topology from X is the trivial topology, then the order topology of Y coincides with its inherited topology from X. Next corollary shows that dense subsets satisfy the hypothesis of Proposition 54(2) in topological posets free of holes.

Corollary 12 *Let X be a topological poset. If X is holefree and Y is a dense subset of X, then the order topology of Y coincides with its inherited topology from X.*

Proof 143 *Fix an arbitrary $y \in Y$. We will show that $\uparrow_{\times} y \cap Y$ is coinitial in $\uparrow_{\times} y$. Indeed, take any $x \in \uparrow_{\times} y$. Then $y < x$ and by hypothesis $(y, x) \neq \varnothing$ because X is free of holes. Thus, (y, x) is a nonempty open subset of X which must intersect Y due to its density. As a consequence, there exists $y' \in (y, x) \cap Y$. Finally, notice that $y' < x$ and $y' \in \uparrow_{\times} y \cap Y$. This shows that $\uparrow_{\times} y \cap Y$ is coinitial in $\uparrow_{\times} y$. In a dual way, it can be shown that $\downarrow^{\times} y \cap Y$ is cofinal in $\downarrow^{\times} y$.*

The following technical lemma, which improves [20, Lemma 3.3], mirrors how to construct a base of neighborhoods of a point for the order topology on the real line.

Lemma 51 *Let X be a topological poset. Let $x_0 \in X$. Then:*

1. *If $\uparrow_{\times} x_0 \neq \varnothing$ is downward directed and $\downarrow^{\times} x_0 \neq \varnothing$ is upward directed, then $\{(a, b) : a < x_0 < b\}$ is a base of neighborhoods of x_0.*

2. *If* $\uparrow_\times x_0 \neq \varnothing$ *is downward directed and* $\downarrow^\times x_0 = \varnothing$*, then* $\{(-\infty, b) : x_0 < b\}$ *is a base of neighborhoods of* x_0*.*

3. *If* $\uparrow_\times x_0 = \varnothing$ *and* $\downarrow^\times x_0 \neq \varnothing$ *is upward directed, then* $\{(a, \infty) : a < x_0\}$ *is a base of neighborhoods of* x_0*.*

4. *If* $\uparrow_\times x_0 = \varnothing$ *and* $\downarrow^\times x_0 = \varnothing$*, then* $\{X\}$ *is a base of neighborhoods of* x_0*.*

Proof 144 *Let* $W \subseteq X$ *be an* x_0*-neighborhood for the order topology. We will distinguish between the four cases above:*

1. *There can be found* $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ *satisfying that* $x_0 \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$*. Observe that* $a_i < x_0$ *and* $b_j > x_0$ *for all* $i \in \{1, \ldots, n\}$ *and all* $j \in \{1, \ldots, m\}$*. Since* $\downarrow^\times x_0$ *is upward directed, there exists* $a_0 \in \downarrow^\times x_0$ *such that* $a_0 \geq a_i$ *for all* $i \in \{1, \ldots, n\}$*. Similarly, since* $\uparrow_\times x_0$ *is downward directed, there exists* $b_0 \in \uparrow_\times x_0$ *such that* $b_0 \leq b_j$ *for all* $j \in \{1, \ldots, m\}$*. Finally, notice that* $x_0 \in (a_0, b_0) \subseteq \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$*.*

2. *There can be found* $b_1, \ldots, b_m \in X$ *satisfying that* $x_0 \in \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$*. Observe that* $b_j > x_0$ *for all* $j \in \{1, \ldots, m\}$*. Since* $\uparrow_\times x_0$ *is downward directed, there exists* $b_0 \in \uparrow_\times x_0$ *such that* $b_0 \leq b_j$ *for all* $j \in \{1, \ldots, m\}$*. Finally, notice that* $x_0 \in (-\infty, b_0) \subseteq \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$*.*

3. *There can be found* $a_1, \ldots, a_n \in X$ *satisfying that* $x_0 \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W$*. Observe that* $a_i < x_0$ *for all* $i \in \{1, \ldots, n\}$*. Since* $\downarrow^\times x_0$ *is upward directed, there exists* $a_0 \in \downarrow^\times x_0$ *such that* $a_0 \geq a_i$ *for all* $i \in \{1, \ldots, n\}$*. Finally, notice that* $x_0 \in (a_0, \infty) \subseteq \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W$*.*

4. *In this situation,* x *is incomparable with all the rest elements of* X*; hence,* X *is the only open set containing* x*.*

Open rays and, consequently, open bounded intervals are open in the order topology. However, closed rays are not necessarily closed in the order topology. If the order is total, then closed rays are closed in the order topology because their complementary is an open ray, and, consequently, closed bounded intervals are closed as well in the order topology. Another sufficient condition for closed rays to be closed in the order topology was unveiled in [20, Lemma 3.2]. Recall that in a poset X, the set of incomparable elements to a certain $x \in X$ is denoted by θ_x. Recall as well that a poset X is said to be cofinal in itself provided that $X = \bigcup_{x \in X} \downarrow^\times x$. Dually, coinitial in itself can be defined.

Lemma 52 *Let* X *be a topological poset. Let* $x \in X$*. Then:*

1. *If* θ_x *is open or cofinal in itself, then* $\uparrow x$ *is closed.*

2. *If* θ_x *is open or coinitial in itself, then* $\downarrow x$ *is closed.*

Proof 145 *Only the first item will be proved, and we will only assume that* θ_x *is cofinal in itself. Notice that* $X \setminus \uparrow x = \downarrow^\times x \cup \theta_x$*. We will show that* $\downarrow^\times x \cup \theta_x$ *is open by showing that it is a neighborhood of each of its points. Indeed, fix an arbitrary* $y \in \downarrow^\times x \cup \theta_x$*. If* $y \in \downarrow^\times x$*, then* $y \in \downarrow^\times x \subseteq \downarrow^\times x \cup \theta_x$*, so* $\downarrow^\times x \cup \theta_x$ *is a neighborhood of* y *by definition of order topology. Next, assume that* $y \in \theta_x$*. By hypothesis,* θ_x *is cofinal in itself, meaning that there exists* $z \in \theta_x$ *such that* $y < z$*. Let us prove that* $y \in \downarrow^\times z \subseteq \downarrow^\times x \cup \theta_x$*. Take any* $w \in \downarrow^\times z$*. If* w *is not comparable to* x*, then* $w \in \theta_x$*. If* w *is comparable to* x*, then we have two options. One is that* $x \leq w$*, which implies that* $x \leq w < z$ *contradicting the fact* z *is not comparable to* x*. This only leaves the other option, that is,* $w < x$*, meaning that* $w \in \downarrow^\times x$*. As a consequence,* $y \in \downarrow^\times z \subseteq \downarrow^\times x \cup \theta_x$*. This shows that* $X \setminus \uparrow x = \downarrow^\times x \cup \theta_x$ *is open because it is a neighborhood of each of its points.*

Next technical lemma compiles [20, Lemma 3.4] and [20, Lemma 3.5] together. Recall that a hole in a poset is an empty nontrivial open interval such as $(x, y) = \varnothing$ for some $x < y$.

Lemma 53 *Let X be a topological poset. Let $a, b \in X$. Then:*

1. *If $\uparrow_{\times} b \neq \varnothing$ is downward directed and $\uparrow b$ is hole free, then $b \notin \operatorname{int}(\downarrow b)$. In particular, if $a < b$, then $b \notin \operatorname{int}([a, b])$.*
2. *If $\downarrow^{\times} a \neq \varnothing$ is upward directed and $\downarrow a$ is hole free, then $a \notin \operatorname{int}(\uparrow a)$. In particular, if $a < b$, then $a \notin \operatorname{int}([a, b])$.*
3. *Suppose $a < b$. If $\downarrow^{\times} b$ is upward directed and $\downarrow b$ is hole free, then $b \in \operatorname{cl}((a, b)) \subseteq \operatorname{cl}(\downarrow^{\times} b)$.*
4. *Suppose $a < b$. If $\uparrow_{\times} a$ is downward directed and $\uparrow a$ is hole free, then $a \in \operatorname{cl}((a, b)) \subseteq \operatorname{cl}(\uparrow_{\times} a)$.*

Proof 146 *Only the first and third items will be proved.*

1. *Assume on the contrary that $b \in \operatorname{int}(\downarrow b)$. There exists $W \subseteq X$ a b-neighborhood for the order topology contained in $\downarrow b$. There are three possibilities:*
   - *There are $a_1, \ldots, a_n \in X$ satisfying that $b \in \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \subseteq W$. Then $\uparrow_{\times} b \subseteq \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \subseteq W \subseteq \downarrow b$, which is a contradiction.*
   - *There are $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $b \in \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$. By hypothesis, $\uparrow_{\times} b \neq \varnothing$ and is downward directed, thus there exists $b_0 \in \uparrow_{\times} b$ such that $b_0 \leq b_j$ for each $j = 1, \ldots, m$. Again, by hypothesis, $\uparrow b$ is hole free, meaning that there exists $c \in \uparrow b$ with $b < c < b_0$. Then we reach the contradiction that $c \in \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W \subseteq \downarrow b$.*
   - *There are $b_1, \ldots, b_m \in X$ satisfying that $b \in \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$. By hypothesis, $\uparrow_{\times} b \neq \varnothing$ and is downward directed, thus there exists $b_0 \in \uparrow_{\times} b$ such that $b_0 \leq b_j$ for each $j = 1, \ldots, m$. Again, by hypothesis, $\uparrow b$ is hole free, meaning that there exists $c \in \uparrow b$ with $b < c < b_0$. Thus, the contradiction that $c \in \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W \subseteq \downarrow b$ is again reached.*
2. *Take any b-neighborhood $W \subseteq X$ for the order topology. There are three possibilities:*
   - *There are $a_1, \ldots, a_n \in X$ satisfying that $b \in \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \subseteq W$. By hypothesis, $\downarrow^{\times} b$ is upward directed, meaning that there exists $a_0 \in \downarrow^{\times} b$ such that $a \leq a_0$ and $a_i \leq a_0$ for each $i = 1, \ldots, n$. Again, by hypothesis, $\downarrow b$ is hole free; therefore, there exists $c \in \downarrow b$ with $a_0 < c < b$. Finally, $c \in (a, b) \cap \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n$.*
   - *There are $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $b \in \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$. By hypothesis, $\downarrow^{\times} b$ is upward directed, meaning that there exists $a_0 \in \downarrow^{\times} b$ such that $a \leq a_0$ and $a_i \leq a_0$ for each $i = 1, \ldots, n$. Again, by hypothesis, $\downarrow b$ is hole free, therefore there exists $c \in \downarrow b$ with $a_0 < c < b$. Finally, $c \in (a, b) \cap \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m$.*
   - *There are $b_1, \ldots, b_m \in X$ satisfying that $b \in \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$. Since $\downarrow b$ is hole free, we can find $c \in \downarrow b$ with $a < c < b$. Then $c \in (a, b) \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m$.*

Lemma 51 and Lemma 53 motivate the following definition.

Definition 151 (Locally directed poset) *A poset X is called:*

- *locally directed provided that, for every $x \in X$, if $\uparrow_{\times} x \neq \varnothing$, then it is downward directed, and if $\downarrow^{\times} x \neq \varnothing$, then it is upward directed.*
- *strongly locally directed provided that, for every $x \in X$, $\uparrow_{\times} x \neq \varnothing$ and is downward directed, and $\downarrow^{\times} x \neq \varnothing$ and is upward directed.*

By relying on the previous definition and by combining together Lemma 51, Lemma 52, and Lemma 53, we obtain the following result.

Lemma 54 *Let X be a topological poset. Suppose that X is holefree and locally directed. Let $x < y$ in X be such that either θ_x and θ_y are open in X or θ_x is cofinal in itself and θ_y is coinitial in itself. Then:*

1. $\mathrm{cl}\,(\uparrow_{\times} x) = \uparrow x$, $\mathrm{cl}\,(\downarrow^{\times} y) = \downarrow y$, *and* $\mathrm{cl}\,((x,y)) = \mathrm{cl}\,((x,y]) = \mathrm{cl}\,([x,y)) = [x,y]$.
2. *Assume* $\downarrow^{\times} x \neq \varnothing$ *and* $\uparrow_{\times} y \neq \varnothing$. *Then* $\mathrm{int}\,(\uparrow x) = \uparrow_{\times} x$, $\mathrm{int}\,(\downarrow y) = \downarrow^{\times} y$, *and* $\mathrm{int}\,([x,y]) = \mathrm{int}\,((x,y]) = \mathrm{int}\,([x,y)) = (x,y)$.

Proof 147 *According to Lemma 52, $\uparrow x, \downarrow y$ are closed; therefore, $[x,y]$ is closed as well. Then $(x,y) \subseteq \mathrm{cl}((x,y)) \subseteq [x,y]$. In virtue of Lemma 53(3,4),* $\mathrm{cl}\,(\uparrow_{\times} x) = \uparrow x$, $\mathrm{cl}\,(\downarrow^{\times} y) = \downarrow y$ *and* $\mathrm{cl}\,((x,y)) = \mathrm{cl}\,((x,y]) = \mathrm{cl}\,([x,y)) = [x,y]$. *Next, $\uparrow_{\times} x, \downarrow^{\times} y$ are open, hence (x,y) is open as well, so $(x,y) \subseteq \mathrm{int}\,([x,y]) \subseteq [x,y]$. Assume $\downarrow^{\times} x \neq \varnothing$ and $\uparrow_{\times} y \neq \varnothing$. In view of Lemma 53(1,2),* $\mathrm{int}\,(\uparrow x) = \uparrow_{\times} x$, $\mathrm{int}\,(\downarrow y) = \downarrow^{\times} y$ *and* $\mathrm{int}\,([x,y]) = \mathrm{int}\,((x,y]) = \mathrm{int}\,([x,y)) = (x,y)$.

Let us continue our study of the order topology by unveiling its most basic separation properties. Recall that the set of isolated points of a topological space X is commonly denoted by $\mathrm{iso}(X)$.

Proposition 55 *Let X be a topological poset. Let $x \neq y$ in X. Then:*

1. *If $x < y$, then x, y can be separated by disjoint open sets. In particular, if X is totally ordered, then it is Hausdorff.*
2. *If X is locally directed and hole free, then* $\mathrm{iso}(X) = \varnothing$.
3. *If $X \setminus \{x,y\}, \downarrow^{\times} x, \downarrow^{\times} y$ are upward directed and $\uparrow_{\times} x = \uparrow_{\times} y = \varnothing$, then x, y cannot be separated by disjoint open sets. Hence X is not Hausdorff.*

Proof 148 *The proof will follow as itemized as the statement of the proposition.*

1. *If $(x,y) = \varnothing$, then $y \in (x,\infty)$, $x \in (-\infty, y)$, and $(-\infty, y) \cap (x,\infty) = (x,y) = \varnothing$. If there exists $z \in X$ with $x < z < y$, then $x \in (-\infty, z)$, $y \in (z,\infty)$, and $(-\infty, z) \cap (z,\infty) = \varnothing$.*
2. *Assume that $x \in \mathrm{iso}(X)$. Then $\{x\}$ is open. According to Lemma 51, $\{x\}$ is of the form (a,b) or $(-\infty, a)$ or $(b, +\infty)$. In any of the three prior cases, we reach the contradiction that X has a hole.*
3. *First off, note that the fact that $\uparrow_{\times} x = \uparrow_{\times} y = \varnothing$ implies that x, y are incomparable. Fix arbitrary neighborhoods U_x, U_y of x, y, respectively, in the order topology. In accordance with Lemma 51(3), there exist $x_0 \in \downarrow^{\times} x$ and $y_0 \in \downarrow^{\times} y$ such that $x \in (x_0,\infty) \subseteq U_x$ and $y \in (y_0,\infty) \subseteq U_y$. Since $X \setminus \{x,y\}$ is upward directed, there exists $z \in X$ with $z \geq x_0$ and $z \geq y_0$. At this point, let us distinguish between the following cases:*

- $z = x_0$ *and* $z \neq y_0$. *Then* $x_0 \geq y_0$ *so* $x \in (x_0, \infty) \subseteq (x_0, \infty) \cap (y_0, \infty) \subseteq U_x \cap U_y$.
- $z \neq x_0$ *and* $z = y_0$. *Then* $y_0 \geq x_0$ *so* $y \in (y_0, \infty) \subseteq (x_0, \infty) \cap (y_0, \infty) \subseteq U_x \cap U_y$.
- $z \neq x_0, y_0$. *Then* $z \in (x_0, \infty) \cap (y_0, \infty) \subseteq U_x \cap U_y$.

Next theorem characterizes separability of the order topology in locally directed holefree posets. This result is known in the classic literature for totally ordered sets.

Theorem 36 *Let X be a locally directed holefree topological poset. Then X is separable if and only if X is second countable.*

Proof 149 *It is well known that every topological space is separable if it is second countable. So, assume that X is separable and fix any countable dense subset $D \subseteq X$. We will show that*

$$\mathcal{B} := \{(-\infty, d), (c, d), (c, +\infty) : d, c \in D\} \cup \{X\}$$

is a countable base of X. It is clear that $\mathcal{B}$ is countable. Fix an arbitrary nonempty open subset U of X and an arbitrary $x \in U$. According to Lemma 51, there exists an interval of the form $(a, b), (-\infty, b), (a, +\infty), (-\infty, \infty)$ containing x and contained in U. Let us distinguish between the four cases:

- $x \in (a, b) \subseteq U$. *Since X has no holes, $(a, x), (x, b) \neq \varnothing$, so, the density of D allows $c \in (a, x), d \in (x, b)$. Then $x \in (c, d) \subseteq (a, b) \subseteq U$.*
- $x \in (-\infty, b) \subseteq U$. *Since X has no holes, $(x, b) \neq \varnothing$, so, the density of D allows $d \in (x, b)$. Then $x \in (-\infty, d) \subseteq (-\infty, b) \subseteq U$.*
- $x \in (a, +\infty) \subseteq U$. *Since X has no holes, $(a, x) \neq \varnothing$, so, the density of D allows $c \in (a, x)$. Then $x \in (c, +\infty) \subseteq (a, +\infty) \subseteq U$.*
- $x \in (-\infty, \infty) \subseteq U$. *Then $x \in X = (-\infty, \infty) \subseteq U$.*

Recall that a poset is said to have the sup (inf) property if every bounded above (below) subset has a supremum (infimum). At this stage, it is worth recalling that a poset satisfies the sup property if and only if it satisfies the inf property. Our next step is to introduce a topological version of this property. But first, a technical lemma is needed.

Lemma 55 *Let X be a topological poset. Let $A \subseteq X$. If x is an upper bound of A, $y \in \mathrm{cl}(A)$, and x and y are comparable, then $y \leq x$. In particular, if X is totally ordered, then $\mathrm{cl}(A)$ is bounded above if so is A, and $\sup(A)$ exists if and only if $\sup(\mathrm{cl}(A))$ exists, in which case $\sup(A) = \sup(\mathrm{cl}(A)) \in \mathrm{cl}(A)$.*

Proof 150 *Suppose on the contrary that $x < y$. Then $\uparrow_{\times} x$ is an open neighborhood of y, meaning that $\uparrow_{\times} x \ \cap A \neq \varnothing$. As a consequence, there exists $a \in A$ with $x < a$, which contradicts the fact that x is an upper bound of A.*

Lemma 55 has its dual version for lower bounds and infimums. It also motivates the following definition.

Definition 152 (Strong sup (inf) property) *A topological poset X is said to have the strong sup (inf) property provided that it verifies the sup (inf) property and $\sup(A) \in \mathrm{cl}(A)$ ($\inf(A) \in \mathrm{cl}(A)$) for every bounded above (below) subset $A \subseteq X$.*

Under the settings of the previous definition, closed subsets of X are sup (inf) closed. A direct consequence of Lemma 55 is that every topological toset with the sup (inf) property enjoys the strong sup (inf) property.

Lemma 56 *Let X be a topological poset. If X is upward directed and satisfies the strong sup property, then X is totally ordered.*

Proof 151 *Fix arbitrary elements $x_1 \neq x_2$ in X. We will show that they are comparable. Since X is upward directed, there exists $x_3 \in X$ such that $x_i \leq x_3$ for $i = 1, 2$. Then $\{x_1, x_2\}$ is bounded above; hence there exists $x_1 \vee x_2$ and $x_1 \vee x_2 \in \mathrm{cl}(\{y_1, y_2\})$. We distinguish now between two cases:*

- *$x_1 \vee x_2 = x_1$. In this case, $x_2 \leq x_1 \vee x_2 = x_1$.*
- *$x_1 \vee x_2 \neq x_1$. In this case, $x_1 \vee x_2 \in \uparrow_{\times} x_1$. Then $\uparrow_{\times} x_1$ is a neighborhood of $x_1 \vee x_2$ in the order topology, so $\uparrow_{\times} x_1 \cap \{x_1, x_2\} \neq \varnothing$, meaning that $x_1 < x_2$.*

As expected, the previous lemma has its dual version for downward directed topological posets enjoying the strong inf property.

Theorem 37 *Let X be a topological toset. Then X is connected if and only if X has no holes and satisfies the sup property.*

Proof 152 *The proof will follow as itemized as the statement of the theorem.*

$\Rightarrow$ *We will show first that X is free of holes. Suppose to the contrary that there are $x, y \in X$ with $x < y$ such that $(x, y) = \varnothing$. Then $\{(-\infty, y), (x, \infty)\}$ is an open partition of X, contradicting that X be connected. Let us prove now that X satisfies the sup property. Again, assume to the contrary that there exists $A \subseteq X$ bounded above, but $\sup(A)$ does not exist. Let $B := \bigcup_{a \in A}(-\infty, a)$ and $C := \bigcup \{(x, +\infty) : x \text{ is an upper bound of } A\}$. It is clear that B, C are open and disjoint. Let us show that $B, C \neq \varnothing$ and $B \cup C = X$. If $B = \varnothing$, then A is a singleton, hence $\sup(A)$ exists. So, $B \neq \varnothing$. If $C = \varnothing$, then there is only one upper bound for A, hence $\sup(A)$ exists. Therefore, $C \neq \varnothing$. Finally, let us prove that $X = B \cup C$. Fix an arbitrary $x_0 \in X$. If $x_0 \notin B$, then x_0 is an upper bound of A. If, in addition, $x_0 \notin C$, then $x_0 \leq x$ for each upper bound x of A. This shows that $x_0 = \sup(A)$, which does not exist. As a consequence, $x_0 \in B \cup C$. The arbitrariness of $x_0 \in X$ means that $X = B \cup C$. We then reach the contradiction that X is not connected.*

$\Leftarrow$ *Let $\{B, C\}$ be an open partition of X. Fix $b_0 \in B, c_0 \in C$. We may assume, without any loss of generality, that $b_0 < c_0$. Define $B_0 := \{b \in B : b \leq c_0\} = B \cap (-\infty, c_0]$. Note that B_0 is closed, $b_0 \in B_0$ and c_0 is an upper bound for B_0. By hypothesis, $\sup(B_0)$ exists, and $\sup(B_0) \leq c_0$. By Lemma 55, $\sup(B_0) \in \mathrm{cl}(B_0) = B_0$. Since $B \cap C = \varnothing$ and $\sup(B_0) \in B$, we have that $\sup(B_0) \notin C$, so $\sup(B_0) < c_0$. Since X has no holes, $(\sup(B_0), c_0) \neq \varnothing$. We will show that $(\sup(B_0), c_0) \subseteq C$. Take any $x \in (\sup(B_0), c_0)$. Since $X = B \cup C$, if $x \notin C$, then $x \in B$. Since $x < c_0$, we have that $x \in B_0$, which is impossible because $x > \sup(B_0)$. As a consequence, $(\sup(B_0), c_0) \subseteq C$. Observe that C is closed, meaning, by bearing in mind Lemma 54, that $\sup(B_0) \in [\sup(B_0), c_0] = \mathrm{cl}\left((\sup(B_0), c_0)\right) \subseteq \mathrm{cl}(C) = C$, which is not possible.*

Our next step is to introduced the one-sided order topologies, which allow the conception of lower and upper semicontinuity.

Definition 153 (One-sided order topologies) *Let X be a poset. The right order topology on X is defined as the topology generated by the subbase of upper open rays $\{\uparrow_{\times} x : x \in X\} \cup \{X\}$ and we call X a right topological poset. Dually, the left order topology on X is defined as the topology generated by the subbase of lower open rays $\{\downarrow^{\times} x : x \in X\} \cup \{X\}$ and we call X called a left topological poset.*

Notice that both the left and right order topologies are coarser than the order topology.

Proposition 56 *Let X be a left topological toset. If $K \subseteq X$ is compact, then there exists* $\max(K)$.

Proof 153 *Consider the family $\{\uparrow k \cap K : k \in K\}$ of closed subsets of K. Let us prove that the previous family satisfies the finite intersection property. Fix arbitrary elements $k_1, \ldots, k_p$ in K. Since K is upward directed, there exists $k_0 \in K$ such that $k_0 \geq k_j$ for all $j = 1, \ldots, p$. Then $k_0 \in \bigcap_{j=1}^{p} (\uparrow k_j \cap K) \neq \varnothing$. The compacity of K allows the existence of $k' \in \bigcap_{k \in K} (\uparrow k \cap K)$. Notice that $k' = \max(K)$.*

The following corollary is a direct consequence of Proposition 56, which has its dual version for the right order topology.

Corollary 13 *If X is a topological toset and $K \subseteq X$ is compact, then there exist* $\max(K)$ *and* $\min(K)$.

Proof 154 *On the one hand, K is closed because the order topology is Hausdorff in totally ordered sets. On the other hand, K is compact as well for the left and right order topologies, since the one-sided order topologies are coarser than the order topology. Therefore, by Proposition 56 and its dual version, there exist* $\max(K)$ *and* $\min(K)$.

1.2.6.6 Uniform spaces

Uniform spaces are a generalization of pseudometric spaces and topological groups. These spaces are endowed with the necessary structure to define closedness of points, hence uniform continuity and uniform convergence. The notion of uniformity involves concepts proper to Order Theory, so it is not precisely a universal algebra or subalgebra.

Definition 154 (Uniform space) *Let X be a set. A uniformity on X is a filter $\mathcal{U}$ of $\mathcal{P}(X \times X)$ satisfying, for every $U \in \mathcal{U}$, the following conditions:*

- *U is a reflexive binary internal relation on X.*
- *There exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.*
- *$U^{-1} \in \mathcal{U}$.*

The pair $(X, \mathcal{U})$ is called a uniform space.

Under the settings of the previous definition, the elements of $\mathcal{U}$ are called entourages or vicinities. Every filter base of $\mathcal{U}$ is called a base of entourages or vicinities. Notice that, if $\mathcal{B}$ is a base of entourages, then $\mathcal{B}_1 := \left\{U \cap U^{-1} : U \in \mathcal{B}\right\}$ and $\mathcal{B}_2 := \left\{U \cup U^{-1} : U \in \mathcal{B}\right\}$ are bases of symmetric entourages. For every $x \in X$ and every $U \in \mathcal{U}$, $U[x] := \{y \in X : (x, y) \in U\}$. If $A \subseteq X$ and $U \in \mathcal{U}$, then $U[A] := \bigcup_{a \in A} U[a]$. Given two uniformities $\mathcal{U}_1, \mathcal{U}_2$ on X, we will say that $\mathcal{U}_1$ is coarser than $\mathcal{U}_2$, or $\mathcal{U}_2$ is finer than $\mathcal{U}_1$, if $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Every subset Y of X inherits its uniform structure by means of the uniformity $\mathcal{U}_Y := \{U \cap (Y \times Y) : U \in \mathcal{U}\}$.

Lemma 57 *Let X be a uniform space. Let U, V be entourages. Let $x \in X$. Then:*

1. *If $V \circ V \subseteq U$, then $V[y] \subseteq U[x]$ for all $y \in V[x]$. In particular, $V[x] \subseteq U[x]$.*
2. *$U \subseteq U^n$ for all $n \in \mathbb{N}$.*

Proof 155 *Let us itemize the proof by following the itemization of the statement of the lemma.*

1. *Simply take any $z \in V[y]$ and notice that $(y,z) \in V$ and $(x,y) \in V$, therefore, $(x,z) \in V \circ V \subseteq U$. As a consequence, $z \in U[x]$.*

2. *Fix an arbitrary $(x,y) \in U$ and choose any $n > 1$. It only suffices to observe that $(x,x), \overset{n-1}{\ldots}, (x,x), (x,y) \in U$, meaning that $(x,y) \in U^n$.*

If X is a set and $\mathcal{B}$ is a filter base of $\mathcal{P}(X \times X)$ satisfying the three conditions of the definition of uniformity, then $\mathcal{U} := \mathcal{J}(\mathcal{B})$ is a uniformity.

Example 36 *Let X be a set. Fix a filter $\mathcal{J}$ in X. Then $\mathcal{B}_{\mathcal{J}} := \{\mathsf{V}_Y : Y \in \mathcal{J}\}$ is a uniformity base of $\mathcal{P}(X)$, where $\mathsf{V}_Y := \{(A,B) \in \mathcal{P}(X)^2 : A\Delta B \subseteq Y\}$ for all $Y \in \mathcal{J}$. Indeed:*

1. *$\mathsf{V}_Y \cap \mathsf{V}_Z = \mathsf{V}_{Y \cap Z}$ for all $Y, Z \in \mathcal{J}$, meaning that $\mathcal{B}_{\mathcal{J}}$ is a filter base of $\mathcal{P}(X)^2$.*

2. *$(A,A) \in \mathsf{V}_Y$ for all $A \in \mathcal{P}(X)$ and all $Y \in \mathcal{J}$.*

3. *$\mathsf{V}_Y \circ \mathsf{V}_Y = \mathsf{V}_Y$ for all $Y \in \mathcal{J}$. Indeed, it always occurs that $\mathsf{V}_Y \circ \mathsf{V}_Y \supseteq \mathsf{V}_Y$. Fix an arbitrary element $(A,C) \in \mathsf{V}_Y \circ \mathsf{V}_Y$ and let $B \in \mathcal{P}(X)$ such that $(A,B),(B,C) \in \mathsf{V}_Y$. Notice that*

$$\begin{aligned} A\Delta C &= [(A\Delta C) \cap B] \cup [(A\Delta C) \cap (X \setminus B)] \\ &= [(A \setminus C \cup C \setminus A) \cap B] \cup [(A \setminus C \cup C \setminus A) \cap (X \setminus B)] \\ &= [(A \cap B) \setminus C \cup (C \cap B) \setminus A] \cup [(A \setminus B) \setminus C \cup (C \setminus B) \setminus A)] \\ &\subseteq [B \setminus C \cup B \setminus A] \cup [A \setminus B \cup C \setminus B] \\ &= A\Delta B \cup B\Delta C \\ &\subseteq Y. \end{aligned}$$

4. *V_Y is clearly symmetric.*

The following concept, isobasism, plays the role of isomorphism in the category of uniform spaces.

Definition 155 (Isobasism) *Let X, Y be uniform spaces. A function $f : X \to Y$ is said to be an isobasism if it is bijective and the uniformity of Y coincides with $\{(f \times f)(U) : U$ entourage in $X\}$.*

Under the settings of the previous definition,

$$(f \times f)(U) := \{(f(x_1), f(x_2)) \in Y \times Y : (x_1, x_2) \in U\}.$$

Definition 156 (Separated uniformity) *A uniform space X is called separated provided that the intersection of all entourages is Δ_X.*

1.2.6.7 Uniform topologies

Every uniform space becomes a topological space by defining the topology by means of the entourages.

Theorem 38 (Uniform topology) *Let X be a uniform space. Then*

$$\tau := \{A \subseteq X : \forall a \in A\ \exists U \subseteq X \times X \text{ entourage with } U[a] \subseteq A\} \cup \{\varnothing\}$$

is a topology on X that turns it into a regular topological space. Also, for every $A \subseteq X$, $\operatorname{int}(A) = \{a \in A : \exists U \subseteq X \times X$ entourage such that $U[a] \subseteq A\}$. As a consequence, $\{U[x] : U$ entourage$\}$ is base of neighborhoods of every $x \in X$.

Proof 156 *It is trivial to check that τ defines a topology on X. We will prove first that, for every $A \subseteq X$, $\mathrm{int}(A) = \{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$. Indeed, if $a \in \mathrm{int}(A)$, then by definition of τ, there exists an entourage $U \subseteq X$ such that $U[a] \subseteq \mathrm{int}(A) \subseteq A$. This shows that $\mathrm{int}(A) \subseteq \{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$. Conversely, we will prove that $\mathrm{int}(A) \supseteq \{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$. To accomplish this, we will show first that $\{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$ is open. Indeed, fix an arbitrary $b \in A$ for which there exists an entourage $V \subseteq X$ with $V[b] \subseteq A$. In accordance with the definition of τ, we need to find an entourage $W \subseteq X$ such that $W[b] \subseteq \{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$. Indeed, take any entourage $W \subseteq X$ such that $W \circ W \subseteq V$. According to Lemma 57, $W[c] \subseteq V[b] \subseteq A$ for every $c \in W[b]$. This shows that $W[b] \subseteq \{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$. Hence, $\{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$ is open, thus, it is contained in $\mathrm{int}(A)$. As a consequence, $\mathrm{int}(A) = \{a \in A : \exists U \subseteq X \text{ entourage } U[a] \subseteq A\}$. From this, it is not hard to understand that $\{U[x] : U \text{ entourage}\}$ is base of neighborhoods of every $x \in X$. Finally, let us check that this topology is regular. Fix an arbitrary $x \in X$ and take any entourage U. There exists an entourage V such that $V \circ V \subseteq U$. We will show that $\mathrm{cl}\,(V[x]) \subseteq U[x]$. Let $y \in \mathrm{cl}\,(V[x])$. Then $V^{-1}[y] \cap V[x] \neq \varnothing$. Take $z \in V^{-1}[y] \cap V[x]$. Then $(x, z) \in V$ and $(y, z) \in V^{-1}$, so $(x, z), (z, y) \in V$, hence $(x, y) \in V \circ V \subseteq U$. As a consequence, $y \in U[x]$.*

A direct consequence of Theorem 38 establishes that if $\mathcal{B}$ is a base of entourages, then $\{U[x] : U \in \mathcal{B}\}$ is base of neighborhoods of x. Notice also that if a uniformity is coarser than another uniformity, then the uniform topo-logy induced by the coarsest uniformity is coarser than the uniform topology endowed by the finest uniformity.

Definition 157 (Vitali class) *Let X be a uniform space. A collection $\mathbb{V} \subseteq \mathcal{P}(X)$ is called a Vitali class for a subset $E \subseteq X$ provided that for every $e \in E$ and every entourage U there exists $S \in \mathbb{V}$ such that $e \in S$ and $S \times S \subseteq U$.*

The following technical lemma establishes basic properties satisfied by Vitali classes.

Lemma 58 *Let X be a uniform space. Let $E \subseteq X$. Let $\mathbb{V}$ be a Vitali class for E. Then:*

1. *$\overline{\mathbb{V}} := \{\mathrm{cl}(S) : S \in \mathbb{V}\}$ is a Vitali class for E as well.*
2. *If Y is closed in X and $x \in E \setminus Y$, then there exists $S \in \mathbb{V}$ with $x \in S \subseteq X \setminus Y$.*

Proof 157 *The proof follows as itemized as the statement of the lemma.*

1. *Fix an arbitrary $e \in S$ and an arbitrary entourage U. Take another symmetric entourage W such that $W \circ W \circ W \subseteq U$. There esists $S \in \mathbb{V}$ such that $e \in S$ and $S \times S \subseteq W$. We will show that $\mathrm{cl}(S) \times \mathrm{cl}(S) \subseteq U$. Indeed, take $(x, y) \in \mathrm{cl}(S) \times \mathrm{cl}(S)$. Then $W[x] \cap S \neq \varnothing$ and $W[y] \cap S \neq \varnothing$. Take $s \in W[x] \cap S$ and $t \in W[y] \cap S$. Then $(x, s), (s, t), (t, y) \in W$ (note that $(s, t) \in S \times S \subseteq W$). As a consequence, $(x, y) \in W \circ W \circ W \subseteq U$.*
2. *Since $X \setminus Y$ is open and $x \in X \setminus Y$, there exists an entourage U such that $U[x] \subseteq X \setminus Y$. Next, $\mathbb{V}$ is a Vitali class for E, meaning that there exists $S \in \mathbb{V}$ with $x \in S$ and $S \times S \subseteq U$. It only remains to show that $S \subseteq X \setminus Y$. Indeed, if $s \in S$, then $(x, s) \in S \times S \subseteq U$, which implies that $s \in U[x] \subseteq X \setminus Y$.*

1.2.7 Pseudometrics and seminorms

Pseudometric spaces have always been studied as part of Analysis, more concretely, Real Analysis. However, they became fundamental structures in General Topology and Geometry. Here, we will need them because of the topology that they induce, which is very rich in the sense that it verifies crucial properties.

1.2.7.1 Pseudometric spaces

We will begin by defining pseudometric spaces whose pseudometric takes values in an ordered ring.

Definition 158 (Pseudometric) *Let $((R, \leq), \{+, 0, -, \cdot, 1\}, \varnothing)$ be a unital ordered ring. A pseudometric on a set X is a binary external operation*

$$\begin{array}{rccc} d: & X \times X & \to & R^+ \\ & (x, y) & \mapsto & d(x, y) \end{array} \tag{1.30}$$

such that for all $x, y, z \in X$:

- *Identity preserving: $d(x, x) = 0$.*
- *Symmetricity: $d(x, y) = d(y, x)$.*
- *Triangular inequality: $d(x, y) \leq d(x, z) + d(z, y)$.*

The universal algebra $(X, \varnothing, \{d\})$ is called a pseudometric space. If the pseudometric d satisfies that $d(x, y) > 0$ whenever $x \neq y$, then we call d a metric and the universal algebra $(X, \varnothing, \{d\})$ is called a metric space.

From now on and unless otherwise stated explicitly, we will only consider real pseudometrics.

Definition 159 (Isometry) *Let X, Y be pseudometric spaces. A map $T : X \to Y$ is called an isometry provided that $d(T(x), T(y)) = d(x, y)$ for all $x, y \in X$.*

Every pseudometric induces a uniformity, hence a topology as well. These topologies satisfy nice properties that simplify, considerably, the study of limits and continuity.

Lemma 59 *Let X be a pseudometric space. Let $x, y, z \in X$ Then $|d(x, z) - d(y, z)| \leq d(x, y)$. In particular, if $d(x, y) = 0$, then $d(x, z) = d(y, z)$.*

Proof 158 *On the one hand, $d(x, z) \leq d(x, y) + d(y, z)$ so $d(x, z) - d(y, z) \leq d(x, y)$. On the other hand, $d(y, z) \leq d(y, x) + d(x, z)$ so $d(y, z) - d(x, z) \leq d(y, x)$. As a consequence, $|d(x, z) - d(y, z)| \leq d(x, y)$.*

Notable subsets of pseudometric spaces are the balls and the spheres. Balls and spheres of null radius will be dealt with later on.

Definition 160 *Let X be a pseudometric space. Let $x \in X$ and $\delta > 0$. We define the following sets:*

- *Closed ball $\mathsf{B}_X(x, \delta) := \{y \in X : d(y, x) \leq \delta\}$ of center x and radius δ.*
- *Open ball $\mathsf{U}_X(x, \delta) := \{y \in X : d(y, x) < \delta\}$ of center x and radius δ.*
- *Sphere $\mathsf{S}_X(x, \delta) := \{y \in X : d(y, x) = \delta\}$ of center x and radius δ.*

Under the settings of the previous definition, if there is no confusion with X, then we will simply write $\mathsf{B}(x, \delta), \mathsf{U}(x, \delta), \mathsf{S}(x, \delta)$. The following crucial theorem describes pseudometric uniformities and pseudometric topologies and shows that every pseudometric space is regular.

Theorem 39 (Pseudometric topology) *Let X be a pseudometric space. The sets of the form $U_\delta := \{(x,y) \in X \times X : d(x,y) < \delta\}$, for every $\delta > 0$, form a base of entourages whose generated filter is called the pseudometric uniformity. For every $x \in X$ and every $\delta > 0$, $\mathsf{U}_\delta[x] = \mathsf{U}(x,\delta)$. As a consequence, the set $\mathcal{B} := \{\mathsf{U}(x,\delta) : x \in X, \delta > 0\}$ forms a base for a topology on X called the pseudometric topology. This topology is given by $\tau := \{U \in \mathcal{P}(X) : \forall x \in U\ \exists \delta > 0\ \mathsf{U}(x,\delta) \subseteq U\}$. For every $x \in X$, the set*

$$\mathcal{B}_x := \left\{ \mathsf{B}\left(x, \frac{1}{n}\right) : x \in X, n \in \mathbb{N} \right\}$$

is a base of neighborhoods of x for the pseudometric topology, hence $X_{(x)} = \mathsf{B}(x,0)$. As a consequence, X is regular and first countable.

Proof 159 *For every $\varepsilon, \delta > 0$, $\mathsf{U}_{\min\{\varepsilon,\delta\}} = \mathsf{U}_\varepsilon \cap \mathsf{U}_\delta$, $\Delta_X \subseteq \mathsf{U}_\varepsilon$, $\mathsf{U}_{\frac{\delta}{2}} \circ \mathsf{U}_{\frac{\delta}{2}} \subseteq \mathsf{U}_\delta$, and $\mathsf{U}_\varepsilon^{-1} = \mathsf{U}_\varepsilon$. This shows that the above sets form a filter base of $\mathcal{P}(X \times X)$, whose generated filter is a uniformity on X. Finally, for every $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$, thus, $\mathsf{B}\left(x, \frac{1}{n}\right) \subseteq \mathsf{U}(x,\delta)$, meaning that $\mathcal{B}_x$ is a base of neighborhoods of x. Therefore, $X_{(x)} = \bigcap \mathcal{N}_x = \bigcap_{n \in \mathbb{N}} \mathsf{B}\left(x, \frac{1}{n}\right) = \mathsf{B}(x,0)$. Observe that $B(x,0)$ is compact in the pseudometric topology, which makes $\{x\}$ relatively compact since $\mathrm{cl}(\{x\}) = B(x,0)$.*

In a pseudometric space, a ball is called degenerate if its radius is zero or infinity. Note that $\mathsf{U}_X(x,0) = \varnothing$ and $\mathsf{U}_X(x,\infty) = \mathsf{B}_X(x,\infty) = X$ for all x in a pseudometric space X. Also, notice that $\bigcap_{n \in \mathbb{N}} \mathsf{B}_X\left(z, r + \frac{1}{n}\right) = \mathsf{B}_X(z,r)$ for all $z \in X$ and all $0 \leq r \leq \infty$. Degenerate balls are not considered when constructing metric uniformities nor metric topologies. As expected, open balls are open in the pseudometric topology, and closed balls and spheres are closed in the pseudometric topology.

Lemma 60 *Let X be a pseudometric space. For every $x \in X$ and every $\delta > 0$, $\mathsf{U}_X(x,\delta)$ is open and $\mathsf{B}_X(x,\delta)$ and $\mathsf{S}_X(x,\delta)$ are closed. As a consequence, $\mathsf{U}_X(x,\delta) \subseteq \mathrm{int}\left(\mathsf{B}_X(x,\delta)\right)$ and $\mathrm{cl}\left(\mathsf{U}_X(x,\delta)\right) \subseteq \mathsf{B}_X(x,\delta)$.*

Proof 160 *Fix any arbitrary $y \in \mathsf{U}_X(x,\delta)$. Take $\varepsilon := \delta - d(y,x) > 0$. If $z \in \mathsf{U}_X(y,\varepsilon)$, then $d(z,x) \leq d(z,y) + d(y,x) < \varepsilon + d(y,x) = \delta$. This shows that $\mathsf{U}_X(y,\varepsilon) \subseteq \mathsf{U}_X(x,\delta)$. The arbitrariness of y proves that $\mathsf{U}_X(x,\delta)$ is open in the pseudometric topology. In order to show that $\mathsf{B}_X(x,\delta)$ is closed, we will prove that $X \setminus \mathsf{B}_X(x,\delta)$ is open. Take any arbitrary $y \in X \setminus \mathsf{B}_X(x,\delta)$ and $\varepsilon := d(y,x) - \delta > 0$. If $z \in \mathsf{U}_X(y,\varepsilon)$, then $d(x,y) \leq d(x,z) + d(z,y)$, thus $d(x,z) \geq d(x,y) - d(z,y) > \varepsilon + \delta - \varepsilon = \delta$. This shows that $\mathsf{U}_X(y,\varepsilon) \subseteq X \setminus \mathsf{B}_X(x,\delta)$. The arbitrariness of y proves that $X \setminus \mathsf{B}_X(x,\delta)$ is open in the pseudometric topology and thus $\mathsf{B}_X(x,\delta)$ is closed. Finally, $\mathsf{B}_X(x,\delta)$ is equal to the disjoint union of $\mathsf{U}_X(x,\delta)$ and $\mathsf{S}_X(x,\delta)$. Therefore, $\mathsf{S}_X(x,\delta)$ is closed in $\mathsf{B}_X(x,\delta)$ and thus closed in X.*

Under the settings of Lemma 60, and by bearing in mind Proposition 51 together with Lemma 46, if $\mathsf{U}_X(x,\delta) = \mathrm{int}\left(\mathsf{B}_X(x,\delta)\right)$, then $\mathsf{U}_X(x,\delta)$ is regular open, and if $\mathrm{cl}\left(\mathsf{U}_X(x,\delta)\right) = \mathsf{B}_X(x,\delta)$, then $\mathsf{B}_X(x,\delta)$ is regular closed. In both cases, $\mathrm{int}\left(\mathsf{S}_X(x,\delta)\right) = \varnothing$. The following example serves to show that the condition $\mathrm{cl}\left(\mathsf{U}_X(x,\delta)\right) = \mathsf{B}_X(x,\delta)$ does not necessarily imply that $\mathsf{U}_X(x,\delta) = \mathrm{int}\left(\mathsf{B}_X(x,\delta)\right)$.

Example 37 *Take $X := [-1,1]$, $x := 0$ and $\delta := 1$. Then $\mathrm{cl}\left(\mathsf{U}_X(x,\delta)\right) = \mathrm{cl}((-1,1)) = [-1,1] = \mathsf{B}_X(x,\delta)$, but $\mathrm{int}\left(\mathsf{B}_X(x,\delta)\right) = \mathrm{int}([-1,1]) = [-1,1] \neq (-1,1) = \mathsf{U}_X(x,\delta)$.*

On the contrary, next example shows that the condition $\mathsf{U}_X(x,\delta) = \mathrm{int}\left(\mathsf{B}_X(x,\delta)\right)$ does not necessarily imply that $\mathrm{cl}\left(\mathsf{U}_X(x,\delta)\right) = \mathsf{B}_X(x,\delta)$.

Example 38 *Take* $X := \mathbb{R} \setminus (1,2)$, $x := 0$ *and* $\delta := 2$. *Then* $\operatorname{int}(\mathsf{B}_X(x,\delta)) = \operatorname{int}([-2,1] \cup \{2\}) = (-2,1] = \mathsf{U}_X(x,\delta)$, *but* $\operatorname{cl}(\mathsf{U}_X(x,\delta)) = \operatorname{cl}((-2,1]) = [-2,1] \neq [-2,1] \cup \{2\} = \mathsf{B}_X(x,\delta)$.

It is intuitively obvious that a pseudometric space is Hausdorff if and only if the pseudometric is a metric.

Theorem 40 *A pseudometric space X is Hausdorff if and only if X is metric.*

Proof 161 *Suppose first that X is metric. Consider $x, y \in X$ with $x \neq y$. Then $d(x,y) > 0$, so,*

$$\mathsf{U}_X\left(x, \frac{d(x,y)}{2}\right) \cap \mathsf{U}_X\left(y, \frac{d(x,y)}{2}\right) = \varnothing.$$

This shows that X is Hausdorff. Conversely, suppose that X is Hausdorff. Fix arbitrary $x, y \in X$ with $x \neq y$. There are $\varepsilon, \delta > 0$ such that $\mathsf{U}_X(x,\varepsilon) \cap \mathsf{U}_X(y,\delta) = \varnothing$. Then $d(y,x) \geq \max\{\varepsilon, \delta\} > 0$. This shows that X is metric.

The following theorem unveils a very important characterization of separable pseudometric spaces.

Theorem 41 *Let X be a pseudometric space. The following conditions are equivalent:*

1. *X is second countable.*
2. *X is Lindelöf.*
3. *X is separable.*

Proof 162 *The proof will be itemized according to the statement.*

1⇒2 We already know it from Remark 52(5).

2⇒3 For every $n \in \mathbb{N}$, consider the open covering $\left\{\mathsf{U}_X\left(x, \frac{1}{n}\right) : x \in X\right\}$. By hypothesis, there exists a countable subcovering $\left\{\mathsf{U}_X\left(x_{nm}, \frac{1}{n}\right) : x \in X\right\}$. It is not hard to check that the set $\{x_{nm} : n, m \in \mathbb{N}\}$ is dense in X.

3⇒1 If D is a countable dense subset of X, then it is clear that $\mathcal{B} = \left\{\mathsf{U}_X\left(d, \frac{1}{n}\right) : d \in D, n \in \mathbb{N}\right\}$ is base for the pseudometric topology of X.

A very important consequence of Theorem 41 is that separability is hereditary in pseudometric spaces.

Definition 161 (Set distance) *Let X be a pseudometric space. If $A, B \subseteq X$ are nonempty subsets of X, then the distance between A and B is defined as*

$$d(A,B) := \inf\{d(a,b) : a \in A,\ b \in B\}. \tag{1.31}$$

Set distance is preserved by taking closures.

Lemma 61 *Let X be a pseudometric space. Let $A, B \subseteq X$ be nonempty subsets of X. Then:*

1. *$d(A,B) = d(\operatorname{cl}(A), \operatorname{cl}(B))$.*
2. *If A is compact, B is closed, and $A \cap B = \varnothing$, then $d(A,B) > 0$.*

Proof 163 *The proof will follow as itemized as the statement of the lemma.*

1. *First off,* $d\left(\mathrm{cl}(A), \mathrm{cl}(B)\right) \leq d(A,B)$. *Fix arbitrary elements* $a \in \mathrm{cl}(A)$ *and* $b \in \mathrm{cl}(B)$. *Fix also an arbitrary* $\varepsilon > 0$. *There are* $a_0 \in A$ *and* $b_0 \in B$ *such that* $d(a,a_0) < \frac{\varepsilon}{2}$ *and* $d(b,b_0) < \frac{\varepsilon}{2}$. *Then*

$$d(A,B) \leq d(a,b) \leq d(a,a_0) + d(a_0,b_0) + d(b_0,b) < d(a_0,b_0) + \varepsilon,$$

   meaning that $d(A,B) \leq d\left(\mathrm{cl}(A), \mathrm{cl}(B)\right)$.

2. *Assume to the contrary that* $d(A,B) = 0$. *For every* $n \in \mathbb{N}$, *there exist* $a_n \in A, b_n \in B$ *such that* $d(a_n,b_n) < \frac{1}{n}$. *For every* $n \in \mathbb{N}$, *denote* $A_n := \{a_m : m \geq n\}$. *The family* $\{\mathrm{cl}_A(A_n) : n \in \mathbb{N}\}$ *is decreasing and trivially satisfies the finite intersection property, therefore the compactness of* A *allows the existence of* $a \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_A(A_n)$. *For every* $n \in \mathbb{N}$, *there exists* $m_n \geq n$ *such that* $a_{m_n} \in A_n \cap \mathsf{U}_A\left(a, \frac{1}{n}\right)$. *Then* $d\left(a, b_{m_n}\right) \leq d\left(a, a_{m_n}\right) + d\left(a_{m_n}, b_{m_n}\right) < \frac{1}{n} + \frac{1}{m_n} \leq \frac{2}{n}$. *Then, for each* $n \in \mathbb{N}$, $b_{m_n} \in B \cap \mathsf{U}_X\left(a, \frac{2}{n}\right)$, *meaning that* $a \in \mathrm{cl}(B) = B$, *which contradicts the fact that* $A \cap B = \varnothing$.

1.2.7.2 Boundedness; centrality; diametrality

The notion of boundedness is proper from Order Theory. Here, we will transport it to the scope of pseudometric spaces, because pseudometrics are functions valued in an ordered ring.

Definition 162 (Diameter) *If* X *is a pseudometric space, then the diameter of* X *is defined as* $\mathrm{d}(X) := \sup d(X \times X) = \sup\{d(x,y) : x, y \in X\}$.

Notice that the diameter constitutes a unary external operation on the power set of a pseudometric space X

$$\begin{array}{rccc} \mathrm{d}: & \mathcal{P}(X) & \to & [0,\infty] \\ & A & \mapsto & \mathrm{d}(A). \end{array}$$

Definition 163 (Bounded pseudometric space) *A pseudometric space* X *is said to be bounded provided that* $\mathrm{d}(X) < \infty$.

The most elementary examples of bounded sets are the balls. Notice that for every x, y in a pseudometric space X and for every $\delta > 0$, $\mathsf{B}_X(x,\delta) \subseteq \mathsf{B}_X(y, \delta + d(x,y))$ and $\mathrm{d}\left(\mathsf{B}_X(x,\delta)\right) \leq 2\delta$.

Lemma 62 *Let* X *be a pseudometric space. Let* A, B *be subsets of* X. *Then:*

1. *If* A *is bounded and* $B \subseteq A$, *then* B *is bounded.*
2. *If* A, B *are bounded, then* $A \cup B$ *is bounded.*
3. A *is bounded if and only if it is contained in a ball* $\mathsf{B}_X(x,\delta)$.
4. *If* A *is bounded, then* $\mathrm{cl}(A)$ *is also bounded and* $\mathrm{d}(A) = \mathrm{d}(\mathrm{cl}(A))$.

Proof 164 *All properties are trivial except for, maybe, the last one, which is the only one we will prove. First off,* $A \subseteq \mathrm{cl}(A)$, *so, it is clear that* $\mathrm{d}(A) \leq \mathrm{d}(\mathrm{cl}(A))$. *Fix an arbitrary* $\varepsilon > 0$. *Fix arbitrary elements* $x, y \in \mathrm{cl}(A)$. *We can find* $a \in \mathsf{B}_X\left(x, \frac{\varepsilon}{2}\right) \cap A$ *and* $b \in \mathsf{B}_X\left(y, \frac{\varepsilon}{2}\right) \cap A$. *Note that* $d(x,y) \leq d(x,a) + d(a,b) + d(b,y) \leq \varepsilon + \mathrm{d}(A)$. *The arbitrariness of* $x, y \in \mathrm{cl}(A)$ *means that* $\mathrm{d}(\mathrm{cl}(A)) \leq \varepsilon + \mathrm{d}(A)$. *The arbitrariness of* $\varepsilon > 0$ *shows that* $\mathrm{d}(\mathrm{cl}(A)) \leq \mathrm{d}(A)$. *As a consequence,* $\mathrm{d}(\mathrm{cl}(A)) = \mathrm{d}(A)$.

The collection of bounded sets of a pseudometric space X, $\mathcal{B}_{\mathtt{d}}(X)$, forms an example of a bornology of $\mathcal{P}(X)$.

Remark 51 *Let X be an unbounded pseudometric space. The set of all bounded subsets of X, $\mathcal{B}_{\mathtt{d}}(X)$, is a bornology of $\mathcal{P}(X)$ in view of Lemma 62. The collection of all closed balls is a bornology base of $\mathcal{B}_{\mathtt{d}}(X)$. In fact, for every $x_0 \in X$, the collection of all closed balls centered at x_0 is also a bornology base of $\mathcal{B}_{\mathtt{d}}(X)$.*

Likewise there is the notion of diameter in a pseudometric space, there is also the notion of radius.

Definition 164 (Radius) *Let X be a bounded pseudometric space. Let $x \in X$. Then:*

- *The x-radius of X is defined as $\mathrm{r}_x(X) := \sup\{d(x,y) : y \in X\}$.*
- *The radius of X is defined as $\mathrm{r}(X) := \inf\{\mathrm{r}_x(X) : x \in X\}$.*
- *The center of X is defined as $X_{\mathrm{c}} := \{x \in X : \mathrm{r}_x(X) = \mathrm{r}(X)\}$. The points of X_{c} will be called central points.*
- *The diametrality of X is defined as $X_{\mathrm{d}} := \{x \in X : \mathrm{r}_x(X) = \mathrm{d}(X)\}$. The points of X_{d} will be called diametral points. And X is called diametral when $X = X_{\mathrm{d}}$.*

Notice that, under the settings of the previous definition, $\mathrm{d}(X) = \sup\{\mathrm{r}_x(X) : x \in X\}$. A direct consequence of Lemma 59 is the fact that if X is a bounded pseudometric space and $d(x,y) = 0$ for some $x, y \in X$, then $\mathrm{r}_x(X) = \mathrm{r}_y(X)$.

Lemma 63 *Let X be a bounded pseudometric space. Then:*

1. $X = \bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}_y(X)) = \bigcap_{y\in X} \left(\bigcap_{n\in\mathbb{N}} \mathsf{B}_X\left(y, \mathrm{r}_y(X) + \frac{1}{n}\right)\right)$.
2. $X_{\mathrm{c}} = \bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}(X)) = \bigcap_{y\in X} \left(\bigcap_{n\in\mathbb{N}} \mathsf{B}_X\left(y, \mathrm{r}(X) + \frac{1}{n}\right)\right)$.
3. *If $X_{\mathrm{d}} \neq X$ and $X_{\mathrm{c}} \neq \varnothing$, then $\mathrm{d}(X_{\mathrm{c}}) < \mathrm{d}(X)$.*
4. $\mathrm{r}(X) = 0 \Leftrightarrow \mathrm{d}(X) = 0$.
5. $\mathrm{d}(X) \leq 2\mathrm{r}(X)$.

Proof 165 *The proof will follow as itemized as the statement of the lemma.*

1. *It only suffices to show that $X = \bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}_y(X))$. Indeed, note that $X \subseteq \mathsf{B}_X(y, \mathrm{r}_y(X))$ for every $y \in X$. Therefore, $X \subseteq \bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}_y(X))$.*
2. *In accordance with Lemma 63(1), we have that $\bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}(X)) = \bigcap_{y\in X} \left(\bigcap_{n\in\mathbb{N}} \mathsf{B}_X\left(y, \mathrm{r}(X) + \frac{1}{n}\right)\right)$. So, it only remains to show that $X_{\mathrm{c}} = \bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}(X))$. Indeed, note that $X_{\mathrm{c}} \subseteq \mathsf{B}_X(y, \mathrm{r}(X))$ for every $y \in X$. Therefore, $X_{\mathrm{c}} \subseteq \bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}(X))$. Conversely, take any $x \in \bigcap_{y\in X} \mathsf{B}_X(y, \mathrm{r}(X))$. Then $d(x,y) \leq \mathrm{r}(X)$ for each $y \in X$, meaning that $\mathrm{r}_x(X) \leq \mathrm{r}(X) \leq \mathrm{r}_x(X)$. As a consequence, $x \in X_{\mathrm{c}}$.*
3. *By hypothesis, X contains a nondiametral point x. Then $\mathrm{r}_x(X) < \mathrm{d}(X)$. For every $y, z \in X_{\mathrm{c}}$, $d(y,z) \leq \mathrm{r}_y(X) = \mathrm{r}(X)$. Finally,*

$$\mathrm{d}(X_{\mathrm{c}}) = \sup\{d(y,z) : y, z \in X_{\mathrm{c}}\} \leq \mathrm{r}(X) \leq \mathrm{r}_x(X) < \mathrm{d}(X).$$

4. *If* $\mathrm{r}(X) = 0$, *then for each* $n \in \mathbb{N}$ *there exists* $x_n \in X$ *such that* $\mathrm{r}_{x_n}(X) \leq \frac{1}{n}$, *implying that* $X \subseteq \mathsf{B}_X\left(x_n, \frac{1}{n}\right)$ *and* $\mathrm{d}(X) \leq \frac{2}{n}$. *Hence* $\mathrm{d}(X) = 0$. *Conversely, if* $\mathrm{d}(X) = 0$, *then* $\mathrm{r}_x(X) = 0$ *for every* $x \in X$ *(because* $\mathrm{r}_x(X) \leq \mathrm{d}(X)$*), so that* $\mathrm{r}(X) = 0$.

5. *For every* $\varepsilon > 0$ *there exists* $x_0 \in X$ *such that* $\mathrm{r}(X) > \mathrm{r}_{x_0}(X) - \frac{\varepsilon}{2}$, *thus, for every* $x, y \in X$, $d(x, y) \leq d(x, x_0) + d(x_0, y) \leq \mathrm{r}_{x_0}(X) + \mathrm{r}_{x_0}(X) < 2\mathrm{r}(X) + \varepsilon$, *meaning that* $\mathrm{d}(X) \leq 2\mathrm{r}(X)$.

Under the settings of the previous lemma and following Lemma 59, notice that if $x, y \in X$, then $|\mathrm{r}_x(X) - \mathrm{r}_y(X)| \leq d(x, y)$.

Lemma 64 *Let* X *be a pseudometric space. Let* $A \subseteq X$ *bounded and* $a \in A$. *Then* $\mathrm{r}_a(A) = \mathrm{r}_a(\mathrm{cl}(A))$ *and* $\mathrm{r}(A) = \mathrm{r}(\mathrm{cl}(A))$.

Proof 166 *First off,* $\mathrm{r}_a(A) \leq \mathrm{r}_a(\mathrm{cl}(A))$ *because* $A \subseteq \mathrm{cl}(A)$. *Fix an arbitrary* $a' \in \mathrm{cl}(A)$. *For every* $\varepsilon > 0$, *there exists* $a_\varepsilon \in A \cap \mathsf{U}_X(a', \varepsilon)$. *Then* $d(a, a') \leq d(a, a_\varepsilon) + d(a_\varepsilon, a') < \mathrm{r}_a(A) + \varepsilon$. *This shows that* $\mathrm{r}_a(\mathrm{cl}(A)) \leq \mathrm{r}_a(A) + \varepsilon$. *Therefore,* $\mathrm{r}_a(A) = \mathrm{r}_a(\mathrm{cl}(A))$. *Let us finally prove that* $\mathrm{r}(A) = \mathrm{r}(\mathrm{cl}(A))$. *Observe that* $\mathrm{r}(\mathrm{cl}(A))$ *is a lower bound for* $\{\mathrm{r}_a(A) : a \in A\}$, *which means that* $\mathrm{r}(\mathrm{cl}(A)) \leq \mathrm{r}(A)$. *Finally, fix an arbitrary* $\varepsilon > 0$ *and take* $a' \in \mathrm{cl}(A)$ *such that* $\mathrm{r}_{a'}(\mathrm{cl}(A)) < \mathrm{r}(\mathrm{cl}(A)) + \varepsilon$. *There is* $a_\varepsilon \in A$ *such that* $d(a_\varepsilon, a') < \varepsilon$, *meaning that* $|\mathrm{r}_{a_\varepsilon}(\mathrm{cl}(A)) - \mathrm{r}_{a'}(\mathrm{cl}(A))| \leq d(a_\varepsilon, a') < \varepsilon$. *Therefore,* $\mathrm{r}(A) \leq \mathrm{r}_{a_\varepsilon}(A) = \mathrm{r}_{a_\varepsilon}(\mathrm{cl}(A)) < \mathrm{r}_{a'}(\mathrm{cl}(A)) + \varepsilon < \mathrm{r}(\mathrm{cl}(A)) + 2\varepsilon$.

The notion of an admissible set is motivated by [21, 22, 23].

Definition 165 (Admissible set) *Let* X *be a pseudometric space. A subset* $A \subseteq X$ *is called admissible if* A *is an intersection of a family of nondegenerate closed balls. The family of all admissible subsets of* X *is denoted as* $\mathcal{A}_X := \{A \in \mathcal{P}(X) \setminus \{\varnothing\} : \exists \mathcal{T} \subseteq \{\mathsf{B}_X(x, r) : x \in X, r > 0\} \text{ with } A = \bigcap \mathcal{T}\}$.

Observe that admissible sets are closed and bounded.

Remark 52 *Let* X *be a pseudometric space. Then:*

1. $\mathcal{A}_X$ *is closed under nonempty intersections. Indeed, let* $\mathcal{B} \subseteq \mathcal{A}_X$ *be a family of admissible subsets of* X *such that* $\bigcap \mathcal{B} = \bigcap_{B \in \mathcal{B}} B \neq \varnothing$. *For every* $B \in \mathcal{B}$, *there exists* $\mathcal{T}_B \subseteq \{\mathsf{B}_X(x, r) : x \in X, r > 0\}$ *such that* $B = \bigcap \mathcal{T}_B$. *Let* $\mathcal{T} := \bigcup_{B \in \mathcal{B}} \mathcal{T}_B$. *Then* $\bigcap \mathcal{B} = \bigcap_{B \in \mathcal{B}} B = \bigcap \mathcal{T} \in \mathcal{A}$.

2. *Trivial admissible set:* $\mathsf{B}_X(x, 0)$ *is admissible for all* $x \in X$. *Indeed,* $\mathsf{B}_X(x, 0) = \bigcap_{n \in \mathbb{N}} \mathsf{B}_X\left(x, \frac{1}{n}\right)$.

3. *If* X *is bounded, then* X *is admissible.*

4. *If the pseudometric of* X *is not null, then* $\varnothing$ *is admissible. Indeed, there are* $x, y \in X$ *for which* $d(x, y) > 0$, *meaning that* $\varnothing = \mathsf{B}_X\left(x, \frac{d(x,y)}{3}\right) \cap \mathsf{B}_X\left(y, \frac{d(x,y)}{3}\right)$ *is admissible.*

Admissible sets of the form described in Remark 52(2), that is, degenerate closed balls of null radius, are usually called trivial admissible sets. In metric spaces, the only trivial admissible sets are exactly the singletons. Notice that trivial admissible sets are central and diametral at the same time.

Theorem 42 *Let* τ *be a topology on the pseudometric space* X *coarser than the pseudometric topology and such that every closed ball is* τ*-closed. Then* $A_{\mathrm{c}} \neq \varnothing$ *and* A_{c} *is* τ*-closed for every nonempty bounded* τ*-compact and* τ*-closed subset* A *of* X.

Proof 167 *In view of Lemma 63(2),* $A_{\mathrm{c}} = \bigcap_{a\in A} \mathsf{B}_A\left(a, \mathrm{r}(A)\right) = A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A)\right)$, *therefore,* A_{c} *is* τ*-closed. Observe that*

$$\begin{aligned} A_{\mathrm{c}} &= A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A)\right) = A \cap \bigcap_{a\in A} \left(\bigcap_{n\in\mathbb{N}} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right)\right) \\ &= \bigcap_{n\in\mathbb{N}} \left(A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right)\right). \end{aligned}$$

Next, for each $n \in \mathbb{N}$, *there exists* $a_n \in A$ *such that* $\mathrm{r}_{a_n}(A) \leq \mathrm{r}(A) + \frac{1}{n}$, *so* $a_n \in \bigcap_{b\in A} \mathsf{B}_A\left(b, \mathrm{r}(A) + \frac{1}{n}\right)$. *Therefore, for every* $n \in \mathbb{N}$, $A \cap \bigcap_{b\in A} \mathsf{B}_X\left(b, \mathrm{r}(A) + \frac{1}{n}\right) \neq \varnothing$. *As a consequence,*

$$\left\{ A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

is a decreasing family of τ*-closed subsets of* A *satisfying the finite intersection property. The* τ*-compactness of* A *allows that*

$$A_{\mathrm{c}} = \bigcap_{n\in\mathbb{N}} \left(A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right)\right) \neq \varnothing.$$

The above theorem motivates the next definition.

Definition 166 (Admissible pseudometric topology) *Let* X *be a pseudometric space. A topology* τ *on* X *is called admissible provided that* τ *is coarser than the pseudometric topology and all the closed balls* $\mathsf{B}_X(x, \delta)$ *are* τ*-closed for all* $x \in X$ *and all* $\delta > 0$.

According to the above definition, all admissible subsets of X are τ-closed (in particular, degenerate closed balls of null radius). Since the whole space X is trivially τ-closed, we conclude that all degenerate closed balls are τ-closed.

Definition 167 (Normal structure) *Let* X *be a pseudometric space. A subset* $A \subseteq X$ *is called normal provided that* $A \setminus A_{\mathrm{d}} \neq \varnothing$. *Then* X *is said to have normal structure if every nontrivial admissible subset of* X *is normal.*

Refer to [24, 25] for the origins of the notion of normal structure.

1.2.7.3 Convexity and proximinality

Definition 165, Remark 52, and Definition 167 motivate the following notion [26, 27].

Definition 168 (Convexity structure) *Let* X *be a pseudometric space. A convexity structure in* X *is a convex structure* $\mathcal{F} \subseteq \mathcal{P}(X)$ *that contains all the admissible subsets. A convexity structure* $\mathcal{F}$ *in* X *is called:*

- *Closed when all of its elements are closed in the pseudometric topology.*
- *Bounded when all of its elements are bounded.*
- *Normal when all of its nontrivial admissible elements are normal.*
- *Central when it is bounded and stable under the centers, that is, the center of each element of the convexity structure belongs to the convexity structure.*

- *Compacted if every subfamily of $\mathcal{F}$ satisfying the finite intersection property has nonempty intersection.*

In a compact pseudometric space, every closed convexity structure is compacted. On the other hand, the intersection of any family of convexity structures is again a convexity structure. Hence, the convexity structure generated by a family of subsets is, by definition, the intersection of all convexity structures containing such a family.

Proposition 57 *Let X be a pseudometric space. Let $\mathcal{F}$ be a convexity structure in X. If $A \in \mathcal{F}$ is bounded and $A_{\mathrm{c}} \neq \varnothing$, then $A_{\mathrm{c}} \in \mathcal{F}$. If, in addition, $\mathcal{F}$ is compacted, then $A_{\mathrm{c}} \neq \varnothing$.*

Proof 168 *First off, Lemma 63(2) assures that $A_{\mathrm{c}} = \bigcap_{a\in A} \mathsf{B}_A\left(a, \mathrm{r}(A)\right) = A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A)\right)$, so $A_{\mathrm{c}} \in \mathcal{F}$ because convexity structures are closed under nonempty intersections. Observe that*

$$\begin{aligned} A_{\mathrm{c}} &= A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A)\right) = A \cap \bigcap_{a\in A} \left(\bigcap_{n\in\mathbb{N}} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right)\right) \\ &= \bigcap_{n\in\mathbb{N}} \left(A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right)\right). \end{aligned}$$

Next, for each $n \in \mathbb{N}$, there exists $a_n \in A$ such that $\mathrm{r}_{a_n}(A) \leq \mathrm{r}(A) + \frac{1}{n}$, so $a_n \in \bigcap_{b\in A} \mathsf{B}_A\left(b, \mathrm{r}(A) + \frac{1}{n}\right)$. Therefore, for every $n \in \mathbb{N}$, $A \cap \bigcap_{b\in A} \mathsf{B}_X\left(b, \mathrm{r}(A) + \frac{1}{n}\right) \neq \varnothing$. As a consequence,

$$\left\{ A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

is a decreasing subfamily of $\mathcal{F}$ satisfying the finite intersection property. If $\mathcal{F}$ is compacted,

$$A_{\mathrm{c}} = \bigcap_{n\in\mathbb{N}} \left(A \cap \bigcap_{a\in A} \mathsf{B}_X\left(a, \mathrm{r}(A) + \frac{1}{n}\right)\right) \neq \varnothing.$$

As a direct consequence of Proposition 57, we obtain the following sufficient condition for a convexity structure to be central.

Corollary 14 *Let X be a pseudometric space. If $\mathcal{F}$ is a bounded and compacted convexity structure in X, then $\mathcal{F}$ is central.*

The following definition generalizes the notion of convexity to metric spaces.

Definition 169 (Convex pseudometric space) *Let $(X, \varnothing, \{d\})$ be a pseudometric space. A convex structure is a mapping $w : X \times X \times [0,1] \to X$ satisfying that $d(z, w(x,y,t)) \leq (1-t)d(z,x) + td(z,y)$ for all $x, y, z \in X$ and all $t \in [0,1]$. The universal algebra $(X, \varnothing, \{d, w\})$ is called a convex pseudometric space.*

Keep in mind that, under the settings of the previous definition, the pseudometric d takes values on a unital ordered ring, which provides sense to the closed interval $[0,1]$. Nevertheless, in all upcoming results we will still be dealing with real pseudometrics.

Definition 170 (Convex set) *Let X be a convex pseudometric space. A nonempty subset A of X is said to be convex provided that $w(A \times A \times [0,1]) \subseteq A$.*

Notice that, under the settings of the previous definition, the nonempty intersection of any family of convex sets is trivially convex. Hence, the convex hull of a subset A of X can be defined as the intersection of all convex subsets of X containing A and denoted by $\mathrm{co}(A)$. In fact, the union of any upward directed family of convex subsets of X is convex as well, meaning that the family of all convex subsets of X (together with the empty set) is a convex structure.

Lemma 65 *Let X be a convex pseudometric space. Let $x, y \in X$, $t \in [0,1]$, and $r \geq 0$. Then:*

1. *$w(x,y,0) \in \mathsf{B}_X(x,0)$, $w(x,y,1) \in \mathsf{B}_X(y,0)$, and $w(x,x,t) \in \mathsf{B}_X(x,0)$.*
2. *$\mathsf{B}_X(x,r)$ is convex, and if $r > 0$, then $\mathsf{U}_X(x,r)$ is convex as well.*
3. *X is convex.*
4. *If $d(x,y) > 0$, then $w(x,y,t) \in \mathsf{B}_X(x,r)$ for all $0 \leq t \leq \frac{r}{d(x,y)}$.*

Proof 169 *The proof will follow as itemized as the statement of the lemma.*

1. *Note that $d(x, w(x,y,0)) \leq d(x,x) = 0$, $d(y, w(x,y,1)) \leq d(y,y) = 0$, and $d(x, w(x,x,t)) \leq (1-t)d(x,x) + td(x,x) = d(x,x) = 0$.*
2. *Note that $d(x, w(a,b,t)) \leq td(x,a) + (1-t)d(x,b) \leq r$ for all $a, b \in \mathsf{B}_X(x,r)$.*
3. *By definition, $w(X \times X \times [0,1]) \subseteq X$.*
4. *Observe that $d(x, w(x,y,t)) \leq (1-t)d(x,x) + td(x,y) \leq r$ for all $0 \leq t \leq \frac{r}{d(x,y)}$.*

A direct consequence of the previous technical lemma is the fact that the family of all convex subsets of a convex pseudometric space conforms a convexity structure since it contains all the admissible sets. The next two definitions deal with Approximation Theory. The first notion is that of proximinality.

Definition 171 (Proximinality) *Let X be a pseudometric space. A non-empty subset $A \subseteq X$ is called proximinal when, for every element $x \in X$, the distance from x to A is always attained at some $a \in A$.*

The most classic example of proximinal sets is the compact subsets. Bear in mind that finite sets and degenerate closed balls of null radius are compact in the pseudometric topology (simply notice that, by Lemma 59, any open set intersecting a degenerate closed ball of null radius must contain the whole ball).

Proposition 58 *Let X be a pseudometric space. Every compact subset $K \subseteq X$ is proximinal. Conversely, if X is metric and $A \subseteq X$ is proximinal, then A is closed.*

Proof 170 *Let us show first that K is proximinal. Fix $x \in X$. For every $n \in \mathbb{N}$, there exists $k_n \in K$ such that $d(x,k_n) < d(x,K) + \frac{1}{n}$. For every $n \in \mathbb{N}$, denote $A_n := \{k_m : m \geq n\}$. The family $\{\mathrm{cl}_K(A_n) : n \in \mathbb{N}\}$ is decreasing and trivially satisfies the finite intersection property, therefore the compactness of K allows the existence of $k \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_K(A_n)$. For every $n \in \mathbb{N}$, there exists $m_n \geq n$ such that $k_{m_n} \in A_n \cap \mathsf{U}_K\left(k, \frac{1}{n}\right)$. Then $d(x,k) \leq d(x,k_{m_n}) + d(k_{m_n},k) < d(x,K) + \frac{1}{m_n} + \frac{1}{n} \leq d(x,K) + \frac{2}{n}$ for each $n \in \mathbb{N}$. As a consequence, $d(x,K) = d(x,k)$. Conversely, let us prove that A is closed. If not, there exists $b \in \mathrm{cl}(A) \setminus A$, meaning that $d(b,A) = d(b,\mathrm{cl}(A)) = 0$, so $d(b,A)$ is attained at some $a \in A$, reaching the contradiction that $b = a \in A$.*

The next one is that of antiproximinality. This notion is defined in the pseudometric ambience, although most of the results make sense only in metric spaces.

Definition 172 (Antiproximinality) *Let X be a pseudometric space. A nonempty proper subset $A \subseteq X$ is called antiproximinal when, for every element $x \in X \setminus A$, the distance from e to A is never attained at any $a \in A$.*

The following proposition unveils the most basic properties satisfied by antiproximinal sets.

Proposition 59 *Let X be a metric space. Let $A, B \subseteq X$. Then:*

1. *If A is antiproximinal and $\mathrm{cl}(B) = A$, then B is also antiproximinal.*
2. *If $\mathrm{cl}(A) = X$, then A is antiproximinal.*
3. *If X is convex and A is open, then A is antiproxminal.*

Proof 171 *The proof will follow as itemized as the statement of the proposition.*

1. *Fix an arbitrary $x \in X \setminus B$. If $x \in A$, then $d(x, B) = d(x, A) = 0$, so $d(x, B)$ can never be attained. If $x \notin A$, then $d(x, B) = d(x, A)$, so if $d(x, B)$ is attained, then so is $d(x, A)$, which is impossible by hypothesis.*
2. *If $x \in X \setminus A$, then $d(x, A) = d(x, X) = 0$, so $d(x, A)$ can never be attained.*
3. *Fix an arbitrary $x \in X \setminus A$ and assume to the contrary that there exists $a \in A$ such that $d(x, A) = d(x, a)$. Since A is open, there exists $\delta > 0$ such that $\mathsf{B}_X(a, \delta) \subseteq A$. According to Lemma 65(4), $w(a, x, t) \in \mathsf{B}_X(a, \delta)$ for each $0 \leq t \leq \frac{\delta}{d(a,x)}$. So, if we fix any $0 < t \leq \frac{\delta}{d(a,x)}$, then $d(x, w(a, x, t)) \leq (1 - t)d(x, a) + td(x, x) < d(x, a)$, which contradicts that $d(x, A) = d(x, a)$.*

1.2.7.4 Group seminorms

Seminorms can be thought of as linear pseudometrics. Seminorms will endow groups, rings, modules, and algebras with the structure of pseudometric spaces, hence with the structure of regular, first countable topological spaces. We will begin by defining invariant pseudometrics, which generalize the notion of group seminorm.

Definition 173 (Pseudometric group) *Let $((R, \leq), \{+, 0, -, \cdot, 1\}, \varnothing)$ be an ordered ring. A pseudometric $d : G \times G \to R^+$ on a group $\left(G, \left\{\cdot, 1, {}^{-1}\right\}, \varnothing\right)$ is said to be left invariant provided that $d(hg_1, hg_2) = d(g_1, g_2)$ for every $h, g_1, g_2 \in G$. In a similar way, right invariance is defined. If d is left and right invariant, then we simply call it invariant and in such case the universal algebra $\left(G, \left\{\cdot, 1, {}^{-1}\right\}, \{d\}\right)$ is called a pseudometric group.*

Under the settings of the previous definition, if d is left-invariant, then G is called a left-pseudometric group. Similarly, for right invariance.

Definition 174 (Seminorm) *Let $((R, \leq), \{+, 0, -, \cdot, 1\}, \varnothing)$ be an ordered ring. A seminorm on a group $\left(G, \left\{\cdot, 1, {}^{-1}\right\}, \varnothing\right)$ is a unary external operation $\| \ \| : G \to R^+$ verifying the following for all $g, f \in G$:*

- *Identity preserving: $\|1\| = 0$.*
- *Symmetricity: $\left\|g^{-1}\right\| = \|g\|$.*

- *Triangular inequality:* $\|gf\| \leq \|g\| + \|f\|$.

The universal algebra $\left(G, \left\{\cdot, 1, {}^{-1}\right\}, \{\|\ \|\}\right)$ *is called a seminormed group. If, in addition, it is verified that*

- $\|g\| = 0 \Rightarrow g = 1$, *then the seminorm is called a norm and the universal algebra is called a normed group;*
- $\forall g, f \in G \quad \left\|fgf^{-1}\right\| = \|g\|$, *then the seminorm is called conjugation-invariant.*

The symmetricity and the triangular inequality are in fact equivalent to the symmetric triangular inequality:

$$\forall g, f \in G \quad \|gf\| \leq \|g\| + \left\|f^{-1}\right\|.$$

With additive notation, the triangular inequality is often called subadditivity. From now on and unless otherwise stated explicitly, we will only consider real seminorms.

Lemma 66 *Let* G *be a seminormed group and* $g, h, g_0, h_0 \in G$. *Then* $|\|g\| - \|h\|| \leq \left\|gh^{-1}\right\|$. *If, in addition, the seminorm is conjugation-invariant, then* $\|(gh)(h_0 g_0)\| \leq \|gg_0\| + \|hh_0\|$ *and* $\|gh\| = \left\|g^{-1}h^{-1}\right\|$.

Proof 172 *On the one hand,* $\|g\| = \left\|\left(gh^{-1}\right)h\right\| \leq \left\|gh^{-1}\right\| + \|h\|$ *therefore,* $\|g\| - \|h\| \leq \left\|gh^{-1}\right\|$. *On the other hand,* $\|h\| = \left\|\left(hg^{-1}\right)g\right\| \leq \left\|hg^{-1}\right\| + \|g\|$, *therefore,* $\|h\| - \|g\| \leq \left\|hg^{-1}\right\| = \left\|gh^{-1}\right\|$. *Thus,* $|\|g\| - \|h\|| \leq \left\|gh^{-1}\right\|$. *Assume now that the seminorm is conjugation-invariant. Notice that* $\|(gh)(h_0 g_0)\| = \left\|(gg_0)\left(g_0^{-1}hh_0g_0\right)\right\| \leq \|gg_0\| + \left\|g_0^{-1}(hh_0)g_0\right\| = \|gg_0\| + \|hh_0\|$. *Finally,* $\|gh\| = \left\|(gh)^{-1}\right\| = \left\|h^{-1}g^{-1}\right\| = \left\|h^{-1}\left(g^{-1}h^{-1}\right)h\right\| = \left\|g^{-1}h^{-1}\right\|$.

Every seminorm induces a pseudometric, which is going to be invariant if the seminorm is conjugation-invariant. Conversely, every (invariant) pseudometric induces a (conjugation-invariant) seminorm.

Proposition 60 *Let* G *be a group. Then:*

1. *If* G *is a seminormed group, then*

$$\begin{array}{rccl} d: & G \times G & \to & [0, \infty) \\ & (g, f) & \mapsto & d(g, f) := \left\|gf^{-1}\right\| \end{array} \tag{1.32}$$

   is a pseudometric on G. *If the seminorm is conjugation-invariant, then the pseudometric is invariant.*

2. *If* G *is a left-pseudometric group, then*

$$\begin{array}{rccl} \|\ \|: & G & \to & [0, \infty) \\ & g & \mapsto & \|g\| := d(g, 1) \end{array} \tag{1.33}$$

   is a seminorm on G. *If, in addition, the pseudometric is also right invariant, then the seminorm is conjugation-invariant.*

Proof 173 *We have to check that all three properties of pseudometrics are satisfied:*

- *Identity preserving:* $d(g, g) = \left\|gg^{-1}\right\| = \|1\| = 0$.
- *Symmetricity:* $d(g, f) = \left\|gf^{-1}\right\| = \left\|\left(fg^{-1}\right)^{-1}\right\| = \left\|fg^{-1}\right\| = d(f, g)$.

- *Triangular inequality:* $d(g,f) = \left\|gf^{-1}\right\| = \left\|\left(gh^{-1}\right)\left(hf^{-1}\right)\right\| \leq \left\|gh^{-1}\right\| + \left\|hf^{-1}\right\| = d(g,h) + d(h,f)$.

Next, assume that the seminorm is conjugation-invariant. Fix arbitrary elements $h, g_1, g_2 \in G$. *Note that* $d(hg_1, hg_2) = \left\|(hg_1)(hg_2)^{-1}\right\| = \left\|h\left(g_1 g_2^{-1}\right)h^{-1}\right\| = \left\|g_1 g_2^{-1}\right\| = d(g_1, g_2)$. *Similarly,* $d(g_1 h, g_2 h) = d(g_1, g_2)$. *Conversely, let us check that all three properties of seminorms are verified:*

- *Identity preserving:* $\|1\| = d(1,1) = 0$.
- *Symmetricity:* $\|g\| = d(g,1) = d\left(1, g^{-1}\right) = d\left(g^{-1}, 1\right) = \left\|g^{-1}\right\|$.
- *Triangular inequality:* $\|gf\| = d(gf,1) = d\left(f, g^{-1}\right) \leq d(f,1) + d\left(1, g^{-1}\right) = d(f,1) + d(g,1) = \|f\| + \|g\|$.

Finally, let us assume that the pseudometric is also right invariant. Fix arbitrary elements $g, f \in G$. *Then* $\left\|fgf^{-1}\right\| = d\left(fgf^{-1}, 1\right) = d\left(gf^{-1}, f^{-1}\right) = d(g,1) = \|g\|$.

The (closed) ball of center 1 and radius 1 is called the (closed) unit ball and is denoted as $\mathsf{B}_G := \{g \in G : \|g\| \leq 1\}$. The open ball of center 1 and radius 1 is called the open unit ball and is denoted as $\mathsf{U}_G := \{g \in G : \|g\| < 1\}$. The sphere of center 1 and radius 1 is called the unit sphere and is denoted as $\mathsf{S}_G := \{g \in G : \|g\| = 1\}$.

Lemma 67 *Let* G *be a left-pseudometric group. Let* $A \subseteq G$ *be bounded and consider* $g \in G$. *Then* gA *is also bounded and* $\mathrm{d}(A) = \mathrm{d}(gA)$. *Also,* $\mathrm{r}_a(A) = \mathrm{r}_{ga}(gA)$ *for all* $a \in A$. *Therefore,* $(gA)_{\mathrm{c}} = gA_{\mathrm{c}}$ *and* $(gA)_{\mathrm{d}} = gA_{\mathrm{d}}$.

Proof 174 *Fix an arbitrary* $a \in A$. *It only suffices to notice that* $d(a,b) = d(ga, gb)$ *for all* $b \in A$. *As a consequence,* $\mathrm{r}_a(A) = \mathrm{r}_{ga}(gA)$, *meaning that* $\mathrm{d}(A) = \mathrm{d}(gA)$ *and* $\mathrm{r}(A) = \mathrm{r}(gA)$. *Finally,* $(gA)_{\mathrm{c}} = gA_{\mathrm{c}}$ *and* $(gA)_{\mathrm{d}} = gA_{\mathrm{d}}$.

The metric notions of center and diametrality are invariant under multiplication in a pseudometric group.

Lemma 68 *Let* G *be a left-pseudometric group. For every* $g, h \in G$ *and every* $n \in \mathbb{N}$, $d\left(g^n, h^n\right) \leq nd(g,h)$ *if* g, h *commute.*

Proof 175 *We will prove it first for* $h = 1$. *Note that* $d(g^2, 1) \leq d(g^2, g) + d(g,1) = d(g,1) + d(g,1) = 2d(g,1)$. *By induction, we obtain the desired result for* $h = 1$. *Finally, for any* $g, h \in G$ *that commute,* $d\left(g^n, h^n\right) = d\left(\left(h^{-1}g\right)^n, 1\right) \leq nd\left(h^{-1}g, 1\right) = nd(g,h)$.

Notice that, under the settings of Lemma 68, if $d\left(g^n, 1\right) = nd\left(g, 1\right)$ for all $g \in G$ and all $n \in \mathbb{N}$, then $d\left(g^n, g^n\right) = nd(g,h)$ for every $g, h \in G$ that commute and every $n \in \mathbb{N}$.

Definition 175 (Homogeneous pseudometric) *Let* G *be a group. A left-pseudometric* d *on* G *satisfying that* $d\left(g^n, g^n\right) = nd(g,h)$ *for every* $g, h \in G$ *that commute and every* $n \in \mathbb{N}$ *will be called a homogeneous left-pseudometric.*

The next result in this subsubsection shows that bounded symmetric sets containing the unity in a homogeneous pseudometric group satisfy "round" properties.

Theorem 43 *Let* G *be a homogeneous left-pseudometric group. If* $A \subseteq G$ *is bounded and symmetric and* $1 \in A$, *then* $A_{\mathrm{c}} \supseteq \mathsf{B}_A(1,0)$ *and* $\mathrm{r}(A) = \frac{1}{2}\mathrm{d}(A)$.

Proof 176 *We will first prove that* $\mathrm{r}_1(A) \leq \mathrm{r}_a(A)$ *for every* $a \in A$. *Indeed, fix an arbitrary* $a \in A$. *Take any* $b \in A$. *If* $d(a,b) \leq d(1,b)$, *then*

$$2d(1,b) = d(1,bb) = d\left(b^{-1},b\right) \leq d\left(b^{-1},a\right) + d(a,b) \leq d\left(b^{-1},a\right) + d(1,b),$$

meaning that $d(1,b) \leq d\left(b^{-1},a\right) \leq \mathrm{r}_a(A)$. *Otherwise,* $d(1,b) < d(a,b) \leq \mathrm{r}_a(A)$. *In any case,* $d(1,b) \leq \mathrm{r}_a(A)$, *hence* $\mathrm{r}_1(A) \leq \mathrm{r}_a(A)$. *As a consequence,* $\mathrm{r}(A) = \mathrm{r}_1(A)$ *and* $1 \in A_{\mathrm{c}}$. *Next, according to Lemma 59,* $\mathrm{r}_a(A) = \mathrm{r}_1(A)$ *for all* $a \in A$ *with* $d(a,1) = 0$. *As a consequence,* $\mathsf{B}_A(1,0) \subseteq A_{\mathrm{c}}$. *Finally, for every* $\varepsilon > 0$ *we can find* $b \in A$ *such that* $\mathrm{r}_1(A) < d(1,b) + \varepsilon$, *meaning that*

$$\mathrm{r}(A) = \mathrm{r}_1(A) < d(1,b) + \varepsilon = \frac{1}{2}d\left(b^{-1},b\right) + \varepsilon \leq \frac{1}{2}\mathrm{d}(A) + \varepsilon,$$

which implies that $\mathrm{r}(A) \leq \frac{1}{2}\mathrm{d}(A)$. *Finally, simply note that the reverse inequality,* $\mathrm{r}(A) \geq \frac{1}{2}\mathrm{d}(A)$, *holds in any bounded pseudometric space in view of Lemma 63(6), reaching the desired result.*

The final observation in this subsubsection is a simple application of intersection of closed balls (see Theorem 39).

Remark 53 *Let* G *be a group endowed with a pseudometric. Then* $G_{(1)} = \mathsf{B}_G(1,0)$.

1.2.7.5 Ring seminorms and absolute semivalues

It is turn now for seminorms on rings. Here, we will use the typical notations for rings, that is, additivity and multiplicative.

Definition 176 (Seminormed ring) *A seminorm on a ring* $(R, \{+,0,-,\cdot, 1\}, \varnothing)$ *is a group seminorm* $\|\ \|$ *on its commutative additive group* $(R, \{+,0,-\}, \varnothing)$, *which is submultiplicative, that is,* $\|rs\| \leq \|r\|\|s\|$ *for all* $r,s \in R$. *The universal algebra* $(R, \{,+,0,-,\cdot,1\}, \{\|\ \|\})$ *is called seminormed ring.*

Absolute semivalues are very particular cases of ring seminorms, satisfying essential properties to develop an Operator Theory on normed modules.

Definition 177 (Absolute semivalue) *An absolute semivalue on a ring* $(R, \{+,0,-,\cdot,1\}, \varnothing)$ *is a ring seminorm* $|\ |$, *which is multiplicative, that is,* $|rs| = |r||s|$ *for all* $r,s \in R$. *The universal algebra* $(R, \{,+,0,-,\cdot,1\}, \{|\ |\})$ *is called absolutely semivalued ring.*

The most basic and essential properties of ring seminorms are displayed in the next lemma.

Lemma 69 *If* R *is a seminormed ring, then:*

1. *Either* $\|1\| \geq 1$ *or* $\|1\| = 0$.
2. *If* $\|1\| = 0$, *then* $\|\ \| = 0$.
3. *If* $\|1\| \geq 1$, *then* $\|r\| > 0$ *and* $\left\|r^{-1}\right\| \geq \|r\|^{-1}$ *for all* $r \in \mathcal{U}(R)$.
4. $\|rs - r_0s_0\| \leq \|r\|\|s - s_0\| + \|r - r_0\|\|s_0\|$ *for all* $r,s,r_0,s_0 \in R$.
5. $\left\|s^{-1} - r^{-1}\right\| \leq \left\|s^{-1}\right\| \|r - s\| \left\|r^{-1}\right\|$ *for all* $r,s \in \mathcal{U}(R)$.
6. $\|r\| \left\|s^{-1}\right\|^{-1} \leq \|rs\|$ *and* $\left\|s^{-1}\right\|^{-1} \|r\| \leq \|sr\|$ *for all* $r \in R$ *and all* $s \in \mathcal{U}(R)$.

Proof 177 *Let us itemize the proof by following the statement of the lemma.*

1. $\|1\| = \|1 \cdot 1\| \leq \|1\|\|1\|$. *This implies that either* $\|1\| \geq 1$ *or* $\|1\| = 0$.
2. $\|r\| = \|1 \cdot r\| \leq \|1\|\|r\| = 0$ *for all* $r \in R$.
3. $1 \leq \|1\| = \left\|r^{-1}r\right\| \leq \left\|r^{-1}\right\| \|r\|$ *for all* $r \in \mathcal{U}(R)$.
4. *Simply note that* $\|rs - r_0s_0\| = \|rs - rs_0 + rs_0 - r_0s_0\| \leq \|rs - rs_0\| + \|rs_0 - r_0s_0\| = \|r(s - s_0)\| + \|(r - r_0)s_0\| \leq \|r\|\|s - s_0\| + \|r - r_0\|\|s_0\|$.
5. *Observe that* $\left\|s^{-1} - r^{-1}\right\| = \left\|s^{-1}(r - s)r^{-1}\right\| \leq \left\|s^{-1}\right\| \|r - s\| \left\|r^{-1}\right\|$.
6. *It suffices to notice that* $\|r\| = \left\|(rs)s^{-1}\right\| \leq \|rs\| \left\|s^{-1}\right\|$. *The other inequality is similar.*

Motivated by Lemma 69(1), a seminormed ring is called unital provided that $\|1\| = 1$. The following definition is taken from [28, 29].

Definition 178 (Absolutely invertible) *Let* R *be a seminormed ring. We say that* $u \in \mathcal{U}(R)$ *is absolutely invertible provided that* $\left\|u^{-1}\right\| = \|u\|^{-1}$. *The subset of absolutely invertibles of* R *is denoted by* $\mathcal{U}_1(R)$.

Later on we will check that every invertible in an absolutely semivalued ring is absolutely invertible.

Lemma 70 *Let* R *be a seminormed ring. Let* $u, v \in R$. *Then:*

1. *If* $u \in \mathcal{U}_1(R)$, *then* $-u, u^{-1} \in \mathcal{U}_1(R)$.
2. *If* $v \in \mathcal{U}_1(R)$, *then* $\|uv\| = \|vu\| = \|u\|\|v\|$.
3. *If* $u, v \in \mathcal{U}_1(R)$, *then* $uv \in \mathcal{U}_1(R)$.
4. *If* $v \in \mathcal{U}(R)$, $\|u\| > 0$, $\|v\| \leq \|u\|$ *and* $\left\|v^{-1}\right\| \leq \|u\|^{-1}$, *then* $v \in \mathcal{U}_1(R)$, $\|u\| = \|v\|$, *and* $\left\|v^{-1}\right\| = \|u\|^{-1}$.
5. $\|1\| = 1$ *and only if* $1 \in \mathcal{U}_1(R)$.
6. *If* $\mathcal{U}_1(R) \neq \varnothing$, *then* $\|1\| = 1$.

In particular, $\mathcal{U}_1(R)$ *is an additively symmetric multiplicative subgroup of* $\mathcal{U}(R)$ *if* R *is unital.*

Proof 178 *The proof will follow as itemized as the statement of the lemma.*

1. *Trivial.*
2. *By applying Lemma 69(6),* $\|u\|\|v\| = \|u\| \left\|v^{-1}\right\|^{-1} \leq \|uv\| \leq \|u\|\|v\|$. *Similarly,* $\|vu\| = \|u\|\|v\|$.
3. $\left\|(uv)^{-1}\right\| = \left\|v^{-1}u^{-1}\right\| = \left\|v^{-1}\right\| \left\|u^{-1}\right\| = \|v\|^{-1}\|u\|^{-1} = (\|v\|\|u\|)^{-1} = \|uv\|^{-1}$.
4. *Note that* $\|u\|^{-1} \leq \|v\|^{-1} \leq \left\|v^{-1}\right\| \leq \|u\|^{-1}$. *Thus,* $\|u\|^{-1} = \|v\|^{-1} = \left\|v^{-1}\right\|$.
5. *If* $\|1\| = 1$, *then it is trivial that* $1 \in \mathcal{U}_1(R)$. *Conversely, if* $1 \in \mathcal{U}_1(R)$, *then* $\|1\| = \|1\|^{-1}$ *so* $\|1\| = 1$.
6. *Fix any arbitrary* $v \in \mathcal{U}_1(R)$. *Then* $\|1\| = \left\|vv^{-1}\right\| = \|v\| \left\|v^{-1}\right\| = \|v\|\|v\|^{-1} = 1$.

Recall that a $*$-ring is a ring endowed with an involution, that is, an additive and antimultiplicative involutive selfmap.

Definition 179 ($*$-Multiplicativity) *Let R be a seminormed $*$-ring. We will say that the seminorm of R is strongly $*$-multiplicative provided that the condition $\|rr^*\| = \|r\|^2$ holds for all $r \in R$; and is $*$-multiplicative if $\|rr^*\| = \|r\| \, \|r^*\|$ for all $r \in R$.*

The most basic and essential properties of $*$-ring seminorms are displayed in the next lemma.

Lemma 71 *Let R be a seminormed $*$-ring. Let $u, v \in R$. Then:*

1. *If $\|r\| \leq \|r^*\|$ for all $r \in R$, then $*$ is an isometry.*
2. *The seminorm of R is strongly $*$-multiplicative if and only if $*$ is an isometry and the seminorm is $*$-multiplicative.*

Proof 179 *We will itemize the proof by following the itemization of the statement.*

1. *Notice that $\|r^*\| \leq \left\|(r^*)^*\right\| = \|r\|$ for all $r \in R$. Therefore, $\|r^*\| = \|r\|$ for all $r \in R$, meaning that $*$ is an isometry due to its additivity.*
2. *If $*$ is an isometry and the seminorm is $*$-multiplicative, then it is clear that the seminorm is strongly $*$-multiplicative. Conversely, suppose that the seminorm is strongly $*$-multiplicative. Then $\|r\|^2 = \|rr^*\| \leq \|r\| \, \|r^*\|$ for all $r \in R$. Therefore, if $\|r\| > 0$, then $\|r\| \leq \|r^*\|$. If $\|r\| = 0$, then we clearly have that $\|r\| \leq \|r^*\|$. As a consequence, $\|r\| \leq \|r^*\|$ for all $r \in R$, meaning that $*$ is an isometry by Lemma 71(1) and that the seminorm is $*$-multiplicative.*

Recall that in a $*$-ring R an invertible element u is said to be an unitary element provided that $u^* = u^{-1}$. The subset of unitary elements is denoted by $\mathcal{U}_{\mathfrak{u}}(R)$ is an $*$-invariant additively symmetric subgroup of $\mathcal{U}(R)$.

Lemma 72 *Let R be a unital seminormed $*$-ring. Let $u, v \in R$. Then:*

1. *If the involution $*$ is an isometry, then $\mathcal{U}_1(R)^* = \mathcal{U}_1(R)$ and $\mathcal{U}_1(R) \cap \mathsf{S}_R$ is an $*$-invariant additively symmetric subgroup of $\mathcal{U}_1(R)$.*
2. *If the seminorm is $*$-multiplicative, then $\mathcal{U}_{\mathfrak{u}}(R) \subseteq \mathcal{U}_1(R)$.*
3. *If the seminorm is strongly $*$-multiplicative, then $\mathcal{U}_{\mathfrak{u}}(R) \subseteq \mathcal{U}_1(R) \cap \mathsf{S}_R$.*
4. *$\mathcal{U}_1(R) \cap \mathsf{S}_R$ is a normal subgroup of $\mathcal{U}_1(R)$.*

Proof 180 *As always, the proof follows with the same itemization of the statement of the lemma.*

1. *Fix any arbitrary $r \in \mathcal{U}_1(R)$. Observe that $(r^*)^{-1} = \left(r^{-1}\right)^*$, so*

$$\left\|(r^*)^{-1}\right\| = \left\|\left(r^{-1}\right)^*\right\| = \left\|r^{-1}\right\| = \|r\|^{-1} = \|r^*\|^{-1},$$

hence $r^* \in \mathcal{U}_1(R)$. This shows that $\mathcal{U}_1(R)^* \subseteq \mathcal{U}_1(R)$. Since $*$ is involutive, we obtain that $\mathcal{U}_1(R)^* = \mathcal{U}_1(R)$. As a consequence, $\mathcal{U}_1(R) \cap \mathsf{S}_R$ is an $*$-invariant additively symmetric subgroup of $\mathcal{U}_1(R)$ due to the assumption that $*$ is an isometry.

2. *If* $u \in \mathcal{U}_{\mathfrak{u}}(R)$, *then* $1 = \|1\| = \left\|uu^{-1}\right\| = \|uu^*\| = \|u\|\,\|u^*\|$, *meaning that* $\|u\| = \|u^*\|^{-1} = \left\|u^{-1}\right\|^{-1}$ *and hence* $u \in \mathcal{U}_1(R)$.

3. *From the previous item, we already know that* $\mathcal{U}_{\mathfrak{u}}(R) \subseteq \mathcal{U}_1(R)$, *so it only remains to show that* $\mathcal{U}_{\mathfrak{u}}(R) \subseteq \mathsf{S}_R$. *Indeed, if* $u \in \mathcal{U}_{\mathfrak{u}}(R)$, *then* $\|u\| = \|u^*\| = \left\|u^{-1}\right\| = \|u\|^{-1}$, *meaning that* $\|u\| = \left\|u^{-1}\right\| = 1$. *Therefore,* $u \in \mathsf{S}_R$.

4. *Take arbitrary elements* $u \in \mathcal{U}_1(R) \cap \mathsf{S}_R$ *and* $r \in \mathcal{U}_1(R)$ *and we will show that* $r^{-1}ur \in \mathcal{U}_1(R) \cap \mathsf{S}_R$. *Indeed, it is clear that* $r^{-1}ur \in \mathcal{U}_1(R)$, *so it only remains to show that* $r^{-1}ur \in \mathsf{S}_R$, *which is a direct consequence of the properties satisfied by the absolutely invertibles because* $\left\|r^{-1}ur\right\| = \left\|r^{-1}\right\|\|u\|\|r\| = \|r\|^{-1}\|r\| = 1$.

A direct consequence of Lemma 72(4) is the fact that $\mathcal{U}_{\mathfrak{u}}(R)$ is a normal subgroup of $\mathcal{U}_1(R)$ provided that $\mathcal{U}_{\mathfrak{u}}(R) = \mathcal{U}_1(R) \cap \mathsf{S}_R$. However, next example shows this assertion is not always true.

Example 39 *Consider the product ring* $R := \mathbb{R} \times \mathbb{C}$ *endowed with the* 1*-norm* $\|(x,z)\|_1 := |x| + |z|$ *and the involution given by right-conjugation* $(x,z)^* := (x,\overline{z})$. *It is not hard to check that this involution is an isometry but it is not* $*$*-multiplicative. Also,* $\mathcal{U}_1(R) = \varnothing$ *and* $\mathcal{U}_{\mathfrak{u}}(R) = \{\pm 1\} \times \mathrm{bd}(\mathbb{D})$, *where* $\mathrm{bd}(\mathbb{D})$ *stands for the boundary of the unit disk, also denoted by* $\mathbb{T}$.

A slight modification of Example 39 provides the existence of absolutely invertible elements.

Example 40 *Consider the unital seminormed ring* $R := \mathbb{C}$ *endowed with the* 1*-norm* $\|z\|_1 := |\Re z| + |\Im z|$ *and the involution given by conjugation* $z^* := \overline{z}$. *It is not hard to check that this involution is an isometry but not* $*$*-multiplicative. Also,* $\mathcal{U}_1(R) = \{z \in \mathbb{C} \setminus \{0\} : \Re z = 0 \text{ or } \Im z = 0\}$ *and* $\mathcal{U}_{\mathfrak{u}}(R) = \mathrm{bd}(\mathbb{D})$.

Observe that if R is $\mathbb{C}$ endowed with its absolute value and involution given by conjugation, then the involution is clearly an isometry, the absolute value is $*$-multiplicative, $\mathcal{U}_{\mathfrak{u}}(R) = \mathsf{S}_{\mathbb{C}}$, and $\mathcal{U}_1(R) = \mathcal{U}(R) = \mathbb{C} \setminus \{0\}$, meaning that $\mathcal{U}_{\mathfrak{u}}(R) = \mathcal{U}_1(R) \cap \mathsf{S}_R$.

Lemma 73 *Let* R *be a unital seminormed ring* R. *The following conditions are equivalent:*

1. $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$.

2. $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \cap \mathsf{U}_R)$

3. $1 \in \mathrm{cl}(\mathcal{U}_1(R) \setminus \mathsf{B}_R)$.

In addition, under any of the above conditions, $1 \in \mathrm{cl}(\mathsf{U}_R)$.

Proof 181 *In the first place, let us observe that* $\mathcal{U}_1(R) \setminus \mathsf{S}_R = (\mathcal{U}_1(R) \cap \mathsf{U}_R) \cup (\mathcal{U}_1(R) \setminus \mathsf{B}_R)$, *meaning that* $\mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{S}_R) = \mathrm{cl}\,(\mathcal{U}_1(R) \cap \mathsf{U}_R) \cup \mathrm{cl}(\mathcal{U}_1(R) \setminus \mathsf{B}_R)$. *As a consequence, it only suffices to show that the first item implies the other two. Indeed, assume that* $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$. *Let us prove first that* $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \cap \mathsf{U}_R)$. *Fix an arbitrary* $\varepsilon > 0$. *By hypothesis, there exists* $u \in \mathsf{U}_R(1,\varepsilon) \cap (\mathcal{U}_1(R) \setminus \mathsf{S}_R)$. *If* $\|u\| < 1$, *then we are done. If not, then* $\left\|u^{-1}\right\| = \|u\|^{-1} < 1$, *meaning that* $\left\|1 - u^{-1}\right\| = \left\|u^{-1}(u-1)\right\| = \left\|u^{-1}\right\|\|u-1\| < \varepsilon$, *that is,* $u^{-1} \in \mathsf{U}_R(1,\varepsilon)$. *Next, we will show that* $1 \in \mathrm{cl}(\mathcal{U}_1(R) \setminus \mathsf{B}_R)$. *Fix an arbitrary* $0 < \varepsilon < 1$. *By hypothesis, there exists* $u \in \mathsf{U}_R(1,\tau) \cap (\mathcal{U}_1(R) \setminus \mathsf{S}_R)$, *where* $\tau > 0$ *is sufficiently small so that* $\tau < \varepsilon$ *and* $\frac{\tau}{1-\tau} < \varepsilon$. *If* $\|u\| > 1$, *then we are done. If not, then* $\left\|u^{-1}\right\| = \|u\|^{-1} > 1$. *Then* $1 - \|u\| \leq \|1-u\| < \tau$, *hence* $\|u\| > 1 - \tau$, *so* $\|u\|^{-1} < \frac{1}{1-\tau}$. *Then* $\left\|1 - u^{-1}\right\| = \left\|u^{-1}(u-1)\right\| = \left\|u^{-1}\right\|\|u-1\| = \|u\|^{-1}\|u-1\| < \frac{1}{1-\tau}\tau < \varepsilon$, *meaning that* $u^{-1} \in \mathsf{U}_R(1,\varepsilon)$.

Next proposition reproduces, in totally ordered rings, the real absolute value.

Proposition 61 *Let R be a totally ordered ring. The following is an absolute value on R:*

$$\begin{array}{rccl} |\,|: & R & \to & R^+ \\ & r & \mapsto & |r| := \left\{ \begin{array}{ll} r & r \geq 0, \\ -r & r < 0. \end{array} \right. \end{array} \tag{1.34}$$

Proof 182 *Let us show one by one all the properties that $|\,|$ needs to verify to be an absolute value:*

- *Positivity: Indeed, if $r < 0$, then $-r > 0$ in view of Proposition 28(3).*
- *Strict positivity: If $|r| = 0$ and $r < 0$, then $-r = 0$, so $r = 0$.*
- *Identity preserving: Clearly, $0 \geq 0$, so $|0| = 0$.*
- *Symmetricity: If $r \geq 0$, then $-r \leq 0$ by Proposition 28(3), so $|r| = r = -(-r) = |-r|$. If $r < 0$, then $-r > 0$ by Proposition 28(3), so $|r| = -r = |-r|$.*
- *Triangular inequality: Take $r, s \in R$. If $r, s \geq 0$, then $r + s \geq 0$, so $|r + s| = r + s = |r| + |s|$. If If $r, s < 0$, then $r + s < 0$, so $|r + s| = -r - s = |r| + |s|$. Suppose now that $r \geq 0$ and $s < 0$. We will distinguish between two cases:*
- *Multiplicativity: Take $r, s \in R$. If $r, s \geq 0$, then $rs \geq 0$, so $|rs| = rs = |r||s|$. If If $r, s < 0$, then $rs \geq 0$, so $|rs| = rs = (-r)(-s) = |r||s|$. If $r \geq 0$ and $s < 0$, then $rs \leq 0$, so $|rs| = -rs = r(-s) = |r||s|$.*

The following lemma compiles the most essential properties satisfied by absolute semivalues. It is noticeable that all invertibles in an absolutely semivalued ring are absolutely invertibles.

Lemma 74 *If R is an absolutely semivalued ring, then:*

1. *Either $|1| = 1$ or $|1| = 0$.*
2. *If $|1| = 1$, then $\left|r^{-1}\right| = |r|^{-1}$ for all $r \in \mathcal{U}(R)$.*
3. *$\left|r^{-1} - s^{-1}\right| = |s|^{-1} |s - r| |r|^{-1}$ for all $r, s \in \mathcal{U}(R)$.*
4. *If R is absolutely valued, then it is an integral domain.*

Proof 183 *The proof follow the same itemization as the statement of the lemma.*

1. *$|1| = |1 \cdot 1| = |1||1|$. This implies that either $|1| = 1$ or $|1| = 0$.*
2. *$1 = |1| = \left|r^{-1} r\right| = \left|r^{-1}\right| |r|$ for all $r \in \mathcal{U}(R)$.*
3. *If $r, s \in \mathcal{U}(R)$, then*

$$\left|r^{-1} - s^{-1}\right| = \left|s^{-1}(s - r)r^{-1}\right| = \left|s^{-1}\right| |s - r| \left|r^{-1}\right| = |s|^{-1} |s - r| |r|^{-1}.$$

4. *If $|rs| = 0$, then $|r||s| = 0$, so either $|r| = 0$ or $|s| = 0$, which implies that either $r = 0$ or $s = 0$.*

1.2.7.6 Module and algebra seminorms

It is turn now for module seminorms. In order to define a module seminorm, we first need the underlying ring to be equipped with a ring seminorm.

Definition 180 (Seminormed module) *A seminorm on a module* $(M, \{+, 0, -\}, \{\cdot\})$ *over a seminormed ring* $(R, \{, +, 0, -, \cdot, 1\}, \{\| \|\})$ *is a group seminorm* $\| \|$ *on its commutative additive group* $(M, \{+, 0, -\}, \varnothing)$*, which is submutiplicative, that is,* $\|rm\| \leq \|r\|\|m\|$ *for all* $r \in R$ *and all* $m \in M$*. The universal algebra* $(M, \{+, 0, -\}, \{\cdot, \| \|\})$ *is called a seminormed module.*

Under the settings of the previous definition, in case R is absolutely semivalued, then the module seminorm is asked to be absolutely homogeneous, that is, $\|rm\| = |r|\|m\|$ for all $r \in R$ and all $m \in M$.

Definition 181 (Seminormed bimodule) *A seminorm on a bimodule* $(M, \{+, 0, -\}, \{\cdot, \cdot\})$ *over seminormed rings* $(R, \{, +, 0, -, \cdot, 1\}, \{\| \|\})$ *and* $(S, \{, +, 0, -, \cdot, 1\}, \{\| \|\})$ *is a module seminorm* $\| \|$ *on the left* $(R, \{, +, 0, -, \cdot, 1\}, \{\| \|\})$*-module* $(M, \{+, 0, -\}, \{\cdot\})$ *and on the right* $(S, \{, +, 0, -, \cdot, 1\}, \{\| \|\})$*-module* $(M, \{+, 0, -\}, \cdot)$*. The universal algebra* $(M, \{+, 0, -\}, \{\cdot, \cdot, \| \|\})$ *is called a seminormed bimodule.*

Algebra seminorms are essentially a particular case of bimodule seminorms.

Definition 182 (Seminormed algebra) *A seminorm on an algebra* $(A, \{+, 0, -, \cdot, 1\}, \{\cdot\})$ *over a seminormed ring* $(R, \{, +, 0, -, \cdot, 1\}, \{\| \|\})$ *is a ring seminorm* $\| \|$ *on* $(A, \{+, 0, -, \cdot, 1\}, \varnothing)$ *as well as a module seminorm on the left* $(R, \{, +, 0, -, \cdot, 1\}, \{\| \|\})$*-module* $(A, \{+, 0, -\}, \{\cdot\})$*. The universal algebra* $(A, \{+, 0, -, \cdot, 1\}, \{\cdot, \| \|\})$ *is called a seminormed algebra.*

The following lemma compiles essential properties of module seminorms.

Lemma 75 *Let* M *be a seminormed module over a seminormed ring* R*. Consider* $m \in M$*,* $r \in R$*,* $u \in \mathcal{U}_1(R)$*, and* $t \geq 0$*. Then:*

1. $\|um\| = \|u\|\|m\|$.
2. $\mathsf{B}_M(m, t) \subseteq \mathsf{B}_M(0, \|m\| + t)$.
3. $r\mathsf{B}_M(m, t) \subseteq \mathsf{B}_M(rm, \|r\|t)$.
4. $u\mathsf{B}_M(m, t) = \mathsf{B}_M(um, \|u\|t)$.
5. *If* R *is absolutely valued and* M *is normed, then* M *is torsionfree.*
6. *If* R *is a seminormed integral domain and* M *is normed with* $M \setminus \mathrm{tor}(M) \neq \varnothing$*, then* R *is normed.*

Proof 184 *We will itemize the proof in view of the statement of the lemma.*

1. *Simply observe that* $\|m\| = \left\|u^{-1}um\right\| \leq \left\|u^{-1}\right\| \|um\| = \|u\|^{-1}\|um\|$*, so* $\|u\|\|m\| \leq \|um\| \leq \|u\|\|m\|$.
2. *If* $p \in \mathsf{B}_M(m, t)$*, then* $\|p\| = \|m + (p - m)\| \leq \|m\| + \|p - m\| \leq \|m\| + t$.
3. *If* $p \in r\mathsf{B}_M(m, t)$*, then* $p = rq$ *with* $q \in \mathsf{B}_M(m, t)$*, so* $\|p - rm\| = \|rq - rm\| \leq \|r\|\|q - m\| \leq \|r\|t$.

4. *By above,* $u\mathsf{B}_M(m,t) \subseteq \mathsf{B}_M(um, \|u\|t)$ *and* $u^{-1}\mathsf{B}_M(um, \|u\|t) \subseteq \mathsf{B}_M\left(m, \left\|u^{-1}\right\| \|u\|t\right) = \mathsf{B}_M(m,t)$.

5. *If* $0 = \|rm\| = |r|\|m\|$, *then either* $r = 0$ *or* $m = 0$.

6. *Let* $r \in R$ *with* $\|r\| = 0$. *Fix any* $m \in M \setminus \mathrm{tor}(M)$ *and observe that* $\|rm\| \leq \|r\|\|m\| = 0$, *so* $rm = 0$ *because* M *is normed. Since* $m \notin \mathrm{tor}(M)$, *we deduce that* r *is a zero divisor; hence* $r = 0$ *because* R *is an integral domain.*

By gluing together Lemma 62(3) and Lemma 75(2), we immediately deduce a characterization of seminorm-boundedness. Let M be a seminormed module over a seminormed ring R. A subset $A \subseteq M$ is seminorm-bounded if and only if there exists $t > 0$ such that $A \subseteq \mathsf{B}_M(0,t)$.

Lemma 76 *Let* M *be a seminormed module over a seminormed ring* R. *Consider* $m \in M$ *and* $\delta > 0$. *Then:*

1. *If* $1 \in \mathrm{cl}\,(\mathsf{U}_R)$, *then* $\mathrm{cl}\,(\mathsf{U}_M(m,\delta)) = \mathsf{B}_M(m,\delta)$.

2. *If* $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{B}_R)$, *then* $\mathrm{int}(\mathsf{B}_M(m,\delta)) = \mathsf{U}_M(m,\delta)$.

Proof 185 *Let us itemize the proof via the itemization if the statement.*

1. *Fix an arbitrary* $p \in \mathsf{S}_M(m,\delta)$ *and an arbitrary* $\varepsilon > 0$. *We will prove that* $\mathsf{U}_M(p,\varepsilon) \cap \mathsf{U}_M(m,\delta) \neq \varnothing$. *Indeed, there exists* $u \in \mathsf{U}_R(1,\varepsilon/\delta) \cap \mathsf{U}_R$. *Then* $m + u(p-m) \in \mathsf{U}_M(p,\varepsilon) \cap \mathsf{U}_M(m,\delta)$.

2. *Fix an arbitrary* $p \in \mathrm{int}(\mathsf{B}_M(m,\delta))$ *and suppose to the contrary that* $\|p-m\| = \delta$. *There exists* $0 < \varepsilon < \delta$ *such that* $\mathsf{U}_M(p,\varepsilon) \subseteq \mathsf{B}_M(m,\delta)$. *By hypothesis, there exists* $u \in \mathsf{U}_R(1,\varepsilon/\delta) \cap (\mathcal{U}_1(R) \setminus \mathsf{B}_R)$. *This way,* $m + u(p-m) \in \mathsf{U}_M(p,\varepsilon)$, *however* $m + u(p-m) \notin \mathsf{B}_M(m,\delta)$.

Under the settings of Lemma 76, $\mathsf{B}_M(m,\delta)$ is regular closed and $\mathsf{U}_M(m,\delta)$ is regular open, for every $m \in M$ and every $\delta > 0$. Note that $1 \in \mathrm{cl}(\mathsf{U}_R)$ implies that R is a unital seminormed ring, that is, $\|1\| = 1$. On the other hand, according to Lemma 59, if $n \in M$ and $\|n\| = 0$, then $\|m+n\| = \|m\|$ for all $m \in M$.

Lemma 77 *Let* M *be a module over a ring* R. *Assume that* M *is endowed with a translation-invariant metric* d. *If* $A \subseteq M$ *is antiproximinal, then* $A + B$ *is antiproximinal as well for all subsets* $B \subseteq M$ *such that* $A + B$ *is proper. If, in addition,* d *is a norm and* $C \subseteq \mathcal{U}_1(R)$, *then* CA *is antiproximinal as well provided that* $CA \neq M$.

Proof 186 *Indeed, suppose on the contrary that* $A + B$ *is not antiproximinal. There exists* $m \in M \setminus (A+B)$ *such that* $d(m, A+B)$ *is attained at a certain element* $a_0 + b_0 \in A + B$. *Then* $m - b_0 \in M \setminus A$ *and* $d(m-b_0, A) = d(m-b_0, a_0)$, *contradicting the fact that* A *is antiproximinal. Indeed,* $d(m-b_0,a_0) = d(m, a_0+b_0) \leq d(m, a+b_0) = d(m-b_0, a)$ *for all* $a \in A$. *Next, suppose that* d *is a norm and* $C \subseteq \mathcal{U}_1(R)$. *If* CA *is not antiproximinal, there exists* $m \in M \setminus CA$ *such that* $d(m, CA)$ *is attained at some* $c_0 a_0 \in CA$, *which implies that* $d\left(c_0^{-1}m, A\right) = \left\|c_0^{-1}m - a_0\right\|$, *contradicting that* A *is antiproximinal. Indeed,*

$$\begin{aligned} \left\|c_0^{-1}m - a_0\right\| &= \left\|c_0^{-1}(m - c_0a_0)\right\| = \left\|c_0^{-1}\right\| \|m - c_0a_0\| \\ &\leq \left\|c_0^{-1}\right\| \|m - c_0a\| = \left\|c_0^{-1}(m-c_0a)\right\| = \left\|c_0^{-1}m - a\right\| \end{aligned}$$

for every $a \in A$.

The last result in this subsubsection is the abstraction of real convexity.

Proposition 62 *Let R be a totally ordered ring endowed with its absolute value* (1.34)*. Let M be an R-module endowed with a translation-invariant and R^+-homogeneous pseudometric d. The function*

$$\begin{array}{rccl} w: & M \times M \times [0,1] & \to & M \\ & (m,n,r) & \mapsto & w(m,n,r) := (1-r)m + rn \end{array} \tag{1.35}$$

is a convex structure on M.

Proof 187 *Let $m, n, l \in M$. Then $d(l, w(m,n,r)) = d(l, (1-r)m + rn) = d((1-r)l + rl, (1-r)m + rn) = d((1-r)(l-m), r(n-l)) \leq d((1-r)(l-m), 0) + d(0, r(n-l)) = |1-r|d(l-m, 0) + |r|d(0, n-l) = (1-r)d(l,m) + rd(l,n)$.*

Chapter 2

Limits

Augustin Louis Cauchy

2.1 Limits of prefilters and functions

The concept of limit constitutes the basis of other relevant concepts in Calculus and Analysis such as differentiability, integrability, and summability. Here, we will approach limits from the most general viewpoint, provided by filters and nets.

2.1.1 Limits of filter bases

All the different concepts of limits can actually be reduced to only one: the concept of limit of filter base.

2.1.1.1 Limits and agglomeration

Recall that a filter base of $\mathcal{P}(X)$ is a subset $\mathcal{B}$ of $\mathcal{P}(X)$ satisfying that $\mathcal{B} \neq \varnothing$, $\varnothing \notin \mathcal{B}$, and if $A, C \in \mathcal{B}$, there exists $D \in \mathcal{B}$ such that $D \subseteq A \cap C$.

Definition 183 (Limit & agglomeration) *Let X be a topological space. Let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. We define the following sets:*

- *The limit of $\mathcal{B}$ is defined as* $\lim(\mathcal{B}) := \{x \in X : \mathcal{N}_x \subseteq \mathcal{J}(\mathcal{B})\}$.
- *The agglomeration of $\mathcal{B}$ is defined as* $\operatorname{agl}(\mathcal{B}) := \bigcap \{\operatorname{cl}(A) : A \in \mathcal{B}\}$.

Under the settings of the previous definition,

$$\begin{aligned} \lim(\mathcal{B}) &= \{x \in X : \forall U \in \mathcal{N}_x \ \exists A \in \mathcal{B} \text{ with } A \subseteq U\}, \\ \operatorname{agl}(\mathcal{B}) &= \{x \in X : \forall U \in \mathcal{N}_x \ \forall A \in \mathcal{B} \ A \cap U \neq \varnothing\}. \end{aligned}$$

If, instead, $\mathcal{B}$ is an ideal base of $\mathcal{P}(X)$, then

$$\begin{aligned} \lim(\mathcal{B}) &:= \lim\left(\mathscr{F}_{\mathcal{B}}\right) = \{x \in X : \forall U \in \mathcal{N}_x \ \exists A \in \mathcal{B} \text{ with } X \setminus U \subseteq A\}, \\ \operatorname{agl}(\mathcal{B}) &:= \operatorname{agl}\left(\mathscr{F}_{\mathcal{B}}\right) = \{x \in X : \forall U \in \mathcal{N}_x \ \forall A \in \mathcal{B} \ (X \setminus A) \cap U \neq \varnothing\}. \end{aligned}$$

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Proposition 63 *Let X be a topological space. Let $\mathcal{B},\mathcal{C}$ be filter bases of $\mathcal{P}(X)$. Then:*

1. $\lim(\mathcal{B}) \subseteq \operatorname{agl}(\mathcal{B})$.
2. *If $\mathcal{B} \subseteq \mathcal{C}$, then* $\lim(\mathcal{B}) \subseteq \lim(\mathcal{C})$ *and* $\operatorname{agl}(\mathcal{C}) \subseteq \operatorname{agl}(\mathcal{B})$.
3. $\lim(\mathcal{B}) = \lim(\mathcal{J}(\mathcal{B}))$ *and* $\operatorname{agl}(\mathcal{B}) = \operatorname{agl}(\mathcal{J}(\mathcal{B}))$.
4. *If X is Hausdorff and* $\lim(\mathcal{B}) \neq \varnothing$, *then* $\lim(\mathcal{B}) = \operatorname{agl}(\mathcal{B})$.

Proof 188 *The proof follows as itemized in the statement.*

1. *Let $x \in \lim(\mathcal{B})$ and fix an arbitrary $C \in \mathcal{B}$. Let $U \in \mathcal{N}_x$. Then $U \in \mathcal{J}(\mathcal{B})$ by hypothesis, so there exists $D \in \mathcal{B}$ such that $D \subseteq U$. Since $\mathcal{B}$ is downward directed, there exists $A \in \mathcal{B}$ such that $A \subseteq C \cap D$. Note that $A \neq \varnothing$, hence $C \cap D \neq \varnothing$ and thus $C \cap U \neq \varnothing$. As a consequence, $x \in \operatorname{cl}(C)$.*
2. *Note that $\mathcal{J}(\mathcal{B}) \subseteq \mathcal{J}(\mathcal{C})$, hence* $\lim(\mathcal{B}) \subseteq \lim(\mathcal{C})$. *On the other hand, since $\mathcal{B} \subseteq \mathcal{C}$, it is clear that* $\operatorname{agl}(\mathcal{C}) \subseteq \operatorname{agl}(\mathcal{B})$.
3. *It is sufficient to observe that $\mathcal{J}(\mathcal{J}(\mathcal{B})) = \mathcal{J}(\mathcal{B})$.*
4. *Fix $x_0 \in \lim(\mathcal{B})$. If $x \in \operatorname{agl}(\mathcal{B})$ and $x \neq x_0$, then we can find $U \in \mathcal{N}_{x_0}$ and $V \in \mathcal{N}_x$ such that $U \cap V = \varnothing$. There exists $A \in \mathcal{B}$ with $A \subseteq U$. However, $A \cap V \neq \varnothing$, which implies the contradiction that $U \cap V \neq \varnothing$. As a consequence, $x = x_0 \in \lim(\mathcal{B})$. This shows that* $\operatorname{agl}(\mathcal{B}) \subseteq \lim(\mathcal{B})$.

If $\mathcal{B}$ is a filter base in a set X and $\tau_1 \subseteq \tau_2$ are topologies in X, then $\lim_{\tau_2}(\mathcal{B}) \subseteq \lim_{\tau_1}(\mathcal{B})$ and $\operatorname{agl}_{\tau_2}(\mathcal{B}) \subseteq \operatorname{agl}_{\tau_1}(\mathcal{B})$.

Lemma 78 *Let X be a regular topological space. If $\mathcal{B}$ is a filter base of $\mathcal{P}(X)$, then* $\lim(\mathcal{B}) = \bigcap \mathcal{N}_x$ *for all $x \in \lim(\mathcal{B})$.*

Proof 189 *The proof is itemized in accordance with the statement.*

- $\subseteq$ *Take $y \in \lim(\mathcal{B})$. Suppose to contrary that there exists $U \in \mathcal{N}_x$ with $y \notin U$. There exists $V \in \mathcal{N}_x$ with $\operatorname{cl}(V) \subseteq U$. Then $y \in X \setminus \operatorname{cl}(V)$. We can find $A_1 \in \mathcal{B}$ such that $A_1 \subseteq X \setminus \operatorname{cl}(V)$. We can also find $A_2 \in \mathcal{B}$ such that $A_2 \subseteq V$. Take $A_0 \in \mathcal{B}$ with $A_0 \subseteq A_1 \cap A_2$. Then $A_0 \subseteq X \setminus \operatorname{cl}(V)$ and $A_0 \subseteq V$. This is impossible.*
- $\supseteq$ *Take $y \in \bigcap \mathcal{N}_x$. By relying on Lemma 49, we have that $x \in \bigcap \mathcal{N}_y$. Fix an arbitrary $U \in \mathcal{N}_y$. Then $x \in U$, therefore, there exists $A_0 \in \mathcal{B}$ such that $A_0 \subseteq U$. This shows that $y \in \lim(\mathcal{B})$.*

The following characterization of the Hausdorff separation property is crucial in the literature.

Theorem 44 *A topological space X is Hausdorff if and only if* $\lim(\mathcal{B})$ *is either empty or a singleton for every filter base $\mathcal{B}$ of $\mathcal{P}(X)$.*

Proof 190 *The proof is itemized in accordance with the statement.*

- $\Rightarrow$ *Suppose to the contrary that there exist $x \neq y$ in* $\lim(\mathcal{B})$. *By hypothesis, we can find $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $U \cap V = \varnothing$. There are $A, C \in \mathcal{B}$ with $A \subseteq U$ and $C \subseteq V$. Then $A \cap C \subseteq U \cap V = \varnothing$. Since $\mathcal{B}$ is downward directed, there exists $D \in \mathcal{B}$ with $D \subseteq A \cap C = \varnothing$. However, $D = \varnothing$, which contradicts that $D \in \mathcal{B}$.*
- $\Leftarrow$ *Suppose to the contrary that there exist $x \neq y$ in X such that $U \cap V \neq \varnothing$ for every $U \in \mathcal{N}_x$ and every $V \in \mathcal{N}_y$. Observe that $\mathcal{B} := \{U \cap V : U \in \mathcal{N}_x, V \in \mathcal{N}_y\}$ is a filter base of $\mathcal{P}(X)$ such that $\{x, y\} \subseteq \lim(\mathcal{B})$.*

2.1.1.2 Ultrafilter limits

Ultrafilters satisfy a very important property in compact topological spaces, which is very useful to define certain functions like, for instance, in [30, Theorem 1].

Lemma 79 *Let X be a topological space. If $\mathcal{F}$ is an ultrafilter of $\mathcal{P}(X)$, then* $\lim(\mathcal{F}) = \operatorname{agl}(\mathcal{F})$.

Proof 191 *Note that* $\lim(\mathcal{F}) \subseteq \operatorname{agl}(\mathcal{F})$. *Take* $x \in \operatorname{agl}(\mathcal{F}) \setminus \lim(\mathcal{F})$. *Then we can find a neighborhood* $U \in \mathcal{N}_x$ *such that* $A \nsubseteq U$ *for every* $A \in \mathcal{F}$. *By hypothesis,* $U \cap A \neq \varnothing$ *for all* $A \in F$. *Consider the filter base* $\mathcal{B} := \mathcal{F} \cup \{A \cap U : A \in \mathcal{F}\}$. *Notice that* $\mathcal{F} \subsetneq \mathcal{J}(\mathcal{B})$ *since* $U \in \mathcal{J}(\mathcal{B}) \setminus \mathcal{F}$. *This contradicts the fact that $\mathcal{F}$ is maximal.*

The following is a crucial characterization of compactness.

Theorem 45 *A topological space X is compact if and only if* $\operatorname{agl}(\mathcal{B}) \neq \varnothing$ *for every filter base $\mathcal{B}$ of $\mathcal{P}(X)$.*

Proof 192 *Let us itemize the proof by following the statement.*

- $\Rightarrow$ *Note that* $\operatorname{agl}(\mathcal{B}) = \bigcap \{\operatorname{cl}(A) : A \in \mathcal{B}\}$. *We will show that the family of closed sets* $\{\operatorname{cl}(A) : A \in \mathcal{B}\}$ *satisfies the finite intersection property. Indeed, if* $A_1, \ldots, A_n \in \mathcal{B}$, *then there exists* $A \in \mathcal{B}$ *such that* $A \subseteq A_1 \cap \cdots \cap A_n$, *so* $A_1 \cap \cdots \cap A_n \neq \varnothing$ *because* $A \neq \varnothing$. *Thus,* $\operatorname{cl}(A_1 \cap \cdots \cap A_n) \neq \varnothing$ *and* $\operatorname{cl}(A_1 \cap \cdots \cap A_n) \subseteq \operatorname{cl}(A_1) \cap \cdots \cap \operatorname{cl}(A_n)$. *This shows that the family of closed sets* $\{\operatorname{cl}(A) : A \in \mathcal{B}\}$ *satisfies the finite intersection property. Since X is compact, we have that* $\operatorname{agl}(\mathcal{B}) = \bigcap \{\operatorname{cl}(A) : A \in \mathcal{B}\} \neq \varnothing$.
- $\Leftarrow$ *Suppose to the contrary that X is not compact. There exists a family $\mathcal{B}$ of closed subsets of X such that $\mathcal{B}$ satisfies the finite intersection property but* $\bigcap \mathcal{B} = \varnothing$. *Notice that $\mathcal{B}$ is itself a filter base and* $\operatorname{agl}(\mathcal{B}) = \bigcap \mathcal{B} = \varnothing$. *This contradicts our hypothesis.*

As a direct consequence of the previous theorem, if a topological space X is compact, then $\lim(\mathcal{F}) \neq \varnothing$ for every ultrafilter $\mathcal{F}$ of $\mathcal{P}(X)$.

2.1.1.3 Limits superior/inferior

Recall that every poset can be endowed with the order topology, which we call a topological poset.

Definition 184 (Limits superior/inferior) *Let X be a topological poset. Let $\mathcal{B}$ be a filter base in X. We define the following:*

- *The limit superior of $\mathcal{B}$ is defined as* $\limsup(\mathcal{B}) := \sup(\operatorname{agl}(\mathcal{B}))$.
- *The limit inferior of $\mathcal{B}$ is defined as* $\liminf(\mathcal{B}) := \inf(\operatorname{agl}(\mathcal{B}))$.

Under the settings of the previous definition, $\limsup(\mathcal{B}) \geq \liminf(\mathcal{B})$ if they exist. If $\mathcal{B}$ is an ideal base of $\mathcal{P}(X)$, then $\limsup(\mathcal{B}) := \limsup(\mathscr{F}_{\mathcal{B}})$ and $\liminf(\mathcal{B}) := \liminf(\mathscr{F}_{\mathcal{B}})$.

Proposition 64 *Let X be a Dedekind complete topological toset. Consider a filter base $\mathcal{B}$ of $\mathcal{P}(X)$. Then:*

1. *If each $A \in \mathcal{B}$ is bounded above, then* $\limsup(\mathcal{B}) = \inf\{\sup(A) : A \in \mathcal{B}\}$.
2. *If each $A \in \mathcal{B}$ is bounded below, then* $\liminf(\mathcal{B}) = \sup\{\inf(A) : A \in \mathcal{B}\}$.

Proof 193 *We will prove only the first item since the second one follows a similar argument. Observe that* $\operatorname{agl}(\mathcal{B}) \subseteq \operatorname{cl}(A)$ *for every* $A \in \mathcal{B}$*, meaning that* $\limsup(\mathcal{B}) = \sup(\operatorname{agl}(\mathcal{B})) \leq \sup(\operatorname{cl}(A)) = \sup(A)$*. As a consequence,* $\limsup(\mathcal{B}) \leq \inf\{\sup(A) : A \in \mathcal{B}\}$*. Next, fix an arbitrary* $A_0 \in \mathcal{B}$*. Let* (x, y) *be an open neighborhood of* $\inf\{\sup(A) : A \in \mathcal{B}\}$*. Since the order of* X *is total, there exists* $A_1 \in \mathcal{B}$ *such that* $x < \inf\{\sup(A) : A \in \mathcal{B}\} \leq \sup(A_1) < y$*. Let* $A_2 \in \mathcal{B}$ *such that* $A_2 \subseteq A_0 \cap A_1$*. Note that* $x < \inf\{\sup(A) : A \in \mathcal{B}\} \leq \sup(A_2) \leq \sup(A_1) < y$*. There can be found* $a \in A_2$ *such that* $x < a \leq \sup(A_2) \leq \sup(A_1) < y$*. Finally,* $a \in (x, y) \cap A_2 \subseteq (x, y) \cap A_0$*. As a consequence,* $\inf\{\sup(A) : A \in \mathcal{B}\} \in \operatorname{cl}(A_0)$*. The arbitrariness of* $A_0 \in \mathcal{B}$ *implies that* $\inf\{\sup(A) : A \in \mathcal{B}\} \in \operatorname{agl}(\mathcal{B})$*, resulting in* $\inf\{\sup(A) : A \in \mathcal{B}\} \leq \sup(\operatorname{agl}(\mathcal{B})) = \limsup(\mathcal{B})$.

2.1.1.4 Cauchy prefilters and completeness

Uniform spaces provide the most general setting where completeness and Cauchy prefilters can be naturally defined.

Definition 185 (Cauchy prefilter) *Let* X *be a uniform space. A Cauchy prefilter in* X *is a prefilter* $\mathcal{B}$ *of* $\mathcal{P}(X)$ *such that* $\mathcal{J}(\{B \times B : B \in \mathcal{B}\})$ *contains the uniformity of* X.

Under the settings of the previous definition, $\mathcal{B}$ is a Cauchy prefilter if and only if for every entourage $U \subseteq X \times X$ there exists $B \in \mathcal{B}$ with $B \times B \subseteq U$. This is known as the topological/analytical version of the Cauchy prefilter notion. Notice that if $\mathcal{B}, \mathcal{C}$ are prefilters in a uniform space X such that $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{B}$ is Cauchy, then $\mathcal{C}$ is Cauchy as well.

Lemma 80 *Let* X *be a uniform space. A prefilter* $\mathcal{B}$ *in* X *is convergent if and only if* $\mathcal{B}$ *is a Cauchy prefilter and* $\operatorname{agl}(\mathcal{B}) \neq \varnothing$*. In this situation,* $\lim(\mathcal{B}) = \operatorname{agl}(\mathcal{B})$.

Proof 194 *Let us itemize the proof by following the statement.*

- $\Rightarrow$ *First off,* $\lim(\mathcal{B}) \subseteq \operatorname{agl}(\mathcal{B})$*, meaning that* $\operatorname{agl}(\mathcal{B}) \neq \varnothing$*. Next, fix an arbitrary* $x \in \lim(\mathcal{B})$*. Take any entourage* U *of* X*. We can consider a symmetric entourage* V *of* X *such that* $V \circ V \subseteq U$*. There exists* $B \in \mathcal{B}$ *such that* $B \subseteq V[x]$*. Notice that* $B \times B \subseteq V \circ V \subseteq U$.
- $\Leftarrow$ *Fix an arbitrary* $x_0 \in \operatorname{agl}(\mathcal{B})$*. Take any symmetric entourage* U *in* X*. There exists another symmetric entourage* V *such that* $V \circ V \subseteq U$*. Since* $\mathcal{B}$ *is a Cauchy prefilter, there exists* $B \in \mathcal{B}$ *such that* $B \times B \subseteq V$*. Since* $x_0 \in \operatorname{agl}(\mathcal{B})$*, we have that* $x_0 \in \operatorname{cl}(B)$*; therefore, we can fix* $b' \in B \cap V[x_0]$*. Finally, if* $b \in B$*, then* $(b, b') \in B \times B \subseteq V$ *and* $(b', x_0) \in V$*, meaning that* $(b, x_0) \in V \circ V \subseteq U$*. As a consequence,* $B \subseteq U[x_0]$*, hence* $x_0 \in \lim(\mathcal{B})$.

The converse to Lemma 80 is by definition the notion of completeness.

Definition 186 (Completeness) *A uniform space is said to be complete provided that every Cauchy prefilter is convergent.*

2.1.2 Limits of functions

What we are about to deal with in this subsection is crucial towards the development of the differentiability chapter.

2.1.2.1 Prefilter notion

Recall that if $f : X \to Y$ is a function between sets X, Y and $\mathcal{B}$ is a filter base of $\mathcal{P}(X)$, then $f(\mathcal{B}) := \{f(A) : A \in \mathcal{B}\}$ is a filter base of $\mathcal{P}(Y)$.

Definition 187 *Let $f : X \to Y$ be a function between a nonempty set X and a topological space Y. Let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. Then:*

- *The limit of f by $\mathcal{B}$ is defined as* $\lim_{\mathcal{B}}(f) := \lim(f(\mathcal{B}))$.
- *The agglomeration of f by $\mathcal{B}$ is defined as* $\operatorname{agl}_{\mathcal{B}}(f) := \operatorname{agl}(f(\mathcal{B}))$.

Under the settings of the previous definition,

$$\begin{aligned}
\lim_{\mathcal{B}} f &= \{y \in Y : \forall U \in \mathcal{N}_y \ \exists A \in \mathcal{B} \text{ with } f(A) \subseteq U\} \\
&= \left\{y \in Y : \forall U \in \mathcal{N}_y \ \exists A \in \mathcal{B} \text{ with } A \subseteq f^{-1}(U)\right\} \\
&= \left\{y \in Y : \forall U \in \mathcal{N}_y \ f^{-1}(U) \in \mathcal{J}(\mathcal{B})\right\}, \\
\operatorname*{agl}_{\mathcal{B}} f &= \{x \in X : \forall U \in \mathcal{N}_x \ \forall A \in \mathcal{B} \ f(A) \cap U \neq \varnothing\} \\
&= \{x \in X : \forall U \in \mathcal{N}_x \ \forall A \in \mathcal{B} \ A \cap f^{-1}(U) \neq \varnothing\}.
\end{aligned}$$

If, instead, $\mathcal{B}$ is an ideal base of $\mathcal{P}(X)$, then

$$\begin{aligned}
\lim_{\mathcal{B}}(f) &:= \lim_{\mathscr{F}_{\mathcal{B}}}(f) = \{y \in Y : \forall U \in \mathcal{N}_y \ \exists A \in \mathcal{B} \text{ with } f(X \setminus A) \subseteq U\} \\
&= \left\{y \in Y : \forall U \in \mathcal{N}_y \ \exists A \in \mathcal{B} \text{ with } X \setminus A \subseteq f^{-1}(U)\right\} \\
&= \left\{y \in Y : \forall U \in \mathcal{N}_y \ \exists A \in \mathcal{B} \text{ with } X \setminus f^{-1}(U) \subseteq A\right\} \\
&= \left\{y \in Y : \forall U \in \mathcal{N}_y \ \exists A \in \mathcal{B} \text{ with } f^{-1}(Y \setminus U) \subseteq A\right\} \\
&= \left\{y \in Y : \forall U \in \mathcal{N}_y \ f^{-1}(Y \setminus U) \in \mathcal{I}(\mathcal{B})\right\}, \\
\operatorname*{agl}_{\mathcal{B}}(f) &:= \operatorname*{agl}_{\mathscr{F}_{\mathcal{B}}}(f) = \{x \in X : \forall U \in \mathcal{N}_x \ \forall A \in \mathcal{B} \ f(X \setminus A) \cap U \neq \varnothing\} \\
&= \{x \in X : \forall U \in \mathcal{N}_x \ \forall A \in \mathcal{B} \ (X \setminus A) \cap f^{-1}(U) \neq \varnothing\}.
\end{aligned}$$

Proposition 65 *Let $f : X \to Y$, $g : Y \to Z$ be functions between a nonempty set X and topological spaces Y, Z. Let $\mathcal{B}, \mathcal{C}$ be filter bases of $\mathcal{P}(X)$. Then:*

1. $\lim_{\mathcal{B}}(f) = \lim_{\mathcal{J}(\mathcal{B})}(f)$ *and* $\operatorname*{agl}_{\mathcal{B}}(f) = \operatorname*{agl}_{\mathcal{J}(\mathcal{B})}(f)$.
2. *If $\mathcal{B} \subseteq \mathcal{C}$, then* $\lim_{\mathcal{B}}(f) \subseteq \lim_{\mathcal{C}}(f)$ *and* $\operatorname*{agl}_{\mathcal{C}}(f) \subseteq \operatorname*{agl}_{\mathcal{B}}(f)$.
3. $\lim_{\mathcal{B}}(f) \subseteq \operatorname*{agl}_{\mathcal{B}}(f)$.
4. $\lim_{\mathcal{B}}(g \circ f) = \lim_{f(\mathcal{B})}(g)$ *and* $\operatorname*{agl}_{\mathcal{B}}(g \circ f) = \operatorname*{agl}_{f(\mathcal{B})}(g)$.
5. *If Y is regular, then* $\lim_{\mathcal{B}}(f) = \bigcap \mathcal{N}_y$ *for all* $y \in \lim_{\mathcal{B}}(f)$.

Proof 195 *Let us itemize the proof the same way as the statement is.*

1. *Since* $\mathcal{J}(f(\mathcal{B})) = \mathcal{J}(f(\mathcal{J}(\mathcal{B})))$, *we have that* $\lim_{\mathcal{B}}(f) = \lim(f(\mathcal{B})) = \lim(\mathcal{J}(f(\mathcal{B}))) = \lim(\mathcal{J}(f(\mathcal{J}(\mathcal{B})))) = \lim(f(\mathcal{J}(\mathcal{B}))) = \lim_{\mathcal{J}(\mathcal{B})}(f)$. *In a similar way,* $\operatorname{agl}_{\mathcal{B}}(f) = \operatorname{agl}_{\mathcal{J}(\mathcal{B})}(f)$.

2. *Since* $f(\mathcal{B}) \subseteq f(\mathcal{C})$, $\lim_{\mathcal{B}}(f) = \lim(f(\mathcal{B})) \subseteq \lim(f(\mathcal{C})) = \lim_{\mathcal{C}}(f)$ *and* $\operatorname{agl}_{\mathcal{C}}(f) = \operatorname{agl}(f(\mathcal{C})) \subseteq \operatorname{agl}(f(\mathcal{B})) = \operatorname{agl}_{\mathcal{B}}(f)$.

3. $\lim\limits_{\mathcal{B}}(f) = \lim(f(\mathcal{B})) \subseteq \operatorname{agl}(f(\mathcal{B})) = \operatorname{agl}\limits_{\mathcal{B}}(f)$.

4. $\lim_{\mathcal{B}}(g \circ f) = \lim((g \circ f)(\mathcal{B})) = \lim(g(f(\mathcal{B}))) = \lim_{f(\mathcal{B})}(g)$ *and* $\operatorname{agl}_{\mathcal{B}}(g \circ f) = \operatorname{agl}((g \circ f)(\mathcal{B})) = \operatorname{agl}(g(f(\mathcal{B}))) = \operatorname{agl}_{f(\mathcal{B})}(g)$.

5. $\lim\limits_{\mathcal{B}}(f) = \lim(f(\mathcal{B})) = \bigcap \mathcal{N}_y$ *for all* $y \in \lim(f(\mathcal{B}))$.

In every product $\prod_{\alpha\in A} X_\alpha$, the coordinate projections are defined by

$$\begin{array}{rccl} \pi_\alpha : & \prod_{\alpha\in A} X_\alpha & \to & X_\alpha \\ & (x_\alpha)_{\alpha\in A} & \mapsto & \pi_\alpha\left((x_\alpha)_{\alpha\in A}\right) := x_\alpha. \end{array}$$

Lemma 81 *Let* $(X_\alpha)_{\alpha\in A}$ *be a family of topological spaces. For every function* $f : Y \to \prod_{\alpha\in A} X_\alpha$ *and every filter base* $\mathcal{B}$ *of* $\mathcal{P}(Y)$,

$$\lim_{\mathcal{B}}(f) = \left\{ x \in \prod_{\alpha\in A} X_\alpha : \forall \alpha \in A \; x_\alpha \in \lim_{\mathcal{B}}(\pi_\alpha \circ f) \right\}.$$

Proof 196 *Let us itemize the proof in accordance with the statement.*

$\subseteq$ *Fix an arbitrary* $x \in \lim_{\mathcal{B}}(f)$. *Fix also an arbitrary* $\alpha_0 \in A$. *Take* $U \in \mathcal{N}_{x_{\alpha_0}}(X_{\alpha_0})$. *Consider* $\prod_{\alpha\in A} V_\alpha$, *where*

$$V_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha_0, \\ U & \alpha = \alpha_0, \end{cases}$$

for every $\alpha \in A$. *Notice that* $\prod_{\alpha\in A} V_\alpha \in \mathcal{N}_x\left(\prod_{\alpha\in A} X_\alpha\right)$. *Therefore, there exists* $A \in \mathcal{B}$ *such that* $f(A) \subseteq \prod_{\alpha\in A} V_\alpha$. *As a consequence,* $\pi_{\alpha_0}(f(A)) \subseteq U$. *This shows that* $x_{\alpha_0} \in \lim_{\mathcal{B}}(\pi_{\alpha_0} \circ f)$.

$\supseteq$ *Fix an arbitrary* $x \in \prod_{\alpha\in A} X_\alpha$ *such that* $x_\alpha \in \lim(\pi_\alpha \circ f)$ *for each* $\alpha \in A$. *Take* $U \in \mathcal{N}_x\left(\prod_{\alpha\in A} X_\alpha\right)$. *There exists a finite number of open sets* $U_{\alpha_1}, \ldots, U_{\alpha_p}$ *of* $X_{\alpha_1}, \ldots, X_{\alpha_p}$, *respectively, such that* $x \in \prod_{\alpha\in A} V_\alpha \subseteq U$, *where*

$$V_\alpha = \begin{cases} X_\alpha & \alpha \notin \{\alpha_1, \ldots, \alpha_p\}, \\ U_{\alpha_q} & \alpha = \alpha_q \text{ for some } q \in \{1, \ldots, p\}. \end{cases}$$

For every $q \in \{1, \ldots, p\}$, *there exists* $A_q \in B$ *such that* $\pi_{\alpha_q}(f(A_q)) \subseteq U_{\alpha_q}$. *There exists* $A_0 \in \mathcal{B}$ *with* $A_0 \subseteq A_1 \cap \cdots \cap A_p$. *Thus,* $f(A_0) \subseteq \prod_{\alpha\in A} V_\alpha \subseteq U$. *This shows that* $x \in \lim_{\mathcal{B}}(f)$.

2.1.2.2 Analytical notion

Recall that, given a topological space X and a subset A of X, the set of limit (or cluster or accumulation) points of A is defined as $\operatorname{ac}(A) := \{x \in X : x \in \operatorname{cl}(A \setminus \{x\})\}$. If $x \in \operatorname{ac}(X)$, then $\mathcal{N}_x^\times$ is a filter base of $\mathcal{P}(X)$ and a filter of $\mathcal{P}(X \setminus \{x\})$.

Definition 188 *Let* X, Y *be topological spaces. Let* x_0 *be a limit point of* X. *Let* $f : X \setminus \{x_0\} \to Y$ *be a function. Then:*

- *The limit of* f *at* x_0 *is defined as* $\lim\limits_{x\to x_0}(f) := \lim\limits_{\mathcal{N}_{x_0}^\times}(f)$.

- *The agglomeration of f at x_0 is defined as* $\operatorname{agl}_{x\to x_0}(f) := \operatorname{agl}_{\mathcal{N}_{x_0}^\times}(f)$.

Under the settings of the previous definition,

$$\begin{aligned}\lim_{x\to x_0}(f) &= \left\{y\in Y : \forall V\in\mathcal{N}_y\ \exists U\in\mathcal{N}_{x_0}^\times \text{ with } f(U)\subseteq V\right\},\\ \operatorname*{agl}_{x\to x_0}(f) &= \left\{y\in Y : \forall V\in\mathcal{N}_y\ \forall U\in\mathcal{N}_{x_0}^\times\ f(U)\cap V\neq\varnothing\right\}.\end{aligned}$$

If X, Y are topological spaces, x_0 is a limit point of X, and $f : X\setminus\{x_0\}\to Y$ is a function, then $\lim_{x\to x_0}(f)\subseteq\operatorname{agl}_{x\to x_0}(f)$. If Y is regular, then $\lim_{x\to x_0}(f)=\bigcap\mathcal{N}_y$ for all $y\in\lim_{x\to x_0}(f)$.

2.1.2.3 One-sided limits; limits superior/inferior

If X is a topological poset, then the set of left limit (or left cluster or left accumulation) points of a subset $A\subseteq X$ is defined as

$$\operatorname{ac}^-(A) := \left\{x\in X : x\in\operatorname{cl}(A\cap{\downarrow^\times}x)\right\}.$$

In a dual manner, the set of right limit (or right cluster or right accumulation) points, $\operatorname{ac}^+(A)$, is defined. For each $x\in\operatorname{ac}(X)^-$, the set $\mathcal{N}_x^- := \{U\cap{\downarrow^\times}x : U\in\mathcal{N}_x\}$ is a filter base of $\mathcal{P}(X)$ and a filter of $\mathcal{P}({\downarrow^\times}x)$. Similarly, for each $x\in\operatorname{ac}(X)^+$, the set $\mathcal{N}_x^+ := \{U\cap{\uparrow_\times}x : U\in\mathcal{N}_x\}$ is a filter base of $\mathcal{P}(X)$ and a filter of $\mathcal{P}({\uparrow_\times}x)$.

Definition 189 (One-sided limit & agglomeration) *Let X be a topological poset. Let Y be a topological space. Let x_0 be a left limit point of X. Consider a function $f : X\setminus\{x_0\}\to Y$. Then:*

- *The limit of f at x_0 from the left is defined as* $\lim_{x\to x_0^-}(f) := \lim_{\mathcal{N}_{x_0}^-}(f)$.
- *The agglomeration of f at x_0 from the left is defined as* $\operatorname{agl}_{x\to x_0^-}(f) := \operatorname{agl}_{\mathcal{N}_{x_0}^-}(f)$.

Similarly, the one-sided limit from the right and the one-sided agglomeration from the right can be defined.

Proposition 66 *Let X be a topological poset. Let Y be a topological space. Let x_0 be a left and right limit point of X. Consider a function $f : X\setminus\{x_0\}\to Y$. Then*

$$\lim_{x\to x_0}(f)\subseteq\lim_{x\to x_0^-}(f)\cap\lim_{x\to x_0^+}(f).$$

If, in addition, X is totally ordered, then $\lim_{x\to x_0}(f)=\lim_{x\to x_0^-}(f)\cap\lim_{x\to x_0^+}(f)$.

Proof 197 *Let us itemize the proof by following the statement.*

- $\subseteq$ *Take any $y\in\lim_{x\to x_0}(f)$ and any $V\in\mathcal{N}_y(Y)$. There exists a neighborhood U of x_0 in X such that $f(U\setminus\{x_0\})\subseteq V$. Then $U\cap{\downarrow^\times}x_0\in\mathcal{N}_{x_0}^-(X), U\cap{\uparrow_\times}x_0\in\mathcal{N}_{x_0}^+(X)$, and $f(U\cap{\downarrow^\times}x_0)\subseteq V, f(U\cap{\uparrow_\times}x_0)\subseteq V$. This shows that $y\in\lim_{x\to x_0^-}(f)\cap\lim_{x\to x_0^+}(f)$.*
- $\supseteq$ *Take any $y\in\lim_{x\to x_0^-}(f)\cap\lim_{x\to x_0^+}(f)$ and any $V\in\mathcal{N}_y(Y)$. There exist neighborhoods U, U' of x_0 in X such that $f(U\cap{\downarrow^\times}x_0)\subseteq V, f(U'\cap{\uparrow_\times}x_0)\subseteq V$. Note that $(U\cap U')\setminus\{x_0\}\in\mathcal{N}_{x_0}^\times(X)$ and $(U\cap U')\setminus\{x_0\}\subseteq(U\cap{\downarrow^\times}x_0)\cup(U'\cap{\uparrow_\times}x_0)$, meaning that $f((U\cap U')\setminus\{x_0\})\subseteq f(U\cap{\downarrow^\times}x_0)\cup f(U'\cap{\uparrow_\times}x_0)\subseteq V$. This shows that $y\in\lim_{x\to x_0}(f)$.*

Limits superior and inferior also make sense for limits of functions through filter bases.

Definition 190 *Let X be a topological space. Let B be a filter base in X. Let Y be a topological poset. Consider a function $f : X \to Y$. Then:*

- *The limit superior of f by B is defined as* $\limsup_{B}(f) := \limsup(f(B))$.
- *The limit inferior of f by B is defined as* $\liminf_{B}(f) := \liminf(f(B))$.

If X is a topological space, x_0 is a limit point of X, Y is a topological poset, and $f : X \setminus \{x_0\} \to Y$ is a function, then:

- The limit superior of f at x_0 is defined as $\limsup_{x \to x_0}(f) := \limsup_{\mathcal{N}_{x_0}^{\times}}(f)$.
- The limit inferior of f at x_0 is defined as $\liminf_{x \to x_0}(f) := \liminf_{\mathcal{N}_{x_0}^{\times}}(f)$.

Finally, if X is a topological poset, x_0 is a left (right) limit point of X, Y is a topological poset, and $f : X \setminus \{x_0\} \to Y$ is a function, then:

- The limit superior of f at x_0 from the left (right) is defined as $\limsup_{x \to x_0^-}(f) := \limsup_{\mathcal{N}_{x_0}^-}(f)$ $\left(\limsup_{x \to x_0^+}(f) := \limsup_{\mathcal{N}_{x_0}^+}(f) \right)$.
- The limit inferior of f at x_0 from the left (right) is defined as $\liminf_{x \to x_0^-}(f) := \liminf_{\mathcal{N}_{x_0}^-}(f)$ $\left(\liminf_{x \to x_0^+}(f) := \liminf_{\mathcal{N}_{x_0}^+}(f) \right)$.

2.2 Limits of nets and sequences

Limits of sets and sequences will be referred to limits of functions, therefore, everything that works for limits of functions also does for limits of nets and sequences.

2.2.1 Limits of nets

Recall that a net is simply a map $\gamma : D \to X$, where D is a directed set and X is a nonempty set.

2.2.1.1 Equivalence of limits

Recall that, if D is a directed set, then the subset of upward closed rays $\mathcal{B}_D := \{\uparrow d : d \in D\}$ is a filter base in D.

Definition 191 *Let D be a directed set. Let X be a topological space. Consider a net $\gamma : D \to X$. We define the following sets:*

- *The limit of γ is defined as* $\lim(\gamma) := \lim_{\mathcal{B}_D}(\gamma)$.

- *The agglomeration of γ is defined as* $\operatorname{agl}(\gamma) := \operatorname{agl}_{\mathcal{B}_D}(\gamma)$.

Under the settings of the previous definition,

$$\begin{aligned}\lim(\gamma) &= \{x \in X : \forall U \in \mathcal{N}_x\ \exists d \in D\ \gamma(\uparrow d) \subseteq U\} \\ &= \{x \in X : \forall U \in \mathcal{N}_x\ \exists d \in D\ \forall e \geq d\ \gamma(e) \in U\}, \\ \operatorname{agl}(\gamma) &= \{x \in X : \forall U \in \mathcal{N}_x\ \forall d \in D\ \gamma(\uparrow d) \cap U \neq \varnothing\} \\ &= \{x \in X : \forall U \in \mathcal{N}_x\ \forall d \in D\ \exists e \geq d\ \gamma(e) \in U\}.\end{aligned}$$

If X is regular, by Lemma 78, $\lim(\gamma) = \bigcap \mathcal{N}_x$ for all $x \in \lim(\gamma)$. According to Remark 14, if X is a topological space and $\mathcal{B}$ is a filter base of $\mathcal{P}(X)$, then $D_{\mathcal{B}} := \{(x, A) \in X \times \mathcal{B} : x \in A\}$ is a directed set endowed with the partial order given by $(x, A) \leq (y, C) \Leftrightarrow C \subseteq A$. Thus,

$$\begin{array}{rccl}\gamma_{\mathcal{B}}: & D_{\mathcal{B}} & \to & X \\ & (x, A) & \mapsto & \gamma_{\mathcal{B}}(x, A) := x\end{array}$$

is a net in X.

Proposition 67 *Let X be a topological space. Let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. Then:*

1. $\lim(\mathcal{B}) = \lim(\gamma_{\mathcal{B}})$.
2. $\operatorname{agl}(\mathcal{B}) = \operatorname{agl}(\gamma_{\mathcal{B}})$.

Proof 198 *The proof follows as itemized as the statement is.*

1. *Take $x \in \lim(\mathcal{B})$ and fix an arbitrary $U \in \mathcal{N}_X$. There exists $A \in \mathcal{B}$ such that $A \subseteq U$. Consider any $a \in A$. Then $\gamma_{\mathcal{B}}(y, C) = y \in C \subseteq A \subseteq U$ for all $(y, C) \geq (a, A)$. This shows that $x \in \lim(\gamma_{\mathcal{B}})$. Hence, $\lim(\mathcal{B}) \subseteq \lim(\gamma_{\mathcal{B}})$. Conversely, take $x \in \lim(\gamma_{\mathcal{B}})$ and fix an arbitrary $U \in \mathcal{N}_X$. There exists $(a, A) \in D_{\mathcal{B}}$ such that if $(y, C) \geq (a, A)$, then $\gamma_{\mathcal{B}}(y, C) \in U$. Observe that $(b, A) \geq (a, A)$ for all $b \in A$, therefore, $b = \gamma_{\mathcal{B}}(b, A) \in U$, which implies that $U \subseteq A$. As a consequence, $x \in \lim(\mathcal{B})$. Hence, $\lim(\mathcal{B}) \supseteq \lim(\gamma_{\mathcal{B}})$.*
2. *Take $x \in \operatorname{agl}(\mathcal{B})$ and fix an arbitrary $U \in \mathcal{N}_X$. For every $A \in \mathcal{B}$, $A \cap U \neq \varnothing$. Consider any $(a, A) \in D_{\mathcal{B}}$. Since $A \cap U \neq \varnothing$, we can take $b \in A \cap U$. Note that $(b, A) \geq (a, A)$ and $\gamma_{\mathcal{B}}(b, A) = b \in U$. This shows that $x \in \operatorname{agl}(\gamma_{\mathcal{B}})$. Hence, $\operatorname{agl}(\mathcal{B}) \subseteq \operatorname{agl}(\gamma_{\mathcal{B}})$. Conversely, take $x \in \operatorname{agl}(\gamma_{\mathcal{B}})$ and fix an arbitrary $U \in \mathcal{N}_X$. For every $(a, A) \in D_{\mathcal{B}}$, there exists $(y, C) \geq (a, A)$ with $\gamma_{\mathcal{B}}(y, C) \in U$. Consider any $A \in \mathcal{B}$. Take any $a \in A$. There exists $(y, C) \geq (a, A)$ with $y = \gamma_{\mathcal{B}}(y, C) \in U$. In other words, $U \cap A \supseteq U \cap C \neq \varnothing$. As a consequence, $x \in \operatorname{agl}(\mathcal{B})$. Hence, $\operatorname{agl}(\mathcal{B}) \supseteq \operatorname{agl}(\gamma_{\mathcal{B}})$.*

As a consequence of the previous result, compactness can also be characterized in terms of agglomerations of nets.

Theorem 46 *A topological space X is compact if and only if $\operatorname{agl}(\gamma) \neq \varnothing$ for every net $\gamma : D \to X$.*

Proof 199 *The proof relies on Theorem 45. Indeed, assume first that X is compact. If $\gamma : D \to X$ is a net, then $\operatorname{agl}(\gamma) = \operatorname{agl}(\mathcal{B}_D) \neq \varnothing$. Conversely, let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. By hypothesis, $\operatorname{agl}(\gamma_{\mathcal{B}}) \neq \varnothing$. However, by bearing in mind Proposition 67(2), we conclude that $\operatorname{agl}(\mathcal{B}) = \operatorname{agl}(\gamma_{\mathcal{B}}) \neq \varnothing$.*

Next technical result versions Lemma 81 for nets and will be relied on in the further chapters and sections.

Lemma 82 *Let $(X_\alpha)_{\alpha\in A}$ be a family of topological spaces. For every net $\gamma : D \to \prod_{\alpha\in A} X_\alpha$,*

$$\lim(\gamma) = \left\{ x \in \prod_{\alpha\in A} X_\alpha : \forall \alpha \in A \; x_\alpha \in \lim\left(\pi_\alpha \circ \gamma\right) \right\}.$$

Proof 200 *This proof is going to be itemized according to the statement of the lemma.*

- $\subseteq$ *Fix an arbitrary $x \in \lim(\gamma)$. Fix also an arbitrary $\alpha_0 \in A$. Take $U \in \mathcal{N}_{x_{\alpha_0}}(X_{\alpha_0})$. Consider $\prod_{\alpha\in A} V_\alpha$, where*

$$V_\alpha = \begin{cases} X_\alpha & \alpha \neq \alpha_0, \\ U & \alpha = \alpha_0, \end{cases}$$

  for every $\alpha \in A$. Notice that $\prod_{\alpha\in A} V_\alpha \in \mathcal{N}_x\left(\prod_{\alpha\in A} X_\alpha\right)$. Therefore, there exists $d_0 \in D$ such that if $d \in D$ and $d \geq d_0$, then $\gamma(d) \in \prod_{\alpha\in A} V_\alpha$. As a consequence, $\pi_{\alpha_0}(\gamma(d)) \in U$ for every $d \in D$ with $d \geq d_0$. This shows that $x_{\alpha_0} \in \lim\left(\pi_{\alpha_0} \circ \gamma\right)$.

- $\supseteq$ *Fix an arbitrary $x \in \prod_{\alpha\in A} X_\alpha$ such that $x_\alpha \in \lim\left(\pi_\alpha \circ \gamma\right)$ for each $\alpha \in A$. Take $U \in \mathcal{N}_x\left(\prod_{\alpha\in A} X_\alpha\right)$. There exists a finite number of open sets $U_{\alpha_1}, \ldots, U_{\alpha_p}$ of $X_{\alpha_1}, \ldots, X_{\alpha_p}$, respectively, such that $x \in \prod_{\alpha\in A} V_\alpha \subseteq U$, where*

$$V_\alpha = \begin{cases} X_\alpha & \alpha \notin \{\alpha_1, \ldots, \alpha_p\}, \\ U_{\alpha_q} & \alpha = \alpha_q \text{ for some } q \in \{1, \ldots, p\}. \end{cases}$$

  For every $q \in \{1, \ldots, p\}$, there exists $d_q \in D$ such that if $d \in D$ and $d \geq d_q$, then $\pi_{\alpha_q}(\gamma(d)) \in U_{\alpha_q}$. If we take $d_0 \in D$ with $d_0 \geq d_q$ for all $q = 1, \ldots, p$, then every $d \in D$ with $d \geq d_0$ satisfies that $\gamma(d) \in \prod_{\alpha\in A} V_\alpha \subseteq U$. This shows that $x \in \lim\left(\gamma\right)$.

Likewise compactness is characterized through agglomerations of nets, topological closure is characterized via limits of nets.

Theorem 47 *Let X be a topological space. Consider $A \subseteq X$ nonempty. Then*

$$\mathrm{cl}(A) = \bigcup \{ \lim(\gamma) : \gamma : D \to A \text{ is a net } \}.$$

Proof 201 *The proof is itemized by following the statement.*

- $\subseteq$ *Take $x \in \mathrm{cl}(A)$. Fix an element $a_U \in U \cap A$ for every $U \in \mathcal{N}_x(X)$. The Axiom of Choice allows us to consider the net*

$$\begin{array}{rccc} \gamma : & \mathcal{N}_x(X) & \to & A \\ & U & \mapsto & a_U. \end{array}$$

  It is obvious that γ converges to x.

- $\supseteq$ *Let D be a directed set and $\gamma : D \to A$ be a net. Take $x \in \lim(\gamma)$. If $U \in \mathcal{N}_x(X)$, then there exists $d_0 \in D$ with $\gamma\left(\uparrow d_0\right) \subseteq U$. Since $\gamma\left(\uparrow d_0\right) \subseteq A$, we conclude that $A \cap U \neq \varnothing$.*

Characterizing topological closure through convergent nets has important consequences, such as, for instance, the next technical lemma.

Lemma 83 *Let X be a Hausdorff topological space. Let $T \subseteq X$. If $x, y \in \mathrm{cl}(T)$ with $x \neq y$, then there exists a subset $S \subseteq T$ such that $x \in \mathrm{cl}(S)$ and $y \notin \mathrm{cl}(S)$.*

Proof 202 *Since X is Hausdorff, we can take U, V neighborhoods of x, y, respectively, such that $U \cap V = \varnothing$. Notice that $I := \{W \in \mathcal{N}_x : W \subseteq U\}$ is a directed set when endowed with the partial order $W_1 \leq W_2 \iff W_2 \subseteq W_1$. Since $x \in \mathrm{cl}(T)$, for every $W \in I$ there exists $t_W \in T \cap W$. It is clear that the net $(t_W)_{W \in I}$ converges to x. However, by construction, $y \notin \mathrm{cl}(\{t_W : W \in I\})$. Finally, simply take $S := \{t_W : W \in I\}$.*

Limits of functions can be characterized through limits of nets. This is very useful when it comes to work with continuous functions.

Proposition 68 *Let X, Y be topological spaces. Let x_0 be a limit point of X. Consider a function $f : X \setminus \{x_0\} \to Y$. Then*

$$\lim_{x \to x_0}(f) = \bigcap \{\lim(f \circ \gamma) : \gamma : D \to X \text{ is a net and } x_0 \in \lim(\gamma) \setminus \gamma(D)\}.$$

Proof 203 *The proof is structured by following the statement.*

- $\subseteq$ *Let $y \in \lim\limits_{x \to x_0}(f)$ and $\gamma : D \to X$ a net converging to x_0 with $x_0 \notin \gamma(D)$. Fix an arbitrary $V \in \mathcal{N}_y$. There exist $U \in \mathcal{N}_{x_0}^{\times}$ such that $f(U) \subseteq V$ and $d \in D$ with $\gamma(\uparrow d) \subseteq U$. Thus, $f(\gamma(\uparrow d)) \subseteq f(U) \subseteq V$. As a consequence, $y \in \lim(f \circ \gamma)$.*

- $\supseteq$ *Let $y \in \lim(f \circ \gamma)$ for every net $\gamma : D \to X$ converging to x_0 with $x_0 \notin \gamma(D)$. Suppose to the contrary that $y \notin \lim\limits_{x \to x_0}(f)$. There exists $V \in \mathcal{N}_y$ such that $f(U) \nsubseteq V$ for every $U \in \mathcal{N}_{x_0}^{\times}$. Fix an element $x_U \in U \setminus f^{-1}(V)$ for every $U \in \mathcal{N}_{x_0}^{\times}$. The Axiom of Choice allows to consider the net*

$$\begin{array}{rccc} \gamma : & \mathcal{N}_{x_0}^{\times} & \to & X \\ & U & \mapsto & x_U. \end{array}$$

It is obvious that γ converges to x_0 and x_0 is not in the range of γ but $y \notin \lim(f \circ \gamma)$.

The following technical lemma has plenty of applications in Approximation Theory and it is the precursor of the famous Squeeze Theorem.

Lemma 84 *Let X be a topological toset. Let $\gamma, \lambda : D \to X$ be two nets in X. Then:*

1. *If γ is increasing (decreasing), then $x \in \lim(\gamma) \Leftrightarrow x = \sup(\gamma(D))$ $(x = \inf(\gamma(D)))$.*
2. *If X is holefree, $\gamma \leq \lambda$, and both nets are convergent, then $\lim(\gamma) \leq \lim(\lambda)$.*

Proof 204 *First off, keep in mind that both $\lim(\gamma), \lim(\lambda)$ are either empty or singletons in view of the fact that X is Hausdorff (Proposition 55).*

1. *Assume that $x \in \lim(\gamma)$. We will show first that x is an upper bound for the net. If not, there exists $d_1 \in D$ with $x < \gamma(d_1)$. Then $\downarrow^{\times}\gamma(d_1)$ is a neighborhood of x, meaning that there exists $d_0 \in \mathbb{N}$ such that $\gamma(d) \in \downarrow^{\times}\gamma(d_1)$ for all $d \geq d_0$, contradicting that γ be increasing because we can choose $d_2 \in D$ with $d_2 \geq d_0, d_1$. Next, take any upper bound y for the net. If $y < x$, then $\uparrow_{\times} y$ is a neighborhood of x not containing any element of the net. As a consequence, $x = \sup(\gamma(D))$. Conversely, assume that $x = \sup(\gamma(D))$. Any ray of the form $\downarrow^{\times} z$ containing x also contains the whole sequence. If a ray of the form $\uparrow_{\times} y$, or an interval of the form (y, z), contains x, then there are elements of the sequence in (y, x) because $x = \sup(\gamma(D))$ and X is totally ordered. As a consequence, the net converges to x.*

2. *Suppose on the contrary that $\lim(\gamma) > \lim(\lambda)$. Since X is free of holes, there exists $z \in X$ with $\lim(\gamma) > z > \lim(\lambda)$. Then $\uparrow_{\times} z, \downarrow^{\times} z$ are open neighborhoods of $\lim(\gamma), \lim(\lambda)$, respectively, meaning that there are $d_1, d_2 \in D$ such that if $d \geq d_1$, then $\gamma(d) > z$, and if $d \geq d_2$, then $\lambda(d) < z$. Finally, choose $d_0 \in D$ with $d_0 \geq d_1, d_2$, reaching the contradiction that $\gamma(d_0) \leq \lambda(d_0) < z < \gamma(d_0)$.*

2.2.1.2 Limits superior/inferior

Limits superior and inferior make sense as well for nets.

Definition 192 *Let D be a directed set. Let X be a topological poset. Consider a net $\gamma : D \to X$. Then:*

- *The limit superior of γ is defined as* $\limsup(\gamma) := \limsup(\gamma(\mathcal{B}_D))$.
- *The limit inferior of γ is defined as* $\liminf(\gamma) := \liminf(\gamma(\mathcal{B}_D))$.

Under the settings of the previous definition, $\liminf(\gamma) \leq \limsup(\gamma)$ if they exist.

Proposition 69 *Let D be a directed set. Let X be a topological toset satisfying the sup property. Consider a net $\gamma : D \to X$. Let $x \in X$. Let $d_0 \in D$. Then:*

1. *If $x \leq \gamma(e)$ for all $e \in D$ with $e \geq d_0$, then $x \leq \liminf(\gamma)$.*
2. *If $x \geq \gamma(e)$ for all $e \in D$ with $e \geq d_0$, then $\limsup(\gamma) \leq x$.*

Proof 205 *We will only prove the first item since the second one follows a similar proof. Notice that*

$$\liminf(\gamma) := \liminf(\gamma(\mathcal{B}_D)) = \inf\left(\operatorname{agl}(\gamma(\mathcal{B}_D))\right) = \inf \bigcap_{d \in D} \operatorname{cl}\left(\{\gamma(e) : e \geq d\}\right).$$

We will prove that x is a lower bound for $\bigcap_{d\in D} \operatorname{cl}(\{\gamma(e) : e \geq d\})$. Indeed, suppose that there exists $z \in \bigcap_{d\in D} \operatorname{cl}(\{\gamma(e) : e \geq d\})$ such that $z < x$. We can consider the open neighborhood of z given by $(-\infty, x)$. Since $z \in \operatorname{cl}(\{\gamma(e) : e \geq d_0\})$, we can find $e \geq d_0$ such that $\gamma(e) \in (-\infty, x)$. However, this contradicts the hypothesis on x. As a consequence, x is a lower bound for $\bigcap_{d\in D} \operatorname{cl}(\{\gamma(e) : e \geq d\})$. Finally, the fact that X satisfies the sup property implies that $\inf \bigcap_{d\in D} \operatorname{cl}(\{\gamma(e) : e \geq d\})$ exists and $x \leq \inf \bigcap_{d\in D} \operatorname{cl}(\{\gamma(e) : e \geq d\})$.

2.2.1.3 Limits of subnets

Given a net $\gamma : D \to X$, another net $\lambda : E \to Y$ is called a subnet of γ provided that, for every $d \in D$, there exists $e \in E$ with $\lambda(\uparrow e) \subseteq \gamma(\uparrow d)$.

Remark 54 *Let D, E be directed sets. Let X a topological space. Consider a net $\gamma : D \to X$. Consider a subnet $\lambda : E \to X$. Then $\gamma(\mathcal{B}_D) \subseteq \mathcal{J}(\lambda(\mathcal{B}_E))$, so $\mathcal{J}(\gamma(\mathcal{B}_D)) \subseteq \mathcal{J}(\lambda(\mathcal{B}_E))$.*

The previous remark has a converse in some sense.

Remark 55 *If X is a topological space and $\mathcal{B}, \mathcal{C}$ are filter bases of $\mathcal{P}(X)$ such that $\mathcal{B} \subseteq \mathcal{C}$, then $D_{\mathcal{B}} \subseteq D_{\mathcal{C}}$ and thus $\gamma_{\mathcal{C}}$ is a subnet of $\gamma_{\mathcal{B}}$. Indeed, take any $(a, A) \in D_{\mathcal{B}}$. Note that $(a, A) \in D_{\mathcal{C}}$ because $A \in \mathcal{B} \subseteq \mathcal{C}$. If $(y, F) \in D_{\mathcal{C}}$ is so that $(y, F) \geq (a, A)$, then $y \in F \subseteq A$, so $(y, A) \in D_{\mathcal{B}}$ and $(y, A) \geq (a, A)$. Finally, $\gamma_{\mathcal{C}}(y, F) = y = \gamma_{\mathcal{B}}(y, A)$. This shows that $\gamma_{\mathcal{C}}$ is a subnet of $\gamma_{\mathcal{B}}$.*

We recall to the reader that for every net $\gamma : D \to X$ and for every $d \in D$, $\uparrow d$ is cofinal in D and $\gamma|_{\uparrow d}$ is a subnet of γ.

Lemma 85 *Let D, E be directed sets. Let X be a topological space. Consider a net $\gamma : D \to X$ a net. Consider a subnet $\lambda : E \to X$. Let $d_0 \in D$. Then:*

1. $\lim(\gamma) \subseteq \lim(\lambda)$ *and* $\operatorname{agl}(\lambda) \subseteq \operatorname{agl}(\gamma)$.

2. $\lim(\gamma) = \lim(\gamma|_{\uparrow d_0})$ *and* $\operatorname{agl}(\gamma) = \operatorname{agl}(\gamma|_{\uparrow d_0})$.

Proof 206 *The proof will be itemized in accordance with the statement.*

1. *Take* $x \in \lim(\gamma)$ *and* $U \in \mathcal{N}_x$. *There exists* $d \in D$ *such that* $\gamma(\uparrow d) \subseteq U$. *Since* $\lambda : E \to X$ *is a subnet of* $\gamma : D \to X$, *there exists* $e \in E$ *with* $\lambda(\uparrow e) \subseteq \gamma(\uparrow d) \subseteq U$, *meaning that* $x \in \lim(\lambda)$. *Next, take* $x \in \operatorname{agl}(\lambda)$ *and* $U \in \mathcal{N}_x$. *For every* $d \in D$ *there exists* $e \in E$ *such that* $\lambda(\uparrow e) \subseteq \gamma(\uparrow d)$. *Since* $\lambda(\uparrow e) \cap U \neq \varnothing$, *we conclude that* $\gamma(\uparrow d) \cap U \neq \varnothing$, *meaning that* $x \in \operatorname{agl}(\gamma)$.

2. *By Remark 54,* $\lim(\gamma) \subseteq \lim(\gamma|_{\uparrow d_0})$. *Fix an arbitrary* $x \in \lim(\gamma|_{\uparrow d_0})$ *and take any* $U \in \mathcal{N}_x$. *There exists* $d' \in \uparrow d_0$ *such that* $\gamma(\uparrow d') \subseteq U$, *meaning that* $x \in \lim(\gamma)$. *Again, by Remark 54,* $\operatorname{agl}(\gamma) \supseteq \operatorname{agl}(\gamma|_{\uparrow d_0})$. *Fix an arbitrary* $x \in \operatorname{agl}(\gamma)$ *and take any* $U \in \mathcal{N}_x$. *For every* $d' \in \uparrow d_0$, $\gamma(\uparrow d') \cap U \neq \varnothing$, *meaning that* $x \in \operatorname{agl}(\gamma|_{\uparrow d_0})$.

The following technical lemma has a very extended use in Approximation Theory.

Lemma 86 *Let* D *be a directed set. Let* X *be a topological space. Consider a net* $\gamma : D \to X$. *Let* $x \in X$. *Then:*

1. $X \setminus \lim(\gamma) = \{x \in X : \exists U \in \mathcal{N}_x \ \exists \lambda : E \to X \setminus U \text{ subnet of } \gamma\}$.

2. *If every subnet* $\lambda : E \to X$ *of* γ *has a further subnet* $\alpha : F \to X$ *such that* $x \in \lim(\alpha)$, *then* $x \in \lim(\gamma)$.

Proof 207 *We will itemize the proof like the statement is.*

1. *Assume that* $x \in X \setminus \lim(\gamma)$. *There exists* $U \in \mathcal{N}_x$ *such that, for every* $d \in D$, *there exists* $e \in \uparrow d$ *verifying that* $\gamma(e) \notin U$. *Take*

$$E := \bigcup_{d \in D} \left(\gamma^{-1}(X \setminus U) \cap \uparrow d\right).$$

   Observe that E *is clearly cofinal in* D, *so* E *is directed and* $\gamma|_E$ *is a subnet of* γ *in view of Corollary 6. Finally, notice that* $\gamma(E) \subseteq X \setminus U$. *Conversely, if* $x \in X$ *is so that there exist* $U \in \mathcal{N}_x$ *and a subnet* $\lambda : E \to X$ *o* γ *with* $\lambda(E) \cap U = \varnothing$, *then* $x \notin \lim(\lambda)$, *so* $x \notin \lim(\gamma)$ *by bearing in mind Lemma 85(1).*

2. *Suppose to the contrary that* $x \notin \lim(\gamma)$. *According to Lemma 86, there exists a subnet* $\lambda : E \to X$ *of* γ *such that* $\lambda(E) \cap U = \varnothing$. *By hypothesis,* λ *has a further subnet* $\alpha : F \to X$ *such that* $x \in \lim(\alpha)$. *Then we can find* $f_0 \in F$ *such that* $\alpha(\uparrow f_0) \subseteq U$. *Fix any* $e \in E$. *Since* α *is a subnet of* λ, *there exists* $f_1 \in F$ *such that* $\alpha(\uparrow f_1) \subseteq \lambda(\uparrow e)$. *Since* F *is directed, we can find* $f_2 \in F$ *with* $f_2 \geq f_1$ *and* $f_2 \geq f_0$. *Then* $\alpha(f_2) \in \alpha(\uparrow f_0) \subseteq U$ *and* $\alpha(f_2) \in \alpha(\uparrow f_1) \subseteq \lambda(\uparrow e)$. *This way, we reach the contradiction that* $\lambda(E) \cap U \neq \varnothing$.

The final theorem of this subsubsection relates limits and agglomerations of nets with those of their subnets, respectively.

Theorem 48 *Let* D *be a directed set. Let* X *be a topological space. Consider a net* $\gamma : D \to X$. *Then*

$$\begin{aligned} \lim(\gamma) &= \bigcap \{\lim(\lambda) : \lambda : E \to X \text{ is a subnet of } \gamma\}, \\ \operatorname{agl}(\gamma) &= \bigcup \{\lim(\lambda) : \lambda : E \to X \text{ is a subnet of } \gamma\}. \end{aligned}$$

Proof 208 *By combining together Lemma 85(1) with Lemma 86, we directly conclude that* $\lim(\gamma) = \bigcap \{\lim(\lambda) : \lambda : E \to X \text{ is a subnet of } \gamma\}$. *Thus, we will show that* $\operatorname{agl}(\gamma) = \bigcup \{\lim(\lambda) : \lambda : E \to X \text{ is a subnet of } \gamma\}$.

- $\subseteq$ *Take* $x \in \operatorname{agl}(\gamma) = \bigcap \{\operatorname{cl}(\gamma(\uparrow d)) : d \in D\}$. *Consider the set* $E := \{(d, U) \in D \times \mathcal{N}_x(X) : \gamma(d) \in U\}$ *endowed with the preorder* $(d, U) \leq (e, V) \Leftrightarrow d \leq e$ *and* $V \subseteq U$. *Notice that* E *is directed. Consider the net*

$$\begin{array}{rccl} \lambda : & E & \to & X \\ & (d, U) & \mapsto & \lambda(d, U) := \gamma(d). \end{array}$$

  Fix any $U \in \mathcal{N}_x(X)$. *For every* $d \in D$, *there exists* $d' \in \uparrow d$ *such that* $\gamma(d') \in U$. *Therefore,* $(d', U) \in E$ *and* $\lambda(\uparrow(d', U)) \subseteq \gamma(\uparrow d)$. *Thus,* λ *is a subnet of* γ. *Finally, for every* $U \in \mathcal{N}_x(X)$, $\lambda(\uparrow(d, U)) \subseteq U$, *which implies that* λ *converges to* x.

- $\supseteq$ *In accordance with Proposition 63(1) and Lemma 85(2), if* $\lambda : E \to X$ *is a subnet of* γ, *then* $\lim(\lambda) \subseteq \operatorname{agl}(\lambda) \subseteq \operatorname{agl}(\gamma)$.

2.2.1.4 Cauchy nets and completeness

Cauchy nets only make sense to be defined in uniform spaces. Actually, the most natural example of a Cauchy prefilter is a Cauchy net.

Definition 193 (Cauchy net) *Let* X *be a uniform space. A net* $(x_\lambda)_{\lambda \in \Lambda} \subseteq X$ *is said to be a Cauchy net provided that the prefilter in* X *given by* $\mathcal{B}_{(x_\lambda)_{\lambda \in \Lambda}} := \{\{x_\lambda : \lambda \geq \gamma\} : \gamma \in \Lambda\}$ *is a Cauchy prefilter.*

In view of Remark 54, every subnet of a Cauchy net is Cauchy as well. The following result displays the classical characterization of a Cauchy net.

Proposition 70 *Let* X *be a uniform space. A net* $(x_\lambda)_{\lambda \in \Lambda} \subseteq X$ *is Cauchy if and only if for every entourage* $U \subseteq X \times X$ *there exists* $\lambda_U \in \Lambda$ *such that* $(x_\lambda, x_\gamma) \in U$ *whenever* $\lambda, \gamma \in \Lambda$ *with* $\lambda, \gamma \geq \lambda_U$.

Proof 209 *Let us itemize the proof the same way the statement is.*

- $\Rightarrow$ *Fix an arbitrary entourage* U. *By assumption, there exists* $\lambda_U \in \Lambda$ *satisfying that* $\{x_\lambda : \lambda \geq \lambda_U\} \times \{x_\lambda : \lambda \geq \lambda_U\} \subseteq U$, *meaning that* $(x_\lambda, x_\gamma) \in U$ *whenever* $\lambda, \gamma \in \Lambda$ *with* $\lambda, \gamma \geq \lambda_U$.

- $\Leftarrow$ *Fix an arbitrary entourage* U. *By assumption, there exists* $\lambda_U \in \Lambda$ *such that* $(x_\lambda, x_\gamma) \in U$ *whenever* $\lambda, \gamma \in \Lambda$ *with* $\lambda, \gamma \geq \lambda_U$. *Then* $\{x_\lambda : \lambda \geq \lambda_U\} \times \{x_\lambda : \lambda \geq \lambda_U\} \subseteq U$, *meaning that* $\mathcal{B}_{(x_\lambda)_{\lambda \in \Lambda}} := \{\{x_\lambda : \lambda \geq \gamma\} : \gamma \in \Lambda\}$ *is a Cauchy prefilter.*

As a direct consequence of Lemma 80 comes the next remark.

Remark 56 *Let* X *be a uniform space. A net* $(x_\lambda)_{\lambda \in \Lambda} \subseteq X$ *is convergent in* X *if and only if* $(x_\lambda)_{\lambda \in \Lambda}$ *is Cauchy and* $\operatorname{agl}((x_\lambda)_{\lambda \in \Lambda}) \neq \varnothing$. *In this situation,* $\lim((x_\lambda)_{\lambda \in \Lambda}) = \operatorname{agl}((x_\lambda)_{\lambda \in \Lambda})$.

Cauchy nets characterize completeness of uniform spaces as well as Cauchy prefilters do.

Proposition 71 *A uniform space* X *is complete if and only if every Cauchy net in* X *is convergent.*

Proof 210 *The proof is itemized in accordance with the statement.*

$\Rightarrow$ *If* $(x_\lambda)_{\lambda\in\Lambda}$ *is a Cauchy net, then* $\mathcal{B}_{(x_\lambda)_{\lambda\in\Lambda}} := \{\{x_\lambda : \lambda \geq \gamma\} : \gamma \in \Lambda\}$ *is a Cauchy prefilter, which is covergent, and so is the net because* $\lim\left((x_\lambda)_{\lambda\in\Lambda}\right) = \lim\left(\mathcal{B}_{(x_\lambda)_{\lambda\in\Lambda}}\right)$ *by definition of limit of a net.*

$\Leftarrow$ *Fix an arbitrary Cauchy filter base* $\mathcal{B}$ *in* X*. It induces the following Cauchy net: Let* $\Lambda := \{(x, B) \in X \times \mathcal{B} : x \in B\}$ *endowed with the directed preorder given by* $(x, B) \leq (y, C) \Leftrightarrow C \subseteq B$ *and consider the net*

$$\begin{array}{rccl} \gamma : & \Lambda & \to & X \\ & (x, B) & \mapsto & \gamma(x, B) := x. \end{array} \tag{2.1}$$

For every entourage U *of* X*, there exists* $B \in \mathcal{B}$ *with* $B \times B \subseteq U$*. Now, fix any* $x \in B$*. If* $(x_1, B_1), (x_2, B_2) \in \Lambda$ *and* $(x_1, B_1), (x_2, B_2) \geq (x, B)$*, then* $B_1, B_2 \subseteq B$*, meaning that* $(\gamma(x_1, B_1), \gamma(x_2, B_2)) = (x_1, x_2) \in B_1 \times B_2 \subseteq B \times B \subseteq U$*. This shows that* γ *is a Cauchy net. By assumption,* γ *is convergent to some* $x_0 \in X$*. We will finally prove that* $\mathcal{B}$ *converges to* x_0*. Indeed, fix an arbitrary entourage* U *of* X*. There exists* $(x_U, B_U) \in \Lambda$ *in such a way that if* $(x, B) \in \Lambda$ *and* $(x, B) \geq (x_U, B_U)$*, then* $\gamma(x, B) \in U[x_0]$*. At this stage, simply observe that* $(x, B_U) \geq (x_U, B_U)$ *for every* $x \in B_U$*, meaning that* $x = \gamma(x, B_U) \in U[x_0]$*, which implies that* $B_U \subseteq U[x_0]$*.*

2.2.2 Limits of sequences

Sequences are particular cases of nets when $\mathbb{N}$ is endowed with its total order. However, $\mathbb{N}$ is also a discrete topological space, which turns $\mathbb{N}$ into a Hausdorff locally compact topological space. Thus, we can consider the one-point compactification of $\mathbb{N}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. This way, sequences can also be seen as functions on a topological space $\mathbb{N}$, where ∞ is a limit point of $\mathbb{N}$.

2.2.2.1 Equivalence of limits

The classical definition of sequence convergence is precisely the analytical definition of the limit of a function when seeing sequences as functions on $\overline{\mathbb{N}}$.

Proposition 72 $\mathcal{N}_\infty^\times\left(\overline{\mathbb{N}}\right) = \mathcal{F}_{\mathbb{N}} = \mathcal{J}\left(\mathcal{B}_{\mathbb{N}}\right)$*. As a consequence, if* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in a topological space* X*, then*

$$\lim_{n\to\infty} x_n = \lim\left((x_n)_{n\in\mathbb{N}}\right) \text{ and } \operatorname*{agl}_{n\to\infty} x_n = \operatorname{agl}\left((x_n)_{n\in\mathbb{N}}\right).$$

Proof 211 *Simply note that* $A \in \mathcal{N}_\infty^\times\left(\overline{\mathbb{N}}\right)$ *if and only if* $\mathbb{N} \setminus A$ *is compact in* $\mathbb{N}$ *if and only if* $\mathbb{N} \setminus A$ *is finite if only if* $A \in \mathcal{J}\left(\mathcal{B}_{\mathbb{N}}\right)$*. This shows that* $\mathcal{N}_\infty^\times\left(\overline{\mathbb{N}}\right) = \mathcal{F}_{\mathbb{N}} = \mathcal{J}\left(\mathcal{B}_{\mathbb{N}}\right)$*. The rest is a direct consequence of Proposition 65.*

Next technical lemma does not work for nets in general, but it does for sequences.

Lemma 87 *Let* X *be a topological poset such that* X *is a lattice. If* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *is convergent, then it is bounded.*

Proof 212 *We may assume that* X *has more than one point. Take* $x \in \lim_{n\to\infty} x_n$*. Note that* x *is not incomparable with all the rest elements of* X *because* X *is a lattice. Then there are three possibilities:*

- $\uparrow_\times x = \varnothing$*. In this case,* $x = \max(X)$ *and* $\downarrow^\times x \neq \varnothing$*. So, there exists* $y \in X \setminus \{x\}$ *with* $y < x$*. Since* $\uparrow_\times y$ *is an open neighborhood of* x*, we can find* $n_0 \in \mathbb{N}$ *such that* $x_n \in \uparrow_\times y$ *for all* $n \geq n_0$*. Since* X *is a lattice, there exist* $y_0, z_0 \in X$ *such that* $y_0 \leq x_n \leq z_0$ *for all* $n \in \{1, \ldots, n_0 - 1\}$*. Finally,* $y_0 \wedge y$ *is a lower bound for* $(x_n)_{n\in\mathbb{N}}$ *and* $z_0 \vee x = x$ *is an upper bound for* $(x_n)_{n\in\mathbb{N}}$*.*

- $\downarrow^{\times} x = \varnothing$. *In this case,* $x = \min(X)$ *and* $\uparrow_{\times} x \neq \varnothing$. *A dual proof applies.*
- $\uparrow_{\times} x \neq \varnothing$ *and* $\downarrow^{\times} x \neq \varnothing$. *Take* $y, z \in X \setminus \{x\}$ *with* $y < x < z$. *Since* (y, z) *is an open neighborhood of* x, *we can find* $n_0 \in \mathbb{N}$ *such that* $x_n \in (y, z)$ *for all* $n \geq n_0$. *Since* X *is a lattice,* $y \wedge x_1 \wedge \cdots \wedge x_{n_0 - 1}$ *is a lower bound for* $(x_n)_{n \in \mathbb{N}}$ *and* $z \vee x_1 \vee \cdots \vee x_{n_0 - 1}$ *is an upper bound for* $(x_n)_{n \in \mathbb{N}}$.

Under the settings of the previous lemma, if X is not a lattice and x is incomparable with all the rest elements of X, then the only neighborhood of x is the whole of X, meaning that any sequence in X converges to x.

Remark 57 *Let* X *be a pseudometric space. If* $(x_n)_{n \in \mathbb{N}} \subseteq X$ *is convergent, then it is bounded.*

2.2.2.2 Limits of subsequences

A subsequence of a sequence $(x_n)_{n \in \mathbb{N}}$ is a sequence of the form $(x_{n_k})_{k \in \mathbb{N}}$, where $(n_k)_{k \in \mathbb{N}}$ is strictly increasing. Subsequences are special cases of subnets. Some of the previous results for subnets, such as Lemma 86 and Theorem 48, can be reformulated in a more refined way for subsequences.

Remark 58 *Let* X *be a topological space. Consider a sequence* $(x_n)_{n \in \mathbb{N}} \subseteq X$. *Let* $x \in X$. *Then:*

- $X \setminus \lim\limits_{n \to \infty} x_n = \left\{ x \in X : \exists U \in \mathcal{N}_x \; \exists \left(x_{n_k}\right)_{k \in \mathbb{N}} \subseteq X \setminus U \right\}$.
- *If every subsequence* $(x_{n_k})_{k \in \mathbb{N}}$ *has a further subsequence* $\left(x_{n_{k_l}}\right)_{l \in \mathbb{N}}$ *such that* $x \in \lim\limits_{l \to \infty} x_{n_{k_l}}$, *then* $x \in \lim\limits_{n \to \infty} x_n$.
- $\lim\limits_{n \to \infty} x_n = \bigcap \left\{ \lim\limits_{k \to \infty} x_{n_k} : (x_{n_k})_{k \in \mathbb{N}} \text{ is a subsequence of } (x_n)_{n \in \mathbb{N}} \right\}$.
- $\operatorname{agl}\limits_{n \to \infty} x_n = \bigcup \left\{ \lim\limits_{k \to \infty} x_{n_k} : (x_{n_k})_{k \in \mathbb{N}} \text{ is a subsequence of } (x_n)_{n \in \mathbb{N}} \right\}$.

2.2.2.3 Cauchy sequences and completeness

Cauchy sequences are typically considered when a metric structure comes into play.

Example 41 (Cauchy sequence) *Let* X *be a pseudometric space. A sequence* $(x_n)_{n \in \mathbb{N}}$ *in* X *is a Cauchy sequence for the pseudometric uniformity provided that for every* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $d(x_p, x_q) < \varepsilon$ *for all* $p, q \in \mathbb{N}$ *with* $p, q \geq n_0$.

Remark 56 can also be reformulated in a more refined manner for Cauchy sequences.

Remark 59 *Let* X *be a pseudometric space. A sequence* $(x_n)_{n \in \mathbb{N}} \subseteq X$ *is convergent in* X *if and only if* $(x_n)_{n \in \mathbb{N}}$ *is Cauchy and* $\operatorname{agl}\left((x_n)_{n \in \mathbb{N}}\right) \neq \varnothing$. *In this situation,* $\lim\left((x_n)_{n \in \mathbb{N}}\right) = \operatorname{agl}\left((x_n)_{n \in \mathbb{N}}\right)$.

Next technical lemma is crucial towards constructing the completion of a metric space.

Lemma 88 *Let* X *be a pseudometric space. Let* $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subseteq X$. *Then:*

1. *If* $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ *are Cauchy, then* $(d(x_n, y_n))_{n \in \mathbb{N}}$ *is convergent.*

2. If $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are convergent to $x, y \in X$, respectively, then

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

Proof 213 *We will only prove the second item, since the first follows a similar proof. So, assume that $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are convergent to $x, y \in X$, respectively. It only suffices to take into account that, for all $n \in \mathbb{N}$,*

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y) + d(y, y_n), \\ d(x, y) &\leq d(x, x_n) + d(x_n, y_n) + d(y_n, y), \end{aligned}$$

which means that $|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y, y_n)$ for all $n \in \mathbb{N}$.

A pseudometric space is complete if so is with the pseudometric uniformity. The following definition is motivated by Proposition 71.

Definition 194 (Sequential completeness) *A uniform space X is said to be sequentially complete provided that all of its Cauchy sequences are convergent in X.*

Every complete uniform space is sequentially complete. However, the converse does not hold in general. The pseudometric character serves to assure that sequentially complete pseudometric spaces are complete. In fact, whenever a uniformity has a countable base of entourages, completeness is equivalent to sequential completeness.

Theorem 49 *Let X be a uniform space. Suppose that X a countable base of entourages. If X is sequentially complete, then it is complete.*

Proof 214 *Let $\gamma : D \to X$ be a Cauchy net. Consider a nested countable base of symmetric entourages $(V_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, there exists $d_n \in D$ such that $\gamma(\uparrow d_n) \times \gamma(\uparrow d_n) \subseteq V_n$. Since D is upward directed, we may assume that $(d_n)_{n\in\mathbb{N}}$ is increasing. Consider the Cauchy sequence $(\gamma(d_n))_{n\in\mathbb{N}}$. We cannot guarantee that $(\gamma(d_n))_{n\in\mathbb{N}}$ be a subnet of γ. However, we know there exists $x \in \lim_{n\to\infty} \gamma(d_n)$. We will show that $x \in \operatorname{agl}(\gamma)$. Indeed, fix an arbitrary $d \in D$. Let U be any symmetric entourage. Take W another symmetric entourage such that $W \circ W \subseteq U$. Consider $n \in \mathbb{N}$ with $V_n \subseteq W$. There exists $k_n \geq n$ in such a way that $\gamma(d_k) \in V_n[x]$ for $k \geq k_n$. Let $e \in D$ be such that $e \geq d$ and $e \geq d_{k_n}$. We will prove that $\gamma(e) \in U[x] \cap \gamma(\uparrow d)$. Indeed, on the one hand, $e \geq d$, so $\gamma(e) \in \gamma(\uparrow d)$. On the other hand, $e \geq d_{k_n} \geq d_n$, so $\gamma(e) \in \gamma(\uparrow d_n)$. This way $(\gamma(e), \gamma(d_{k_n})) \in \gamma(\uparrow d_n) \times \gamma(\uparrow d_n) \subseteq V_n \subseteq W$ and $(\gamma(d_{k_n}), x) \in V_n \subseteq W$, meaning that $(\gamma(e), x) \in W \circ W \subseteq U$. As a consequence, $\gamma(e) \in U[x] \cap \gamma(\uparrow d)$. This means that $x \in \operatorname{cl}(\gamma(\uparrow d))$. The arbitrariness of $d \in D$ allows that $x \in \bigcap_{d\in D} \operatorname{cl}(\gamma(\uparrow d)) = \operatorname{agl}(\gamma)$. Finally, we just need to call on Remark 56 to conclude that γ is convergent.*

The following two results are considered fundamental theorems of Metric Space Theory.

Theorem 50 (Cantor Theorem) *Let X be a pseudometric space. The following conditions are equivalent:*

- *X is complete.*
- *For every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of X such that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$, it is satisfied that $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$.*

Proof 215 *The proof is itemized likewise the statement.*

⇓ *For every $n \in \mathbb{N}$, choose $x_n \in A_n$. Note that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X; therefore, it is convergent to some $x \in X$. For every $n \in \mathbb{N}$, $\{x_k : k \geq n\} \subseteq A_n$ because $(A_n)_{n\in\mathbb{N}}$ is decreasing. Since A_n is closed for every $n \in \mathbb{N}$, we deduce that $a \in A_n$ for all $n \in \mathbb{N}$. This means that $x \in \bigcap_{n\in\mathbb{N}} A_n$.*

⇑ *Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$. For every $n \in \mathbb{N}$, define $A_n := \mathrm{cl}(\{x_k : k \geq n\})$. It is not hard to check that $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of closed subsets of X such that $\lim_{n\to\infty} \mathrm{diam}(A_n) = 0$. By hypothesis, $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. It can be checked that*

$$\lim_{n\to\infty} x_n = \bigcap_{n\in\mathbb{N}} A_n.$$

After Cantor Theorem, it is time now for the Baire (Category) Theorem.

Theorem 51 (Baire Theorem) *Every complete pseudometric space X is a Baire space.*

Proof 216 *Let $(U_n)_{n\in\mathbb{N}}$ be a countable family of open dense subsets of X. We will show that $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X. Indeed, fix an arbitrary nonempty open subset $U \subseteq X$. We will follow an inductive process:*

1. *Fix $u_1 \in U$ and $0 < \delta_1 < 1$ with $\mathsf{B}_X(u_1, \delta_1) \subseteq U$.*
2. *Fix $u_2 \in \mathsf{U}_X(u_1, \delta_1) \cap U_1$ and $0 < \delta_2 < \frac{1}{2}$ with $\mathsf{B}_X(u_2, \delta_2) \subseteq \mathsf{U}_X(u_1, \delta_1) \cap U_1$.*
3. *Fix $u_3 \in \mathsf{U}_X(u_2, \delta_2) \cap U_2$ and $0 < \delta_3 < \frac{1}{3}$ with $\mathsf{B}_X(u_3, \delta_3) \subseteq \mathsf{U}_X(u_2, \delta_2) \cap U_2$.*

   ⋮

We end up with a decreasing sequence $\left(\mathsf{B}_X(u_n, \delta_n)\right)_{n\in\mathbb{N}}$ of closed balls such that $\lim_{n\to\infty} \mathrm{diam}\left(\mathsf{B}_X(u_n, \delta_n)\right) = 0$. According to the Cantor Theorem,

$$\varnothing \neq \bigcap_{n\in\mathbb{N}} \mathsf{B}_X(u_n, \delta_n) \subseteq U \cap \bigcap_{n\in\mathbb{N}} U_n.$$

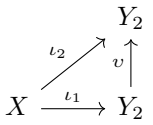
2.2.2.4 Completion of a metric space

Completion of metric spaces is a fundamental concept of Approximation Theory, one of the most important branches of classical Calculus.

Definition 195 (Completion) *Let X be a (pseudo)metric space. A completion of X is a pair (Y, ι), where Y is a complete (pseudo)metric space and $\iota : X \hookrightarrow Y$ is an isometry such that $\iota(X)$ is dense in Y.*

In metric spaces, completions are unique up to surjective isometries.

Proposition 73 *Let X be a metric space. If $(Y_1, \iota_1), (Y_2, \iota_2)$ are completions of X, then there exists a unique surjective isometry $\upsilon : Y_1 \to Y_2$ such that $\upsilon \circ \iota_1 = \iota_2$:*



Proof 217 *In the first place, if $\upsilon, \nu : Y_1 \to Y_2$ are surjective isometries such that $\upsilon \circ \iota_1 = \iota_2 = \nu \circ \iota_1$, then $\upsilon|_{\iota_1(X)} = \nu|_{\iota_1(X)}$, so υ and ι coincide on a dense subset of Y_1. Fix an arbitrary $y_1 \in Y_1$. There exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $(\iota_1(x_n))_{n\in\mathbb{N}}$ converges to y_1. Then $d(\iota_2(x_n), \upsilon(y_1)) = d(\upsilon(\iota_1(x_n)), \upsilon(y_1)) = d(\iota_1(x_n), y_1) = d(\nu(\iota_1(x_n)), \nu(y_1)) = d(\iota_2(x_n), \nu(y_1))$. Thus, $(\iota_2(x_n))_{n\in\mathbb{N}}$ converges to both $\upsilon(y_1), \nu(y_1)$. Since Y_2 is metric, $\upsilon(y_1) = \nu(y_1)$. The arbitrariness of y_1 shows that $\upsilon = \nu$. This proves the uniqueness of the surjective isometry $\upsilon : Y_1 \to Y_2$ satisfying $\upsilon \circ \iota_1 = \iota_2$. Let us show the existence of such a surjective isometry. Fix again an arbitrary $y_1 \in Y_1$. There exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $(\iota_1(x_n))_{n\in\mathbb{N}}$ converges to y_1. For every $p, q \in \mathbb{N}$,*

$$d(\iota_2(x_p), \iota_2(x_q)) = d(x_p, x_q) = d(\iota_1(x_p), \iota_1(x_q)).$$

This implies that $(\iota_2(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y_2; therefore, it converges to some $y_2 \in Y_2$. Observe that, if $(x'_n)_{n\in\mathbb{N}} \subseteq X$ is another sequence such that $(\iota_1(x'_n))_{n\in\mathbb{N}}$ converges to y_1, then the sequence

$$(z_n)_{n\in\mathbb{N}} := (x_1, x'_1, x_2, x'_2, \dots, x_n, x'_n, \dots)$$

satisfies that $(\iota_1(z_n))_{n\in\mathbb{N}}$ also converges to y_1. Therefore, $(\iota_2(z_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y_2. Now, $(\iota_2(x_n))_{n\in\mathbb{N}}$ is clearly a subsequence of $(\iota_2(z_n))_{n\in\mathbb{N}}$, therefore $(\iota_2(z_n))_{n\in\mathbb{N}}$ must be convergent to y_2, which implies that $(\iota_2(x'_n))_{n\in\mathbb{N}}$ must also be convergent to y_2. As a consequence of all these, we obtain a well-defined map $\upsilon : Y_1 \to Y_2$. This map is, in fact, an isometry. Indeed, take $y_1, y'_1 \in Y_1$ and consider sequences $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}} \subseteq X$ such that $(\iota_1(x_n))_{n\in\mathbb{N}}, (\iota_1(x'_n))_{n\in\mathbb{N}}$ converge to y_1, y'_1, respectively. Notice that

$$d(\iota_1(x_n), \iota_1(x'_n)) = d(x_n, x'_n) = d(\iota_2(x_n), \iota_2(x'_n))$$

for all $n \in \mathbb{N}$. Since $(\iota_2(x_n))_{n\in\mathbb{N}}, (\iota_2(x'_n))_{n\in\mathbb{N}}$ converge to $\upsilon(y_1), \upsilon(y'_1)$, respectively, in virtue of Lemma 88, we conclude that $d(y_1, y'_1) = d(\upsilon(y_1), \upsilon(y'_1))$. It only remains to prove that υ is surjective. Indeed, for every $y_2 \in Y_2$, we can consider $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $(\iota_2(x_n))_{n\in\mathbb{N}}$ converges to y_2. Notice that $(x_n)_{n\in\mathbb{N}} \subseteq X$ is Cauchy, and so is $(\iota_1(x_n))_{n\in\mathbb{N}}$. Therefore, $(\iota_1(x_n))_{n\in\mathbb{N}}$ converges to some $y_1 \in Y_1$. It is clear that $\upsilon(y_1) = y_2$.

Given a pseudometric space X, we denote by $\mathscr{CS}(X)$ to the set of all Cauchy sequences of X. An equivalence relation can be defined on $\mathscr{CS}(X)$ given by

$$\mathcal{R} := \left\{ ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \in \mathscr{CS}(X) \times \mathscr{CS}(X) : \lim_{n\to\infty} d(x_n, y_n) = 0 \right\}.$$

Lemma 89 *Let X be a pseudometric space. If Y is a dense subspace of X such that every Cauchy sequence in Y is convergent in X, then X is complete.*

Proof 218 *If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, for every $n \in \mathbb{N}$, we can find $y_n \in Y$ such that $d(x_n, y_n) < \frac{1}{n}$. Then $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, which is convergent in X. Finally, $(x_n)_{n\in\mathbb{N}}$ is convergent as well.*

We are finally in the right position to state and prove the completion theorem.

Theorem 52 *For every metric space X, there exists a completion (Z, ι).*

Proof 219 *We will take $Z := {}^{\mathscr{CS}(X)}/_{\mathcal{R}}$ and*

$$\begin{array}{rccl} \iota : & X & \to & Z \\ & x & \mapsto & \iota(x) := [(x)_{n\in\mathbb{N}}]_{\mathcal{R}}, \end{array}$$

where $(x)_{n\in\mathbb{N}}$ clearly denotes the constant sequence of general term equal to x. Lemma 88 assures that

$$d\left([(x_n)_{n\in\mathbb{N}}]_{\mathcal{R}}, [(y_n)_{n\in\mathbb{N}}]_{\mathcal{R}}\right) := \lim_{n\to\infty} d(x_n, y_n)$$

defines a metric in Z. On the other hand, for every $x, y \in X$,

$$d\left([(x)_{n\in\mathbb{N}}]_{\mathcal{R}}, [(y)_{n\in\mathbb{N}}]_{\mathcal{R}}\right) = d(x, y),$$

meaning that ι defines an isometry. In order to see that $\iota(X)$ is dense in Z, it only suffices to notice that, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, then $(\iota(x_n))_{n\in\mathbb{N}}$ converges to $[(x_n)_{n\in\mathbb{N}}]_{\mathcal{R}}$. Finally, in order to check that Z is complete, we just need to call on Lemma 89 and note that every Cauchy sequence $(\iota(x_n))_{n\in\mathbb{N}}$ in $\iota(X)$ is clearly convergent in Z to $[(x_n)_{n\in\mathbb{N}}]_{\mathcal{R}}$.

2.3 Convergence

The most general and abstract notion of convergence will be introduced and covered here, providing a solid ground for Summability Theory.

2.3.1 Convergence topologies and uniformities

Once the notion of limit has been properly introduced and studied, our next step is to construct specific topologies in function spaces prioritizing the power of convergence.

2.3.1.1 Convergence topologies

Convergence topologies will be crucial towards the development of the upcoming chapters, which deal with limits and summability.

Proposition 74 *Let I, X be nonempty sets. Consider $\mathcal{G} \subseteq \mathcal{P}(I)$ upward directed and $\mathcal{T} \subseteq \mathcal{P}(X)$ downward directed. Consider the family $\{\mathcal{U}(G,T) : G \in \mathcal{G}, T \in \mathcal{T}\}$, where $\mathcal{U}(G,T) := \{f \in X^I : f(G) \subseteq T\}$. Then:*

1. *If $\mathcal{U}(G,T) \neq \varnothing$ for all $G \in \mathcal{G}$ and all $T \in \mathcal{T}$, then $\{\mathcal{U}(G,T) : G \in \mathcal{G}, T \in \mathcal{T}\}$ is a filter base in X^I.*
2. *If $\bigcup\{\mathcal{U}(G,T) : G \in \mathcal{G}, T \in \mathcal{T}\} = X^I$, then $\{\mathcal{U}(G,U) : G \in \mathcal{G}, T \in \mathcal{T}\}$ is a subbase for a topology on X^I.*

Proof 220 *Suppose first that $\mathcal{U}(G,T) \neq \varnothing$ for all $G \in \mathcal{G}$ and all $T \in \mathcal{T}$. If $G_1, G_2 \in \mathcal{G}$ and $T_1, T_2 \in \mathcal{T}$, then there exist $G_0 \in \mathcal{G}$ and $T_0 \in \mathcal{T}$ such that $G_1 \cup G_2 \subseteq G_0$ and $T_0 \subseteq T_1 \cap T_2$, meaning that $\mathcal{U}(G_0, T_0) \subseteq \mathcal{U}(G_1, T_1) \cap \mathcal{U}(G_2, T_2)$. Finally, for the family of finite intersections of $\{\mathcal{U}(G,T) : G \in \mathcal{G}, T \in \mathcal{T}\}$ to become a topology base it only suffices that $\bigcup\{\mathcal{U}(G,T) : G \in \mathcal{G}, T \in \mathcal{T}\} = X^I$.*

Under the settings of Proposition 74, notice that $\mathcal{U}(G,T) = \prod_{i\in I} T_i$, where

$$T_i := \begin{cases} X & i \in I \setminus G, \\ T & i \in G, \end{cases}$$

and $\mathcal{U}(G_1,T_1) \cap \cdots \cap \mathcal{U}(G_k,T_k) = \prod_{i\in I} S_i$, where

$$S_i := \begin{cases} X & i \in I \setminus (G_1 \cup \cdots \cup G_k), \\ \bigcap_{i\in G_j} T_j & i \in G_1 \cup \cdots \cup G_k, \end{cases}$$

when X^I is seen as a product. Previous proposition motivates the following definition.

Definition 196 (Convergence topology) *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let X be a topological space. The convergence topology of X^I generated by $\mathcal{G}$ is defined as the topology generated by the subbase $\{\mathcal{U}(G,U) : G \in \mathcal{G}, U \subseteq X \text{ open}\}$ and denoted by $\sigma\left(X^I, \mathcal{G}\right)$.*

The following remark serves to bear in mind important details about convergence topologies.

Remark 60 *Let I be a nonempty set. Let $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{P}(I)$ be upward directed. Let X be a topological space. Let $\mathcal{F} \subseteq X^I$ be a subset. Then:*

- *If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then the convergence topology generated by $\mathcal{G}_1$ is clearly coarser than the convergence topology generated by $\mathcal{G}_2$.*
- *If for every $G_2 \in \mathcal{G}_2$ and every $f \in X^I$ there exists $G_1 \in \mathcal{G}_1$ such that $G_2 \subseteq G_1$ and $f(G_2) = f(G_1)$, then the convergence topology generated by $\mathcal{G}_1$ is finer than the convergence topology generated by $\mathcal{G}_2$. Indeed, for every $G_2 \in \mathcal{G}_2$, every $U \subseteq X$ open, and every $f \in \mathcal{U}(G_2, U)$, we have that $f \in \mathcal{U}(G_1, U) \subseteq \mathcal{U}(G_2, U)$, where $G_1 \in \mathcal{G}_1$ satisfies that $G_2 \subseteq G_1$ and $f(G_2) = f(G_1)$.*
- *If $\mathcal{B}$ is a topology base for X and $\mathcal{P}_{\mathtt{f}}(I) = \mathcal{G}$, then $\{\mathcal{U}(G,U) : G \in \mathcal{G}, U \in \mathcal{B}\}$ is also a subbase for $\sigma\left(X^I, \mathcal{G}\right)$. Indeed, the topology generated by $\{\mathcal{U}(G,U) : G \in \mathcal{G}, U \in \mathcal{B}\}$ is clearly coarser than $\sigma\left(X^I, \mathcal{G}\right)$. However, for every finite subset G of I, every open subset U of X, and every $f \in \mathcal{U}(G,U)$, we have that $f \in \bigcap_{i \in G} \mathcal{U}(\{i\}, V_i) \subseteq \mathcal{U}(G,U)$, where, for every $i \in G$, $V_i \in \mathcal{B}$ and $f(i) \in V_i \subseteq U$.*
- *The inherited $\mathcal{G}$-convergence topology of $\mathcal{F}$ from X^I is generated by the subbase $\{\mathcal{U}(G,U) \cap \mathcal{F} : G \in \mathcal{G}, U \subseteq X \text{ open}\}$ and denoted by $\sigma(\mathcal{F}, \mathcal{G})$.*

The most representative example of a convergence topology is the compact-open topology.

Example 42 (Compact-open topology) *Let Y, X be topological spaces. If $\mathcal{K}_Y$ stands for the family of compact subsets of Y, then $\sigma\left(X^Y, \mathcal{K}_Y\right)$ is called the compact-open topology on X^Y.*

The compact-open topology is a classic example of a convergence topology in the literature of General Topology. Recall that a bornology of a bounded poset is an ideal whose sup is 1.

Theorem 53 *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be a bornology. Let X be a topological space. If X is Hausdorff, then the convergence topology on X^I induced by $\mathcal{G}$ is Hausdorff as well.*

Proof 221 *Take any $f, g \in X^I$ with $f \neq g$. There exists $i \in I$ such that $f(i) \neq g(i)$. Since X is Hausdorff, we can find open sets $U \in \mathcal{N}_{f(i)}(X), V \in \mathcal{N}_{g(i)}(X)$ with $U \cap V = \varnothing$. By hypothesis, since $\bigcup \mathcal{G} = I$, there exists $G \in \mathcal{G}$ such that $i \in G$. By hypothesis again, since $\mathcal{G}$ is downward closed, $G \cap f^{-1}(U) \in \mathcal{G}$ and $G \cap g^{-1}(V) \in \mathcal{G}$. Finally, $f \in \mathcal{U}\left(G \cap f^{-1}(U), U\right)$, $g \in \mathcal{U}\left(G \cap g^{-1}(V), V\right)$ and $\mathcal{U}\left(G \cap f^{-1}(U), U\right) \cap \mathcal{U}\left(G \cap g^{-1}(V), V\right) = \varnothing$.*

The pointwise convergence topology is a particular case of convergence topology generated by the finite subsets.

Definition 197 (Pointwise convergence topology) *Let I be a nonempty set. Let X be a topological space. The pointwise convergence topology of X^I is the convergence topology generated by $\mathcal{G} := \mathcal{P}_{\mathtt{f}}(I)$, that is, $\sigma\left(X^I, \mathcal{P}_{\mathtt{f}}(I)\right)$.*

Next example shows that the pointwise convergence topology coincides with the product topology.

Example 43 (Pointwise convergence topology) *Let I be a nonempty set. Let X be a topological space. The pointwise convergence topology on X^I is precisely the convergence topology generated by $\mathcal{G} := \mathcal{P}_{\mathtt{f}}(I)$. If we look at X^I as a direct product, then, for every finite subset $G \subseteq I$ and every open subset $U \subseteq X$, we have*

$$\mathcal{U}(G,U) = \left\{f \in X^I : f(G) \subseteq U\right\} = \prod_{i\in I} U_i,$$

where

$$U_i = \begin{cases} U & i \in G, \\ X & i \notin G, \end{cases}$$

for all $i \in I$.

An immediate consequence of Theorem 53 is the fact that the pointwise convergence topology is Hausdorff if so is the corresponding topological space. Now, it is possible to formulate in an abstract manner the classical analytical definition of pointwise convergence.

Remark 61 *Let I be a nonempty set. Let X be a topological space. Consider the pointwise convergence topology on X^I, that is, $\sigma\left(X^I, \mathcal{P}_{\mathtt{f}}(I)\right)$. Then:*

- *For each filter base $\mathcal{B}$ of X^I, $\lim(\mathcal{B}) = \{f_0 \in X^I : \forall i \in I\ f_0(i) \in \lim(\mathcal{B}(i))\}$, where $\mathcal{B}(i) := \{B(i) : B \in \mathcal{B}\}$ and $B(i) := \{f(i) : f \in B\}$, for every $i \in I$.*
- *For every function $T : H \setminus \{h_0\} \to X^I$, where H is a topological space and h_0 is a limit point of H, $\lim_{h\to h_0} T(h) = \{f_0 \in X^I : \forall i \in I\ f_0(i) \in \lim_{h\to h_0} T_i(h)\}$, where, for every $i \in I$, $T_i : H \setminus \{h_0\} \to X$ is defined by $T_i(h) := T(h)(i)$ for every $h \in H \setminus \{h_0\}$.*
- *For every net $(f_\lambda)_{\lambda\in\Lambda}$ of X^I, $\lim_{\lambda\in\Lambda} f_\lambda = \{f_0 \in X^I : \forall i \in I\ f_0(i) \in \lim_{\lambda\in\Lambda} f_\lambda(i)\}$.*

Next, it is natural to introduce the notion of Function Space.

Definition 198 (Function space) *Let I be a nonempty set. Let X be a topological space. A subset $\mathcal{F} \subseteq X^I$ is called a function space if $\mathcal{F}$ is endowed with a convergence topology induced by a bornology of subsets of I containing the finite subsets.*

As a consequence of the previous definition, the convergence topology of a function space is always finer than the pointwise convergence topology.

Example 44 (Finest convergence topology) *Let I be a nonempty set. Let X be a topological space. The finest convergence topology on X^I is precisely the convergence topology generated by $\mathcal{G} := \mathcal{P}(I)$. The finest convergence topology on X^I is obviously finer than the pointwise convergence topology. If we look at X^I as a direct product, then, for every subset $G \subseteq I$ and every open subset $U \subseteq X$, we have*

$$\mathcal{U}(G,U) = \left\{f \in X^I : f(G) \subseteq U\right\} = \prod_{i\in I} U_i,$$

where

$$U_i = \begin{cases} U & i \in G, \\ X & i \notin G, \end{cases}$$

for all $i \in I$.

Under the settings of the previous example, observe that the finest convergence topology of X^I is coarser than the uniform product topology of X^I.

Theorem 54 *Let I be a nonempty set. Let X be a topological space. The finest convergence topology on X^I coincides with the uniform product topology if and only if for every family $(U_i)_{i\in I}$ of nonempty open subsets of X and for every $(x_i)_{i\in I} \in \prod_{i\in I} U_i$ there exists a finite number of nonempty open subsets $O_1, \ldots, O_l$ of X such that for each $i \in I$ there exists $j_i \in \{1, \ldots, l\}$ with $x_i \in O_{j_i} \subseteq U_i$.*

Proof 222 *We will itemize the proof by following the statement.*

- $\Rightarrow$ *Suppose first that the finest convergence topology on X^I coincides with the uniform product topology. Let $(U_i)_{i\in I}$ be a family of nonempty open subsets of X and fix an arbitrary $(x_i)_{i\in I} \in \prod_{i\in I} U_i$. By hypothesis, there exist $G_1, \ldots, G_k \in \mathcal{G}$ and nonempty open subsets $V_1, \ldots, V_k$ of X such that that $(x_i)_{i\in I} \in \mathcal{U}(G_1, V_1) \cap \cdots \cap \mathcal{U}(G_k, V_k) \subseteq \prod_{i\in I} U_i$. Notice that $\mathcal{U}(G_1, V_1) \cap \cdots \cap \mathcal{U}(G_k, V_k) = \prod_{i\in I} W_i$, where*

$$W_i := \begin{cases} X & i \in I \setminus (G_1 \cup \cdots \cup G_k), \\ \bigcap_{i\in G_j} V_j & i \in G_1 \cup \cdots \cup G_k. \end{cases}$$

  Obviously, for every $i \in I$, $x_i \in W_i \subseteq U_i$. Next, let us denote by $O_1, \ldots, O_l$ to all the nonempty finite intersections of $V_1, \ldots, V_k$. Since $\mathcal{U}(G, X) = X^I$ for any $G \in \mathcal{G}$, we may assume without any loss of generality that $V_k = X$. Fix an arbitrary $i_0 \in I$. If $i_0 \in I \setminus (G_1 \cup \cdots \cup G_k)$, then $U_{i_0} = W_{i_0} = X$, so $x_{i_0} \in V_k \subseteq U_{i_0}$. If $i_0 \in G_1 \cup \cdots \cup G_k$, then $x_{i_0} \in W_{i_0} \subseteq U_{i_0}$ and $W_{i_0} = \bigcap_{i_0\in G_j} V_j$, meaning that there exists $j_{i_0} \in \{1, \ldots, l\}$ with $O_{j_{i_0}} = W_{i_0}$.

- $\Leftarrow$ *Conversely, fix an arbitrary open set $\prod_{i\in I} U_i$ of the base of the uniform product topology and take any $(x_i)_{i\in I} \in \prod_{i\in I} U_i$. By assumption, there exists a finite number of nonempty open subsets $O_1, \ldots, O_l$ of X such that for each $i \in I$ there exists $j_i \in \{1, \ldots, l\}$ with $x_i \in O_{j_i} \subseteq U_i$. Then it only suffices to notice that*

$$(x_i)_{i\in I} \in \mathcal{U}(G_1, O_1) \cap \cdots \cap \mathcal{U}(G_l, O_l) \subseteq \prod_{i\in I} U_i,$$

  where $G_h := \{i \in I : j_i = h\}$ for every $h \in \{1, \ldots, l\}$.

2.3.1.2 Convergence uniformities

Every topological space induces a convergence topology on a set of functions valued on that topological space. In a similar way, every uniform space induces a uniformity on a set of functions valued on that uniform space.

Theorem 55 *Let X be a uniform space. Let I be a nonempty set. Take $\mathcal{G} \subseteq \mathcal{P}(I)$ upward directed. For every entourage V of X and every $G \in \mathcal{G}$, let*

$$\begin{aligned} \mathcal{V}(G, V) &:= \left\{(f, g) \in X^I \times X^I : (f \times g)(G) \subseteq V\right\} \\ &= \left\{(f, g) \in X^I \times X^I : \forall i \in G \ \ (f(i), g(i)) \in V\right\}. \end{aligned}$$

Then $\{\mathcal{V}(G, V) : G \in \mathcal{G} \text{ and } V \text{ is an entourage of } X\}$ is a uniformity base on X^I. Furthermore, $\mathcal{V}(G, V)[f] = \bigcap_{i\in G} \mathcal{U}(\{i\}, V[f(i)])$ for each $f \in X^I$.

Proof 223 *Notice that $\{\mathcal{V}(G, V) : G \in \mathcal{G} \text{ and } V \text{ is an entourage of } X\}$ is a filter base in $X^I \times X^I$ in view of the fact that $\mathcal{G}$ is upward directed and the uniformity of X is downward directed. We have to check that all three conditions for being a uniformity are satisfied:*

1. *$\mathcal{V}(G,V)$ is a reflexive binary internal relation in X^I. Indeed, for every $f \in X^I$ and every $i \in G$, $(f(i),f(i)) \in V$ because V is a reflexive binary internal relation in X; hence, $(f,f) \in \mathcal{V}(G,V)$.*

2. *There exists an entourage U in X such that $U \circ U \subseteq V$. Then $\mathcal{V}(G,U) \circ \mathcal{V}(G,U) \subseteq \mathcal{V}(G,V)$. Indeed, take $(f,g) \in \mathcal{V}(G,U) \circ \mathcal{V}(G,U)$. There exists $h \in X^I$ such that $(f,h) \in \mathcal{V}(G,U)$ and $(h,g) \in \mathcal{V}(G,U)$. For every $i \in I$, $(f(i),h(i)) \in U$ and $(h(i),g(i)) \in U$, so $(f(i),g(i)) \in U \circ U \subseteq V$. This shows that $(f,g) \in \mathcal{V}(G,V)$.*

3. *V^{-1} is an entourage in X, so $\mathcal{V}(G,V)^{-1} = \mathcal{V}\left(G,V^{-1}\right)$ is an entourage in X^I. Indeed, let $(f,g) \in \mathcal{V}(G,V)$. For every $i \in G$, $(g(i),f(i)) \in V^{-1}$. Thus, $(g,f) \in \mathcal{V}\left(G,V^{-1}\right)$. This shows that $\mathcal{V}(G,V)^{-1} \subseteq \mathcal{V}\left(G,V^{-1}\right)$. Conversely, if $(g,f) \in \mathcal{V}\left(G,V^{-1}\right)$, then $(g(i),f(i)) \in V^{-1}$ for all $i \in G$, hence $(f(i),g(i)) \in V$ for all $i \in G$, meaning that $(f,g) \in \mathcal{V}(G,V)$ and $(g,f) \in \mathcal{V}(G,V)^{-1}$.*

Finally, let us prove the equality $\mathcal{V}(G,V)[f] = \bigcap_{i\in G} \mathcal{U}\left(\{i\},V[f(i)]\right)$. Fix an arbitrary $g \in F$. Observe that $g \in \mathcal{V}(G,V)[f]$ if and only if $(f,g) \in \mathcal{V}(G,V)$, if and only if $(f(i),g(i)) \in V$ for all $i \in G$, if and only if $g(i) \in V[f(i)]$ for all $i \in G$, if and only if $g \in \mathcal{U}\left(\{i\},V[f(i)]\right)$ for all $i \in G$.

Theorem 55 motivates the following definition.

Definition 199 (Convergence uniformity) *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let X be a uniform space. The convergence uniformity of X^I generated by $\mathcal{G}$ is defined as the uniformity given by the uniformity base $\{\mathcal{V}(G,V) : G \in \mathcal{G}$ and V is an entourage of $X\}$ and denoted by $\upsilon(X^I,\mathcal{G})$.*

The following remark serves to bear in mind important details about convergence uniformities.

Remark 62 *Let I be a nonempty set. Let $\mathcal{G},\mathcal{G}_1,\mathcal{G}_2 \subseteq \mathcal{P}(I)$ be upward directed. Let X be a uniform space. Let $\mathcal{F} \subseteq X^I$ be a subset. Then:*

- *If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then the convergence uniformity generated by $\mathcal{G}_1$ is clearly coarser than the convergence uniformity generated by $\mathcal{G}_2$.*
- *If for every $G_2 \in \mathcal{G}_2$ there exists $G_1 \in \mathcal{G}_1$ such that $G_2 \subseteq G_1$, then the convergence uniformity generated by $\mathcal{G}_1$ is finer than the convergence uniformity generated by $\mathcal{G}_2$.*
- *If $\mathcal{B}$ is a base of entourages of X, then $\{\mathcal{V}(G,V) : G \in \mathcal{G},\ V \in \mathcal{B}\}$ is a base of entourages of X^I.*
- *The inherited $\mathcal{G}$-convergence uniformity of $\mathcal{F}$ from X^I is generated by the uniformity base $\{\mathcal{V}(G,V) \cap (\mathcal{F} \times \mathcal{F}) : G \in \mathcal{G}$ and V is an entourage of $X\}$.*

Theorem 55 makes possible to formulate in an abstract manner the classical analytical definition of uniform convergence.

Remark 63 *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let X be a uniform space. Then:*

- *For each filter base $\mathcal{B}$ of X^I, $\lim(\mathcal{B}) = \{f_0 \in X^I : \forall G \in \mathcal{G}\ \forall V \in X \times X$ entourage $\exists B_{G,V} \in \mathcal{B}\ \forall f \in B_{G,V}\ \forall i \in G\ (f_0(i),f(i)) \in V\}$.*

- *For every function $T : H \setminus \{h_0\} \to X^I$, where H is a topological space and h_0 is a limit point of H,* $\lim_{h\to h_0} T(h) = \{f_0 \in X^I : \forall G \in \mathcal{G}\ \forall V \in X \times X$ *entourage* $\exists W_{G,V} \in \mathcal{N}_{h_0}^{\times}(H)\ \ \forall h \in W_{G,V}\ \forall i \in G\ \ (f_0(i), T(h)(i)) \in V\}$.
- *For every net $(f_\lambda)_{\lambda\in\Lambda}$ of X^I,* $\lim_{\lambda\in\Lambda} f_\lambda = \{f_0 \in X^I : \forall G \in \mathcal{G}\ \forall V \in X \times X$ *entourage* $\exists \lambda_{G,V} \in \Lambda\ \ \forall \lambda \geq \lambda_{G,V}\ \forall i \in G\ \ (f_0(i), f_\lambda(i)) \in V\}$.

Therefore, now it is more natural to redefine the notion of Function Space provided in Definition 198.

Definition 200 (Function space) *Let I be a nonempty set. Let X be a uniform space. A subset $\mathcal{F} \subseteq X^I$ is called a function space if $\mathcal{F}$ is endowed with a convergence uniformity induced by a bornology of subsets of I containing the finite subsets.*

Next theorem establishes that the convergence uniformity endowed by the finite subsets induces the pointwise convergence topology.

Theorem 56 *Let I be a nonempty set. Let X be a uniform space. The pointwise convergence topology $\sigma\left(X^I, \mathcal{P}_{\mathtt{f}}(I)\right)$ on X^I coincides with the uniform topology induced by the convergence uniformity on X^I endowed by $\mathcal{P}_{\mathtt{f}}(I)$.*

Proof 224 *We will show that, for every $f_0 \in X^I$, the neighborhoods of f_0 in both topologies coincide. So, fix an arbitrary $f_0 \in X^I$. A generic neighborhood of f_0 in $\tau\left(\mathcal{P}_{\mathtt{f}}(I), X\right)$ contains a set of the form $\mathcal{U}(G_1, U_1) \cap \cdots \cap \mathcal{U}(G_k, U_k)$ where, for every $i \in \{1, \ldots, k\}$, $G_i \subseteq I$ is finite, $U_i \subseteq X$ is open, and $f_0(G_i) \subseteq U_i$. For every $i \in \{1, \ldots, k\}$ and each $g_i \in G_i$, there exists an entourage V_{g_i} in X such that $V_{g_i}[f_0(g_i)] \subseteq U_i$. Consider the entourage in X given by*

$$V := \bigcap_{i=1}^{k} \bigcap_{g_i \in G_i} V_{g_i}$$

and $G := G_1 \cup \cdots \cup G_k$. Note that $\mathcal{V}(G, V)[f_0] \subseteq \mathcal{U}(G_1, U_1) \cap \cdots \cap \mathcal{U}(G_k, U_k)$. Indeed, if $f_1 \in \mathcal{V}(G, V)[f_0]$, then for every $i \in \{1, \ldots, k\}$ and each $g_i \in G_i$, $(f_0(g_i), f_1(g_i)) \in V \subseteq V_{g_i}$, meaning that $f_1(g_i) \in V_{g_i}[f_0(g_i)] \subseteq U_i$, that is, $f_1 \in \mathcal{U}(G_i, U_i)$. Conversely, a generic neighborhood of f_0 in the uniform topology of X^I contains a set of the form $\mathcal{V}(G, V)[f_0]$, where $G \subseteq I$ is finite and V is an entourage in X. According to Theorem 55, $\mathcal{V}(G, V)[f_0] = \bigcap_{i\in G} \mathcal{U}\left(\{i\}, V[f_0(i)]\right)$. As a consequence, $\mathcal{V}(G, V)[f_0]$ is a neighborhood of f_0 for the pointwise convergence topology.

The following theorem shows that the uniform topology induced by a convergence uniformity is Hausdorff if the convergence uniformity is endowed by a bornology.

Theorem 57 *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be a bornology. Let X be a uniform space. If X is Hausdorff, then the uniform topology induced by the convergence uniformity on X^I endowed by $\mathcal{G}$ is Hausdorff as well.*

Proof 225 *Take any $f_1, f_2 \in X^I$ with $f_1 \neq f_2$. There exists $i \in I$ such that $f_1(i) \neq f_2(i)$. Since X is Hausdorff, we can find a symmetric entourage V in X with $V[f_1(i)] \cap V[f_2(i)] = \varnothing$. By hypothesis, since $\bigcup \mathcal{G} = I$, there exists $G \in \mathcal{G}$ such that $i \in G$. Finally, $\mathcal{V}\left(G, V\right)[f_1] \cap \mathcal{V}\left(G, V\right)[f_2] = \varnothing$.*

The final result in this subsubsection provides a sufficient condition for the entourages of a convergence uniformity to be open entourages. Recall that an entourage V of a uniform space X is called an open entourage provided that $V[x]$ is open in the uniform topology of X for all $x \in X$.

Theorem 58 *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let X be a uniform space. Consider $E \in \mathcal{G}$ and V an open entourage in X. Let $f \in X^I$ be a constant function. If $g(E)$ is compact in X for each $g \in \mathcal{V}(E,V)[f]$, then $\mathcal{V}(E,V)[f]$ is open in the uniform topology of X^I.*

Proof 226 *Fix an arbitrary $g \in \mathcal{V}(E,V)[f]$. For every $i \in E$, there exists an entourage U_i in X with $U_i[g(i)] \subseteq V[f(i)]$. For each $i \in I$, there exists an entourage W_i in X such that $W_i \circ W_i \subseteq U_i$. Since $g(E)$ is compact in X, there is a finite number of entourages $W_{i_1}, \ldots, W_{i_k}$ covering $g(E)$. Let $W := W_{i_1} \cap \cdots \cap W_{i_k}$. We will prove that $\mathcal{V}(E,W)[g] \subseteq \mathcal{V}(E,V)[f]$. For this, it only suffices to show that $W[g(i)] \subseteq V[f(i)]$ for all $i \in I$. Choose an arbitrary $i \in E$. Take any $x \in W[g(i)]$. There exists $j \in \{1, \ldots, k\}$ such that $g(i) \in W_{i_j}[g(i_j)]$. Note that $(g(i_j), g(i)) \in W_{i_j}$ and $(g(i), x) \in W \subseteq W_{i_j}$, implying that $(g(i_j), x) \in W_{i_j} \circ W_{i_j} \subseteq U_{i_j}$. Therefore, $W[g(i)] \subseteq U_{i_j}[g(i_j)] \subseteq V[f(i_j)] = V[f(i)]$ because f is constant.*

2.3.2 Double limits

Double limits can be naturally approached in different ways. One of the most common ways is by relying on convergence topologies. Other ways are related to the product order.

2.3.2.1 Prefilter double limits

If X, Y are sets and $\mathcal{B}, \mathcal{C}$ are filter bases of $\mathcal{P}(X), \mathcal{P}(Y)$, respectively, then $\mathcal{B} \times \mathcal{C} := \{A \times D : A \in \mathcal{B}, D \in \mathcal{C}\}$ is a filter base of $\mathcal{P}(X \times Y)$. On the other hand, recall that a topological space X is called regular provided that, for every closed subset $A \subseteq X$ and every $x \in X \setminus A$, there exist open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $x \in V$. Next, if X, Y, Z are nonempty sets and $f : X \times Y \to Z$ is a function, then the marginal maps allow

$$\begin{array}{rccccccc} f(\bullet,\ ): & X & \to & Z^Y & & & & \\ & x & \mapsto & f(x,\ ): & Y & \to & Z \\ & & & & y & \mapsto & f(x,y), \end{array}$$

and

$$\begin{array}{rccccccc} f(\ ,\bullet): & Y & \to & Z^X & & & & \\ & y & \mapsto & f(\ ,y): & X & \to & Z \\ & & & & x & \mapsto & f(x,y). \end{array}$$

Theorem 59 *Let X, Y be nonempty sets. Let Z be a topological space. Let $f : X \times Y \to Z$ be a function. Let $\mathcal{B}_X, \mathcal{B}_Y$ be filter bases of $\mathcal{P}(X), \mathcal{P}(Y)$, respectively. Then:*

1. *Suppose that Z is regular and that both Z^X and Z^Y are endowed with the pointwise convergence topology. Then*

$$\lim_{\mathcal{B}_X \times \mathcal{B}_Y}(f) \subseteq \lim_{\mathcal{B}_X}(g) \cap \lim_{\mathcal{B}_Y}(h)$$

for every $h \in \lim_{\mathcal{B}_X}(f(\bullet,\ ))$ and every $g \in \lim_{\mathcal{B}_Y}(f(\ ,\bullet))$.

2. *Suppose that both Z^X and Z^Y are endowed with the uniform convergence topology. Then*

$$\lim_{\mathcal{B}_X \times \mathcal{B}_Y}(f) \supseteq \lim_{\mathcal{B}_X}(g) \cup \lim_{\mathcal{B}_Y}(h)$$

for every $h \in \lim_{\mathcal{B}_X}(f(\bullet,\ ))$ and every $g \in \lim_{\mathcal{B}_Y}(f(\ ,\bullet))$. If, in addition, Z is regular, then

$$\lim_{\mathcal{B}_X \times \mathcal{B}_Y}(f) = \lim_{\mathcal{B}_X}(g) = \lim_{\mathcal{B}_Y}(h)$$

for every $h \in \lim_{\mathcal{B}_X}(f(\bullet,\ ))$ and every $g \in \lim_{\mathcal{B}_Y}(f(\ ,\bullet))$.

Proof 227 *The proof is itemized by following the itemization of the statement.*

1. *In the first place, by relying on Lemma 81, we have that*

$$\begin{aligned}\lim_{\mathcal{B}_X}(f(\bullet,\ )) &= \left\{h\in Z^Y : \forall y\in Y\ h(y)\in\lim_{\mathcal{B}_X}(\pi_y\circ f(\bullet,\ ))\right\}\\ &= \left\{h\in Z^Y : \forall y\in Y\ h(y)\in\lim_{\mathcal{B}_X}(f(\ ,y))\right\},\\ \lim_{\mathcal{B}_Y}(f(\ ,\bullet)) &= \left\{g\in Z^X : \forall x\in X\ g(x)\in\lim_{\mathcal{B}_Y}(\pi_x\circ f(\ ,\bullet))\right\}\\ &= \left\{g\in Z^X : \forall x\in X\ g(x)\in\lim_{\mathcal{B}_Y}(f(x,\ ))\right\}.\end{aligned}$$

   Take $z\in\lim_{\mathcal{B}_X\times\mathcal{B}_Y}(f)$. We will only show that $z\in\lim_{\mathcal{B}_X}(g)$ for an arbitrary $g\in\lim_{\mathcal{B}_Y}(f(\ ,\bullet))$, since, in a similar manner, it can be shown that $z\in\lim_{\mathcal{B}_Y}(h)$ for every $h\in\lim_{\mathcal{B}_X}(f(\bullet,\ ))$. Let $U\in\mathcal{N}_z(Z)$. Since Z is regular, there exists $V\in\mathcal{N}_z(Z)$ such that $\mathrm{cl}(V)\subseteq U$. There are $A_0\in\mathcal{B}_X$ and $C_0\in\mathcal{B}_Y$ such that $f(A_0\times C_0)\subseteq V$. We will show that $g(A_0)\subseteq U$. Indeed, suppose to the contrary that there exists $a_0\in A_0$ such that $g(a_0)\notin U$. Then $g(a_0)\in X\setminus\mathrm{cl}(V)$. Since $g\in\lim_{\mathcal{B}_Y}(f(\ ,\bullet))$, we have that $g(a_0)\in\lim_{\mathcal{B}_Y}(f(a_0,\ ))$, thus, there exists $C_1\in\mathcal{B}_Y$ such that $f(a_0,c_1)\in X\setminus\mathrm{cl}(V)$ whenever $c_1\in C_1$. Fix $C_2\in\mathcal{B}_Y$ such that $C_2\subseteq C_0\cap C_1$. Note that, for all $c_2\in C_2$, $f(a_0,c_2)\in V$ and $f(a_0,c_2)\in X\setminus\mathrm{cl}(V)$, which is a contradiction. This shows that $z\in\lim_{\mathcal{B}_X}(g)$. As a consequence, $\lim_{\mathcal{B}_X\times\mathcal{B}_Y}(f)\subseteq\lim_{\mathcal{B}_X}(g)$.

2. *Fix an arbitrary $z\in\lim_{\mathcal{B}_Y}(h)$ for an arbitrary $h\in\lim_{\mathcal{B}_X}(f(\bullet,\ ))$. Take $U\in\mathcal{N}_z(Z)$. There exists $C_0\in\mathcal{B}_Y$ such that $h(C_0)\subseteq U$. Define $\prod_{y\in Y}U_y$, where*

$$U_y := \begin{cases} U & y\in C_0,\\ Y & \text{otherwise,}\end{cases}$$

   for every $y\in Y$. Observe that $h\in\prod_{y\in Y}U_y=\mathcal{U}(C_0,U)$. Since $h\in\lim_{\mathcal{B}_X}(f(\bullet,\ ))$, we can find $A_0\in\mathcal{B}_X$ so that $f(A_0,\ )\subseteq\mathcal{U}(C_0,U)$. In other words, $f(a_0,c_0)\in U$ for all $a_0\in A_0$ and all $c_0\in C_0$. This means that $f(A_0\times C_0)\subseteq U$ with $A_0\times C_0\in\mathcal{B}_X\times\mathcal{B}_Y$. This proves that $z\in\lim_{\mathcal{B}_X\times\mathcal{B}_Y}(f)$. As a consequence, $\lim_{\mathcal{B}_Y}(h)\subseteq\lim_{\mathcal{B}_X\times\mathcal{B}_Y}(f)$. Following a similar procedure, it can be proved that $\lim_{\mathcal{B}_X}(g)\subseteq\lim_{\mathcal{B}_X\times\mathcal{B}_Y}(f))$. Finally, if Z is regular, $\lim_{\mathcal{B}_X\times\mathcal{B}_Y}(f)\subseteq\lim_{\mathcal{B}_X}(g)\cap\lim_{\mathcal{B}_Y}(h)$ for every $h\in\lim_{\mathcal{B}_X}(f(\bullet,\ ))$ and every $g\in\lim_{\mathcal{B}_Y}(f(\ ,\bullet))$, reaching the conclusion that $\lim_{\mathcal{B}_X\times\mathcal{B}_Y}(f)=\lim_{\mathcal{B}_X}(g)=\lim_{\mathcal{B}_Y}(h)$ for every $h\in\lim_{\mathcal{B}_X}(f(\bullet,\ ))$ and every $g\in\lim_{\mathcal{B}_Y}(f(\ ,\bullet))$.

The following theorem is the version of the previous result for limits of functions at accumulation points.

Theorem 60 *Let X,Y,Z be topological spaces. Let $x_0\in\mathrm{ac}(X), y_0\in\mathrm{ac}(Y)$. Let $f: X\setminus\{x_0\}\times Y\setminus\{y_0\}\to Z$ be a function. Then:*

1. *Suppose that Z is regular and that both $Z^{X\setminus\{x_0\}}$ and $Z^{Y\setminus\{y_0\}}$ are endowed with the pointwise convergence topology. Then*

$$\lim_{(x,y)\to(x_0,y_0)}(f)\subseteq\lim_{x\to x_0}(g)\cap\lim_{y\to y_0}(h)$$

   for every $h\in\lim_{x\to x_0}(f(\bullet,\ ))$ and every $g\in\lim_{y\to y_0}(f(\ ,\bullet))$.

2. *Suppose that both* $Z^{X\setminus\{x_0\}}$ *and* $Z^{Y\setminus\{y_0\}}$ *are endowed with the uniform convergence topology. Then*

$$\lim_{(x,y)\to(x_0,y_0)}(f)\supseteq\lim_{x\to x_0}(g)\cup\lim_{y\to y_0}(h)$$

for every $h\in\lim_{x\to x_0}(f(\bullet,\ ))$ *and every* $g\in\lim_{y\to y_0}(f(\ ,\bullet))$*. If, in addition,* Z *is regular, then*

$$\lim_{(x,y)\to(x_0,y_0)}(f)=\lim_{x\to x_0}(g)=\lim_{y\to y_0}(h)$$

for every $h\in\lim_{x\to x_0}(f(\bullet,\ ))$ *and every* $g\in\lim_{y\to y_0}(f(\ ,\bullet))$*.*

Proof 228 *It is only sufficient to bear in mind Theorem 59 together with the fact that* $(x_0,y_0)\in\mathrm{ac}(X\setminus\{x_0\}\times Y\setminus\{y_0\})$ *and*

$$\mathcal{J}\left(\mathcal{N}_{x_0}^{\times}(X)\times\mathcal{N}_{y_0}^{\times}(Y)\right)=\mathcal{N}_{(x_0,y_0)}^{\times}\left((X\setminus\{x_0\}\times Y\setminus\{y_0\})\cup\{(x_0,y_0)\}\right).$$

2.3.2.2 Net double limits

If I,J are preordered sets, then $I\times J$ can be endowed with the product preorder; hence, $I\times J$ becomes a preordered set. If I,J are directed sets, then $I\times J$ is also directed. For every $i_0\in I$ and every j_0 J, $\uparrow i_0\times\uparrow j_0=\uparrow(i_0,j_0)$, which means that $\mathcal{B}_I\times\mathcal{B}_J=\mathcal{B}_{I\times J}$. A double net is simply a function of the form $c:I\times J\to X$. The marginal nets allow

$$\begin{array}{rccccccc} c(\bullet,\ ): & I & \to & X^J & & & & \\ & i & \mapsto & c(i,\ ): & J & \to & X \\ & & & & j & \mapsto & c(i,j), \end{array}$$

and

$$\begin{array}{rccccccc} c(\ ,\bullet): & J & \to & X^I & & & & \\ & j & \mapsto & c(\ ,j): & I & \to & X \\ & & & & i & \mapsto & c(i,j). \end{array}$$

Theorem 61 *Let* X *be a topological space. Let* I,J *be directed sets. Let* $c:I\times J\to X$ *be a double net. Then:*

1. *Suppose that* X *is regular and* X^I *and* X^J *are endowed with the pointwise convergence topology. If* $e\in\lim(c(\bullet,\ ))$ *and* $d\in\lim(c(\ ,\bullet))$*, then* $\lim(c)\subseteq\lim(d)\cap\lim(e)$*.*

2. *Suppose that* X^I *and* X^J *are endowed with the uniform convergence topology. If* $e\in\lim(c(\bullet,\ ))$ *and* $d\in\lim(c(\ ,\bullet))$*, then* $\lim(c)\supseteq\lim(d)\cup\lim(e)$*. If, in addition,* X *is regular, then* $\lim(c)=\lim(d)=\lim(e)$*.*

Proof 229 *The proof is itemized like the statement is.*

1. *In the first place, by relying on Lemma 82, we have that*

$$\begin{aligned} \lim(c(\bullet,\ )) &= \left\{e\in X^J:\forall j\in J\ e(j)\in\lim(\pi_j\circ c(\bullet,\ ))\right\} \\ &= \left\{e\in X^J:\forall j\in J\ e(j)\in\lim(c(\ ,j))\right\}, \\ \lim(c(\ ,\bullet)) &= \left\{d\in X^I:\forall i\in I\ d(i)\in\lim(\pi_i\circ c(\ ,\bullet))\right\} \\ &= \left\{d\in X^I:\forall i\in I\ d(i)\in\lim(c(i,\ ))\right\}. \end{aligned}$$

Take $x\in\lim(c)$*. We will only show that* $x\in\lim(d)$*, since, in a similar manner, it can be shown that* $x\in\lim(d)$*. Let* $U\in\mathcal{N}_x(X)$*. Since* X *is regular, there exists* $V\in\mathcal{N}_x(X)$ *such that* $\mathrm{cl}(V)\subseteq U$*. There are* $i_0\in I$ *and* $j_0\in J$ *such that* $c(i,j)\in V$

whenever $i \geq i_0$ and $j \geq j_0$. We will show that $d(i) \in U$ if $i \geq i_0$. Indeed, suppose to the contrary that there exists $i \geq i_0$ such that $d(i) \notin U$. Then $d(i) \in X \setminus \mathrm{cl}(V)$. There exists $j_1 \in J$ such that $c(i,j) \in X \setminus \mathrm{cl}(V)$ whenever $j \geq j_1$. Fix $j_2 \in J$ such that $j_2 \geq j_1$ and $j_2 \geq j_0$. Note that $c(i,j_2) \in V$ and $c(i,j_2) \in X \setminus \mathrm{cl}(V)$, which is a contradiction. This shows that $x \in \lim(d)$.

2. *Fix an arbitrary $x \in \lim(e)$. Take $U \in \mathcal{N}_x(X)$. There exists $j_0 \in J$ such that, if $j \in J$ and $j \geq j_0$, then $e(j) \in U$. Define $\prod_{j\in J} U_j$, where*

$$U_j := \begin{cases} U & j \geq j_0, \\ X & \text{otherwise}, \end{cases}$$

for every $j \in J$. Observe that $e \in \prod_{j\in J} U_j = \mathcal{U}(\uparrow j_0, U)$. Thus, we can find $i_0 \in I$ so that if $i \in I$ and $i \geq i_0$ and $j \in J$, then $c(i,j) \in U_j$. In other words, if $(i,j) \geq (i_0,j_0)$, then $c(i,j) \in U$. This proves that $\lim(e) \subseteq \lim(c)$. Following a similar procedure, it can be proved that $\lim(d) \subseteq \lim(c)$. Finally, if X is regular, $\lim(c) \subseteq \lim(d) \cap \lim(e)$. So, $\lim(c) = \lim(d) = \lim(e)$.

Chapter 3

Continuity

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3.1 Types of continuity

Continuity is an essential concept in classical Calculus. It can be studied from an analytical viewpoint [31] or from a topological viewpoint [32]. Here, we will begin with the topological approach to end up with an analytical study of continuity.

3.1.1 Pointwise continuity

There are several distinctions between pointwise continuity and uniform continuity. One of them, which is also one of the most relevant distinctions, is the fact that pointwise continuity can be defined for general topological spaces, whereas uniform continuity only makes sense with additional structure such as metric or linear structures.

3.1.1.1 Continuity in topological spaces

Recall that, for every function $f : X \to Y$ between sets X, Y and for every filter base B of $\mathcal{P}(X)$, $f(B)$ is a filter base of $\mathcal{P}(Y)$.

Definition 201 (Continuous map) *Let X, Y be topological spaces. A function $f : X \to Y$ is said to be continuous at $x_0 \in X$ provided that $\mathcal{N}_{f(x_0)}(Y) \subseteq \mathcal{J}\left(f(\mathcal{N}_{x_0}(X))\right)$. The set of continuity points of f is denoted by*

$$\operatorname{cont}(f) := \left\{x \in X : \mathcal{N}_{f(x)}(Y) \subseteq \mathcal{J}\left(f(\mathcal{N}_x(X))\right)\right\}.$$

We will say that f is continuous if $\operatorname{cont}(f) = X$. The set of continuous functions from X to Y is denoted by $\mathcal{C}(X, Y)$. If $X = Y$, then we will simply denote $\mathcal{C}(X) := \mathcal{C}(X, Y)$.

The following remark highlights the analytical definition of continuity, which corresponds with the classical definition.

Remark 64 *Let X, Y be topological spaces. A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for every $U \in \mathcal{N}_{f(x_0)}(Y)$ there exists $V \in \mathcal{N}_{x_0}(X)$ such that $f(V) \subseteq U$. In*

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particular, f is continuous if and only if $f^{-1}(U)$ is open in X whenever U is open in Y, and if and only if $f^{-1}(C)$ is closed in X whenever C is closed in Y. If τ_X, τ_Y denote the topologies of X, Y, respectively, then the previous condition turns into $f^{-1}(\tau_Y) \subseteq \tau_X$.

On the other hand, notice the similitudes between the limit points of a function and the continuity points. In fact, continuity points can be characterized in terms of limit points.

Lemma 90 *Let X, Y be topological spaces. For any function $f : X \to Y$,*

$$\begin{aligned} & \operatorname{cont}(f) = \operatorname{iso}(X) \cup \left\{ x_0 \in \operatorname{ac}(X) : \lim_{x \to x_0} f(x) = f(x_0) \right\} \\ = \quad & \{ x_0 \in X : \forall (x_i)_{i \in I} \subseteq X \;\; x_0 \in \lim ((x_i)_{i \in I}) \Rightarrow f(x_0) \in \lim ((f(x_i))_{i \in I}) \} . \end{aligned}$$

Proof 230 *The second equality above is a direct consequence of Proposition 68. Thus, we will only need to prove the first equality. In the first place, we will show that* $\operatorname{iso}(X) \subseteq \operatorname{cont}(X)$. *Indeed, fix an arbitrary $x \in \operatorname{iso}(X)$. We know that $\{x\}$ is open. Therefore, if $U \in \mathcal{N}_{f(x)}(Y)$, then $\{x\} \in \mathcal{N}_x(X)$ and $f(\{x\}) \subseteq U$. This shows that $x \in \operatorname{cont}(f)$. As a consequence,* $\operatorname{iso}(X) \subseteq \operatorname{cont}(X)$. *Next, we will show that, if $x_0 \in \operatorname{ac}(X)$ and $\lim_{x \to x_0} f(x) = f(x_0)$, then $x_0 \in \operatorname{cont}(f)$. Indeed, take $U \in \mathcal{N}_{f(x_0)}(Y)$. There exists $V \in \mathcal{N}_{x_0}^{\times}(X)$ such that $f(V) \subseteq U$. Since $f(x_0) \in U$, we conclude that $f(V \cup \{x_0\}) \subseteq U$ with $V \cup \{x_0\} \in \mathcal{N}_{x_0}(X)$. This shows that $x_0 \in \operatorname{cont}(f)$. Finally, take an arbitrary $x_0 \in \operatorname{cont}(f)$ and suppose that $x_0 \notin \operatorname{iso}(X)$. Then $x_0 \in \operatorname{ac}(X)$. For every $U \in \mathcal{N}_{f(x_0)}(Y)$ there exists $V \in \mathcal{N}_{x_0}(X)$ such that $f(V) \subseteq U$. Next, $V \setminus \{x_0\} \in \mathcal{N}_{x_0}^{\times}(X)$ and $f(V) \subseteq U$. This shows that $\lim_{x \to x_0} f(x) = f(x_0)$.*

The following lemma is somehow a direct consequence of Lemma 90.

Lemma 91 *Let X, Y be topological spaces. Let $f : X \to Y$ be bijective. Let $x \in X$. Suppose that for every net $(x_i)_{i \in I} \subseteq X$ such that $(f(x_i))_{i \in I}$ converges to $f(x)$, there exists a subnet $(z_j)_{j \in J}$ of $(x_i)_{i \in I}$ convergent to x. Then f^{-1} is continuous at $f(x)$.*

Proof 231 *Assume to the contrary that f^{-1} is not continuous at $f(x)$. Then there exists a net $(x_i)_{i \in I} \subseteq X$ such that $(f(x_i))_{i \in I}$ converges to $f(x)$ but $(x_i)_{i \in I}$ does not converge to x. There exists a neighborhood U of x and a subnet $(z_j)_{j \in J}$ of $(x_i)_{i \in I}$ such that $z_j \notin U$ for all $j \in J$. Note that $(f(z_j))_{j \in J}$ converges to $f(x)$. Thus, by hypothesis, there exists a subnet $(p_k)_{k \in K}$ of $(z_j)_{j \in J}$ convergent to x, which is a contradiction in view of the fact that $z_j \notin U$ for all $j \in J$.*

A useful characterization of continuity is given in the next lemma in terms of closures.

Lemma 92 *Let X, Y be topological spaces. A map $f : X \to Y$ is continuous if and only if $f(\operatorname{cl}(A)) \subseteq \operatorname{cl}(f(A))$ for all $A \subseteq X$, and if and only if $f^{-1}(\operatorname{int}(A)) \subseteq \operatorname{int}\left(f^{-1}(A)\right)$ for all $A \subseteq X$*

Proof 232 *We will only prove it for the closure. Suppose first that f is continuous. Fix an arbitrary $A \subseteq X$. Take any $x \in \operatorname{cl}(A)$. If U is a neighborhood of $f(x)$, then there exists a neighborhood V of x such that $f(V) \subseteq U$. There exists $a \in V \cap A$, meaning that $f(x) \in U \cap f(A)$. Conversely, fix an arbitrary closed subset F of Y. If $A := f^{-1}(F)$, then $f(\operatorname{cl}(A)) \subseteq \operatorname{cl}(f(A))$, that is,* $\operatorname{cl}(A) \subseteq f^{-1}(\operatorname{cl}(f(A))) \subseteq f^{-1}(F) = A$.

Observe that the composition of continuous functions is continuous, that is, if X, Y, Z are topological spaces and $f : X \to Y, g : Y \to Z$ are continuous functions, then $g \circ f : X \to Z$ is continuous. This turns $\mathcal{C}(X)$ into a submonoid of X^X under the composition.

Lemma 93 *Let X, Y, Z be topological spaces. Let x_0, y_0 be limit points of X, Y, respectively. Let $f : X \setminus \{x_0\} \to Y$, $g : Y \to Z$ be functions. If Z is regular, g is continuous at y_0, and $y_0 \in \lim_{x \to x_0}(f)$, then $\lim_{x \to x_0}(g \circ f) = \lim_{y \to y_0}(g)$*

Proof 233 *Let us itemize the proof by following the statement.*

- $\subseteq$ *Take any $z \in \lim_{x \to x_0}(g \circ f)$. Let V be any open neighborhood of z. Then $g^{-1}(V)$ is open in Y. Since g is continuous at y_0, then $g(y_0) \in \lim_{x \to x_0}(g \circ f)$. Since Z is regular, then $g(y_0) \in V$. As a consequence, $g^{-1}(V) \in \mathcal{N}_{y_0}$ and $g\left(g^{-1}(V)\right) \subseteq V$.*
- $\supseteq$ *Take any $z \in \lim_{y \to y_0}(g)$. Let $V \in \mathcal{N}_z$. There exists $U \in \mathcal{N}_{y_0}$ such that $g(U \setminus \{y_0\}) \subseteq V$. Since g is continuous at y_0, then $g(y_0) \in \lim_{y \to y_0}(g)$. Since Z is regular, then $g(y_0) \in V$. As a consequence, $g(U) \subseteq V$. Next, there exists $W \in \mathcal{N}_{x_0}$ with $f(W \setminus \{x_0\}) \subseteq U$. Finally. $(g \circ f)(W \setminus \{x_0\}) \subseteq V$.*

What follows is the notion of homeomorphism, that is, isomorphisms in the category of topological spaces.

Definition 202 (Homeomorphism) *A function $f : X \to Y$ between topological spaces X, Y is said to be a homeomorphism provided that f is continuous, bijective and f^{-1} is continuous. In this situation, we say that X, Y are homeomorphic.*

The composition of homeomorphisms is also a homeomorphism, which turns the subset of homeomorphisms on X into a subgroup of the monoid $\mathcal{C}(X)$.

Theorem 62 (Intermediate Value Theorem) *Let X, Y be topological spaces. Let $T : X \to Y$ be a continuous map. Then:*

1. *If X is compact, then $T(X)$ is compact.*
2. *If X is connected, then $T(X)$ is connected.*

Proof 234 *The proof is itemized according to the statement.*

1. *Let $\{V_i : i \in I\}$ be an open covering of $T(X)$ in Y. The continuity of T means that $\left\{T^{-1}(V_i) : i \in I\right\}$ is an open covering of X. By hypothesis, X is compact, so there exists a finite subcovering $\left\{T^{-1}\left(V_{i_k}\right) : k \in \{1, \ldots, n\}\right\}$. Notice that $\{V_{i_k} : k \in \{1, \ldots, n\}\}$ is a finite subcovering of $T(X)$.*
2. *Suppose to the contrary that $T(X)$ is not connected and let $\{V_1, V_2\}$ be a nontrivial open partition of $T(X)$, that is, V_1, V_2 are nonempty, disjoint, and open in $T(X)$ and $T(X) = V_1 \cup V_2$. By the continuity of T, $\left\{T^{-1}(V_1), T^{-1}(V_2)\right\}$ is a nontrivial open partition of X. This contradicts the fact that X is connected.*

The following notion is crucial in Functional Analysis and Measure Theory.

Definition 203 (Homogeneous topological space) *A topological space X is said to be homogeneous if for every $x, y \in X$ there exists a homeomorphism $T : X \to Y$ such that $T(x) = y$.*

The set of homeomorphisms between two topological spaces X, Y is usually denoted by $\text{homeo}(X, Y)$. When $X = Y$, we simply write $\text{homeo}(X)$.

Definition 204 (Homotopy) *Let X, Y be topological spaces. Let $A \subseteq X$. Let $f_0, f_1 \in \mathcal{C}(X,Y)$ be such that $f_0|_A = f_1|_A$. We say that f_0, f_1 are homotopic relative to A, denoted by $f_0 \sim_A f_1$, if there exists an homotopy F of f_0 to f_1 such that $F(a,t) = f_0(a) = f_1(a)$ for all $t \in [0,1]$ and all $a \in A$.*

In the above definition, A might be empty and then the condition $f_0|_A = f_1|_A$ is trivially satisfied.

Lemma 94 *Let X, Y be topological spaces. Let $A \subseteq X$. Then $\sim_A$ is an equivalence relation in $\mathcal{C}(X,Y)$.*

Proof 235 *Fix arbitrary $f, g, h \in \mathcal{C}(X,Y)$.*

- *Reflexivity: $f \sim_A f$ by $F(x,t) := f(x)$.*
- *Symmetricity: If $f \sim_A g$ by $F(x,t)$, then $g \sim_A f$ by $G(x,t) := F(x, 1-t)$.*
- *Transitivity: If $f \sim_A g$ by $F(x,t)$ and $g \sim_A h$ by $G(x,t)$, then $f \sim_A h$ by*

$$H(x,t) := \begin{cases} F(x,2t) & 0 \leq t \leq \frac{1}{2}, \\ G(x,2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The previous lemma together with the next definition are crucial towards constructing the Fundamental Group.

Definition 205 (Path) *A path in a topological space X is a continuous function $\gamma : [0,1] \to X$.*

The final result in this subsubsection assures that a homotopy is simply a path of continuous functions.

Theorem 63 *Let X, Y be topological spaces. Suppose that $\mathcal{C}(X,Y)$ is endowed with the compact-open topology. Every homotopy $F : X \times [0,1] \to Y$ induces a path $\gamma_F : [0,1] \to \mathcal{C}(X,Y)$ by $\gamma_F(t)(x) := F(x,t)$. Conversely, if X is locally compact and Hausdorff, then every path $\gamma : [0,1] \to \mathcal{C}(X,Y)$ induces a homotopy $F_\gamma : X \times [0,1] \to Y$ by $F_\gamma(x,t) := \gamma(t)(x)$.*

Proof 236 *The proof will be itemized to fit the statement.*

- $\Rightarrow$ *Fix a homotopy $F : X \times [0,1] \to Y$ and consider $\gamma_F : [0,1] \to \mathcal{C}(X,Y)$ by $\gamma_F(t)(x) := F(x,t)$. Fix $t_0 \in [0,1]$ and take an arbitrary neighborhood $A \subseteq \mathcal{C}(X,Y)$ of $\gamma_F(t_0)$. There are $K_1, \ldots, K_n$ compact in X and $U_1, \ldots, U_n$ open in Y with $\gamma_F(t_0) \in \bigcap_{i=1}^n \mathcal{U}(K_i, U_i) \subseteq A$. For each $i = 1, \ldots, n$, $F^{-1}(U_i)$ is open in $X \times [0,1]$ and $F(K_i \times \{t_0\}) = \gamma_F(t_0)(K_i) \subseteq U_i$. For every $k \in K_i$, $(k, t_0) \in F^{-1}(U_i)$; hence, there are V_k open in X and W_{k,t_0} open in $[0,1]$ such that $(k,t_0) \in V_k \times W_{k,t_0}$ and $F(V_k \times W_{k,t_0}) \subseteq U_i$. Since K_i is compact and $\{V_k : k \in K_i\}$ is an open covering of K_i, there is a finite number $k_1^i, \ldots, k_{p_i}^i \in K_i$ such that $\{V_{k_1^i}, \ldots, V_{k_{p_i}^i}\}$ is an open subcovering of K_i. Let $W_0 = \bigcap_{i=1}^n \bigcap_{j=1}^{p_i} W_{k_j^i, t_0}$, which is an open neighborhood of t_0 in $[0,1]$. Then $\gamma_F(t)(K_i) \subseteq U_i$ for all $t \in W_0$ and for all $i = 1, \ldots, n$, that is, $\gamma_F(t) \in \bigcap_{i=1}^n \mathcal{U}(K_i, U_i) \subseteq A$. As a consequence, $\gamma_F(W_0) \subseteq \bigcap_{i=1}^n \mathcal{U}(K_i, U_i) \subseteq A$.*
- $\Leftarrow$ *Fix a path $\gamma : [0,1] \to \mathcal{C}(X,Y)$ and consider $F_\gamma : X \times [0,1] \to Y$ by $F_\gamma(x,t) := \gamma(t)(x)$. Fix $(x_0, t_0) \in X \times [0,1]$ and take an arbitrary open neighborhood U of $F_\gamma(x_0, t_0)$. Next, $\gamma(t_0)^{-1}(U)$ is an open neighborhood of x_0. Let K be a compact neighborhood of x_0 contained in $\gamma(t_0)^{-1}(U)$. Then $\mathcal{U}(K,U)$ is an open neighborhood of $\gamma(t_0)$. By the continuity of γ at t_0, there exists a neighborhood W of t_0 in $[0,1]$ satisfying that $\gamma(W) \subseteq \mathcal{U}(K,U)$. Finally, $K \times W$ is a neighborhood of (x_0, t_0) such that $F_\gamma(K \times W) = \gamma(W)(K) \subseteq U$.*

3.1.1.2 Open maps

Open maps are essential in advanced Calculus, more specifically, in Functional Analysis, since they serve to characterize homeomorphisms.

Definition 206 (Open map) *Let X, Y be topological spaces. A map $T : X \to Y$ is said to be open provided that $T(U)$ is open in Y whenever U is open in X.*

The previous definition can also be described as T is open provided that $T(\tau_X) \subseteq \tau_Y$, where τ_X, τ_Y stand for the topologies of X, Y, respectively. The following is an easy characterization of open maps.

Remark 65 *Let X, Y be topological spaces. Let $T : X \to Y$ be a map. The following conditions are equivalent:*

- *T is open.*
- *For every $x \in X$, whenever $W \subseteq X$ is a neighborhood of $x \in X$, $T(W)$ is a neighborhood of $T(x)$.*

The previous remark motivates the local notion of openness for maps in the sense that a map $T : X \to Y$, between topological spaces X, Y, is open at $x \in X$ provided that $T(W)$ is a neighborhood of $T(x)$ whenever $W \subseteq X$ is a neighborhood of x.

Remark 66 *Let X, Y be topological spaces. A map $T : X \to Y$ is continuous and open if and only if, for each $A \subseteq X$, $T^{-1}(\text{int}(A)) = \text{int}\left(T^{-1}(A)\right)$ and $\text{cl}\left(T^{-1}(A)\right) = T^{-1}(\text{cl}(A))$.*

The expected characterization of homeomorphisms in terms of open maps is a simple note that does not precise of any proof. Indeed, a map $T : X \to Y$ between topological spaces X, Y is a homeomorphism if and only if T is continuous, open and bijective.

Theorem 64 *Let X, Y be topological spaces. Let $T : X \to Y$ be a bijective map. If X is compact, Y is Hausdorff, and T is continuous, then T is open, hence a homeomorphism.*

Proof 237 *Let $U \subseteq X$ be open. Notice that $X \setminus U$ is closed in X, so it is compact by Proposition 52(3). According to Theorem 62(1), $T(X \setminus U)$ is compact in Y. Since Y is Hausdorff, Proposition 52(4) assures that $Y \setminus T(U) = T(X) \setminus T(U) = T(X \setminus U)$ is closed in $Y = T(X)$. Finally, $T(U)$ is open in Y.*

Let X be a topological space. Let $A \subseteq X$ with $\text{int}(A) \neq \varnothing$. If $B \subseteq \text{int}(A)$ is open in the relative topology of A, then B is open in X. Indeed, there exists W open in X with $B = W \cap A$. Then $B = W \cap \text{int}(A)$.

Proposition 75 *Let X, Y be topological spaces such that Y is Hausdorff. Let $x_0 \in X$. Prove that if $T : X \to Y$ is continuous in x_0, bijective, and there exists an open neighborhood V_0 de $T(x_0)$ such that $\text{cl}(T^{-1}(V_0))$ is compact, then T^{-1} is continuous in x_0.*

Proof 238 *According to the previous theorem, $T|_{\text{cl}(T^{-1}(V_0))} : \text{cl}(T^{-1}(V_0)) \to T\left(\text{cl}(T^{-1}(V_0))\right)$ is a homeomorphism. Fix any arbitrary open neighborhood U_0 of x_0. All we need to show is that $T(U_0)$ is open in Y. First, $T^{-1}(V_0)$ is open in X, so $U_0 \cap T^{-1}(V_0)$ is open in X, hence open in the relative topology of $\text{cl}(T^{-1}(V_0))$. Therefore $T(U_0 \cap T^{-1}(V_0)) = T(U_0) \cap V_0$ is open in the relative topology of $T\left(\text{cl}(T^{-1}(V_0))\right)$. Next, V_0 is open in Y, so it is open in the relative topology of $T\left(\text{cl}(T^{-1}(V_0))\right)$. Note that $V_0 = T\left(T^{-1}(V_0)\right) \subseteq T\left(\text{cl}(T^{-1}(V_0))\right)$ and V_0 is open in Y, so $T(U_0 \cap T^{-1}(V_0)) = T(U_0) \cap V_0 \subseteq V_0 \subseteq \text{int}\left(T\left(\text{cl}(T^{-1}(V_0))\right)\right)$. By the above observation, $T(U_0 \cap T^{-1}(V_0))$ is open in Y and contained in $T(U_0)$.*

3.1.1.3 One-sided continuity; semicontinuity

One-sided continuity involves the order topology in its full meaning (like one-sided limits), whereas semicontinuity involves either the left or the right order topology (like limits superior and inferior).

Definition 207 (One-sided continuity) *Let X be a topological poset. Let Y be a topological space. A function $f : X \to Y$ is said to be continuous from the right (left) at $x \in X$ if so is $f|_{\uparrow x}$ ($f|_{\downarrow x}$) when $\uparrow x$ ($\downarrow x$) is endowed with its inherited topology.*

According to Remark 153, the right order topology on a poset X is given by the subbase of right open rays $\{(x, \infty) : x \in X\} \cup \{(-\infty, \infty)\}$. In a dual way, the left order topology is given by the subbase of left open rays $\{(-\infty, y) : y \in X\} \cup \{(-\infty, \infty)\}$. If X is totally ordered, then the previous subbases are, in fact, bases.

Definition 208 (Semicontinuous map) *Let X be a topological space. Let Y be a poset. A function $f : X \to Y$ is said to be upper (lower) semicontinuous at $x \in X$ if f is continuous at x when Y is endowed with the left (right) order topology. The function f is called upper (lower) semicontinuous if so is f at every $x \in X$. The set of upper semicontinuous maps from X to A is denoted by $\mathcal{C}_{\mathtt{u}}(X, A)$. Similarly, $\mathcal{C}_{\mathtt{l}}(X, A)$ stands for the set of lower semicontinuous maps from X to A.*

The following remark shows that maximal and minimal elements are always points of upper semicontinuity and lower semicontinuity, respectively.

Remark 67 *Let X be a topological space. Let Y be a poset. Let $f : X \to Y$ be a function. Let $x_0 \in X$. If $f(x_0)$ is maximal, then f is upper semicontinuous at x_0. Indeed, since $\uparrow_{\times} f(x_0) = \varnothing$, the only open set of the left order topology containing $f(x_0)$ is Y. If U is any neighborhood of x_0, then $f(U) \subseteq Y$. In a dual way, if $f(x_0)$ is minimal, then f is lower semicontinuous at x_0.*

The following proposition characterizes upper semicontinuity under certain unimposing conditions.

Proposition 76 *Let X be a topological space. Let Y be a toset with no holes verifying the sup property. The following conditions are equivalent for a function $f : X \to Y$ and a point $x_0 \in \mathrm{ac}(X)$:*

1. *f is upper semicontinuous at x_0.*
2. *For every net $(x_i)_{i \in I} \subseteq X$ converging to x_0, $\limsup_{i \in I} f(x_i) \leq f(x_0)$, when Y is endowed with the order topology.*
3. *$\limsup_{x \to x_0} f(x) \leq f(x_0)$, when Y is endowed with the order topology.*

Proof 239 *We will only show the equivalence of (1) and (2). The equivalence of (1) and (3) follows similar ideas.*

1⇒2 Take a net $(x_i)_{i \in I} \subseteq X$ converging to x_0. Recall that

$$\limsup_{i \in I} f(x_i) := \sup\left(\mathrm{agl}\left(f(x_i)_{i \in I}\right)\right) = \sup \bigcap_{i \in I} \mathrm{cl}\left(\{f(x_j) : j \geq i\}\right).$$

Therefore, by taking into consideration that Y satisfies the sup property, it will be sufficient to prove that $y \leq f(x_0)$ for each $y \in \bigcap_{i\in I} \mathrm{cl}\,(\{f(x_j) : j \geq i\})$. Indeed, fix an arbitrary $y \in \bigcap_{i\in I} \mathrm{cl}\,(\{f(x_j) : j \geq i\})$ and suppose to the contrary that $f(x_0) < y$. Since Y has no holes, there exists $z \in Y$ with $f(x_0) < z < y$. The upper semicontinuity of f at x_0 means that there exists $U \in \mathcal{N}_{x_0}(X)$ such that $f(U) \subseteq (-\infty, z)$. Since $(x_i)_{i\in I} \subseteq X$ converges to x_0, there exists $i_0 \in I$ such that $x_j \in U$ for all $j \geq i_0$. Next, $y \in \mathrm{cl}\,(\{f(x_j) : j \geq i_0\})$. Bear in mind that Y is endowed with the order topology, so $(z, +\infty)$ is an open neighborhood of y. Then we can find $j_0 \geq i_0$ such that $f(x_{j_0}) \in (z, +\infty)$. However, $x_{j_0} \in U$ so $f(x_{j_0}) \in f(U) \subseteq (-\infty, z)$. This is a contradiction. As a consequence, $y \leq f(x_0)$. The arbitrariness of $y \in \bigcap_{i\in I} \mathrm{cl}\,(\{f(x_j) : j \geq i\})$, together with the fact that Y verifies the sup property, proves that $\limsup_{i\in I} f(x_i) \leq f(x_0)$.

2⇒1 Suppose to the contrary that f is not upper semicontinuous at x_0. There exists an interval of the form $(-\infty, y)$ containing $f(x_0)$ in such a way that, for every $U \in \mathcal{N}_{x_0}(X)$, we can find $x_U \in U$ with $f(x_U) \notin (-\infty, y)$. Consider the net $(x_U)_{U\in\mathcal{N}_{x_0}(X)}$. Observe that y is a lower bound for the net $(f(x_U))_{U\in\mathcal{N}_{x_0}(X)}$. In view of Proposition 69(1), we conclude that

$$f(x_0) < y \leq \liminf_{i\in I} f(x_i) \leq \limsup_{i\in I} f(x_i).$$

In a dual way, the previous proposition has its version for lower semicontinuity.

Lemma 95 *Let X be a topological space. Let Y be a topological poset satisfying the strong sup property. Let $\{f_i\}_{i\in I}$ be a family of functions from X to Y. Then:*

1. *If $\{f_i\}_{i\in I}$ is pointwise bounded below and f_i is upper semicontinuous for all $i \in I$, then $\inf_{i\in I} f_i$ is upper semicontinuous.*
2. *If $\{f_i\}_{i\in I}$ is pointwise bounded above and f_i is lower semicontinuous for all $i \in I$, then $\sup_{i\in I} f_i$ is lower semicontinuous.*

Proof 240 *We will prove only the second item. Denote $f := \sup_{i\in I} f_i$. Let V be a neighborhood of $f(x_0)$ in Y for the right order topology. We will find a neighborhood U of $x_0 \in X$ such that $f(U) \subseteq V$. By construction of the right order topology, if $\downarrow^{\times} f(x_0) = \varnothing$, then $V = Y$, hence we can take $U := X$. If $\downarrow^{\times} f(x_0) \neq \varnothing$, then there are $y_1, \ldots, y_n \in Y$ satisfying that $f(x_0) \in \uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n \subseteq V$. By hypothesis, $f(x_0) \in \mathrm{cl}(\{f_i(x_0) : i \in I\})$ in the order topology, meaning that $(\uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n) \cap \{f_i(x_0) : i \in I\} \neq \varnothing$, that is, there exists $j \in I$ such that $y_k < f_j(x_0)$ for all $k = 1, \ldots, n$. Since f_j is lower semicontinuous at x_0, there exists a neighborhood U of x_0 such that $f_j(U) \subseteq \uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n$. Finally, $y_k < f_j(u) \leq f(u)$ for all $u \in U$, meaning that $f(U) \subseteq \uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n \subseteq V$.*

The following remark, which will be called on when working on convex optimization, is a technical remark that allows to compute global sups. And, obviously, it has its dual version for infs.

Remark 68 *Let X be a set. Let Y be a poset. Let $\{f_i\}_{i\in I}$ be a pointwise bounded above family of functions from X to Y such that $\sup_{i\in I} f_i(x)$ exists for all $x \in X$. Let $f := \sup_{i\in I} f_i$. If $\sup f_i(X)$ exists for all $i \in I$ and $\sup\{\sup f_i(X) : i \in I\}$ exists as well, then $\sup f(X)$ exists and $\sup f(X) = \sup\{\sup f_i(X) : i \in I\}$. Indeed, denote $y := \sup\{\sup f_i(X) : i \in I\}$. For an arbitrarily fixed $x \in X$, $y \geq \sup f_i(X) \geq f_i(x)$ for all $i \in I$, thus y is an upper bound for $\{f_i(x) : i \in I\}$, hence $y \geq f(x)$. Then y is an upper bound for $f(X)$. Let $z \in Y$ be another upper bound for $f(X)$. Fix an arbitrary $i \in I$. For every $x \in X$, $z \geq f(x) \geq f_i(x)$. This means that z is an upper bound of $f_i(X)$, meaning that $\sup f_i(X) \leq z$. Therefore, z is an upper bound of $\{\sup f_i(X) : i \in I\}$, resulting in $y \leq z$. This shows that $y = \sup f(X)$.*

Notice that, in a topological toset X, $\sup(A) = \sup(\mathrm{cl}(A))$ for each subset $A \subseteq X$ whenever those sups exist. Also, it is worth mentioning that every continuous map $f : X \to Y$ between topological spaces X, Y satisfies that $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ for each subset $A \subseteq X$.

Lemma 96 *Let X be a topological space. Let Y be a topological toset. Let $A \subseteq X$. Let $f : X \to Y$ be a lower semicontinuous function. Then* $\sup f(A)$ *exists if and only if* $\sup f(\mathrm{cl}(A))$ *exists and, in this situation,* $\sup f(A) = \sup f(\mathrm{cl}(A))$.

Proof 241 *Suppose first that* $\sup f(A)$ *exists. If y is an upper bound for $f(\mathrm{cl}(A))$, then y is also an upper bound for $f(A)$, meaning that* $\sup f(A) \leq y$. *Let us show that* $\sup f(A)$ *is a upper bound for $f(\mathrm{cl}(A))$. If not, there exists $b \in \mathrm{cl}(A)$ such that* $\sup f(A) < f(b)$. *Then* $\uparrow_{\times} \sup f(A)$ *is an open neighborhood of $f(b)$ in the right order topology. Since f is lower semicontinuous, there exists a neighborhood U of b in X such that $f(U) \subseteq \uparrow_{\times} \sup f(A)$. Since $U \cap A \neq \varnothing$, by considering any $a \in U \cap A$, we reach the contradiction that* $\sup f(A) < f(a)$. *This shows that* $\sup f(A) = \sup f(\mathrm{cl}(A))$. *Conversely, suppose that* $\sup f(\mathrm{cl}(A))$ *exists. Observe that* $\sup f(\mathrm{cl}(A))$ *is an upper bound for $f(A)$. Let $y \in Y$ be an upper bound for $f(A)$. Let us show that y is an upper bound for $f(\mathrm{cl}(A))$. If not, there exists $b \in cl(A)$ such that $y < f(b)$. Then $\uparrow_{\times} y$ is an open neighborhood of $f(b)$ in the right order topology. Since f is lower semicontinuous, there exists a neighborhood U of b in X such that $f(U) \subseteq \uparrow_{\times} y$. Since $U \cap A \neq \varnothing$, by considering any $a \in U \cap A$, we reach the contradiction that $y < f(a)$. Finally, since y is an upper bound for $f(\mathrm{cl}(A))$, we obtain that* $\sup f(\mathrm{cl}(A)) \leq y$. *This shows that* $\sup f(\mathrm{cl}(A)) = \sup f(A)$.

In a dual way, the previous lemma has its version for lower semicontinuity. The final theorem in this subsubsection constitutes a topological generalization of the famous and classical Extreme Value Theorem.

Theorem 65 (Extreme Value Theorem) *Let X be a compact topological space. Let Y be a toset. If $f : X \to Y$ is upper (lower) semicontinuous, then f is bounded above (below) and attains its maximum (minimum).*

Proof 242 *By hypothesis, f is continuous when Y is endowed with the left order topology. As a consequence, $f(X)$ is compact in Y for the left order topology. By applying Proposition 56, there exists* $\max f(X)$.

3.1.2 Uniform continuity

As we have seen in the previous subsection, pointwise continuity lives in the scope of topological spaces. However, uniform continuity lives in the scope of uniform spaces.

3.1.2.1 Uniform continuity in uniform spaces

In the first place, observe that if X, Y are pseudometric spaces and $f : X \to Y$ is continuous at $x \in X$, then $d(f(x), f(y)) = 0$ whenever $y \in X$ and $d(x, y) = 0$.

Definition 209 (Uniformly continuous function) *Let X, Y be uniform spaces. A function $f : X \to Y$ is said to be uniformly continuous provided that, for every entourage V in Y, there exists an entourage U in X such that $(f \times f)(U) \subseteq V$.*

Under the settings of the previous definition,

$$(f \times f)(U) := \{(f(x_1), f(x_2)) \in Y \times Y : (x_1, x_2) \in U\}.$$

Trivial examples of uniformly continuous functions are the identity and the constant functions as well.

Remark 69 *Let X, Y, Z be uniform spaces. Let $f : X \to Y, g : Y \to Z$ be uniformly continuous functions. Then:*

1. *$g \circ f$ is uniformly continuous.*
2. *f is continuous when X, Y are endowed with the uniform topology.*

There is a famous theorem in classical Calculus that assures that the uniform limit of continuous functions is continuous. We will prove such a theorem in the abstract settings we are dealing with in this book.

Theorem 66 *Let Y be a uniform space. Let X be a topological space. Suppose that Y^X is endowed with the finest convergence uniformity. Consider $x_0 \in X$. Let $(f_i)_{i \in I} \subseteq Y^X$ be a net converging to $f \in Y^X$. If f_i is continuous at x_0 for every $i \in I$, then f is continuous at x_0 as well.*

Proof 243 *Fix an arbitrary symmetric entourage $U \subseteq Y$. Take a symmetric entourage W with $W \circ W \circ W \subseteq U$. There exists $i_W \in I$ such that, if $i \in I$ and $i \geq i_W$, then $f_i \in \mathcal{U}(X, W)[f]$. Since f_{i_0} is continuous at x_0, there exists $V \in \mathcal{N}_{x_0}(X)$ such that $f_{i_0}(V) \subseteq W[f_{i_0}(x_0)]$. Let us show that $f(V) \subseteq U[f(x_0)]$, which proves the continuity of f at x_0 due to the arbitrariness of the entourage U. If $v \in V$, then $(f(v), f_{i_0}(v)), (f_{i_0}(v), f_{i_0}(x_0)), (f_{i_0}(x_0), f(x_0)) \in W$, meaning that $(f(v), f(x_0)) \in W \circ W \circ W \subseteq U$.*

Another classical Calculus results asserts that continuous functions defined on a compact uniform space must be uniformly continuous.

Proposition 77 *Let X, Y be uniform spaces. Let $f : X \to Y$ be a continuous function. If X is compact, then f is uniformly continuous.*

Proof 244 *Fix an arbitrary entourage V in Y. Let W be a symmetric entourage in Y such that $W \circ W \subseteq V$. For every $x \in X$, there exists an entourage U_x in X such that $f(U_x[x]) \subseteq W[f(x)]$. For every $x \in X$, we can find another entourage U'_x in X such that $U'_x \circ U'_x \subseteq U_x$. Consider the covering $\{U'_x[x] : x \in X\}$ and extract a finite subcovering $\{U'_{x_i}[x_i] : i = 1, \ldots, l\}$. Let $U' := U_{x_1} \cap \cdots \cap U_{x_l}$. Take any $(x, z) \in U'$. There exists $i \in \{1, \ldots, l\}$ such that $x \in U'_{x_i}[x_i]$. On the one hand, $(x_i, x) \in U'_{x_i} \subseteq U_{x_i}$, meaning that $(f(x_i), f(x)) \in W$. On the other hand, $(x_i, z) \in U'_{x_i} \circ U' \subseteq U'_{x_i} \circ U'_{x_i} \subseteq U_{x_i}$, therefore, $(f(x_i), f(z)) \in W$. Finally, $(f(x), f(z)) \in W \circ W \subseteq V$*

3.1.2.2 Index of uniform continuity

Recall that, if X, Y are pseudometric spaces and $f : X \to Y$ is a function, then, for $\varepsilon, \delta > 0$, we denote:

- $\mathsf{U}_\delta := \{(x_1, x_2) \in X \times X : d(x_1, x_2) < \delta\}$.
- $f(\mathsf{U}_\delta) := \{(f(x_1), f(x_2)) \in Y \times Y : (x_1, x_2) \in \mathsf{U}_\delta\}$.
- $\mathsf{V}_\varepsilon := \{(y_1, y_2) \in Y \times Y : d(y_1, y_2) < \varepsilon\}$.

In order for the following definition to be consistent, we will use the notation $\mathsf{U}_0 := \varnothing =: \mathsf{V}_0$.

Definition 210 (Index of uniform continuity) *Let X, Y be pseudometric spaces. Let $f : X \to Y$ be a function. The index of uniform continuity of f is defined as*

$$\begin{array}{rccl} \delta_f : & [0, \infty) & \to & [0, \infty] \\ & \varepsilon & \mapsto & \delta_f(\varepsilon) := \sup\{\delta \in [0, \infty) : f(\mathsf{U}_\delta) \subseteq \mathsf{V}_\varepsilon\}. \end{array}$$

Under the settings of the previous definition, it is clear that $\delta_f(0) = 0$ and $f\left(\mathsf{U}_{\delta_f(\varepsilon)}\right) \subseteq \mathsf{V}_\varepsilon$ for every $\varepsilon > 0$, in other words, the above sup is attained if it is finite. In fact, if f is constant, then $\delta_f(\varepsilon) = \infty$ for all $\varepsilon > 0$.

Remark 70 *Let X, Y be pseudometric spaces. The following conditions are equivalent for a function $f : X \to Y$:*

1. *f is uniformly continuous.*
2. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon$ whenever $d(x_1, x_2) < \delta$.*
3. *$\delta_f(\varepsilon) > 0$ for every $\varepsilon > 0$.*

The previous remark highlights the classical definition of uniform continuity, which is precisely the analytical definition, reformulated by means of the index of uniform continuity.

3.1.2.3 Lipschitz functions

The most representative example of uniformly continuous function is given by the Lipschitz functions (although not every uniformly continuous function is Lipschitz). If X, Y are pseudometric spaces and $f : X \to Y$ is a function, then we define the following set for every $K \geq 0$:

$$L_X(f, K) := \{(x_1, x_2) \in X \times X : d(f(x_1), f(x_2)) \leq K d(x_1, x_2)\}.$$

Note that $\Delta_X \subseteq L_X(f, K)$. We also define the set

$$L(f) := \{K \geq 0 : L_X(f, K) = X \times X\}.$$

Definition 211 (Lipschitz constant) *Let X, Y be pseudometric spaces. The Lipschitz constant of a function $f : X \to Y$ is defined as*

$$\mathrm{Lip}(f) := \begin{cases} \infty & L(f) = \varnothing, \\ \inf(L(f)) & L(f) \neq \varnothing. \end{cases}$$

The function f is said to be Lipschitz provided that $\mathrm{Lip}(f) < \infty$.

Observe that, if $L(f) \neq \varnothing$, then $\mathrm{Lip}(f) \in L(f)$. Also, it is clear that, if f is constant, then $\mathrm{Lip}(f) = 0$.

Lemma 97 *Let X, Y be metric spaces. Let $f : X \to Y$ be a function. If* $\mathrm{Lip}(f) = 0$*, then f is constant.*

Proof 245 *For every $x_1, x_2 \in X$, $(x_1, x_2) \in X \times X = L_X(f, \mathrm{Lip}(f))$. Thus, $d(f(x_1), f(x_2)) \leq \mathrm{Lip}(f) d(x_1, x_2) = 0$. Since Y is metric, $f(x_1) = f(x_2)$. The arbitrariness of $x_1, x_2 \in X$ allows to deduce that f is constant.*

The following proposition relates Lipschitz constants with the index of uniform continuity.

Proposition 78 *Let X, Y be pseudometric spaces. Consider a nonconstant Lipschitz function $f : X \to Y$. Then*

$$\frac{\varepsilon}{\mathrm{Lip}(f)} \leq \delta_f(\varepsilon)$$

for every $\varepsilon \geq 0$.

Proof 246 *Simply notice that* $f\left(\mathsf{U}_{\varepsilon \mathrm{Lip}(f)^{-1}}\right) \subseteq \mathsf{V}_{\varepsilon}$, *which implies that* $\dfrac{\varepsilon}{\mathrm{Lip}(f)} \leq \delta_f(\varepsilon)$.

Next technical lemma provides a sufficient condition to obtain an admissible pseudometric topology.

Lemma 98 *Let* X *be a pseudometric space endowed with a topology* τ *coarser than the pseudometric topology and such that the mapping*

$$\begin{array}{rccc} d(\bullet, y): & X & \to & \mathbb{R} \\ & x & \mapsto & d(x,y) \end{array} \tag{3.1}$$

is τ*-lower semicontinuous for every* $y \in X$. *Then* $\mathsf{B}_X(x,r)$ *is* τ*-closed for every* $x \in X$ *and every* $r > 0$. *In other words,* τ *is an admissible pseudometric topology.*

Proof 247 *The* τ*-lower semicontinuity of the mapping* (3.1) *is equivalent to asserting that, for every* $x \in X$, *if* $(x_i)_{i \in I}$ *is a net in* X *which is* τ*-convergent to* x, *then*

$$d(x,y) \leq \liminf_{i \in I} d(x_i, y)$$

for every $y \in X$. *Let us prove now that* $\mathsf{B}_X(x,r)$ *is* τ*-closed for every* $x \in X$ *and every* $r > 0$. *Indeed, fix arbitrary elements* $x \in X$ *and* $r > 0$. *Let* $(y_i)_{i \in I}$ *be a net in* $\mathsf{B}_X(x,r)$ *which is* τ*-convergent to some* $y \in X$. *By hypothesis,*

$$d(y,x) \leq \liminf_{i \in I} d(y_i, x) \leq r.$$

As a consequence, $y \in \mathsf{B}_X(x,r)$.

With the help of Lemma 98, we will construct nontrivial admissible pseudometric topologies. But first, we need to introduce the pointed Lipschitz dual.

Definition 212 (Pointed Lipschitz dual) *Let* X *be a pseudometric space. Fix* $x_0 \in X$. *The* x_0*-pointed Lipschitz dual of* X *is defined as* $\mathrm{Lip}(X, x_0) := \{T : X \to \mathbb{R} : T(x_0) = 0$ *and* T *is Lipschitz*$\}$.

Now we are in the right position to construct nontrivial admissible pseudometric topologies.

Theorem 67 *Let* X *be a pseudometric space with at least two points. Fix* $x_0 \in X$. *The initial topology* $\sigma\left(X, \mathrm{Lip}(X, x_0)\right)$ *of* X *generated by* $\mathrm{Lip}(X, x_0)$ *is an admissible pseudometric topology on* X, *which is Hausdorff if* X *is metric.*

Proof 248 *Let us recall that the initial topology* $\sigma\left(X, \mathrm{Lip}(X, x_0)\right)$ *of* X *generated by* $\mathrm{Lip}(X, x_0)$ *is the coarsest topology* τ *on* X *satisfying that all the mappings* $T \in \mathrm{Lip}(X, x_0)$ *are continuous from* (X, τ) *to* $\mathbb{R}$. *A subbase of* τ *is*

$$\mathcal{B}_\tau := \left\{T^{-1}(G) : T \in \mathrm{Lip}(X, x_0),\ G \subseteq \mathbb{R} \text{ open}\right\}.$$

We will prove in the next two items that τ *is an admissible pseudometric topology on* X*:*

- τ *is coarser than the metric topology. Indeed, since every element of* $\mathrm{Lip}(X, x_0)$ *is continuous when* X *is endowed with its on metric topology,* τ *is by definition coarser than the metric topology.*

- *The mapping $d(\bullet, y)$ given in (3.1) is τ-lower semicontinuous for all $y \in Y$. In fact, we will show that $d(\bullet, y)$ is τ-continuous for all $y \in X$. Indeed, fix an arbitrary $y \in X$. The mapping ϕ_y described in (3.2) is an element of $\mathrm{Lip}(X, x_0)$. As a consequence, ϕ_y is τ-continuous. Since $d(\bullet, y) = \phi_y + d(x_0, y)$, we conclude that $d(\bullet, y)$ is simply a translation of ϕ_y, hence $d(\bullet, y)$ is τ-continuous.*

Next, it only suffices to call on Lemma 98. Finally, let us prove that τ is Hausdorff provided that X is metric. Indeed, fix arbitrary elements $x, y \in X$ with $x \neq y$. Consider the mapping

$$\begin{array}{rccl} \phi_y : & X & \to & \mathbb{R} \\ & z & \mapsto & \phi_y(z) := d(z, y) - d(x_0, y). \end{array} \tag{3.2}$$

Notice that $\phi_y(x_0) = 0$ and $|\phi_y(a) - \phi_y(b)| = |d(a, y) - d(b, y)| \leq d(a, b)$ for all $a, b \in X$. As a consequence, $\phi_y \in \mathrm{Lip}(X, x_0)$. Observe that $\phi_y(y) = -d(x_0, y)$ and $\phi_y(x) = d(x, y) - d(x_0, y)$. Since $d(x, y) > 0$ because $x \neq y$, we conclude that $\phi_y(x) \neq \phi_y(y)$. Since ϕ_y is τ-continuous, it suffices to consider $0 < \varepsilon < \frac{|\phi_y(x) - \phi_y(y)|}{2} = \frac{d(x,y)}{2}$ to construct two disjoint τ-neighborhoods $U := \phi_y^{-1}(\phi_y(x) - \varepsilon, \phi_y(x) + \varepsilon)$ and $V := \phi_y^{-1}(\phi_y(y) - \varepsilon, \phi_y(y) + \varepsilon)$ of x and y, respectively.

3.1.2.4 Nonexpansive and contractive functions

Nonexpansive functions and contractive functions are a very specific type of Lipschitz functions, which are extremely important for their fixed-point properties.

Definition 213 (Nonexpansive & contractive functions) *Let X, Y be pseudometric spaces. A function $f : X \to Y$ is said to be:*

- *Nonexpansive provided that $\mathrm{Lip}(f) \leq 1$.*
- *Contractive provided that $\mathrm{Lip}(f) < 1$.*

Recall that a fixed point of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. Also, note as well that if X, Y are pseudometric spaces and $f : X \to Y$ is a Lipschitz function, then

$$d\left(f^n(x_1), f^n(x_2)\right) \leq \mathrm{Lip}(f)^n d(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Theorem 68 (Fixed-Point Banach Theorem) *Let X be a complete metric space. If $f : X \to X$ is contractive, then there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.*

Proof 249 *Assume first that there are two different fixed points x_0 and x_1. Then we obtain the following contradiction:*

$$d(x_0, x_1) = d\left(f(x_0), f(x_1)\right) \leq \mathrm{Lip}(f) d(x_0, x_1) < d(x_0, x_1).$$

Fix an arbitrary $x \in X$ and consider the sequence $(f^n(x))_{n \in \mathbb{N}}$. If $p > q$, then

$$\begin{array}{rcl} d\left(f^p(x), f^q(x)\right) & \leq & d\left(f^p(x), f^{p-1}(x)\right) + \cdots + d\left(f^{q+1}(x), f^q(x)\right) \\ & \leq & \mathrm{Lip}(f)^{p-1} d\left(f(x), x\right) + \cdots + \mathrm{Lip}(f)^q d\left(f(x), x\right) \\ & = & d\left(f(x), x\right) \displaystyle\sum_{i=q}^{p-1} \mathrm{Lip}(f)^i \\ & = & d\left(f(x), x\right) \dfrac{\mathrm{Lip}(f)^q - \mathrm{Lip}(f)^p}{1 - \mathrm{Lip}(f)}. \end{array}$$

This shows that $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, $(f^n(x))_{n\in\mathbb{N}}$ converges to some $x_0 \in X$. Finally,

$$f(x_0) = f\left(\lim_{n\to\infty} f^n(x)\right) = \lim_{n\to\infty} f^{n+1}(x) = x_0.$$

3.1.3 Universal topologies

There are certain types of topologies which are universal in the sense that they verify universal properties. The most representative universal topologies are the initial topology and the final topology. Both topologies are defined by means of continuous functions.

3.1.3.1 Initial topology

The initial topology is the coarsest topology that makes continuous a certain family of functions. The pointwise convergence topology is a particular case of the initial topology.

Definition 214 (Initial topology) *Let X be a set. Let $\mathcal{Y}$ be a family of topological spaces. Let $\mathcal{F}$ be a family of maps from X into a topological space of $\mathcal{Y}$, that is, $\mathcal{F} \subseteq \bigcup_{Y\in\mathcal{Y}} Y^X$. The initial topology of X generated by $\mathcal{F}$ is defined as the coarsest topology on X that makes all the functions in $\mathcal{F}$ continuous. This topology is denoted by $\sigma(X,\mathcal{F})$.*

The initial topology is universal because it verifies the universal property given in the following theorem.

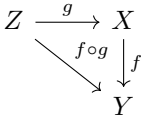
Theorem 69 *Let X be a set. Let $\mathcal{Y}$ be a family of topological spaces. Let $\mathcal{F} \subseteq \bigcup_{Y\in\mathcal{Y}} Y^X$. Then:*

1. *$\sigma(X,\mathcal{F})$ is generated by the subbase*

$$\mathcal{S} := \left\{f^{-1}(U) : f \in \mathcal{F},\ U \text{ open in } Y \in \mathcal{Y},\ \text{where } f : X \to Y\right\}.$$

2. *A net $(x_j)_{j\in J} \subseteq X$ is $\sigma(X,\mathcal{F})$-convergent to $x \in X$ if and only if $(f(x_j))_{j\in J}$ converges to $f(x)$ for all $f \in \mathcal{F}$.*

3. *For every topological space Z and every map $g : Z \to X$, g is continuous if and only if $f \circ g$ is continuous for all $f \in \mathcal{F}$.*



Proof 250 *The proof is itemized following the statement.*

1. *It is trivial that $\mathcal{S}$ is a subbase for a topology on X and this topology clearly makes all the elements of $\mathcal{F}$ continuous. Conversely, if there exists a topology on X for which every element of $\mathcal{F}$ is continuous, then $f^{-1}(U)$ is open in that topology for every $f \in \mathcal{F}$ and every U open in $Y \in \mathcal{Y}$ (where $f : X \to Y$). As a consequence, the topology generated by the subbase $\mathcal{S}$ is the coarsest topology making all the elements of $\mathcal{F}$ continuous.*

2. *If $(x_j)_{j\in J} \subseteq X$ is $\sigma(X,\mathcal{F})$-convergent to x, then $(f(x_j))_{j\in J}$ converges to $f(x)$ for all $f \in \mathcal{F}$ due to the continuity of every $f \in \mathcal{F}$. Conversely, suppose that $(f(x_j))_{j\in J}$ converges to $f(x)$ for all $f \in \mathcal{F}$. Let $V \subseteq X$ be a neighborhood of x in the initial topology.*

We can find $f_1, \ldots, f_k \in \mathcal{F}$ and $U_1, \ldots, U_k$ open in $Y_1, \ldots, Y_k \in \mathcal{Y}$, respectively, where $f_i : X \to Y_i$ for $i = 1, \ldots, k$, such that $x \in f_1^{-1}(U_1) \cap \cdots \cap f_k^{-1}(U_k) \subseteq V$. Notice that U_i is a neighborhood of $f(x)$ in Y_i for $i = 1, \ldots, k$. Thus, for $i = 1, \ldots, k$, there exists $j_i \in J$ such that $f_i(x_j) \in U_i$ whenever $j \geq j_i$. We can find $j_0 \in J$ in such a way that $j_0 \geq j_i$ for $i = 1, \ldots, k$. Then $f_i(x_j) \in U_i$ for all $j \geq j_0$ and for every $i = 1, \ldots, k$. Finally, $x_j \in f_1^{-1}(U_1) \cap \cdots \cap f_k^{-1}(U_k) \subseteq V$ for all $j \geq j_0$. This shows that $(x_j)_{j \in J}$ is $\sigma(X, \mathcal{F})$-convergent to $x \in X$.

3. *If $g : Z \to X$ is continuous, then $f \circ g$ is continuous for all $f \in \mathcal{F}$ due to the continuity of every $f \in F$. Conversely, suppose that $f \circ g$ is continuous for all $f \in \mathcal{F}$. In order to show that g is continuous, it only suffices to prove that the preimage under g of every element of the subbase $\mathcal{S}$ is open in Z. So, let $f \in \mathcal{F}$ and U open in $Y \in \mathcal{Y}$, where $f : X \to Y$. Observe that $g^{-1}\left(f^{-1}(U)\right) = (f \circ g)^{-1}(U)$ is open in Z by hypothesis.*

There are three representative examples of initial topologies: the *product topology*, the *pointwise convergence topology*, and the *weak topology*.

Example 45 (Product topology) *If $(X_i)_{i \in I}$ is a family of topological spaces, then the product topology on $\prod_{i \in I} X_i$ is given by $\sigma\left(\prod_{i \in I} X_i, \{\pi_i\}_{i \in I}\right)$, where π_i is the projection on the i^{th}-coordinate. Indeed, if $J \subseteq I$ is finite and V_j is open in X_j for all $j \in J$, then $\bigcap_{j \in J} \pi_j^{-1}(V_j) = \prod_{i \in I} V_i$, where $V_i = X_i$ for all $i \in I \setminus J$.*

According to Remark 6, if X, Y are sets and $\mathcal{F} \subseteq Y^X$, then every $x \in X$ induces a function from $\mathcal{F}$ to Y via

$$\begin{array}{rccccccc} \widehat{\ }: & X & \to & Y^{\mathcal{F}} & & & & \\ & & & \widehat{x}: & \mathcal{F} & \to & Y \\ & x & \mapsto & & f & \mapsto & \widehat{x}(f) := f(x). \end{array}$$

If $\mathcal{F}$ is separating, then the above mapping is injective, meaning that X can be identified with $\widehat{X}$.

Theorem 70 (Pointwise convergence topology) *Let X be a set. Let Y be a topological space. Let $\mathcal{F} \subseteq Y^X$. Then:*

1. *$\sigma\left(\mathcal{F}, \widehat{X}\right) = \sigma\left(\mathcal{F}, \mathcal{P}_{\mathtt{f}}(X)\right)$.*

2. *If $\mathcal{F}$ separates points of X, then $\widehat{\ }: (X, \sigma(X, \mathcal{F})) \to \left(\widehat{X}, \sigma\left(\widehat{X}, \mathcal{P}_{\mathtt{f}}(\mathcal{F})\right)\right)$ is a homeomorphism.*

Proof 251 *The proof is itemized the same way as the statement is.*

1. *For every finite subset $G = \{x_1, \ldots, x_n\}$ of X, every open subset U of Y, $\mathcal{U}(G, U) = \bigcap_{x \in G} \widehat{x}^{\,-1}(U)$. This shows that $\sigma\left(\mathcal{F}, \widehat{X}\right) \supseteq \sigma\left(\mathcal{F}, \mathcal{P}_{\mathtt{f}}(X)\right)$. Conversely, if $x \in X$ and U is an open subset of Y, then $\widehat{x}^{\,-1}(U) = \mathcal{U}(\{x\}, U)$. This implies that $\sigma\left(\mathcal{F}, \widehat{X}\right) \subseteq \sigma\left(\mathcal{F}, \mathcal{P}_{\mathtt{f}}(X)\right)$.*

2. *Since $\mathcal{F}$ separates points of X, we have that $\widehat{\ }: X \to \widehat{X}$ is a bijection. Fix an arbitrary element of the subbase of $\sigma(X, \mathcal{F})$, that is, of the form $f^{-1}(U)$, where $f \in \mathcal{F}$ and U is open in Y. Then $\widehat{f^{-1}(U)} = \mathcal{U}(\{f\}, U)$. Conversely, fix an arbitrary element of the subbase of $\sigma\left(\widehat{X}, \mathcal{P}_{\mathtt{f}}(\mathcal{F})\right)$, that is, of the form $\mathcal{U}(G, U)$, where G is a finite subset of $\mathcal{F}$ and U is open in Y. Then $\mathcal{U}(G, U) = \widehat{\bigcap_{f \in G} f^{-1}(U)}$.*

The weak topology is another particular case of initial topology.

Example 46 (Weak topology) *If X, Y are topological spaces, then the Y-weak topology on X is defined as the initial topology of X relative to $\mathcal{C}(X,Y)$, that is, $\sigma\left(X, \mathcal{C}(X,Y)\right)$, or even to a special subset of $\mathcal{C}(X,Y)$. By definition, the weak topology is contained in the original topology.*

Recall that a family $\mathcal{F}$ of functions on a set X is said to separate points of X provided that, for every $x, y \in X$ with $x \neq y$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

Theorem 71 *Let X be a set. Let $\mathcal{Y}$ be a family of Hausdorff topological spaces. Let $\mathcal{F} \subseteq \bigcup_{Y \in \mathcal{Y}} Y^X$. If $\mathcal{F}$ separates points of X, then $\sigma(X, \mathcal{F})$ is Hausdorff.*

Proof 252 *Fix arbitrary elements $x, z \in X$ with $x \neq z$. By hypothesis, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(z)$. Let $Y \in \mathcal{Y}$ such that $f \in Y^X$. Since Y is Hausdorff, we can find $U \in \mathcal{N}_{f(x)}(Y), V \in \mathcal{N}_{f(z)}(Y)$ such that $U \cap V = \varnothing$. Then $f^{-1}(U) \cap f^{-1}(V) = \varnothing$ and $f^{-1}(U)$ is a neighborhood of x in X and $f^{-1}(V)$ is a neighborhood of z in X when X is endowed with $\sigma(X, \mathcal{F})$.*

As an immediate corollary, we obtain that the product topology of Hausdorff spaces is Hausdorff (Theorem 32). The final lemma of this subsubsection establishes that the initial topology in a product space is the product topology of the initial topologies.

Lemma 99 *Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $\mathcal{Y}$ be another family of topological spaces. For every $i \in I$, let $\mathcal{F}_i \subseteq \bigcup_{Y \in \mathcal{Y}} Y^{X_i}$. Let $\mathcal{F} := \{f \circ \pi_j : j \in I,\ f \in \mathcal{F}_j\}$. Then $\sigma\left(\prod_{i \in I} X_i, \mathcal{F}\right)$ coincides with the product topology of $\sigma(X_i, \mathcal{F}_i)$ for $i \in I$.*

Proof 253 *Fix arbitrary elements $j \in I$ and $f \in \mathcal{F}_j$. If V is any open set in the range of f, then $(f \circ \pi_j)^{-1}(V) = \prod_{i \in I} W_i$, where $W_i := X_i$ if $i \in I \setminus \{j\}$ and $W_j := f^{-1}(V)$. Therefore, every element of $\mathcal{F}$ is continuous when $\prod_{i \in I} X_i$ is endowed with the product topology of $\sigma(X_i, \mathcal{F}_i)$ for $i \in I$. This shows that $\sigma\left(\prod_{i \in I} X_i, \mathcal{F}\right)$ is contained in the product topology of $\sigma(X_i, \mathcal{F}_i)$ for $i \in I$. Conversely, the elements of the form $\prod_{i \in I} W_i$, where $W_i := X_i$ if $i \in I \setminus \{j\}$ and $W_j := f^{-1}(V)$, for some $j \in I$, some $f \in \mathcal{F}_j$, and some open set V in the range of f, form a subbase of the product topology of $\sigma(X_i, F_i)$ for $i \in I$. However, $\prod_{i \in I} W_i = (f \circ \pi_j)^{-1}(V)$, meaning that $\prod_{i \in I} W_i$ is open in $\sigma\left(\prod_{i \in I} X_i, \mathcal{F}\right)$. As a consequence, the product topology of $\sigma(X_i, \mathcal{F}_i)$ for $i \in I$ is contained in $\sigma\left(\prod_{i \in I} X_i, \mathcal{F}\right)$.*

3.1.3.2 Final topology

The final topology is constructed in a dual manner with respect to the initial topology.

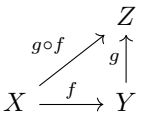
Definition 215 (Final topology) *Let Y be a set. Let $\mathcal{X}$ be a family of topological spaces. Let $\mathcal{F}$ be a family of functions from a topological space of $\mathcal{X}$ to Y, that is, $\mathcal{F} \subseteq \bigcup_{X \in \mathcal{X}} Y^X$. The final topology of Y for $\mathcal{F}$ is defined as the finest topology on Y that makes all the elements of $\mathcal{F}$ continuous. This topology is denoted by $\tau(Y, \mathcal{F})$.*

The final topology is universal because it verifies the universal property given in the following theorem.

Theorem 72 *Let Y be a set. Let $\mathcal{X}$ be a family of topological spaces. Let $\mathcal{F} \subseteq \bigcup_{X \in \mathcal{X}} Y^X$. Then:*

1. $\tau(Y, \mathcal{F}) = \left\{U \subseteq Y : \forall f \in \mathcal{F}\ \ f^{-1}(U) \text{ open in } X \in \mathcal{X}, \text{ with } f : X \to Y\right\}$.

2. *For every topological space Z and every map $g : Y \to Z$, g is continuous if and only if $g \circ f$ is continuous for all $f \in \mathcal{F}$.*



Proof 254 *Let us itemize the proof by following the statement.*

1. *Note that $\left\{U \subseteq Y : \forall f \in \mathcal{F} \;\; f^{-1}(U) \text{ open in } X \in \mathcal{X}, \text{ with } f : X \to Y\right\}$ is a topology on Y that makes every function $f \in \mathcal{F}$ continuous. Conversely, if there exists a topology on Y making every element of $\mathcal{F}$ continuous, then such topology must be contained in $\left\{U \subseteq Y : \forall f \in \mathcal{F} \;\; f^{-1}(U) \text{ open in } X \in \mathcal{X}, \text{ with } f : X \to Y\right\}$.*

2. *If $g : Y \to Z$ is continuous, then $g \circ f$ is continuous for all $f \in \mathcal{F}$ due the continuity of every $f \in \mathcal{F}$. Conversely, suppose that $g \circ f$ is continuous for all $f \in \mathcal{F}$. Let W be any open subset of Z. Fix an arbitrary $f \in \mathcal{F}$ and write $f : X \to Y$ with $X \in \mathcal{X}$. By hypothesis, $f^{-1}\left(g^{-1}(W)\right) = (g \circ f)^{-1}(W)$ is open in X. The arbitrariness of $f \in \mathcal{F}$ allows that $g^{-1}(W)$ be open in Y.*

The most representative example of a final topology is, undoubtedly, the quotient topology.

Example 47 (Quotient topology) *Let X be a topological space. Consider an equivalence relation $\mathcal{R}$ on X. Consider the canonical projection*

$$\begin{array}{rccc} [\ ]_{\mathcal{R}} : & X & \to & {}^{X}\!/_{\mathcal{R}} \\ & x & \mapsto & [x]_{\mathcal{R}}. \end{array}$$

The quotient topology on ${}^{X}\!/_{\mathcal{R}}$ is defined as the final topology on ${}^{X}\!/_{\mathcal{R}}$ given by the family $\{[\ ]_{\mathcal{R}}\}$, that is, $\tau\left({}^{X}\!/_{\mathcal{R}}, \{[\ ]_{\mathcal{R}}\}\right)$.

Another representative example of final topology is given in the disjoint union of a family of topological spaces.

Example 48 (Disjoint union topology) *Let $(X_i)_{i \in I}$ be a family of topological spaces. The disjoint union $\bigsqcup_{i \in I} X_i$ is endowed with the final topology induced by the canonical injections $\iota_i : X_i \to \bigsqcup_{i \in I} X_i$ for all $i \in I$, that is, $\tau\left(\bigsqcup_{i \in I} X_i, \{\iota_i\}_{i \in I}\right)$.*

3.2 Topological operations

By providing continuity to operations in universal algebras endowed with a topology, we end up having a much richer mathematical structure: topological universal algebras. Topology and Algebra merge up into this notion that allows to develop Analysis and Geometry.

3.2.1 Topological universal algebras

The concept of topological universal algebra simply pretends to englobe, in just one notion, many different mathematical structures that have something in common: a set of continuous

multiary operations. Examples of topological universal algebras are the topological magmas, topological semigroups, topological monoids, topological groups, topological rings, topological fields, topological modules, and topological algebras. We will bring our attention to topological modules.

3.2.1.1 Continuous multiary operations

If a universal algebra is endowed with a topology, then it is natural to wonder whether the multiary operations involved in the universal algebra are continuous. A topology that makes them continuous will be called a universal algebra topology.

Definition 216 (Topological universal algebra) *A topological universal algebra is a 3-tuple $((A,\tau),I,E)$, where (A,τ) is a topological space and (A,I,E) is a universal algebra such that:*

- *The external sets in the operations of E are topological spaces and all the direct products in the operations of I and E are endowed with the product topology.*
- *All the multiary internal operations of I and all the multiary external operations of E are continuous.*

Every universal subalgebra of a topological universal algebra becomes a topological universal algebra with the inherited topology.

3.2.1.2 Continuous operators

Continuous operators generalize the concept of continuous linear maps to the context of topological universal algebras.

Definition 217 (Continuous operator) *Let $((A,\tau),I,E),((B,\eta),J,F)$ be compatible topological universal algebras. A continuous operator between topological universal algebras is an operator $(f,\hat{},^{-})$ between the universal algebras $(A,I,E),(B,J,F)$ such that $f:A\to B$ is continuous.*

The composition of continuous operators between topological universal algebras is again a continuous operator. Recall that a projection on a set A is an idempotent of the monoid $\left(A^A,\{\circ,I_A\},\varnothing\right)$.

Definition 218 (Complemented subalgebra) *Let $((A,\tau),I,E)$ be a topological universal algebra. A subalgebra B of $((A,\tau),I,E)$ is said to be complemented if B is the range a continuous projection operator. In other words, $B=f(A)$, where (f,I_I,I_E) is a continuous operator on $((A,\tau),I,E)$ such that $f:A\to A$ is a projection.*

3.2.2 Topological groups

In this subsection, we will take care of topological groups [33], which are representative examples of topological universal algebras.

3.2.2.1 Pregroup topologies

Recall that a magma is a universal algebra with only one binary internal operation and no other internal or external operations. Essentially, a topological magma is a magma endowed with a topology that makes the binary internal operation continuous. Such a topology is called a magma topology.

Definition 219 (Topological magma) *A topological magma is a topological universal algebra $((G,\tau),\{\cdot\},\varnothing)$, where $(G,\{\cdot\},\varnothing)$ is a magma and (G,τ) is a topological space.*

A magma topology is simply a topology on a magma that turns it into a topological magma, that is, a topology that makes the binary internal operation continuous.

Remark 71 *A topology τ on a magma G is a magma topology if and only if for every $g,h\in G$ and every neighborhood of U of gh there are neighborhoods V,W of g,h, respectively, such that $VW\subseteq U$.*

It is immediate to check that, given a topological magma, the topology inherited by every submagma is also a magma topology.

Lemma 100 *Let G be a topological magma. Let H be a submagma of G. Let $A,B\subseteq G$. Let $p\in G$. Then:*

1. $\mathrm{cl}(H)$ *is a submagma of* G.
2. $\mathrm{cl}(A)\mathrm{cl}(B)\subseteq\mathrm{cl}(AB)$.
3. *The unary internal operations given by the left- and right-multiplication operators in* (1.14), L_p *and* R_p, *are continuous.*

Proof 255 *In the first place, L_p and R_p are clearly continuous because they are the restriction of the magma operation to the sets $\{p\}\times G$ and $G\times\{p\}$. Second, the fact that $\mathrm{cl}(A)\mathrm{cl}(B)\subseteq\mathrm{cl}(AB)$ is a direct consequence of the continuity of the magma operation. So, let us prove that $\mathrm{cl}(H)$ is a submagma of G. Indeed, let $g_1,g_2\in\mathrm{cl}(H)$ and fix an arbitrary neighborhood V of g_1g_2. By continuity of the magma operation, there can be found two neighborhoods V_1,V_2 of g_1,g_2, respectively, such that $V_1V_2\subseteq V$. There are $h_1\in H\cap V_1$ and $h_2\in H\cap V_2$. Then $h_1h_2\in H\cap V_1V_2\subseteq H\cap V$. As a consequence, $g_1g_2\in\mathrm{cl}(H)$.*

A semigroup is a magma whose binary internal operation satisfies the associative property. Thus, a topological semigroup is a semigroup endowed with a magma topology.

Definition 220 (Topological semigroup) *A topological semigroup is a topological universal algebra $((G,\tau),\{\cdot\},\varnothing)$, where $(G,\{\cdot\},\varnothing)$ is a semigroup and (G,τ) is a topological space.*

A semigroup topology is a magma topology on a semigroup. The topology inherited by every subsemigroup is also a semigroup topology. A monoid is a semigroup endowed with an identity element. So, as expected, a topological monoid is a monoid endowed with a magma topology.

Definition 221 (Topological monoid) *A topological monoid is a topological universal algebra $((G,\tau),\{\cdot,1\},\varnothing)$, where $(G,\{\cdot,1\},\varnothing)$ is a monoid and (G,τ) is a topological space.*

A monoid topology is simply a semigroup topology on a monoid. Notice that the inherited topology by a submonoid is also a monoid topology.

Remark 72 *If G is a topological monoid, for every neighborhood U of 1, there exist neighborhoods V and W of 1 such that $VW\subseteq U$. If we take $U_0:=V\cap W$, then $U_0U_0\subseteq U$.*

The unary internal operations (1.14) are called translations with additive notation. They are also called left- and right-multiplication operators, respectively, with multiplicative notation.

Lemma 101 *Let G be a topological monoid. For every $p \in G^{\times}$, the unary internal operations given in* (1.14) *by L_p and R_p are homeomorphisms mapping $G^{\times}$ to itself, whose inverses are $\mathsf{L}_{p^{-1}}$ and $\mathsf{R}_{p^{-1}}$, respectively. As a consequence:*

1. *If U is open in G and $A \subseteq G^{\times}$, then AU and UA are both open.*
2. *If $\operatorname{int}(G^{\times}) \neq \varnothing$, then $G^{\times}$ is open.*
3. *$G_{(1)}$ is a submomoid of G.*

Proof 256 *We know that the multiplication operators are continuous by Lemma 100. The fact that $G^{\times}$ is invariant under both unary internal operations is a direct consequence of Proposition 16(3).*

1. *Simply notice that $AU = \bigcup_{a \in A} aU = \bigcup_{a \in A} \mathsf{L}_a(U)$ is open because L_a is a homeomorphism. In a similar way, UA is also open.*
2. *Fix an element $g \in \operatorname{int}(G^{\times})$. For every $h \in G^{\times}$, consider the translation $\mathsf{L}_{hg^{-1}}$, which is a homeomorphisms mapping $G^{\times}$ to itself. The above translation maps g to h. Therefore, $h \in \operatorname{int}(G^{\times})$. As a consequence, $G^{\times}$ is open.*
3. *Note that, by definition, $1 \in G_{(1)}$. Fix arbitrary elements $g, h \in G_{(1)}$. Take any neighborhood U of 1. According to Remark 72, there exist neighborhoods V and W of 1 such that $VW \subseteq U$. By construction, $g \in V$ and $h \in W$, meaning that $gh \in VW \subseteq U$. This shows that $G_{(1)}$ is a submonoid.*

If G is a Hausdorff topological monoid, then $\operatorname{tor}_n(G) := \{g \in G : g^n = 1\}$ is closed in G for all $n \geq 1$. Indeed, the map $G \to G$ given by $g \mapsto g^n$ is continuous and $\{1\}$ is closed in G.

Definition 222 (Topological kernel) *Let G, H be topological monoids. The topological kernel of a unital-multiplicative operator $T : G \to H$ is defined as* $\ker_{\mathtt{t}}(T) := T^{-1}\left(H_{(1)}\right)$.

Under the settings of the previous definition, $H_{(1)}$ stands for the intersection of all neighborhoods of 1 in H. Continuity of unital-multiplicative operators between topological monoids is subject to local continuity at 1.

Theorem 73 *Let G, H be topological monoids. Consider a unital-multiplicative operator $T : G \to H$. Then:*

1. *$\ker_{\mathtt{t}}(T)$ is a submonoid of G.*
2. *If T is continuous, then $G_{(1)} \subseteq \ker_{\mathtt{t}}(T)$.*
3. *If T is continuous and H is Hausdorff, then T vanishes on $G_{(1)}$.*
4. *If T is continuous at 1, then T is continuous at every $g \in G^{\times}$.*

Proof 257 *The proof will follow as itemized as the statement of the theorem.*

1. *In the first place, $\ker_{\mathtt{t}}(T)$ is a submonoid of G because it is the preimage of a submonoid under a unital-multiplicative map.*
2. *The continuity of T allows that T^{-1} preserve neighborhoods of 1, hence $\left\{T^{-1}(V) : V \in \mathcal{N}_1(H)\right\} \subseteq \mathcal{N}_1(G)$. Therefore, $G_{(1)} = \bigcap \mathcal{N}_1(G) \subseteq \bigcap_{V \in \mathcal{N}_1(H)} T^{-1}(V) = T^{-1}\left(\bigcap \mathcal{N}_1(H)\right) = \ker_{\mathtt{t}}(T)$.*

3. *If T is continuous and H is Hausdorff, then $G_{(1)} \subseteq \ker_{\mathtt{t}}(T) = \ker(T)$.*

4. *Let V be a neighborhood of $T(g)$. Then $T(g) \in H^{\times}$ and $T(g)^{-1} = T\left(g^{-1}\right)$, hence $T(g)^{-1}V$. Since T is continuous at 1, there exists a neighborhood W of 1 in G such that $T(W) \subseteq T(g)^{-1}V$. Now, gW is a neighborhood of g and $T(gW) = T(g)T(W) \subseteq T(g)\left(T(g)^{-1}V\right) = V$.*

We will pay special attention to topological monoids with continuous inversion, called topological inversion monoids.

Definition 223 (Topological inversion monoid) *A topological inversion monoid is a topological universal algebra $\left((G,\tau), \left\{\cdot, 1, {}^{-1}\right\}, \varnothing\right)$, where $(G, \{\cdot, 1\}, \varnothing)$ is a monoid, ${}^{-1} : G^{\times} \to G^{\times}$ is its inversion, and (G,τ) is a topological space.*

Recall that a subset A of a monoid G is said to be symmetric provided that $(A \cap G^{\times})^{-1} \subseteq A$, in other words, A is a universal subalgebra of the universal algebra $\left(G, \left\{ {}^{-1}\right\}, \varnothing\right)$.

Lemma 102 *Let G be a topological inversion monoid. Then the inversion map ${}^{-1} : G^{\times} \to G^{\times}$ is a homeomorphism. As a consequence:*

1. *If $B \subseteq G^{\times}$ is symmetric, then $\operatorname{int}_{G^{\times}}(B)$ and $\operatorname{cl}_{G^{\times}}(B)$ are symmetric as well.*

2. *$G_{(1)}$ is symmetric. If, in addition, $G^{\times}$ contains a closed neighborhood of 1, then $G_{(1)}$ is closed.*

Proof 258 *The inversion map is a homeomorphism because it is an involution. A direct consequence of this fact is that if $B \subseteq G^{\times}$ is symmetric, then $\operatorname{int}_{G^{\times}}(B)$ and $\operatorname{cl}_{G^{\times}}(B)$ are symmetric as well. Next, let us show that $G_{(1)}$ is symmetric. Indeed, let $g \in G_{(1)} \cap G^{\times}$. Let $V \in \mathcal{N}_1(G)$. We will prove that $g^{-1} \in V$. Note that $V \cap G^{\times} \in \mathcal{N}_1(G^{\times})$, therefore, $(V \cap G^{\times})^{-1} \in \mathcal{N}_1(G^{\times})$. There exists $U \in \mathcal{N}_1(G)$ such that $U \cap G^{\times} = (V \cap G^{\times})^{-1}$. Then $g \in U \cap G^{\times} = (V \cap G^{\times})^{-1}$, that is, $g^{-1} \in V$. As a consequence, $g^{-1} \in \bigcap_{V \in \mathcal{N}_1(G)} V = G_{(1)}$, resulting in $G_{(1)}$ being symmetric. Finally, let us assume that $G^{\times}$ contains a closed neighborhood W of 1. We will show then that $G_{(1)}$ is closed. For this, it only suffices to show that for any neighborhood U of 1 there exists a closed neighborhood of 1 contained in U. Indeed, by Remark 72, there exists $V \in \mathcal{N}_1(G)$ such that $VV \subseteq U$. We will show that $\operatorname{cl}(V \cap W) \subseteq U$. Indeed, take any $h \in \operatorname{cl}(V \cap W)$. Note that $h \in \operatorname{cl}(V \cap W) \subseteq \operatorname{cl}(W) = W \subseteq G^{\times}$. Then $(V \cap W)^{-1}$ is a neighborhood of 1 because the inversion map is continuous and $G^{\times}$ is open, and so $h(V \cap W)^{-1}$ is a neighborhood of h. As a consequence, $h(V \cap W)^{-1} \cap V \neq \varnothing$, meaning that we can find $v_1 \in V \cap W$ and $v_2 \in V$ such that $hv_1^{-1} = v_2$, that is, $h = v_2 v_1 \in VV \subseteq U$.*

As a consequence of the previous lemma, if $T : G \to H$ is a continuous unital-multiplicative operator between topological inversion monoids G, H such that $H^{\times}$ contains a closed neighborhood of 1, then $\ker_{\mathtt{t}}(T)$ is closed.

3.2.2.2 Group topologies

A group is a monoid in which every element has a symmetric. In the language of topological universal algebras, topological groups are defined the same way as topological monoids with continuous inversion.

Definition 224 (Topological group) *A topological group is a topological universal algebra $\left((G,\tau), \left\{\cdot, {}^{-1}\right\}, \varnothing\right)$, where $\left(G, \left\{\cdot, {}^{-1}\right\}, \varnothing\right)$ is a group and (G,τ) is a topological space.*

Essentially, a topological group is a group endowed with a topology that makes both the multiplication and the inversion continuous. Such a topology is called a group topology. It is trivial to check that the inherited topology on every subgroup is a group topology. The following fundamental theorem characterizes group topologies. It also settles the foundations for ring and module topologies.

Theorem 74 *Let G be a group. If τ is a group topology on G and $\mathcal{B}$ is a base of neighborhoods of 1, then the following is verified:*

1. *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $UU \subseteq V$.*
2. *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $U^{-1} \subseteq V$.*
3. *For every $V \in \mathcal{B}$ and every $g \in G$, there exists $U \in \mathcal{B}$ with $gU \subseteq Vg$.*

Conversely, if $\mathcal{B}$ is a filter base of $\mathcal{P}(G)$ verifying the above properties, then there exists a unique group topology on G for which $\mathcal{B}$ is a base of neighborhoods of 1. This topology is given by

$$\tau := \{A \subseteq G : \forall a \in A \ \ \exists U \in \mathcal{B} \ \ aU \subseteq A\} \cup \{\varnothing\}.$$

Proof 259 *Suppose first that τ is a group topology on G and $\mathcal{B}$ is a base of neighborhoods of 1. We will check that the three properties above are verified:*

1. *This a direct consequence of the continuity of the multiplication.*
2. *This is a direct consequence of the continuity of the inversion.*
3. *Take $V \in \mathcal{B}$ and $g \in G$. In view of Lemma 101(2), $\{gB : B \in \mathcal{B}\}$ is a base of neighborhoods of g. On the other hand, Vg is a neighborhood of g by relying again on Lemma 101(2). Therefore, there exists $U \in \mathcal{B}$ such that $gU \subseteq Vg$.*

Conversely, assume that $\mathcal{B}$ is a filter base of $\mathcal{P}(G)$ verifying the above properties and define

$$\tau := \{A \subseteq G : \forall a \in A \ \ \exists U \in \mathcal{B} \ \ aU \subseteq A\} \cup \{\varnothing\}.$$

First off, let us check that τ is indeed a topology on G:

- *Clearly, $\varnothing, G \in \tau$.*
- *Let $(A_i)_{i\in I} \subseteq \tau$. If $a \in \bigcup_{i\in I} A_i$, then there exists $i_0 \in I$ such that $a \in A_{i_0}$, thus there exists $B \in \mathcal{B}$ with $aB \subseteq A_{i_0} \subseteq \bigcup_{i\in I} A_i$.*
- *Let $A, B \in \tau$. If $a \in A \cap B$, then there exist $U, V \in \mathcal{B}$ with $aU \subseteq A$ and $aV \subseteq B$. Since $\mathcal{B}$ is a filter base of G, it is downward directed, so there exists $W \in \mathcal{B}$ with $W \subseteq U \cap V$. Then $aW \subseteq aU \cap aV \subseteq A \cap B$.*

The next step is to prove that $\mathcal{B}$ is a base of neighborhoods of 1:

- *For every $B \in \mathcal{B}$, $1 \in B$. Indeed, there exists $C \in \mathcal{B}$ with $CC \subseteq B$. Now there exists $D \in \mathcal{B}$ such that $D^{-1} \subseteq C$. Next, since $\mathcal{B}$ is a filter base in G, we have that $\varnothing \notin \mathcal{B}$; therefore, $D \cap C \neq \varnothing$ and thus we can fix $d \in D \cap C$. Finally, $1 = dd^{-1} \in (D \cap C)D^{-1} \subseteq CC \subseteq B$.*
- *For every $B \in \mathcal{B}$, $\operatorname{int}(B) = \{b \in B : \exists C \in \mathcal{B}\ bC \subseteq B\}$. Indeed, denote $U := \{b \in B : \exists C \in \mathcal{B}\ bC \subseteq B\}$. We will first show that $U \in \tau$. Fix an arbitrary $b \in U$. There exists $C \in \mathcal{B}$ with $bC \subseteq B$. Let $D \in \mathcal{B}$ with $DD \subseteq C$. We will prove that $bD \subseteq U$. Fix an arbitrary $d \in D$. Note that $bd = b(d1) \in b(DD) \subseteq bC \subseteq B$. Let $E \in \mathcal{B}$ with $dE \subseteq Dd$.*

Then $(bd)E = b(dE) \subseteq b(Dd) \subseteq b(DD) \subseteq bC \subseteq B$. *This implies that* $bd \in U$, *so* $bE \subseteq U$, *and hence* $U \in \tau$. *It is clear that* $1 \in U \subseteq \operatorname{int}(B)$, *so* B *is a neighborhood of* 1. *Conversely, if* $b \in \operatorname{int}(B)$, *since* $\operatorname{int}(B) \in \tau$ *by default, there exists* $F \in \mathcal{B}$ *such that* $bF \subseteq \operatorname{int}(B) \subseteq B$. *Therefore,* $b \in U$. *The arbitrariness of* $b \in \operatorname{int}(B)$ *shows that* $\operatorname{int}(B) \subseteq U$.

- *If* V *is a neighborhood of* 1, *then there exists* $W \in \tau$ *with* $1 \in W \subseteq V$. *By definition of* τ, *there exists* $B \in \mathcal{B}$ *with* $1B \subseteq W$. *This trivially implies that* $\mathcal{B}$ *is a base of neighborhoods of* B.

- *For every* $g \in G$ *and every* $U \in \tau$, *we have that* $\{gU, Ug\} \subseteq \tau$. *Indeed, if* $u \in U$, *then there exists* $B \in \mathcal{B}$ *such that* $uB \subseteq U$, *so* $(gu)B = g(uB) \subseteq gU$. *As a consequence,* $gU \in \tau$. *Next, there exist* $C, D \in \mathcal{B}$ *such that* $u^{-1}C \subseteq Bu^{-1}$ *and* $(ug)D \subseteq C(ug)$. *Then* $(ug)D \subseteq C(ug) = (Cu)g \subseteq (uB)g \subseteq Ug$. *As a consequence,* $Ug \in \tau$. *In particular, if* W *is a neighborhood of* $h \in G$, *then* gW *and* Wg *are neighborhoods of* gh *and* hg, *respectively. This implies that the left- and right-multiplication operators are homeomorphisms.*

All that remains to show is that multiplication and the inversion are both continuous:

- *If* $g, h \in G$ *and* V *is a neighborhood of* gh, *then there exists* $B \in \mathcal{B}$ *with* $(gh)B \subseteq V$. *Take* $C, E \in \mathcal{B}$ *so that* $CC \subseteq B$ *and* $h^{-1}E \subseteq Ch^{-1}$. *Then* gE *and* hC *are neighborhoods of* g *and* h, *respectively, and besides* $(gE)(hC) = g(Eh)C \subseteq g(hC)C \subseteq (gh)(CC) \subseteq (gh)B \subseteq V$.

- *If* $g \in G$ *and* V *is a neighborhood of* g^{-1}, *then there exists* $B \in \mathcal{B}$ *with* $g^{-1}B \subseteq V$. *Let* $C \in \mathcal{B}$ *such that* $C^{-1} \subseteq B$. *Take* $E \in \mathcal{B}$ *with* $gE \subseteq Cg$. *Finally,* gE *is a neighborhood of* g *and* $(gE)^{-1} \subseteq (Cg)^{-1} = g^{-1}C^{-1} \subseteq g^{-1}B$.

The pseudometric topology induced by an invariant pseudometric on a group is an example of group topology.

Example 49 *Let* G *be a pseudometric group. Then the pseudometric topology on* G *is a group topology. Take* $(g_i)_{i\in I}, (h_i)_{i\in I}$ *nets in* G *converging to* g, h *in* G, *respectively. We have to show that* $(g_i h_i)_{i\in I}$ *converges to* gh. *Indeed, it only suffices to observe that* $d(g_i h_i, gh) = d\left(g^{-1}g_i, hh_i^{-1}\right) \leq d\left(g^{-1}g_i, 1\right) + d\left(1, hh_i^{-1}\right) = d(g_i, g) + d(h_i, h)$ *for all* $i \in I$. *Finally,* $d\left(g_i^{-1}, g^{-1}\right) = d(g, g_i)$ *for all* $i \in I$, *meaning that* $\left(g_i^{-1}\right)_{i\in I}$ *converges to* g^{-1}.

Every seminorm on a group induces a pseudometric (1.32), which is not necessarily invariant. Therefore, every seminormed group can be endowed with the corresponding pseudometric topology.

Example 50 *If* G *is a conjugation-invariant seminormed group, then the induced pseudometric is invariant by Proposition 60, hence the pseudometric topology induced on* G *is a group topology. Furthermore, the seminorm is a nonexpansive function because* $|\|g\| - \|h\|| \leq \left\|gh^{-1}\right\|$ *for all* $g, h \in G$ *in view of Lemma 66.*

Group topologies are uniform topologies without the necessity of an invariant pseudometric.

Proposition 79 *Let* G *be a topological group. For every* $V \in \mathcal{N}_1$, *the sets* $\mathsf{U}_V := \left\{(g, h) \in G \times G : gh^{-1} \in V\right\}$ *constitute a base of entourages whose generated filter is called the group uniformity. Even more, for every* $g \in G$ *and every* $V, W, Z \in \mathcal{N}_1$:

1. $\Delta_G \subseteq \mathsf{U}_V$, $(\mathsf{U}_V)^{-1} = \mathsf{U}_{V^{-1}}$, $\mathsf{U}_{V\cap W} = \mathsf{U}_V \cap \mathsf{U}_W$, $\mathsf{U}_V \circ \mathsf{U}_W \subseteq \mathsf{U}_{WV}$, *and* $\mathsf{U}_V[g] = V^{-1}g$.
2. *If* Z *is symmetric and* $ZZ \subseteq V \cap W$, *then* $\mathsf{U}_Z \subseteq \mathsf{U}_V \circ \mathsf{U}_W$.

Proof 260 *The following properties are trivially satisfied for arbitrary* $V, W, Z \in \mathcal{N}_1$*:* $\Delta_G \subseteq \mathsf{U}_V$, $(\mathsf{U}_V)^{-1} = \mathsf{U}_{V^{-1}}$, $\mathsf{U}_{V\cap W} = \mathsf{U}_V \cap \mathsf{U}_W$, $\mathsf{U}_V \circ \mathsf{U}_W \subseteq \mathsf{U}_{WV}$, *and* $\mathsf{U}_V[g] = V^{-1}g$. *Suppose now that* Z *is symmetric and* $ZZ \subseteq V \cap W$. *Take any* $(g,h) \in \mathsf{U}_Z$ *and consider* $k := hg^{-1}h$. *Then* $gk^{-1} = \left(gh^{-1}\right)\left(gh^{-1}\right) \in ZZ \subseteq V \cap W \subseteq W$, *and* $kh^{-1} = \left(hg^{-1}\right)\left(hh^{-1}\right) = hg^{-1} \in Z^{-1} = Z \subseteq ZZ \subseteq V \cap W \subseteq V$. *As a consequence,* $(g,h) \in \mathsf{U}_V \circ U_W$. *All these properties show that the* U_V*'s, for* $V \in \mathcal{N}_1$*, form a base of entourages. Let us prove that* 1 *has a base of closed neighborhoods. Indeed, fix an arbitrary* $V \in \mathcal{N}_1$. *By applying Theorem 74(1), there exists* $U \in \mathcal{N}_1$ *with* $UU \subseteq V$. *Take* $W := U \cap U^{-1} \in \mathcal{N}_1$. *We will show that* $\mathrm{cl}(W) \subseteq V$. *Take any* $g \in \mathrm{cl}(W)$. *Then* $gW \cap W \neq \varnothing$, *thus, there are* $w_1, w_2 \in U$ *with* $gw_1 = w_2$. *Note that* $g = w_2 w_1^{-1} \in WW \subseteq UU \subseteq V$.

Recall that, if X is a topological space and $x \in X$, then

$$X_{(x)} := \bigcap \mathcal{N}_x(X) = \bigcap_{V \in \mathcal{N}_x(X)} V.$$

Observe that, if X is regular, then $X_{(x)} = \mathrm{cl}(\{x\})$ in view of Lemma 49.

Corollary 15 *Let* G *be a topological group. Let* H *be a subgroup of* G. *Let* $g, h \in G$. *Then:*

1. $G_{(1)} = \mathrm{cl}(\{1\})$ *is a closed subgroup of* G *whose inherited topology is the trivial topology and which is contained in any closed subgroup of* G.
2. $hG_{(g)} = G_{(hg)}$ *and* $G_{(g)}h = G_{(gh)}$.
3. G *is Hausdorff if and only if* $G_{(1)} = \{1\}$. *As a consequence,* H *is Hausdorff if and only if* $H \cap G_{(1)} = \{1\}$.
4. G *is discrete if and only if* $\{1\}$ *is open.*
5. *If* $\mathcal{B}$ *is a base of neighborhoods of* 1, *then* $\mathcal{B}_1 := \left\{V \cap V^{-1} : V \in \mathcal{B}\right\}$ *and* $\mathcal{B}_2 := \left\{V \cup V^{-1} : V \in \mathcal{B}\right\}$ *are bases of symmetric neighborhoods of* 1.
6. $\mathrm{cl}(H)$ *is a subgroup of* G.
7. G *is a homogeneous topological space.*

Proof 261 *The proof follows as itemized in the statement.*

1. *Let* $g, h \in G_{(1)}$ *and fix an arbitrary neighborhood* $V \in \mathcal{N}_1$. *By Theorem 74(1), there exists* $U \in \mathcal{N}_1$ *such that* $UU \subseteq V$. *Next,* $g, h \in U$, *so* $gh \in UU \subseteq V$. *This shows that* $G_{(1)}G_{(1)} \subseteq G_{(1)}$. *Take now* $g \in G_{(1)}$ *and fix an arbitrary neighborhood* $V \in \mathcal{N}_1$. *Then* $V^{-1} \in \mathcal{N}_1$, *therefore* $g \in V^{-1}$ *so* $g^{-1} \in V$. *This shows that* $\left(G_{(1)}\right)^{-1} \subseteq G_{(1)}$. *As a consequence,* $G_{(1)}$ *is a subgroup of* G. *The fact the* $G_{(1)}$ *is closed follows immediately from Corollary 15(1) together with Lemma 49 because* $G_{(1)} = \mathrm{cl}(\{1\})$. *Next, let us prove that the inherited topology of* $G_{(1)}$ *is the trivial topology. Indeed, any open neighborhood of* 1 *in* $G_{(1)}$ *is given by* $V \cap G_{(1)}$ *with* $V \in \mathcal{N}_1$ *open. However, simply notice that* $V \cap G_{(1)} = G_{(1)}$, *which suffices to assure that* $G_{(1)}$ *has the trivial topology.*

2. *Fix any arbitrary $g' \in G_{(g)}$. Take any $V \in \mathcal{N}_1$. Note that $hg' \in h(gV) = (hg)V$ and $(hg)V \in \mathcal{N}_1$. The arbitrariness of $V \in \mathcal{N}_1$ implies that $hg' \in G_{(hg)}$. This shows that $hG_{(g)} \subseteq G_{(hg)}$. The other inclusions follow similar reasonings.*

3. *If G is Hausdorff, then $G_{(g)} = \{g\}$ for all $g \in G$ in view of Remark 52(1). Conversely, suppose that $G_{(1)} = \{1\}$. Let $g, h \in G$ with $g \neq h$. There exists $U \in \mathcal{N}_1$ such that $h^{-1}g \notin U$. Let $W, V \in \mathcal{N}_1$ such that $VV \subseteq U$ and $W^{-1} \subseteq V$. Suppose that $gW \cap hV \neq \varnothing$. Then there are $v \in V$ and $w \in W$ with $gw = hv$, so $h^{-1}g = vw^{-1} \in VW^{-1} \subseteq VV \subseteq U$, which is a contradiction. Next, it only suffices to take into consideration that $G_{(1)} \cap H = H_{(1)}$.*

4. *If $\{1\}$ is open, then $\{g\} = \mathsf{L}_g(\{1\}) = \mathsf{R}_g(\{1\})$ is open for all $g \in G$.*

5. *Simply observe that $V \cap V^{-1}$ is a neighborhood of 1.*

6. *We already know that $\mathrm{cl}(H)$ is a submagma of G. It only remains to show that $\mathrm{cl}(H)$ is symmetric, which is immediate since H is symmetric and the inversion map is a homeomorphism.*

7. *For every $g, h \in G$, $\mathsf{L}_h \circ \mathsf{L}_{g^{-1}} = \mathsf{L}_{hg^{-1}}$ and $\mathsf{R}_h \circ \mathsf{R}_{g^{-1}} = \mathsf{R}_{g^{-1}h}$ are homeomorphisms on G mapping g to h.*

Group topologies have a strong influence on the continuity of multiplicative maps.

Lemma 103 *Let G, H be topological groups. Let $f : G \to H$ be multiplicative. Then:*

1. *If f is continuous at 1, then f is uniformly continuous and $\ker_{\mathfrak{t}}(f)$ is closed in G.*

2. *If f is open at 1, then f is open.*

Proof 262 *Suppose first that f is continuous at 1. Let $V \in \mathcal{N}_1(H)$. There exists $U \in \mathcal{N}_1(G)$ such that $f(U) \subseteq V$. It is not hard to check that $(f \times f)(\mathsf{U}_U) \subseteq \mathsf{U}_V$, where $\mathsf{U}_U := \left\{(g_1, g_2) \in G \times G : g_1 g_2^{-1} \in U\right\}$ and $\mathsf{U}_V := \left\{(h_1, h_2) \in H \times H : h_1 h_2^{-1} \in V\right\}$. To see that $\ker_{\mathfrak{t}}(f)$ is closed in G it only suffices to realize that $\ker_{\mathfrak{t}}(f) = f^{-1}(H_{(1)})$ and that $H_{(1)}$ is closed in H. Next, assume that f is open at 1. If $g \in G$ and $V \in \mathcal{N}_g(G)$, then $g^{-1}V \in \mathcal{N}_1(G)$, meaning that $f(g)^{-1}f(V) = f\left(g^{-1}V\right) \in \mathcal{N}_1(H)$, that is, $f(V) \in \mathcal{N}_{f(g)}(H)$.*

The final results in this subsubsection deal with the product of group topologies and with projections.

Theorem 75 *Let $(G_i)_{i \in I}$ be a family of topological groups. Then $\prod_{i \in I} G_i$ is a topological group endowed with the product topology.*

Proof 263 *Let $((g_j, h_j))_{j \in J}$ be a net of $\prod_{i \in I} G_i \times \prod_{i \in I} G_i$ converging to some $(g, h) \in \prod_{i \in I} G_i \times \prod_{i \in I} G_i$. We know that $(g_j)_{j \in J}, (h_j)_{j \in J}$ converge to g, h, respectively, in $\prod_{i \in I} G_i$. Fix an arbitrary $i \in I$. We have that $(g_j(i))_{j \in J}, (h_j(i))_{j \in J}$ converge to $g(i), h(i)$, respectively, in G_i. Since G_i is a topological group, $(g_j(i)h_j(i))_{j \in J}$ converges to $g(i)h(i)$. This shows that $(g_j h_j)_{j \in J}$ converges to gh. In a similar way, it can be shown that $\left(g_j^{-1}\right)_{j \in J}$ converges to g^{-1}.*

As a direct consequence of the previous theorem, the pointwise convergence topology on G^I is always a group topology. The product topology is directly related to the notion of projection. Recall that a projection is an idempotent element of the monoid A^A, where A is a set, that is, a map $p : A \to A$ such that $p^2 = p$.

Lemma 104 *If G is a topological group and $p : G \to G$ is a multiplicative projection, then p is an open map over its range. If, in addition, G is Hausdorff and p is continuous, then $p(G)$ is closed.*

Proof 264 *Let U be open in G. Let $u \in U$ and consider an open neighborhood V of 1 such that $uV \subseteq U$. We will show that $p(u)V \cap p(G) \subseteq p(U)$. Indeed, let $v \in V$ and $g \in G$ such that $p(u)v = p(g)$. Note that $p(g) = p(p(g)) = p(p(u)v) = p(p(u))p(v) = p(u)p(v) = p(uv) \in p(U)$. Finally, suppose that G is Hausdorff and p is continuous. Let $(p(g_i))_{i\in I}$ be a net in $p(G)$ converging to some $g \in G$. By the continuity of p we have that $(p(p(g_i)))_{i\in I}$ converges to $p(g)$. Since $p(p(g_i)) = p(g_i)$ for all $i \in I$, the Hausdorff character of G makes us conclude that $g = p(g)$.*

The previous lemma motivates the following definition.

Definition 225 (Topological complement) *A subgroup of a topological group is said to be topologically complemented provided that it is the range of a continuous multiplicative projection.*

Whenever a subgroup is topologically complemented, its complement is defined as the kernel of the corresponding projection.

Lemma 105 *Let G be a topological group. If $G_{(1)}$ is complemented, then its complement is dense in G.*

Proof 265 *Fix an arbitrary $g \in G$. Fix also an arbitrary $V \in \mathcal{N}_1$. Note that $p\left(g^{-1}\right) \in p(G) = G_{(1)} \subseteq V$, hence $gp\left(g^{-1}\right) \in gV$. Finally, $p\left(gp\left(g^{-1}\right)\right) = p(g)p\left(p\left(g^{-1}\right)\right) = p(g)p\left(g^{-1}\right) = p(g)p(g)^{-1} = 1$, meaning that $gp\left(g^{-1}\right) \in \ker(p)$.*

Next theorem shows that the quotient topology on a quotient group is a group topology. However, a technical lemma is needed first.

Lemma 106 *Let G be a congruent group. If $A \subseteq G$, then $\pi_{\mathcal{R}}^{-1}\left(\pi_{\mathcal{R}}(A)\right) = A[1]_{\mathcal{R}} = [1]_{\mathcal{R}}A$.*

Proof 266 *First, recall that $\pi_{\mathcal{R}} : G \to {}^{G}\!/_{\mathcal{R}}$ stands for the canonical projection. Next, observe that $\pi_{\mathcal{R}}\left(A[1]_{\mathcal{R}}\right) = \pi_{\mathcal{R}}\left(A\right)$, thus $A[1]_{\mathcal{R}} \subseteq \pi_{\mathcal{R}}^{-1}\left(\pi_{\mathcal{R}}(A)\right)$. Take any $g \in G$ such that $\pi_{\mathcal{R}}(g) \in \pi_{\mathcal{R}}(A)$. There exists $a \in A$ such that $[g]_{\mathcal{R}} = [a]_{\mathcal{R}}$, that is, $g \in [a]_{\mathcal{R}} = a[1]_{\mathcal{R}} \subseteq A[1]_{\mathcal{R}}$ in virtue of Remark 25. In a dual way, $\pi_{\mathcal{R}}^{-1}\left(\pi_{\mathcal{R}}(A)\right) = [1]_{\mathcal{R}}A$.*

Under the settings of the previous lemma and assuming that G is a topological group, notice that for every open subset $V \subseteq G$, $\pi_{\mathcal{R}}(V)$ is open in the quotient topology of ${}^{G}\!/_{\mathcal{R}}$ due to the fact that $\pi_{\mathcal{R}}^{-1}\left(\pi_{\mathcal{R}}(V)\right) = V[1]_{\mathcal{R}} = \bigcup_{g\in[1]_{\mathcal{R}}} Vg$. This shows that $\{\pi_{\mathcal{R}}(V) : V \text{ is a neighborhood of } 1 \text{ in } G\}$ is a base of neighborhoods of $[1]_{\mathcal{R}}$ for the quotient topology.

Theorem 76 *Let G be a topological group. Let $\mathcal{R}$ be a compatible equivalence relation on G. The quotient topology is a group topology on the quotient group ${}^{G}\!/_{\mathcal{R}}$ and the canonical projection $\pi : G \to {}^{G}\!/_{\mathcal{R}}$ is an open map.*

Proof 267 *All is needed to do is check that the collection $\{\pi_{\mathcal{R}}(V) : V$ is a neighborhood of 1 in $G\}$ satisfies the conditions displayed in Theorem 74 for group topologies.*

1. *Fix an arbitrary neighborhood V of 1 in G. There exists another neighborhood U of 1 in G such that $UU \subseteq V$. Then $\pi_{\mathcal{R}}(U)\pi_{\mathcal{R}}(U) = \pi_{\mathcal{R}}(UU) \subseteq \pi_{\mathcal{R}}(V)$ in view of the fact that $[g]_{\mathcal{R}}[h]_{\mathcal{R}} = [gh]_{\mathcal{R}}$ for all $g, h \in G$.*

2. *Fix an arbitrary neighborhood V of 1 in G. There exists another neighborhood U of 1 in G such that $U^{-1} \subseteq V$. Then $\pi_{\mathcal{R}}(U)^{-1} = \pi_{\mathcal{R}}\left(U^{-1}\right) \subseteq \pi_{\mathcal{R}}(V)$ in view of the fact that $\left[g^{-1}\right]_{\mathcal{R}} = [g]_{\mathcal{R}}^{-1}$ for all $g \in G$.*

3. *Fix an arbitrary neighborhood V of 1 in G and an arbitrary element $g \in G$. There exists another neighborhood U of 1 in G such that $gU \subseteq Vg$. Then $\pi_{\mathcal{R}}(g)\pi_{\mathcal{R}}(U) = \pi_{\mathcal{R}}(gU) \subseteq \pi_{\mathcal{R}}(Vg) = \pi_{\mathcal{R}}(V)\pi_{\mathcal{R}}(g)$ in view of the fact that $[g]_{\mathcal{R}}[h]_{\mathcal{R}} = [gh]_{\mathcal{R}}$ for all $g, h \in G$.*

Notice that every finite subset A of a topological group satisfies the following property in view of Theorem 74(3): For every $U \in \mathcal{N}_1$, there exists $V \in \mathcal{N}_1$ such that $aV \subseteq Ua$ for every $a \in A$.

Definition 226 (Topological commutant) *Let G be a topological group. A subset $A \subseteq G$ is said to be topologically commutant if, for every $U \in \mathcal{N}_1$, there exists $V \in \mathcal{N}_1$ such that $aV \subseteq Ua$ for every $a \in A$.*

Topologically commutant subsets will help construct convergence group topologies.

3.2.2.3 Topological actions

As expected, topological actions are continuous actions, for which it will be necessary to endow the involved magma and set with a topology.

Definition 227 (Topological action) *An action $G \times M \to M$ of a topological magma G on a topological space M is said to be a topological action if it is continuous when $G \times M$ is endowed with the product topology. In this sense, M is called a topological G-set.*

If M is a topological G-set, for $((G, \tau), \{\cdot\}, \varnothing)$ a topological magma, and we denote the action by $\cdot$, then $((M, \tau), \varnothing, \{\cdot\})$ is a topological universal algebra.

Remark 73 *Let G be a topological magma continuously acting on a topological space M. Let N be a G-subset of N. Let $A \subseteq G$, $B \subseteq M$. Let $g \in G$, $m \in M$. Then:*

1. $\mathrm{cl}(N)$ *is a G-subset of M.*

2. $\mathrm{cl}(A)\mathrm{cl}(B) \subseteq \mathrm{cl}(AB)$.

3. *The unary internal operations given by the left- and right-multiplication operators in* (1.16), L_g *and* R_m, *are continuous.*

4. *If G is a topological monoid, then for every $g \in G^{\times}$, the unary internal operation given in* (1.16) *by L_g is a homeomorphism on M whose inverse is $\mathsf{L}_{g^{-1}}$. As a consequence, if U is open in M and $A \subseteq G^{\times}$, then AU is open in M.*

5. *If G is a topological group and M is Hausdorff, then $\mathrm{tor}_g(M) := \{m \in M : gm = m\}$ is closed in M for all $g \in G \setminus \{1\}$. Indeed, L_g is continuous and $\{m\}$ is closed in M.*

An immediate consequence of Example 36 is that if G is a topological group, then $\mathcal{P}(G)$ is a uniform space endowed with the uniformity whose uniformity base is $\mathcal{B}_{\mathcal{N}_1}$, where $\mathcal{N}_1$ stands for the filter of neighborhoods of 1 in G.

Definition 228 (Strongly continuous self-action) *Let G be a topological group acting continuously on itself. We say that the action is strongly continuous if the map*

$$\begin{array}{ccc} G & \to & \mathcal{P}(G) \\ g & \mapsto & G_g \end{array} \tag{3.3}$$

is uniformly continuous.

3.2.2.4 Topological ordered groups

Let G be a left (or right) ordered monoid. If $\uparrow_\times 1 \neq \emptyset$, then $\uparrow_\times g \neq \emptyset$ for all $g \in G^\times$. Similarly, if $\downarrow^\times 1 \neq \emptyset$, then $\downarrow^\times g \neq \emptyset$ for all $g \in G^\times$. In particular, if G is an ordered group, then $\uparrow_\times 1 \neq \emptyset$ or $\downarrow^\times 1 \neq \emptyset$ implies that $\uparrow_\times g \neq \emptyset$ and $\downarrow^\times g \neq \emptyset$ for all $g \in G$.

Lemma 107 *Let G be an ordered monoid. If $\downarrow^\times g$ is upward directed for all $g \in G$, then the order topology on $\uparrow g$ coincides with its relative topology from G for all $g \in G$.*

Proof 268 *Fix $g_0 \in G$. According to Proposition 54, the order topology of $\uparrow g_0$ and its relative topology from G coincide if we show that $\uparrow_\times g \cap \uparrow g_0$ is coinitial in $\uparrow_\times g$ and $\downarrow^\times g \cap \uparrow g_0$ is cofinal in $\downarrow^\times g$ for all $g \in \uparrow g_0$. Indeed, fix an arbitrary $g \in \uparrow g_0$. On the one hand, $\uparrow_\times g \cap \uparrow g_0 = \uparrow_\times g$, which is clearly coinitial in $\uparrow_\times g$. On the other hand, if $g = g_0$, then $\downarrow^\times g \cap \uparrow g_0 = \varnothing$, so there is nothing to prove. Suppose that $g > g_0$. Take any $h \in \downarrow^\times g$. By hypothesis, $\downarrow^\times g$ is upward directed, meaning that there exists $h' \in \downarrow^\times g$ such that $h \leq h'$ and $g_0 \leq h'$. Finally, simply observe that $h' \in \downarrow^\times g \cap \uparrow g_0$.*

Under the settings of the previous definition, by duality, if $\uparrow_\times g$ is downward directed for all $g \in G$, then the order topology on $\downarrow g$ coincides with its relative topology from G for all $g \in G$.

Lemma 108 *Let G be an ordered monoid. Suppose that $\uparrow_\times 1 \neq \emptyset$, $\uparrow_\times 1$ is downward directed, and $\uparrow_\times 1 \cap G^\times$ is coinitial in $\uparrow_\times 1$. Suppose as well that $\downarrow^\times 1 \neq \emptyset$, $\downarrow^\times 1$ is upward directed, and $\downarrow^\times 1 \cap G^\times$ is cofinal in $\downarrow^\times 1$. Then:*

1. *A base of neighborhoods of 1 is given by sets of the form $\uparrow_\times k^{-1} \cap \downarrow^\times k$ for $k > 1$.*
2. *If $1 \in \mathrm{cl}\,(\uparrow_\times 1) \cap \mathrm{cl}\,(\downarrow^\times 1)$, then for any $g_0, h_0 \in G$ such that $g_0 < 1 < h_0$ there exist $g_1, g_2 < 1 < h_1, h_2$ so that $(g_1, h_1)(g_2, h_2) \subseteq (g_0, h_0)$.*
3. *The order topology on $G^\times$ coincides with its relative topology from G.*

Proof 269 *The proof will follow as itemized as the statement of the proposition.*

1. *For every neighborhood U of 1 in G there exist $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ satisfying that $1 \in \uparrow_\times g_1 \cap \cdots \cap \uparrow_\times g_n \cap \downarrow^\times h_1 \cap \cdots \cap \downarrow^\times h_m \subseteq U$. Therefore, $g_i < 1 < h_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus, there exist $g, h \in G^\times$ such that $g_i \leq g < 1 < h \leq h_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Next, $g^{-1}, h \in \uparrow_\times 1$, meaning that there exists $k \in \uparrow_\times 1 \cap G^\times$ with $k \leq g^{-1}$ and $k \leq h$. As a consequence, $1 \in \uparrow_\times k^{-1} \cap \downarrow^\times k \subseteq \uparrow_\times g_1 \cap \cdots \cap \uparrow_\times g_n \cap \downarrow^\times h_1 \cap \cdots \cap \downarrow^\times h_m \subseteq U$.*
2. *By hypothesis, there exists $h_1 \in G^\times$ with $1 < h_1 \leq h_0$, meaning that $1 < h_1^{-1} h_0$. Now, by repeating the process, there exists $h_2 \in \uparrow_\times 1 \cap \downarrow^\times h_1^{-1} h_0$. Therefore, $1 < h_2 < h_1^{-1} h_0$. Consequently, $h_1 h_2 < h_0$. Dually, $g_0 < g_1 g_2$. Finally, $(g_1, h_1)(g_2, h_2) \subseteq (g_1 g_2, h_1 h_2) \subseteq (g_0, h_0)$.*
3. *According to Proposition 54, the order topology of $G^\times$ and its relative topology from G coincide if we show that $\uparrow_\times g \cap G^\times$ is coinitial in $\uparrow_\times g$ and $\downarrow^\times g \cap G^\times$ is cofinal in $\downarrow^\times g$ for all $g \in G^\times$, which is precisely a direct consequence of our hypotheses.*

The following technical lemma shows that if the order topology on a ordered group is well defined, then it is close to become a group topology.

Lemma 109 *Let G be an ordered group. The inversion map is continuous for the order topology, hence a homeomorphism. As a consequence, $\mathrm{cl}\,(\uparrow_\times g) = \mathrm{cl}\left(\left[\downarrow^\times (g^{-1})\right]^{-1}\right) = \mathrm{cl}\left(\downarrow^\times (g^{-1})\right)^{-1}$ for all $g \in G$.*

Proof 270 *Fix an arbitrary $g \in G$. For each neighborhood U of g^{-1} in G there are $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ satisfying that $g^{-1} \in \uparrow_{\times} g_1 \cap \cdots \cap \uparrow_{\times} g_n \cap \downarrow^{\times} h_1 \cap \cdots \cap \downarrow^{\times} h_m \subseteq U$. Then $g_i < g^{-1} < h_j$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Next, $h_j^{-1} < g < g_i^{-1}$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Thus, $\uparrow_{\times} (h_1^{-1}) \cap \cdots \cap \uparrow_{\times} (h_n^{-1}) \cap \downarrow^{\times} (g_1^{-1}) \cap \cdots \cap \downarrow^{\times} (g_n^{-1})$ is a neighborhood of g in G whose image under the inversion map is precisely $\uparrow_{\times} g_1 \cap \cdots \cap \uparrow_{\times} g_n \cap \downarrow^{\times} h_1 \cap \cdots \cap \downarrow^{\times} h_m$.*

Now are finally in the right position to state and prove a sufficient condition for the order topology on a ordered group to be a group topology.

Theorem 77 *Let G be an ordered group. Suppose that $\uparrow_{\times} 1 \neq \emptyset$ and downward directed, and $1 \in \mathrm{cl}(\uparrow_{\times} 1)$. Then the order topology is a group topology.*

Proof 271 *We will prove that all three conditions of Theorem 74 are satisfied for $\mathcal{B}$ the base of neighborhoods of 1 in the order topology given by sets of the form $\uparrow_{\times} h^{-1} \cap \downarrow^{\times} h$ for $h > 1$.*

1. *It is a direct consequence of Proposition 108(3).*
2. *It is a direct consequence of Lemma 109.*
3. *We will show that for every $V \in \mathcal{B}$ there exists $U \in \mathcal{B}$ such that $gU \subseteq Vg$ for every $g \in G$. Indeed, fix an arbitrary $g \in G$. Let $h > 1$ such that $\uparrow_{\times} h^{-1} \cap \downarrow^{\times} h = V$. Note that $\downarrow^{\times} h$ is a neighborhood of 1. Since $1 \in \mathrm{cl}(\uparrow_{\times} 1)$, there exists $k \in \downarrow^{\times} h \cap \uparrow_{\times} 1$. Hence $1 < k < h$. As $h > 1$, then $hg > g$. Therefore, $1 < g^{-1}hg$. By using again that $1 \in \mathrm{cl}(\uparrow_{\times} 1)$, there is $k \in \downarrow^{\times} (g^{-1}hg) \cap \uparrow_{\times} 1$. So, $1 < k < g^{-1}hg$. Thus, $gk < hg$. Consequently, $g\left(\uparrow_{\times} k^{-1} \cap \downarrow^{\times} k\right) \subseteq \left(\uparrow_{\times} h^{-1} \cap \downarrow^{\times} h\right) g$.*

In the next theorem, we prove that the sufficient condition given by $1 \in \mathrm{cl}(\uparrow_{\times} 1)$ in Theorem 77 is not compulsory for the given context.

Theorem 78 *Let G be a group. Assume that there exists a torsionfree element $g_0 \in \mathsf{Z}(G)$. Comsider the ordering on G given by $g \leq h$ if and if $h = g_0^n g$, for some $n \geq 0$. Then this ordering is compatible with the multiplication on G and its corresponding order topology is the discrete topology.*

Proof 272 *Notice that if $g_1, g_2, h \in G$ and $g_1 \leq g_2$, then $g_2 = g_0^n g_1$, for some $n \geq 0$. Keep in mind that $hg_2 = hg_0^n g_1 = g_0^n (hg_1)$, then $hg_1 \leq hg_2$, hence this ordering is left compatible with G. In a dual way, it is also right compatible. Next, we claim that $\uparrow_{\times} g_0^{-1} \cap \downarrow^{\times} g_0 = \{1\}$. Let $g \in \uparrow_{\times} g_0^{-1} \cap \downarrow^{\times} g_0$, then $g = g_0^{n_1} g_0^{-1}$ and $g_0 = g_0^{n_2} g$, for some $n_1, n_2 \geq 1$, and so $g_0^{n_1 - 1} = g_0^{1 - n_2}$. Since g_0 is torsionfree, $n_1 - 1 = 1 - n_2$, that is, $n_1 + n_2 = 1 + 1$. By hypothesis, $n_1, n_2 \geq 1$, which means that $n_1 = n_2 = 1$, hence $g = 1$. Finally, we show that the order topology induced by this ordering is the discrete topology. As a matter of fact, for every $g \in G$, the equality $\{1\} = \uparrow_{\times} g_0^{-1} \cap \downarrow^{\times} g_0$ concludes that $\{g\} = \uparrow_{\times} g g_0^{-1} \cap \downarrow^{\times} g g_0$, as required.*

The previous theorems motivate the notion of topological ordered group.

Definition 229 (Topological ordered group) *A topological ordered group is an ordered group in which the order topology is a group topology.*

We finish this subsubsection by providing a version of the previous Theorem 77 to ordered monoids.

Corollary 16 *Let G be an ordered monoid. Suppose that $1 \in \mathrm{cl}\,(\uparrow_\times 1) \cap \mathrm{cl}\,(\downarrow^\times 1)$, $\uparrow_\times 1$ is downward directed, $\downarrow^\times 1$ is upward directed, $\uparrow_\times 1 \cap G^\times$ is coinitial in $\uparrow_\times 1$, and $\downarrow^\times 1 \cap G^\times$ is cofinal in $\downarrow^\times 1$. Then multiplication on G is continuous at the unity and the inversion map on $G^\times$ is continuous.*

Proof 273 *Continuity of multiplication at the unity is given by Lemma 108(2). Continuity of the inversion map on $G^\times$ is given by plugging together Lemma 108(3) and Lemma 109.*

3.2.2.5 Convergence group topologies

Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let H be a topological group. The uniform topology induced by the convergence uniformity of H^I (induced itself by the group uniformity of H) is not necessarily a group topology on H^I. However, if we restrict to certain subgroups of H^I, then the uniform topology is also a group topology. Notice that, for every $G \in \mathcal{G}$, every $U \in \mathcal{N}_1(H)$ and every $f_0 \in H^I$,

$$\begin{aligned}
\mathcal{V}(G, \mathsf{U}_U)[\mathbf{1}] &= \left\{f \in H^I : (\mathbf{1}, f) \in \mathcal{V}(G, \mathsf{U}_U)\right\} \\
&= \left\{f \in H^I : \forall g \in G \;\; (1, f(g)) \in \mathsf{U}_U\right\} \\
&= \left\{f \in H^I : \forall g \in G \;\; f(g)^{-1} \in U\right\} \\
&= \left\{f \in H^I : \forall g \in G \;\; f(g) \in U^{-1}\right\} \\
&= \left\{f \in H^I : f(G) \subseteq U^{-1}\right\} \\
&= \mathcal{U}(G, U^{-1}),
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}(G, \mathsf{U}_U)[f_0] &= \left\{f \in H^I : (f_0, f) \in \mathcal{V}(G, \mathsf{U}_U)\right\} \\
&= \left\{f \in H^I : \forall g \in G \;\; (f_0(g), f(g)) \in \mathsf{U}_U\right\} \\
&= \left\{f \in H^I : \forall g \in G \;\; f_0(g) f(g)^{-1} \in U\right\} \\
&= \left\{f \in H^I : \forall g \in G \;\; f(g) f_0(g)^{-1} \in U^{-1}\right\} \\
&= \left\{f \in H^I : \left(f f_0^{-1}\right)(G) \subseteq U^{-1}\right\} \\
&= \mathcal{U}(G, U^{-1}) f_0
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{U}_{\mathcal{U}(G,U)} &= \left\{(f_1, f_2) \in H^I \times H^I : f_1 f_2^{-1} \in \mathcal{U}(G, U)\right\} \\
&= \left\{(f_1, f_2) \in H^I \times H^I : \left(f_1 f_2^{-1}\right)(G) \subseteq U\right\} \\
&= \left\{(f_1, f_2) \in H^I \times H^I : \forall g \in G \;\; f_1(g) f_2^{-1}(g) \in U\right\} \\
&= \left\{(f_1, f_2) \in H^I \times H^I : \forall g \in G \;\; (f_1(g), f_2(g)) \in \mathsf{U}_U\right\} \\
&= \mathcal{V}(G, \mathsf{U}_U).
\end{aligned}$$

If $\mathcal{F} \subseteq H^I$ is a subgroup, then for the sake of simplification, we will still denote $\mathcal{U}(G, U) \cap \mathcal{F}$ simply by $\mathcal{U}(G, U)$.

Theorem 79 (Convergence group topology) *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let H be a topological group. Let $\mathcal{F} \subseteq H^I$ be a subgroup. Consider the filter base*

$$\mathcal{B} := \{\mathcal{U}(G, U) : G \in \mathcal{G}, U \in \mathcal{N}_1(H) \text{ symmetric}\}.$$

Then:

1. *If $f(G)$ is topologically commutant for every $f \in \mathcal{F}$ and every $G \in \mathcal{G}$, then $\mathcal{B}$ is a base of 1-neighborhoods for a group topology on $\mathcal{F}$.*

2. *If $\mathcal{F}$ contains the constant functions and $\mathcal{B}$ is a base of 1-neighborhoods for a group topology on $\mathcal{F}$, then $f(G)$ is topologically commutant for every $f \in \mathcal{F}$ and every $G \in \mathcal{G}$.*

Proof 274 *The proof follows as itemized as the statement of the theorem.*

1. *We will rely on Theorem 74. We have to verify that the filter base $\mathcal{B} := \{\mathcal{U}(G,U) : G \in \mathcal{G}, U \in \mathcal{N}_1(H)\}$ satisfies all three conditions of Theorem 74.*

   (a) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find a 1-neighborhood $V \subseteq H$ such that $VV \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $\mathcal{U}(G,V)\mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*

   (b) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find a 1-neighborhood $V \subseteq H$ such that $V^{-1} \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $\mathcal{U}(G,V)^{-1} \subseteq \mathcal{U}(G,U)$.*

   (c) *Take $\mathcal{U}(G,U) \in \mathcal{B}$ and $f \in \mathcal{F}$. Since $f(G)$ is topologically commutant, there exists a symmetric 1-neighborhood $V \subseteq H$ such that $f(g)V \subseteq Uf(g)$ for all $g \in G$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $f\mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)f$.*

2. *Conversely, suppose that $\mathcal{F}$ contains the constant functions and $\mathcal{B}$ is a base of 1-neighborhoods for a group topology on $\mathcal{F}$. Fix arbitrary element $f \in \mathcal{F}, G \in \mathcal{G}, U \in \mathcal{N}_1(G)$. In virtue of Theorem 74(3), there exist $J \in \mathcal{G}, V \in \mathcal{N}_1(H)$ such that $f\mathcal{U}(J,V) \subseteq \mathcal{U}(G,U)f$. We have to check that $f(g)V \subseteq Uf(g)$ for all $g \in G$. Take any $g \in G$ and any $v \in V$. Let us denote by $v\mathbf{1} \in \mathcal{F}$ to the constant function mapping I to v. Note that $v\mathbf{1} \in \mathcal{U}(J,V)$. There exists $h \in \mathcal{U}(G,U)$ satisfying that $fv\mathbf{1} = hf$. In particular, $f(g)v = h(g)f(g) \in Uf(g)$. The arbitrariness of $v \in V$ assures that $f(g)V \subseteq Uf(g)$.*

The convergence group topology can still be obtained if the sets $\mathcal{U}(G,U)$ run over U in a base of 1-neighborhoods of H (even if they are not symmetric). When dealing with additive notation, the convergence group topology will be referred to as "convergence additive topology", and when dealing with multiplicative notation, it will be referred to as "convergence multiplicative topology".

Theorem 80 *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let H be a topological group. Let $\mathcal{F} \subseteq H^I$ be a subgroup. Assume that $f(G)$ is topologically commutant for every $f \in \mathcal{F}$ and every $G \in \mathcal{G}$. If U is an open neighborhood of 1 in G and $E \in \mathcal{G}$ is so that $g(E)$ is compact in G for all $g \in \mathcal{U}(E,U)$, then $\mathcal{U}(E,U)$ is an open neighborhood of $\mathbf{1}$ in $\mathcal{F}$.*

Proof 275 *Fix an arbitrary $g \in \mathcal{U}(E,U)$. For every $i \in E$, there exists an open neighborhood V_i of 1 in G such that $g(i)V_i \subseteq U$ and $V_iV_i \subseteq U$. For each $i \in I$, we can find an open neighborhood W_i of 1 in G satisfying that $W_iW_i \subseteq V_i$. Since $g(E)$ is compact in G, there are $i_1, \ldots, i_k \in E$ such that $\{g(i_1)W_{i_1}, \ldots, g(i_k)W_{i_k}\}$ is a finite subcovering of $g(E)$. Take $W := W_{i_1} \cap \cdots \cap W_{i_k}$. We will show next that $g(i)W \subseteq U$ for each $i \in E$. Indeed, take any $i \in E$. There exists $j \in \{1, \ldots, k\}$ in such a way that $g(i) \in g(i_j)W_{i_j}$. Thus, if $w \in W$, $g(i)w = g(i_j)g(i_j)^{-1}g(i)w \in g(i_j)W_{i_j}w \subseteq g(i_j)W_{i_j}W_{i_j} \subseteq g(i_j)V_{i_j} \subseteq U$. Finally, we will prove that $g\mathcal{U}(E,W) \subseteq \mathcal{U}(E,U)$. Indeed, fix an arbitrary $h \in g\mathcal{U}(E,W)$. Then $g^{-1}h \in \mathcal{U}(E,W)$, meaning that $g(i)^{-1}h(i) = \left(g^{-1}h\right)(i) \in W$ for all $i \in E$, that is, $h(i) \in g(i)W \subseteq U$ for each $i \in E$.*

3.2.3 Topological rings and fields

It is easy to understand that every ring can be turned into a topological ring by endowing it with the discrete topology or the trivial topology. This way, the category of topological rings somehow contains the category of rings. Thus, everything that applies to topological rings also applies, in some sense, to rings in general. We refer the reader to [34, 35] for a wider perspective on topological rings and fields.

3.2.3.1 Ring topologies

As often here we will assume that all rings have a unity 1. Most of the results that we will present are actually valid, in some cases with the same proof, for nonunitary rings. However, in classical Calculus, all rings are always assumed to be unitary. This is why, in this modern Calculus book, we keep making the same assumption.

Definition 230 (Topological ring) *A topological ring is a topological universal algebra* $((R,\tau),\{+,0,-,\cdot,1\},\varnothing)$*, where* $(R,\{+,0,-,\cdot,1\},\varnothing)$ *is a ring and* (R,τ) *is a topological space.*

The notion of topological universal algebra in the settings of the previous definition means that the internal operations, $+$, 0, $-$, $\cdot$, 1, are continuous. A ring topology is a topology on a ring that makes it a topological ring. The additive group of a topological ring is trivially a topological group, and its multiplicative monoid is a topological monoid. The inherited topology on every subring is a ring topology.

Theorem 81 *Let* R *be a ring. If* τ *is a ring topology on* R *and* $\mathcal{B}$ *is a base of neighborhoods of* 0*, then the following is satisfied:*

1. *For every* $V\in\mathcal{B}$*, there exists* $U\in\mathcal{B}$ *with* $U+U\subseteq V$*.*
2. *For every* $V\in\mathcal{B}$*, there exists* $U\in\mathcal{B}$ *with* $-U\subseteq V$*.*
3. *For every* $V\in\mathcal{B}$*, there exists* $U\in\mathcal{B}$ *with* $UU\subseteq V$*.*
4. *For every* $V\in\mathcal{B}$ *and every* $r\in R$*, there exists* $U\in\mathcal{B}$ *with* $rU\cup Ur\subseteq V$*.*

Conversely, if $\mathcal{B}$ *is a filter base of* $\mathcal{P}(R)$ *verifying all four properties above, then there exists a unique ring topology on* R *such that* $\mathcal{B}$ *is a base of* 0*-neighborhoods. This topology is given by*

$$\tau:=\{A\subseteq R:\forall a\in A\ \ \exists U\in\mathcal{B}\ \ a+U\subseteq A\}\cup\{\varnothing\}.$$

Proof 276 *Suppose first that* τ *is a ring topology on* R *and* $\mathcal{B}$ *is a base of neighborhoods of* 0*. The above properties are direct consequences of the continuity of the internal operations* $+$*,* $-$*,* $\cdot$*. In particular, the last property is a direct consequence of the continuity of the left- and right-multiplication operators*

$$\begin{array}{ccc} R & \to & R \\ s & \mapsto & rs \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \to & R \\ s & \mapsto & sr. \end{array}$$

Conversely, assume that $\mathcal{B}$ *is a filter base of* $\mathcal{P}(R)$ *verifying the above properties. In view of Theorem 74,*

$$\tau:=\{A\subseteq R:\forall a\in A\ \ \exists U\in\mathcal{B}\ \ \ a+U\subseteq A\}\cup\{\varnothing\}$$

is a group topology on R *and it is the only group topology on* R *for which* $\mathcal{B}$ *is a base of* 0*-neighborhoods. It only remains to show* τ *is a ring topology on* R *and is the unique ring topology on* R *for which* $\mathcal{B}$ *is a base of* 0*-neighborhoods. Let us prove first that multiplication*

on R is continuous. Indeed, fix arbitrary $r, s \in R$. Take U a 0-neighborhood in R, which we may assume without any loss to belong to $\mathcal{B}$. There exists $W \in \mathcal{B}$ with $W + W + W \subseteq U$. Since $\mathcal{B}$ is a filter base and satisfies (3),(4), we can find $V \in \mathcal{B}$ such that $VV \subseteq W$ and $rV \cup Vs \subseteq W$. As a consequence,

$$(r + V)(s + V) \subseteq rs + (rV + Vs + VV) \subseteq rs + (W + W + W) \subseteq rs + U.$$

This shows that multiplication on R is continuous and thus τ is a ring topology. Finally, if τ' is another ring topology on R for which $\mathcal{B}$ is a base of 0-neighborhoods, then τ' is a group topology on R for which $\mathcal{B}$ is a base of 0-neighborhoods, so $\tau' = \tau$.

Representative examples of topological rings are the seminormed rings with the topology induced by the seminorm.

Example 51 *If R is a seminormed ring, then the topology induced by the pseudometric, $d(r, s) := \|r - s\|$ for all $r, s \in R$, induced by the seminorm of R, is a ring topology. Indeed, the seminorm of R is conjugation-invariant on the commutative additive group of R, which makes it a topological group. It only remains to show that multiplication on R is continuous. If $(r_i)_{i\in I}, (s_i)_{i\in I}$ are nets on R converging to $r, s \in R$, respectively, then*

$$\|r_i s_i - rs\| \leq \|r_i\|\|s_i - s\| + \|r_i - r\|\|s\|$$

according to Lemma 69(4). By calling again on Example 50, the seminorm on R is a non-expansive map; therefore, it is continuous, so $(\|r_i\|)_{i\in I}$ converges to $\|r\|$. As a consequence, $(\|r_i s_i - rs\|)_{i\in I}$ converges to 0 and then $(r_i s_i)_{i\in I}$ converges to rs.

If multiplicative inversion on a topological ring R is continuous, then we will call R a topological inversion ring. The following definition expresses this concept in the language of topological universal algebras.

Definition 231 (Topological inversion ring) *A topological inversion ring is a topological universal algebra $\left((R, \tau), \left\{+, 0, -, \cdot, 1, {}^{-1}\right\}, \varnothing\right)$, where $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$ is a ring, ${}^{-1} : \mathcal{U}(R) \to \mathcal{U}(R)$ is its inversion, and (R, τ) is a topological space.*

The multiplicative monoid of a topological inversion ring is a topological inversion monoid, and its multiplicative group of invertibles is a topological group.

Lemma 110 *Let M be a module over a topological inversion ring R. Let B^+ be a unit segment of R such that $B^+ \setminus \{0\} \subseteq \mathcal{U}(R)$ and $0 \in \mathrm{cl}\,(B^+ \setminus \{0, 1\})$. For every $a \neq b$ in M and every neighborhood V of 0 in R, there exists $r_0 \in B^+ \setminus \{0, 1\}$ such that $v_0 := -(1 - r_0)^{-1} r_0 \in V$ and*

$$a = r_0 b + (1 - r_0)(a + v_0(b - a)) \in \Delta(b, a + v_0(b - a)).$$

Proof 277 *The following function*

$$\begin{array}{rcl} B^+ \setminus \{1\} & \to & R \\ r & \mapsto & -(1 - r)^{-1} r \end{array}$$

is well-defined and continuous at 0. Therefore, there exists a neighborhood U of 0 in R such that, if $r \in U \cap (B^+ \setminus \{1\})$, then $-(1 - r)^{-1} r \in V$. Since $0 \in \mathrm{cl}\,(B^+ \setminus \{0, 1\})$, we can find $r_0 \in U \cap (B^+ \setminus \{0, 1\})$, hence $v_0 := -(1 - r_0)^{-1} r_0 \in V$. Then $a = a + (r_0 + (1 - r_0)v_0)\,(b - a) = a + r_0(b - a) + (1 - r_0)v_0(b - a) = r_0 b + (1 - r_0)\,(a + v_0(b - a)) \in \Delta(b, a + v_0(b - a))$.

On topological rings for which inversion is not continuous, it is often considered a special topology on its multiplicative group of invertibles.

Proposition 80 *Let R be a topological ring whose inversion is not continuous. Then $\{U(V) : V \in \mathcal{N}_0(R)\}$ is a base for a group topology on $\mathcal{U}(R)$, where $U(V) := \left\{r \in \mathcal{U}(R) : r, r^{-1} \in 1 + V\right\}$.*

Proof 278 *All is needed to check is that the three conditions of Theorem 74 are satisfied:*

1. *For every $V \in \mathcal{N}_0(R)$ there exists $W \in \mathcal{N}_0(R)$ such that $W + W + WW \subseteq V$. Let us show that $U(W)U(W) \subseteq U(V)$. Indeed, let $r_1, r_2 \in U(W)$. There are $w_1, w_2, w_1', w_2' \in W$ such that $r_1 = 1 + w_1$, $r_1^{-1} = 1 + w_1'$, $r_2 = 1 + w_2$, and $r_2^{-1} = 1 + w_2'$. Then $r_1 r_2 = 1 + w_1 + w_2 + w_1 w_2 \in 1 + W + W + WW \subseteq 1 + V$ and $r_1^{-1} r_2^{-1} = 1 + w_1' + w_2' + w_1' w_2' \in 1 + W + W + WW \subseteq 1 + V$.*

2. *For every $V \in \mathcal{N}_0(R)$, $U(V)^{-1} = U(V)$.*

3. *For every $V \in \mathcal{N}_0(R)$ and every $u \in \mathcal{U}(R)$, there exists $W \in \mathcal{N}_0(R)$ such that $uW \subseteq Vu$. Let us show that $uU(W) \subseteq U(V)u$. Indeed, let $r \in U(W)$. There are $w, w' \in W$ such that $r = 1 + w$ and $r^{-1} = 1 + w'$. Then $ur = u + uw = u + vu = (1 + v)u$ for some $v \in V$ and $ur^{-1} = u + uw' = u + v' = (1 + v')\, u$ for some $v' \in V'$. Finally, $1 + v = uru^{-1}$ and $1 + v' = ur^{-1}u^{-1}$, meaning that $(1 + v)^{-1} = 1 + v'$.*

Particular cases of topological inversion rings are the topological division rings.

Definition 232 (Topological division ring) *A topological division ring is a topological universal algebra $\left((R, \tau), \left\{+, 0, -, \cdot, 1, {}^{-1}\right\}, \varnothing\right)$, where $\left(R, \{+, 0, -, \cdot, 1, {}^{-1}\}, \varnothing\right)$ is a division ring and (R, τ) is a topological space.*

Informally speaking, a topological division ring is a division topological inversion ring, that is, a division topological ring with a continuous multiplicative inversion.

Corollary 17 *Let R be a topological ring. Then $R_{(0)} := \bigcap \mathcal{N}_0(R)$ is a closed two-sided ideal of R.*

Proof 279 *According to Corollary 15(2), we have that $\bigcap \mathcal{N}_0(R)$ is a closed additive subgroup of R. Let us show that $\bigcap \mathcal{N}_0(R)$ is also a two-sided ideal. Fix arbitrary elements $r, s \in R$ and $t \in \bigcap \mathcal{N}_0(R)$. Take any 0-neighborhood $U \subseteq R$. In view of Theorem 81(4), there exists $V \in \mathcal{N}_0(R)$ such that $rV \cup Vs \subseteq U$. Then $rt, ts \in U$. The arbitrariness of $U \in \mathcal{N}_0(R)$ shows that $rt, ts \in \bigcap \mathcal{N}_0(R)$. As a consequence, $\bigcap \mathcal{N}_0(R)$ is a two-sided ideal.*

Next result is a direct consequence of the fact that topological rings are regular topological spaces.

Corollary 18 *Let R be a topological ring. The ring topology of R is trivial if and only if any of the following is satisfied:*

- $0 \in \mathrm{cl}(\{1\}) = \bigcap \mathcal{N}_1$.
- $1 \in \mathrm{cl}(\{0\}) = \bigcap \mathcal{N}_0$.

Proof 280 *The equivalences of the two items are a direct consequence of Corollary 15(1) together with Lemma 49. Next, if the ring topology of R is trivial, then $0 \in R = \mathrm{cl}(\{1\})$. Conversely, if $1 \in \bigcap \mathcal{N}_0$, then $\bigcap \mathcal{N}_0 = R$ because $\bigcap \mathcal{N}_0$ is a closed two-sided ideal of R in view of Corollary 17. Hence, R has the trivial topology by calling again on Corollary 15(2).*

Another consequence of the previous result follows next.

Corollary 19 *Let R be a topological ring not endowed with the trivial topology. If $(u_i)_{i\in I} \subseteq \mathcal{U}(R)$ is a net of invertibles converging to 0, then no subnet of $\left(u_i^{-1}\right)_{i\in I}$ is convergent.*

Proof 281 *Assume on the contrary that there exists a subnet $\left(u_{i_j}^{-1}\right)_{j\in J}$ which converges to some $r \in R$. Then the net $\left(u_{i_j}u_{i_j}^{-1}\right)_{J\in J}$ converges to $0 \cdot r = 0$. However, $u_{i_j}u_{i_j}^{-1} = 1$ for every $j \in J$, reaching a contradiction with the fact that $0 \notin \mathrm{cl}(\{1\})$ in view of Corollary 18.*

Next result establishes that the closure of a subring or ideal is still a subring or ideal, respectively.

Corollary 20 *Let R be a topological ring. Then:*

1. *For every subring S of R, $\mathrm{cl}(S)$ is also a subring.*
2. *For every left, right, or two-sided ideal I of R, $\mathrm{cl}(I)$ is also a left, right, or two-sided ideal, respectively.*

Proof 282 *We will only prove the first item since the second follows a similar proof. So, let S be a subring of R. According to Corollary 15(8), $\mathrm{cl}(S)$ is an additive subgroup of R. Fix arbitrary elements $r_1, r_2 \in R$. Take any $U \in \mathcal{N}_0$. There exists $W \in \mathcal{N}_0$ such that $W+W+W \subseteq U$. We can find a 0-neighborhood $V \in \mathcal{N}_0$ verifying that $VV \subseteq W$, $r_1V \subseteq W$, and $Vr_2 \subseteq W$. Next,*

$$(r_1+V)(r_2+V) \subseteq r_1r_2 + r_1V + Vr_2 + VV \subseteq r_1r_2 + W + W + W \subseteq r_1r_2 + U.$$

Since $(r_1+V)\cap S \neq \varnothing$ and $(r_2+V)\cap S \neq \varnothing$ and S is closed under multiplication, we conclude that $(r_1r_2+U)\cap S \neq \varnothing$.

Boundedness is a notion that will be studied with more depth in the scope of topological modules. Here, we will briefly taste it because it is necessary to construct convergence ring topologies.

Definition 233 (Bounded set) *Let R be a topological ring. Let $A \subseteq R$. Then:*

1. *We say that A is left bounded if, for every $U \in \mathcal{N}_0$, there exists $u \in \mathcal{U}(R)$ with $A \subseteq uU$. In a similar way, we define right boundedness.*
2. *We say that A is bounded if it is left and right bounded.*

In every topological ring R, a 0-neighborhood $W \subseteq R$ is left bounded if and only if $\{uW : u \in \mathcal{U}(R)\}$ is a base of neighborhoods of 0. Similarly, a 0-neighborhood $W \subseteq R$ is right bounded if and only if $\{Wu : u \in \mathcal{U}(R)\}$ is a base of neighborhoods of 0.

Corollary 21 *Let R be a topological $*$-ring. If there exists a left or right bounded and $*$-invariant neighborhood W of 0, then $*$ is a homeomorphism.*

Proof 283 *Since $*^{-1} = *$, it only suffices to prove is that if $V \subseteq R$ is a neighborhood of 0, then V^* is also a neighborhood of 0. Indeed, there exists an invertible $u \in \mathcal{U}(R)$ such that $uW \subseteq V$ (here we are assuming that W is left bounded, a similar proof applies for the right bounded case). Then $Wu^* \subseteq V^*$, meaning that V^* is a neighborhood of 0 in R because u^* is also invertible.*

The final remark in this subsubsection deals with the product of ring topologies, which turns out to be a ring topology as well.

Remark 74 *Let $(R_i)_{i\in I}$ be a family of topological rings. Then $\prod_{i\in I} R_i$ is a topological ring endowed with the product topology.*

As a direct consequence of the previous theorem, the pointwise convergence topology on R^I is always a ring topology.

3.2.3.2 Topological ordered rings

A topological ordered ring [18] is simply an ordered ring in such a way that the order topology is well defined and is a ring topology.

Definition 234 (Topological ordered ring) *An ordered ring R is called a topological ordered ring whenever the order topology on R is well defined and a ring topology.*

Not every order topology on a unital ordered ring is a ring topology.

Example 52 *Let $R := \mathbb{R}[x]$ and define $p \leq q$ whenever $p = q$ or the leading coefficient of $q - p$ is positive. Notice that this is a total ring ordering. According to the properties of ring topologies, for every neighborhood V of 0 in R and every $r \in R$ there exists another neighborhood W of 0 in R satisfying that $rW \subseteq V$, which cannot hold for $r := x$ and $V := (-1, 1)$.*

In order to find sufficient conditions to assure that the order topology in a ordered ring is a ring topology, we will strongly rely on Theorem 34.

Lemma 111 *Let R be an ordered ring. Let $\mathcal{S} := \{\uparrow_\times r, \downarrow^\times r : r \in R\}$. Then:*

1. *$\mathcal{S}$ is a subbase for a topology on R if and only if $R^+ \neq \{0\}$. In this situation, $\downarrow^\times r = r + R^- \setminus \{0\} \neq \varnothing$ and $\uparrow_\times r = r + R^+ \setminus \{0\} \neq \varnothing$ for all $r \in R$.*
2. *If $R^+ \neq \{0\}$ and $R^+ \setminus \{0\}$ is downward directed, then $\{(-r, r) : r > 0\}$ is a base of neighborhoods of 0 for the order topology.*

Proof 284 *The proof will follow as itemized as the statement of the lemma.*

1. *According to Theorem 34, $\mathcal{S}$ is a subbase for a topology on R if and only if, for all $r \in R$, either $\uparrow_\times r \neq \varnothing$ or $\downarrow^\times r \neq \varnothing$, which is equivalent to $R^+ \neq \{0\}$ by bearing in mind that $\downarrow r = r + R^-$ and $\uparrow r = r + R^+$ for all $r \in R$.*
2. *Let $W \subseteq R$ be a 0-neighborhood for the order topology. There can be found $r_1, \ldots, r_n, s_1, \ldots, s_m \in R$ satisfying that $0 \in \uparrow_\times r_1 \cap \cdots \cap \uparrow_\times r_n \cap \downarrow^\times s_1 \cap \cdots \cap \downarrow^\times s_m \subseteq W$. Observe that $r_i < 0$ and $s_j > 0$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Since $R^+ \setminus \{0\}$ is downward directed, there exists $r_0 \in R^+ \setminus \{0\}$ such that $r_0 \leq -r_i$ for all $i \in \{1, \ldots, n\}$ and $r_0 \leq s_j$ for all $j \in \{1, \ldots, m\}$. Notice that $(-r_0, r_0) \subseteq \uparrow_\times r_1 \cap \cdots \cap \uparrow_\times r_n \cap \downarrow^\times s_1 \cap \cdots \cap \downarrow^\times s_m \subseteq W$.*

We are finally in the right position to find a sufficient condition for the order topology induced by a ring ordering to be a ring topology.

Theorem 82 *Let R be an ordered ring. Suppose that $R^+ \neq \{0\}$, $R^+ \setminus \{0\}$ is downward directed, and $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$. Then:*

1. *The order topology is an additive group topology.*

2. *If, in addition, R is a division ring, $0, 1$ are comparable, $R^+ \setminus \{0\}$ is multiplicatively symmetric, and $R = R^+ + R^-$, then the order topology is a ring topology.*

Proof 285 *By Lemma 111(1), $\mathcal{S} := \{\uparrow_\times r, \downarrow^\times r : r \in R\}$ is a subbase for a topology on R (the order topology). According to Lemma 111(2), $\{(-r, r) : r > 0\}$ is a base of neighborhoods of 0 for the order topology. Let $\mathcal{B}_0(\mathcal{S})$ denote the family of all finite intersections of elements of $\mathcal{S}$ containing 0. We will show that $\mathcal{B}(\mathcal{S})$ satisfies the first two conditions of Theorem 81.*

1.1. Let $W \in \mathcal{B}_0(\mathcal{S})$. There exists $r_0 \in R^+ \setminus \{0\}$ such that $(-r_0, r_0) \subseteq W$. Since $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, there exists $t_0 \in R$ such that $0 < t_0 < r_0$. By using again that $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, we can find $s_0 \in R$ such that $0 < s_0 < r_0 - t_0$. Finally, we will prove that $(-s_0, s_0) + (-t_0, t_0) \subseteq W$. Indeed, let us observe first that $(-s_0, s_0) + (-t_0, t_0) \subseteq (-s_0 - t_0, s_0 + t_0)$. Next, take $a \in R$ with $-s_0 - t_0 < a < s_0 + t_0$. Note that $a < s_0 + t_0 < r_0$. Also, $-s_0 - t_0 < a$ so $-a < s_0 + t_0 < r_0$. We have just shown that $(-s_0, s_0) + (-t_0, t_0) \subseteq (-s_0 - t_0, s_0 + t_0) \subseteq (-r_0, r_0) \subseteq W$ with $(-s_0, s_0), (-t_0, t_0) \in \mathcal{B}_0(\mathcal{S})$.

1.2. Let $W \in \mathcal{B}_0(\mathcal{S})$. There exists $r_0 \in R^+ \setminus \{0\}$ such that $(-r_0, r_0) \subseteq W$. Now, it only suffices to notice that $(-r_0, r_0)$ is an additively symmetric element of $\mathcal{B}_0(\mathcal{S})$.

Next, let us assume, in addition, that R is a division ring, $0, 1$ are comparable, $R^+ \setminus \{0\}$ is multiplicatively symmetric, and $R = R^+ + R^-$. So, let us finally check that $\mathcal{B}_0(\mathcal{S})$ also satisfies the third and fourth condition of Theorem 81.

2.3. Let $W \in \mathcal{B}_0(\mathcal{S})$. There exists $r_0 \in R^+ \setminus \{0\}$ such that $(-r_0, r_0) \subseteq W$. Since $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, there exists $t_0 \in R$ such that $0 < t_0 < r_0$. By using again that $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, we can find $p_0 \in R$ such that $0 < p_0 < r_0 - t_0$. Again, by relying on the fact that $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, there exists $q_0 \in R$ satisfying that $0 < q_0 < r_0 - t_0 - p_0$. Next, since $R^+ \setminus \{0\}$ is downward directed, $h_0 \in R^+ \setminus \{0\}$ can be found in such a way that $h_0 \leq q_0$, $h_0 \leq p_0$, and $h_0 \leq t_0$. Finally, we will prove that $(-h_0, h_0)(-1, 1) \subseteq (-3h_0, 3h_0) \subseteq (-r_0, r_0) \subseteq W$. Indeed, take arbitrary elements $a, b \in R$ with $-h_0 < a < h_0$ and $-1 < b < 1$. Note that $h_0 - a > 0$ and $1 - b > 0$, meaning that $h_0 - h_0 b - a + ab = (h_0 - a)(1 - b) \geq 0$. Then $ab \geq -h_0 + h_0 b + a$. Since $b > -1$ and $h_0 > 0$, we have that $h_0 b \geq -h_0$. Also, $-h_0 < a$. As a consequence, $ab \geq -h_0 + h_0 b + a > -h_0 - h_0 - h_0$. By repeating the same reasoning for a and $-b$, we conclude that $-ab = a(-b) > -h_0 - h_0 - h_0$, that is, $ab < 3h_0$. We have just shown that $(-h_0, h_0)(-1, 1) \subseteq (-3h_0, 3h_0)$. Let us finally prove that $(-3h_0, 3h_0) \subseteq (-r_0, r_0) \subseteq W$. Indeed, if $c \in R$ and $-3h_0 < c < 3h_0$, then $c < h_0 + h_0 + h_0 \leq q_0 + p_0 + t_0 < r_0$, and $-c < h_0 + h_0 + h_0 \leq q_0 + p_0 + t_0 < r_0$, so $-r_0 < c$, reaching the desired result.

2.4. Let $W \in \mathcal{B}_0(\mathcal{S})$ and fix an arbitrary element $s \in R$. There exists $r \in R^+ \setminus \{0\}$ such that $(-r, r) \subseteq W$. By hypothesis, $s = s^+ + s^-$ with $s^+ \in R^+$ and $s^- \in R^-$. We may assume without any loss of generality that $s^+ > 0$ and $s^- < 0$. First off, since $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, there exists $r_1 \in R$ such that $0 < r_1 < r$. By using again that $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, we can find $r_2 \in R$ such that $0 < r_2 < r - r_1$. Then $(-r_1, r_1) + (-r_2, r_2) \subseteq (-(r_1 + r_2), r_1 + r_2) \subseteq (-r, r)$. Next, $R^+ \setminus \{0\}$ is multiplicatively symmetric, therefore $(s^+)^{-1} > 0$ and $(-s^-)^{-1} > 0$, and since the two of them are invertible, it occurs that $(s^+)^{-1} r_1 > 0$ and $(-s^-)^{-1} r_2 > 0$. Since $R^+ \setminus \{0\}$ is downward directed, there exists $t \in R^+ \setminus \{0\}$ such that $t \leq (s^+)^{-1} r_1$ and $t \leq (-s^-)^{-1} r_2$. Then $s(-t, t) \subseteq s^+(-t, t) + s^-(-t, t) = (-s^+ t, s^+ t) + (s^- t, -s^- t) \subseteq (-r_1, r_1) + (-r_2, r_2) \subseteq (-(r_1 + r_2), r_1 + r_2) \subseteq (-r, r) \subseteq (-r, r) \subseteq W$. Similarly, by choosing $t \in R^+ \setminus \{0\}$ with $t \leq r_1 (s^+)^{-1}$ and $t \leq r_2 (-s^-)^{-1}$, we obtain $(-t, t)s \subseteq (-r, r) \subseteq W$.

Observe that the proof of the previous theorem can be easily readapted to prove the following corollary.

Corollary 22 *If R is a totally ordered division ring, then its order topology is a ring topology.*

Proof 286 *The total order of R allows that $R = R^+ \cup R^-$ and $R^+ \setminus \{0\}$ be downward directed and multiplicatively symmetric. Finally, the condition that $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$ is obtained by the fact that R has no holes because $(1+1)^{-1} > 0$.*

The final result in this subsubsection shows that the converse to Theorem 82 does not hold.

Theorem 83 *Let R be a ring of characteristic 0. Consider the ordering set $S := \{0\} \cup \left\{1+ \overset{n}{\cdots} +1 : n \in \mathbb{N}\right\}$. The ring ordering induced by S is faithful and its order topology is the discrete topology, hence a ring topology.*

Proof 287 *First off, $S \cup -S = \mathbb{Z}1$, which is a subring of R, hence the ring ordering induced by S is faithful. Notice that $\uparrow_\times (-1) = -1 + S \setminus \{0\}$ and $\downarrow^\times 1 = 1 - S \setminus \{0\}$. If $r \in \uparrow_\times(-1) \cap \downarrow^\times 1$, then there are $s_1, s_2 \in S \setminus \{0\}$ such that $r = -1+s_1$ and $r = 1-s_2$. Then $s_1 + s_2 = 1 + 1$. There are $n_1, n_2 \in \mathbb{N}$ such that $s_1 = 1+ \overset{n_1}{\cdots} +1$ and $s_2 = 1+ \overset{n_2}{\cdots} +1$. Thus, $1+ \overset{n_1+n_2}{\cdots} +1 = 1 + 1$. Since $\mathrm{char}(R) = \{0\}$, we necessarily have that $n_1 = n_2 = 1$, hence $s_1 = s_2 = 1$, meaning that $r = 0$. As a consequence, $\{0\} = \uparrow_\times(-1) \cap \downarrow^\times 1$. Next, by Lemma 111(1), $\mathcal{S} := \{\uparrow_\times r, \downarrow^\times r : r \in R\}$ is a subbase for a topology on R (the order topology). We will prove that this topology is the discrete topology. Fix any $r \in R$. We will prove that $\{r\} = \uparrow_\times(r-1) \cap \downarrow^\times(r+1)$. Indeed, $0 < 1$ and $-1 < 0$, so $r < r+1$ and $r-1 < r$, meaning that $r \in \uparrow_\times(r-1) \cap \downarrow^\times(r+1)$. Conversely, take any $s \in \uparrow_\times(r-1) \cap \downarrow^\times(r+1)$. Notice that $r - 1 < s < r + 1$, hence $-1 < s - r < 1$, so $s - r \in (-1, 1) = \{0\}$, meaning that $s = r$.*

Modulus functions were originally introduced in [36] for the positive real line.

Definition 235 (Modulus function) *Let R be a unital ordered ring. A modulus function is a function $f : R^+ \to R^+$ satisfying the following conditions for all $r, s \in R^+$:*

- *Nullity: $f(r) = 0$ if and only if $r = 0$.*
- *Subadditivity: $f(r + s) \leq f(r) + f(s)$.*
- *Increasingness: $f(r) \leq f(s)$ if $r \leq s$.*
- *Continuity: f is continuous at 0 for the order topology of R^+.*

Basic properties of modulus functions are described in the following proposition.

Proposition 81 *Let R be a unital ordered ring. Let $f : R^+ \to R^+$ be a modulus function. Let $n \in \mathbb{N}$. Then:*

1. *$f(nr) \leq nf(r)$ for every $r \in R^+$.*
2. *If $n1 \in \mathcal{U}(R)$, then $(n1)^{-1} f(r) \leq f\left((n1)^{-1} r\right)$ for every $r \in R^+$.*
3. *f is continuous from the right for the order topology of R^+.*

Proof 288 *Let us itemize the proof via the itemization of the statement.*

1. *Simply note that* $f(nr) = f\left(r+\overset{n}{\cdots}+r\right) \leq f(r)+\overset{n}{\cdots}+f(r) = nf(r)$.

2. *By Proposition 81(1),* $f(r) = f\left(n\left((n1)^{-1}r\right)\right) \leq nf\left((n1)^{-1}r\right)$, *hence* $n^{-1}f(r) \leq f\left((n1)^{-1}r\right)$.

3. *Fix an arbitrary* $r \in R^+ \setminus \{0\}$. *Let* W *be a neighborhood of* $f(r)$ *for the order topology of* R^+. *Then there are* $a_1, \ldots, a_k, b_1, \ldots, b_l \in R^+$ *in such a way that* $f(r) \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_k \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_l \subseteq W$. *Notice that* $\downarrow^\times (b_1 - f(r)) \cap \cdots \cap \downarrow^\times (b_l - f(r))$ *is a neighborhood of* 0 *in the order topology of* R^+. *Therefore, the continuity of* f *at* 0 *assures the existence of elements* $c_1, \ldots, c_j \in R^+$ *such that* $0 \in \downarrow^\times c_1 \cap \cdots \cap \downarrow^\times c_j$ *and*

$$f\left(\uparrow 0 \cap \downarrow^\times c_1 \cap \cdots \cap \downarrow^\times c_j\right) \subseteq \downarrow^\times (b_1 - f(r)) \cap \cdots \cap \downarrow^\times (b_l - f(r)).$$

Let us prove that $f\left(\uparrow r \cap \downarrow^\times (r+c_1) \cap \cdots \cap \downarrow^\times (r+c_j)\right) \subseteq \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_k \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_l$. *Take an element* $y \in \uparrow r \cap \downarrow^\times (r+c_1) \cap \cdots \cap \downarrow^\times (r+c_j)$. *Then* $y \geq r$ *and* $y - r \in \downarrow^\times c_1 \cap \cdots \cap \downarrow^\times c_j$, *meaning that* $f(y-r) \in \downarrow^\times (b_1 - f(r)) \cap \cdots \cap \downarrow^\times (b_l - f(r))$. *Then* $a_j < f(r) \leq f(y) \leq f(y-r) + f(r) < b_i - f(r) + f(r) = b_i$ *for all* $i \in \{1, \ldots, l\}$ *and all* $j \in \{1, \ldots, k\}$.

Modulus functions can be bounded or unbounded. For example, $f(x) = \dfrac{x}{x+1}$ is bounded, whereas $f(x) = x^p$, for $0 < p < 1$, is unbounded. The notion of compatible moduli was first introduced in [37].

Definition 236 (Compatible modulus function) *Let* R *be a unital ordered ring. Let* $f : R^+ \to R^+$ *be a modulus function. We say that* f *is compatible if the following conditions are met:*

- $f(n1) \in \mathcal{U}(R)$ *for all* $n \in \mathbb{N}$.
- *For any* $r > 0$ *there exists* $s > 0$ *and* $n_0 = n_0(r)$ *such that* $f(n1)^{-1} f(ns) < r$ *for all* $n \geq n_0$.

Examples [37] of compatible modulus functions are $f(x) = x + \log(x+1)$ and $f(x) = x + \dfrac{x}{x+1}$. Examples of noncompatible modulus functions are $f(x) = \log(x+1)$ and $f(x) = W(x)$, where W is the W-Lambert function restricted to $[0, \infty)$, that is, the inverse of xe^x.

3.2.3.3 Unit zero-neighborhoods

The beginning of this subsubsection is a motivational technical lemma, which establishes the bases for the generalization of absolutely invertible to the scope of topological rings.

Lemma 112 *Let* R *be a unital seminormed ring. Then:*

1. *The absolutely inversion map*

$$\begin{array}{ccc} \mathcal{U}_1(R) & \to & \mathcal{U}_1(R) \\ u & \mapsto & u^{-1} \end{array}$$

is continuous. In fact, $\mathrm{cl}(\mathcal{U}_1(R)) \cap \mathcal{U}(R) \subseteq \mathcal{U}_1(R)$.

2. *If* B_R *is compact and* $1 \in \mathrm{cl}\left(\{\|u\| : u \in \mathcal{U}_1(R) \setminus \mathsf{S}_R\}\right)$, *then* $1 \in \mathrm{cl}(\mathsf{U}_R)$. *If, in addition,* R *is normed, then* $1 \in \mathrm{cl}\left(\mathcal{U}_1(R) \setminus \mathsf{S}_R\right)$.

Proof 289 *The proof will follow as itemized as the statement.*

1. *If $(r_i)_{i\in\mathbb{N}} \subseteq \mathcal{U}_1(R)$ is a sequence converging to $s \in \mathcal{U}_1(R)$, then $(\|r_i\|)_{i\in\mathbb{N}}$ converges to $\|s\|$, so $\left(\|r_i\|^{-1}\right)_{i\in\mathbb{N}}$ converges to $\|s\|^{-1}$. Finally,*

$$\left\|r_i^{-1} - s^{-1}\right\| \leq \left\|s^{-1}\right\| \|s - r_i\| \left\|r_i^{-1}\right\| = \|s\|^{-1} \|s - r_i\| \|r_i\|^{-1} \to 0$$

as $i \to \infty$. Next, take $v \in \mathrm{cl}(\mathcal{U}_1(R)) \cap \mathcal{U}(R)$. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R)$ be convergent to v. Then $(\|u_n\|)_{n\in\mathbb{N}}$ converges to $\|v\|$ and $\left(\|u_n\|^{-1}\right)_{n\in\mathbb{N}}$ converges to $\|v\|^{-1}$. This means that $\left(\|u_n\|^{-1}\right)_{n\in\mathbb{N}}$ is bounded. Therefore,

$$\left\|u_n^{-1} - v^{-1}\right\| \leq \left\|v^{-1}\right\| \|v - u_n\| \left\|u_n^{-1}\right\| = \left\|v^{-1}\right\| \|v - u_n\| \|u_n\|^{-1} \to 0$$

as $n \to \infty$, meaning that $\left(u^{-1}\right)_{n\in\mathbb{N}}$ converges to v^{-1}. Finally, $\left(\|u_n\|^{-1}\right)_{n\in\mathbb{N}}$ converges to $\left\|v^{-1}\right\|$, resulting in $\left\|v^{-1}\right\| = \|v\|^{-1}$ for being $\mathbb{R}$ Hausdorff.

2. *Consider a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R) \setminus \mathsf{S}_R$ such that $(\|u_n\|)_{n\in\mathbb{N}}$ converges to 1. For each $n \in \mathbb{N}$, either $\|u_n\| < 1$ or $\|u_n\| > 1$, thus, $\left\|u_n^{-1}\right\| = \|u_n\|^{-1} > 1$ or $\left\|u_n^{-1}\right\| = \|u_n\|^{-1} < 1$. Therefore, by passing to a subsequence if necessary, we may assume without any loss of generality that $\|u_n\| > 1$ for all $n \in \mathbb{N}$ and that $(\|u_n\|)_{n\in\mathbb{N}}$ is decreasing to 1. Thus, $(u_n)_{n\in\mathbb{N}} \subseteq \mathsf{B}_R(0, \|u_1\|) = u_1\mathsf{B}_R$, which is compact because so is B_R by hypothesis. This compactness allows to assume, also without any loss by passing to a subsequence if neccesary, that $(u_n)_{n\in\mathbb{N}}$ converges to some $r \in \mathsf{S}_R$. For every $n \in \mathbb{N}$, $\left\|1 - ru_n^{-1}\right\| = \left\|u_n u_n^{-1} - ru_n^{-1}\right\| \leq \|u_n - r\| \left\|u_n^{-1}\right\| = \|u_n - r\| \|u_n\|^{-1} \to 0$ as $n \to \infty$. Next, simply observe that $\left\|ru_n^{-1}\right\| \leq \|r\| \left\|u_n^{-1}\right\| = \|u_n\|^{-1} < 1$ for every $n \in \mathbb{N}$. Finally, if we extra assume that R is Hausdorff, then we can assume that $\left(u_n^{-1}\right)$ is convergent to some $s \in \mathsf{S}_R$, which will satisfy that $rs = sr = 1$, ending up with a sequence $\left(ru_n^{-1}\right)_{n\in\mathbb{N}}$ of absolutely invertibles of norm strictly less than 1 converging to 1.*

The following technical lemma establishes sufficient conditions for the absolutely invertibles of norm 1 to be conform a closed set.

Lemma 113 *Let R be a unital normed ring. Then $\mathcal{U}_1(R) \cap \mathsf{S}_R$ is closed if either one of the following conditions hold:*

1. *R is complete.*
2. *S_R is compact.*

Proof 290 *Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R) \cap \mathsf{S}_R$ be a sequence converging to an element $r \in \mathsf{S}_R$. We will distinguish between the two conditions above:*

1. *R is complete. In this case, the equality $u_p^{-1} - u_q^{-1} = u_q^{-1}(u_q - u_p)u_p^{-1}$, for all $p, q \in \mathbb{N}$, shows that $\left(u_n^{-1}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence; hence, it is converging to an element $s \in \mathsf{S}_R$. Now it is clear that $rs = sr = 1$.*
2. *S_R is compact. In this case, the compacity of S_R allows us to deduce the existence of a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and an element $s \in \mathsf{S}_R$ such that $u_{n_k}^{-1} \to s$ as $k \to \infty$. Finally, $rs = 1 = sr$.*

Before introducing unit zero-neighborhoods, also known as unit neighborhoods of zero, we will present a weaker notion: unit ball. First, notice that an additively symmetric and multiplicatively idempotent open set in a topological ring does not necessarily contain zero

and the unity. Indeed, if R is a discrete ring, then $\{-1,1\}$ is an additively symmetric and multiplicatively idempotent open subset not containing 0, and $\{0\}$ is an additively symmetric and multiplicatively idempotent open subset not containing 1. Even more, in $\mathbb{R}$, $(-1,1)\setminus\{0\}$ is open, additively symmetric, multiplicatively idempotent, and does not contain 0 nor 1.

Definition 237 (Unit ball) *Let R be a topological ring. A unit ball in R is an additively symmetric and multiplicatively idempotent closed neighborhood of 0 in R containing 1.*

The unit ball of a unital seminormed ring is the most representative example of the above notion. Obviously, the entire ring is trivially a unit ball. In fact, if a ring R is endowed with the trivial topology, then R is the only unit ball. However, if R is endowed with the discrete topology, then $\{-1,0,1\}$ is the smallest unit ball. A unit ball B in a topological ring R is said to be unital provided that $1\in\mathrm{bd}(B)$. This fact implies that $B\neq R$ because R is trivially open and its boundary is empty. A nontrivial example of a nonunital unit ball is $\{-1,0,1\}$ in a discrete ring. For instance, $\mathbb{Z}$ endowed with its absolute value provides an example of a unital seminormed ring whose unit ball is not unital.

Definition 238 (Total unit ball) *Let R be a topological ring. A unit ball B of R is called total provided that, for every $u\in\mathcal{U}(R)$, either $u\in B$ or $u^{-1}\in B$.*

Examples of total unit balls are the closed unit balls of absolutely semivalued rings. The previous concept motivates the following definition.

Definition 239 (Absolutely invertible) *Let R be a topological ring endowed with a unit ball B. An invertible element $u\in\mathcal{U}(R)$ is said to be absolutely invertible provided that $u,u^{-1}\in B$. The set of absolutely invertibles of R is denoted again by $\mathcal{U}_1(R)$.*

Notice that the notion of absolutely invertible depends on the unit ball. Like mentioned above, in a unital seminormed ring R its unit ball B_R is an additively symmetric and multiplicatively idempotent closed neighborhood of 0 containing 1. As a consequence, the definition of absolutely invertible in topological rings extends properly that of seminormed rings.

Lemma 114 *Let R be a topological ring endowed with a unit ball B. Then:*

1. *$\mathcal{U}_1(R)$ is an additively symmetric subgroup of $\mathcal{U}(R)$*
2. *If B is unital, then $\mathcal{U}_1(R)\subseteq\mathrm{bd}(B)$.*
3. *If R is a $*$-ring, then $\mathcal{U}_{\mathfrak{u}}(R)\cap B\cap B^*\subseteq\mathcal{U}_1(R)$.*

Proof 291 *The proof is itemized via the statement of the lemma.*

1. *Observe that $\mathcal{U}_1(R)$ is trivially an additively symmetric subgroup of $\mathcal{U}(R)$ in view of the fact that B is additively symmetric and multiplicatively idempotent.*
2. *Fix an arbitrary $u\in\mathcal{U}_1(B)$. Suppose on the contrary that $u\in\mathrm{int}(B)$. Note that $u^{-1}\mathrm{int}(B)$ is an open neighborhood of 0 contained in $BB=B$, hence $u^{-1}\mathrm{int}(B)\subseteq\mathrm{int}(B)$, meaning that $1=u^{-1}u\in\mathrm{int}(B)$, which contradicts the fact that B is unital. As a consequence, $u\in\mathrm{bd}(B)$. Therefore, $\mathcal{U}_1(R)\subseteq\mathrm{bd}(B)$.*
3. *Fix an arbitrary $u\in\mathcal{U}_{\mathfrak{u}}(R)\cap B\cap B^*$. All we need to show is that $u,u^{-1}\in B$. By assumption, $u\in B\cap B^*$, meaning that $u^*\in B\cap B^*$. Finally, since $u\in\mathcal{U}_{\mathfrak{u}}(R)$, we obtain that $u^{-1}=u^*\in B$.*

Under the assumptions of Lemma 114(2), if B is $*$-invariant ($B^* = B$), then $\mathcal{U}_{\mathfrak{u}}(R) \cap B \subseteq \mathcal{U}_1(R)$. The following example shows that, in general, $\mathcal{U}_{\mathfrak{u}}(R) \nsubseteq \mathcal{U}_1(R)$.

Example 53 *Consider the topological ring $R := \mathbb{C}$ endowed with the involution given by conjugation $z^* := \overline{z}$. According to [38], the unit ball of the 1-norm $B := \{z \in \mathbb{C} : \|z\|_1 \leq 1\}$ is in fact a closed unit neighborhood of 0 in R. It is not hard to check that B is $*$-invariant. Also, $\mathcal{U}_1(R) = \{1, -1, i, -i\}$ and $\mathcal{U}_{\mathfrak{u}}(R) = \mathrm{bd}(\mathbb{D})$.*

Special attention will be paid to a particular class of unit balls: the closed unit neighborhoods of zero. Unit zero-neighborhoods constitute a new concept in Associative Ring Theory with multiple applications in topological modules. We refer the reader to [9, 10, 11, 39, 40] for a wider perspective on this notion.

Definition 240 (Unit zero-neighborhood) *Let R be a topological ring. Let U, B be subsets of R. Then:*

- *U is called an open unit neighborhood of 0 provided that U is an additively symmetric, $U = -U$, and multiplicatively idempotent, $UU = U$, regular open neighborhood of 0 such that $1 \in \mathrm{cl}(U)$.*
- *B is called a closed unit neighborhood of 0 provided that B is regular closed and its interior is an open unit neighborhood of 0.*

Under the settings of the previous definition, if U is an open unit neighborhood of 0, then $\mathrm{cl}(U)$ is a closed unit neighborhood of 0. Alternatively (and by definition), if B is a closed unit neighborhood of 0, then $\mathrm{int}(B)$ is an open unit neighborhood of 0. In [10, Theorem 3.7], it is proved that the only proper closed unit neighborhood of 0 in $\mathbb{R}$ is $[-1, 1]$ and the only proper closed unit neighborhood of 0 in $\mathbb{C}$ containing the unit complex numbers $\mathbb{T}$ is the closed unit disk $\overline{\mathbb{D}}$. In [38], other proper closed unit neighborhoods of 0 in $\mathbb{C}$ are provided.

Proposition 82 *Let R be a topological ring. The following conditions are equivalent for a subset B of R:*

1. *B is a closed unit neighborhood of 0.*
2. *B is a regular closed unit ball such that $\mathrm{int}(B)$ is multiplicatively idempotent.*
3. *B is an additively symmetric regular closed neighborhood of 0 containing 1 such that $\mathrm{int}(B)$ is multiplicatively idempotent.*

Proof 292 *The proof follows as itemized as the statement is.*

1⇒2 *By definition, B is regular closed and $\mathrm{int}(B)$ is an open unit neighborhood of 0; therefore, B is a neighborhood of 0. Since $\mathrm{int}(B)$ is additively symmetric we have that*

$$B = \mathrm{cl}(\mathrm{int}(B)) = \mathrm{cl}(-\mathrm{int}(B)) = -\mathrm{cl}(\mathrm{int}(B)) = -B,$$

which implies that B is additively symmetric as well. Next, $1 \in \mathrm{cl}(\mathrm{int}(B)) = B$, so $B \subseteq BB$. On the other hand, $\mathrm{int}(B)$ is multiplicatively idempotent because it is an open unit neighborhood of 0; therefore, the continuity of the multiplication allows that

$$BB = \mathrm{cl}(\mathrm{int}(B))\mathrm{cl}(\mathrm{int}(B)) \subseteq \mathrm{cl}(\mathrm{int}(B)\mathrm{int}(B)) = \mathrm{cl}(\mathrm{int}(B)) = B,$$

so B is multiplicatively idempotent.

2⇒3 *Immediate.*

3⇒1 Since B is regular closed, all that remains to prove is that $\mathrm{int}(B)$ is an open unit neighborhood of 0. Since B is closed, we have that $\mathrm{int}(B)$ is regular open (see Proposition 51). Also, B is a neighborhood of 0, so $\mathrm{int}(B)$ is a neighborhood of 0 as well. Next, since B is additively symmetric, we have that $\mathrm{int}(B) = \mathrm{int}(-B) = -\mathrm{int}(B)$; therefore, $\mathrm{int}(B)$ is additively symmetric as well. Also, $\mathrm{int}(B)$ is multiplicatively idempotent by hypothesis. Finally, $1 \in B = \mathrm{cl}(\mathrm{int}(B))$. As a consequence, $\mathrm{int}(B)$ is an open unit neighborhood of 0.

Recall that if multiplicative inversion on a topological ring is continuous, then we will call it a topological inversion ring.

Lemma 115 *Let R be a topological ring. Let U be an open subset of R. Then:*

1. *If $\mathcal{U}(R)$ is open and $1 \in \mathrm{cl}(U)$, then $1 \in \mathrm{cl}(U \cap \mathcal{U}(R))$.*
2. *If multiplicative inversion is continuous and $1 \in \mathrm{cl}(U \cap \mathcal{U}(R))$, then $U \subseteq UU$.*

Proof 293 *Let us itemize the proof in the same way as the statement.*

1. *Let V be any open subset containing 1. Then $V \cap \mathcal{U}(R)$ is an open neighborhood of 1, therefore by hypothesis, $(V \cap \mathcal{U}(R)) \cap U \neq \varnothing$, meaning that $V \cap (\mathcal{U}(R) \cap U) \neq \varnothing$. As a consequence, $1 \in \mathrm{cl}(U \cap \mathcal{U}(R))$.*
2. *Fix an arbitrary $u \in U$. Let $(u_i)_{i \in I} \subseteq U \cap \mathcal{U}(R)$ be a net converging to 1. By the continuity of the inversion, we deduce that $\left(u_i^{-1}\right)_{i \in I}$ also converges to 1; therefore $\left(u_i^{-1}u\right)_{i \in I}$ converges to u, which means that we can find $j \in I$ with $u_j^{-1}u \in U$. Finally, $u = u_j\left(u_j^{-1}u\right) \in UU$.*

As a direct consequence of Proposition 82 together with Lemma 115, we find the following sufficient condition for being a closed unit neighborhood of zero.

Proposition 83 *Let R be a topological inversion ring. Then a subset B of R is a closed unit neighborhood of 0 if B is an additively symmetric regular closed neighborhood of 0 satisfying that $\mathrm{int}(B)^2 \subseteq \mathrm{int}(B)$ and $1 \in \mathrm{cl}(\mathrm{int}(B) \cap \mathcal{U}(R))$.*

The following definition expresses this concept in the language of topological universal algebras.

Definition 241 (Topological absolute-inversion ring) *A topological universal algebra $\left((R,\tau), \left\{+, 0, -, \cdot, 1, {}^{-1}\right\}, \varnothing\right)$ is called a topological absolute-inversion ring provided that $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$ is a ring endowed with a unit ball, ${}^{-1} : \mathcal{U}_1(R) \to \mathcal{U}_1(R)$ is its absolute inversion, and (R,τ) is a topological space.*

Notice that Lemma 115 still holds for topological absolute-inversion rings by relying on absolutely invertibles.

Lemma 116 *Let R be a topological absolute-inversion ring (provided R is endowed with a unit ball). Let U be an open subset of R. If $1 \in \mathrm{cl}(U \cap \mathcal{U}_1(R))$, then $U \subseteq UU$.*

Proof 294 *Fix an arbitrary $u \in U$. Let $(u_i)_{i \in I} \subseteq U \cap \mathcal{U}_1(R)$ be a net converging to 1. By the continuity of the absolute inversion, we deduce that $\left(u_i^{-1}\right)_{i \in I}$ also converges to 1, therefore $\left(u_i^{-1}u\right)_{i \in I}$ converges to u, which means that we can find $j \in I$ with $u_j^{-1}u \in U$. Finally, $u = u_j\left(u_j^{-1}u\right) \in UU$.*

One more technical lemma is needed to achieve the desired characterization of closed unit neighborhood of zero in topological division rings.

Lemma 117 *Let R be a topological ring. Let B be an additively symmetric regular closed neighborhood of 0 containing 1. Then $1 \in \mathrm{cl}\,(\mathrm{int}(B) \setminus \{0\})$.*

Proof 295 *Observe that $1 \in B = \mathrm{cl}(\mathrm{int}(B))$ because B is regular closed. We will distinguish between two cases:*

- *$1 \in \mathrm{cl}(\{0\})$. Then $0 \in 1 + \mathrm{int}(B)$, so $-1 \in \mathrm{int}(B)$ and hence $1 \in \mathrm{int}(B)$ because $\mathrm{int}(B)$ is additively symmetric since so is B. Thus $1 \in \mathrm{int}(B) \setminus \{0\} \subseteq \mathrm{cl}(\mathrm{int}(B) \setminus \{0\})$.*
- *$1 \notin \mathrm{cl}(\{0\})$. Since $1 \in \mathrm{cl}(\mathrm{int}(B)) = \mathrm{cl}(\mathrm{int}(B) \setminus \{0\}) \cup \mathrm{cl}(\{0\})$, we conclude that $1 \in \mathrm{cl}(\mathrm{int}(B) \setminus \{0\})$.*

We are finally in the right position to state and prove a characterization of closed unit neighborhood of zero in topological division rings. This characterization will be improved further in the upcoming subsections.

Corollary 23 *Let R be a topological division ring. A subset B of R is a closed unit neighborhood of 0 if and only if B is an additively symmetric and multiplicatively idempotent regular closed neighborhood of 0 containing 1.*

Proof 296 *The proof follows as itemized as the statement.*

- $\Rightarrow$ *By taking into consideration Proposition 82, if B is a closed unit neighborhood of 0, then B is an additively symmetric and multiplicatively idempotent regular closed neighborhood of 0 containing 1.*
- $\Leftarrow$ *By bearing in mind again Proposition 82, all we need to show is that $\mathrm{int}(B)$ is multiplicatively idempotent. In first place, the multiplication of two open sets is open in topological division rings, therefore $\mathrm{int}(B)\mathrm{int}(B)$ is open. Since $\mathrm{int}(B)\mathrm{int}(B) \subseteq BB = B$, we deduce that $\mathrm{int}(B)\mathrm{int}(B) \subseteq \mathrm{int}(B)$. Next, according to Lemma 117,*

$$1 \in \mathrm{cl}\,(\mathrm{int}(B) \setminus \{0\}) = \mathrm{cl}(\mathrm{int}(B) \cap \mathcal{U}(R)).$$

In view of Lemma 115, $\mathrm{int}(B) \subseteq \mathrm{int}(B)\mathrm{int}(B)$. Now, by Proposition 82, we conclude that B is a closed unit neighborhood of 0.

One more corollary can be derived from Lemma 115, which has potential applications towards constructing more unit neighborhoods of zero.

Corollary 24 *Let R be a topological inversion ring. Let S be a dense subring of R. If $U \subseteq R$ is an open unit neighborhood of 0 in R such that $1 \in \mathrm{cl}(U \cap \mathcal{U}(S))$, then $U \cap S$ is an open unit neighborhood of 0 in S.*

Proof 297 *According to Corollary 11, $U \cap S$ is a regular open neighborhood of 0 in S and $\mathrm{cl}_S(U \cap S) = \mathrm{cl}(U) \cap S$. Notice that $U \cap S$ is additively symmetric and $(U \cap S)(U \cap S) \subseteq U \cap S$. Finally, since $1 \in \mathrm{cl}(U \cap \mathcal{U}(S))$ by hypothesis, Lemma 115 guarantees that $(U \cap S)(U \cap S) = U \cap S$.*

Next theorem unveils sufficient conditions for the unit ball of a seminorm ring to be a closed unit neighborhood of zero.

Theorem 84 *Let R be a unital seminormed ring. If $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$, then $\mathsf{B}_R = \mathrm{cl}(\mathsf{U}_R)$, $\mathsf{U}_R = \mathrm{int}(\mathsf{B}_R)$, B_R is a closed unit neighborhood of 0 and U_R is an open unit neighborhood of 0.*

Proof 298 *According to Lemma 76(2), $\mathsf{B}_R = \mathrm{cl}(\mathsf{U}_R)$ and $\mathsf{U}_R = \mathrm{int}(\mathsf{B}_R)$, meaning that B_R is regular closed and U_R is regular open. At this stage, it only suffices to prove that U_R is an open unit neighborhood of 0. First off, U_R is clearly additively symmetric because so is the norm of R. Next, by hypothesis, $1 \in \mathsf{B}_R = \mathrm{cl}(\mathsf{U}_R)$. Thus, it only remains to show that U_R is multiplicatively idempotent, that is, $\mathsf{U}_R\mathsf{U}_R = \mathsf{U}_R$. Observe that $\mathsf{U}_R\mathsf{U}_R \subseteq \mathsf{U}_R$ because of the submultiplicative character of the norm of R. Let us show that $\mathsf{U}_R \subseteq \mathsf{U}_R\mathsf{U}_R$. To accomplish this, we will prove that $1 \in \mathrm{cl}(\mathsf{U}_R \cap \mathcal{U}_1(R))$ and apply Lemma 115. Indeed, this is just a direct consequence of the fact that $1 \in \mathrm{cl}(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$.*

The following technical lemma unveils nontrivial examples satisfying the hypothesis of the previous theorem.

Lemma 118 *Let K be an absolutely semivalued division ring. If K is not discrete and $1 \in \mathrm{ext}_2(\mathsf{B}_K)$, then $1 \in \mathrm{cl}(\mathcal{U}_1(K) \setminus \mathsf{S}_K)$.*

Proof 299 *Since K is not discrete, $\{0\}$ is not open, so $K \setminus \{0\}$ is not closed, so we can take a sequence $(k_n)_{n\in\mathbb{N}}$ of nonzero elements of K converging to 0 satisfying that $|k_n| < 1$ for all . If there exists $n \in \mathbb{N}$ such that $|1-k_n| = |1+k_n| = 1$, then $1+1 = (1-k_n)+(1+k_n)$ reaching a contradiction with $1 \in \mathrm{ext}_2(\mathsf{B}_K)$. Finally, the sequence $(l_n)_{n\in\mathbb{N}}$, given by*

$$l_n := \begin{cases} 1-k_n & |1-k_n| \neq 1 \\ 1+k_n & |1-k_n| = 1 \end{cases}$$

for every $n \in \mathbb{N}$, is contained in $\mathcal{U}_1(K) \setminus \mathsf{S}_K$ and converges to 1.

The next results show how to obtain unit zero-neighborhoods in topological ordered rings. We begin with unit balls.

Proposition 84 *Let R be a unital ordered ring. If the order topology is a ring topology and $[-1,1]$ is closed, then $B := [-1,1]$ is a unit ball.*

Proof 300 *First off, B is additively symmetric, multiplicatively idempotent, and contains 1. In the order topology, $(-1,1)$ is open so B is a neighborhood of 0.*

The following proposition is simply a direct application of Proposition 82 to topological ordered rings.

Proposition 85 *Let R be a topological unital ordered ring. If $\mathrm{int}([-1,1]) = (-1,1)$, $\mathrm{cl}((-1,1)) = [-1,1]$, and $(-1,1)(-1,1) = (-1,1)$, then $[-1,1]$ is a closed unit neighborhood of 0.*

Proof 301 *By assumption, $[-1,1]$ is closed, therefore it is a unit ball since it is trivially additively symmetric, multiplicatively idempotent and contains 1. Also, by assumption, $[-1,1]$ is regular closed and its interior is multiplicatively idempotent. By Proposition 82, $[-1,1]$ is a closed unit neighborhood of 0.*

Proposition 85 settles the solid ground for the following crucial theorem.

Theorem 85 *Let R be a topological unital ordered ring with continuous inversion. If $\mathcal{U}(R)$ is open, $R^+ \setminus \{0\}$ is downward directed, and R^+ is closed and hole free, then $[-1,1]$ is a closed unit neighborhood of 0 and $(-1,1)$ is an open unit neighborhood of 0.*

Proof 302 *In accordance with Proposition 85, it only suffices to show that* $\mathrm{int}([-1,1]) = (-1,1)$, $\mathrm{cl}((-1,1)) = [-1,1]$ *and* $(-1,1)(-1,1) = (-1,1)$. *By hypothesis,* R^+ *is closed, meaning that the sets* $R^- = -R^+$, $\downarrow 1 = 1 + R^-$ *and* $\uparrow -1 = -1 + R^+$ *are closed as well since the order topology is a ring topology. In particular,* $[-1,1] = \uparrow -1 \cap \downarrow 1$ *is closed. Also, by hypothesis,* $R^+ \setminus \{0\}$ *is downward directed, meaning that* $\uparrow_\times 1 = 1 + R^+ \setminus \{0\}$ *and* $\uparrow_\times -1 = -1 + R^+ \setminus \{0\}$ *are downward directed by translation. Also,* $R^+ \setminus \{0\}$ *being downward directed implies that* $R^- \setminus \{0\}$ *is upward directed by additive symmetricity, so again by translation,* $\downarrow^\times 1 = 1 + R^- \setminus \{0\}$ *and* $\downarrow^\times -1 = R^- \setminus \{0\}$ *are upward directed. Since* R^+ *is hole free, then so is* R^- *by additive symmetricity, hence so are* $\uparrow 1$, $\uparrow -1$, $\downarrow 1$ *and* $\downarrow -1$ *as well by translation. In virtue of Lemma 53 together with the fact that* $[-1,1]$ *is closed, we obtain that* $\mathrm{int}([-1,1]) = (-1,1)$ *and* $\mathrm{cl}((-1,1)) = [-1,1]$. *By bearing in mind Lemma 29, it only remains to show that* $(-1,1) \subseteq (-1,1)(-1,1)$. *But this is just a direct application of Lemma 115.*

As a direct consequence of Theorem 85, we obtain corollary, which improves [9, Theorem 3.7] and [20, Corollary 3.1] because totally ordered division rings have no holes.

Corollary 25 *Let* R *be a topological totally ordered division ring. Then* $[-1,1]$ *is a closed unit neighborhood of* 0 *and* $(-1,1)$ *is an open unit neighborhood of* 0.

Proof 303 *By definition,* R *is a topological ordered ring (whose order is total) as well as a topological division ring (hence multiplicative inversion is continuous by default). In totally ordered sets, the order topology is always Hausdorff. Therefore,* R *is Hausdorff, so* $\{0\}$ *is closed and* $\mathcal{U}(R) = R \setminus \{0\}$ *is open. Next,* $R^+ \setminus \{0\}$ *is trivially downward directed because the order is total. It only remains to show that* R^+ *is closed and hole free. It is hole free because the whole of* R *is free of holes. Finally,* $R^+ = R \setminus \downarrow^\times 0$ *due to the fact that the order is total, so* R^+ *is closed since its complementary is open in the order topology.*

The following results in this subsection are aimed at obtaining unit segments from unit balls. For this, the next technical lemma will be crucial.

Lemma 119 *Let* R *be a topological ring. Let* $B \subseteq R$ *a unit ball. Then:*

1. $2(1+B) \subseteq (1+B)(1+B)$.
2. $(1+B)(1+B) = 2(1+B)$ *if and only if* $(b-1) + B(b+1) \subseteq 2B$ *for all* $b \in B$.
3. *If* $(b-1) + B(b+1) \subseteq 2B$ *for all* $b \in B$, *then* $1 + B \subseteq 2B$.

Proof 304 *Let us itemize the proof in accordance with the statement.*

1. *Fix any arbitrary* $b \in B$. *Since* $1 \in B$, *we clearly have that* $2(1+b) = (1+1)(1+b) \in (1+B)(1+B)$.
2. *Suppose first that* $(b-1) + B(b+1) \subseteq 2B$ *for all* $b \in B$. *We only need to show that* $(1+B)(1+B) \subseteq 2(1+B)$. *Let* $b, c \in B$. *By hypothesis, there exists* $a \in B$ *such that* $(c-1) + b(c+1) = 2a$. *Then* $b + c + bc = 1 + 2a$, *so* $(1+b)(1+c) = 2 + 2a = 2(1+a) \in 2(1+B)$. *Conversely, assume that* $(1+B)(1+B) = 2(1+B)$. *Let* $b, c \in B$. *By hypothesis, there exists* $a \in B$ *such that* $(1+b)(1+c) = 2(1+a)$. *Then* $(c-1) + b(c+1) = -2 + (1+b)(1+c) = -2 + 2(1+a) = 2a \in 2B$.
3. *Fix an arbitrary* $b \in B$. *Then* $-(1+b) = (-b-1) + 0(-b+1) \in 2B$ *by hypothesis, hence* $1 + b \in -2B = 2(-B) = 2B$.

After Lemma 119, and under its settings, it is expected to take $B^+ := 2^{-1}(1+B)$ provided that $2 \in \mathcal{U}(R)$.

Remark 75 *Let R be a topological ring. Let $B^+ \subseteq R$ be a unit segment. Notice that $0 \in \operatorname{cl}(B^+ \setminus \{0,1\}) \Leftrightarrow 1-0 \in 1-\operatorname{cl}(B^+ \setminus \{0,1\}) \Leftrightarrow 1 \in \operatorname{cl}(1-B^+ \setminus \{0,1\}) \Leftrightarrow 1 \in \operatorname{cl}(B^+ \setminus \{0,1\})$ because $1-B^+ = B^+$ and $1-\{0,1\} = \{0,1\}$.*

Remark 75 motivates the following definition.

Definition 242 (Practical unit segment) *Let R be a topological ring. A unit segment $B^+ \subseteq R$ is called practical provided that $0 \in \operatorname{cl}(B^+ \setminus \{0,1\})$.*

Next result provides sufficient conditions for a unit ball to generate a unit segment.

Theorem 86 *Let R be a topological ring such that $2 \in \mathcal{U}(R)$. Let B be a unit ball of 0 in R. If $0 \in \operatorname{ext}_2(1+B)$ and $(b-1)+B(b+1) \subseteq 2B$ for all $b \in B$, then $B^+ := 2^{-1}(1+B)$ is a centered unit segment of R contained in B. If, in addition, $1 \in \operatorname{cl}(\operatorname{int}(B)) \setminus \operatorname{int}(B)$, then B^+ is practical.*

Proof 305 *In the first place, $1-B^+ = 2^{-1}(2-(1+B)) = 2^{-1}(1-B) = 2^{-1}(1+B) = B^+$. Lemma 35 assures that $B^+ \cap B^- = \{0\}$. By lemma 119(2), $B^+B^+ \subseteq B^+$. Because of Lemma 119(3), $B^+ \subseteq B$. On the other hand, it is clear that $2^{-1} \in B^+$. If, in addition, $1 \in \operatorname{cl}(\operatorname{int}(B)) \setminus \operatorname{int}(B)$, then $-1 \in \operatorname{cl}(\operatorname{int}(B)) \setminus \operatorname{int}(B)$, meaning that $1 = 2^{-1}(1+1) \in 2^{-1}(1+\operatorname{cl}(\operatorname{int}(B))) = \operatorname{cl}\left(2^{-1}(1+\operatorname{int}(B))\right) \subseteq \operatorname{cl}\left(2^{-1}(1+B \setminus \{-1,1\})\right) = \operatorname{cl}(B^+ \setminus \{0,1\})$. Finally, Remark 75 assures that $0 \in \operatorname{cl}(B^+ \setminus \{0,1\})$.*

Next example shows that not all closed unit zero-neighborhoods satisfy the hypotheses of the previous theorem.

Example 54 *Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed unit disk, which is a closed unit neighborhood of 0 in $\mathbb{C}$ in view of [10, Theorem 3.7]. Denote $\overline{\mathbb{D}}^+ := 2^{-1}\left(1+\overline{\mathbb{D}}\right) \subseteq \overline{\mathbb{D}}$. The additive symmetricity of $\overline{\mathbb{D}}$ allows $1+\overline{\mathbb{D}}^- = \overline{\mathbb{D}}^+$. Also, $0 \in \operatorname{ext}_2\left(\overline{\mathbb{D}}^+\right)$, hence $\overline{\mathbb{D}}^+ \cap \overline{\mathbb{D}}^- = \{0\}$ in view of Lemma 35. However, $\overline{\mathbb{D}}^+$ is not multiplicatively idempotent and thus it is not a unit segment of $\mathbb{C}$. Indeed,*

$$\left(\frac{1+i}{2}\right)^2 = \frac{2i}{4} = \frac{i}{2}.$$

If $z \in \mathbb{C}$ is so that $\frac{1+z}{2} = \frac{i}{2}$, then $z = i-1$ so $|z| = |i-1| = \sqrt{2} > 1$.

Notice that the existence of a unit segment contained in the unit ball of a seminormed ring implies that the seminormed ring is necessarily unital.

Definition 243 (Regular unit segment) *Let R be a unital seminormed ring. A unit segment $B^+ \subseteq \mathsf{B}_R$ is said to be:*

- *Regular if, for every $r_1, \ldots, r_k \in B^+$ with $\sum_{i=1}^k r_i = 1$, we have that $\sum_{i=1}^k \|r_i\| = 1$.*
- *Strongly regular if $\|\ \|_{B^+}$ is additive, that is, if $r, s \in B^+$ and $r+s \in B^+$, then $\|r+s\| = \|r\| + \|s\|$.*

To conclude this subsubsection, we will prove next that strongly regular full unit segments are regular.

Proposition 86 *Let R be a unital seminormed ring. Let $B^+ \subseteq \mathsf{B}_R$ be a unit segment. If B^+ is strongly regular and full, then B^+ is regular.*

Proof 306 *Fix arbitrary elements $r_1, \ldots, r_k \in B^+$ with $\sum_{i=1}^{k} r_i = 1$. Since B^+ is full, we have that $\sum_{j \in J} r_j \in B^+$ for every $J \subseteq \{1, \ldots, k\}$. Since $\| \|_{B^+}$ is additive, we finally conclude that*

$$\sum_{i=1}^{k} \|r_i\| = \left\| \sum_{i=1}^{k} r_i \right\| = \|1\| = 1.$$

3.2.3.4 Practical rings

Practical rings are topological rings for which the invertibles approach zero. These rings play a fundamental role in transporting classical Operator Theory towards the scope of topological nmodules.

Definition 244 (Practical ring) *A topological ring R is called (absolutely) practical provided that $0 \in \mathrm{cl}(\mathcal{U}(R))$ $(0 \in \mathrm{cl}(\mathcal{U}_1(R)))$.*

Trivially, nonzero discrete rings are not practical. These are probably the most simple examples of nonpractical rings.

Lemma 120 *If R is a practical topological ring, then $\bigcap \mathcal{N}_0 \subseteq \mathrm{cl}(\mathcal{U}(R))$.*

Proof 307 *Fix an arbitrary $w \in \bigcap \mathcal{N}_0$. Take $W \in \mathcal{N}_w$. There exists an additively symmetric 0-neighborhood V such that $w + V \subseteq \mathrm{int}(W)$. However, $w \in \bigcap \mathcal{N}_0 \subseteq V$, so $-w \in V$ and hence $0 \in \mathrm{int}(W)$, that is, $W \in \mathcal{N}_0$. Since R is practical by hypothesis, $W \cap \mathcal{U}(R) \neq \varnothing$. This shows that $w \in \mathrm{cl}(\mathcal{U}(R))$.*

Following the discussion about discrete rings, we find that division topological rings will be practical as long as they are not discrete.

Lemma 121 *A division topological ring R is practical if and only it is not discrete.*

Proof 308 *Suppose that R is a nonpractical division topological ring. Then $\mathcal{U}(R) = R \setminus \{0\}$ is closed, so $\{0\}$ is open. By translations, R is discrete.*

Let $R \subseteq S$ be a unital extension of topological rings, that is, a continuous additive and unital-multiplicative injective map $R \hookrightarrow S$. If R is practical, then S is also practical.

Lemma 122 *Let R be a topological ring endowed a 0-neighborhood $W \subseteq R$ such that $W \setminus \{0\} \subseteq \mathcal{U}(R)$. Then:*

1. *If R is practical, then R is a division topological ring.*
2. *If R is not discrete, then R is practical, hence a division topological ring.*

Proof 309 *Let us itemize the proof likewise the statement of the lemma.*

1. *Fix an arbitrary $r \in R \setminus \{0\}$. There exists a 0-neighborhood $V \subseteq R$ such that $Vr \subseteq W$. Since R is practical, we can find $v \in \mathcal{U}(R) \cap V$. Let $w \in W$ such that $vr = w$. Notice that v is invertible; hence, it cannot be a zero divisor, thus $w \neq 0$. Then, by hypothesis, $w \in \mathcal{U}(R)$. Finally, $r = v^{-1}w \in \mathcal{U}(R)$.*

2. *Fix an arbitrary* 0*-neighborhood* $U \subseteq R$*. Notice that* $U \cap W$ *is a neighborhood of* 0 *in* R*. By hypothesis,* R *is not discrete, thus* $U \cap W \neq \{0\}$*. As a consequence* $(U \cap W) \setminus \{0\} \subseteq W \setminus \{0\} \subseteq \mathcal{U}(R)$*. This shows that* R *is practical. By Lemma 122(1),* R *is a division topological ring.*

The upcoming example in this subsubsection unveils that, in Lemma 122(2), the hypothesis of not being discrete cannot be disregarded.

Example 55 $\mathbb{Z}$ *is a topological ring which is not a division ring but there exists a* 0*-neighborhood* $W \subseteq R$ *such that* $W \setminus \{0\} \subseteq \mathcal{U}(R)$*. Indeed, it suffices to take* $W := \{-1, 0, 1\}$*.*

In unital seminormed rings, any other proper unit ball must be, in part, contained in the seminorm unit ball. This is shown in the next lemma, improving [10, Lemma 3.2].

Lemma 123 *Let* R *be a unital seminormed ring. If* B *is a proper unit ball in* R*, then* $B \cap \mathcal{U}_1(R) \subseteq \mathsf{B}_R$*.*

Proof 310 *Suppose that there exists* $u \in B \cap \mathcal{U}_1(R)$ *such that* $\|u\| > 1$*. Since* B *is a neighborhood of* 0 *in* R*, there exists* $t > 0$ *such that* $\mathsf{B}_R(0, t) \subseteq B$*. Then*

$$R = \bigcup_{n \in \mathbb{N}} \mathsf{B}_R\left(0, \|u\|^n t\right) = \bigcup_{n \in \mathbb{N}} \mathsf{B}_R\left(0, \|u^n\| t\right) = \bigcup_{n \in \mathbb{N}} u^n \mathsf{B}_R(0, t) \subseteq B,$$

which implies that B *is not proper.*

Next theorem, which is a refinement of [40, Theorem 7] and is motivated by Lemma 123, provides a characterization of practical ring in the context of seminormed rings.

Theorem 87 *Let* R *be a unital seminormed ring. The following statements are equivalent:*

1. R *is practical.*
2. B_R *is right bounded.*
3. B_R *is left bounded.*

Proof 311 *Since* $1 \in \mathsf{B}_R$*, if* B_R *is left or right bounded, then* R *is practical. Conversely, suppose that* R *is practical. We will show that* $\{u\mathsf{B}_R : u \in \mathcal{U}(R)\}$ *is a base of neighborhoods of* 0*. Take* $t > 0$ *and consider* $\mathsf{B}_R(0, t)$*. By assumption, there exists* $u \in \mathcal{U}(R)$ *such that* $\|u\| < t$*. Then* $u\mathsf{B}_R \subseteq \mathsf{B}_R(0, u) \subseteq \mathsf{B}_R(0, t)$*. In a dual way, it can be shown that* $\{\mathsf{B}_R u : u \in \mathcal{U}(R)\}$ *is a base of neighborhoods of* 0*.*

A simpler characterization of practical ring than the previous theorem is displayed in the next lemma.

Lemma 124 *Let* R *be a seminormed ring. Then* R *is practical if and only if there exists* $r \in \mathcal{U}(R) \cap \mathsf{U}_R$*.*

Proof 312 *If* R *is practical, then by definition there exists* $r \in \mathcal{U}(R) \cap \mathsf{U}_R$ *since* U_R *is a neighborhood of* 0*. Conversely, let* $r \in \mathcal{U}(R) \cap \mathsf{U}_R$*. Then* $\|r\| < 1$ *and* $\|r^n\| \leq \|r\|^n \to 0$ *as* $n \to \infty$*, so* $0 \in \mathrm{cl}(\mathcal{U}(R))$*.* □

In the context of seminormed rings, there is a notion which is stronger than the one of absolutely practical.

Definition 245 (Hyperpractical ring) *Let* R *be a seminormed ring. We will say that* R *is hyperpractical provided that* $\{\|u\| : u \in \mathcal{U}_1(R)\}$ *is dense in* $[0, \infty)$*.*

3.2.3.5 Topological factorization

Topological divisors are fundamental objects in the development of classical theories such as Spectral Theory.

Definition 246 (Topological divisor) *Let R be a topological ring. Let $s \in R$. An element $r \in R$ is said to be a topological left divisor of s provided that there exists a subset $T \subseteq R$ such that $0 \notin \mathrm{cl}(T)$ and $s \in \mathrm{cl}(rT)$. The set of topological left divisors of s is denoted by $\mathtt{tld}(s)$.*

In a similar way and under the settings of the previous definition, the topological right divisors of s can be defined and they are denoted by $\mathtt{trd}(s)$. The following lemma, motivated by Lemma 83, establishes that the condition that $0 \notin \mathrm{cl}(T)$ is only necessary when $s = 0$.

Lemma 125 *Let R be a Hausdorff topological ring. Let $s \in R \setminus \{0\}$. Then $\mathtt{tld}(s) = \{r \in R : \exists T \subseteq R \;\; s \in \mathrm{cl}(rT)\}$.*

Proof 313 *By definition, it is clear that $\mathtt{tld}(s) \subseteq \{r \in R : \exists T \subseteq R \;\; s \in \mathrm{cl}(rT)\}$. Suppose that $r \in R$ is so that there exists $T \subseteq R$ with $s \in \mathrm{cl}(rT)$. Since R is Hausdorff, we can take U, V neighborhoods of $s, 0$, respectively, such that $U \cap V = \varnothing$. Notice that $I := \{W \subseteq U : W$ is a neighborhood of $s\}$ is a directed set when endowed with the partial order $W_1 \leq W_2 \iff W_2 \subseteq W_1$. Since $s \in \mathrm{cl}(rT)$, for every $W \in I$ there exists $t_w \in T$ such that $rt_w \in W$. It is clear that the net $(rt_W)_{W \in I}$ converges to s. However, by construction, $0 \notin \mathrm{cl}(\{t_W : W \in I\})$. Finally, simply take $S := \{t_W : W \in I\}$ and we have that $0 \notin \mathrm{cl}(S)$ but $s \in \mathrm{cl}(rS)$, meaning that $r \in \mathtt{tld}(s)$.*

The following proposition unveils basic properties satisfied by topological divisors. Special attention will be paid to the topological zero-divisors.

Proposition 87 *Let R be a topological ring. Then:*

1. $\mathtt{rd}(1) \cap \mathtt{tld}(0) = \varnothing = \mathtt{ld}(1) \cap \mathtt{trd}(0)$. *In particular,* $(\mathtt{tld}(0) \cup \mathtt{trd}(0)) \cap \mathcal{U}(R) = \varnothing$.
2. *If the ring topology of R is trivial, then* $\mathtt{tld}(0) = \mathtt{trd}(0) = \varnothing$.
3. *If the ring topology of R is not trivial, then* $\bigcap \mathcal{N}_0(R) \subseteq \mathtt{tld}(0) \cap \mathtt{trd}(0)$.
4. *If R is Hausdorff, then* $\mathtt{ld}(0) \subseteq \mathtt{tld}(0)$ *and* $\mathtt{rd}(0) \subseteq \mathtt{trd}(0)$.
5. *If R is discrete, then* $\mathtt{ld}(0) = \mathtt{tld}(0)$ *and* $\mathtt{rd}(0) = \mathtt{trd}(0)$.

Proof 314 *As always, we will itemize the proof via the statement.*

1. *Suppose, for instance, that there exists $r \in \mathtt{rd}(1) \cap \mathtt{tld}(0)$. Let $t \in R$ with $tr = 1$ and take $S \subseteq R$ such that $0 \notin \mathrm{cl}(S)$ and $0 \in \mathrm{cl}(rS)$. Let $(s_i)_{i \in I} \subseteq S$ be a net such that $(rs_i)_{i \in I}$ converges to 0. Since $s_i = t(rs_i)$ for all $i \in I$, we conclude that $(s_i)_{i \in I}$ also converges to 0, which contradicts that $0 \notin \mathrm{cl}(S)$. As a consequence,* $\mathtt{rd}(1) \cap \mathtt{tld}(0) = \varnothing$. *Similarly,* $\mathtt{ld}(1) \cap \mathtt{trd}(0) = \varnothing$.
2. *If the ring topology of R is trivial, then every nonempty subset S of R satisfies that $0 \in R = \mathrm{cl}(S)$. As a consequence,* $\mathtt{tld}(0) = \mathtt{trd}(0) = \varnothing$.
3. *Take an arbitrary element $r \in \bigcap \mathcal{N}_0(R)$ and consider $S := \{1\}$. Since the ring topology of R is not trivial, by Corollary 18, we have that $0 \notin \mathrm{cl}(\{1\})$. However, in virtue of Lemma 49, $0 \in \mathrm{cl}(\{r\}) = \mathrm{cl}(rS)$. As a consequence, $r \in \mathtt{tld}(0)$. In a similar way, $r \in \mathtt{trd}(0)$.*

4. *Take, for instance,* $r \in \mathtt{ld}(0)$. *There exists* $t \in R \setminus \{0\}$ *such that* $rt = 0$. *Take now* $S := \{t\}$. *Since* R *is Hausdorff,* $\bigcap \mathcal{N}_0 = \{0\}$ *by taking into consideration Corollary 15(4). Therefore,* $0 \notin \mathrm{cl}(\{t\})$ *in virtue of Lemma 49. Finally,* $0 \in \mathrm{cl}(\{0\}) = \mathrm{cl}(rS)$. *As a consequence,* $r \in \mathtt{tld}(0)$. *Hence* $\mathtt{ld}(0) \subseteq \mathtt{tld}(0)$. *In a similar way,* $\mathtt{rd}(0) \subseteq \mathtt{trd}(0)$.

5. *Discrete topological rings are trivially Hausdorff. As a consequence,* $\mathtt{ld}(0) \subseteq \mathtt{tld}(0)$ *and* $\mathtt{rd}(0) \subseteq \mathtt{trd}(0)$. *Conversely, if* $r \in \mathtt{tld}(0)$ *and* $S \subseteq R$ *is so that* $0 \notin \mathrm{cl}(S) = S$ *but* $0 \in \mathrm{cl}(rS) = rS$, *then there exists* $s \in S$ *such that* $0 = rs$, *so* $r \in \mathtt{ld}(0)$ *because* $s \neq 0$. *Thus,* $\mathtt{ld}(0) = \mathtt{tld}(0)$. *The equality* $\mathtt{rd}(0) = \mathtt{trd}(0)$ *can be proved similarly.*

The following theorem relates absolutely invertibles with topological divisors.

Theorem 88 *Let* R *be a Hausdorff topological ring endowed with a left-bounded unit ball* B. *Let* $s \in R \setminus \{0\}$. *Then* $\mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\} \subseteq \mathtt{tld}(s)$.

Proof 315 *Let* V_0 *be a fixed neighborhood of* 0. *By hypothesis, there exist a* 0*-neighborhood* W *such that* $WW \subseteq V_0$ *and* $u_0 \in \mathcal{U}(R)$ *such that* $u_0 B \subseteq W$. *Let* $x \in \mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\}$ *and let denote* $T := \mathcal{U}_1(R)$. *It follows that* $(x + Wu_0) \cap sT \neq \emptyset$. *As a consequence, there exists* $t \in \mathcal{U}_1(R)$ *such that* $st - x \in Wu_0$ *or equivalently* $s - xt^{-1} \in W(u_0 t^{-1}) \subseteq W(u_0 B) \subseteq WW \subseteq V_0$. *This implies* $(s + V_0) \cap xT \neq \emptyset$ *and then* $x \in \mathtt{tld}(s)$ *by Lemma 125.*

The final result in this subsubsection is the version of the previous theorem in the context of seminormed rings.

Theorem 89 *Let* R *be a unital normed ring. Let* $s \in R \setminus \{0\}$. *Then* $\mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\} \subseteq \mathtt{tld}(s)$.

Proof 316 *Fix an arbitrary* $r \in \mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\}$. *Take a sequence* $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R)$ *such that* $(su_n)_{n\in\mathbb{N}}$ *converges to* r. *Since* $r \neq 0$, *no subsequence of* $(u_n)_{n\in\mathbb{N}}$ *converges to* 0. *Hence there exist* n_0 *and* $K > 0$ *such that* $\|u_n\| \geq K$ *for all* $n \geq n_0$. *Then* $\left\|u_n^{-1}\right\| = \|u_n\|^{-1} \leq \frac{1}{K}$ *for all* $n \geq n_0$. *Finally, for every* $n \geq n_0$,

$$\left\|ru_n^{-1} - s\right\| \leq \|r - su_n\| \left\|u_n^{-1}\right\| \leq \frac{\|r - su_n\|}{K},$$

meaning that $\left(ru_n^{-1}\right)_{n\in\mathbb{N}}$ *converges to* s. *In view of Lemma 125,* $r \in \mathtt{tld}(s)$.

3.2.3.6 Convergence ring topologies

There is also a version of the convergence group topology for topological rings. The idea is that, if R is a topological ring and I is a nonempty set, then R^I can be endowed with a convergence additive topology due to the commutativity character of the ring addition. Thus, it is natural to wonder whether a convergence additive topology on R^I is actually a ring topology. In general, it is not; however, it will be for certain subrings of R^I.

Theorem 90 *Let* I *be a nonempty set. Let* $\mathcal{G} \subseteq \mathcal{P}(I)$ *be upward directed. Let* R *be a topological ring. Let* $\mathcal{F} \subseteq R^I$ *be a subring. Consider the filter base*

$$\mathcal{B} := \{\mathcal{U}(G, U) : G \in \mathcal{G}, U \in \mathcal{N}_0(R) \text{ additively symmetric}\}.$$

Then:

1. *If* $f(G)$ *is bounded in* R *for all* $f \in \mathcal{F}$ *and all* $G \in \mathcal{G}$, *then* $\mathcal{B}$ *is a base of* 0*-neighborhoods for a ring topology on* $\mathcal{F}$.

2. *If R is practical, $\mathcal{F}$ contains the constant functions, and $\mathcal{B}$ is a base of 0-neighborhoods for a ring topology on $\mathcal{F}$, then $f(G)$ is bounded in R for all $f \in \mathcal{F}$ and all $G \in \mathcal{G}$.*

Proof 317 *The proof follows as itemized in the statement.*

1. *Suppose first that $f(G)$ is bounded in R for all $f \in \mathcal{F}$ and all $G \in \mathcal{G}$. We will rely on Theorem 81. We have to verify that the filter base $\mathcal{B}$ satisfies all four conditions of Theorem 81:*
   (a) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find a 0-neighborhood $V \subseteq R$ such that $V + V \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $\mathcal{U}(G,V) + \mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*
   (b) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find a 0-neighborhood $V \subseteq R$ such that $-V \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $-\mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*
   (c) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find a 0-neighborhood $V \subseteq R$ such that $VV \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $\mathcal{U}(G,V)\mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*
   (d) *Take $\mathcal{U}(G,U) \in \mathcal{B}$ and $f \in F$. We can find a 0-neighborhood $V \subseteq R$ such that $VV \subseteq U$. Since $f(G)$ is bounded in R, there exist invertibles $r, s \in \mathcal{U}(R)$ such that $f(G) \subseteq rV \cap Vs$. Then $Vr^{-1}, s^{-1}V$ are 0-neighborhoods in R and $f\mathcal{U}\left(G, s^{-1}V\right) \cup \mathcal{U}\left(G, Vr^{-1}\right) f \subseteq \mathcal{U}(G,U)$.*
2. *Conversely, suppose that R is practical, $\mathcal{F}$ contains the constant functions, and $\mathcal{B}$ is a base of 0-neighborhoods for a ring topology on $\mathcal{F}$. Fix arbitrary elements $f \in \mathcal{F}$ and $G \in \mathcal{G}$. Take any additively symmetric 0-neighborhood $U \subseteq M$. In view of Theorem 81(4), there exist an additively symmetric 0-neighborhood $W \subseteq R$ and $K \in \mathcal{G}$ such that $f\mathcal{U}(K,W) \cup \mathcal{U}(K,W)f \subseteq \mathcal{U}(G,U)$. Next, since R is practical, there exists $u \in \mathcal{U}(R)$ such that $u \in W$. By hypothesis, $\mathcal{F}$ contains the constant function equal to u, which we will denote by $u\mathbf{1}$. It is clear that $u\mathbf{1} \in \mathcal{U}(K,W)$. As a consequence, $uf, fu \in \mathcal{U}(G,U)$, that is, $(uf)(G) \cup (fu)(G) \subseteq U$, hence $f(G) \subseteq u^{-1}U \cap Uu^{-1}$. This shows that $f(G)$ is bounded in R.*

Under the settings of Theorem 90, if $\mathcal{F}$ is a unitary subring of R^I and an R-submodule of R^I, then $\mathcal{F}$ clearly contains the constant functions.

3.2.4 Topological modules and algebras

Recall that every ring can be turned into a topological ring by endowing it with the discrete topology or the trivial topology. Therefore, every module can be turned into a topological module by endowing both the module and the underlying ring with the discrete topology or the trivial topology. We refer the reader to [41, 42, 43, 44, 45] for a wider perspective on topological modules.

3.2.4.1 Module topologies

A module topology is essentially a topology on a module that makes the all the involved operations continuous, that is, the binary internal operation $+$, the unary internal operation $-$ and the binary external operation $\cdot$.

Definition 247 (Topological module) *A topological module over a topological ring is a topological universal algebra $((M,\tau),\{+,0,-\},\{\cdot\})$, where $(M,\{+,0,-\},\{\cdot\})$ is a module over a topological ring $((R,\tau),\{+,0,-,\cdot,1\},\varnothing)$ and (M,τ) is a topological space.*

The additive group of a topological module is trivially a topological group. It is also trivial to check that the inherited topology on every submodule is a module topology. On the other hand, the discrete topology of $\mathbb{Z}$ turns every commutative additive topological group into a topological $\mathbb{Z}$-module.

Theorem 91 *Let R be a topological ring. Let M be an R-module. If τ is a module topology on R and $\mathcal{B}$ is a base of neighborhoods of 0 in M, then the following is verified:*

1. *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $U + U \subseteq V$.*
2. *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $-U \subseteq V$.*
3. *For every $V \in \mathcal{B}$, there exist $U \in \mathcal{B}$ and $W \in \mathcal{N}_0(R)$ with $WU \subseteq V$.*
4. *For every $V \in \mathcal{B}$ and every $r \in R$, there exists $U \in \mathcal{B}$ with $rU \subseteq V$.*
5. *For every $V \in \mathcal{B}$ and every $m \in M$, there exists $W \in \mathcal{N}_0(R)$ with $Wm \subseteq V$.*

Conversely, if $\mathcal{B}$ is a filter base of $\mathcal{P}(M)$ verifying all five properties above, then there exists a unique module topology on M for which $\mathcal{B}$ is a base of zero-neighborhoods. This topology is given by

$$\tau := \{A \subseteq M : \forall a \in A \;\; \exists U \in \mathcal{B} \;\; a + U \subseteq A\} \cup \{\varnothing\}.$$

Proof 318 *Suppose first that τ is a module topology on R and $\mathcal{B}$ is a base of neighborhoods of 0. The above properties are direct consequences of the continuity of the internal operations, $+$, $-$, and the external operation, $\cdot$. In particular, the last two properties are direct consequences of the continuity of the left- and right-multiplication operators*

$$\begin{array}{ccc} M & \to & M \\ m & \mapsto & rm \end{array} \quad \textit{and} \quad \begin{array}{ccc} R & \to & M \\ r & \mapsto & rm. \end{array}$$

Conversely, assume that $\mathcal{B}$ is a filter base of $\mathcal{P}(M)$ verifying the above properties. In view of Theorem 74,

$$\tau := \{A \subseteq M : \forall a \in A \;\; \exists U \in \mathcal{B} \;\; a + U \subseteq A\} \cup \{\varnothing\}$$

is a group topology on M and it is the only group topology on M for which $\mathcal{B}$ is a base of zero-neighborhoods. It only remains to show τ is a module topology on M and is the unique module topology on M for which $\mathcal{B}$ is a base of 0-neighborhoods. Let us prove first that the action of R on M is continuous. Indeed, fix arbitrary $r \in R, m \in M$. Take U a 0-neighborhood in M, which we may assume without any loss to belong to $\mathcal{B}$. There exists $V \in \mathcal{B}$ with $V + V + V \subseteq U$. Since $\mathcal{B}$ is a filter base and satisfies (3),(4),(5), we can find $V_0 \in \mathcal{B}, W \in \mathcal{N}_0(R)$ such that $WV_0 \subseteq V$, $rV_0 \subseteq V$, and $Wm \subseteq V$. As a consequence,

$$(r + W)(m + V_0) \subseteq rm + (rV_0 + Wm + WV_0) \subseteq rm + (V + V + V) \subseteq rm + U.$$

This shows that the action of R on M is continuous and thus τ is a module topology. Finally, if τ' is another module topology on M for which $\mathcal{B}$ is a base of 0-neighborhoods, then τ' is a group topology on M for which $\mathcal{B}$ is a base of 0-neighborhoods, so $\tau' = \tau$.

Every module seminorm on a module induces a translation-invariant pseudometric. Thus, every seminormed module can be endowed with the corresponding pseudometric topology.

Example 56 *If M is a seminormed module over a seminormed ring R, then the topology induced by the seminorm is a module topology. Indeed, the seminorm on M is conjugation-invariant on the commutative additive group of M, which makes it an additive group topology in virtue of Example 50. It only remains to show that multiplication by elements of R is continuous. If $(r_i)_{i\in I}, (m_i)_{i\in I}$ are nets in R, M converging to r, m in R, M, respectively, then*

$$\|r_i m_i - rm\| \leq \|r_i\|\|m_i - m\| + \|r_i - r\|\|m\|.$$

By calling again on to Example 50, we have that the seminorms on R, M are both nonexpansive maps, therefore, continuous, so $(\|r_i\|)_{i\in I}$ converges to $\|r\|$ and $(\|m_i\|)_{i\in I}$ converges to $\|m\|$. As a consequence, $(\|r_i m_i - rm\|)_{i\in I}$ converges to 0 and then $(r_i m_i)_{i\in I}$ converges to rm.

Let M be a topological module over a topological ring R. For every subset $A \subseteq M$, $\overline{\operatorname{span}}(A)$ stands for the intersection of closed submodules of M containing A. Since the closure of a submodule is trivially a submodule, we have that $\overline{\operatorname{span}}(A) = \operatorname{cl}(\operatorname{span}(A))$. It is immediate to check that $\overline{\operatorname{span}}$ is a hull operator on $\mathcal{P}(M)$. The next results highlight several basic topological properties of module topologies.

Proposition 88 *Let M be a topological module over a topological ring R. If A is a dense subset of R and G is a spanning system of M, then the set $S := \{\sum_{i=1}^{n} a_i g_i : n \in \mathbb{N}, a_1, \ldots, a_n \in A, g_1, \ldots, g_n \in G\}$ is dense in M.*

Proof 319 *Fix an arbitrary nonempty open subset $U \subseteq M$. Take any $u \in U$. We can write $u = \sum_{i=1}^{n} r_i g_i$ with $r_1, \ldots, r_n \in R$ and $g_1, \ldots, g_n \in G$. There exists a 0-neighborhood V of M such that $u + V \subseteq U$. There also exists a 0-neighborhood $W \subseteq M$ satisfying that $W + \overset{n}{\cdots} + W \subseteq V$. For every $i \in \{1, \ldots, n\}$, we can find a 0-neighborhood $O_i \subseteq R$ satisfying that $O_i g_i \subseteq W$. For every $i \in \{1, \ldots, n\}$, let $a_i \in A \cap (r_i + O_i)$. Then*

$$\sum_{i=1}^{n} a_i g_i = u + \sum_{i=1}^{n} (a_i - r_i) g_i \in u + \sum_{i=1}^{n} O_i g_i \subseteq u + \left(W + \overset{n}{\cdots} + W\right) \subseteq u + V \subseteq U.$$

A topological module is said to be topologically countably (finitely) generated if there exists a countable (finite) subset whose closed linear span is the whole module.

Corollary 26 *Let M be a topological module over a topological ring R. If R is separable and M is topologically countably generated, then M is separable.*

Proof 320 *According to Proposition 88, the set*

$$S := \left\{ \sum_{i=1}^{n} a_i g_i : n \in \mathbb{N}, a_1, \ldots, a_n \in A, g_1, \ldots, g_n \in G \right\}$$

is dense in M, where A is a countable dense subset of R and G is a countable $\overline{\operatorname{span}}(G) = M$. Finally, notice that S is clearly countable.

Recall that a topological module M is Hausdorff if and only if $\bigcap \mathcal{N}_0(M) = \{0\}$, and M has the trivial topology if and only if $\bigcap \mathcal{N}_0(M) = M$. Even more, a submodule N of M is Hausdorff if and only if $N \cap \bigcap \mathcal{N}_0(M) = \{0\}$. Recall also the notation $M_{(0)} := \bigcap \mathcal{N}_0(M)$. Another notation for $\bigcap \mathcal{N}_0(M)$ is $\mathscr{O}_M$, that is, $\mathscr{O}_M := M_{(0)} := \bigcap \mathcal{N}_0(M)$.

Lemma 126 *Let R be a topological ring. Let M be a topological R-module. Then:*

1. *$\mathscr{O}_M$ is a closed submodule of M such that $\mathscr{O}_R M \subseteq \mathscr{O}_M$.*
2. *If U is an open neighborhood of 0 in M, then $v + U = U$ for all $v \in \mathscr{O}_M$.*
3. *If U is a regular closed neighborhood of 0 in M, then $v + U = U$ for all $v \in \mathscr{O}_M$.*

Proof 321 *The proof will follow the same itemization of the statement.*

1. *According to Corollary 15(2), we have that $\mathscr{O}_M$ is a closed additive subgroup of M. Let us show that $\mathscr{O}_M$ is also a submodule. Fix arbitrary elements $r \in R$ and $m \in \mathscr{O}_M$. Take any 0-neighborhood $U \subseteq M$. In view of Theorem 91(4), there exists $V \in \mathcal{N}_0(M)$ such that $rV \subseteq U$. Then $rm \in rV \subseteq U$. The arbitrariness of $U \in \mathcal{N}_0(M)$ shows that $rm \in \mathscr{O}_M$. As a consequence, $\mathscr{O}_M$ is a submodule. Finally, let us show that $\mathscr{O}_R M \subseteq \mathscr{O}_M$. If $r \in \mathscr{O}_R$ and $m \in M$, then for every 0-neighborhood $U \subseteq M$ there exists $W \in \mathcal{N}_0(R)$ with $Wm \subseteq U$. Then $rm \in Wm \subseteq U$. By the arbitrariness of U in $\mathcal{N}_0(M)$, we conclude that $rm \in \mathscr{O}_M$.*
2. *For every $u \in U$, $U - u \in \mathcal{N}_0(M)$, hence $v \in U - u$, meaning that $v + U \subseteq U$. By the same argument, $-v + U \subseteq U$. As a consequence, we obtain the desired result.*
3. *Since $\mathrm{int}(U)$ is open, we know that $\mathrm{int}(U + v) = \mathrm{int}(U) + v = \mathrm{int}(U)$. By taking closures, we obtain that $U + v = \mathrm{cl}(\mathrm{int}(U)) + v = \mathrm{cl}(\mathrm{int}(U + v)) = \mathrm{cl}(\mathrm{int}(U)) = U$.*

Lemma 126 has strong consequences pertinent as to the lack of extreme points for certain zero-neighborhoods.

Theorem 92 *Let M be a topological module over a topological ring R. If M is not Hausdorff, then every regular closed 0-neighborhood of M is free of 2-extreme points.*

Proof 322 *Let U be any regular closed 0-neighborhood of M. Fix any arbitrary $u \in U$. We will show that u is not a 2-extreme point of U. Take any $v \in \bigcap_{V \in \mathcal{N}_0(M)} V$ such that $v \neq 0$. By Lemma 126(3), $v+u, -v+u \in U$. Then $2u = (v+u)+(-v+u)$ with $u \neq v+u$, meaning that $u \notin \mathrm{ext}_2(U)$.*

In a topological module M over a topological ring R, if $M_{(0)}$ is linearly complemented in M and we let $P : M \to M$ denote a linear projection of M onto $M_{(0)}$, then P is continuous due to the fact that $M_{(0)}$ is endowed with the trivial topology. The following technical and useful lemma shows that, under these settings, $I - P$ is a closed map over its range $\ker(P)$.

Lemma 127 *Let M be a topological module over a topological ring R. Assume that $M_{(0)}$ is linearly complemented in M. Consider a continuous linear projection $P : M \to M$ of M onto $M_{(0)}$. Let $B \subseteq M$. If $z \in \mathrm{cl}(B)$, then $z - P(z) \in \mathrm{cl}(B)$. As a consequence, if B is closed in M, then $(I - P)(B)$ is closed in $\ker(P)$.*

Proof 323 *Let us show first that $z - P(z) \in \mathrm{cl}(B)$. Fix any arbitrary 0-neighborhood V of M. There exists another 0-neighborhood W of M such that $W + W \subseteq V$. Observe that $z+W \subseteq (z-P(z))+V$. Indeed, if $w \in W$, then $z+w = (z-P(z))+P(z)+w \in (z-P(z))+W+W \subseteq (z-P(z))+V$. At this point, it is sufficient to realize that $(z+W) \cap B \neq \varnothing$ by hypothesis, meaning that $((z-P(z))+V) \cap B \neq \varnothing$. This shows that $z - P(z) \in \mathrm{cl}(B)$. Next, assume that B is closed in M. For every $b \in B$, $b - P(b) \in \mathrm{cl}(B) = B$, that is, $(I - P)(B) \subseteq B$, meaning that $(I - P)(B) = B \cap \ker(P)$, hence $(I - P)(B)$ is closed in $\ker(P)$.*

Let M, N be topological modules over a topological ring R. If $T : M \to N$ is linear and continuous, then there is no guarantee that $\ker(T)$ be closed in M unless N is Hausdorff. This fact motivates the notion of topological kernel, which has already been introduced for topological monoids.

Theorem 93 *Let M, N be topological modules over a topological ring R. Consider a linear operator $T : M \to N$. Then $\ker_{\mathtt{t}}(T)$ is a submodule of M. Even more, if T is continuous, then $\ker_{\mathtt{t}}(T)$ is closed and contains $\mathscr{O}_M$.*

Proof 324 *In view of Lemma 126(1), $\mathscr{O}_N$ is a closed submodule of N, so $\ker_{\mathtt{t}}(T) := T^{-1}(\mathscr{O}_N)$ is a submodule of M. Suppose that T is continuous. Then $\ker_{\mathtt{t}}(T)$ is closed in M because $\mathscr{O}_N$ is a closed submodule of N. On the other hand, the continuity of T allows that T^{-1} preserve neighborhoods of 0, hence $\left\{T^{-1}(V) : V \in \mathcal{N}_0(N)\right\} \subseteq \mathcal{N}_0(M)$. Therefore, $\mathscr{O}_M = \bigcap \mathcal{N}_0(M) \subseteq \bigcap_{V \in \mathcal{N}_0(N)} T^{-1}(V) = T^{-1}\left(\bigcap \mathcal{N}_0(N)\right) = \ker_{\mathtt{t}}(T)$.*

Note that, under the settings of the previous theorem, if N is Hausdorff and T is continuous, then T vanishes on $\mathscr{O}_M$.

Theorem 94 *Let R be a practical topological ring. Let M, N be topological R-modules. Then:*

1. *If $\mathcal{B}_R$ is a base of 0-neighborhoods in R and $\mathcal{B}_M$ is a base of 0-neighborhoods in M, then $\mathcal{B} := \{WV : W \in \mathcal{B}_R, V \in \mathcal{B}_M\}$ is a base of 0-neighborhoods in M.*
2. *If N is a proper submodule of M, then N has empty interior.*
3. *If U is a neighborhood of 0 in M and $T, S : M \to N$ are linear maps such that $T|_U = S|_U$, then $T = S$.*
4. *M does not have the discrete topology.*

Proof 325 *Let us itemize the proof by following the statement.*

1. *If $W_1, W_2 \in \mathcal{B}_R$ and $V_1, V_2 \in \mathcal{B}_M$, then there are $W_3 \in \mathcal{B}_R$ and $V_3 \in \mathcal{B}_M$ such that $W_3 \subseteq W_1 \cap W_2$ and $V_3 \subseteq V_1 \cap V_2$, meaning that $W_3V_3 \subseteq W_1V_1 \cap W_2V_2$. As a consequence, $\mathcal{B}$ is a filter base. In view of Theorem 91(3), $\mathcal{B}$ is a filter base of $\mathcal{N}_0(M)$. Finally, it only suffices to observe that if $W \in \mathcal{B}_R$ and $V \in \mathcal{B}_M$, then there exists $w \in \mathcal{U}(R) \cap W$, therefore, $wV \subseteq WV$ and wV is a neighborhood of 0 in M because w is invertible.*
2. *Suppose on the contrary that N has nonempty interior. Then there exists an open set $V \subseteq N$. If we take any $p \in V$, then $V - p \subseteq N$ and $V - p$ is an open neighborhood of 0. So, let us denote $U := V - p$. Fix any arbitrary $m \in M$. There is a 0-neighborhood W in R with $Wm \subseteq U$. Next, R is practical, so there is an invertible $u \in \mathcal{U}(R)$ verifying that $u \in W$. Then $m = u^{-1}(um) \in u^{-1}Wm \subseteq u^{-1}N = N$. The arbitrariness of $m \in M$ implies that $M \subseteq N$ and thus N is not proper.*
3. *Fix an arbitrary $m \in M$. There exists $W \in \mathcal{N}_0(R)$ such that $Wm \subseteq U$. We can find an invertible $u \in \mathcal{U}(R) \cap W$. Then $um \in Wm \subseteq U$. Finally,*

$$T(m) = u^{-1}T(um) = u^{-1}S(um) = S(m).$$

The arbitrariness of $m \in M$ assures that $T = S$.

4. *Indeed, if* $\{0\}$ *is open in* M*, then it suffices to consider any* $m \in M \setminus \{0\}$ *(recall that by default we are assuming all modules to be nonzero unless we explicitly state otherwise). There must exist a* 0*-neighborhood* W *in* R *in such a way that* $Wm \subseteq \{0\}$*. By hypothesis, there exists an invertible* $w \in \mathcal{U}(R) \cap W$*, meaning that* $wm = 0$*, which is impossible.*

Theorem 94(1) does not hold if the topological ring is not practical. Indeed, take $R := \mathbb{Z}$ endowed with the inherited topology from $\mathbb{R}$ and $M := \mathbb{R}$. If $U := \{0\}$ and $V := [-1, 1]$, then $UV = \{0\}$ is not a neighborhood of 0 in M.

Lemma 128 *Let* R *be a topological ring. Let* M, N *be topological* R*-modules. Consider continuous linear mappings* $T : M \to N$ *and* $m^* : M \to R$*. Then:*

1. T *is open if and only if* $T(W)$ *is a* 0*-neighborhood in* N *for every* 0*-neighborhood* W *in* M*.*

2. *If* $m^*(M) \cap \mathcal{U}(R) \neq \varnothing$*, then* m^* *is open and surjective.*

Proof 326 *The proof is itemized in accordance with the statement of the lemma.*

1. *Let* U *be a nonempty open subset of* M*. We will show that* $T(U)$ *is a neighborhood of* $T(u)$ *for all* $u \in U$*. Fix then an arbitrary* $u \in U$*. Notice that* $U - u$ *is a* 0*-neighborhood in* M*. Thus* $T(U - u) = T(U) - T(u)$ *is a* 0*-neighborhood in* N*. This means that* $T(U)$ *is a neighborhood of* $T(u)$*.*

2. *Fix* $m \in M$ *with* $m^*(m) \in \mathcal{U}(R)$*. In view of Theorem 91(5), there is a* 0*-neighborhood* $U \subseteq R$ *verifying that* $Um \subseteq V$*. Then* $Um^*(m) \subseteq m^*(V)$*. Since* $m^*(m)$ *is invertible, we have that* $Um^*(m)$ *is a neighborhood of* 0 *in* R *and so is* $m^*(V)$*. This shows that* m^* *is open. Finally, if* $r \in R$*, then* $rm^*(m)^{-1}m$ *is mapped to* r *by* m^**. Thus,* m^* *is surjective.*

For the upcoming proposition, we recall the reader that a nonmeager set in a topological space is a set of second category.

Proposition 89 *Let* M *be a nontrivial topological module over a sequentially practical topological ring* R*. Then* M *is a Baire space if and only if* M *is nonmeager (of second category).*

Proof 327 *If* M *is Baire, then* M *has to be nonmeager since* M *is open. Conversely, assume that* M *is nonmeager. By hypothesis,* M *does not have the trivial topology. Next, fix any arbitrary open* 0*-neighborhood* V *in* M*. Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of invertibles in* R *converging to* 0*. Let us prove that* $M = \bigcup_{n\in\mathbb{N}} u_n^{-1}V$*. If* $m \in M$*, then there exists a neighborhood* W *of* 0 *in* R *such that* $Wm \subseteq V$*. Since* $(u_n)_{n\in\mathbb{N}}$ *converges to* 0*, there exists* $n_W \in \mathbb{N}$ *such that* $u_{n_W} \in W$*, meaning that* $u_{n_W}m \in Wm \subseteq V$*. Hence* $m \in u_{n_W}^{-1}V$*. As a consequence,* $M = \bigcup_{n\in\mathbb{N}} u_n^{-1}V$*. Since* M *is nonmeager, then* V *is nonmeager too. Since translations are homeomorphisms, any nonempty open set is nonmeager too. As a consequence,* M *is a Baire space.*

The upcoming theorem unveils that the addition of two Hausdorff submodules is not necessarily Hausdorff.

Theorem 95 *Let* M *be a topological module over a division topological ring. If* M *is not Hausdorff and is not endowed with the trivial topology, then there are two Hausdorff submodules* P, Q *of* M *such that* $P \cap Q = \{0\}$ *and* $P + Q$ *is not Hausdorff.*

Proof 328 *Since M is not Hausdorff, we can find $m_0 \in \mathscr{O}_M \setminus \{0\}$. On the other hand, M is not endowed with the trivial topology, thus we can find $n_0 \in M \setminus \mathscr{O}_M$. Take $l_0 := m_0 - n_0$. Note that $l_0 \notin \mathscr{O}_M$ since otherwise $n_0 = m_0 - l_0 \in \mathscr{O}_M$. Consider the submodules of M given by $P := Rl_0$ and $Q := Rn_0$. Notice that $l_0 + n_0 = m_0 \in \mathscr{O}_M \setminus \{0\}$, meaning that $P + Q = Rl_0 + Rn_0$ is not Hausdorff since $(P + Q) \cap \mathscr{O}_M \neq \{0\}$. Let us prove that both Rl_0, Rn_0 are Hausdorff. Suppose for instance that there exists $r_0 \in R$ with $r_0 l_0 \in \mathscr{O}_M \setminus \{0\}$. Then $r_0 \neq 0$, obtaining the contradiction that $l_0 = r_0^{-1}(r_0 l_0) \in \mathscr{O}_M$. As a consequence, Rl_0 is Hausdorff. Similarly, so is Rn_0. Finally, let us prove that $P \cap Q = \{0\}$. Assume in the contrary that there exists $m \in (Rl_0 \cap Rn_0) \setminus \{0\}$. Then we can find $r_1, r_2 \in R \setminus \{0\}$ satisfying that $m = r_1 l_0 = r_2 n_0$. Therefore, $r_1(m_0 - n_0) = r_2 n_0$, in other words, $(r_2 + r_1)n_0 = r_1 m_0$. If $r_2 + r_1 = 0$, then $r_1 m_0 = 0$, meaning the contradiction that $m_0 = 0$ because $r_1 \neq 0$ so it is invertible. As a consequence, $r_1 + r_2 \neq 0$, hence $n_0 = (r_1 + r_2)^{-1} r_1 m_0 \in \mathscr{O}_M$, which contradicts the choice of $n_0 \in M \setminus \mathscr{O}_M$.*

The following notion is an extension of a classical concept in the literature in real or complex Banach spaces.

Definition 248 (Klee module) *Let R be a topological ring. Let M be a topological R-module. We will say that M is Klee if for every spanning system L of M there exists a base $\mathcal{B}$ of 0-neighborhoods in M such that* $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(L)$.

The following proposition serves to easily construct plenty of examples of Klee modules in the context of real or complex Banach spaces.

Proposition 90 *Let R be a topological ring. Let M be a topological R-module. If M is first countable but not finitely spanned, then M is Klee.*

Proof 329 *Let $L \subseteq M$ be a spanning system. Since M is not finitely spanned, we have that L is infinite. Since M is first countable, there exists a base $\mathcal{B}$ of 0-neighborhoods in M such that* $\operatorname{card}(\mathcal{B}) \leq \aleph_0$. *As a consequence,* $\operatorname{card}(\mathcal{B}) \leq \aleph_0 \leq \operatorname{card}(L)$.

Klee free modules allow the construction of noncontinuous linear functionals.

Theorem 96 *Let R be a topological ring. Let M be a topological free R-module. Then:*

1. *If R is not endowed with the trivial topology and there exists a nonclosed base B of M as free R-module, then there exists a noncontinuous linear map $f : M \to R$.*
2. *If R is practical and M is Klee, then there exists a nonclosed basis of M as free R-module.*

Proof 330 *The proof follows as itemized as the statement of the theorem.*

1. *Let $m \in \operatorname{cl}(B) \setminus B$. By considering the basis $B' := B - m$, we may assume without any loss of generality that $m = 0$. We will begin by defining f on B as $f(B) = \{1\}$, that is, $f(b) := 1$ for all $b \in B$. Since B is a basis of M as free R-module, we can extend f to the whole of M by R-linearity. Finally, f is not continuous because $0 \notin \operatorname{cl}(\{1\})$ in virtue of Corollary 18.*
2. *Let $\mathcal{B}$ be a base of 0-neighborhoods of M and let $B \subseteq M$ be a basis of M as free R-module such that* $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(B)$. *Consider a one-to-one function*

$$\begin{array}{ccc} \mathcal{B} & \to & B \\ U & \mapsto & b_U. \end{array} \tag{3.4}$$

For every $U \in \mathcal{B}$, there exists a 0-neighborhood $V_U \subseteq R$ satisfying that $V_U b_U \subseteq U$. Since R is practical, we can find an invertible $v_U \in V_U$ for every $U \in \mathcal{B}$. Notice then that $v_U b_U \in V_U b_U \subseteq U$ for every $U \in \mathcal{B}$. As a consequence, the net $(v_U b_U)_{U \in \mathcal{B}}$ converges to 0. Finally, the new basis $B' := \{v_U b_U : U \in \mathcal{B}\} \cup B \setminus \{b_U : U \in \mathcal{B}\}$ satisfies that $0 \in \mathrm{cl}(B') \setminus B'$.

The following corollary is a direct consequence of Theorem 96 and it will be used later on.

Corollary 27 *Let R be a Hausdorff practical topological ring. Let M be a Klee topological free R-module. There exists a nonclosed basis of M as free R-module and a noncontinuous linear map $f : M \to R$.*

Next, we will find examples of equivalence-free subsets of topological modules satisfying that their closure contains zero.

Theorem 97 *Let R be a topological ring. Let M be a topological R-module. If R is practical and M is Klee, then there exists a maximal equivalence-free $Q \subseteq M$ such that $0 \in \mathrm{cl}(Q)$.*

Proof 331 *In view of Lemma 26, there exists a maximal equivalence-free $P \subseteq M$. Observe that, by maximality, P contains exactly one element of each equivalent class $\mathcal{U}(R)m$ for $m \neq 0$. In other words,*

$$\mathrm{card}(P) = \mathrm{card}\left(\{\mathcal{U}(R)m : m \in M \setminus \{0\}\}\right).$$

Since P is trivially a spanning system in virtue of Lemma 26, by hypothesis, there exists a base $\mathcal{B}$ of neighborhoods of 0 such that $\mathrm{card}(\mathcal{B}) \leq \mathrm{card}(P)$. Then we can take a one-to-one function $f : \mathcal{B} \to P$. Since R is practical by hypothesis, for every $U \in \mathcal{B}$, there exists an invertible $u_U \in \mathcal{U}(R)$ verifying that $u_U f(U) \in \mathrm{int}(U)$. Consider now

$$Q := \{u_U f(U) : U \in \mathcal{B}\} \cup (P \setminus f(\mathcal{B})).$$

It is clear that Q is a maximal equivalence-free subset of M satisfying that $0 \in \mathrm{cl}(Q)$.

It is the turn now for the topological version of torsion. For this, it is necessary to recall that a subset A of a topological ring R is called topologically regular provided that its closure is made of regular elements, that is, $\mathrm{cl}(A) \subseteq R \setminus (\mathtt{ld}(0) \cup \mathtt{rd}(0))$.

Definition 249 (Strong torsion) *Let M be a topological module over a topological ring R. An element $m \in M$ is said to be a strongly torsion element of M provided that there exists a topologically regular subset $A \subseteq R$ such that $0 \in \mathrm{cl}(Am)$. The set of strongly torsion elements of M is denoted by $\mathrm{tor}_{\mathtt{s}}(M)$.*

Under the settings of the previous definition, the strongly torsionfree elements of M are exactly those in $M \setminus \mathrm{tor}_{\mathtt{s}}(M)$. For the following lemma, it is useful to recall that the right-multiplication operator

$$\begin{array}{rccc} \mathsf{R}_m : & R & \to & Rm \subseteq M \\ & r & \mapsto & \mathsf{R}_m(r) := rm \end{array}$$

for m in a module M over a ring R, is an R-linear operator.

Lemma 129 *Let M be a topological module over a topological ring R. Then:*

1. *If $m \in \mathrm{tor}_{\mathsf{s}}(M)$, then $\mathrm{cl}(\{m\}) \subseteq \mathrm{tor}_{\mathsf{s}}(M)$. Thus, if $\{1\}$ is topologically regular, then $\mathrm{cl}(\{0\}) \in \mathrm{tor}_{\mathsf{s}}(M)$. This way, M is called strongly torsionfree provided that $\mathrm{tor}_{\mathsf{s}}(M) = \mathrm{cl}(\{0\}) = \mathscr{O}_M$.*
2. *If R is Hausdorff, then $\mathrm{tor}(M) \subseteq \mathrm{tor}_{\mathsf{s}}(M)$.*
3. *If R is a nontrivial integral domain and M is torsionfree, then $M \setminus \mathrm{tor}_{\mathsf{s}}(M) = \{m \in M : \mathsf{R}_m$ is a topological isomorphism over its image$\}$.*
4. *If R is an integral domain, $M \setminus \mathrm{tor}(M) \neq \varnothing$, and M is discrete, then R is discrete as well.*
5. *If R is a nontrivial division topological ring, then R is strongly torsionfree over itself.*
6. *If R is absolutely valued and M is normed, then M is strongly torsionfree.*

Proof 332 *Let us itemize the proof like the statement of the lemma is.*

1. *Fix an arbitrary $n \in \mathrm{cl}(\{m\})$. Let $A \subseteq R$ topologically regular such that $0 \in \mathrm{cl}(Am)$. Then $m \in \mathrm{cl}(\{n\})$, so $0 \in \mathrm{cl}(A)m \subseteq \mathrm{cl}(A)\mathrm{cl}(\{n\}) \subseteq \mathrm{cl}(An)$.*
2. *If $m \in \mathrm{tor}(M)$ and r is a regular element in R such that $rm = 0$, then $0 \in \mathrm{cl}(\{r\}m)$ and $\{r\}$ is topologically regular.*
3. *Suppose first that $m \in M$ satisfies that R_m is a topological isomorphism over its image. Assume on the contrary that $m \in \mathrm{tor}_{\mathsf{s}}(M)$. There exists $A \subseteq R$ such that $0 \in \mathrm{cl}(Am)$ but $0 \notin \mathrm{cl}(A)$. However, this fact contradicts that R_m be an homeomorphism since $\mathsf{R}_m(A) = Am$. Conversely, assume that $m \in M \setminus \mathrm{tor}_{\mathsf{s}}(M)$. Let us show first that $m \neq 0$. Note that $1 \notin \mathrm{cl}(\{0\})$ because R does not have the trivial topology, meaning that $0 \notin \mathrm{cl}(\{1\})$, that is, $\{1\}$ is topologically regular, so $0 \in \mathrm{tor}_{\mathsf{s}}(M)$. As a consequence, $m \neq 0$ and R_m is an algebraic isomorphism over its image. Next, assume to the contrary thar R_m^{-1} is not continuous at 0. Then there exists $U \in \mathcal{N}_0(R)$ such that Um is not a neighborhood of 0 in Rm. This means that for any $V \in \mathcal{N}_0(M)$, $V \cap Rm \nsubseteq Um$, so we can consider $r_V m \in (V \cap Rm) \setminus Um$. Then $(r_V m)_{V \in \mathcal{N}_0(M)}$ is a net in M converging to 0. Finally, $0 \in \mathrm{cl}(\{r_V : V \in \mathcal{N}_0(M)\}m)$ but $0 \notin \mathrm{cl}(\{r_V : V \in \mathcal{N}_0(M)\}$ because $\{r_V : V \in \mathcal{N}_0(M)\} \cap U = \varnothing$, contradicting that $m \in M \setminus \mathrm{tor}_{\mathsf{s}}(M)$.*
4. *Fix any arbitrary $m \in M \setminus \mathrm{tor}(M)$. In view of Theorem 91(5), there exists $W \in \mathcal{N}_0(R)$ such that $Wm \subseteq \{0\}$. If $r \in W$, then $rm = 0$, so $r = 0$ because $m \in M \setminus \mathrm{tor}(M)$ and R is an integral domain. As a consequence, $W = \{0\}$ and R is discrete.*
5. *Notice that R is a nontrivial integral domain and torsionfree over itself. Fix $s \in R \setminus \{0\}$. All we have to show is that*
$$\begin{array}{rccc} \mathsf{R}_s : & R & \to & R \\ & r & \mapsto & rs \end{array} \tag{3.5}$$
has a continuous inverse. Since s is invertible in R, we have that $\mathsf{R}_s^{-1} = \mathsf{R}_{s^{-1}}$, which is clearly continuous since R is a topological ring.
6. *Fix an arbitrary $m \in M \setminus \{0\}$. Take $A \subseteq R$ such that $0 \in \mathrm{cl}(Am)$. Let $(r_i)_{i \in I}$ be a net in A such that $(r_i m)_{i \in I}$ converges to 0. Then*
$$|r_i| = \frac{|r_i|\|m\|}{\|m\|} = \frac{\|r_i m\|}{\|m\|} \to 0,$$
meaning that A cannot be topologically regular.

The following series of results deal with limits in topological modules. Observe that, if $\mathcal{B}_1, \mathcal{B}_2$ are filter bases of $\mathcal{P}(M)$, for M a module over a ring R, then $\mathcal{B}_1 + \mathcal{B}_2 := \{A_1 + A_2 : A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2\}$ is a filter base of $\mathcal{P}(M)$. Also, $r\mathcal{B}_1 := \{rA_1 : A_1 \in \mathcal{B}_1\}$ is a filter base of $\mathcal{P}(M)$ for all $r \in R$.

Proposition 91 *Let M be a topological module over a topological ring R. Let $\mathcal{B}_1, \mathcal{B}_2$ be filter bases of $\mathcal{P}(M)$. Consider $r \in R$. Then:*

1. $\lim(\mathcal{B}_1) + \lim(\mathcal{B}_2) = \lim(\mathcal{B}_1 + \mathcal{B}_2)$.
2. $r\lim(\mathcal{B}_1) \subseteq \lim(r\mathcal{B}_1)$.
3. *If $r \in \mathcal{U}(R)$, then* $r\lim(\mathcal{B}_1) = \lim(r\mathcal{B}_1)$.

Proof 333 *Let us again itemize the proof according to the statement.*

1. *Take $m_i \in \lim(\mathcal{B}_i)$ for $i = 1, 2$. Fix an arbitrary 0-neighborhood $U \subseteq M$. Let $V \subseteq M$ another 0-neighborhood such that $V + V \subseteq U$. There are $A_i \in \mathcal{B}_i$, $i = 1, 2$, with $A_i \subseteq m_i + V$. Then $A_1 + A_2 \in \mathcal{B}_1 + \mathcal{B}_2$ and $A_1 + A_2 \subseteq (m_1 + V) + (m_2 + V) = (m_1 + m_2) + (V + V) \subseteq (m_1 + m_2) + U$. This shows that $m_1 + m_2 \in \lim(\mathcal{B}_1 + \mathcal{B}_2)$. As a consequence, $\lim(\mathcal{B}_1) + \lim(\mathcal{B}_2) \subseteq \lim(\mathcal{B}_1 + \mathcal{B}_2)$. Following a similar reasoning, it can be proved that $\lim(\mathcal{B}_1 + \mathcal{B}_2) - \lim(\mathcal{B}_1) \subseteq \lim(\mathcal{B}_2)$. In view of Remark 36, we conclude that $\lim(\mathcal{B}_1) + \lim(\mathcal{B}_2) = \lim(\mathcal{B}_1 + \mathcal{B}_2)$.*
2. *Take $m_1 \in \lim(\mathcal{B}_1)$. Fix an arbitrary 0-neighborhood $U \subseteq M$. Let $V \subseteq M$ another 0-neighborhood such that $rV \subseteq U$. There is $A_1 \in \mathcal{B}_1$ with $A_1 \subseteq m_1 + V$. Then $rA_1 \in r\mathcal{B}_1$ and $rA_1 \subseteq r(m_1 + V) = rm_1 + rV \subseteq rm_1 + U$. This shows that $rm_1 \in \lim(r\mathcal{B}_1)$. As a consequence, $r\lim(\mathcal{B}_1) \subseteq \lim(r\mathcal{B}_1)$.*
3. *If $r \in \mathcal{U}(R)$, then $r^{-1}\lim(r\mathcal{B}_1) \subseteq \lim\left(r^{-1}(r\mathcal{B}_1)\right) = \lim(\mathcal{B}_1)$. Therefore, $\lim(r\mathcal{B}_1) \subseteq r\lim(\mathcal{B}_1)$.*

In order to provide a version of the previous proposition for limits of functions through prefilters, the following technical lemma is necessary.

Lemma 130 *Let X be a nonempty set. Let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. Let M be a module over a ring R. Let $f_1, f_2 : X \to M$ be functions. Then $\mathcal{J}\left((f_1 + f_2)(\mathcal{B})\right) \supseteq \mathcal{J}\left(f_1(\mathcal{B}) + f_2(\mathcal{B})\right)$.*

Proof 334 *Since $\mathcal{B}$ is a filter base, for every $A, C \in \mathcal{B}$, there exists $D \in \mathcal{B}$ with $D \subseteq A \cap C$, which implies that $(f_1 + f_2)(D) \subseteq f_1(A) + f_2(C)$. This shows that $\mathcal{J}\left((f_1 + f_2)(\mathcal{B})\right) \supseteq \mathcal{J}\left(f_1(\mathcal{B}) + f_2(\mathcal{B})\right)$.*

Proposition 91 has its version for limits of functions through prefilters by relying upon the previous lemma.

Corollary 28 *Let X be a nonempty set. Let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. Let M be a topological module over a topological ring R. Let $f_1, f_2 : X \to M$ be functions. Consider $r \in R$. Then:*

1. $\lim_{\mathcal{B}}(f_1) + \lim_{\mathcal{B}}(f_2) = \lim_{\mathcal{B}}(f_1 + f_2)$.
2. $r\lim_{\mathcal{B}}(f_1) \subseteq \lim_{\mathcal{B}}(rf_1)$.
3. *If $r \in \mathcal{U}(R)$, then* $r\lim_{\mathcal{B}}(f_1) = \lim_{\mathcal{B}}(rf_1)$.

Proof 335 *The proof is as itemized as the statement.*

1. *According to Lemma 130,* $\mathcal{J}\left((f_1+f_2)(\mathcal{B})\right) \supseteq \mathcal{J}\left(f_1(\mathcal{B})+f_2(\mathcal{B})\right)$. *As a consequence, by relying on Proposition 91(1),*

$$\begin{aligned}\lim_{\mathcal{B}}(f_1)+\lim_{\mathcal{B}}(f_2) &= \lim(f_1(\mathcal{B}))+\lim(f_2(\mathcal{B}))=\lim(f_1(\mathcal{B})+f_2(\mathcal{B}))\\ &= \lim\left(\mathcal{J}\left(f_1(\mathcal{B})+f_2(\mathcal{B})\right)\right)\subseteq\lim\left(\mathcal{J}\left((f_1+f_2)(\mathcal{B})\right)\right)\\ &= \lim\left((f_1+f_2)(\mathcal{B})\right)=\lim_{\mathcal{B}}(f_1+f_2).\end{aligned}$$

   Following a similar reasoning, it can be proved that $\lim_{\mathcal{B}}(f_1+f_2)-\lim_{\mathcal{B}}(f_1)\subseteq\lim_{\mathcal{B}}(f_2)$. *In view of Remark 36, we conclude that* $\lim_{\mathcal{B}}(f_1)+\lim_{\mathcal{B}}(f_2)=\lim_{\mathcal{B}}(f_1+f_2)$.

2. *In accordance with Proposition 91(1),* $r\lim_{\mathcal{B}}(f_1)=r\lim(f_1(\mathcal{B}))\subseteq\lim(rf_1(\mathcal{B}))=\lim_{\mathcal{B}}(rf_1)$.

3. *If* $r\in\mathcal{U}(R)$, *then* $r^{-1}\lim_{\mathcal{B}}(rf_1)\subseteq\lim_{\mathcal{B}}\left(r^{-1}(rf_1)\right)=\lim_{\mathcal{B}}(f_1)$. *Therefore,* $\lim_{\mathcal{B}}(rf_1)\subseteq r\lim_{\mathcal{B}}(f_1)$.

The following corollary is an immediate consequence of Corollary 28, thus, it does not precise of any proof.

Corollary 29 *Let* X *be a topological space. Let* x_0 *be a limit point of* X. *Let* M *be a topological module over a topological ring* R. *Let* $f_1, f_2 : X\setminus\{x_0\}\to M$ *be functions. Consider* $r\in R$. *Then:*

1. $\lim_{x\to x_0}(f_1)+\lim_{x\to x_0}(f_2)=\lim_{x\to x_0}(f_1+f_2)$.
2. $r\lim_{x\to x_0}(f_1)\subseteq\lim_{x\to x_0}(rf_1)$.
3. *If* $r\in\mathcal{U}(R)$, *then* $r\lim_{x\to x_0}(f_1)=\lim_{x\to x_0}(rf_1)$.

Next corollary is again a direct consequence of Corollary 28, therefore, its proof will be omitted and left as exercise for the reader.

Corollary 30 *Let* D *be a directed set. Let* M *be a topological module over a topological ring* R. *Let* $\gamma_1,\gamma_2 : D\to M$ *be nets. Consider* $r\in R$. *Then:*

1. $\lim(\gamma_1)+\lim(\gamma_2)=\lim(\gamma_1+\gamma_2)$.
2. $r\lim(\gamma_1)\subseteq\lim(r\gamma_1)$.
3. *If* $r\in\mathcal{U}(R)$, *then* $r\lim(\gamma_1)=\lim(r\gamma_1)$.

The final result in this subsubsection deals with the product of module topologies. We omit the proof of the following theorem since it is extremely similar to the one of Theorem 75.

Theorem 98 *Let* R *be a topological ring. Let* $(M_i)_{i\in I}$ *be a family of topological* R*-modules. Then* $\prod_{i\in I}M_i$ *is a topological* R*-module endowed with the product topology.*

As a direct consequence of the previous theorem, the pointwise convergence topology on M^I is always a module topology for any topological module M.

3.2.4.2 Topological ordered modules

A topological ordered module [46] is simply an ordered module in such a way that the order topology is a module topology.

Definition 250 (Topological ordered module) *A topological ordered module is an ordered module M over a topological ordered ring R in such a way that the order topology on M is a module topology.*

In order to find sufficient conditions to assure that the order topology in an ordered module is a module topology, we will strongly rely on Theorem 34.

Lemma 131 *Let R be a topological ordered ring. Let M be an ordered R-module. Let $\mathcal{S} := \{\uparrow_\times m, \downarrow^\times m : m \in M\}$. Then:*

1. *$\mathcal{S}$ is a subbase for a topology on M if and only if $M^+ \neq \{0\}$. In this situation, $\downarrow^\times m = m + M^- \setminus \{0\} \neq \varnothing$ and $\uparrow_\times m = m + M^+ \setminus \{0\} \neq \varnothing$ for all $m \in M$.*
2. *If $M^+ \neq \{0\}$ and $M^+ \setminus \{0\}$ is downward directed, then $\{(-m, m) : m > 0\}$ is a base of neighborhoods of 0 for the order topology.*

Proof 336 *The proof is itemized in view of the statement.*

1. *According to Theorem 34, $\mathcal{S}$ is a subbase for a topology on M if and only if, for all $m \in M$, either $\uparrow_\times m \neq \varnothing$ or $\downarrow^\times m \neq \varnothing$, which is equivalent to $M^+ \neq \{0\}$ by bearing in mind that $\downarrow m = m + M^-$ and $\uparrow m = m + M^+$ for all $m \in M$.*
2. *Let $W \subseteq M$ be a 0-neighborhood for the order topology. There can be found $p_1, \ldots, p_n, q_1, \ldots, q_m \in M$ satisfying that $0 \in \uparrow_\times p_1 \cap \cdots \cap \uparrow_\times p_n \cap \downarrow^\times q_1 \cap \cdots \cap \downarrow^\times q_m \subseteq W$. Observe that $p_i < 0$ and $q_j > 0$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Since $M^+ \setminus \{0\}$ is downward directed, there exists $p_0 \in M^+ \setminus \{0\}$ such that $p_0 \leq -p_i$ for all $i \in \{1, \ldots, n\}$ and $p_0 \leq q_j$ for all $j \in \{1, \ldots, m\}$. Notice that $(-p_0, p_0) \subseteq \uparrow_\times p_1 \cap \cdots \cap \uparrow_\times p_n \cap \downarrow^\times q_1 \cap \cdots \cap \downarrow^\times q_m \subseteq W$.*

We are finally in the right position to provide a sufficient condition for the order topology induced by a module ordering to be a module topology.

Theorem 99 *Let R be a topological ordered ring. Let M be an ordered R-module. Suppose that $M^+ \neq \{0\}$, $M^+ \setminus \{0\}$ is downward directed, and $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$. Then:*

1. *The order topology is an additive group topology.*
2. *If, in addition, R is a division ring, $0, 1$ are comparable, $R = R^+ + R^-$, $R^+ \setminus \{0\}$ is downward directed and multiplicatively symmetric, $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, $M = M^+ + M^-$, and $0 \in \mathrm{cl}\,(R^+ m \setminus \{0\})$ for all $m \in M^+ \setminus \{0\}$, then the order topology is a module topology.*

Proof 337 *By Lemma 131(1), $\mathcal{S} := \{\uparrow_\times m, \downarrow^\times m : m \in M\}$ is a subbase for a topology on M (the order topology). According to Lemma 131(2), $\{(-m, m) : m > 0\}$ is a base of neighborhoods of 0 for the order topology. Let $\mathcal{B}_0(\mathcal{S})$ denote the family of all finite intersections of elements of $\mathcal{S}$ containing 0. We will show first that $\mathcal{B}_0(\mathcal{S})$ satisfies the first two conditions of Theorem 91.*

1.1. Let $W \in \mathcal{B}_0(\mathcal{S})$. There exists $m_0 \in M^+ \setminus \{0\}$ such that $(-m_0, m_0) \subseteq W$. Since $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$, there exists $n_0 \in M$ such that $0 < n_0 < m_0$. By using again that $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$, we can find $l_0 \in M$ such that $0 < l_0 < m_0 - n_0$. Finally, we will prove that $(-l_0, l_0) + (-n_0, n_0) \subseteq W$. Indeed, let us observe first that $(-l_0, l_0) + (-n_0, n_0) \subseteq (-l_0 - n_0, l_0 + n_0)$. Next, take $m \in M$ with $-l_0 - n_0 < m < l_0 + n_0$. Note that $m < l_0 + n_0 < m_0$. Also, $-l_0 - n_0 < m$ so $-m < l_0 + n_0 < m_0$. We have just shown that $(-l_0, l_0) + (-n_0, n_0) \subseteq (-l_0 - n_0, l_0 + n_0) \subseteq (-m_0, m_0) \subseteq W$ with $(-l_0, l_0), (-n_0, n_0) \in \mathcal{B}_0(\mathcal{S})$.

1.2. Let $W \in \mathcal{B}_0(\mathcal{S})$. There exists $m_0 \in M^+ \setminus \{0\}$ such that $(-m_0, m_0) \subseteq W$. Now, it only suffices to notice that $(-m_0, m_0)$ is an additively symmetric element of $\mathcal{B}_0(\mathcal{S})$.

Next, let us assume, in addition, that R is a division ring, $0, 1$ are comparable, $R = R^+ + R^-$, $R^+ \setminus \{0\}$ is downward directed and multiplicatively symmetric, $0 \in \mathrm{cl}\,(R^+ \setminus \{0\})$, $M = M^+ + M^-$, and $0 \in \mathrm{cl}\,(R^+ m \setminus \{0\})$ for all $m \in M^+ \setminus \{0\}$. So, let us finally check that $\mathcal{B}_0(\mathcal{S})$ also satisfies the last three conditions of Theorem 91.

2.3. Let $W \in \mathcal{B}_0(\mathcal{S})$. There exists $m_0 \in M^+ \setminus \{0\}$ such that $(-m_0, m_0) \subseteq W$. Since $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$, there exists $n_0 \in M$ such that $0 < n_0 < m_0$. By using again that $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$, we can find $p_0 \in M$ such that $0 < p_0 < m_0 - n_0$. Again, by relying on the fact that $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$, there exists $q_0 \in M$ satisfying that $0 < q_0 < m_0 - n_0 - p_0$. Next, since $M^+ \setminus \{0\}$ is downward directed, $h_0 \in M^+ \setminus \{0\}$ can be found in such a way that $h_0 \leq q_0$, $h_0 \leq p_0$, and $h_0 \leq n_0$. Finally, we will prove that $(-1, 1)(-h_0, h_0) \subseteq (-3h_0, 3h_0) \subseteq (-m_0, m_0) \subseteq W$. Indeed, take arbitrary elements $r \in R$ with $-1 < r < 1$ and $a \in M$ with $-h_0 < a < h_0$. Note that $h_0 - a > 0$ and $1 - r > 0$, meaning that $h_0 - rh_0 - a + ra = (1 - r)(h_0 - a) \geq 0$. Then $ra \geq -h_0 + rh_0 + a$. Since $r > -1$ and $h_0 > 0$, we have that $rh_0 \geq -h_0$. Also, $-h_0 < a$. As a consequence, $ra \geq -h_0 + rh_0 + a > -h_0 - h_0 - h_0$. By repeating the same reasoning for $-r$ and a, we conclude that $-ra = (-r)a > -h_0 - h_0 - h_0$, that is, $ra < 3h_0$. We have just shown that $(-1, 1)(-h_0, h_0) \subseteq (-3h_0, 3h_0)$. Let us finally prove that $(-3h_0, 3h_0) \subseteq (-m_0, m_0) \subseteq W$. Indeed, if $c \in M$ and $-3h_0 < c < 3h_0$, then $c < h_0 + h_0 + h_0 \leq q_0 + p_0 + n_0 < m_0$, and $-c < h_0 + h_0 + h_0 \leq q_0 + p_0 + n_0 < m_0$, so $-m_0 < c$, reaching the desired result.

2.4. Let $W \in \mathcal{B}_0(\mathcal{S})$ and fix an arbitrary element $r \in R$. There exists $m \in M^+ \setminus \{0\}$ such that $(-m, m) \subseteq W$. By hypothesis, $r = r^+ + r^-$ with $r^+ \in R^+$ and $r^- \in R^-$. We may assume without any loss of generality that $r^+ > 0$ and $r^- < 0$. First off, since $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$, there exists $m_1 \in M$ such that $0 < m_1 < m$. By using again that $0 \in \mathrm{cl}\,(M^+ \setminus \{0\})$, we can find $m_2 \in M$ such that $0 < m_2 < m - m_1$. Then $(-m_1, m_1) + (-m_2, m_2) \subseteq (-(m_1 + m_2), m_1 + m_2) \subseteq (-m, m)$. Next, $R^+ \setminus \{0\}$ is multiplicatively symmetric, therefore $(r^+)^{-1} > 0$ and $(-r^-)^{-1} > 0$, and since the two of them are invertible, it occurs that $(r^+)^{-1} m_1 > 0$ and $(-r^-)^{-1} m_2 > 0$. Since $M^+ \setminus \{0\}$ is downward directed, there exists $p \in M^+ \setminus \{0\}$ such that $p \leq (r^+)^{-1} m_1$ and $p \leq (-r^-)^{-1} m_2$. Then $r(-p, p) \subseteq r^+(-p, p) + r^-(-p, p) = (-r^+ p, r^+ p) + (r^- p, -r^- p) \subseteq (-m_1, m_1) + (-m_2, m_2) \subseteq (-(m_1 + m_2), m_1 + m_2) \subseteq (-m, m) \subseteq W$.

2.5. Let $W \in \mathcal{B}_0(\mathcal{S})$ and fix an arbitrary element $n \in M$. There exists $m \in M^+ \setminus \{0\}$ such that $(-m, m) \subseteq W$. By hypothesis, $n = n^+ + n^-$ with $n^+ \in M^+$ and $n^- \in M^-$. We may assume without any loss of generality that $n^+ > 0$ and $n^- < 0$. As before, since $0 \in \mathrm{cl}\,(R^+ n^+ \setminus \{0\})$ and $0 \in \mathrm{cl}\,(R^+(-n^-) \setminus \{0\})$, there exist $s_1, s_2 \in R^+$ such that $0 < s_1 n^+ < m$ and $0 < s_2(-n^-) < m - s_1 n^+$. Then $(-s_1 n^+, s_1 n^+) + (s_2 n^-, -s_2 n^-) \subseteq (-s_1 n^+ + s_2 n^-, s_1 n^+ - s_2 n^-) \subseteq (-m, m)$. Next,

$R^+\backslash\{0\}$ is downward directed, therefore there exists $s \in R^+\backslash\{0\}$ such that $s \leq s_1$ and $s \leq s_2$, meaning that $(-s,s)n \subseteq (-s,s)n^+ + (-s,s)n^- \subseteq (-s_1,s_1)n^+ + (-s_2,s_2)n^- \subseteq (-s_1n^+, s_1n^+) + (s_2n^-, -s_2n^-) \subseteq (-s_1n^+ + s_2n^-, s_1n^+ - s_2n^-) \subseteq (-m,m) \subseteq W$.

The following remark unveils a sufficient condition to obtain the condition given in the previous theorem by $0 \in \mathrm{cl}\,(R^+m\backslash\{0\})$ for all $m \in M^+\backslash\{0\}$.

Remark 76 *Let R be an ordered ring. Let M be an ordered R-module. If M^+ has no holes, $M^+\backslash\{0\}$ is downward directed, and $R^+m\backslash\{0\}$ is coinitial in $M^+\backslash\{0\}$ for all $m \in M^+\setminus\{0\}$, then $0 \in \mathrm{cl}\,(R^+m\backslash\{0\})$ for all $m \in M^+\backslash\{0\}$. Indeed, the sets of the form $(-p,p)$ with $p > 0$ form a base of neighborhoods of 0 in M, so fix an arbitrary $p > 0$. Next, there exists $q \in (0,p)$ in M because M^+ is holefree. Since $R^+m\backslash\{0\}$ is coinitial in $M^+\backslash\{0\}$, there exists $t \in R^+$ such that $0 < tm \leq q < p$. Finally, $tm \in (-p,p) \cap R^+m\backslash\{0\}$.*

As an immediate consequence of the previous theorem, there is the following corollary.

Corollary 31 *Let R be a totally ordered division ring. Let M be a totally ordered R-module. If $R^+m\backslash\{0\}$ is coinitial in $M^+\backslash\{0\}$ for all $m \in M^+\backslash\{0\}$, then the order topology is a module topology.*

Proof 338 *First off, Corollary 22 guarantees that $R = R^+ + R^-$, $R^+\backslash\{0\}$ be downward directed and multiplicatively symmetric, R be holefree, and $0 \in \mathrm{cl}\,(R^+\backslash\{0\})$. Next, $M = M^+ \cup M^-$ and $M^+\backslash\{0\}$ is downward directed because M is totally ordered. On the other hand, M is free of holes by Proposition 38(9). As a consequence, $0 \in \mathrm{cl}\,(M^+\backslash\{0\})$ because M is free of holes and $M^+\backslash\{0\}$ is downward directed. Finally, Remark 76 allows that $0 \in \mathrm{cl}\,(R^+m\backslash\{0\})$ for all $m \in M^+\backslash\{0\}$.*

The next result of this subsubsection, which slightly improves [46, Theorem 14], is aimed at showing that the converse to Theorem 99 does not hold. In order to achieve this, it is worth drawing our attention to Theorem 83, where it was proved that, in every ring R of characteristic 0, the ring ordering given by

$$r \leq s \Leftrightarrow \exists n \in \mathbb{N}\ s - r = 1 + \overset{n}{\cdots} + 1 \tag{3.6}$$

is strong and faithful, and the order topology generated by the subbase $\{\uparrow_\times r, \downarrow^\times r : r \in R\}$ is the discrete topology, hence a ring topology.

Theorem 100 *Let R be an integral domain ring of characteristic 0 endowed with the ring ordering given by (3.6). Let M be an R-module. Let $m_1 \in M\backslash\mathrm{tor}(M)$. Consider the ordering set $S := R^+m_1$. The module ordering induced by S is strong and faithful and its order topology is the discrete topology, hence a module topology.*

Proof 339 *First off, $S \cup -S = (R^+ \cup R^-)\,m_1$, which is a $(R^+ \cup R^-)$-submodule of M; hence, the module ordering induced by S is faithful. Next, we will prove that $\uparrow_\times(-m_1) \cap \downarrow^\times m_1 = \{0\}$. Indeed, if $m \in \uparrow_\times(-m_1) \cap \downarrow^\times m_1$, then there are $r_1, r_2 \in R^+ \setminus \{0\}$ such that $m = -m_1 + r_1m_1$ and $m = m_1 - r_2m_1$. Then $(r_1 + r_2 - 1 - 1)m_1 = 0$. Since m_1 is torsionfree and R is an integral domain, $r_1 + r_2 = 1 + 1$. By hypothesis, $r_1, r_2 \geq 1$, meaning that $r_1 = r_2 = 1$, hence $m = 0$. Therefore, the module ordering induced by S is trivially strong since for every $m \in M^+ \setminus \{0\}$ and every $s \in R$, $s(-m_1, m_1) = \{0\} \subseteq \downarrow^\times m$. Next, by Lemma 131(1), $\mathcal{S} := \{\uparrow_\times m, \downarrow^\times m : m \in M\}$ is a subbase for a topology on M because $m_1 \in M^+ \setminus \{0\}$. We will prove that this topology is the discrete topology. Fix any $m \in M$. We will prove that $\{m\} = \uparrow_\times(m - m_1) \cap \downarrow^\times(m + m_1)$. Indeed, $0 < m_1$ and $-m_1 < 0$, so $m < m + m_1$ and $m - m_1 < m$, meaning that $m \in \uparrow_\times(m - m_1) \cap \downarrow^\times(m + m_1)$. Conversely, take any $p \in \uparrow_\times (m - m_1) \cap \downarrow^\times (m + m_1)$. Notice that $m - m_1 < p < m + m_1$, hence $-m_1 < p - m < m_1$, so $p - m \in (-m_1, m_1) = \{0\}$, meaning that $p = m$.*

Let us finally highlight, as a direct consequence of Lemma 107, that the positive cone of an ordered module satisfies that its relative order topology coincides with its own order topology.

Theorem 101 *Let R be an ordered ring. Let M be an ordered R-module. If $M^+\backslash\{0\}$ is downward directed, then the order topology of $\uparrow m$ and of $\downarrow m$ coincides with their inherited topology from M, for all $m \in M$.*

3.2.4.3 Boundedness: convergence linear topologies

Let us deal now with boundedness in topological modules. The concept of boundedness that is defined in this setting is based upon the metric concept of boundedness, but it does not rely directly on it.

Definition 251 (Bounded set) *Let M be a topological module over a topological ring R. A subset $A \subseteq X$ is said to be bounded if for each 0-neighborhood U in M there is an invertible $u \in \mathcal{U}(R)$ such that $A \subseteq uU$.*

The collection of all bounded subsets of a topological module M over a topological ring is usually denoted by $\mathcal{B}_0(M)$. We will show that, in the next two technical lemmas, $\mathcal{B}_0(M)$ is a bornology of $\mathcal{P}(M)$.

Lemma 132 *Let R be a topological ring. Let M, N be topological R-modules. Let $T : M \to N$ be linear. Let $B \subseteq M$ be bounded. Let $A \subseteq R$ be left bounded. Then:*

1. *If T is continuous, then $T(B)$ is bounded in N.*
2. *If there exists a 0-neighborhood $V \subseteq M$ such that $T(V)$ is bounded, then T is continuous.*
3. *If $r \in \mathcal{U}(R)$, then rB is bounded.*
4. *$\mathrm{cl}(B)$ is bounded.*
5. *If $C \subseteq B$, then C is bounded.*
6. *$\mathscr{O}_M$ is bounded.*
7. *M is bounded if and only if the module topology of M is trivial.*
8. *If M is discrete, then $\{0\}$ is the only bounded set.*
9. *AB is bounded.*

Proof 340 *We will only prove the first five items since the rest are trivial.*

1. *Let $U \in \mathcal{N}_0(N)$. Then $T^{-1}(U) \in \mathcal{N}_0(M)$. Since $B \subseteq M$ is bounded, there exists $u \in \mathcal{U}(R)$ with $B \subseteq uT^{-1}(U)$. Thus, $T(B) \subseteq uU$. This implies that $T(B)$ is bounded.*
2. *Fix any arbitrary 0-neighborhood $W \subseteq N$. By hypothesis, there exists an invertible $u \in \mathcal{U}(R)$ in such a way that $T(V) \subseteq uW$, meaning that $u^{-1}V \subseteq T^{-1}(W)$, so $T^{-1}(W)$ is a neighborhood of 0 in M.*
3. *For every $U \in \mathcal{N}_0(M)$, there exists $u \in \mathcal{U}(R)$ with $B \subseteq uU$, so $rB \subseteq (ru)U$ with $ru \in \mathcal{U}(R)$.*

4. *Fix an arbitrary $U \in \mathcal{N}_0(M)$. Take $V \in \mathcal{N}_0(M)$ additively symmetric such that $V + V \subseteq U$. There exists $u \in \mathcal{U}(R)$ such that $B \subseteq uV$. We will prove that $\mathrm{cl}(B) \subseteq uU$. Indeed, take any $x \in \mathrm{cl}(B)$. Since $(x + uV) \cap B \neq \varnothing$, there exist $b \in B, v_1 \in V$ with $x + uv_1 = b$. Since $B \subseteq uV$, we can find $v_2 \in V$ with $b = uv_2$. Finally,*

$$x = b - uv_1 = u(v_2 - v_1) \in u(V - V) = u(V + V) \subseteq uU.$$

5. *Fix any arbitrary 0-neighborhood $V \subseteq M$. There are 0-neighborhoods $W_0 \subseteq R$ and $V_0 \subseteq M$ such that $W_0V_0 \subseteq V$. Since B is bounded in M, an invertible $u_0 \in \mathcal{U}(R)$ can be found so that $B \subseteq u_0V_0$. Since A is left-bounded in R, an invertible $u_1 \in \mathcal{U}(R)$ can be found so that $A \subseteq u_1\left(W_0u_0^{-1}\right)$. Finally, $AB \subseteq u_1\left(W_0u_0^{-1}\right)u_0V_0 \subseteq u_1W_0V_0 \subseteq u_1V$.*

The hypothesis that the ring be practical is crucial in the second technical lemma.

Lemma 133 *Let R be a practical topological ring. Let M be a topological R-module. Let $A, B \subseteq M$ be bounded. Let $K \subseteq M$ be compact. Let $\{m_1, \ldots, m_k\} \subseteq M$. Then:*

1. *$\{m_1, \ldots, m_k\}$ is bounded.*
2. *$A + B$ is bounded.*
3. *$A \cup B$ is bounded. In particular, $\mathrm{sym}(A) = A \cup -A$ is bounded.*
4. *K is bounded.*

Proof 341 *The proof follows as itemized as the statement of the lemma.*

1. *Fix an arbitrary 0-neighborhood $U \subseteq M$. For every $i \in \{1, \ldots, k\}$, there exists a 0-neighborhood $V_i \subseteq R$ such that $V_im_i \subseteq U$. Take $V := V_1 \cap \cdots \cap V_k$. Since R is practical, there exists an invertible $v \in V \cap \mathcal{U}(R)$. Then $vm_i \in V_im_i \subseteq U$ for every $i \in \{1, \ldots, k\}$, meaning that $m_i \in v^{-1}U$ for every $i \in \{1, \ldots, k\}$.*
2. *Fix an arbitrary 0-neighborhood U in M. Take V a 0-neighborhood in M with $V + V \subseteq U$. There are neighborhoods W_1, W_2 of 0 in R and M, respectively, such that $W_1W_2 \subseteq V$. Since A, B are bounded, we can find invertibles $r, s \in R$ with $A \subseteq rW_2$ and $B \subseteq sW_2$. Next, there exist neighborhoods E, F of 0 in R such that $Er \subseteq W_1$ and $Fs \subseteq W_1$. Since R is practical, there exists $t \in \mathcal{U}(R)$ with $t \in E \cap F$. Then $tr \in Er \subseteq W_1$ and $ts \in Fs \subseteq W_1$. This means that*

$$t(rW_2 + sW_2) = (tr)W_2 + (ts)W_2 \subseteq W_1W_2 + W_1W_2 \subseteq V + V \subseteq U,$$

in other words,

$$A + B \subseteq rW_2 + sW_2 \subseteq t^{-1}U.$$

This proves that $A + B$ is bounded.

3. *Observe that $A \cup \{0\}$ and $B \cup \{0\}$ are trivially bounded. Next, $A \cup B \subseteq (A \cup \{0\}) + (B \cup \{0\})$ and $(A \cup \{0\}) + (B \cup \{0\})$ is bounded.*
4. *Fix an arbitrary 0-neighborhood U of M. There exists another 0-neighborhood V of M such that $V + V \subseteq U$. There can be found a 0-neighborhood W_0 in R and a 0-neighborhood V_0 in M such that $W_0V_0 \subseteq V$. Notice that $\{k + \mathrm{int}(V_0) : k \in K\}$ is an open covering of K, meaning that we can extract a finite subcovering $k_1 + \mathrm{int}(V_0), \ldots, k_l + \mathrm{int}(V_0)$. Then $K \subseteq \{k_1, \ldots, k_l\} + V_0$. For every $i \in \{1, \ldots, l\}$, there exists a 0-neighborhood $W_i \subseteq R$ such that $W_ik_i \subseteq V$. Take $W := W_0 \cap W_1 \cap \cdots \cap W_l$. Since R is practical, there exists an invertible $w \in W \cap \mathcal{U}(R)$. Then $wk_i \in W_ik_i \subseteq V$ for every $i \in \{1, \ldots, l\}$, meaning that $k_i \in w^{-1}V$ for every $i \in \{1, \ldots, l\}$. Next, $K \subseteq \{k_1, \ldots, k_l\} + V_0 \subseteq w^{-1}V + V_0 = w^{-1}(V + wV_0) \subseteq w^{-1}(V + W_0V_0) \subseteq w^{-1}(V + V) \subseteq w^{-1}U$.*

Lemma 133(4) does not hold without the practical hypothesis. Indeed, it only suffices to consider any discrete ring R and a nonzero element $r \in R \setminus \{0\}$. Notice that $\{r\}$ is clearly compact, but it cannot be bounded since $\{0\}$ is a zero-neighborhood in R and no invertible $u \in \mathcal{U}(R)$ satisfies that $\{r\} \subseteq u\{0\}$.

Lemma 134 *Let R be a topological ordered ring such that $\mathcal{U}(R) \subseteq R^+ \cup R^-$. Let M be a topological ordered R-module such that $M^+ \setminus \{0\} \neq \varnothing$. If $A \subseteq M$ is bounded in the module topology, then A is ordered bounded.*

Proof 342 *By hypothesis, there exists $m \in M^+ \setminus \{0\}$. Note that $\uparrow_\times (-m) \cap \downarrow^\times m$ is an open neighborhood of 0 in M for the order topology, therefore there exists an invertible $u \in \mathcal{U}(R)$ such that $uA \subseteq \uparrow_\times (-m) \cap \downarrow^\times m$. Let us assume without any loss of generality that $u > 0$. Then $u^{-1} > 0$, meaning that $A \subseteq \uparrow_\times \left(-u^{-1}m\right) \cap \downarrow^\times u^{-1}m$. As a consequence, A is ordered-bounded.*

Previous lemma relates topological-module boundedness with ordered boundedness. Next results are aimed at relating topological-module boundedness with seminormed boundedness.

Lemma 135 *Let M be a seminormed module over a practical seminormed ring R. Consider $m \in M$ and $t, p > 0$. Then:*

1. *There exists $r \in \mathcal{U}(R)$ such that $\mathsf{B}_M(m,t) \subseteq \mathsf{B}_M(0, \|r\|)$.*
2. *There exists $v \in \mathcal{U}(R)$ such that $\mathsf{B}_M(m,t) \subseteq v\mathsf{B}_M(0,p)$.*

Proof 343 *The proof is itemized likewise the statement of the lemma.*

1. *Since R is practical, there exists $s \in \mathcal{U}(R)$ such that $\|s\| \leq \frac{1}{\|m\|+t}$. According to Lemma 69(3), $\left\|s^{-1}\right\| \geq \|s\|^{-1} \geq \|m\| + t$. In virtue of Lemma 75(2),*

$$\mathsf{B}_M(m,t) \subseteq \mathsf{B}_M(0, \|m\| + t) \subseteq \mathsf{B}_M\left(0, \|s\|^{-1}\right) \subseteq \mathsf{B}_M\left(0, \left\|s^{-1}\right\|\right),$$

   so it only suffices to take $r := s^{-1}$.

2. *Since R is practical, there exists $u \in \mathcal{U}(R)$ such that $\|u\| \leq \frac{p}{\|m\|+t}$. In virtue of Lemma 75(2,3),*

$$\begin{aligned} \mathsf{B}_M(m,t) &\subseteq \mathsf{B}_M(0, \|m\| + t) \subseteq u^{-1}u\mathsf{B}_M\left(0, \|m\| + t\right) \\ &\subseteq u^{-1}\mathsf{B}_M\left(0, \|u\|(\|m\| + t)\right) \subseteq u^{-1}\mathsf{B}_M\left(0, p\right), \end{aligned}$$

   so it only suffices to take $v := u^{-1}$.

The upcoming proposition shows that seminorm boundedness and topological-module boundedness coincide when the underlying ring is practical.

Proposition 92 *Let R be a seminormed ring. Let M be a seminormed R-module. If $A \subseteq M$ is bounded, then A is seminorm bounded. Conversely, if R is practical and $A \subseteq M$ is seminorm bounded, then it is bounded.*

Proof 344 *Assume first that $A \subseteq M$ is bounded. There is $u \in \mathcal{U}(R)$ with $A \subseteq u\mathsf{B}_M \subseteq \mathsf{B}_M(0, \|u\|)$, which means that A is seminorm bounded. Conversely, assume that R is practical and $A \subseteq M$ is seminorm bounded. Fix any arbitrary 0-neighborhood V in M. Take $t > 0$ with $\mathsf{B}_M(0,t) \subseteq V$. Since A is seminorm bounded, there exists $s > 0$ with $A \subseteq \mathsf{B}_M(0,s)$. According to Lemma 135(2), we can find an invertible $v \in R$ with $A \subseteq \mathsf{B}_M(0,s) \subseteq v\mathsf{B}_M(0,t) \subseteq vV$.*

Let R be a practical topological ring. Let M be a nontrivial topological R-module. The set of all bounded subsets of M, $\mathcal{B}_0(M)$, is a bornology of $\mathcal{P}(M)$ in view of Lemma 132 and Lemma 133. The collection

$$\{uV : u \in \mathcal{U}(R), V \in \mathcal{N}_0(M) \cap \mathcal{B}_0(M)\}$$

is clearly a bornology base of $\mathcal{B}_0(M)$ provided that it is nonempty. In fact, if there exists $V \in \mathcal{N}_0(M) \cap \mathcal{B}_0(M)$, then $\{uV : u \in \mathcal{U}(R)\}$ is both a base of the filter $\mathcal{N}_0(M)$ and a base of the bornology $\mathcal{B}_0(M)$.

Definition 252 (Compatible bimodule topology) *Let R, S be topological rings. Let M be a topological (R, S)-bimodule. We say that the bimodule topology of M is compatible if the R-bounded subsets of M coincide with the S-bounded subsets, that is, $\mathcal{B}_0({}_RM) = \mathcal{B}_0(M_S)$.*

The following technical lemma is an auxiliary result which becomes helpful in many situations.

Lemma 136 *Let M be a topological module over a topological ring R. Let $(r_i)_{i\in I} \subseteq R$ and $(m_i)_{i\in I} \subseteq M$ be nets. Consider $r \in R$, $C \subseteq R$, $m, n \in M$, and $D, E \subseteq M$. Then:*

1. *If $(r_i)_{i\in I}$ is convergent to r, $(r_i m_i)_{i\in I}$ is convergent to m, and $(m_i)_{i\in I}$ is bounded, then $(rm_i)_{i\in I}$ converges to m.*
2. *If $(m_i)_{i\in I}$ is convergent to m, $(r_i m_i)_{i\in I}$ is convergent to n, and $(r_i)_{i\in I}$ is right bounded, then $(r_i m)_{i\in I}$ converges to n.*
3. *If D is closed and bounded and E is closed and compact, then $D + E$ is closed.*
4. *If M is Hausdorff, R is a division ring, C is closed and compact, and D is closed and bounded, then CD is closed in M.*

Proof 345 *Let us itemize the proof in accordance with the statement of the lemma.*

1. *It suffices to show that $(rm_i - r_i m_i)_{i\in I}$ converges to 0. Fix an arbitrary 0-neighborhood $U \subseteq M$. There are 0-neighborhoods $V \subseteq R$ and $U_1 \subseteq M$ such that $VU_1 \subseteq U$. By hypothesis, there exists an invertible $u \in \mathcal{U}(R)$ such that $m_i \in uU_1$ for all $i \in I$. There exists $i_0 \in I$ such that $r - r_i \in Vu^{-1}$ for all $i \geq i_0$. Then, for all $i \geq i_0$, $rm_i - r_i m_i = (r - r_i)m_i \in Vu^{-1}uU_1 = VU_1 \subseteq U$.*
2. *It suffices to show that $(r_i m - r_i m_i)_{i\in I}$ converges to 0. Fix an arbitrary 0-neighborhood $U \subseteq M$. There are 0-neighborhoods $V \subseteq R$ and $U_1 \subseteq M$ such that $VU_1 \subseteq U$. By hypothesis, there exists an invertible $w \in \mathcal{U}(R)$ such that $r_i \in Vw$ for all $i \in I$. There exists $i_0 \in I$ such that $m - m_i \in w^{-1}U_1$ for all $i \geq i_0$. Then, for all $i \geq i_0$, $r_i m - r_i m_i = r_i(m - m_i) \in Vww^{-1}U_1 = VU_1 \subseteq U$.*
3. *Take a net $(d_i + e_i)_{i\in I} \subseteq D + E$ converging to some $m \in M$, where $(d_i)_{i\in I} \subseteq D$ and $(e_i)_{i\in I} \subseteq E$. Since E is closed and compact, there exists a subnet $(e_{i_j})_{j\in J}$ convergent to some $e \in E$. Then $\left(d_{i_j} + e_{i_j}\right)_{j\in J}$ converges to m. Therefore, $\left(d_{i_j}\right)_{j\in J}$ converges to $m - e$, so $m - e \in D$ because D is closed, hence $m = (m - e) + e \in D + E$.*
4. *Take a net $(r_i m_i)_{i\in I} \subseteq CD$ converging to some $m \in M$, where $(r_i)_{i\in I} \subseteq C$ and $(m_i)_{i\in I} \subseteq D$. Since C is closed and compact, there exists a subnet $(r_{i_j})_{j\in J}$ convergent to some $r \in C$. Then $\left(r_{i_j} m_{i_j}\right)_{j\in J}$ converges to m. Therefore, $\left(rm_{i_j}\right)_{j\in J}$ converges to m. At this stage, if $r = 0$, then $m = 0 \in CD$. If $r \neq 0$, then $\left(m_{i_j}\right)_{j\in J}$ converges to $r^{-1}m$, so $r^{-1}m \in D$ because D is closed, hence $m = r\left(r^{-1}m\right) \in CD$.*

Next technical lemma also has plenty of applications, some of them towards constructing biorthogonal systems.

Lemma 137 *Let M be a first countable topological module over a practical topological ring. Let $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ be a countable nested base of 0-neighborhoods of M. If $u_n \in U_n$ for each $n \in \mathbb{N}$, then $\{u_n : n \in \mathbb{N}\}$ is bounded.*

Proof 346 *Fix any arbitrary 0-neighborhood V of M. There are $W_0 \subseteq R$ a 0-neighborhood in R and $V_0 \subseteq M$ a 0-neighborhood in M satisfying that $W_0V_0 \subseteq V$. There also exists $W_{00} \subseteq R$ a 0-neighborhood in R with $W_{00}W_{00} \subseteq W_0$. Since $\mathcal{B}$ is nested, there exists $n_0 \in \mathbb{N}$ such that $U_n \subseteq V_0$ for all $n \geq n_0$. For every $n \in \{1, \ldots, n_0 - 1\}$, there exists a 0-neighborhood $W_n \subseteq R$ such that $W_n u_n \subseteq V_0$. Since R is practical, there exists $r \in \mathcal{U}(R) \cap (W_{00} \cap W_0 \cap W_1 \cap \cdots \cap W_{n_0-1})$. Finally, observe that if $n \in \{1, \ldots, n_0 - 1\}$, then $r^2u_n = r(ru_n) \in r(W_nu_n) \subseteq rV_0 \subseteq W_0V_0 \subseteq V$, and if $n \geq n_0$, then $r^2u_n \in W_{00}W_{00}U_n \subseteq W_0U_n \subseteq W_0V_0 \subseteq V$, in other words, $r^2\{u_n : n \in \mathbb{N}\} \subseteq V$, reaching the conclusion that $\{u_n : n \in \mathbb{N}\}$ is bounded because $r^2 \in \mathcal{U}(R)$.*

Lemma 137 does not remain true if $\mathcal{B}$ is not nested. Indeed, consider $\mathcal{B} := \left\{\left[-\frac{1}{n}, \frac{1}{n}\right] : n \in \mathbb{N}\right\} \cup \{[-n, n] : n \in \mathbb{N}\}$ is a countable base of 0-neighborhoods in $\mathbb{R}$ but taking $u_n := n$ for all $n \in \mathbb{N}$ we conclude that $\{u_n : n \in \mathbb{N}\}$ is unbounded.

Theorem 102 *Let M be a topological module over a practical topological ring R. Let W_0 be a right-bounded neighborhood of 0 in R. Let V_0 be a bounded neighborhood of 0 in M. Then:*

1. *If $\mathcal{B}$ is a base of neighborhoods of 0 in M, then $\mathcal{B}_{W_0} := \{W_0V : V \in \mathcal{B}\}$ is a base of neighborhoods of 0 in M.*
2. *If $\mathcal{B}$ is a base of neighborhoods of 0 in R, then $\mathcal{B}_{V_0} := \{WV_0 : W \in \mathcal{B}\}$ is a base of neighborhoods of 0 in M.*

Proof 347 *As usual, the proof will be itemized via the statement.*

1. *On the hand, $\mathcal{B}_{W_0}$ is a filter base of $\mathcal{N}_0(M)$. Indeed, $\mathcal{B}_{W_0}$ is a nonempty subset of $\mathcal{N}_0(M)$ in view of Theorem 94(1), and if $V_1, V_2 \in \mathcal{B}$, there exists $V_3 \in \mathcal{B}$ such that $V_3 \subseteq V_1 \cap V_2$, meaning that $W_0V_3 \subseteq W_0(V_1 \cap V_2) \subseteq W_0V_1 \cap W_0V_2$. On the other hand, let U be a neighborhood of 0 in M. There exist a neighborhood W of 0 in R and a neighborhood V of 0 in M such that $WV \subseteq U$. Since W_0 is right-bounded, there exists $u \in \mathcal{U}(R)$ such that $W_0 \subseteq Wu$. Next, there exists $V_1 \in \mathcal{B}$ such that $V_1 \subseteq u^{-1}V$. Finally, $W_0V_1 \subseteq Wuu^{-1}V = WV \subseteq U$ and $W_0V_1 \in \mathcal{B}_{W_0}$.*
2. *In the first place, following similar reasoning as above, $\mathcal{B}_{V_0}$ is a filter base of $\mathcal{N}_0(M)$. Now, let U be a neighborhood of 0 in M. There exist a neighborhood W of 0 in R and a neighborhood V of 0 in M such that $WV \subseteq U$. Since V_0 is bounded, there exists an invertible $u \in \mathcal{U}(R)$ such that $V_0 \subseteq uV$. By hypothesis, we can find $W_1 \in \mathcal{B}$ with $W_1 \subseteq Wu^{-1}$. As a consequence, $W_1V_0 \subseteq Wu^{-1}uV = WV \subseteq U$ and $W_1V_0 \in \mathcal{B}_{V_0}$.*

An important application of the previous theorem allows a sufficient condition for continuity of linear operators.

Corollary 32 *Let M be a topological module over a topological division ring R. Let $B \subseteq R$ be a total right-bounded unit ball. If $f \in \mathcal{L}(M, R)$ is so that $\ker(f)$ is closed, then f is continuous.*

Proof 348 *Assume to contrary that f is not continuous. There exists $W \in \mathcal{N}_0(R)$ such that $f^{-1}(W) \notin \mathcal{N}_0(M)$. By hypothesis, there exists $r \in \mathcal{U}(R)$ such that $Br \subseteq W$. Fix $m \in M$ such that $f(m) \neq 0$ (notice that such m must exist since constant functions are continuous). We will reach the contradiction that $rf(m)^{-1}m \in \mathrm{cl}(\ker(f)) \setminus \ker(f)$. Fix an arbitrary additively symmetric neighborhood U of 0 in M. There are 0-neighborhoods $W_0 \subseteq R$ and $V_0 \subseteq M$ such that $W_0V_0 \subseteq U$. There can be found $r_0 \in \mathcal{U}(R)$ such that $Br_0 \subseteq W_0$. There exists $n \in r_0V_0 \setminus f^{-1}(W)$. Then $f(n)r^{-1} \notin B$. Since B is total, $rf(n)^{-1} \in B$, meaning that $rf(n)^{-1}n \in B(r_0V_0) = (Br_0)V_0 \subseteq W_0V_0 \subseteq U$. Finally, $rf(m)^{-1}m - rf(n)^{-1}n \in \ker(f) \cap \left(rf(m)^{-1}m + U\right)$.*

There is also a version of the convergence group topology for topological modules. The idea is that, if M is a topological module and I is a nonempty set, then M^I can be endowed with a convergence additive topology due the commutativity character of the module addition. Thus, it is natural to wonder whether a convergence additive topology on M^I is actually a module topology. In general, it is not; however, it will be for certain submodules of M^I.

Theorem 103 *Let I be a nonempty set. Let $\mathcal{G} \subseteq \mathcal{P}(I)$ be upward directed. Let M be a topological R-module for R a topological ring. Let $\mathcal{F} \subseteq M^I$ be a submodule. Consider the filter base*

$$\mathcal{B} := \{\mathcal{U}(G,U) : G \in \mathcal{G}, U \in \mathcal{N}_0(M) \text{ additively symmetric}\}.$$

Then:

1. *If $f(G)$ is bounded in M for all $f \in \mathcal{F}$ and all $G \in \mathcal{G}$, then $\mathcal{B}$ is a base of 0-neighborhoods for a module topology on $\mathcal{F}$.*
2. *If R is practical and $\mathcal{B}$ is a base of 0-neighborhoods for a module topology on $\mathcal{F}$, then $f(G)$ is bounded in M for all $f \in \mathcal{F}$ and all $G \in \mathcal{G}$.*

Proof 349 *The proof follows the itemization of the statement.*

1. *Suppose first that $f(G)$ is bounded in M for all $f \in \mathcal{F}$ and all $G \in \mathcal{G}$. We will rely on Theorem 91. We have to verify the filter base $\mathcal{B}$ satisfies all five conditions of Theorem 91:*
   (a) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find a 0-neighborhood $V \subseteq M$ such that $V + V \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $\mathcal{U}(G,V) + \mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*
   (b) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find a 0-neighborhood $V \subseteq M$ such that $-V \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $-\mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*
   (c) *Take $\mathcal{U}(G,U) \in \mathcal{B}$. We can find 0-neighborhoods $V \subseteq M$ and $W \subseteq R$ such that $WV \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $W\mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*
   (d) *Take $\mathcal{U}(G,U) \in \mathcal{B}$ and $r \in R$. We can find a 0-neighborhood $V \subseteq M$ such that $rV \subseteq U$. Then $\mathcal{U}(G,V) \in \mathcal{B}$ and $r\mathcal{U}(G,V) \subseteq \mathcal{U}(G,U)$.*
   (e) *Take $\mathcal{U}(G,U) \in \mathcal{B}$ and $f \in \mathcal{F}$. We can find 0-neighborhoods $V \subseteq M$ and $W \subseteq R$ such that $WV \subseteq U$. Since $f(G)$ is bounded in M, there exists an invertible $u \in \mathcal{U}(R)$ such that $f(G) \subseteq uV$. Then Wu^{-1} is a 0-neighborhood in R and $\left(Wu^{-1}\right) f \subseteq \mathcal{U}(G,U)$.*
2. *Conversely, suppose that R is practical and $\mathcal{B}$ is a base of 0-neighborhoods for a module topology on $\mathcal{F}$. Fix arbitrary elements $f \in \mathcal{F}$ and $G \in \mathcal{G}$. Take any additively symmetric 0-neighborhood $U \subseteq M$. In view of Theorem 91(5), there exists a 0-neighborhood $W \subseteq R$ such that $Wf \subseteq \mathcal{U}(G,U)$. Since R is practical, there exists $u \in \mathcal{U}(R)$ such that $u \in W$. Finally, $uf \in \mathcal{U}(G,U)$, that is, $(uf)(G) \subseteq U$, hence $f(G) \subseteq u^{-1}U$. This shows that $f(G)$ is bounded in M.*

3.2.4.4 Function spaces: operators and duality

If M is a topological module over a topological ring R and Γ is a nonempty set, then M^Γ is an R-module. The product topology (or pointwise convergence topology) is a module topology on it.

Definition 253 (Function space) *Let R be a topological ring. Let M be a topological R-module. Let Γ be a nonempty set. A function space is a submodule $\mathcal{F} \subseteq M^\Gamma$ endowed with a module topology finer than the pointwise convergence linear topology.*

A function-space topology is simply a topology that makes a submodule of functions become a function space. We will see several examples of function spaces, such as spaces of bounded functions.

Example 57 *Let M be a topological module over a topological ring R. Let Γ be a nonempty set. Let $\mathcal{F} \subseteq M^\Gamma$ be a submodule. Consider $\mathcal{P}_\mathtt{f}(\Gamma) \subseteq \mathcal{G} \subseteq \mathcal{P}(\Gamma)$ upward directed. If $f(G)$ is bounded in M for every $G \in \mathcal{G}$ and every $f \in \mathcal{F}$, then, by Theorem 103, the $\mathcal{G}$-convergence topology on $\mathcal{F}$ is a module topology, which is finer than the pointwise convergence linear topology because $\mathcal{P}_\mathtt{f}(\Gamma) \subseteq \mathcal{G}$. As a consequence, $\mathcal{F}$ is a function space.*

Let R be a topological ring. Consider topological R-modules M, N. The set of continuous R-linear operators from M to N is denoted by $\mathcal{CL}(M,N)$. In case $M = N$, then we will simply denote $\mathcal{CL}(M) := \mathcal{CL}(M,M)$. We know that $\mathcal{L}(M,N)$ acquires structure of commutative additive group (in fact, it is a $\mathsf{Z}(R)$-module). It is not hard to check that $\mathcal{CL}(M,N)$ is a subgroup of $\mathcal{L}(M,N)$. If, in addition, N is a topological (R,S)-bimodule, for S another topological ring, then $\mathcal{L}(M,N)$ acquires structure of right S-module, hence $\mathcal{CL}(M,N)$ becomes an S-submodule of the right S-module $\mathcal{L}(M,N)$. In this situation, the pointwise convergence linear topology on $\mathcal{CL}(M,N)$ is an S-module topology.

Definition 254 (Operator topology) *Let R, S be topological rings. Let M be a topological R-module. Let N be a topological (R,S)-bimodule. An operator topology on $\mathcal{CL}(M,N)$ is a function-space topology on $\mathcal{CL}(M,N)$.*

Given a nontrivial topological module M over a practical topological ring R, since $\mathcal{B}_\mathsf{0}(M)$ is a bornology of $\mathcal{P}(M)$, it generates a convergence linear topology.

Theorem 104 *Let R be a practical topological ring. Let M be a nontrivial topological R-module. Let N be a nontrivial topological (R,S)-bimodule. If $\mathcal{B}_\mathsf{0}({}_RN) \subseteq \mathcal{B}_\mathsf{0}(N_S)$, then the $\mathcal{B}_\mathsf{0}(M)$-convergence linear topology on $\mathcal{CL}(M,N)$ is an operator topology.*

Proof 350 *All we need to check is that the $\mathcal{B}_\mathsf{0}(M)$-convergence linear topology on $\mathcal{CL}(M,N)$ is an S-module topology finer than the pointwise convergence linear topology. First off, since R is practical, $\mathcal{P}_\mathtt{f}(M) \subseteq \mathcal{B}_\mathsf{0}(M)$, meaning that $\mathcal{B}_\mathsf{0}(M)$-convergence linear topology is finer than the pointwise convergence linear topology. Therefore, all is left to prove is that the $\mathcal{B}_\mathsf{0}(M)$-convergence linear topology on $\mathcal{CL}(M,N)$ is an S-module topology. For this, it only suffices to show that $T(G)$ is S-bounded in N for every $G \in \mathcal{B}_\mathsf{0}(M)$ and every $T \in \mathcal{CL}(M,N)$, and apply Theorem 103. Indeed, let $G \in \mathcal{B}_\mathsf{0}(M)$ and $T \in \mathcal{CL}(M,N)$. By Lemma 132, $T(G)$ is R-bounded in N. Finally, by hypothesis, $T(G)$ is S-bounded in N.*

Keep in mind that topological modules over commutative topological rings automatically become topological bimodules.

Corollary 33 *Let R be a commutative practical topological ring. Let M, N be nontrivial topological R-modules. Then $\mathcal{B}_\mathsf{0}(M)$ induces a convergence linear topology on $\mathcal{CL}(M,N)$ that is an operator topology.*

Now, it is the turn for the definition of spaces of bounded functions. It is actually natural to define this kind of spaces by relying on bornologies.

Definition 255 (ℓ_∞-space) *If Γ, X are nonempty sets such that X is endowed with a bornology $\mathcal{B} \subseteq \mathcal{P}(X)$, then the ℓ_∞-space is defined as*

$$\ell_\infty(\Gamma, X) := \left\{ f \in X^\Gamma : f(\Gamma) \in \mathcal{B} \right\}.$$

In particular, if R is a practical topological ring and M is a nontrivial topological R-module, then $\mathcal{B}_0(M)$ is a bornology of $\mathcal{P}(M)$, thus

$$\ell_\infty(\Gamma, M) := \left\{ f \in M^\Gamma : f(\Gamma) \text{ is bounded in } M \right\}.$$

Under the settings of the previous definition, a direct consequence of Lemma 133 is the fact that $\ell_\infty(\Gamma, M)$ is a commutative additive group; and if, in addition, R is a division ring, then $\ell_\infty(\Gamma, M)$ is an R-module by Lemma 132.

Theorem 105 *Let R be a practical topological ring. Let M be a nontrivial topological R-module. If Γ is a nonempty set, then $\ell_\infty(\Gamma, M)$ acquires structure of commutative additive group with the regular addition of M-valued functions. If, in addition, R is a division ring, then $\ell_\infty(\Gamma, M)$ acquires structure of R-module.*

Proof 351 *If $f, g \in \ell_\infty(\Gamma, M)$, then $(f+g)(\Gamma) \subseteq f(\Gamma) + g(\Gamma)$ and $f(\Gamma) + g(\Gamma)$ is bounded in virtue of Lemma 133. Finally, if R is a division ring, according to Lemma 132, $rf \in \ell_\infty(\Gamma, M)$ for all $r \in R$ and all $f \in \ell_\infty(\Gamma, M)$.*

The following result in this subsubsection relates continuous linear operators with bounded functions.

Proposition 93 *Let R be a practical topological ring. Let M, N be a nontrivial topological R-modules. Consider a bounded subset $B \subseteq M$. The additive map*

$$\begin{array}{ccc} \mathcal{CL}(M,N) & \to & \ell_\infty(B,N) \\ T & \mapsto & T|_B \end{array} \tag{3.7}$$

is continuous when $\mathcal{CL}(M,N)$ is endowed with the convergence additive topology induced by $\mathcal{B}_0(M)$ and $\ell_\infty(B,N)$ is endowed with the uniform convergence additive topology (the one induced by $\mathcal{P}(B)$).

Proof 352 *In accordance with Lemma 132,* (3.7) *is well defined. For every $C \subseteq B$ and each $V \in \mathcal{N}_0(N)$, $\{f \in \ell_\infty(B,N) : f(C) \subseteq V\}$ is a 0-neighborhood in $\ell_\infty(B,N)$ containing the image under* (3.7) *of the 0-neighborhood of $\mathcal{CL}(M,N)$ given by $\{T \in \mathcal{CL}(M,N) : T(C) \subseteq V\}$.*

Let M be a seminormed module over a seminormed ring R. If Γ is nonempty set, then it makes sense to consider

$$\ell_\infty(\Gamma, M) := \left\{ f \in M^\Gamma : f(\Gamma) \text{ is seminorm-bounded in } M \right\}$$

because $\mathcal{B}_{\mathfrak{d}}(M)$ is a bornology of $\mathcal{P}(M)$ (provided that M is seminorm-unbounded). Observe that if R is practical and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{U}(R)$ converging to 0, then $\|u_n\|^{-1} \leq \left\|u_n^{-1}\right\|$ for each $n \in \mathbb{N}$, meaning that M is seminorm-unbounded because $\left(\left\|u_n^{-1}\right\|\right)_{n\in\mathbb{N}}$ diverges to ∞.

Proposition 94 *Let M be a seminormed module over a seminormed ring R. If M is seminorm-unbounded, then $\ell_\infty(\Gamma, M)$ acquires structure of seminormed R-module, where $\|f\|_\infty := \sup\{\|f(\tau)\| : \tau \in \Gamma\}$. Even more, the seminorm topology on $\ell_\infty(\Gamma, M)$ coincides with the uniform convergence linear topology.*

Proof 353 *If $f, g \in \ell_\infty(\Gamma, M)$ and $r, s \in R$, then $\|rf(\tau) + sg(\tau)\| \leq \|rf(\tau)\| + \|sg(\tau)\| \leq \|r\|\|f\|_\infty + \|s\|\|g\|_\infty$ for all $\tau \in \Gamma$, which implies that $\|rf + sg\|_\infty \leq \|r\|\|f\|_\infty + \|s\|\|g\|_\infty$. This shows that $\ell_\infty(\Gamma, M)$ acquires structure of seminormed R-module. We will prove next that the uniform convergence linear topology coincides with the seminorm topology on $\ell_\infty(\Gamma, M)$ induced by the sup norm $\| \ \|_\infty$. Indeed, on the one hand, for every $\delta > 0$,*

$$\mathsf{B}_{\ell_\infty(\Gamma,M)}(0, \delta) = \mathcal{U}\left(\Gamma, \mathsf{B}_M(0, \delta)\right),$$

and, on the other hand, for every $G \in \mathcal{P}(\Gamma)$ and every $U \in \mathcal{N}_0(M)$,

$$\mathsf{B}_{\ell_\infty(\Gamma,M)}(0, \rho) \subseteq \mathcal{U}\left(G, \mathsf{B}_M(0, \rho)\right) \subseteq \mathcal{U}(G, U),$$

where $\rho > 0$ is so that $\mathsf{B}_M(0, \rho) \subseteq U$. This shows that the two topologies coincide.

The dual of a topological module M over a topological ring R is defined as $M^* := \mathcal{CL}(M, R)$. Note that M^* is a right R-module. The pointwise convergence linear topology on M^* is called w^*-topology and is usually denoted by $\sigma\left(M^*, \widehat{M}\right)$, because the w^*-topology coincides with the initial topology on M^* given by the family of functions $\widehat{M} := \{\widehat{m} : m \in M\}$, where

$$\begin{array}{rccl} \widehat{m}: & M^* & \to & R \\ & m^* & \mapsto & \widehat{m}(m^*) := m^*(m). \end{array}$$

Notice also that the w^*-topology is the topology inherited by M^* from the product topology of R^M.

Definition 256 (Dual topology) *Let M be a topological module over a topological ring R. A dual topology on M^* is by definition an operator topology on $\mathcal{CL}(M, R)$.*

Observe that a dual topology is simply a module topology on the dual module finer than the w^*-topology.

Remark 77 (w^*-Topology) *Let M be a topological module over a topological ring R. A base of w^*-open neighborhoods of an element $m^* \in M^*$ is given by the sets*

$$\begin{array}{rcl} \mathsf{U}_{M^*}\left(m^*; \{m_i\}_{i=1}^k; U\right) & := & \{n^* \in M^* : \forall i = 1, \ldots, k \ n^*(m_i) - m^*(m_i) \in U\} \\ & = & m^* + \mathcal{U}(\{m_1, \ldots, m_k\}, U) \end{array}$$

where $\{m_i\}_{i=1}^k \subseteq M$ and U is an open neighborhood of 0 in R. The w^*-topology is characterized by the property that a net $(m_d^*)_{d\in D} \subseteq M^*$ is w^*-convergent to an element $m^* \in M^*$ if and only if $(m_d^*(m))_{d\in D}$ is convergent to $m^*(m)$ for all $m \in M$. The w^*-topology is a module topology on M^* since it is the inherited topology from the product topology of R^M. Observe that, if R is Hausdorff, then so is R^M with the product topology and thus so is M^* endowed with any dual topology.

Another module topology can be defined in a topological module over a topological ring, called the w-topology.

Remark 78 (w-Topology) *Let M be a topological module over a topological ring R. The w-topology on M is $\sigma(M, M^*)$, that is, the initial topology on M for M^*, in other words, the coarsest topology on M for which all the elements of M^* are continuous. By definition, the w-topology is contained in the module topology. The w-topology is characterized by the property that a net $(m_d)_{d\in D} \subseteq M$ is w-convergent to an element $m \in M$ if and only if $(m^*(m_d))_{d\in D}$ is convergent to $m^*(m)$ for all $m^* \in M^*$. Suppose now that M^* is endowed with a dual topology and consider M^{**}. By definition, the dual topology is finer than the w^*-topology, therefore, $\widehat{M} \subseteq M^{**}$. As a consequence, for every $m \in M$, $\widehat{m} : M^* \to R$ is continuous when M^* is endowed with the w-topology. Thus, the w^*-topology is coarser than the w-topology on M^*. To summarize, if τ^* stands for a dual topology on M^*, then $w^* \subseteq w \subseteq \tau^*$.*

Under the settings of the previous remark and by assuming that M^* is endowed with a dual topology, we can also consider the following natural linear map from M to M^{**}:

$$\begin{array}{rccccc} \widehat{\ }: & M & \to & M^{**} & & \\ & & & \widehat{m}: & M^* & \to & R \\ & m & \mapsto & & m^* & \mapsto & \widehat{m}(m^*) := m^*(m). \end{array}$$

Lemma 138 *Let R be a topological ring. Let M, N be topological R-modules whose duals M^*, N^* are endowed with dual topologies. Let $T : M \to N$ be a continuous linear operator. Then*

$$\begin{array}{rccc} \widehat{T}: & \widehat{M} & \to & \widehat{N} \\ & \widehat{m} & \mapsto & \widehat{T}(\widehat{m}) := \widehat{T(m)} \end{array}$$

is a well-defined linear operator which is continuous when $\widehat{M}, \widehat{N}$ are endowed with the inherited w^*-topology of M^{**}, N^{**}, respectively.

Proof 354 *The linearity of $\widehat{T}$ is a direct consequence of the linearity of T and $\widehat{\ }: M \to M^{**}$. Let us prove now that $\widehat{T}$ is well defined. Indeed, if $m_1, m_2 \in M$ and $\widehat{m_1} = \widehat{m_2}$, then, for every $n^* \in N^*$,*

$$\begin{aligned} \widehat{T(m_1)}(n^*) &= n^*(T(m_1)) = (n^* \circ T)(m_1) = \widehat{m_1}(n^* \circ T) \\ &= \widehat{m_2}(n^* \circ T) = (n^* \circ T)(m_2) = n^*(T(m_2)) = \widehat{T(m_2)}(n^*), \end{aligned}$$

meaning that $\widehat{T(m_1)} = \widehat{T(m_2)}$. This shows that $\widehat{T}$ is well defined. Finally, let us show the continuity of $\widehat{T}$. Fix an arbitrary net $(\widehat{m_i})_{i\in I} \subseteq \widehat{M}$ w^*-convergent to some $\widehat{m} \in \widehat{M}$. We have to check that $\left(\widehat{T}(\widehat{m_i})\right)_{i\in I} \subseteq \widehat{M}$ w^*-converges to $\widehat{T}(\widehat{m})$. Fix an arbitrary $n^* \in N^*$. Then $\widehat{T}(\widehat{m_i})(n^*) = (n^* \circ T)(m_i) = \widehat{m_i}(n^* \circ T)$ for all $i \in I$. Since $(\widehat{m_i})_{i\in I} \subseteq \widehat{M}$ is w^*-convergent to $\widehat{m} \in \widehat{M}$ and $n^* \circ T \in M^*$, we conclude that $(\widehat{m_i}(n^* \circ T))_{i\in I}$ converges to $\widehat{m}(n^* \circ T) = (n^* \circ T)(m) = \widehat{T(m)}(n^*)$. This proves the result.

Several concepts involved in the following definition make also sense in modules without having to rely on a module topology. However, we know that every module can be turned into a topological module by endowing both the underlying ring and the module with the discrete topology. In this situation, $\mathcal{L}(M, R) = \mathcal{CL}(M, R) = M^*$.

Definition 257 (Annihilator) *Let M be a topological module over a topological ring R. Consider $A \subseteq M$ and $C \subseteq M^*$. Then:*

- *The annihilator of A is defined as $A^\perp := \{m^* \in M^* : A \subseteq \ker_{\mathfrak{t}}(m^*)\}$.*

- *The preannihilator of C is defined as* $C^{\top} := \bigcap_{m^* \in C} \ker_{\mathtt{t}}(m^*)$.

Under the settings of the previous definition, if $m^* \in M^*$, then $\ker_{\mathtt{t}}(m^*) = (m^*)^{-1}(\mathscr{O}_R)$ is a w-closed submodule of M, which does not occur for $\ker(m^*)$ in general, unless, for instance R is Hausdorff (in which case, we obtain the classical algebraical definition of annihilator).

Remark 79 *Let M be a topological module over a topological ring R. Consider $A, B \subseteq M$ and $C, D \subseteq M^*$. Then:*

1. $B \subseteq A \Rightarrow A^{\perp} \subseteq B^{\perp}$ *and* $D \subseteq C \Rightarrow C^{\top} \subseteq D^{\top}$.
2. $A \subseteq \left(A^{\perp}\right)^{\top}$ *and* $C \subseteq \left(C^{\top}\right)^{\perp}$.
3. $A^{\perp} \cap B^{\perp} \subseteq (A+B)^{\perp}$ *and* $A^{\perp} + B^{\perp} \subseteq (A \cap B)^{\perp}$.
4. $C^{\top} \cap D^{\top} \subseteq (C+D)^{\top}$ *and* $C^{\top} + D^{\top} \subseteq (C \cap D)^{\top}$.
5. $0 \in A \cap B \Rightarrow A^{\perp} \cap B^{\perp} = (A+B)^{\perp}$.
6. $0 \in C \cap D \Rightarrow C^{\top} \cap D^{\top} = (C+D)^{\top}$.
7. $A^{\perp}$ *is a w^*-closed submodule of M^* with* $A^{\perp} = \overline{\operatorname{span}}(A)^{\perp} = \overline{\operatorname{span}}^{w}(A)^{\perp}$.
8. $C^{\top}$ *is a w-closed submodule of M satisfying that* $C^{\top} = \overline{\operatorname{span}}^{w^*}(C)^{\top}$.

The notions of annihilator and preannihilator induce that one of torsion for subsets of a topological module.

Definition 258 (Torsion) *Let M be a topological module over a topological ring R. Consider $A \subseteq M$ and $C \subseteq M^*$. Then:*

- *A is said to be a torsion subset of M provided that $A^{\perp} = M^*$. The collection of torsion subsets of M is denoted by* $\operatorname{tor}_{\mathtt{t}}(M)$.
- *A is said to be a strongly torsion subset of M provided that there exists $m \in M \setminus \overline{\operatorname{span}}(A)$ satisfying that $A^{\perp} \subseteq \{m\}^{\perp}$. The collection of strongly torsion subsets of M is denoted by* $\operatorname{tor}_{\mathtt{st}}(M)$.
- *C is said to be a w^*-strongly torsion subset of M^* provided that there exists $m^* \in M^* \setminus \overline{\operatorname{span}}^{w^*}(C)$ satisfying that $C^{\top} \subseteq \{m^*\}^{\top}$. The collection of w^*-strongly torsion subsets of M is denoted by* $\operatorname{tor}_{\mathtt{st}}^{w^*}(M)$.

Under the settings of the previous definition, every subset of $\mathscr{O}_M$ is a torsion subset. This way, M is said to be torsionless if the only torsion subsets of M are the subsets of $\mathscr{O}_M$. Also, M is said to be strongly torsionless provided that M is free of strongly torsion subsets. Finally, M is said to be w^*-strongly torsionless provided that M^* is free of w^*-strongly torsion subsets. As above, if R is Hausdorff, then $\ker_{\mathtt{t}} = \ker$, so, if M is Hausdorff as well, then we turn to the classical (algebraical) notions of torsionless.

Lemma 139 *Let M be a topological module over a topological ring R. If M is strongly torsionless, then M is torsionless and* $\operatorname{cl}(N) = \operatorname{cl}_w(N)$ *for every submodule of M.*

Proof 355 *Take a torsion subset A of M and suppose to the contrary that there exists $a \in A \setminus \mathscr{O}_M$. Since $\mathscr{O}_M$ is a closed submodule of M and M is strongly torsionless, there exists $m^* \in (\mathscr{O}_M)^\perp \setminus \{a\}^\perp$. By assumption, A is a torsion subset, so $m^* \in M^* = A^\perp \subseteq \{a\}^\perp$. This is a contradiction. Next, let N be a submodule of M. Since the w-topology is coarser than the module topology on M, we conclude that $\mathrm{cl}(N) \subseteq \mathrm{cl}_w(N)$. Suppose to the contrary that there exists $m \in \mathrm{cl}_w(N) \setminus \mathrm{cl}(N)$. By hypothesis, we can find $m^* \in N^\perp \setminus \{m\}^\perp$. Take a net $(n_i)_{i\in I} \subseteq N$ w-convergent to m. Then $(m^*(n_i))_{i\in I}$ converges to $m^*(m)$. At this stage, note that $m^*(n_i) \in \mathscr{O}_R$ for all $i \in I$ and $\mathscr{O}_R$ is closed in R, meaning that $m^*(m) \in \mathscr{O}_R$, that is, $m \in \ker_{\mathfrak{t}}(m^*)$, so $m^* \in \{m\}^\perp$, which is a contradiction.*

The following technical lemma requires the hypothesis of Hausdorff for both the module and the underlying ring.

Lemma 140 *Let M be a Hausdorff topological module over a Hausdorff topological ring R. Then:*

1. *$\widehat{\ }$ is injective if and only if M is torsionless.*
2. *If M is torsionless, then M is torsionfree and the w-topology on M is Hausdorff.*
3. *If M is torsionless and R is absolutely valued, then M is strongly torsionfree.*

Proof 356 *The proof is itemized according to the itemization of the statement.*

1. *Simply observe that, for every $m \in M$, $\widehat{m} = 0$ if and only if $\{m\}^\perp = M^*$.*
2. *Let $rm = 0$ with r not a zero divisor. Suppose to the contrary that $m \neq 0$. We will show that $\{m\}^\perp = M^*$, which contradicts that M be torsionless. Fix any arbitrary $m^* \in M^*$. Note that $rm^*(m) = m^*(rm) = 0$. Since r is not a zero divisor, $m^*(m) = 0$. Next, since M is torsionless, M^* separates points of M. According to Theorem 71, $\sigma(M, M^*)$ is Hausdorff.*
3. *Let $(r_i)_{i\in I}$ be a net in R and $m \in M \setminus \{0\}$ such that $(r_i m)_{i\in I}$ converges to 0. Since M is torsionless, there exists $m^* \in M^*$ such that $m^*(m) \neq 0$. Since m^* is continuous, $(r_i m^*(m))_{i\in I}$ converges to 0 in R. Since R is absolutely valued, $|m^*(m)| \neq 0$. Finally,*

$$|r_i| = \frac{|r_i||m^*(m)|}{|m^*(m)|} = \frac{|r_i m^*(m)|}{|m^*(m)|} \to 0.$$

A topological module over a topological ring is said to be topologically finitely spanned if there exists a finite subset whose closed linear span is the whole module.

Theorem 106 *Let M be a strongly torsionless Hausdorff topological module over a discrete topological ring R. The following conditions are equivalent:*

- *M is topologically finitely spanned.*
- *The w^*-topology on M^* is discrete.*

Proof 357 *First off, R is discrete, therefore, a base of w^*-open neighborhoods of an element $m^* \in M^*$ consists of sets of the form $\mathsf{U}_{M^*}(m^*; m_1, \ldots, m_k) = \{n^* \in M^* : \forall i \in \{1, \ldots, k\}\ n^*(m_i) = m^*(m_i)\} = m^* + \mathcal{U}(\{m_1, \ldots, m_k\}, \{0\})$.*

⇓ *Assume first that M is topologically finitely spanned, that is, $M = \overline{\mathrm{span}}\{m_1, \ldots, m_k\}$ for some $m_1, \ldots, m_k \in M$. Since R is discrete, by the observation above and by taking into consideration that M is Hausdorff, we have that $\mathsf{U}_{M^*}(0; m_1, \ldots, m_k) = \{0\}$. Since $\mathsf{U}_{M^*}(0; m_1, \ldots, m_k)$ is w^*-open, we deduce that the w^*-topology on M^* is discrete.*

⇑ *Conversely, assume now that the w^*-topology on M^* is discrete. Since $\{0\}$ is w^*-open, there exist $m_1, \ldots, m_k \in M$ such that $\mathsf{U}_{M^*}(0; m_1, \ldots, m_k) = \{0\}$. Assume that there exists $m \in M \setminus \overline{\text{span}}\{m_1, \ldots, m_k\}$. Since M is strongly torsionless by hypothesis, there exists $p^* \in M^*$ such that $p^*(\overline{\text{span}}\{m_1, \ldots, m_k\}) = \{0\}$ and $p^*(m) \neq 0$. This implies that $p^* \in \mathsf{U}_{M^*}(n^*; m_1, \ldots, m_k) = \{0\}$, which contradicts that $p^*(m) \neq 0$.*

Note that the hypothesis of being strongly torsionless in the previous theorem has only been used in one implication.

Proposition 95 *Let M be a topological module over a topological ring R. Then:*

1. *M is strongly torsionless if and only if $\overline{\text{span}}(P) = \overline{\text{span}}^w(P) = \left(P^\perp\right)^\top$ for every $P \subseteq M$.*
2. *M is w^*-strongly torsionless if and only if $\overline{\text{span}}^{w^*}(Q) = \left(Q^\top\right)^\perp$ for every $Q \subseteq M^*$.*

Proof 358 *In view of Remark 79, it always occurs that $\overline{\text{span}}(P) \subseteq \overline{\text{span}}^w(P) \subseteq \left(P^\perp\right)^\top$ and $\overline{\text{span}}^{w^*}(Q) \subseteq \left(Q^\top\right)^\perp$ for every $P \subseteq M$ and $Q \subseteq M^*$.*

1. *Assume first that M is strongly torsionless. Suppose to the contrary that there exists $m \in \left(P^\perp\right)^\top \setminus \overline{\text{span}}(P)$. By hypothesis, there exists $m^* \in M^*$ such that $m^* \in \overline{\text{span}}(P)^\perp \setminus \{m\}^\perp$. Then $m^* \in P^\perp$ and, since $m \in \left(P^\perp\right)^\top$, we have that $m^*(m) \in \mathscr{O}_R$, that is, $m^* \in \{m\}^\perp$, which is a contradiction. Conversely, assume that $\overline{\text{span}}(P) = \overline{\text{span}}^w(P) = \left(P^\perp\right)^\top$ for every subset P of M. Consider $A \subseteq M$ and $m \in M \setminus \overline{\text{span}}(A)$. By hypothesis, $\overline{\text{span}}(A) = \left(A^\perp\right)^\top$, so there exists $m^* \in A^\perp$ such that $m^*(m) \notin \mathscr{O}_R$, that is, $m^* \in A^\perp \setminus \{m\}^\perp$. This shows that M is strongly torsionless.*
2. *Assume first that M is w^*-strongly torsionless. Suppose to the contrary that there exists $m^* \in \left(Q^\top\right)^\perp \setminus \overline{\text{span}}^{w^*}(Q)$. By hypothesis, there exists $m \in M$ such that $m \in \overline{\text{span}}^{w^*}(Q)^\top \setminus \{m^*\}^\top$. Then $m \in Q^\top$ and, since $m^* \in \left(Q^\top\right)^\perp$, we conclude that $m^*(m) \in \mathscr{O}_R$, that is, $m \in \{m^*\}^\top$, which is a contradiction. Conversely, assume that $\overline{\text{span}}^{w^*}(Q) = \left(Q^\top\right)^\perp$ for every subset Q of M^*. Consider $C \subseteq M^*$ and $m^* \in M^* \setminus \overline{\text{span}}^{w*}(C)$. By hypothesis, $\overline{\text{span}}^{w*}(C) = \left(C^\top\right)^\perp$, so there exists $m \in C^\top$ such that $m^*(m) \notin \mathscr{O}_R$, that is, $m \in C^\top \setminus \{m^*\}^\top$. This shows that M is w^*-strongly torsionless.*

Recall that the principal submodules of a module are those submodules spanned by one element. The following lemma establishes sufficient conditions for the principal submodules to be closed.

Lemma 141 *Let M be a topological module over a topological ring R. Let $m_0 \in M$ such that there exists $m_0^* \in M^*$ with $m_0^*(m_0) \in \mathcal{U}(R)$. Then $m - m_0^*(m)m_0^*(m_0)^{-1}m_0 \in M_{(0)}$ for all $m \in \text{cl}(Rm_0)$*

Proof 359 *Fix an arbitrary $U_0 \in \mathcal{N}_0(M)$. Take an additively symmetric $U_1 \in \mathcal{N}_0(M)$ with $U_1 + U_1 \subseteq U_0$. Next, there exists $W \in \mathcal{N}_0(R)$ such that $Wm_0^*(m_0)^{-1}m_0 \subseteq U_1$. Now, take an additively symmetric $V_0 \in \mathcal{N}_0(M)$ such that $m^*(V_0) \subseteq W$. Finally, take $V := V_0 \cap U_1$, which is an additively symmetric neighborhood of 0 in M. If $m \in \text{cl}(Rm_0)$, then there exists $v \in V_0$ with $m + v \in Rm_0$, that is, there exists $r \in R$ with $m + v = rm_0$. Then $m_0^*(m) + m_0^*(v) = rm_0^*(m_0)$, meaning that, $r = m_0^*(m)m_0^*(m_0)^{-1} + m_0^*(v)m_0^*(m_0)^{-1}$. Thus, $m + v = rm_0 = m_0^*(m)m_0^*(m_0)^{-1}m_0 + m_0^*(v)m_0^*(m_0)^{-1}m_0$. Therefore, $m - m_0^*(m)m_0^*(m_0)^{-1}m_0 = -v + m_0^*(v)m_0^*(m_0)^{-1}m_0 \in V_0 + m_0^*(V_0)m_0^*(m_0)^{-1}m_0 \subseteq U_1 + Wm_0^*(m_0)^{-1}m_0 \subseteq U_1 + U_1 \subseteq U_0$.*

Observe that, under the settings of Lemma 141, if M is Hausdorff, then Rm_0 is closed.

Theorem 107 *Let M be a Hausdorff topological module over a Hausdorff topological division ring R. Then:*

1. *If M is torsionless, then the principal submodules of M are closed.*
2. *If M is strongly torsionless, then the finitely spanned submodules of M are closed.*

Proof 360 *We will proceed by induction on the number of spanning generators.*

1. *Fix an arbitrary $m \in M$ and consider Rm. If $Rm = \{0\}$, then it is closed because M is Hausdorff. Assume that $Rm \neq \{0\}$. Since M is torsionless, there exists $m^* \in M^*$ such that $m^*(Rm) \neq \{0\}$. Now, if $(r_i m)_{i\in I}$ is a net of Rm converging to some $n \in M$, then $(r_i m^*(m))_{i\in I}$ converges to $m^*(n)$. Note that $m^*(m) \neq 0$, hence $(r_i)_{i\in I}$ converges to $m^*(n)m^*(m)^{-1}$. As a consequence, $(r_i m)_{i\in I}$ converges to $m^*(n)m^*(m)^{-1}m$. Since R is Hausdorff, we conclude that $n = m^*(n)m^*(m)^{-1}m$.*
2. *According to Lemma 140(2), M is torsionless, so the principal submodules of M are closed. Let $m_1, \ldots, m_k \in M$ with $k > 1$. We will show that $\operatorname{span}\{m_1, \ldots, m_k\}$ is closed. By induction on k, we assume that $\operatorname{span}\{m_1, \ldots, m_{k-1}\}$ is closed. We also may assume that $m_k \notin \operatorname{span}\{m_1, \ldots, m_{k-1}\}$. Since M is strongly torsionless, there exists $m^* \in M^*$ with $m^*(m_k) \neq 0$ and $m^*(\operatorname{span}\{m_1, \ldots, m_{k-1}\}) = \{0\}$. Next, take a net $(r_{1i}m_1 + \cdots + r_{ki}m_k)_{i\in I}$ of $\operatorname{span}\{m_1, \ldots, m_k\}$ converging to some $n \in M$. The net $(r_{ki}m^*(m_k))_{i\in I}$ converges to $m^*(n)$, hence $(r_{ki})_{i\in I}$ converges to $m^*(n)m^*(m_k)^{-1}$. As a consequence, $(r_{ki}m_k)_{i\in I}$ converges to $m^*(n)m^*(m_k)^{-1}m_k$. Then $(r_{1i}m_1+\cdots+r_{k-1i}m_k)_{i\in I}$ converging to $n-m^*(n)m^*(m_k)^{-1}m_k$, in other words, $n-m^*(n)m^*(m_k)^{-1}m_k \in \operatorname{span}\{m_1, \ldots, m_{k-1}\}$, meaning that $n \in \operatorname{span}\{m_1, \ldots, m_k\}$. As a consequence, $\operatorname{span}\{m_1, \ldots, m_k\}$ is closed.*

The following proposition describes the dual of a topological ring as a topological module over itself.

Lemma 142 *Let R be a topological ring. Consider $M := R$, that is, R seen as a left R-module. Then $M^* = \{\mathrm{R}_s : s \in R\}$, where, for each $s \in R$ R_s denotes the right-multiplication operator*

$$\begin{array}{rccc} \mathrm{R}_s: & R & \to & R \\ & r & \mapsto & \mathrm{R}_s(r) := rs. \end{array}$$

Proof 361 *Note that $M^* \supseteq \{\mathrm{R}_s : s \in R\}$. Conversely, if $m^* \in M^*$, then $m^*(r) = m^*(r1) = rm^*(1) = \mathrm{R}_{m^*(1)}(r)$ for every $r \in R$, so $m^* = \mathrm{R}_{m^*(1)}$.*

If M is a topological module over a topological ring R and $r_0 \in R$ is fixed, then the left-multiplication operator

$$\begin{array}{rccc} \mathsf{L}_{r_0}: & M & \to & M \\ & m & \mapsto & \mathsf{L}_{r_0}(m) := r_0 m \end{array} \tag{3.8}$$

is $\mathsf{Z}(R)$-linear and continuous. If $r_0 \in \mathsf{Z}(R)$, then (3.8) is R-linear. If $r_0 \in \mathcal{U}(R)$, then (3.8) is bijective.

Lemma 143 *Let R be an integral domain Hausdorff topological ring for which there exists $s_0 \in R \setminus \{0\}$ such that $\operatorname{int}(Rs_0) = \emptyset$. Let M be a strongly torsionfree topological R-module such that there exists $m_0 \in M \setminus \{0\}$ and a continuous linear projection $P : M \to M$ with $P(M) = Rm_0$. Then:*

1. *The map*
$$\begin{array}{rccc} Q: & Rm_0 & \to & R \\ & rm_0 & \mapsto & rs_0 \end{array}$$
is well-defined, continuous and linear.

2. *The continuous linear functional $m^* := Q \circ P$ is not open.*

Proof 362 *The proof is itemized via the statement of the lemma.*

1. *Let us show first that Q is well-defined. Indeed, if $rm_0 = sm_0$, then $(r-s)m_0 = 0$, so the constant net $r-s$ converges to a zero divisor because $m_0 \neq 0$ and M is strongly torsionfree. Since R is an integral domain and Hausdorff, we conclude that $r-s=0$ and hence $r=s$. This shows that Q is well defined. Note that Q is linear. The continuity of Q follows from the fact that M is strongly torsionfree and R is Hausdorff. Indeed, let $(r_i m_0)_{i\in I}$ be a net convergent to some rm_0. Then $((r_i - r)m_0)_{i\in I}$ is convergent to 0. Since $m_0 \neq 0$ and M is strongly torsionfree, $(r_i - r)_{i\in I}$ converges to a zero divisor. However, R is an integral domain, thus $(r_i - r)_{i\in I}$ converges to 0 and hence $(r_i)_{i\in I}$ converges to r, so $(r_i s_0)_{i\in I}$ converges to rs_0. This proves the continuity of Q.*

2. *Since P is linear and continuous by hypothesis, we have that $m^* \in M^*$. It only remains to show that m^* is not open. Notice that $m^*(M) = Q(P(M)) = Q(Rm_0) = Rs_0$, which implies that $\operatorname{int}(m^*(M)) = \operatorname{int}(Rs_0) = \emptyset$. Thus, m^* cannot be open.*

An example of a module satisfying the hypotheses of Lemma 143 follows next.

Example 58 *Let R be an absolutely valued ring. Take $M := R \times R$ endowed with the product topology. Observe that M is strongly torsionfree because R is absolutely valued. If we take $m_0 := (1,0)$, then*
$$\begin{array}{rccc} P: & R\times R & \to & R\times R \\ & (r,s) & \mapsto & (r,0) \end{array}$$
is a continuous linear projection such that $P(R\times R) = R\times\{0\} = R(1,0)$.

An example of a ring satisfying the hypotheses of Lemma 143 is displayed next.

Example 59 *Take $R := \mathbb{Q}[\pi]$ with the usual absolute value and $s_0 := \pi$. Note that π is not invertible in $\mathbb{Q}[\pi]$ because π is transcendental over $\mathbb{Q}$. Also, $\mathbb{Q}[\pi]\pi$ has empty interior in $\mathbb{Q}[\pi]$ since $\mathbb{Q}\backslash\{0\} \subseteq \mathbb{Q}[\pi]\backslash\mathbb{Q}[\pi]\pi$. Indeed, if there are $q_0 \in \mathbb{Q}\backslash\{0\}$ and $q(x) \in \mathbb{Q}[x]$ such that $q_0 = q(\pi)\pi$, then $q_0 - q(\pi)\pi = 0$, so if we consider the polynomial $a(x) := q_0 - q(x)x$, then $a(x) \in \mathbb{Q}[x]$ and $a(\pi) = 0$. Since π is transcendental over $\mathbb{Q}$, it must occur that $a(x) = 0$, that is, $q_0 = q(x)x$, which implies that $q_0 = q(x) = 0$. This contradicts the fact that $q_0 \neq 0$. This shows that $\mathbb{Q}\setminus\{0\} \subseteq \mathbb{Q}[\pi]\setminus\mathbb{Q}[\pi]\pi$ and then $\mathbb{Q}[\pi]\setminus\mathbb{Q}[\pi]\pi$ is dense in $\mathbb{Q}[\pi]$ which means that $\mathbb{Q}[\pi]\pi$ has empty interior in $\mathbb{Q}[\pi]$.*

Next proposition deals with the w^* structure of the canonical injection of a module into its bidual.

Proposition 96 *Let M be a topological module over a Hausdorff topological ring R. Then:*

1. *If M^* is endowed with a dual topology for which it is w^*-strongly torsionless, then $\operatorname{cl}_{w^*}\left(\widehat{M}\right) = M^{**}$.*

2. *If R is a topological division ring and M is w^*-strongly torsionless, then $\mathcal{C}_{w^*}\mathcal{L}(M^*, R) = \widehat{M}$.*

Proof 363 *We will itemize the proof the same way the statement is.*

1. *Assume to the contrary that there exists* $m^{**} \in M^{**} \setminus \mathrm{cl}_{w^*}\left(\widehat{M}\right)$. *Since* M^* *is* w^**-strongly torsionless, there exists* $m^* \in M^*$ *such that* $m^{**}(m^*) \neq 0$ *and* $m^* \in \mathrm{cl}_{w^*}\left(\widehat{M}\right)^\top$. *In particular,* $m^* \in M^\perp = \{0\}$, *which contradicts the fact that* $m^{**}(m^*) \neq 0$.

2. *It is immediate to see, by using nets for instance, that* $\widehat{M} \subseteq \mathcal{C}_{w^*}\mathcal{L}(M^*, R)$. *Assume to the contrary that there exists*

$$f \in \mathcal{C}_{w^*}\mathcal{L}(M^*, R) \setminus \widehat{M}.$$

Note that $f \neq 0$. *By Lemma 27(1), we deduce that* $\ker(f)$ *is a maximal submodule of* M^*. *Since* R *is Hausdorff,* $\ker(f)$ *is* w^**-closed. We will show now that* $\ker(f)^\top = \{0\}$. *If there exists* $m \in \ker(f)^\top \setminus \{0\}$, *then* $\ker(f) \subseteq \ker(\widehat{m})$. *By the maximality of* $\ker(f)$, *we obtain that* $\ker(f) = \ker(\widehat{m})$, *which implies the contradiction that*

$$f = r\widehat{m} = \widehat{rm} \in \widehat{M}$$

in virtue of Lemma 27(2). As a consequence, $\ker(f)^\top = \{0\}$. *Then Proposition 95(2) comes into play to assure that*

$$\ker(f) = \overline{\mathrm{span}}^{w^*}(\ker(f)) = \left(\ker(f)^\top\right)^\perp = \{0\}^\perp = M^*.$$

This contradicts the fact that $f \neq 0$.

The upcoming definition is the one of dual operator, playing a crucial role in Functional Analysis and Quantum Mechanics.

Definition 259 (Dual operator) *Let* $T : M \to N$ *be a continuous linear operator between topological modules* M, N *over a topological ring* R. *The dual operator of* T *is defined as*

$$\begin{array}{rccl} T^* : & N^* & \to & M^* \\ & n^* & \mapsto & T^*(n^*) := n^* \circ T. \end{array}$$

Next proposition unveils the most basic and principal properties satisfied by dual operators, also known as dual mappings.

Proposition 97 *Let* R *be a topological ring. Let* M, N, P *be topological* R*-modules whose duals are endowed with dual topologies. Consider* $T, L \in \mathcal{CL}(M, N)$ *and* $S \in \mathcal{CL}(N, P)$. *Then:*

1. $T^* \in \mathcal{C}_{w^*}\mathcal{L}(N^*, M^*)$.
2. $(I_M)^* = I_{M^*}$, $(T + L)^* = T^* + L^*$, *and* $(S \circ T)^* = T^* \circ S^*$.
3. *If* $T(M)$ *is dense in* N *and* N *is Hausdorff, then* T^* *is injective.*
4. *If* R *is Hausdorff,* T^* *is injective, and* N *is strongly torsionless, then* $T(M)$ *is dense in* N.
5. *If* T *has a left inverse* $S : N \to M$, *then* T^* *is surjective.*
6. *If* T *is an isomorphism, then so is* T^* *and* $(T^*)^{-1} = \left(T^{-1}\right)^*$.
7. $T^{**}|_{\widehat{M}} = \widehat{T}$.
8. T *is* w*-*w *continuous.*

Proof 364 *Let us itemize the proof the same way the statement is.*

1. *The linearity of T^* is left to the reader. Let us show the w^*-w^* continuity. Let $(n_d^*)_{d\in D}$ be a net in N^* that is w^*-convergent to $n^* \in N^*$. Fix an arbitrary $m \in M$. Notice that $T^*(n_d^*)(m) = (n_d^* \circ T)(m) = n_d^*(T(m))$ for all $d \in D$, therefore $(n_d^*(T(m)))_{d\in D}$ converges to $n^*(T(m)) = T^*(n^*)(m)$. This shows that $(T^*(n_d^*))_{d\in D}$ w^*-converges to $T^*(n^*)$.*

2. *First off, it is clear that $(I_M)^* = I_{M^*}$. Next, $(T+L)^*(m^*) = m^* \circ (T+L) = m^* \circ T + m^* \circ L = T^*(m^*) + L^*(m^*)$ for all $m^* \in M^*$, and $(S\circ T)^*(p^*) = p^* \circ (S \circ T) = (p^* \circ S) \circ T = S^*(p^*) \circ T = T^*(S^*(p^*)) = (T^* \circ S^*)(p^*)$ for all $p^* \in P^*$.*

3. *Let $n^* \in N^*$ such that $T^*(n^*) = 0$. Then n^* vanishes on the dense submodule $T(M)$. Since N is Hausdorff and n^* is continuous, we deduce that $n^* = 0$.*

4. *Suppose to the contrary that $T(M)$ is not dense in N. Fix any $n \in N \setminus \mathrm{cl}(T(M))$. Since N is strongly torsionless, there exists $n^* \in \mathrm{cl}(T(M))^\perp \setminus \{n\}^\perp$. Since R is Hausdorff, we obtain the contradiction that $T^*(n^*) = 0$ but $n^* \neq 0$.*

5. *Fix an arbitrary $m^* \in M^*$. Take $n^* := m^* \circ S \in N^*$ because S is linear and continuous. Finally, $T^*(n^*) = n^* \circ T = m^* \circ S \circ T = m^* \circ I_M = m^*$.*

6. *It suffices to observe that $T^* \circ \left(T^{-1}\right)^* = \left(T^{-1} \circ T\right)^* = (I_M)^* = I_{M^*}$ and $\left(T^{-1}\right)^* \circ T^* = \left(T \circ T^{-1}\right) = (I_N)^* = I_{N^*}$.*

7. *Let $m \in M$. All we need to show is that $T^{**}(\widehat{m}) = \widehat{T(m)}$. Fix an arbitrary $n^* \in N^*$. Then $T^{**}(\widehat{m})(n^*) = n^*(T(m)) = \widehat{T(m)}(n^*)$.*

8. *Let $(m_i)_{i\in I}$ be a net in M w-convergent to $m \in M$. Fix an arbitrary $n^* \in M^*$. Notice that $T^*(n^*) \in M^*$, therefore $(T^*(n^*)(m_i))_{i\in I}$ converges to $T^*(n^*)(m)$. Since $T^*(n^*) = n^* \circ T$, we conclude that $(T(m_i))_{i\in I}$ w-converges to $T(m)$.*

Theorem 97 only assures the w^*-continuity of the dual map, hence it motivates the upcoming definition.

Definition 260 (Feasible dual topology) *Let M be a topological module over a topological ring R. A dual topology on M^* is called feasible if, for every $T \in \mathcal{CL}(M)$, T^* is continuous.*

Projections were studied in the context of topological groups (with multiplicative notation). In the context of topological modules, projections are described the same way but using additive notation.

Definition 261 (Topological complement) *Let M be a topological module over a topological ring R. A submodule N of M is said to be complemented if there exists a continuous linear projection $P : M \to M$ such that $P(M) = N$.*

If M, N are topological modules over a topological ring R and $M \times N$ is endowed with the product topology, then

$$\begin{array}{ccc} M \times N & \to & M \times \{0\} \\ (m,n) & \mapsto & (m,0) \end{array} \tag{3.9}$$

is a continuous linear projection. Next theorem is somehow a converse of this fact.

Theorem 108 *Let M be a topological module over a topological ring R. Let $P : M \to M$ be a continuous linear projection. Then:*

1. *M is isomorphic to $P(M) \times \ker(P)$ endowed with the product topology.*
2. *$P^* : M^* \to M^*$ is also a projection.*

Proof 365 *The proof is itemized in accordance with the statement of the theorem.*

1. *We already know that the map*

$$\begin{array}{ccc} M & \to & P(M) \times \ker(P) \\ m & \mapsto & (P(m), m - P(m)) \end{array} \tag{3.10}$$

is a linear isomorphism. So, let us check the continuity of the above map and its inverse. Indeed, if $(m_i)_{i\in I}$ is a net of M converging to $m \in M$, then $(P(m_i))_{i\in I}$ converges to $P(m)$ due to the continuity of P, so $(m_i - P(m_i))_{i\in I}$ also converges to $m-P(m)$, meaning that $((P(m_i), m_i-P(m_i)))_{i\in I}$ converges to $(P(m), m-P(m))$ due to the product topology. Conversely, take a net $(m_i)_{i\in I}$ of M and an element $m \in M$ in such a way that $((P(m_i), m_i - P(m_i)))_{i\in I}$ converges to $(P(m), m - P(m))$ in the product topology. Then $(P(m_i))_{i\in I}$ converges to $P(m)$ and $(m_i - P(m_i))_{i\in I}$ converges to $m - P(m)$. By adding the two nets, we end up concluding that $(m_i)_{i\in I}$ converges to $m \in M$.

2. *If $m^* \in M^*$, then $P^*(P^*(m^*)) = m^* \circ P \circ P = m^* \circ P = P^*(m^*)$.*

3.2.4.5 Absorbance and balancedness

Internal point is a geometrical concept originally introduced in [47, 48] for real or complex topological vector spaces to study the geometry of convex bodies. For generalizations of internal points in the setting of topological vector spaces, we refer the reader to [49, 50, 51, 52]. Here, we will extended it to topological modules.

Definition 262 (Inner point) *Let M be a module over a topological ring R. Let $A \subseteq M$. A point $a \in A$ is said to be an inner point of A provided that for every $m \in \mathrm{aff}(A)$ there exists a neighborhood V of 0 in R such that $a + V(m - a) \subseteq A$. The set of internal points of A is denoted by $\mathrm{inn}(A)$.*

Notice that, under the settings of the previous definition, if $A = \{a\}$ is a singleton, then $\mathrm{inn}(A) = A$. Indeed, $\mathrm{aff}(A) = A$; hence, every $m \in \mathrm{aff}(A)$ satisfies that $a + R(m - a) \subseteq A$ because $m = a$. The following proposition highlights basic properties verified by inner points.

Proposition 98 *Let M, N be modules over a topological ring R. Let $T : M \to N$ be a linear map. Let $A, B \subseteq M$, $C \subseteq N$, $m \in M$, and $r \in R$. Then:*

1. $\mathrm{inn}(m + A) = m + \mathrm{inn}(A)$.
2. $r\mathrm{inn}(A) = \mathrm{inn}(rA)$ *if* $r \in \mathcal{U}(R)$.
3. $T(\mathrm{inn}(A)) \subseteq \mathrm{inn}(T(A))$.
4. $T^{-1}(\mathrm{inn}(C)) \subseteq \mathrm{inn}\left(T^{-1}(C)\right)$.
5. *If $A \subseteq B$ and $\mathrm{aff}(A) = \mathrm{aff}(B)$, then $\mathrm{inn}(A) \subseteq \mathrm{inn}(B)$.*

6. $\text{inn}(A) \cap \text{inn}(B) \subseteq \text{inn}(A \cap B)$.

7. *If* $\text{aff}(A) = \text{aff}(B) = \text{aff}(A \cap B)$, *then* $\text{inn}(A \cap B) = \text{inn}(A) \cap \text{inn}(B)$.

8. *If* R *is practical,* $\text{inn}(A) \neq \varnothing$, *and* $\text{inn}(A) = \text{inn}(\text{inn}(A))$, *then* $\text{aff}(\text{inn}(A)) = \text{aff}(A)$.

Proof 366 *The proof will follow as itemized as the statement of the proposition.*

1. *Take any* $c \in \text{inn}(m + A)$. *We will show first that* $c - m \in \text{inn}(A)$. *Fix an arbitrary* $b \in \text{aff}(A)$. *Note that* $b + m \in m + \text{aff}(A) = \text{aff}(m + A)$ *by Lemma 28(3). There exists a neighborhood* V *of* 0 *in* R *such that* $c + V((b + m) - c) \subseteq m + A$. *Then* $(c - m) + V(b - (c - m)) \subseteq A$. *This shows that* $c - m \in \text{inn}(A)$, *meaning that* $c \in m + \text{inn}(A)$. *As a consequence,* $\text{inn}(m + A) \subseteq m + \text{inn}(A)$. *By reproducing the same argument,* $\text{inn}(A) = \text{inn}(-m + (m + A)) \subseteq -m + \text{inn}(m + A)$, *meaning that* $m + \text{inn}(A) \subseteq \text{inn}(m + A)$.

2. *Fix an arbitrary* $a \in \text{inn}(A)$. *We will show first that* $ra \in \text{inn}(rA)$. *Take any* $m \in \text{aff}(rA)$. *By bearing in mind Lemma 28(4),* $m = rn$ *for some* $n \in \text{aff}(A)$. *There exists* $V \in \mathcal{N}_0(R)$ *such that* $a + V(n - a) \subseteq A$. *Then* $ra + rV(n - a) \subseteq rA$. *Next, there exists* $W \in \mathcal{N}_0(R)$ *such that* $Wr \subseteq V$. *Thus,* $ra + (rW)(m - ra) = ra + r(Wr)(n - a) \subseteq ra + rV(n - a) \subseteq rA$. *Since* $r \in \mathcal{U}(R)$, *we have that* $rW \in \mathcal{N}_0(R)$, *meaning that* $ra \in \text{inn}(rA)$. *As a consequence,* $r\text{inn}(A) \subseteq \text{inn}(rA)$. *By using the same argument,* $r^{-1}\text{inn}(rA) \subseteq \text{inn}(A)$, *hence* $\text{inn}(rA) \subseteq r\text{inn}(A)$.

3. *Take an element* $a \in \text{inn}(A)$. *We will show first that* $T(a) \in \text{inn}(T(A))$. *Take any* $n \in \text{aff}(T(A))$. *By bearing in mind Lemma 28(5),* $n = T(m)$ *for some* $m \in \text{aff}(A)$. *There exists* $V \in \mathcal{N}_0(R)$ *such that* $a + V(m - a) \subseteq A$. *Then* $T(a) + V(n - T(a)) \subseteq T(A)$. *As a consequence,* $T(\text{inn}(A)) \subseteq \text{inn}(T(A))$.

4. *Fix an element* $a \in T^{-1}(\text{inn}(C))$. *We will show first that* $a \in \text{inn}\left(T^{-1}(C)\right)$. *Take any* $m \in \text{aff}\left(T^{-1}(C)\right)$. *By bearing in mind Lemma 28(6),* $T(m) \in \text{aff}(C)$. *There exists* $V \in \mathcal{N}_0(R)$ *such that* $T(a) + V(T(m) - T(a)) \subseteq C$. *Then* $T(a + V(m - a)) \subseteq C$, *meaning that* $a + V(m - a) \subseteq T^{-1}(C)$, *that is,* $a \in \text{inn}\left(T^{-1}(C)\right)$. *As a consequence,* $T^{-1}(\text{inn}(C)) \subseteq \text{inn}\left(T^{-1}(C)\right)$.

5. *For every* $a \in \text{inn}(A)$ *and every* $m \in \text{aff}(B) = \text{aff}(A)$, *there exists* $V \in \mathcal{N}_0(R)$ *such that* $a + V(m - a) \subseteq A \subseteq B$.

6. *Let* $a \in \text{inn}(A) \cap \text{inn}(B)$. *If* $m \in \text{aff}(A \cap B) \subseteq \text{aff}(A) \cap \text{aff}(B)$, *then there are* $V, W \in \mathcal{N}_0(R)$ *with* $a + V(m - a) \subseteq A$ *and* $a + W(m - a) \subseteq B$, *therefore,* $a + (V \cap W)(m - a) \subseteq A \cap B$ *with* $V \cap W \in \mathcal{N}_0(R)$.

7. *It only suffices to apply Proposition 98(5) to conclude that* $\text{inn}(A \cap B) \subseteq \text{inn}(A) \cap \text{inn}(B)$.

8. *First off, since* $\text{inn}(A) \subseteq A$, *we have that* $\text{aff}(\text{inn}(A)) \subseteq \text{aff}(A)$ *because* aff *is a hull operator. Fix an arbitrary* $m \in \text{aff}(A)$. *By hypothesis,* $\text{inn}(A) \neq \varnothing$; *therefore, we can take any* $a \in \text{inn}(A)$. *There exists* $W \in \mathcal{N}_0(R)$ *such that* $a + W(m - a) \subseteq \text{inn}(A)$ *because* $\text{inn}(A) = \text{inn}(\text{inn}(A))$. *Since* R *is practical, there exists an invertible* $w \in W \cap \mathcal{U}(R)$. *Then* $a + w(m - a) = a'$ *for some* $a' \in \text{inn}(A)$. *Finally,* $m = a + w^{-1}(a' - a) \in a + \text{span}(\text{inn}(A) - a) = \text{aff}(\text{inn}(A))$ *in view of Lemma 28(1).*

The following theorem establishes that, under certain conditions, inner points may not be extreme points.

Theorem 109 *Let M be a module over a topological ring R. Assume that M is torsionfree, R is nondiscrete, an integral domain, and* $\mathrm{char}(R) \neq 2$. *If $A \subseteq M$ is not a singleton, then* $\mathrm{inn}(A) \cap \mathrm{ext}_2(A) = \varnothing$.

Proof 367 *Suppose that there exists $a \in \mathrm{inn}(A) \cap \mathrm{ext}_2(A)$. Since A is not a singleton, there exists $m \in \mathrm{aff}(A) \setminus \{a\}$. Since $a \in \mathrm{inn}(A)$, there exists an additively symmetric neighborhood V of 0 in R such that $a + V(m-a) \subseteq A$. Since R is nondiscrete, we can find $v \in V \setminus \{0\}$. Observe that*

$$2a = (a + v(m-a)) + (a - v(m-a)).$$

Since $a \in \mathrm{ext}_2(A)$, we conclude that $a + v(m-a) = a - v(m-a)$, which implies that $(2v)(m-a) = 0$. Since $m - a \neq 0$, M is torsionfree, and R is an integral domain, we have that $2v = 0$. Now, $v \neq 0$, so, we use again that R is an integral domain to obtain $2 = 0$. However, this contradicts the fact that $\mathrm{char}(R) \neq 2$.

Next theorem generalizes Theorem 109.

Theorem 110 *Let M be a module over a topological inversion ring R. Let B^+ be a practical completely full unit segment of R. Let $E \subseteq A \subseteq M$. If E is extremal in A and $E \cap \mathrm{inn}(A) \neq \varnothing$, then $E = A$.*

Proof 368 *We may assume that A is not a singleton. Let $a \in E \cap \mathrm{inn}(A)$. Fix an arbitrary $b \in A \setminus \{a\}$. We will prove that $b \in E$. Since $b \in \mathrm{aff}(A)$, there exists a neighborhood V of 0 in R such that $a + V(b-a) \subseteq A$. According to Lemma 110, there exists $r_0 \in B^+ \setminus \{0,1\}$ such that $v_0 := -(1-r_0)^{-1} r_0 \in V$ and*

$$a = r_0 b + (1 - r_0)\,(a + v_0(b-a)) \in \Delta(b, a + v_0(b-a)).$$

Notice that $\{b, a + v_0(b-a)\} \subseteq A$ and $E \cap \Delta(b, a + v_0(b-a)) \neq \varnothing$. Since E is extremal in A, we conclude that $\{b, a + v_0(b-a)\} \subseteq E$. The arbitrariness of b implies that $A = E$.

As an immediate consequence of Theorem 110, we obtain the following corollary, whose proof we omit for simplicity.

Corollary 34 *Let M be a module over a topological inversion ring R. Let B^+ be a practical completely full unit segment of R. If $A \subseteq M$ is not a singleton, then* $\mathrm{inn}(A) \cap \mathrm{extl}(A) = \varnothing$.

Under the settings of the previous corollary, notice that, if A is a singleton, then, by definition, we clearly have that $A = \mathrm{inn}(A) = \mathrm{ext}_2(A) = \mathrm{extl}(A)$.

Definition 263 (Internal point) *Let M be a module over a topological ring R. Let $A \subseteq M$. A point $a \in A$ is said to be an internal point of A provided that for every $m \in M$ there exists a neighborhood V of 0 in R such that $a + V(m-a) \subseteq A$. The set of internal points of A is denoted by* $\mathrm{inter}(A)$.

The following theorem highlights close relationships between inner and internal points.

Theorem 111 *Let M be a topological module over a topological ring R. If $A \subseteq M$, then* $\mathrm{int}(A) \subseteq \mathrm{inter}(A) \subseteq \mathrm{inn}(A)$. *If, in addition, R is practical and* $\mathrm{inter}(A) \neq \varnothing$, *then* $\mathrm{aff}(A) = M$ *and* $\mathrm{inter}(A) = \mathrm{inn}(A)$.

Proof 369 *It is trivial that* $\mathrm{inter}(A) \subseteq \mathrm{inn}(A)$. *If $a \in \mathrm{int}(A)$, then we can find a neighborhood U of 0 in M such that $a + U \subseteq A$. Now, let $m \in M$. There exists V a neighborhood of 0 in R such that $V(m-a) \subseteq U$. Then $a + V(m-a) \subseteq a + U \subseteq A$. This shows that*

$\operatorname{int}(A) \subseteq \operatorname{inter}(A)$. *Suppose next that* R *is practical and* $\operatorname{inter}(A) \neq \varnothing$. *Fix* $a \in \operatorname{inter}(A)$. *If* $m \in M$, *then there exists a neighborhood* V *of* 0 *in* R *such that* $a + V(m-a) \subseteq A$. *By hypothesis,* R *is practical, therefore we can find* $u \in \mathcal{U}(R)$ *with* $u \in V$. *Then* $a + u(m-a) \in A$ *so take* $b \in A$ *with* $a + u(m-a) = b$. *Note that* $m = a + u^{-1}(b-a) \in a + \operatorname{span}(A-a) = \operatorname{aff}(A)$ *by Lemma 28(1). As a consequence,* $\operatorname{aff}(A) = M$.

Under the settings of Theorem 111, if $A \subseteq M$ is open, then $\operatorname{int}(A) = \operatorname{inter}(A) = \operatorname{inn}(A)$. Theorem 111 motivates the following definition.

Definition 264 (Linearly open set) *Let* M *be a module over a topological ring* R. *A subset* A *of* M *is said to be linearly open provided that* $A = \operatorname{inter}(A)$.

According to Theorem 111, every open subset of a topological module is linearly open. The converse is not true as shown together by Theorem 112 and Corollary 35.

Theorem 112 *Let* R *be a topological ring. Let* M *be a topological* R*-module. For every linear map* $f : M \to R$, $M \setminus \ker_{\mathtt{t}}(f)$ *is linearly open.*

Proof 370 *We will prove that* $M \setminus \ker_{\mathtt{t}}(f) = \operatorname{inter}(M \setminus \ker_{\mathtt{t}}(f))$. *Take any* $m \in M \setminus \ker_{\mathtt{t}}(f)$. *Fix an arbitrary* $n \in M$. *Since* $f(m) \notin \mathscr{O}_R$, *there exists a* 0*-neighborhood* $U \subseteq R$ *such that* $f(m) \notin U$. *There exists another* 0*-neighborhood* W *in* R *such that* $W + W \subseteq U$. *There also exists an additively symmetric* 0*-neighborhood* $V \subseteq R$ *such that* $Vf(n) \subseteq W$. *Let us show that* $m + Vn \subseteq M \setminus \ker_{\mathtt{t}}(f)$. *Suppose on the contrary that there exists* $v \in V$ *such that* $m + vn \in \ker_{\mathtt{t}}(f)$. *Then* $f(m + vn) \in \mathscr{O}_R \subseteq W$, *meaning that, for some* $w \in W$, $f(m) = -vf(n) + w \in Vf(n) + W \subseteq W + W \subseteq U$, *which is a contradiction. As a consequence,* $m + Vn \subseteq M \setminus \ker(f)$, *hence* $m \in \operatorname{inter}(M \setminus \ker(f))$. *Thus,* $M \setminus \ker(f) = \operatorname{inter}(M \setminus \ker(f))$, *meaning that* $M \setminus \ker(f)$ *is linearly open.*

Next corollary shows the existence of nonopen linearly open sets.

Corollary 35 *Let* R *be a Hausdorff topological division ring endowed with a total right-bounded unit ball* B. *Let* M *be a topological* R*-module such that there exists a noncontinuous linear map* $f : M \to R$. *Then* $M \setminus \ker(f)$ *has empty interior in* M *but is linearly open.*

Proof 371 *According to Lemma 27(1),* $\ker(f)$ *is a maximal submodule of* M. *Therefore,* $\operatorname{cl}(\ker(f))$ *is either* $\ker(f)$ *or* M. *If* $\operatorname{cl}(\ker(f)) = \ker(f)$, *then* $\ker(f)$ *is closed in* M, *hence* f *is continuous in view of Corollary 32, which is a contradiction. As a consequence,* $M = \operatorname{cl}(\ker(f))$. *This implies that* $\ker(f)$ *is dense in* M, *meaning that* $M \setminus \ker(f)$ *has empty interior in* M. *Finally, Theorem 112 assures that* $M \setminus \ker(f)$ *is linearly open.*

The notion of absorbing set is extremely related to the one of internal point.

Definition 265 (Absorbing set) *Let* M *be a module over a topological ring* R. *A nonempty subset* A *of* M *is said to be absorbing provided that* $0 \in \operatorname{inter}(A)$.

Trivial examples of absorbing sets are the zero-neighborhoods in any topological module. In fact, it can be shown that a subset of a topological ring is absorbing if and only if it is a zero-neighborhood. Next theorem is a generalization of Theorem 94.

Theorem 113 *Let* R *be a practical topological ring. Let* M, N *be* R*-modules. Then:*

1. *If* $W \subseteq R$ *is a* 0*-neighborhood and* $A \subseteq M$ *is absorbing, then* WA *is absorbing.*
2. *If* $N \subsetneq M$ *is a submodule of* M, *then* N *is free of internal points.*

3. *If $A \subseteq M$ is absorbing and $T, S : M \to N$ are linear maps such that $T|_A = S|_A$, then $T = S$.*

Proof 372 *The proof will follow as itemized as the statement of the lemma.*

1. *Take $w \in \mathcal{U}(R)$ with $w \in W$. Now fix an arbitrary $m \in M$. There exists a 0-neighborhood $V \subseteq R$ with $Vm \subseteq A$. Finally, keep in mind that $(wV)m \subseteq WVm \subseteq WA$ and wV is a neighborhood of 0 in R because w is invertible.*

2. *Assume on the contrary that $\text{inter}(N) \neq \varnothing$ and choose any $n \in \text{inter}(N)$. Consider an arbitrary $m \in M$. A neighborhood $W \subseteq R$ of 0 exists with $n + Wm \subseteq N$. Since R is practical, an invertible $w \in \mathcal{U}(R)$ exists satisfying that $w \in W$. Then $n + wm \in n + Wm \subseteq N$, meaning that $m \in u^{-1}(N - n) = N$. The arbitrariness of $m \in M$ implies that $M \subseteq N$, hence N is not proper.*

3. *Take any arbitrary $m \in M$. Since $0 \in \text{inter}(A)$, there can be found a 0-neighborhood $W \subseteq R$ verifying that $0 + Wm \subseteq A$. The fact that R is practical allows to find an invertible element $u \in \mathcal{U}(R) \cap W$. Then $um \in Wm \subseteq A$. Finally, $T(m) = u^{-1}T(um) = u^{-1}S(um) = S(m)$. The arbitrariness of $m \in M$ assures that $T = S$.*

Our next results are aimed at showing the existence of additively symmetric absorbing sets which are not zero-neighborhoods.

Theorem 114 *Let R be a topological ring. Let $W \subseteq R$ be an additively symmetric 0-neighborhood. Let M be a topological R-module such that there exists a noncontinuous linear map $f : M \to R$. Then $f^{-1}(W)$ is additively symmetric and absorbing. If W is left-bounded, then $f^{-1}(W)$ is not a 0-neighborhood.*

Proof 373 *First off, $f^{-1}(W)$ is additively symmetric because so is W. Let us show now that $f^{-1}(W)$ is absorbing. Indeed, fix an arbitrary $m \in M$. There exists a 0-neighborhood $V \subseteq R$ such that $Vf(m) \subseteq W$. Then $Vm \subseteq f^{-1}(W)$. This shows that $0 \in \text{inter}\left(f^{-1}(W)\right)$, meaning that $f^{-1}(W)$ is absorbing. Finally, let us assume that W is left-bounded. Suppose on the contrary that $f^{-1}(W)$ is a 0-neighborhood. Then $f\left(f^{-1}(W)\right) \subseteq W$ is left-bounded, meaning that f is continuous in virtue of Lemma 132, reaching a contradiction.*

By combining together Theorem 114 and Corollary 27, we obtain the following result.

Corollary 36 *Let R be a Hausdorff practical topological ring with a left-bounded zero-neighborhood. If M is a Klee topological free R-module, then there exists an additively symmetric absorbing subset of M which is not a zero-neighborhood.*

By bearing in mind Proposition 90, we obtain the following corollary as a direct consequence of Corollary 36.

Corollary 37 *Let R be a Hausdorff practical topological ring with a left-bounded zero-neighborhood. If M is a first countable, but not finitely spanned, topological free R-module, then there exists an additively symmetric absorbing subset of M which is not a zero-neighborhood.*

We will illustrate next an example of a topological module satisfying the hypotheses of Corollary 37.

Example 60 *Let $R := \mathbb{Q}[\pi]$ endowed with the inherited topology from $\mathbb{R}$. It is immediate that R is Hausdorff. Also, R is a $\mathbb{Q}$-algebra, so it is clearly practical. Next, $B := [-1,1] \cap R$ is a bounded closed unit 0-neighborhood of R since $[-1,1]$ is a bounded closed unit 0-neighborhood of $\mathbb{R}$ in view of [10, Theorem 3.7] and R is dense in $\mathbb{R}$ (see Proposition 24). Finally, $M := \bigoplus_{n\in\mathbb{N}} R$ is a free R-module, which is not finitely spanned. However, in view of Theorem 33, it is first countable when endowed with the product topology since R is first countable.*

A different way of constructing additively symmetric absorbing sets which are not zero-neighborhoods follows now, this time without relying on the existence of noncontinuous linear functionals. However, we first need the following technical lemma.

Lemma 144 *Let R be a topological ring such that $\mathcal{U}(R)$ is open. Let $W \subseteq R$ be a 0-neighborhood such that $1 \notin \operatorname{int}(W)$. Let M be a topological R-module. For every subset $G \subseteq M$ and every $g \in \operatorname{inter}(G)$, there exists $s \in \mathcal{U}(R) \setminus W$ such that $sg \in G$.*

Proof 374 *There exists a 0-neighborhood $V_1 \subseteq R$ verifying that $1 + V_1 \subseteq \mathcal{U}(R)$. There can be found a 0-neighborhood $V_2 \subseteq R$ satisfying that $g + V_2 g \subseteq G$. Consider $V := V_1 \cap V_2$. Then $1 + V \subseteq \mathcal{U}(R)$ and $g + Vg \subseteq G$. If $1 + V \subseteq W$, then $1 \in \operatorname{int}(W)$, reaching a contradiction with our hypothesis. As a consequence, $1 + V \nsubseteq W$, meaning that there exists $v \in V$ with $1 + v \notin W$. Finally, it only suffices to take $s := 1 + v$ because $sg = g + vg \in g + Vg \subseteq G$.*

Next example reveals that the hypothesis that $1 \notin \operatorname{int}(W)$ is crucial in Lemma 144.

Example 61 *Consider an integral domain R and M a torsionfree R-module. Assume that both R and M are endowed with the discrete topology. Take $W := \{-1, 0, 1\}$. If $m \in M \setminus \{0\}$, then $\{m\}$ is open in M. If $s \in R \setminus W$, then $sm \notin \{m\}$ because M is torsionfree and R is an integral domain.*

Finally, next theorem shows another way to construct additively symmetric absorbing sets which are not zero-neighborhoods.

Theorem 115 *Let R be a topological ring and let M be a topological R-module. For every additively symmetric 0-neighborhood $W \subseteq R$ and every maximal equivalence-free subset $P \subseteq M$, let $A(W, P) := \bigcup_{p\in P} Wp$. The following holds:*

1. *$A(W, P)$ is additively symmetric and absorbing.*
2. *If $\mathcal{U}(R)$ is open, $0 \in \operatorname{cl}(P)$, $1 \notin \operatorname{int}(W)$, and $W \setminus \{0\} \subseteq \mathcal{U}(R)$, then $A(W, P)$ is not a neighborhood of 0.*

Proof 375 *The proof will follow as itemized as the statement of the theorem.*

1. *For every $w \in W$ and every $p \in P$, $-wp = (-w)p \in Wp \subseteq A(W, P)$, hence $A(W, P)$ is additively symmetric. Let us prove next that $A(W, P)$ is absorbing. Indeed, take any $m \in M \setminus \{0\}$. Since P is maximal equivalence free, there exists a unique $p \in P$ and an invertible $u \in \mathcal{U}(R)$ such that $p = um$. There exists a 0-neighborhood $V \subseteq R$ satisfying that $Vu^{-1} \subseteq W$. Then $0 + Vm = 0 + \left(Vu^{-1}\right) p \subseteq 0 + Wp = Wp \subseteq A(W, P)$. This shows that $0 \in \operatorname{inter}(A(W, P))$, meaning that $A(W, P)$ is absorbing.*
2. *Assume on the contrary that $A(W, P)$ is a neighborhood of 0. Since $0 \in \operatorname{cl}(P)$, we can find $p_0 \in P$ with $p_0 \in \operatorname{int}(A(W, P))$. By bearing in mind Lemma 144, we can find $s \in \mathcal{U}(R) \setminus W$ such that $sp_0 \in A(W, P)$. There exist $w \in W$ and $p_1 \in P$ such that $sp_0 = wp_1$. Notice that $p_0, p_1 \in P \cap \mathcal{U}(R)p_0$. By hypothesis, P is maximal equivalence*

free, meaning that $P \cap \mathcal{U}(R)p_0$ *is a singleton. As a consequence,* $p_0 = p_1$*. In other words,* $sp_0 = wp_0$*, or equivalently,* $(s-w)p_0 = 0$*. At this stage, observe that the fact that* $0 \notin P$ *and* $0 \in \mathrm{cl}(P)$ *implies that* R *is not discrete. Then we call on Lemma 122 to conclude that* R *is a division topological ring. Since* $s \notin W$*, we have that* $s - w \neq 0$*, so we obtain the contradiction that* $p_0 = (s-w)^{-1}(s-w)p_0 = 0$*. As a consequence,* $A(W, P)$ *is not a neighborhood of* 0*.*

We will illustrate next an example of a topological module satisfying the hypotheses of Theorem 115.

Example 62 *Let* $R := \mathbb{Q}\left[\sqrt{2}\right]$ *endowed with the inherited topology from* $\mathbb{R}$*. Note that* R *is a division ring since* $\sqrt{2}$ *is algebraic over* $\mathbb{Q}$ *in view of Theorem 19. It is immediate that* R *is Hausdorff, meaning that* $\mathcal{U}(R) = R \setminus \{0\}$ *is open. Also,* R *is a* $\mathbb{Q}$*-algebra, so it is clearly practical. Next,* $W := (-1, 1) \cap R$ *is an additively symmetric open* 0*-neighborhood of* R *satisfying that* $1 \notin W$ *and* $W \setminus \{0\} \subseteq \mathcal{U}(R)$*. Finally,* $M := \bigoplus_{n\in\mathbb{N}} R$ *is a free* R*-module, which is not finitely spanned. However, in view of Theorem 33, it is first countable when endowed with the product topology since* R *is first countable. According to Proposition 90,* M *is Klee. Finally, by applying Theorem 97, there exists a maximal equivalence-free subset* $P \subseteq M$ *such that* $0 \in \mathrm{cl}(P)$*.*

The notion of balancedness is proper from Banach Space Theory. This concept has also been profoundly studied in the scope of topological vector space. In this book, we will transport it to topological modules. We refer the reader to [10, 53, 54, 55, 39, 11] for a more specific perspective.

Definition 266 (Balanced set) *Let* M *be a module over a topological ring* R*. Let* $W \subseteq R$ *be an additively symmetric* 0*-neighborhood such that* $1 \in W$*. A subset* A *of* M *is said to be* W*-balanced provided that* $WA = A$*.*

Under the settings of the previous definition, A is symply called balanced when there exists such W for which A is W-balanced. Special attention will be paid to the B-balanced sets for B a unit ball or a closed unit neighborhood of zero. Notice that, by definition, balanced sets are additively symmetric.

Proposition 99 *Let* M, N *be modules over a topological ring* R*. Let* $T : M \to N$ *be a linear operator. Let* $W \subseteq R$ *be an additively symmetric* 0*-neighborhood such that* $1 \in W$*. Let* B *be a unit ball in* R*. Let* $A \subseteq M$*,* $(A_i)_{i\in I} \subseteq \mathcal{P}(M)$*, and* $C \subseteq N$*. Then:*

1. *If* A_i *is* W*-balanced for all* $i \in I$*, then* $\bigcup_{i\in I} A_i$ *and* $\bigcap_{i\in I} A_i$ *are* W*-balanced. If* I *is finite and* A_i *is balanced for all* $i \in I$*, then* $\bigcup_{i\in I} A_i$ *and* $\bigcap_{i\in I} A_i$ *are balanced.*
2. *The* B*-balanced hull, defined as the intersection of all* B*-balanced sets containing a given set, is a hull operator in* M*. The* B*-balanced hull of* A *is* BA*.*
3. *If* A *is bounded and* B *is left-bounded, then* BA *is bounded.*
4. *If* A *is* W*-balanced and* $r \in R$ *is so that* $rW = Wr$*, then* rA *is also* W*-balanced.*
5. *If* A *is* W*-balanced, then* $\mathrm{cl}(A)$ *is* $\mathrm{cl}(W)$*-balanced. In particular, if* M *is Hausdorff,* R *is a division ring,* A *is closed and bounded and* B *is compact, then* BA *is closed.*
6. *If* A *is* W*-balanced,* $0 \in \mathrm{int}(A)$*, and* $W \setminus \{0\} \subseteq \mathcal{U}(R)$*, then* $\mathrm{int}(A)$ *is* W*-balanced. In particular, if* A *is open and contains* 0 *and* $B \setminus \{0\} \subseteq \mathcal{U}(R)$*, then* BA *is open.*

7. *If $M = R$ and it is a division ring, $1 \in \mathrm{cl}(\mathrm{int}(W))$, and A is a W-balanced closed neighborhood of 0 in R, then A is regular closed.*

8. *If A is W-balanced, then $T(A)$ is W-balanced.*

9. *If C is W-balanced, then $T^{-1}(C)$ is W-balanced.*

Proof 376 *The proof will follow as itemized as the statement of the proposition.*

1. *Suppose first that I is finite. For every $i \in I$, take $W_i \subseteq R$ an additively symmetric 0-neighborhood in such a way that $1 \in W_i$ and $W_i A_i = A_i$. Then $W_0 := \bigcap_{i\in I} W_i$ is an additively symmetric 0-neighborhood in R containing 1 satisfying that $W_0 \bigcup_{i\in I} A_i = \bigcup_{i\in I} A_i$ and $W_0 \bigcap_{i\in I} A_i = \bigcap_{i\in I} A_i$. The rest follows a similar proof.*

2. *Note that BA is B-balanced and contains A; therefore, BA contains the B-balanced hull of A. Next, if C is any B-balanced subset of M containing A, then C clearly contains BA, thus BA is contained in the B-balanced hull.*

3. *Fix an arbitrary neighborhood U of 0 in M. There are $W \subseteq R$ and $V \subseteq M$ neighborhoods of 0 in R and M, respectively, such that $WV \subseteq U$. There exists an invertible $r \in \mathcal{U}(R)$ such that $A \subseteq rV$. There exists another invertible $s \in \mathcal{U}(R)$ such that $sB \subseteq Wr^{-1}$. Finally, $s(BA) = (sB)A \subseteq Wr^{-1}A \subseteq WV \subseteq U$.*

4. *Simply note that $W(rA) = (Wr)A = (rW)A = r(WA) = rA$.*

5. *Notice that $\mathrm{cl}(W)$ is also an additively symmetric 0-neighborhood in R containing 1 satisfying that $\mathrm{cl}(W)\mathrm{cl}(A) \subseteq \mathrm{cl}(WA) = \mathrm{cl}(A)$. Suppose now that M is Hausdorff, R is a division ring, A is closed and bounded and B is compact. Then BA is closed as a direct application of Lemma 136.*

6. *By hypothesis, $W\mathrm{int}(A) = (W \setminus \{0\})\,\mathrm{int}(A)$. Since every $w \in W \setminus \{0\}$ is invertible, we have that $W\mathrm{int}(A)$ is open in M. Therefore, $W\mathrm{int}(A) \subseteq \mathrm{int}(A)$.*

7. *Let us prove that $\mathrm{cl}(\mathrm{int}(A)) = A$. Fix an arbitrary $a \in A$ and an arbitrary $U \subseteq R$ neighborhood of 0. There exists a 0-neighborhood $V \subseteq R$ such that $Va \subseteq U$. By hypothesis, there exists $v \in V$ such that $1 + v \in \mathrm{int}(W)$. On the one hand, $a + va \in a + Va \subseteq a + U$. On the other hand, $a + va = (1 + v)a \in \mathrm{int}(W)A \subseteq \mathrm{int}(A)$. Finally, $a + va \in (a + U) \cap \mathrm{int}(A)$, meaning that $a \in \mathrm{cl}(\mathrm{int}(A))$.*

8. *For every $w \in W$ and every $a \in A$, $wT(a) = T(wa) \in T(WA) = T(A)$, meaning that $WT(A) \subseteq T(A)$.*

9. *For every $w \in W$ and every $d \in T^{-1}(C)$, $T(wd) = wT(d) \in WC = C$, meaning that $WT^{-1}(C) \subseteq T^{-1}(C)$.*

Proposition 99 has strong consequences in the creation of closed unit neighborhoods of zero in topological division rings. Next corollary improves considerably Corollary 23.

Corollary 38 *Let R be a topological division ring. A subset B of R is a closed unit neighborhood of 0 if and only if B is an additively symmetric and multiplicatively idempotent closed neighborhood of 0 such that $1 \in \mathrm{cl}(\mathrm{int}(B))$.*

Proof 377 *According to Corollary 23, all we need to prove is that B is regular closed. However, this is a direct consequence of Proposition 99(8).*

Notice that, if $\mathcal{B}$ is a base of neighborhoods of 0 in a topological module M over a topological ring, then $\mathcal{B}_s := \{A \cap -A : A \in \mathcal{B}\}$ and $\mathcal{B}_S := \{A \cup -A : A \in \mathcal{B}\}$ are both bases of additively symmetric neighborhoods of 0 in M. It is a simple observation that a subset A of a topological module M over a topological ring R is balanced if and only if A is additively symmetric and there exists a 0-neighborhood W in R such that $WA \subseteq A$.

Lemma 145 *Let M be a topological module over a topological ring R. Then:*

1. *If V is an additively symmetric bounded neighborhood of 0 in M, then V is balanced.*
2. *If $A \subseteq M$ and $P \subseteq R$ is left balanced in R, then PA is balanced.*
3. *If $A \subseteq M$ and $W \subseteq R$ is an additively symmetric left-bounded 0-neighborhood, then WA is balanced.*

Proof 378 *The proof will follow as itemized as the statement of the lemma.*

1. *There are $W_0 \subseteq R$ and $V_0 \subseteq M$ neighborhoods of 0 in R and M, respectively, in such a way that $W_0V_0 \subseteq V$. By hypothesis, V is bounded, so there exists an invertible $u \in \mathcal{U}(R)$ satisfying that $V \subseteq uV_0$. Then W_0u^{-1} is a neighborhood of 0 in R and $\left(W_0u^{-1}\right)V \subseteq W_0u^{-1}uV_0 = W_0V_0 \subseteq V$.*
2. *Note that PA is additively symmetric because so is P. By hypothesis, there exists a 0-neighborhood W_1 in R such that $W_1P \subseteq P$. Then $W_1(PA) \subseteq PA$.*
3. *By Lemma 145(1), W is left-balanced in R. By Lemma 145(2), WA is balanced.*

By taking into consideration Proposition 99(1), the collection of all balanced zero-neighborhoods in a topological module is a filter base.

Theorem 116 *Let M be a topological module over a practical topological ring R. Then:*

1. *If there exists a bounded neighborhood of 0 in R, then the collection of all balanced neighborhoods of 0 in M is a base of 0-neighborhoods in M.*
2. *If there exists a right-bounded closed unit neighborhood B of 0 in R, then the collection of all B-balanced neighborhoods of 0 in M is a base of 0-neighborhoods in M.*

Proof 379 *Let $\mathcal{B}$ denote the collection of all balanced neighborhoods of 0 in M. Notice that $\mathcal{B}$ is a filter base by Proposition 99(1). Then:*

1. *We may assume that W_0 is an additively symmetric bounded neighborhood of 0 in R. Next, Theorem 102(1) together with Lemma 145(3) assure that $\mathcal{B}$ contains $\{W_0V : V \in \mathcal{N}_0(M)\}$, which is a base of neighborhoods of 0 in M.*
2. *Theorem 102(1) assures that $\{BV : V \in \mathcal{N}_0(M)\}$ is a base of neighborhoods of 0 in M. Finally, it is clear that BV is B-balanced for each $V \in \mathcal{N}_0(M)$.*

The following corollary is a very interesting consequence of the previous theorem about the continuity of the inverse of (1.24). Recall that (1.24) can be seen as the right-multiplication operator $\mathsf{R}_m : R \to Rm \subseteq M$ acting via $\mathsf{R}_m(r) := rm$.

Corollary 39 *Let M be a topological module over a division topological ring R. Let B be a total bounded closed unit neighborhood of 0 in R. If $m \notin \mathscr{O}_M$, then (1.24) is an isomorphism over its image. Under these circumstances, M is Hausdorff if and only if M is strongly torsionfree and R is Hausdorff.*

Proof 380 *First off, R is an integral domain and M is torsionfree because R is a division ring (Remark 38). Let us check the continuity of the inverse of* (1.24). *Take $U \in \mathcal{N}_0(R)$. Since B is left-bounded, we can find $u \in \mathcal{U}(R)$ such that $uB \subseteq U$. By hypothesis, we can fix $V \in \mathcal{N}_0(M)$ such that $m \notin V$. In virtue of Theorem 116(2), we can assume, without loss of generality, that V is B-balanced. We will prove now that, if $rm \in V$, then $r \in B$. Indeed, if $r = 0$, then it is obvious. If $r \neq 0$, then r is invertible, thus, since B is total, if $r \notin B$, then $r^{-1} \in B$, meaning that $m = r^{-1}(rm) \in BV = V$, which contradicts that $m \notin V$. Next, note that $Rm \cap uV$ is a neighborhood of 0 in Rm since u is invertible. Next, we will show that the image of $Rm \cap uV$ under the inverse of* (1.24) *is contained in U. Indeed, take an element rm in $Rm \cap uV$ and notice that $\left(u^{-1}r\right) m \in V$, therefore, by our prior observation, $u^{-1}r \in B$, meaning that $r \in uB \subseteq U$. This proves the continuity of the inverse of* (1.24). *Finally, assume that M is also Hausdorff. Then $\mathscr{O}_M = \{0\}$, meaning that M is strongly torsionfree and R is Hausdorff because so is Rm for each $m \in M \setminus \{0\}$. Conversely, if R is Hausdorff and M is strongly torsionfree, then $\mathscr{O}_M = \{0\}$ by Lemma 129(2).*

The following result deals with maximal additively symmetric absorbing subsets. Notice that every subset of a module over a topological ring containing zero has a maximal additively symmetric subset in view of Zorn's Lemma. Also, every subset of a module over a topological ring containing zero has a maximal W-balanced subset also in view of Zorn's Lemma.

Theorem 117 *Let M be a module over a topological ring R. Let $A \subseteq M$ be absorbing. Then:*

1. *Every maximal additively symmetric subset $D \subseteq A$ is also absorbing.*
2. *If there exists an additively symmetric neighborhood $W \subseteq R$ of 0 containing 1 and a base of neighborhoods of 0 in R which are left W-balanced, then every maximal W-balanced subset $D \subseteq A$ is also absorbing.*

Proof 381 *The proof will follow as itemized as the statement of the theorem.*

1. *We will assume that A is not additively symmetric, since otherwise there is nothing to prove. Let $\mathcal{L} := \{C \subseteq A : C = -C\}$. Clearly, $\mathcal{L}$ is nonempty because A contains 0 and $\{0\}$ is trivially additively symmetric. Observe that $\mathcal{L}$ can be partially ordered by the inclusion. Every chain $(C_i)_{i\in I}$ of $\mathcal{L}$ satisfies that $\bigcup_{i\in I} C_i \in \mathcal{L}$. Then Zorn's Lemma assures the existence of a maximal element $D \in \mathcal{L}$. Let us finally prove that $0 \in \operatorname{inter}(D)$. Fix an arbitrary $m \in M$. Since $0 \in \operatorname{inter}(A)$, there exists an additively symmetric 0-neighborhood $V \subseteq R$ such that $0 + Vm \subseteq A$. Notice that Vm is additively symmetric, hence so is $D \cup Vm$. By maximality, $D \cup Vm = D$, meaning that $Vm \subseteq D$. This shows that $0 \in \operatorname{inter}(D)$.*
2. *We will reproduce a similar argument as above, so we will begin by assuming that A is not W-balanced. Let $\mathcal{L} := \{C \subseteq A : C \text{ is } W\text{-balanced}\}$. Clearly, $\mathcal{L}$ is nonempty because A contains 0 and $\{0\}$ is trivially W-balanced. Notice that $\mathcal{L}$ can be partially ordered by the inclusion. If $(C_i)_{i\in I}$ is a chain of $\mathcal{L}$, then $\bigcup_{i\in I} C_i \in \mathcal{L}$ in virtue of Proposition 99(1). By Zorn's Lemma, there exists a maximal element $D \in \mathcal{L}$. Let us finally prove that D is absorbing. Fix an arbitrary $m \in M$. Since A is absorbing, there exists a 0-neighborhood $V \subseteq R$ such that $0 + Vm \subseteq A$. By hypothesis, we may assume that V is left W-balanced, hence $D \cup Vm$ is W-balanced as well. Also, $D \subseteq D \cup Vm \subseteq A$. By maximality, $D \cup Vm = D$, which implies that $Vm \subseteq D$. This shows that $0 \in \operatorname{inter}(D)$.*

Next series of results are aimed at constructing the finest locally convex module topology. We refer to Proposition 86 for sufficient conditions to obtain a unit segment from a unit ball.

Proposition 100 *Let M a module over a topological ring R such that $2 \in \mathcal{U}(R)$. Let B be a unit ball in R such that $B^+ := 2^{-1}(1+B)$ is a unit segment of R. Then:*

1. *For every $a \in R$, $\Delta[-a,a] = Ba$.*
2. *If $A \subseteq M$ is convex and additively symmetric, then A is B-balanced.*

Proof 382 *The proof will follow as itemized as the statement of the proposition.*

1. *If $r \in B$, then $ra = 2^{-1}(1+r)a + 2^{-1}(1-r)(-a) \in \Delta[-a,a]$. If $s \in B^+$, then there exists $r \in B$ with $s = 2^{-1}(1+r)$, meaning that $sa + (1-s)(-a) = 2^{-1}(1+r)a + 2^{-1}(1-r)(-a) = ra \in Ba$.*
2. *Fix arbitrary elements $a \in A$ and $r \in B$. Then $ra \in Ba = \Delta[-a,a] \subseteq A$. As a consequence, $BA = A$; hence, A is B-balanced.*

By bearing in mind Proposition 100, a base of zero-neighborhoods which are balanced can be constructed if there exists a base of zero-neighborhoods which are convex.

Corollary 40 *Let M be a topological module over a topological ring R such that $2 \in \mathcal{U}(R)$. Let B be a unit ball in R such that $B^+ := 2^{-1}(1+B)$ is a unit segment of R. If M has a base of neighborhoods of 0 which are convex, then M has a base of neighborhoods of 0 which are convex and B-balanced.*

Proof 383 *Let $\mathcal{B}$ be a base of neighborhoods of 0 which are convex. Notice that $\mathcal{B}_1 := \{A \cap -A : A \in \mathcal{B}\}$ is a base of neighborhoods of 0 which are additively symmetric and convex, so by Proposition 100 they are also B-balanced.*

We are finally on the right spot to define locally convex module topologies, which in some sense is based upon Corollary 40.

Definition 267 (Locally convex module topology) *Let M be a topological module over a topological ring R. Let B^+ be a unit segment of R. We say that the module topology of M is locally convex if there exists a base of neighborhoods of 0 which are convex.*

Recall that completely full unit segments satisfy that all of their nonzero elements are invertible.

Lemma 146 *Let M be a topological module over a topological ring R. Let B^+ be a completely full unit segment of R. Let A be a convex subset of M with $\operatorname{int}(A) \neq \varnothing$. If $a \in \operatorname{int}(A)$ and $b \in A$, then*

$$\Delta[a,b] \setminus \{b\} \subseteq \{ra + sb : r,s \in B^+, r+s=1, r \neq 0\} \subseteq \operatorname{int}(A).$$

In particular, $\operatorname{int}(A)$ is convex.

Proof 384 *Let U be a neighborhood of 0 in M such that $a+U \subseteq A$. Fix arbitrary elements $r,s \in B^+$ with $r+s=1$ and $r \neq 0$. Note that $B^+ \setminus \{0\} \subseteq \mathcal{U}(R)$ by hypothesis. Then rU is a neighborhood of 0 in M because r is invertible. We will prove that $(ra+sb)+rU \subseteq A$. If $u \in U$, then*

$$(ra+sb)+ru = r(a+u)+sb \in \Delta[a+u,b] \subseteq A$$

because $a+u, b \in A$. Finally, in order to see that $\operatorname{int}(A)$ is convex, it only suffices to call on Proposition 41.

Next lemma unveils topological properties of convex hulls related to closedness, openness, boundedness, and balancedness.

Lemma 147 *Let M be a topological module over a topological ring R. Let B^+ be a unit segment of R. Let $A, C \subseteq M$. Then:*

1. *If A is convex, then $\mathrm{cl}(A)$ is convex as well.*
2. *Suppose that M is locally convex and B^+ is commuting. If A is bounded, then $\mathrm{co}(A)$ is bounded as well.*
3. *Suppose that R is an integral domain, $W \subseteq R$ is an additively symmetric neighborhood of 0 with $1 \in W$, and B^+ is commuting. If A is W-balanced, then $\mathrm{co}(A)$ is W-balanced as well.*
4. *Suppose that B^+ is completely full. If A is open, then $\mathrm{co}(A)$ is open as well.*

Proof 385 *The proof will follow as itemized as the statement of the lemma.*

1. *Take $m_1, \ldots, m_k \in \mathrm{cl}(A)$ and $r_1, \ldots, r_k \in B^+$ with $\sum_{i=1}^{k} r_i = 1$. Fix an arbitrary 0-neighborhood $U_0 \subseteq M$. We can find another 0-neighborhood $U \subseteq M$ such that $U + \overset{k}{\cdots} + U \subseteq U_0$. We can find a 0-neighborhood V of M in such a way that $r_i V \subseteq U$ for all $i \in \{1, \ldots, k\}$. For every $i \in \{1, \ldots, k\}$, there exists $a_i \in A$ with $a_i \in m_i + V$. Then $r_1 a_1 + \cdots + r_k a_k \in r_1 m_1 + \cdots + r_k m_k + (r_1 V + \cdots + r_k V) \subseteq r_1 m_1 + \cdots + r_k m_k + \left(U + \overset{k}{\cdots} + U\right) \subseteq r_1 m_1 + \cdots + r_k m_k + U_0$. This shows that $r_1 m_1 + \cdots + r_k m_k \in \mathrm{cl}(A)$.*
2. *Let $U \subseteq M$ be a 0-neighborhood. Since M is locally convex, we may assume that U is convex. By hypothesis, there exists an invertible $r \in \mathcal{U}(R)$ such that $A \subseteq rU$. Then $\mathrm{co}(A) \subseteq \mathrm{co}(rU) = r\mathrm{co}(U) = rU$ by Corollary 5(3).*
3. *We will show that $W\mathrm{co}(A) \subseteq \mathrm{co}(A)$. Fix an arbitrary $w \in W$. Then $w\mathrm{co}(A) = \mathrm{co}(wA) \in \mathrm{co}(WA) = \mathrm{co}(A)$ by Corollary 5(3).*
4. *We may assume that A is not empty. Take $b \in \mathrm{co}(A)$ and write $b = r_1 a_1 + \cdots + r_k a_k$ with $a_1, \ldots, a_k \in A$ and $r_1, \ldots, r_k \in B^+$ with $r_1 + \cdots + r_k = 1$. Observe that $\mathrm{co}(A)$ is convex and $A \subseteq \mathrm{int}(\mathrm{co}(A))$, meaning that $\mathrm{int}(\mathrm{co}(A)) \neq \varnothing$. According to Lemma 146, $\mathrm{int}(\mathrm{co}(A))$ is convex, therefore, $b \in \Delta[a_1, \ldots, a_k] \subseteq \mathrm{int}(\mathrm{co}(A))$. As a consequence, $\mathrm{co}(A) \subseteq \mathrm{int}(\mathrm{co}(A))$, resulting in $\mathrm{co}(A)$ being open.*

Under the settings of the previous proposition, if C is a convex component of A, then C is closed in A. The following corollary is the version of Theorem 117 for convexity.

Corollary 41 *Let M be a module over a topological ring R such that R is an integral domain. Let $B^+ \subseteq R$ be a commuting unit segment. Let $A \subseteq M$ be convex and absorbing. Then:*

1. *Every maximal additively symmetric subset $D \subseteq A$ is also convex and absorbing.*
2. *If there exists an additively symmetric neighborhood $W \subseteq R$ of 0 containing 1 and a base of neighborhoods of 0 in R which are left W-balanced, then every maximal W-balanced subset $D \subseteq A$ is also convex and absorbing.*

Proof 386 *The proof will follow as itemized as the statement of the corollary.*

1. *Zorn's Lemma assures the existence of a maximal additively symmetric subset $D \subseteq A$. Notice that D is absorbing in view of Theorem 117(1). Finally, $D \subseteq \mathrm{co}(D) \subseteq A$ and $\mathrm{co}(D)$ is trivially additively symmetric, meaning by maximality that $D = \mathrm{co}(D)$.*

2. *Zorn's Lemma assures the existence of a maximal W-balanced subset $D \subseteq A$. Note that D is absorbing according to Theorem 117(2). Finally, $D \subseteq \mathrm{co}(D) \subseteq A$ and $\mathrm{co}(D)$ is W-balanced by Lemma 147(3), meaning by maximality that $D = \mathrm{co}(D)$.*

Recall that the commutant of a subset A of a ring R is defined as $A' := \{r \in R : \forall a \in A \;\; ra = ar\}$. The center of R is precisely defined as R' and usually denoted by $\mathsf{Z}(R)$.

Theorem 118 *Let M be a topological module over a topological ring R. Let B^+ be a self-commuting completely full unit segment of R. Let A be a convex subset of M with $\mathrm{int}(A) \neq \varnothing$. If $a \in \mathrm{int}(A)$ and $b \in \mathrm{cl}(A)$, then*

$$\Delta[a,b] \setminus \{b\} \subseteq \{ra + sb : r,s \in B^+, r+s=1, r \neq 0\} \subseteq \mathrm{int}(A).$$

Proof 387 *Fix arbitrary elements $r,s \in B^+$ with $r+s=1$ and $r \neq 0$. If $r=1$, then $ra+sb = a \in \mathrm{int}(A)$, so we will suppose that $r \neq 1$. Denote $z := ra+sb$. We will show that $z \in \mathrm{int}(A)$. Consider the function*

$$\begin{array}{rccl} g: & M & \to & M \\ & p & \mapsto & g(p) := z - rs^{-1}(p-z) \end{array}$$

Notice that g is a homeomorphism because rs^{-1} is invertible. Also, $g(z) = z$. On the other hand,

$$g(a) = ra + sb - rs^{-1}(a - ra - sb) = ra + sb - rs^{-1}s(a-b) = ra + sb - r(a-b) = b.$$

Since r and s commute, then so do r and s^{-1}, thus, for all $p \in M$,

$$\begin{array}{rcl} rp + sg(p) & = & rp + s\left(z - rs^{-1}(p-z)\right) = rp + sz - srs^{-1}(p-z) \\ & = & rp + sz - r(p-z) = z. \end{array}$$

Next, since $a \in \mathrm{int}(A)$, there exists an open neighborhood U of a contained in A. Then $g(U)$ is an open neighborhood of b; therefore, $g(U) \cap A \neq \varnothing$. Fix $u \in U$ with $g(u) \in A$. By relying first on Lemma 146, we conclude that $z = ru + sg(u) \in \mathrm{int}(A)$.

Completely full commuting unit segments satisfy the hypotheses of Theorem 118.

Corollary 42 *Let M be a module over a topological ring R. Let B^+ be a practical completely full unit segment of R. Let A be a convex subset of M with $\mathrm{int}(A) \neq \varnothing$. Then:*

1. $\mathrm{cl}(\mathrm{int}(A)) = \mathrm{cl}(A)$.

2. *If multiplicative inversion is continuous and B^+ is self-commuting, then $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(A)$.*

Proof 388 *The proof will follow as itemized as the statement of the corollary.*

1. *Since* $\text{int}(A) \subseteq A$ *we have that* $\text{cl}(\text{int}(A)) \subseteq \text{cl}(A)$. *Fix an arbitrary* $b \in \text{cl}(A)$. *Let* U *be an open neighborhood of* b. *Notice that* $U \cap A \neq \varnothing$. *Take* $a \in U \cap A$. *Since* $\text{int}(A) \neq \varnothing$, *we can fix an element* $a_0 \in \text{int}(A)$. *The function*

$$\begin{array}{ccc} R & \to & M \\ r & \mapsto & (1-r)a_0 + ra \end{array} \tag{3.11}$$

is well defined, continuous, and maps 1 *to* a. *Then there exists a neighborhood* W *of* 1 *in* R *such that if* $r \in W$, *then* $(1-r)a_0 + ra \in U$. *In accordance with Lemma 146, the image of* $B^+ \setminus \{1\}$ *under* (3.11) *is contained in* $\text{int}(A)$. *Since* $1 \in \text{cl}\,(B^+ \setminus \{1\})$, *we can find* $r_0 \in W \cap (B^+ \setminus \{1\})$. *Hence,* $(1-r_0)a_0 + r_0 a \in \text{int}(A) \cap U$. *As a consequence,* $b \in \text{cl}(\text{int}(A))$.

2. *Since* $A \subseteq \text{cl}(A)$ *we have that* $\text{int}(A) \subseteq \text{int}(\text{cl}(A))$. *Fix an arbitrary* $a \in \text{int}(\text{cl}(A))$ *and let* U *be an open neighborhood of* 0 *such that* $a + U \subseteq \text{cl}(A)$. *Since* $\text{int}(A) \neq \varnothing$, *we can fix an element* $a_0 \in \text{int}(A)$. *There exists a neighborhood* V *of* 0 *in* R *such that* $V(a_0 - a) \subseteq U$. *According to Lemma 110, there exists* $r_0 \in B^+ \setminus \{0,1\}$ *such that* $v_0 := -(1-r_0)^{-1} r_0 \in V$ *and*

$$a = r_0 a_0 + (1-r_0)\,(a + v_0(a_0 - a)) \in \Delta(a_0, a + v_0(a_0 - a)).$$

Note that $a + v_0(a_0 - a) \in a + V(a_0 - a) \subseteq a + U \subseteq \text{cl}(A)$. *Since* $r_0 \in B^+ \setminus \{0\}$, *Theorem 118 assures that* $a \in \text{int}(A)$.

Next corollary precises of the hypotheses of totally fullness for unit segments.

Corollary 43 *Let* M *be a topological module over a topological ring* R. *Let* B^+ *be a self-commuting completely and totally full unit segment of* R. *Let* A *be a convex subset of* M *and* $a, b \in \text{cl}(A)$. *If* $\Delta[a,b] \cap \text{int}(A) \neq \varnothing$, *then* $\Delta[a,b] \setminus \{a,b\} \subseteq \text{int}(A)$.

Proof 389 *Let* $z \in \Delta[a,b] \cap \text{int}(A)$. *In accordance with Lemma 34(2),*

$$\Delta[a,b] = \Delta[a,z] \cup \Delta[z,b].$$

By bearing in mind Theorem 118, we have that

$$\Delta[a,b] \setminus \{a,b\} \subseteq (\Delta[a,z] \setminus \{a\}) \cup (\Delta[z,b] \setminus \{b\}) \subseteq \text{int}(A).$$

Notice that in a nonzero ring it can never happen that $2^{-1} = 1$, since otherwise $1 = 1+1$, meaning that $1 = 0$.

Corollary 44 *Let* M *be a topological module over a topological ring* R *such that* $2 \in \mathcal{U}(R)$. *Let* B^+ *be a centered and completely full unit segment of* R. *Let* A *be a convex subset of* M. *If* $A \subseteq M$ *is convex, additively symmetric, and* $\text{int}(A) \neq \varnothing$, *then* $0 \in \text{int}(A)$.

Proof 390 *Let* $a \in \text{int}(A)$. *Notice that* $-a \in \text{int}(A)$. *According to Lemma 146, we have that*

$$0 = 2^{-1}a + 2^{-1}(-a) \in \Delta(a, -a) \subseteq \text{int}(A).$$

After the definition of locally convex module topologies, we can finally state and prove the finest locally convex module topology result.

Theorem 119 (Finest locally convex module topology) *Let M be a module over a practical topological ring R. Suppose that R is an integral domain, $2 \in \mathcal{U}(R)$, there exists a bounded neighborhood of 0 in R, and R is endowed with a commuting and centered full unit segment B^+. Then*

$$\mathcal{B}_M := \{A \subseteq M : A \text{ is balanced, convex, and absorbing}\}$$

is base of neighborhoods of 0 for a module topology τ_M in M in such a way that if τ is a locally convex module topology in M, then $\tau \subseteq \tau_M$.

Proof 391 *In the first place, we will check that $\mathcal{B}_M$ verifies the requirements to be a filter base in M:*

- *$\mathcal{B}_M$ is clearly nonempty and $\varnothing \notin \mathcal{B}_M$. Indeed, M is trivially convex, additively symmetric, absorbing, and $WM \subseteq M$ for every 0-neighborhood $W \subseteq R$. On the other hand, if $A \subseteq M$ is balanced, convex, and absorbing, then $0 \in A$, hence $A \neq \varnothing$.*
- *If $C, D \in \mathcal{B}_M$, then there exists $A \in \mathcal{B}_M$ with $A \subseteq C \cap D$. Indeed, note that $C \cap D$ is convex and additively symmetric. Also, $0 \in \operatorname{inter}(C) \cap \operatorname{inter}(D) = \operatorname{inter}(C \cap D)$, meaning that $C \cap D$ is absorbing. Finally, if W_1, W_2 are 0-neighborhoods in R such that $W_1 C \subseteq C$ and $W_2 D \subseteq D$, then $(W_1 \cap W_2)(C \cap D) \subseteq (W_1 \cap W_2)C \cap (W_1 \cap W_2)D \subseteq C \cap D$. Therefore, $A := C \cap D$ verifies the desired properties.*

Next, we will check that $\mathcal{B}_M$ verifies Theorem 91(1,2,3,4,5):

1. *If $A \in \mathcal{B}_M$, then there exists $C \in \mathcal{B}_M$ with $C + C \subseteq A$. Indeed, if we take $C := 2^{-1}A$, then C is convex in view of Corollary 4(3), C is trivially additively symmetric, and C is absorbing because $\operatorname{inter}\left(2^{-1}A\right) = 2^{-1}\operatorname{inter}(A)$. Also, if W is a 0-neighborhood in R such that $WA \subseteq A$, then $WC = W2^{-1}A = 2^{-1}WA \subseteq 2^{-1}A = C$. As a consequence, $C \in \mathcal{B}_M$. Finally, the convexity of A together with the fact that $2^{-1} \in B^+$ allow that $C + C = 2^{-1}A + 2^{-1}A \subseteq A$.*
2. *If $A \in \mathcal{B}_M$, then there exists $C \in \mathcal{B}_M$ with $-C \subseteq A$. Indeed, since A is additively symmetric, it suffices to take $C := A$.*
3. *If $A \in \mathcal{B}_M$, then there exist $C \in \mathcal{B}_M$ and $W \in \mathcal{N}_0(R)$ such that $WC \subseteq A$. Indeed, since A is balanced, there exists a 0-neighborhood W in R such that $WA \subseteq A$.*
4. *If $A \in \mathcal{B}_M$ and $r \in R$, then there exists $C \in \mathcal{B}_M$ such that $rC \subseteq A$. Indeed, let W_0 be a neighborhood of 0 in R such that $W_0 A \subseteq A$. There exists an additively symmetric neighborhood W_1 of 0 in R such that $rW_1 \subseteq W_0$. We may assume without any loss that W_1 is left-bounded. Let us take $C := \operatorname{co}(W_1 A)$. By taking into account Corollary 5, $rC = r\operatorname{co}(W_1 A) = \operatorname{co}(rW_1 A) \subseteq \operatorname{co}(W_0 A) \subseteq \operatorname{co}(A) = A$. It remains to show that $C \in \mathcal{B}_M$. On the one hand, C is clearly convex and additively symmetric. Next, A is absorbing and R is practical, hence $W_1 A$ is also absorbing in view of Lemma 113, meaning that C is absorbing since $W_1 A \subseteq \operatorname{co}(W_1 A) = C$. Finally, $W_1 A$ is balanced in view of Lemma 145(3), thus $C = \operatorname{co}(W_1 A)$ is balanced in accordance with Lemma 147(3).*
5. *If $A \in \mathcal{B}_M$ and $m \in M$, then there exists $W \in \mathcal{N}_0(R)$ such that $Wm \subseteq A$. Indeed, since A is absorbing, there exists $W \in \mathcal{N}_0(R)$ such that $Wm \subseteq A$.*

What follows is a nontrivial example of a topological module over a topological ring satisfying the hypotheses of Theorem 119.

Example 63 *Take* $R := \mathbb{Z}\left[\frac{1}{2}\right]$ *with the inherited topology from* $\mathbb{R}$*. Notice that* R *is practical since* $\frac{1}{2^n} \to 0$ *as* $n \to \infty$*, an integral domain, and* $2 \in \mathcal{U}(R)$*. Observe that* $B := [-1,1] \cap R$ *is a bounded neighborhood of* 0 *in* R*. Also,* $B^+ := [0,1] \cap R$ *is a commuting and centered full unit segment of* R*. According to Theorem 119, if* M *is an* R*-module, then there exists the finest locally convex module topology in* M*. A nontrivial example of* R*-module is, for instance, the module of bounded sequences with coefficients in* R*,* $\ell_\infty\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$*.*

Another nontrivial example of a topological module over a topological ring satisfying the hypotheses of Theorem 119 is displayed to conclude this subsubsection.

Example 64 *Take* $R := \mathbb{Q}[\pi]$ *with the inherited topology from* $\mathbb{R}$*. Notice that* R *is practical, an integral domain, and* $2 \in \mathcal{U}(R)$*. Also notice that* π *is transcendental over* $\mathbb{Q}$*; therefore,* π *is not invertible in* $\mathbb{Q}[\pi]$*, hence* $\mathbb{Q}[\pi]$ *is not a field. Observe that* $B := [-1,1] \cap R$ *is a bounded neighborhood of* 0 *in* R*. Also,* $B^+ := [0,1] \cap R$ *is a commuting and centered full unit segment of* R*. According to Theorem 119, if* M *is an* R*-module, then there exists the finest locally convex module topology in* M*. A nontrivial example of* R*-module is, for instance, the module of null sequences with coefficients in* R*,* $c_0(\mathbb{Q}[\pi])$*.*

3.2.4.6 Stereographic projection

Let us begin by bearing in mind the trivial fact that if M is a module over a ring R, then $Ra = \mathrm{stl}(0, a)$ for every $a \in M$.

Definition 268 (Fit body) *Let* M *be a topological module over a topological ring* R*. A subset* $A \subseteq M$ *is said to be a fit body if* A *is closed,* $0 \in \mathrm{int}(A)$*, and every* $a \in \mathrm{bd}(A)$ *satisfies that* $Ra \cap \mathrm{bd}(A) \subseteq \{-a, a\}$*.*

Under the settings of the previous definition, it is immediate to realize that if a fit body A is additively symmetric, then $Ra \cap \mathrm{bd}(A) = \{-a, a\}$ for every $a \in \mathrm{bd}(A)$. Next, we will provide nontrivial examples of fit bodies.

Proposition 101 *Let* R *be a topological totally ordered ring with no holes. Then* $A := [-1,1]$ *is a fit body.*

Proof 392 *By assumption, the ring topology of* R *is the order topology, thus* $(-1,1)$ *is open. Let us show next that* $\mathrm{bd}(A) = \{-1,1\}$*. Indeed, fix an arbitrary open interval* (a,b) *containing* 1*. Notice that* $1+1>1$*, so we may assume that* $b \leq 1+1$*. By hypothesis,* A *has no holes, therefore* $(1,b)$ *is not empty, meaning that* $(a,b) \cap (R \setminus A) \neq \varnothing$*. As a consequence,* $1 \in \mathrm{bd}(A)$*. In a similar way,* $-1 \in \mathrm{bd}(A)$*. On the other hand,* $[-1,1]$ *is closed in the order topology, which assures that* $\mathrm{bd}(A) = \{-1,1\}$*. Then* A *is an additively symmetric body whose interior contains* 0*. Finally,* $R1 \cap \mathrm{bd}(A) = \{-1,1\} = R(-1) \cap \mathrm{bd}(A)$*, meaning that* A *is a fit body.*

$\mathbb{Q}[\pi]$, endowed with the inherited topology from $\mathbb{R}$, is a topological totally ordered ring with no holes in view of Corollary 12.

Theorem 120 *Let* M *be a torsionfree topological module over a Hausdorff topological-inversion integral domain ring* R *of* $\mathrm{char}(R) \neq 2$ *such that* $\mathcal{U}(R)$ *is open. Consider* $m^* \in M^*$*,* $A \subseteq M$ *a fit body, and* $s \in \mathcal{U}(R)$ *such that* $(m^*)^{-1}(\{s\}) \cap \mathrm{int}(A) = \varnothing$ *but* $(m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A) \neq \varnothing$*. Then*

$$\mathrm{int}_{(m^*)^{-1}(\{s\})}\left((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\right) \subseteq \mathrm{int}_{\mathrm{bd}(A)}\left((m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A)\right).$$

Proof 393 *Fix an arbitrary* $a \in \operatorname{int}_{(m^*)^{-1}(\{s\})}\left((m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)\right)$. *A* 0*-neighborhood* $V_0 \subseteq M$ *exists such that* $(a+V_0) \cap (m^*)^{-1}(\{s\}) \subseteq \operatorname{bd}(A)$. *Another* 0*-neighborhood* $V_1 \subseteq M$ *can be found in such a way that* $V_1+V_1+V_1 \subseteq V_0$. *Next, there are* 0*-neighborhoods* $W_1 \subseteq R$ *and* $V_2 \subseteq M$ *satisfying that* $W_1V_2 \subseteq V_1$. *Take* W_2 *a* 0*-neighborhood in* R *so that* $W_2a \subseteq V_1$. *Let* W_3 *be a* 0*-neighborhood in* R *such that* $s+W_3 \subseteq \mathcal{U}(R)$. *Since* $\operatorname{char}(R) \neq 2$ *and* s *is invertible, we have that* $s+s \neq 0$, *thus, since* R *is Hausdorff, there exists a* 0*-neighborhood* $W_4 \subseteq R$ *such that* $-s-s \notin W_4$. *Consider the continuous function (recall that* R *is a topological-inversion ring):*

$$\begin{array}{rcl} V_3 & \to & R \\ m & \mapsto & s\left(s+m^*(m)\right)^{-1}, \end{array} \tag{3.12}$$

where V_3 *is a* 0*-neighborhood in* M *such that* $V_3 \subseteq (m^*)^{-1}(W_3 \cap W_4)$. *Note that* (3.12) *maps* 0 *to* 1. *Thus, we can find* $V_4 \subseteq M$ *a* 0*-neighborhood verifying that* $s\left(s+m^*(m)\right)^{-1} \in 1+(W_1 \cap W_2)$ *for all* $m \in V_4 \cap V_3$. *Finally, let* $V_5 := V_4 \cap V_3 \cap V_2 \cap V_1$. *We will show that* $(a+V_5) \cap \operatorname{bd}(A) \subseteq (m^*)^{-1}(\{s\})$. *Indeed, fix an arbitrary* $m \in V_5$ *with* $a+m \in \operatorname{bd}(A)$. *Denote* $t := s\left(s+m^*(m)\right)^{-1}$ *and* $v := (t-1)a+(t-1)m+m$. *Notice that* $v \in W_2a+W_1V_2+V_1 \subseteq V_1+V_1+V_1 \subseteq V_0$. *Therefore,* $t(a+m) = a+v \in a+V_0$. *Also,* $t(a+m) \in (m^*)^{-1}(\{s\})$. *As a consequence,* $t(a+m) \in \operatorname{bd}(A)$. *Since* A *is a fit body, either* $t(a+m) = a+m$ *or* $t(a+m) = -(a+m)$. *Next,* $a+m \neq 0$ *because* $a+m \in \operatorname{bd}(A)$ *and* $0 \in \operatorname{int}(A)$. *Also,* M *is torsionfree and* R *is an integral domain, hence either* $t=1$ *or* $t=-1$. *In the latter case,* $m^*(m) = -s-s$ *which is impossible by bearing in mind that* $m^*(m) \in W_4$ *but* $-s-s \notin W_4$. *As a consequence,* $t=1$ *meaning that* $m^*(m)=0$, *so* $a+m \in (m^*)^{-1}(\{s\})$.

Let M be a topological module over a topological ring R. Let $A \subseteq M$ be an absorbing subset. If $m \in M$, then there exists a 0-neighborhood $W \subseteq R$ such that $Wm \subseteq A$. If R is practical, then we can find an invertible $u \in W \cap \mathcal{U}(R)$. This fact motivates the following definition, based also upon the classical Minkowski functional [47] in real or complex topological vector spaces.

Definition 269 (Minkowski functional) *Let* M *be a topological module over a practical topological-inversion ring* R. *Let* $A \subseteq M$ *be an absorbing subset. A Minkowski functional is a function*

$$\begin{array}{rccl} j_A: & M & \to & \mathcal{U}(R) \cup \{0\} \\ & m & \mapsto & j_A(m) \end{array} \tag{3.13}$$

satisfying that $j_A(0)=0$ *and, for every* $m \in M \setminus \{0\}$, $j_A(m) \in \mathcal{U}(R)$ *and* $j_A(m)^{-1}m \in A$.

Under the settings of the previous definition, if $u \in \mathcal{U}(R)$, then $g_A(m) := uj_A\left(u^{-1}m\right)$ for every $m \in M$ also defines a Minkowski functional.

Proposition 102 *Let* M *be a topological module over a practical topological-inversion ring* R. *Let* $A \subseteq M$ *be an absorbing subset. There exists a Minkowski functional* j_A *for* A.

Proof 394 *According to the Axiom of Choice, there exist choice functions* $\phi : \mathcal{P}(\mathcal{P}(R)) \setminus \{\varnothing\} \to \mathcal{P}(R)$ *and* $\chi : \mathcal{P}(\mathcal{U}(R)) \setminus \{\varnothing\} \to \mathcal{U}(R)$. *They satisfy that* $\phi(D) \in D$ *for all* $D \in \mathcal{P}(\mathcal{P}(R)) \setminus \{\varnothing\}$ *and* $\chi(Z) \in Z$ *for all* $Z \in \mathcal{P}(\mathcal{U}(R)) \setminus \{\varnothing\}$. *Let us then construct the Minkowski functional for* A *by relying on the choice functions* ϕ *and* χ. *Indeed, fix an arbitrary* $m \in M \setminus \{0\}$. *There exists a* 0*-neighborhood* $W \subseteq R$ *such that* $Wm \subseteq A$. *Therefore, the set* $B_m := \{W \in \mathcal{P}(R) : W \text{ is a } 0\text{-neighborhood such that } Wm \subseteq A\}$ *is not empty, hence* $B_m \in \mathcal{P}(\mathcal{P}(R)) \setminus \{\varnothing\}$. *Observe that* $\phi(B_m) \in B_m$ *for all* $m \in M \setminus \{0\}$. *On the other hand, since* R *is practical, for every* $m \in M \setminus \{0\}$ *and every* $W \in B_m$, *we have that* $W \cap \mathcal{U}(R) \neq \varnothing$, *meaning that* $W \cap \mathcal{U}(R) \in \mathcal{P}(\mathcal{U}(R)) \setminus \{\varnothing\}$. *Note that* $\chi(W \cap \mathcal{U}(R)) \in W \cap \mathcal{U}(R)$. *Then it suffices to define* $j_A(m) := \chi(\phi(B_m) \cap \mathcal{U}(R))^{-1}$ *for* $m \in M \setminus \{0\}$ *and* $j_A(0)=0$.

For the next definition, bear in mind that every body of a topological module which is also a neighborhood of zero is absorbing.

Definition 270 (Minkowski body) *Let M be a topological module over a practical topological-inversion ring R. A subset $A \subseteq M$ is said to be a Minkowski body if A is closed, $0 \in \operatorname{int}(A)$, and there exists a continuous Minkowski functional j_A satisfying that $j_A(m)^{-1}m \in \operatorname{bd}(A)$ for every $m \in M \setminus \{0\}$ and $j_A(a) = 1$ for all $a \in \operatorname{bd}(A)$.*

The simplest example of Minkowski body is the unit ball of a real or complex normed space. The following is a nontrivial example of a Minkowski body.

Example 65 *Let $R := \mathbb{Q}\left[\sqrt{2}\right]$ be endowed with the inherited topology from $\mathbb{R}$. Notice that R is a division ring because $\sqrt{2}$ is algebraic over $\mathbb{Q}$. In fact, R is a nondiscrete topological division ring, so in particular it is a practical topological-inversion ring. Take $A := [-1,1] \cap R$. Then A is a Minkowski body. Indeed, A is closed and $0 \in \operatorname{int}(A)$, and $j_A(x) := |x|$ is a continuous Minkowski functional satisfying that $j_A(x)^{-1}x \in \operatorname{bd}(A)$ for every $x \in R \setminus \{0\}$ and $j_A(a) = 1$ for all $a \in \operatorname{bd}(A)$.*

Next result is a version of Theorem 120 for the case of Minkowski bodies with slightly different theses.

Theorem 121 *Let M be a topological module over a practical topological-inversion ring R. Consider $m^* \in M^*$, $A \subseteq M$ a Minkowski body, and $s \in R \setminus \mathtt{rd}(0)$ such that $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) \neq \varnothing$. Then*

$$\operatorname{int}_{(m^*)^{-1}(\{s\})}\left((m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)\right) \supseteq \operatorname{int}_{\operatorname{bd}(A)}\left((m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)\right).$$

Proof 395 *Fix an arbitrary $a \in \operatorname{int}_{\operatorname{bd}(A)}\left((m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)\right)$. There exists a 0-neighborhood $V_0 \subseteq M$ such that $(a + V_0) \cap \operatorname{bd}(A) \subseteq (m^*)^{-1}(\{s\})$. Another 0-neighborhood $V_1 \subseteq M$ can be found in such a way that $V_1 + V_1 + V_1 \subseteq V_0$. Next, there are 0-neighborhoods $W_1 \subseteq R$ and $V_2 \subseteq M$ satisfying that $W_1V_2 \subseteq V_1$. Take W_2 a 0-neighborhood in R so that $W_2a \subseteq V_1$. Notice that the map*

$$\begin{array}{rcl} M \setminus \{0\} & \to & \mathcal{U}(R) \\ m & \mapsto & j_A(m)^{-1} \end{array} \tag{3.14}$$

is continuous and maps a to 1 (keep in mind that $a \neq 0$ because $a \in \operatorname{bd}(A)$ and $0 \in \operatorname{int}(A)$). Let V_3 be a 0-neighborhood in M such that $j_A\left((a + V_3) \cap (M \setminus \{0\})\right)^{-1} \subseteq 1 + (W_1 \cap W_2)$. Finally, let $V_4 := V_3 \cap V_2 \cap V_1$. We will show that $(a + V_4) \cap (m^*)^{-1}(\{s\}) \subseteq \operatorname{bd}(A)$. Indeed, fix an arbitrary $m \in V_4$ with $a + m \in (m^*)^{-1}(\{s\})$. Observe that $a + m \neq 0$ because $s \neq 0$, hence $j_A(a + m)$ is invertible. Denote $t := j_A(a + m)^{-1}$ and $v := (t - 1)a + (t - 1)m + m$. Notice that $v \in W_2a + W_1V_2 + V_1 \subseteq V_1 + V_1 + V_1 \subseteq V_0$. Therefore, $t(a+m) = a+v \in a+V_0$. Also, $t(a + m) \in \operatorname{bd}(A)$. As a consequence, $t(a + m) \in (m^*)^{-1}(\{s\})$. Then we obtain that $ts = s$. Since s is not a right 0-divisor, we conclude that $t = 1$ meaning that $a+m \in \operatorname{bd}(A)$.

Let M be a topological module over a topological-inversion ring R such that $2 := 1 + 1 \in \mathcal{U}(R)$. Let $m^* \in M^*$ and $s \in \mathcal{U}(R)$. Let $a \in (m^*)^{-1}(\{s\})$. Take $b \in M$. If there exists an element $c \in \operatorname{stl}_{\mathcal{U}}(-a, b) \cap (m^*)^{-1}(\{s\})$ with $c \neq -a$, then $s + m^*(b) \in \mathcal{U}(R)$ and $c = -a+2s(m^*(b)+s)^{-1}(b+a)$. Indeed, there exists $u \in \mathcal{U}(R)$ such that $c = -a+u(b+a)$, so $s = m^*(c) = -s+u(m^*(b)+s)$, hence $2s = u(m^*(b)+s)$, meaning that $s+m^*(b) \in \mathcal{U}(R)$ and $u = 2s(m^*(b) + s)^{-1}$. This motivates the definition of the following function:

$$\begin{array}{rcl} (m^*)^{-1}(-s+\mathcal{U}(R)) & \to & (m^*)^{-1}(\{s\}) \\ b & \mapsto & -a + 2s(m^*(b) + s)^{-1}(b + a). \end{array} \tag{3.15}$$

Definition 271 (Stereographic projection) *Let M be a topological module over a topological-inversion ring R such that $2 := 1+1 \in \mathcal{U}(R)$. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. Consider $m^* \in M^*$ and $s \in \mathcal{U}(R)$ such that $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) \neq \varnothing$. Let $a \in (m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)$. The following map is known as a stereographic projection:*

$$\begin{array}{rcl} \operatorname{bd}(A) \cap (m^*)^{-1}(-s+\mathcal{U}(R)) & \to & (m^*)^{-1}(\{s\}) \\ b & \mapsto & -a + 2s(m^*(b)+s)^{-1}(b+a). \end{array} \tag{3.16}$$

The stereographic projection (3.16) is well defined and continuous (note that multiplicative inversion on R is continuous).

Remark 80 *Let M be a topological module over a topological-inversion ring R such that $2 := 1+1 \in \mathcal{U}(R)$. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. Consider $m^* \in M^*$ and $s \in \mathcal{U}(R)$ such that $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$ but $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) \neq \varnothing$. Let $a \in (m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A)$. Let ϕ be the stereographic projection* (3.16)*. Then:*

1. $\{a\} \cup (\operatorname{bd}(A) \cap \ker(m^*)) \subseteq \operatorname{bd}(A) \cap (m^*)^{-1}(-s+\mathcal{U}(R)) \subseteq \operatorname{bd}(A) \setminus \{-a\}$.
2. $\operatorname{bd}(A) \cap (m^*)^{-1}(-s+\mathcal{U}(R)) = \operatorname{bd}(A) \setminus \{-a\}$ *if and only if* $s + m^*(\operatorname{bd}(A) \setminus \{-a\}) \subseteq \mathcal{U}(R)$.
3. $\phi(b) \in \operatorname{stl}_{\mathcal{U}}(-a, b)$ *for all* $b \in \operatorname{bd}(A) \cap (m^*)^{-1}(-s+\mathcal{U}(R))$.
4. *If R is not discrete, $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) = \{a\}$, and $m^*(\operatorname{bd}(A))$ is closed in R, then $m^*(\operatorname{bd}(A) \setminus \{-a\}) = \operatorname{cl}(m^*(\operatorname{bd}(A) \setminus \{-a\})) \setminus \{-s\}$. Indeed, note that $m^*(\operatorname{bd}(A) \setminus \{-a\}) = m^*(\operatorname{bd}(A)) \setminus \{-s\}$. Since R is not discrete, $\{-s\}$ has empty interior, meaning that $\operatorname{cl}(m^*(\operatorname{bd}(A) \setminus \{-a\})) = \operatorname{cl}(m^*(\operatorname{bd}(A)) \setminus \{-s\}) = m^*(\operatorname{bd}(A))$. Finally, $\operatorname{cl}(m^*(\operatorname{bd}(A) \setminus \{-a\})) \setminus \{-s\} = m^*(\operatorname{bd}(A)) \setminus \{-s\} = m^*(\operatorname{bd}(A) \setminus \{-a\})$.*
5. *If R is a division ring and $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) = \{a\}$, then $s + m^*(\operatorname{bd}(A) \setminus \{-a\}) \subseteq \mathcal{U}(R)$. Indeed, if $b \in \operatorname{bd}(A) \setminus \{-a\}$ satisfies that $s + m^*(b) = 0$, then $-b \in (m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) = \{a\}$, meaning the contradiction that $b = -a$.*

The following definition, which is partly motivated by the previous remark, is based upon the classical notion of strongly rotund point [56] from the Geometry of real or complex Banach Spaces.

Definition 272 (Strongly rotund point) *Let M be a topological module over a topological-inversion ring R. Let $A \subseteq M$ be an additively symmetric body with $0 \in \operatorname{int}(A)$. An element $a \in \operatorname{bd}(A)$ is said to be a strongly rotund point of A provided that there exist $m^* \in M^*$ and $s \in \mathcal{U}(R)$, called supporting functional and support value, respectively, satisfying the following conditions:*

- $m^*(A)$ *is relatively compact in* R.
- $(m^*)^{-1}(\{s\}) \cap \operatorname{int}(A) = \varnothing$.
- $(m^*)^{-1}(\{s\}) \cap \operatorname{bd}(A) = \{a\}$.
- $s + \operatorname{cl}(m^*(\operatorname{bd}(A) \setminus \{-a\})) \setminus \{-s\} \subseteq \mathcal{U}(R)$.
- $\operatorname{stl}_{\mathcal{U}}(-a, b) \cap \operatorname{bd}(A) = \{-a, b\}$ *for all* $b \in \operatorname{bd}(A)$.
- $\operatorname{stl}_{\mathcal{U}}(-a, c) \cap (\operatorname{bd}(A) \setminus \{-a\}) \neq \varnothing$ *for all* $c \in (m^*)^{-1}(\{s\})$.
- *If $(c_j)_{j \in J} \subseteq \operatorname{bd}(A) \setminus \{-a\}$ is a net converging to $-a$, then $\left((m^*(c_j)+s)^{-1}(c_j+a)\right)_{j \in J}$ is not convergent.*

Notice that strongly rotund points are trivially preserved by isomorphisms of topological modules. The following remark gathers several considerations about strongly rotund points.

Remark 81 *Let M be a topological module over a topological-inversion ring R. Let $A \subseteq M$ be an additively symmetric body with $0 \in \mathrm{int}(A)$. Let $a \in \mathrm{bd}(A)$ be a strongly rotund point of A with supporting functional $m^* \in M^*$ and support value $s \in \mathcal{U}(R)$. Observe the following:*

- *$s \neq 0$ because $0 \in \mathrm{int}(A)$ and $(m^*)^{-1}(\{s\}) \cap \mathrm{int}(A) = \varnothing$. Also, if $b \in \mathrm{bd}(A) \setminus \{-a\}$, then $m^*(b) \neq -s$ because otherwise we conclude that $b = -a$ in view of the fact that $(m^*)^{-1}(\{s\}) \cap \mathrm{bd}(A) = \{a\}$. As a consequence, $s + m^*(\mathrm{bd}(A) \setminus \{-a\}) \subseteq s + \mathrm{cl}\,(m^*(\mathrm{bd}(A) \setminus \{-a\})) \setminus \{-s\} \subseteq \mathcal{U}(R)$.*
- *If $\mathrm{char}(R) \neq 2$, then $-a \notin (m^*)^{-1}(\{s\})$ or equivalently $-a \neq a$. Indeed, if $-a \in (m^*)^{-1}(\{s\})$, then $-s = s$, hence $(1+1)s = 0$, so $1 + 1 = 0$ because s is invertible, contradicting that $\mathrm{char}(R) \neq 2$.*

At this stage, we are finally in the right position to construct the stereographic projection.

Theorem 122 *Let M be a topological module over a topological-inversion ring R satisfying that $2 := 1+1 \in \mathcal{U}(R)$. Let $A \subseteq M$ be an additively symmetric body with $0 \in \mathrm{int}(A)$. If there exists a strongly rotund point $a \in \mathrm{bd}(A)$ with supporting functional $m^* \in M^*$ and support value $s \in \mathcal{U}(R)$, then the stereographic projection* (3.16) *is a homeomorphism.*

Proof 396 *First off, let us denote by ϕ to the stereographic projection* (3.16)*. We already know that ϕ is well defined, $\phi(b) \in \mathrm{stl}_{\mathcal{U}}(-a, b)$ for all $b \in \mathrm{bd}(A) \setminus \{-a\}$, and ϕ is trivially continuous due to the fact that R is a topological-inversion ring. Let us check now that ϕ is surjective. Fix an arbitrary $c \in (m^*)^{-1}(\{s\})$. If $c = a$, then $\phi(a) = a$. So let us assume that $c \neq a$. Since a is a strongly rotund point of A, by definition we have that $\mathrm{stl}_{\mathcal{U}}(-a, c) \cap (\mathrm{bd}(A) \setminus \{-a\}) \neq \varnothing$. Let $u \in \mathcal{U}(R)$ such that $b := -a + u(c+a) \in \mathrm{bd}(A) \setminus \{-a\}$. We will show that $\phi(b) = c$. Indeed*

$$\begin{aligned}
\phi(b) &= -a + 2s(m^*(b) + s)^{-1}(b + a) \\
&= -a + 2s(m^*(-a + u(c+a)) + s)^{-1}(-a + u(c+a) + a) \\
&= -a + 2s(-s + u2s + s)^{-1}u(c+a) \\
&= -a + (2s)(2s)^{-1}u^{-1}u(c+a) \\
&= -a + c + a \\
&= c.
\end{aligned}$$

Next step is to prove that ϕ is one-to-one. Indeed, take $b_1, b_2 \in \mathrm{bd}(A) \setminus \{-a\}$ with $\phi(b_1) = \phi(b_2)$. Then $b_2 = -a + (m^*(b_2) + s)(m^*(b_1) + s)^{-1}(b_1 + a) \in \mathrm{stl}_{\mathcal{U}}(-a, b_1) \cap \mathrm{bd}(A) = \{-a, b_1\}$, meaning that $b_1 = b_2$. Let us finally prove that ϕ^{-1} is continuous. We will rely on Lemma 91. Fix an arbitrary $b \in \mathrm{bd}(A) \setminus \{-a\}$. Take a net $(b_i)_{i \in I} \subseteq \mathrm{bd}(A) \setminus \{-a\}$ such that $(\phi(b_i))_{i \in I}$ converges to $\phi(b)$. We will show the existence of a subnet $(c_j)_{j \in J}$ of $(b_i)_{i \in I}$ convergent to b. Indeed, $m^*(A)$ is relatively compact in R; therefore, there exists a subnet $(c_j)_{j \in J}$ of $(b_i)_{i \in I}$ such that $(m^*(c_j))_{j \in J}$ is convergent to some $r \in R$. Then $(\phi(c_j))_{j \in J}$ converges to $\phi(b)$. This is equivalent to saying that $\left((m^*(c_j) + s)^{-1}(c_j + a)\right)_{j \in J}$ converges to $(m^*(b) + s)^{-1}(b + a)$. Since $(m^*(c_j) + s)_{j \in J}$ is convergent to $r + s$, we conclude that $\left((m^*(c_j) + s)(m^*(c_j) + s)^{-1}(c_j + a)\right)_{j \in J}$ converges to $(r+s)(m^*(b) + s)^{-1}(b + a)$, in other words, $(c_j + a)_{j \in J}$ converges to $(r+s)(m^*(b) + s)^{-1}(b + a)$, which is equivalent to stating that $(c_j)_{j \in J}$ converges to $-a + (r + s)(m^*(b) + s)^{-1}(b + a)$. At this point, observe that

$r \neq -s$ since otherwise we obtain that $(c_j)_{j\in J}$ converges to $-a$, reaching the contradiction that $\left((m^*(c_j)+s)^{-1}(c_j+a)\right)_{j\in J}$ is not convergent by bearing in mind Definition 272. As a consequence, $r+s \in s+\operatorname{cl}\left(m^*(\operatorname{bd}(A)\setminus\{-a\})\right)\setminus\{-s\} \subseteq \mathcal{U}(R)$, meaning that $-a+(r+s)(m^*(b)+s)^{-1}(b+a) \in \operatorname{stl}_{\mathcal{U}}(-a,b)$. On the other hand, $\operatorname{bd}(A)$ is closed, hence $-a+(r+s)(m^*(b)+s)^{-1}(b+a) \in \operatorname{bd}(A)$. Since a is a strongly rotund point of A, $\operatorname{stl}_{\mathcal{U}}(-a,b)\cap\operatorname{bd}(A)=\{-a,b\}$. As a consequence, either $-a+(r+s)(m^*(b)+s)^{-1}(b+a)=-a$ or $-a+(r+s)(m^*(b)+s)^{-1}(b+a)=b$. If $-a+(r+s)(m^*(b)+s)^{-1}(b+a)=-a$, then $b=-a$, which is impossible. Thus, $-a+(r+s)(m^*(b)+s)^{-1}(b+a)=b$. By relying on Lemma 91, we conclude that ϕ^{-1} is continuous at b.

In the final results of this subsubsection, we will provide nontrivial examples of stereographic projections. Observe that if M is a module over a ring R, $S\subseteq R$ is a subring of R, and $m^*\in M^*$, then $(m^*)^{-1}(S)$ is an S-submodule of M.

Lemma 148 *Let M be a topological module over a division topological ring R. Let $m^*\in M^*$. If $S\subseteq R$ is a dense subring of R, then $(m^*)^{-1}(S)$ is dense in M.*

Proof 397 *We may assume that $m^*\neq 0$. Then $m^*(R)\cap\mathcal{U}(R)\neq\varnothing$. By Lemma 128, m^* is open. Fix an arbitrary open subset U of M. Then $m^*(U)$ is open in R, meaning that $m^*(U)\cap S\neq\varnothing$. Therefore, $(m^*)^{-1}(S)\cap U\neq\varnothing$.*

The upcoming theorem allows to restrict given stereographic projections via dense subrings.

Theorem 123 *Let M be a topological module over a topological division ring R with $\operatorname{char}(R)\neq 2$. Let $A\subseteq M$ be an additively symmetric body with $0\in\operatorname{int}(A)$. Consider $m^*\in M^*$ and $s\in\mathcal{U}(R)$ such that $(m^*)^{-1}(\{s\})\cap\operatorname{int}(A)=\varnothing$ but $(m^*)^{-1}(\{s\})\cap\operatorname{bd}(A)\neq\varnothing$. Let $a\in(m^*)^{-1}(\{s\})\cap\operatorname{bd}(A)$ such that $s+m^*(\operatorname{bd}(A)\setminus\{-a\})\subseteq\mathcal{U}(R)$. Consider the stereographic projection (3.16). If $S\subseteq R$ is a dense subring of R such that $\mathcal{U}(R)\cap S=\mathcal{U}(S)$ and $s\in\mathcal{U}(S)$, then $B:=A\cap(m^*)^{-1}(S)$ is an additively symmetric body of the S-module $(m^*)^{-1}(S)$ with $0\in\operatorname{int}_{(m^*)^{-1}(S)}(B)$ and the following stereographic projection is well defined and continuous:*

$$\begin{array}{rcl}\operatorname{bd}_{(m^*)^{-1}(S)}(B)\setminus\{-a\} & \to & (m^*)^{-1}(\{s\})\\ b & \mapsto & -a+2s(m^*(b)+s)^{-1}(b+a).\end{array} \tag{3.17}$$

Proof 398 *Note that B is trivially closed in $(m^*)^{-1}(S)$ since A is closed in M. In fact, $0\in\operatorname{int}(A)\cap(m^*)^{-1}(S)\subseteq\operatorname{int}_{(m^*)^{-1}(S)}(B)$. As a consequence, B is an additively symmetric body of the S-module $(m^*)^{-1}(S)$ with $0\in\operatorname{int}_{(m^*)^{-1}(S)}(B)$. It only remains to show that*

$$s+m^*\left(\operatorname{bd}_{(m^*)^{-1}(S)}(B)\setminus\{-a\}\right)\subseteq\mathcal{U}(S).$$

Indeed, $\operatorname{bd}_{(m^*)^{-1}(S)}(B)\subseteq\operatorname{bd}(A)\cap(m^*)^{-1}(S)$, hence

$$\begin{array}{rcl}s+m^*\left(\operatorname{bd}_{(m^*)^{-1}(S)}(B)\setminus\{-a\}\right) & \subseteq & (s+m^*(\operatorname{bd}(A)\setminus\{-a\}))\cap S\\ & \subseteq & \mathcal{U}(R)\cap S=\mathcal{U}(S).\end{array}$$

Finally, notice that, since S is dense in R and R is a topological division ring, then $(m^*)^{-1}(S)$ is dense in M by Lemma 148, which implies, together with the fact that A is closed in M, that $\operatorname{int}(A)\cap(m^*)^{-1}(S)=\operatorname{int}_{(m^*)^{-1}(S)}(B)$ and $\operatorname{bd}(A)\cap(m^*)^{-1}(S)=\operatorname{bd}_{(m^*)^{-1}(S)}(B)$ in view of Lemma 47.

Under the settings of Theorem 123, if S is a proper subring of R, then $(m^*)^{-1}(S)$ is a proper submodule of M (Lemma 128), hence the stereographic projection (3.17) might not be necessarily surjective. Theorem 123 allows plenty of examples of nontrivial stereographic projections.

Example 66 *Let X be a real Banach space of dimension strictly greater than 1. Consider $x \in \mathsf{S}_X$ and $x^* \in \mathsf{S}_{X^*}$ such that $\{x\} = (x^*)^{-1}(\{1\}) \cap \mathsf{S}_X$. According to Remark 80, the stereographic projection*

$$\begin{array}{rcl} \mathsf{S}_X \setminus \{-x\} & \to & (x^*)^{-1}(\{1\}) \\ y & \mapsto & -x + 2(x^*(y)+1)^{-1}(y+x) \end{array} \tag{3.18}$$

is well defined and continuous. Here, $R := \mathbb{R}$, $M := X$, $s = 1$, $m^* = x^*$, and $A := \mathsf{B}_X$. Now, take $S := \mathbb{Q}\left[\sqrt{2}\right]$ and let N denote $(x^*)^{-1}(S)$ as S-module. Also, let $B := \mathsf{B}_X \cap N$. In view of Theorem 123, the stereographic projection

$$\begin{array}{rcl} \mathrm{bd}_N(B) \setminus \{-x\} & \to & (x^*)^{-1}(\{1\}) \\ y & \mapsto & -x + 2(x^*(y)+1)^{-1}(y+x) \end{array} \tag{3.19}$$

is well-defined and continuous.

3.2.4.7 Topological manifolds

In the Differential Calculus Part of this book, we will define differential manifolds. In order to carry out such task successfully, it is desirable to go over topological manifolds first. This is the right place for topological manifolds to be defined and studied.

Definition 273 (Locally linear space) *A topological space X is said to be locally linear if, for every $x \in X$, there exist an open neighborhood U of x, a topological ring R, and a topological R-module M, such that U is homeomorphic to an open subset of M.*

Under the settings of the previous definition, if $\varphi : U \to M$ is a homeomorphism over an open subset $\varphi(U)$ of M, then the pair (U, φ) is called a chart. If $(U, \varphi), (V, \psi)$ are charts with $U \cap V \neq \varnothing$, then the homeomorphism

$$\psi|_{U\cap V} \circ \varphi^{-1}|_{\varphi(U\cap V)} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called a transition map.

Definition 274 (Locally M-linear space) *Let R be a topological ring. Let M be a topological R-module. A topological space X is said to be locally M-linear if, for every $x \in X$, there exists an open neighborhood U of x homeomorphic to an open subset of M.*

Locally M-linear spaces are particular cases of the locally linear ones, which, for obvious reasons, are much harder to deal with.

Definition 275 (Atlas) *Let R be a topological ring. Let M be a topological R-module. Let X be a locally M-linear topological space. An atlas in X is a set $\{(U_\gamma, \varphi_\gamma) : \gamma \in \Gamma\}$ such that $X = \bigcup_{\gamma\in\Gamma} U_\gamma$ and, for every $\gamma \in \Gamma$, $(U_\gamma, \varphi_\gamma)$ is a chart.*

The definition of atlas is the previous step towards the crucial notion of topological manifold, which concludes this brief subsubsection.

Definition 276 (Topological manifold) *Let R be a Hausdorff topological ring. Let M be a Hausdorff topological R-module. A topological space X is said to be a topological M-manifold if X is Hausdorff and locally M-linear.*

3.2.4.8 Algebra topologies

We have mentioned several times already that every ring R is, trivially, an (R, R)-bimodule and an $(\mathsf{Z}(\mathsf{R}), R)$-algebra, where $\mathsf{Z}(R)$ stands for the center of R. Obviously, if R is commutative, then R is an (R, R)-algebra. Let us begin by defining and characterizing topological bimodules.

Definition 277 (Topological bimodule) *A topological bimodule over topological rings* $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$, $(S, \{+, 0, -, \cdot, 1\}, \varnothing)$ *is a topological universal algebra such as* $((M, \tau), \{+, 0, -\}, \{\cdot, \bullet\})$, *where* $(M, \{+, 0, -\}, \{\cdot, \bullet\})$ *is an* (R, S)*-bimodule,* $(M, \{+, 0, -\}, \{\cdot\})$ *is a topological left* R*-module, and* $(M, \{+, 0, -\}, \{\bullet\})$ *is a topological right* S*-module.*

The following theorem, whose proof we omit since it is a direct consequence of Theorem 91, characterizes bimodule topologies.

Theorem 124 *Let* R, S *be topological rings. Let* M *be an* (R, S)*-bimodule. If* τ *is a bimodule topology on* R *and* $\mathcal{B}$ *is a base of neighborhoods of* 0 *in* M*, then the following is verified:*

1. *For every* $V \in \mathcal{B}$*, there exists* $U \in \mathcal{B}$ *with* $U + U \subseteq V$.
2. *For every* $V \in \mathcal{B}$*, there exists* $U \in \mathcal{B}$ *with* $-U \subseteq V$.
3. *For every* $V \in \mathcal{B}$*, there exist* $U \in \mathcal{B}$ *and* $W \in \mathcal{N}_0(R)$ *with* $WU \subseteq V$.
4. *For every* $V \in \mathcal{B}$*, there exist* $U \in \mathcal{B}$ *and* $W \in \mathcal{N}_0(S)$ *with* $UW \subseteq V$.
5. *For every* $V \in \mathcal{B}$ *and every* $r \in R$*, there exists* $U \in \mathcal{B}$ *with* $rU \subseteq V$.
6. *For every* $V \in \mathcal{B}$ *and every* $s \in S$*, there exists* $U \in \mathcal{B}$ *with* $Us \subseteq V$.
7. *For every* $V \in \mathcal{B}$ *and every* $m \in M$*, there exists* $W \in \mathcal{N}_0(R)$ *with* $Wm \subseteq V$.
8. *For every* $V \in \mathcal{B}$ *and every* $m \in M$*, there exists* $W \in \mathcal{N}_0(S)$ *with* $mW \subseteq V$.

Conversely, if $\mathcal{B}$ *is a filter base of* $\mathcal{P}(M)$ *verifying all eight properties above, then there exists a unique bimodule topology on* M *such that* $\mathcal{B}$ *is a base of zero-neighborhoods. This topology is given by*

$$\tau := \{A \subseteq M : \forall a \in A \ \ \exists U \in \mathcal{B} \ \ a + U \subseteq A\} \cup \{\varnothing\}.$$

Observe that the inherited topology on every subbimodule is a bimodule topology. Note also that topological modules over commutative topological rings automatically become topological bimodules. Also, every topological ring is a topological bimodule over itself.

Definition 278 (Topological algebra) *A topological algebra over a topological ring* $(R, \{+, 0, -, \cdot, 1\}, \varnothing)$ *is a topological universal algebra such as* $((A, \tau), \{+, 0, -, \cdot, 1\}, \{\cdot\})$, *where* $(A, \{+, 0, -, \cdot, 1\}, \{\cdot\})$ *is an* R*-algebra and* (A, τ) *is a topological space.*

An algebra topology is, by definition, a topology on an algebra that makes it a topological algebra, that is, that makes continuous the addition, the subtraction, the multiplication, and the external multiplication. Topological algebras are topological rings and topological modules at the same time. It is also immediate to notice that the inherited topology on every subalgebra is an algebra topology.

Theorem 125 *Let* R *be a topological ring. Let* A *be an* R*-algebra. If* τ *is an algebra topology on* A *and* $\mathcal{B}$ *is a base of neighborhoods of* 0*, then the following is verified:*

1. *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $U + U \subseteq V$.*
2. *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $-U \subseteq V$.*
3. *For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $UU \subseteq V$.*
4. *For every $V \in \mathcal{B}$ and every $a \in A$, there exists $U \in \mathcal{B}$ with $aU \cup Ua \subseteq V$.*
5. *For every $V \in \mathcal{B}$, there exist $U \in \mathcal{B}$ and $W \in \mathcal{N}_0(R)$ with $WU \subseteq V$.*
6. *For every $V \in \mathcal{B}$ and every $r \in R$, there exists $U \in \mathcal{B}$ with $rU \subseteq V$.*
7. *For every $V \in \mathcal{B}$ and every $a \in A$, there exists $W \in \mathcal{N}_0(R)$ with $Wa \subseteq V$.*

Conversely, if $\mathcal{B}$ is a filter base of $\mathcal{P}(A)$ verifying all seven properties above, then there exists a unique algebra topology on A such that $\mathcal{B}$ is a base of zero-neighborhoods. This topology is given by

$$\tau := \{U \subseteq A : \forall u \in U \ \ \exists V \in \mathcal{B} \ \ a + V \subseteq U\} \cup \{\varnothing\}.$$

The proof of Theorem 125 will be omitted and left as an exercise since it can immediately obtained by combining together Theorem 81 and Theorem 91.

Example 67 *If A is a seminormed algebra over a seminormed ring R, then the topology induced by the seminorm is an algebra topology.*

The product of any family of topological algebras is a topological algebra with the product topology. As a consequence, if A is a topological algebra and I is a nonempty set, then the pointwise convergence topology on A^I is always an algebra topology.

Theorem 126 *Let A be a unital seminormed algebra over a unital seminormed ring R. If $1 \in \mathrm{cl}(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$, then $\mathsf{U}_A = \mathrm{int}(\mathsf{B}_A)$ is an open unit neighborhood of 0 and $\mathsf{B}_A = \mathrm{cl}(\mathsf{U}_A)$ is a closed unit neighborhood of 0.*

Proof 399 *We will strongly rely on Theorem 84. Thus, it only suffices to show that $1 \in \mathrm{cl}(\mathcal{U}_1(A) \setminus \mathsf{S}_A)$. Indeed, let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R) \setminus \mathsf{S}_R$ be converging to 1 in R. Then the sequence $(u_n 1)_{n\in\mathbb{N}}$ is contained in $\mathcal{U}_1(A) \setminus \mathsf{S}_A$ and converges to 1 in A.*

In virtue of Theorem 126, more examples of unit neighborhoods of zero can be constructed.

Corollary 45 *Let A be a unital seminormed algebra over an absolutely semivalued division ring K. If K is not discrete and $1 \in \mathrm{ext}_2(\mathsf{B}_K)$, then $\mathsf{U}_A = \mathrm{int}(\mathsf{B}_A)$ is an open unit neighborhood of 0 and $\mathsf{B}_A = \mathrm{cl}(\mathsf{U}_A)$ is a closed unit neighborhood of 0.*

Proof 400 *Lemma 118 assures that $1 \in \mathrm{cl}(\mathcal{U}_1(K) \setminus \mathsf{S}_K)$. Theorem 126 comes in to accomplish the desired result.*

Unital seminormed algebras over the reals satisfy the conditions imposed in Corollary 45.

Scholium 1 *If A is a unital seminormed real algebra, then $\mathsf{U}_A = \mathrm{int}(\mathsf{B}_A)$ is an open unit 0-neighborhood and $\mathsf{B}_A = \mathrm{cl}(\mathsf{U}_A)$ is a closed unit 0-neighborhood.*

If $\mathcal{B}_1, \mathcal{B}_2$ are filter bases of $\mathcal{P}(A)$, for A an algebra over a ring R, then $\mathcal{B}_1\mathcal{B}_2 := \{A_1A_2 : A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2\}$ is a filter base of $\mathcal{P}(A)$.

Proposition 103 *Let A be a topological algebra over a topological ring R. Let $\mathcal{B}_1, \mathcal{B}_2$ be filter bases of $\mathcal{P}(A)$. Then $\lim(\mathcal{B}_1)\lim(\mathcal{B}_2) \subseteq \lim(\mathcal{B}_1\mathcal{B}_2)$.*

Proof 401 *Take $a_i \in \lim(\mathcal{B}_i)$ for $i = 1, 2$. Fix an arbitrary 0-neighborhood $U \subseteq A$. Let $W \subseteq A$ be another 0-neighborhood such that $W + W + W \subseteq U$. We can find 0-neighborhoods $V_1, V_2, V_3 \subseteq A$ such that $V_1 a_2 \subseteq W$, $a_1 V_2 \subseteq W$, and $V_3 V_3 \subseteq W$. Take $V := V_1 \cap V_2 \cap V_3$. There are $A_i \in \mathcal{B}_i$, $i = 1, 2$, with $A_i \subseteq a_i + V$. Then $A_1 A_2 \in \mathcal{B}_1\mathcal{B}_2$ and $A_1 A_2 \subseteq (a_1 + V)(a_2 + V) \subseteq a_1 a_2 + (V a_2 + a_1 V + VV) \subseteq a_1 a_2 + (W + W + W) \subseteq a_1 a_2 + U$. This shows that $a_1 a_2 \in \lim(\mathcal{B}_1\mathcal{B}_2)$. As a consequence, $\lim(\mathcal{B}_1)\lim(\mathcal{B}_2) \subseteq \lim(\mathcal{B}_1\mathcal{B}_2)$.*

In order to adapt the previous proposition to limits of functions, the following technical lemma is helpful.

Lemma 149 *Let X be a nonempty set. Let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. Let A be a topological algebra over a topological ring R. Let $f_1, f_2 : X \to A$ be functions. Then $\mathcal{J}((f_1 f_2)(\mathcal{B})) \supseteq \mathcal{J}(f_1(\mathcal{B}) f_2(\mathcal{B}))$.*

Proof 402 *Since $\mathcal{B}$ is a filter base, for every $C, D \in \mathcal{B}$, there exists $E \in \mathcal{B}$ with $E \subseteq C \cap D$, which implies that $(f_1 f_2)(E) \subseteq f_1(C) f_2(D)$. This shows that $\mathcal{J}((f_1 f_2)(\mathcal{B})) \supseteq \mathcal{J}(f_1(\mathcal{B}) f_2(\mathcal{B}))$.*

Lemma 149 allows the following adaption of Proposition 103 to limits of functions.

Corollary 46 *Let X be a nonempty set. Let $\mathcal{B}$ be a filter base of $\mathcal{P}(X)$. Let A be a topological algebra over a topological ring R. Let $f_1, f_2 : X \to A$ be functions. Then $\lim\limits_{\mathcal{B}}(f_1)\lim\limits_{\mathcal{B}}(f_2) \subseteq \lim\limits_{\mathcal{B}}(f_1 f_2)$.*

Proof 403 *According to Lemma 149, $\mathcal{J}((f_1 f_2)(\mathcal{B})) \supseteq \mathcal{J}(f_1(\mathcal{B}) f_2(\mathcal{B}))$. As a consequence, by relying on Proposition 103,*

$$\begin{aligned}
\lim_{\mathcal{B}}(f_1)\lim_{\mathcal{B}}(f_2) &= \lim(f_1(\mathcal{B}))\lim(f_2(\mathcal{B})) \subseteq \lim(f_1(\mathcal{B}) f_2(\mathcal{B})) \\
&= \lim\left(\mathcal{J}(f_1(\mathcal{B}) f_2(\mathcal{B}))\right) \subseteq \lim\left(\mathcal{J}((f_1 f_2)(\mathcal{B}))\right) \\
&= \lim((f_1 f_2)(\mathcal{B})) = \lim_{\mathcal{B}}(f_1 f_2).
\end{aligned}$$

Next corollary is an immediate consequence of the previous results; therefore, its proof is omitted and left as exercise.

Corollary 47 *Let X be a topological space. Let x_0 be a limit point of X. Let A be a topological algebra over a topological ring R. Let $f_1, f_2 : X \setminus \{x_0\} \to A$ be functions. Then $\lim\limits_{x \to x_0}(f_1) \lim\limits_{x \to x_0}(f_2) \subseteq \lim\limits_{x \to x_0}(f_1 f_2)$.*

Likewise the previous corollary, the following one is also a direct consequence and its proof is left as an exercise.

Corollary 48 *Let D be a directed set. Let A be a topological algebra over a topological ring R. Let $\gamma_1, \gamma_2 : D \to A$ be nets. Then $\lim(\gamma_1)\lim(\gamma_2) \subseteq \lim(\gamma_1\gamma_2)$.*

We will show next that the product of bounded subsets is bounded in every topological algebra.

Proposition 104 *Let R be a topological ring. Let A be a topological R-algebra. If $C, D \subseteq A$ are bounded, then CD is also bounded.*

Proof 404 *Fix an arbitrary $U \in \mathcal{N}_0(A)$. We can find $V \in \mathcal{N}_0(A)$ such that $VV \subseteq U$. There are invertibles $r, s \in \mathcal{U}(R)$ such that $C \subseteq rV$ and $D \subseteq sV$. Then $CD \subseteq (rV)(sV) = (rs)VV \subseteq (rs)U$.*

Let R be a practical topological ring. Let A be a nontrivial topological R-algebra. Then $\ell_\infty(\Gamma, A)$ acquires structure of ring for every nonempty set Γ. If, in addition, R is a division ring, then $\ell_\infty(\Gamma, A)$ acquires structure of R-algebra.

Proposition 105 *Let A be a seminormed algebra over a seminormed ring R. Then $\ell_\infty(\Gamma, A)$ acquires structure of seminormed R-algebra for every nonempty set Γ.*

Proof 405 *According to Proposition 94, $\ell_\infty(\Gamma, A)$ acquires structure of seminormed R-module. If $f, g \in \ell_\infty(\Gamma, A)$, then $\|f(\tau)g(\tau)\| \leq \|f(\tau)\|\|g(\tau)\| \leq \|f\|_\infty \|g\|_\infty$ for all $\tau \in \Gamma$, which implies that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$.*

Next proposition deals with the extremal properties of the unity of an ℓ_∞-algebra.

Proposition 106 *Let A be a seminormed algebra over a seminormed ring R. Let Γ be a nonempty set. Then:*

1. *If R, A are both unital, $1 \in \operatorname{ext}_2^w(\mathsf{B}_A)$ if and only if $1 \in \operatorname{ext}_2^w\left(\mathsf{B}_{\ell_\infty(\Gamma,A)}\right)$.*
2. *If R, A are both unital, $1 \in \operatorname{ext}_2(\mathsf{B}_A)$ if and only if $1 \in \operatorname{ext}_2\left(\mathsf{B}_{\ell_\infty(\Gamma,A)}\right)$.*
3. *$\mathcal{U}\left(\ell_\infty(\Gamma, A)\right) \subseteq \left\{f \in \ell_\infty(\Gamma, A) : \forall \tau \in \Gamma\ f(\tau) \in \mathcal{U}(A) \text{ and } \inf\limits_{\tau\in\Gamma} \|f(\tau)\| > 0\right\}$.*
4. *If R, A are both absolutely semivalued, then*

$$\mathcal{U}\left(\ell_\infty(\Gamma, A)\right) = \left\{f \in \ell_\infty(\Gamma, A) : \forall \tau \in \Gamma\ f(\tau) \in \mathcal{U}(A),\ \inf_{\tau\in\Gamma} |f(\tau)| > 0\right\}.$$

Proof 406 *The proof is itemized by following the statement.*

1. *Simply observe that, for every $f, g \in \ell_\infty(\Gamma, A)$, $f, g \in \mathsf{B}_{\ell_\infty(\Gamma,A)}$ if and only if $f(\tau), g(\tau) \in \mathsf{B}_A$ for all $\tau \in \Gamma$, and $2 = f + g$ if and only if $2 = f(\tau) + g(\tau)$ for all $\tau \in \Gamma$.*
2. *It is essentially the same proof as above.*
3. *Take any $f \in \mathcal{U}\left(\ell_\infty(\Gamma, A)\right)$. There exists $g \in \ell_\infty(\Gamma, A)$ such that $fg = gf = 1$, that is, $f(\tau)g(\tau) = g(\tau)f(\tau) = 1$ for all $\tau \in \Gamma$, so $f(\tau) \in \mathcal{U}(A)$ for all $\tau \in \Gamma$. Next, notice that, for every $\tau \in \Gamma$,*

$$1 \leq \|1\| \leq \|f(\tau)g(\tau)\| \leq \|f(\tau)\|\|g(\tau)\|,$$

meaning that

$$\|f(\tau)\| \geq \frac{1}{\|g(\tau)\|} \geq \frac{1}{\|g\|_\infty}$$

for every $\tau \in \Gamma$. As a consequence,

$$\inf_{\tau\in\Gamma} \|f(\tau)\| \geq \frac{1}{\|g\|_\infty} > 0.$$

4. *If $f \in \ell_\infty(\Gamma, A)$ satisfies that, for all $\tau \in \Gamma$, $f(\tau) \in \mathcal{U}(A)$, and $\inf_{\tau\in\Gamma} |f(\tau)| > 0$, then it suffices to define $g(\tau) := f(\tau)^{-1}$. It is clear that $fg = gf = 1$ and*

$$\|g\|_\infty \leq \frac{1}{\inf_{\tau\in\Gamma} |f(\tau)|}.$$

We will conclude this section with the definition of a crucial concept in Spectral Theory: the spectrum of an algebra.

Definition 279 (Spectrum of an algebra) *Let R be a topological ring. Let A be a topological R-algebra. The spectrum of an element $a \in A$ is defined as*

$$\sigma(a) := \{r \in R : a - r1 \notin \mathcal{U}(A)\}.$$

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Part II

Differential Calculus

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Chapter 4

Differentiability

Gottfried Wilhelm Leibniz

4.1 Derivations

A derivation is a special class of operators motivated by Leibniz's Rule. Derivations are the abstract natural generalizations of the classical differential operators.

4.1.1 Leibniz's derivations

Leibniz's Rule, also known as the Product Rule, has more implications in the development of Differential Calculus than it can be thought. In fact, the Quotient Rule and the Chain Rule can be obtained as direct consequences of Leibniz's Rule.

4.1.1.1 Product Rule

The Product Rule is the principal basis upon which derivations are constructed. We will show next the formal definition of derivation along with several familiar examples.

Definition 280 (Derivation) *Let M be a bimodule over a ring R. A derivation is an additive map $D : R \to M$ that satisfies Leibniz's Rule or Product Rule: $D(rs) = D(r)\, s + rD(s)$ for all $r, s \in R$.*

Under the settings of the above definition, if $r \in R$, then $D(r)$ is often called the derivative of r.

Example 68 (Polynomial derivation) *Let R be a ring. The map*

$$\begin{array}{ccc} R[x] & \to & R[x] \\ p(x) & \mapsto & p'(x) \end{array} \tag{4.1}$$

is a derivation, where $p'(x) = na_n x^{n-1} + \cdots + 2a_2 x + a_1$ provided that $p(x) = a_n x^n + \cdots + a_1 x + a_0$.

The derivative of a polynomial is a classic example. Inner derivations are classic examples as well.

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Example 69 (Inner derivation) *Let M be a bimodule over a ring R. Fix $m \in M$. The map*

$$\begin{array}{rccc} D_m: & R & \to & M \\ & r & \mapsto & mr - rm \end{array} \tag{4.2}$$

is a derivation called an inner derivation.

Recall that if $I \subseteq \mathbb{R}$ is an open interval, then $\mathcal{C}^1(I)$ stands for the algebra of real-valued continuously differentiable functions on I, which is a subalgebra of $\mathcal{C}(I)$, and thus $\mathcal{C}(I)$ is a bimodule over $\mathcal{C}^1(I)$.

Example 70 *Let $I \subseteq \mathbb{R}$ be an open interval. The map*

$$\begin{array}{ccc} \mathcal{C}^1(I) & \to & \mathcal{C}(I) \\ f & \mapsto & f' \end{array}$$

is a derivation.

The set of derivations from a ring R into an (R,R)-bimodule M is denoted by $\mathcal{D}(R,M)$, which acquires structure of commutative additive group.

Proposition 107 *Let M be a bimodule over a ring R. Let $D, E : R \to M$ be derivations. Let $t \in \mathrm{Z}(R)$. Then:*

1. *$D+E$ is a derivation.*
2. *tD is a derivation.*

In particular, $\mathcal{D}(R,M)$ is a commutative additive group.

Proof 407 *$D+E$ and tD are both additive maps. Fix arbitrary $r, s \in R$. Then $(D+E)(rs) = D(rs) + E(rs) = rD(s) + D(r)s + rE(s) + E(r)s = r(D+E)(s) + (D+E)(r)s$ and $(tD)(rs) = tD(rs) = trD(s) + tD(r)s = r(tD)(s) + (tD)(r)s$.*

4.1.1.2 Constants

Constants play a crucial role in Derivation Theory. The following definition generalizes the usual concept of constant.

Definition 281 (Constant) *Let M be a bimodule over a ring R. Consider a derivation $D : R \to M$. The elements of $\ker(D)$ will be called constants.*

Notice that $\ker(D)$ is a subring of R, so R is a bimodule over $\ker(D)$, and D becomes $\ker(D)$-linear, that is, $D(sr) = sD(r)$ and $D(rs) = D(r)s$ for all $r \in R$ and $s \in \ker(D)$.

Theorem 127 *Let M be a bimodule over a ring R. Consider a derivation $D : R \to M$. Then:*

1. *1 and -1 are constants.*
2. *$\mathcal{U}(R) \cap \ker(D) = \mathcal{U}(\ker(D))$.*

Proof 408 *The proof follows as itemized in the statement.*

1. *Simply observe that $D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = D(1) + D(1)$. Thus, $D(1) = 0$. Note that $D(-1) = -D(1) = 0$, since D is additive.*

2. *It is trivial that* $\mathcal{U}(\ker(D)) \subseteq \mathcal{U}(R) \cap \ker(D)$. *Let* $r \in \mathcal{U}(R) \cap \ker(D)$. *All we need to show is that* r^{-1} *is a constant. According to Theorem 127(1),*

$$0 = r^{-1}D(1) = r^{-1}D(rr^{-1}) = D(r^{-1}).$$

Let R be a ring. Consider the derivation $D : R[x] \to R[x]$ given in (4.1). Notice that $R \subseteq \ker(D)$.

Theorem 128 *Let* R *be a ring. Consider the derivation* $D : R[x] \to R[x]$ *given in* (4.1). *The following conditions are equivalent:*

1. $R = \ker(D)$.
2. $n1$ *is not a zero divisor in* R *for every* $n \in \mathbb{N}$.

Proof 409 *The proof will follow as itemized as the statement of the theorem.*

1⇒2 *Suppose on the contrary that there exist* $n \in \mathbb{N}$ *and* $a \in R \setminus \{0\}$ *such that* $na = 0$. *Consider* $p(x) = ax^n \in R[x]$. *Since* $a \neq 0$ *and* $n \neq 0$, *we have that* $p(x) \notin R$. *However,* $D(p(x)) = p'(x) = nax^{n-1} = 0$. *This contradicts our hypothesis. Therefore,* n *is not a zero divisor in* R.

2⇒1 *Let* $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x]$ *such that* $p'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} = 0$. *Then* $ka_k = 0$ *for each* $1 \leq k \leq n$. *By hypothesis, we conclude that* $a_k = 0$ *for all* $1 \leq k \leq n$. *As a consequence,* $p(x) = a_0 \in R$.

4.1.2 Rules of derivations

As we mentioned earlier, Quotient Rule and Chain Rule can be obtained from Leibniz's Rule.

4.1.2.1 Quotient Rule

Recall that, in a bimodule M over a ring R, two elements $m \in M$ and $r \in R$ are said to commute provided that $rm = mr$.

Theorem 129 (Quotient Rule) *Let* M *be a bimodule over a ring* R. *Consider a derivation* $D : R \to M$. *Let* $s \in R$ *and* $r \in \mathcal{U}(R)$. *Then:*

$$D\left(sr^{-1}\right) = \left(D(s)\,r - sr^{-1}D(r)\,r\right)r^{-2}.$$

If $D(r)$ *and* r *commute, then* $D\left(sr^{-1}\right) = \left(D(s)\,r - sD(r)\right)r^{-2}$.

Proof 410 *On the one hand,* $0 = D(1) = D\left(rr^{-1}\right) = D(r)\,r^{-1} + rD\left(r^{-1}\right)$, *thus* $D\left(r^{-1}\right) = -r^{-1}D(r)\,r^{-1}$. *On the other hand,* $D\left(sr^{-1}\right) = D(s)\,r^{-1} + sD\left(r^{-1}\right) = D(s)\,r^{-1} - sr^{-1}D(r)\,r^{-1} = \left(D(s)\,r - sr^{-1}D(r)\,r\right)r^{-2}$.

4.1.2.2 Chain Rule

In a bimodule M over a ring R, if two elements $m \in M$ and $r \in \mathcal{U}(R)$ commute, then m and r^{-1} also commute.

Theorem 130 (Polynomial Chain Rule) *Let* M *be a bimodule over a ring* R. *Consider a derivation* $D : R \to M$. *Let* $p \in \ker(D)[x]$ *and* $r \in R$. *If* $D(r)$ *and* r *commute, then*

$$D(p(r)) = p'(r)\,D(r). \tag{4.3}$$

If, in addition, $r \in \mathcal{U}(R)$, *then* $D(r^n) = nr^{n-1}D(r)$ *for all* $n \in \mathbb{Z}$.

Proof 411 *It is sufficient to show that* $D(r^n) = nr^{n-1}D(r)$ *for all* $n \geq 0$*, which we will do by induction. This is obvious for* $n \in \{0,1\}$*, so assume* $n > 1$*. Then* $D(r^n) = D\left(r^{n-1}r\right) = D\left(r^{n-1}\right)r + r^{n-1}D(r) = (n-1)r^{n-2}D(r)r + r^{n-1}D(r) = nr^{n-1}D(r)$*. Suppose next that* $r \in \mathcal{U}(R)$*. By the Quotient Rule, we have that* $D(r^{-1}) = -r^{-2}D(r)$*. Therefore, if* $n < 0$*, then, by Chain Rule,* $D(r^n) = D\left(\left(r^{-1}\right)^{-n}\right) = -n\left(r^{-1}\right)^{-n-1}D(r^{-1}) = nr^{n+1}r^{-2}D(r) = nr^{n-1}D(r)$*.*

A direct consequence of the previous theorem applies to conclude that roots of constants are constants as well.

Corollary 49 *Let* M *be a torsionfree bimodule over an integral domain* R *of characteristic* 0*, Consider a derivation* $D : R \to M$*. If* $D(r)$ *and* r *commute and* $r \in R$ *is an* n^{th}*-root of a constant* $s \in \ker(D)$ *for some* $n \geq 1$*, then* r *is also a constant.*

Proof 412 *If* $s = 0$*, then* $r = 0 \in \ker(D)$ *because* R *is an integral domain and* $r^n = s$*. So, let us assume that* $s \neq 0$*. By Polynomial Chain Rule,*

$$0 = D(s) = D(r^n) = nr^{n-1}D(r).$$

By hypothesis, $\text{char}(R) = 0$*, thus,* $n1 \neq 0$ *in* R*. If* $r^{n-1} = 0$*, then* $s = rr^{n-1} = 0$*, which contradicts our assumption that* $s \neq 0$*. Therefore,* $r^{n-1} \neq 0$*. By using again that* R *is an integral domain,* $nr^{n-1} \neq 0$*. Finally, the fact that* M *is torsionfree assures that* $D(r) = 0$*.*

As an immediate consequence of the previous corollary, we deduce that nilpotent elements and n^{th}-roots of unity are also constants.

Definition 282 *A derivation* $D : R \to R$ *in an extended ring* R *is said to verify the Chain Rule provided that* $D(r \circ s) = (D(r) \circ s)\, D(s)$ *for all* $r, s \in R$*.*

This kind of derivations satisfy basic, yet expected, properties.

Remark 82 *Let* R *be an extended ring and* $D : R \to R$ *a derivation verifying the Chain Rule. Let* $r \in R$*. Then:*

1. $D(r) = D(r)D(I) = (D(I) \circ r)\, D(r)$*. Indeed,* $D(r) = D(r \circ I) = (D(r) \circ I)\, D(I) = D(r)D(I)$ *and* $D(r) = D(I \circ r) = (D(I) \circ r)\, D(r)$*. In particular, if* $D(r) \in \mathcal{U}(R)$*, then* $D(I) = 1$*.*

2. $1 \circ r$ *and* $r \circ 1$ *are constants. Indeed,* $D(1 \circ r) = (D(1) \circ r)\, D(r) = (0 \circ r)D(r) = 0D(r) = 0 and D(r \circ 1) = (D(r) \circ 1)\, D(1) = (D(r) \circ 1)\, 0 = 0$*.*

3. *If* $D(I) = 1$ *and* $r \in \mathcal{U}(R)$*, then* $D(r^{-1})$ *is left-invertible for the multiplication and its left-inverse is* $D(r) \circ r^{-1}$*. Indeed,* $1 = D(I) = D(r \circ r^{-1}) = \left(D(r) \circ r^{-1}\right) D(r^{-1})$*.*

4.1.3 Antiderivatives

Antiderivatives, which are proper objects of Differential Calculus, can also be defined in the context of Derivations.

4.1.3.1 Indefinite integral

Indefinite integral is simply another name for antiderivatives, according to the historical development of Differential and Integral Calculus.

Definition 283 (Antiderivative) *Let M be a bimodule over a ring R. Consider a derivation $D : R \to M$. The indefinite integral or set of antiderivatives of $m \in M$ is defined as*

$$\int m := D^{-1}(m).$$

Under the settings of the previous definition,

$$\int D(r) = r + \ker(D)$$

for all $r \in R$.

4.1.3.2 Linearity

Linearity is a typical property of classical antiderivatives. As expected, the indefinite integral defined here is also linear in some sense.

Proposition 108 *Let M be a bimodule over a ring R. Let $D : R \to M$ be a derivation. Let $u, v \in R$ and $s, t \in \ker(D)$. Then*

$$\int (sD(u) + tD(v)) \supseteq s \int D(u) + t \int D(v).$$

If, in addition, $s \in \mathcal{U}(\ker(D))$, then

$$\int (sD(u) + tD(v)) = s \int D(u) + t \int D(v).$$

Proof 413 *If $k_1, k_2 \in \ker(D)$, then $s(u + k_1) + t(v + k_2) = su + tv + (sk_1 + sk_2) \in su + tv + \ker(D) = \int D(su + tv) = \int (sD(u) + tD(v))$. This shows that $\int (sD(u) + tD(v)) \supseteq s \int D(u) + t \int D(v)$. Conversely, assume that $s \in \mathcal{U}(\ker(D))$ and take $r \in \int (sD(u) + tD(v)) = \int D(su + tv)$. Then $r = su + tv + k$ with $k \in \ker(D)$. Note that $r = s(u + s^{-1}k) + tv \in s \int D(u) + t \int D(v)$, since $s^{-1}k \in \ker(D)$ in virtue of Theorem 127(2).*

4.1.3.3 Integration by parts

Integration by parts is a classical integration technique. As expected, the indefinite integral defined here also satisfies integration by parts.

Proposition 109 *Let M be a bimodule over a ring R. Let $D : R \to M$ be a derivation. Then*

$$\int uD(v) = uv - \int D(u)v$$

for all $u, v \in R$.

Proof 414 *Let $r \in \int uD(v)$. We can write $r = uv - (uv - r)$, so all we need to prove is that $uv - r \in \int D(u)v$. Indeed,*

$$D(uv - r) = D(uv) - D(r) = D(u)v + uD(v) - uD(v) = D(u)v.$$

Conversely, let $r \in \int D(u)v$. We have to show that $uv - r \in \int uD(v)$. Indeed,

$$D(uv - r) = D(uv) - D(r) = D(u)v + uD(v) - D(u)v = uD(v).$$

4.1.3.4 Change of variable

Changes of variables are also possible in derivations that satisfy the Chain Rule on extended rings.

Remark 83 *Let R be an extended ring. Let $D : R \to R$ be a derivation. If D satisfies the Chain Rule, then*

$$\int (D(r) \circ s)\, D(s) = r \circ s + \ker(D)$$

for all $r, s \in R$.

4.2 Derivative

In this section, the concept of real or complex derivative will be extended to the scope of topological modules over topological rings [57].

4.2.1 Differentiability

Differentiability is one of the most important concepts in Mathematics. Here, we will give a categorical and algebraic treatment to this crucial concept, without losing an iota of generality.

4.2.1.1 Directional derivative

We begin by defining a fundamental concept in classical Calculus: Rate of change. This notion constitutes the basis of Differential Calculus.

Definition 284 (Rate of change) *Let M, N be topological modules over a practical topological ring R. Let $A \subseteq M$ be nonempty and open. Consider a map $f : A \to N$. Take $U \subseteq M$ a 0-neighborhood such that $m + U \subseteq A$. Let $W \subseteq R$ be another 0-neighborhood such that $Wl \subseteq U$. The rate of change of f at $m \in M$ in the direction of $l \in M$ by $r \in \mathcal{U}(R) \cap W$ is defined as*

$$\Delta_l f(m)(r) := r^{-1} \left(f\left(m + rl\right) - f\left(m\right) \right).$$

Under the settings of the previous definition, the following associated function will be considered:

$$\begin{array}{rccc} \Delta_l f(m): & \mathcal{U}(R) \cap W & \to & N \\ & r & \mapsto & \Delta_l f(m)(r). \end{array} \tag{4.4}$$

Notice that there can be found $W \in \mathcal{N}_0(R), V \in \mathcal{N}_0(M)$ such that $WV \subseteq U$, hence the following associated functions will also be considered:

$$\begin{array}{rccc} \Delta f(m)(r): & V & \to & N \\ & l & \mapsto & \Delta_l f(m)(r) \end{array} \tag{4.5}$$

for $r \in W$, and

$$\begin{array}{rccc} \Delta f(m): & \mathcal{U}(R) \cap W & \to & N^V \\ & r & \mapsto & \Delta f(m)(r). \end{array} \tag{4.6}$$

Recall that if R is a practical topological ring, then 0 is not an isolated point of $\mathcal{U}(R) \cup \{0\}$ and thus $\mathcal{N}_0^{\times}\left(\mathcal{U}(R) \cup \{0\}\right)$ constitutes a filter. Therefore, it makes sense to consider the limit as $r \in \mathcal{U}(R)$ tends to 0.

Definition 285 (Directional derivative) *Let M, N be topological modules over a practical topological ring R. Let $A \subseteq M$ be nonempty and open. Consider a map $f : A \to N$. Take $W \subseteq R$ a 0-neighborhood such that $m + rl \in A$ for all $r \in W$. We say that f is differentiable at $m \in A$ in the direction of $l \in M$ if $\lim_{r\to 0} \Delta_l f(m)(r) \neq \varnothing$. The previous set is called the derivative of f at m in the direction of l and is usually denoted by $f'_l(m)$ or $\frac{\partial f}{\partial l}(m)$ or $\mathrm{D}_l f(m)$, that is,*

$$f'_l(m) := \frac{\partial f}{\partial l}(m) := \mathrm{D}_l f(m) := \lim_{r\to 0} \Delta_l f(m)(r).$$

Under the settings of Definition 285, $\lim_{r\to 0} \Delta_l f(m)(r)$ does not change if another 0-neighborhood $W' \subseteq R$ is chosen satisfying that $m + W'l \subseteq A$. If N is Hausdorff, then $\lim_{r\to 0} \Delta_l f(m)(r)$ is either empty or a singleton. Therefore, in case that N is Hausdorff and f is differentiable at m in the direction of l, we will identify $\frac{\partial f}{\partial l}(m)$ with its only element.

Example 71 *Let M, N be topological modules over a practical topological ring R. Let $A \subseteq M$ be nonempty and open. Consider a map $f : A \to N$. Then:*

- *If f is constant, then f is differentiable at every $m \in A$ in the direction of every $l \in M$ and $0 \in \mathrm{D}_l f(m)$. Indeed, for all $r \in \mathcal{U}(R)$, all $m \in A$ and all $l \in M$, $r^{-1}(f(m + rl) - f(m)) = 0$.*
- *If $f : M \to N$ is linear, then f is differentiable at every $m \in M$ in the direction of every $l \in M$ and $f(l) \in \mathrm{D}_l f(m)$. Indeed, for all $r \in \mathcal{U}(R)$ and all $m, l \in M$, $r^{-1}(f(m + rl) - f(m)) = f(l)$.*

A not so trivial example of differentiable function for topological rings follows. Recall that in any ring R, $r^{-1} - s^{-1} = r^{-1}(s - r)s^{-1}$ for all $r, s \in R$.

Theorem 131 *Let R be a topological ring. Let S be a subring of R such that $\mathcal{U}(S) \subseteq \mathsf{Z}(\mathcal{U}(R))$. If $0 \in \mathrm{cl}(\mathcal{U}(S))$, $\mathcal{U}(R)$ is open, and the inversion map on R is continuous, then the inversion map on R is S-differentiable and its derivative at $u \in \mathcal{U}(R)$ in the direction of $v \in R$ is given by $-u^{-1}vu^{-1} \in \mathrm{D}_v\left(\ ^{-1}\right)(u)$.*

Proof 415 *Fix arbitrary elements $u \in \mathcal{U}(R)$ and $v \in R$. Let $W \subseteq R$ be a 0-neighborhood. We can find a 0-neighborhood $U \subseteq R$ such that $Uvu^{-1} \subseteq W$. Take $U_0 \subseteq R$ a 0-neighborhood such that $u + U_0 \subseteq \mathcal{U}(R)$. Since inversion on R is continuous, there exists an additively symmetric 0-neighborhood $U_2 \subseteq R$ so that if $w \in U_2 \cap U_0$, then $(u+w)^{-1} - u^{-1} \in U$. There exists a 0-neighborhood $U_1 \subseteq R$ such that $U_1 v \subseteq U_2 \cap U_0$. For every $s \in U_1 \cap \mathcal{U}(S)$, we have that $u + sv \in u + U_1 v \subseteq u + U_0 \subseteq \mathcal{U}(R)$ and*

$$\begin{aligned} & s^{-1}\left((u+sv)^{-1} - u^{-1}\right) - \left(-u^{-1}vu^{-1}\right) \\ = \ & s^{-1}(u+sv)^{-1}(-sv)u^{-1} + u^{-1}vu^{-1} \\ = \ & -(u+sv)^{-1}vu^{-1} + u^{-1}vu^{-1} \\ = \ & \left(-(u+sv)^{-1} + u^{-1}\right)vu^{-1} \in Uvu^{-1} \subseteq W. \end{aligned}$$

As a consequence, $-u^{-1}vu^{-1} \in \mathrm{D}_v\left(\ ^{-1}\right)(u)$.

Under certain conditions, next lemma assures that differentiability implies some continuity.

Lemma 150 *Let R be an integral domain, a practical topological ring. Let M be a strongly torsionfree topological R-module. Consider $A \subseteq M$ nonempty and open. Let N be a topological R-module. If $f : A \to N$ is differentiable at $m \in A$ in the direction of $l \in M \setminus \{0\}$, then $f|_B$ is continuous at m, where $B := m + (W \cap (\mathcal{U}(R) \cup \{0\}))\, l$ and $W \subseteq R$ is a 0-neighborhood such that $m + Wl \subseteq A$.*

Proof 416 *In the first place, the fact that the right-multiplication operator $\mathsf{R}_l : R \to Rl$ be a homeomorphism over its image implies that the neighborhoods of m in the relative topology of B are of the form $m + (W' \cap W \cap (\mathcal{U}(R) \cup \{0\}))\, l$ for W' a 0-neighborhood of R. Fix an arbitrary 0-neighborhood V of N. We can consider 0-neighborhoods $W_1 \subseteq R$, $V_0 \subseteq N$ and $V_1 \subseteq N$ such that $V_0 + V_0 \subseteq V$ and $W_1 V_1 \subseteq V_0$. Take any $n \in \mathrm{D}_l f(m)$. Another 0-neighborhood $W_2 \subseteq R$ can be found so that $W_2 n \subseteq V_0$. There exists a 0-neighborhood $W_3 \subseteq R$ so that if $r \in W_3 \cap W \cap \mathcal{U}(R)$, then $r^{-1}(f(m + rl) - f(m)) - n \in V_1$. Finally, if $r \in W \cap W_1 \cap W_2 \cap W_3 \cap \mathcal{U}(R)$, then $f(m + rl) - f(m) = r\left(r^{-1}(f(m + rl) - f(m)) - n\right) + rn \in W_1 V_1 + W_2 n \subseteq V_0 + V_0 \subseteq V$.*

As an immediate corollary, we obtain the following.

Corollary 50 *Let K be a nondiscrete topological division ring. Let Y be a topological vector space over K. If $f : K \to Y$ is K-differentiable, then it is continuous.*

Proof 417 *In the first place, K is practical because it is not discrete. Since K is a division ring, K is trivially an integral domain as well as strongly torsionfree over itself. Finally, we can take $W := K$ and $B := k + Kl = K$, for any arbitrarily fixed $k \in K$ and $l \in K^\times$, and directly apply Lemma 150.*

The following lemma displays basic properties of directional derivatives that relate them to R-linear mappings.

Lemma 151 *Let M, N be topological modules over a practical topological ring R. Let $A \subseteq M$ be nonempty and open. Consider a map $f : A \to N$. Then:*

1. *f is differentiable at any $m \in A$ in the direction of $0 \in M$ and $0 \in \mathrm{D}_0 f(m)$.*
2. *If $r \in \mathcal{U}(R)$ and f is differentiable at $m \in A$ in the direction of $l \in M$, then f is differentiable at m in the direction of rl and*

$$\mathrm{D}_{rl} f(m) = r\, \mathrm{D}_l f(m). \tag{4.7}$$

Proof 418 *The proof follows as itemized in the statement.*

1. *It trivially follows from the fact that $r^{-1}(f(m + r0) - f(m)) = 0$ for all $r \in \mathcal{U}(R)$.*
2. *We will prove first that $r\, \mathrm{D}_l f(m) \subseteq \mathrm{D}_{rl} f(m)$. Fix an arbitrary $n \in \mathrm{D}_l f(m)$. Let $V \subseteq N$ be a 0-neighborhood. There exists a 0-neighborhood $V_0 \subseteq N$ such that $rV_0 \subseteq V$. Let $U \subseteq M$ be a 0-neighborhood such that $m + U \subseteq A$. There exists a 0-neighborhood $W \subseteq R$ such that $Wl \subseteq U$. Since f is differentiable at m in the direction of l, there exists a 0-neighborhood $W_1 \subseteq R$ so that if $t \in W_1 \cap W \cap \mathcal{U}(R)$, then $m + tl \in m + Wl \subseteq m + U \subseteq A$ and $t^{-1}(f(m + tl) - f(m)) - n \in V_0$. We can find a 0-neighborhood $W_2 \subseteq R$ such that $W_2 r \subseteq W_1 \cap W$. Fix an arbitrary $s \in W_2 \cap \mathcal{U}(R)$. Note that $sr \in W_2 r \subseteq W_1 \cap W$ and $m + s(rl) = m + (sr)l \in M + Wl \subseteq M + U \subseteq A$. Also,*

$$\begin{aligned} & s^{-1}(f(m + s(rl)) - f(m)) - rn \\ = \;& r\left((sr)^{-1}(f(m + (sr)l) - f(m)\right) - rn \\ = \;& r\left[\left((sr)^{-1}(f(m + (sr)l) - f(m)\right) - n\right] \in rV_0 \subseteq V. \end{aligned}$$

As a consequence, $rn \in \mathrm{D}_{rl} f(m)$*, meaning that* $r \mathrm{D}_l f(m) \subseteq \mathrm{D}_{rl} f(m)$*. Since* r *is invertible,* $\mathrm{D}_{rl} f(m) = rr^{-1} \mathrm{D}_{rl} f(m) \subseteq r \mathrm{D}_{r^{-1}rl} f(m) = r \mathrm{D}_l f(m)$*.*

The following proposition establishes R-linearity of directional derivatives.

Proposition 110 *Let* M, N *be topological modules over a practical topological ring* R*. Let* $A \subseteq M$ *be nonempty and open. If* $f, g : A \to N$ *are differentiable at* $m \in A$ *in the direction of* $l \in M$*, then* $rf + sg$ *is differentiable at* m *in the direction of* l *for every* $r, s \in \mathsf{Z}(R)$ *and*

$$\mathrm{D}_l(rf + sg)(m) \supseteq r \mathrm{D}_l f(m) + s \mathrm{D}_l g(m). \tag{4.8}$$

If, in addition, N *is Hausdorff, then* (4.8) *is an equality.*

Proof 419 *Fix arbitrary elements* $n_1 \in \mathrm{D}_l f(m)$ *and* $n_2 \in \mathrm{D}_l g(m)$*. Let* $V \subseteq N$ *be a* 0*-neighborhood. There exists a* 0*-neighborhood* $V_0 \subseteq N$ *such that* $V_0 + V_0 \subseteq V$*. We can find another* 0*-neighborhood* $V_1 \subseteq N$ *such that* $rV_1 \cup sV_1 \subseteq V_0$*. Take a* 0*-neighborhood* $W \subseteq R$ *such that* $m + Wl \subseteq A$*. Since* f, g *are differentiable at* m *in the direction of* l*, there exists a* 0*-neighborhood* $W_1 \subseteq R$ *so that if* $t \in W_1 \cap W \cap \mathcal{U}(R)$*, then* $m + tl \in m + Wl \subseteq A$*,* $t^{-1}(f(m + tl) - f(m)) - n_1 \in V_1$ *and* $t^{-1}(g(m + tl) - g(m)) - n_2 \in V_1$*. Then for every* $t \in W_1 \cap W \cap \mathcal{U}(R)$*, we obtain that*

$$\begin{aligned}
& t^{-1}((rf + sg)(m + tl) - (rf + sg)(m)) - (rn_1 + sn_2) \\
= \; & t^{-1}r\left(f(m + tl) - f(m)\right) + t^{-1}s\left(g(m + tl) - g(m)\right) - (rn_1 + sn_2) \\
= \; & r\left(t^{-1}(f(m + tl) - f(m))\right) + s\left(t^{-1}(g(m + tl) - g(m))\right) - (rn_1 + sn_2) \\
= \; & r\left(t^{-1}(f(m + tl) - f(m)) - n_1\right) + s\left(t^{-1}(g(m + tl) - g(m)) - n_2\right) \\
\in \; & rV_1 + sV_1 \subseteq V_0 + V_0 \subseteq V.
\end{aligned}$$

As a consequence, $rn_1 + sn_2 \in \mathrm{D}_l(rf + sg)(m)$*, meaning that* $r \mathrm{D}_l f(m) + s \mathrm{D}_l g(m) \subseteq \mathrm{D}_l(rf + sg)(m)$*.*

Under the settings of Proposition 110, it is not hard to check that if, in addition, $r, s \in \mathcal{U}(R)$, then by relying on Lemma 151(2), equality in (4.8) can also be accomplished.

Proposition 111 *Let* R *be a practical topological ring. Let* M *be a topological* R*-module. Let* N *be a topological* R*-algebra. Let* $A \subseteq M$ *be nonempty and open. If* $f, g : A \to N$ *are differentiable at* $m \in A$ *in the direction of* $l \in M$ *and either* f *or* g *is continuous at* m*, then* fg *is differentiable at* m *in the direction of* l *and*

$$\mathrm{D}_l(fg)(m) \supseteq \mathrm{D}_l f(m) g(m) + f(m) \mathrm{D}_l g(m). \tag{4.9}$$

If, in addition, N *is Hausdorff, then* (4.9) *is an equality.*

Proof 420 *First off, we may assume without any loss of generality that* g *is continuous at* m *because the other case follows a similar proof. Fix arbitrary elements* $n_1 \in \mathrm{D}_l f(m)$ *and* $n_2 \in \mathrm{D}_l g(m)$*. Let* $V \subseteq N$ *be a* 0*-neighborhood. There exists a* 0*-neighborhood* $V_0 \subseteq N$ *such that* $V_0 + V_0 + V_0 + V_0 \subseteq V$*. We can find another* 0*-neighborhood* $V_1 \subseteq N$ *with* $V_1 V_1 \subseteq V_0$*,* $f(m)V_1 \subseteq V_0$*,* $n_1 V_1 \subseteq V_0$ *and* $V_1 g(m) \subseteq V_0$*. Take a* 0*-neighborhood* $W \subseteq R$ *such that* $m + Wl \subseteq A$*. Since* f, g *are differentiable at* m *in the direction of* l*, there is a* 0*-neighborhood* $W_1 \subseteq R$ *so that if* $t \in W_1 \cap W \cap \mathcal{U}(R)$*, then* $m + tl \in m + Wl \subseteq A$*,* $t^{-1}(f(m + tl) - f(m)) - n_1 \in V_1$ *and* $t^{-1}(g(m + tl) - g(m)) - n_2 \in V_1$*. Continuity of* g *at* m *implies the existence of a*

0-*neighborhood* $U \subseteq M$ *such that* $g(m+m_U)-g(m) \in V_1$ *for all* $m_U \in U$. *Take a* 0-*neighborhood* $W_2 \subseteq R$ *such that* $W_2 l \subseteq U$. *For every* $t \in W_2 \cap W_1 \cap W \cap \mathcal{U}(R)$, *we have that*

$$\begin{aligned}
& t^{-1}((fg)(m+tl)-(fg)(m))-(n_1 g(m)+f(m)n_2) \\
= \;& t^{-1}(f(m+tl)-f(m))g(m+tl)+f(m)t^{-1}(g(m+tl)-g(m)) \\
- \;& n_1 g(m+tl)+n_1 g(m+tl)-n_1 g(m)-f(m)n_2 \\
= \;& \left[t^{-1}(f(m+tl)-f(m))-n_1\right] g(m+tl) \\
+ \;& f(m)\left[t^{-1}(g(m+tl)-g(m))-n_2\right] \\
+ \;& n_1\left(g(m+tl)-g(m)\right)-\left[t^{-1}(f(m+tl)-f(m))-n_1\right] g(m) \\
+ \;& \left[t^{-1}(f(m+tl)-f(m))-n_1\right] g(m) \\
= \;& \left[t^{-1}(f(m+tl)-f(m))-n_1\right]\left(g(m+tl)-g(m)\right) \\
+ \;& f(m)\left[t^{-1}(g(m+tl)-g(m))-n_2\right] \\
+ \;& n_1\left(g(m+tl)-g(m)\right)+\left[t^{-1}(f(m+tl)-f(m))-n_1\right] g(m) \\
\in \;& V_1V_1+f(m)V_1+n_1V_1+V_1g(m) \subseteq V_0+V_0+V_0+V_0 \subseteq V.
\end{aligned}$$

As a consequence of all the above, $n_1 g(m)+f(m)n_2 \in \mathrm{D}_l(fg)(m)$, *meaning that* $\mathrm{D}_l f(m)g(m)+f(m)\,\mathrm{D}_l g(m) \subseteq \mathrm{D}_l(fg)(m)$.

4.2.1.2 Differentiable functions

The following definition establishes the notion of differential in the topological module setting. From this step on, we will begin assuming some Hausdorff conditions on the final module.

Definition 286 (Differential) *Let* M, N *be topological modules over a practical topological ring* R *such that* N *is Hausdorff. Let* $A \subseteq M$ *be nonempty and open. Consider a map* $f : A \to N$. *We say that:*

- f *is differentiable at* $m \in A$ *provided that* f *is differentiable at* m *in the direction of every* $l \in M$ *and the following operator (called the differential of* f *at* m*)*

$$\begin{array}{rccc} \mathrm{D}\,f(m): & M & \to & N \\ & l & \mapsto & \mathrm{D}\,f(m)(l) := \mathrm{D}_l\,f(m) \end{array} \tag{4.10}$$

  is R*-linear and continuous.*

- f *is differentiable provided that* f *is differentiable at every* $m \in A$.

Under the settings of Definition 286, the R-linearity of $\mathrm{D}\,f(m)$ means that $\mathrm{D}_{r_1 l_1 + r_2 l_2}\,f(m) = \mathrm{D}\,f(m)(r_1 l_1 + r_2 l_2) = r_1\,\mathrm{D}\,f(m)(l_1) + r_2\,\mathrm{D}\,f(m)(l_2) = r_1\,\mathrm{D}_{l_1}\,f(m) + r_2\,\mathrm{D}_{l_2}\,f(m)$, for all $r_1, r_2 \in R$ and all $l_1, l_2 \in M$. In case f is differentiable, the map

$$\begin{array}{rccc} \mathrm{D}\,f: & A \times M & \to & N \\ & (m,l) & \mapsto & \mathrm{D}\,f(m,l) := \mathrm{D}_l\,f\,(m) \end{array} \tag{4.11}$$

is called the differential of f. In the context of Banach Space Theory, some authors call f a Gâteaux differentiable function.

Definition 287 *Let* M, N *be topological modules over a practical topological ring* R *such that* N *is Hausdorff. Let* $A \subseteq M$ *be nonempty and open. We define the sets*

$$\mathcal{D}(A,N) := \{f : A \to N : f \text{ is differentiable}\}$$

and

$$\mathcal{CD}(A,N) := \{f : A \to N : f \text{ is continuous and differentiable}\}.$$

In accordance with Proposition 110, $\mathcal{D}(A,N)$ is a $\mathsf{Z}(R)$-module. According to Proposition 111 (and under its settings), $\mathcal{CD}(A,N)$ is a $\mathsf{Z}(R)$-algebra. Note that $N^{A\times M}$ can be endowed with structure of $\mathcal{CD}(A,N)$-bimodule as follows: if $f \in \mathcal{CD}(A,N)$ and $g \in N^{A\times M}$, then $fg \in N^{A\times M}$, where $(fg)(m,l) = f(m)g(m,l)$, and $gf \in N^{A\times M}$, where $(gf)(m,l) = g(m,l)f(m)$.

Theorem 132 *Let R be a practical topological ring. Let M be a topological R-module. Let N be a Hausdorff topological R-algebra. Let $A \subseteq M$ be nonempty and open. Then*

$$\begin{array}{rccc} \mathrm{D}: & \mathcal{CD}(A,N) & \to & N^{A\times M} \\ & f & \mapsto & \mathrm{D}\, f \end{array}$$

is a derivation.

Proof 421 *Fix arbitrary elements $f, g \in \mathcal{CD}(A,N)$. In view of Proposition 110, we have that $\mathrm{D}(f+g) = \mathrm{D}\, f + \mathrm{D}\, g$, that is, D is additive. In fact, Proposition 110 assures that D is $\mathsf{Z}(R)$-linear. Finally, by taking into consideration Proposition 111 together with the structure of $\mathcal{CD}(A,N)$-bimodule $N^{A\times M}$ is endowed with, we conclude that $\mathrm{D}(fg) = (\mathrm{D}\, f)g + f\,\mathrm{D}\, g$, that is, D satisfies Leibniz's Product Rule.*

Under the settings of Proposition 111, $N^{A\times M}$ does not acquire structure of $\mathcal{CD}(A,N)$-algebra, in general. Indeed, if $f \in \mathcal{CD}(A,N)$ and $g_1, g_2 \in N^{A\times M}$, then $f(g_1g_2) = (fg_1)g_2$ but $f(g_1g_2) \neq g_1(fg_2)$ unless N is commutative.

4.2.1.3 Uniformly differentiable functions

Let M, N be topological modules over a practical topological ring R such that N is Hausdorff. Let $A \subseteq M$ be nonempty and open. Consider a function $f : A \to N$ differentiable at certain $m \in A$. Uniform differentiability is about uniform convergence of $\Delta_l f(m)(r)$ to $\mathrm{D}_l\, f(m)$ on l in an upward directed collection of subsets of M as $r \to 0$. There exists $U \in \mathcal{N}_0(M)$ such that $m + U \subseteq A$. There are $W \in \mathcal{N}_0(R), V \in \mathcal{N}_0(M)$ so that $WV \subseteq U$. Notice that for every bounded subset C of M there exists an invertible $u_C \in \mathcal{U}(R)$ such that $u_C C \subseteq V$, meaning that $W u_C C \subseteq WV \subseteq U$, so, if we take $W_C := W u_C$, then we are essentially saying that there exists a 0-neighborhood $W_C \subseteq R$ such that $m + W_C C \subseteq A$.

Definition 288 *Let M, N be topological modules over a practical topological ring R such that N is Hausdorff. Let $A \subseteq M$ be nonempty and open. Consider a function $f : A \to N$. We say that:*

- *f is uniformly differentiable at $m \in A$ if f is differentiable at m and there exist $W \in \mathcal{N}_0(R), V \in \mathcal{N}_0(M)$ such that $m + WV \subseteq A$ and*

  $$\lim_{r\to 0} \Delta f(m)(r) = \mathrm{D}\, f(m)$$

  in N^V endowed with the finest uniform convergence additive topology.

- *f is uniformly differentiable if it is uniformly differentiable at every $m \in A$.*

In the context of Banach Space Theory, some authors call f a Fréchet differentiable function. Under the settings of previous definition, there is no guarantee that $\Delta f(m)(r)(V)$ is a bounded subset of N for every $r \in W$, this is why the convergence topology on N^V is called additive instead of linear. The following remark expresses the uniform differentiability concept in classical analytical terms.

Remark 84 *Let M, N be topological modules over a practical topological ring R such that N is Hausdorff. Let $A \subseteq M$ be nonempty and open. Consider a uniformly differentiable function $f : A \to N$ at $m \in A$. Let $W \in \mathcal{N}_0(R), V \in \mathcal{N}_0(M)$ be so that $m + WV \subseteq A$. The expression*

$$\lim_{r \to 0} \Delta f(m)(r) = \mathrm{D}\, f(m)$$

in N^V endowed with the finest uniform convergence additive topology means that, for every $U \in \mathcal{N}_0(N)$, there exists $W_U \in \mathcal{N}_0(R)$ such that

$$r^{-1}(f(m + rl) - f(m)) \in \mathrm{D}_l\, f(m) + U$$

for all $r \in W_U \cap W \cap \mathcal{U}(R)$ and every $l \in V$.

Constant functions and continuous linear maps are examples of uniformly differentiable functions.

Example 72 *Let M, N be topological modules over a practical topological ring R such that N is Hausdorff. Let $A \subseteq M$ be nonempty and open. Consider a map $f : A \to N$. Then:*

- *If f is constant, then f is uniformly differentiable.*
- *If $f : M \to N$ is linear and continuous, then f is uniformly differentiable.*

4.2.2 Fundamental theorems

This subsection is devoted to extend classical fundamental theorems from Differential Calculus to the scope of topological modules.

4.2.2.1 Critical points

A point x of a topological space X is called a relative extreme of a function $f : X \to A$, where A is a poset, provided that there exists a neighborhood U of X such that either $f(x) = \max f(U)$ (relative maximum) or $f(x) = \min f(U)$ (relative minimum).

Definition 289 *Let M, N be topological modules over a practical topological ring R such that N is Hausdorff. Let $A \subseteq M$ be nonempty and open. Consider a function $f : A \to N$. Then:*

- *$m \in A$ is said to be a critical point of f in the direction of $l \in M$ provided that f is differentiable at m in the direction of l and $\mathrm{D}_l\, f(m) = 0$.*
- *$m \in A$ is said to be a critical point of f provided that it is a critical point of f in every direction.*

Recall that by topological totally ordered ring, we mean a totally ordered ring in such a way that the order topology is a ring topology. Recall also that tosets endowed with the order topology are always Hausdorff.

Proposition 112 *Let R be a practical topological totally ordered ring. Let M be a topological R-module. Let $A \subseteq M$ be nonempty and open. If $f : A \to R$ is differentiable at $m \in A$ in the direction of $l \in M$ and m is a relative extreme of f, then m is a critical point of f in the direction of l.*

Proof 422 *Suppose to the contrary that* $\mathrm{D}_l f(m) \neq 0$. *We may assume without any loss of generality that* $\mathrm{D}_l f(m) < 0$ *and* m *is a relative maximum of* f. *The other cases follow a similar proof. Let* U *be a neighborhood of* m *in such a way that* $m \in \arg\max f(U \cap A)$. *Take an interval* $(-t,t) \subseteq R$ *such that* $m + (-t,t)l \subseteq A$ *and satisfying that, if* $r \in (-t,t) \cap \mathcal{U}(R)$, *then* $r^{-1}(f(m+rl) - f(m)) \in (-\infty, 0)$. *Notice that, if* $r \in (-t,0)$, *then we obtain the contradiction that* $r^{-1}(f(m+rl) - f(m)) \geq 0$.

4.2.2.2 Rolle's Theorem

Famous Rolle's Theorem also holds for the differentiability settings that we have been developing here.

Theorem 133 (Rolle's Theorem) *Let* R *be a Dedekind complete, practical topological totally ordered ring. Let* M *be a topological* R*-module. Let* A *be a nonempty open subset of* M. *If* $f : \mathrm{cl}(A) \to R$ *is continuous,* $f(\mathrm{cl}(A))$ *is closed and bounded,* $f|_A$ *is differentiable in the direction of* $l \in M$, *and* $f|_{\partial A}$ *is constant, then there exists* $a \in A$ *such that* $\mathrm{D}_l f(a) = 0$.

Proof 423 *By hypothesis,* $f(\mathrm{cl}(A))$ *has a supremum and an infimum. Since* $f(\mathrm{cl}(A))$ *is closed, the supremum and the infimum belong to* $f(\mathrm{cl}(A))$, *becoming* $\max f(\mathrm{cl}(A))$ *and* $\min f(\mathrm{cl}(A))$, *respectively. This makes possible the existence of elements* $a \in \arg\max f(\mathrm{cl}(A))$ *and* $b \in \arg\min f(\mathrm{cl}(A))$. *If both* $a, b \in \partial A$, *then the condition that* $f|_{\partial A}$ *be constant implies that* $f|_A$ *is also constant, thus* $\mathrm{D}_l f(m) = 0$ *for all* $m \in A$. *So, assume for instance that* $a \in A$. *In accordance with Proposition 112, we have that* $\mathrm{D}_l f(a) = 0$ *by bearing in mind that* a *is a relative maximum of* f.

4.2.2.3 Chain Rule

In order for the Chain Rule to work in the scope of topological modules, we need to make use of uniform differentiability.

Theorem 134 (Chain Rule) *Let* M, N, P *be topological modules over a practical topological ring* R *such that* N, P *are Hausdorff. Let* $A \subseteq M$ *and* $B \subseteq N$ *be nonempty and open. Let* $f : B \to P$ *and* $g : A \to N$ *such that* $g(A) \subseteq B$, g *is differentiable at* $m \in A$ *in the direction of* $l \in M$, *and* f *is uniformly differentiable at* $g(m)$. *Then* $f \circ g$ *is differentiable at* m *in the direction of* l *and*

$$\mathrm{D}_l (f \circ g)(m) = \mathrm{D}_{\mathrm{D}_l g(m)} f(g(m)).$$

Proof 424 *Fix an arbitrary* $U \in \mathcal{N}_0(P)$ *and take* $U_1 \in \mathcal{N}_0(P)$ *with* $U_1 + U_1 \subseteq U$. *Since* f *is uniformly differentiable at* $g(m)$, *there exist open* 0*-neighborhoods* $W_1 \in \mathcal{N}_0(R), V_1 \in \mathcal{N}_0(N)$ *such that* $g(m) + W_1 V_1 \subseteq B$ *and, for every* $U' \in \mathcal{N}_0(P)$, *there exists* $W' \in \mathcal{N}_0(R)$ *such that*

$$r^{-1}(f(g(m) + rk) - f(g(m))) - \mathrm{D}_k f(g(m)) \in U'$$

for all $r \in W' \cap W_1 \cap \mathcal{U}(R)$ *and all* $k \in V_1$. *There exists* $W_2 \in \mathcal{N}_0(R)$ *such that* $W_2 \mathrm{D}_l g(m) \subseteq V_1$. *Since* R *is practical, we can fix* $w_2 \in W_2 \cap \mathcal{U}(R)$ *such that* $w_2 \mathrm{D}_l g(m) \in V_1$. *Since* V_1 *is open, we can find* $V_2 \in \mathcal{N}_0(N)$ *with* $w_2 \mathrm{D}_l g(m) + V_2 \subseteq V_1$. *There exists* $V_3 \in \mathcal{N}_0(N)$ *such that* $w_2 V_3 \subseteq V_2$. *Since* $\mathrm{D} f(g(m)) : N \to P$ *is* R*-linear and continuous, there exists* $V_4 \in \mathcal{N}_0(N)$ *such that* $\mathrm{D} f(g(m))(V_4) \subseteq U_1$. *Since* g *is differentiable at* m *in the direction of* l, *there exists* $W_3 \in \mathcal{N}_0(R)$ *such that* $m + W_3 l \subseteq A$ *and*

$$r^{-1}(g(m + rl) - g(m)) - \mathrm{D}_l g(m) \in V_3 \cap V_4$$

for all $r \in W_3 \cap \mathcal{U}(R)$. *The uniform differentiability of* f *at* $g(m)$ *allows to find* $W_4 \in \mathcal{N}_0(R)$ *satisfying that*

$$r^{-1}(f(g(m) + rk) - f(g(m))) - \mathrm{D}_k f(g(m)) \in w_2 U_1$$

for all $r \in W_4 \cap W_1 \cap \mathcal{U}(R)$ *and all* $k \in V_1$. *Next, there exists* $W_5 \in \mathcal{N}_0(R)$ *with* $W_5 w_2^{-1} \subseteq W_1 \cap W_4$. *Finally, fix an arbitrary* $r_0 \in W_0 \cap \mathcal{U}(R)$, *where* $W_0 := W_1 \cap W_2 \cap W_3 \cap W_4 \cap W_5$. *Let* $n_{r_0} \in V_3 \cap V_4$ *be such that*

$$r_0^{-1}(g(m + r_0 l) - g(m)) = \mathrm{D}_l\, g(m) + n_{r_0}.$$

Note that $r_0 w_2^{-1} \in W_5 w_2^{-1} \subseteq W_1 \cap W_4 \cap \mathcal{U}(R)$ *and* $w_2 \mathrm{D}_l\, g(m) + w_2 n_{r_0} \in w_2 \mathrm{D}_l\, g(m) + w_2 V_3 \subseteq w_2 \mathrm{D}_l\, g(m) + V_2 \subseteq V_1$. *Thus, we can take* $p_{r_0} \in U_1$ *with* $\left(r_0 w_2^{-1}\right)^{-1} \left[f\left(g(m) + \left(r_0 w_2^{-1}\right)\left(w_2 \mathrm{D}_l\, g(m) + w_2 n_{r_0}\right)\right) - f(g(m))\right] = \mathrm{D}_{w_2 \mathrm{D}_l\, g(m) + w_2 n_{r_0}} f(g(m)) + w_2 p_{r_0}$. *Then*

$$\begin{aligned}
& r_0^{-1}\left[(f \circ g)(m + r_0 l) - (f \circ g)(m))\right] - \mathrm{D}_{\mathrm{D}_l\, g(m)} f\left(g\left(m\right)\right) \\
= \;& r_0^{-1}\left[f\left(g(m) + r_0 r_0^{-1}(g(m + r_0 l) - g(m))\right) - f(g(m))\right] \\
- \;& \mathrm{D}_{\mathrm{D}_l\, g(m)} f(g(m)) \\
= \;& r_0^{-1}\left[f\left(g(m) + r_0(\mathrm{D}_l\, g(m) + n_{r_0})\right) - f(g(m))\right] - \mathrm{D}_{\mathrm{D}_l\, g(m)} f(g(m)) \\
= \;& w_2^{-1}\left(r_0 w_2^{-1}\right)^{-1}\left[f\left(g(m) + \left(r_0 w_2^{-1}\right)\left(w_2 \mathrm{D}_l\, g(m) + w_2 n_{r_0}\right)\right) - f(g(m))\right] \\
- \;& \mathrm{D}_{\mathrm{D}_l\, g(m)} f(g(m)) \\
= \;& w_2^{-1}\left(\mathrm{D}_{w_2 \mathrm{D}_l\, g(m) + w_2 n_{r_0}} f(g(m)) + w_2 p_{r_0}\right) - \mathrm{D}_{\mathrm{D}_l\, g(m)} f(g(m)) \\
= \;& \mathrm{D}_{\mathrm{D}_l\, g(m) + n_{r_0}} f(g(m)) + w_2^{-1} w_2 p_{r_0} - \mathrm{D}_{\mathrm{D}_l\, g(m)} f(g(m)) \\
= \;& \mathrm{D}_{\mathrm{D}_l\, g(m)} f(g(m)) + \mathrm{D}_{n_{r_0}} f(g(m)) + p_{r_0} - \mathrm{D}_{\mathrm{D}_l\, g(m)} f(g(m)) \\
= \;& \mathrm{D}\, f(g(m))(n_{r_0}) + p_{r_0} \in \mathrm{D}\, f(g(m))(V_4) + U_1 \subseteq U_1 + U_1 \subseteq U.
\end{aligned}$$

Since the composition of continuous linear operators is a continuous linear operator, we trivially conclude the following result.

Corollary 51 *Let* M, N, P *be topological modules over a practical topological ring* R *such that* N, P *are Hausdorff. Let* $A \subseteq M$ *and* $B \subseteq N$ *be nonempty and open. Let* $f : B \to P$ *and* $g : A \to N$ *such that* $g(A) \subseteq B$, g *is differentiable at* $m \in A$ *and* f *is uniformly differentiable at* $g\left(m\right)$. *Then* $f \circ g$ *is differentiable at* m *and*

$$\mathrm{D}\left(f \circ g\right)(m) = \mathrm{D}\, f\left(g\left(m\right)\right) \circ \mathrm{D}\, g\left(m\right).$$

By performing slight modifications to the proof of Theorem 134, we obtain the uniformly differentiable version of the Chain Rule.

Corollary 52 (Uniform Chain Rule) *Let* M, N, P *be topological modules over a practical topological ring* R *such that* N, P *are Hausdorff. Let* $A \subseteq M$ *and* $B \subseteq N$ *be nonempty and open. Let* $f : B \to P$ *and* $g : A \to N$ *such that* $g(A) \subseteq B$, g *is uniformly differentiable at* $m \in A$ *and* f *is uniformly differentiable at* $g\left(m\right)$. *Then* $f \circ g$ *is uniformly differentiable at* m *and*

$$\mathrm{D}\left(f \circ g\right)(m) = \mathrm{D}\, f\left(g\left(m\right)\right) \circ \mathrm{D}\, g\left(m\right).$$

Proof 425 *Consider open* 0*-neighborhoods* $W_1 \in \mathcal{N}_0\left(R\right), V_1 \in \mathcal{N}_0(N)$ *that fulfill the condition of uniform differentiability of* f *at* $g(m)$ *as in Remark 84. Also, consider* 0*-neighborhoods* $W_2 \in \mathcal{N}_0\left(R\right), L_2 \in \mathcal{N}_0(M)$ *that fulfill the condition of uniform differentiability of* g *at* m *again as in Remark 84. The continuity and* R*-linearity of* $\mathrm{D}\, g(m)$ *allows the existence of* $L_1 \in \mathcal{N}_0(M)$ *such that* $\mathrm{D}\, g(m)(L_1) \subseteq V_2$, *where* $V_2 \in \mathcal{N}_0(N)$ *satisfies* $V_2 + V_2 \subseteq V_1$. *We will prove the uniform differentiability of* $f \circ g$ *at* m *with the* 0*-neighborhoods* $W_2 \in \mathcal{N}_0\left(R\right), L_1 \cap L_2 \in \mathcal{N}_0(M)$. *Indeed, fix an arbitrary* $U \in \mathcal{N}_0(P)$ *and take* $U_1 \in \mathcal{N}_0(P)$

with $U_1 + U_1 \subseteq U$. *Since* $\mathrm{D}\, f(g(m)) : N \to P$ *is* R*-linear and continuous, there exists* $V_3 \in \mathcal{N}_0(N)$ *such that* $\mathrm{D}\, f(g(m))(V_3) \subseteq U_1$. *Since* g *is uniformly differentiable at* m*, there exists* $W_3 \in \mathcal{N}_0(R)$ *such that*

$$r^{-1}(g(m + rl) - g(m)) - \mathrm{D}_l\, g(m) \in V_2 \cap V_3$$

for all $r \in W_2 \cap W_3 \cap \mathcal{U}(R)$ *and all* $l \in L_2$. *The uniform differentiability of* f *at* $g(m)$ *allows to find* $W_4 \in \mathcal{N}_0(R)$ *satisfying that*

$$r^{-1}(f(g(m) + rk) - f(g(m))) - \mathrm{D}_k\, f\,(g\,(m)) \in U_1$$

for all $r \in W_1 \cap W_4 \cap \mathcal{U}(R)$ *and all* $k \in V_1$. *Finally, fix an arbitrary* $r_0 \in W_0 \cap \mathcal{U}(R)$*, where* $W_0 := W_1 \cap W_2 \cap W_3 \cap W_4$. *Fix also an arbitrary* $l_{12} \in L_1 \cap L_2$. *Let* $n_{r_0 l_{12}} \in V_2 \cap V_3$ *be such that*

$$r_0^{-1}(g(m + r_0 l_{12}) - g(m)) = \mathrm{D}_{l_{12}}\, g(m) + n_{r_0 l_{12}}.$$

Notice that $\mathrm{D}_{l_{12}}\, g(m) + n_{r_0 l_{12}} \in \mathrm{D}\, g(m)(L_1) + V_2 \subseteq V_2 + V_2 \subseteq V_1$. *Thus, we can take* $p_{r_0 l_{12}} \in U_1$ *with* $r_0^{-1}\left[f\,(g(m) + r_0(\mathrm{D}_{l_{12}}\, g(m) + n_{r_0 l_{12}})) - f(g(m))\right] = \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m) + n_{r_0 l_{12}}}\, f(g(m)) + p_{r_0 l_{12}}$. *Then*

$$\begin{aligned}
& r_0^{-1}\left[(f \circ g)(m + r_0 l_{12})) - (f \circ g)(m))\right] - \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m)}\, f\,(g\,(m)) \\
= \;& r_0^{-1}\left[f\left(g(m) + r_0 r_0^{-1}(g(m + r_0 l_{12}) - g(m))\right) - f(g(m))\right] \\
- \;& \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m)}\, f(g(m)) \\
= \;& r_0^{-1}\left[f\,(g(m) + r_0(\mathrm{D}_{l_{12}}\, g(m) + n_{r_0 l_{12}})) - f(g(m))\right] - \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m)}\, f(g(m)) \\
= \;& \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m) + n_{r_0 l_{12}}}\, f(g(m)) + p_{r_0 l_{12}} - \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m)}\, f(g(m)) \\
= \;& \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m)}\, f(g(m)) + \mathrm{D}_{n_{r_0 l_{12}}}\, f(g(m)) + p_{r_0 l_{12}} - \mathrm{D}_{\mathrm{D}_{l_{12}}\, g(m)}\, f(g(m)) \\
= \;& \mathrm{D}\, f(g(m))\,(n_{r_0 l_{12}}) + p_{r_0 l_{12}} \in \mathrm{D}\, f(g(m))(V_3) + U_1 \subseteq U_1 + U_1 \subseteq U.
\end{aligned}$$

4.3 Differential manifolds

Topological manifolds were extended to the scope of topological modules in Subsubsection 3.2.4.7. In a similar manner, differential manifolds will be transported also to the scope of topological modules. In this section, we will only generalize the basic concepts of classical Differential Geometry.

4.3.1 Differential local linearity

Recall that a topological space is said to be locally linear provided that every point has an open neighborhood homeomorphic to an open subset of a topological module. If this module remains the same for all points, say M, then the topological space is called locally M-linear. A topological M-manifold is a Hausdorff locally M-linear topological space for M a Hausdorff topological module over a Hausdorff topological ring.

4.3.1.1 Differentiable atlases

Let us recall that an atlas in a topological M-manifold X is a collection $\{(U_\gamma, \varphi_\gamma) : \gamma \in \Gamma\}$ such that $X = \bigcup_{\gamma \in \Gamma} U_\gamma$ and, for every $\gamma \in \Gamma$, $(U_\gamma, \varphi_\gamma)$ is a chart, in other words, U_γ is an open subset of X and $\varphi_\gamma : U_\gamma \to M$ is a homeomorphism from U_γ onto an open subset $\varphi_\gamma\,(U_\gamma)$ of M.

Definition 290 (Smooth atlas) *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. Let X be a topological M-manifold. An atlas $\{(U_\gamma, \varphi_\gamma) : \gamma \in \Gamma\}$ is called smooth provided that all the transition maps*

$$\varphi_\gamma|_{U_\gamma \cap U_\lambda} \circ \varphi_\lambda^{-1}|_{\varphi_\lambda(U_\gamma \cap U_\lambda)} : \varphi_\lambda(U_\gamma \cap U_\lambda) \to \varphi_\gamma(U_\gamma \cap U_\lambda)$$

are uniformly differentiable for all $\lambda, \gamma \in \Gamma$ such that $U_\lambda \cap U_\gamma \neq \varnothing$.

Proposition 113 *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. Let X be a topological M-manifold. Every smooth atlas of X is contained in a maximal smooth atlas.*

Proof 426 *Consider the order relation on the set of all smooth atlases of X given by the inclusion. The union of all the smooth atlases in a chain is a smooth atlas. Therefore, Zorn's Lemma assures the existence of a maximal smooth atlas. Next, we will construct a maximal smooth atlas in a different way. Consider the following equivalence relation on the set of all smooth atlases of X: two smooth atlases are equivalent if their union is a smooth atlas. The union of all the atlases in an equivalent class is clearly a maximal smooth atlas. The transitive property is the only property of the previous equivalent class which is not completely trivial. Let $\{(U_\gamma, \varphi_\gamma) : \gamma \in \Gamma\}$, $\{(V_\lambda, \phi_\lambda) : \lambda \in \Lambda\}$ and $\{(W_\delta, \psi_\delta) : \delta \in \Delta\}$ be smooth atlases such that the first is equivalent to the second, and the second is equivalent to the third. We will show that the first is equivalent to the third. Indeed, take $\gamma \in \Gamma$ and $\delta \in \Delta$ such that $U_\gamma \cap W_\delta \neq \varnothing$. We need to check that the transition map*

$$\varphi_\gamma|_{U_\gamma \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap W_\delta)} : \psi_\delta(U_\gamma \cap W_\delta) \to \varphi_\gamma(U_\gamma \cap W_\delta)$$

is uniformly differentiable. Fix an arbitrary $m \in \psi_\delta(U_\gamma \cap W_\delta)$. There exists $\lambda \in \Lambda$ such that $\psi_\delta^{-1}(m) \in V_\lambda$. Keep in mind that $U_\gamma \cap V_\lambda \cap W_\delta \neq \varnothing$ since it contains $\psi_\delta^{-1}(m)$. Then $\varphi_\gamma|_{U_\gamma \cap V_\lambda \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap V_\lambda \cap W_\delta)} = \left(\varphi_\gamma|_{U_\gamma \cap V_\lambda \cap W_\delta} \circ \phi_\lambda^{-1}|_{\phi_\lambda(U_\gamma \cap V_\lambda \cap W_\delta)}\right) \circ \left(\phi_\lambda|_{U_\gamma \cap V_\lambda \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap V_\lambda \cap W_\delta)}\right)$. Since the composition of uniformly differentiable functions is uniformly differentiable by the Uniform Chain Rule, we conclude that $\varphi_\gamma|_{U_\gamma \cap V_\lambda \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap V_\lambda \cap W_\delta)}$ is uniformly differentiable at m. Next, $\psi_\delta(U_\gamma \cap V_\lambda \cap W_\delta)$ is an open subset of $\psi_\delta(U_\gamma \cap W_\delta)$ and $\left(\varphi_\gamma|_{U_\gamma \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap W_\delta)}\right)|_{\psi_\delta(U_\gamma \cap V_\lambda \cap W_\delta)} = \varphi_\gamma|_{U_\gamma \cap V_\lambda \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap V_\lambda \cap W_\delta)}$. As a consequence, the transition map $\varphi_\gamma|_{U_\gamma \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap W_\delta)}$ is uniformly differentiable at m. The arbitrariness of m allows to conclude that the transition map $\varphi_\gamma|_{U_\gamma \cap W_\delta} \circ \psi_\delta^{-1}|_{\psi_\delta(U_\gamma \cap W_\delta)}$ is uniformly differentiable. This shows that the first and the third atlas are equivalent.

Definition 291 (Differential manifold) *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. A topological M-manifold X is said to be a differential M-manifold when X is endowed with a maximal smooth atlas.*

Example 73 *If A is a nonempty open subset of a Hausdorff topological module M over a practical Hausdorff topological ring R, then A can be endowed with structure of differential M-manifold. Indeed, it suffices to consider the smooth atlas $\{(U, I_U) : U \subseteq A \text{ is open}\}$, which is contained in a maximal smooth atlas in view of Proposition 113. In fact, this maximal smooth atlas can be described as the set of all charts (U, φ) of A such that $\varphi : U \to \varphi(U)$ and $\varphi^{-1} : \varphi(U) \to U$ are uniformly differentiable between topological modules.*

Proposition 114 *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. If X is a differential M-manifold, then every nonempty open subset A of X inherits structure of differential M-manifold.*

Proof 427 *Let* $\{(U_\gamma, \varphi_\gamma) : \gamma \in \Gamma\}$ *be a maximal smooth atlas of* X*. Then*

$$\left\{\left(U_\gamma \cap A, \varphi_\gamma|_{U_\gamma \cap A}\right) : \gamma \in \Gamma,\ U_\gamma \cap A \neq \varnothing\right\}$$

is a maximal smooth atlas of A*, which is contained in a maximal smooth atlas in view of Proposition 113.*

A differential submanifold of M is a subset Y of X which inherits structure of differential N-manifold, for N a submodule of M.

Definition 292 *Let* M *be a Hausdorff topological module over a practical Hausdorff topological ring* R*. Let* X *be a differential* M*-manifold. We say that* $Y \subseteq X$ *is a differential submanifold of* X *if there exists a submodule* N *of* M *such that* $\varphi(U \cap Y)$ *is open in* N *for every chart* (U, φ) *of* X*.*

4.3.1.2 Differentiable maps

The concept of differentiable maps between differential manifolds strongly relies on the concept of differentiable function on topological modules over practical topological rings.

Definition 293 (Differentiable map) *Let* M, N *be Hausdorff topological modules over a practical Hausdorff topological ring* R*. Let* X, Y *be differential* M*-,*N*-manifolds, respectively. A continuous function* $f : X \to Y$ *is said to be differentiable provided that*

$$\psi_{f(U)\cap V} \circ f|_{U\cap f^{-1}(V)} \circ \varphi^{-1}|_{\varphi(U\cap f^{-1}(V))} : \varphi\left(U \cap f^{-1}(V)\right) \to \psi\left(f(U) \cap V\right)$$

is uniformly differentiable for every chart (U, φ) *of* X *and every chart* (V, ψ) *of* Y *such that* $V \cap f(U) \neq \varnothing$*.*

Trivial examples of differentiable maps are the identity I_X on a differential M-manifold X or the inclusion $\iota_Y : Y \hookrightarrow X$ of a differential submanifold.

Proposition 115 *Let* M, N *be Hausdorff topological modules over a practical Hausdorff topological ring* R*. Let* $A \subseteq M$ *be open and nonempty. If* $f : A \to N$ *is continuous, then* f *is differentiable between differential manifolds if and only if* f *is a uniformly differentiable function between topological modules.*

Proof 428 *The proof will be itemized in accordance with the statement.*

- $\Rightarrow$ *Notice that* $(A, I_A), (N, I_N)$ *are charts of* A, N*, respectively. By hypothesis,* $I_N \circ f \circ I_A = f$ *is uniformly differentiable between topological modules.*
- $\Leftarrow$ *Take* $(U, \varphi), (V, \psi)$ *charts of* A, N*, respectively, such that* $V \cap f(U) \neq \varnothing$*. We have to check that* $\psi_{f(U)\cap V} \circ f|_{U\cap f^{-1}(V)} \circ \varphi^{-1}|_{\varphi(U\cap f^{-1}(V))} : \varphi\left(U \cap f^{-1}(V)\right) \to \psi\left(f(U) \cap V\right)$ *is uniformly differentiable between topological modules. In view of Example 73,* $\varphi : U \to \varphi(U)$*,* $\varphi^{-1} : \varphi(U) \to U$*,* $\psi : V \to \psi(V)$*,* $\psi^{-1} : \psi(V) \to V$ *are uniformly differentiable between topological modules. Since* f *is uniformly differentiable between topological modules by hypothesis, we conclude the desired result by relying on the Uniform Chain Rule.*

Proposition 116 *Let* M *be a Hausdorff topological module over a practical Hausdorff topological ring* R*. If* X *is a differential* M*-manifold, then every chart* (W, ϕ) *of* X *satisfies that* $\phi : W \to \phi(W)$ *and* $\phi^{-1} : \phi(W) \to W$ *are differentiable when* W *and* $\phi(W)$ *are seen as differential* M*-manifolds.*

Proof 429 *Take arbitrary charts $(U,\varphi),(V,\psi)$ of $W,\phi(W)$, respectively, such that $V\cap\phi(U)\neq\varnothing$. In the first place, according to Proposition 114, there exists a chart (U_0,φ_0) of X such that $U=U_0\cap W$ and $\varphi=\varphi_0|_U$. In the second place, in accordance with Example 73, the chart (V,ψ) satisfies that $\psi:V\to\psi(V)$ and $\psi^{-1}:\psi(V)\to V$ are uniformly differentiable. Observe that $\psi|_{\phi(U)\cap V}\circ\phi|_{U\cap\phi^{-1}(V)}\circ\varphi^{-1}|_{\varphi(U\cap\phi^{-1}(V))}=\psi|_{\phi(U)\cap V}\circ\left(\phi|_{U\cap\phi^{-1}(V)}\circ\varphi_0^{-1}|_{\varphi(U\cap\phi^{-1}(V))}\right)$ is uniformly differentiable because $\phi|_{U\cap\phi^{-1}(V)}\circ\varphi_0^{-1}|_{\varphi(U\cap\phi^{-1}(V))}$ is a transition map in X and ψ is uniformly differentiable. Finally, $\varphi|_{U\cap\phi^{-1}(V)}\circ\phi^{-1}|_{V\cap\phi(U)}\circ\psi^{-1}|_{\psi(V\cap\phi(U))}=\left(\varphi_0|_{U\cap\phi^{-1}(V)}\circ\phi^{-1}|_{V\cap\phi(U)}\right)\circ\psi^{-1}|_{\psi(V\cap\phi(U))}$ is uniformly differentiable because $\varphi_0|_{U\cap\phi^{-1}(V)}\circ\phi^{-1}|_{V\cap\phi(U)}$ is a transition map in X and ψ^{-1} is uniformly differentiable.*

Definition 294 (Diffeomorphism) *Let M,N be Hausdorff topological modules over a practical Hausdorff topological ring R. Let X,Y be differential M-,N-manifolds, respectively. A differentiable function $f:X\to Y$ is said to be a diffeomorphism if there exists another differentiable function $g:Y\to X$ such that $f\circ g=I_Y$ and $g\circ f=I_X$. Under these conditions, we say that X,Y are diffeomorphic.*

Notice that every diffeomorphism is a homeomorphism. Proposition 116 shows that, for every chart (W,ϕ), $\phi:W\to\phi(W)$ is a diffeomorphism.

4.3.2 Tangent space

The geometrical concept of tangent line is one of the origins of classical Differential Calculus.

4.3.2.1 Tangent vectors

Tangent vector is a relatively abstract concepts in the scope of differential manifolds. In order to define tangent vectors, we first need the notion of tangent element, which is much closer to the classical concept of tangency.

Definition 295 *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. Let X be a differential M-manifold. A tangent element of X is an element of the form $\mathrm{D}_l\left(\psi|_{U\cap V}\circ\varphi^{-1}|_{\varphi(U\cap V)}\right)(\varphi(x))$, where $(U,\varphi),(V,\psi)$ are charts of X, $x\in U\cap V$ and $l\in M$.*

The concept of tangent element depends on the charts $(U,\varphi),(V,\psi)$ containing x. In order to avoid this dependency, we introduce the notion of tangent vector.

Definition 296 (Tangent vector) *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. Let X be a differential M-manifold. Consider $x\in X$. Let $\{(U_\alpha,\varphi_\alpha):\alpha\in A\}$ be the atlas of X. Consider $A_x:=\{\alpha\in A:x\in U_\alpha\}$. A tangent vector of X at x is a function*

$$\begin{array}{rccc} v: & A_x & \to & M \\ & \alpha & \mapsto & v_\alpha \end{array}$$

satisfying that

$$v_\beta=\mathrm{D}_{v_\alpha}\left(\varphi_\beta|_{U_\alpha\cap U_\beta}\circ\varphi_\alpha^{-1}|_{\varphi(U_\alpha\cap U_\beta)}\right)(\varphi_\alpha(x))),$$

for all $\alpha,\beta\in A_x$.

4.3.2.2 Tangent space

As expected, the tangent space will be the set of all tangent vectors. A linear structure will be endowed on the tangent space.

Definition 297 (Tangent space) *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. Let X be a differential M-manifold. Consider $x \in X$. The tangent space to X at x is defined as the set $\mathcal{T}_x(X)$ consisting of all the tangent vectors at x.*

If there is no confusion with X, then we will simply denote the tangent space as $\mathcal{T}_x$. The linear structure of the tangent space is a direct consequence of the linearity of the differential.

Proposition 117 *Let M be a Hausdorff topological module over a practical Hausdorff topological ring R. Let X be a differential M-manifold. Consider $x \in X$. The tangent space to X at x can be endowed with structure of topological R-module.*

Proof 430 *Consider tangent vectors v, w and $r \in R$. For every $\alpha, \beta \in A_x$, $\mathrm{D}\left(\varphi_\beta|_{U_\alpha \cap U_\beta} \circ \varphi_\alpha^{-1}|_{\varphi(U_\alpha \cap U_\beta)}\right)$ is R-linear and continuous; therefore,*

$$\begin{aligned} rv_\beta &= r\,\mathrm{D}_{v_\alpha}\left(\varphi_\beta|_{U_\alpha \cap U_\beta} \circ \varphi_\alpha^{-1}|_{\varphi(U_\alpha \cap U_\beta)}\right)(\varphi_\alpha(x))) \\ &= \mathrm{D}_{rv_\alpha}\left(\varphi_\beta|_{U_\alpha \cap U_\beta} \circ \varphi_\alpha^{-1}|_{\varphi(U_\alpha \cap U_\beta)}\right)(\varphi_\alpha(x))), \end{aligned}$$

and

$$\begin{aligned} v_\beta + w_\beta &= \mathrm{D}_{v_\alpha}\left(\varphi_\beta|_{U_\alpha \cap U_\beta} \circ \varphi_\alpha^{-1}|_{\varphi(U_\alpha \cap U_\beta)}\right)(\varphi_\alpha(x))) \\ &+ \mathrm{D}_{w_\alpha}\left(\varphi_\beta|_{U_\alpha \cap U_\beta} \circ \varphi_\alpha^{-1}|_{\varphi(U_\alpha \cap U_\beta)}\right)(\varphi_\alpha(x))) \\ &= \mathrm{D}_{v_\alpha + w_\alpha}\left(\varphi_\beta|_{U_\alpha \cap U_\beta} \circ \varphi_\alpha^{-1}|_{\varphi(U_\alpha \cap U_\beta)}\right)(\varphi_\alpha(x))). \end{aligned}$$

This shows that the operations

$$\begin{array}{rccc} v + w: & A_x & \to & M \\ & \alpha & \mapsto & v_\alpha + w_\alpha \end{array}$$

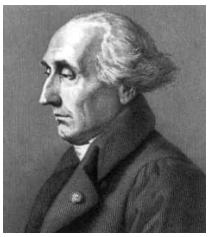
and

$$\begin{array}{rccc} rv: & A_x & \to & M \\ & \alpha & \mapsto & rv_\alpha \end{array}$$

confers $\mathcal{T}_x(X)$ with the structure of R-module.

Chapter 5

Optimization



Joseph-Louis Lagrange

5.1 Multiobjective optimization

Optimization Theory deals with the resolution of optimization problems. Most of the optimization problems that arise in real life have more than one objective function, that is, there are more than one dependent variable to optimize. For example, if we want to buy something, there are several variables that we need to take into account, such as price, quality, duration, etc. This is why it is relevant to study multiobjective optimization problems. We refer the reader to [58, 59, 60, 61, 62].

5.1.1 Formulation

If $f : X \to A$ is a function from a nonempty set X into a poset A, then $\max_X(f)$ and $\min_X(f)$ stand for $\max f(X)$ and $\min f(X)$, respectively. Similarly, $\arg\max_X(f)$ and $\arg\min_X(f)$ stand for $\arg\max f(X)$ and $\arg\min f(X)$, respectively, that is, the set of points of X at which $\max_X(f)$ and $\min_X(f)$ are attained, respectively. If there is no confusion with X, then we will simply write $\max(f), \min(f), \arg\max(f), \arg\min(f)$.

5.1.1.1 General form

A multiobjective optimization problem has a particular form described in the following definition. Proper notions from Order Theory are obviously involved.

Problem 1 (Multiobjective optimization) *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. The problem*

$$\begin{cases} \max f(x) \ \ \forall f \in M, \\ \min g(x) \ \ \forall g \in N, \\ x \in \mathcal{R}, \end{cases} \tag{5.1}$$

is called a multiobjective optimization problem.

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The elements of M, N are called *objective functions*. The set $\mathcal{R}$ is called *feasible region*, *region of constrains/restrictions*, or *set of feasible solutions*, and it is often denoted as fea(5.1).

Example 74 *Let R be a seminormed ring. Let M, N be seminormed R-modules. Let $T : M \to N$ be a linear map. Consider $n \in M$ and $\delta > 0$. The following is a multiobjective optimization problem:*

$$\begin{cases} \max \|T(m)\|, \\ \min \|m\|, \\ m \in \mathsf{B}_M(n, \delta). \end{cases} \tag{5.2}$$

The previous example falls into a particular type of multiobjective optimization problems called operator multiobjective optimization problems, in which the objective functions involve linear operators between seminormed modules.

Problem 2 (Operator multiobjective optimization) *Let R be a seminormed ring. Let M, N_i, N_j be seminormed R-modules, $i \in I, j \in J$. Let $T_i : M \to N_i$, $S_j : M \to N_j$ be linear operators, $i \in I, j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. The problem*

$$\begin{cases} \max \|T_i(m)\| & i \in I, \\ \min \|S_j(m)\| & j \in J, \\ m \in \mathcal{R}, \end{cases} \tag{5.3}$$

is called an operator multiobjective optimization problem.

5.1.1.2 Optimal solution

The set of optimal solutions of a multiobjective optimization problem is, as expected, the subset of feasible solutions at which all the objective functions are optimized at once.

Definition 298 (Optimal solution) *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. The set of optimal solutions of* (5.1) *is defined as*

$$\operatorname{sol}(5.1) := \{x \in \mathcal{R} : \forall f \in M \ \forall g \in N \ \forall y \in \mathcal{R} \ f(x) \geq f(y) \text{ and } g(x) \leq g(y)\}.$$

Any multiobjective optimization problem can be rewritten as the intersection of single optimization problems.

Remark 85 *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. Notice that* (5.1) *can be decomposed into the following optimization problems:*

$$\begin{cases} \max f(x), \\ x \in \mathcal{R}, \end{cases} \quad \text{and} \quad \begin{cases} \min g(x), \\ x \in \mathcal{R}, \end{cases}$$

for all $f \in M$ and all $g \in N$. Then

$$\operatorname{sol}(5.1) = \bigcap_{f \in M} \arg\max_{\mathcal{R}} (f) \cap \bigcap_{g \in N} \arg\max_{\mathcal{R}} (g).$$

The following remark serves to construct the set of optimal solutions of a multiobjective optimization problem in a more abstract way.

Remark 86 *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. Consider in $\mathcal{R}$ the equivalence relation given by*

$$\mathcal{S} = \{(x, y) \in \mathcal{R}^2 : \forall f \in M \ \forall g \in N \ \ f(x) = f(y) \text{ and } g(x) = g(y)\}.$$

Consider, in the quotient set of $\mathcal{R}$ by $\mathcal{S}$, ${}^{\mathcal{R}}/_{\mathcal{S}}$, the order relation given by

$$[x]_{\mathcal{S}} \leq [y]_{\mathcal{S}} \Leftrightarrow \forall f \in M \ \forall g \in N \ \ f(x) \leq f(y) \text{ and } g(y) \leq g(x).$$

Notice that the above equivalence relation is precisely the one described in Example 1 for the family of functions $\{f|_{\mathcal{R}} : f \in M\} \cup \{g|_{\mathcal{R}} : g \in N\}$.

Theorem 135 *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. Then*

$$\text{sol}(5.1) = \{x \in \mathcal{R} : [x]_{\mathcal{S}} \text{ is the maximum of } {}^{\mathcal{R}}/_{\mathcal{S}}\}.$$

Proof 431 *The proof will be itemized in accordance with the statement.*

- $\subseteq$ *Fix an arbitrary $x_0 \in \text{sol}(5.1)$. For every $y \in \mathcal{R}$, $f(x_0) \geq f(y)$ and $g(x_0) \leq g(y)$ for all $f \in M$ and all $g \in N$. Then $[x_0]_{\mathcal{S}} \geq [y]_{\mathcal{S}}$. The arbitrariness of $y \in \mathcal{R}$ assures that $[x_0]_{\mathcal{S}}$ is the maximum of ${}^{\mathcal{R}}/_{\mathcal{S}}$.*
- $\supseteq$ *Conversely, fix an arbitrary $x_0 \in \mathcal{R}$ such that $[x_0]_{\mathcal{S}}$ is the maximum of ${}^{\mathcal{R}}/_{\mathcal{S}}$. For every $y \in \mathcal{R}$, $[y]_{\mathcal{S}} \leq [x_0]_{\mathcal{S}}$, therefore $f(x_0) \geq f(y)$ and $g(x_0) \leq g(y)$ for all $f \in M$ and all $g \in N$. The arbitrariness of $y \in \mathcal{R}$ proves that $x_0 \in \text{sol}(5.1)$.*

The following example justifies the introduction of Pareto optimality in the literature of Optimization Theory.

Example 75 *Let R be a normed ring. Let M, N be normed R-modules. Let $T : M \to N$ be a nonzero linear map. The operator multiobjective optimization problem*

$$\begin{cases} \max \|T(m)\|, \\ \min \|m\|, \\ m \in M. \end{cases} \tag{5.4}$$

lacks any optimal solutions. Indeed, if m_0 is an optimal solution, then $\|m_0\| \leq \|0\| = 0$, which implies that $m_0 = 0$. Next, if we consider any $m \in M \setminus \ker(T)$, then we reach the contradiction that $0 = \|T(m_0)\| \geq \|T(m)\| > 0$.

Next proposition generalizes the previous example and serves to justify even more the irruption of Pareto optimality.

Proposition 118 *Let R be a seminormed ring. Let M, N_i, N_j be seminormed R-modules, $i \in I, j \in J$. Let $T_i : M \to N_i$, $S_j : M \to N_j$ be linear operators, $i \in I, j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. Then $\text{sol}(5.3) = \varnothing$ in any of the following situations:*

1. *There exist $i_0 \in I$ and $j_0 \in J$ such that $\ker_{\mathfrak{t}}(S_{j_0}) \cap \mathcal{R} \subseteq \ker_{\mathfrak{t}}(T_{i_0}) \neq \mathcal{R}$.*
2. *R is practical, $\mathcal{U}(R)\mathcal{R} \subseteq \mathcal{R}$, and there exist $i_0 \in I$ and $m_0 \in \mathcal{R}$ such that $\|T_{i_0}(m_0)\| > 0$.*

Proof 432 *The proof follows the itemization displayed in the statement.*

1. *If there exists* $m_0 \in \text{sol}(5.3)$*, then* $\|S_{j_0}(m_0)\| \leq \|S_{j_0}(0)\| = 0$*, which implies that* $m_0 \in \ker_{\mathtt{t}}(S_{j_0}) \cap \mathcal{R} \subseteq \ker_{\mathtt{t}}(T_{i_0})$*. Next, if we consider any* $m \in \mathcal{R} \setminus \ker_{\mathtt{t}}(T_{i_0})$*, then we reach the contradiction that* $0 = \|T_{i_0}(m_0)\| \geq \|T_{i_0}(m)\| > 0$*.*

2. *Take a sequence* $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(R)$ *of invertibles converging to* 0*. Following Lemma 69(6), we have that*

$$\|r_n\|^{-1} \|T_{i_0}(m_0)\| \leq \left\|r_n^{-1} T_{i_0}(m_0)\right\| = \left\|T_{i_0}\left(r_n^{-1} m_0\right)\right\| \to \infty$$

as $n \to \infty$*, since* $\|r_n\|^{-1} \to \infty$ *as* $n \to \infty$*. As a consequence,* $\text{sol}(5.3) \subseteq \arg\max_{m\in\mathcal{R}} \|T_{i_0}(m)\| = \varnothing$*.*

5.1.1.3 Pareto optimality

Many problems that arise in Bioengineering, Physics, or Statistics have no optimal solution when formulated as multiobjective optimization problems. This is why it is important the necessity of weakening the concept of optimal solution. The set of Pareto optimal solutions of a multiobjective optimization problem is defined as follows.

Definition 299 (Pareto optimal solution) *Let* X *be a nonempty set. Let* A, B *be posets. Consider* $M \subseteq A^X$ *and* $N \subseteq B^X$ *nonempty. Let* $\mathcal{R}$ *be a nonempty subset of* X*. The set of Pareto optimal solutions of* (5.1) *is defined as*

$$\begin{aligned}\text{Par}(5.1) := \{x \in \mathcal{R} : \forall y \in \mathcal{R},\ &\text{if } \exists f \in M\ f(y) > f(x)\ \text{ or } \exists g \in N\ g(y) < g(x),\\ &\text{then } \exists f' \in M\ f'(y) < f'(x)\ \text{ or } \exists g' \in N\ g'(x) < g'(y)\}.\end{aligned}$$

It is trivial that $\text{sol}(5.1) \subseteq \text{Par}(5.1)$. In fact, if $\text{sol}(5.1) \neq \varnothing$, then $\text{sol}(5.1) = \text{Par}(5.1)$. Next remark highlights an important observation related to Pareto optimal solutions

Remark 87 *Let* X *be a nonempty set. Let* A, B *be posets. Consider* $M \subseteq A^X$ *and* $N \subseteq B^X$ *nonempty. Let* $\mathcal{R}$ *be a nonempty subset of* X*. Suppose that* $x \in \text{Par}(5.1)$*. If* $y \in \mathcal{R}$ *satisfies that* $f(y) \geq f(x)$ *for all* $f \in M$ *and* $g(y) \leq g(x)$ *for all* $g \in N$*, then* $f(y) = f(x)$ *for all* $f \in M$ *and* $g(y) = g(x)$ *for all* $g \in N$*, hence* $y \in \text{Par}(5.1)$*.*

We will rely again on Remark 86 to show a more abstract way to construct the set of Pareto optimal solutions.

Theorem 136 *Let* X *be a nonempty set. Let* A, B *be posets. Consider* $M \subseteq A^X$ *and* $N \subseteq B^X$ *nonempty. Let* $\mathcal{R}$ *be a nonempty subset of* X*. Then* $\text{Par}(5.1) \subseteq \{x \in \mathcal{R} : [x]_{\mathcal{S}}$ *is a maximal element of* $\mathcal{R}/\mathcal{S}\}$*. If* A, B *are tosets, then* $\text{Par}(5.1) = \{x \in \mathcal{R} : [x]_{\mathcal{S}}$ *is a maximal element of* $\mathcal{R}/\mathcal{S}\}$*.*

Proof 433 *Let us itemize the proof like the statement of the theorem is.*

- $\subseteq$ *Fix an arbitrary* $x_0 \in \text{Par}(5.1)$ *and suppose to the contrary that there exists* $y \in \mathcal{R}$ *such that* $[x_0]_{\mathcal{S}} < [y]_{\mathcal{S}}$*. Then* $f(x_0) \leq f(y)$ *for all* $f \in M$ *and* $g(x_0) \geq g(y)$ *for all* $g \in N$*. However,* $[x_0]_{\mathcal{S}} \neq [y]_{\mathcal{S}}$*, therefore there exists* $f_0 \in M$ *or* $g_0 \in N$ *such that* $f_0(x_0) < f_0(y)$ *or* $g_0(x_0) > g_0(y)$*, respectively. Since* $x_0 \in \text{Par}(5.1)$ *by assumption, there exists* $f_1 \in M$ *or* $g_1 \in N$ *such that* $f_1(x_0) > f_1(y)$ *or* $g_1(x_0) < g_1(y)$*, respectively, which is a contradiction. Therefore,* $[x_0]_{\mathcal{S}}$ *is a maximal element of* $\mathcal{R}/\mathcal{S}$*.*
- $\supseteq$ *Conversely, suppose that* A, B *are tosets. Fix an arbitrary* $x_0 \in \mathcal{R}$ *such that* $[x_0]_{\mathcal{S}}$ *is a maximal element of* $\mathcal{R}/\mathcal{S}$*. Take* $y \in \mathcal{R}$ *satisfying that there exists* $f_0 \in M$ *with* $f_0(y) > f_0(x_0)$ *or* $g_0 \in N$ *with* $g_0(y) < g_0(x_0)$*. Since* A, B *are totally ordered,* $f(y), f(x_0)$ *and*

$g(y), g(x_0)$ are comparable for all $f \in M$ and all $g \in N$. Assume that $f(x_0) \leq f(y)$ for all $f \in M$ and $g(x_0) \geq g(y)$ for all $g \in N$. Then $[x_0]_{\mathcal{S}} < [y]_{\mathcal{S}}$, which contradicts the assumption that $[x_0]_{\mathcal{S}}$ is a maximal element of ${}^{\mathcal{R}}/_{\mathcal{S}}$. As a consequence, there must exist $f_1 \in M$ or $g_1 \in N$ such that $f_1(x_0) > f_1(y)$ or $g_1(x_0) < g_1(y)$, respectively, meaning that $x_0 \in \mathrm{Par}(5.1)$.

Under the settings of Remark 86, $\pi_{\mathcal{S}} : \mathcal{R} \to {}^{\mathcal{R}}/_{\mathcal{S}}$ stands for the quotient projection of $\mathcal{R}$ onto the quotient set ${}^{\mathcal{R}}/_{\mathcal{S}}$.

Lemma 152 *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. Then:*

1. *If there exist $f \in M$ and $x \in \mathcal{R}$ such that $[x]_{\mathcal{S}}$ is a maximal element of $\pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f)$, then $[x]_{\mathcal{S}}$ is a maximal element of ${}^{\mathcal{R}}/_{\mathcal{S}}$. Hence, $x \in \mathrm{Par}(5.1)$.*
2. *If there exist $g \in N$ and $x \in \mathcal{R}$ such that $[x]_{\mathcal{S}}$ is a maximal element of $\pi_{\mathcal{S}}(\arg\min_{\mathcal{R}} g)$, then $[x]_{\mathcal{S}}$ is a maximal element of ${}^{\mathcal{R}}/_{\mathcal{S}}$. Hence, $x \in \mathrm{Par}(5.1)$.*

Proof 434 *We will only prove the first item since the other follows a dual proof. Suppose to the contrary that $[x]_{\mathcal{S}}$ is not a maximal element of ${}^{\mathcal{R}}/_{\mathcal{S}}$. Then we can find $y \in \mathcal{R}$ in such a way that $[x]_{\mathcal{S}} < [y]_{\mathcal{S}}$. In particular, $f(x) \leq f(y)$, therefore, $f(y) = \max_{\mathcal{R}} f$, hence $y \in \arg\max_{\mathcal{R}} f$. As a consequence, $[y]_{\mathcal{S}} \in \pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f)$, contradicting that $[x]_{\mathcal{S}}$ be a maximal element of $\pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f)$.*

As an immediate consequence of Lemma 152, if A, B are tosets and there exists $f \in M$ or $g \in N$ such that $\arg\max_{\mathcal{R}}(f)$ or $\arg\min_{\mathcal{R}}(g)$ is a singleton, respectively, then $\arg\max_{\mathcal{R}}(f) \subseteq \mathrm{Par}(5.1)$ or $\arg\min_{\mathcal{R}}(g) \subseteq \mathrm{Par}(5.1)$, respectively.

Theorem 137 *Let X be a topological space. Let A, B be topological tosets with no holes satisfying the sup property. Consider $M \subseteq \mathcal{C}_{\mathtt{u}}(X, A)$ and $N \subseteq \mathcal{C}_{\mathtt{l}}(X, B)$ be nonempty. Let $\mathcal{R}$ be a nonempty compact subset of X. Then $\mathrm{Par}(5.1) \neq \varnothing$.*

Proof 435 *Fix any $f_0 \in M$. According to the Extreme Value Theorem, $\arg\max_{\mathcal{R}} f_0$ is not empty. In view of Lemma 152 and Theorem 136, it only suffices to find a maximal element of $\pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f_0)$. We will rely on the Zorn's Lemma. Consider a chain in $\pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f_0)$, that is, a totally ordered subset of elements $[x_k]_{\mathcal{S}}$ of $\pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f_0)$, with k ranging a totally ordered set K, in such a way that $k_1 < k_2$ if and only if $[x_{k_1}]_{\mathcal{S}} < [x_{k_2}]_{\mathcal{S}}$. Since K is totally ordered, we have that $(x_k)_{k \in K}$ is a net in $\mathcal{R}$. The compactness of $\mathcal{R}$ allows to extract a subnet $(y_h)_{h \in H}$ of $(x_k)_{k \in K}$ convergent to some $x_0 \in \mathcal{R}$. We will prove, in two steps, that $[x_0]_{\mathcal{S}}$ is an upper bound for the chain $\{[x_k]_{\mathcal{S}} : k \in K\}$.*

- *Let us show first that $x_0 \in \arg\max_{\mathcal{R}} f_0$. Suppose to the contrary that $f_0(x_0) < \max_{\mathcal{R}} f_0$. The upper semicontinuity of f_0 assures the existence of $U \in \mathcal{N}_{x_0}(X)$ such that $f_0(U) \subseteq (-\infty, \max_{\mathcal{R}} f_0)$. There exists $h_U \in H$ such that, if $h \geq h_U$, then $y_h \in U$. Fix any $k_0 \in K$. There exists $h_0 \in H$ such that $\{y_h : h \geq h_0\} \subseteq \{x_k : k \geq k_0\}$. Since H is a directed set, we can find $h_1 \in H$ with $h_1 \geq h_U$ and $h_1 \geq h_0$. There exists $k_1 \in K$ with $k_1 \geq k_0$ such that $y_{h_1} = x_{k_1}$. Next, $f_0(y_{h_1}) = f_0(x_{k_1}) = \max_{\mathcal{R}} f_0$. Now, observe that $y_{h_1} \in U$, so $f_0(y_{h_1}) \in f_0(U) \subseteq (-\infty, \max_{\mathcal{R}} f_0)$. This is a contradiction.*
- *Finally, we will prove that $[x_0]_{\mathcal{S}}$ is an upper bound for the chain $\{[x_k]_{\mathcal{S}} : k \in K\}$. Fix an arbitrary $k_0 \in K$. In order to prove that $[x_{k_0}]_{\mathcal{S}} \leq [x_0]_{\mathcal{S}}$, we have to check that $f(x_{k_0}) \leq f(x_0)$ for all $f \in M$ and $g(x_{k_0}) \geq g(x_0)$ for all $g \in N$. Indeed, fix an arbitrary $f \in M$ and suppose to the contrary that $f(x_{k_0}) > f(x_0)$. The upper semicontinuity of f assures the existence of $U \in \mathcal{N}_{x_0}(X)$ such that $f(U) \subseteq (-\infty, f(x_{k_0}))$.*

There exists $h_U \in H$ such that, if $h \geq h_U$, then $y_h \in U$. We can find $h_0 \in H$ such that $\{y_h : h \geq h_0\} \subseteq \{x_k : k \geq k_0\}$. Since H is a directed set, we can find $h_1 \in H$ with $h_1 \geq h_U$ and $h_1 \geq h_0$. There exists $k_1 \in K$ with $k_1 \geq k_0$ such that $y_{h_1} = x_{k_1}$. Since $k_0 \leq k_1$, we have that $[x_{k_0}]_{\mathcal{S}} \leq [x_{k_1}]_{\mathcal{S}}$. Thus, $f(y_{h_1}) = f(x_{k_1}) \geq f(x_{k_0})$, contradicting that $f(y_{h_1}) \in f(U) \subseteq (-\infty, f(x_{k_0}))$. In a similar way, it can be shown that $g(x_{k_0}) \geq g(x_0)$ for all $g \in N$. As a consequence, $[x_{k_0}]_{\mathcal{S}} \leq [x_0]_{\mathcal{S}}$. In other words, $[x_0]_{\mathcal{S}}$ is an upper bound for the chain $\{[x_k]_{\mathcal{S}} : k \in K\}$.

Since every chain of $\pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f_0)$ has an upper bound, Zorn's Lemma assures the existence of maximal elements in $\pi_{\mathcal{S}}(\arg\max_{\mathcal{R}} f_0)$.

The following example shows the existence of Pareto optimal solutions in a multiobjective optimization problem which lacks optimal solutions.

Example 76 *Let R be a normed ring. Let M be a nonzero normed R-module. The multiobjective optimization problem*

$$\begin{cases} \max \|m\|, \\ \min \|m\|, \\ m \in M, \end{cases} \tag{5.5}$$

lacks any optimal solutions because $M \neq \{0\}$. However, every $m \in M$ is a Pareto optimal solution.

From now on until the end of this subsubsection, we will only focus on operator multiobjective optimizations.

Theorem 138 *Let R be a practical seminormed ring. Let M, N_i, N_j be seminormed R-modules, $i \in I, j \in J$. Let $T_i : M \to N_i$, $S_j : M \to N_j$ be linear operators, $i \in I, j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. Suppose that the following conditions are satisfied:*

- *$\{\|T_i(m)\| : i \in I\}$ is bounded below for every $m \in \bigcap_{j \in J} \ker_{\mathfrak{t}}(S_j) \cap \mathcal{R}$.*
- *$\{\|T_i(m)\| : i \in I\}$ is bounded above for some $m_0 \in \mathrm{Par}(5.3)$.*
- *$\mathcal{U}(R)\mathcal{R} \subseteq \mathcal{R}$.*

Then $\bigcap_{j \in J} \ker_{\mathfrak{t}}(S_j) \cap \mathcal{R} \subseteq \bigcup_{i \in I} \ker_{\mathfrak{t}}(T_i)$.

Proof 436 *Suppose to the contrary that there exists $m_1 \in \bigcap_{j \in J} \ker_{\mathfrak{t}}(S_j) \cap \mathcal{R}$ with $m_1 \notin \bigcup_{i \in I} \ker_{\mathfrak{t}}(T_i)$. In particular, $\|S_j(m_1)\| = 0$ for all $j \in J$ and $\|T_i(m_1)\| > 0$ for all $i \in I$. Let $L > 0$ be a lower bound for $\{\|T_i(m_1)\| : i \in I\}$ and $K > 0$ an upper bound for $\{\|T_i(m_0)\| : i \in I\}$. Since R is practical, we can find a sequence $(r_k)_{k \in \mathbb{N}} \subseteq R$ such that $(\|r_k\|)_{k \in \mathbb{N}}$ converges to 0. For every $i \in I$ and each $k \in \mathbb{N}$, we have that*

$$\|r_k\|^{-1} L \leq \|r_k\|^{-1} \|T_i(m_1)\| \leq \left\|r_k^{-1} T_i(m_1)\right\| = \left\|T_i\left(r_k^{-1} m_1\right)\right\|.$$

We can then find $k_1 \in \mathbb{N}$ such that $\|r_{k_1}\| < \frac{L}{K}$. This way

$$\left\|T_i\left(r_{k_1}^{-1} m_1\right)\right\| \geq \|r_{k_1}\|^{-1} L > K \geq \|T_i(m_0)\|$$

for all $i \in I$. Since $r_{k_1}^{-1} m_1 \in \mathcal{U}(R)\mathcal{R} \subseteq \mathcal{R}$ and $m_0 \in \mathrm{Par}(5.3)$, there must exist $j_0 \in J$ such that

$$\|S_{j_0}(m_0)\| < \left\|S_{j_0}\left(r_{k_1}^{-1} m_1\right)\right\| \leq \left\|r_{k_1}^{-1}\right\| \|S_{j_0}(m_1)\| = 0,$$

which is a contradiction.

The following theorem unveils sufficient conditions for the Pareto optimal solutions to be invariant under multiplication by absolutely invertibles.

Theorem 139 *Let R be a unital seminormed ring. Let M, N_i, N_j be seminormed R-modules, $i \in I, j \in J$. Let $T_i : M \to N_i$, $S_j : M \to N_j$ be linear operators, $i \in I, j \in J$. Let $\mathcal{R}$ be a nonempty subset of M such that $\mathcal{U}_1(R)\mathcal{R} \subseteq \mathcal{R}$. Then* $\mathrm{Par}(5.3) = \mathcal{U}_1(R)\mathrm{Par}(5.3)$.

Proof 437 *Since $1 \in \mathcal{U}_1(R)$, we trivially have that* $\mathrm{Par}(5.3) \subseteq \mathcal{U}_1(R)\mathrm{Par}(5.3)$. *Conversely, fix arbitrary elements $u \in \mathcal{U}_1(R)$ and $m \in \mathrm{Par}(5.3)$. We have to prove that $um \in \mathrm{Par}(5.3)$. Suppose not. There exists $m_0 \in M$ satisfying one of the following conditions:*

- *There exists $i_0 \in I$ such that $\|T_{i_0}(m_0)\| > \|T_{i_0}(um)\|$, $\|T_i(m_0)\| \geq \|T_i(um)\|$ for all $i \in I \setminus \{i_0\}$, and $\|S_j(m_0)\| \leq \|S_j(um)\|$ for all $j \in J$.*
- *There exists $j_0 \in J$ such that $\|S_{j_0}(m_0)\| < \|S_{j_0}(um)\|$, $\|S_j(m_0)\| \leq \|S_j(um)\|$ for all $j \in J \setminus \{j_0\}$, and $\|T_i(m_0)\| \geq \|T_i(um)\|$ for all $i \in I$.*

We may assume without any loss of generality that the first condition above is satisfied. Then $\left\|T_{i_0}\left(u^{-1}m_0\right)\right\| = \left\|u^{-1}\right\| \|T_{i_0}(m_0)\| > \left\|u^{-1}\right\| \|T_{i_0}(um)\| = \|T_{i_0}(m)\|$. Since $m \in \mathrm{Par}(5.3)$, there must exist $i \in I \setminus \{i_0\}$ or $j \in J$ such that either $\|T_i(m)\| > \left\|T_i\left(u^{-1}m_0\right)\right\|$ or $\|S_j(m)\| < \left\|S_j\left(u^{-1}m_0\right)\right\|$. Each condition implies that $\|T_i(um)\| > \|T_i(m_0)\|$ or $\|S_j(um)\| < \|S_j(m_0)\|$, respectively, reaching then a contradiction with the first condition above.

Let R be a seminormed ring. Let M, N_i, N_j be seminormed R-modules, $i \in I, j \in J$. Let $T_i : M \to N_i$, $S_j : M \to N_j$ be linear operators, $i \in I, j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. Fix $j' \in J$ and $t \geq 0$. The following operator multioptimization problem

$$\begin{cases} \max \|T_i(m)\| & i \in I, \\ \min \|S_j(m)\| & j \in J \setminus \{j'\}, \\ \|S_{j'}(m)\| \leq t, \\ m \in \mathcal{R}. \end{cases} \tag{5.6}$$

will be considered in the next results. Notice that the feasible region of (5.6) is $\mathcal{R}' := \{m \in \mathcal{R} : \|S_{j'}(m)\| \leq t\}$.

Theorem 140 *Let R be a seminormed ring. Let M, N_i, N_j be seminormed R-modules, $i \in I, j \in J$. Let $T_i : M \to N_i$, $S_j : M \to N_j$ be linear operators, $i \in I, j \in J$. Let $\mathcal{R}$ be a nonempty subset of M. Fix $j' \in J$ and $t \geq 0$. Then* $\mathrm{Par}(5.3) \cap \{m \in \mathcal{R} : \|S_{j'}(m)\| = t\} \subseteq \mathrm{Par}(5.6)$.

Proof 438 *Fix an arbitrary $m \in \mathrm{Par}(5.3)$ such that $\|S_{j'}(m)\| = t$. Suppose to the contrary that $m \notin \mathrm{Par}(5.6)$. There exists $m_0 \in \mathcal{R}'$ satisfying one of the following conditions:*

- *There exists $i_0 \in I$ such that $\|T_{i_0}(m_0)\| > \|T_{i_0}(m)\|$, $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I \setminus \{i_0\}$, and $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $j \in J \setminus \{j'\}$.*
- *There exists $j_0 \in J \setminus \{j'\}$ such that $\|S_{j_0}(m_0)\| < \|S_{j_0}(m)\|$, $\|S_j(m_0)\| \leq \|S_j(m)\|$ for all $j \in J \setminus \{j_0, j'\}$, and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$.*

If any one of the two previous conditions is satisfied, then, by bearing in mind that $m \in \mathrm{Par}(5.3)$, the only possibility left is that $\|S_{j'}(m)\| < \|S_{j'}(m_0)\|$. However, this means the contradiction that $t = \|S_{j'}(m)\| < \|S_{j'}(m_0)\| \leq t$ since $m_0 \in \mathcal{R}'$.

The following theorem is an approach to the converse of Theorem 140. Let R be a seminormed ring. Let M, N_i, N be seminormed R-modules, $i \in I$. Let $T_i : M \to N_i$, $S : M \to N$ be linear operators, $i \in I$. Let $\mathcal{R}$ be a nonempty subset of M. Fix $t \geq 0$. The following operator multioptimization problems

$$\begin{cases} \max \|T_i(m)\| \ \ i \in I, \\ \min \|S(m)\|, \\ m \in \mathcal{R}. \end{cases} \tag{5.7}$$

and

$$\begin{cases} \max \|T_i(m)\| \ \ i \in I, \\ \|S(m)\| \leq t, \\ m \in \mathcal{R}. \end{cases} \tag{5.8}$$

will be considered in the next result. Notice that the feasible region of (5.8) is $\mathcal{R}' := \{m \in \mathcal{R} : \|S(m)\| \leq t\}$.

Theorem 141 *Let R be a hyperpractical seminormed ring. Let M, N_i, N be seminormed R-modules, $i \in I$. Let $T_i : M \to N_i$, $S : M \to N$ be linear operators, $i \in I$. Let $\mathcal{R}$ be a nonempty subset of M such that $\mathcal{U}_1(R)\mathcal{R} \subseteq \mathcal{R}$ and $\mathcal{R}' \nsubseteq \bigcap_{i\in I} \ker_{\mathfrak{t}}(T_i)$. Fix $t \geq 0$. Then* $\mathrm{Par}(5.7) \cap \{m \in M : \|S(m)\| = t\} = \mathrm{Par}(5.8)$.

Proof 439 *Following a similar proof as in Theorem 140, we conclude that* $\mathrm{Par}(5.7) \cap \{m \in M : \|S(m)\| = t\} \subseteq \mathrm{Par}(5.8)$. *Fix an arbitrary $m \in \mathrm{Par}(5.8)$. We will show first that $\|S(m)\| = t$. Indeed, suppose to the contrary that $\|S(m)\| < t$. By hypothesis, R is hyperpractical, therefore, there exists $u \in \mathcal{U}_1(R)$ with $1 < \|u\| < \frac{t}{\|S(m)\|}$, meaning, by Lemma 75(1), that $\|S(m)\| < \|S(um)\| < t$. In case $\|S(m)\| = 0$, then it suffices to take any $u \in \mathcal{U}_1(R)$ with $\|u\| > 1$. Then $um \in \mathcal{R}'$ and $\|T_i(um)\| = \|u\|\|T_i(m)\| > \|T_i(m)\|$ for all $i \in I$, meaning a contradiction with the fact that $m \in \mathrm{Par}(5.8)$. As a consequence, $\|S(m)\| = t$. Let us prove now that $m \in \mathrm{Par}(5.7)$. Suppose not. There exists $m_0 \in \mathcal{R}$ satisfying one of the following conditions:*

- *There exists $i_0 \in I$ such that $\|T_{i_0}(m_0)\| > \|T_{i_0}(m)\|$, $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I \setminus \{i_0\}$, and $\|S(m_0)\| \leq \|S(m)\|$.*
- *$\|S(m_0)\| < \|S(m)\|$ and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$.*

If the first condition holds, then $m_0 \in \mathcal{R}'$, meaning a contradiction with the fact that $m \in \mathrm{Par}(5.8)$. So, let us assume that the second condition holds. In this situation, we have that $\|S(m_0)\| < \|S(m)\|$ and $\|T_i(m_0)\| \geq \|T_i(m)\|$ for all $i \in I$. Since $m \in \mathrm{Par}(5.8)$, we actually have that $\|S(m_0)\| < \|S(m)\|$ and $\|T_i(m_0)\| = \|T_i(m)\|$ for all $i \in I$. This shows that $m_0 \in \mathrm{Par}(5.8)$, implying the contradiction that $t = \|S(m_0)\| < \|S(m)\| = t$.

5.1.2 Reformulation

Sometimes, it is not an easy task to find the set of Pareto optimal solutions of a multiobjective optimization problem. In this case, it might result of a lot of help to reformulate the problem in order to reduce the number of objective functions.

5.1.2.1 Necessity

The following theorem, which is a generalization of Example 75, shows the necessity of reformulating certain multiobjective optimization problems.

Theorem 142 *Let R be a normed ring. Let M, N be normed R-modules. Let $T, S : M \to N$ be nonzero linear maps. If R is absolutely valued and practical or $\ker(S) \subseteq \ker(T)$, then the multiobjective optimization problem*

$$\begin{cases} \max \|T(m)\|, \\ \min \|S(m)\|, \\ m \in M. \end{cases} \tag{5.9}$$

lacks any optimal solutions.

Proof 440 *Suppose first that R is absolutely valued and practical. Take $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(R)$ a sequence of invertibles converging to 0. For every $n \in \mathbb{N}$, $\left|r_n^{-1}\right| = |r_n|^{-1}$, therefore, $\left(\left|r_n^{-1}\right|\right)_{n\in\mathbb{N}}$ diverges to ∞. Take $m \in M \setminus \ker(T)$. Then $\left\|T\left(r_n^{-1}m\right)\right\| = |r_n|^{-1}\|T(m)\| \to \infty$ as $n \to \infty$. As a consequence, $\mathrm{sol}(5.9) \subseteq \arg\max_{m\in M} \|T(m)\| = \varnothing$. Finally, assume that $\ker(S) \subseteq \ker(T)$. If $m_0 \in \mathrm{sol}(5.9)$, then $\|S(m_0)\| \leq \|S(0)\| = 0$, which implies that $m_0 \in \ker(S) \subseteq \ker(T)$. Next, if we consider any $m \in M \setminus \ker(T)$, then we reach the contradiction that $0 = \|T(m_0)\| \geq \|T(m)\| > 0$.*

5.1.2.2 Simplification

We will present three general ways to reformulate multiobjective optimization problems in such a way that the solutions are somehow preserved. The first way, displayed in the following remark, shows that every multiobjective optimization problem can be rewritten as a single-objective optimization problem.

Remark 88 *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. Let $C := A^M \times B^N$ be endowed with the skew-product order (that is, inverting the order in B^N). In other words, if $c = (c_1, c_2), d = (d_1, d_2) \in C$, then*

$$c \leq d \Leftrightarrow \begin{cases} c_1(f) \leq d_1(f) & \forall f \in M, \\ d_2(g) \leq c_2(g) & \forall g \in N. \end{cases}$$

Consider the single-objective optimization problem

$$\begin{cases} \max h(x), \\ x \in \mathcal{R}, \end{cases} \tag{5.10}$$

where

$$\begin{array}{rccl} h: & X & \to & C \\ & x & \mapsto & h(x) := \left(\left(f(x)\right)_{f\in M}, \left(g(x)\right)_{g\in N}\right). \end{array}$$

Then $\mathrm{sol}(5.1) = \mathrm{sol}(5.10)$.

The second way, presented in the following theorem, also reformulates multiobjective optimization problems as single-objective optimization problems.

Theorem 143 *Let X be a nonempty set. Let A, B be posets. Consider $M \subseteq A^X$ and $N \subseteq B^X$ nonempty. Let $\mathcal{R}$ be a nonempty subset of X. Let R be a unital ordered ring such that R^+ is multiplicatively symmetric. Suppose that $F : A^M \to R^+, G : B^N \to R^+ \cap \mathcal{U}(R)$ are strictly increasing when A^M, B^N are endowed with the product order. Consider the single-objective optimization problem*

$$\begin{cases} \max F\left(\left(f(x)\right)_{f\in M}\right) G\left(\left(g(x)\right)_{g\in N}\right)^{-1}, \\ x \in \mathcal{R}. \end{cases} \tag{5.11}$$

Then:

1. $\mathrm{sol}(5.11) \subseteq \mathrm{Par}(5.1)$.
2. *If* $\mathrm{sol}(5.1) \neq \emptyset$, *then* $\mathrm{sol}(5.1) = \mathrm{sol}(5.11)$.

Proof 441 *Let us itemize the proof via the statement.*

1. *Fix an arbitrary* $x \in \mathrm{sol}(5.11)$. *Suppose to the contrary that* $x \notin \mathrm{Par}(5.1)$. *Then there exists* $y \in \mathcal{R}$ *satisfying at least one of the following two conditions:*

   - *There exists* $f_0 \in M$ *with* $f_0(y) > f_0(x)$, $f(y) \geq f(x)$ *for all* $f \in M \setminus \{f_0\}$, *and* $g(y) \leq g(x)$ *for all* $g \in N$.
   - *There exists* $g_0 \in N$ *with* $g_0(y) < g_0(x)$, $g(y) \leq g(x)$ *for all* $g \in N \setminus \{g_0\}$, *and* $f(y) \geq f(x)$ *for all* $f \in M$.

   We may assume, without any loss of generality, that the first condition holds. Notice that $(f(x))_{f\in M} < (f(y))_{f\in M}$ *and* $(g(y))_{g\in N} \leq (g(x))_{g\in N}$. *Since* F, G *are strictly increasing, we obtain that* $0 \leq F((f(x))_{f\in M}) < F((f(y))_{f\in M})$ *and* $0 < G((g(y))_{g\in N}) < G((g(x))_{g\in N})$. *Next,* $0 < G((g(x))_{g\in N})^{-1} < G((g(y))_{g\in N})^{-1}$ *due to the fact that* R^+ *is multiplicatively symmetric, meaning that* $F((f(x))_{f\in M})\, G((g(x))_{g\in N})^{-1} < F((f(y))_{f\in M})\, G((g(y))_{g\in N})^{-1}$. *This contradicts the fact that* $x \in \mathrm{sol}(5.11)$.

2. *First off, since* $\mathrm{sol}(5.1) \neq \emptyset$, *we have that* $\mathrm{Par}(5.1) = \mathrm{sol}(5.1)$. *Next, according to Theorem 143(1),* $\mathrm{sol}(5.11) \subseteq \mathrm{Par}(5.1) = \mathrm{sol}(5.1)$. *It only remains to prove that* $\mathrm{sol}(5.1) \subseteq \mathrm{sol}(5.11)$. *Take any* $x \in \mathrm{sol}(5.1)$. *Let* $y \in \mathcal{R}$. *Then* $f(x) \geq f(y)$ *for all* $f \in M$ *and* $g(x) \leq g(y)$ *for all* $g \in N$. *Therefore,* $(f(x))_{f\in M} \geq (f(y))_{f\in M}$ *and* $(g(x))_{g\in N} \leq (g(y))_{g\in N}$. *Since* F, G *are both strictly increasing, we have that* $F((f(x))_{f\in M}) \geq F((f(y))_{f\in M})$ *and* $G((g(x))_{g\in N}) \leq G((g(y))_{g\in N})$. *As a consequence,* $F((f(y))_{f\in M})\, G((g(y))_{g\in N})^{-1} \leq F((f(x))_{f\in M})\, G((g(x))_{g\in N})^{-1}$. *The arbitrariness of* y *shows that* $x \in \mathrm{sol}(5.11)$.

The third and final way transforms the multiobjective optimization problem (5.1) into another multiobjective optimization problem with only two objective functions.

Theorem 144 *Let* X *be a nonempty set. Let* A, B *be tosets. Consider* $M \subseteq A^X$ *and* $N \subseteq B^X$ *nonempty. Let* $\mathcal{R}$ *be a nonempty subset of* X. *Suppose that* C, D *are tosets and* $F : A^M \to C, G : B^N \to D$ *are strictly increasing when* A^M, B^N *are endowed with the product order. Consider the multiobjective optimization problem*

$$\begin{cases} \max F((f(x))_{f\in M}), \\ \min G((g(x))_{g\in N}), \\ x \in \mathcal{R}. \end{cases} \tag{5.12}$$

Then:

1. $\mathrm{Par}(5.12) \subseteq \mathrm{Par}(5.1)$.
2. *If* $\mathrm{sol}(5.1) \neq \emptyset$, *then* $\mathrm{sol}(5.1) = \mathrm{sol}(5.12)$.

Proof 442 *The proof will follow the same itemization as of the statement.*

1. *Fix an arbitrary* $x \in \mathrm{Par}(5.12)$. *Suppose to the contrary that* $x \notin \mathrm{Par}(5.1)$. *Then there exists* $y \in \mathcal{R}$ *satisfying at least one of the following two conditions:*

- *There exists* $f_0 \in M$ *with* $f_0(y) > f_0(x)$, $f(y) \geq f(x)$ *for all* $f \in M \setminus \{f_0\}$ *and* $g(y) \leq g(x)$ *for all* $g \in N$.
- *There exists* $g_0 \in N$ *with* $g_0(y) < g_0(x)$, $g(y) \leq g(x)$ *for all* $g \in N \setminus \{g_0\}$ *and* $f(y) \geq f(x)$ *for all* $f \in M$.

We may assume, without any loss of generality, that the first condition holds. Notice that $(f(x))_{f\in M} < (f(y))_{f\in M}$ *and* $(g(y))_{g\in N} \leq (g(x))_{g\in N}$. *Since* F, G *are strictly increasing, we obtain that* $F((f(x))_{f\in M}) < F((f(y))_{f\in M})$ *and* $G((g(y))_{g\in N}) \leq G((g(x))_{g\in N})$. *This contradicts the fact that* $x \in \text{Par}(5.12)$.

2. *First off, since* $\text{sol}(5.1) \neq \emptyset$, *we have that* $\text{Par}(5.1) = \text{sol}(5.1)$. *Next, according to Theorem 144(1),* $\text{sol}(5.12) \subseteq \text{Par}(5.12) \subseteq \text{Par}(5.1) = \text{sol}(5.1)$. *It only remains to prove that* $\text{sol}(5.1) \subseteq \text{sol}(5.12)$. *Take any* $x \in \text{sol}(5.1)$. *Let* $y \in \mathcal{R}$. *Then* $f(x) \geq f(y)$ *for all* $f \in M$ *and* $g(x) \leq g(y)$ *for all* $g \in N$. *Therefore,* $(f(x))_{f\in M} \geq (f(y))_{f\in M}$ *and* $(g(x))_{g\in N} \leq (g(y))_{g\in N}$. *Since* F, G *are both strictly increasing, we have that* $F((f(x))_{f\in M}) \geq F((f(y))_{f\in M})$ *and* $G((g(x))_{g\in N}) \leq G((g(y))_{g\in N})$. *The arbitrariness of* y *shows that* $x \in \text{sol}(5.12)$.

5.2 Convex optimization

Convex optimization is a branch of Optimization Theory that focuses on the study of multiobjective optimization problems whose objective functions are convex.

5.2.1 Convex functions

The concept of convex function is originally from the scope of real vector spaces. Here, we will transport it to the scope of modules. We refer the reader to [9, 63, 64].

5.2.1.1 Epigraph

The concept of convexity in modules allows the definition of convex functions in the same setting.

Definition 300 (Affine function) *Let* M *be a module over a ring* R. *Let* B^+ *be a unit segment of* R. *Consider a convex subset* $C \subseteq M$. *A function* $f : C \to R$ *is said to be:*

- *Weakly affine if* $f(\Delta[m_1, \ldots, m_k]) \subseteq \Delta[f(m_1), \ldots, f(m_k)]$ *for all* $m_1, \ldots, m_k \in C$.
- *Affine provided that* $f(r_1m_1 + \cdots + r_km_k) = r_1f(m_1) + \cdots + r_kf(m_k)$ *for all* $m_1, \ldots, m_k \in C$ *and all* $r_1, \ldots, r_k \in B^+$ *with* $r_1 + \cdots + r_k = 1$.

Observe that linear functionals are trivially affine functions, and affine functions are trivially weakly affine.

Definition 301 (Convex function) *Let* M *be a module over a unital ordered ring* R. *Consider a convex subset* $C \subseteq M$. *A function* $f : C \to R$ *is said to be convex provided that* $f(r_1m_1 + \cdots + r_km_k) \leq r_1f(m_1) + \cdots + r_kf(m_k)$ *for all* $m_1, \ldots, m_k \in C$ *and all* $r_1, \ldots, r_k \in [0, 1]$ *with* $r_1 + \cdots + r_k = 1$.

Under the settings of the previous definition, affine functions are also convex.

Definition 302 (Epigraph) *Let R be a unital ordered ring. Let M be an R-module. Let $A \subseteq M$ and $f : A \to R$ a function. The epigraph of f is defined as* $\operatorname{epi}(f) := \{(m, r) \in A \times R : f(m) \leq r\}$.

The convexity character of a function can be studied through its epigraph.

Proposition 119 *Let M be a module over a unital ordered ring R. Consider a convex subset $C \subseteq M$. A function $f : C \to R$ is convex if and only if its epigraph,* $\operatorname{epi}(f)$*, is convex in the R-module $M \times R$.*

Proof 443 *Suppose first that $f : C \to R$ is convex. Fix arbitrary elements $(m_1, r_1), \ldots, (m_k, r_k) \in \operatorname{epi}(f)$. We will prove that $\Delta[(m_1, r_1), \ldots, (m_k, r_k)] \subseteq \operatorname{epi}(f)$. Let $a_1, \ldots, a_k \in [0, 1]$ with $a_1 + \cdots + a_k = 1$. Note that $a_1(m_1, r_1) + \cdots + a_k(m_k, r_k) = (a_1 m_1 + \cdots + a_k m_k, a_1 r_1 + \cdots + a_k r_k) \in C \times R$ because $a_1 m_1 + \cdots + a_k m_k \in \Delta[m_1, \ldots, m_k] \subseteq C$. On the other hand, $f(a_1 m_1 + \cdots + a_k m_k) \leq a_1 f(m_1) + \cdots + a_k f(m_k) \leq a_1 r_1 + \cdots + a_k r_k$ because f is convex and $a_i \geq 0$ for each $i \in \{1, \ldots, k\}$. Conversely, assume that* $\operatorname{epi}(f)$ *is convex. Fix arbitrary elements $m_1, \ldots, m_k \in C$ and $a_1, \ldots, a_k \in [0, 1]$ with $a_1 + \cdots + a_k = 1$. Notice that $(a_1 m_1 + \cdots + a_k m_k, a_1 f(m_1) + \cdots + a_k f(m_k)) = a_1(m_1, f(m_1)) + \cdots + a_k(m_k, f(m_k)) \in \Delta[(m_1, f(m_1)), \ldots, (m_k, f(m_k))] \subseteq \operatorname{epi}(f)$, which implies that $f(a_1 m_1 + \cdots + a_k m_k) \leq a_1 f(m_1) + \cdots + a_k f(m_k)$.*

In the upcoming theorem, π_M stands for the projection on the first coordinate of $M \times R$, that is,

$$\begin{array}{rccl} \pi_M : & M \times R & \to & M \\ & (m, r) & \mapsto & \pi_M(m, r) := m. \end{array}$$

Theorem 145 *Let M be a module over a unital ordered ring R. Let D be a subset of $M \times R$. The following conditions are equivalent:*

1. *There exists a subset C of M and a function $f : C \to R$ such that $D = \operatorname{epi}(f)$.*
2. *For every $(c, r) \in D$, $\{c\} \times \uparrow r \subseteq D$ and there exists* $\min\{s \in R : (c, s) \in D\}$.

Proof 444 *The proof will follow as itemized as the statement of the theorem.*

1⇒2 *Fix an arbitrary $(c, r) \in D$. Then $(c, r) \in \operatorname{epi}(f)$, thus if $s \geq r$, then $f(c) \leq r \leq s$, meaning that $(c, s) \in \operatorname{epi}(f) = D$. Next, if $s \in R$ and $(c, s) \in D$, then $(c, s) \in \operatorname{epi}(f)$, so $f(c) \leq s$. Finally, $(c, f(c)) \in \operatorname{epi}(f) = D$. This way, we conclude that $f(c) = \min\{s \in R : (c, s) \in D\}$.*

2⇒1 *Take $C := \pi_M(D)$ and define the following function:*

$$\begin{array}{rccl} f : & C & \to & R \\ & c & \mapsto & f(c) := \min\{s \in R : (c, s) \in D\}. \end{array}$$

We will prove that $D = \operatorname{epi}(f)$. Fix an arbitrary $(c, r) \in \operatorname{epi}(f)$. Observe that, by hypothesis, $f(c) := \min\{s \in R : (c, s) \in D\}$, thus $(c, f(c)) \in D$. Then, also by hypothesis, $(c, r) \in \{c\} \times \uparrow f(c) \subseteq D$ because $r \geq f(c)$. This shows that $D \supseteq \operatorname{epi}(f)$. Conversely, suppose that $(c, r) \in D$. Note that $c \in \pi_M(D) = C$ and, by hypothesis, $f(c) := \min\{s \in R : (c, s) \in D\} \leq r$. This shows that $D \subseteq \operatorname{epi}(f)$.

Under the settings of the previous theorem, observe that π_M is a linear map; therefore, if $D \subseteq M \times R$ is convex, then $\pi_M(D)$ is convex as well in virtue of Corollary 4(2). Therefore, by taking into consideration Proposition 119, we obtain the following immediate corollary.

Corollary 53 *Let M be a module over a unital ordered ring R. Let D be a subset of $M \times R$. The following conditions are equivalent:*

1. *There exists a convex subset C of M and a convex function $f : C \to R$ such that $D = \operatorname{epi}(f)$.*
2. *D is convex and for every $(c, r) \in D$, $\{c\} \times \uparrow r \subseteq D$ and there exists $\min\{s \in R : (c, s) \in D\}$.*

The following technical lemma makes the notion of convex function meaningful.

Lemma 153 *Let M be a module over a unital ordered ring R. Let $C \subseteq M$ be convex. If $f : C \to R$ is convex, then $f^{-1}(\downarrow r)$ is convex for all $r \in R$.*

Proof 445 *Fix arbitrary elements $m_1, \ldots, m_k \in f^{-1}(\downarrow r)$. We will show that $\Delta[m_1, \ldots, m_k] \subseteq f^{-1}(\downarrow r)$. Take $r_1, \ldots, r_k \in [0, 1]$ with $r_1 + \cdots + r_k = 1$. Notice that $f(r_1 m_1 + \cdots + r_k m_k) \leq r_1 f(m_1) + \cdots + r_k f(m_k) \leq r_1 r + \cdots + r_k r = r$ because $r_i \geq 0$ for all $i \in \{1, \ldots, k\}$. Therefore, $r_1 m_1 + \cdots + r_k m_k \in f^{-1}(\downarrow r)$ and hence $\Delta[m_1, \ldots, m_k] \subseteq f^{-1}(\downarrow r)$ obtaining the convexity of $f^{-1}(\downarrow r)$.*

Recall that a unital ordered topological ring is a topological ring which is also endowed with a unital ring ordering in such a way that there is not necessarily any relation between the ring topology and the ring ordering.

Theorem 146 *Let R be an integral domain, unital ordered topological ring R such that $0 \in \operatorname{cl}((0, 1])$. Let M be an R-module. Let $f : M \to R$ be a convex function. If $\operatorname{inter}\left(f^{-1}(0)\right) \neq \varnothing$, then $f^{-1}(\downarrow^{\times} 0) = \varnothing$ and $f^{-1}(0)$ is convex.*

Proof 446 *Suppose on the contrary that there exists $y \in M$ such that $f(y) < 0$. Take any $x \in \operatorname{inter}\left(f^{-1}(0)\right)$. There exists an additively symmetric neighborhood $V \subseteq R$ of 0 such that $x + V(y - x) \subseteq f^{-1}(0)$. Since $0 \in \operatorname{cl}((0, 1])$, we can find $v \in V \cap (0, 1]$. Then $vy + (1 - v)x = x + v(y - x) \in f^{-1}(0)$, therefore, we obtain the contradiction that $0 = f(vy + (1 - v)x) \leq v f(y) + (1 - v) f(x) = v f(y) < 0$. Finally, $f^{-1}(0) = f^{-1}(\downarrow 0)$ is convex in view of Lemma 153.*

The following lemma is an adaptation of Lemma 95 to the scope of topological modules over ordered topological rings.

Lemma 154 *Let R be a unital ordered topological ring satisfying the sup property. Let M be a topological R-module. Let $\{m_i^*\}_{i \in I} \subseteq M^*$ be a pointwise bounded above family of continuous linear functionals on M. Consider the following function*

$$\begin{array}{rccl} f: & M & \to & R \\ & m & \mapsto & f(m) := \sup\limits_{i \in I} m_i^*(m). \end{array} \tag{5.13}$$

Then:

1. *f is convex.*
2. *Suppose that R satisfies the strong sup property. If m_i^* is lower semicontinuous for all $i \in I$, then f is lower semicontinuous as well.*
3. *Suppose that $\mathcal{U}(R)$ is open, $0 \in \operatorname{cl}(R^+ \setminus \{0\})$, and R^+ is multiplicatively symmetric. If U is an open subset of M and $r \in R^+ \setminus \mathtt{rd}(0)$ is an upper bound for $f(U)$, then $f(u) < r$ for all $u \in U$.*

Proof 447 *The proof will follow as itemized as the statement of the lemma.*

1. *Take $m_1, \ldots, m_k \in M$ and $r_1, \ldots, r_k \in [0,1]$ with $r_1 + \cdots + r_k = 1$. Fix an arbitrary $i \in I$. Since $r_i \geq 0$ for each $i \in \{1, \ldots, k\}$, we have that $m_i^*(r_1 m_1 + \cdots + r_k m_k) = r_1 m_i^*(m_1) + \cdots + r_k m_i^*(m_k) \leq r_1 f(m_1) + \cdots + r_k f(m_k)$. The arbitrariness of $i \in I$ shows that $f(r_1 m_1 + \cdots + r_k m_k) \leq r_1 f(m_1) + \cdots + r_k f(m_k)$. This shows that f is convex.*

2. *Lemma 95 guarantees that if the m_i^*'s are lower semicontinuous, then f is lower semicontinuous as well.*

3. *Fix an arbitrary $u \in U$. Consider a 0-neighborhood $V \subseteq R$ satisfying $Vu \subseteq -u + U$ and $1 + V \subseteq \mathcal{U}(R)$ (recall that $\mathcal{U}(R)$ is open by hypothesis). Since $0 \in \mathrm{cl}(R^+ \setminus \{0\})$, we can choose $v \in V$ with $v > 0$. Then $u + vu \in u + Vu \subseteq U$, so $(1+v)m_i^*(u) = m_i^*(u+vu) \leq f(u+vu) \leq r$ for all $i \in I$. Next, R^+ is multiplicatively symmetric, hence $(1+v)^{-1} > 0$. Therefore, $m_i^*(u) \leq (1+v)^{-1} r$ for all $i \in I$, hence $f(u) \leq (1+v)^{-1} r$. Now, observe that $v > 0$, so $1 + v > 1$ and then $(1+v)^{-1} < 1$, which implies that $(1+v)^{-1} r < r$ because r is strictly positive and not a right divisor of zero. Thus $f(u) < r$.*

Note that, in the settings of Lemma 154, if the ring topology of R coincides with the order topology, then the continuity of the m_i^*'s implies their lower semicontinuity, hence the lower semicontinuity of f.

5.2.1.2 Slices

Geometrically speaking, a slice is the intersection of a convex set with a halfplane. Analytically speaking, a slice is a generalization of $\arg\max$.

Definition 303 (Slice) *Let M be a module over a unital ordered ring R. Let $C \subseteq M$, $f : M \to R$ a convex function, and $r \in R$. The slice of C determined by f and r is defined by* $\mathrm{slc}(C, f, r) := f^{-1}(\uparrow r) \cap C$.

Under the settings of the previous definition, keep in mind that if $\max_C(f)$ exists in R, then

$$\mathrm{slc}\left(C, f, \max_C(f)\right) = \arg\max_C(f).$$

Exposed faces and exposed points are defined as particular cases of slices.

Definition 304 (Exposed face) *Let M be a module over a unital ordered ring R. Let $F \subseteq C \subseteq M$. We say that F is exposed in C if there exists a convex function $f : M \to R$ such that $\max_C(f)$ exists in R and*

$$F = \mathrm{slc}\left(C, f, \max_C(f)\right) = \arg\max_C(f).$$

Under the settings of the previous definition, if f is continuous and R is Hausdorff, then $\mathrm{slc}(C, f, \max_C(f))$ is closed in C because $\mathrm{slc}(C, f, \max_C(f)) = f^{-1}(\{\max_C(f)\}) \cap C$. Exposed faces are examples of extremal sets.

Proposition 120 *Let R be an integral domain, unital ordered ring. Let M be an R-module. Let $F \subseteq C \subseteq M$. If F is exposed in C, then F is extremal in C.*

Proof 448 *Let $c, d \in C$ such that $F \cap \Delta(c,d) \neq \varnothing$. There exists $r \in (0,1)$ such that $rc + (1-r)d \in F$. Let $f : M \to R$ be a convex function such that $\max_C(f)$ exists in R and*

$$F = \operatorname{slc}\left(C, f, \max_C(f)\right) = \arg\max_C(f).$$

Suppose to the contrary that, for instance, $c \notin F$. Then $f(c) < \max(f)$ so $rf(c) < r\max(f)$ because $0 < r < 1$ and R is an integral domain. This implies the contradiction that $\max(f) = f(rc + (1-r)d) \leq rf(c) + (1-r)f(d) < r\max(f) + (1-r)\max(f) = \max(f)$. In a similar way, $d \in F$.

Exposed faces determined by linear functionals satisfy interesting properties, which are consistent with the fact that extremal sets do not intersect internal points.

Theorem 147 *Let M be a topological module over an ordered topological ring R. Let $m^* \in M^*$ and $A \subseteq M$. If $m^*(M) \cap (R^+ \setminus \{0\}) \neq \varnothing$, $0 \in \operatorname{cl}(R^+ \setminus \{0\})$, and $(R^+ \setminus \{0\})(R^+ \setminus \{0\}) \subseteq R^+ \setminus \{0\}$, then $\arg\max_A(m^*) \cap \operatorname{inter}(A) = \varnothing$.*

Proof 449 *Suppose on the contrary that there exists $a \in \arg\max_A(m^*) \cap \operatorname{inter}(A)$. Take $m \in M$ such that $m^*(m) \in R^+ \setminus \{0\}$. Since a is an internal point of A, there exists a 0-neighborhood $V \subseteq R$ such that $a + Vm \subseteq A$. Take any $v \in V \cap (R^+ \setminus \{0\})$. Notice that $vm^*(m) > 0$, hence $m^*(a + vm) = m^*(a) + vm^*(m) > m^*(a)$ reaching a contradiction with the fact that $m^*(a) = \max m^*(A)$.*

Under the settings of Theorem 147, observe that $\arg\max_A(m^*) \cap \operatorname{int}(A) = \varnothing$ because $\operatorname{int}(A) \subseteq \operatorname{inter}(A)$ in view of Theorem 111. Also, under these settings, if m^* is an open map and $0 \in \operatorname{cl}(R^+ \setminus \{0\})$, then $m^*(M) \cap (R^+ \setminus \{0\}) \neq \varnothing$.

Definition 305 (Exposed point) *Let M be a module over a unital ordered ring R. Let $C \subseteq M$ be a subset. A point $c \in C$ is said to be an exposed point of C if $\{c\}$ is exposed in C. The set of exposed points of C is denoted as $\exp(C)$.*

The final results in this subsubsection are aimed at obtaining the boundary of a nonempty proper open convex subset of a topological module via the slice of its closure by a convenient lower semicontinuous convex function.

Definition 306 (Hahn-Banach Separation Property) *Let M be a topological module over a (unital) ordered topological ring R. We say that M has the (unital) Hahn-Banach Separation Property if for every nonempty proper open convex subset U of M and every $m \in M \setminus U$, there exists a lower semicontinuous $m^* \in M^*$ such that $(1 =) m^*(m)$ is an upper bound for $m^*(U)$.*

Under the settings of the previous definition, if R enjoys the sup property, then we may state that $m^*(m) \geq \sup m^*(U)$. Nevertheless, we cannot assure that $\sup m^*(U) = \sup m^*(\operatorname{cl}(U))$ because in general the ring ordering is not necessarily total. Nevertheless, if R enjoys the strong sup property, then at least we can assure that $\sup m^*(U)$ is in the closure of $m^*(U)$ for the order topology. And if the ring topology is the order topology, then the continuity of m^* implies its lower semicontinuity.

Theorem 148 *Let R be a topological unital ordered ring R verifying the strong sup property and such that $\mathcal{U}(R)$ is open, $0 \in \operatorname{cl}(R^+ \cap \mathcal{U}(R))$, and R^+ is multiplicatively symmetric. Let M be a topological R-module satisfying the unital Hahn-Banach separation property. Let U be a nonempty proper open convex subset of M. There exists a lower semicontinuous convex function $f : M \to R$ such that $\operatorname{bd}(U) = \operatorname{slc}(\operatorname{cl}(U), f, 1)$.*

Proof 450 *In the first place, we will make the assumption that* $0 \in U$. *Since* M *satisfies the unital Hahn-Banach separation property, for every* $x \in \mathrm{bd}(U)$ *there exists a lower semicontinuous* $x^* \in M^*$ *such that* $1 = x^*(x)$ *is an upper bound for* $x^*(U)$. *Since* R *enjoys the sup property,* $\sup x^*(U)$ *exists in* R, *thus* $1 = x^*(x) \geq \sup x^*(U)$. *The Axiom of Choice allows to define the function*

$$\begin{array}{rccl} f: & M & \to & R \\ & m & \mapsto & f(m) := \sup\limits_{x \in \mathrm{bd}(U)} x^*(m). \end{array} \tag{5.14}$$

Let us verify the desired properties for f:

- f *is well defined. Indeed, choose an arbitrary* $m \in M$. *There exists a neighborhood* $W \subseteq R$ *of* 0 *such that* $Wm \subseteq U$. *Since* $0 \in \mathrm{cl}(R^+ \cap \mathcal{U}(R))$, *we can consider* $w_m \in W \cap \mathcal{U}(R)$ *with* $w_m > 0$. *Note that* $(w_m)^{-1} > 0$ *because* R^+ *is multiplicatively symmetric. Then* $w_m m \in U$ *so* $w_m x^*(m) = x^*(w_m m) \leq 1$ *for every* $x \in \mathrm{bd}(U)$. *That is,* $x^*(m) \leq (w_m)^{-1}$ *for every* $x \in \mathrm{bd}(U)$. *This means that* $(w_m)^{-1}$ *is an upper bound for the set* $\{x^*(m) : x \in \mathrm{bd}(U)\}$. *Since* R *satisfies the sup property,* $\{x^*(m) : x \in \mathrm{bd}(U)\}$ *has a supremum in* R, *which will be denoted by* $f(m)$ *in accordance with* (5.14). *This shows that* $\{x^*\}_{x \in \mathrm{bd}(U)}$ *is pointwise bounded above and* f *is well defined.*

- f *is convex and lower semicontinuous. Indeed, convexity and lower semicontinuity of* f *are guaranteed by Lemma 154(1,2).*

- $\sup f(U) \leq 1$. *Indeed, in accordance with Remark 68,* $\sup f(U) = \sup\{\sup x^*(U) : x \in \mathrm{bd}(U)\}$, *thus, it only suffices to show that* $\sup x^*(U) \leq 1$ *for all* $x \in \mathrm{bd}(U)$. *So, fix an arbitrary* $x \in \mathrm{bd}(U)$. *We already know that* 1 *is an upper bound for* $x^*(U)$, *meaning that* $\sup x^*(U) \leq 1$.

- $\mathrm{bd}(U) \subseteq \{m \in \mathrm{cl}(U) : f(m) \geq 1\}$. *Indeed, if* $x \in \mathrm{bd}(U)$, *then* $f(x) \geq x^*(x) = 1$.

- $\mathrm{bd}(U) \supseteq \{m \in \mathrm{cl}(U) : f(m) \geq 1\}$. *Fix an arbitrary* $m \in \mathrm{cl}(U)$ *such that* $f(m) \geq 1$. *Since* $\mathrm{cl}(U)$ *is the disjoint union of* U *and* $\mathrm{bd}(U)$, *all is needed to be shown is that* $m \notin U$. *Indeed, assume to the contrary that* $m \in U$. *Following Lemma 154(3),* $f(m) < 1$, *contradicting our initial assumption.*

Finally, suppose that $0 \notin U$. *Then fix any* $u_0 \in U$ *and consider* $U - u_0$. *Now,* $0 \in U - u_0$ *so there exists a lower semicontinuous convex function* $f : M \to R$ *satisfying* $\mathrm{bd}(U - u_0) = \{m \in \mathrm{cl}(U - u_0) : f(m) \geq 1\}$. *Consider the function*

$$\begin{array}{rccl} g: & M & \to & R \\ & m & \mapsto & g(m) := f(m - u_0). \end{array}$$

It is not hard to check that g *is convex, lower semicontinuous and* $\mathrm{bd}(U) = \{m \in \mathrm{cl}(U) : g(m) \geq 1\}$.

The following corollary is immediately derived from Theorem 54.

Corollary 54 *Let* R *be a topological totally ordered division ring* R *verifying the sup property. Let* M *be a topological* R*-module satisfying the unital Hahn-Banach separation property. Let* U *be a nonempty proper open convex subset of* M. *There exists a lower semicontinuous convex function* $f : M \to R$ *such that* $\mathrm{bd}(U) = \mathrm{slc}(\mathrm{cl}(U), f, 1)$.

Proof 451 *The unique tricky aspect is to verify that* $0 \in \mathrm{cl}(R^+ \cap \mathcal{U}(R))$. *In the first place, notice that the order topology in a totally ordered set is always Hausdorff, hence* $\{0\}$ *is closed so* $\mathcal{U}(R) := R \setminus \{0\}$ *is open. Then* $R^+ \cap \mathcal{U}(R) = R^+ \setminus \{0\}$. *Next, since the order is total,* $n^{-1} \in R^+ \setminus \{0\}$ *for every* $n := 1 + \overset{n}{\cdots} + 1$. *Since the ring topology of* R *is the order topology, every neighborhood* U *of* 0 *contains an open interval of the form* $(-r, r)$ *for some* $r > 0$. *It is not hard to check that* $0 < 2^{-1}r < r$, *meaning that* $0 \in \mathrm{cl}(R^+ \setminus \{0\})$.

Under the settings of Corollary 54, Lemma 96 applies to obtain that $\sup f(U) = \sup f(\mathrm{cl}(U))$, hence to conclude that $\mathrm{bd}(U) = \{m \in \mathrm{cl}(U) : f(m) = 1\}$.

5.2.2 Fundamental theorems

There are two fundamental theorems in Optimization Theory: Krein-Milman Theorem [65, 66] and Bauer Minimum Principle [67, 68]. These two theorems constitute the basis of most of the optimization methods employed in linear programming such as the famous Simplex Method.

5.2.2.1 Krein-Milman Theorem

The hypotheses of Hausdorff and torsionless are crucial for the Krein-Milman Theorem to work.

Lemma 155 *Let* M *be a torsionless topological module over a topological ring* R. *If* $F \subseteq M$ *satisfies that every* $m^* \in M^*$ *is constant on* A, *then* A *is a singleton.*

Proof 452 *If we fix any* $m \in F$, *then* $(F - \{m\})^{\perp} = M^*$. *By hypothesis,* M^* *is torsionless, which means that* $F - \{m\} = \{0\}$, *that is,* $F = \{m\}$.

Recall that a topological space is compact if and only if every family of closed subsets verifying the finite intersection property has nonempty intersection.

Theorem 149 (Krein-Milman Theorem) *Let* R *be an integral domain, Hausdorff practical topological ordered ring verifying the strong sup property and such that* $\mathcal{U}(R) \subseteq R^+ \cup R^-$. *Let* M *be a torsionless topological* R*-module. If* C *is a compact subset of* M, *then* $\mathrm{extl}(C) \neq \varnothing$.

Proof 453 *Let us begin by considering the inductive set* $\mathcal{F} := \{F \subseteq C : F$ *is nonempty, extremal, and closed in* $C\}$ *partially ordered by the inclusion. Notice that* $\mathcal{F} \neq \varnothing$ *since* $C \in \mathcal{F}$. *If* L *is a chain of* $\mathcal{F}$, *then* L *clearly verifies the finite intersection property, therefore the compactness of* C *allows that* $\bigcap L \neq \varnothing$. *Thus,* $\bigcap L \in \mathcal{F}$, *hence* $\bigcap L$ *is a lower bound for* L. *According to Zorn's Lemma,* $\mathcal{F}$ *possesses minimal elements. Let* F *be a minimal element of* $\mathcal{F}$. *We will prove next that every* $m^* \in M^*$ *is constant on* F. *Indeed, fix an arbitrary* $m^* \in M^*$. *Lemma 133 guarantees that* $m^*(F)$ *is bounded in the ring topology of* R *since it is compact in* R *due to the continuity of* m^*. *Next, Lemma 134 assures that* $m^*(F)$ *is ordered-bounded in* R. *Since* R *satisfies the sup property, there exists* $\sup m^*(F)$. *By hypothesis, the order topology of* R *is Hausdorff, therefore* $m^*(F)$ *is closed in* R *because it is compact in* R. *By assumption,* R *satisfies the strong sup property, so* $\sup m^*(F)$ *exists in* R *and* $\sup m^*(F) \in m^*(F)$, *meaning that* $\arg\max_F(m^*)$ *is not empty. On the other hand,* $\arg\max_F(m^*)$ *is extremal in* C *by Proposition 120. Finally, by assumption, the ring topology on* R *coincides with the order topology, which is Hausdorff, and* m^* *is continuous, meaning that* $\arg\max_F(m^*)$ *is closed in* F, *hence in* C. *As a consequence,* $\arg\max_F(m^*) \in \mathcal{L}$. *Since* $\arg\max_F(m^*) \subseteq F$, *the minimality of* F *allows us to conclude that* $\arg\max_F(m^*) = F$, *which means that* m^* *is constant on* F. *Then Lemma 155 assures that* F *is a singleton. Finally, since* F *is extremal in* C, *we have that* $m \in \mathrm{extl}(C)$.

Notice that the order topology is always Hausdorff on a totally ordered set, but not necessarily on a partial ordered set. Also, as pointed out in the Methodology Section, if a totally ordered set satisfies the sup property, then the order topology satisfies the strong sup property. On the other hand, every totally ordered ring R satisfies that $\mathcal{U}(R) \subseteq R^+ \cup R^-$. For all these reasons, the following corollary is immediately derived.

Corollary 55 *Let R be an integral domain, practical topological totally ordered ring verifying the sup property. Let M be a torsionless topological R-module. If C is a compact subset of M, then* $\text{extl}(C) \neq \varnothing$.

5.2.2.2 Bauer Minimum Principle

The Bauer Minimum Principle is the basis upon which the Simplex Method relies.

Theorem 150 (Bauer Minimum Principle) *Let R be an integral domain, Hausdorff practical topological ordered ring verifying the strong sup property and such that $\mathcal{U}(R) \subseteq R^+ \cup R^-$. Let M be a torsionless topological R-module. If C is a compact subset of M and $f : M \to R$ is continuous and convex, then f attains its maximum on C at an extremal point of C.*

Proof 454 *By following a similar argument as above, $F := \arg\max_C(f) \neq \varnothing$. Even more, F is closed is C, so F is compact as well. By definition, F is an exposed face of C. In view of Proposition 120, F verifies the extremal condition with respect to C. By the Krein-Milman Theorem, we deduce that* $\text{extl}(F) \neq \varnothing$. *Finally, since F is extremal in C, we conclude that* $\text{extl}(F) \subseteq \text{extl}(C)$.

As we did with the Krein-Milman Theorem, an immediate corollary is derived.

Corollary 56 *Let R be an integral domain, practical topological totally ordered ring verifying the sup property. Let M be a torsionless topological R-module. If C is a compact subset of M and $f : M \to R$ is continuous and convex, then f attains its maximum on C at an extremal point of C.*

5.3 Normed optimization

Normed optimization is a branch of Optimization Theory that focuses on the study of multiobjective optimization problems whose objective functions are norms of linear operators. It is a particular case of convex optimization since the norm of a linear operator is a convex function.

5.3.1 Operator Theory

Two seminormed modules induce naturally a seminorm on the additive group of linear operators between them. We refer the reader to [40] for the basis of the results of this subsection.

5.3.1.1 Normalizing rings

Even though an Operator Theory approach can be done on seminormed modules over seminormed rings, we have to restrict to a special class of seminormed rings if we want that

approach to be successful. Otherwise, many pathological phenomena would dramatically appear.

Definition 307 (Normalizing ring) *A R seminormed ring is called*

- *strongly practical if* $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \operatorname{cl}(\{\|r\| : r \in \mathcal{U}(R)\})$.
- *normalizing if for each seminormed R-module M and for each $m \in M$ with $\|m\| > 0$,* $1 \in \operatorname{cl}\left(\left\{\|rm\| : r \in \mathcal{U}(R), \|r\| \leq \|m\|^{-1}\right\}\right)$;
- *strongly normalizing if for each seminormed R-module M and for each $m \in M$ with* $\|m\| > 0$, $\|m\| \in \operatorname{cl}\left(\left\{\left\|r^{-1}\right\| : r \in \mathcal{U}(R), \|r\| \leq \|m\|^{-1}\right\}\right)$.

The following theorem establishes the implications between the above notions together with the ones of hyperpractical and practical ring.

Theorem 151 *Let R be a seminormed ring. Then:*

1. *If R is hyperpractical, then it is strongly normalizing.*
2. *If R is strongly normalizing, then it is normalizing.*
3. *If R is normalizing, then it is strongly practical.*
4. *If R is strongly practical, then it is practical.*

Proof 455 *Let us itemize the proof in accordance with the statement.*

1. *By hypothesis, $\{\|r\| : r \in \mathcal{U}_1(R)\}$ is dense in $(0, \infty)$. Let M be a seminormed R-module and consider $m \in M$ with $\|m\| > 0$. There exists a sequence $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R)$ such that $\|r_n\| \to \|m\|^{-1}$ as $n \to \infty$ and $\|r_n\| < \|m\|^{-1}$ for all $n \in \mathbb{N}$. Next, $\left\|r_n^{-1}\right\| = \|r_n\|^{-1}$ for all $n \in \mathbb{N}$, meaning that $\left\|r_n^{-1}\right\| \to \|m\|$ as $n \to \infty$. This shows that $\|m\| \in \operatorname{cl}\left(\left\{\left\|r^{-1}\right\| : r \in \mathcal{U}(R), \|r\| \leq \|m\|^{-1}\right\}\right)$.*
2. *Let M be a seminormed R-module and consider $m \in M$ with $\|m\| > 0$. There exists a sequence $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(R)$ such that $\left\|r_n^{-1}\right\| \to \|m\|$ as $n \to \infty$ and $\|r_n\| \leq \|m\|^{-1}$ for all $n \in \mathbb{N}$. The inequality*

$$\left\|r_n^{-1}\right\|^{-1} \|m\| \leq \|r_n m\| \leq \|r_n\| \|m\| \leq 1$$

   implies that $\|r_n m\| \to 1$ as $n \to \infty$.
3. *Take $M := \bigoplus_{n\in\mathbb{N}} R$ endowed with the seminorm $\|(r_n)_{n\in\mathbb{N}}\|_1 := \frac{1}{\|1\|} \sum_{n\in\mathbb{N}} \|r_n\|$. For every $p \in \mathbb{N}$, we will let $m_p := e_1 + \cdots + e_p \in M$, where the e_p's are the elements of the canonical basis of M. Note that $\|m_p\|_1 = p$ for each $p \in \mathbb{N}$. Since R is normalizing and M is a seminormed module over R, we conclude that, for every $p \in \mathbb{N}$, there exists a sequence $(r_n)_{n\in\mathbb{N}}$ of $\mathcal{U}(R)$ such that*

$$\|r_n\| \leq \frac{1}{\|m_p\|_1} = \frac{1}{p}$$

   for every $n \in \mathbb{N}$ and $\|r_n m_p\|_1 \to 1$, which also implies that $\|r_n\| \to \frac{1}{p}$.
4. *It is a direct consequence of the fact that $0 \in \operatorname{cl}(\{\frac{1}{n} : n \in \mathbb{N}\}) \subseteq \operatorname{cl}(\{\|r\| : r \in \mathcal{U}(R)\}$, meaning that $0 \in \operatorname{cl}(\mathcal{U}(R))$.*

The notions of hyperpractical, strongly normalizing, normalizing, and strongly practical coincide on absolutely semivalued rings.

Corollary 57 *Let R be a seminormed ring with $\mathcal{U}_1(R) = \mathcal{U}(R)$. Then R is strongly practical if and only if it is hyperpractical.*

Proof 456 *Assume that R is strongly practical. By hypothesis, $\{\|r\| : r \in \mathcal{U}(R)\}$ approaches the inverse of the naturals. Since $\left\|r^{-1}\right\| = \|r\|^{-1}$ for all $r \in \mathcal{U}(R)$, we have that $\{\|r\| : r \in \mathcal{U}(R)\}$ approaches the positive rationals. Therefore, $\{\|r\| : r \in \mathcal{U}(R)\}$ is dense in $(0, \infty)$.*

5.3.1.2 Operator norms

The classical Operator Theory between complex Banach spaces is transported here to the scope of seminormed modules over seminormed rings.

Definition 308 (Operator norm) *Let R be a seminormed ring. Let M, N be seminormed R-modules. Let $T : M \to N$ be a linear map. The norm of T is defined as $\|T\| := \sup\{\|T(m)\| : m \in \mathsf{B}_M\}$.*

We will denote $\mathcal{B}(M, N) := \{T \in \mathcal{L}(M, N) : \|T\| < \infty\}$. The elements of $\mathcal{B}(M, N)$ are usually called bounded operators (because they are bounded on B_M). If $M = N$, then we will simply denote $\mathcal{B}(M) := \mathcal{B}(M, M)$.

Remark 89 *Let R be a seminormed ring. Let M, N be seminormed R-modules. For all $T, S \in \mathcal{L}(M, N)$,*

- $\|0\| = 0$,
- $\| - T\| = \|T\|$,
- $\|T + S\| \leq \|T\| + \|S\|$.

These properties turn $\mathcal{B}(M, N)$ into a seminormed subgroup of $\mathcal{L}(M, N)$. If, in addition, N is a seminormed (R, S)-bimodule for S another seminormed ring, then it is further verified that $\|Ts\| \leq \|T\|\|s\|$ for all $T \in \mathcal{L}(M, N)$ and all $s \in S$, and thus $\mathcal{B}(M, N)$ becomes a seminormed right S-module (in case R is absolutely semivalued, then it holds that $\|Ts\| = \|T\||s|$ for all $T \in \mathcal{L}(M, N)$ and all $s \in S$).

Recall that, in a seminormed module M over a seminormed ring R, the collection of all seminorm-bounded subsets of M, $\mathcal{B}_{\mathsf{d}}(M)$, is upward directed for the inclusion, eve more, it is a bornology of $\mathcal{P}(M)$ if M is unbounded. This implies that $\mathcal{B}_{\mathsf{d}}(M)$ generates a convergence additive topology on the subgroup $\mathcal{B}(M, N)$.

Theorem 152 *Let R be a seminormed ring. Let M, N be seminormed R-modules. Then:*

1. *The seminormed topology of $\mathcal{B}(M, N)$ is coarser than the $\mathcal{B}_{\mathsf{d}}(M)$-convergence additive topology.*
2. *If R is practical, then the seminormed topology of $\mathcal{B}(M, N)$ is an operator topology.*
3. *If R is absolutely semivalued and practical, then the seminormed topology of $\mathcal{B}(M, N)$ coincides with the $\mathcal{B}_0(M)$-convergence linear topology.*

Proof 457 *The proof is itemized via the itemization of the statement.*

1. *Simply observe that, for every $\delta > 0$, $\mathsf{B}_{\mathcal{B}(M,N)}(0, \delta) = \mathcal{U}\left(\mathsf{B}_M, \mathsf{B}_N(0, \delta)\right)$.*

2. *We know that the seminormed topology of $\mathcal{B}(M,N)$ induces a group topology on $\mathcal{B}(M,N)$. It only suffices to show that it is finer than the pointwise convergence linear topology. Indeed, let $(T_i)_{i\in I}\subseteq\mathcal{B}(M,N)$ be a net convergent to T in the seminormed topology. We will prove that $(T_i)_{i\in I}\subseteq\mathcal{B}(M,N)$ is pointwise convergent to T. Indeed, fix an arbitrary $m\in M$. There exists $W\in\mathcal{N}_0(R)$ such that $Wm\subseteq\mathsf{B}_M$. Since R is practical, we can find $u\in\mathcal{U}(R)\cap W$. Then $um\in Wm\subseteq\mathsf{B}_M$. Next, fix an arbitrary $\varepsilon>0$. There exists $i_0\in I$ such that, if $i\in I$ and $i\geq i_0$, then $\|T_i-T\|<\frac{\varepsilon}{\|u^{-1}\|}$. Thus, $\|T_i(m)-T(m)\|=\left\|u^{-1}(T_i(um)-T(m))\right\|\leq\left\|u^{-1}\right\|\|T_i(um)-T(um)\|<\left\|u^{-1}\right\|\|T_i-T\|<\varepsilon$.*

3. *First of all, $\mathcal{B}_0(M)=\mathcal{B}_{\mathfrak{d}}(M)$ in view of Proposition 92. Fix arbitrary $A\in\mathcal{B}_0(M),U\in\mathcal{N}_0(N)$. There exists $u\in\mathcal{U}(R)$ such that $A\subseteq u\mathsf{B}_M$. Then*

$$\mathsf{B}_{\mathcal{B}(M,N)}\left(0,\frac{\rho}{|u|}\right)\subseteq\mathcal{U}\left(A,\mathsf{B}_N(0,\rho)\right)\subseteq\mathcal{U}(A,U),$$

where $\rho>0$ is so that $\mathsf{B}_N(0,\rho)\subseteq U$. This shows, together with the first item, that the two topologies coincide.

Next proposition shows that if the underlying ring is practical, then the seminorm of a bounded linear operator between normed modules is in fact a norm.

Proposition 121 *Let M,N be normed modules over a normed ring R. If R is practical, then $\mathcal{B}(M,N)$ is a normed group. If, in addition, N is a normed (R,S)-bimodule for S another normed ring, then $\mathcal{B}(M,N)$ is a normed right S-module.*

Proof 458 *All we need to show is that $T=0$ if $T\in\mathcal{B}(M,N)$ and $\|T\|=0$. Indeed, fix an arbitrary $m\in M$ and consider a sequence $(r_n)_{n\in\mathbb{N}}\subseteq\mathcal{U}(R)$ converging to 0. Since $(r_nm)_{n\in\mathbb{N}}$ converges to 0, there exists $k\in\mathbb{N}$ such that $\|r_km\|\leq 1$. Therefore $\|T(r_km)\|\leq\|T\|=0$, so $T(r_km)=0$ and*

$$T(m)=r_k^{-1}r_kT(m)=r_k^{-1}T(r_km)=0.$$

Practical rings make linear operators be bounded if and only if they are continuous.

Theorem 153 *Let M,N be seminormed modules over a practical seminormed ring R. The following conditions are equivalent for a linear map $T:M\to N$:*

1. *$\|T\|=\infty$.*
2. *There exists a sequence $(m_n)_{n\in\mathbb{N}}\subseteq M$ converging to 0 such that $\|T(m_n)\|\to\infty$ as $n\to\infty$.*
3. *T is not continuous.*

In particular, $\mathcal{B}(M,N)=\mathcal{CL}(M,N)$.

Proof 459 *Assume first that $\|T\|=\infty$. Then there exists a sequence $(l_n)_{n\in\mathbb{N}}\subseteq\mathsf{B}_M$ such that $\|T(l_n)\|\to\infty$ as $n\to\infty$. Consider a sequence $(r_n)_{n\in\mathbb{N}}\subseteq\mathcal{U}(R)$ converging to 0. By passing again to subsequences if necessary, we may assume without any loss that $\|T(l_n)\|\geq\left\|r_n^{-1}\right\|^2$ for all $n\in\mathbb{N}$. Notice that $\|r_nl_n\|\leq\|r_n\|\|l_n\|\leq\|r_n\|$ for all $n\in\mathbb{N}$, thus $(r_nl_n)_{n\in\mathbb{N}}$ converges to 0. Also, observe that*

$$\|T(r_nl_n)\|=\|r_nT(l_n)\|\geq\left\|r_n^{-1}\right\|^{-1}\|T(l_n)\|\geq\left\|r_n^{-1}\right\|\geq\|r_n\|^{-1}.$$

Next, suppose T is not continuous. Then, by translations, T is not continuous at 0; therefore,

there exist a sequence $(m_n)_{n\in\mathbb{N}}$ *converging to* 0 *in* M *and* $t>0$ *such that* $\|T(m_n)\|\geq t$ *for all* $n\in\mathbb{N}$. *Consider a sequence* $(r_n)_{n\in\mathbb{N}}\subseteq\mathcal{U}(R)$ *converging to* 0. *By passing to subsequences if necessary, we may assume without any loss that* $\|m_n\|\leq\left\|r_n^{-1}\right\|^{-1}$ *for all* $n\in\mathbb{N}$. *Note that*

$$\left\|r_n^{-1}m_n\right\|\leq\left\|r_n^{-1}\right\|\|m_n\|\leq 1$$

for all $n\in\mathbb{N}$, *thus* $\left(r_n^{-1}m_n\right)_{n\in\mathbb{N}}\subseteq\mathsf{B}_M$. *Also,*

$$\left\|T\left(r_n^{-1}m_n\right)\right\|=\left\|r_n^{-1}T(m_n)\right\|\geq\|r_n\|^{-1}\|T(m_n)\|\geq\|r_n\|^{-1}t$$

for all $n\in\mathbb{N}$, *therefore* $\left\|T\left(r_n^{-1}m_n\right)\right\|\to\infty$ *as* $n\to\infty$. *As a consequence,* $\|T\|=\infty$.

The above theorem directly applies to dual seminorms in the dual of a seminormed module.

Corollary 58 *Let* M *be a seminormed module over a practical seminormed ring* R. *Then:*

1. M^* *is a seminormed right* R*-module. If, in addition,* M *is normed and* R *is normed as well, then* M^* *is also normed.*

2. *The seminorm topology on* M^* *is a dual topology, that is, finer than the* w^**-topology.*

3. *The dual seminorm on* M^* *is* w^**-lower semicontinuous.*

Proof 460 *The proof will follow the same itemization as the statement.*

1. *Since* R *is a seminormed* R*-bimodule, then, by combining together Remark 89 and Theorem 153, we conclude that* M^* *is a seminormed right* R*-module. If, in addition,* M *is normed and* R *is normed as well, then* M^* *is also normed in view of Proposition 121.*

2. *Let* $(m_i^*)_{i\in I}$ *be a net in* M^* *converging to an element* $m^*\in M^*$. *We have to prove that* $(m_i^*)_{i\in I}$ *is* w^**-convergent to* $m^*\in M^*$. *Fix an arbitrary* $m\in M$. *If* $\|m\|\leq 1$, *then* $\|m_i^*(m)-m^*(m)\|\leq\|m_i^*-m^*\|$ *for all* $i\in I$, *hence* $(m_i^*(m))_{i\in I}$ *converges to* $m^*(m)$. *Suppose now that* $\|m\|>1$. *By hypothesis, there exists* $r\in\mathcal{U}(R)$ *such that* $\|r\|\leq\|m\|^{-1}$. *Now, for every* $i\in I$, *since* $\|rm\|\leq 1$, *we have that* $\|m_i^*(m)-m^*(m)\|=\left\|(m_i^*-m^*)\left(r^{-1}rm\right)\right\|\leq\left\|r^{-1}\right\|\|(m_i^*-m^*)(rm)\|\leq\left\|r^{-1}\right\|\|m_i^*-m^*\|$. *Therefore,* $(m_i^*(m))_{i\in I}$ *converges to* $m^*(m)$. *The arbitrariness of* $m\in M$ *proves that* $(m_i^*)_{i\in I}$ *is* w^**-convergent to* m^*.

3. *Fix an arbitrary* $m^*\in M^*$. *Let* $(m_i^*)_{i\in I}\subseteq M^*$ *be a net* w^**-converging to* m^*. *For every* $\varepsilon>0$, *we can find* $m_\varepsilon\in\mathsf{B}_M$ *such that* $\|m^*\|\leq\|m^*(m_\varepsilon)\|+\varepsilon$. *Since* $(m_i^*(m_\varepsilon))_{i\in I}$ *converges to* $m^*(m_\varepsilon)$, *we have that the net* $(\|m_i^*(m_\varepsilon)\|)_{i\in I}$ *converges to* $\|m^*(m_\varepsilon)\|$. *Thus,*

$$\|m^*\|\leq\|m^*(m_\varepsilon)\|+\varepsilon=\lim_{i\in I}\|m_i^*(m_\varepsilon)\|+\varepsilon\leq\liminf_{i\in I}\|m_i^*\|+\varepsilon.$$

By the arbitrariness of ε, *we conclude that* $\|m^*\|\leq\liminf_{i\in I}\|m_i^*\|$.

Theorem 153 does not assure that the continuity of T implies that T be a Lipschitz map. In other words, from Theorem 153, we cannot conclude the existence of $K>0$ verifying that $\|T(m)\|\leq K\|m\|$ for all $m\in M$.

Definition 309 (Supporting sequence) *Let* M,N *be seminormed modules over a practical seminormed ring* R. *Let* $T:M\to N$ *be a continuous linear operator. A* supporting sequence *for* T *is a sequence* $(m_n)_{n\in\mathbb{N}}\subseteq\mathsf{B}_M$ *such that* $(\|T(m_n)\|)_{n\in\mathbb{N}}$ *converges to* $\|T\|$.

For the upcoming results, recall that normalizing seminormed rings are practical in view of Theorem 151.

Proposition 122 *Let M, N be seminormed modules over a normalizing seminormed ring R. Let $T : M \to N$ be a continuous linear operator. Then:*

1. *$\|T(m)\| \leq \|T\|\|m\|$ for all $m \in M$.*
2. *If $\|T\| > 0$, then every supporting sequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathsf{B}_M$ satisfies that $(\|m_n\|)_{n\in\mathbb{N}}$ converges to 1.*
3. *If R is absolutely semivalued, then $\|T\| = \sup\{\|T(m)\| : m \in \mathsf{U}_M\}$.*
4. *If $\mathrm{cl}(T(\mathsf{B}_M)) = \mathsf{B}_N$, then $\mathrm{cl}(T(M)) = N$.*

Proof 461 *The proof is itemized via the statement.*

1. *Fix an arbitrary $m \in M$. If $\|m\| = 0$, then the constant sequence $(m)_{n\in\mathbb{N}}$ converges to 0, therefore, by the continuity of T, the constant sequence $(T(m))_{n\in\mathbb{N}}$ converges to $T(0) = 0$, which implies that $\|T(m)\| = 0 = \|T\|\|m\|$. Suppose now that $\|m\| > 0$. Let $(r_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(R) \cap \mathsf{B}_R\left(0, \|m\|^{-1}\right)$ such that $\left\|r_n^{-1}\right\| \to \|m\|$ as $n \to \infty$. For every $n \in \mathbb{N}$, we have that $\|T(m)\| = \left\|r_n^{-1}T(r_n m)\right\| \leq \left\|r_n^{-1}\right\| \|T(r_n m)\| \leq \left\|r_n^{-1}\right\| \|T\|$. Since $\left(\left\|r_n^{-1}\right\|\right)_{n\in\mathbb{N}}$ converges to $\|m\|$, we conclude that $\|T(m)\| \leq \|T\|\|m\|$.*
2. *Simply observe that, for all $n \in \mathbb{N}$,*

$$1 \geq \|m_n\| \geq \frac{\|T(m_n)\|}{\|T\|} \to 1,$$

   which means that $(\|m_n\|)_{n\in\mathbb{N}}$ converges to 1.
3. *Let $(m_n)_{n\in\mathbb{N}}$ be a supporting sequence. Since R is normalizing, by Corollary 57, we can find $r_n \in \mathcal{U}(R)$ such that $|r_n| \in \left(\frac{1}{\|m_n\|} - \frac{1}{n}, \frac{1}{\|m_n\|}\right)$ for every $n \in \mathbb{N}$. In accordance with Proposition 122(2), we have that $(\|m_n\|)_{n\in\mathbb{N}}$ converges to 1. Thus, so does $(|r_n|)_{n\in\mathbb{N}}$. Therefore, $(r_n m_n)_{n\in\mathbb{N}}$ is a supporting sequence for T entirely contained in U_M.*
4. *Let $n \in N$ with $\|n\| \geq 1$. Since R is normalizing, we can fix a sequence $(r_k)_{k\in\mathbb{N}} \subseteq \mathcal{U}(R)$ such that $\|r_k\| \leq \|n\|^{-1}$ for all $k \in \mathbb{N}$ and $\left\|r_k^{-1}\right\| \to \|n\|$ as $k \to \infty$. Note that, for every $k \in \mathbb{N}$, $\|r_k n\| \leq \|r_n\|\|n\| \leq 1$, thus we can find $m_k \in \mathsf{B}_M$ such that $\|T(m_k) - r_k n\| \leq \frac{1}{k}$. Finally, for every $k \in \mathbb{N}$, $\left\|T\left(r_k^{-1}m_k\right) - n\right\| \leq \left\|r_k^{-1}\right\| \|T(m_k) - r_k n\| \to \|n\| \cdot 0 = 0$.*

If M, N, P are seminormed modules over a seminormed ring R and $T : M \to N, S : N \to P$ are bounded linear operators such that $\|T\| \leq 1$, then $\|S \circ T\| \leq \|S\|$. In particular, if $T \in \mathcal{B}(M)$, then $\left\|T^{n+1}\right\| \leq \|T^n\|$ for each $n \in \mathbb{N}$ if $\|T\| \leq 1$. Bear in mind that $\|I_M\| \leq 1$. If R is normalizing and M is not trivial (there exists $m \in M$ with $\|m\| > 0$), then $\|I_M\| = 1$

Corollary 59 *Let M, N, P be seminormed modules over a normalizing seminormed ring R. If $T : M \to N$ and $S : N \to P$ are continuous linear operators, then $\|S \circ T\| \leq \|S\|\|T\|$. In particular, $\mathcal{B}(M)$ is a (unital) seminormed ring (if M is not trivial).*

Proof 462 *It suffices to apply Proposition 122 to conclude that*

$$\|S(T(m))\| \leq \|S\|\|T(m)\| \leq \|S\|\|T\|\|m\| \leq \|S\|\|T\|$$

for all $m \in \mathsf{B}_M$.

Keeping in mind Proposition 121, if M is a normed module over a normalizing normed ring R, then $\mathcal{B}(M)$ is a (unital) normed ring (if M is not trivial).

Lemma 156 *If M is a seminormed module over a normalizing seminormed ring R, then $\|\widehat{m}\| \leq \|m\|$ for all $m \in M$.*

Proof 463 *Fix an arbitrary $m^* \in \mathsf{B}_{M^*}$. By Proposition 122(1), we have that*

$$|\widehat{m}(m^*)| = |m^*(m)| \leq \|m^*\|\|m\| \leq \|m\|.$$

The following notion is a transportation of topological torsion to the context of seminorm modules.

Definition 310 (Isometric torsion) *A seminormed module over a normalizing seminormed ring is said to be isometrically torsionless if $\widehat{\ }$ is an isometry.*

The following technical proposition highlights the behaviour of dual operators with respect to seminorms.

Proposition 123 *Let $T : M \to N$ be a continuous linear operator between seminormed modules M, N over a normalizing seminormed ring R. Then:*

1. *$\|T^*\| \leq \|T\|$.*
2. *If both M, N are isometrically torsionless, then $\|T^*\| = \|T\|$.*
3. *If $T(\mathsf{B}_M)$ is dense in B_N, then T^* is an isometry. In particular, if T is a surjective isometry, then T^* is an isometry.*
4. *If R, M, N are normed and T is a surjective isometry, then so is T^*.*

Proof 464 *The proof is itemized via the statement of the proposition.*

1. *Let $n^* \in \mathsf{B}_{N^*}$. In accordance with Corollary 59, we have that*

$$\|T^*(n^*)\| = \|n^* \circ T\| \leq \|n^*\|\|T\| \leq \|T\|,$$

   therefore $\|T^*\| \leq \|T\|$.

2. *It suffices to apply Proposition 97 and to note that*

$$\|T\| = \left\|\widehat{\ } \circ T \circ \widehat{\ }^{-1}\right\| = \left\|\widehat{T}\right\| = \left\|T^{**}|_{\widehat{M}}\right\| \leq \|T^{**}\| \leq \|T^*\|.$$

3. *If $n^* \in N^*$, then $\|n^*\| = \|n^* \circ T\| = \|T^*(n^*)\|$ because $T(\mathsf{B}_M)$ is dense in B_N. If, in particular, T is a surjective isometry, then $T(\mathsf{B}_M) = \mathsf{B}_N$.*
4. *It only suffices to show that T^* is surjective, which is a consequence of Proposition 97 because T is an isomorphism.*

At this stage, we refer the reader to Definition 260 for the concept of feasible dual topology.

Corollary 60 *Let M be a seminormed module over a normalizing seminormed ring R. The norm topology on M^* is a feasible dual topology on M^*.*

Proof 465 *By applying Theorem 151, we have that R is practical. In view of Corollary 58, we have that the dual norm on M^* generates a dual topology. Now, if $T \in \mathcal{B}(M)$, by Proposition 123, we conclude that $T^* \in \mathcal{B}(M^*) = \mathcal{CL}(M^*)$ (see Proposition 121 for this last equality).*

5.3.1.3 Banach-Alaoglu Theorem

This subsubsection is devoted to obtain a version of the Banach-Alaoglu Theorem for seminormed modules.

Lemma 157 *Let M be a seminormed module over a practical seminormed ring R. Then $\mathsf{B}_{M^*}(m^*,t)$ is w^*-closed for every $m^* \in M^*$ and all $t \in [0,\infty)$.*

Proof 466 *Let $(m_i^*)_{i\in I}$ be a w^*-convergent net of $\mathsf{B}_{M^*}(m^*,t)$ to some $n^* \in M^*$. Fix an arbitrary $m \in \mathsf{B}_M$. For every $i \in I$, we have that*

$$\begin{aligned}\|n^*(m)-m^*(m)\| &\leq \|n^*(m)-m_i^*(m)\|+\|m_i^*(m)-m^*(m)\|\\ &\leq \|n^*(m)-m_i^*(m)\|+\|m_i^*-m^*\|\\ &\leq \|n^*(m)-m_i^*(m)\|+t.\end{aligned}$$

Since $(m_i^*(m))_{i\in I}$ converges to $n^*(m)$, we conclude that $\|n^*(m)-m^*(m)\| \leq t$. Finally, the arbitrariness of $m \in \mathsf{B}_{M^*}$ allows $\|n^*-m^*\| \leq t$.

Notice that the previous lemma still holds if we do not require the underlying ring to be practical and we restrict to $\mathcal{B}(M,R)$.

Theorem 154 (Banach-Aloaglu Theorem) *Let M be a seminormed mo-dule over a practical seminormed ring R. If B_R is compact, then B_{M^*} is w^*-compact.*

Proof 467 *Observe that*

$$\mathsf{B}_{M^*} \subseteq \prod_{m\in M} \mathsf{B}_R(0,\|m\|) \subseteq R^M.$$

The inherited topology of $\prod_{m\in M}\mathsf{B}_R(0,\|m\|)$ coincides with its product topology. Observe that if $\|m\|=0$, then $\mathsf{B}_R(0,\|m\|) \subseteq \mathsf{B}_R$, which implies that $\mathsf{B}_R(0,\|m\|)$ is clearly compact because it is closed. If $\|m\|>0$, then, since R is practical, we can find $r_m \in \mathcal{U}(R)$ with $\|r_m\| \leq \|m\|^{-1}$, which implies that

$$\mathsf{B}_R(0,\|m\|) \subseteq r_m^{-1}\mathsf{B}_R.$$

Since B_R is compact, then so are $r_m^{-1}\mathsf{B}_R$ and $\mathsf{B}_R(0,\|m\|)$. Therefore, $\prod_{m\in M}\mathsf{B}_R(0,\|m\|)$ is compact in virtue of the Tychonov Theorem. Finally, by Lemma 157, we have that B_{M^*} is w^*-closed and thus w^*-compact.

By plugging together the Banach-Aloaglu Theorem with Lemma 157, the following corollary is obtained.

Corollary 61 *Let M be a topological module over a practical seminormed ring R. If B_R is compact, then $\mathsf{B}_{M^*}(m^*,t)$ is w^*-compact for every $m^* \in M^*$ and all $t \in [0,\infty)$.*

Proof 468 *By applying Theorem 154, we have that B_{M^*} is w^*-compact. If $\|m^*\|=0$ and $t=0$, then $\mathsf{B}_{M^*}(m^*,t) \subseteq \mathsf{B}_{M^*}$, which implies that $\mathsf{B}_{M^*}(m^*,t)$ is clearly w^*-compact since it is w^*-closed in view of Lemma 157. So, we may assume that $\|m^*\|+t>0$. Since R is practical, we can find $r \in \mathcal{U}(R)$ such that $\|r\| \leq (\|m^*\|+t)^{-1}$. Then*

$$\mathsf{B}_{M^*}(m^*,t) \subseteq \mathsf{B}_{M^*}(0,\|m^*\|+t) \subseteq \mathsf{B}_{M^*}r^{-1}.$$

Since $\mathsf{B}_{M^*}r^{-1}$ is a w^*-continuous image of the w^*-compact set B_{M^*}, we deduce that $\mathsf{B}_{M^*}r^{-1}$ is also w^*-compact. Finally, Lemma 157 assures that $\mathsf{B}_{M^*}(m^*,t)$ is w^*-compact.

The final theorem in this subsubsection deals with uniform differentiability of differentiable Lipschitz functions provided that the unit ball of the module is compact.

Theorem 155 *Let R be a normalizing absolutely semivalued ring R. Let M, N be seminormed R-modules such that B_M is compact. Let $A \subseteq M$ be open. If $T : A \to N$ is Lipschitz and differentiable at a certain $a \in A$, then T is uniformly differentiable at a.*

Proof 469 *First off, we will be assuming that $\|\operatorname{D} T(a)\| > 0$ and $\mathsf{Lip}(T) > 0$, since otherwise the proof is simpler. There exists $0 < t < 1$ so that $a + \mathsf{B}_R(0,t)\mathsf{B}_M(0,t) \subseteq A$. We will rely on Remark 84 for $W := \mathsf{B}_R(0,t)$ and $V := \mathsf{B}_M(0,t)$. Fix an arbitrary $\varepsilon > 0$. Since B_M is compact and $\mathsf{B}_M(0,t)$ is closed and contained in B_M, we conclude that $\mathsf{B}_M(0,t)$ is compact as well. Therefore, there can be found $l_1, \ldots, l_k \in \mathsf{B}_M(0,t)$ and such that $\{\mathsf{U}_M(l_i, \tau) : 1 \leq i \leq k\}$ is a finite open covering of $\mathsf{B}_M(0,t)$, where*

$$\tau := \min\left\{\frac{\varepsilon}{3\mathsf{Lip}(T)}, \frac{\varepsilon}{3\|\operatorname{D} T(a)\|}\right\}.$$

For each $i = 1, \ldots, k$, there exists $0 < \delta(\varepsilon, l_i) < t$ in such a way that

$$\left\|r^{-1}\left(T(a + rl_i) - T(a)\right) - \operatorname{D}_{l_i} T(a)\right\| < \frac{\varepsilon}{3}$$

whenever $r \in \mathcal{U}(R)$ and $|r| < \delta(\varepsilon, l_i)$. Choose $\delta(\varepsilon) := \min\{\delta(\varepsilon, l_i) : i = 1, \ldots, k\}$. Fix arbitrary elements $r \in \mathcal{U}(R)$ with $|r| < \delta(\varepsilon)$ and $l \in \mathsf{B}_M(0,t)$. There exists $i \in \{1, \ldots, k\}$ in such a way that $l \in \mathsf{U}_M(l_i, \tau)$. Thus,

$$\begin{aligned}
& \left\|r^{-1}\left(T(a + rl) - T(a)\right) - \operatorname{D}_l T(a)\right\| \leq \left\|r^{-1}\left(T(a + rl) - T(a)\right)\right. \\
- \quad & \left. r^{-1}\left(T(a + rl_i) - T(a)\right)\right\| + \left\|r^{-1}\left(T(a + rl_i) - T(a)\right) - \operatorname{D}_{l_i} T(a)\right\| \\
+ \quad & \|\operatorname{D}_{l_i} T(a) - \operatorname{D}_l T(a)\| \\
\leq \quad & |r|^{-1}\|T(a + rl) - T(a + rl_i)\| + \frac{\varepsilon}{3} + \|\operatorname{D} T(a)\|\|l_i - l\| \\
\leq \quad & |r|^{-1}\mathsf{Lip}(T)|r|\|l - l_i\| + \frac{\varepsilon}{3} + \|\operatorname{D} T(a)\|\|l_i - l\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

5.3.1.4 Complementation

Recall that a submodule of a (topological) module is said to be (topologically) complemented provided that it is the range of a (continuous) linear projection.

Remark 90 *If $P : M \to M$ is a continuous linear projection on a seminormed module M over a normalizing seminormed ring R, then either $\|P\| = 0$ or $\|P\| \geq 1$. Indeed, according to Corollary 59, we have that*

$$\|P\| = \|P \circ P\| \leq \|P\|^2.$$

Another trivial property is that

$$\max\left\{\frac{\|P(m)\|}{\|P\|}, \frac{\|(I - P)(m)\|}{\|I - P\|}\right\} \leq \|m\| \leq \|P(m)\| + \|(I - P)(m)\|$$

whenever P is a nontrivial projection, that is, $\|P\| > 0$ and $\|I - P\| > 0$.

A 1-projection is a projection P of norm 1. A $(1,1)$-projection is a 1-projection P whose complementary projection, $I - P$, also has norm 1.

Lemma 158 *Let M be a seminormed module over a practical seminormed ring R. Let $P : M \to M$ be a continuous linear projection. The following conditions are equivalent:*

1. $\|P(m)\| = 0$ *for all* $m \in M$.
2. $\|m - P(m)\| = \|m\|$ *for all* $m \in M$.

Proof 470 *Assume first that $\|P(m)\| = 0$ for all $m \in M$. Fix an arbitrary $m \in M$. Then $\|m\| = |\|m\| - \|P(m)\|| \leq \|m - P(m)\| \leq \|m\| + \|P(m)\| = \|m\|$ for all $m \in M$. Conversely, suppose that $\|m - P(m)\| = \|m\|$ for all $m \in M$. Then $\|P(m)\| = \|P(m) - P(P(m)\| = \|P(m) - P(m)\| = 0$ for all $m \in M$.*

Next technical lemma allows a sufficient condition for a submodule to be topologically complemented.

Lemma 159 *Let M be a normed module over a normed ring R. Let V, W be submodules of M satisfying that there exists $K > 0$ with $\|v\| \leq K \|v + w\|$ for all $v \in V$ and $w \in W$. Then $V \cap W = \{0\}$ and thus V is complemented in $V + W$.*

Proof 471 *If $m \in V \cap W$, then, by hypothesis, $0 \leq \|m\| \leq K\|m - m\| = 0$, so $m = 0$. Since $V \cap W = \{0\}$, the map*

$$\begin{array}{ccc} V + W & \to & V \\ v + w & \mapsto & v \end{array}$$

is well-defined and thus it is a continuous linear projection on $V + W$.

Finally, to conclude this section, Remark 90 motivates the following two concepts.

Definition 311 *Let M be a seminormed module over a seminormed ring R. A linear projection $P : M \to M$ is said to be a:*

- *M-projection whenever $\forall m \in M \ \|m\| = \max\{\|P(m)\|, \|(I - P)(m)\|\}$.*
- *L-projection whenever $\forall m \in M \ \|m\| = \|P(m)\| + \|(I - P)(m)\|$.*

Under the settings of the previous definition, if P is an M- or an L-projection, then so is $I - P$ and $\|P\| \leq 1$ and $\|I - P\| \leq 1$. If, in addition, R is normalizing and $\|P\| > 0$, then $\|P\| = 1$ by Remark 90. As a consequence, if R is normalizing and P is a nontrivial M- or L-projection, then P is a $(1, 1)$-projection.

5.3.2 Representations

We will begin by exploring the notions of supporting vectors and smoothness before applying them to linear and isometric representations.

5.3.2.1 Supporting vectors

The concept of supporting vector was formerly introduced for the first time in [69], although it appeared implicitly and scattered throughout the literature of Banach Space Theory [70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80]. In [81, 82, 83, 84], the algebraic and topological nature of supporting vectors was studied. Supporting vectors were successfully applied to solve certain multiobjective optimization problems that typically arise in Bioengineering, Physics, and Statistics [58, 59, 60, 61, 62].

Definition 312 (Supporting vector) *Let M, N be seminormed modules over a seminormed ring R. The set of supporting vectors of a bounded linear operator $T : M \to N$ is defined as*

$$\operatorname{suppv}(T) := \arg\max_{\|m\|\leq 1} \|T(m)\| = \{m \in \mathsf{B}_M : \|T(m)\| = \|T\|\}.$$

Under the settings of the previous definition, if R is normalizing, T is continuous, and $\|T\| > 0$, then $\operatorname{suppv}(T) \subseteq \mathsf{S}_M$. If, in addition, R is absolutely semivalued, then $\mathsf{S}_R\operatorname{suppv}(T) = \operatorname{suppv}(T)$. If, in addition again, N is a seminormed (R, S)-bimodule for S another absolutely semivalued ring, then $\operatorname{suppv}(T) = \operatorname{suppv}(Ts)$ for all $s \in \mathsf{S}_S$. Notice that $\operatorname{suppv}(T)$ might be empty.

Theorem 156 *Let M, N be seminormed modules over a seminormed ring R. Let $T : M \to N$ be a linear operator. Consider*

$$\begin{cases} \max \|T(m)\|, \\ \min \|m\|, \\ m \in M. \end{cases} \tag{5.15}$$

Then:

1. $\operatorname{suppv}(T) \supseteq \operatorname{Par}(5.15) \cap \mathsf{S}_M$.
2. *If R is hyperpractical and $\|T\| > 0$, then* $\operatorname{suppv}(T) = \operatorname{Par}(5.15) \cap \mathsf{S}_M$.
3. *If M is normed, then* $0 \in \operatorname{Par}(5.15)$.

Proof 472 *According to Theorem 140,* $\operatorname{suppv}(T) \supseteq \operatorname{Par}(5.15) \cap \mathsf{S}_M$. *If R is hyperpractical and $\|T\| > 0$, then* $\operatorname{suppv}(T) = \operatorname{Par}(5.15) \cap \mathsf{S}_M$ *by relying on Theorem 141. Finally, if M is normed, then* $\arg\min_{m\in M} \|m\| = \{0\}$, *so, by taking into account Theorem 136, we conclude that* $0 \in \operatorname{Par}(5.15)$.

There is also the notion of w^*-supporting functionals, which will be related later with the previous one of supporting vectors.

Definition 313 (w^*-Supporting functional) *Let M be a seminormed module over a seminormed ring R. The set of w^*-supporting functionals of an element $m \in M$ is defined as*

$$\operatorname{suppv}^{w^*}(m) := \{m^* \in \mathsf{S}_{M^*} : \|m^*(m)\| = \|m\|\}.$$

Under the settings of the previous definition, $\operatorname{suppv}^{w^*}(m)$ might be empty. Next lemma relates the previous two definitions.

Lemma 160 *Let M be a normed module over a normalizing seminormed ring R. Consider $m \in M$. Then:*

1. *If* $\|m\| = \|\widehat{m}\|$, *then* $\operatorname{suppv}^{w^*}(m) = \operatorname{suppv}(\widehat{m})$.
2. *If* $\operatorname{suppv}^{w^*}(m) \neq \varnothing$, *then* $\|m\| = \|\widehat{m}\|$.
3. *If M is isometrically torsionless, then, for all $m \in M$,* $\operatorname{suppv}^{w^*}(m) = \operatorname{suppv}(\widehat{m})$.

Proof 473 *The proof is itemized via the statement of the lemma.*

1. *Take* $m^* \in \operatorname{suppv}^{w^*}(m)$. *Then* $\|\widehat{m}(m^*)\| = \|m^*(m)\| = \|m\| = \|\widehat{m}\|$. *As a consequence,* $m^* \in \operatorname{suppv}(\widehat{m})$. *Hence,* $\operatorname{suppv}^{w^*}(m) \subseteq \operatorname{suppv}(\widehat{m})$. *Conversely, take* $m^* \in \operatorname{suppv}(\widehat{m})$. *Then* $\|m^*(m)\| = \|\widehat{m}(m^*)\| = \|\widehat{m}\| = \|m\|$. *As a consequence,* $m^* \in \operatorname{suppv}^{w^*}(m)$. *Thus,* $\operatorname{suppv}^{w^*}(m) \supseteq \operatorname{suppv}(\widehat{m})$.

2. *Let* $m^* \in \operatorname{suppv}^{w^*}(m)$. *Then* $\|\widehat{m}(m^*)\| = \|m^*(m)\| = \|m\|$, *which means that* $\|m\| \leq \|\widehat{m}\|$. *In virtue of Lemma 156, we have* $\|m\| \geq \|\widehat{m}\|$ *and thus* $\|m\| = \|\widehat{m}\|$.

3. *It is a direct consequence of the previous two items.*

Previous lemma motivates the upcoming definition, which is an extension of a classical notion of Banach Space Theory.

Definition 314 (Hahn-Banach module) *Let* M *be a seminormed module over a normalizing seminormed ring* R. *We say that* M *is Hahn-Banach provided that* $\operatorname{suppv}^{w^*}(m) \neq \varnothing$ *for all* $m \in M$.

As a direct consequence of Lemma 160, every Hahn-Banach seminormed module is isometrically torsionless. On the other hand, if M is a seminormed module over a normalizing seminormed ring R, then

$$m \in \operatorname{suppv}(m^*) \Leftrightarrow m^* \in \operatorname{suppv}^{w^*}(m)$$

for all $m \in \mathsf{S}_M$ and $m^* \in \mathsf{S}_{M^*}$. This motivates the definition of the following set, often called set of supporting pairs of M:

$$\Pi_M := \{(m, m^*) \in \mathsf{S}_M \times \mathsf{S}_{M^*} : m \in \operatorname{suppv}(m^*)\}.$$

More generally, we have the following definition.

Definition 315 (Supporting pairs) *Let* M, N *be seminormed modules over a normalizing seminormed ring* R. *We define the set of supporting pairs of* M, N *as* $\Pi_{M,N} := \left\{(m, T) \in \mathsf{S}_M \times \mathsf{S}_{\mathcal{B}(M,N)} : m \in \operatorname{suppv}(T)\right\}$.

In virtue of Proposition 123, if $T : M \to M$ is a surjective linear isometry on a seminormed module M over a normalizing seminormed ring R, then $T^* : M^* \to M^*$ is also a surjective linear isometry. As a consequence, $T^*\left(\operatorname{suppv}^{w^*}(m)\right) = \operatorname{suppv}^{w^*}\left(T^{-1}(m)\right)$ for all $m \in M$, and $T\left(\operatorname{suppv}(m^*)\right) = \operatorname{suppv}\left((T^*)^{-1}(m^*)\right)$ for all $m^* \in M^*$.

5.3.2.2 Smoothness

The concept of smoothness is proper from the Geometry of Banach Spaces [85, 86, 56]. Here, we transport it to the scope of seminormed modules.

Definition 316 (Smooth point) *Let* M *be a seminormed module over a normalizing absolutely semivalued ring* R. *Then:*

- *The set of smooth points of* M *is defined as*

$$\operatorname{smo}(M) := \left\{m \in M : \exists m^* \in \operatorname{suppv}^{w^*}(m) \text{ with } \operatorname{suppv}^{w^*}(m) = m^* \mathsf{S}_R\right\}.$$

- *The set of* w^**-smooth points of* M^* *is defined as*

$$\operatorname{smo}^{w^*}(M^*) := \{m^* \in M^* : \exists m \in \operatorname{suppv}(m^*) \text{ with } \operatorname{suppv}(m^*) = \mathsf{S}_R m\}.$$

For the upcoming technical lemma, it is convenient to remember that absolutely valued rings are integral domains as well as Hausdorff.

Lemma 161 *Let M be a seminormed module over a normalizing absolutely valued ring R. Then:*

1. *If $m \in \mathrm{smo}(M)$ with $\|m\| > 0$ and $m_1^*, m_2^* \in \mathrm{suppv}(m)$ are so that $m_1^*(m) = m_2^*(m)$, then $m_1^* = m_2^*$.*
2. *If $m^* \in \mathrm{smo}^{w^*}(M^*)$ with $\|m^*\| > 0$ and $m_1, m_2 \in \mathrm{suppv}^{w^*}(m^*)$ are so that $m^*(m_1) = m^*(m_2)$, then $m_1 = m_2$.*

Proof 474 *Let us itemize the proof likewise the statement of the lemma is.*

1. *Let $r_1, r_2 \in \mathsf{S}_R$ such that $m_1^* = n^* r_1$ and $m_2^* = n^* r_2$, where $n^* \in \mathrm{suppv}^{w^*}(m)$ verifies that $\mathrm{suppv}^{w^*}(m) = n^* \mathsf{S}_R$. Then $n^*(m) r_1 = m_1^*(m) = m_2^*(m) = n^*(m) r_2$, which implies that $r_1 = r_2$ because R is an integral domain and $n^*(m) \neq 0$ because $|n^*(m)| = \|m\| > 0$.*
2. *Let $r_1, r_2 \in \mathsf{S}_R$ such that $m_1 = r_1 n$ and $m_2 = r_2 n$, where $n \in \mathrm{suppv}(m^*)$ verifies that $\mathrm{suppv}(m^*) = \mathsf{S}_R n$. Then $r_1 m^*(n) = m^*(m_1) = m^*(m_2) = r_2 m^*(n)$, which implies that $r_1 = r_2$ because R is an integral domain and $m^*(n) \neq 0$ because $|m^*(n)| = \|m^*\| > 0$.*

Under the settings of the previous lemma, if m is a smooth point of norm 1, then the only element of $\mathrm{suppv}^{w^*}(m)$ taking the value 1 at m is denoted by $\mathsf{J}(m)$ (it if exists) and is called the unit supporting functional of m.

Proposition 124 *Let M be a seminormed module over a normalizing absolutely valued ring R. If m is a smooth point of M of norm 1 and $m_0^* \in \mathrm{suppv}^{w^*}(m)$ is so that $\mathrm{suppv}^{w^*}(m) = m_0^* \mathsf{S}_R$, then $\mathsf{J}(m)$ exists if and only if $m_0^*(m)$ has a right-inverse.*

Proof 475 *Suppose that $\mathsf{J}(m)$ exists and let $r \in \mathsf{S}_R$ with $\mathsf{J}(m) = m_0^* r$. Then $1 = \mathsf{J}(m)(m) = m_0^*(m) r$. Conversely, assume that there exists $r \in R$ with $m_0^*(m) r = 1$. Note that $1 = |1| = |m_0^*(m)||r| = \|m\||r| = |r|$, so $r \in \mathsf{S}_R$ and hence $m_0^* r \in m_0^* \mathsf{S}_R = \mathrm{suppv}^{w^*}(m)$ and thus $m_0^* r = \mathsf{J}(\mathsf{m})$.*

5.3.2.3 Isometric representations

Isometric representations are a special class of linear representations (see Subsubsection A.1.2.2) that typically arise in Banach Space Theory [87, 88].

Example 77 *Let H be an infinite-dimensional complex Hilbert space. Let V be any infinite-dimensional complex vector space. Suppose that V is endowed with the null seminorm, that is, $\|v\| = 0$ for all $v \in V$. Consider $X := V \oplus_2 H$, that is, $\|x\|_2 := \sqrt{\|v\|^2 + \|h\|^2}$, where $x \in X$ and $x = (v, h)$ with $v \in V$ and $h \in H$. Consider the linear operator $T := P \oplus I_H : X \to X$ defined by $T(v, h) := (P(v), h)$ for all $v \in V$ and all $h \in H$, where $P : V \to V$ is linear and surjective but not injective. Notice that T is a surjective linear isometry which is not an isomorphism.*

If M is a normed module over a normed ring R, then the set $\mathcal{G}_M$ of surjective linear isometries of M is a group under composition.

Definition 317 (Isometric representation) *An isometric representation of a group G in a normed module M over a normed ring R is a multiplicative map $\pi : G \to \mathcal{G}_M$.*

Under the settings of the previous definition, $\mathcal{G}_M \subseteq \mathsf{TGL}(M)$, where $\mathsf{TGL}(M) := \mathcal{U}(\mathcal{CL}(M))$ stands for the Topological General Linear Group of M. On the other hand, if R is a normalizing normed ring, then, by Theorem 153, $\mathcal{B}(M) = \mathcal{CL}(M)$, and, in view of Corollary 58 together with Proposition 123, the dual representation of π, given by

$$\begin{array}{rccl} \pi^* : & G & \to & \mathcal{G}_{M^*} \\ & g & \mapsto & \pi^*(g) := \pi(g^{-1})^*, \end{array} \tag{5.16}$$

is an isometric representation as well.

Part III

Integral Calculus

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Chapter 6

Summability

Leonhard Euler

6.1 Sequences and series

We will explore summability in its most general setting, that is, the one provided by topological modules. Commutativity turns out to be an essential property, very much needed to extend the classical results. This is why we will not stick to topological groups.

6.1.1 Summability types

There are different types of summability as we know from the classical Summability Theory [89].

6.1.1.1 Convergence

We will begin with the formal definition of convergence of series in topological modules over topological rings.

Definition 318 (Series) *A series in a topological module M over a topological ring R is a sequence of the form $\left(\sum_{i=1}^{k} m_i\right)_{k\in\mathbb{N}}$ where $(m_i)_{i\in\mathbb{N}} \subseteq M$ is called the sequence of general terms. The set of limit points of $\left(\sum_{i=1}^{k} m_i\right)_{k\in\mathbb{N}}$ is denoted by*

$$\sum_{i=1}^{\infty} m_i := \lim_{k\to\infty} \sum_{i=1}^{k} m_i.$$

If this set is not empty, then we will say that the series is convergent or that the sequence $(m_i)_{i\in\mathbb{N}}$ is summable.

Under the settings of the previous definition, for simplification purposes, we will often identify a series with the set of its limit points, $\sum_{i=1}^{\infty} m_i$. This set is also called the summation of the sequence $(m_i)_{i\in\mathbb{N}}$.

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Example 78 (Geometric series) *If R is a topological ring and $r \in R \setminus \{0\}$, then the series $\sum_{n=0}^{\infty} r^n$ is called geometric series.*

The following technical lemma shows what operations can be performed with series in topological modules.

Lemma 162 *Let M be a topological module over a topological ring R. Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ be convergent series in M. Let $r \in R$. Then:*

1. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
2. $r\sum_{n=1}^{\infty} a_n \subseteq \sum_{n=1}^{\infty} ra_n$.
3. *If $r \in \mathcal{U}(R)$, then* $\sum_{n=1}^{\infty} ra_n = r\sum_{n=1}^{\infty} a_n$.
4. $0 \in \lim_{n\to\infty} a_n$.
5. *For every $k > 1$,* $\sum_{n=k}^{\infty} a_n$ *is convergent and* $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{k-1} a_n + \sum_{n=k}^{\infty} a_n$.
6. *For every $k \in \mathbb{N}$ and every* $b_k \in \sum_{n=k}^{\infty} a_n$, $0 \in \lim_{k\to\infty} b_k$.

Proof 476 *The proof is itemized as the statement is.*

1. *Fix arbitrary elements $a \in \sum_{n=1}^{\infty} a_n$ and $b \in \sum_{n=1}^{\infty} b_n$. Take a 0-neighborhood $W \subseteq M$. Let $V \subseteq M$ be an additively symmetric 0-neighborhood such that $V + V \subseteq W$. There exist $k_1, k_2 \in \mathbb{N}$ such that, if $k \geq k_1$, then $\sum_{n=1}^{k} a_n \in a + V$, and if $k \geq k_2$, then $\sum_{n=1}^{k} b_n \in b + V$. Thus, if $k \geq k_0 := \max\{k_1, k_2\}$, then*

$$\sum_{n=1}^{k}(a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n \in (a + V) + (b + V) \subseteq (a + b) + W.$$

This shows that $a + b \in \sum_{n=1}^{\infty}(a_n + b_n)$. The arbitrariness of $a \in \sum_{n=1}^{\infty} a_n$ and $b \in \sum_{n=1}^{\infty} b_n$ implies that $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \subseteq \sum_{n=1}^{\infty}(a_n + b_n)$. Following a similar reasoning, it can be shown that $\sum_{n=1}^{\infty}(a_n + b_n) - \sum_{n=1}^{\infty} a_n \subseteq \sum_{n=1}^{\infty} b_n$. In view of Remark 36(1), we conclude that $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

2. *Fix an arbitrary element $a \in \sum_{n=1}^{\infty} a_n$. Take a 0-neighborhood $W \subseteq M$. Let $V \subseteq M$ be an additively symmetric 0-neighborhood such that $rV \subseteq W$. There exists $k_0 \in \mathbb{N}$ such that, if $k \geq k_0$, then $\sum_{n=1}^{k} a_n \in a + V$. Thus, if $k \geq k_0$, then*

$$\sum_{n=1}^{k} ra_n = r\sum_{n=1}^{k} a_n \in r\,(a + V) = ra + rV \subseteq ra + W.$$

This shows that $ra \in \sum_{n=1}^{\infty} ra_n$. The arbitrariness of $a \in \sum_{n=1}^{\infty} a_n$ implies that $r\sum_{n=1}^{\infty} a_n \subseteq \sum_{n=1}^{\infty} ra_n$.

3. *We know from the previous item that* $r\sum_{n=1}^{\infty} a_n \subseteq \sum_{n=1}^{\infty} ra_n$ *and* $r^{-1}\sum_{n=1}^{\infty} ra_n \subseteq \sum_{n=1}^{\infty} a_n$. *As a consequence,* $r\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ra_n$.

4. *Take any* $a \in \sum_{n=1}^{\infty} a_n$. *Fix any* 0*-neighborhood* $W \subseteq M$. *Let* $V \subseteq M$ *be an additively symmetric* 0*-neighborhood such that* $V + V \subseteq W$. *There exists* $k_0 \in \mathbb{N}$ *such that, if* $k \geq k_0$, *then* $\sum_{n=1}^{k} a_n \in a + V$. *For every* $k \geq k_0$,

$$a_k = \sum_{n=1}^{k+1} a_n - \sum_{n=1}^{k} a_n \in a + V - a - V = V - V = V + V \subseteq W.$$

5. *Fix* $j > 1$. *Fix an arbitrary element* $a \in \sum_{n=1}^{\infty} a_n$. *Take a* 0*-neighborhood* $V \subseteq M$. *There exists* $k_0 \in \mathbb{N}$ *such that, if* $k \geq k_0$, *then* $\sum_{n=1}^{k} a_n \in a + V$. *Next, for every* $k \geq \max\{k_0, j\}$,

$$\sum_{n=j}^{k} a_n = \sum_{n=1}^{k} a_n - \sum_{n=1}^{j-1} a_n \in a + V - \sum_{n=1}^{j-1} a_n = \left(a - \sum_{n=1}^{j-1} a_n\right) + V.$$

This shows that $a - \sum_{n=1}^{j-1} a_n \in \sum_{n=j}^{\infty} a_n$. *In fact, the arbitrariness of* $a \in \sum_{n=1}^{\infty} a_n$ *means that* $\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{j-1} a_n \subseteq \sum_{n=j}^{\infty} a_n$. *Thus,* $\sum_{n=1}^{\infty} a_n \subseteq \sum_{n=1}^{j-1} a_n + \sum_{n=j}^{\infty} a_n$. *Conversely, we will show now that* $\sum_{n=1}^{\infty} a_n \supseteq \sum_{n=1}^{j-1} a_n + \sum_{n=j}^{\infty} a_n$. *Indeed, fix an arbitrary* $b_j \in \sum_{n=j}^{\infty} a_n$. *Take a* 0*-neighborhood* $V \subseteq M$. *There exists* $k_0 \geq j$ *such that, if* $k \geq k_0$, *then* $\sum_{n=j}^{k} a_n \in b_j + V$. *Finally, for every* $k \geq k_0$,

$$\sum_{n=1}^{k} a_n = \sum_{n=1}^{j-1} a_n + \sum_{n=j}^{k} a_n \in \left(\sum_{n=1}^{j-1} a_n + b_j\right) + V.$$

This shows that $\sum_{n=1}^{j-1} a_n + b_j \in \sum_{n=1}^{k} a_n$, *hence, by the arbitrariness of* $b_j \in \sum_{n=j}^{\infty} a_n$, $\sum_{n=1}^{j-1} a_n + \sum_{n=j}^{\infty} a_n \subseteq \sum_{n=1}^{\infty} a_n$.

6. *Take any* $a \in \sum_{n=1}^{\infty} a_n$. *Fix any* 0*-neighborhood* $W \subseteq M$. *Let* $V \subseteq M$ *be an additively symmetric* 0*-neighborhood such that* $V + V \subseteq W$. *There exists* $k_0 \in \mathbb{N}$ *such that, if* $k \geq k_0$, *then* $\sum_{n=1}^{k} a_n \in a + V$. *Observe that*

$$c_k := \sum_{n=1}^{k-1} a_n + b_k \in \sum_{n=1}^{\infty} a_n$$

for every $k > 1$. *Since* M *is regular and* $c_k, a \in \sum_{n=1}^{\infty} a_n$, *we conclude that* $c_k \in \bigcap \mathcal{N}_a \subseteq a + V$ *for every* $k > 1$. *Finally, if* $k \geq k_0 + 1$, *then*

$$b_k = c_k - \sum_{n=1}^{k-1} a_n \in a + V - a - V = V - V = V + V \subseteq W.$$

A direct consequence of the previous lemma applies to the geometric series in order to determine its convergence.

Corollary 62 *In a Hausdorff topological ring* R, *if an element* $r \in R \setminus \{0\}$ *verifies that* $\sum_{n=0}^{\infty} r^n$ *is convergent, then* $1 - r \in \mathcal{U}(R)$ *and* $(1-r)^{-1} = \sum_{n=0}^{\infty} r^n$.

Proof 477 *Take into consideration that* $(r^k)_{k\in\mathbb{N}}$ *converges to* 0 *in virtue of Lemma 162(4). Thus*

$$(1-r)\sum_{n=0}^{k} r^n = \left(\sum_{n=0}^{k} r^n\right)(1-r) = 1 - r^{k+1}$$

for all $k \in \mathbb{N}$*. Since* R *is Hausdorff, we conclude that*

$$(1-r)\sum_{n=0}^{\infty} r^n = \left(\sum_{n=0}^{\infty} r^n\right)(1-r) = 1.$$

6.1.1.2 Unconditional convergence

The famous of Dvoretzky-Rogers Theorem [90] asserts that every infinite-dimensional Banach space admits an unconditionally convergent series that is not absolutely convergent. Since we are working in the topological module setting, we are entitled to extend the concept of unconditionally convergent series to that scope.

Definition 319 (Unconditional convergence) *A series* $\sum_{i=1}^{\infty} m_i$ *in a topological module* M *over a topological ring* R *is said to be unconditionally convergent if, for every permutation* $\pi : \mathbb{N} \to \mathbb{N}$*,* $\sum_{i=1}^{\infty} m_{\pi(i)}$ *is convergent and* $\sum_{i=1}^{\infty} m_{\pi(i)} = \sum_{i=1}^{\infty} m_i$*.*

Recall that $\Pi(\mathbb{N})$ stands for the set of all permutations of $\mathbb{N}$. Using monoid notation, $\Pi(\mathbb{N}) = \left(\mathbb{N}^{\mathbb{N}}\right)^{\times}$.

Theorem 157 *Let* M *be a topological module over a topological ring* R*. Let* $\sum_{i=1}^{\infty} m_i$ *be a series in* M*. Then*

$$\lim(\gamma) = \bigcap_{\pi\in\Pi(\mathbb{N})} \sum_{i=1}^{\infty} m_{\pi(i)},$$

where γ *is the net*

$$\begin{array}{rccc} \gamma: & \mathcal{P}_{\mathtt{f}}(I) & \to & M \\ & A & \mapsto & \displaystyle\sum_{i\in A} m_i. \end{array} \tag{6.1}$$

Proof 478 *Let us itemize the proof the same way the statement is.*

- $\subseteq$ *Fix an arbitrary* $m \in \lim(\gamma)$*. Take any* $\pi \in \Pi(\mathbb{N})$*. Let* U *be a* 0*-neighborhood in* M*. There exists* $A_U \in \mathcal{P}_{\mathtt{f}}(\mathbb{N})$ *such that, if* $A \in \mathcal{P}_{\mathtt{f}}(\mathbb{N})$ *and* $A \supseteq A_U$*, then* $\sum_{i\in A} m_i \in m+U$*. Let* $j_U := \max\left(\pi^{-1}(A_U)\right)$*. If* $j \geq j_U$*, then* $\{\pi(1),\ldots,\pi(j)\} \supseteq \{\pi(1),\ldots,\pi(j_U)\} \supseteq A_U$*. Thus,*

  $$\sum_{i=1}^{j} m_{\pi(i)} \in m + U.$$

  This shows that $m \in \sum_{i=1}^{\infty} m_{\pi(i)}$*. As a consequence,*

  $$\lim(\gamma) \subseteq \bigcap_{\pi\in\Pi(\mathbb{N})} \sum_{i=1}^{\infty} m_{\pi(i)}.$$

- $\supseteq$ *Conversely, fix an arbitrary* $m \in \bigcap_{\pi\in\Pi(\mathbb{N})} \sum_{i=1}^{\infty} m_{\pi(i)}$*. Suppose on the contrary that* $m \notin \lim(\gamma)$*. There exists a* 0*-neighborhood* U *in* M *such that, for every finite subset* $A \subseteq \mathbb{N}$*, a finite subset* $B \subseteq \mathbb{N}$ *can be found in such a way that* $A \subseteq B$ *and* $\sum_{i\in B} m_i \notin m + U$*. We will inductively construct a permutation* $\pi \in \Pi(\mathbb{N})$ *for which* $m \notin \sum_{i=1}^{\infty} m_{\pi(i)}$*. Indeed:*

1. *For* $A := \{1\}$, *there exists* $A_1 = \{a_{11}, a_{12}, \ldots, a_{1n_1}\} \subseteq \mathbb{N}$ *such that* $\sum_{i\in A_1} m_i \notin m + U$, *where* $a_{11} := 1$.
2. *For* $A := A_1$, *there is* $A_2 = \{a_{11}, a_{12}, \ldots, a_{1n_1}, a_{21}, a_{22}, \ldots, a_{2n_2}\} \subseteq \mathbb{N}$ *such that* $\sum_{i\in A_2} m_i \notin m + U$.

$\vdots$ *And so on.*

By repeating this process, we obtain a permutation $\pi \in \Pi(\mathbb{N})$ *for which* $m \notin \sum_{i=1}^{\infty} m_{\pi(i)}$.

As a direct consequence of Theorem 157, we obtain the classical characterization of unconditional convergence.

Corollary 63 *Let* M *be a topological module over a topological ring* R. *Let* $\sum_{i=1}^{\infty} m_i$ *be a series in* M. *Let* γ *be as in* (6.1). *If* $\sum_{i=1}^{\infty} m_i$ *is unconditionally convergent, then* $\lim(\gamma) = \sum_{i=1}^{\infty} m_{\pi(i)}$ *for each* $\pi \in \Pi(\mathbb{N})$. *Conversely, if* M *is Hausdorff and* $\lim(\gamma) \neq \varnothing$, *then* $\sum_{i=1}^{\infty} m_i$ *is unconditionally convergent.*

Proof 479 *If* $\sum_{i=1}^{\infty} m_i$ *is unconditionally convergent, then* $\sum_{i=1}^{\infty} m_{\pi(i)} = \sum_{i=1}^{\infty} m_i$ *for each* $\pi \in \Pi(\mathbb{N})$, *meaning that* $\lim(\gamma) = \sum_{i=1}^{\infty} m_{\pi(i)}$ *for each* $\pi \in \Pi(\mathbb{N})$. *Conversely, if* M *is Hausdorff and* $\lim(\gamma) \neq \varnothing$, *then* $\sum_{i=1}^{\infty} m_{\pi(i)}$ *is a singleton for each* $\pi \in \Pi(\mathbb{N})$, *which forces that* $\lim(\gamma) = \sum_{i=1}^{\infty} m_{\pi(i)}$ *for each* $\pi \in \Pi(\mathbb{N})$, *implying that* $\sum_{i=1}^{\infty} m_i$ *be unconditionally convergent.*

The context of unconditionally convergent series is very extensive, overall in Operator Theory and Banach Space Theory. In the further subsubsections, we will prove that absolutely convergent series in seminormed modules are unconditionally convergent.

6.1.1.3 Subseries convergence

Given any series $\sum_{i=1}^{\infty} m_i$ in a topological module M, a subseries of $\sum_{i=1}^{\infty} m_i$ is a series of the form $\sum_{j=1}^{\infty} m_{i_j}$, where $(m_{i_j})_{j\in\mathbb{N}}$ is a subsequence of $(m_i)_{i\in\mathbb{N}}$.

Lemma 163 *Let* M *be a topological module over a topological ring* R. *Let* $\sum_{n\in\mathbb{N}} a_n$ *be a convergent series in* M. *If* $I \subseteq \mathbb{N}$ *satisfies that* $\sum_{i\in I} a_i$ *converges, then* $\sum_{j\in J} a_j$ *converges as well, where* $J := \mathbb{N} \setminus I$. *In fact,*

$$\sum_{j\in J} a_j = \sum_{n\in\mathbb{N}} a_n - \sum_{i\in I} a_i.$$

Proof 480 *We will prove the case where both* I, J *are infinite. Therefore, we can write* $I := (i_p)_{p\in\mathbb{N}}$ *and* $J := (j_q)_{q\in\mathbb{N}}$ *as strictly increasing sequences of naturals. Fix arbitrary values* $a \in \sum_{n=1}^{\infty} a_n$ *and* $b \in \sum_{p=1}^{\infty} a_{i_p}$. *Let* U *be a neighborhood of* 0 *in* M. *We can fix an additively symmetric neighborhood* V *of* 0 *such that* $V + V \subseteq U$. *There are* $n_0, p_0 \in \mathbb{N}$ *such that* $i_{p_0} \geq n_0$ *and if* $n' \geq n_0$ *and* $p' \geq p_0$, *then*

$$\sum_{n=1}^{n'} a_n \in a + V \quad \textit{and} \quad \sum_{p=1}^{p'} a_{i_p} \in b + V.$$

Take $l_0 \in \mathbb{N}$ *such that* $j_{l_0} > i_{p_0}$. *Fix an arbitrary* $l \geq l_0$. *We will prove that*

$$\sum_{q=1}^{l} a_{j_q} \in (a - b) + U.$$

Take $t := \max\{p \in \mathbb{N} : i_p < j_l\}$. *Notice that* $p_0 \leq t$ *because* $i_{p_0} < j_{l_0} \leq j_l$. *Since* $j_l \geq j_{l_0} > i_{p_0} \geq n_0$ *and* $t \geq p_0$,

$$\sum_{q=1}^{l} a_{j_q} = \sum_{n=1}^{j_l} a_n - \sum_{p=1}^{t} a_{i_p} \in (a+V) + (-b-V) = (a-b) + (V+V) \subseteq (a-b) + U.$$

This shows that $a - b \in \sum_{q=1}^{\infty} a_{j_q}$, *so the arbitrariness of* $a \in \sum_{n=1}^{\infty} a_n$ *and* $b \in \sum_{p=1}^{\infty} a_{i_p}$ *means*

$$\sum_{n=1}^{\infty} a_n - \sum_{p=1}^{\infty} a_{i_p} \subseteq \sum_{q=1}^{\infty} a_{j_q}.$$

Following a similar reasoning, it can be proved that

$$\sum_{n=1}^{\infty} a_n - \sum_{q=1}^{\infty} a_{j_q} \subseteq \sum_{p=1}^{\infty} a_{i_p}.$$

As a consequence, Remark 36(2) assures that

$$\sum_{q=1}^{\infty} a_{j_q} = \sum_{n=1}^{\infty} a_n - \sum_{p=1}^{\infty} a_{i_p}.$$

Previous lemma motivates the notion of subseries convergence, which plays an important role in constructing counting measures.

Definition 320 (Subseries convergence) *A series* $\sum_{i=1}^{\infty} m_i$ *in a topological module* M *over a topological ring* R *is said to be subseries convergent if every subseries* $\sum_{j=1}^{\infty} m_{i_j}$ *is convergent.*

If I is a finite subset of $\mathbb{N}$, then $\sum_{i\in I} m_i$ is well defined independently of the order chosen in I due to the commutativity of the addition in a module M. However, if I is infinite, then we will be assuming by default that $\sum_{i\in I} m_i$ is a series where I has the natural order inherited from $\mathbb{N}$, as we did in Lemma 163, unless we explicitly state something different.

Lemma 164 *Let* M *be a topological module over a topological ring* R. *Let* $\sum_{n\in\mathbb{N}} a_n$ *be a series in* M. *If* $I_1, \ldots, I_p$ *are pairwise disjoint subsets of* $\mathbb{N}$ *and* $\sum_{n\in I_1} a_n, \ldots, \sum_{n\in I_p} a_n$ *are convergent, then* $\sum_{n\in I} a_n$ *is convergent as well and*

$$\sum_{n\in I} a_n = \sum_{n\in I_1} a_n + \cdots + \sum_{n\in I_p} a_n,$$

where $I := I_1 \cup \cdots \cup I_p$.

Proof 481 *Fix arbitrary elements* $b_1 \in \sum_{n\in I_1} a_n, \ldots, b_p \in \sum_{n\in I_p} a_n$. *Let* U *be a neighborhood of* 0 *in* M. *We can fix a neighborhood* V *of* 0 *in* M *such that* $V \overset{p}{+\cdots+} V \subseteq U$. *For every* $l \in \{1, \ldots, p\}$, *we can find* $k_l \in \mathbb{N}$ *such that if* $k \geq k_l$, *then* $b_l - \sum_{n\in I_l^k} a_n \in V$, *where* $I_l^k := \{n \in I_l : n \leq k\}$. *Take* $k_0 := \max\{k_1, \ldots, k_p\}$. *If* $k \geq k_0$, *then*

$$\begin{aligned}(b_1 + \cdots + b_p) - \sum_{n\in I^k} a_n &= \left(b_1 - \sum_{n\in I_1^k} a_n\right) + \cdots + \left(b_p - \sum_{n\in I_p^k} a_n\right) \\ &\in V \overset{p}{+\cdots+} V \subseteq U,\end{aligned}$$

where, as expected, $I^k := \{n \in I : n \leq k\}$*. This shows that* $b_1 + \cdots + b_p \in \sum_{n\in I} a_n$*. As a consequence,*

$$\sum_{n\in I_1} a_n + \cdots + \sum_{n\in I_p} a_n \subseteq \sum_{n\in I} a_n.$$

Following a similar reasoning, it can be proved that

$$\sum_{n\in I} a_n - \sum_{n\in I_1} a_n \subseteq \sum_{n\in I_2} a_n + \cdots + \sum_{n\in I_p} a_n.$$

A simple generalization of Remark 36 implies that

$$\sum_{n\in I} a_n = \sum_{n\in I_1} a_n + \cdots + \sum_{n\in I_p} a_n.$$

Previous technical lemma serves to prove the following one, which is a bit more technical.

Lemma 165 *Let* M *be a topological module over a topological ring* R*. Let* $\sum_{n\in\mathbb{N}} a_n$ *be a series in* M*. If* $I_1, I_2, \ldots$ *is a sequence of pairwise disjoint subsets of* $\mathbb{N}$ *ordered in such a way that* $\min(I_l) < \min(I_{l+1})$ *for all* $l \in \mathbb{N}$*, and* $\sum_{n\in I_l} a_n$ *is convergent for every* $l \in \mathbb{N}$*, then*

$$\sum_{n\in I} a_n \subseteq \sum_{l=1}^{\infty} b_l,$$

where $I := \bigcup_{l=1}^{\infty} I_l$ *and* $b_l \in \sum_{n\in I_l} a_n$ *for every* $l \in \mathbb{N}$*.*

Proof 482 *Fix an arbitrary element* $c \in \sum_{n\in I} a_n$*. Let* U *be a neighborhood of* 0 *in* M*. We can choose an additively symmetric neighborhood* V *of* 0 *in* M *such that* $V + V \subseteq U$*. There exists* $k_1 \in \mathbb{N}$ *such that if* $k \geq k_1$*, then* $c - \sum_{n\in I^k} a_n \in V$*, where* $I^k := \{n \in I : n \leq k\}$*. The fact that* $\min(I_l) < \min(I_{l+1})$ *for all* $l \in \mathbb{N}$ *means that* $\min(I_l) \geq l$ *for all* $l \in \mathbb{N}$*, meaning that, for every* $k \in \mathbb{N}$*,*

$$\sum_{n\in I^k} a_n = \sum_{n\in J_k^k} a_n,$$

where $J_k := I_1 \cup \cdots \cup I_k$ *and* $J_k^k := \{n \in J_k : n \leq k\}$*. By Lemma 164,* $\sum_{n\in J_k} a_n$ *is convergent and* $\sum_{l=1}^{k} b_l \in \sum_{n\in J_k} a_n$ *for every* $k \in \mathbb{N}$*. Also, by Lemma 162(5),*

$$\sum_{n\in J_k} a_n = \sum_{n\in J_k^k} a_n + \sum_{n\in\mathbb{N}\setminus J_k^k} a_n$$

for every $k \in \mathbb{N}$*. This means that, for every* $k \in \mathbb{N}$*, there exists* $d_k \in \sum_{n\in\mathbb{N}\setminus J_k^k} a_n$ *such that*

$$\sum_{l=1}^{k} b_l = \sum_{n\in J_k^k} a_n + d_k.$$

According to Lemma 162(6), d_k *converges to* 0 *as* $k \to \infty$*. Therefore, there exists* $k_2 \in \mathbb{N}$ *such that if* $k \geq k_2$*, then* $d_k \in V$*. Finally, if* $k \geq k_0 := \max\{k_1, k_2\}$*, then*

$$c - \sum_{l=1}^{k} b_l = \left(c - \sum_{n\in J_k^k} a_n\right) - d_k = \left(c - \sum_{n\in I^k} a_n\right) - d_k \in V + V \subseteq U.$$

6.1.1.4 Absolute convergence

In order to introduce absolute convergence, we need to jump to seminormed modules.

Definition 321 (Absolute convergence) *A series $\sum_{i=1}^{\infty} m_i$ in a seminormed R-module M over a seminormed ring R is said to be absolutely convergent provided that $\sum_{i=1}^{\infty} \|m_i\|$ is convergent.*

The geometric series is one of the most representative examples of absolutely convergent series under certain conditions.

Example 79 *If R is a seminormed ring and $r \in R$ with $\|r\| < 1$, then the geometric series $\sum_{n=0}^{\infty} r^n$ is clearly absolutely convergent since*

$$\sum_{n=0}^{\infty} \|r^n\| \leq \sum_{n=0}^{\infty} \|r\|^n = \frac{1}{1-\|r\|}.$$

The following proposition is a very important result in Functional Analysis, which characterizes completeness in terms of absolute convergence.

Proposition 125 *A seminormed module M over a seminormed ring R is complete if and only if every absolutely convergent series on M is convergent.*

Proof 483 *Te proof is itemized in view of the statement.*

- $\Rightarrow$ *Let $\sum_{i=1}^{\infty} m_i$ be an absolutely convergent series in M. Notice that the sequence $\left(\sum_{i=1}^{k} m_i\right)_{k\in\mathbb{N}}$ is a Cauchy sequence in M because $\left(\sum_{i=1}^{k} \|m_i\|\right)_{k\in\mathbb{N}}$ is obviously a Cauchy sequence and*

$$\left\|\sum_{i=1}^{p} m_i - \sum_{i=1}^{q} m_i\right\| = \left\|\sum_{i=q+1}^{p} m_i\right\| \leq \sum_{i=q+1}^{p} \|m_i\| = \sum_{i=1}^{p} \|m_i\| - \sum_{i=1}^{q} \|m_i\|$$

  for all $p \geq q$ in $\mathbb{N}$. Since M is complete, we have that $\sum_{i=1}^{\infty} m_i$ is convergent.

- $\Leftarrow$ *Let $(a_i)_{i\in\mathbb{N}} \subseteq M$ be a Cauchy sequence. For every $k \in \mathbb{N}$, there exists $i_k \in \mathbb{N}$ such that if $p, q \geq i_k$, then $\|a_p - a_q\| < \frac{1}{2^k}$. We can trivially choose $(i_k)_{k\in\mathbb{N}}$ to be strictly increasing. For every $k \in \mathbb{N}$, define $m_k := a_{i_{k+1}} - a_{i_k}$. The series $\sum_{k=1}^{\infty} m_k$ is absolutely convergent because*

$$\sum_{k=1}^{\infty} \|m_k\| = \sum_{k=1}^{\infty} \left\|a_{i_{k+1}} - a_{i_k}\right\| \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

  Next,

$$\sum_{k=1}^{j} m_k = a_{i_{j+1}} - a_{i_1}$$

  for every $j \in \mathbb{N}$. By hypothesis, $\left(a_{i_{j+1}} - a_{i_1}\right)_{j\in\mathbb{N}}$ is convergent, and so is $\left(a_{i_{j+1}}\right)_{j\in\mathbb{N}}$. By Proposition 59, $(a_i)_{i\in\mathbb{N}}$ is convergent as well.

Absolutely convergent series on complete seminormed modules are subseries convergent and unconditionally convergent, as we can see in the next lemma.

Lemma 166 *Let M be a complete seminormed module over a seminormed ring R. If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series in M, then it is subseries convergent and unconditionally convergent.*

Proof 484 *We will prove the result in two steps:*

1. *Subseries convergence: Let $(i_p)_{p\in\mathbb{N}} \subseteq \mathbb{N}$ be strictly increasing. For every $p_0 \in \mathbb{N}$,*

$$\sum_{p=1}^{p_0} \|a_{i_p}\| \leq \sum_{n=1}^{i_{p0}} \|a_n\|.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, we deduce that $\sum_{p=1}^{\infty} a_{i_p}$ is absolutely convergent as well. The completeness of M assures that $\sum_{p=1}^{\infty} a_{i_p}$ is convergent.

2. *Unconditional convergence: Fix an arbitrary permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and an arbitrary $m \in \sum_{k=1}^{\infty} m_k$. Fix also an arbitrary $\varepsilon > 0$. We can find $n_\varepsilon \in \mathbb{N}$ such that $\|\sum_{k=1}^{n} m_k - m\| < \frac{\varepsilon}{2}$ and $\sum_{k=1}^{n} \|m_n\| < \frac{\varepsilon}{2}$ for each $n \geq n_\varepsilon$. Take $n_0 := \max(N_{\sigma,\varepsilon})$, where $N_{\sigma,\varepsilon} := \sigma^{-1}(\{1, 2, \ldots, n_\varepsilon\})$. For every $n \geq n_0$, denote $K_{n,\sigma,\varepsilon} := \{1, \ldots, n\} \setminus N_{\sigma,\varepsilon}$. If $n \geq n_0$, then*

$$\begin{aligned}
\left\|\sum_{k=1}^{n} m_{\sigma(k)} - m\right\| &= \left\|\sum_{k\in N_{\sigma,\varepsilon}} m_{\sigma(k)} - m + \sum_{k\in K_{n,\sigma,\varepsilon}} m_{\sigma(k)}\right\| \\
&\leq \left\|\sum_{l=1}^{n_\varepsilon} m_l - m\right\| + \sum_{k\in K_{n,\sigma,\varepsilon}} \left\|m_{\sigma(k)}\right\| \\
&\leq \left\|\sum_{l=1}^{n_\varepsilon} m_l - m\right\| + \sum_{l=n_\varepsilon+1}^{\infty} \|m_l\| \\
&< \varepsilon.
\end{aligned}$$

The arbitrariness of $\varepsilon > 0$ shows that $m \in \sum_{k=1}^{\infty} m_{\sigma(k)}$. As a consequence, $\sum_{k=1}^{\infty} m_k \subseteq \sum_{k=1}^{\infty} m_{\sigma(k)}$. Following the same reasoning, $\sum_{k=1}^{\infty} m_{\sigma(k)} \subseteq \sum_{k=1}^{\infty} m_{\sigma^{-1}(\sigma(k))} = \sum_{k=1}^{\infty} m_k$.

The following technical lemma deals with additivity of the norm and has several applications when dealing with double series.

Lemma 167 *Let M be a complete normed module over a normed ring R. If $\sum_{i=1}^{\infty} a_i$ is a series in M such that $\|\sum_{i=1}^{\infty} a_i\| = \sum_{i=1}^{\infty} \|a_i\| < \infty$, then $\left\|\sum_{i\in I} a_i\right\| = \sum_{i\in I} \|a_i\|$ for all $I \subseteq \mathbb{N}$.*

Proof 485 *In the first place, since $\sum_{i=1}^{\infty} a_i$ is absolutely convergent, we know that it is subseries convergent (see Lemma 166). Next, according to Lemma 163, $\sum_{i=1}^{\infty} a_i = \sum_{i\in I} a_i + \sum_{i\notin I} a_i$ and $\sum_{i=1}^{\infty} \|a_i\| = \sum_{i\in I} \|a_i\| + \sum_{i\notin I} \|a_i\|$. Notice that*

$$\begin{aligned}
\sum_{i=1}^{\infty} \|a_i\| &= \left\|\sum_{i=1}^{\infty} a_i\right\| = \left\|\sum_{i\in I} a_i + \sum_{i\notin I} a_i\right\| \leq \left\|\sum_{i\in I} a_i\right\| + \left\|\sum_{i\notin I} a_i\right\| \\
&\leq \sum_{i\in I} \|a_i\| + \sum_{i\notin I} \|a_i\| = \sum_{i=1}^{\infty} \|a_i\|.
\end{aligned}$$

In particular, $\left\|\sum_{i\in I} a_i\right\| + \left\|\sum_{i\notin I} a_i\right\| = \sum_{i\in I} \|a_i\| + \sum_{i\notin I} \|a_i\|$. This forces that $\left\|\sum_{i\in I} a_i\right\| = \sum_{i\in I} \|a_i\|$ and $\left\|\sum_{i\notin I} a_i\right\| = \sum_{i\notin I} \|a_i\|$.

Since seminormed rings are pseudometric spaces, it makes sense to talk about complete seminormed rings, that is, seminormed rings which are complete with respect to the induced pseudometric.

Theorem 158 *Let R be a complete normed ring. If $r \in \mathcal{U}(R)$ and $s \in \mathsf{U}_R\left(r, \left\|r^{-1}\right\|^{-1}\right)$, then $s \in \mathcal{U}(R)$ and*

$$s^{-1} = r^{-1}\sum_{n=0}^{\infty}\left(1 - sr^{-1}\right)^n \text{ and } \left\|s^{-1}\right\| \leq \frac{\left\|r^{-1}\right\|}{1 - \|r - s\|\left\|r^{-1}\right\|}.$$

In addition,

$$\left\|r^{-1} - s^{-1}\right\| \leq \frac{\left\|r^{-1}\right\|^2}{1 - \left\|r^{-1}\right\|\|s - r\|}\|s - r\|,$$

so $\mathcal{U}(R)$ is open and the inversion map

$$\begin{array}{ccc} \mathcal{U}(R) & \to & \mathcal{U}(R) \\ u & \mapsto & u^{-1} \end{array}$$

is continuous.

Proof 486 *First of all, $\left\|1 - sr^{-1}\right\| = \left\|(r-s)r^{-1}\right\| \leq \|r - s\|\left\|r^{-1}\right\| < 1$ because $s \in \mathsf{U}_R\left(r, \left\|r^{-1}\right\|^{-1}\right)$. In view of Example 79, the series $\sum_{n=0}^{\infty}\left(1 - sr^{-1}\right)^n$ is absolutely convergent. Since R is complete, Theorem 125 assures that $\sum_{n=0}^{\infty}\left(1 - sr^{-1}\right)^n$ is convergent. By Lemma 62, $sr^{-1} \in \mathcal{U}(R)$ and so does s. Again by applying Lemma 62, it follows that $\left(sr^{-1}\right)^{-1} = \sum_{n=0}^{\infty}\left(1 - sr^{-1}\right)^n$ and thus*

$$s^{-1} = r^{-1}\sum_{n=0}^{\infty}\left(1 - sr^{-1}\right)^n.$$

Therefore,

$$\begin{array}{rcl} \left\|s^{-1}\right\| & \leq & \left\|r^{-1}\right\|\sum_{n=0}^{\infty}\left\|1 - sr^{-1}\right\|^n \leq \left\|r^{-1}\right\|\sum_{n=0}^{\infty}\left(\|r - s\|\left\|r^{-1}\right\|\right)^n \\ & = & \dfrac{\left\|r^{-1}\right\|}{1 - \|r - s\|\left\|r^{-1}\right\|}. \end{array}$$

Now, in virtue of Lemma 69(5),

$$\begin{array}{rcl} \left\|r^{-1} - s^{-1}\right\| & = & \left\|r^{-1}(s - r)s^{-1}\right\| \leq \left\|r^{-1}\right\|\|s - r\|\left\|s^{-1}\right\| \\ & \leq & \left\|r^{-1}\right\|\|s - r\|\dfrac{\left\|r^{-1}\right\|}{1 - \|r - s\|\left\|r^{-1}\right\|} \\ & = & \dfrac{\left\|r^{-1}\right\|^2}{1 - \left\|r^{-1}\right\|\|s - r\|}\|s - r\|. \end{array}$$

This proves the continuity of the inverse map at r.

The following result is an immediate corollary of the above theorem and applies to the construction of topological divisors of zero.

Corollary 64 *Let R be a complete normed ring. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}(R)$ be convergent to some $r \in R$. The following conditions are equivalent:*

1. $r \notin \mathcal{U}(R)$.
2. *Every subsequence of* $(u_n^{-1})_{n\in\mathbb{N}}$ *is unbounded.*
3. $(u_n^{-1})_{n\in\mathbb{N}}$ *is unbounded.*

Proof 487 *Assume first that* $r \notin \mathcal{U}(R)$. *Suppose to the contrary that there exists a bounded subsequence* $(u_{n_k}^{-1})_{k\in\mathbb{N}}$ *with bound* $M > 0$. *Notice then that*

$$\begin{aligned}\|u_{n_i}^{-1} - u_{n_j}^{-1}\| &= \|u_{n_i}^{-1}(u_{n_j} - u_{n_i})u_{n_j}^{-1}\| \leq \left\|u_{n_i}^{-1}\right\| \|u_{n_j} - u_{n_i}\| \left\|u_{n_j}^{-1}\right\| \\ &\leq 2M\|u_{n_j} - u_{n_i}\|\end{aligned}$$

for every $i, j \in \mathbb{N}$. *This implies that* $(u_{n_k}^{-1})_{k\in\mathbb{N}}$ *is a Cauchy sequence and thus convergent to some* $s \in R$. *In this case, it is trivial that* $rs = sr = 1$, *reaching the contradiction that* $r \in \mathcal{U}(R)$. *Conversely, assume that* $(u_n^{-1})_{n\in\mathbb{N}}$ *is unbounded. Suppose to the contrary that* $r \in \mathcal{U}(R)$. *The continuity of the inverse map (see Theorem 158) forces that* $(u_n^{-1})_{n\in\mathbb{N}}$ *converges to* r^{-1}. *This contradicts the fact that* $(u_n^{-1})_{n\in\mathbb{N}}$ *is unbounded.*

Next technical lemma serves to the construction of unit neighborhoods of zero.

Lemma 168 *Let* S *be a practical complete unital normed ring. If* $\mathcal{U}(S) = \mathcal{U}_1(S)$ *and* 1 *is a* 2*-extreme point of* B_S, *then* $1 \in \mathrm{cl}(\mathcal{U}_1(S) \setminus \mathsf{S}_S)$.

Proof 488 *There exists a sequence* $(s_k)_{k\in\mathbb{N}} \subseteq \mathcal{U}(S)$ *convergent to* 0 *such that* $\|s_k\| < 1$ *for all* $k \in \mathbb{N}$. *In view of Theorem 158, for every* $k \in \mathbb{N}$ *we have that* $1 - s_k$ *and* $1 + s_k$ *are invertible in* S *and their inverses are* $(1-s_k)^{-1} = \sum_{n=0}^{\infty} s_k^n$ *and* $(1+s_k)^{-1} = \sum_{n=0}^{\infty}(-s_k)^n$, *respectively. By hypothesis,* $\mathcal{U}(S) = \mathcal{U}_1(S)$, *meaning that* $(1 - s_k)^{-1}, (1 + s_k)^{-1} \in \mathcal{U}_1(S)$ *for all* $k \in \mathbb{N}$. *Suppose that there exists* $k_0 \in \mathbb{N}$ *such that* $\|1 - s_{k_0}\| = \|1 + s_{k_0}\| = 1$. *Since* $1 + 1 = (1 + s_{k_0}) + (1 - s_{k_0})$ *and* 1 *is a* 2*-extreme point of* B_S, *we deduce that* $1 = 1+s_{k_0} = 1-s_{k_0}$, *which implies the contradiction that* $s_{k_0} = -s_{k_0} = 0$. *As a consequence, for every* $k \in \mathbb{N}$, $1 - s_k$ *and* $1 + s_k$ *cannot have both norm* 1 *at the same time. Now, we construct the sequence* $(u_k)_{k\in\mathbb{N}}$ *defined by*

$$u_k := \begin{cases} 1 - s_k & \text{if } \|1 - s_k\| < 1, \\ (1 - s_k)^{-1} & \text{if } \|1 - s_k\| > 1, \\ 1 + s_k & \text{if } \|1 - s_k\| = 1 \ \&\ \|1 + s_k\| < 1, \\ (1 + s_k)^{-1} & \text{if } \|1 - s_k\| = 1 \ \&\ \|1 + s_k\| > 1. \end{cases}$$

In virtue of Theorem 158, the inverse map $u \mapsto u^{-1}$ *in* $\mathcal{U}(S)$ *is continuous, so both sequences* $\left((1 - s_k)^{-1}\right)_{k\in\mathbb{N}}$ *and* $\left((1 + s_k)^{-1}\right)_{k\in\mathbb{N}}$ *converge to* 1. *As a consequence,* $(u_k)_{k\in\mathbb{N}}$ *is a sequence in* $\mathsf{U}_S \cap \mathcal{U}_1(S)$ *converging to* 1.

Before stating and proving the next theorem, we recall to the reader that all subrings of a ring are considered unitary subrings, that is, sharing the same unity.

Theorem 159 *Let* R *be a complete unital normed ring. Suppose that* R *contains a closed practical subring* S *such that* $\mathcal{U}(S) = \mathcal{U}_1(S)$ *and* 1 *is a* 2*-extreme point of* B_S. *Then* $\mathsf{B}_R = \mathrm{cl}(\mathsf{U}_R)$, $\mathsf{U}_R = \mathrm{int}(\mathsf{B}_R)$, B_R *is a closed unit neighborhood of* 0 *and* U_R *is an open unit neighborhood of* 0.

Proof 489 *Note that* S *satisfies the hypotheses of Lemma 168, meaning that* $1 \in \mathrm{cl}(\mathcal{U}_1(S) \setminus \mathsf{S}_S) \subseteq \mathrm{cl}(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$. *Therefore, according to Theorem 84,* $\mathsf{B}_R = \mathrm{cl}(\mathsf{U}_R)$, $\mathsf{U}_R = \mathrm{int}(\mathsf{B}_R)$, B_R *is a closed unit neighborhood of* 0 *and* U_R *is an open unit neighborhood of* 0.

In [91, Theorem 1.1.23], it is proved that the inverse map on a complex Banach algebra is $\mathbb{C}$-differentiable. Here, this result is improved.

Theorem 160 *If R is a practical complete normed ring, then the inverse map $^{-1} : \mathcal{U}(R) \to \mathcal{U}(R)$ is differentiable at every $u \in \mathsf{Z}(\mathcal{U}(R))$ in the direction of any $v \in R$ and $\mathrm{D}_v\left(\ ^{-1}\right)(u) = -u^{-1}vu^{-1}$.*

Proof 490 *Let $u, w \in \mathcal{U}(R)$ and $v \in R$. If $\|w\| < \|v\|^{-1}\|u^{-1}\|^{-1}$, then $\left\|-wvu^{-1}\right\| = \left\|wvu^{-1}\right\| \leq \|w\|\|v\|\left\|u^{-1}\right\| < 1$, therefore $1 + wvu^{-1} \in \mathcal{U}(R)$ and $\left(1+wvu^{-1}\right)^{-1} = \sum_{n=0}^{\infty}\left(-wvu^{-1}\right)^n$. Therefore,*

$$\begin{aligned}
& w^{-1}\left((u+wv)^{-1}-u^{-1}\right) = w^{-1}\left(u^{-1}\left(1+wvu^{-1}\right)^{-1}-u^{-1}\right) \\
= \ & w^{-1}u^{-1}\left(\left(1+wvu^{-1}\right)^{-1}-1\right) = w^{-1}u^{-1}\left(\sum_{n=0}^{\infty}\left(-wvu^{-1}\right)^n - 1\right) \\
= \ & w^{-1}u^{-1}\sum_{n=1}^{\infty}\left(-wvu^{-1}\right)^n = w^{-1}u^{-1}(-wvu^{-1})\sum_{n=0}^{\infty}\left(-wvu^{-1}\right)^n \\
= \ & u^{-1}w^{-1}(-wvu^{-1})\sum_{n=0}^{\infty}\left(-wvu^{-1}\right)^n = -u^{-1}vu^{-1}\sum_{n=0}^{\infty}\left(-wvu^{-1}\right)^n \\
= \ & -u^{-1}vu^{-1}\left(1+\sum_{n=1}^{\infty}\left(-wvu^{-1}\right)^n\right) \\
= \ & -u^{-1}vu^{-1} - u^{-1}vu^{-1}\sum_{n=1}^{\infty}\left(-wvu^{-1}\right)^n.
\end{aligned}$$

Finally,

$$\begin{aligned}
\left\|-u^{-1}vu^{-1}\sum_{n=1}^{\infty}\left(-wvu^{-1}\right)^n\right\| &\leq \|v\|\|u^{-1}\|^2\sum_{n=1}^{\infty}\left(\|w\|\|v\|\|u^{-1}\|\right)^n \\
&= \|v\|\|u^{-1}\|^2\frac{\|w\|\|v\|\|u^{-1}\|}{1-\|w\|\|v\|\|u^{-1}\|} \to 0
\end{aligned}$$

as $w \to 0$, which implies that

$$\lim_{w\to 0} w^{-1}\left((u+wv)^{-1}-u^{-1}\right) = -u^{-1}vu^{-1}.$$

6.1.1.5 Generalized series

Observe that if I is a set, then $\mathcal{P}_{\mathtt{f}}(I)$ is an upward directed set when endowed with the inclusion.

Definition 322 (Generalized series) *Let M be a topological module over a topological ring R. Let I be a set. Consider an element $(a_i)_{i\in I} \in M^I$. The generalized series $\sum_{i\in I} a_i$ is defined as the net*

$$\begin{array}{rcl} \mathcal{P}_{\mathtt{f}}(I) & \to & M \\ A & \mapsto & \displaystyle\sum_{i\in A} a_i. \end{array}$$

The set of limit points of this net is called the summation of the generalized series $\sum_{i\in I} a_i$.

For simplification purposes and under the settings of the previous definition, we will let $\sum_{i\in I} a_i$ denote the set of its limit points.

Theorem 161 *Let M be a topological module over a topological ring R. Consider two convergent generalized series $\sum_{i\in I} a_i$ and $\sum_{i\in I} b_i$ in M. Let $r \in R$. Let $J \subseteq I$. Then:*

1. $\displaystyle\sum_{i\in I}(a_i + b_i) = \sum_{i\in I} a_i + \sum_{i\in I} b_i$.
2. $\displaystyle r\sum_{i\in I} a_i \subseteq \sum_{i\in I} ra_i$.
3. *If $r \in \mathcal{U}(R)$, then* $\displaystyle r\sum_{i\in I} a_i = \sum_{i\in I} ra_i$.
4. *If* $\displaystyle\sum_{j\in J} a_j$ *is convergent, then so is* $\displaystyle\sum_{k\in I\setminus J} a_k$ *and* $\displaystyle\sum_{k\in I\setminus J} a_k = \sum_{i\in I} a_i - \sum_{j\in J} a_j$.

Proof 491 *As always, the proof is itemized by following the statement.*

1. *Fix arbitrary elements $a \in \sum_{i\in I} a_i$ and $b \in \sum_{i\in I} b_i$. Take a 0-neighborhood $W \subseteq M$. Let $V \subseteq M$ be an additively symmetric 0-neighborhood such that $V + V \subseteq W$. There exist $A_1, A_2 \in \mathcal{P}_{\mathtt{f}}(I)$ such that, if $A \in \mathcal{P}_{\mathtt{f}}(I)$ and $A \supseteq A_1$, then $\sum_{i\in A} a_i \in a + V$, and if $B \in \mathcal{P}_{\mathtt{f}}(I)$ and $B \supseteq A_2$, then $\sum_{i\in B} b_i \in b + V$. Thus, if $C \in \mathcal{P}_{\mathtt{f}}(I)$ and $C \supseteq A_0 := A_1 \cup A_2$, then*

$$\sum_{i\in C}(a_i + b_i) = \sum_{i\in C} a_i + \sum_{i\in C} b_i \in (a + V) + (b + V) \subseteq (a + b) + W.$$

   This shows that $a+b \in \sum_{i\in I}(a_i+b_i)$. The arbitrariness of $a \in \sum_{i\in I} a_i$ and $b \in \sum_{i\in I} b_i$ implies that $\sum_{i\in I} a_i+\sum_{i\in I} b_i \subseteq \sum_{i\in I}(a_i+b_i)$. Following a similar reasoning, it can be shown that $\sum_{i\in I}(a_i + b_i) - \sum_{i\in I} a_i \subseteq \sum_{i\in I} b_i$. In view of Remark 36(1), we conclude that $\sum_{i\in I}(a_i + b_i) = \sum_{i\in I} a_i + \sum_{i\in I} b_i$.

2. *Fix an arbitrary element $a \in \sum_{i\in I} a_i$. Take a 0-neighborhood $W \subseteq M$. Let $V \subseteq M$ be an additively symmetric 0-neighborhood such that $rV \subseteq W$. There exists $A_0 \in \mathcal{P}_{\mathtt{f}}(I)$ such that, if $A \in \mathcal{P}_{\mathtt{f}}(I)$ and $A \supseteq A_0$, then $\sum_{i\in A} a_i \in a + V$. Thus, if $A \in \mathcal{P}_{\mathtt{f}}(I)$ and $A \supseteq A_0$, then*

$$\sum_{i\in A} ra_i = r\sum_{i\in A} a_i \in r\,(a + V) = ra + rV \subseteq ra + W.$$

   This shows that $ra \in \sum_{i\in I} ra_i$. The arbitrariness of $a \in \sum_{i\in I} a_i$ implies that $r\sum_{i\in I} a_i \subseteq \sum_{i\in I} ra_i$.

3. *We know from the previous item that $r\sum_{i\in I} a_i \subseteq \sum_{i\in I} ra_i$ and $r^{-1}\sum_{i\in I} ra_i \subseteq \sum_{i\in I} a_i$. As a consequence, $r\sum_{i\in I} a_i = \sum_{i\in I} ra_i$.*

4. *Fix arbitrary values $a \in \sum_{i\in I} a_i$ and $b \in \sum_{j\in J} a_j$. Let U be a neighborhood of 0 in M. We can fix an additively symmetric neighborhood V of 0 such that $V + V \subseteq U$. There are $C_0 \in \mathcal{P}_{\mathtt{f}}(I), D_0 \in \mathcal{P}_{\mathtt{f}}(J)$ such that, if $C \in \mathcal{P}_{\mathtt{f}}(I), D \in \mathcal{P}_{\mathtt{f}}(J)$ and $C \supseteq C_0, D \supseteq D_0$, then*

$$\sum_{i\in C} a_i \in a + V \quad \text{and} \quad \sum_{j\in J} a_j \in b + V.$$

   Take $K_0 := C_0 \setminus J \in \mathcal{P}_{\mathtt{f}}(I \setminus J)$. Observe that, if $K \in \mathcal{P}_{\mathtt{f}}(I \setminus J)$ with $K \supseteq K_0$ and

$D := (C_0 \cap J) \cup D_0$, then $D \in \mathcal{P}_{\mathtt{f}}(J)$, $D \supseteq D_0$, $K \cup D \in \mathcal{P}_{\mathtt{f}}(I)$ and $K \cup D \supseteq K_0 \cup (C_0 \cap J) = (C_0 \setminus J) \cup (C_0 \cap J) = C_0$. As a consequence,

$$\begin{aligned}\sum_{k\in K} a_k &= \sum_{i\in K\cup D} a_i - \sum_{j\in D} a_j \in (a+V)+(-b-V)\\ &= (a-b)+(V+V) \subseteq (a-b)+U.\end{aligned}$$

This shows that $a-b \in \sum_{k\in I\setminus J} a_k$, so the arbitrariness of $a \in \sum_{i\in I} a_i$ and $b \in \sum_{j\in J} a_j$ means

$$\sum_{i\in I} a_i - \sum_{j\in J} a_j \subseteq \sum_{k\in I\setminus J} a_k.$$

Following a similar reasoning, it can be proved that

$$\sum_{i\in I} a_i - \sum_{k\in I\setminus J} a_k \subseteq \sum_{j\in J} a_j.$$

As a consequence, Remark 36(2) assures that

$$\sum_{k\in I\setminus J} a_k = \sum_{i\in I} a_i - \sum_{j\in J} a_j.$$

The final remark in this subsection highlights that the convergence in Definition 322 is much stronger than the convergence in Definition 318 when $I := \mathbb{N}$. In fact, as highlighted in Theorem 157, convergence in Definition 322 is, in some sense, unconditional convergence. The following remark is then Theorem 157 reformulated in terms of generalized series.

Remark 91 *Let M be a topological module over a topological ring R. If $\sum_{i\in\mathbb{N}} m_i$ is a generalized series in M, then*

$$\sum_{i\in\mathbb{N}} m_i = \bigcap_{\pi\in\Pi(\mathbb{N})} \sum_{i=1}^{\infty} m_{\pi(i)},$$

where, here, $\sum_{i=1}^{\infty} m_{\pi(i)}$ stands for the usual series.

The classical characterizations of unconditional convergence via unital multipliers is versioned now in the scope of uniform spaces.

Theorem 162 *Let M a topological module over a topological ring R. A generalized series $\sum_{i\in\mathbb{N}} m_i$ in M is Cauchy (as a net) if and only if $\sum_{i=1}^{\infty} \varepsilon_i m_i$ is Cauchy (as a sequence) for each $(\varepsilon_i)_{i\in\mathbb{N}} \in \{-1,0,1\}^{\mathbb{N}}$.*

Proof 492 *The proof is itemized by following the itemization of the statement.*

⇒ *Take any $(\varepsilon_i)_{i\in\mathbb{N}} \in \{-1,0,1\}^{\mathbb{N}}$. Fix an arbitrary additively symmetric neighborhood V of 0 in M. There exists a finite $A_V \subseteq \mathbb{N}$ such that $\sum_{i\in B} m_i - \sum_{i\in C} m_i \in V$ whenever B, C are finite subsets of $\mathbb{N}$ containing A_V. Take $n_V := \max(A_V)$. If $p \geq q \geq n_V$, then by taking $B := A_V \cup \{i \in \mathbb{N} : q+1 \leq i \leq p,\ \varepsilon_i \in \{0,1\}\}$ and $C := A_V \cup \{i \in \mathbb{N} : q+1 \leq i \leq p,\ \varepsilon_i \in \{-1,0\}\}$, we obtain that*

$$\sum_{i=1}^{p} \varepsilon_i m_i - \sum_{i=1}^{q} \varepsilon_i m_i = \sum_{i=q+1}^{p} \varepsilon_i m_i = \sum_{i\in B} m_i - \sum_{i\in C} m_i \in V.$$

As a consequence, $\sum_{i=1}^{\infty} \varepsilon_i m_i$ is Cauchy.

$\Leftarrow$ *Suppose on the contrary that* $\sum_{i\in\mathbb{N}} m_i$ *is not Cauchy as a generalized series. There exists an additively symmetric* 0*-neighborhood* U *in* M *such that, for every finite subset* $A \subseteq \mathbb{N}$*, two finite subsets* $B, C \subseteq \mathbb{N}$*, containing* A*, can be found in such a way that* $\sum_{i\in B} m_i - \sum_{i\in C} m_i \notin U$*. We will inductively construct* $(\varepsilon_i)_{i\in\mathbb{N}} \in \{-1,0,1\}^{\mathbb{N}}$ *for which* $\sum_{i=1}^{\infty} \varepsilon_i m_i$ *is not Cauchy. Indeed:*

1. *For* $A := \{1\}$*, there are finite subsets* $B_1, C_1 \subseteq \mathbb{N}$ *containing* A *such that* $\sum_{i\in B_1} m_i - \sum_{i\in C_1} m_i \notin U$*. Here, we take*

$$\varepsilon_i := \begin{cases} 1 & i \in B_1 \setminus C_1 \\ -1 & i \in C_1 \setminus B_1 \\ 0 & \text{otherwise} \end{cases}$$

   for $1 \leq i \leq \max(B_1 \cup C_1)$*.*

2. *For* $A := \{1, \ldots, \max(B_1 \cup C_1)\}$*, there are finite subsets* $B_2, C_2 \subseteq \mathbb{N}$ *containing* A *such that* $\sum_{i\in B_2} m_i - \sum_{i\in C_2} m_i \notin U$*. Here, we take*

$$\varepsilon_i := \begin{cases} 1 & i \in B_2 \setminus C_2 \\ -1 & i \in C_2 \setminus B_2 \\ 0 & \text{otherwise} \end{cases}$$

   for $\max(B_1 \cup C_1) + 1 \leq i \leq \max(B_2 \cup C_2)$*.*

$\vdots$ *And so on.*

By repeating this process, $(\varepsilon_i)_{i\in\mathbb{N}} \in \{-1,0,1\}^{\mathbb{N}}$ *is constructed in such a way that* $\sum_{i=1}^{\infty} \varepsilon_i m_i$ *is not Cauchy. Indeed, fix any* $n \in \mathbb{N}$*. Let* $m \in \mathbb{N}$ *with* $n \leq \max(B_m \cup C_m)$*. Take* $p := \max(B_{m+1} \cup C_{m+1})$ *and* $q := \max(B_m \cup C_m)\}$*. Then*

$$\begin{aligned} \sum_{i=1}^{p} \varepsilon_i m_i - \sum_{i=1}^{q} \varepsilon_i m_i &= \sum_{i=q+1}^{p} \varepsilon_i m_i = \sum_{i\in B_{m+1}\setminus C_{m+1}} m_i \\ - \sum_{i\in C_{m+1}\setminus B_{m+1}} m_i &= \sum_{i\in B_{m+1}} m_i - \sum_{i\in C_{m+1}} m_i \notin U. \end{aligned}$$

6.1.2 Biorthogonal systems

Schauder bases in Banach spaces and orthonormal bases in Hilbert spaces are particular cases of biorthogonal systems and constitute the foundations of Fourier Analysis. This is why it is important to study them. We refer the reader to [92, 93] for a wider perspective on biorthogonal systems in real or complex normed spaces. Here, we will deal with biorthogonal systems in topological modules.

6.1.2.1 Markushevich bases

In the study of biorthogonal systems, the series span is a crucial notion that must be previously introduced.

Definition 323 (Series span) *Let* M *be a topological module over a topological ring* R*. For every* $G \subseteq M$*, the series span of* G *is defined as*

$$\operatorname{sspan}(G) := \left\{ m \in M : \exists (r_n)_{n\in\mathbb{N}} \subseteq R \; \exists (g_n)_{n\in\mathbb{N}} \subseteq G \;\; m \in \sum_{n=1}^{\infty} r_n g_n \right\}.$$

Under the settings of the previous definition, $\operatorname{span}(G) \subseteq \operatorname{sspan}(G) \subseteq \overline{\operatorname{span}}(G)$. Recall that δ_{ij} is the Kronecker δ, that is

$$\delta_{ij} := \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Definition 324 (Biorthogonal system) *Let R be a topological ring. Let M be a topological R-module. A biorthogonal system is a family of pairs $(e_i, e_i^*)_{i\in I} \subseteq M \times M^*$ such that $e_i^*(e_j) = \delta_{ij}$ for all $i, j \in I$. A biorthogonal system is said to be:*

- *expanding if $M = \operatorname{span}\{e_i : i \in I\}$;*
- *series-expanding if $M = \operatorname{sspan}\{e_i : i \in I\}$;*
- *fundamental if $M = \overline{\operatorname{span}}\{e_i : i \in I\}$; and*
- *total if $M^* = \overline{\operatorname{span}}^{w^*}\{e_i^* : i \in I\}$.*

The most immediate property satisfied by biorthogonal systems is the fact that they are linearly independent sets.

Lemma 169 *Let R be a topological ring. Let M be a topological R-module. If $(e_i, e_i^*)_{i\in I} \subseteq M \times M^*$ is a biorthogonal system, then $\{e_i : i \in I\}$ and $\{e_i^* : i \in I\}$ are linearly independent.*

Proof 493 *Consider $i_1, \ldots, i_k \in I$ all different and $r_1, \ldots, r_k \in R$ such that $r_1 e_{i_1} + \cdots + r_k e_{i_k} = 0$. For every $j \in \{1, \ldots, k\}$, we have that*

$$0 = e_{i_j}^* \left(r_1 e_{i_1} + \cdots + r_k e_{i_k}\right) = r_j.$$

This shows the linear independence of $\{e_i : i \in I\}$. In a similar way, $\{e_i^* : i \in I\}$ is also linearly independent.

The following proposition unveils the most basic properties satisfied by biorthogonal systems.

Proposition 126 *Let M be a Hausdorff topological module over a Hausdorff topological ring R. Let $(e_i, e_i^*)_{i\in I} \subseteq M \times M^*$ be a biorthogonal system. Then:*

1. *If $(e_i, e_i^*)_{i\in I}$ is fundamental, then $\{e_i : i \in I\}^\perp = \{0\}$.*
2. *If M is strongly torsionless and $\{e_i : i \in I\}^\perp = \{0\}$, then $(e_i, e_i^*)_{i\in I}$ is fundamental.*
3. *If M is torsionless and $(e_i, e_i^*)_{i\in I}$ is total, then $\{e_i^* : i \in I\}^\top = \{0\}$.*
4. *If $(e_i, e_i^*)_{i\in I}$ is series-expanding, then $\{e_i^* : i \in I\}^\top = \{0\}$.*
5. *If M is w^*-strongly torsionless and $\{e_i^* : i \in I\}^\top = \{0\}$, then $(e_i, e_i^*)_{i\in I}$ is total.*

Proof 494 *The proof is as itemized as the statement is.*

1. *By Proposition 95(1), $M = \overline{\operatorname{span}}\{e_i : i \in I\} \subseteq \left(\{e_i : i \in I\}^\perp\right)^\top \subseteq M$. Therefore, $\left(\{e_i : i \in I\}^\perp\right)^\top = M$. As a consequence, $\{e_i : i \in I\}^\perp = \{0\}$.*
2. *In view of Proposition 95(3), $\overline{\operatorname{span}}\{e_i : i \in I\} = \left(\{e_i : i \in I\}^\perp\right)^\top = \{0\}^\top = M$.*

3. *According to Proposition 95(2),* $M^* = \overline{\text{span}}^{w^*}\{e_i^* : i \in I\} \subseteq \left(\{e_i^* : i \in I\}^\top\right)^\perp \subseteq M^*$. *Therefore,* $\left(\{e_i^* : i \in I\}^\top\right)^\perp = M^*$. *By hypothesis,* M *is torsionless and Hausdorff, which implies that* $\{e_i^* : i \in I\}^\top = \{0\}$.

4. *Choose any* $m \in M$ *with* $m \in \{e_i^* : i \in I\}^\top = \bigcap_{i\in I} \ker(e_i^*)$. *By hypothesis,* $m \in \text{sspan}\{e_i : i \in I\}$ *and* M *is Hausdorff, hence there exists a sequence* $(r_n)_{n\in\mathbb{N}} \subseteq R$ *and a sequence* $(i_n)_{n\in\mathbb{N}} \subseteq I$ *such that* $m = \sum_{n=1}^{\infty} r_n e_{i_n}$. *Next, since* R *is Hausdorff and* $e_{i_k}^*$ *is continuous for all* $k \in \mathbb{N}$, *we conclude that* $0 = e_{i_k}^*(m) = \sum_{n=1}^{\infty} r_n e_{i_k}^*(e_{i_n}) = r_k$ *for all* $k \in \mathbb{N}$. *As a consequence,* $m = 0$, *meaning that* $\{e_i^* : i \in I\}^\top = \{0\}$.

5. *In view of Proposition 95(4),* $\overline{\text{span}}^{w^*}\{e_i^* : i \in I\} = \left(\{e_i^* : i \in I\}^\top\right)^\perp = \{0\}^\perp = M^*$.

If M is a topological module over a division ring R and $(e_i)_{i\in I} \subseteq M$ satisfies the condition that $e_j \notin \text{span}\{e_i : i \in I \setminus \{j\}\}$ for all $j \in I$, then $\{e_i : i \in I\}$ is linearly independent. This fact, together with the notion of biorthogonal system, motivate the concept of minimal system.

Definition 325 (Minimal system) *Let* M *be a topological module over a topological ring* R. *A minimal system in* M *is a linearly independent subset* $\{e_i : i \in I\} \subseteq M$ *in such a way that* $e_j \notin \overline{\text{span}}\{e_i : i \in I \setminus \{j\}\}$ *for all* $j \in I$.

The relation between minimal systems and biorthogonal systems is described in the upcoming proposition.

Proposition 127 *Let* M *be a topological module over a Hausdorff topological ring* R. *Then:*

1. *If* $(e_i, e_i^*)_{i\in I} \subseteq M \times M^*$ *is a biorthogonal system, then* $\{e_i : i \in I\}$ *is a minimal system.*

2. *If* R *is a division ring,* M *is strongly torsionless, and* $\{e_i : i \in I\} \subseteq M$ *is a minimal system, then there exists a subset* $\{e_i^* : i \in I\} \subseteq M^*$ *such that* $(e_i, e_i^*)_{i\in I}$ *is a biorthogonal system.*

Proof 495 *Let us itemize the proof as the statement is.*

1. *Fix an arbitrary* $j \in I$. *Notice that* $\text{span}\{e_i : i \in I \setminus \{j\}\} \subseteq \ker(e_j^*)$. *Since* R *is Hausdorff,* $\{0\}$ *is closed in* R, *and so is* $\ker(e_j^*)$ *in* M; *hence* $\overline{\text{span}}\{e_i : i \in I \setminus \{j\}\} \subseteq \ker(e_j^*)$, *meaning that* $e_j \notin \overline{\text{span}}\{e_i : i \in I \setminus \{j\}\}$.

2. *Fix an arbitrary* $j \in I$. *Since* M *is strongly torsionless, the condition that* $e_j \notin \overline{\text{span}}\{e_i : i \in I \setminus \{j\}\}$ *implies the existence of* $m_j^* \in \overline{\text{span}}\{e_i : i \in I \setminus \{j\}\}^\perp \setminus \{e_j\}^\perp$. *Next, it only suffices to take* $e_j^* := m_j^* m_j^*(e_j)^{-1}$. *As a consequence,* $(e_i, e_i^*)_{i\in I}$ *is a biorthogonal system.*

Following the classical literature on biorthogonal systems in real or complex normed spaces, a fundamental and total biorthogonal system is called a Markushevich basis.

Definition 326 (Markushevich basis) *Let* R *be a topological ring. Let* M *be a topological* R*-module. A Markushevich basis is a fundamental and total biorthogonal system* $(e_i, e_i^*)_{i\in I} \subseteq M \times M^*$.

Next result provides sufficient conditions to assure the existence of countable biorthogonal systems.

Theorem 163 *Let R be a Hausdorff topological division ring. Let M be a Hausdorff topological R-module. If M is strongly torsionless and countably spanned, then there exists a countable biorthogonal system $(e_i, e_i^*)_{i\in N} \subseteq M \times M^*$ such that $M = \operatorname{span}\{e_i : i \in N\}$. If, in addition, M is w^*-strongly torsionless, then $M^* = \overline{\operatorname{span}}^{\,\omega^*}\{e_i^* : i \in N\}$.*

Proof 496 *Since R is a division ring, Zorn's Lemma assures the existence of a countable basis $(u_n)_{n\in N} \subseteq M$ in the sense that it is linearly independent and a spanning system for M, where $N \subseteq \mathbb{N}$ can be finite or the whole of $\mathbb{N}$. According to Theorem 107, the finitely spanned submodules of M are closed. We will construct the biorthogonal system inductively according to the following steps:*

1. *Take $e_1 := u_1$. Obviously, $\operatorname{span}\{e_1\} = \operatorname{span}\{u_1\}$. Since M is Hausdorff, we have that $\{0\}$ is closed; hence, by hypothesis, there exists $u_1^* \in \{0\}^\perp$ such that $u_1^*(e_1) \neq 0$. Then $e_1^* := u_1^* u_1^*(e_1)^{-1}$. This way $e_1^*(e_1) = 1$.*

2. *Take $e_2 := u_2 - e_1^*(u_2)\, e_1$. Note that $\operatorname{span}\{e_1, e_2\} = \operatorname{span}\{u_1, u_2\}$. Since $\operatorname{span}\{e_1\}$ is a closed submodule, by hypothesis we can find $u_2^* \in \operatorname{span}\{e_1\}^\perp$ such that $u_2^*(e_2) \neq 0$. Then $e_2^* := u_2^* u_2^*(e_2)^{-1}$. This way $e_i^*(e_j) = \delta_{ij}$ for $i, j \leq 2$.*

3. *Take $e_3 := u_3 - e_1^*(u_3)\, e_1 - e_2^*(u_3)\, e_2$. Observe that $\operatorname{span}\{e_1, e_2, e_3\} = \operatorname{span}\{u_1, u_2, u_3\}$. Since $\operatorname{span}\{e_1, e_2\}$ is closed, by hypothesis there exists $u_3^* \in \operatorname{span}\{e_1, e_2\}^\perp$ such that $e_3^*(e_3) \neq 0$. Then $e_3^* := u_3^* u_3^*(e_3)^{-1}$. This way $e_i^*(e_j) = \delta_{ij}$ for $i, j \leq 3$.*

⋮ *And so on.*

The above inductive process assures that $(e_i, e_i^*)_{i\in N} \subseteq M \times M^*$ is a biorthogonal system such that $M = \operatorname{span}\{e_i : i \in N\}$. Finally, let us assume that M is w^*-strongly torsionless. According to Proposition 95(4) and by bearing in mind that $M = \operatorname{span}\{e_i : i \in N\}$, we conclude that

$$\overline{\operatorname{span}}^{\,\omega^*}\{e_i^* : i \in N\} = \left(\{e_i^* : i \in N\}^\top\right)^\perp = \{0\}^\perp = M^*.$$

6.1.2.2 Schauder bases

Schauder bases were introduced for the first time in [94] in the setting of real or complex Banach spaces. In [95], Schauder bases were transported to the scope of topological vector spaces over absolutely valued division rings. Here, we will extend PreSchauder bases to the scope of topological modules.

Definition 327 (PreSchauder basis) *Let R be a topological ring. Let M be a topological R-module. A preSchauder basis is a sequence $(e_n)_{n\in\mathbb{N}} \subseteq M$ such that, for every $m \in M$, there exists a unique sequence $(r_n)_{n\in\mathbb{N}} \subseteq R$ with $m \in \sum_{n=1}^{\infty} r_n e_n$.*

For every $n \in \mathbb{N}$, the coefficient functionals and the canonical projections are, respectively, defined by

$$\begin{array}{rccc} e_n^* : & M & \to & R \\ & m & \mapsto & r_n \end{array} \tag{6.2}$$

and

$$\begin{array}{rccc} p_n : & M & \to & \operatorname{span}\{e_1, \ldots, e_n\} \\ & m & \mapsto & \sum_{i=1}^{n} r_i e_i. \end{array} \tag{6.3}$$

Proposition 128 *Let R be a topological ring. Let M be a topological R-module. Let $(e_n)_{n\in\mathbb{N}} \subseteq M$ be a preSchauder basis. Let $m \in M$. Then:*

1. *For every $n \in \mathbb{N}$, e_n^* and p_n are R-linear.*

2. *$(e_n, e_n^*)_{n\in\mathbb{N}}$ is a fundamental biorthogonal system (except for the continuity of the e_i^*'s).*

3. *If R, M are Hausdorff, then $\{e_i^* : i \in I\}^\top = \{0\}$.*

4. *$\{e_i^* : i \in I\}^\top \subseteq \mathscr{O}_M$. If e_i^* is continuous for all $i \in \mathbb{N}$, then $\{e_i^* : i \in I\}^\top = \mathscr{O}_M$.*

5. *If $m \in \operatorname{span}\{e_1, \ldots, e_k\}$ and $m = r_1e_1 + \cdots + r_ke_k$ with $r_1, \ldots, r_k \in R$, then*

$$e_n^*(m) = \begin{cases} r_k & 1 \leq n \leq k, \\ 0 & n > k. \end{cases}$$

6. *If $e_n \in M^*$ for all $n \in \mathbb{N}$, then p_n is continuous for all $n \in \mathbb{N}$.*

7. *$\bigcap \mathcal{N}_m = \{p \in M : \forall n \in \mathbb{N}\ e_n^*(p) = e_n^*(m)\}$. Therefore, $\bigcap \mathcal{N}_0 \cap \operatorname{span}\{e_n : n \in \mathbb{N}\} = \{0\}$. Hence, $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is Hausdorff.*

Proof 497 *The proof is itemized via the itemization of the statement.*

1. *Take $m_1, m_2 \in M$. Let $(r_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subseteq R$ be the only sequences such that $m_1 \in \sum_{n=1}^\infty r_ne_n, m_2 \in \sum_{n=1}^\infty s_ne_n$, respectively. In view of Lemma 162(1),*

$$m_1 + m_2 \in \sum_{n=1}^\infty r_ne_n + \sum_{n=1}^\infty s_ne_n = \sum_{n=1}^\infty (r_ne_n + s_ne_n) = \sum_{n=1}^\infty (r_n + s_n)e_n.$$

The uniqueness in the PreSchauder condition means that $e_n^*(m_1 + m_2) = r_n + s_n = e_n^*(m_1) + e_n^*(m_2)$ for all $n \in \mathbb{N}$. Now, take $r \in R$. By applying Lemma 162(2), we have that

$$rm_1 \in r\sum_{n=1}^\infty r_ne_n \subseteq \sum_{n=1}^\infty r(r_ne_n) = \sum_{n=1}^\infty (rr_n)e_n.$$

By relying again on the uniqueness in the PreSchauder condition, we obtain that

$$e_n^*(rm_1) = rr_n = re_n^*(m_1)$$

for all $n \in \mathbb{N}$. This shows the R-linearity of e_n^* for all $n \in \mathbb{N}$. Finally, by taking into account that

$$p_n(m) = e_1^*(m)e_1 + \cdots + e_n^*(m)e_n$$

for all $n \in \mathbb{N}$ and all $m \in M$, we conclude that p_n is R-linear for all $n \in \mathbb{N}$.

2. *Indeed, for every $n_0 \in \mathbb{N}$, it is clear that $e_{n_0} \in \sum_{n=1}^\infty r_ne_n$ where $r_n = \delta_{nn_0}$. The uniqueness of the sequence $(r_n)_{n\in\mathbb{N}} \subset R$ implies that $e_i^*(e_j) = \delta_{ij}$ for all $i, j \in \mathbb{N}$. It is immediate that $(e_n, e_n^*)_{n\in\mathbb{N}}$ is fundamental because $m \in \sum_{n=1}^\infty e_n^*(m)e_n \subseteq \overline{\operatorname{span}}\{e_i : i \in I\}$.*

3. *Take $m \in \{e_i^* : i \in I\}^\top$. Then $e_n^*(m) = 0$ for every $n \in \mathbb{N}$ because R is Hausdorff. Since M is Hausdorff, $m = 0$.*

4. *Fix an arbitrary $m \in \{e_i^* : i \in I\}^\top = \bigcap_{i\in\mathbb{N}} \ker_{\mathfrak{t}}(e_i^*)$. Then $e_i^*(m) \in \mathscr{O}_R$ for all $i \in \mathbb{N}$. Therefore, $e_i^*(m)m \in \mathscr{O}_M$ for all $i \in \mathbb{N}$. Since $\mathscr{O}_M$ is a submodule of M, we have that $\sum_{i=1}^n e_i^*(m)m \in \mathscr{O}_M$ for each $n \in \mathbb{N}$. Next, $\mathscr{O}_M$ is closed in M, meaning that $m \in \sum_{i=1}^\infty e_i^*(m)m \subseteq \mathscr{O}_M$. Finally, if we assume the continuity of e_i^* for all $i \in \mathbb{N}$, then $\mathscr{O}_M \subseteq \bigcap_{i\in\mathbb{N}} \ker_{\mathfrak{t}}(e_i^*) = \{e_i^* : i \in I\}^\top$.*

5. *It is a direct consequence of the PreSchauder uniqueness.*

6. *It suffices to notice that* $p_n(m) = e_1^*(m)e_1 + \cdots + e_n^*(m)e_n$ *for all* $n \in \mathbb{N}$ *and all* $m \in M$.

7. *Fix an arbitrary* $p \in \bigcap \mathcal{N}_m$. *Notice that* $m \in \bigcap \mathcal{N}_p$ *in view of Lemma 49. Therefore,* $\sum_{n=1}^{\infty} e_n^*(m)e_n$ *converges to* p *as well. Hence,* $e_n^*(p) = e_n^*(m)$ *for all* $n \in \mathbb{N}$ *in view of the PreSchuder uniqueness. Conversely, assume that* $p \in M$ *verifies that* $e_n^*(p) = e_n^*(m)$ *for all* $n \in \mathbb{N}$. *This means that* $\sum_{n=1}^{\infty} e_n^*(m)e_n$ *converges to* p, *meaning that* $p \in \bigcap \mathcal{N}_m$. *As a consequence,*

$$\bigcap \mathcal{N}_0 = \{m \in M : \forall n \in \mathbb{N} \;\; e_n^*(m) = 0\}. \tag{6.4}$$

Finally, let us show that $\bigcap \mathcal{N}_0 \cap \operatorname{span}\{e_n : n \in \mathbb{N}\} = \{0\}$. *If* $m \in \bigcap \mathcal{N}_0 \cap \operatorname{span}\{e_n : n \in \mathbb{N}\}$, *then there are* $r_1, \ldots, r_k \in R$ *such that* $m = r_1e_1 + \cdots + r_ke_k$. *By* (6.4), $0 = e_n^*(m) = r_n$ *for all* $1 \leq n \leq k$. *Thus,* $m = 0$. *The fact that* $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ *is Hausdorff is now a direct consequence of Corollary 15(6).*

A consequence of the previous theorem is the equivalence of the continuity of coefficient functionals and canonical projections.

Corollary 65 *Let* M *be a strongly torsionfree topological module over an integral domain, Hausdorff topological ring* R. *Let* $(e_n)_{n\in\mathbb{N}} \subseteq M$ *be a preSchauder basis. The following conditions are equivalent:*

1. *The coefficient functionals are all continuous.*

2. *The canonical projections are all continuous.*

Proof 498 *First off, we already know from Proposition 128 that, if the coefficient functionals are all continuous, then the canonical projections are all continuous. So, suppose conversely that the canonical projections are all continuous. Observe that* $e_1^*(m)e_1 = p_1(m)$ *and* $e_n^*(m)e_n = p_n(m) - p_{n-1}(m)$ *for all* $m \in M$ *and all* $n > 1$. *In other words, the map* $m \mapsto e_n^*(m)e_n$ *is continuous for all* $n \in \mathbb{N}$. *Since* M *is strongly torsionfree and* R *is a Hausdorff integral domain,* $re_n \mapsto r$ *is continuous for all* $n \in \mathbb{N}$. *Finally, the composition* $m \mapsto e_n^*(m)e_n \mapsto e_n^*(m)$ *is continuous for each* $n \in \mathbb{N}$. *However, the previous composition is nothing else but* e_n^*. *As a consequence,* $e_n^* \in M^*$ *for all* $n \in \mathbb{N}$.

Next result is a technical proposition that deals with boundedness conditions for the canonical projections.

Proposition 129 *Let* M *be a seminormed module over a unital seminormed ring* R. *Let* $B^+ \subseteq \mathsf{B}_R$ *be a regular unit segment. Consider a preSchauder basis* $(e_n)_{n\in\mathbb{N}} \subseteq M$. *Let* $V := \{m \in M : \forall n \in \mathbb{N} \; \|p_n(m)\| \leq 1\}$. *Then* V *is* B_R*-balanced and convex. If, in addition,* R *is practical, then* V *is absorbing.*

Proof 499 *We will follow several steps:*

- V *is* B_R*-balanced. Indeed, if* $m \in V$ *and* $r \in \mathsf{B}_R$, *then* $\|p_n(rm)\| = \|rp_n(m)\| \leq \|r\|\|p_n(m)\| \leq 1$ *for all* $n \in \mathbb{N}$. *This shows that* $\mathsf{B}_R V \subseteq V$, *hence* V *is* B_R*-balanced.*

- V *is convex. Indeed, take* $r_1, \ldots, r_k \in B^+$ *with* $r_1 + \cdots + r_k = 1$ *and* $m_1, \ldots, m_k \in M$. *Since* B^+ *is regular, we have that* $\|r_1\| + \cdots + \|r_k\| = 1$. *Thus,* $\|p_n(r_1m_1 + \cdots + r_km_k)\| = \|r_1p_n(m_1) + \cdots + r_kp_n(m_k)\| \leq \|r_1\|\|p_n(m_1)\| + \cdots + \|r_k\|\|p_n(m_k)\| \leq \|r_1\| + \cdots + \|r_k\| = 1$. *This shows that* $\Delta(m_1, \ldots, m_k) \subseteq V$. *As a consequence,* V *is convex.*

- *V is absorbing (assuming that R is practical). Indeed, fix an arbitrary $m \in M$. If $\|p_n(m)\| = 0$ for all $n \in \mathbb{N}$, then $m \in V$ and hence $\mathsf{B}_R m \subseteq V$. Thus, we may assume that $\|p_n(m)\| > 0$ for some $n \in \mathbb{N}$. Since $(p_n(m))_{n\in\mathbb{N}}$ is a convergent sequence to m, we have that $(\|p_n(m)\|)_{n\in\mathbb{N}}$ converges to $\|m\|$, so $(\|p_n(m)\|)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Since R is practical, there exists $u \in \mathcal{U}(R)$ with*

$$\|u\| \leq \frac{1}{\sup\{\|p_n(m)\| : n \in \mathbb{N}\}}.$$

Let us show that $u\mathsf{B}_R m \subseteq V$, which proves that V is absorbing. Indeed, if $r \in B$ and $n \in \mathbb{N}$, then $\|p_n(urm)\| \leq \|u\|\|r\|\|p_n(m)\| \leq 1$.

Under the settings of the previous definition, if R is practical and the canonical projections are all continuous, then V is closed, hence a barrel. The end of this section is reserved for the definition of Schauder basis, one of the most remarkable notion of classical Functional Analysis.

Definition 328 (Schauder basis) *Let R be a topological ring. Let M be a topological R-module. A preSchauder basis $(e_n)_{n\in\mathbb{N}} \subseteq M$ is called a Schauder basis if the coefficient functionals are all continuous, that is, $e_n^* \in M^*$ for all $n \in \mathbb{N}$.*

6.2 Convergence and summability methods

This section is devoted to the study of the different methods of convergence and summability. In the classical Summability Theory, there are multiple methods of convergence and summability, involving different objects such as filters, operators, etc. Here, we will try to unify all of them as much as possible. This unification will be carried out in the context of topological modules.

6.2.1 Sequence spaces

Classical sequence spaces, such as c_0, c, ℓ_p, constitute a fundamental arm of classical Functional Analysis [96, 97, 98]. These spaces can be transported to the topological vector space setting. Here, we will extend them to the topological module framework.

6.2.1.1 Coordinate maps

Sequence spaces are the main objects on which convergence and summability methods are defined and studied. What characterizes sequence spaces are the coordinate maps along with the convergence linear topologies. In fact, sequence spaces are inspired by function spaces where the domain of the functions is the naturals.

Definition 329 (Sequence space) *Let R be a topological ring. Let M be a topological R-module. An M-sequence space is simply a submodule $\mathcal{V} \subseteq M^{\mathbb{N}}$ endowed with a module topology for which the coordinate maps*

$$\begin{array}{rccl} \delta_k : & \mathcal{V} & \to & M \\ & (m_n)_{n\in\mathbb{N}} & \mapsto & \delta_k\left((m_n)_{n\in\mathbb{N}}\right) := m_k \end{array}$$

are continuous for all $k \in \mathbb{N}$.

Under the settings of Definition 329, any module topology on $\mathcal{V} \subseteq M^{\mathbb{N}}$ for which the coordinate maps are continuous is precisely any module topology on $\mathcal{V}$ finer than the inherited product topology on $\mathcal{V}$, that is, the pointwise convergence topology on $\mathcal{V}$, or, in other words, the initial topology $\sigma(\mathcal{V}, \{\delta_n : n \in \mathbb{N}\})$ of $\mathcal{V}$ generated by $\{\delta_n : n \in \mathbb{N}\}$. Notice that the coordinate maps are precisely the restriction to $\mathcal{V}$ of the coordinate projections $\pi_n : M^{\mathbb{N}} \to M$, that is, $\delta_n = \pi_n|_{\mathcal{V}}$ for all $n \in \mathbb{N}$.

Remark 92 *Let R be a topological ring. Let M be a topological R-module. If M is Hausdorff, then so is $M^{\mathbb{N}}$ with the product topology, and hence, any M-sequence space $\mathcal{V} \subseteq M^{\mathbb{N}}$ is also Hausdorff because its module topology is finer than the inherited product topology.*

As mentioned above, function spaces on the naturals are representative examples of sequence spaces.

Example 80 *Let R be a topological ring. Let M be a topological R-module. Let $\mathcal{V} \subseteq M^{\mathbb{N}}$ be a submodule which admits a convergence linear topology generated by an upward directed subset $\mathcal{P}_{\mathtt{f}}(\mathbb{N}) \subseteq \mathcal{G} \subseteq \mathcal{P}(\mathbb{N})$, that is, we are assuming that $\{m_n : n \in G\}$ is bounded in M for every $(m_n)_{n\in\mathbb{N}} \in \mathcal{V}$ and every $G \in \mathcal{G}$. Then $\mathcal{V}$ is an M-sequence space. Indeed, the convergence linear topology on $\mathcal{V}$ is a module topology in view of Theorem 103(1). Finally, since $\mathcal{P}_{\mathtt{f}}(\mathbb{N}) \subseteq \mathcal{G}$, the convergence linear topology on $\mathcal{V}$ is finer than the pointwise convergence topology on $\mathcal{V}$, and thus, the coordinate maps are continuous.*

6.2.1.2 Some sequence spaces

As we mentioned at the beginning of this section, there are basically four classical sequence spaces: c_{00} (space of eventually null sequences), c_0 (space of null sequences), c (space of convergent sequences), and ℓ_p (space of absolutely p-summable sequences). The last sequence space, ℓ_p, is the only one that requires the existence of a norm to be defined, except for the case $p = \infty$, where we obtain the space of bounded sequences.

Definition 330 (Classical sequence spaces) *Let M be a topological module over a topological ring R. We define the following sets:*

- $c_{00}(M) := \big\{(m_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}} : \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \ m_n = 0\big\}$.
- $c_0(M) := \Big\{(m_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}} : 0 \in \lim_{n\to\infty} m_n\Big\}$.
- $c(M) := \Big\{(m_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}} : \lim_{n\to\infty} m_n \neq \varnothing\Big\}$.
- $\ell_\infty(M) := \big\{(m_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}} : (m_n)_{n\in\mathbb{N}}$ *is bounded*$\big\}$.

Under the settings of the previous definition, it is immediate to check that $c_{00}(M) \subseteq c_0(M) \subseteq c(M)$. Even more, by bearing in mind Corollary 30, $c_{00}(M), c_0(M), c(M)$ are clearly submodules of $M^{\mathbb{N}}$. According to Theorem 105, if R is practical, then $\ell_\infty(M)$ is a commutative additive group.

Theorem 164 *Let M be a topological module over a practical topological ring R. Then $c(M) \subseteq \ell_\infty(M)$.*

Proof 500 *Let $U \in \mathcal{N}_0(M)$. We can find 0-neighborhoods $V \subseteq R$ and $W \subseteq M$ such that $VW \subseteq U$. On the one hand, there exists $n_0 \in \mathbb{N}$ such that $m_n \in W$ for all $n \geq n_0$. On the other hand, for every $n \in \{1, \ldots, n_0 - 1\}$, we can find a 0-neighborhood $V_n \subseteq R$ such that $V_n m_n \subseteq U$. Take $V_0 := V \cap V_1 \cap \cdots \cap V_{n_0-1}$. Since R is practical, there exists*

$r \in \mathcal{U}(R) \cap V$. We will prove now that $\{m_n : n \in \mathbb{N}\} \subseteq r^{-1}U$. Indeed, if $n \in \{1, \ldots, n_0 - 1\}$, then $rm_n \in V_n m_n \subseteq U$, so $m_n \in r^{-1}U$. Finally, if $n \geq n_0$, then $rm_n \in VW \subseteq U$, that is, $m_n \in r^{-1}U$.

By bearing in mind Theorem 105, if M is a topological module over a practical division topological ring R, then $\ell_\infty(M)$ is a submodule of $M^{\mathbb{N}}$. Therefore, if $\ell_\infty(M)$ is endowed with a module topology finer than the product topology, then $c_{00}(M), c_0(M), c(M), \ell_\infty(M)$ all become sequence spaces.

6.2.2 Methods

There are multiple methods of convergence [99, 100, 101, 102, 103, 104, 105]. In this subsection, we will try to collect them in a unified manner, as much as possible.

6.2.2.1 Convergence through free filters

Recall that a free filter in $\mathbb{N}$ is a filter in $\mathbb{N}$ containing the Fréchet filter

$$\mathcal{F}_{\mathbb{N}} := \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}.$$

Definition 331 (Filter convergence) *Let $\mathcal{F}$ be a free filter in $\mathbb{N}$. Let R be a topological ring. Let M, N be topological R-modules. Let $\mathcal{V}, \mathcal{W}$ be M-,N-sequence spaces, respectively. Consider $T \in \mathcal{CL}(\mathcal{V}, \mathcal{W})$. We define the space of $(\mathcal{F}, T)$-convergence as*

$$c_{(\mathcal{F},T)}(M) := \left\{(m_n)_{n\in\mathbb{N}} \in \mathcal{V} : \lim_{\mathcal{F}} T\left((m_n)_{n\in\mathbb{N}}\right) \neq \varnothing\right\}.$$

The $(\mathcal{F}, T)$-limit operator is defined as

$$\begin{array}{rccl} (\mathcal{F},T)\lim: & c_{(\mathcal{F},T)}(M) & \to & \mathcal{P}(N) \\ & (m_n)_{n\in\mathbb{N}} & \mapsto & (\mathcal{F},T)\lim(m_n)_{n\in\mathbb{N}} := \lim\limits_{\mathcal{F}} T\left((m_n)_{n\in\mathbb{N}}\right). \end{array}$$

Under the settings of the previous definition, if N is Hausdorff, then the $(\mathcal{F}, T)$-limit operator $(\mathcal{F}, T)\lim$ is assumed to be N-valued.

Theorem 165 *Let $\mathcal{F}$ be a free filter in $\mathbb{N}$. Let R be a topological ring. Let M, N be topological R-modules. Let $\mathcal{V}, \mathcal{W}$ be M-,N-sequence spaces, respectively. Consider $T \in \mathcal{CL}(\mathcal{V}, \mathcal{W})$. Then:*

1. *$c_{(\mathcal{F},T)}(M)$ is a submodule of $\mathcal{V}$.*
2. *If N is Hausdorff, then $(\mathcal{F}, T)\lim$ is R-linear. If, in addition, $\mathcal{W} \subseteq \ell_\infty(N)$ and it is endowed with the uniform convergence linear topology, then $(\mathcal{F}, T)\lim$ is continuous.*
3. *If $\mathcal{G}$ is another filter in $\mathbb{N}$ containing $\mathcal{F}$, then $c_{(\mathcal{F},T)}(M) \subseteq c_{(\mathcal{G},T)}(M)$. If, in addition, N is Hausdorff, then $(\mathcal{G}, T)\lim|_{c_{(\mathcal{F},T)}(M)} = (\mathcal{F}, T)\lim$.*
4. *If $c(M) \subseteq \mathcal{V}$ and $T(c(M)) \subseteq c(N)$, then $c(M) \subseteq c_{(\mathcal{F},T)}(M)$. If, in addition, N is Hausdorff, then $(\mathcal{F}, T)\lim|_{c(M)} = \lim \circ \left(T|_{c(M)}\right)$.*
5. *If R is practical, $\ell_\infty(N) \subseteq \mathcal{W}$ and $T^{-1}(\ell_\infty(N)) \subseteq \ell_\infty(M)$, then $c_{(\mathcal{F}_{\mathbb{N}},T)}(M) \subseteq \ell_\infty(M)$.*

Proof 501 *The proof will follow as itemized as the statement is.*

1. *Take* $(p_n)_{n\in\mathbb{N}}, (q_n)_{n\in\mathbb{N}} \in c_{(\mathcal{F},T)}(M)$. *Notice that* $T\left((p_n)_{n\in\mathbb{N}}\right), T\left((q_n)_{n\in\mathbb{N}}\right)$ *are sequences in* N. *According to Corollary 28(1),*

$$\begin{aligned}\lim_{\mathcal{F}} T\left((p_n)_{n\in\mathbb{N}}\right) + \lim_{\mathcal{F}} T\left((q_n)_{n\in\mathbb{N}}\right) &= \lim_{\mathcal{F}} \left(T\left((p_n)_{n\in\mathbb{N}}\right) + T\left((q_n)_{n\in\mathbb{N}}\right)\right) \\ &= \lim_{\mathcal{F}} T\left((p_n+q_n)_{n\in\mathbb{N}}\right).\end{aligned}$$

As a consequence, $\lim_{\mathcal{F}} T\left((p_n+q_n)_{n\in\mathbb{N}}\right) \neq \varnothing$, *meaning that* $(p_n+q_n)_{n\in\mathbb{N}} \in c_{(\mathcal{F},T)}(M)$. *Take now* $r \in R$. *By relying now on Corollary 28(2),*

$$r\lim_{\mathcal{F}} T\left((p_n)_{n\in\mathbb{N}}\right) \subseteq \lim_{\mathcal{F}} rT\left((p_n)_{n\in\mathbb{N}}\right) = \lim_{\mathcal{F}} T\left((rp_n)_{n\in\mathbb{N}}\right).$$

As a consequence, $\lim_{\mathcal{F}} T\left((rp_n)_{n\in\mathbb{N}}\right) \neq \varnothing$, *meaning that* $(rp_n)_{n\in\mathbb{N}} \in c_{(\mathcal{F},T)}(M)$.

2. *If* N *is Hausdorff, by applying Corollary 28, we obtain that*

$$\begin{aligned}(\mathcal{F},T)\lim(p_n+q_n)_{n\in\mathbb{N}} &= \lim_{\mathcal{F}} T\left((p_n+q_n)_{n\in\mathbb{N}}\right) \\ &= \lim_{\mathcal{F}} \left(T\left((p_n)_{n\in\mathbb{N}}\right) + T\left((q_n)_{n\in\mathbb{N}}\right)\right) \\ &= \lim_{\mathcal{F}} T\left((p_n)_{n\in\mathbb{N}}\right) + \lim_{\mathcal{F}} T\left((q_n)_{n\in\mathbb{N}}\right) \\ &= (\mathcal{F},T)\lim(p_n)_{n\in\mathbb{N}} + (\mathcal{F},T)\lim(q_n)_{n\in\mathbb{N}}\end{aligned}$$

and

$$\begin{aligned}(\mathcal{F},T)\lim(rp_n)_{n\in\mathbb{N}} &= \lim_{\mathcal{F}} T\left((rp_n)_{n\in\mathbb{N}}\right) = \lim_{\mathcal{F}} rT\left((p_n)_{n\in\mathbb{N}}\right) \\ &= r\lim_{\mathcal{F}} T\left((p_n)_{n\in\mathbb{N}}\right) = r(\mathcal{F},T)\lim(p_n)_{n\in\mathbb{N}}\end{aligned}$$

for all $(p_n)_{n\in\mathbb{N}}, (q_n)_{n\in\mathbb{N}} \in c_{(\mathcal{F},T)}(M)$ *and all* $r \in R$. *This proves that* $(\mathcal{F},T)\lim$ *is* R*-linear. Let us assume now that* $\mathcal{W} \subseteq \ell_\infty(N)$. *Theorem 103(1) assures that* $\mathcal{W}$ *can be endowed with the uniform convergence linear topology, that is, the convergence linear topology generated by* $\mathcal{G} := \mathcal{P}(\mathbb{N})$. *Fix an arbitrary neighborhood* U *of* 0 *in* N. *Since all topological modules are regular, we may assume without loss that* U *is closed. Take* $W := U^{\mathbb{N}} \cap \mathcal{W}$, *which is a neighborhood of* 0 *in* $\mathcal{W}$. *Take* $V := T^{-1}(W)$, *which is a neighborhood of* 0 *in* $\mathcal{V}$. *We will show that* $(\mathcal{F},T)\lim(V) \subseteq U$. *Let* $(m_n)_{n\in\mathbb{N}} \in V$. *Then* $(q_n)_{n\in\mathbb{N}} := T\left((m_n)_{n\in\mathbb{N}}\right) \in W$. *Denote* $q := \lim_{\mathcal{F}}(q_n)_{n\in\mathbb{N}} = (\mathcal{F},T)\lim(m_n)_{n\in\mathbb{N}}$. *Suppose that* $q \notin U$. *Since* U *is closed, we can find an open neighborhood* A *of* q *such that* $A \cap U = \varnothing$. *Note that* $\{n \in \mathbb{N} : q_n \in A\} \in \mathcal{F}$. *Since* $(q_n)_{n\in\mathbb{N}} \in W$, *we have that* $q_n \in U$ *for every* $n \in \mathbb{N}$. *Since* $U \cap A = \varnothing$, *we conclude that* $\{n \in \mathbb{N} : q_n \in A\} = \varnothing$, *which implies the contradiction that* $\varnothing \in \mathcal{F}$.

3. *Since* $\mathcal{F} \subseteq \mathcal{G}$, *we have that* $\lim_{\mathcal{F}} T\left((m_n)_{n\in\mathbb{N}}\right) \subseteq \lim_{\mathcal{G}} T\left((m_n)_{n\in\mathbb{N}}\right)$ *for every* $(m_n)_{n\in\mathbb{N}} \in \mathcal{V}$. *Therefore,* $c_{(\mathcal{F},T)}(M) \subseteq c_{(\mathcal{G},T)}(M)$. *If, in addition,* N *is Hausdorff, then it is clear that* $(\mathcal{G},T)\lim|_{c_{(\mathcal{F},T)}(M)} = (\mathcal{F},T)\lim$.

4. *If* $(m_n)_{n\in\mathbb{N}} \in c(M)$, *by hypothesis, we have that* $T\left((m_n)_{n\in\mathbb{N}}\right) \in c(N)$, *so*

$$\lim_{n\to\infty} T\left((m_n)_{n\in\mathbb{N}}\right) = \lim_{\mathcal{F}_{\mathbb{N}}} T\left((m_n)_{n\in\mathbb{N}}\right) \subseteq \lim_{\mathcal{F}} T\left((m_n)_{n\in\mathbb{N}}\right),$$

which implies that $(m_n)_{n\in\mathbb{N}} \in c_{(\mathcal{F},T)}(M)$. *If, in addition,* N *is Hausdorff, then* $(\mathcal{F},T)\lim|_{c(M)} = \lim \circ \left(T|_{c(M)}\right)$ *by the previous item.*

5. *Fix an arbitrary* $(m_n)_{\in\mathbb{N}} \in c_{(\mathcal{F}_{\mathbb{N}},T)}(M)$*, then* $\lim\limits_{\mathcal{F}_{\mathbb{N}}} T\left((m_n)_{n\in\mathbb{N}}\right)$ *is not empty, but we know that* $\lim\limits_{\mathcal{F}_{\mathbb{N}}} T\left((m_n)_{n\in\mathbb{N}}\right) = \lim\limits_{n\to\infty} T\left((m_n)_{n\in\mathbb{N}}\right)$*. As a consequence,* $T\left((m_n)_{n\in\mathbb{N}}\right) \in c(N)$*. In accordance with Theorem 164,* $c(N) \subseteq \ell_\infty(N)$*. Finally, by hypothesis,* $(m_n)_{n\in\mathbb{N}} \in T^{-1}\left(\ell_\infty(N)\right) \subseteq \ell_\infty(M)$*.*

What follows is the classical f-statistical convergence, which is a particular case of ideal convergence [37, 106, 107, 108, 109].

Example 81 (f-Statistical convergence) *Let R be a topological unital ordered ring. Let M a topological R-module. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an unbounded modulus function. Consider the ideal $\mathcal{D}_0^f(\mathbb{N})$ of subsets of $\mathbb{N}$ with null f-density. Since $\mathcal{D}_0^f(\mathbb{N})$ is a bornology, $\mathcal{D}_0^f(\mathbb{N})^{\mathsf{c}}$ is a free filter. As a consequence, we can consider*

$$c_{\left(\mathcal{D}_0^f(\mathbb{N})^{\mathsf{c}},I_{\mathcal{V}}\right)}(M) := \left\{(m_n)_{n\in\mathbb{N}} \in \mathcal{V} : \lim_{\mathcal{D}_0^f(\mathbb{N})^{\mathsf{c}}} (m_n)_{n\in\mathbb{N}} \neq \varnothing\right\},$$

where $\mathcal{V}$ is a given M-sequence space. Analytically speaking, a sequence $(m_n)_{n\in\mathbb{N}} \subseteq M$ is f-statistically convergent to $m \in M$ if the set $\{n \in \mathbb{N} : m_n \notin m + U\}$ has f-density 0 for every 0-neighborhood U in M. In other words,

$$\lim_{n\to\infty} \frac{f\left(\operatorname{card}\left(\{k \leq n : m_k \notin m + U\}\right)\right)}{f(n)} = 0$$

for every 0-neighborhood U in M.

We will use next the famous and classical Cesàro mean convergence to illustrate an example of convergence through free filters. However, we will need first a couple of technical lemmas.

Lemma 170 *Let R be a Hausdorff practical division topological ring of characteristic 0 endowed with a completely full unit segment $B^+ \subseteq R$. Let M be a Hausdorff topological R-module. If the module topology of M is locally convex, then for every $(m_n)_{n\in\mathbb{N}} \in \ell_\infty(M)$, $\left(n^{-1}\left(m_1 + \cdots + m_n\right)\right)_{n\in\mathbb{N}} \in \ell_\infty(M)$.*

Proof 502 *Fix an arbitrary $U \in \mathcal{N}_0(M)$. We can find an invertible $u \in \mathcal{U}(R) = R \setminus \{0\}$ such that $\{m_n : n \in \mathbb{N}\} \subseteq uU$. By hypothesis, we may assume without any loss of generality that U is convex. For every $n \in \mathbb{N}$, $n^{-1} \in B^+$ because B^+ is completely full, and $n^{-1} + \overset{n)}{\cdots} + n^{-1} = 1$. Therefore,*

$$\begin{aligned} u^{-1}\left(n^{-1}\left(m_1 + \cdots + m_n\right)\right) &= n^{-1}\left(u^{-1}m_1\right) + \cdots + n^{-1}\left(u^{-1}m_n\right) \\ &\in \Delta\left(u^{-1}m_1, \ldots, u^{-1}m_n\right) \subseteq U. \end{aligned}$$

This shows that $\left\{n^{-1}\left(m_1 + \cdots + m_n\right) : n \in \mathbb{N}\right\} \subseteq uU$.

By taking into account Lemma 170, and under its settings, the following R-linear operator is well defined:

$$\begin{array}{ccc} \ell_\infty(M) & \to & \ell_\infty(M) \\ (m_n)_{n\in\mathbb{N}} & \mapsto & \left(n^{-1}\left(m_1 + \cdots + m_n\right)\right)_{n\in\mathbb{N}}. \end{array}$$

This operator is called Cesàro mean operator.

Lemma 171 *Let R be a Hausdorff practical division topological ring of characteristic 0 endowed with a completely full unit segment $B^+ \subseteq R$. Let M be a Hausdorff topological R-module. If the module topology of M is locally convex, then the Cesàro mean operator on M is continuous when $\ell_\infty(M)$ is endowed with the uniform convergence linear topology.*

Proof 503 *Fix an arbitrary $\mathcal{U}(G,U) \in \mathcal{N}_0(\ell_\infty(M))$, where $G \in \mathcal{P}(\mathbb{N})$ and $U \in \mathcal{N}_0(M)$. By hypothesis, we may assume without any loss of generality that U is convex. Let us prove that the Cesàro mean operator maps $\mathcal{U}(\mathbb{N},U) = U^{\mathbb{N}} \cap \ell_\infty(M)$ inside $\mathcal{U}(G,U)$. Indeed, take $(m_n)_{n\in\mathbb{N}} \in \mathcal{U}(\mathbb{N},U) = U^{\mathbb{N}} \cap \ell_\infty(M)$. Then $m_n \in U$ for all $n \in \mathbb{N}$. Since U is convex, B^+ is completely full (meaning that $n^{-1} \in B^+$ for all $n \in \mathbb{N}$), and $n^{-1} + \overset{n)}{\cdots} + n^{-1} = 1$ for all $n \in \mathbb{N}$, we obtain that*

$$n^{-1}(m_1 + \cdots + m_n) = n^{-1}m_1 + \cdots + n^{-1}m_n \in \Delta(m_1, \ldots, m_n) \subseteq U.$$

This shows that $\left(n^{-1}(m_1 + \cdots + m_n)\right)_{n\in\mathbb{N}} \in \mathcal{U}(\mathbb{N},U) \subseteq \mathcal{U}(G,U)$.

Now, we have gathered all the necessary ingredients to construct the famous Cesàro mean convergence.

Example 82 (Cesàro mean convergence) *Let R be a Hausdorff practical division topological ring of characteristic 0 endowed with a completely full unit segment $B^+ \subseteq R$. Let M be a Hausdorff topological R-module. Assume that the module topology of M is locally convex. Assume that $\ell_\infty(M)$ is endowed with the uniform convergence linear topology. Consider the Cesàro mean operator on M,*

$$\begin{array}{rccc} \mathscr{C}: & \ell_\infty(M) & \to & \ell_\infty(M) \\ & (m_n)_{n\in\mathbb{N}} & \mapsto & \left(n^{-1}(m_1 + \cdots + m_n)\right)_{n\in\mathbb{N}}, \end{array}$$

which is R-linear and continuous in view of Lemma 170 and Lemma 171. We can now consider the sequence space $c_{(\mathcal{F}_{\mathbb{N}},\mathscr{C})}(M)$.

6.2.2.2 Convergence through operators

Consider a Hausdorff practical division topological ring R together with a Hausdorff topological R-module M. Consider the sequence space $\ell_\infty(M)$ endowed with a module topology finer than the pointwise convergence topology. Notice that $c(M) \subseteq \ell_\infty(M)$. The limit operator on $c(M)$ is defined as

$$\begin{array}{rccc} \lim: & c(M) & \to & M \\ & (m_n)_{n\in\mathbb{N}} & \mapsto & \lim\limits_{n\to\infty} m_n. \end{array}$$

We will rely on the set

$$\mathcal{HB}(\lim) := \left\{T \in \mathcal{CL}(\ell_\infty(M), M) : T|_{c(M)} = \lim\right\}$$

to construct the convergence through operators.

Definition 332 (Operator convergence) *Let R be a Hausdorff practical division topological ring. Let M be a Hausdorff topological R-module. Consider the sequence space $\ell_\infty(M)$ endowed with a module topology finer than the pointwise convergence topology. Let $C \subseteq \mathcal{HB}(\lim)$ be not empty. We define the space of C-convergence as*

$$c_C(M) := \left\{(m_n)_{n\in\mathbb{N}} \in \ell_{\infty(M)} : \exists m \in M\ \forall T \in C\ \ T((m_n)_{n\in\mathbb{N}}) = m\right\}.$$

The C-limit operator is defined as

$$\begin{array}{rccl} C\lim: & c_C(M) & \to & M \\ & (m_n)_{n\in\mathbb{N}} & \mapsto & C\lim\left((m_n)_{n\in\mathbb{N}}\right) := T\left((m_n)_{n\in\mathbb{N}}\right), \end{array}$$

where T is any element of C.

The following proposition unveils the most basic properties satisfied by the sequence space defined above.

Proposition 130 *Let R be a Hausdorff practical division topological ring. Let M be a Hausdorff topological R-module. Consider the sequence space $\ell_\infty(M)$ endowed with a module topology finer than the pointwise convergence topology. Let $C, D \subseteq \mathcal{HB}(\lim)$ be not empty. Then:*

1. *$c_C(M)$ is a submodule of $\ell_\infty(M)$ and $C\lim$ is R-linear and continuous.*
2. *$c(M) \subseteq c_C(M)$ and $C\lim|_{c(M)} = \lim$.*
3. *If $C \subseteq D$, then $c_D(M) \subseteq c_C(M)$ and $C\lim|_{c_D(M)} = D\lim$.*

Proof 504 *All items are easy to verify, except for, maybe, the continuity of $C\lim$. Fix an arbitrary $U \in \mathcal{N}_0(M)$. Take any $T_0 \in C$. Notice that $V := T_0^{-1}(U) \cap c_C(M) \in \mathcal{N}_0(c_C(M))$. We will prove that $C\lim(V) \subseteq U$. If $(m_n)_{n\in\mathbb{N}} \in V$, there exists $m \in M$ such that $T\left((m_n)_{n\in\mathbb{N}}\right) = m$ for all $T \in C$. By definition, $m := C\lim\left((m_n)_{n\in\mathbb{N}}\right)$. Also, $T_0\left((m_n)_{n\in\mathbb{N}}\right) = m$. Since $(m_n)_{n\in\mathbb{N}} \in T_0^{-1}(U)$, we have that $m = T_0\left((m_n)_{n\in\mathbb{N}}\right) \in U$. The arbitrariness of $(m_n)_{n\in\mathbb{N}} \in V$ shows that $C\lim(V) \subseteq U$. As a consequence, $C\lim$ is continuous.*

The following example illustrates the well-known concept of Banach limit [30, 100, 101, 110, 111, 112, 113, 114, 115].

Example 83 (Banach limits) *Let R be a normalizing absolutely valued division ring. Let M be a normed R-module. Consider $\ell_\infty(M)$ endowed with the sup norm. The set of Banach limits on M is defined as*

$$\mathscr{BL}(M) := \left\{T \in \mathcal{B}\left(\ell_\infty(M), M\right) : T|_{c(M)} = \lim \text{ and } \|T\| = 1\right\}.$$

If $\mathscr{BL}(M)$ is not empty, then we can consider the sequence space $c_{\mathscr{BL}(M)}(M)$.

6.2.2.3 Multipliers

The concept of multiplier is essential in Summability Theory. Given a topological ring R and topological R-modules M, N, a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{CL}(M, N)$ is said to be a multiplier of a sequence $(m_n)_{n\in\mathbb{N}} \subseteq M$ provided that $\sum_{n=1}^{\infty} T_n(m_n)$ is convergent through a certain predetermined summability method. We will first consider multipliers for the summability method described by convergence through free filters.

Definition 333 (Multiplier) *Let $\mathcal{F}$ be a free filter in $\mathbb{N}$. Let R be a topological ring. Let M, N, P be topological R-modules. Let $\mathcal{V}, \mathcal{W}$ be M-,N-sequence spaces, respectively. Consider $T \in \mathcal{CL}(\mathcal{V}, \mathcal{W})$. The $(\mathcal{F}, T)$-summability spaces associated to sequences $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{CL}(P, M), (p_n)_{n\in\mathbb{N}} \subseteq P$ are defined as*

$$\mathcal{S}_{(\mathcal{F},T)}\left((T_n)_{n\in\mathbb{N}}\right) := \left\{(q_n)_{n\in\mathbb{N}} \in P^{\mathbb{N}} : \sum_{n=1}^{\infty} T_n(q_n) \in c_{(\mathcal{F},T)}(M)\right\},$$

$$\mathcal{S}_{(\mathcal{F},T)}\left((p_n)_{n\in\mathbb{N}}\right) := \left\{(S_n)_{n\in\mathbb{N}} \in \mathcal{CL}(P,M)^{\mathbb{N}} : \sum_{n=1}^{\infty} S_n(p_n) \in c_{(\mathcal{F},T)}(M)\right\},$$

respectively.

Under the settings of the previous definition and by bearing in mind Theorem 165(1), it is not hard to check that both $\mathcal{S}_{(\mathcal{F},T)}\left((T_n)_{n\in\mathbb{N}}\right), \mathcal{S}_{(\mathcal{F},T)}\left((p_n)_{n\in\mathbb{N}}\right)$ are submodules of $P^{\mathbb{N}}, \mathcal{CL}(P,M)^{\mathbb{N}}$, respectively. Next, we will consider multipliers for the summability method described by convergence through operators.

Definition 334 (Multiplier) *Let R be a Hausdorff practical division topological ring. Let M, P be Hausdorff topological R-modules. Consider the sequence space $\ell_\infty(M)$ endowed with a module topology finer than the pointwise convergence topology. Let $C \subseteq \mathcal{HB}(\lim)$ be not empty. The C-summability spaces associated with sequences $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{CL}(P,M), (p_n)_{n\in\mathbb{N}} \subseteq P$ are defined as*

$$\mathcal{S}_C\left((T_n)_{n\in\mathbb{N}}\right) := \left\{(q_n)_{n\in\mathbb{N}} \in P^{\mathbb{N}} : \sum_{n=1}^{\infty} T_n(q_n) \in c_C(M)\right\},$$

$$\mathcal{S}_C\left((p_n)_{n\in\mathbb{N}}\right) := \left\{(S_n)_{n\in\mathbb{N}} \in \mathcal{CL}(P,M)^{\mathbb{N}} : \sum_{n=1}^{\infty} S_n(p_n) \in c_C(M)\right\},$$

respectively.

By bearing in mind Proposition 130(1) and under the settings of the previous definition, it is not hard to check that both $\mathcal{S}_C\left((T_n)_{n\in\mathbb{N}}\right), \mathcal{S}_C\left((p_n)_{n\in\mathbb{N}}\right)$ are submodules of $P^{\mathbb{N}}, \mathcal{CL}(P,M)^{\mathbb{N}}$, respectively.

Chapter 7

Integrability

Isaac Newton

7.1 Measures

The concept of integral was conceived in an implicit manner much earlier than the concept of derivative. In fact, integrals are the successor of the method of exhaustion invented by the ancient Greek astronomer Eudoxus, which sought to find areas and volumes by breaking them up into an infinite number of divisions for which the area or volume was known. The idea behind the concept of integral is a "continuous" summation of lines (or rectangles of infinitesimal width) to obtain an area. The symbol of integral, introduced by Leibniz, is a deformation of the letter "S" of summation.

7.1.1 Measures on effect algebras

In modern Mathematics, the concept of integral relies on the notion of measure. In classical Measure Theory, measures are studied on the scope of Boolean algebras of sets. Here, we will study measures in its most abstract environment, that is, defined on effect algebras or Boolean algebras and with values in a topological module. In [116, 13], measures on effect algebras are studied with values on real or complex Banach spaces. We refer the reader to [117, 118, 119] for a wider perspective on measures on Boolean algebras of sets.

7.1.1.1 Classification of measures

We will define measures on effect algebras, which are universal algebras more general than Boolean algebras, and with values in a topological module. This way, vector measures studied in [118] are particular cases.

Definition 335 (Measure) *Let L be an effect algebra. Let M be a module over a ring R. Let $\mu : L \to M$ be a map. We say that μ is a measure if it is an orthogonally additive map, that is, $\mu(a \oplus b) = \mu(a) + \mu(b)$ for every orthogonal $a, b \in L$.*

Measures are also called charges and they are not assumed to be countably additive by default.

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Definition 336 *Let L be an effect algebra. Let M be a topological module over a topological ring. A measure $\mu : L \to M$ is said to be:*

- *Orthogonally countably additive, or orthogonally σ-additive, if $\mu\left(\bigoplus_{n\in\mathbb{N}} a_n\right) \in \sum_{n\in\mathbb{N}} \mu(a_n)$ for every orthogonal sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$ such that $\bigoplus_{n\in\mathbb{N}} a_n$ exists.*
- *Strongly additive if $\sum_{n\in\mathbb{N}} \mu(a_n)$ converges for every orthogonal sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$.*
- *Exhaustive or strongly bounded if $0 \in \lim_{n\to\infty} \mu(a_n)$ for every orthogonal sequence $(a_n)_{n\in\mathbb{N}} \subseteq L$.*

In the previous definition, and throughout the rest of this chapter, $\sum_{n\in\mathbb{N}}$ is a summation in the regular sense, not in the sense of uncountable series. Several basic, but also interesting, properties about the previous concepts are highlighted in the upcoming results.

Proposition 131 *Let L be an effect algebra. Let M be a module over a ring R. Let $\mu : L \to M$ be a measure. Let $a, b, u, u', v, v' \in L$. Then:*

1. $\mu(0) = 0$.
2. *If $a \leq b$, then $\mu(b \ominus a) = \mu(b) - \mu(a)$.*
3. *If M is ordered, $a \leq b$, and $\mu(b \ominus a) \geq 0$, then $\mu(a) \leq \mu(b)$.*
4. *If L is a distributive complemented lattice, then $\mu(a \vee b) = \mu(a) + \mu(b) - \mu(a \wedge b)$. In particular, if M is ordered and μ is positive, then $\mu(a \vee b) \leq \mu(a) + \mu(b)$.*
5. *If L is a distributive complemented lattice, M is ordered, μ is positive, and $u \leq u'$, then*

$$\begin{aligned} \mu(u' \wedge a) - \mu(u \wedge a) &\leq \mu(u') - \mu(u), \\ \mu(u' \vee a) - \mu(u \vee a) &\leq \mu(u') - \mu(u). \end{aligned}$$

6. *If L is a distributive complemented lattice, M is ordered, μ is positive, and $u \leq u'$ and $v \leq v'$, then*

$$\mu(u \wedge v) - \mu(u, v') - \mu(u', v) + \mu(u', v') \geq 0. \tag{7.1}$$

Proof 505 *The proof will follow as itemized in the statement.*

1. *Observe that $\mu(0) = \mu(0 \oplus 0) = \mu(0) + \mu(0)$. Therefore, $\mu(0) = 0$.*
2. *Simply notice that $\mu(b) = \mu(a \oplus (b \ominus a)) = \mu(a) + \mu(b \ominus a)$. Therefore, $\mu(b \ominus a) = \mu(b) - \mu(a)$.*
3. *Observe that $\mu(b) - \mu(a) = \mu(b \ominus a) \geq 0$.*
4. *Note that*

$$\begin{aligned} \mu(a \vee b) &= \mu((a \setminus (a \wedge b)) \vee (b \setminus (a \wedge b)) \vee (a \wedge b)) \\ &= \mu((a \ominus (a \wedge b)) \oplus (b \ominus (a \wedge b)) \oplus (a \wedge b)) \\ &= \mu(a \ominus (a \wedge b)) + \mu(b \ominus (a \wedge b)) + \mu(a \wedge b) \\ &= \mu(a) - \mu(a \wedge b) + \mu(b) - \mu(a \wedge b) + \mu(a \wedge b) \\ &= \mu(a) + \mu(b) - \mu(a \wedge b). \end{aligned}$$

5. *By applying Theorem 3(5),* $(u' \wedge a) \setminus (u \wedge a) \leq u' \setminus u$ *and* $(u' \vee a) \setminus (u \vee a) \leq u' \setminus u$. *Then*

$$\mu(a \wedge u') - \mu(a \wedge u) = \mu\left((u' \wedge a) \setminus (u \wedge a)\right) \leq \mu\left(u' \setminus u\right) = \mu(u') - \mu(u)$$

and

$$\mu(a \vee u') - \mu(a \vee u) = \mu\left((u' \vee a) \setminus (u \vee a)\right) \leq \mu\left(u' \setminus u\right) = \mu(u') - \mu(u).$$

6. *On the one hand,* $u' \wedge v \leq u' \wedge v'$, *hence*

$$\mu(u' \wedge v') - \mu(u' \wedge v) = \mu\left((u' \wedge v') \setminus (u' \wedge v)\right).$$

On the other hand, $u \wedge v \leq u \wedge v'$, *hence*

$$\mu(u \wedge v') - \mu(u \wedge v) = \mu\left((u \wedge v') \setminus (u \wedge v)\right).$$

Next, $(u \wedge v') \setminus (u \wedge v) \leq (u' \wedge v') \setminus (u' \wedge v)$, *meaning that*

$$\begin{array}{rl} & \mu\left((u' \wedge v') \setminus (u' \wedge v)\right) - \mu\left((u \wedge v') \setminus (u \wedge v)\right) \\ = & \mu\left(\left((u' \wedge v') \setminus (u' \wedge v)\right) \setminus \left((u \wedge v') \setminus (u \wedge v)\right)\right). \end{array}$$

Finally, since μ *is positive, we have that*

$$\begin{array}{rl} & \mu(u \wedge v) - \mu(u \wedge v') - \mu(u' \wedge v) + \mu(u' \wedge v') \\ = & [\mu(u' \wedge v') - \mu(u' \wedge v)] - [\mu(u \wedge v') - \mu(u \wedge v)] \\ = & \mu\left((u' \wedge v') \setminus (u' \wedge v)\right) - \mu\left((u \wedge v') \setminus (u \wedge v)\right) \\ = & \mu\left(\left((u' \wedge v') \setminus (u' \wedge v)\right) \setminus \left((u \wedge v') \setminus (u \wedge v)\right)\right) \\ \geq & 0. \end{array}$$

It is not difficult to check that the set of all measures on an effect algebra L with values on a topological module M over a topological ring R is a submodule of M^L. Notice also that, by bearing in mind Lemma 162, the previous sentence holds for σ-additive measures, strongly additive measures, exhaustive measures, absolutely additive measures, and bounded measures.

Remark 93 *Let* L *be an effect algebra. Let* M *be a Hausdorff topological module over a topological ring* R. *Let* $\mu : L \to M$ *be a map. Then:*

1. *If* μ *is a* σ*-additive measure, then* $\sum_{n\in\mathbb{N}} \mu(a_n)$ *is unconditionally convergent for all orthogonal sequences* $(a_n)_{n\in\mathbb{N}}$ *for which* $\bigoplus_{n\in\mathbb{N}} a_n$ *exists.*

2. *If* μ *satisfies the condition of* σ*-additivity, then* μ *is, in fact, a measure, and thus, a* σ*-additive measure. Indeed, in order to obtain* $\mu(a) + \mu(b)$ *from an orthogonal pair* $a \oplus b$, *it only suffices to consider the orthogonal sequence* $a \oplus b \oplus 0 \oplus \cdots \oplus 0 \oplus \cdots$.

3. *If* μ *is a* σ*-additive measure and* L *is a* σ*-effect algebra, then* μ *is strongly additive. Indeed, if* $(a_n)_{n\in\mathbb{N}} \subseteq L$ *is an orthogonal sequence, then* $\bigoplus_{n\in\mathbb{N}} a_n$ *exists and* $\sum_{n\in\mathbb{N}} \mu(a_n)$ *converges to* $\mu\left(\bigoplus_{n\in\mathbb{N}} a_n\right)$.

4. *If* μ *is a strongly additive measure, then* μ *is exhaustive. Indeed, if* $(a_n)_{n\in\mathbb{N}} \subseteq L$ *is an orthogonal sequence, then* $\sum_{n\in\mathbb{N}} \mu(a_n)$ *converges, meaning that* $0 \in \lim_{n\to\infty} \mu(a_n)$ *in view of Lemma 162(4).*

Taking into consideration Remark 93, it is important to notice that, by definition, σ-additive measures, strongly additive measures, and exhaustive measures are, in fact measures, by default even the corresponding module is not Hausdorff.

Example 84 *Consider the Boolean algebra of subsets of* $\mathbb{N}$ *given by*

$$\mathcal{A} := \{A \subseteq \mathbb{N} : \text{ either } A \text{ or } \mathbb{N} \setminus A \text{ is finite}\}$$

and the measure

$$\begin{array}{rcl} \mu : \ \mathcal{A} & \to & [0,2] \subset \mathbb{R} \\ A & \mapsto & \mu(A) := \begin{cases} 0 & A = \varnothing, \\ \displaystyle\sum_{i \in A} \frac{1}{2^i} & A \in \mathcal{P}_{\mathfrak{f}}(\mathbb{N}) \setminus \{\varnothing\}, \\ \displaystyle 2 - \sum_{i \in \mathbb{N} \setminus A} \frac{1}{2^i} & \mathbb{N} \setminus A \in \mathcal{P}_{\mathfrak{f}}(\mathbb{N}) \setminus \{\varnothing\}, \\ 2 & A = \mathbb{N}. \end{cases} \end{array}$$

If $A_n := \{n\}$ *for every* $n \in \mathbb{N}$*, then*

$$\sum_{n=1}^{\infty} \mu(A_n) = 1 < 2 = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right).$$

The previous example shows a very curious and interesting measure which is not σ-additive.

Theorem 166 *Let* L *be an effect algebra. Let* M *be a topological module over a topological ring* R*. Let* $\mu : L \to M$ *be a* σ*-additive measure. If* $(a_n)_{n \in \mathbb{N}} \subseteq L$ *is an increasing sequence such that* $\bigvee_{n \in \mathbb{N}} a_n$ *exists, then* $(\mu(a_n))_{n \in \mathbb{N}}$ *converges to* $\mu\left(\bigvee_{n \in \mathbb{N}} a_n\right)$*.*

Proof 506 *On the one hand, in virtue of Lemma 40(1),*

$$\bigvee_{n \in \mathbb{N}} a_n = a_1 \oplus \bigoplus_{n \geq 2} (a_n \ominus a_{n-1}).$$

On the other hand, for every $k \geq 2$*,*

$$\sum_{n=2}^{k} (\mu(a_n) - \mu(a_{n-1})) = \mu(a_k) - \mu(a_1).$$

Finally,

$$\begin{array}{rcl} \mu\left(\displaystyle\bigvee_{n \in \mathbb{N}} a_n\right) & = & \mu\left(a_1 \oplus \displaystyle\bigoplus_{n \geq 2} (a_n \ominus a_{n-1})\right) = \mu(a_1) + \mu\left(\displaystyle\bigoplus_{n \geq 2} (a_n \ominus a_{n-1})\right) \\ & \in & \mu(a_1) + \displaystyle\sum_{n=2}^{\infty} (\mu(a_n) - \mu(a_{n-1})) = \mu(a_1) \\ & + & \displaystyle\lim_{n \to \infty} (\mu(a_n) - \mu(a_1)) \subseteq \lim_{n \to \infty} \mu(a_n). \end{array}$$

Two relevant corollaries can be derived from the previous theorem. Here comes the first one, which deals with countably subadditivity.

Corollary 66 *Let L be a distributive complemented lattice. Let M be a topological totally ordered module over a topological totally ordered ring R. Let $\mu : L \to M^+$ be a positive and σ-additive measure. If $(a_n)_{n\in\mathbb{N}} \subseteq L$ satisfies that $\bigvee_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} \mu(a_n)$ exist, then $\mu\left(\bigvee_{n\in\mathbb{N}} a_n\right) \leq \sum_{n\in\mathbb{N}} \mu\left(a_n\right)$.*

Proof 507 *On the one hand, for every $k \in \mathbb{N}$, we have that $\mu\left(\bigvee_{n=1}^{k} a_n\right) \leq \sum_{n=1}^{k} \mu\left(a_n\right) \leq \sum_{n\in\mathbb{N}} \mu\left(a_n\right)$ by Proposition 131(4) together with Lemma 84. On the other hand, $\left(\bigvee_{n=1}^{k} a_n\right)_{k\in\mathbb{N}}$ is an increasing sequence whose supremum exists, which is $\bigvee_{n\in\mathbb{N}} a_n$. Therefore, Theorem 166 applies to conclude that $\left(\mu\left(\bigvee_{n=1}^{k} a_n\right)\right)_{k\in\mathbb{N}}$ converges to $\mu\left(\bigvee_{n\in\mathbb{N}} a_n\right)$. As a consequence, $\mu\left(\bigvee_{n\in\mathbb{N}} a_n\right) \leq \sum_{n\in\mathbb{N}} \mu\left(a_n\right)$ in view of Lemma 55.*

Theorem 166 finds its dual version in the following corollary. This is its second corollary.

Corollary 67 *Let L be an effect algebra. Let M be a topological module over a topological ring R. Let $\mu : L \to M$ be a σ-additive measure. If $(a_n)_{n\in\mathbb{N}} \subseteq L$ is a decreasing sequence such that $\bigwedge_{n\in\mathbb{N}} a_n$ exists, then $\left(\mu\left(a_n\right)\right)_{n\in\mathbb{N}}$ converges to $\mu\left(\bigwedge_{n\in\mathbb{N}} a_n\right)$.*

Proof 508 *In accordance with Lemma 40(2), $\left(1 \ominus a_n\right)_{n\in\mathbb{N}}$ is an increasing sequence such that $\bigvee_{n\in\mathbb{N}}(1 \ominus a_n) = 1 \ominus \bigwedge_{n\in\mathbb{N}} a_n$ exists. Therefore, by Theorem 166, $\left(\mu\left(1 \ominus a_n\right)\right)_{n\in\mathbb{N}}$ converges to*

$$\mu\left(\bigvee_{n\in\mathbb{N}} \left(1 \ominus a_n\right)\right) = \mu\left(1 \ominus \bigwedge_{n\in\mathbb{N}} a_n\right) = \mu\left(1\right) - \mu\left(\bigwedge_{n\in\mathbb{N}} a_n\right).$$

Finally, it suffices to observe that $\mu\left(1 \ominus a_n\right) = \mu\left(1\right) - \mu\left(a_n\right)$ for all $n \in \mathbb{N}$.

Recall that a subset B of a poset L is called a generator of L if, for every $a \in L$, there exists an increasing sequence $(b_n)_{n\in\mathbb{N}} \subseteq B$ such that $a = \bigvee_{n\in\mathbb{N}} b_n$.

Corollary 68 *Let L be an effect algebra. Let M be a topological module over a topological ring R. Let $\mu : L \to M$ be a σ-additive measure. Then $\mu(L)$ is bounded in M if and only if $\mu(B)$ is bounded in M, for B a generator of L.*

Proof 509 *The proof will follow as itemized as the statement.*

- $\Rightarrow$ *If $\mu(L)$ is bounded in M and B is a generator of L, then $\mu(B)$ is trivially bounded because $\mu(B) \subseteq \mu(L)$.*
- $\Leftrightarrow$ *Assume that there exists a generator B of L such that $\mu(B)$ is bounded in M. Fix an arbitrary $a \in L$ and choose, by hypothesis, an increasing sequence $(b_n)_{n\in\mathbb{N}} \subseteq B$ such that $a = \bigvee_{n\in\mathbb{N}} b_n$. In accordance with Theorem 166, we have that $\left(\mu(b_n)\right)_{n\in\mathbb{N}}$ converges to $\mu\left(a\right)$. As a consequence, $\mu(a) \in \mathrm{cl}\left(\mu(B)\right)$. The arbitrariness of $a \in L$ shows that $\mu(L) \subseteq \mathrm{cl}\left(\mu(B)\right)$. Finally, in accordance with Theorem 132(3), $\mathrm{cl}\left(\mu(B)\right)$ is bounded in M, hence so is $\mu(L)$.*

Theorem 166 can be generalized to uncountable directed sets with countable cofinal subsets.

Corollary 69 *Let L be an effect algebra. Let M be a topological module over a topological ring R. Let $\mu : L \to M$ be a σ-additive measure. Let I be a directed set that has a countable cofinal subset $J \subseteq I$. If $(a_i)_{i\in I} \subseteq L$ is an increasing net such that $\bigvee_{i\in I} a_i$ exists, then $\left(\mu\left(a_i\right)\right)_{i\in I}$ converges to $\mu\left(\bigvee_{i\in I} a_i\right)$.*

Proof 510 *Suppose on the contrary that* $(\mu(a_i))_{i\in I}$ *does not converge to* $\mu\left(\bigvee_{i\in I} a_i\right)$. *Then we can find a neighborhood* W *of* $\mu\left(\bigvee_{i\in I} a_i\right)$ *such that for all* $i \in I$ *there exists* $k \in I$ *with* $k \geq i$ *such that* $\mu(a_k) \notin W$. *In view of Proposition 3(3), we can write* $J = (j_n)_{n\in\mathbb{N}}$ *with* $n \leq m \Leftrightarrow j_n \leq j_m$. *For every* $n \in \mathbb{N}$, *we can find* $k_n \geq j_n$ *such that* $\mu(A_{k_n}) \notin W$. *Notice that we can choose* $(k_n)_{n\in\mathbb{N}}$ *such that* $n \leq m \Leftrightarrow k_n \leq k_m$. *By Proposition 4(2),* $\bigvee_{i\in I} a_i = \bigvee_{n\in\mathbb{N}} a_{k_n}$. *Also,* $(a_{k_n})_{n\in\mathbb{N}}$ *is increasing. By Theorem 166,*

$$\mu\left(\bigvee_{n\in\mathbb{N}} a_{k_n}\right) \in \lim_{n\to\infty} \mu(a_{k_n}).$$

However, the previous equality contradicts the fact that $\mu(a_{k_n}) \notin W$ *for every* $n \in \mathbb{N}$.

The following corollary can be proved by either readapting the proof of Corollary 69, or by relying on it taking orthocomplements as in Corollary 67.

Corollary 70 *Let* L *be an effect algebra. Let* M *be a topological module over a topological ring* R. *Let* $\mu : L \to M$ *be a* σ*-additive measure. Let* I *be a directed set that has a countable cofinal subset* $J \subseteq I$. *If* $(a_i)_{i\in I} \subseteq L$ *is a decreasing net such that* $\bigwedge_{i\in I} a_i$ *exists, then* $(\mu(a_i))_{i\in I}$ *converges to* $\mu\left(\bigwedge_{i\in I} a_i\right)$.

7.1.1.2 Variation of a measure

The variation of a measure is an essential concept in Measure Theory since it endows measures with a norm. This is why we will work now with seminormed modules over seminormed rings.

Definition 337 (Absolutely additive measure) *Let* L *be an effect algebra. Let* M *be a seminormed module over a seminormed ring* R. *Let* $\mu : L \to M$ *be a measure. We say that* μ *is absolutely additive if* $\sum_{n\in\mathbb{N}} \|\mu(a_n)\| < +\infty$ *for every orthogonal sequence* $(a_n)_{n\in\mathbb{N}} \subseteq L$.

The next theorem shows that, under the settings of Definition 337 with somehow restrictive conditions on M, σ-additive measures on σ-effect algebras are absolutely additive.

Theorem 167 *Let* L *be an effect algebra. Let* M *be a seminormed module over a seminormed ring* R. *Let* $\mu : L \to M$ *be a measure. Then:*

1. *If* L *is an effect* σ*-algebra,* $R = \mathbb{R}$, M *is finite-dimensional and normed, and* μ *is* σ*-additive, then* μ *is absolutely additive.*
2. *If* M *is complete and* μ *is absolutely additive, then* μ *is strongly additive.*
3. *If* μ *is absolutely additive and* L *satisfies the sRDP and has a natural basis* $(b_n)_{n\in\mathbb{N}}$, *then* $\mu(L)$ *is seminorm-bounded.*

Proof 511 *The proof will follow as itemized as the statement.*

1. *Let* $(a_n)_{n\in\mathbb{N}} \subseteq L$ *be an orthogonal sequence. Since* L *is a* σ*-effect algebra, we have that* $\bigoplus_{n\in\mathbb{N}} a_n$ *exists in* L. *Next,* μ *is* σ*-additive and* M *is Hausdorff, so* $\mu\left(\bigoplus_{n\in\mathbb{N}} a_n\right) = \sum_{n\in\mathbb{N}} \mu(a_n)$. *By applying Remark 93(1), we have that* $\sum_{n\in\mathbb{N}} \mu(a_n)$ *is unconditionally convergent. Finally, since* M *is a finite-dimensional real normed space, we conclude that* $\sum_{n\in\mathbb{N}} \mu(a_n)$ *is absolutely convergent.*

2. *Let $(a_n)_{n\in\mathbb{N}} \subseteq L$ be an orthogonal sequence. By hypothesis, $\sum_{n\in\mathbb{N}} \|\mu(a_n)\| < +\infty$. This means that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Since M is complete, Proposition 125 assures that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. This proves that μ is strongly additive.*

3. *Since $(b_n)_{n\in\mathbb{N}}$ is orthogonal, by hypothesis we can consider $K := \sum_{n\in\mathbb{N}} \|\mu(b_n)\| < +\infty$. In virtue of Proposition 48,*

$$B := \{b_{i_1} \oplus \cdots \oplus b_{i_k} : i_1, \ldots, i_k \in \mathbb{N}\}$$

is a generator of L. According to Corollary 68 adapted to seminorm-boundedness, it only suffices to show that $\mu(B)$ is seminorm-bounded. For each $b_{i_1} \oplus \cdots \oplus b_{i_k} \in B$,

$$\|\mu\left(b_{i_1} \oplus \cdots \oplus b_{i_k}\right)\| = \left\|\sum_{j=1}^{k} \mu\left(b_{i_j}\right)\right\| \leq \sum_{j=1}^{k} \left\|\mu\left(b_{i_j}\right)\right\| \leq K.$$

As a consequence, $\mu(B)$ is seminorm-bounded.

Now, it is time to introduce the concept of variation of a measure.

Definition 338 (Variation of a measure) *Let L be an effect algebra. Let M be a seminormed module over a seminormed ring R. Let $\mu : L \to M$ be a measure. Then:*

- *We say that $\pi = \{e_1, e_2, \ldots, e_n\} \subseteq L$ is a decomposition of $e \in L$ if π is an orthogonal set and $e = e_1 \oplus \cdots \oplus e_n$.*
- *We define the variation of μ in $e \in L$ as the map*

$$\begin{array}{rccl} |\mu| : & L & \to & [0, +\infty] \\ & e & \mapsto & |\mu|(e) := \sup_{\pi\in\Pi} \sum_{e_i\in\pi} \|\mu(e_i)\|, \end{array}$$

  where the supremum is taken over the family Π of all decompositions $\pi = \{e_1, e_2, \ldots, e_n\}$ of e.

- *We say that μ is of bounded variation if $|\mu|(1) < +\infty$.*

The set of measures of bounded variation on an effect algebra L with values on a seminormed module M over a seminormed ring R is denoted by $\mathrm{ba}(L, M)$. It is not difficult to check that $\mathrm{ba}(L, M)$ is a submodule of M^L.

Theorem 168 *Let L be an effect algebra. Let M be a seminormed module over a seminormed ring R. Let $\mu : L \to M$ be a measure. Let $e, f \in L$ be orthogonal. Then:*

1. *$|\mu|(e) + |\mu|(f) \leq |\mu|(e \oplus f)$.*
2. *If L is RDP, then $|\mu|(e \oplus f) \leq |\mu|(e) + |\mu|(f)$.*

Proof 512 *The proof will follow as itemized as the statement.*

1. *Fix an arbitrary $\varepsilon > 0$ and let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ be decompositions of e and f, respectively, such that*

$$|\mu|(e) - \sum_{i=1}^{n} \|\mu(e_i)\| < \frac{\varepsilon}{2}$$

and

$$|\mu|(f) - \sum_{i=1}^{m} \|\mu(f_i)\| < \frac{\varepsilon}{2}.$$

Then $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is a decomposition of $e \oplus f$, therefore,

$$|\mu|(e) + |\mu|(f) < \sum_{i=1}^{n} \|\mu(e_i)\| + \sum_{i=1}^{m} \|\mu(f_i)\| + \varepsilon \leq |\mu|(e \oplus f) + \varepsilon.$$

The arbitrariness of ε implies that $|\mu|(e) + |\mu|(f) \leq |\mu|(e \oplus f)$.

2. *If L is RDP, given a decomposition $e \oplus f = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, since $e \leq e \oplus f = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, there exist $e_1, e_2, \ldots, e_n \in L$ such that $e_1 \leq a_1$, $e_2 \leq a_2$, $\ldots$ $e_n \leq a_n$ and $e = e_1 \oplus e_2 \oplus \cdots \oplus e_n$. There exist $f_1, \ldots, f_n \in L$ such that $e_i \oplus f_i = a_i$ for all $i \in \{1, \ldots, n\}$. Then*

$$\begin{aligned} e \oplus f &= a_1 \oplus a_2 \oplus \cdots \oplus a_n \\ &= (e_1 \oplus f_1) \oplus (e_2 \oplus f_2) \oplus \cdots \oplus (e_n \oplus f_n) \\ &= (e_1 \oplus e_2 \oplus \cdots \oplus e_n) \oplus (f_1 \oplus f_2 \oplus \cdots \oplus f_n), \end{aligned}$$

so $f = f_1 \oplus f_2 \oplus \cdots \oplus f_n$ and $f_i \leq f$ for all $i \in \{1, \ldots, n\}$. This shows that $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are decompositions of e and f, respectively. Thus,

$$\begin{aligned} \sum_{i=1}^{n} \|\mu(a_i)\| &= \sum_{i=1}^{n} \|\mu(e_i \oplus f_i)\| = \sum_{i=1}^{n} \|\mu(e_i) + \mu(f_i)\| \\ &\leq \sum_{i=1}^{n} \|\mu(e_i)\| + \sum_{i=1}^{n} \|\mu(f_i)\| \leq |\mu|(e) + |\mu|(f), \end{aligned}$$

Since this inequality is valid for every decomposition of $e \oplus f$, we conclude that

$$|\mu|(e \oplus f) \leq |\mu|(e) + |\mu|(f).$$

A direct consequence of Theorem 168(1) is the fact that, if $\mu : L \to M$ is a measure on an effect algebra L with values in a seminormed module M over a seminormed ring R, then $|\mu| : L \to [0, +\infty]$ is an increasing function, that is, if $a, b \in L$ with $a \leq b$, then $|\mu(a)| \leq |\mu(b)|$.

Corollary 71 *Let L be an effect algebra. Let M be a seminormed module over a seminormed ring R. Let $\mu : L \to M$ be a measure. If L satisfies the RDP, then the variation of μ is also a measure on L.*

The above corollary is another direct consequence of Theorem 168.

Proposition 132 *Let L be an effect algebra. Let M be a seminormed module over a seminormed ring R. Let $\mu : L \to M$ be a measure. Then:*

1. *If μ has bounded variation, then $\mu(L)$ is seminormed bounded.*
2. *If M is complete and μ has bounded variation, then μ is absolutely additive and strongly additive.*
3. *If $M = \mathbb{R}$ and $\mu(L)$ is bounded, then μ has bounded variation.*

Proof 513 *The proof will follow as itemized as the statement.*

1. *Fix an arbitrary $a \in L$. Then*

$$\|\mu(a)\| \leq \|\mu(a)\| + \left\|\mu\left(a^{\perp}\right)\right\| \leq |\mu|(1).$$

2. *If $(a_n)_{n\in\mathbb{N}} \subseteq L$ is an orthogonal sequence, then*

$$\sum_{n=1}^{k} \|\mu(a_n)\| \leq |\mu|\left(\bigoplus_{n=1}^{k} a_n\right) \leq |\mu|(1).$$

Thus, $\sum_{n\in\mathbb{N}} \|\mu(a_n)\| \leq |\mu|(1) < \infty$, so $\sum_{n\in\mathbb{N}} \mu(a_n)$ is absolutely convergent, hence convergent in virtue of Proposition 125.

3. *Let $K > 0$ be such that $\|\mu(a)\| \leq K$ for all $a \in L$. Fix an arbitrary decomposition $\pi = \{e_1, e_2, \ldots, e_n\}$ of 1. Let us denote $I^+ := \{i \in \{1, \ldots, n\} : \mu(e_i) \geq 0\}$ and $I^- := \{1, \ldots, n\} \setminus I^+$. Then*

$$\begin{aligned} \sum_{i=1}^{n} |\mu(e_i)| &= \sum_{i\in I^+} \mu(e_i) - \sum_{i\in I^-} \mu(e_i) = \mu\left(\bigoplus_{i\in I^+} e_i\right) - \mu\left(\bigoplus_{i\in I^-} e_i\right) \\ &\leq \sup \mu(L) - \inf \mu(L). \end{aligned}$$

As a consequence, $|\mu(1)| \leq \sup \mu(L) - \inf \mu(L)$.

As a direct consequence of the previous proposition, we obtain the following corollary.

Corollary 72 *Let L be an sRDP σ-effect algebra with a natural basis. If $\mu : L \to \mathbb{R}$ is a countably additive measure, then μ has bounded variation.*

Proof 514 *Since L is a σ-effect algebra and μ is σ-additive, we conclude, by means of Theorem 167(1), that μ is absolutely additive. According to Theorem 167(3), $\mu(L)$ is bounded. Finally, Proposition 132(3) assures that μ has bounded variation.*

We will conclude this subsection by proving that the variation of a measure induces a seminorm.

Theorem 169 *Let L be an effect algebra. Let M be a seminormed module over a seminormed ring R. The map*

$$\begin{array}{rccc} \| \|_1 : & \mathrm{ba}(L, M) & \to & [0, +\infty) \\ & \mu & \mapsto & \|\mu\|_1 := |\mu|(1) \end{array} \tag{7.2}$$

defines a seminorm in $\mathrm{ba}(L, M)$.

Proof 515 *Take $\mu, \nu \in \mathrm{ba}(L, M)$ and $r \in R$. Fix an arbitrary partition $\{e_1, \ldots, e_n\}$ of 1. Notice that*

$$\sum_{i=1}^{n} \|(\mu+\nu)(e_i)\| \leq \sum_{i=1}^{n} \|\mu(e_i)\| + \sum_{i=1}^{n} \|\nu(e_i)\| \leq |\mu|(1) + |\nu|(1).$$

The arbitrariness of the partition $\{e_1, \ldots, e_n\}$ of 1 shows that $|\mu+\nu|(1) \leq |\mu|(1) + |\nu|(1)$. This proves the triangular inequality. Next,

$$\sum_{i=1}^{n} \|(r\mu)(e_i)\| = \sum_{i=1}^{n} \|r\mu(e_i)\| \leq \|r\| \sum_{i=1}^{n} \|\mu(e_i)\| \leq \|r\||\mu|(1).$$

Again, the arbitrariness of the partition $\{e_1, \ldots, e_n\}$ of 1 shows that $|r\mu|(1) \leq \|r\||\mu|(1)$. We then conclude that $\| \ \|_1$ is a seminorm on $\mathrm{ba}(L, M)$. Finally, if R is absolutely semivalued, for every $\varepsilon > 0$, we can fix a partition $\{e_1, \ldots, e_n\}$ of 1satisfying that

$$|\mu|(1) - \varepsilon < \sum_{i=1}^{n} \|\mu(e_i)\|,$$

meaning that

$$|r|\,(|\mu|(1) - \varepsilon) \leq |r| \sum_{i=1}^{n} \|\mu(e_i)\| = \sum_{i=1}^{n} \|r\mu(e_i)\| \leq |r\mu|(1),$$

which implies that $|r||\mu|(1) = |r\mu|(1)$ by bearing in mind the arbitrariness of $\varepsilon > 0$.

7.1.1.3 Atoms and outer measures

This subsubsection is devoted to explore atoms and outer measures. We will begin by introducing the notion of atoms for measures.

Definition 339 (Atom) *Let L be an effect algebra. Let M be an ordered module over a unital ordered ring R. An element $e \in L$ is said to be an atom of a measure $\mu : L \to M$ provided that $\mu(e) > 0$ and if $f \in L$ with $f \leq e$, either $\mu(f) = 0$ or $\mu(f) = \mu(e)$.*

Next, we will work with outer measures. To start off, if R is a unital ordered ring, then the positive cone R^+ is, in fact, a commutative additive monoid (and a multiplicative monoid as well). If we add an outer point ∞ to R^+, then the addition operation can be extended to $[0, \infty]$ by defining $r + \infty := \infty =: \infty + r$ for all $r \in R^+$. This turns $[0, \infty]$ into a commutative additive monoid. The partial ordering of R^+ can also be extended by setting $r \leq \infty$ for all $r \in R^+$.

Definition 340 (Outer measure) *Let L be a Dedekind σ-complete distributive complemented lattice. Let R be a topological totally ordered ring. An outer measure is a mapping $\mu : L \to [0, \infty]$ satisfying the following conditions for all $a, b \in L$ and all $(a_n)_{n\in\mathbb{N}} \subseteq L$:*

1. *Nullity: $\mu(0) = 0$.*
2. *Monotonicity: $a \leq b \Rightarrow \mu(a) \leq \mu(b)$.*
3. *Countably subadditivity: $\mu\left(\bigvee_{n\in\mathbb{N}} a_n\right) \leq \sum_{n\in\mathbb{N}} \mu(a_n)$.*

Every outer measure induces a countably additive measure in view of the famous Carathéodory's Theorem.

Theorem 170 (Carathéodory) *Let L be a Dedekind σ-complete distributive complemented lattice. Let R be a topological totally ordered ring. Consider an outer measure $\mu : L \to [0,\infty]$. Then:*

1. *The set $E := \{e \in L : \forall a \in L \ \ \mu(a) = \mu(a \wedge e) + \mu\left(a \wedge (1 \setminus e)\right)\}$ is a σ-complete sublattice closed for complements and containing $0, 1$ (in other words, a Boolean σ-subalgebra of L).*

2. *$\mu|_E$ is a countably additive measure.*

3. *$\{e \in L : \mu(e) = 0\} \subseteq E$.*

Proof 516 *First off, notice that $\mu(a) \leq \mu(a \wedge e) + \mu\left(a \wedge (1 \setminus e)\right)$ for all $a, e \in L$ in virtue of the countably subadditivity of the outer measure μ. If $\mu(e) = 0$, then $0 \leq \mu(a \wedge e) \leq \mu(e) = 0$, so $\mu(a) \leq \mu(a \wedge e) + \mu\left(a \wedge (1 \setminus e)\right) = \mu\left(a \wedge (1 \setminus e)\right) \leq \mu(a)$ for all $a \in L$. At a first glance, $0, 1 \in E$. Also, if $e \in E$, then $1 \setminus e \in E$. Next, we will show that if $e, f \in E$, then $e \vee f \in E$. Indeed, fix an arbitrary $a \in L$. Since $f, e \in E$,*

$$\begin{aligned}
& \mu(a \wedge (e \vee f)) + \mu(a \wedge (1 \setminus (e \vee f))) = \mu((a \wedge e) \vee (a \wedge f)) \\
+ \quad & \mu(a \wedge (1 \setminus e) \wedge (1 \setminus f)) \\
= \quad & \mu((a \wedge e) \vee (a \wedge f \wedge e) \vee (a \wedge f \wedge (1 \setminus e))) + \mu(a \wedge (1 \setminus e) \wedge (1 \setminus f)) \\
= \quad & \mu((a \wedge e) \vee (a \wedge f \wedge (1 \setminus e))) + \mu(a \wedge (1 \setminus e) \wedge (1 \setminus f)) \\
\leq \quad & \mu(a \wedge e) + \mu(a \wedge (1 \setminus e) \wedge f) + \mu(a \wedge (1 \setminus e) \wedge (1 \setminus f)) \\
= \quad & \mu(a \wedge e) + \mu(a \wedge (1 \setminus e)) = \mu(a).
\end{aligned}$$

This shows that E is a sublattice of L. At this stage, let us show that $\mu\left(\bigvee_{n\in\mathbb{N}} e_n\right) = \sum_{n\in\mathbb{N}} \mu(e_n)$ if $(e_n)_{n\in\mathbb{N}} \subseteq E$ and they are pairwise disjoint. Indeed, on the one hand, $\mu\left(\bigvee_{n\in\mathbb{N}} e_n\right) \geq \mu\left(\bigvee_{n=1}^{N} e_n\right)$ for all $N \in \mathbb{N}$. Next, let us inductively prove that $\mu\left(\bigvee_{n=1}^{N} e_n\right) = \sum_{n=1}^{N} \mu(e_n)$ for all $N \in \mathbb{N}$. Suppose that it is already proven for $N \geq 1$. Then

$$\begin{aligned}
\mu\left(\bigvee_{n=1}^{N+1} e_n\right) &= \mu\left(\left(\bigvee_{n=1}^{N+1} e_n\right) \wedge e_{N+1}\right) + \mu\left(\left(\bigvee_{n=1}^{N+1} e_n\right) \wedge (1 \setminus e_{N+1})\right) \\
&= \mu(e_{N+1}) + \mu\left(\bigvee_{n=1}^{N} e_n\right) = \mu(e_{N+1}) + \sum_{n=1}^{N} \mu(e_n) \\
&= \sum_{n=1}^{N+1} \mu(e_n).
\end{aligned}$$

As a consequence, $\mu\left(\bigvee_{n\in\mathbb{N}} e_n\right) \geq \mu\left(\bigvee_{n=1}^{N} e_n\right) = \sum_{n=1}^{N} \mu(e_n)$ for all $N \in \mathbb{N}$, which implies that $\mu\left(\bigvee_{n\in\mathbb{N}} e_n\right) \geq \sum_{n\in\mathbb{N}} \mu(e_n)$. The reverse inequality is just the countably subadditivity. Thus, $\mu\left(\bigvee_{n\in\mathbb{N}} e_n\right) = \sum_{n\in\mathbb{N}} \mu(e_n)$ if $(e_n)_{n\in\mathbb{N}} \subseteq E$ and they are pairwise disjoint. It only remains to verify that if $(e_n)_{n\in\mathbb{N}} \subseteq E$, then $\bigvee_{n\in\mathbb{N}} e_n \in E$. In appliance of Theorem 3(9), there exists a pairwise disjoint sequence $(f_n)_{n\in\mathbb{N}} \subseteq E$ such that $\bigvee_{n\in\mathbb{N}} e_n = \bigvee_{n\in\mathbb{N}} f_n$. Fix an arbitrary $a \in L$. For every $N \geq 1$, $\bigvee_{n=1}^{N} f_n \in L$. Suppose that for $N \geq 1$ it is already

proven that $\mu\left(a \wedge \bigvee_{n=1}^{N+1} f_n\right) = \sum_{n=1}^{N} \mu(a_n \wedge f_n)$. *Then*

$$\begin{aligned}
\mu\left(a \wedge \bigvee_{n=1}^{N+1} f_n\right) &= \mu\left(\left(a \wedge \bigvee_{n=1}^{N+1} f_n\right) \wedge f_{N+1}\right) \\
&+ \mu\left(\left(a \wedge \bigvee_{n=1}^{N+1} f_n\right) \wedge (1 \setminus f_{N+1})\right) \\
&= \mu(a \wedge f_{N+1}) + \mu\left(a \wedge \bigvee_{n=1}^{N} f_n\right) \\
&= \mu(a \wedge f_{N+1}) + \sum_{n=1}^{N} \mu(a \wedge f_n) = \sum_{n=1}^{N+1} \mu(a \wedge f_n).
\end{aligned}$$

Next, for every $N \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{n=1}^{N} \mu(a \wedge f_n) + \mu\left(a \wedge \left(1 \setminus \bigvee_{n\in\mathbb{N}} f_n\right)\right) \\
\leq\ & \mu\left(a \wedge \bigvee_{n=1}^{N} f_n\right) + \mu\left(a \wedge \left(1 \setminus \bigvee_{n=1}^{N} f_n\right)\right) = \mu(a).
\end{aligned}$$

Therefore,

$$\sum_{n\in\mathbb{N}} \mu(a \wedge f_n) + \mu\left(a \wedge \left(1 \setminus \bigvee_{n\in\mathbb{N}} f_n\right)\right) \leq \mu(a).$$

Finally,

$$\begin{aligned}
\mu(a) &\leq \mu\left(a \wedge \bigvee_{n\in\mathbb{N}} f_n\right) + \mu\left(a \wedge \left(1 \setminus \bigvee_{n\in\mathbb{N}} f_n\right)\right) \\
&= \mu\left(\bigvee_{n\in\mathbb{N}} (a \wedge f_n)\right) + \mu\left(a \wedge \left(1 \setminus \bigvee_{n\in\mathbb{N}} f_n\right)\right) \\
&\leq \sum_{n\in\mathbb{N}} \mu(a \wedge f_n) + \mu\left(a \wedge \left(1 \setminus \bigvee_{n=1}^{N} f_n\right)\right) \leq \mu(a).
\end{aligned}$$

Carathéodory Theorem motivates the following notion, crucial in Measure Theory.

Definition 341 (Outer regular outer measure) *Let* L *be a Dedekind* σ*-complete distributive complemented lattice. Let* R *be a topological totally ordered ring. An outer measure* $\mu : L \to [0, \infty]$ *is called outer regular provided that* $\mu(a) = \inf\{\mu(e) : e \text{ is measurable and } a \leq e\}$.

7.1.1.4 Probabilities

Probability measures are a very special kind of measure, with multiple applications to the real world since they constitute the theoretical basis of Statistics. Recall that every distributive complemented lattice can be endowed with a structure of Boolean algebra, hence with a structure of effect algebra.

Definition 342 (Probability) *Let L be an effect algebra. Let R be a unital ordered ring. A measure $\mu : L \to R$ is said to be a probability measure if $\mu(L) \subseteq R^+$ and $\mu(1) = 1$.*

Under the settings of the previous definition, note that $\mu(L) \subseteq [0,1]$, where the classical interval notation $[0,1]$ stands as usually for $\downarrow 1 \cap \uparrow 0$. Also, notice that if $a, b \in L$ and $a \leq b$, then $\mu(a) \leq \mu(b)$.

Remark 94 *Let L be a distributive complemented lattice. Let R be a unital ordered ring. Let $\mu : L \to R$ be a probability measure. Then $\mu(a) = \mu(a \wedge b)$ for all $a, b \in L$ whenever $\mu(b) = 1$. Indeed, $1 = b \vee (1 \setminus b)$, so $a = a \wedge 1 = (a \wedge b) \vee (a \wedge (1 \setminus b))$, meaning that $\mu(a) = \mu(a \wedge b) + \mu(a \wedge (1 \setminus b)) = \mu(a \wedge b)$ because $\mu(a \wedge (1 \setminus b)) \leq \mu(1 \setminus b) = 0$. In a dual way, $\mu(a) = \mu(a \vee b)$ for all $a, b \in L$ whenever $\mu(b) = 0$.*

The above remark is somehow related to the notion of independence.

Remark 95 (Conditional probability) *Let L be a distributive complemented lattice. Let R be a unital ordered ring. Consider a probability measure $\mu : L \to R$. Fix $a \in L$ with $\mu(a) \in \mathcal{U}(R)$ and $\mu(a)^{-1} > 0$. The map*

$$\begin{array}{rccc} \mu(\ |a) : & L & \to & R \\ & b & \mapsto & \mu(b|a) := \mu(a)^{-1} \mu(a \wedge b) \end{array} \tag{7.3}$$

is a probability measure.

Notice that, under the settings of the previous remark, $\mu(\ |a)$ is still a probability measure if we restrict it to the sublattice $L_a := \{a \wedge b : b \in L\}$.

Definition 343 (Independent events) *Let L be a distributive complemented lattice. Let R be a unital ordered ring. Consider a probability measure $\mu : L \to R$. Two elements $a, b \in L$ are said to be independent when $\mu(a)$ and $\mu(b)$ commute and $\mu(a \wedge b) = \mu(a)\mu(b)$.*

Under the settings of the previous definition, if $\mu(a), \mu(b) \in \mathcal{U}(R)$ and $\mu(a)^{-1}, \mu(b)^{-1} > 0$, then a, b are independent if and only if $\mu(b|a) = \mu(b)$ and $\mu(a|b) = \mu(a)$.

Proposition 133 *Let L be a distributive complemented lattice. Let R be a unital ordered ring. Consider a probability measure $\mu : L \to R$. If $a, b \in L$ are independent, then so are $1 \setminus a$ and b.*

Proof 517 *It suffices to take into consideration that $\mu(1 \setminus a)\mu(b) = (\mu(1) - \mu(a))\mu(b) = (1 - \mu(a))\mu(b) = \mu(b) - \mu(a)\mu(b) = \mu(b) - \mu(a \wedge b) = \mu(b \setminus (a \wedge b))) = \mu((1 \setminus a) \wedge b)$. In a dual way, $\mu(b)\mu(1 \setminus a) = \mu((1 \setminus a) \wedge b)$.*

The following proposition characterizes those elements of a distributive complemented lattices which are independent of all other elements.

Proposition 134 *Let L be a distributive complemented lattice. Let R be a unital ordered ring. Consider a probability measure $\mu : L \to R$. Let $a \in L$. Then:*

1. *If either $\mu(a) = 0$ or $\mu(a) = 1$, then a is independent of every $b \in L$.*
2. *Assume that R is an integral domain. If a is independent of every $b \in L$, then either $\mu(a) = 0$ or $\mu(b) = 1$.*

Proof 518 *The proof will follow as itemized as the statement of the proposition.*

1. *In virtue of Proposition 133, we may assume without any loss of generality that* $\mu(a) = 1$. *This way, we conclude that* $\mu(b \wedge (1 \setminus a)) \leq \mu(1 \setminus a) = \mu(1) - \mu(a) = 1 - 1 = 0$. *Thus,* $\mu(b)\mu(a) = \mu(b) = \mu((b \wedge a) \vee (b \wedge (1 \setminus a))) = \mu(b \wedge a) + \mu(b \wedge (1 \setminus a)) = \mu(b \wedge a)$.

2. *Suppose, for instance, that* $\mu(a) \neq 1$. *Then* $\mu(1 \setminus a) \neq 0$. *Thus,* $0 = \mu(0) = \mu(a \wedge (1 \setminus a)) = \mu(a)\mu(1 \setminus a)$, *so* $\mu(a) = 0$. *Similarly, if* $\mu(a) \neq 0$, *it can be shown that* $\mu(a) = 1$.

7.1.1.5 Radon measures

The notion of Radon measure is a crucial in Functional Analysis. It is typically defined for real or complex Banach spaces [118], however, we will transport it here to the scope of topological modules.

Definition 344 (Radon measure) *Let* L *be an effect algebra endowed with a compactness structure* K. *Let* M *be a topological module over a topological ring* R. *Let* $\mu : L \to M$ *be a measure. An element* $a \in L$ *is called a Radon element if for every* 0*-neighborhood* $W \subseteq M$ *there exists a compact element* $f \leq a$ *such that* $\mu(b) \in W$ *for every* $b \leq a \ominus f$. *The measure* μ *is called a Radon measure if every element is Radon.*

Under the settings of the previous definition, if M is an ordered module, μ is positive, and $a \in L$ with $\mu(a) = 0$, then a is trivially Radon. The well-known notion of regular measure can also be transported to the scope of topological modules [18] and effect algebras.

Definition 345 (Regular measure) *Let* L *be an effect algebra endowed with a compactness structure* K *and an openness structure* T. *Let* M *be an ordered module over an ordered ring* R. *Let* $\mu : L \to M^+$ *be a measure. Let* $a \in L$. *Then:*

- a *is called inner regular if* $\mu(a) = \sup\{\mu(f) : f \leq a \text{ compact}\}$. *Then* μ *is called inner regular if every element is inner regular.*
- a *is called outer regular if* $\mu(a) = \inf\{\mu(u) : u \geq a \text{ open}\}$. *Then* μ *is called outer regular if every element is outer regular.*
- a *is called regular if it is inner and outer regular. Then* μ *is called regular if it is inner and outer regular.*

Under the settings of the previous definition, if $a \in L$ with $\mu(a) = 0$, then a is trivially inner regular. If $\mu(a) = \mu(1)$, then a is trivially outer regular.

Theorem 171 *Let* L *be an effect algebra endowed with a compactness structure* K. *Let* M *be a topological faithful ordered module over a topological faithful ordered ring* R. *Let* $\mu : L \to M^+$ *be a measure. An element* $a \in L$ *is Radon if and only if* a *is inner regular.*

Proof 519 *Let us itemize the proof in accordance with the statement of the theorem.*

⇒ *Suppose first that* a *is Radon. If* $\mu(a) = 0$, *then* a *is trivially inner regular. So, suppose that* $\mu(a) > 0$. *Notice that* $\mu(a) \geq \mu(f)$ *for all* $f \leq a$ *compact. Take an upper bound* $m \in M$ *for* $\{\mu(f) : f \leq a, f \in K\}$. *We will prove that* $\mu(a) \leq m$. *Notice that* $m \geq \mu(0) = 0$ *because* $0 \in K$. *By hypothesis,* M^+ *is totally ordered; therefore, either* $\mu(a) \leq m$ *or* $m < \mu(a)$. *If* $m < \mu(a)$, *then* $W := (-\infty, \mu(a) - m)$ *is a* 0*-neighborhood in* M. *By hypothesis, there exists a compact element* $f \leq a$ *such that, for every element* $b \leq a \ominus f$, $\mu(b) \in W$. *In particular,* $\mu(a) - \mu(f) = \mu(a \ominus f) \in W$, *meaning the contradiction that* $m < \mu(f) \leq m$.

⇐ *Conversely, suppose that a is inner regular. If $\mu(a) = 0$, then a is clearly Radon, so let us assume that $\mu(a) > 0$. Take $W \subseteq M$ a 0-neighborhood. Since $M^+ \setminus \{0\}$ is downward directed, there exists $m \in M$, $m > 0$, such that $(-m, m) \subseteq W$. By using again that $M^+ \setminus \{0\}$ is downward directed, we can find $p > 0$ such that $p \leq \mu(a)$ and $p \leq m$. Note that $0 \leq \mu(a) - p < \mu(a)$. Since M^+ is totally ordered, $\mu(a) - p$ is comparable with $\mu(f)$ for each $f \leq a$ compact. As a consequence, there exists a compact element $f \leq a$ such that $\mu(f) > \mu(a) - p$. Finally, for every element $b \leq a \ominus f$, $0 \leq \mu(b) \leq \mu(a \ominus f) = \mu(a) - \mu(f) < p \leq m$, thus $\mu(b) \in W$.*

As a consequence of Theorem 171 and under its settings, a positive measure is Radon if and only if it is inner regular.

Theorem 172 *Let L be an effect algebra endowed with a compactness structure K and an openness structure T such that $T = 1 \ominus K$. Let M be an ordered module over an ordered ring R. Let $\mu : L \to M^+$ be a measure. An element $a \in L$ is inner regular if and only if $1 \ominus a$ is outer regular.*

Proof 520 *Let us itemize the proof in accordance with the statement of the theorem.*

⇒ *Let us assume first that a is inner regular. Then $\mu(1 \ominus a) = \mu(1) - \mu(a) = \mu(1) - \sup\{\mu(f) : f \leq a, f \in K\} = \inf\{\mu(1) - \mu(f) : f \leq a, f \in K\} = \inf\{\mu(1 \ominus f) : f \leq a, f \in K\} = \inf\{\mu(u) : 1 \ominus a \leq u, u \in T\}$.*

⇐ *Conversely, suppose that $1 \ominus a$ is outer regular. Then $\mu(a) = \mu(1 \ominus (1 \ominus a)) = \mu(1) - \mu(1 \ominus a) = \mu(1) - \inf\{\mu(u) : u \geq 1 \ominus a, u \in T\} = \sup\{\mu(1) - \mu(u) : u \geq 1 \ominus a, u \in T\} = \sup\{\mu(1 \ominus u) : 1 \ominus u \leq a, u \in T\} = \sup\{\mu(f) : f \leq a, f \in K\}$.*

The upcoming theorem relies on the notion of conditional probability (7.3) previously defined and overviewed.

Theorem 173 *Let L be a distributive effect algebra endowed with a compactness structure K and an openness structure T. Let R be a unital ordered ring. Let $\mu : L \to R^+$ be a probability measure. Fix $a \in L$ with $\mu(a) \in \mathcal{U}(R)$ and $\mu(a)^{-1} > 0$. Let $b \in L$. Then:*

1. *If $a \wedge b$ is inner regular, then b is inner $\mu(\,|a)$-regular.*
2. *If $a \wedge b$ is outer regular and $1 \ominus a \in T$, then b is outer $\mu(\,|a)$-regular.*
3. *If b is outer regular, then b is outer $\mu(\,|a)$-regular.*

Proof 521 *The proof will be as itemized as the statement.*

1. *Notice that $\mu(\,|a)(f) \leq \mu(\,|a)(b)$ for all $f \leq b$ compact. Let $r \in R$ be an upper bound for $\{\mu(\,|a)(f) : f \leq b, f \in K\}$. We will prove that $\mu(\,|a)(b) \leq r$. For every $f \leq a \wedge b$ compact, we have that $\mu(f) = \mu(f \wedge a) = \mu(a)\mu(\,|a)(f) \leq \mu(a)r$, that is, $\mu(a)r$ is an upper bound for $\{\mu(f) : f \leq a \wedge b, f \in K\}$. Therefore, $\mu(a \wedge b) \leq \mu(a)r$, hence $\mu(\,|a)(b) = \mu(a)^{-1}\mu(a \wedge b) \leq \mu(a)^{-1}\mu(a)r = r$.*
2. *Again, $\mu(\,|a)(u) \geq \mu(\,|a)(b)$ for all $u \geq b$ open. Let $r \in R$ be a lower bound for $\{\mu(\,|a)(u) : u \geq b, u \in T\}$. We will prove that $\mu(\,|a)(b) \geq r$. For every $u \geq a \wedge b$ open, $\mu(u) \geq \mu(u \wedge a) = \mu((u \wedge a) \oplus 0) = \mu((u \wedge a) \oplus ((1 \ominus a) \wedge a)) = \mu(((u \wedge a) \oplus (1 \ominus a)) \wedge a) = \mu(a)\mu(\,|a)((u \wedge a) \oplus (1 \ominus a)) = \mu(a)\mu(\,|a)(u \vee (1 \ominus a)) \geq \mu(a)r$ because $u \vee (1 \ominus a) \in T$ (T is finitely sup closed) and $u \vee (1 \ominus a) \geq b$ since $b = b \wedge 1 = b \wedge (a \oplus (1 \ominus a)) = (b \wedge a) \oplus (b \wedge (1 \ominus a)) \leq (u \wedge a) \oplus (1 \ominus a) = u \vee (1 \ominus a)$. Thus, $\mu(a)r$ is a lower bound for $\{\mu(u) : u \geq a \wedge b, u \in T\}$. Therefore, $\mu(a \wedge b) \geq \mu(a)r$, hence $\mu(\,|a)(b) = \mu(a)^{-1}\mu(a \wedge b) \geq \mu(a)^{-1}\mu(a)r = r$.*

3. *Once again,* $\mu(\ |a)(u) \geq \mu(\ |a)(b)$ *for all* $u \geq b$ *open. Let* $r \in R$ *be a lower bound for* $\{\mu(\ |a)(u) : u \geq b, u \in T\}$*. We will prove that* $\mu(\ |a)(b) \geq r$*. For each* $u \in T$ *with* $b \leq u$*,*

$$\begin{aligned} r - \mu(\ |a)(b) + \mu(a)^{-1}\mu(b) &\leq \mu(\ |a)(u) - \mu(\ |a)(b) + \mu(a)^{-1}\mu(b) \\ &= \mu(a)^{-1}\left(\mu(a \wedge u) - \mu(a \wedge b) + \mu(b)\right) \\ &= \mu(a)^{-1}\mu\left((a \wedge u) \ominus (a \wedge b) + \mu(b)\right) \\ &\leq \mu(a)^{-1}\left(\mu(u \ominus b) + \mu(b)\right) \\ &= \mu(a)^{-1}(\mu(u) - \mu(b) + \mu(b)) = \mu(a)^{-1}\mu(u). \end{aligned}$$

Therefore, $\mu(a)\left(r - \mu(\ |a)(b) + \mu(a)^{-1}\mu(b)\right)$ *is a lower bound for* $\{\mu(u) : u \geq b, u \in T\}$*, meaning that* $\mu(a)\left(r - \mu(\ |a)(b) + \mu(a)^{-1}\mu(b)\right) \leq \mu(b)$*, that is,* $r \leq \mu(\ |a)(b)$ *as desired.*

7.1.2 Measure spaces

Integration of functions with respect to a measure is possible when the measure is defined on a Boolean algebra of sets. This is why, from now on throughout the end of this chapter, we will consider measures on Boolean algebras of sets [120].

7.1.2.1 Measurable spaces

The characteristic function of a subset A of a set Ω is defined as

$$\begin{array}{rccl} \chi_A : & \Omega & \to & \{0,1\} \\ & \omega & \mapsto & \chi_A(\omega) := \left\{ \begin{array}{ll} 1 & \omega \in A, \\ 0 & \omega \notin A. \end{array} \right. \end{array}$$

Easily verifiable properties satisfied by characteristic functions follow for all $A, B \in \mathcal{P}(\Omega)$:

- $\chi_\varnothing = 0$ and $\chi_\Omega = 1$.
- $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$.
- $\chi_{A \cap B} = \chi_A \chi_B$.
- $\chi_A = \chi_B$ if and only if $A = B$.

Definition 346 (Simple function) *Let* X, Y *be nonempty sets. A function* $f \in X^Y$ *is said to be simple provided that* $f(X)$ *is finite.*

If Ω is a set and R is a ring, every simple function $f : \Omega \to R$ can be written as a linear combination of characteristic functions,

$$f = \sum_{i=1}^{k} a_i \chi_{A_i},$$

where $f(\Omega) = \{a_1, \ldots, a_k\}$ and $A_i := f^{-1}(\{a_i\})$ for all $i \in \{1, \ldots, k\}$.

Lemma 172 *Let* Ω *be a set. Let* R *be a ring. Then the set* $\mathcal{S}(\Omega, R) := \left\{f \in R^\Omega : f \text{ is simple}\right\}$ *is a subring and an* (R, R)*-subbimodule of* R^Ω*.*

Proof 522 *It is trivial to realize that, if $f, g \in R^{\Omega}$ are simple functions and $r \in R$, then $f+g, fg, rf, fr$ are also simple functions. However, we will provide explicit expressions of the previous operations. Assume that $f = \sum_{i=1}^{k} a_i \chi_{A_i}, g = \sum_{j=1}^{l} b_j \chi_{B_j}$, where $\{A_1, \ldots, A_k\}, \{B_1, \ldots, B_l\}$ are partitions of Ω. For every $i \in \{1, \ldots, k\}, j \in \{1, \ldots, l\}$, define $C_{ij} := A_i \cap B_j$. Then, for every $i \in \{1, \ldots, k\}, j \in \{1, \ldots, l\}$, $\{C_{i1}, \ldots, C_{il}\}$ is a partition of A_i and $\{C_{1j}, \ldots, C_{kj}\}$ is a partition of B_i. This way,*

$$\chi_{A_i} = \sum_{j=1}^{l} \chi_{C_{ij}} \text{ and } \chi_{B_j} = \sum_{i=1}^{k} \chi_{C_{ij}}$$

for every $i \in \{1, \ldots, k\}, j \in \{1, \ldots, l\}$. We obtain that

$$f = \sum_{i=1}^{k} a_i \sum_{j=1}^{l} \chi_{C_{ij}} = \sum_{i=1}^{k} \sum_{j=1}^{l} a_i \chi_{C_{ij}} \text{ and } g = \sum_{j=1}^{l} b_j \sum_{i=1}^{k} \chi_{C_{ij}} = \sum_{i=1}^{k} \sum_{j=1}^{l} b_j \chi_{C_{ij}}.$$

Hence

$$f + g = \sum_{i=1}^{k} \sum_{j=1}^{l} (a_i + b_j) \chi_{C_{ij}} \text{ and } fg = \sum_{i=1}^{k} \sum_{j=1}^{l} a_i b_j \chi_{C_{ij}}.$$

Finally, it is clear that

$$rf = \sum_{i=1}^{k} r a_i \chi_{A_i} \text{ and } fr = \sum_{i=1}^{k} a_i r \chi_{A_i}.$$

The notion of measurable space is partially motivated by the following result about simple functions. Recall that a field of sets is a pair $(\Omega, \mathcal{A})$ where Ω is a set and $\mathcal{A}$ is a Boolean algebra of subsets of Ω.

Lemma 173 *Let $(\Omega, \mathcal{A})$ be a field of sets. Let R be a ring. Let $f : \Omega \to R$ be a simple function written as $f = \sum_{i=1}^{k} a_i \chi_{A_i}$, where $f(\Omega) = \{a_1, \ldots, a_k\}$ and $A_i := f^{-1}(\{a_i\})$ for all $i \in \{1, \ldots, k\}$. The following conditions are equivalent:*

1. *$A_i \in \mathcal{A}$ for all $i \in \{1, \ldots, k\}$.*
2. *$f^{-1}(F) \in \mathcal{A}$ for every subset F of R.*
3. *$f^{-1}(F) \in \mathcal{A}$ for every singleton subset F of R.*

Proof 523 *The proof will follow as itemized as the statement.*

1⇒2 *Let F be a subset of R. If $F \cap \{a_1, \ldots, a_k\} = \varnothing$, then $f^{-1}(F) = \varnothing \in \mathcal{A}$. If $F \cap \{a_1, \ldots, a_k\} \neq \varnothing$, then $f^{-1}(F) = A_{j_1} \cup \cdots \cup A_{j_p} \in \mathcal{A}$, where $F \cap \{a_1, \ldots, a_k\} = \{a_{j_1}, \ldots, a_{j_p}\}$.*

2⇒3 *Trivial.*

3⇒1 *By hypothesis, $A_i = f^{-1}(\{a_i\}) \in \mathcal{A}$ for all $i \in \{1, \ldots, k\}$.*

By bearing in mind that singletons are closed subsets in every Hausdorff topological space, Lemma 173 implies the following remark.

Remark 96 *Let $(\Omega, \mathcal{A})$ be a field of sets. Let R be a Hausdorff topological ring. Let $f : \Omega \to R$ be a simple function written as $f = \sum_{i=1}^{k} a_i \chi_{A_i}$, where $f(\Omega) = \{a_1, \ldots, a_k\}$ and $A_i := f^{-1}(\{a_i\})$ for all $i \in \{1, \ldots, k\}$. The following conditions are equivalent:*

1. $A_i \in \mathcal{A}$ *for all* $i \in \{1, \ldots, k\}$.
2. $f^{-1}(F) \in \mathcal{A}$ *for every closed subset* F *of* R.

Measurable spaces are historically defined in the context of σ-field of sets (although they can be perfectly defined for field of sets).

Definition 347 (Measurable space) *A measurable space is a* σ*-field of sets, in other words, a pair* (Ω, Σ)*, where* Ω *is a set and* Σ *is a Boolean* σ*-algebra of subsets of* Ω.

With the notion of measurable space comes the one of measurable function.

Definition 348 (Measurable function) *Let* $(\Omega_1, \Sigma_1), (\Omega_2, \Sigma_2)$ *be measurable spaces. A function* $f : \Omega_1 \to \Omega_2$ *is said to be measurable provided that* $f^{-1}(\Sigma_2) \subseteq \Sigma_1$*. The set of measurable functions from* Ω_1 *to* Ω_2 *is denoted by* $\mathsf{L}^0(\Omega_1, \Omega_2)$.

It is fairly trivial that the composition of measurable functions is measurable. According to Remark 3 and Notation 2, in the above definition, $f^{-1}(\Sigma_2) := \left\{f^{-1}(B) : B \in \Sigma_2\right\}$. On the other hand, if τ is a collection of subsets of a set Y, then the Boolean σ-algebra generated by τ is denoted by $\Sigma(\tau)$. Using universal algebras notation, $\Sigma(\tau)$ is precisely $\mathrm{span}(\tau)$ in the Boolean σ-algebra $\mathcal{P}(Y)$. Thus, by bearing in mind Remark 49 and Remark 8, if (Ω, Σ) is a measurable space, then a function $f : \Omega \to Y$ is measurable if and only if $f^{-1}(\tau) \subseteq \Sigma$.

Definition 349 (Borel set) *Let* X *be a topological space. The Boolean* σ*-algebra generated by the closed subsets of* X *is called the Borel* σ*-algebra of* X *and is denoted by* $\mathcal{B}(X)$.

If $L := \mathcal{B}(X)$ for X a Hausdorff topological space, then the collection of compact subsets of X is a compactness structure in L, and the collection of open subsets of X is an openness structure in L. Continuous functions between topological spaces are Borel-measurable. Now, Remark 96 can be rewritten as follows.

Remark 97 *Let* (Ω, Σ) *be a measurable space. Let* R *be a Hausdorff topological ring. Let* $f : \Omega \to R$ *be a simple function written as* $f = \sum_{i=1}^{k} a_i \chi_{A_i}$*, where* $f(\Omega) = \{a_1, \ldots, a_k\}$ *and* $A_i := f^{-1}(\{a_i\})$ *for all* $i \in \{1, \ldots, k\}$*. Then* $f \in \mathsf{L}^0(\Omega, R)$ *if and only if* $f^{-1}(F) \in \mathcal{A}$ *for every closed subset* F *of* R.

In the next results, we will study measurable functions with values on a totally ordered topological ring.

Theorem 174 *Let* (Ω, Σ) *be a measurable space. Let* R *be a separable topological totally ordered ring with no holes. Let* $f, g \in R^{\Omega}$*. Let* $r \in R$*. Then:*

1. $f \in \mathsf{L}^0(\Omega, R)$ *if and only if* $f^{-1}((-\infty, d)) \in \Sigma$ *for all* d *in a countable dense subset* D *of* R.
2. *If* $f, g \in \mathsf{L}^0(\Omega, R)$*, then* $f \pm g, fg, rf, fr \in \mathsf{L}^0(\Omega, R)$*. As a consequence,* $\mathsf{L}^0(\Omega, R)$ *is a subring and an* (R, R)*-subbimodule of* R^{Ω}.
3. *If* R^{Ω} *is endowed with the pointwise convergence ring topology, then* $\mathsf{L}^0(\Omega, R)$ *is sequentially closed in* R^{Ω}.

Proof 524 *Since* R *is separable and has no holes, Theorem 36 assures that* $\mathcal{B} := \{(-\infty, d), (c, d), (c, +\infty) : c, d \in D\}$ *is a countable base of* R*, where* D *is a countable dense subset of* R*. The proof follows, itemized like the statement of the theorem.*

1. *First off, if $f : \Omega \to R$ is measurable, then $f^{-1}((-\infty, s)) \in \Sigma$ for all $s \in R$ because $(-\infty, s)$ is a Borel subset of R. Conversely, suppose that $f^{-1}((-\infty, r)) \in \Sigma$ for all $r \in D$. Since $\mathcal{B}$ is countable, it only suffices to check that $f^{-1}(\mathcal{B}) \subseteq \Sigma$. Indeed, by hypothesis, $f^{-1}((-\infty, d)) \in \Sigma$ for all $d \in D$. Therefore, by taking complementaries, $f^{-1}([d, \infty)) \in \Sigma$ for all $d \in D$. Fix now an arbitrary $c \in D$. Since X has no holes, $(c, +\infty) = \bigcup \{[d, +\infty) : d \in D, c < d\}$. Thus $f^{-1}((c, +\infty)) = \bigcup \left\{f^{-1}([d, +\infty)) : d \in D, c < d\right\} \in \Sigma$. By taking complementaries, $f^{-1}((-\infty, c]) \in \Sigma$ for all $c \in D$. Finally, fix arbitrary elements $c, d \in D$ with $c < d$. Note that $X \setminus (c, d) = (-\infty, c] \cup [d, +\infty)$, meaning that $f^{-1}(X \setminus (c, d)) = f^{-1}((-\infty, c]) \cup f^{-1}([d, +\infty)) \in \Sigma$. By taking complementaries again, $f^{-1}((c, d)) \in \Sigma$.*

2. *Fix an arbitrary $c \in R$. Since R has no holes, $(f + g)^{-1}((-\infty, c)) = \bigcup_{d \in D} f^{-1}((-\infty, d)) \cap g^{-1}((-\infty, c - d)) \in \Sigma$. This shows that $f \pm g \in \mathsf{L}^0(\Omega, R)$. Next, observe that rf is the composition of f with the left-multiplication operator on R, which is a continuous function on R, hence Borel-measurable. Since the composition of measurable functions is measurable, we conclude that rf is measurable. Similarly, fr is measurable. Now, keep in mind that fg is equal to the following composition:*

$$\begin{array}{ccccc} \Omega & \overset{f \times g}{\to} & R \times R & \overset{\cdot}{\to} & R \\ \omega & \mapsto & (f(\omega), g(\omega)) & \mapsto & f(\omega)g(\omega). \end{array}$$

Since multiplication on R is continuous by definition of topological ring (and, thus, Borel measurable), it only remains to check that

$$\begin{array}{rccc} f \times g : & \Omega & \to & R \times R \\ & \omega & \mapsto & (f(\omega), g(\omega)) \end{array}$$

is Borel measurable when $R \times R$ is endowed with the product topology. This is essentially trivial. Indeed, $\mathcal{B} \times \mathcal{B}$ is a base of $R \times R$. Then, for every $U, W \in \mathcal{B}$, $(f \times g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \in \Sigma$. Finally, every nonempty open subset of $R \times R$ is a countable union of elements of $\mathcal{B} \times \mathcal{B}$, meaning that $(f \times g)^{-1}(\mathcal{B} \times \mathcal{B}) \subseteq \Sigma$. As a consequence, $f \times g$ is Borel measurable, and so is fg.

3. *Let $(f_n)_{n \in \mathbb{N}} \subseteq \mathsf{L}^0(\Omega, R)$ be pointwise convergent to $f \in R^\Omega$. Simply note that*

$$f^{-1}((-\infty, c)) = \bigcup \left\{ \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} f_m^{-1}((-\infty, d)) : d \in D, d < c \right\} \in \Sigma$$

for all $c \in D$.

Under the settings of Theorem 174 and as a direct consequence of it, together with Lemma 172,

$$\mathsf{E}(\Omega, R) := \mathcal{S}(\Omega, R) \cap \mathsf{L}^0(\Omega, R) = \left\{f \in R^\Omega : f \text{ is simple and measurable}\right\}$$

is a subring and an (R, R)-subbimodule of R^Ω.

7.1.2.2 Measure spaces

Measure spaces constitute the main object in classical Measure Theory. They are obtained by combining measurable spaces with measures.

Definition 350 (Measure space) *Let (Ω, Σ) be a measurable space. Let M be a Hausdorff topological module over a Hausdorff topological ring R. Let $\mu : \Sigma \to M$ be a countably additive measure. We call (Ω, Σ, μ) a measure space.*

In the next subsubsection, we will deal with Haar measures, which are invariant for translations in a topological group. This translation invariance can, in fact, be generalized to topological spaces.

Definition 351 (Invariant measure) *Let X be a topological space. Let $\mathcal{H} \subseteq \text{homeo}(X)$ be a class of homeomorphisms on X. Let M be a topological module over a topological ring. A measure $\mu : \mathcal{B}(X) \to M$ is said to be $\mathcal{H}$-invariant provided that $\mu(h(A)) = \mu(A)$ for all $A \in \mathcal{B}(X)$ and all $h \in \mathcal{H}$.*

Recall that a subgroup of homeomorphisms on a topological space is said to be isotropic if any two points in the space can be mapped to each other by a homeomorphism of the subgroup. Equivalently, the subgroup acts transitively on the space.

Theorem 175 *Let X be a homogeneous compact topological space. Let $\mathcal{H} \subseteq \text{homeo}(X)$ be an isotropic subgroup of homeomorphisms on X. Let M be an ordered module over an ordered ring R. Let $\mu : \mathcal{B}(X) \to M^+$ be a finitely additive positive measure. If $\mu(X) > 0$, then $\mu(U) > 0$ for each nonempty open subset U of X.*

Proof 525 *Suppose on the contrary that there exists a nonempty open subset U for which $\mu(U) = 0$. Since $\mathcal{H}$ is isotropic, the set $\{h(U) : h \in \mathcal{H}\}$ is an open covering of X. Since X is compact, a finite subcovering can be extracted $\{h_1(U), \ldots, h_l(U)\}$. Then $0 < \mu(X) \leq \mu(h_1(U)) + \cdots + \mu(h_l(U)) = \mu(U) + \overset{l}{\cdots} + \mu(U) = 0$, which is a contradiction.*

Haar measures [121] are extremely important mathematical objects. Here, we will work with Haar measures valued on a module.

Definition 352 (Haar measure) *Let R be a ring. Let M be an R-module. Let G be a topological group. A left Haar measure on G is a measure $\mu : \mathcal{B}(G) \to M$ which is left invariant, that is, $\mu(gA) = \mu(A)$ for all $A \in \mathcal{B}(G)$ and all $g \in G$.*

Under the settings of the previous definition and in view of the continuity of the group multiplication, if A is a Borel subset of G, then so is gA for all $g \in G$. Right Haar measures are dually defined. A Haar measure is a left and right Haar measure.

Example 85 (Counting measure) *Let R be a ring. Let G be a finite discrete group. Then $\mathcal{B}(G) = \mathcal{P}(G)$ and the counting measure is trivially left and right invariant, hence a Haar measure,*

$$\begin{array}{rcl} \mathcal{P}(G) & \to & R \\ A & \mapsto & \text{card}(A)1, \end{array} \tag{7.4}$$

simply because if $a_1, \ldots, a_k$ are distinct elements of G and $g \in G$, then $ga_1, \ldots, ga_k$ and $a_1 g, \ldots, a_k g$ are, respectively, distinct elements of G as well.

Under the settings of the previous example, if R has characteristic p and $A \subseteq G$ is composed of p elements, then A is a set of null measure. To avoid these pathological cases, it only suffices to take R a unital ordered ring, which, in particular, has characteristic 0. Counting measures taking values on a unital ordered ring have atoms (the singletons).

Lemma 174 *Let R be a ring. Let M be an R-module. Let G be a topological group. Let $\mu : \mathcal{B}(G) \to M$ be a measure. Suppose that G acts continuously on a topological space S. Then:*

1. *If $s \in S$ and $\{s\}$ is closed, then G_s is closed, hence a Borel subset of G. In particular, if S is Hausdorff, then all estabilizers are closed, hence Borel subsets.*

2. *If S is Hausdorff, then* $\mathrm{fix}(S)$ *is closed.*

3. *If S is Hausdorff and the action is uniform, then all stabilizers have the same measure.*

4. *If S is Hausdorff, then* $\{\mu(G_s) : s \in S\} = \{\mu(G_s) : s \in \mathscr{K}(S)\}$.

5. *If μ is a left or right Haar measure and H is a subgroup of G of finite index, then* $\mu(G) = [G : H]\mu(H)$.

6. *If μ is a left or right Haar measure and $\mathcal{R}$ is a congruence in G such that ${}^{G}/_{\mathcal{R}}$ is finite, then* $\mu(G) = \mathrm{card}\left({}^{G}/_{\mathcal{R}}\right)\mu\left([1]_{\mathcal{R}}\right)$.

Proof 526 *The proof will follow as itemized as the statement of the lemma.*

1. *Note that $G_s = \mathsf{R}_s^{-1}(\{s\})$. At this stage, keep in mind that $\{s\}$ is closed by hypothesis. Also, the right multiplication operator R_s is a continuous function because it is a marginal map of the continuous action. As a consequence, G_s is closed, hence a Borel subset of G.*

2. *Let $(s_\alpha)_{\alpha\in A}$ be a net in* $\mathrm{fix}(S)$ *converging to s. Fix an arbitrary $g \in G$. By continuity of the action, $(gs_\alpha)_{\alpha\in A}$ converges to gs. Since $gs_\alpha = s_\alpha$ for all $\alpha \in A$, we conclude that $(s_\alpha)_{\alpha\in A}$ converges to gs. Finally, the fact that S is Hausdorff allows that $gs = s$. The arbitrariness of $g \in G$ implies that $s \in \mathrm{fix}(S)$.*

3. *For every $s, t \in S$ there exists $g \in G$ such that $G_s = g^{-1}G_t g$. Since μ is left and right invariant, $\mu(G_s) = \mu\left(g^{-1}G_t g\right) = \mu(G_t)$ for all $s, t \in S$, meaning that all estabilizers have the same measure.*

4. *For every $s \in S$ there exists $t \in \mathscr{K}(S)$ such that $s \in Gt$, so there is $g \in G$ with $s = gt$ and then $G_t = g^{-1}G_s g$. Since μ is left and right invariant, $\mu(G_t) = \mu\left(g^{-1}G_s g\right) = \mu(G_s)$.*

5. *Simply observe that G is the disjoint union of the cosets of H. Then, by using left cosets for instance, we can write $G = \bigcup_{i=1}^{[G:H]} g_i H$ for some $g_1, \ldots, g_{[G:H]} \in G$. Since μ is multiplicatively invariant, we have that*

$$\mu(G) = \sum_{i=1}^{[G:H]} \mu(g_i H) = \sum_{i=1}^{[G:H]} \mu(H) = [G : H]\mu(H).$$

6. *Again, G is the disjoint union of the equivalence classes of $\mathcal{R}$, say $G = \bigcup_{i=1}^{k} [g_i]_{\mathcal{R}}$ for some $g_1, \ldots, g_k \in G$, where $k = \mathrm{card}\left({}^{G}/_{\mathcal{R}}\right)$. Since μ is multiplicatively invariant, we have that*

$$\mu(G) = \sum_{i=1}^{k} \mu\left([g_i]_{\mathcal{R}}\right) = \sum_{i=1}^{k} \mu\left(g_i\,[1]_{\mathcal{R}}\right) = \sum_{i=1}^{k} \mu\left([1]_{\mathcal{R}}\right) = \mathrm{card}\left({}^{G}/_{\mathcal{R}}\right)\mu\left([1]_{\mathcal{R}}\right).$$

An ordered ring R is said to verify the Archimedean property provided that for any positive elements r, s in R, there exists $n \in \mathbb{N}$ such that $nr > s$.

Lemma 175 *Let R be an Archimedean totally ordered field. Let G be a topological group. Let $\mu : \mathcal{B}(G) \to R^+$ be a left or right Haar measure. Then:*

1. *If H is a subgroup of G of infinite index, then* $\mu(H) = 0$.

2. *If $\mathcal{R}$ is a congruence in G such that ${}^{G}/_{\mathcal{R}}$ is infinite, then $\mu([1]_{\mathcal{R}}) = 0$.*

Proof 527 *The proof will follow as itemized as the statement of the lemma.*

1. *We use again that G is the disjoint union of the cosets of H. Fix an arbitrary $k \in \mathbb{N}$. Then $G \supseteq \bigcup_{i=1}^{k} g_i H$, where $g_1 H, g_2 H, \ldots, g_k H$ are k different cosets of H, which implies that*

$$\mu(G) \geq \sum_{i=1}^{k} \mu(g_i H) = \sum_{i=1}^{k} \mu(H) = k\mu(H).$$

If $\mu(H) \neq 0$, then the above inequality contradicts the fact that R be Archimedean.

2. *Again, G is the disjoint union of the equivalence classes of $\mathcal{R}$. Fix an arbitrary $k \in \mathbb{N}$. If $[g_1]_{\mathcal{R}}, [g_2]_{\mathcal{R}}, \ldots, [g_k]_{\mathcal{R}}$ are k different equivalence classes, then $G \supseteq \bigcup_{i=1}^{k} [g_i]_{\mathcal{R}}$, meaning that*

$$\mu(G) \geq \bigcup_{i=1}^{k} \mu([g_i]_{\mathcal{R}}) = \sum_{i=1}^{k} \mu(g_i\,[1]_{\mathcal{R}}) = \sum_{i=1}^{k} \mu([1]_{\mathcal{R}}) = k\mu([1]_{\mathcal{R}}).$$

If $\mu([1]_{\mathcal{R}}) \neq 0$, then the above inequality contradicts the fact that R be Archimedean.

The notion of stabilizing probability is a generalization of that of commuting probability [122, 123] for pseudoconjugation actions.

Definition 353 (Stabilizing probability) *Let R be a unital ordered ring. Let G be a Hausdorff topological group. Let $\mu : \mathcal{B}(G) \to R^{+}$ be a probability measure. Suppose that G acts continuously on itself. The stabilizing probability of G is defined as $\Pr(G) := (\mu \times \mu)(\mathcal{S})$, where $\mathcal{S} := f^{-1}(\{1\})$ and*

$$\begin{array}{rccc} f: & G \times G & \to & G \\ & (x,y) & \mapsto & f(x,y) := x^{y} x^{-1}. \end{array} \tag{7.5}$$

Under the settings of the previous definition, note that (7.5) is trivially a continuous function, which implies that $\mathcal{S}$ is closed, hence Borel, since $\{1\}$ is closed because G is Hausdorff.

Theorem 176 *Let R be a topological totally ordered ring. Let G be a Hausdorff topological group. Let $\mu : \mathcal{B}(G) \to R^{+}$ be a nonatomic outer-regular measure. Then μ is uniformly continuous when $\mathcal{P}(G)$ is endowed with the uniformity whose base is $\mathcal{B}_{\mathcal{N}_1}$ (Example 36).*

Proof 528 *Fix an arbitrary neighborhood U_0 of 0 in R. There exists $r_0 > 0$ in R such that $(-r_0, r_0) \subseteq U_0$. Since μ is outer regular,*

$$\mu(\{1\}) = \inf\{\mu(W) : W \text{ is an open neighborhood of } 1 \text{ in } G\}. \tag{7.6}$$

Next, μ is nonatomic; therefore, $\mu(\{1\}) = 0$, which implies that there exists an open neighborhood W_0 of 1 in G such that $0 \leq \mu(W_0) < r_0$. Now, if $A, B \in \mathcal{B}(G)$ and $A \Delta B \subseteq W_0$, then

$$\begin{array}{rcl} \mu(A) - \mu(B) & = & [\mu(A) - \mu(A \cap B)] - [\mu(B) - \mu(A \cap B)] \\ & = & \mu(A \setminus (A \cap B)) - \mu(B \setminus (A \cap B)) \\ & = & \mu(A \setminus B) - \mu(B \setminus A) \leq \mu(A \setminus B) + \mu(B \setminus A) \\ & = & \mu(A \Delta B) \leq \mu(W_0) < r_0, \end{array}$$

and the same argument shows that $\mu(B) - \mu(A) < r_0$, resulting in $\mu(A) - \mu(B) \in (-r_0, r_0) \subseteq U_0$.

Since the composition of uniformly continuous functions is uniformly continuous, strongly continuous self-actions allow the measurability of the stabilizing map (7.7).

Corollary 73 *Let R be a topological totally ordered ring. Let G be a Hausdorff topological group. Let $\mu : \mathcal{B}(G) \to R^+$ be a nonatomic outer-regular measure. Suppose that G acts strongly continuously on itself. The mapping*

$$\begin{array}{ccc} G & \to & R \\ g & \mapsto & \mu(G_g) \end{array} \tag{7.7}$$

is uniformly continuous, hence Borel-measurable.

Proof 529 *Simply observe that* (7.7) *is the composition of* (3.3) *with μ.*

7.2 Integration

The concept of integration forms the second fundamental pillar of classical Calculus (remember that the first fundamental pillar was the notion of derivation).

7.2.1 The definite integral

In the fourth chapter of this book, we defined indefinite integrals as the inverse operation to derivations. Here, we will introduce definite integrals a special type of linear operators defined on a very specific domain called integrable domains.

7.2.1.1 Integrable domains

If R is a topological ring and Ω is a set, then R^Ω, or maybe a subring of it, can be endowed with a convergence ring topology provided that we have an upward directed subset $\mathcal{G}$ of $\mathcal{P}(\Omega)$. If $\mathcal{G}$ contains the finite subsets of Ω, then the convergence ring topology induced by $\mathcal{G}$ is finer than the pointwise convergence ring topology.

Definition 354 (Integrable domain) *Let (Ω, Σ) be a measurable space. Let R be a Hausdorff topological ring. Consider R^Ω endowed with a ring topology finer than the pointwise convergence topology. We say that $\mathcal{I}$ is an integrable domain provided that:*

- *$\mathcal{I}$ is a (unitary) subring and an (R, R)-subbimodule of R^Ω.*
- *$\mathsf{E}(\Omega, R) \subseteq \mathcal{I} \subseteq \mathsf{L}^0(\Omega, R)$.*
- *$\mathcal{I}$ is sequentially closed in $\mathsf{L}^0(\Omega, R)$.*

Under the settings of the above definition, since $\mathcal{I}$ is a unitary subring of R^Ω and an (R, R)-submodule of R^Ω, then $\mathcal{I}$ clearly contains the constant functions. Theorem 174, together with Lemma 172, provides us with examples of integrable domain.

Example 86 *Let (Ω, Σ) be a measurable space. Let R be a separable topological totally ordered ring with no holes. Consider R^Ω endowed with the pointwise convergence ring topology. Then:*

- $\mathsf{L}^0(\Omega, R)$ *is a subring and an* (R, R)*-subbimodule of* R^Ω*. As a consequence,* $\mathsf{L}^0(\Omega, R)$ *is an integrable domain. In addition,* $\mathsf{L}^0(\Omega, R)$ *is sequentially closed in* R^Ω *for the pointwise convergence ring topology.*
- $\mathsf{E}(\Omega, R)$ *is also a subring and an* (R, R)*-subbimodule of* R^Ω*, but it is not necessarily sequentially closed in* $\mathsf{L}^0(\Omega, R)$*. Nevertheless,* $\mathrm{cl}(\mathsf{E}(\Omega, R)) \cap \mathsf{L}^0(\Omega, R)$ *is still a subring and an* (R, R)*-subbimodule of* R^Ω*, and it is sequentially closed in* $\mathsf{L}^0(\Omega, R)$*.*

7.2.1.2 Integral operators

In classical Integral Calculus, a very well-known property of integrals is their linearity properties. Here, we will go the other way around and define integrals by means of continuous linear operators.

Definition 355 (Integrable operator) *Let* (Ω, Σ) *be a measurable space. Let* R *be a Hausdorff topological ring. Let* M *be a Hausdorff topological* R*-module. Consider* R^Ω *endowed with a ring topology finer than the pointwise convergence topology. An integral operator is a continuous linear operator* $T : \mathcal{I} \to M$*, where* $\mathcal{I}$ *is an integrable domain of* $\mathsf{L}^0(\Omega, R)$*.*

Every integral operator induces a measure. However, first a technical lemma is needed.

Lemma 176 *Let* (Ω, Σ) *be a measurable space. Let* R *be a Hausdorff topological ring. Let* M *be a Hausdorff topological* R*-module. Consider* R^Ω *endowed with the pointwise convergence ring topology. If* $(A_i)_{i\in\mathbb{N}}$ *is a sequence of pairwise-disjoint measurable subsets of* Ω*, then* $\left(\chi_{\bigcup_{i=1}^n A_i}\right)_{n\in\mathbb{N}}$ *converges to* $\left(\chi_{\bigcup_{i=1}^\infty A_i}\right)_{n\in\mathbb{N}}$*.*

Proof 530 *Fix an arbitrary* $U \in \mathcal{N}_0(R)$ *and an arbitrary* $w \in \Omega$*. If* $w \notin \bigcup_{i=1}^\infty A_i$*, then*

$$\chi_{\bigcup_{i=1}^n A_i}(w) - \chi_{\bigcup_{i=1}^\infty A_i}(w) \in U$$

for all $n \in \mathbb{N}$*. If* $w \in \bigcup_{i=1}^\infty A_i$*, then there exists* $n_0 \in \mathbb{N}$ *such that* $w \in A_{n_0}$*, meaning that*

$$\chi_{\bigcup_{i=1}^n A_i}(w) - \chi_{\bigcup_{i=1}^\infty A_i}(w) \in U$$

for all $n \geq n_0$*.*

Now we are in the right position to state and prove the main result of this subsubsection.

Theorem 177 *Let* (Ω, Σ) *be a measurable space. Let* R *be a Hausdorff topological ring. Let* M *be a Hausdorff topological* R*-module. Consider* R^Ω *endowed with the pointwise convergence ring topology. Let* $\mathcal{I}$ *be an integrable domain of* $\mathsf{L}^0(\Omega, R)$*. Let* $T : \mathcal{I} \to M$ *be an integral operator. Then*

$$\begin{array}{rccl} \mu : & \Sigma & \to & M \\ & A & \mapsto & \mu(A) := T(\chi_A) \end{array}$$

is a countably additive measure on Ω*.*

Proof 531 *If* A, B *are disjoint measurable subsets of* Ω*, then*

$$\mu(A \cup B) = T(\chi_{A\cup B}) = T(\chi_A + \chi_B) = T(\chi_A) + T(\chi_B) = \mu(A) + \mu(B).$$

This shows that μ *is a measure. Let us prove now that* μ *is countably additive. Take any sequence* $(A_i)_{i\in\mathbb{N}}$ *of disjoint measurable subsets of* Ω*. By taking into consideration Lemma*

176, $\left(\chi_{\bigcup_{i=1}^{n} A_i}\right)_{n\in\mathbb{N}}$ is pointwise convergent to $\left(\chi_{\bigcup_{i=1}^{\infty} A_i}\right)_{n\in\mathbb{N}}$. Since T is continuous when $\mathcal{I}$ is endowed with the pointwise convergent ring topology, $\left(T\left(\chi_{\bigcup_{i=1}^{n} A_i}\right)\right)_{n\in\mathbb{N}}$ is convergent to $T\left(\chi_{\bigcup_{i=1}^{\infty} A_i}\right)_{n\in\mathbb{N}}$ in M. As a consequence,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= T\left(\chi_{\bigcup_{i=1}^{\infty} A_i}\right) = \lim_{n\to\infty} T\left(\chi_{\bigcup_{i=1}^{n} A_i}\right) = \lim_{n\to\infty} T\left(\sum_{i=1}^{n} \chi_{A_i}\right)\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} T\left(\chi_{A_i}\right) = \sum_{i=1}^{\infty} T\left(\chi_{A_i}\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right).\end{aligned}$$

7.2.1.3 Stabilizing probability

This subsubsection is aimed at finding the stabilizing probability for continuous selfactions [122].

Theorem 178 *Let G be a Hausdorff compact topological group endowed with a real-valued probability left or right Haar measure μ. Suppose that G acts continuously on itself. Then*

$$\Pr(G) \leq \frac{1}{2}\mu\left(\operatorname{fix}(G)\right) + \frac{1}{2}. \tag{7.8}$$

Proof 532 *Let $\chi_{\mathcal{S}}$ be the characteristic function of $\mathcal{S} := f^{-1}(\{1\})$, where f is as (7.5). Notice that $\mathcal{S}$ is closed because f is trivially continuous. Then $\mathcal{S}$ is measurable in $G\times G$ and so is $\chi_{\mathcal{S}}$. Since the action is continuous and G is Hausdorff, all stabilizers are closed and $\operatorname{fix}(G)$ is closed as well, then measurable, by Lemma 174. On the one hand, if $x \in \operatorname{fix}(G)$, then $G_x = G$ and $\mu(G_x) = \mu(G) = 1$. On the other hand, if $x \in G \setminus \operatorname{fix}(G)$, then $[G : G_x] \geq 2$, meaning that either $\mu(G_x) = 0$ if $[G : G_x]$ is infinite (Lemma 175), or $\mu(G_x) = \frac{1}{[G:G_x]} \leq \frac{1}{2}$ if $[G : G_x]$ is finite (Lemma 174). Next, $\int_G \chi_{\mathcal{S}}(x,y)d\mu(y) = \mu(G_x)$ for every $x \in G$. Finally, Foubini's Theorem [124, p. 148] assures that*

$$\begin{aligned}\Pr(G) &:= (\mu\times\mu)(\mathcal{S}) = \iint_{G\times G} \chi_{\mathcal{S}} d(\mu\times\mu) = \int_G\int_G \chi_{\mathcal{S}}(x,y)d\mu(y)d\mu(x)\\ &= \int_G \mu(G_x)d\mu(x) = \int_{\operatorname{fix}(G)} \mu(G_x)d\mu(x) + \int_{G\setminus\operatorname{fix}(G)} \mu(G_x)d\mu(x)\\ &\leq \mu\left(\operatorname{fix}(G)\right) + \frac{1}{2}\mu\left(G\setminus\operatorname{fix}(G)\right) = \mu\left(\operatorname{fix}(G)\right) + \frac{1}{2}\left(1-\mu\left(\operatorname{fix}(G)\right)\right)\\ &= \frac{1}{2}\mu\left(\operatorname{fix}(G)\right) + \frac{1}{2}.\end{aligned}$$

Next corollary is a direct consequence of Theorem 178 and provides important estimates of the stabilizing probability.

Corollary 74 *Let G be a Hausdorff compact topological group endowed with a real-valued probability left or right Haar measure μ. Suppose that G acts continuously on itself. Then:*

1. *If $[G : \operatorname{fix}(G)]$ is finite, then*

$$\Pr(G) \leq \frac{1}{2}\mu\left(\operatorname{fix}(G)\right) + \frac{1}{2} \leq \frac{1}{2}\frac{1}{[G:\operatorname{fix}(G)]} + \frac{1}{2}. \tag{7.9}$$

2. *If* $[G : \mathrm{fix}(G)]$ *is infinite, then*

$$\Pr(G) \leq \frac{1}{2}. \tag{7.10}$$

3. *If the action is transitive, then*

$$\Pr(G) = \mu(G_1). \tag{7.11}$$

Proof 533 *In accordance with Lemma 174, if* $[G : \mathrm{fix}(G)]$ *is finite, then we have that* $\mu(\mathrm{fix}(G)) = \frac{1}{[G:\mathrm{fix}(G)]}$*, leading to the desired inequality* (7.9)*. Nevertheless, if* $[G : \mathrm{fix}(G)]$ *is infinite, then Lemma 175 assures that* $\mu(\mathrm{fix}(G)) = 0$*, leading to the desired inequality* (7.10)*. Finally, if the action is transitive, then it is uniform (Lemma 10), thus all stabilizers have the same measure (Lemma 174), leading to*

$$\Pr(G) = \int_G \mu(G_x) d\mu(x) = \mu(G_1).$$

Part IV

Appendix

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Appendix A

Category Theory

Alexander Grothendieck

A.1 Categories

Bourbaki School has been present in the mathematical community since it was founded in 1934. Actually, their methods and ways of thinking are still present as a methodology for many research papers. Nevertheless, a new methodology came into play nearly ten years after the foundation of the Bourbaki School: Category Theory. This new methodology focuses on global relations between mathematical objects, whereas Bourbaki methodology emphasizes more on local relations.

A.1.1 Objects and morphisms

Categories are composed of objects and morphisms between the objects. Objects are always sets but morphisms may not necessarily be maps or relations between the objects.

A.1.1.1 Classes of objects and morphisms

The following definition utilizes the notion of class, which will be reviewed in Appendix B after the introduction of Set Theory.

Definition 356 (Category) *A category $\mathcal{C}$ is a pair of classes of sets $(\mathrm{ob}(\mathcal{C}), \mathrm{mor}(\mathcal{C}))$ consisting of:*

- *A class of objects, $\mathrm{ob}(\mathcal{C})$, whose elements are sets but $\mathrm{ob}(\mathcal{C})$ is not a set itself, it is a class.*
- *A class of morphisms, $\mathrm{mor}(\mathcal{C})$, whose elements are nonempty sets denoted by $\mathrm{hom}_{\mathcal{C}}(A, B)$ or $\mathcal{C}(A, B)$ for all $A, B \in \mathrm{ob}(\mathcal{C})$, that is,*

$$\mathrm{mor}(\mathcal{C}) := \{\mathrm{hom}_{\mathcal{C}}(A, B) : A, B \in \mathrm{ob}(\mathcal{C})\}.$$

The elements of $\mathrm{hom}_{\mathcal{C}}(A, B)$ are not necessarily maps or relations from A to B. Again, $\mathrm{mor}(\mathcal{C})$ is not a set but a class.

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It must be verified that for any three objects $A, B, C \in \mathrm{ob}(\mathcal{C})$, a binary external operation exists

$$\begin{array}{rcl} \hom_{\mathcal{C}}(A,B) \times \hom_{\mathcal{C}}(B,C) & \to & \hom_{\mathcal{C}}(A,C) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

called composition of morphisms, verifying the following two properties:

- *Associativity:* $\forall A,B,C,D \in \mathrm{ob}(\mathcal{C})\ \forall f \in \hom_{\mathcal{C}}(A,B)\ \forall g \in \hom_{\mathcal{C}}(B,C)\ \forall h \in \hom_{\mathcal{C}}(C,D)\ (h \circ g) \circ f = h \circ (g \circ f)$.
- *Identity:* $\forall A \in \mathrm{ob}(C)\ \exists I_A \in \hom_{\mathcal{C}}(A,A)\ \forall B,C \in \mathrm{ob}(\mathcal{C})\ \forall f \in \hom_{\mathcal{C}}(A,B)\ \forall g \in \hom_{\mathcal{C}}(C,A)\ f \circ I_A = f \ \wedge \ I_A \circ g = g$.

Some of the most relevant and useful categories are displayed in the following example.

Example 87 *The following are examples of categories:*

- `Sets`*: This is the category whose objects are the sets and whose morphisms are the maps.*
- `Groups`*: This is the category whose objects are the groups and whose morphisms are the multiplicative maps.*
- `Abe`*: This is the category whose objects are the commutative groups and whose morphisms are the additive maps.*
- `Rings`*: This is the category whose objects are the rings and whose morphisms are the additive and multiplicative maps.*
- $\texttt{Rings}_1$*: This is the category whose objects are the unitary rings and whose morphisms are the additive and unital-multiplicative maps.*
- ${}_R\texttt{Mod}$*: This is the category whose objects are the R-modules and whose morphisms are the R-linear maps.*
- ${}_R\texttt{Mod}_S$*: This is the category whose objects are the (R,S)-bimodules and whose morphisms are the (R,S)-linear maps.*
- ${}_R\texttt{Alg}$*: This is the category whose objects are the R-algebras and whose morphisms are the multiplicative R-linear maps.*
- `Top`*: This is the category whose objects are the topological spaces and whose morphisms are the continuous maps.*
- $\texttt{Top}_R\texttt{Mod}$*: This is the category whose objects are the topological R-modules and whose morphisms are the continuous R-linear maps.*
- ${}_R\texttt{Mani}$*: This is the category whose objects are the M-differentiable manifolds and whose morphisms are the differentiable maps.*

A.1.1.2 Special morphisms

Notice that, given an object A in a category $\mathcal{C}$, the identity I_A is unique and $\hom_{\mathcal{C}}(A,A)$ endowed with composition becomes a monoid being I_A its identity element.

Definition 357 *Let $\mathcal{C}$ be a category. Let $A, B \in \mathrm{ob}(\mathcal{C})$. A morphism $f \in \hom_{\mathcal{C}}(A,B)$ is said to be a:*

- *Monomorphism if* $\forall C \in \mathrm{ob}(\mathcal{C})\ \forall g, h \in \hom_{\mathcal{C}}(C, A)\ f \circ g = f \circ h \Rightarrow g = h$. *The set of monomorphisms from* A *into* B *is denoted by* $\mathrm{mono}_{\mathcal{C}}(A, A)$.
- *Epimorphism if* $\forall C \in \mathrm{ob}(\mathcal{C})\ \forall g, h \in \hom_{\mathcal{C}}(B, C)\ g \circ f = h \circ f \Rightarrow g = h$. *The set of epimorphisms from* A *onto* B *is denoted by* $\mathrm{epi}_{\mathcal{C}}(A, A)$.
- *Bimorphism if* f *is both a monomorphism and an epimorphism.*
- *Section if* f *is left-invertible:* $\exists g \in \hom_{\mathcal{C}}(A, B)\ \ g \circ f = I_A$.
- *Retraction if* f *is right-invertible:* $\exists g \in \hom_{\mathcal{C}}(B, A)\ \ f \circ g = I_B$.
- *Isomorphism if* f *is invertible:* $\exists g \in \hom_{\mathcal{C}}(B, A)\ f \circ g = I_B \quad \wedge \quad g \circ f = I_A$. *The morphism* g *is unique and is called the inverse of* f *and is also an isomorphism. The objects* A *and* B *are said to be isomorphic if there exists an isomorphism from* A *to* B. *The set of isomorphisms from* A *to* B *is denoted by* $\mathrm{iso}_{\mathcal{C}}(A, A)$.
- *Endomorphism if* $A = B$. *Now,* $\mathrm{end}_{\mathcal{C}}(A) := \hom_{\mathcal{C}}(A, A)$.
- *Automorphism if* f *is an endomorphism and an isomorphism. Now,* $\mathrm{aut}_{\mathcal{C}}(A) := \mathrm{iso}_{\mathcal{C}}(A, A)$.
- *Projection if* f *is an endomorphism such that* $f \circ f = f$.
- *Involution if* f *is an endomorphism such that* $f \circ f = I_A$.

Under the settings of the previous definition, if f is a retraction and g is its right-inverse, then $g \circ f$ is a projection. Every isomorphism is a bimorphism but the converse is not true. The identity is both a projection and an involution. For every object A, $\mathrm{aut}_{\mathcal{C}}(A)$ is a group under composition. In fact, $\mathrm{aut}_{\mathcal{C}}(A) = \mathrm{end}_{\mathcal{C}}(A)^{\times}$.

Remark 98 *Let* $\mathcal{C}$ *be a category. Let* $f \in \hom_{\mathcal{C}}(A, B)$ *where* $A, B \in \mathrm{ob}(\mathcal{C})$.

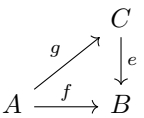
- *If there exist* $C \in \mathrm{ob}(\mathcal{C})$ *and* $g \in \hom_{\mathcal{C}}(B, C)$ $\left(g \in \hom_{\mathcal{C}}(C, A)\right)$ *such that* $g \circ f$ $(f \circ g)$ *is a monomorphism (epimorphism), then* f *is a monomorphism (epimorphism). In particular, every section is a monomorphism and every retraction is an epimorphism.*
- *The following conditions are equivalent for* f*:*
  - $\star$ f *is an isomorphism.*
  - $\star$ f *is a section and a retraction.*
  - $\star$ f *is an epimorphism and a section.*
  - $\star$ f *is a monomorphism and a retraction.*

A.1.1.3 Special objects

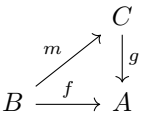
The following notions and ideas have been historically imported from Ring and Module Theory.

Definition 358 *Let* $\mathcal{C}$ *be a category. An object* $A \in \mathrm{ob}(\mathcal{C})$ *is said to be:*

- *Projective if* $\forall B \in \mathrm{ob}(\mathcal{C})\ \forall f \in \hom_{\mathcal{C}}(A, B)\ \forall C \in \mathrm{ob}(\mathcal{C})\ \forall e \in \mathrm{epi}_{\mathcal{C}}(C, B)\ \exists g \in \hom_{\mathcal{C}}(A, C)\ f = e \circ g$*:*



- *Injective if* $\forall B \in \mathrm{ob}(\mathcal{C})\ \forall f \in \hom_{\mathcal{C}}(B,A)\ \forall C \in \mathrm{ob}(\mathcal{C})\ \forall m \in \mathrm{mono}_{\mathcal{C}}(B,C)\ \exists g \in \hom_{\mathcal{C}}(C,A)\ f = g \circ m$*:*



The following concepts have been historically imported from Topology and Set Theory.

Definition 359 (Universal objects) *Let* $\mathcal{C}$ *be a category. An object* $A \in \mathrm{ob}(\mathcal{C})$ *is said to be:*

- *Initial if* $\forall B \in \mathrm{ob}(\mathcal{C})\ \exists! f \in \hom_{\mathcal{C}}(A,B)$*, denoted by* $A \to B$*.*
- *Final if* $\forall B \in \mathrm{ob}(\mathcal{C})\ \exists! f \in \hom_{\mathcal{C}}(B,A)$*, denoted by* $B \to A$*.*
- *Null if* A *is both initial and final.*

Initial and final objects are unique up to isomorphisms. If O is a null object, then, for every $A, B \in \mathrm{ob}(\mathcal{C})$, there exists a morphism $0_{AB} \in \hom_{\mathcal{C}}(A,B)$ defined as $0_{AB} := (O \to B) \circ (A \to O) = A \to O \to B$.

Proposition 135 *Let* $\mathcal{C}$ *be a category. Let* $A, B, C \in \mathrm{ob}(\mathcal{C})$*. Then:*

1. *If* A *is an initial (final) object, then* $A \to B$ *(*$B \to A$*) is a section (retraction).*
2. *If* A, B *are both initial or both final, then they are isomorphic.*
3. *If there exists a null object* O*, then* $0_{BC} \circ 0_{AB} = 0_{AC}$*.*
4. *If* O *and* $\widehat{O}$ *are null objects of* $\mathcal{C}$*, then* $0_{AB} = \widehat{0}_{AB}$*.*

Proof 534 *The proof will follow as itemized as the statement.*

1. *Let* $f \in \hom_{\mathcal{C}}(B,A)$*. Then* $f \circ (A \to B) \in \mathrm{end}_{\mathcal{C}}(A) = \{I_A\}$*, which means that* $A \to B$ *is a section.*
2. *Suppose that* A, B *are both initial (if they are both final, then similar reasoning applies). Then* $(A \to B) \circ (B \to A) \in \mathrm{end}_{\mathcal{C}}(B) = \{I_B\}$ *and* $(B \to A) \circ (A \to B) \in \mathrm{end}_{\mathcal{C}}(A) = \{I_A\}$*, meaning that* A *and* B *are isomorphic.*
3. *Indeed,* $0_{BC} \circ 0_{AB} = A \to O \to B \to O \to C = (A \to O \to B \to O) \to C = A \to O \to C = 0_{AC}$*.*
4. *Indeed,* $0_{AB} = A \to O \to B = \left(A \to \widehat{O} \to O\right) \to B = A \to \left(\widehat{O} \to O \to B\right) = A \to \widehat{O} \to B = \widehat{0}_{AB}$*.*

The following concepts will sound familiar to the reader from Linear and Multilinear Algebra.

Definition 360 *Let* $\mathcal{C}$ *be a category. Consider an object* $A \in \mathrm{ob}(\mathcal{C})$*. Then:*

- *A subobject of* A *is a pair* (B,i) *where* $B \in \mathrm{ob}(\mathcal{C})$ *and* $i \in \hom_{\mathcal{C}}(B,A)$ *is a monomorphism.*
- *A proper subobject of* A *is a subobject* (B,i) *of* A *such that* B *is not null and* i *is not an isomorphism.*
- *A quotient object of* A *is a pair* (C,p) *where* $B \in \mathrm{ob}(\mathcal{C})$ *and* $p \in \hom_{\mathcal{C}}(A,C)$ *is an epimorphism.*

A.1.1.4 Preadditive categories

Preadditive categories are somehow a generalization of the category of Abelian groups.

Definition 361 (Preadditive category) *A category $\mathcal{C}$ is called preadditive provided that, for every $A, B \in \mathrm{ob}(\mathcal{C})$, there exists a binary internal operation $+ : \hom_{\mathcal{C}}(A,B) \times \hom_{\mathcal{C}}(A,B) \to \hom_{\mathcal{C}}(A,B)$ that turns $\hom_{\mathcal{C}}(A,B)$ into an Abelian group and verifies the following distributive property with respect to $\circ$: $\forall C, D \in \mathrm{ob}(\mathcal{C})$ $\forall f \in \hom_{\mathcal{C}}(C,A)$ $\forall g \in \hom_{\mathcal{C}}(B,D)$ $\forall h, k \in \hom_{\mathcal{C}}(A,B)$*

$$(h+k) \circ f = (h \circ f) + (k \circ f) \text{ and } g \circ (h+k) = (g \circ h) + (g \circ k).$$

Note that, in preadditive categories, $\mathrm{end}_{\mathcal{C}}(A)$ becomes a ring whose multiplicative group of invertibles is precisely $\mathrm{aut}_{\mathcal{C}}(A)$.

Proposition 136 *Let $\mathcal{C}$ be a preadditive category. Then:*

1. *The composition of the neutral element with a morphism is again the neutral element.*
2. *An object A is initial if and only if it is final.*
3. *If $\mathcal{C}$ has a null object O, then 0_{AB} is precisely the neutral element of $\hom_{\mathcal{C}}(A,B)$.*

Proof 535 *The proof will follow as itemized as the statement.*

1. *If $f \in \hom_{\mathcal{C}}(B,C)$, $g \in \hom_{\mathcal{C}}(A,B)$ and we let 0 denote the neutral elements of the previous homsets, then*

$$f \circ g = (f+0) \circ g = (f \circ g) + (0 \circ g) \Rightarrow 0 \circ g = 0$$

$$f \circ g = f \circ (g+0) = (f \circ g) + (f \circ 0) \Rightarrow f \circ 0 = 0.$$

2. *Assume that A is initial. Let $B \in \mathrm{ob}(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(B,A)$. If we let 0 denote the neutral element of $\mathrm{end}_{\mathcal{C}}(A)$, then by uniqueness $\mathrm{end}_{\mathcal{C}}(A) = \{I_A\} = \{0\}$ and thus $f = I_A \circ f = 0 \circ f = 0$ in virtue of Proposition 136(1). If A is final, then a similar proof applies.*
3. *Indeed, by uniqueness, 0_{AO} and 0_{OB} are the neutral elements of $\hom_{\mathcal{C}}(A,O)$ and $\hom_{\mathcal{C}}(O,B)$, respectively. Therefore, by Proposition 136(1), $0_{AB} = 0_{OB} \circ 0_{AO}$ is the neutral element of $\hom_{\mathcal{C}}(A,B)$.*

A.1.2 Functors

Functors were originally defined as natural transformations between categories. They represent the most iconic example of transversality in mathematics since classical areas of mathematics like Analysis, Algebra, Topology or Geometry can be related to each other by means of functors.

A.1.2.1 Functors and cofunctors

There are two types of functors: covariant and contravariant. These two versions of functors are actually dual to each other in some sense.

Definition 362 (Functor) *Given categories $\mathcal{C}, \mathcal{D}$, we will say that $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a covariant (contravariant) functor if*

- $\forall A \in \text{ob}(\mathcal{C})\ \mathcal{F}(A) \in \text{ob}(\mathcal{D})$.
- $\forall X, Y \in \text{ob}(\mathcal{C})\ \forall f \in \hom_{\mathcal{C}}(X, Y)\ \mathcal{F}(f) \in \hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ *(*$\mathcal{F}(f) \in \hom_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))$*) in such a way that*
  - $\star\ \forall A \in \text{ob}(\mathcal{C})\ \ \mathcal{F}(I_A) = I_{\mathcal{F}(A)}$.
  - $\star\ \forall X, Y, Z \in \text{ob}(\mathcal{C})\ \forall f \in \hom_{\mathcal{C}}(X, Y)\ \forall g \in \hom_{\mathcal{C}}\ (Y, Z)$

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)\ \ \Big(\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)\Big).$$

Under the settings of the previous definition, every functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ induces a map, for every $X, Y \in \text{ob}(\mathcal{C})$, denoted by $\mathcal{F}_{X,Y} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$, if $\mathcal{F}$ is covariant, and by $\mathcal{F}_{X,Y} : \hom_{\mathcal{C}}(X, Y) \to \hom_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))$, if F is contravariant.

Definition 363 (Faithful functor) *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}, \mathcal{D}$. We say that $\mathcal{F}$ is:*

- *Faithful if $\mathcal{F}_{X,Y}$ is injective for every $X, Y \in \text{ob}(\mathcal{C})$.*
- *Full if $\mathcal{F}_{X,Y}$ is surjective for every $X, Y \in \text{ob}(\mathcal{C})$.*
- *Fully faithful if $\mathcal{F}_{X,Y}$ is bijective for every $X, Y \in \text{ob}(\mathcal{C})$.*

Faithful functors allow to consider an important kind of category with relevant importance towards the existence of free objects.

Definition 364 (Concrete category) *Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is concrete if there exists a faithful functor $\mathcal{F} : \mathcal{C} \to \texttt{Sets}$.*

A.1.2.2 Representations

A very important application of functors is given by representations. Representation Theory is one of the most relevant theories in Pure Mathematics since it permits to study the properties of a certain object by representing it in a different context. For instance, the famous Riesz Representation Theorem [125, 126, 127] establishes an important connection between a Hilbert space and its continuous dual space.

Definition 365 (Representation) *Let $\mathcal{C}, \mathcal{D}$ be categories. Let $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ be a functor. A representation of an object $C \in \text{ob}(\mathcal{C})$ in an object $D \in \text{ob}(\mathcal{D})$ is any element $f \in \hom_{\mathcal{C}}(C, \mathcal{F}(D))$.*

The most represented objects are the groups, via the assignment between a given category and the category of groups which sends an object to its group of automorphisms. Nevertheless, notice that this assignment is not a functor.

Definition 366 (Representation) *Let G be a group. A representation of G in an object A of a category $\mathcal{C}$ is an element $f \in \hom_{\texttt{Groups}}(G, \text{aut}_{\mathcal{C}}(A))$.*

Intuitively speaking, a representation of a group in an object is simply a multiplicative map from the group into the group of automorphisms of the object. The most common representations of groups are given by the algebraic representations, that is, representations where the object is a ring.

Definition 367 (Algebraic representation) *An algebraic representation of a group G on a ring R is simply a group homomorphism $G \to \mathcal{U}(R)$.*

Although Definition 366 does not fit in Definition 365, Definition 367 does fit in Definition 365 because the assignment that takes every ring to its multiplicative group of invertibles is a covariant functor. If $\pi : G \to \mathcal{U}(R)$ is an algebraic representation of a group G on a $*$-ring R, then

$$\begin{array}{rccl} \pi^* : & G & \to & \mathcal{U}(R) \\ & g & \mapsto & \pi^*(g) := \pi(g^{-1})^* \end{array} \tag{A.1}$$

is also a representation of G called the $*$-representation of π.

Definition 368 (Linear representation) *A linear representation of a group G in a topological module M over a topological ring R is an algebraic representation of G in $\mathcal{CL}(M)$, that is, a multiplicative map $\pi : G \to \mathsf{TGL}(M)$.*

In the above definition, the Topological General Linear Group of M is defined as $\mathsf{TGL}(M) := \mathcal{U}(\mathcal{CL}(M))$. Note that $M^* := \mathcal{CL}(M, R)$ is a right R-module (we are assuming that M is a left R-module) that can be endowed with a feasible dual topology. Inspired by (A.1), the dual representation of π is given by

$$\begin{array}{rccl} \pi^* : & G & \to & \mathsf{TGL}(M^*) \\ & g & \mapsto & \pi^*(g) := \pi(g^{-1})^*. \end{array} \tag{A.2}$$

On the other hand, π naturally induces an action

$$\begin{array}{ccl} G \times M & \to & M \\ (g, m) & \mapsto & \pi(g)(m). \end{array} \tag{A.3}$$

Therefore, the notion of invariance can be transported to the scope of linear representations.

A.2 Universal properties

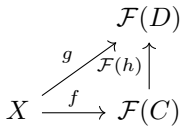
Here, we will see precise definitions of very-well algebraic concepts, such as free objects, direct products, direct sums, or tensor products.

A.2.1 Universal objects

In Definition 359, two types of universal objects are described: initial and final objects.

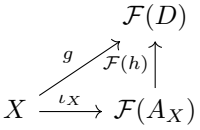
A.2.1.1 Free objects

Consider a concrete category $(\mathcal{C}, \mathcal{F})$, where $\mathcal{F} : \mathcal{C} \to \mathtt{Sets}$ stands for a faithful functor. Let $X \in \mathrm{ob}(\mathtt{Sets})$. We can construct a new category, denoted $\mathcal{C}_X$, whose objects are $\mathrm{ob}(\mathcal{C}_X) := \{(C, f) : C \in \mathcal{C}, f \in \hom_{\mathtt{Sets}}(X, \mathcal{F}(C))\}$, and whose morphisms are $\hom_{\mathcal{C}_X}((C, f), (D, g)) := \{h \in \hom_{\mathcal{C}}(C, D) : g = \mathcal{F}(h) \circ f\}$ for $(C, f), (D, g) \in \mathrm{ob}(\mathcal{C}_X)$:



The identity element of every object $(C, f) \in \mathrm{ob}(\mathcal{C}_X)$ is $I_{(C,f)} = I_C$.

Definition 369 (Free object) *Let $(\mathcal{C}, \mathcal{F})$ be a concrete category, where $\mathcal{F} : \mathcal{C} \to \texttt{Sets}$ is a faithful functor. Let $X \in \text{ob}(\texttt{Sets})$. If the category $\mathcal{C}_X$ has an initial object, then we call it a free object of $(\mathcal{C}, \mathcal{F})$ with respect to X and will denote it by (A_X, ι_X). The set X is called the basis of A_X.*

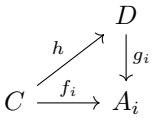


Lemma 177 *Let $(\mathcal{C}, \mathcal{F})$ be a concrete category, where $\mathcal{F} : \mathcal{C} \to \texttt{Sets}$ is a faithful functor. Let $X \in \text{ob}(\texttt{Sets})$. If (A_X, ι_X) exists and there exists an object D such that $\mathcal{F}(D) = X$, then ι_X is a section.*

Proof 536 *Since (A_X, ι_X) is initial and $(D, I_X) \in \text{ob}(\mathcal{C}_X)$, there exists a unique element h in $\hom_{\mathcal{C}_X}((A_X, \iota_X), (D, I_X))$. Note that $I_X = \mathcal{F}(h) \circ \iota_X$, meaning that ι_X is a section.*

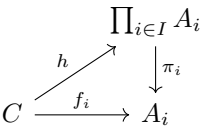
A.2.1.2 Product

Given a category $\mathcal{C}$ and a family of objects $(A_i)_{i \in I}$, a new category $\mathcal{C}_\pi((A_i)_{i \in I})$ is constructed whose objects $\text{ob}(\mathcal{C}_\pi((A_i)_{i \in I})) := \{(C, \{f_i\}_{i \in I}) : C \in \mathcal{C},\ \forall i \in I\ f_i \in \hom_{\mathcal{C}}(C, A_i)\}$, and whose morphisms are $\hom_{\mathcal{C}_\pi((A_i)_{i \in I})}((C, \{f_i\}_{i \in I}), (D, \{g_i\}_{i \in I})) := \{h \in \hom_{\mathcal{C}}(C, D) : \forall i \in I\ g_i \circ h = f_i\}$ for $(C, \{f_i\}_{i \in I}), (D, \{g_i\}_{i \in I}) \in \text{ob}(\mathcal{C}_\pi((A_i)_{i \in I}))$:



The identity element of every object $(C, \{f_i\}_{i \in I}) \in \text{ob}(\mathcal{C}_\pi((A_i)_{i \in I}))$ is $I_{(C, \{f_i\}_{i \in I})} = I_C$.

Definition 370 (Product) *Let $\mathcal{C}$ be a category. Consider a family of objects $(A_i)_{i \in I}$. If the category $\mathcal{C}_\pi((A_i)_{i \in I})$ has a final object, then we call it the product of the family $(A_i)_{i \in I}$ and we denote it by $\left(\prod_{i \in I} A_i, \{\pi_i\}_{i \in I}\right)$. The morphisms π_i's are called the projection morphisms.*



Lemma 178 *Let $\mathcal{C}$ be a category. Consider a family of objects $(A_i)_{i \in I}$. If $\left(\prod_{i \in I} A_i, \{\pi_i\}_{i \in I}\right)$ exists, then π_i is a retraction for all $i \in I$.*

Proof 537 *Fix an arbitrary $i_0 \in I$ and consider the object $(A_{i_0}, \{f_i\}_{i \in I})$ of $\mathcal{C}_\pi((A_i)_{i \in I})$, where f_i is an element of $\hom_{\mathcal{C}}(A_{i_0}, A_i)$ if $i \in I \setminus \{i_0\}$ and $f_{i_0} = I_{A_{i_0}}$. There exists a unique*

$$h \in \hom_{\mathcal{C}_\pi((A_i)_{i \in I})}\left((A_{i_0}, \{f_i\}_{i \in I}), \left(\prod_{i \in I} A_i, \{\pi_i\}_{i \in I}\right)\right).$$

This morphism h verifies, in particular, that $\pi_{i_0} \circ h = I_{A_{i_0}}$. Thus, π_{i_0} is a retraction.

Essentially, in all the classical categories (sets, groups, rings, modules and algebras), the product of a family of objects is given by the corresponding direct product of sets. Special attention can be paid to the category of topological spaces, where the product of a family of topological spaces is endowed with the product topology, which is an initial topology.

A.2.1.3 Coproduct

Given a category $\mathcal{C}$ and a family of objects $(A_i)_{i\in I}$, a new category $\mathcal{C}_\iota\left((A_i)_{i\in I}\right)$ is constructed whose objects are $\mathrm{ob}\left(\mathcal{C}_\iota\left((A_i)_{i\in I}\right)\right) := \left\{(C,\{f_i\}_{i\in I}) : C\in\mathcal{C},\ \forall i\in I\ f_i\in\hom_{\mathcal{C}}(A_i,C)\right\}$, and whose morphisms are $\hom_{\mathcal{C}_\iota((A_i)_{i\in I})}\left((C,\{f_i\}_{i\in I}),(D,\{g_i\}_{i\in I})\right) := \{h\in\hom_{\mathcal{C}}(C,D) : \forall i\in I\ h\circ f_i = g_i\}$ for $(C,\{f_i\}_{i\in I}),(D,\{g_i\}_{i\in I})\in\mathrm{ob}\left(\mathcal{C}_\iota\left((A_i)_{i\in I}\right)\right)$:

$$\begin{array}{ccc} & & D \\ & \overset{g_i}{\nearrow} & \uparrow{\scriptstyle h} \\ A_i & \xrightarrow{f_i} & C \end{array}$$

The identity element of every object $(C,\{f_i\}_{i\in I})\in\mathrm{ob}\left(\mathcal{C}_\iota\left((A_i)_{i\in I}\right)\right)$ is $I_{(C,\{f_i\}_{i\in I})} = I_C$.

Definition 371 (Coproduct) *Let $\mathcal{C}$ be a category. Consider a family of objects $(A_i)_{i\in I}$. If the category $\mathcal{C}_\iota\left((A_i)_{i\in I}\right)$ has an initial object, then we call it the coproduct of the family $(A_i)_{i\in I}$ and we denote it by $\left(\coprod_{i\in I}A_i,\{\iota_i\}_{i\in I}\right)$. The morphisms ι_i's are called the injection morphisms.*

$$\begin{array}{ccc} & & D \\ & \overset{g_i}{\nearrow} & \uparrow{\scriptstyle h} \\ A_i & \xrightarrow{\iota_i} & \coprod_{i\in I}A_i \end{array}$$

Lemma 179 *Let $\mathcal{C}$ be a category. Consider a family of objects $(A_i)_{i\in I}$. If $\left(\coprod_{i\in I}A_i,\{\iota_i\}_{i\in I}\right)$ exists, then ι_i is a section for all $i\in I$.*

Proof 538 *Fix an arbitrary $i_0\in I$ and consider the object $(A_{i_0},\{g_i\}_{i\in I})$ of $\mathcal{C}_\iota\left((A_i)_{i\in I}\right)$, where g_i is an element of $\hom_{\mathcal{C}}(A_i,A_{i_0})$ if $i\in I\setminus\{i_0\}$ and $g_{i_0} = I_{A_{i_0}}$. There exists a unique*

$$h\in\hom_{\mathcal{C}_\iota((A_i)_{i\in I})}\left(\left(\coprod_{i\in I}A_i,\{\iota_i\}_{i\in I}\right),(A_{i_0},\{g_i\}_{i\in I})\right).$$

This morphism h verifies, in particular, that $h\circ\iota_{i_0} = I_{A_{i_0}}$. Thus, ι_{i_0} is a section.

A representative example of coproduct is given by the direct sum in the category of modules. Since commutative groups are $\mathbb{Z}$-modules, the direct sum also gives the coproduct in the category of commutative groups. It is important to observe that in the general category of groups, the direct sum is not necessarily the coproduct.

Example 88 (Direct sum) *Let $(M_i)_{i\in I}$ be a family of R-modules. There exists $\coprod_{i\in I}M_i$ and it is $\bigoplus_{i\in I}M_i := \{(m_i)_{i\in I}\in\prod_{i\in I}M_i : \exists J\in\mathcal{P}_{\mathtt{f}}(I)\ m_i = 0\ \forall i\in I\setminus J\}$ with $\iota_i : M_i\to\bigoplus_{i\in I}M_i$ being the canonical injection for all $i\in I$. Indeed, let N be an R-module with R-linear maps $h_i : M_i\to N$ for all $i\in I$. Define*

$$\begin{array}{rccc} h: & \bigoplus_{i\in I}M_i & \to & N \\ & (m_i)_{i\in I} & \mapsto & \sum_{i\in I}h_i(m_i). \end{array}$$

Then h is unique such that $h\circ\iota_i = h_i\ \forall i\in I$

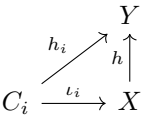
$$\begin{array}{ccc} & & N \\ & \overset{h_i}{\nearrow} & \uparrow{\scriptstyle h} \\ M_i & \xrightarrow{\iota_i} & \bigoplus_{i\in I}M_i \end{array}$$

Another interesting example of coproduct happens in the category of topological spaces. Let us first see a beginners' example.

Example 89 (Connected components) *Let X be a locally connected topological space. Let $(C_i)_{i\in I}$ be the family of its connected components. There exists $\coprod_{i\in I} C_i$ and it is X with $\iota_i : C_i \hookrightarrow X$ being the inclusion for all $i \in I$. Indeed, let Y be a topological space with continuous maps $h_i : C_i \to Y$ for all $i \in I$. Define*

$$\begin{array}{rccl} h: & X & \to & Y \\ & x & \mapsto & h_i(x) \;\; x \in C_i. \end{array}$$

Then h is unique such that $h \circ \iota_i = h_i \;\forall i \in I$

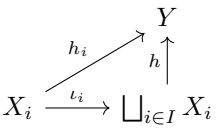


After this motivational example, it is time now for unveiling the coproduct in the category of topological spaces.

Example 90 (Disjoint union topology) *Let $(X_i)_{i\in I}$ be a family of topological spaces. There exists $\coprod_{i\in I} X_i$ and it is $\bigsqcup_{i\in I} X_i$ endowed with the final topology induced by the canonical injections $\iota_i : X_i \to \bigsqcup_{i\in I} X_i$ for all $i \in I$. Indeed, let Y be a topological space with continuous maps $h_i : X_i \to Y$ for all $i \in I$. Define*

$$\begin{array}{rccl} h: & \bigsqcup_{i\in I} X_i & \to & Y \\ & (x,i) & \mapsto & h_i(x) \;\; x \in X_i. \end{array}$$

Then h is unique such that $h \circ \iota_i = h_i \;\forall i \in I$



A.2.1.4 Biproduct

Let $\mathcal{C}$ be a category with a null object O. Let $(A_i)_{i\in I}$ be a family of objects for which the product $\left(\prod_{i\in I} A_i, \{\pi_i\}_{i\in I}\right)$ exists. For every $i, j \in I$, we can define a morphism in $\hom_{\mathcal{C}}(A_i, A_j)$ as follows

$$\delta_{ij} := \begin{cases} I_{A_i} & i = j, \\ 0_{A_i A_j} & i \neq j. \end{cases}$$

By the universality of the product, there exists a morphism $\iota_i \in \hom_{\mathcal{C}}\left(A_i, \prod_{i\in I} A_i\right)$ such that $\pi_j \circ \iota_i = \delta_{ij}$ for every $i, j \in I$. If $\mathcal{C}$ does not have a null object but is preadditive, then δ_{ij} is defined using the neutral element of $\hom_{\mathcal{C}}(A_i, A_j)$.

Proposition 137 (Biproduct) *Let $\mathcal{C}$ be a preadditive category. Let $\{A_i\}_{i=1}^n \subseteq \mathrm{ob}(\mathcal{C})$ such that $(\prod_{i=1}^n A_i, \{\pi_i\}_{i=1}^n)$ exists. Then:*

1. *$\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n = I_{\prod_{i=1}^n A_i}$.*
2. *$(\prod_{i=1}^n A_i, \{\iota_i\}_{i=1}^n)$ is the coproduct of the family $\{A_i\}_{i=1}^n$.*

Proof 539 *The proof will follow as itemized as the statement.*

1. *For every $i, j \in \{1, \ldots, n\}$, we have that $\pi_j \circ \iota_i = \delta_{ij}$, therefore*

$$\pi_i \circ (\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n) = \pi_i$$

and $\pi_i \circ I_{\prod_{i=1}^n A_i} = \pi_i$. By the universality of $(\prod_{i=1}^n A_i, \{\pi_i\}_{i=1}^n)$, we conclude the result.

2. *Let $A \in \mathrm{ob}(\mathcal{C})$ and $f_i \in \hom_{\mathcal{C}}(A_i, A)$ for every $i \in I$. In the first place, if $f, g \in \hom_{\mathcal{C}}(\prod_{i=1}^n A_i, A)$ are so that $f \circ \iota_i = f_i = g \circ \iota_i$ for every $i \in I$, then*

$$\begin{aligned} f &= f \circ I_{\prod_{i=1}^n A_i} \\ &= f \circ (\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n) \\ &= (f \circ \iota_1) \circ \pi_1 + \cdots + (f \circ \iota_n) \circ \pi_n \\ &= f_1 \circ \pi_1 + \cdots + f_n \circ \iota_n \circ \pi_n \\ &= (g \circ \iota_1) \circ \pi_1 + \cdots + (g \circ \iota_n) \circ \pi_n \\ &= g \circ (\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n) \\ &= g \circ I_{\prod_{i=1}^n A_i} \\ &= g. \end{aligned}$$

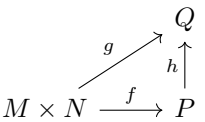
Finally, define $f := f_1 \circ \pi_1 + \cdots + f_n \circ \pi_n$. Since $\pi_j \circ \iota_i = \delta_{ij}$ for every $i, j \in \{1, \ldots, n\}$, it suffices to observe that $f \circ \iota_i = (f_1 \circ \pi_1 + \cdots + f_n \circ \pi_n) \circ \iota_i = f_1 \circ (\pi_1 \circ \iota_i) + \cdots + f_n \circ (\pi_n \circ \iota_i) = f_i$.

A.2.2 Tensor product

We will not construct the tensor product in general categories, like we did it with the product and the coproduct. We will construct the tensor product on the category of modules.

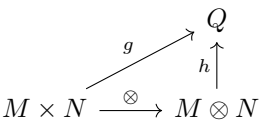
A.2.2.1 Noncommutative (bilinear) case

If M, N are a right module and a left module over a ring R, respectively, then we can construct a category denoted by $\mathcal{C}_{\otimes}(M, N)$ whose objects are $\mathrm{ob}\left(\mathcal{C}_{\otimes}(M, N)\right) := \{(P, f) : P \in \mathrm{ob}(\mathtt{Abe}), f \in \mathcal{L}^{\mathtt{a}}(M \times N, P)\}$ and whose morphisms are $\hom_{\mathcal{C}_{\otimes}(M,N)}\left((P, f), (Q, g)\right) := \{h \in \hom_{\mathtt{Abe}}(P, Q) : h \circ f = g\}$:



The identity element of every object $(P, f) \in \mathrm{ob}\left(\mathcal{C}_{\otimes}\left(M, N\right)\right)$ is $I_{(P,f)} = I_P$.

Definition 372 (Tensor product) *Let M, N be a right module and a left module over a ring R, respectively. If the category $\mathcal{C}_{\otimes}(M, N)$ has an initial element, then we call it the tensor product of M, N and we denote it by $(M \otimes N, \otimes)$. The elements $m \otimes n := \otimes(m, n)$, for $(m, n) \in M \times N$, are called elementary tensors.*



Proposition 138 *Let R be a ring. Let M, N be a right module and a left module over R, respectively. Then:*

1. *If $(P,f),(Q,g) \in \operatorname{ob}(\mathcal{C}_{\otimes}(M,N))$ with $\operatorname{span}(f(M\times N)) = P$ and $h_1, h_2 \in \hom_{\mathcal{C}_{\otimes}(M,N)}((P,f),(Q,g))$, then $h_1 = h_2$.*
2. *There exists a tensor product of M and N.*

Proof 540 *The proof will follow as itemized as the statement.*

1. *Since $h_i \circ f = g$ for $i = 1, 2$, we have that $h_1|_{f(M\times N)} = h_2|_{f(M\times N)}$. Since h_1 and h_2 are R-linear and $\operatorname{span}(f(M\times N)) = P$, we conclude that $h_1 = h_2$.*
2. *Consider F the free $\mathbb{Z}$-module with basis $M \times N$, that is,*

$$F := \bigoplus_{M\times N} \mathbb{Z}.$$

We let $\iota_{(m,n)}$ denote the canonical injection of $\mathbb{Z}$ into F. Let

$$\begin{aligned} S \;:=\; & \operatorname{span}\{\iota_{(m,n+n')}(1) - \iota_{(m,n)}(1) - \iota_{(m,n')}(1), \\ & \iota_{(m+m',n)}(1) - \iota_{(m,n)}(1) - \iota_{(m',n)}(1), \\ & \iota_{(mr,n)}(1) - \iota_{(m,rn)}(1) : m, m' \in M, n, n' \in N\}. \end{aligned}$$

Denote $M \otimes N := \frac{F}{S}$ and

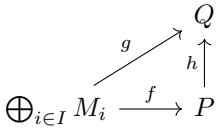
$$\begin{aligned} \otimes : \; M \times N &\to M \otimes N \\ (m,n) &\mapsto m \otimes n := \iota_{(m,n)}(1) + S. \end{aligned}$$

It is not hard to check that $\otimes$ is balanced and $\operatorname{span}\{m \otimes n : m \in M, n \in N\} = M \otimes N$. Therefore, $(M \otimes N, \otimes) \in \operatorname{ob}(\mathcal{C}_{\otimes}(M,N))$ and it only suffices to check that $\hom_{\mathcal{C}_{\otimes}(M,N)}((M\otimes N, \otimes),(Q,g)) \neq \varnothing$ for all $(Q,g) \in \operatorname{ob}(\mathcal{C}_{\otimes}(M,N))$. So, since F is the coproduct of the family $\{\mathbb{Z}\}_{(m,n)\in M\times N}$, there exists $f : F \to Q$ $\mathbb{Z}$-linear such that $f(\iota_{(m,n)}(k)) = kg(m,n)$ for all $k \in \mathbb{Z}$ and all $(m,n) \in M \times N$. In particular, $f(\iota_{(m,n)}(1)) = g(m,n)$. As a consequence, since g is balanced, we have that $S \subseteq \ker(f)$. Therefore, there exists $h : \frac{F}{S} \to Q$ such that $h(x+S) = f(x)$ and thus $h \circ \otimes = g$ because

$$h(m \otimes n) = h\left(\iota_{(m,n)}(1) + S\right) = f\left(\iota_{(m,n)}(1)\right) = g(m,n).$$

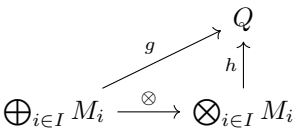
A.2.2.2 Commutative (multilinear) case

If $\mathcal{C} := {}_R\texttt{Mod}$ stands for the category of R-modules over a commutati-ve ring R and $(M_i)_{i\in I}$ is a family of objects of $\mathcal{C}$, then we can construct another category denoted by $\mathcal{C}_{\otimes}((M_i)_{i\in I})$ whose objects are $\operatorname{ob}(\mathcal{C}_{\otimes}((M_i)_{i\in I})) := \left\{(P,f) : P \in \operatorname{ob}(\mathcal{C}), f \in \mathcal{L}^I\left(\bigoplus_{i\in I} M_i, P\right)\right\}$ and whose morphisms are $\hom_{\mathcal{C}_{\otimes}((M_i)_{i\in I})}((P,f),(Q,g)) := \{h \in \hom_{\mathcal{C}}(P,Q) : h \circ f = g\}$:



The identity element of every object $(P,f) \in \operatorname{ob}(\mathcal{C}_{\otimes}((M_i)_{i\in I}))$ is $I_{(P,f)} = I_P$.

Definition 373 (Tensor product) *Let $(M_i)_{i\in I}$ be a family of R-modules over a commutative ring R. If the category $\mathcal{C}_{\otimes}((M_i)_{i\in I})$ has an initial element, then we call it the tensor product of $(M_i)_{i\in I}$ and we denote it by $\left(\bigotimes_{i\in I} M_i, \otimes\right)$. The elements $\otimes_{i\in I} m_i := \otimes((m_i)_{i\in I})$, for $(m_i)_{i\in I} \in \bigoplus_{i\in I} M_i$, are called elementary tensors.*



We can define

$$\begin{array}{rccc} \otimes & : \bigoplus_{i\in I} M_i & \to & \mathcal{L}^I\left(\bigoplus_{i\in I} M_i\right)^* \\ & (m_i)_{i\in I} & \mapsto & \otimes_{i\in I} m_i, \end{array}$$

where

$$\begin{array}{rccc} \otimes_{i\in I} m_i : & \mathcal{L}^I\left(\bigoplus_{i\in I} M_i\right) & \to & R \\ & g & \mapsto & (\otimes_{i\in I} m_i)(g) := g\left((m_i)_{i\in I}\right). \end{array}$$

Observe that $\otimes \in \mathcal{L}^I\left(\bigoplus_{i\in I} M_i, \mathcal{L}^I\left(\bigoplus_{i\in I} M_i\right)^*\right)$ and $\left(\bigotimes_{i\in I} M_i, \otimes\right) \in \mathrm{ob}\left(\mathcal{C}_{\otimes}((M_i)_{i\in I})\right)$, where now we define $\bigotimes_{i\in I} M_i := \mathrm{span}\left(\otimes\left(\bigoplus_{i\in I} M_i\right)\right)$.

Proposition 139 *Let R be a commutative ring. Let $(M_i)_{i\in I}$ be a family of R-modules. Then:*

1. *If $(P,f),(Q,g) \in \mathrm{ob}\left(\mathcal{C}_{\otimes}((M_i)_{i\in I})\right)$ with $\mathrm{span}\left(f\left(\bigoplus_{i\in I} M_i\right)\right) = P$ and $h_1, h_2 \in \hom_{\mathcal{C}_{\otimes}((M_i)_{i\in I})}((P,f),(Q,g))$, then $h_1 = h_2$.*
2. *If Q is torsionless, there is $h \in \hom_{\mathcal{C}_{\otimes}((M_i)_{i\in I})}\left(\left(\bigotimes_{i\in I} M_i, \otimes\right),(Q,g)\right)$.*
3. *If R is a field, then $\left(\bigotimes_{i\in I} M_i, \otimes\right)$ is the tensor product of $(M_i)_{i\in I}$.*

Proof 541 *The proof will follow as itemized as the statement.*

1. *Note that $h_1|_{f(\oplus_{i\in I} M_i)} = h_2|_{f(\oplus_{i\in I} M_i)}$ because $h_1 \circ f = g = h_2 \circ f$. Since h_1, h_2 are R-linear and $\mathrm{span}\left(f\left(\bigoplus_{i\in I} M_i\right)\right) = P$, we conclude that $h_1 = h_2$.*
2. *We will show that $\sum_{p=1}^{k} r_p \otimes_{i\in I} m_{i,p} = \sum_{j=1}^{l} r'_j \otimes_{i\in I} m'_{i,j}$ implies that $\sum_{p=1}^{k} r_p g\left((m_i,p)_{i\in I}\right) = \sum_{j=1}^{l} r'_j g\left((m'_{i,j})_{i\in I}\right)$. Suppose to the contrary that $\sum_{p=1}^{k} r_p g\left((m_{i,p})_{i\in I}\right) \neq \sum_{j=1}^{l} r'_j g\left((m'_{i,j})_{i\in I}\right)$. By hypothesis, since*

$$\left\{\sum_{p=1}^{k} r_p g\left((m_{i,p})_{i\in I}\right) - \sum_{j=1}^{l} r'_j g\left((m'_{i,j})_{i\in I}\right)\right\}^{\perp} \neq Q^*,$$

therefore there exists $q^* \in Q^*$ such that

$$q^*\left(\sum_{p=1}^{k} r_p g\left((m_i,p)_{i\in I}\right)\right) \neq q^*\left(\sum_{j=1}^{l} r'_j g\left((m'_i,j)_{i\in I}\right)\right).$$

However, notice that $q^* \circ g \in \mathcal{L}^I\left(\bigoplus_{i\in I} M_i\right)^*$, which means that we reach the following

contradiction:

$$q^*\left(\sum_{p=1}^{k} r_p g\left((m_{i,p})_{i\in I}\right)\right) = \left(\sum_{i=p}^{k} r_p \otimes_{i\in I} m_{i,p}\right)(q^*\circ g)$$

$$= \left(\sum_{j=1}^{l} r'_j \otimes_{i\in I} m'_{i,j}\right)(q^*\circ g) = q^*\left(\sum_{j=1}^{l} r'_j g\left((m'_{i,j})_{i\in I}\right)\right).$$

Finally, in order to construct h, it simply suffices to define

$$h\left(\sum_{i=1}^{k} r_i \otimes_{i\in I} m_i\right) := \sum_{i=1}^{k} r_i g\left((m_i)_{i\in I}\right),$$

which is well defined by the argument right above, and extend h by linearity.

3. *It only suffices to show that every nonzero R-vector space Q is torsionless. Indeed, if $q \in Q\setminus\{0\}$, then by the Zorn's Lemma we can find a vector subspace V of Q such that $Q = Rq \oplus V$. Now, it only suffices to define $q^* \in Q$ with $\ker(q^*) = Q$ and $q^*(q) = 1$.*

Appendix B

Set Theory

Ernst Zermelo

B.1 Predicate logic

Set Theory is the formal language upon which modern mathematics relies. Basically, almost every book on mathematical logic describes formal languages and axiomatic systems. Here, we will only provide few brushstrokes on this topic. From a simple viewpoint, every formal language is mainly composed of three parts: syntax, semantics, and pragmatics.

B.1.1 Syntax

Informally speaking, the syntax of any language is nothing but the symbols and the rules to construct words and sentences. In the particular case that we are dealing with, the syntax of the mathematical language is what we usually call Pure Mathematics. This syntax can be decomposed in two main parts:

- The symbols: Letters of the mathematical alphabet (the operators are also called logical operators or Boolean operators).
  - Variables: $\varnothing, x, y, z$, etc.
  - Quantifiers: $\forall, \exists, \exists!$.
  - Connectors: $=, \neq, \in, \notin, \subseteq, \nsubseteq, \subsetneq$.
  - Operators: unary ($\neg$) and binary ($\wedge, \vee, \Rightarrow, \Leftrightarrow$).
- The rules: To construct the sentences or predicates of the mathematical language. These predicates can be classified into two types: axioms and theorems. Later on, we will review axioms and theorems with more precision. The following laws are basic syntax rules by default.
  - Propositional logic rules:
    - Commutativity:

$$\begin{aligned} p \vee q &\Leftrightarrow q \vee p \\ p \wedge q &\Leftrightarrow q \wedge p \end{aligned}$$

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* Associativity:

$$\begin{aligned}(p \vee q) \vee r &\Leftrightarrow p \vee (q \vee r)\\ (p \wedge q) \wedge r &\Leftrightarrow p \wedge (q \wedge r)\end{aligned}$$

* Distributivity:

$$\begin{aligned}(p \wedge q) \vee r &\Leftrightarrow (p \vee r) \wedge (q \vee r)\\ (p \vee q) \wedge r &\Leftrightarrow (p \wedge r) \vee (q \wedge r)\end{aligned}$$

* Negation laws:

$$\begin{aligned}p \Rightarrow q &\Leftrightarrow \neg p \vee q\\ p \Rightarrow q &\Leftrightarrow \neg q \Rightarrow \neg p\\ \neg(\neg p) &\Leftrightarrow p\\ \neg(p \vee q) &\Leftrightarrow (\neg p \wedge \neg q)\\ \neg(p \wedge q) &\Leftrightarrow (\neg p \vee \neg q)\end{aligned}$$

* Identities:

$$\begin{aligned}p &\Leftrightarrow p\\ p \vee p &\Leftrightarrow p\\ p \wedge p &\Leftrightarrow p\end{aligned}$$

* Modus ponendo ponens:

$$p \wedge (p \Rightarrow q) \Rightarrow q$$

* Modus ponendo tollens:

$$\neg q \wedge (p \Rightarrow q) \Rightarrow \neg p$$

– Predicate logic rules:

* Distributivity of quantifiers:

$$\begin{aligned}\forall x\,(p(x) \wedge q(x)) &\Leftrightarrow \forall x\; p(x) \wedge \forall x\; q(x)\\ \exists x\,(p(x) \vee q(x)) &\Leftrightarrow \exists x\; p(x) \vee \exists x\; q(x)\end{aligned}$$

* Connectors rules:

$$\begin{aligned}\exists! x\; p(x) &\Leftrightarrow \exists x\,(p(x) \wedge \forall y(p(y) \Rightarrow x = y))\\ x \in y &\Leftrightarrow \exists z\,((z \in y) \wedge (x = z))\\ x = \varnothing &\Leftrightarrow \neg\exists y\,(y \in x)\\ x \subseteq y &\Leftrightarrow \forall z\,(z \in x \Rightarrow z \in y)\end{aligned}$$

* Negation laws:

$$\begin{aligned}\neg\,\forall x\; p(x) &\Leftrightarrow \exists x\; \neg p(x)\\ \neg\,\exists x\; p(x) &\Leftrightarrow \forall x\; \neg p(x)\\ x \neq y &\Leftrightarrow \neg(x = y)\\ x \notin y &\Leftrightarrow \neg(x \in y)\\ x \not\subseteq y &\Leftrightarrow \neg(x \subseteq y)\\ x \subsetneq y &\Leftrightarrow (x \subseteq y) \wedge (x \neq y)\end{aligned}$$

B.1.2 Semantics

Informally speaking, the semantics of any language is the meaning in the real world of the sentences of the language. The semantics of the mathematical language is what we usually call Applied Mathematics. Modeling the real world by using mathematical models is a way of providing significance to the mathematical sentences constructed by means of the syntax.

B.1.3 Pragmatics

Informally speaking, the pragmatics of any language is, essentially, the ability of using the language for communication purposes between individuals or beings.

B.2 Axiomatic systems

An axiomatic system is a collection of axioms from which we infer the rest of the sentences (theorems) of our formal language based on the syntax rules.

B.2.1 Axioms and theorems

We have already lied down, in an implicit way, the notions of axiom and theorem. Here, we will define them with rigor.

Definition 374 (Axiom) *Given a formal language, an axiom is a sentence or predicate, written in the syntax of the formal language, from which the rest of the sentences of the formal language are inferred, always following the syntax rules.*

A collection of axioms satisfying certain conditions will be called an axiomatic system.

Definition 375 (Contradiction) *Given a formal language, a contradiction is a predicate of the form* $p \wedge \neg p$.

The notion of logical contradiction is necessary before introducing axiomatic systems.

Definition 376 (Axiomatic system) *Given a formal language, an axiomatic system is a collection of axioms satisfying the following conditions:*

- *Consistency: No contradiction can be inferred from the axioms of the collection.*
- *Independence: No axiom of the collection can be inferred by the rest of the axioms of the collection.*

A usual way to check whether a certain axiom of a collection of axioms is independent or not of the rest of the axioms of the collection is by checking if the new collection formed by the negation of the axiom and the rest of the axioms is consistent.

Definition 377 (Completeness) *Given a formal language, an axiomatic system is said to be complete provided that, for any sentence or predicate written in the syntax of the formal language, either the predicate or its negation can be inferred from the axiomatic system, always following the syntax rules.*

Gödel Incompleteness Theorem [128, 129, 130] assures that any axiomatic system that models basic arithmetics is incomplete.

Definition 378 (Undecidable proposition) *Given a formal language endowed with an incomplete axiomatic system, an undecidable proposition is a predicate, written in the syntax of the formal language, such that the collection of axioms formed by the axiomatic system together with the proposition is consistent, and the collection of axioms formed by the axiomatic system together with the negation of the proposition is also consistent.*

The following example displays a very simple axiomatic system, written in the language of predicate logic.

Example 91 *Consider the axiomatic system whose axioms are:*

A1 $\exists x.$

A2 $\forall u \exists v \, (u \neq v).$

Like we have already indicated, the predicates of the mathematical language that can be deduced from a system of axioms are called theorems.

Definition 379 (Theorem) *Given a formal language endowed with an axiomatic system, a theorem is a predicate, written in the syntax of the formal language, that is inferred from the axioms of the axiomatic system, always following the syntax rules.*

The notion of proof is essentially a logic sequence or net of theorems which finishes at a certain statement, which is the desired theorem to be proved.

Definition 380 (Proof) *Given a formal language endowed with an axiomatic system, and given a theorem, a proof of that theorem is a logic sequence of inferred theorems that end up in the given theorem, always following the syntax rules.*

B.2.2 Zermelo-Fraenkel Axiomatic System

The predominant system of axioms for the mathematical language is the Zermelo-Fraenkel Axiomatic System.

Axiom 1 (Extensionality) $\forall x \forall y \left[(x = y) \Leftrightarrow ((x \subseteq y) \wedge (y \subseteq x))\right].$

The Axiom of Extensionality is equivalent to $\forall x \forall y [(x = y) \Leftrightarrow \forall z ((z \in x) \Leftrightarrow (z \in y))]$.

Axiom 2 (Specification) $\forall z \forall w_1 \cdots \forall w_n \left[\exists y \forall x \left((x \in y) \Leftrightarrow ((x \in z) \wedge p(x, z, w_1, \ldots, w_n))\right)\right]$.

The Axiom of Specification means that the set of all $x \in z$ satisfying $p(x, z, w_1, \ldots, w_n)$ can be constructed. It is denoted as $\{x \in z : p(x, z, w_1, \ldots, w_n)\}$. Another common notation for it is $\{x \in z \mid p(x, z, w_1, \ldots, w_n)\}$.

Theorem 179 (Empty set) *If x is a set, then there exists the (empty) set $\varnothing := \{a \in x : a \neq a\}$.*

The Axiom of Specification also allows us to construct the difference of two sets.

Theorem 180 (Difference of sets) *If x, y are sets, then there exists the set $x \setminus y := \{a \in x : a \notin y\}$.*

Notice that there is a correspondence between logical operators and set operators. In this correspondence, the difference of sets matches the negation operator.

Axiom 3 (Pairing) $\forall x \forall y \exists z \left((x \in z) \wedge (y \in z) \right)$.

In particular, with the Axiom of Specification and the Axiom of Pairing, we can construct the following sets.

Theorem 181 (Singletons & ordered pairs) *If x, y are sets and z is the set given by the Axiom of Pairing, then there exist the sets*

$$\begin{aligned} \{x\} &:= \{u \in z : u = x\}, \\ \{x, y\} &:= \{u \in z : (u = x) \vee (u = y)\}, \\ (x, y) &:= \{\{x\}, \{x, y\}\}. \end{aligned}$$

Similarly, other sets can be constructed by combining the Axiom of Specification and the Axiom of Pairing.

Axiom 4 (Union) $\forall x \exists y \left[\forall u \forall v \left(((v \in u) \wedge (u \in x)) \Rightarrow (v \in y) \right) \right]$.

With the Axiom of Union and the Axiom of Specification, we can construct the union of the elements of a given set.

Theorem 182 *If x is a set and y is the set given by the Axiom of Union, then there exist the sets $\cup x := \{v \in y : \exists u \left((v \in u) \wedge (u \in x) \right)\}$ and $\cap x := \{v \in y : \forall u \left((u \in x) \Rightarrow (v \in u) \right)\}$.*

With the Axiom of Union and the Axiom of Pairing, we can also construct the union and intersection of two given sets.

Theorem 183 (Union & intersection) *If x, y are sets and z is the set given by the Axiom of Union for $\{x, y\}$, then there exist $x \cup y := \cup\{x, y\} = \{v \in z : (v \in x) \vee (v \in y)\}$ and $x \cap y := \cap\{x, y\} = \{v \in z : (v \in x) \wedge (v \in y)\}$.*

The union and intersection of arbitrary families of sets will be constructed later on, precisely after introducing the notion of family.

Axiom 5 (Replacement) $\forall z \forall w_1 \cdots \forall w_n [\forall x ((x \in z) \Rightarrow \exists! y \ p(x, y, z, w_1, \ldots, w_n)) \Rightarrow \exists u \forall x ((x \in z) \Rightarrow \exists v ((v \in u) \wedge p(x, v, z, w_1, \ldots, w_n)))]$.

The Axiom of Replacement is an auxiliary axiom that allows the substitution of a variable in a sentence.

Definition 381 (Successor of a set) *For every set w, we define the successor of w as $\mathtt{s}(w) := w \cup \{w\}$.*

Successors are crucial for the next axiom: Axiom of Infinity.

Axiom 6 (Infinity) $\exists x \left[(\varnothing \in x) \wedge \forall y \left((y \in x) \Rightarrow (\mathtt{s}(y) \in x) \right) \right]$.

The Axiom of Infinity makes possible the definition of inductive set.

Definition 382 (Inductive set) *A set x is called inductive if $\varnothing \in x$ and satisfies that $\forall y \left((y \in x) \Rightarrow (\mathtt{s}(y) \in x) \right)$.*

With the Axiom of Infinity and the Axiom of Specification, it is possible to construct the set of natural numbers:

$$\mathbb{N} := \{n \in x : \forall z \left([(\varnothing \in z) \wedge \forall y \left((y \in z) \Rightarrow (\mathtt{s}(y) \in z)\right)] \Rightarrow (n \in z)\right)\}.$$

Axiom 7 (Regularity) $\forall x[(x \neq \varnothing) \Rightarrow \exists y((y \in x) \wedge \neg\exists z((z \in y) \wedge (z \in x)))]$.

The Axiom of Regularity is equivalent to $\forall x\ [(x \neq \varnothing) \Rightarrow \exists y\ ((y \in x) \wedge (y \cap x = \varnothing))]$.

Axiom 8 (Power set) $\forall x \exists y\, [\forall z((z \subseteq x) \Rightarrow (z \in y))]$.

With the Axiom of Power set and the Axiom of Specification, the power set of a given set can be constructed.

Theorem 184 *If x is a set and y is the set given by the Axiom of Power set, then there exists the power set $\mathcal{P}(x) := \{z \in y : z \subseteq x\}$.*

By relying again on the Axiom of Power Set and the Axiom of Specification, the product of two sets can also be constructed.

Theorem 185 (Product of sets) *If x, y are sets, then there exists the set*

$$x \times y := \{z \in \mathcal{P}\left(\mathcal{P}(x \cup y)\right) : \exists u \exists v\, [(u \in x) \wedge (v \in y) \wedge (z = (u, v))]\}.$$

Relations and maps will be defined and studied later on. However, we need them to state the last axiom: Axiom of Choice. This is why we define next relations and maps.

Definition 383 (Binary relation) *A binary relation from a set x to a set y is a subset of $x \times y$.*

As the next definition reveals, functions are particular cases of binary relations.

Definition 384 (Function) *The set of maps or functions from a set x to a set y is defined as*

$$y^x := \{f \in \mathcal{P}(x \times y) : \forall u\, [(u \in x) \Rightarrow \exists! v\, ((v \in y) \wedge ((u, v) \in f))]\}.$$

Choice functions are very important objects in mathematics, without which many other structures could not be constructed.

Definition 385 (Choice function) *The set of choice functions on a non-empty set x is defined as*

$$ch(x) := \left\{f \in x^{\mathcal{P}(x)\setminus\{\varnothing\}} : \forall y \in \mathcal{P}(x) \setminus \{\varnothing\}\ \ f(y) \in y\right\}.$$

The Axiom of Choice establishes the existence of choice functions. There are equivalent reformulations of the Axiom of Choice, such as Zorn's Lemma, which will be reviewed further on.

Axiom 9 (Choice) $\forall x[(x \neq \varnothing) \Rightarrow ch(x) \neq \varnothing]$.

Bibliography

[1] P. J. Campbell. The origin of "Zorn's Lemma". *Hist. Math.*, 5(1):77–89, 1978.

[2] T. J. Jech. *The axiom of choice*, volume 75 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[3] L. R. J. Westermann. On the hull operator. *Nederl. Akad. Wetensch. Proc. Ser. A*, 38(2):179–184, 1976.

[4] F. J. García-Pacheco. Hull operators in bounded posets. *Contrib. Math.*, 10:58–62, 2024.

[5] F. J. García-Pacheco. Chaos in topological modules. *Axioms*, 11(10):526, 2022.

[6] M. L. J. van de Vel. *Theory of convex structures*, volume 50 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1993.

[7] T. L. B. Almeida, A. R. Moghaddamfar, and I. N. Nakaoka. On pseudo-conjugation actions of finite groups and related topics. *Internat. J. Algebra Comput.*, 35(2):223–241, 2025.

[8] M. Bahrami-Taghanaki and A. R. Moghaddamfar. Toward pseudo-conjugation actions of groups. *Sib. Èlektron. Mat. Izv.*, 22(1):532–539, 2025.

[9] F. J. García-Pacheco. Convex functions on topological modules. *Linear Multilinear Algebra*, 0(0):1–14, 2020.

[10] F. J. García-Pacheco and P. Piniella. Unit neighborhoods in topological rings. *Banach J. Math. Anal.*, 9(4):234–242, 2015.

[11] F. J. García-Pacheco and P. Piniella. Geometry of balanced and absorbing subsets of topological modules. *J. Alg. Appl.*, 18(6), 2019.

[12] F. J. García-Pacheco. The finest locally convex module topology. In *Geometry of Banach spaces and related fields*, volume 106 of *Proc. Sympos. Pure Math.*, pages 133–166. Amer. Math. Soc., Providence, RI, [2024] ©2024.

[13] S. Moreno-Pulido. *Generalizations of classical properties of Measure Theory to effect algebras*, volume 2 of *eBooks on Mathematics and Physics applied to Engineering*. Editorial UCA, Cádiz, 2019.

[14] D. J. Foulis and M. K. Bennett. Effect algebras and unsharp quantum logics. *Found. Phys.*, 24(10):1331–1352, 1994.

[15] A. Dvurečenskij and S. Pulmannová. *New trends in quantum structures*, volume 516 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht; Ister Science, Bratislava, 2000.

[16] S. Koppelberg. *Handbook of Boolean algebras. Vol. 1.* North-Holland Publishing Co., Amsterdam, 1989.

[17] J. R. Munkres. *Topology.* Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.

[18] F. J. García-Pacheco, M. A. Moreno-Frías, and M. Murillo-Arcila. Topological ordered rings and measures. *Results Math.*, 78(6):Paper No. 212, 16, 2023.

[19] F. J. García-Pacheco. The order topology in partially ordered sets. *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, 18(1):161–177, 2026.

[20] F. J. Garcia-Pacheco. Unit neighborhoods of zero in topological ordered rings. *Carpathian J. Math.*, 41(4):951–957, 2025.

[21] S. Mazur. Über schwache Konvergenz in den Räumen (L^P). *Studia Math.*, 4:128–133, 1933.

[22] A. S. Granero, M. Jiménez-Sevilla, and J. P. Moreno. Intersections of closed balls and geometry of Banach spaces. *Extracta Math.*, 19(1):55–92, 2004.

[23] A. S. Granero, J. P. Moreno, and R. R. Phelps. Convex sets which are intersections of closed balls. *Adv. Math.*, 183(1):183–208, 2004.

[24] L. P. Belluce, W. A. Kirk, and E. F. Steiner. Normal structure in Banach spaces. *Pacific J. Math.*, 26:433–440, 1968.

[25] M. S. Brodskiĭ and D. P. Milman. On the center of a convex set. *Doklady Akad. Nauk SSSR (N.S.)*, 59:837–840, 1948.

[26] M. L. J. van de Vel. *Theory of convex structures*, volume 50 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1993.

[27] M. A. Khamsi. On metric spaces with uniform normal structure. *Proc. Amer. Math. Soc.*, 106(3):723–726, 1989.

[28] F. J. García-Pacheco, A. Miralles, and M. Murillo-Arcila. Invertibles in topological rings: a new approach. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 116(1):Paper No. 38, 17, 2022.

[29] F. J. García-Pacheco, M. Á. Moreno-Frías, and M. Murillo-Arcila. On absolutely invertibles. *Electron. Res. Arch.*, 32(12):6578–6592, 2024.

[30] R. Armario, F. Kh. Garsiya-Pacheko, and F. Kh. Peres-Fernandes. On vector-valued Banach limits. *Funktsional. Anal. i Prilozhen.*, 47(4):82–86, 2013.

[31] H. L. Royden. *Real analysis.* Macmillan Publishing Company, New York, third edition, 1988.

[32] A. Aizpuru. *Apuntes y notas de topología.* Servicio de Publicaciones de la Universidad de Cádiz, 2009.

[33] F. J. García-Pacheco, M. A. Moreno-Frías, and S. Rahbariyan. Generalized topological ordered groups. *Electron. Res. Arch.*, 34(5):3277–3288, 2026.

[34] S. Warner. *Topological fields*, volume 157 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1989.

[35] S. Warner. *Topological rings*, volume 178 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1993.

[36] H. Nakano. Concave modulars. *J. Math. Soc. Japan*, 5:29–49, 1953.

[37] F. León-Saavedra, M. C. Listán-García, F. J. Pérez Fernández, and M. P. Romero de la Rosa. On statistical convergence and strong Cesàro convergence by moduli. *J. Inequal. Appl.*, pages Paper No. 298, 12, 2019.

[38] P. Piniella. Existence of non-trivial complex unit neighborhoods. *Carpathian J. Math.*, 33(1):107–114, 2017.

[39] F. J. García-Pacheco and P. Piniella. Linear topologies and sequential compactness in topological modules. *Quaest. Math.*, 40(7):897–908, 2017.

[40] F. J. García-Pacheco and S. Sáez-Martínez. Normalizing rings. *Banach J. Math. Anal.*, 14(3):1143–1176, 2020.

[41] G. Köthe. *Topological vector spaces. I*, volume 159 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag New York Inc., New York, 1969.

[42] G. Köthe. *Topological vector spaces. II*, volume 237 of *Fundamental Principles of Mathematical Sciences*. Springer-Verlag, New York-Berlin, 1979.

[43] V. I. Arnautov, S. T. Glavatsky, and A. V. Mikhalev. *Introduction to the theory of topological rings and modules*, volume 197 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1996.

[44] F. J. García-Pacheco. Non-continuous linear functionals on topological vector spaces. *Banach J. Math. Anal.*, 2(1):11–15, 2008.

[45] R. Armario, F. J. García-Pacheco, and F. J. Pérez-Fernández. On the Krein-Milman property and the Bade property. *Linear Alg. Appl.*, 436(5):1489–1502, 2012.

[46] F. J. García-Pacheco, M. Á. Moreno-Frías, M. Murillo-Arcila, and S. Rahbariyan. Topological ordered modules. *Lobachevskii J. Math.*, 47(3):1011–1021, 2026.

[47] N. Bourbaki. *Topological vector spaces. Chapters 1–5*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1987.

[48] L. E. Dubins. On extreme points of convex sets. *J. Math. Anal. Appl.*, 5:237–244, 1962.

[49] F. J. García-Pacheco and E. Naranjo-Guerra. Inner structure in real vector spaces. *Georgian Math. J.*, 27(3):361–366, 2020.

[50] F. J. García-Pacheco. Relative interior and closure of the set of inner points. *Quaest. Math.*, 43(5-6):761–772, 2020.

[51] F. J. García-Pacheco. A Solution to the Faceless Problem. *J. Geom. Anal.*, 30(4):3859–3871, 2020.

[52] F. J. García-Pacheco, S. Moreno-Pulido, E. Naranjo-Guerra, and A. Sánchez-Alzola. Non-linear inner structure of topological vector spaces. *Mathematics*, 9(5), 2021.

[53] F. J. García-Pacheco. Regularity in topological modules. *Mathematics*, 8(9):1580, 2020.

[54] F. J. García-Pacheco and E. Naranjo-Guerra. Balanced and absorbing subsets with empty interior. *Adv. Geom.*, 16(4):477–480, 2016.

[55] F. J. García-Pacheco and E. Naranjo-Guerra. A family of balanced and absorbing sets with empty interior. *J. Nonlinear Func. Anal.*, 2017:1–10, 2017.

[56] R. E. Megginson. *An introduction to Banach space theory*, volume 183 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[57] F. J. García-Pacheco. The chain rule in topological modules. *Lobachevskii J. Math.*, 45(12):6083–6093, 2024.

[58] C. Cobos-Sánchez, F. J. García-Pacheco, J. M. Guerrero-Rodríguez, and J. R. Hill. An inverse boundary element method computational framework for designing optimal TMS coils. *Eng. Anal. Bound. Elem.*, 88:156–169, 2018.

[59] F. J. García-Pacheco, C. Cobos-Sánchez, S. Moreno-Pulido, and A. Sánchez-Alzola. Exact solutions to $\max_{\|x\|=1} \sum_{i=1}^{\infty} \|T_i(x)\|^2$ with applications to Physics, Bioengineering and Statistics. *Commun. Nonlinear Sci. Numer. Simul.*, 82:105054, 2020.

[60] S. Moreno-Pulido, F. J. García-Pacheco, C. Cobos-Sánchez, and A. Sánchez-Alzola. Exact Solutions to the maxmin problem $\max \|Ax\|$ subject to $\|Bx\| \leq 1$. *Mathematics*, 8(1):85, 2020.

[61] C. Cobos-Sánchez, F. J. García-Pacheco, J. M. Guerrero-Rodríguez, and L. García-Barrachina. Solving an IBEM with supporting vector analysis to design quiet TMS coils. *Eng. Anal. Bound. Elem.*, 117:1–12, 2020.

[62] C. Cobos-Sánchez, J. A. Vílchez-Membrilla, A. Campos-Jiménez, and F. J. García-Pacheco. Pareto optimality for multioptimization of continuous linear operators. *Symmetry*, 13(4), 2021.

[63] F. J. García-Pacheco. Convex components and multi-slices in real topological vector spaces. *Ann. Funct. Anal.*, 6(3):73–86, 2015.

[64] F. J. García-Pacheco. The boundary of an open convex set in a topological module. *J. Anal.*, 34(1):217–228, 2026.

[65] M. Krein and D. Milman. On extreme points of regular convex sets. *Studia Math.*, 9:133–138, 1940.

[66] F. J. García-Pacheco. The krein–milman theorem in topological modules. *Integral Transforms Spec. Funct.*, 0(0):1–12, 2026.

[67] H. Bauer. Minimalstellen von Funktionen und Extremalpunkte. *Arch. Math.*, 9:389–393, 1958.

[68] H. Bauer. Minimalstellen von Funktionen und Extremalpunkte. II. *Arch. Math.*, 11:200–205, 1960.

[69] C. Cobos-Sánchez, F. J. García-Pacheco, S. Moreno-Pulido, and S. Sáez-Martínez. Supporting vectors of continuous linear operators. *Ann. Funct. Anal.*, 8(4):520–530, 2017.

[70] E. Bishop and R. R. Phelps. A proof that every Banach space is subreflexive. *Bull. Amer. Math. Soc.*, 67:97–98, 1961.

[71] E. Bishop and R. R. Phelps. The support functionals of a convex set. In *Proc. Sympos. Pure Math., Vol. VII*, pages 27–35. Amer. Math. Soc., Providence, R.I., 1963.

[72] R. C. James. Characterizations of reflexivity. *Studia Math.*, 23:205–216, 1963/64.

[73] R. C. James. Reflexivity and the supremum of linear functionals. *Ann. of Math. (2)*, 66:159–169, 1957.

[74] J. Lindenstrauss. On operators which attain their norm. *Israel J. Math.*, 1:139–148, 1963.

[75] J. R. Partington. Norm attaining operators. *Israel J. Math.*, 43(3):273–276, 1982.

[76] M. D. Acosta, A. Aizpuru, R. M. Aron, and F. J. García-Pacheco. Functionals that do not attain their norm. *Bull. Belg. Math. Soc. Simon Stevin*, 14(3):407–418, 2007.

[77] F. J. García-Pacheco and D. Puglisi. Lineability of functionals and operators. *Studia Math.*, 201(1):37–47, 2010.

[78] F. J. García-Pacheco and D. Puglisi. A short note on the lineability of norm-attaining functionals in subspaces of ℓ_∞. *Arch. Math. (Basel)*, 105(5):461–465, 2015.

[79] F. J. García-Pacheco and D. Puglisi. Renormings concerning the lineability of the norm-attaining functionals. *J. Math. Anal. Appl.*, 445(2):1321–1327, 2017.

[80] F. J. García-Pacheco and D. Puglisi. Lineability of functionals and renormings. *Bull. Belg. Math. Soc. Simon Stevin*, 25(1):141–147, 2018.

[81] F. J. García-Pacheco and E. Naranjo-Guerra. Supporting vectors of continuous linear projections. *International J. Functional Analysis, Operator Theory and Applications*, 9(3):85–95, 2017.

[82] F. J. García-Pacheco. Lineability of the set of supporting vectors. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 115(2):41, 2021.

[83] A. Sánchez-Alzola, F. J. García-Pacheco, E. Naranjo-Guerra, and S. Moreno-Pulido. Supporting vectors for the ℓ_1-norm and the ℓ_∞-norm and an application. *Math. Sci.*, 15:173–187, 2021.

[84] Almudena P. Márquez, Francisco Javier García-Pacheco, Míriam Mengibar-Rodríguez, and Alberto Sánchez-Alzola. Supporting vectors vs. principal components. *AIMS Math.*, 8(1):1937–1958, 2023.

[85] M. M. Day. *Normed linear spaces*. Springer-Verlag, New York-Heidelberg, third edition, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 21.

[86] J. Diestel. *Geometry of Banach spaces—selected topics*. Lecture Notes in Mathematics, Vol. 485. Springer-Verlag, Berlin-New York, 1975.

[87] F. J. García-Pacheco. The subspace of G-invariant vectors in isometric representations. *Functional Analysis: Theory, Methods & Applications*, 1(2):69–79, 2015.

[88] F. J. García-Pacheco. Complementation of the subspace of G-invariant vectors. *J. Algebra Appl.*, 16(7):1750124, 7, 2017.

[89] J. Diestel. *Sequences and series in Banach spaces*, volume 92 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984.

[90] A. Dvoretzky and C. A. Rogers. Absolute and unconditional convergence in normed linear spaces. *Proc. Nat. Acad. Sci. U.S.A.*, 36:192–197, 1950.

[91] M. Cabrera-García and A. Rodríguez-Palacios. *Non-associative normed algebras. Vol. 1*, volume 154 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2014.

[92] M. Fabian, P. Habala, P. Hájek, V. Montesinos-Santalucía, J. Pelant, and V. Zizler. *Functional analysis and infinite-dimensional geometry*, volume 8 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2001.

[93] P. Hájek, V. Montesinos-Santalucía, J. Vanderwerff, and V. Zizler. *Biorthogonal systems in Banach spaces*, volume 26 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer, New York, 2008.

[94] J. Schauder. Zur Theorie stetiger Abbildungen in Funktionalräumen. *Math. Z.*, 26(1):47–65, 1927.

[95] F. J. García-Pacheco and F. J. Pérez-Fernández. Pre-Schauder bases in topological vector spaces. *Symmetry*, 11(8):1026, 2019.

[96] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces*, volume 338 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1973.

[97] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. I*, volume 92 of *Results in Mathematics and Related Areas*. Springer-Verlag, Berlin-New York, 1977.

[98] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. II*, volume 97 of *Results in Mathematics and Related Areas*. Springer-Verlag, Berlin-New York, 1979.

[99] P. Orlicz. Beiträge zur theorie der orthogonalentwicklungen. *Studia Math.*, 1(1):1–39, 1929.

[100] S. Banach. *Theory of linear operations*, volume 38 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1987.

[101] G. G. Lorentz. A contribution to the theory of divergent sequences. *Acta Math.*, 80:167–190, 1948.

[102] P. Antosik and C. Swartz. *Matrix methods in analysis*, volume 1113 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985.

[103] F. J. Pérez-Fernández, F. Benítez-Trujillo, and A. Aizpuru. Characterizations of completeness of normed spaces through weakly unconditionally Cauchy series. *Czechoslovak Math. J.*, 50(125)(4):889–896, 2000.

[104] C. Swartz. *Multiplier convergent series*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.

[105] F. J. García-Pacheco, R. Kama, and M. C. Listán-García. General methods of convergence and summability. *J. Inequal. Appl.*, 2021(1):62, 2021.

[106] H. Fast. Sur la convergence statistique. *Colloq. Math.*, 2:241–244 (1952), 1951.

[107] J. A. Fridy. On statistical convergence. *Analysis*, 5(4):301–313, 1985.

[108] E. Kolk. The statistical convergence in Banach spaces. *Tartu Ül. Toimetised*, (928):41–52, 1991.

[109] A. Aizpuru, M. C. Listán-García, and F. Rambla-Barreno. Density by moduli and statistical convergence. *Quaest. Math.*, 37(4):525–530, 2014.

[110] E. M. Semenov and F. A. Sukochev. Invariant Banach limits and applications. *J. Funct. Anal.*, 259(6):1517–1541, 2010.

[111] A. Aizpuru, R. Armario, F. J. García-Pacheco, and F. J. Pérez-Fernández. Banach limits and uniform almost summability. *J. Math. Anal. Appl.*, 379(1):82–90, 2011.

[112] A. Aizpuru, R. Armario, F. J. García-Pacheco, and F. J. Pérez-Fernández. Vector-valued almost convergence and classical properties in normed spaces. *Proc. Indian Acad. Sci. Math. Sci.*, 124(1):93–108, 2014.

[113] F. J. García-Pacheco. Extremal properties of the set of vector-valued Banach limits. *Open Math.*, 13(1):757–767, 2015.

[114] F.-Kh. Garsiya-Pacheko and F.-Kh. Peres-Fernandes. Fundamental aspects of vector-valued Banach limits. *Izv. Ross. Akad. Nauk Ser. Mat.*, 80(2):33–46, 2016.

[115] R. Tanaka. On vector-valued Banach limits with values in $\mathcal{B}(\mathcal{H})$. *Publ. Math. Debrecen*, 92(3-4):471–480, 2018.

[116] G. Barbieri, F. J. García-Pacheco, and S. Moreno-Pulido. Measures on effect algebras. *Math. Slovaca*, 69(1):159–170, 2019.

[117] K. P. S. Bhaskara Rao and M. Bhaskara Rao. *Theory of charges*, volume 109 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[118] J. Diestel and J. J. Uhl, Jr. *Vector measures*. American Mathematical Society, Providence, R.I., 1977.

[119] N. Dunford and J. T. Schwartz. *Linear operators. Part I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988.

[120] J. Diestel, B. Faires, and R. Huff. Boundedness and convergence of measures on non-σ-complete algebras. *Preprint*, 1976.

[121] A. Haar. Der massbegriff in der theorie der kontinuierlichen gruppen. *Annals of Mathematics*, 34(1):147–169, 1933.

[122] Robert M. Guralnick and Geoffrey R. Robinson. On the commuting probability in finite groups. *J. Algebra*, 300(2):509–528, 2006.

[123] W. H. Gustafson. What is the probability that two group elements commute? *Amer. Math. Monthly*, 80:1031–1034, 1973.

[124] P. R. Halmos. *Measure Theory*. D. Van Nostrand Co., Inc., New York, 1950.

[125] M. Fréchet. Sur les opérations linéaires. III. *Trans. Amer. Math. Soc.*, 8(4):433–446, 1907.

[126] F. Riesz. Sur une espèce de géométrie analytique des systèmes de fonctions sommables. *Comptes rendus de l'Académie des Sciences*, 144:1409–1411, 1907.

[127] F. Riesz. Sur les opérations fonctionnelles linéaires. *Comptes rendus de l'Académie des Sciences*, 149:974–977, 1909.

[128] K. Gödel. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. *Monatsh. Math. Phys.*, 37(1):349–360, 1930.

[129] K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatsh. Math. Phys.*, 38(1):173–198, 1931.

[130] K. Gödel. *Logic lectures–Gödel's basic logic course at Notre Dame*. Logical Society, Belgrade; Dosije Studio, Belgrade, 2017. Edited by Miloš Adžić and Kosta Došen.

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