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Mathematical Analysis with Topological Examples

A Narrative Approach for Undergraduates

Omid Zabeti



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Mathematical Analysis with Topological Examples

Mathematical Analysis with Topological Examples: A Narrative Approach for Undergraduates offers an introduction to elementary analysis for undergraduates studying mathematics or any related quantitative science. This book takes a unique approach, adopting a narrative framework to guide students through the material as though it were an unfolding story. Alongside this, the book introduces a vast array of examples from the world of topology to illustrate the core principles of analysis in an engaging and accessible way.

This book is suitable for undergraduate students who are familiar with elementary calculus. It can serve as an introduction to elementary analysis and a supplementary resource for a course in topology.

Features

- Unique narrative structure to engage students
- Numerous examples from topology
- Replete with exercises and solutions.

Mathematical Analysis with Topological Examples A Narrative Approach for Undergraduates

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Dedicated to:

*the heavenly souls of my devoted parents
(**Gholamreza** and **Esmat**),
whose love and sacrifices shaped my path;
to **Tayebeh**, my patient and selfless wife,
the quiet strength behind every step;
and to **Armin** and **Anisa**,
the pure joys and brightest reasons of my life.*

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Preface

Let me begin with an example from our Iranian culture. You may have heard of famous poets like Hafez, Khayyam, and Ferdowsi. Although they lived several centuries ago, their works remain relevant and meaningful today and will undoubtedly continue to resonate in the future. This raises an important question:

Why are these poems and their messages indelible?

Certainly, the poets' ingenious minds and natural talents play a crucial role in this phenomenon. However, one of the most noteworthy aspects is their ability to communicate with a broad audience using the language of their time. They skillfully articulated significant ideas and principles in a way that was accessible to the masses, which is key to their lasting popularity and impact over the centuries.

Now, let us return to the subject of our discussion: mathematics. One of the advantages of math compared to other sciences is its capacity to serve as a translator and validator for various fields of study. Every theory in any branch of science requires mathematical validation. Additionally, in our daily lives, we often rely on examples to illustrate, interpret, judge, and reflect on various concepts and individuals. Therefore, examples are fundamental to our existence.

We believe that mathematics is the “queen of sciences,” and we expect it to play a significant role in our daily lives as well. Consequently, math holds a central position about examples. In other words, mathematical examples provide everyone with the opportunity to understand “what is happening” in the world and, more specifically, “how we can make the universe more comprehensible.” This book aims to present a comprehensive perspective to illustrate these statements and explore additional ideas that may arise throughout the text.

We strive to maintain a narrative style from the beginning of the analysis to its more advanced topics to engage readers. Additionally, topology serves as a foundation for critical thinking, as every analytical concept can have a counterpart in topology. The primary advantage of topology is its broad

scope and expansive vision, which complements analysis. For example, topology introduces concepts like “Hausdorff spaces,” which validate the notion of “convergence” in analysis. Topology also examines “nets,” which provide a suitable generalization of sequences and encompass various types of convergence. Moreover, we can characterize important concepts like “continuity” in terms of these ideas.

Thus, this rich material is well-suited to enhance our understanding of analysis, and we take this valuable opportunity to write this book. Finally, we would like to highlight three distinctive features of this work:

1. It employs a narrative style to facilitate a better understanding of analysis.
2. It utilizes topology as an advanced resource for presenting examples and offering a broad perspective on concepts.
3. It provides a unique selection of examples to illustrate the concepts of analysis.

I hope you savor this unique flavor throughout your adventure.

Here, the rationale for writing this book is exemplified. We firmly believe that “topology” is a part of “analysis” and is a versatile tool for understanding it better. Likewise, Mathematics is alive with examples, and these are examples (and counterexamples) that assist mathematicians in the soundest understanding of the mysterious and unknown worlds. Suppose “Mathematics” to be an outlandish restaurant with fantastical yummy foods. You intend to taste as much as you can eat but because of the diversity and cost of the foods, you are forced to choose by trying a few morsels of several foods that appear to be more delicious and then order them. Here, the role of examples in Mathematics is “morsels to try.” Indeed, we should ensure that every subject and every part of the theory matches our intuition and demonstrates its best performance. In other fields, every statement with a story language is more likely to be perceived and memorized by a broader readership. Story language affords an attractive way of understanding the statements better and perceiving the importance of the subjects that act as heroes of a novel (besides lovely Mathematics objects). Accordingly, by merging all these matters, we arranged to write a book about topological analysis enriched with relevant examples. To draft an engaging book that contains long-enduring and memorable materials, we benefit from a story language to devise a platform for the readers to learn about the importance

of analysis objects (as heroes of our novel) as well as to pursue the story until the end to complete a course (story) about analysis and preliminary topology. This book is suitable for undergraduate students who are familiar with elementary Calculus. Additionally, it can serve as an advanced source for elementary analysis and an elementary source for topology. Again, it presents a multitude of examples and counterexamples to illustrate the analysis satisfactorily. For this goal, this book uses topology for a comprehensive view of analysis. Ultimately, “topology” is a taste utilized across the book to make our “analysis” more delectable. We make this yummy food memorable by beginning a journey with boats and submarines. Thus, let us start our trip to the “math ocean” and “analysis island” while you enjoy your food.

Omid Zabeti

Author Bios

Omid Zabeti is an Associate Professor of Mathematics specializing in functional analysis. He received his PhD from Ferdowsi University of Mashhad in 2012 and has been a faculty member at the University of Sistan and Baluchestan since then. His research interests include Banach lattices, locally solid vector lattices, topology, and operators. He believes that mathematics underpins all sciences and plays a vital role in everyday life. In this book, he combines mathematical analysis with a narrative style and a topological perspective to illustrate the beauty and importance of mathematics in the modern world.

A Different Story About Mathematics

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1.1 IMPORTANCE OF MATHEMATICS IN THE WORLD

The analysis presented is sufficient to cover many analytical concepts and enhance various methods, making them easier and more accurate. However, this sufficiency sometimes feels incomplete. For instance, in the upcoming chapters, we will explore metric spaces and normed spaces, which you may already be familiar with. These analytical constructs are applicable across all domains of analysis and other branches of mathematics. Nevertheless, they don't fully encapsulate the importance of mathematics.

To illustrate this, imagine that each branch of mathematics is an island in the vast ocean of thinking—referred to as “the math ocean,” as thinking is a hallmark of mathematics. In this metaphor, “analysis” is a powerful island rich in natural resources such as sequences, metrics, norms, compactness, and connectedness. Thus, analysis is indeed a very prosperous island.

On this island, some inhabitants engage in fishing and agriculture (we will learn about other occupations and characteristics of this island throughout the book). They produce various fruits, such as theorems, lemmas, and propositions, which they trade with neighboring islands like “numerical analysis” and “linear algebra.” The former uses these fruits to develop more efficient methods, which are then sold to the “application” island. The latter combines them with specific algebraic structures to create compotes for several other islands. The fishing aspect involves gathering crucial resources from the math ocean, resulting in vital theorems and

axioms that serve as fertilizer for maintaining fertile soil across all mathematical islands.

Before we delve deeper, let's describe the math ocean more precisely. All the sciences in the world comprise a continent called “the world,” which is lush, vibrant, and pioneering in sustaining the planet amidst the other planets, referred to as “the parallel worlds,” orbiting a star named “God.” Within this continent, each science occupies its territory, while mathematics spans an ocean. More specifically, all sciences lie along a coastline that borders this ocean. This arrangement signifies that mathematics is the leader and queen of all sciences, necessitating access to each one. The intriguing aspect lies in the dynamics of how this continent functions. As mentioned earlier, within the vast “math ocean,” there are various islands representing different branches of mathematics. Additionally, all sciences have their territories along the shores of this ocean. To navigate between these islands and the scientific lands, we use two types of boats: “pure” and “applied.”

There are also established routes between the scientific territories. For instance, a well-defined connection exists between “physics land” and “chemistry land,” where trucks transport items such as “chemicals” and “energy boxes.” To sustain life in these lands—providing fuel to factories, equipment to hospitals, and educational resources—materials from the diverse islands in the math ocean are essential. Each island has its coast guard command overseeing the boats along its shoreline.

In certain key areas, such as the basic sciences, there are wharves to facilitate the transfer of materials, as the volume of goods is often substantial and requires additional ports. The primary role of the “pure” boats is to transport significant theorems and results between the islands of the math ocean, while the “applied” boats typically operate along the coastline, connecting specific lands or traversing between various territories.

A question that may arise is: Why do we need these math boats to transfer information between the scientific lands? This question becomes more complex when we recognize that the most valuable data on the land is transported by math boats rather than trucks. However, the answer is relatively straightforward. Mathematics serves as the foundation and validator of other sciences, meaning that every land relies on mathematical boats to fulfill its essential needs. As mentioned earlier, the Math islands export delightful fruits and unique compotes to other lands and islands. But this is just part of the narrative.

Each island also houses factories, mines, refineries, and other facilities that produce various materials. For instance, in the mines, there is a quest for valuable old results that can either be used to create something new or sold directly as significant antiquities that are highly sought after in certain islands or lands. Refineries often employ “new ideas” to generate valuable materials for export. Meanwhile, factories develop algorithms, methods, and many other innovations. Thus, the entire ecosystem of mathematics and its surrounding shores operates meticulously and harmoniously.

Now, what materials do the islands import? Primarily, they import a multitude of questions and complex problems from other lands to solve. This overview sketches the larger story that we will explore in this book. With each step and concept, we will enrich our narrative, ultimately providing you with insights into analytical concepts as well. By the end of this book, readers will feel as though they have experienced both a novel and a mathematics textbook.

1.2 WHY DO WE NEED TO STUDY ANALYSIS WITH TOPOLOGY?

We continue our previous story. The ocean was home to numerous sea creatures and mysterious materials, offering new ideas and life forms to discover every day and everywhere. This rich ocean was a treasure for neighboring lands. Over time, the analysis island learned about some peculiar new entities, such as “different types of unknown convergences,” “separability axioms,” and “strange new vector spaces.” While many types of convergences and vector spaces already existed, these new ones were unfamiliar to the analysts. They tried to impose norms or metrics on them in their factories, but these entities were unruly, necessitating a new approach.

Additionally, the nets analysts cast into the ocean were too loose, causing many new creatures and plankton to slip through and remain unreachable. In the space between the analysis and geometry, islands lay a fertile island rich in scarce and valuable minerals. Both islands utilized these minerals under a signed agreement, though beneath the surface, a quiet rivalry persisted over who benefited more from the arrangement. As an analyst, I firmly believe that the majority of topology is inherently linked to analysis.

Near the beach of the topology island grew special algae ideal for crafting ropes and nets. These algae could also be used as preservatives for various compotes. Consequently, the analysts decided to harness the algae to create preservers, ropes, and nets for capturing the undefined and

unknown plankton. This collaboration between the topology and analysis islands aimed to produce longer-lasting compotes and a special sauce for the many branches of “functional analysis” that extended across various islands and lands.

Thus, when studying any concept in analysis, having a solid understanding of topology—serving as a valuable resource—can be incredibly beneficial. For instance, many topological vector spaces are neither metrizable nor normable (resources that are abundant on the analysis island), yet they hold significant value in their own right. The uniqueness of limits analysis is attributed to topology and separation axioms, particularly the Hausdorff property. With the support of topology, analysts can navigate faster and further in their boats, discovering new sea creatures and delivering superior, customer-friendly products to other islands and lands.

1.3 HOW DOES TOPOLOGY ILLUSTRATE ANALYSIS BETTER?

The ocean's waves are moderate in height and speed, creating fertile soil for the islands and delivering valuable resources to their shores without the need for boating. Every ten years, the “thinking and analytic” volcanoes located in the depths of the math ocean—though seemingly calm—release new ideas that rise to the surface, forming new islands. It's important to note that these new ideas can also originate from certain algae beneath the ocean, which will be discussed later. A special force in the math islands called novelty allows these new ideas to connect with existing islands, enhancing their strength and efficiency in the world. This process is a routine and ingenious aspect of the ocean and its islands.

Topology is one such island that has emerged through this method. The ingenious nature of mathematics recognizes every problem across all islands, and by refreshing its subjects every ten years, it helps these islands reform themselves. Topology has a fascinating history, having arisen from a need for analysis. There exists a hidden competition among the islands of the math ocean, striving for greater exports than imports. Increased exports translate to a more powerful position among the islands and with other lands. While cooperation among the islands is evident, the underlying competition is paramount.

As mentioned earlier, strange objects of significant importance emerged, although analysts struggled to discover and analyze them. Conversely, these significant objects gained popularity in other lands, leading to the establishment of several factories and refineries dependent on them for raw

materials. The lands themselves also experienced competition, as they required more efficient and cost-effective raw materials to enhance their business. Although previous products may have provided some raw materials, they were not as efficient as the new ones. Although analysts held important concepts such as “convergences” and “big theorems,” these alone were insufficient to secure their position in global markets. They needed to better illustrate and promote their products, as well as develop new offerings using “strange popular new objects.” Topology proved to be a valuable tool in discovering, analyzing, and constructing these new materials. For example, certain plankton in the ocean were useful in communication equipment and formed the basis of a programming language designed to enhance data export and import between land and islands.

The conventional analytical tool known as “sequences” was inadequate for capturing these plankton. Analysts required a more effective “net.” Topology provided this capability through an extension of sequences, known as “nets,” allowing them to trap and analyze these objects and produce new materials for export.

Furthermore, there were numerous uneven pathways between the analysis island and other mathematical islands, including some new and problematic ones dubbed “the thoughtless islands.” These islands disrupted communication between the mathematical territories and contained stalls that sold “useless” materials, making travelers sick and discouraging them from continuing on their journey. Analysts could only recognize these obstacles from the front and back, which meant many remained hidden and mobile. In contrast, topology utilized aerial shots to survey the surroundings, enabling it to identify all potential disruptive islands or stalls.

After illustrating the importance of topology for analysis through a narrative, we shift to mathematical language. Building on our previous understanding of analysis, we know that sequences are useful tools for characterizing various types of convergence. However, they do not encompass all possible convergence directions. Specifically, there exist certain topological spaces that are too large for sequences to fully explore. To address this, topology introduces the concept of “nets,” an extension of sequences that can traverse any space and encompass all its elements.

1.4 EXAMPLES: AN INVALUABLE HELP FOR MATH

In daily life, people often use examples to clarify their points or attract attention. Examples, including both direct examples and counterexamples, aid in understanding and identifying incorrect ideas. In the vast realm of mathematics, examples serve as powerful sources of illumination—much like lightning that strikes across islands, land, cities, and the depths of the ocean.

These examples can be thought of as power stations in various mathematical domains, fueled by unique ideas represented as codes on different types of algae found underwater. Most of these novel ideas thrive in cities or their surrounding areas, which we will explore further soon. Mathematicians, as the distinguished explorers of this mathematical landscape, dive deep into the ocean daily to uncover new algae. This newfound knowledge fuels the power stations, enabling illumination throughout the mathematical world.

With this illumination, mathematicians can navigate their domains, discovering new concepts, producing innovative solutions in factories and refineries, and ultimately advancing mathematics as a whole. Without examples, there is no fuel for this illumination, and without illumination, the exploration of mathematical realms becomes impossible. Thus, examples are an invaluable resource in the field of mathematics.

A Brief Story About Topology

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Method cards: a card for all purposes in this book

Method card, the membership stamp: The ocean of mathematics, with all its components, forms a giant set whose inhabitants live in different subsets, such as the Numbers realm (where the majority live), the cities of Continuous Functions, and the islands of Unknowns. Each citizen who wishes to enter the territory of another realm must have a special stamp on their membership card issued by the mathematics ocean (method card). The philosophy behind labeling it the “method card” is that each set with a special stamp from a realm can only enter a specific realm and use its particular facilities; this way, the set's working method can change in the ocean of mathematics. For example, residents of the Real Number district, with various topologies, can enter the Topology district and use its facilities. However, they are only permitted to enter Banach Street (one of the most luxurious and modern areas of the Topology realm) and use its most exquisite facilities if they have Euclidean topology. Or, as another example, to enter the Function cities, sets require the “Cartesian Product” stamp. In addition, upon arrival at any Function city, a border officer validates the stamp's “originality” with certain set of ordered pairs, and if the set does not match, the officer denies entry to that set. Throughout the vast ocean of mathematics, there are numerous regions with significant facilities. For example, the Topology realm (where most of our story takes place) is a large set that includes cities, islands, and unknown regions.

Topology Realm: A World of Beauty, Elegance, and Power

Within the ocean of mathematics lies a world of reasoning: a set of cities, regions, and islands, and a different world beneath the ocean. The Topology realm is one of the most influential areas of the ocean of mathematics. Any set (region) in the world of mathematics can choose a topological manager for itself. There are three specific basic conditions for a set (region) to enter the Topology realm, and all residents who receive a membership stamp in the Topology realm must comply with them. These conditions are:

1. All points of the chosen set (region) must be located within the Topology realm. Even regions with no members (residents) are considered inside the Topology realm's territory.
2. Each pair of members, in case they have common capabilities and ideas, can utilize the said capability and idea to improve the topology, forming new members with better focus (special attention to common capabilities and teamwork). In other words, the intersection of two regions of the realm is itself a distinct (new) part of the realm.
3. The presence of any number of members beside each other to help progress toward the goals of the intended region leads to advancement and progression of topology (the importance of maintaining cohesion and integrity among members). In other words, any arbitrary union of the intended regions is also deemed a member of the Topology realm. Conditions 2 and three indicate that in the Topology realm, minor parts (intersection of regions) and larger parts (union of regions) belong to the realm, and their residents can simply receive membership stamps on their cards, allowing them to freely trade and exchange across the Topology realm with no restriction. Each region that fulfills the above conditions and complies with certain upstream rules is called a Topologic region. Together, these regions form the Topology realm.

Each manager manages their topological set according to their own way of thinking and capabilities, and the efforts of other members. Therefore, the results obtained by different topological sets are not identical. On the other hand, given the vastness of the regions and the need to deal with the affairs of the inhabitants of the Topology realm in a timely and effective manner, as well as the importance of being able to compete in the export and import market, the managers of the Topology realm select representatives with similar types as the members themselves (called bases) for various matters based on their adequacy to carry out the assigned responsibilities in accordance with the upstream rules and also for the welfare of other members. We call them the topology bases. The base elements have two characteristics:

1. All representative managers (base elements) must form the entirety of the region. That is, there must be enough representatives for all possible areas.
2. For every two representatives with common capabilities and ideas, there must be another representative who has a higher level of mastery of the subject and is capable of supervising, optimizing ideas, and providing counsel. In fact, this rule has two benefits: promoting teamwork spirit among the representative (base) members, just like all other topology members, and the significant role of other representatives in gaining insight, exercising sufficient control, and advancing the objectives of the topology.

Definition 2.1 *Let X be a non-empty set and let $\tau \subseteq \mathcal{P}(X)$. We say that τ is a topology on X if the following conditions hold:*

1. $\emptyset, X \in \tau$;
2. If $A, B \in \tau$, then $A \cap B \in \tau$ (that is, τ is closed under finite intersections);
3. If $(A_\alpha)_{\alpha \in I}$ is an arbitrary family of sets in τ , then

$$\bigcup_{\alpha \in I} A_\alpha \in \tau$$

(that is, τ is closed under arbitrary unions).

In this case, the pair (X, τ) is called a topological space, and the elements of τ are called open sets.

Although this book does not aim to present a comprehensive course in topology, we include several examples to illustrate this notion.

Example 2.1 *Let X be a non-empty set.*

- The family $\tau_1 = \mathcal{P}(X)$ is the largest possible topology on X and is called the discrete topology. In this topology, every subset of X is open.
- The family $\tau_2 = \{\emptyset, X\}$ is the smallest possible topology on X and is called the indiscrete (or trivial) topology.

Example 2.2 Let $X = \mathbb{R}$ and define

$$\tau_{\rightarrow} = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

It is easy to verify that $\tau_{\rightarrow}$ is a topology on $\mathbb{R}$; it is called the right-ray topology.

Similarly, define

$$\tau_{\leftarrow} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

Then $\tau_{\leftarrow}$ is also a topology on $\mathbb{R}$, called the left-ray topology.

Before introducing further examples of topologies on $\mathbb{R}$, we explain the notion of a basis for a topology.

A natural and fundamental question that arises in the study of topological spaces is the following.

Question. Is it possible to generate a topology using a smaller collection of open sets? In other words, does there exist a representative subfamily of open sets that completely determines the topology?

Before answering this question, we need the following fundamental results.

Let $X \neq \emptyset$ be a set and let $\{\tau_{\alpha}\}_{\alpha \in I}$ be a family of topologies on X . It is easy to verify that

$$\tau = \bigcap_{\alpha \in I} \tau_{\alpha}$$

is also a topology on X .

Consequently, we may speak of the topology generated by a given family of sets. More precisely, let $\mathcal{S} \subseteq \mathcal{P}(X)$. The topology generated by $\mathcal{S}$, denoted by $\tau_{\mathcal{S}}$, is defined by

$$\tau_{\mathcal{S}} = \bigcap \{\tau : \tau \text{ is a topology on } X \text{ and } \mathcal{S} \subseteq \tau\}.$$

Note that this intersection is non-empty since $\mathcal{P}(X)$ is a topology containing $\mathcal{S}$. Define

$$\mathcal{S}_{\delta} = \{A \subseteq X : A \text{ is a finite intersection of elements of } \mathcal{S}\}$$

and

$$\mathcal{S}_\sigma = \{A \subseteq X : A \text{ is an arbitrary union of elements of } \mathcal{S}\}.$$

By convention, we denote $(\mathcal{S}_\delta)_\sigma$ by $\mathcal{S}_{\delta\sigma}$. The following result is elementary but fundamental.

Proposition 2.1 *Let (X, τ) be a topological space and let $\mathcal{S} \subseteq \mathcal{P}(X)$. Then the topology generated by $\mathcal{S}$ satisfies*

$$\tau_{\mathcal{S}} = \mathcal{S}_{\delta\sigma}.$$

Proof 2.1 *We first show that $\mathcal{S}_{\delta\sigma}$ is a topology on X containing $\mathcal{S}$.*

By definition, $\mathcal{S} \subseteq \mathcal{S}_\delta \subseteq \mathcal{S}_{\delta\sigma}$, hence $\mathcal{S} \subseteq \mathcal{S}_{\delta\sigma}$. Moreover, $\mathcal{S}_{\delta\sigma}$ is closed under arbitrary unions by construction, and it is closed under finite intersections since finite intersections of finite intersections are again finite intersections.

Furthermore, both $\emptyset$ and X belong to $\mathcal{S}_{\delta\sigma}$. Indeed,

$$\emptyset = \bigcup_{A \in \emptyset} A \in \mathcal{S}_\sigma \subseteq \mathcal{S}_{\delta\sigma},$$

and

$$X = \bigcap_{A \in \emptyset} A \in \mathcal{S}_\delta \subseteq \mathcal{S}_{\delta\sigma},$$

where these equalities follow from the principle of vacuous truth.

Hence, $\mathcal{S}_{\delta\sigma}$ is a topology on X containing $\mathcal{S}$. Since $\tau_{\mathcal{S}}$ is defined as the smallest topology on X containing $\mathcal{S}$, we conclude that

$$\tau_{\mathcal{S}} = \mathcal{S}_{\delta\sigma}.$$

Consequently, the topology generated by $\mathcal{S}$ consists precisely of all arbitrary unions of finite intersections of elements of $\mathcal{S}$. In other words,

$$\tau_{\mathcal{S}} = \left\{ \bigcup_{\alpha \in I} \bigcap_{k=1}^{n_\alpha} S_{\alpha,k} : S_{\alpha,k} \in \mathcal{S} \right\}.$$

Definition 2.2 Let (X, τ) be a topological space. A family $\mathcal{B} \subseteq \tau$ is called a basis for τ if

$$\tau = \tau_{\mathcal{B}}.$$

Equivalently, every open set in (X, τ) can be written as an arbitrary union of elements of $\mathcal{B}$.

Proposition 2.2 Let (X, τ) be a topological space and let $\mathcal{B} \subseteq \tau$. Then $\mathcal{B}$ is a basis for τ if and only if for every $U \in \tau$ and every $x \in U$, there exists $B_x \in \mathcal{B}$ such that

$$x \in B_x \subseteq U.$$

Proof 2.2 Assume first that $\mathcal{B}$ is a basis for τ , so that $U = \bigcup_{\alpha \in I} B_{\alpha}$ for some $B_{\alpha} \in \mathcal{B}$. If $x \in U$, then $x \in B_{\alpha}$ for some α , and clearly $B_{\alpha} \subseteq U$.

Conversely, suppose that for every $U \in \tau$ and every $x \in U$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then

$$U = \bigcup_{x \in U} B_x,$$

and hence $U \in \tau_{\mathcal{B}}$. Therefore $\tau \subseteq \tau_{\mathcal{B}}$. Since $\mathcal{B} \subseteq \tau$, we also have $\tau_{\mathcal{B}} \subseteq \tau$, and thus $\tau = \tau_{\mathcal{B}}$.

Remark 2.1 $\hookleftarrow$ Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis for τ . If $U \in \tau$ and $x \in U$, then there exists $B_x \in \mathcal{B}$ such that

$$x \in B_x \subseteq U.$$

Proposition 2.3 Let X be a set and suppose $\mathcal{B} \subseteq \mathcal{P}(X)$ is a collection of subsets of X . Then $\mathcal{B}$ is a basis for a topology on X if and only if the following conditions hold:

$$1. X = \bigcup_{B \in \mathcal{B}} B;$$

2. For every pair $B_1, B_2 \in \mathcal{B}$ and for every $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that

$$x \in B \subseteq B_1 \cap B_2.$$

Proof 2.3 Suppose first that $\mathcal{B}$ is a basis. Then $X \in \mathcal{B}_{\mathcal{B}}$, hence X is a union of elements of $\mathcal{B}$, and (1) follows.

To prove (2), let $B_1, B_2 \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Since $B_1, B_2 \in \mathcal{B}$, they are open in the topology generated by $\mathcal{B}$. Thus $B_1 \cap B_2$ is open. Hence

$$B_1 \cap B_2 = \bigcup_{\alpha \in I} B_{\alpha}$$

for some subcollection $\{B_{\alpha}\}_{\alpha \in I} \subseteq \mathcal{B}$. Since $x \in B_1 \cap B_2$, there exists $\alpha_0 \in I$ such that $x \in B_{\alpha_0}$. Therefore

$$x \in B_{\alpha_0} \subseteq B_1 \cap B_2,$$

and (2) follows.

Conversely, assume that (1) and (2) hold. We show that $\mathcal{B}$ is a basis. Let $\mathcal{B}_{\mathcal{B}}$ denote the collection of all unions of elements of $\mathcal{B}$. By (1), $X \in \mathcal{B}_{\mathcal{B}}$. Clearly $\emptyset \in \mathcal{B}_{\mathcal{B}}$, and arbitrary unions of members of $\mathcal{B}_{\mathcal{B}}$ are again in $\mathcal{B}_{\mathcal{B}}$.

It remains to show that finite intersections of members of $\mathcal{B}_{\mathcal{B}}$ belong to $\mathcal{B}_{\mathcal{B}}$. Let

$$A_1 = \bigcup_{\alpha \in I} B_{\alpha}, \quad A_2 = \bigcup_{\beta \in J} C_{\beta},$$

where $B_{\alpha}, C_{\beta} \in \mathcal{B}$. Let $x \in A_1 \cap A_2$. Then $x \in A_1$ and $x \in A_2$, so there exist $\alpha_0 \in I$ and $\beta_0 \in J$ such that

$$x \in B_{\alpha_0}, \quad x \in C_{\beta_0}.$$

Thus $x \in B_{\alpha_0} \cap C_{\beta_0} \subseteq A_1 \cap A_2$.

By (2), there exists $B_x \in \mathcal{B}$ such that

$$x \in B_x \subseteq B_{\alpha_0} \cap C_{\beta_0} \subseteq A_1 \cap A_2.$$

Hence for every $x \in A_1 \cap A_2$ there exists $B_x \in \mathcal{B}$ with

$$x \in B_x \subseteq A_1 \cap A_2.$$

Therefore,

$$A_1 \cap A_2 = \bigcup_{x \in A_1 \cap A_2} B_x,$$

which shows that $A_1 \cap A_2 \in \mathcal{B}_{\mathcal{B}}$.

Thus $\mathcal{B}_{\mathcal{B}}$ is a topology on X , and $\mathcal{B}$ is a basis for this topology.

In other words, a basis for a topology is a collection of open sets that is

1. large enough to cover the entire space;
2. sufficiently rich so that every open set can be expressed as a union of its elements.

Hence, if $\mathcal{B}$ is a basis for a topological space (X, τ) , then every open set in τ is an arbitrary union of elements of $\mathcal{B}$.

Example 2.3 Let $X = \mathbb{R}$ and define

$$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}.$$

Then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

Clearly,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n).$$

Let $(a, b), (c, d) \in \mathcal{B}$ and suppose that $x \in (a, b) \cap (c, d)$. Then

$$(a, b) \cap (c, d) = (\max \{a, c\}, \min \{b, d\}),$$

which is again an element of $\mathcal{B}$ containing x . Hence, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

This topology is called the Euclidean topology. In the sequel, we shall see that it is generated by a metric.

Example 2.4 Let $X = \mathbb{R}$ and define

$$\mathcal{B}_u = \{(a, b] : a, b \in \mathbb{R}, a < b\}.$$

Then $\mathcal{B}_u$ forms a basis for another topology on $\mathbb{R}$.

Clearly,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n].$$

Let $(a, b], (c, d] \in \mathcal{B}_u$ and suppose that $x \in (a, b] \cap (c, d]$. Then

$$(a, b] \cap (c, d] = (\max \{a, c\}, \min \{b, d\}],$$

which belongs to $\mathcal{B}_u$ and contains x . Hence, $\mathcal{B}_u$ is a basis for a topology on $\mathbb{R}$.

This topology is called the upper limit topology (or the Sorgenfrey topology) on $\mathbb{R}$. Moreover, $\mathbb{R}$ equipped with this topology is denoted by $\mathbb{R}_u$. In the sequel, we shall see that this topology cannot be generated by any metric.

Example 2.5 Let $X = \mathbb{R}$ and define

$$\mathcal{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}.$$

Then $\mathcal{B}_\ell$ forms another topology on $\mathbb{R}$.

Clearly,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n).$$

Let $[a, b), [c, d) \in \mathcal{B}_\ell$ and suppose that $x \in [a, b) \cap [c, d)$. Then

$$[a, b) \cap [c, d) = [\max \{a, c\}, \min \{b, d\}),$$

which belongs to $\mathcal{B}_\ell$ and contains x . Hence, $\mathcal{B}_\ell$ is a basis for a topology on $\mathbb{R}$.

This topology is called the lower limit topology on $\mathbb{R}$. Furthermore, $\mathbb{R}$ with this topology is denoted by $\mathbb{R}_\ell$. In the sequel, we shall see that this topology cannot be generated by any metric.

As we have seen, there exist several distinct topologies on $\mathbb{R}$. Some of them arise from a metric (or a norm), while others possess only a topological structure and cannot be induced by any metric. So, a comparison of different types of topologies on a set, focusing on their capabilities and strengths, is beneficial.

Definition 2.3 Assume that $X \neq \emptyset$ is a set. Moreover, assume that τ_1 and τ_2 are topologies on X . We say that τ_2 is finer than τ_1 provided that $\tau_1 \subseteq \tau_2$; in this case, we say τ_1 is coarser than τ_2 . Indeed, $\tau_1 = \tau_2$ provided that each one is finer than the other. Furthermore, if neither $\tau_1 \subseteq \tau_2$ nor $\tau_2 \subseteq \tau_1$, we say that these topologies are not comparable.

Since in a topological space we usually work with bases rather than the entire topology, the following result is useful as it characterizes the comparison of topologies in terms of their bases.

Theorem 2.1 $\Leftarrow$ Let (X, τ_1) and (X, τ_2) be topological spaces with bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then τ_2 is finer than τ_1 if and only if for every $B_1 \in \mathcal{B}_1$ and every $x \in B_1$, there exists $B_2 \in \mathcal{B}_2$ such that

$$x \in B_2 \subseteq B_1.$$

Proof 2.4 Assume first that $\tau_1 \subseteq \tau_2$. Let $B_1 \in \mathcal{B}_1$ and $x \in B_1$. Since B_1 is open in (X, τ_1) and $\tau_1 \subseteq \tau_2$, it follows that B_1 is open in (X, τ_2) . Because $\mathcal{B}_2$ is a basis for τ_2 , by [Remark 2.1](#) there exists $B_2 \in \mathcal{B}_2$ such that

$$x \in B_2 \subseteq B_1.$$

Conversely, assume that for every $B_1 \in \mathcal{B}_1$ and every $x \in B_1$, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. Let $U \in \tau_1$ and let $x \in U$. Since $\mathcal{B}_1$ is a basis for τ_1 , there exists $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq U$. By the assumption, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1 \subseteq U$. Hence, U is open in (X, τ_2) . Therefore, $\tau_1 \subseteq \tau_2$, and τ_2 is finer than τ_1 .

Example 2.6 $\Leftarrow$ Consider $X = \mathbb{R}^2$. Assume that $\mathbf{B}_1$ is the collection of all open circles in $\mathbb{R}^2$, $\mathbf{B}_2$ is the collection of all open rhombuses in $\mathbb{R}^2$, and $\mathbf{B}_3$ is the collection of all open squares in $\mathbb{R}^2$. It is easily seen that $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ form topological bases. Let τ_1 , τ_2 , and τ_3 denote the corresponding topologies, respectively. Then τ_1 , τ_2 , and τ_3 coincide. In other words, for each $i, j \in \{1, 2, 3\}$, the topology τ_i is finer than τ_j .

Consider [Theorem 2.1](#) and [Figures 2.1](#), [2.2](#), [2.3](#), and [2.4](#).

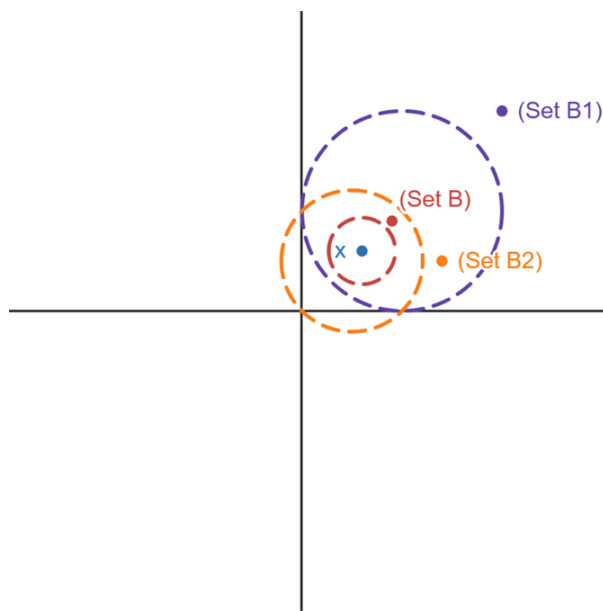


Figure 2.1 Explanation for [Example 2.6](#): Circles form a basis for a topology on $\mathbb{R}^2$. ↩

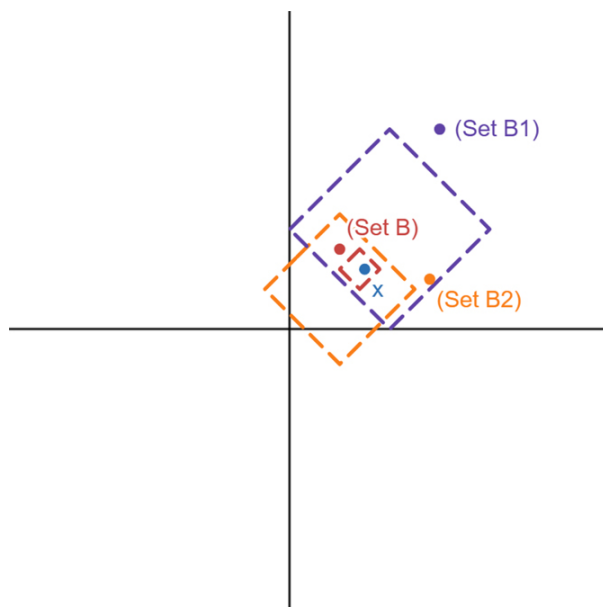


Figure 2.2 Explanation for [Example 2.6](#): Rhombuses form a basis for a topology on $\mathbb{R}^2$. ↩

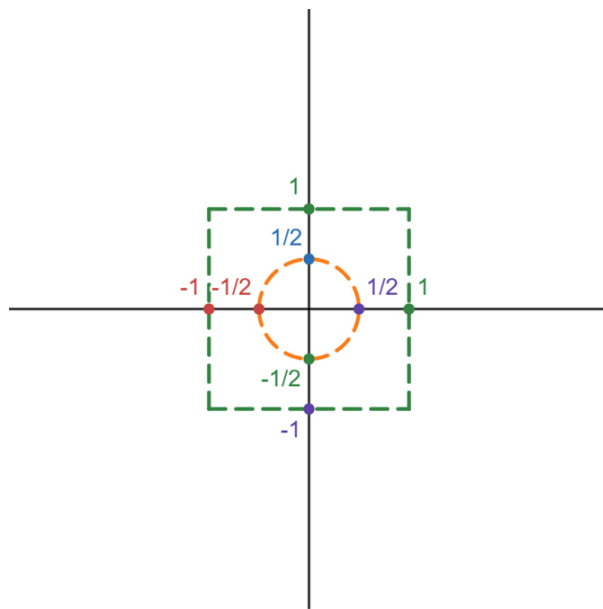


Figure 2.3 Explanation for [Example 2.6](#): τ_1 is finer than τ_3 . ◀

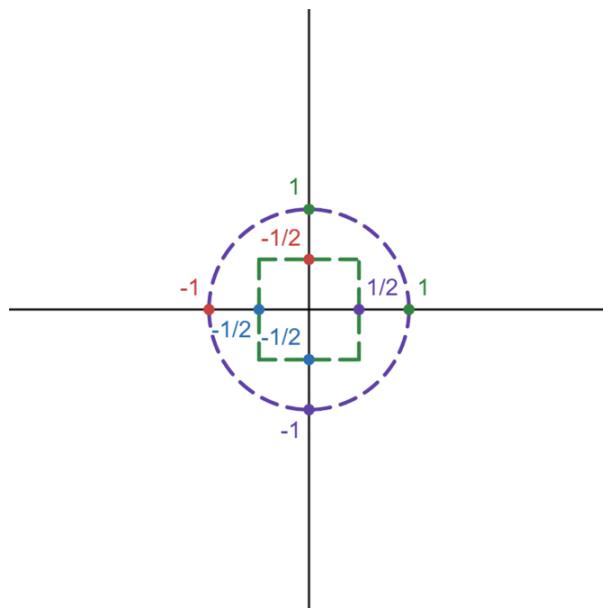


Figure 2.4 Explanation for [Example 2.6](#): τ_3 is finer than τ_1 . ◀

Proposition 2.4 *The lower limit topology $\mathbb{R}_\ell$ and the upper limit topology $\mathbb{R}_u$ on $\mathbb{R}$ are not compatible.*

Proof 2.5 *Recall that the lower limit topology $\mathbb{R}_\ell$ has as a base all intervals of the form $[a, b)$ with $a < b$, while the upper limit topology $\mathbb{R}_u$ has as a base all intervals of the form $(a, b]$ with $a < b$.*

We show that neither topology is finer than the other.

Step 1. $\mathbb{R}_\ell \not\subseteq \mathbb{R}_u$.

Consider the set $[0, 1)$. This set is open in $\mathbb{R}_\ell$. Suppose that $[0, 1)$ were open in $\mathbb{R}_u$. Then, since $0 \in [0, 1)$, there exists a basic open set $(a, b]$ of $\mathbb{R}_u$ such that

$$0 \in (a, b] \subseteq [0, 1).$$

Since $0 \in (a, b]$, we must have $a < 0$, and hence $(a, b]$ contains points less than 0, contradicting $(a, b] \subseteq [0, 1)$. Therefore $[0, 1)$ is not open in $\mathbb{R}_u$, and thus $\mathbb{R}_\ell \not\subseteq \mathbb{R}_u$.

Step 2. $\mathbb{R}_u \not\subseteq \mathbb{R}_\ell$.

Consider the set $(0, 1]$. This set is open in $\mathbb{R}_u$. Suppose that $(0, 1]$ were open in $\mathbb{R}_\ell$. Then, since $1 \in (0, 1]$, there exists a basic open set $[a, b)$ of $\mathbb{R}_\ell$ such that

$$1 \in [a, b) \subseteq (0, 1].$$

Since $1 \in [a, b)$, we must have $b > 1$, and hence $[a, b)$ contains points greater than 1, contradicting $[a, b) \subseteq (0, 1]$. Therefore $(0, 1]$ is not open in $\mathbb{R}_\ell$, and thus $\mathbb{R}_u \not\subseteq \mathbb{R}_\ell$.

Hence neither topology is finer than the other, and consequently $\mathbb{R}_\ell$ and $\mathbb{R}_u$ are not compatible.

Proposition 2.5 The topological spaces $\mathbb{R}_\rightarrow$ and $\mathbb{R}_\leftarrow$ are not compatible.

Proof 2.6 We show that neither topology is finer than the other.

Step 1. $\mathbb{R}_\rightarrow \not\subseteq \mathbb{R}_\leftarrow$.

The set $(0, \infty)$ is open in $\mathbb{R}_\rightarrow$ by definition. Suppose that $(0, \infty)$ were open in $\mathbb{R}_\leftarrow$. Then for any $x \in (0, \infty)$ there exists $b \in \mathbb{R}$ such that

$$x \in (-\infty, b) \subseteq (0, \infty).$$

However, every set of the form $(-\infty, b)$ contains negative real numbers and therefore cannot be contained in $(0, \infty)$. This contradiction shows that $(0, \infty)$ is not open in $\mathbb{R}_\leftarrow$. Hence $\mathbb{R}_\rightarrow \not\subseteq \mathbb{R}_\leftarrow$.

Step 2. $\mathbb{R}_{\leftarrow} \not\subseteq \mathbb{R}_{\rightarrow}$.

The set $(-\infty, 0)$ is open in $\mathbb{R}_{\leftarrow}$ by definition. Suppose that $(-\infty, 0)$ were open in $\mathbb{R}_{\rightarrow}$. Then for any $x \in (-\infty, 0)$ there exists $a \in \mathbb{R}$ such that

$$x \in (a, \infty) \subseteq (-\infty, 0).$$

But every set of the form (a, ∞) contains positive real numbers and thus cannot be contained in $(-\infty, 0)$. This contradiction shows that $(-\infty, 0)$ is not open in $\mathbb{R}_{\rightarrow}$. Hence $\mathbb{R}_{\leftarrow} \not\subseteq \mathbb{R}_{\rightarrow}$.

Since neither topology is finer than the other, the spaces $\mathbb{R}_{\rightarrow}$ and $\mathbb{R}_{\leftarrow}$ are not compatible.

Consider [Figure 2.5](#), which provides a schematic illustration of several topologies on $\mathbb{R}$ and compares them with respect to fineness.

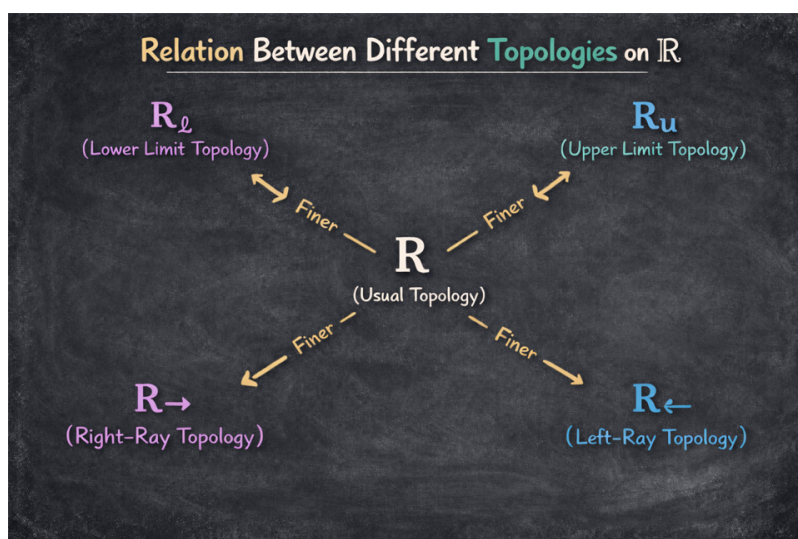


Figure 2.5 Comparison of different topologies on $\mathbb{R}$. ↩

We end this section with a very useful separation property in topological spaces. It is very important in the context of this book, as well; in mathematical language as well as in our story part.

Definition 2.4 Assume that (X, τ) is a topological space. We say that the topology τ is Hausdorff if for every two distinct elements $x, y \in X$, there exist open sets $U \subseteq X$ containing x and $V \subseteq X$ containing y such that $U \cap V = \emptyset$.

In other words, in a Hausdorff topological space, any two distinct points can be separated by disjoint open sets.

Example 2.7 $\mathbb{R}$ with the usual topology is Hausdorff. Take $x \neq y$ in $\mathbb{R}$ and set

$$r = \frac{|x - y|}{3}.$$

Let $U = (x - r, x + r)$ and $V = (y - r, y + r)$. Both sets are open in $\mathbb{R}$. We claim that $U \cap V = \emptyset$.

Suppose, to the contrary, that $z \in U \cap V$. Then $|z - x| < r$ and $|z - y| < r$. Hence

$$|x - y| \leq |x - z| + |z - y| < r + r = 2r = \frac{2}{3}|x - y|,$$

which is a contradiction. Therefore $U \cap V = \emptyset$.

Example 2.8 Consider $\mathbb{R}$ with the right-ray topology. For distinct elements $x \neq y$ (assume without loss of generality that $x < y$), we cannot find disjoint open sets U containing x and V containing y . Indeed, every open set containing x contains a ray of the form (a, ∞) with $a < x$, and similarly every open set containing y contains a ray extending to infinity. Consequently, any two such open sets must intersect. Therefore, the right-ray topology on $\mathbb{R}$ is not Hausdorff.

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The Reals: Almost Everything and Everywhere

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The Real Numbers: Fundamental Ideas and Power

The book's general theme introduces various concepts of analysis through our narrative, illustrating their significance before transitioning into mathematical language to further develop the course. Now, we turn our attention to the world beneath the mathematical ocean, a realm filled with vital objects, cities, and tools. While many perceive “math” as islands scattered across the ocean, the truly intriguing aspects lie below the surface. Our exploration begins in the city of “the reals,” a powerful hub known for its exports of order, convergence, and innovative ideas. To paint a clearer picture of this city, we can describe some of its notable towns. For instance, the town of “natural numbers” is renowned for its industriousness, well-ordering, and fairness. Meanwhile, “integers” town is recognized for its inclusive perspective on both positive and negative aspects of life.

Connecting these cities are submarines that facilitate communication and transport between the towns, islands, and even distant lands. Most islands and territories rely on imports from the city of the real. In addition to a central port for the city, each town boasts multiple ports to streamline the import and export process. The primary export of the city of the real is ideas, which are essential for many complex products found on the islands.

For example, a notable product from the “functional analysis” section of the “analysis” island is vector lattices, which originated from the concepts of the real. Another powerful town within the analysis island is known for its normed space structures, which serve as the foundation for most of the island's constructions. This concept is rooted in the “modulus” from the real, which

plays a critical role in managing the city's operations, including programming and evaluation.

Beneath the surface of the city of the reals lie treasures that contribute to the wealth and allure of both the city and the islands. A special submarine is tasked with transporting these treasures to the islands. In the upcoming chapters, we will explore how these treasures can enrich and empower the analysis island.

3.1 IMPORTANT PROPERTIES OF THE REALS IN MATH

We assume that the readers of this book are familiar with the elementary axioms related to the real numbers. We recall some of them here, once again, using a story-like language.

Definition 3.1 *Suppose that X is a non-empty set. A binary relation $\preceq$ on X is called a partial order if the following conditions hold for all $x, y, z \in X$:*

- $x \preceq x$ (reflexivity);
- if $x \preceq y$ and $y \preceq x$, then $x = y$ (anti-symmetry);
- if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitivity).

In this case, the pair $(X, \preceq)$ is called a partially ordered set. For example, the set of real numbers $\mathbb{R}$ equipped with the usual order is a partially ordered set.

Before continuing our story, we present another example to illustrate the situation more clearly.

Example 3.1 *Suppose that X is a non-empty set and let $P(X)$ denote the power set of X . Consider the binary relation on $P(X)$ defined by set inclusion. That is, for any $A, B \in P(X)$, define*

$$A \preceq B \quad \text{if and only if} \quad A \subseteq B.$$

It is easy to verify that $(P(X), \preceq)$ is a partially ordered set.

Partially Ordered Sets: The First Law of a Better Group Life

Partially ordered sets are another area within the vast realm of mathematics, with the real numbers serving as its capital. In essence, the entire mathematical world can be viewed as a set, which we refer to as the universal set. This land is immense, and although we have identified many sets in mathematics, a significant portion of this territory remains unexplored. Given that mathematics and its concepts are infinite, the world beneath this mathematical ocean is also infinite. As we uncover new areas, we can extract valuable insights and

resources from this landscape, which we might call the land of sets. Every set that has a partial order can gain access to this province; without it, entry is impossible. Fortunately, gaining entry is not particularly challenging. For example, when considering the power set, the inclusion relation serves as a key to this province. However, it is important to note that these keys are not physical objects but rather criteria that sets must satisfy to comply with the rules of the province.

The realm of real numbers plays a distinctive role within this territory, and access to it is restricted even for many general inhabitants of the province. The real numbers possess the property of total order, meaning that every two elements can be compared to one another. For instance, in general, power sets with inclusion cannot enter the realm of real numbers. One of the advantages of this total order is the ability to evaluate the performance of each element not only in isolation but also in others. The mayor of the realm of real numbers is known for his fairness and consistently strives to ensure that each element reaches its potential. This commitment makes the realm of real numbers unique, distinguishing it as a highly developed area with a prominent position among all the regions beneath the mathematical ocean. Now, let us return to the language of mathematics.

Definition 3.2 *A partially ordered set $(X, \preceq)$ is called a totally ordered set if every pair of elements is comparable; that is, for all $x, y \in X$, either $x \preceq y$ or $y \preceq x$ holds.*

Example 3.2 *The set of real numbers $\mathbb{R}$ equipped with the usual order is a totally ordered set.*

In contrast, the power set $P(X)$ of a non-empty set X , ordered by inclusion, is not totally ordered, in general.

Indeed, let $X = \{1, 2\}$. Then $\{1\}, \{2\} \in P(X)$, but neither $\{1\} \subseteq \{2\}$ nor $\{2\} \subseteq \{1\}$ holds. Hence, these two elements are not comparable.

As we said before, the *natural numbers town* has a unique character in the *reals city*.

Principal 3.1 (The well-ordering principle) *Every non-empty subset of the natural numbers $\mathbb{N}$ has a first (minimum) element.*

It is easy to verify that, in general, the real numbers do not possess this property. For instance, the open interval $(0, 1)$ does not contain a minimum element within the real numbers. This characteristic of the natural numbers offers many advantages; as previously mentioned, although export ports in the realm of real numbers are always busy, the natural numbers and integers have specialized export ports that are undoubtedly even more crowded. This is because all

equipment and programming for communication between various parts of the mathematical landscape and the world beneath it, as well as other regions, are designed with them in mind.

Order: The Power Behind Structure

One of the most notable aspects of the realm of real numbers is its sense of order. The mayor of the real number city can manage everything in the best possible way. We will continue our discussion with a few examples. The inhabitants of the mathematical society—comprising those from the mathematical islands, the world beneath the ocean, and the mathematics ambassadors in other lands—can transfer their products along with engaging in export and import with other regions. They utilize domestic boats, ships, or submarines for this purpose. The primary difference between domestic and international transportation equipment lies in the “method card,” which serves as a ticket, visa, key for entry, and financial document for the inhabitants of the mathematical society. More specifically, each member of this society possesses two types of abilities. One is natural and well-known; for instance, the inhabitants of the real number city possess a strong order that enables them to maintain a unique city with a high demand for exports. On the other hand, they can also perform various topologies as part of their capabilities. These abilities are electronically issued on their “method card,” which allows them to travel through different cities, islands, and regions. For example, the inhabitants of the real numbers city with a discrete topology can navigate through the eastern boundaries of the topological territories, including topology islands and related lands. Additionally, they can transfer via stations related to topological aspects. However, with a non-discrete topology, they can only traverse the western frontiers of topological lands or islands. If they equip themselves with Euclidean topology, they gain the ability to navigate through metric, normed, and topological spaces using various transportation methods, thereby facilitating better export and import. Sometimes, combining different abilities—achieved through innovative applications of new ideas—results in a more efficient “method card.” For instance, merging order and topology produces an “order topology” method that allows free access to places with both ordered and topological characteristics. Another example will be more specialized topics. A combination of linear algebra and topology results in “topological vector spaces,” which are fundamental extensions of normed spaces—the most important aspect of analysis. In this context, mathematical objects can acquire various attributes (either naturally or through development), combine them to create new products, and so forth. As previously mentioned, each new step requires innovative ideas to advance. One significant source of these ideas can be found in the realm of real numbers, where order and topological aspects

intersect. For example, concepts related to vector lattices and normed lattices in analysis draw valuable insights from this rich source of ideas.

3.2 BOUNDS: THE GATEWAY TO ANALYSIS

It is important to note that the open interval $(0, 1)$ lacks both a minimum and a maximum element in $\mathbb{R}$. However, a notable feature of the real numbers is their inherent fairness: while the absence of least and greatest elements in $(0, 1)$ may seem like a disadvantage, there is a distinction between $(0, 1)$ and $(1, \infty)$. We can “control” the elements of $(0, 1)$, but the situation is different for $(1, \infty)$.

This brings us to the concept of bounds. In our discussion, although real numbers do not have explicit permission to enter the domain of natural numbers or integers, under certain boundedness conditions, they can still access the remarkable properties of both natural and integer sets. It is worth noting that the notion of bounds can be defined in any partially ordered set, but since our focus in elementary analysis is on the real, we will define it from that perspective initially.

Definition 3.3 *Let A be a subset of the real numbers $\mathbb{R}$.*

- *An element $x \in \mathbb{R}$ is called an upper bound for A if $a \leq x$ for every $a \in A$.*
- *An element $y \in \mathbb{R}$ is called a lower bound for A if $y \leq a$ for every $a \in A$.*

If A has an upper bound, we say that A is bounded above. If A has a lower bound, we say that A is bounded below.

For example, for the interval $(0, 1)$, the set $[1, \infty)$ consists of all upper bounds, while the set $(-\infty, 0]$ consists of all lower bounds. However, merely having upper or lower bounds is not sufficient for many theoretical and practical purposes. Among all possible bounds, those with extremal (universal) properties play a fundamental role. These special elements—when they exist—are unique and deserve particular attention. This motivates the following definitions.

Definition 3.4 (Supremum) *Let $A \subseteq \mathbb{R}$. An element $x \in \mathbb{R}$ is called the least upper bound (or supremum) of A if:*

- *x is an upper bound for A ;*
- *for every upper bound z of A , we have $x \leq z$.*

Definition 3.5 (Infimum) *Let $A \subset \mathbb{R}$. An element $y \in \mathbb{R}$ is called the greatest lower bound (or infimum) of A if:*

- y is a lower bound for A ;
- for every lower bound z of A , we have $z \leq y$.

By definition, the supremum and infimum of a set—if they exist—are unique. We denote them by $\sup A$ and $\inf A$, respectively.

Note that having a bound does not guarantee that individuals will benefit from exports as effectively as the natural ones do. In other words, the supremum and infimum are the essential permissions needed for conducting viable business.

The good news is that, with the efforts of the mayor of the reals city, if the inhabitants obey certain rules (namely boundedness), they are guaranteed to obtain the necessary permits (the supremum and the infimum). This principle is known as the axiom of completeness of the real numbers.

Axiom 3.1 (Axiom of Completeness of $\mathbb{R}$) $\hookrightarrow$ Let $A \subseteq \mathbb{R}$ be non-empty.

1. If A is bounded above, then A has a least upper bound in $\mathbb{R}$; that is, $\sup A \in \mathbb{R}$ exists.
2. If A is bounded below, then A has a greatest lower bound in $\mathbb{R}$; that is, $\inf A \in \mathbb{R}$ exists.

Notably, this property fosters a connection between rational and irrational numbers; neither rational nor irrational numbers alone can issue the permissions needed for export. Consider the following example.

Example 3.3 Let

$$A = \{p \in \mathbb{Q}_+ : p^2 < 2\}.$$

Then A is bounded above in $\mathbb{Q}$, but it has no supremum in $\mathbb{Q}$.

Proof 3.1 The set A is non-empty since $1 \in A$ and $1^2 < 2$. Moreover, A is bounded above in $\mathbb{Q}$; for instance, 2 is an upper bound.

We first show that for every $p \in A$, there exists $q \in A$ such that $q > p$. Let $p \in A$, so $p^2 < 2$. Define

$$q := p - \frac{p^2 - 2}{p + 2} = \frac{2(p + 1)}{p + 2}$$

Since $p \in \mathbb{Q}$, we have $q \in \mathbb{Q}$ and clearly $q > p$.

A direct computation shows

$$q^2 - 2 = \frac{4(p+1)^2}{(p+2)^2} - 2 = \frac{4(p+1)^2 - 2(p+2)^2}{(p+2)^2} = \frac{2(p^2 - 2)}{(p+2)^2} < 0,$$

hence $q \in A$.

Now suppose, for contradiction, that $\alpha = \sup A$ for some $\alpha \in \mathbb{Q}$. Since α is an upper bound of A , we must have $\alpha^2 \geq 2$; otherwise, if $\alpha^2 < 2$, then $\alpha \in A$. Hence, by the previous part, there exists $\beta \in A$ such that $\beta > \alpha$, which contradicts the fact that α is the supremum of A . However, $\alpha^2 = 2$ is impossible because $\sqrt{2} \notin \mathbb{Q}$. Thus $\alpha^2 > 2$.

Define

$$r := \alpha - \frac{\alpha^2 - 2}{\alpha + 2}.$$

Then $r \in \mathbb{Q}$ and $r < \alpha$. A direct calculation shows that $r^2 > 2$, hence r is still an upper bound of A .

This contradicts the minimality of α as the least upper bound. Therefore, A has no supremum in $\mathbb{Q}$.

One of the distinguishing features of real numbers is their ability to address any potential gaps. This characteristic is what makes the realm of real numbers the most remarkable and applicable within the vast ocean of mathematics. We observe that the domain of natural numbers serves as the heart of the realm of real numbers.

Theorem 3.1 (The Archimedean property of the reals) $\hookrightarrow$ Let x and y be real numbers with $x > 0$. Then there exists a natural number n such that $nx > y$.

Proof 3.2 Suppose, to the contrary, that $nx \leq y$ for every $n \in \mathbb{N}$. Define

$$A = \{nx : n \in \mathbb{N}\}.$$

By assumption, A is bounded above. Hence, by [Axiom 3.1](#), A has a supremum, say α .

For each $n \in \mathbb{N}$, we have

$$nx = (n+1)x - x \leq \alpha - x.$$

Thus, $\alpha - x$ is also an upper bound for A . Since $x > 0$, we have $\alpha - x < \alpha$, which contradicts the minimality of α as the least upper bound. This contradiction completes the proof.

Corollary 3.1 $\Leftarrow$ In [Theorem 3.1](#), let $x = 1$. Then for every real number y , there exists $n \in \mathbb{N}$ such that $n > y$.

Corollary 3.2 $\Leftarrow$ In [Theorem 3.1](#), let $y = 1$ and let $x > 0$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $nx > 1$, which implies that $\frac{1}{n} < x$.

These two corollaries highlight fundamental interactions between the natural numbers and the real numbers. Although $\mathbb{N}$ is a discrete and comparatively small subset of $\mathbb{R}$, it is unbounded above ([Corollary 3.1](#)) and accumulates arbitrarily close to zero through reciprocals ([Corollary 3.2](#)). This reflects the central structural role played by the natural numbers within the real numbers system.

The City of Real Numbers: A Place for Progress

In the city of reals, the inhabitants hold strong beliefs in merit and accuracy. They value respect for every member of the mathematical community and eagerly share innovative ideas for the advancement of society as a whole. The well-ordering principle and the completeness of the reals exemplify this commitment. Justice plays a crucial role in the city; when a set lacks a maximum or minimum, inhabitants establish bounds to maintain order. Their goal is to ensure that justice, order, and abilities are distributed fairly. Among various bounds, the least upper bound (supremum) and the greatest lower bound (infimum) hold special significance, as they are closely associated with every set—almost functioning as its members—while also providing clear limits for the set's activities through a precise framework. For instance, consider the set $A = (0, 1)$. We can demonstrate that $\sup A = 1$ and $\inf A = 0$. This allows members to operate freely within the confines of this interval, with zero and one serving as protective boundaries. Additionally, set A includes the Cantor set, which is uncountable with the same cardinality as the reals, yet contains no intervals. (We will explore the Cantor set in detail in [Chapter 4](#), referring to it as the “heaven of set theory.”) The set A can also be equivalent to any interval in various respects—topologically and in terms of order—among other properties. Thus, bounds play a vital role in categorizing the different types of members in the city of reals into distinct groups with desired characteristics. Another important aspect of the real numbers is their density. Specifically, both rational and irrational numbers are dense within the real. While we will define this mathematical concept in the next chapter, for now, density implies that there are numbers of both types interspersed throughout the real line. In the context of our narrative, next to every house that represents a rational number, there exists a house belonging to an irrational number and vice versa. When you aim to travel to a location represented by rational numbers, numerous taxis, metros, buses, and other modes of transportation provided by irrational numbers can easily take

you there, and the reverse holds as well. This interconnectedness facilitates the growth and development of real numbers efficiently and rapidly.

Theorem 3.2 $\hookleftarrow$ Assume that x and y are two distinct real numbers. Then there exists a rational number between them.

Proof 3.3 Assume that $x, y \in \mathbb{R}$ with $y > x$.

Case 1: $x > 0$. By [Corollary 3.1](#), there exists $n \in \mathbb{N}$ such that $n > \frac{1}{y-x}$, i.e., $nx + 1 < ny$. Let

$$A = \{k : k \in \mathbb{N}, k > nx\}.$$

By [Corollary 3.1](#), A is non-empty. By [Theorem 3.1](#), it contains a first element, say p . Then

$$p > nx \geq p - 1 \quad \text{and} \quad ny > nx + 1 \geq p.$$

Thus, $r = \frac{p}{n}$ is the desired rational number between x and y .

Case 2: $x = 0$. Then $0 < \frac{1}{2}y < y$, and by Case 1, there exists a rational r such that

$$0 < \frac{1}{2}y < r < y.$$

Case 3: $x < 0$.

- If $y > 0$, then 0 is a rational number between x and y .
- If $y \leq 0$, then $x < y \leq 0$. Apply the previous arguments to $-x$ and $-y$ to find a rational number $r > 0$ with $-y < r < -x$, so that $x < -r < y$.

Corollary 3.3 Assume that x and y are two distinct real numbers with $x < y$. Then there exists an irrational number t such that $x < t < y$.

Proof 3.4 By [Theorem 3.2](#), there exists a rational number r such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Then $t = r\sqrt{2}$ is irrational and satisfies $x < t < y$.

3.3 EXAMPLES

In this section, we present some examples illustrating the supremum and infimum of subsets of the real numbers.

Example 3.4 Suppose A and B are subsets of $\mathbb{R}$ that are bounded above. Then

$$A + B = \{a + b : a \in A, b \in B\}$$

is also bounded above, and we have

$$\sup (A + B) = \sup (A) + \sup (B).$$

Proof. Let $x = \sup (A)$ and $y = \sup (B)$. For any $a \in A$ and $b \in B$, we have

$$a + b \leq x + y,$$

showing that $x + y$ is an upper bound of $A + B$. Therefore, $\sup (A + B) \leq x + y$.

For the reverse inequality, fix $b \in B$. Then, for all $a \in A$, we have

$$a \leq \sup (A + B) - b,$$

which implies $\sup (A) \leq \sup (A + B) - b$. Since b was arbitrary, it follows that

$$\sup (B) \leq \sup (A + B) - \sup (A),$$

and thus $\sup (A + B) \geq \sup (A) + \sup (B)$. This completes the proof.

Example 3.5  Suppose A and B are non-negative subsets of $\mathbb{R}$ that are bounded above. Then

$$AB = \{ab : a \in A, b \in B\}$$

is also bounded above, and

$$\sup (AB) = \sup (A) \cdot \sup (B).$$

Proof. Let $\alpha = \sup (A)$, $\beta = \sup (B)$, and $\gamma = \sup (AB)$. For any $a \in A$ and $b \in B$, we have

$$ab \leq \alpha\beta,$$

so $\alpha\beta$ is an upper bound for AB , giving $\gamma \leq \alpha\beta$.

For the reverse inequality, fix $b \in B$. Then $a \leq \frac{\gamma}{b}$ for all $a \in A$, which implies $\alpha \leq \frac{\gamma}{b}$. Since b is arbitrary, it follows that $\beta \leq \frac{\gamma}{\alpha}$, hence $\alpha\beta \leq \gamma$. This proves the claim.

Corollary 3.4 Let $A \subseteq \mathbb{R}_+$ be bounded above. Then for each $n \in \mathbb{N}$,

$$A^n = \{a^n : a \in A\}$$

is also bounded above, and

$$\sup (A^n) = (\sup A)^n.$$

Example 3.6 Let

$$S = \left\{ \frac{n}{2^n} : n \in \mathbb{N} \right\}.$$

Determine $\inf (S)$ and $\sup (S)$.

Solution.

Put

$$a_n = \frac{n}{2^n}, \quad n \in \mathbb{N}.$$

First observe that for every $n \geq 1$,

$$2n \geq n + 1 \quad \Longleftrightarrow \quad n \geq \frac{n + 1}{2}.$$

Dividing both sides by $2^n > 0$, we obtain

$$\frac{n}{2^n} \geq \frac{n + 1}{2^{n+1}},$$

that is,

$$a_n \geq a_{n+1}.$$

Thus (a_n) is decreasing (more precisely, non-increasing).

Now compute

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{2}.$$

Since the sequence is non-increasing, for every $n \in \mathbb{N}$,

$$a_n \leq a_1 = \frac{1}{2}.$$

Hence $\frac{1}{2}$ is an upper bound of S . Because $\frac{1}{2} \in S$, it follows that

$$\sup (S) = \max (S) = \frac{1}{2}.$$

Next, we claim that $\inf (S) = 0$. Certainly, 0 is a lower bound for S . Let $\varepsilon > 0$ be arbitrary. Since

$$2^n = 1 + n + \frac{n(n-1)}{2} + \dots \geq \frac{n(n-1)}{2},$$

it follows that, for each $n \geq 2$,

$$\frac{n}{2^n} \leq \frac{2}{n-1}.$$

By the Archimedean property ([Theorem 3.1](#)), there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Take $n = N + 1$. Then

$$\frac{n}{2^n} \leq \frac{2}{N} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, no positive number can serve as a lower bound smaller than all elements of S . Therefore,

$$\inf (S) = 0.$$

The notions of supremum and infimum are dual. More precisely, we have the following results.

Proposition 3.1 Suppose $A \subseteq \mathbb{R}$ is bounded above. Then

$$-A = \{-a : a \in A\}$$

is bounded below, and

$$\inf (-A) = -\sup (A).$$

Proof 3.5 Let $x = \sup (A)$. For each $a \in A$, we have $a \leq x$, hence $-a \geq -x$, showing that $-A$ is bounded below. Therefore, $-x = -\sup (A)$ is a lower bound for $-A$, so $\inf (-A) \geq -\sup (A)$.

Conversely, for each $a \in A$, $-a \geq \inf (-A)$, which implies $a \leq -\inf (-A)$. Hence, $\sup (A) \leq -\inf (-A)$. Combining both inequalities, we conclude

$$\inf (-A) = -\sup (A),$$

as desired.

Similarly, we obtain the following useful fact.

Proposition 3.2 Suppose $A \subseteq \mathbb{R}$ is bounded below. Then

$$-A = \{-a : a \in A\}$$

is bounded above, and

$$\sup (-A) = -\inf (A).$$

Proposition 3.3  Assume $A, B \subseteq \mathbb{R}$ are non-empty with $A \subseteq B$ and B bounded above. Then

$$\sup (A) \leq \sup (B).$$

Proof 3.6 Since $A \subseteq B$ and B is bounded above, A is also bounded above. For each $a \in A$, we have $a \leq \sup (B)$, showing that $\sup (B)$ is an upper bound for A . Hence,

$$\sup (A) \leq \sup (B),$$

as claimed.

Corollary 3.5 Assume $A, B \subseteq \mathbb{R}$ are non-empty with $A \subseteq B$ and B bounded below. Then

$$\inf (B) \leq \inf (A).$$

Proof 3.7 Note that $-B$ is bounded above and $-A \subseteq -B$. By [Proposition 3.3](#), we have

$$\sup (-A) \leq \sup (-B).$$

Using the duality between supremum and infimum, this gives

$$-\inf (A) \leq -\inf (B) \implies \inf (B) \leq \inf (A),$$

as desired.

Corollary 3.6 Assume $A \subseteq \mathbb{R}$ is bounded above and $B \subseteq \mathbb{R}$ is bounded below. Then

$$A - B = \{a - b : a \in A, b \in B\}$$

is bounded above, and

$$\sup (A - B) = \sup (A) - \inf (B).$$

Proof 3.8 That $A - B$ is bounded above is straightforward. Moreover, $-B$ is bounded above, so by previous results,

$$\sup (A - B) = \sup (A) + \sup (-B) = \sup (A) - \inf (B),$$

as claimed.

Now, we consider the quotient of two sets. Assume that $A, B \subseteq \mathbb{R}_+$. Define

$$\frac{A}{B} = \left\{ \frac{a}{b} : a \in A, b \in B \right\}.$$

Then we have the following observations.

Proposition 3.4 Assume that $A, B \subseteq \mathbb{R}_+$ are bounded and that $\inf (B) > 0$. Then:

$$1. \sup \left(\frac{1}{A} \right) = \frac{1}{\inf(A)};$$

$$2. \sup \left(\frac{A}{B} \right) = \frac{\sup A}{\inf B}.$$

Proof 3.9 For the former statement, let $\gamma = \sup \left(\frac{1}{A} \right)$ and $\beta = \inf (A)$. We show that $\gamma = \frac{1}{\beta}$. Note that both γ and β exist by the axiom of completeness, since A is simultaneously bounded above and bounded below. For each $a \in A$, we have

$$a \geq \beta \implies \frac{1}{a} \leq \frac{1}{\beta} \implies \gamma \leq \frac{1}{\beta}.$$

On the other hand, we have

$$\frac{1}{a} \leq \gamma \Rightarrow a \geq \frac{1}{\gamma} \Rightarrow \beta \geq \frac{1}{\gamma} \Rightarrow \frac{1}{\beta} \leq \gamma.$$

Therefore, $\gamma = \frac{1}{\beta}$.

For the latter statement, we use [Example 3.5](#). We obtain

$$\sup \left(\frac{A}{B} \right) = \sup (A) \sup \left(\frac{1}{B} \right) = \frac{\sup (A)}{\inf (B)}.$$

Now, we investigate some results concerning the supremum and the infimum of the modulus of a set. For a set $A \subseteq \mathbb{R}$, we define

$$|A| = \{|a| : a \in A\}.$$

Lemma 3.1 *Let $A \subseteq \mathbb{R}$ be non-empty and bounded. Then $|A|$ is bounded and*

$$|\sup (A)| \leq \sup (|A|).$$

Proof 3.10 *Since A is bounded, there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$. Hence $|A|$ is bounded above and $\sup (|A|)$ exists.*

For every $a \in A$, we have

$$|a| \leq \sup (|A|),$$

which implies

$$a \leq \sup (|A|) \quad \text{and} \quad -a \leq \sup (|A|).$$

Taking the supremum over A in the first inequality gives

$$\sup (A) \leq \sup (|A|).$$

Taking the supremum over $-A$ in the second inequality yields

$$\sup (-A) \leq \sup (|A|).$$

Since $\sup (-A) = -\inf (A) \geq -\sup (A)$, we conclude that

$$-\sup (A) \leq \sup (|A|).$$

Therefore,

$$\sup (A), -\sup (A) \leq \sup (|A|),$$

and hence

$$|\sup (A)| \leq \sup (|A|).$$

Remark 3.1 The boundedness assumption on A is essential. If A is merely bounded above, then $|A|$ need not be bounded above. For example, if $A = (-\infty, 1)$, then A is bounded above but $|A|$ is unbounded.

Lemma 3.2 Let $A \subseteq \mathbb{R}$ be non-empty and bounded below. Then $|A|$ is bounded below and

$$\inf (|A|) \leq |\inf (A)|.$$

Proof 3.11 Since $|a| \geq 0$ for every $a \in A$, the set $|A|$ is bounded below and $\inf (|A|)$ exists.

Let $\alpha = \inf (A)$. Then $\alpha \leq a$ for all $a \in A$. Fix $\varepsilon > 0$. By the definition of infimum, there exists $a_\varepsilon \in A$ such that

$$\alpha \leq a_\varepsilon < \alpha + \varepsilon.$$

Taking absolute values and using the inequality

$$|x| - |y| \leq |x - y| \quad (x, y \in \mathbb{R}),$$

we obtain

$$|a_\varepsilon| \leq |\alpha| + \varepsilon.$$

Since $\inf (|A|) \leq |a_\varepsilon|$, it follows that

$$\inf (|A|) \leq |\alpha| + \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we conclude that

$$\inf (|A|) \leq |\alpha| = |\inf (A)|.$$

Proposition 3.5 Let A be non-empty and let $f, g : A \rightarrow \mathbb{R}$ be real-valued functions. Then:

1. If f and g are bounded above, then $f + g$ is bounded above and

$$\sup (f + g) \leq \sup (f) + \sup (g).$$

2. If f and g are bounded below, then $f + g$ is bounded below and

$$\inf (f) + \inf (g) \leq \inf (f + g).$$

Proof 3.12 For the first statement, let $\alpha = \sup (f)$ and $\beta = \sup (g)$. For each $t \in A$, we have

$$(f + g)(t) = f(t) + g(t) \leq \alpha + \beta.$$

Hence, $\sup (f + g) \leq \alpha + \beta$.

For the second statement, it is easy to see that $f + g$ is bounded below. Now, using the duality between supremum and infimum, we have

$$\inf (f) + \inf (g) = -\sup (-f) - \sup (-g) \leq -\sup (-(f + g)) = \inf (f + g),$$

as claimed.

3.4 EXERCISES

Exercise 3.1

1. Let $A, B \subseteq \mathbb{R}_+$ be bounded below. Show that AB is also bounded below and that

$$\inf (AB) = \inf (A) \cdot \inf (B).$$

2. Let $A, B \subseteq \mathbb{R}_-$ be bounded below. Show that AB is bounded above and that

$$\sup (AB) = \inf (A) \cdot \inf (B).$$

3. Let $A, B \subseteq \mathbb{R}_-$ be bounded above. Show that AB is bounded below and that

$$\inf (AB) = \sup (A) \cdot \sup (B).$$

4. Let $A \subseteq \mathbb{R}$ be non-empty and bounded above (resp. bounded below). Show that for each positive scalar λ , the set

$$\lambda A = \{\lambda a : a \in A\}$$

is also bounded above (resp. bounded below) and that

$$\sup (\lambda A) = \lambda \sup (A), \quad \inf (\lambda A) = \lambda \inf (A).$$

5. Let $A \subseteq \mathbb{R}$ be non-empty and bounded above (resp. bounded below). Show that for each negative scalar λ , the set

$$\lambda A = \{\lambda a : a \in A\}$$

is bounded below (resp. bounded above) and that

$$\inf (\lambda A) = \lambda \sup (A), \quad \sup (\lambda A) = \lambda \inf (A).$$

6. Let X and Y be non-empty sets, and let $h : X \times Y \rightarrow \mathbb{R}$ be a bounded function. Define

$$f(x) = \sup \{h(x, y) : y \in Y\}, \quad g(y) = \inf \{h(x, y) : x \in X\}.$$

Show that

$$\sup \{g(y) : y \in Y\} \leq \inf \{f(x) : x \in X\}.$$

7. Let A be a non-empty set and let $f, g : A \rightarrow \mathbb{R}$ be bounded functions. Show that

$$|\sup (f) - \sup (g)| \leq \sup |f - g|, \quad |\inf (f) - \inf (g)| \leq \sup |f - g|.$$

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Metric Spaces: The Managers of Topological Territories

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Metric Spaces: Principles and Strategies for Success

As we discussed in the previous chapter, the realm of mathematics resembles an ocean, with various islands representing different branches of math and scientific territories located along its shores. Beneath the surface, there are cities, equipment, and other resources, while boats and submarines facilitate the transport of goods between these locations. A defining characteristic of this mathematical ocean and its related areas is their precision. Despite the high volume of activity, including the extensive import and export operations, even a single missing or lost item can go unnoticed. Conversely, each element within this ocean has the freedom to explore and evolve, guided by the axioms that govern them. The potential for growth and the ability to harness the ocean's infinite advantages depend on the merit of each element. This accuracy is maintained by the managers overseeing the ports, islands, and lands from the town of “metric space,” situated on the islands of analysis. These managers vigilantly monitor the rules and positions of each element, creating an optimal environment for the inhabitants of the mathematical ocean to work swiftly, accurately, and harmoniously. They can rely on the assurance that every boat or submarine will deliver or collect products punctually and fairly. For instance, every resident of the town of natural numbers is aware of their designated position at the submarine station for exporting and importing goods, as well as the exact times for departure and arrival. This reliable system is the result of the diligent oversight provided by the managers of the metric spaces.

4.1 METRIC, A NOVEL IDEA, A SIMPLE NATURE

When measuring the distance between two cities, several key factors come into play. Firstly, the distance is zero only if the two cities are in the same location. Additionally, the distance remains the same regardless of the direction of travel between the cities. Furthermore, the triangle inequality must apply. These fundamental properties are essential and represent significant findings from the field of real analysis, as discovered by Fréchet, a notable figure from the realm of metric spaces.

Definition 4.1  Suppose X is a non-empty set. A mapping $d : X \times X \rightarrow \mathbb{R}$ is called a metric on X provided that for each $x, y, z \in X$, we have

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;

$$3. d(x, y) \leq d(x, z) + d(y, z).$$

The third condition is called the *triangle inequality*. There are several practical applications of this rule. For instance, if you try to chase two bunnies at the same time, you probably cannot catch either of them. In other words, the straight line is often the most efficient path to a target.

Example 4.1 *The first example naturally comes from the origin of the idea: the real numbers. The set of real numbers $\mathbb{R}$ with the absolute value function $d(x, y) = |x - y|$ is a metric space. It is easy to verify that it satisfies all the properties in [Definition 4.1](#).*

Example 4.2 $\hookleftarrow$ *Suppose X is a non-empty set. Define a mapping $d : X \times X \rightarrow \mathbb{R}$ by*

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

It is straightforward to check that d satisfies all the metric axioms. The metric space (X, d) is called the discrete metric.

The idea of the real numbers leads naturally to a fascinating construction in higher dimensions.

Example 4.3 $\hookleftarrow$ *Define the mapping d_2 on $\mathbb{R}^n$ by*

$$d_2(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}, \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

It is straightforward to verify the first two axioms of [Definition 4.1](#). The most interesting part is the triangle inequality, which requires a bit of calculation but follows from the Cauchy–Schwarz inequality.

In what follows, besides proving the triangle inequality for [Example 4.3](#), we explore several remarkable inequalities in the real numbers, which are interesting in their own right. We begin with the following lemma.

Lemma 4.1 $\hookleftarrow$ *Assume that $p, q > 1$ are positive conjugate reals; that is, $\frac{1}{p} + \frac{1}{q} = 1$. Then for each non-negative $u, v \in \mathbb{R}$, we have*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Proof 4.1 *For a fixed v , define the function $f : [0, \infty) \rightarrow \mathbb{R}$ by*

$$f(u) = \frac{u^p}{p} + \frac{v^q}{q} - uv.$$

Then

$$f'(u) = u^{p-1} - v.$$

Setting $f'(u) = 0$, we get $u^p = v^q$. Using the relation $\frac{1}{p} + \frac{1}{q} = 1$, we see that $q(p-1) = p$, so $p, q > 1$. Moreover,

$$f''(u) = (p-1)u^{p-2} > 0,$$

showing that the critical point $u^p = v^q$ is a minimum. At this point, $f(u^p = v^q) = 0$, so $f(u) \geq 0$ for all $u, v \geq 0$. Hence, the inequality follows.

Next, we use this lemma to prove the important Hölder inequality.

Theorem 4.1 (Hölder inequality) Let $p, q > 1$ be conjugate exponents. Then for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we have

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}.$$

Proof 4.2 Put

$$a = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, \quad b = \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}.$$

Set $u_i = \frac{|x_i|}{a}$ and $v_i = \frac{|y_i|}{b}$. By [Lemma 4.1](#), we have

$$\sum_{i=1}^n u_i v_i \leq \sum_{i=1}^n \left(\frac{u_i^p}{p} + \frac{v_i^q}{q} \right) = \frac{1}{p} \sum_{i=1}^n \frac{|x_i|^p}{a^p} + \frac{1}{q} \sum_{i=1}^n \frac{|y_i|^q}{b^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying both sides by ab gives

$$\sum_{i=1}^n |x_i y_i| \leq ab = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}.$$

Now we are ready to prove the Minkowski inequality, which establishes the triangle inequality for the p -norm in $\mathbb{R}^n$.

Theorem 4.2 (Minkowski inequality) Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be elements of $\mathbb{R}^n$, and let $1 \leq p \in \mathbb{R}$. Then

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}.$$

Proof 4.3 If $p = 1$, the inequality reduces to the usual triangle inequality in $\mathbb{R}$. Assume $p > 1$ and set $q = \frac{p}{p-1}$ so that $\frac{1}{p} + \frac{1}{q} = 1$.

By Hölder's inequality,

$$\begin{aligned} \sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \leq \sum_{i=1}^n |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| |x_i + y_i|^{p-1} \\ &\leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |x_i + y_i|^{(p-1)q} \right)^{\frac{1}{q}} \end{aligned}$$

$$= \left(\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right) \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{q}}.$$

If $(\sum_{i=1}^n |x_i + y_i|^p)^{1/q} = 0$, the inequality holds trivially. Otherwise, divide both sides by $(\sum_{i=1}^n |x_i + y_i|^p)^{1/q}$ to obtain the desired result.

The Minkowski's inequality completes the argument in [Example 4.3](#) and also allows the construction of a family of metrics on $\mathbb{R}^n$.

Example 4.4 For each real $p \geq 1$, define the mapping d_p on $\mathbb{R}^n$ by

$$d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

By Minkowski's inequality, d_p is a metric on $\mathbb{R}^n$.

4.2 METRIC EQUIPMENT FOR A PERFECT MANAGEMENT

Interior Points, Cluster, Limit, and Boundary: The Upstream Rules of the Topology Realm

We continue our story in the real city. The managers, known as metrics, are selective and diligent. They employ neighborhoods (or balls) as their representatives to manage the points in the set on which the metric operates. In this section, we will investigate the points of a metric space and examine some special subsets within it. In other words, neighborhoods (balls) represent the topology induced by the metric. Indeed, topologies serve as the foundation of every metric space, facilitating effective management. Although all residents of the Topology realm can, depending on their efforts and characteristics, find certain positions and receive different stamps on their method card, they must also follow the upstream rules of the realm. More precisely, they must maintain their topological structures, but at the same time, they are allowed to accept a metric or not. Each manager, considering their ability and possibilities (depending on the type of management and efforts of the members), can attract certain facilities, like selecting a submarine, boat, or even a train as the transportation means between the points of the area (internal connections). In fact, the speed and nature of these connections are essential for any topology and lay the foundation for its development. A resident in a specific area of the accepted Topology realm is referred to as an internal point of the set (relative to the topology rules) if, within a certain radius, we can have residents of the same set (at a certain time). In fact, the internal points of a set are the points loyal to the set (from the perspective of the upstream rules of the realm). A set whose every point is internal is considered an open set; the export and import stamp is issued to all its residents, and they can freely and collectively engage in commercial exchanges.

Limit points are the main corridors of economic exchange and strategic targets for advancing each region in the Topology realm. In the next chapter, we will see that the ambassadors and representatives of the convergence archipelago are stationed in those places to manage important projects and promote economic prosperity in each region. Regions that monitor and control all their limit corridors are called closed regions, and they will play a significant role in advancing the topology's goals. Boundary points are the customs centers for exchanges between regions that belong to both the inside and the outside of the region. In fact, they are responsible for issuing the import and export permit types within the Topology realm. The adherence points for a region are composed of the points of the region itself, along with those in the main corridors. Another definition for a closed set is a set whose customs centers are within the set itself. In fact, it all

comes down to the management of the desired topology and the efforts of the members (residents) of the region to be able to carry out better exchanges with the type of equipment they have.

Boundedness: Although the upstream rules of the Topology realm are considered somewhat of a norm and authenticity for residents, some areas within the realm define their own norms and regulations to enable residents to make optimal use of facilities and improve resource management. In regions managed by metrics, the members' method cards will also be stamped as “bounded. In a metric space, a bounded region is one in which, according to the metric, the distance between the houses of its residents does not exceed a certain value. In other words, these houses are located within a specific town. Typically, this distance is measured by time. For example, the distance of the homes of residents of (0,1) town in the Real Number district is traversed at a maximum of one minute with a Euclidean meter. However, the Natural Number district is not bounded, since it extends endlessly into the depths of the long trenches (in search of an infinite treasure).

We first define these concepts mathematically before returning to our narrative.

Definition 4.2 *Let (X, d) be a metric space, let $x \in X$, and let $r > 0$ be a real number. The open neighborhood (or open ball) centered at x with radius r is defined by*

$$N_r^{(x)} = \{y \in X : d(x, y) < r\}.$$

Similarly, the *closed neighborhood (or closed ball)* centered at x with radius r is defined by

$$N_r^{[x]} = \{y \in X : d(x, y) \leq r\}.$$

Freedom: the Honor of Mathematics

Before delving deeper into these new (yet familiar) ideas, let us illustrate their roles in our story. As previously mentioned, the realm beneath the mathematical ocean represents a significant portion of the entire world of mathematics. Notably, many mathematical concepts branch out in both domains; they exist as islands and also beneath the ocean. For example, we stated that metric spaces serve as the managers of the underwater world. They possess facilities that ensure the completeness or compactness of their products and employ express trains to traverse infinite lands and territories.

However, it is important to note that certain territories, whether under the ocean or as islands, operate under different rules than those found in conventional lands. For instance, some parts of the “analysis island” belong to the topology territory. Due to its unique properties and characteristics, topology exports and imports concepts to the algebra island, geometry island, and application island. The rules governing the topology territory differ significantly; for example, within the “analysis island,” possessing a norm or metric (as a guiding principle) is essential, whereas, in the topology territory, it is not obligatory. The inherent power and flexibility of topology grant it greater freedom compared to other areas.

Now, we return to the concept of balls with a few examples.

Example 4.5 $\hookleftarrow$ *Suppose $X = \mathbb{R}^2$ with the d_2 metric as in [Example 4.3](#). Consider the balls centered at the origin with radius 1. We have*

$$N_1^{(0,0)} = \{(x, y) \in \mathbb{R}^2 : d_2((x, y), (0, 0)) < 1\} = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\},$$

$$N_1^{[(0,0)]} = \{(x, y) \in \mathbb{R}^2 : d_2((x, y), (0, 0)) \leq 1\} = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq 1\}.$$

The former is the interior of a circle centered at the origin with radius 1, while the latter is the interior together with the boundary. As expected, these balls coincide with our geometric intuition. However, as we will see, the situation can be completely different for other, less usual metrics.

Example 4.6 Suppose $X = \mathbb{R}^2$ with the discrete metric d as in [Example 4.2](#). Again, consider the balls centered at the origin with radius 1. We have

$$N_1^{(0,0)} = \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) < 1\} = \{(0, 0)\},$$

$$N_1^{[(0,0)]} = \{(x, y) \in \mathbb{R}^2 : d((x, y), (0, 0)) \leq 1\} = \mathbb{R}^2.$$

Here we see a striking difference: the open unit ball consists of a single point, while the closed unit ball is the entire space.

Next, we consider two other metrics on $\mathbb{R}^2$ whose balls have shapes similar to those in [Example 4.5](#).

Example 4.7 $\hookleftarrow$ Let $X = \mathbb{R}^2$. Define two metrics d_r and d_m on X by

$$d_r((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|, d_m((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Consider the open unit balls centered at the origin; the closed balls are similar to the case in [Example 4.5](#). We have

$$(N_1^{(0,0)})_r = \{(x, y) \in \mathbb{R}^2 : d_r((x, y), (0, 0)) < 1\} = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\},$$

$$(N_1^{(0,0)})_m = \{(x, y) \in \mathbb{R}^2 : d_m((x, y), (0, 0)) < 1\} = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < 1\}.$$

Geometrically, the first ball is a rhombus, while the second one is a square.

For a better illustration, consider the graphs in [Figures 4.1](#), [4.2](#), and [4.3](#).

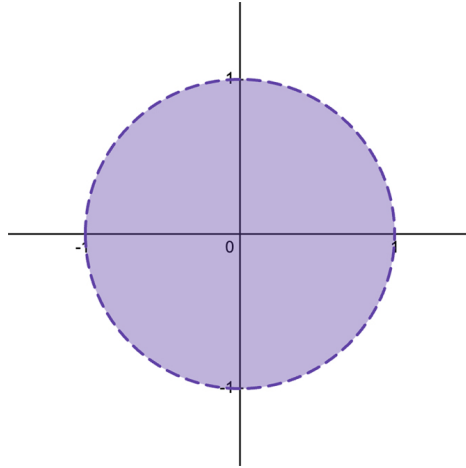


Figure 4.1 [Example 4.7](#): the open unit ball in [Example 4.5](#) (d_2 metric). $\hookleftarrow$

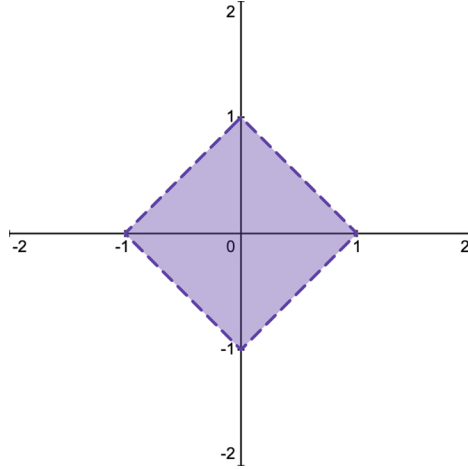


Figure 4.2 [Example 4.7](#): the open unit ball of the metric d_r (sum metric). ↵

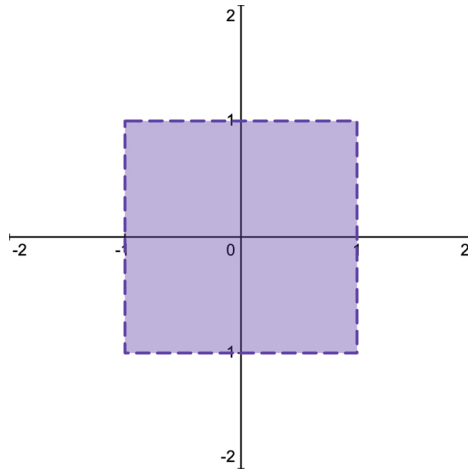


Figure 4.3 [Example 4.7](#): the open unit ball of the metric d_m (maximum metric). ↵

Now, let us explore the world beneath the mathematical ocean more thoroughly. As mentioned earlier, this entire area comprises a set, referred to as the universal set, which contains numerous subsets. Some of these subsets are cities, while others are provinces or towns, among others. In the previous chapter, we saw that ordered sets form a province with a known real city at its center. Metric spaces, in general, also constitute a province. A key point is that certain properties related to mathematics are naturally inherited by every inhabitant. For example, the order is associated with the real city. However, the specific metric or topology applicable to them depends on the tools they possess. For instance, $\mathbb{R}$ with the modulus can traverse the province of metric spaces or normed spaces. Yet, with the right radius topology, while it can freely enter the topology territory, it does not have permission to enter the metric province. Thus, every inhabitant in the world of mathematics has the freedom to carry the necessary tools to access various realms or not. This illustrates the power of mathematics: freedom.

Now, let us explore the subsets of a metric space. We will journey through the province of metric spaces to uncover the secrets behind the effective management provided by the metrics.

Definition 4.3 Suppose (X, d) is a metric space and $A \subseteq X$. We say that $x \in A$ is an interior point of A if there exists $r > 0$ such that

$$N_r^{(x)} \subseteq A.$$

We denote the set of all interior points of A by A° or $\text{int}(A)$. Clearly, $A^\circ \subseteq A$. Moreover, we say that A is open if $A \subseteq A^\circ$, or equivalently, $A = A^\circ$.

Definition 4.4 Suppose (X, d) is a metric space and $A \subseteq X$. We say that $x \in X$ is a cluster point of A if for every $r > 0$, we have

$$N_r^{(x)} \cap A \neq \emptyset.$$

We denote the set of all cluster points of A by $\overline{A}$. Clearly, $A \subseteq \overline{A}$. Moreover, we say that A is closed if $\overline{A} \subseteq A$, or equivalently, $A = \overline{A}$.

Now, we show that every open ball is an open set and every closed ball is a closed set, justifying their names.

Theorem 4.3 Suppose (X, d) is a metric space, $x \in X$, and $r > 0$. Then $N_r^{(x)}$ is an open set.

Proof 4.4 Let $y \in N_r^{(x)}$. We show that y is an interior point. Put

$$r_0 = \frac{1}{2}(r - d(x, y)).$$

Note that $r_0 > 0$ since $d(x, y) < r$. We claim that

$$N_{r_0}^{(y)} \subseteq N_r^{(x)}.$$

Indeed, for any $z \in N_{r_0}^{(y)}$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r_0 = d(x, y) + \frac{1}{2}(r - d(x, y)) = \frac{1}{2}d(x, y) + \frac{1}{2}r < r,$$

showing that $z \in N_r^{(x)}$, as claimed.

Theorem 4.4 Suppose (X, d) is a metric space, $x \in X$, and $r > 0$. Then $N_r^{[x]}$ is a closed set.

Proof 4.5 It suffices to show that if $y \notin N_r^{[x]}$, then y is not a cluster point of $N_r^{[x]}$. Assume $y \notin N_r^{[x]}$ and put

$$r_0 = \frac{1}{2}(d(x, y) - r).$$

Note that $r_0 > 0$ since $y \notin N_r^{[x]}$. We claim that

$$N_{r_0}^{(y)} \cap N_r^{[x]} = \emptyset.$$

Suppose, for contradiction, that this is not true. Choose $z \in N_{r_0}^{(y)} \cap N_r^{[x]}$. Then

$$d(x, z) \leq r \quad \text{and} \quad d(y, z) < r_0.$$

Hence, by the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) < r + r_0 = r + \frac{1}{2}(d(x, y) - r) = \frac{1}{2}r + \frac{1}{2}d(x, y) < d(x, y),$$

which is a contradiction. This completes the proof.

In [Figures 4.4](#) and [4.5](#), we consider the open and closed unit balls in $\mathbb{R}^2$ with the Euclidean metric (the usual modulus), both centered at the origin with radius 1.

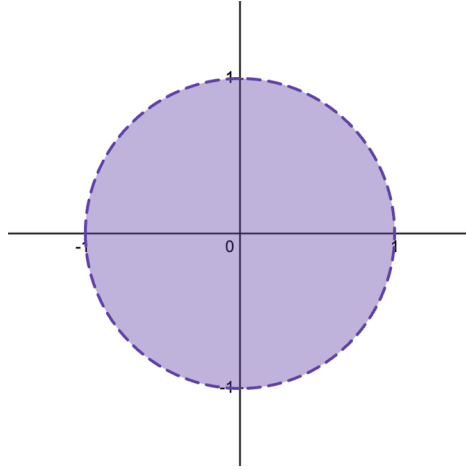


Figure 4.4 The open unit ball centered at the origin with radius 1. [↗](#)

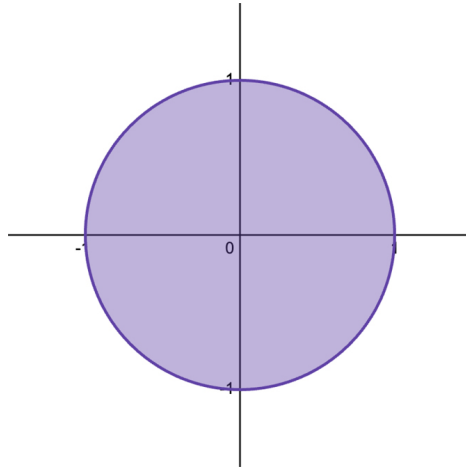


Figure 4.5 The closed unit ball centered at the origin with radius 1. [↗](#)

Definition 4.5 Assume that (X, d) is a metric space. We say that a set $A \subseteq X$ is dense in X if

$$\overline{A} = X.$$

Intuitively, a dense subset of a metric space is “everywhere” in the space in the sense that every open neighborhood contains points of the subset, though the subset may not be the whole space. For example, using the material from [Chapter 3](#), one can show that both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$ when equipped with the standard Euclidean topology (see the first exercise below for more details).

Remark 4.1 [↗](#) Assume that (X, d) is a metric space and $A \subseteq X$. Then A is dense in X if and only if for each non-empty open set $U \subseteq X$, $U \cap A \neq \emptyset$. To see this, first, assume that A is dense

and $U \subseteq X$ is open and non-empty. Take $x \in U$. So, there exists $r > 0$ with $N_r^{(x)} \subseteq U$. Thus,

$$\emptyset \neq N_r^{(x)} \cap A \subseteq U \cap A,$$

showing that $U \cap A \neq \emptyset$. For the other side, Take $x \in X$ and let $r > 0$ is arbitrary. Therefore, by the assumption, $N_r^{(x)} \cap A \neq \emptyset$ so that $x \in \overline{A}$. This shows that $\overline{A} = X$.

Definition 4.6 Assume that (X, d) is a metric space and $A \subseteq X$. A point $x \in X$ is called an accumulation point (or limit point) of A if for each $r > 0$, we have

$$(N_r^{(x)} - \{x\}) \cap A = N_r^{(x)} \cap (A - \{x\}) \neq \emptyset.$$

The set of all limit points of A is denoted by A' .

It is clear that every accumulation point is a cluster point, but the converse is not true in general (see [Figure 4.6](#)).

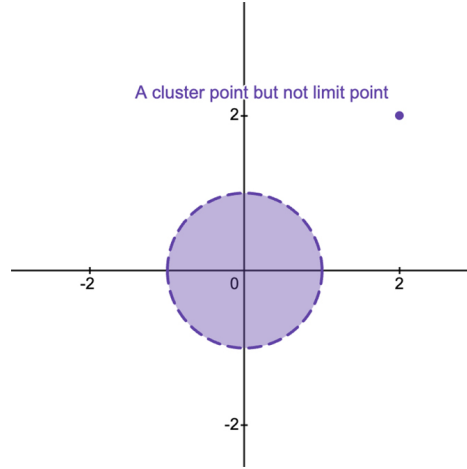


Figure 4.6 A cluster point that is not a limit point. ↵

Definition 4.7 Assume that (X, d) is a metric space and $A \subseteq X$. A point $x \in X$ is called a boundary point (or frontier point) of A if for every $r > 0$,

$$N_r^{(x)} \cap A \neq \emptyset \quad \text{and} \quad N_r^{(x)} \cap A^c \neq \emptyset.$$

The set of all boundary points of A is denoted by ∂A . Equivalently,

$$\partial A = \overline{A} \cap \overline{A^c}.$$

Example 4.8 Consider $X = \mathbb{R}$ with the standard Euclidean topology.

- If $A = \{0\}$, then $x_0 = 0$ is a boundary point of A , although it is not a limit point.
- If $B = [0, 1]$, then $x_0 = 0.5$ is a limit point of B but not a boundary point.

Theorem 4.5 Assume that (X, d) is a discrete space and $A \subseteq X$. Then $\partial A = \emptyset$.

Proof 4.6 In a discrete space, every set is both open and closed. Therefore,

$$\partial A = \overline{A} \cap \overline{A^c} = A \cap A^c = \emptyset.$$

Example 4.9  Consider $X = \mathbb{R}$ with the standard (Euclidean) topology. Let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

We claim that $A' = \{0\}$.

We show that $0 \in A'$. Let $\varepsilon > 0$ be arbitrary. By the consequence of the Archimedean property, there exists $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \varepsilon$. Then

$$(-\varepsilon, \varepsilon) \cap (A \setminus \{0\}) \neq \emptyset.$$

This means that 0 is a limit point of A .

Now, we show that $x \notin A'$ for $x \neq 0$.

- If $x < 0$, let $r_0 = -\frac{x}{2}$. Then $N_{r_0}^{(x)} \cap (A - \{0\}) = \emptyset$.
- If $x > 1$, let $r_0 = \frac{x-1}{2}$. Then $N_{r_0}^{(x)} \cap (A - \{0\}) = \emptyset$.
- If $x \in (0, 1]$ and $x \neq \frac{1}{n}$ for any $n \in \mathbb{N}$, choose $n_1 \in \mathbb{N}$ such that

$$\frac{1}{n_1 + 1} < x < \frac{1}{n_1}.$$

Then, let

$$r_0 = \frac{1}{2} \min \left\{ x - \frac{1}{n_1 + 1}, \frac{1}{n_1} - x \right\}.$$

It follows that $N_{r_0}^{(x)} \cap (A - \{0\}) = \emptyset$.

- If $x = \frac{1}{n_2}$ for some $n_2 \in \mathbb{N}$, choose

$$r_0 = \frac{1}{2n_2(n_2 + 1)}.$$

Then $N_{r_0}^{(x)} \cap (A - \{0\}) = \emptyset$.

This proves that $A' = \{0\}$.

As illustrated in [Example 4.9](#), the sequence of elements in $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ converges to 0. This observation naturally leads us to the intriguing concept of compactness, which will be explored in [Chapter 7](#).

Example 4.10 Consider $X = \mathbb{R}$ with the Euclidean topology and let

$$A = \left\{ \frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N} \right\}.$$

Then

$$A' = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

Indeed, for sufficiently large n, m , the elements $\frac{1}{n} + \frac{1}{m}$ can be arbitrarily close to 0, so $0 \in A'$. Moreover, fixing n and varying m , each point $\frac{1}{n}$ is a limit point of A .

Theorem 4.6 $\hookrightarrow$ Let (X, d) be a metric space and $A \subseteq X$. Then

$$\overline{A} = A \cup A'.$$

Proof 4.7 By definition, every limit point is a cluster point, so $A \cup A' \subseteq \overline{A}$. Conversely, if $x \in \overline{A}$ but $x \notin A$, then for each $r > 0$,

$$N_r^{(x)} \cap (A - \{x\}) = N_r^{(x)} \cap A \neq \emptyset,$$

showing that $x \in A'$. Hence, $\overline{A} = A \cup A'$.

Corollary 4.1 $\hookrightarrow$ Let (X, d) be a metric space and $A \subseteq X$. Then A is closed if and only if $A' \subseteq A$.

Proof 4.8 If A is closed, then $\overline{A} = A$, so by [Theorem 4.6](#), $A' \subseteq A$. Conversely, if $A' \subseteq A$, then

$$\overline{A} = A \cup A' \subseteq A,$$

so A is closed.

Corollary 4.2 $\hookrightarrow$ Let (X, d) be a metric space and $A \subseteq X$ be finite. Then $A' = \emptyset$, so A is closed.

Proof 4.9 Let $A = \{a_1, \dots, a_n\}$ and let $x \in X$. Define

$$r_0 = \frac{1}{2} \min \{d(x, a) : a \in A, a \neq x\}.$$

Then

$$N_{r_0}^{(x)} \cap (A - \{x\}) = \emptyset,$$

so $x \notin A'$. Hence $A' = \emptyset \subseteq A$, and by [Corollary 4.1](#), A is closed.

Example 4.11 Let $X = \mathbb{R}$ with the Euclidean metric and

$$A = \left\{ \frac{k}{2^n} : 1 \leq k \leq 2^n, n \in \mathbb{N} \right\}.$$

Then

$$A' = \overline{A} = [0, 1].$$

Proof 4.10 Since $A \subseteq [0, 1]$ and the interval $[0, 1]$ is closed, we conclude that $A' \subseteq [0, 1]$; otherwise for $x > 1$, take $r_0 = \frac{x-1}{2}$ and for $x < 0$, take $r_0 = -\frac{x}{2}$ to show that $N_{r_0}^{(x)} \cap (A - \{x\}) = N_{r_0}^{(x)} \cap A = \emptyset$ so that $x \notin A'$. Therefore, $A' \subseteq [0, 1]$. Now let $x \in (0, 1]$. By

the Archimedean property, choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$. Thus, $\frac{1}{2^{n_0}} < \frac{1}{n_0} < x$ so that $2^{n_0}x > 1$. Let $n \geq n_0$ and define $k = \lfloor 2^n x \rfloor$, so that

$$\frac{k}{2^n} \leq x < \frac{k+1}{2^n} \implies 0 \leq x - \frac{k}{2^n} < \frac{1}{2^n}.$$

Note that $\frac{1}{2^n} \rightarrow 0$. Hence, every neighborhood of x contains points of A , so $x \in A'$. Furthermore, since $\frac{n}{2^n} \rightarrow 0$, we conclude that every neighborhood of zero, contains some elements of A showing that $0 \in A'$. Therefore, $A' = [0, 1]$.

There is another useful relation between the closure of a set and the interior of its complement, as the following result shows.

Theorem 4.7 $\hookleftarrow$ Assume that (X, d) is a metric space and $A \subseteq X$. Then,

$$(\overline{A})^c = (A^c)^\circ \quad \text{and} \quad (A^\circ)^c = \overline{A^c}.$$

Proof 4.11 First, assume that $x \in (\overline{A})^c$, so that $x \notin \overline{A}$. Therefore, there exists $r_0 > 0$ such that

$$N_{r_0}^{(x)} \cap A = \emptyset,$$

which implies that $N_{r_0}^{(x)} \subseteq A^c$. Hence, $x \in (A^c)^\circ$, and consequently,

$$(\overline{A})^c \subseteq (A^c)^\circ.$$

Similarly, one can verify that

$$(A^c)^\circ \subseteq (\overline{A})^c.$$

Therefore, $(\overline{A})^c = (A^c)^\circ$. Replacing A with A^c in this relation, we obtain

$$((A^c)^\circ)^\circ = (\overline{A^c})^c,$$

which yields

$$(A^\circ)^c = \overline{A^c}.$$

The notions of openness and closeness are dual to each other, as the following result shows.

Theorem 4.8 Assume that (X, d) is a metric space and $A \subseteq X$. Then A is open if and only if A^c is closed.

Proof 4.12 First, suppose that A is open. To show that A^c is closed, it is enough to prove that

$$\overline{A^c} \subseteq A^c.$$

Let $x \in \overline{A^c}$. Then, for each $r > 0$, we have

$$N_r^{(x)} \cap A^c \neq \emptyset,$$

which implies that $N_r^{(x)} \not\subseteq A$. Hence, x cannot be an interior point of A , so

$$x \notin A^\circ = A,$$

and therefore $x \in A^c$.

Conversely, assume that A^c is closed. To show that A is open, it suffices to prove that every point of A is an interior point. Let $x \in A$. Then $x \notin A^c = \overline{A^c}$. By the definition of closure, there exists $r_0 > 0$ such that

$$N_{r_0}^{(x)} \cap A^c = \emptyset,$$

which implies that $N_{r_0}^{(x)} \subseteq A$. Hence, x is an interior point of A , completing the proof.

Example 4.12 $\hookrightarrow$ Consider $X = \mathbb{R}$ with the discrete topology. Put

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then $A' = \emptyset$.

To show this, let $x \in \mathbb{R}$ be arbitrary and put $r_0 = \frac{1}{2}$. We have

$$N_{r_0}^{(x)} \cap (A - \{x\}) = \{x\} \cap (A - \{x\}) = \emptyset,$$

which shows that $x \notin A'$. Since x was arbitrary, we conclude that $A' = \emptyset$.

It seems that in [Example 4.12](#) we did not use any special property of $\mathbb{R}$. In other words, the result appears to hold for every metric space equipped with the discrete topology. Motivated by this observation, we derive the following general result.

Theorem 4.9 $\hookrightarrow$ Assume that (X, d) is a metric space with the discrete topology. Then, for every $A \subseteq X$, we have $A' = \emptyset$.

Proof 4.13 Let $x \in X$ be arbitrary and put $r_0 = \frac{1}{2}$. Then

$$N_{r_0}^{(x)} \cap (A - \{x\}) = \{x\} \cap (A - \{x\}) = \emptyset,$$

which shows that $x \notin A'$. Since this holds for every $x \in X$, we conclude that $A' = \emptyset$.

This procedure is a key step toward many deep and significant results. Consider [Example 4.9](#) and [Example 4.12](#). First, we used the Euclidean topology to determine the limit points of a set. Then, we changed the topology to observe what happens in the discrete case. Finally, by examining the argument in [Example 4.12](#), we noticed that the reasoning does not depend on $\mathbb{R}$, which naturally led to a more general result, stated in [Theorem 4.9](#).

Theorem 4.10 $\hookrightarrow$ Assume that (X, d) is a metric space and $A \subseteq X$. Then both A' and $\overline{A}$ are closed sets.

Proof 4.14 We prove the statement for A' . The proof for $\overline{A}$ is similar. It is enough to show that $(A')' \subseteq A'$. Suppose, to the contrary, that there exists $x \in (A')'$ such that $x \notin A'$. Since $x \notin A'$, there exists $r_0 > 0$ such that

$$N_{r_0}^{(x)} \cap (A - \{x\}) = \emptyset.$$

On the other hand, since $x \in (A')'$, for every $r > 0$ (in particular, for $r = r_0$), we have

$$N_{r_0}^{(x)} \cap (A' - \{x\}) \neq \emptyset.$$

Hence, there exists $y \in A'$ with $y \neq x$ and $y \in N_{r_0}^{(x)}$. Since open balls are open sets, there exists $r_1 > 0$ such that

$$N_{r_1}^{(y)} \subseteq N_{r_0}^{(x)}.$$

Moreover, we may choose $r_1 < d(x, y)$ so that $x \notin N_{r_1}^{(y)}$. Since $y \in A'$, for every $r > 0$ (in particular, for $r = r_1$), we have

$$N_{r_1}^{(y)} \cap (A - \{y\}) \neq \emptyset.$$

Thus, there exists $z \in X$ such that

$$z \in N_{r_1}^{(y)} \subseteq N_{r_0}^{(x)}, \quad z \neq x, \quad z \neq y, \quad z \in A.$$

This implies that

$$N_{r_0}^{(x)} \cap (A - \{x\}) \neq \emptyset,$$

which is a contradiction. Therefore, $(A')' \subseteq A'$, and the proof is complete.

Definition 4.8 Assume that (X, d) is a metric space and $A \subseteq X$. We say that $x \in X$ is an exterior point of A provided that $x \in (A^c)^\circ$. We denote by $\text{ext}(A)$ the set of all exterior points of A . Note that, by [Theorem 4.7](#), we have

$$\text{ext}(A) = (A^c)^\circ = (\overline{A})^c.$$

Now we state the following result.

Theorem 4.11 $\hookleftarrow$ Assume that (X, d) is a metric space and $A \subseteq X$. Then

$$X = A^\circ \cup \text{ext}(A) \cup \partial A.$$

Moreover, these sets are pairwise disjoint.

Proof 4.15 Let $x \in X$ such that $x \notin A^\circ$ and $x \notin \text{ext}(A)$. Since $x \notin A^\circ$, for every $r > 0$, we have $N_r^{(x)} \not\subseteq A$, which implies

$$N_r^{(x)} \cap A^c \neq \emptyset.$$

On the other hand, since $x \notin \text{ext}(A)$, for every $r > 0$, we have

$$N_r^{(x)} \cap A \neq \emptyset.$$

Thus, $x \in \partial A$.

To prove disjointness, assume that $x \in A^\circ \cap \partial A$. Since $x \in A^\circ$, there exists $r_0 > 0$ such that $N_{r_0}^{(x)} \subseteq A$, and hence

$$N_{r_0}^{(x)} \cap A^c = \emptyset.$$

However, since $x \in \partial A$, for every $r > 0$ (in particular, for $r = r_0$), we must have

$$N_{r_0}^{(x)} \cap A^c \neq \emptyset,$$

which is a contradiction. Therefore, $A^\circ \cap \partial A = \emptyset$. Similarly, one can show that $\text{ext}(A) \cap \partial A = \emptyset$. Moreover,

$$A^\circ \cap \text{ext}(A) \subseteq A \cap A^c = \emptyset.$$

This completes the proof.

Theorem 4.12 $\hookleftarrow$ Assume that (X, d) is a metric space. Then:

1. A subset $F \subseteq X$ is closed if and only if $\partial F \subseteq F$;
2. A subset $G \subseteq X$ is open if and only if $\partial G \cap G = \emptyset$.

Proof 4.16 For the first statement, assume that F is closed. Then

$$\partial F = \overline{F} \cap \overline{F^c} \subseteq \overline{F} = F.$$

Conversely, assume that $\partial F \subseteq F$. To show that F is closed, it is enough to prove that $\overline{F} \subseteq F$. Suppose, to the contrary, that there exists $x \in \overline{F}$ such that $x \notin F$. Then $x \in F^c \subseteq \overline{F^c}$. Therefore,

$$x \in \overline{F} \cap \overline{F^c} = \partial F \subseteq F,$$

which is a contradiction.

For the second statement, assume that $G \subseteq X$ is open. By [Theorem 4.11](#), we have

$$\partial G \cap G = \partial G \cap G^\circ = \emptyset.$$

Conversely, assume that $\partial G \cap G = \emptyset$. We show that every point of G is an interior point. Let $x \in G$. Since $\partial G \cap G = \emptyset$, we have

$$x \notin \partial G = \overline{G} \cap \overline{G^c}.$$

Thus, either $x \notin \overline{G}$ or $x \notin \overline{G^c}$. Since $x \in G \subseteq \overline{G}$, it follows that $x \notin \overline{G^c}$. Hence, there exists $r_0 > 0$ such that

$$N_{r_0}^{(x)} \cap G^c = \emptyset,$$

which implies that $N_{r_0}^{(x)} \subseteq G$. This completes the proof.

Lemma 4.2 $\hookleftarrow$ Assume that (X, d) is a metric space and $A \subseteq X$. Then ∂A is closed.

Proof 4.17 Note that $\partial A = \overline{A} \cap \overline{A^c}$. By [Theorem 4.10](#), both $\overline{A}$ and $\overline{A^c}$ are closed sets. Therefore, ∂A is closed.

Proposition 4.1 $\hookleftarrow$ Assume that (X, d) is a metric space and $A \subseteq X$. Then

$$\partial(\partial A) \subseteq \partial A.$$

Proof 4.18 By [Lemma 4.2](#), ∂A is a closed set. Hence, by [Theorem 4.12](#), the desired inclusion follows.

At first glance, it may seem that $\partial(\partial A) = \partial A$. However, the following example shows that the inclusion in [Proposition 4.1](#) can be proper.

Example 4.13 Consider $X = \mathbb{R}$ with the Euclidean topology and let $A = \mathbb{Q}$. Then

$$\partial \mathbb{Q} = \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}.$$

Moreover,

$$\partial(\partial \mathbb{Q}) = \partial \mathbb{R} = \emptyset.$$

Nevertheless, an interesting fact remains true: for every subset A of a metric space (X, d) , we have

$$\partial(\partial(\partial A)) = \partial(\partial A).$$

Before proving this, we state some useful facts, again. The interior points, the cluster points, and the limit points behave well with respect to subsets.

Proposition 4.2 $\hookrightarrow$ Assume that (X, d) is a metric space and $A, B \subseteq X$ such that $A \subseteq B$. Then we have

$$A^\circ \subseteq B^\circ, \quad \overline{A} \subseteq \overline{B}, \quad A' \subseteq B'.$$

Proof 4.19 The proof is straightforward. Take any $x \in A^\circ$. There exists $r_0 > 0$ such that $N_{r_0}^{(x)} \subseteq A$, and by the assumption $A \subseteq B$, we have $N_{r_0}^{(x)} \subseteq B$. Note that $x \in A^\circ \subseteq A \subseteq B$. This shows that $x \in B^\circ$.

Now, take any $x \in \overline{A}$. For each $r > 0$, we have $N_r^{(x)} \cap A \neq \emptyset$, and since $A \subseteq B$, it follows that $N_r^{(x)} \cap B \neq \emptyset$. Thus, $x \in \overline{B}$. The proof for the limit points is similar.

Theorem 4.13 $\hookrightarrow$ Assume that (X, d) is a metric space and $\{A_\alpha\}_{\alpha \in I}$ is a family of subsets of X . Then the following statements hold:

1. $\bigcup_{\alpha \in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in I} A_\alpha}$;
2. $\bigcup_{\alpha \in I} A_\alpha' \subseteq \left(\bigcup_{\alpha \in I} A_\alpha \right)'$;
3. $\left(\bigcap_{\alpha \in I} A_\alpha \right)^\circ \subseteq \bigcap_{\alpha \in I} A_\alpha^\circ$;
4. $\overline{\bigcap_{\alpha \in I} A_\alpha} \subseteq \bigcap_{\alpha \in I} \overline{A_\alpha}$;

$$5. \left(\bigcap_{\alpha \in I} A_\alpha \right)' \subseteq \bigcap_{\alpha \in I} A'_\alpha;$$

$$6. \bigcup_{\alpha \in I} A_\alpha^\circ \subseteq \left(\bigcup_{\alpha \in I} A_\alpha \right)^\circ.$$

Moreover, for a finite family $\{A_1, \dots, A_n\}$ of subsets of X , we have

$$\overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i};$$

$$(2') \quad \left(\bigcup_{i=1}^n A_i \right)' = \bigcup_{i=1}^n A'_i;$$

$$(3') \quad \left(\bigcap_{i=1}^n A_i \right)^\circ = \bigcap_{i=1}^n A_i^\circ.$$

Proof 4.20 To prove (1), for each $\beta \in I$, we have $A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha$; hence, by [Proposition 4.2](#),

$$\overline{A_\beta} \subseteq \overline{\bigcup_{\alpha \in I} A_\alpha}.$$

Since β is arbitrary, it follows that

$$\bigcup_{\alpha \in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in I} A_\alpha}.$$

The proof of (2) is similar.

To prove (3), note that $\bigcap_{\alpha \in I} A_\alpha \subseteq A_\beta$ for each $\beta \in I$. Therefore, by [Proposition 4.2](#),

$$\left(\bigcap_{\alpha \in I} A_\alpha \right)^\circ \subseteq (A_\beta)^\circ.$$

Since β is arbitrary, the desired inclusion follows.

The remaining parts can be proved in a similar manner; we leave them as an exercise for the reader. For the finite case, we prove (3'); the other cases are analogous.

Assume that $x \in \bigcap_{i=1}^n (A_i)^\circ$. Then $x \in A_i^\circ$ for each $i \in \{1, \dots, n\}$, so there exist $r_i > 0$ such that $N_{r_i}^{(x)} \subseteq A_i$. Let

$$r = \min \{r_1, \dots, r_n\}.$$

Then

$$N_r^{(x)} \subseteq N_{r_i}^{(x)} \subseteq A_i, \quad i = 1, \dots, n.$$

Hence,

$$N_r^{(x)} \subseteq \bigcap_{i=1}^n A_i,$$

which implies $x \in \left(\bigcap_{i=1}^n A_i\right)^\circ$.

Example 4.14 Note that the inclusions in [Theorem 4.13](#) can be proper, even in parts (4), (5), and (6), and this may occur even for finite families. We illustrate this with several examples.

For parts (1) and (2), consider $X = \mathbb{R}$ with the Euclidean topology and enumerate $\mathbb{Q} = \{r_n\}_{n \in \mathbb{N}}$. Let $A_n = \{r_n\}$. Since singletons are closed in any metric space (see [Corollary 4.2](#)), we have $\overline{A_n} = A_n$. On the other hand, $\overline{\mathbb{Q}} = \mathbb{R}$ (see the first exercise below). Therefore,

$$\bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \overline{\{r_n\}} = \overline{\mathbb{Q}} = \mathbb{R} \supset \mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\} = \bigcup_{n=1}^{\infty} A_n.$$

The same example shows that the inclusion in part (2) can also be proper.

To give a counterexample for part (3), let $A_n = (-\frac{1}{n}, \frac{1}{n})$. Then

$$\left(\bigcap_{n=1}^{\infty} A_n\right)^\circ = \left(\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})\right)^\circ = \{0\}^\circ = \emptyset \supset \{0\} = \bigcap_{n=1}^{\infty} A_n^\circ.$$

To obtain counterexamples for parts (4), (5), and (6), consider the finite sets $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Then

$$\overline{A \cap B} = \overline{\mathbb{Q} \cap \mathbb{Q}^c} = \overline{\emptyset} = \emptyset \subset \mathbb{R} = \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \overline{A} \cap \overline{B}.$$

Moreover,

$$(A \cap B)' = (\mathbb{Q} \cap \mathbb{Q}^c)' = \emptyset' = \emptyset \subset \mathbb{R} = \mathbb{Q}' \cap (\mathbb{Q}^c)' = A' \cap B'.$$

Finally,

$$(A \cup B)^\circ = (\mathbb{Q} \cup \mathbb{Q}^c)^\circ = \mathbb{R}^\circ = \mathbb{R} \supset \emptyset = \mathbb{Q}^\circ \cup (\mathbb{Q}^c)^\circ = A^\circ \cup B^\circ.$$

Lemma 4.3  Assume that (X, d) is a metric space and $A, B \subseteq X$. Then the following statements hold.

1. $\partial(\overline{A}) \subseteq \partial(A)$;
2. $\partial(A^\circ) \subseteq \partial(A)$;
3. $\partial(A \cap B) \subseteq \partial(A) \cup \partial(B)$;
4. $\partial(A \cup B) \subseteq \partial(A) \cup \partial(B)$.

Proof 4.21 For the first statement, note that

$$\partial(\overline{A}) = \overline{\overline{A}} \cap \overline{(\overline{A})^c} = \overline{A} \cap \overline{(\overline{A})^c} = \overline{A} \cap \overline{(A^c)^\circ} \subseteq \overline{A} \cap \overline{A^c} = \partial(A).$$

To prove (2), observe that

$$\partial(A^\circ) = \overline{A^\circ} \cap \overline{(A^\circ)^c} = \overline{A^\circ} \cap \overline{\overline{A^c}} = \overline{A^\circ} \cap \overline{A^c} \subseteq \overline{A} \cap \overline{A^c} = \partial(A).$$

Now we prove (3). Using basic properties of closures and complements, we have

$$\begin{aligned} \partial(A \cap B) &= \overline{A \cap B} \cap \overline{(A \cap B)^c} \subseteq \overline{A} \cap \overline{B} \cap \overline{(A^c \cup B^c)} \\ &= \overline{A} \cap \overline{B} \cap \overline{(A^c \cup B^c)} = (\overline{A} \cap \overline{B} \cap \overline{A^c}) \cup (\overline{A} \cap \overline{B} \cap \overline{B^c}) \\ &= (\partial(A) \cap \overline{B}) \cup (\overline{A} \cap \partial(B)) \subseteq \partial(A) \cup \partial(B). \end{aligned}$$

Finally, for (4), note that

$$\begin{aligned} \partial(A \cup B) &= \overline{A \cup B} \cap \overline{(A \cup B)^c} = (\overline{A} \cup \overline{B}) \cap \overline{A^c \cap B^c} \\ &\subseteq (\overline{A} \cup \overline{B}) \cap (\overline{A^c} \cap \overline{B^c}) = (\overline{A} \cap \overline{A^c} \cap \overline{B^c}) \cup (\overline{B} \cap \overline{A^c} \cap \overline{B^c}) \\ &= (\partial(A) \cap \overline{B^c}) \cup (\overline{A^c} \cap \partial(B)) \subseteq \partial(A) \cup \partial(B). \end{aligned}$$

Remark 4.2 It may appear that the inclusions in [Lemma 4.3](#) are equalities; however, this is not the case, even in Euclidean spaces. Consider $X = \mathbb{R}$ with the Euclidean metric and let $A = \mathbb{Q}$. Using basic properties, we have

$$\partial(A) = \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}.$$

However,

$$\partial(\overline{A}) = \partial(\overline{\mathbb{Q}}) = \partial(\mathbb{R}) = \mathbb{R} \cap \emptyset = \emptyset.$$

On the other hand,

$$\partial(A^\circ) = \partial(\mathbb{Q}^\circ) = \partial(\emptyset) = \emptyset \subset \mathbb{R} = \partial(A).$$

Furthermore, let $B = \mathbb{Q}^c$. Then

$$\partial(A \cap B) = \partial(\emptyset) = \emptyset \subset \mathbb{R} = \partial(A) \cup \partial(B),$$

and

$$\partial(A \cup B) = \partial(\mathbb{R}) = \emptyset \subset \mathbb{R} = \partial(A) \cup \partial(B).$$

We now present the following significant result.

Theorem 4.14 Assume that (X, d) is a metric space and $A \subseteq X$. Then

$$\partial(\partial(\partial(A))) = \partial(\partial(A)).$$

Proof 4.22 Since $\partial(\partial(A))$ is closed, a direct application of [Theorem 4.12](#) shows that

$$\partial(\partial(\partial(A))) \subseteq \partial(\partial(A)).$$

We now prove the reverse inclusion. First, observe that

$$\partial(\partial(A)) = \overline{\partial(A)} \cap \overline{(\partial(A))^c} = \partial(A) \cap \overline{(\partial(A))^c}.$$

Furthermore,

$$\partial(\partial(\partial(A))) = \partial(\partial(A)) \cap \overline{(\partial(\partial(A)))^c}.$$

On the other hand,

$$\overline{(\partial(\partial(A)))^c} = \overline{(\partial(A) \cap \overline{(\partial(A))^c})^c} = \overline{(\partial(A))^c \cup \overline{(\partial(A))^\circ}},$$

we obtain

$$\partial(\partial(\partial(A))) = (\partial(A) \cap \overline{(\partial(A))^c}) \cap (\overline{(\partial(A))^c} \cup \overline{(\partial(A))^\circ}).$$

Thus,

$$\partial(\partial(\partial(A))) = (\partial(A) \cap \overline{(\partial(A))^c}) \cup (\overline{(\partial(A))^c} \cap \overline{(\partial(A))^\circ}) \supseteq \partial(\partial(A)).$$

This completes the proof.

Definition 4.9 Assume that (X, d) is a metric space and $A \subseteq X$. We say that $x \in A$ is an isolated point of A provided that $x \notin A'$. In other words, $x \in X$ is an isolated point if there exists $r_0 > 0$ such that

$$N_{r_0}^{(x)} \cap A = \{x\}.$$

Let us explain this concept further with an illustrative example.

Example 4.15 We have seen in [Example 4.9](#) that in $\mathbb{R}$ with the Euclidean topology, for the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, we have $A' = \{0\}$, so that every element of A is an isolated point. Finally, observe that the discrete space yields a non-surprising result in this context.

Proposition 4.3 Assume that (X, d) is a discrete space and $A \subseteq X$. Then every element of A is isolated.

Proof 4.23 The proof is easy and straightforward. As seen earlier, we have $A' = \emptyset$, so all elements of A must be isolated.

Now, let us consider subspaces; in other words, we are going to generate many (similar) metric spaces from a given metric space.

Definition 4.10 Assume that (X, d) is a metric space and $Y \subseteq X$. If we restrict the map d to the elements of Y , we do not lose any of the metric properties. In other words, the restriction map $d' : Y \times Y \rightarrow \mathbb{R}$ defined by $d'(x, y) = d(x, y)$ for each $x, y \in Y$ is again a metric. To avoid any confusion, we denote d' by d again and call it the induced metric on Y by (X, d) . We also call the metric space (Y, d) the induced subspace of (X, d) .

With this definition, for any metric space, we are able to generate many subspaces (which are metric spaces in their own right). Let us give an illustrative example.

Example 4.16 Take $X = \mathbb{R}$ with the Euclidean metric and let $Y = [0, 1)$. Then Y is a metric space with the induced Euclidean metric. Put $H = [0, \frac{1}{2})$. Clearly, H is neither open nor closed in X . However, we show that it is open in Y . This demonstrates that open sets may differ between an ambient space and its subspaces.

To prove this, the only subtle point is explaining why 0 is an interior point of H in Y . Note that neighborhoods are computed relative to Y . We have

$$(N_{\frac{1}{4}}^{(0)})_Y = \{x \in Y : |x - 0| < \frac{1}{4}\} = (-\frac{1}{4}, \frac{1}{4}) \cap Y = [0, \frac{1}{4}) \subseteq H.$$

Furthermore, let $F = [\frac{1}{2}, 1)$. Again, F is neither open nor closed in X . However, it is closed in Y . To see this, observe that the only unusual case to verify is whether 1 is a cluster point of F in Y . Since $1 \notin Y$, it cannot belong to the closure of F with respect to Y . Hence, F is closed in Y .

Thus, the main question that naturally arises is the following:

Question. Is there any relation between open (or closed) subsets of an ambient space and those of its subspaces?

The following result provides a satisfactory answer.

Theorem 4.15 $\triangleleft$ Assume that (X, d) is a metric space and (Y, d) is an induced subspace of it. Then a set $H \subseteq Y$ is open with respect to Y if and only if there exists an open set $G \subseteq X$ such that $H = G \cap Y$.

Proof 4.24 Assume that $H \subseteq Y$ is open with respect to Y . Then for each $h \in H$, there exists $r_h > 0$ such that $(N_{r_h}^{(h)})_Y \subseteq H$. But

$$(N_{r_h}^{(h)})_Y = \{x \in Y : d(x, h) < r_h\} = (N_{r_h}^{(h)})_X \cap Y.$$

Define

$$G = \bigcup_{h \in H} (N_{r_h}^{(h)})_X.$$

Clearly, G is open in X . Moreover,

$$H = H \cap Y = \left(\bigcup_{h \in H} \{h\} \right) \cap Y \subseteq \left(\bigcup_{h \in H} (N_{r_h}^{(h)})_X \right) \cap Y = G \cap Y = \bigcup_{h \in H} ((N_{r_h}^{(h)})_Y) \subseteq H.$$

Therefore, $H = G \cap Y$.

Conversely, assume that there exists an open set $G \subseteq X$ such that $H = G \cap Y$. For each $h \in H$, since G is open in X and $h \in G$, there exists $r > 0$ such that $(N_r^{(h)})_X \subseteq G$. Hence,

$$(N_r^{(h)})_Y = (N_r^{(h)})_X \cap Y \subseteq G \cap Y = H.$$

Thus, h is an interior point of H , and so H is open with respect to Y .

We now characterize closed sets in the induced topology as well.

Corollary 4.3 $\triangleleft$ Assume that (X, d) is a metric space and (Y, d) is an induced subspace of it. Then a set $E \subseteq Y$ is closed with respect to Y if and only if there exists a closed set $F \subseteq X$ such that $E = F \cap Y$.

Proof 4.25 Assume that $E \subseteq Y$ is closed with respect to Y . Then E_Y^c is open in Y , where E_Y^c denotes the complement of E relative to Y . By [Theorem 4.15](#), there exists an open set $G \subseteq X$ such that $E_Y^c = G \cap Y$. Thus,

$$E = (E_Y^c)^c = G^c \cap Y.$$

Conversely, assume that $E = F \cap Y$ for some closed set $F \subseteq X$. Then

$$E_Y^c = F^c \cap Y.$$

Since F^c is open in X , [Theorem 4.15](#) implies that E_Y^c is open in Y . Therefore, E is closed in Y .

Next, we establish a result describing the relationship between the interior and the closure of a set in a subspace and in the ambient metric space.

Proposition 4.4 Assume that (X, d) is a metric space, $Y \subseteq X$, and $A \subseteq Y$. Then the following statements hold.

1. $A^\circ = (A^\circ)_Y \cap Y^\circ$;
2. $(\bar{A})_Y = \bar{A} \cap Y$.

Proof 4.26 (1) Take $x \in A^\circ$. Then there exists an open set $U \subseteq X$ such that $x \in U \subseteq A$. Since $A \subseteq Y$, we have $x \in U \cap Y \subseteq A$. By [Theorem 4.15](#), the set $U \cap Y$ is open in Y , and hence $x \in (A^\circ)_Y$. Moreover, since $A \subseteq Y$, we have $A^\circ \subseteq Y^\circ$. Therefore,

$$A^\circ \subseteq (A^\circ)_Y \cap Y^\circ.$$

Conversely, let $x \in (A^\circ)_Y \cap Y^\circ$. Then there exists an open set $G \subseteq X$ such that $x \in G \subseteq Y$. Also, there exists an open set V in Y such that $x \in V \subseteq A$. By [Theorem 4.15](#), there exists an open set $V_1 \subseteq X$ with $V = V_1 \cap Y$. Hence, $G \cap V_1$ is open in X and satisfies $x \in G \cap V_1 \subseteq A$, which completes the proof of (1).

(2) Take any $x \in (\bar{A})_Y$. Let $G \subseteq X$ be any open set containing x . By [Theorem 4.15](#), the set $H = G \cap Y$ is open in Y and contains x . Since $x \in (\bar{A})_Y$, we have $H \cap A \neq \emptyset$, and hence $G \cap A \neq \emptyset$. Thus, $x \in \bar{A} \cap Y$.

Conversely, let $x \in \bar{A} \cap Y$, and let $H \subseteq Y$ be an open set containing x . By [Theorem 4.15](#), there exists an open set $G \subseteq X$ such that $H = G \cap Y$. Since $x \in G$ and $x \in \bar{A}$, we have $G \cap A \neq \emptyset$. Therefore, $H \cap A \neq \emptyset$, which shows that $x \in (\bar{A})_Y$.

Definition 4.11 Assume that (X, d) is a metric space. We say that it is separable provided that there exists a countable set $A \subseteq X$ such that $\bar{A} = X$.

Note that, for example, $\mathbb{R}$ with the Euclidean topology is separable since $\bar{\mathbb{Q}} = \mathbb{R}$ (see [Exercises](#)). Furthermore, $\mathbb{R}$ with the discrete topology is not separable, since in this case, X would have to be countable, which is impossible.

In general, we have the following observation:

Proposition 4.5 Assume that (X, d) is a discrete space. Then X is separable if and only if it is countable.

Proof 4.27 The proof essentially relies on the fact that in a discrete space, the closure of every subset is itself (since every subset is closed).

Furthermore, we have the following simple observation.

Corollary 4.4 Suppose $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ with the Euclidean topology. Then X is separable.

Definition 4.12 Assume that (X, d) is a metric space and $A \subseteq X$. We say that A is a G_δ set provided that we can write

$$A = \bigcap_{n=1}^{\infty} G_n,$$

where each $G_n \subseteq X$ is open; in other words, A can be written as a countable intersection of open sets.

Furthermore, we say that A is an F_σ set if

$$A = \bigcup_{n=1}^{\infty} F_n,$$

where each $F_n \subseteq X$ is closed; that is, A can be written as a countable union of closed sets.

Remark 4.3 Clearly, every open set is a G_δ set, and every closed set is an F_σ set. However, observe that a G_δ set need not be open, and an F_σ set need not be closed in general.

For example, consider $X = \mathbb{R}$ with the Euclidean topology. For each $n \in \mathbb{N}$, let

$$G_n = \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Clearly, each G_n is open. Nevertheless,

$$\bigcap_{n=1}^{\infty} G_n = \{0\},$$

which is not open.

Similarly, in every metric space, singletons are closed. Again, consider $\mathbb{R}$ with the Euclidean topology. We can write

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\},$$

where each r_n is a rational number. Each $\{r_n\}$ is closed, but their union, $\mathbb{Q}$, is not closed.

Definition 4.13 Assume that (X, d) is a metric space and $\emptyset \neq A \subseteq X$. We say that A is bounded provided that its diameter (denoted by $\text{diam}(A)$ or $d(A)$) is finite, where

$$\text{diam}(A) = d(A) = \sup \{d(x, y) : x, y \in A\}.$$

In other words, a subset of a metric space is bounded if the distance between its farthest points is finite. By convention, we also consider $\emptyset$ to be bounded and set $\text{diam}(\emptyset) = 0$.

Example 4.17 In $\mathbb{R}$ with the Euclidean metric, the interval $A = (0, 1)$ is bounded, and we have $\text{diam}(A) = 1$. However, $\mathbb{R}$ itself is not bounded.

On the other hand, if we consider $\mathbb{R}$ with the discrete metric, then $\mathbb{R}$ is bounded and $\text{diam}(\mathbb{R}) = 1$. In fact, in any discrete space, every subset is bounded.

Theorem 4.16 Assume that (X, d) is a metric space and $A \subseteq X$. Then,

$$\text{diam}(A) = \text{diam}(\overline{A}).$$

Proof 4.28 Let $x, y \in \overline{A}$ and fix $r > 0$. By the definition of a cluster point, we have

$$N_{\frac{r}{2}}^{(x)} \cap A \neq \emptyset, \quad N_{\frac{r}{2}}^{(y)} \cap A \neq \emptyset.$$

Therefore, we can find $x', y' \in A$ such that

$$d(x, x') < \frac{r}{2}, \quad d(y, y') < \frac{r}{2}.$$

Hence,

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) \leq \frac{r}{2} + \text{diam}(A) + \frac{r}{2} = r + \text{diam}(A),$$

so $r + \text{diam}(A)$ is an upper bound for the set $\{d(x, y) : x, y \in \overline{A}\}$. Therefore,

$$\text{diam}(\overline{A}) \leq \text{diam}(A) + r.$$

Since $r > 0$ is arbitrary, we conclude that $\text{diam}(\overline{A}) \leq \text{diam}(A)$. On the other hand, since $A \subseteq \overline{A}$, it is clear that $\text{diam}(A) \leq \text{diam}(\overline{A})$. Hence, $\text{diam}(A) = \text{diam}(\overline{A})$.

Definition 4.14 Suppose (X, d) is a metric space, $A, B \subseteq X$, and $x \in X$. The distance between x and A is defined as

$$d(x, A) = \inf \{d(x, a) : a \in A\}.$$

Moreover, the distance between two sets A and B is defined by

$$d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}.$$

By convention, we set $d(x, \emptyset) = 0$.

Intuitively, the distance between a point and a set in a metric space can be interpreted as the least possible distance from the point to any element of the set. Similarly, the distance between two sets is the least distance between any pair of elements from the two sets. Note also that if $x \in A$, then $d(x, A) = 0$. Furthermore, $d(A, A) = 0$.

Now, we illustrate this notion with an example.

Example 4.18 Assume that $X = \mathbb{R}$ with the Euclidean metric. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, $d(0, A) = 0$, although $0 \notin A$.

Moreover, let $B = \{2, 3, \dots\}$ and $C = \{n + \frac{1}{n} : n \in \mathbb{N}\}$. Then $d(B, C) = 0$, even though $B \cap C = \emptyset$ and both B and C are closed subsets of $\mathbb{R}$.

Remark 4.4 Assume that (X, d) is a metric space. Define a real-valued function d' on the Cartesian product $P(X) \times P(X)$ via

$$d'(A, B) = d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}.$$

Indeed, $d'(A, B) \geq 0$, $d'(A, A) = 0$, and $d'(A, B) = d'(B, A)$. However, the triangle inequality fails. For example, in $\mathbb{R}$ with the Euclidean metric, let $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2\}$. Then

$$d'(A, B) = 1 \neq 0 + 0 = d(A, C) + d(B, C),$$

so d' does not form a metric on $P(X)$. Nevertheless, with consideration of the farthest distance (in some reasonable sense), the triangle inequality can hold, as we shall see next.

Definition 4.15 Assume that (X, d) is a metric space and $\emptyset \neq A \subseteq X$ is bounded. For each $x \in X$, the farthest distance between x and A is denoted by $D(x, A)$ and defined as

$$D(x, A) = \sup \{d(x, a) : a \in A\}.$$

Moreover, for two non-empty bounded sets A and B , the farthest distance between them is

$$D(A, B) = \sup \{d(a, b) : a \in A, b \in B\}.$$

Note that boundedness is necessary for these definitions since it guarantees the existence of the supremums. This notion generalizes the diameter of a set, as we have $D(A, A) = \text{diam}(A)$.

Clearly, $D(A, B) \geq 0$ and $D(A, B) = D(B, A)$. However, $D(A, A) = \text{diam}(A) > 0$ unless A is a singleton. Nevertheless, D satisfies the triangle inequality, as shown next.

Proposition 4.6 Let (X, d) be a metric space and A, B, C be non-empty bounded subsets of X . Then

$$D(A, B) \leq D(A, C) + D(B, C).$$

Proof 4.29 Let $a \in A$, $b \in B$, and $c \in C$ be arbitrary elements. Then

$$d(a, b) \leq d(a, c) + d(c, b) \leq D(A, C) + D(B, C),$$

so $D(A, C) + D(B, C)$ is an upper bound for $\{d(a, b) : a \in A, b \in B\}$. Hence, $D(A, B) \leq D(A, C) + D(B, C)$.

Combining the notions D and d for a metric space (X, d) , we obtain a new concept:

Definition 4.16 Assume that (X, d) is a metric space and $A, B \subseteq X$ are non-empty bounded sets. The eccentricity of B with respect to A is denoted by $\tau(A, B)$ and defined as

$$\tau(A, B) = \sup \{d(a, B) : a \in A\}.$$

We now observe some properties of τ :

- $\tau(A, B) \geq 0$.
- $\tau(A, A) = \sup \{d(a, A) : a \in A\} = 0$.

However, from $\tau(A, B) = 0$ we cannot conclude that $A = B$. For example, in $\mathbb{R}$ with the Euclidean metric, let

$$A = \mathbb{Q} \cap [0, 1], \quad B = \mathbb{Q}^c \cap [0, 1].$$

Then

$$\tau(A, B) = \sup \{d(a, B) : a \in A\} = \sup \{0 : a \in \mathbb{Q} \cap [0, 1]\} = 0,$$

even though $A \cap B = \emptyset$.

Moreover, τ is not symmetric. For instance, let $A = \{1, 2\}$ and $B = \{3, 5\}$. Then

$$\tau(A, B) = \sup \{d(a, B) : a \in A\} = 2 \neq 3 = \tau(B, A).$$

Nevertheless, τ satisfies the triangle inequality (which we shall show shortly). As we have seen, these notions defined from a metric space (X, d) — namely, the distance between sets, D , and τ — do not form a metric on $P(X)$ (or its bounded subsets), although they share several metric-like properties.

Interestingly, even though d' , D , and τ do not define metrics, they do induce topologies on $P(X)$ or on the collection of bounded subsets of X . We will explore this after proving the triangle inequality for τ .

Definition 4.17 Assume that (X, d) is a metric space and $\emptyset \neq A \subseteq X$. An open neighborhood centered at A with radius r (denoted by $N_r^{(A)}$) is defined as

$$N_r^{(A)} = \{x \in X : d(x, A) < r\}.$$

Moreover, the closed neighborhood (denoted by $N_r^{[A]}$) is defined as

$$N_r^{[A]} = \{x \in X : d(x, A) \leq r\}.$$

The following lemma will be useful for our purposes.

Lemma 4.4  Assume that (X, d) is a metric space and τ is the eccentricity map associated with it. Then the following assertions hold:

1. $\tau(A, B) < r \Rightarrow A \subseteq N_r^{(B)}$;
2. $A \subseteq N_r^{(B)} \Rightarrow \tau(A, B) \leq r$.

Proof 4.30 (1). Assume that $\tau(A, B) < r$ and let $a \in A$ be arbitrary. Since $\tau(A, B) = \sup \{d(a, B) : a \in A\}$, we have $d(a, B) \leq \tau(A, B) < r$, so $a \in N_r^{(B)}$.

(2). Let $A \subseteq N_r^{(B)}$. Then for each $a \in A$, we have $d(a, B) < r$. Therefore, r is an upper bound for $\{d(a, B) : a \in A\}$, so $\tau(A, B) \leq r$.

Now, we are ready to prove the triangle inequality for the eccentricity map.

Proposition 4.7 Assume that (X, d) is a metric space and τ is the eccentricity map on bounded subsets of X . Then τ satisfies the triangle inequality.

Proof 4.31 Let $A, B, C \subseteq X$ be non-empty and bounded, and suppose that $\tau(A, C) < r$ and $\tau(C, B) < s$ for arbitrary positive r and s . By [Lemma 4.4](#), we have $A \subseteq N_r^{(C)}$ and $C \subseteq N_s^{(B)}$. We claim that $A \subseteq N_{r+s}^{(B)}$, so by [Lemma 4.4](#), $\tau(A, B) \leq r + s$. Letting $r \rightarrow \tau(A, C)$ and $s \rightarrow \tau(C, B)$ gives the desired result.

To prove the claim, take $a \in A$. Then $d(a, C) = \inf \{d(a, c) : c \in C\} < r$, so we can find $c_0 \in C$ with $d(a, c_0) < r$. Furthermore, since $\tau(C, B) = \sup \{d(c, B) : c \in C\} < s$, there exists

$b_0 \in B$ such that $d(c_0, b_0) < s$. Therefore,

$$d(a, B) \leq d(a, b_0) \leq d(a, c_0) + d(c_0, b_0) < r + s,$$

so $a \in N_{r+s}^{(B)}$, proving the claim.

Now let us explore a topology induced by the eccentricity map.

Theorem 4.17 Assume that (X, d) is a metric space and define

$$\mathbf{B} = \{N_r^{(A)} : A \in P(X), r > 0\}.$$

Then $\mathbf{B}$ forms a basis for a topology on X , which we call the distance topology.

Proof 4.32 To show that $\mathbf{B}$ is a basis, first note that

$$X = \bigcup_{B \in \mathbf{B}} B,$$

because $X \in P(X)$ and

$$N_r^{(X)} = \{x \in X : d(x, X) < r\} = X.$$

Next, let $N_r^{(A)}, N_s^{(B)} \in \mathbf{B}$. We need to find $N_t^{(C)} \in \mathbf{B}$ such that

$$N_t^{(C)} \subseteq N_r^{(A)} \cap N_s^{(B)}.$$

Without loss of generality, assume $r \leq s$, so $N_r^{(B)} \subseteq N_s^{(B)}$.

If $A \cap B = \emptyset$, then $N_r^{(\emptyset)} = \emptyset \subseteq N_r^{(A)} \cap N_r^{(B)}$. Otherwise, $N_r^{(A \cap B)} \subseteq N_r^{(A)} \cap N_r^{(B)}$, as desired.

To see this, note that for any $z \in X$,

$$\{d(z, t) : t \in A \cap B\} \subseteq \{d(z, t) : t \in A\} \cap \{d(z, t) : t \in B\},$$

so $N_r^{(A \cap B)} \subseteq N_r^{(A)} \cap N_r^{(B)}$.

Theorem 4.18 Assume that (X, d) is a metric space,

$$Y = \{A \subseteq X : A \text{ is bounded}\},$$

and for $A \in Y$ and $r > 0$, define

$$B_r^{(A)} = \{x \in X : D(x, A) < r\}.$$

Then

$$\mathbf{B} = \{B_r^{(A)} : A \in Y, r > 0\}$$

forms a basis for a topology on bounded subsets of X , which we call the farthest distance topology.

Proof 4.33 Recall that $D(x, A) = \sup \{d(x, a) : a \in A\}$.

First, we show that

$$X = \bigcup_{A \in Y, r > 0} B_r^{(A)}.$$

Indeed, $\bigcup_{A \in Y, r > 0} B_r^{(A)} \subseteq X$. Conversely, for any $x \in X$, the singleton $\{x\}$ is bounded and clearly $x \in B_r^{(\{x\})}$ for any $r > 0$.

Next, let $B_r^{(A)}$ and $B_s^{(C)}$ be arbitrary elements of $\mathbf{B}$. Without loss of generality, assume $r = s$ (if $r < s$, then $B_r^{(C)} \subseteq B_s^{(C)}$).

We claim that

$$B_r^{(A \cup C)} \subseteq B_r^{(A)} \cap B_r^{(C)}.$$

To see this, take any $z \in B_r^{(A \cup C)}$. Then

$$D(z, A) = \sup \{d(z, a) : a \in A\} \leq \sup \{d(z, a) : a \in A \cup C\} < r,$$

and similarly $D(z, C) < r$, so $z \in B_r^{(A)} \cap B_r^{(C)}$. This proves the claim, completing the proof that $\mathbf{B}$ is a basis.

Theorem 4.19 $\blacksquare$ Suppose (X, d) is a metric space and $A \subseteq X$. Then for each $r > 0$,

$$N_r^{(A)} \text{ is open, and } N_r^{[A]} \text{ is closed.}$$

Proof 4.34 Let $z \in N_r^{(A)}$. We show that z is an interior point. Let $\varepsilon > 0$ be arbitrary. By definition of the infimum, there exists $a_0 \in A$ such that

$$d(z, a_0) < d(z, A) + \varepsilon.$$

Put $r_0 = \frac{1}{2}(r - d(z, A)) > 0$. We claim that

$$N_{r_0}^{(z)} \subseteq N_r^{(A)}.$$

Take any $y \in N_{r_0}^{(z)}$. Then

$$d(y, A) \leq d(y, a_0) \leq d(y, z) + d(z, a_0) < r_0 + d(z, A) + \varepsilon =$$

$$\frac{r}{2} - \frac{d(z, A)}{2} + d(z, A) + \varepsilon = \frac{r + d(z, A)}{2} + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ gives

$$d(y, A) \leq \frac{r + d(z, A)}{2} < r,$$

so $y \in N_r^{(A)}$. Hence $N_r^{(A)}$ is open.

Now, we show that $N_r^{(A)}$ is closed. Let $z \notin N_r^{[A]}$. Then $d(z, A) > r$. Put $r_0 = \frac{1}{2}(d(z, A) - r) > 0$. We claim that

$$N_{r_0}^{(z)} \cap N_r^{[A]} = \emptyset.$$

Suppose, for a contradiction, there exists $y \in N_{r_0}^{(z)} \cap N_r^{[A]}$. Then $d(z, y) < r_0$ and $d(y, A) \leq r$. By the definition of the infimum, choose $a_0 \in A$ such that $d(y, a_0) < r + \varepsilon$ for arbitrary small $\varepsilon > 0$. Then

$$d(z, A) \leq d(z, a_0) \leq d(z, y) + d(y, a_0) < r_0 + r + \varepsilon =$$

$$\frac{d(z, A) - r}{2} + r + \varepsilon = \frac{d(z, A) + r}{2} + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ yields $d(z, A) \leq \frac{d(z, A) + r}{2} \Rightarrow d(z, A) \leq r$, a contradiction. Thus $N_r^{[A]}$ is closed.

We have a similar result for the farthest distance topology on bounded sets of X .

Theorem 4.20 Suppose (X, d) is a metric space and $A \subseteq X$ is bounded. Then for each $r > 0$, $B_r^{(A)}$ is an open set and $B_r^{[A]}$ is a closed set.

Proof 4.35 Let $z \in B_r^{(A)}$. We need to show that z is an interior point. Let an arbitrary $\varepsilon > 0$ be given. By the definition of the supremum of a set ($D(z, A) = \sup \{d(z, a) : a \in A\}$), we can find $a_0 \in A$ with $D(z, A) - \varepsilon < d(z, a_0)$. Put $r_0 = \frac{1}{2}(r - D(z, A))$. We claim that $N_{r_0}^{(z)} \subseteq B_r^{(A)}$. Take any $y \in N_{r_0}^{(z)}$. We have

$$D(y, A) - \varepsilon \leq d(y, a_0) \leq d(y, z) + d(z, a_0) < r_0 + D(z, A) = \frac{r}{2} - \frac{D(z, A)}{2} + D(z, A) < r,$$

and letting $\varepsilon \rightarrow 0$, we get $D(y, A) < r$.

To show that $B_r^{[A]}$ is closed, it is enough to show that if $z \notin B_r^{[A]}$, then $z \notin \overline{B_r^{[A]}}$. Since $z \notin B_r^{[A]}$, we have $D(z, A) > r$. Put $r_0 = \frac{1}{2}(D(z, A) - r) > 0$. We claim that $N_{r_0}^{(z)} \cap B_r^{[A]} = \emptyset$. Suppose, for a contradiction, that there exists $y \in N_{r_0}^{(z)} \cap B_r^{[A]}$. By the definition of the supremum, we can find $a_0 \in A$ such that $D(z, A) - \varepsilon < d(z, a_0)$. Then

$$D(z, A) \leq d(z, a_0) + \varepsilon \leq d(z, y) + d(y, a_0) + \varepsilon < r_0 + r + \varepsilon =$$

$$\frac{1}{2}(D(z, A) - r) + r + \varepsilon = \frac{1}{2}(D(z, A) + r) + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ gives $D(z, A) \leq \frac{1}{2}(D(z, A) + r)$, so $D(z, A) \leq r$, a contradiction.

Now, we can present an application of the previous results.

Theorem 4.21 Assume that (X, d) is a metric space and $F \subseteq X$ is closed. Then F is a G_δ -set.

Proof 4.36 For each $n \in \mathbb{N}$, put $G_n = N_{\frac{1}{n}}^{(F)}$. By [Theorem 4.19](#), each G_n is open. We claim that $F = \bigcap_{n \in \mathbb{N}} G_n$. Since $F \subseteq G_n$ for each n , we have $F \subseteq \bigcap_{n \in \mathbb{N}} G_n$.

Conversely, assume $x \in \bigcap_{n \in \mathbb{N}} G_n$, so that $d(x, F) < \frac{1}{n}$ and $d(x, F) \rightarrow 0$. We show that $x \in \overline{F}$, so $x \in F$ since F is closed. Given $r > 0$, by the definition of the infimum, we can find $y \in F$ with $d(x, y) < r$, i.e., $y \in N_r^{(x)}$. Hence, $N_r^{(x)} \cap F \neq \emptyset$.

Definition 4.18 Assume that (X, d) is a metric space and $A \subseteq X$. We say that $x \in X$ is a condensation point of A provided that for each $r > 0$, $N_r^{(x)} \cap A$ is uncountable.

Obviously, by this definition, every condensation point is a limit point (cluster point). Countable metric spaces cannot have condensation points. Moreover, in discrete spaces, we cannot have condensation points since no set has limit points.

Example 4.19 Assume $X = \mathbb{R}$ with the Euclidean topology.

1. Let $A = (0, 1)$. Then the set of all condensation points of A is $[0, 1]$;
2. Let $B = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then the set of condensation points of B is empty since B is countable, although $B' = \{0\}$.

There are some significant results about condensation points in Euclidean spaces.

Theorem 4.22 In $X = \mathbb{R}$ with the Euclidean topology, every uncountable set has condensation points.

Proof 4.37 Suppose, for a contradiction, that a given uncountable set A has no condensation point. Then no rational number x is a condensation point of A .

So, for each $x \in \mathbb{Q}$, we can find $r_x > 0$ such that $N_{r_x}^{(x)} \cap A$ is countable. On the other hand, we have

$$\mathbb{Q} \subseteq \bigcup_{x \in \mathbb{Q}} N_{\frac{r_x}{2}}^{(x)}.$$

Thus,

$$\mathbb{R} = \overline{\mathbb{Q}} \subseteq \overline{\bigcup_{x \in \mathbb{Q}} N_{\frac{r_x}{2}}^{(x)}} \subseteq \bigcup_{x \in \mathbb{Q}} N_{r_x}^{(x)},$$

where the last inclusion holds in the Euclidean topology. Therefore,

$$A = \mathbb{R} \cap A = \left(\bigcup_{x \in \mathbb{Q}} N_{r_x}^{(x)} \right) \cap A = \bigcup_{x \in \mathbb{Q}} (N_{r_x}^{(x)} \cap A).$$

But the right-hand side is a countable union of countable sets, which is countable. This contradicts the fact that A is uncountable.

This result has several significant consequences.

Corollary 4.5 In $X = \mathbb{R}$ with the Euclidean topology, the set of all condensation points of an uncountable set is uncountable.

Proof 4.38 Let A be uncountable, and let B be the set of all condensation points of A . Suppose, for a contradiction, that B is countable. Then $A - B$ is uncountable and has no condensation point (since all condensation points of A are in B). This is impossible.

Moreover, we have another version of this result.

Corollary 4.6 In $X = \mathbb{R}$ with the Euclidean topology, every uncountable set contains uncountably many condensation points.

Proof 4.39 Let A be uncountable, and let B be the set of all condensation points of A . Suppose, for a contradiction, that B is countable. Then $A \cap B$ is countable, so $A - B$ is uncountable. But $A - B$ has no condensation point, which is impossible.

Definition 4.19 We say that a metric space (X, d) is a Lindelöf space provided that if there exists a family $(G_\alpha)_{\alpha \in I}$ of open sets of X such that $X = \bigcup_{\alpha \in I} G_\alpha$, then one can find a countable subfamily of indices, namely, $\{\alpha_n : n \in \mathbb{N}\}$, with

$$X = \bigcup_{n=1}^{\infty} G_{\alpha_n}.$$

We shall show (as a corollary of a theorem) that $\mathbb{R}^k$ with the Euclidean topology is a Lindelöf space. For a counterexample, discrete spaces are the best choice, particularly if they are uncountable.

Example 4.20 $\blacktriangleleft$ Assume that (X, d) is an arbitrary uncountable discrete metric space. Observe that we can write

$$X = \bigcup_{x \in X} N_1^{(x)} = \{\{x\}\}_{x \in X},$$

however, we are not able to write the equality for a countable subfamily, for sure.

Before we proceed further, we need to determine representatives of a topological space.

Proposition 4.8 A subset A of a metric space (X, d) is dense if and only if for any non-empty open set $G \subseteq X$, we have $A \cap G \neq \emptyset$.

Proof 4.40 Assume that A is dense in X , so that $\overline{A} = X$. Let G be an arbitrary non-empty open set in X . Since $G \neq \emptyset$, there exists $x \in G$. By hypothesis, $x \in \overline{A}$, so $A \cap G \neq \emptyset$.

Conversely, assume that for each non-empty open set $G \subseteq X$, $A \cap G \neq \emptyset$. We claim that $X \subseteq \overline{A}$. Take $x \in X$. By hypothesis, for any open set G containing x , $A \cap G \neq \emptyset$, so $x \in \overline{A}$.

Now, we provide a thorough examination of the notion of a “base” in a metric space. These statements complete and complement some of the results presented in [Chapter 2](#).

Definition 4.20 Assume that (X, d) is a metric space. A family $(G_\alpha)_{\alpha \in I}$ of open subsets of X is called a base for (X, d) if for each $x \in X$ and every open set $G \subseteq X$ with $x \in G$, we can find an index α such that

$$x \in G_\alpha \subseteq G.$$

In other words, $(G_\alpha)_{\alpha \in I}$ is a basis for (X, d) if every open set of X can be written as a union of elements of the family.

Proposition 4.9 Assume that (X, d) is a metric space. Then the family $\{N_r^{(x)} : x \in X, r > 0\}$ is a base for the metric topology on X .

Proof 4.41 Let $x \in X$ and let G be an arbitrary open set in X containing x . Since x is an interior point of G , there exists $r > 0$ with

$$N_r^{(x)} \subseteq G.$$

This shows that $\{N_r^{(x)} : x \in X, r > 0\}$ is a base for (X, d) .

When dealing with a special point, local information about the base may be sufficient. This motivates the following definition.

Definition 4.21 Assume that (X, d) is a metric space and $x \in X$. A family $(G_\alpha)_{\alpha \in I}$ of open subsets of X containing x is called a local base for (X, d) if for every open set $G \subseteq X$ with $x \in G$, we can find an index α such that

$$x \in G_\alpha \subseteq G.$$

Theorem 4.23 $\blacktriangleleft$ Assume that (X, d) is a metric space and $x \in X$. Then there exists a countable local base for x .

Proof 4.42 Consider the family $\{N_{\frac{1}{n}}^{(x)}\}_{n \in \mathbb{N}}$. Obviously, this family is countable and each element contains x . We show that it is a local base around x .

Let G be any open set in X containing x . Then there exists $r_0 > 0$ such that $N_{r_0}^{(x)} \subseteq G$. By the Archimedean property, there exists $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < r_0$. This implies that

$$N_{\frac{1}{n_0}}^{(x)} \subseteq N_{r_0}^{(x)} \subseteq G,$$

as claimed.

This property in a topological space is called the *first axiom of countability*. [Theorem 4.23](#) shows that any metric space satisfies the first axiom of countability.

Definition 4.22 Assume that (X, d) is a metric space. We say that it satisfies the second axiom of countability if it contains a countable base.

Example 4.21 To illustrate the difference between the first and second axioms of countability, consider any uncountable set with the discrete metric. Since it is a metric space, by [Theorem 4.23](#), it satisfies the first axiom of countability. However, being uncountable, by using the method of [Example 4.20](#), it cannot have a countable base.

Now, we are ready to state a somewhat surprising, somewhat predictable, and certainly very important result.

Theorem 4.24 For a metric space (X, d) , the following statements are equivalent:

1. X is separable;
2. X satisfies the second axiom of countability;
3. X is Lindelöf.

Proof 4.43 (1) $\rightarrow$ (2). Suppose X is separable, so there exists a countable set $A \subseteq X$ with $\overline{A} = X$. Consider the family

$$\{N_{\frac{1}{n}}^{(x)} : n \in \mathbb{N}, x \in A\}.$$

It is countable since both A and $\mathbb{N}$ are countable. We show that this family is a base for the metric topology.

Take any $x \in X$ and any open set $G \subseteq X$ with $x \in G$. Since G is open, x is an interior point of G , so there exists $r_0 > 0$ with $N_{r_0}^{(x)} \subseteq G$. By the Archimedean property, choose $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < r_0$. Then

$$N_{\frac{1}{n_0}}^{(x)} \subseteq N_{r_0}^{(x)} \subseteq G.$$

Since $\overline{A} = X$, we have $N_{\frac{1}{2n_0}}^{(x)} \cap A \neq \emptyset$. So, there exists $a_0 \in A$ with $a_0 \in N_{\frac{1}{2n_0}}^{(x)}$. We claim that

$$N_{\frac{1}{2n_0}}^{(a_0)} \subseteq G.$$

Take any $y \in N_{\frac{1}{2n_0}}^{(a_0)}$. Then

$$d(x, y) \leq d(x, a_0) + d(a_0, y) < \frac{1}{2n_0} + \frac{1}{2n_0} = \frac{1}{n_0},$$

showing that $y \in N_{\frac{1}{n_0}}^{(x)} \subseteq G$. Therefore,

$$x \in N_{\frac{1}{2n_0}}^{(a_0)} \subseteq N_{\frac{1}{n_0}}^{(x)} \subseteq G,$$

so the family qualifies as a base.

(2) $\rightarrow$ (3). Assume that X has a countable base $\{G_n\}_{n \in \mathbb{N}}$. Let $\{H_\alpha\}_{\alpha \in I}$ be a family of open sets with $X = \bigcup_{\alpha \in I} H_\alpha$. For each $x \in X$, there exists α_x with $x \in H_{\alpha_x}$. Since $\{G_n\}$ is a base, we can find $n_x \in \mathbb{N}$ with

$$x \in G_{n_x} \subseteq H_{\alpha_x}.$$

Thus,

$$X = \bigcup_{x \in X} G_{n_x} \subseteq \bigcup_{x \in X} H_{\alpha_x}.$$

The family $\{G_{n_x}\}_{x \in X}$ is countable since G_{n_x} repeats for several x . For each n_x , choose (by the axiom of choice) an index α_{n_x} with $G_{n_x} \subseteq H_{\alpha_{n_x}}$. Then

$$X = \bigcup_{x \in X} G_{n_x} \subseteq \bigcup_{n_x} H_{\alpha_{n_x}},$$

showing that X is Lindelöf.

(3) $\rightarrow$ (1). Assume X is Lindelöf. Consider the family $\{N_{\frac{1}{n}}^{(x)} : x \in X\}$ for each $n \in \mathbb{N}$ so that

$$X = \bigcup_{x \in X} N_{\frac{1}{n}}^{(x)}.$$

By the Lindelöf property, there exists a countable subfamily $\{N_{\frac{1}{n}}^{(x_{n,m})}\}_{m \in \mathbb{N}}$ with

$$X = \bigcup_{m=1}^{\infty} N_{\frac{1}{n}}^{(x_{n,m})}.$$

Put

$$A = \{x_{n,m} : n, m \in \mathbb{N}\}.$$

We claim that A is dense in X . For a given $r > 0$, choose $n \in \mathbb{N}$ with $\frac{1}{n} < r$. Since $x \in X = \bigcup_{m=1}^{\infty} N_{\frac{1}{n}}^{(x_{n,m})}$, there exists $m \in \mathbb{N}$ with $x \in N_{\frac{1}{n}}^{(x_{n,m})}$, so

$$x_{n,m} \in A \cap N_{\frac{1}{n}}^{(x)} \subseteq A \cap N_r^{(x)},$$

showing that $N_r^{(x)} \cap A \neq \emptyset$. Therefore, A is dense, proving separability.

Now, we are going to compare different metrics on a set.

Definition 4.23 $\hookleftarrow$ Assume $X \neq \emptyset$ and d_1, d_2 are metrics on X . We say d_1 and d_2 are equivalent if there exist positive constants α, β such that for all $x, y \in X$,

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

Remark 4.5 [Definition 4.23](#) shows that two equivalent metrics produce very similar neighborhoods; in other words, they generate the same topology on X . This implies they also have the same convergence.

Take any $x \in X$ and $r > 0$. Put $r_0 = \alpha r$ and $r_1 = \frac{r}{\beta}$. Then

$$(N_{r_1}^{(x)})_1 \subseteq (N_r^{(x)})_2, \quad (N_{r_0}^{(x)})_2 \subseteq (N_r^{(x)})_1,$$

where $(N_r^{(x)})_i$ denotes the open neighborhood centered at x with radius r in the metric d_i . Thus, every open set in one metric contains an open set in the other, i.e., (X, d_1) and (X, d_2) are topologically equivalent.

Example 4.22 Consider $X = \mathbb{R}^2$ with the following metrics:

$$d_2(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2},$$

$$d_r(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$

$$d_M(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|\},$$

for $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

The neighborhoods of radius 1 centered at the origin are:

$$(N_1^{(0,0)})_2 = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\},$$

$$(N_1^{(0,0)})_r = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\},$$

$$(N_1^{(0,0)})_M = \{(x, y) \in \mathbb{R}^2 : \max \{|x|, |y|\} < 1\}.$$

Geometrically:

1. $(N_1^{(0,0)})_2$ is the interior of a circle of radius 1.

2. $(N_1^{(0,0)})_r$ is the interior of a rhombus with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.
3. $(N_1^{(0,0)})_M$ is the interior of a square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$.

Moreover, these metrics are equivalent on $\mathbb{R}^2$:

$$\begin{aligned} d_M(x, y) &\leq d_2(x, y) \leq \sqrt{2}d_M(x, y), \\ d_M(x, y) &\leq d_r(x, y) \leq 2d_M(x, y), \\ d_2(x, y) &\leq d_r(x, y) \leq 2d_2(x, y). \end{aligned}$$

Thus, although we have three different metrics, they are topologically identical: every neighborhood in one metric contains a neighborhood in the others.

We now turn to the notion of a nowhere dense subset, a concept characterized by the interplay between closure and interior.

Definition 4.24 Let (X, d) be a metric space and $A \subseteq X$. The set A is called nowhere dense in X if

$$\text{int}(\overline{A}) = (\overline{A})^\circ = \emptyset.$$

Remark 4.6 Let (X, d) be a metric space and let $A \subseteq X$ be nowhere dense. Then $\overline{A}^c$ is dense in X . More precisely,

$$(\overline{A})^\circ = \emptyset \Leftrightarrow ((\overline{A})^\circ)^c = X \Leftrightarrow \overline{\overline{A}^c} = X.$$

Remark 4.7 $\hookrightarrow$ Assume that (X, d) is a metric space and that $U, V \subseteq X$ are open and dense subsets of X . Then $U \cap V$ is also dense in X .

Indeed, since U and V are dense in X , they are non-empty, and hence $U \cap V$ is also non-empty and open. Using the equivalent characterization of density described in [Remark 4.1](#), we conclude that

$$A \cap (U \cap V) \neq \emptyset,$$

as desired. Note that

$$A \cap (U \cap V) = (A \cap U) \cap V.$$

Proposition 4.10 Let (X, d) be a metric space and let $A_1, \dots, A_n \subseteq X$ be nowhere dense subsets. Then

$$\bigcup_{k=1}^n A_k$$

is nowhere dense in X .

Proof 4.44 We prove the result for $n = 2$; the general finite case follows immediately by induction. Assume that A_1 and A_2 are nowhere dense in (X, d) . We show that

$$(\overline{A_1 \cup A_2})^\circ = \emptyset.$$

By [Theorem 4.13](#), we have

$$\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}.$$

Suppose, to the contrary, that $z \in (\overline{A_1 \cup A_2})^\circ$. Then there exists $r > 0$ such that

$$N_r^{(z)} \subseteq \overline{A_1 \cup A_2}.$$

Therefore,

$$N_r^{(z)} \cap (\overline{A_1 \cup A_2})^c = N_r^{(z)} \cap (\overline{A_1}^c \cap \overline{A_2}^c) = \emptyset.$$

Note that $\overline{A_1}^c$ and $\overline{A_2}^c$ are both open and dense. Hence, by [Remark 4.7](#), their intersection $\overline{A_1}^c \cap \overline{A_2}^c$ is also non-empty, open, and dense. Using [Remark 4.1](#), we obtain

$$N_r^{(z)} \cap (\overline{A_1}^c \cap \overline{A_2}^c) \neq \emptyset,$$

which is a contradiction.

Therefore,

$$(\overline{A_1 \cup A_2})^\circ = \emptyset.$$

Remark 4.8 The conclusion of the above proposition fails in general for infinite unions.

Example 4.23 Let $X = \mathbb{R}$ with the usual metric. Enumerate the rational numbers as $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ and define

$$A_n = \{r_n\}, \quad n \in \mathbb{N}.$$

Then each A_n is nowhere dense in $\mathbb{R}$, since

$$(\overline{A_n})^\circ = \emptyset.$$

However,

$$\bigcup_{n=1}^{\infty} A_n = \mathbb{Q},$$

and since $\overline{\mathbb{Q}} = \mathbb{R}$, we have

$$(\overline{\mathbb{Q}})^\circ = \mathbb{R} \neq \emptyset.$$

Thus, a countable union of nowhere dense sets need not be nowhere dense.

Definition 4.25 (Perfect Set) Let (X, d) be a metric space. A subset $P \subseteq X$ is called perfect if it equals its set of limit points:

$$P = P',$$

where P' denotes the set of all limit points of P in X . Equivalently, a perfect set has no isolated points.

Example 4.24

1. The closed interval $[0, 1] \subseteq \mathbb{R}$ is perfect. Every point in $[0, 1]$ is a limit point.
2. The Cantor set $C \subseteq [0, 1]$ is perfect. All points of C are limit points of C , so $C = C'$; we investigate this set in “Example Section” in this chapter.
3. The whole space $\mathbb{R}$ (or $\mathbb{R}^n$) is perfect. Every point is a limit point.
4. Any non-empty closed interval in $\mathbb{R}^n$, such as $[a, b]$ or $[a_1, b_1] \times \cdots \times [a_n, b_n]$, is perfect.

Theorem 4.25 (Cantor–Bendixson Theorem in Separable Metric Spaces) $\Leftarrow$ Let (X, d) be a separable metric space and let $F \subseteq X$ be closed. Then there exist a perfect set $P \subseteq X$ and a countable set $C \subseteq X$ such that

$$F = P \dot{\cup} C.$$

Proof 4.45 First recall that for a set $A \subseteq X$, A' is closed and that if A is closed then $A' \subseteq A$. Define inductively

$$F^{(0)} = F, \quad F^{(n+1)} = (F^{(n)})'.$$

Then each $F^{(n)}$ is closed and

$$F^{(0)} \supseteq F^{(1)} \supseteq F^{(2)} \supseteq \cdots.$$

Set

$$P = \bigcap_{n=0}^{\infty} F^{(n)}.$$

Since this is a countable intersection of closed sets, P is closed.

We show that P is perfect; that is $P' = P$.

Let $x \in P$. Then $x \in F^{(n)}$ for every n . Thus x is a limit point of $F^{(n-1)}$ for every $n \geq 1$. Hence every open ball around x contains a point of $F^{(n-1)}$ distinct from x . Since $P \subseteq F^{(n-1)}$ for every n , every neighborhood of x contains a point of P different from x . Thus x is not isolated in P , and therefore $P' = P$. Hence P is perfect.

Now, we show that $C = F - P$ is countable.

$$C = \bigcup_{n=0}^{\infty} (F^{(n)} - F^{(n+1)}).$$

Each set $F^{(n)} - F^{(n+1)}$ consists exactly of the isolated points of $F^{(n)}$.

We claim that in a separable metric space, any set of isolated points is at most countable.

Let $D = \{d_k : k \in \mathbb{N}\}$ be a countable dense subset of X . For each isolated point x , choose $r_x > 0$ such that

$$N_{r_x}^{(x)} \cap F^{(n)} = \{x\}.$$

Since D is dense, each ball $N_{r_x}^{(x)}$ contains some d_k . Moreover, distinct isolated points yield disjoint such balls. Thus we obtain an injection from the set of isolated points into D . Hence $F^{(n)} - F^{(n+1)}$ is countable.

Since C is a countable union of countable sets, C is countable.

Corollary 4.7 Let $F \subseteq \mathbb{R}^n$ be closed. Then there exist a perfect set $P \subseteq \mathbb{R}^n$ and a countable set $C \subseteq \mathbb{R}^n$ such that

$$F = P \dot{\cup} C.$$

Proof 4.46 The space $\mathbb{R}^n$ with the Euclidean metric is separable, since $\mathbb{Q}^n$ is a countable dense subset. Therefore the theorem applies directly.

Consider [Figure 4.7](#) for better understanding Cantor–Bendixson Theorem.

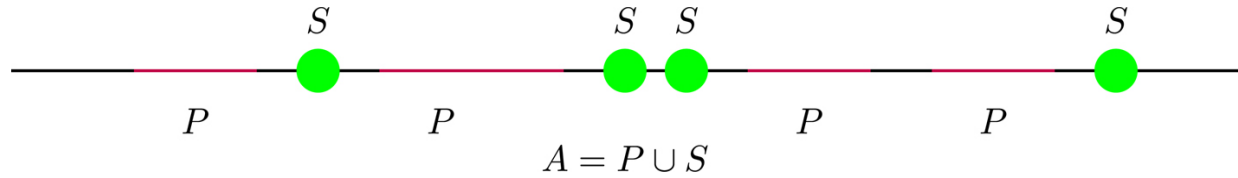


Figure 4.7 Illustration of Cantor–Bendixson decomposition: the closed set A is split into a perfect set P (purple) and a countable set S of isolated points (green). ↵

Remark 4.9 Note that separability is essential in [Theorem 4.25](#) and cannot be omitted. Consider $X = \mathbb{R}$ equipped with the discrete metric. Then every singleton is an isolated point, and hence there is no non-empty perfect subset of X . Consequently, the Cantor–Bendixson Theorem fails in this space.

4.3 CONSTRUCTION OF METRIC SPACES: A NEW COLLABORATION

We Can Construct Together; this Means a Better Life

In general, in the Realm of Topology, there exist certain agreements between the mayors (metrics) of cities for better export and import, as well as for the creation of joint projects in some regions with high productivity potential. For example, two cities with inhabitants consisting of real numbers, which geographically meet at the strategic strait of “Descartes’ Idea,” signed an agreement to establish the “Intelligent Product Control Zone” in order to jointly control the strait and to pave the way for further advancement of their inhabitants and other commercial interactions within the Realm of Topology. From another perspective, in the Realm of Topology, despite the availability of facilities for advancement, artistic freedom is unparalleled and legally binding. This means that inhabitants, if they achieve appropriate progress, can change their city of residence and move to a place with better facilities. Likewise, the mayors of the cities of metric spaces, in the presence of individuals who are more deserving and capable, are replaced in accordance with the higher laws of the Realm of Topology. All these points stem from the capability and commitment of the inhabitants and administrators in the Realm of Topology toward progress and service to mathematics.

4.3.1 Product Spaces

Suppose $\{(X_i, d_i)\}_{i=1}^n$ is a finite family of metric spaces. Put

$$X = \prod_{i=1}^n X_i = \{(x_1, \dots, x_n) : x_i \in X_i\}.$$

It is natural to ask whether we can generate a metric on X using the metrics on the X_i 's. We illustrate two examples.

Example 4.25 $\hookrightarrow$ Suppose $\{(X_i, d_i)\}_{i=1}^n$ is a finite family of metric spaces and put $X = \prod_{i=1}^n X_i$. For each $p \geq 1$ and $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X$, define

$$d_p(x, y) = \left(\sum_{i=1}^n d_i(x_i, y_i)^p \right)^{\frac{1}{p}}.$$

Then (X, d_p) is a metric space. The only nontrivial part to verify is the triangle inequality. Take $x, y, z \in X$. By Minkowski's inequality,

$$\begin{aligned} d_p(x, y) &= \left(\sum_{i=1}^n d_i(x_i, y_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n (d_i(x_i, z_i) + d_i(z_i, y_i))^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_{i=1}^n d_i(x_i, z_i)^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n d_i(z_i, y_i)^p \right)^{\frac{1}{p}} = d_p(x, z) + d_p(z, y). \end{aligned}$$

Example 4.26 $\hookrightarrow$ Suppose $\{(X_i, d_i)\}_{i=1}^n$ is a finite family of metric spaces and put $X = \prod_{i=1}^n X_i$. For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in X$, define

$$d_M(x, y) = \max_{1 \leq i \leq n} d_i(x_i, y_i).$$

Then (X, d_M) is also a metric space. The triangle inequality follows from the individual triangle inequalities of the d_i :

$$d_i(x_i, y_i) \leq d_i(x_i, z_i) + d_i(z_i, y_i) \leq d_M(x, z) + d_M(z, y),$$

and taking the maximum over i yields

$$d_M(x, y) \leq d_M(x, z) + d_M(z, y).$$

We now formalize the notion of a product metric.

Definition 4.26 Suppose $\{(X_i, d_i)\}_{i=1}^n$ is a finite family of metric spaces and put $X = \prod_{i=1}^n X_i$. A metric d on X is called a product metric provided that the collection

$$\{U_1 \times \dots \times U_n : U_i \subseteq X_i \text{ is open}\}$$

forms a base for the topology induced by d .

Proposition 4.11 Suppose $\{(X_i, d_i)\}_{i=1}^n$ is a finite family of metric spaces and put $X = \prod_{i=1}^n X_i$. Then the metrics d_p ($p \geq 1$) and d_M considered in [Examples 4.25](#) and [4.26](#), respectively, are product metrics.

Proof 4.47 For d_p , take an open set $U \subseteq X$ and $x = (x_1, \dots, x_n) \in U$. There exists $r > 0$ such that

$$(N_r^{(x)})_p = \left\{ y = (y_1, \dots, y_n) : \left(\sum_{i=1}^n d_i(x_i, y_i)^p \right)^{\frac{1}{p}} < r \right\} \subseteq U.$$

Choose positive reals $r_1, \dots, r_n$ such that $\left(\sum_{i=1}^n r_i^p \right)^{\frac{1}{p}} < r$, and put

$$U_i = (N_{r_i}^{(x_i)})_i = \{y_i \in X_i : d_i(x_i, y_i) < r_i\}.$$

Then $x \in U_1 \times \dots \times U_n \subseteq U$, and for any $z \in U_1 \times \dots \times U_n$,

$$d_p(x, z) = \left(\sum_{i=1}^n d_i(x_i, z_i)^p \right)^{\frac{1}{p}} < \left(\sum_{i=1}^n r_i^p \right)^{\frac{1}{p}} < r,$$

so $z \in U$.

For d_M , take an open set $U \subseteq X$ and $x \in X$. There exists $r > 0$ such that

$$(N_r^{(x)})_M = \{y = (y_1, \dots, y_n) \in X : \max_{1 \leq i \leq n} d_i(x_i, y_i) < r\} \subseteq U.$$

Set

$$U_i = \{y_i \in X_i : d_i(x_i, y_i) < r\}.$$

Then $x \in U_1 \times \dots \times U_n \subseteq U$, showing that d_M is a product metric.

Now, we establish a result about the relation between the Cartesian product and the closure of sets.

Proposition 4.12  Assume that $(X_1, d_1), \dots, (X_n, d_n)$ are metric spaces, and let $E_1 \subseteq X_1, E_2 \subseteq X_2, \dots, E_n \subseteq X_n$. Then

$$\overline{E_1} \times \dots \times \overline{E_n} = \overline{E_1 \times \dots \times E_n}.$$

Proof 4.48 Consider the Cartesian product $\prod_{i=1}^n X_i$ with the product topology. Since

$$E_1 \times \dots \times E_n \subseteq \overline{E_1} \times \dots \times \overline{E_n}$$

and the right-hand side is closed in the product topology, we conclude that

$$\overline{E_1 \times \dots \times E_n} \subseteq \overline{E_1} \times \dots \times \overline{E_n}.$$

For the other direction, assume $(x_1, \dots, x_n) \in \overline{E_1} \times \dots \times \overline{E_n}$. Let W be an arbitrary neighborhood of $(x_1, \dots, x_n)$ in the product topology $\prod_{i=1}^n X_i$. By the definition of the product topology, we can find neighborhoods $V_1 \subseteq X_1, \dots, V_n \subseteq X_n$ with $(x_1, \dots, x_n) \in \prod_{i=1}^n V_i \subseteq W$. By hypothesis, $E_i \cap V_i \neq \emptyset$ for each i . Therefore,

$$(E_1 \cap V_1) \times \dots \times (E_n \cap V_n) = (E_1 \times \dots \times E_n) \cap (V_1 \times \dots \times V_n) \neq \emptyset,$$

and we are done.

4.3.2 A New Metric Defines a New Life, Yet Preserves the Same Value

In this part, we are going to construct new metric with (possibly) different properties on a set.

Example 4.27 Assume that (X, d) is a metric space. Take any fixed $n \in \mathbb{N}$ and define a new metric d_n on X via

$$d_n(x, y) = \frac{d(x, y)}{1 + nd(x, y)}.$$

Then (X, d_n) is also a metric space. Except for the triangle inequality, all other metric conditions are straightforward.

Consider the function $f(t) = \frac{t}{1+nt}$. We have

$$f'(t) = \frac{1 + nt - nt}{(1 + nt)^2} = \frac{1}{(1 + nt)^2} > 0,$$

so f is increasing. Moreover, for each $r, s \geq 0$, we have

$$f(r + s) = \frac{r + s}{1 + n(r + s)} = \frac{r}{1 + nr + ns} + \frac{s}{1 + nr + ns} \leq \frac{r}{1 + nr} + \frac{s}{1 + ns} = f(r) + f(s).$$

Since d is a metric, for all $x, y, z \in X$,

$$d(x, y) \leq d(x, z) + d(z, y).$$

Because f is increasing and by the above inequality, we obtain

$$f(d(x, y)) \leq f(d(x, z) + d(z, y)) \leq f(d(x, z)) + f(d(z, y)),$$

so that

$$d_n(x, y) \leq d_n(x, z) + d_n(z, y),$$

verifying the triangle inequality.

4.4 NORMED SPACES: A NEW SPIRIT WITH COMPLETE EQUIPMENT

Banach Spaces: a Novel town and an Outstanding street

Let us continue our narrative in this chapter. As mentioned earlier, metric spaces serve as the “managers” within the mathematical realm. They provide a tangible identity to every entity in this mathematical landscape, allowing us to trust, assess, and monitor any advancements. We can accurately gauge the volume of exports and imports and establish long-term plans for these activities.

Within the domain of metric spaces, which only inhabitants with a valid metric card can enter, several unique towns exhibit more sophisticated characteristics. One such town is the normed spaces. Entry to this town is restricted to those holding a method card issued from the island of Linear Algebra or its related territories; not every resident with a metric card is permitted access. The inhabitants of the normed spaces enjoy rapid transportation, social amenities, and educational resources made possible by “scalar multiplication.” The norms that define each individual in this town contribute to a highly developed way of life.

In this town, there is a street called “completeness,” where the residents are known for their determination, precision, and efficiency. Additionally, there are factories within the metric spaces of the analysis island that produce compact or connected materials which are exceedingly valuable; we will discuss these soon. The inhabitants utilize every moment of their lives to create high-quality products or to support the mathematical community.

Every resident of the normed spaces has the opportunity to access this street, provided they possess sufficient motivation and diligence. The methodologies employed by the residents of the normed spaces town are robust, allowing them to export their goods to virtually any land or island. The residents of “Completeness” Street, also known as “Banach” Street, possess a distinguishing star on their method cards that sets them apart from others in the normed space town. This distinction grants them access to the island of “big last-standing theorems,” a secretive location in the mathematical ocean that only a select few can enter. Here, the finest ideas flourish, the best factories for theorem production operate, and the most skilled ships and submarines are found. The inhabitants of this island are dedicated solely to the pursuit of life and thought, making each of them highly valued. This is why the residents of Banach Street take great pride in their identity.

Definition 4.27 Suppose X is a vector space over $\mathbb{R}$. A function $\| \cdot \|: X \rightarrow \mathbb{R}$ is said to be a norm on X if for all $x, y \in X$ and all scalars α :

- $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$,
- $\|\alpha x\| = |\alpha| \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

In this case, $(X, \| \cdot \|)$ is called a *normed space*. Every normed space $(X, \| \cdot \|)$ is naturally a metric space via

$$d(x, y) = \|x - y\|.$$

The converse is not true: a metric can be defined on any set, but a norm requires a vector space structure.

Example 4.28 Consider $X = \mathbb{R}^n$, the space of n -tuples with real coefficients. Define

$$\|x - y\|_2 = \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{\frac{1}{2}}, \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

Then $(X, \| \cdot \|_2)$ is a normed space. By [Example 4.3](#), we see that it is more than a metric space; it is a normed space.

Remark 4.10 Suppose $X = \mathbb{R}^n$. For each $p \geq 1$, we can define the p -norm

$$\|x - y\|_p = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad x, y \in \mathbb{R}^n.$$

This defines a normed space. However, the discrete metric on $\mathbb{N}$ does not arise from a norm since $\mathbb{N}$ is not a vector space.

Definition 4.28 A normed space whose metric (induced by the norm) is complete is called a *Banach space*.

As mentioned earlier, “metrics” play a crucial role in managing sets in a systematic manner. Specifically, when a set has a metric associated with it, it can navigate the realm of metric spaces and their connections to topological spaces. This flexibility allows for advantages in both exporting and importing elements within this framework. However, the question arises: How do metrics manage sets?

To address this, we can break down management by metrics into two key aspects. First, metrics establish rules, as outlined in the definition of a metric space, which help ensure the accuracy of the set. Second, metrics identify representatives of the subsets to facilitate management and evaluation; we refer to these as neighborhoods or balls. In essence, balls or neighborhoods serve as representatives of the metric to organize the points within the set.

For instance, based on the metric, some points may reside on the boundary of the set, while others lie within it. Certain neighborhoods need to be open to enable efficient export and import, while others may be closed for security reasons. Convergence, which we will discuss in the next chapter, is a valuable concept from topology that directs sets toward a targeted outcome (for an improved experience) under the guidance of the defined metric. Limit points are considered potential targets, achieved through the reasonable behaviors of points within a set in cooperation with the metric.

In this part, we study several fundamental properties of product metric spaces.

Theorem 4.26 *Assume that (X_1, d_1) and (X_2, d_2) are metric spaces. Put $X = X_1 \times X_2$ with any product metric. For $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$, we have*

1. $(A_1 \times A_2)^\circ = A_1^\circ \times A_2^\circ$.
2. $\partial(A_1 \times A_2) = (\overline{A_1} \times \partial(A_2)) \cup (\partial(A_1) \times \overline{A_2})$.

Proof 4.49

(1) Take any $(x_1, x_2) \in (A_1 \times A_2)^\circ$. There exists an open set $U \subseteq X$ containing (x_1, x_2) such that $U \subseteq A_1 \times A_2$.

Since the family

$$\{U_1 \times U_2 : U_1 \subseteq X_1 \text{ open}, U_2 \subseteq X_2 \text{ open}\}$$

forms a basis for the product topology, there exist open sets $U_1 \subseteq X_1$ containing x_1 and $U_2 \subseteq X_2$ containing x_2 with

$$U_1 \times U_2 \subseteq U.$$

Hence,

$$U_1 \times U_2 \subseteq A_1 \times A_2 \quad \Leftrightarrow \quad U_1 \subseteq A_1, U_2 \subseteq A_2,$$

so $x_1 \in A_1^\circ$ and $x_2 \in A_2^\circ$. The converse follows similarly.

(2) Take any $z = (z_1, z_2) \in \partial(A_1 \times A_2)$. By definition, for each open set $U \subseteq X_1 \times X_2$ containing z , we have

$$U \cap (A_1 \times A_2) \neq \emptyset \quad \text{and} \quad U \cap (A_1 \times A_2)^c \neq \emptyset.$$

Using the basis of the product topology, choose open sets $U_1 \subseteq X_1$ containing z_1 and $U_2 \subseteq X_2$ containing z_2 such that $U_1 \times U_2 \subseteq U$. Then,

$$(U_1 \times U_2) \cap (A_1 \times A_2) = (U_1 \cap A_1) \times (U_2 \cap A_2) \neq \emptyset,$$

so that $z_1 \in \overline{A_1}$ and $z_2 \in \overline{A_2}$.

Next, note that

$$(A_1 \times A_2)^c = (A_1^c \times A_2) \cup (A_1 \times A_2^c) \cup (A_1^c \times A_2^c),$$

and

$$(U_1 \times U_2) \cap (A_1 \times A_2)^c = ((U_1 \cap A_1^c) \times (U_2 \cap A_2)) \cup ((U_1 \cap A_1) \times (U_2 \cap A_2^c)) \cup ((U_1 \cap A_1^c) \times (U_2 \cap A_2^c)) \neq \emptyset.$$

Thus, either

$$(z_1, z_2) \in (U_1 \cap A_1^c) \times (U_2 \cap A_2) \Rightarrow z_1 \in \overline{A_1^c}, z_2 \in \overline{A_2},$$

or

$$(z_1, z_2) \in (U_1 \cap A_1) \times (U_2 \cap A_2^c) \Rightarrow z_1 \in \overline{A_1}, z_2 \in \overline{A_2^c},$$

or

$$(z_1, z_2) \in (U_1 \cap A_1^c) \times (U_2 \cap A_2^c) \Rightarrow z_1 \in \overline{A_1^c}, z_2 \in \overline{A_2^c}.$$

Since $z_1 \in \overline{A_1}$ and $z_2 \in \overline{A_2}$ (from the first statement), we conclude that either

$$(z_1, z_2) \in \overline{A_1} \times \partial(A_2) \quad \text{or} \quad (z_1, z_2) \in \partial(A_1) \times \overline{A_2}.$$

The converse inclusion follows similarly.

Theorem 4.27 Let (X, d_1) and (Y, d_2) be metric spaces. Then we have the following observations.

1. If $A \subseteq X$ and $B \subseteq Y$ are nowhere dense, then the product $A \times B \subseteq X \times Y$ is nowhere dense in the product space (with any standard product metric).
2. If $P \subseteq X$ and $Q \subseteq Y$ are perfect, then the product $P \times Q \subseteq X \times Y$ is perfect.

Proof 4.50 (1). The closure of the product satisfies

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

The interior of this set is

$$(\overline{A \times B})^\circ = (\overline{A})^\circ \times (\overline{B})^\circ = \emptyset \times \emptyset = \emptyset.$$

Hence, $A \times B$ is nowhere dense.

Now, we prove (2). Since P and Q are perfect, we conclude that they are closed. Furthermore, since $\overline{P \times Q} = \overline{P} \times \overline{Q}$, we conclude that $P \times Q$ is closed so that $(P \times Q)' \subseteq P \times Q$. Now, let $(x_0, y_0) \in P \times Q$. Since $x_0 \in P = P'$ and $y_0 \in Q = Q'$, every neighborhood of x_0 contains $x \in P$, $x \neq x_0$, and every neighborhood of y_0 contains $y \in Q$, $y \neq y_0$.

Consider any neighborhood of (x_0, y_0) in $X \times Y$. By the definition of a product metric, this neighborhood contains a point $(x, y) \in P \times Q$ with $(x, y) \neq (x_0, y_0)$.

Therefore, every point of $P \times Q$ is a limit point, i.e.,

$$(P \times Q)' = P \times Q.$$

Thus, $P \times Q$ is perfect.

4.5 EXAMPLES

Example 4.29 (The Cantor Set) We now introduce the Cantor set, one of the most fundamental examples in real analysis and topology.

We begin with the closed interval $[0, 1]$ and divide it into three equal subintervals:

$$\left[0, \frac{1}{3}\right], \quad \left[\frac{1}{3}, \frac{2}{3}\right], \quad \left[\frac{2}{3}, 1\right].$$

We remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, leaving

$$\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

We now repeat this procedure on each remaining subinterval: at every step, the open middle third of each interval is removed. After n iterations, the remaining set F_n consists of 2^n closed intervals, each of length 3^{-n} . Explicitly, F_n is a finite union of closed intervals of the form

$$\left[\frac{k}{3^n}, \frac{k+1}{3^n}\right], \quad k \in \{0, 2, 6, 8, \dots, 3^n - 1\},$$

where the indices correspond to those subintervals not removed during the construction. In particular, each F_n is a closed subset of $[0, 1]$.

The Cantor set is defined as

$$C = \bigcap_{n=1}^{\infty} F_n.$$

Since it is an intersection of closed sets, the Cantor set C is closed. Equivalently, C consists of all points in $[0, 1]$ that are never removed at any stage of the construction.

We now describe the Cantor set in terms of base-3 expansions. Every real number $x \in [0, 1]$ admits a ternary expansion of the form

$$x = 0.\varepsilon_1\varepsilon_2\varepsilon_3\dots_3, \quad \varepsilon_i \in \{0, 1, 2\}.$$

This expansion is not always unique; for instance,

$$\frac{1}{3} = 0.1_3 = 0.0222\dots_3.$$

Observe that if the first ternary digit of x equals 1, then $\frac{1}{3} \leq x \leq \frac{2}{3}$, so x lies in the middle third removed at the first step. More generally, at the n th stage of the construction, the middle thirds removed correspond exactly to those numbers whose n -th ternary digit equals 1.

Consequently, a point $x \in [0, 1]$ belongs to the Cantor set if and only if it admits a ternary expansion containing only the digits 0 and 2. Thus,

$$C = \{x \in [0, 1] : x = 0.\varepsilon_1\varepsilon_2\varepsilon_3 \dots_3 \text{ with } \varepsilon_i \in \{0, 2\}\}.$$

From this characterization it follows that the Cantor set is uncountable, closed, contains no nontrivial interval, and has empty interior. In [Figure 4.8](#), we see the Cantor set construction up to Step 5.

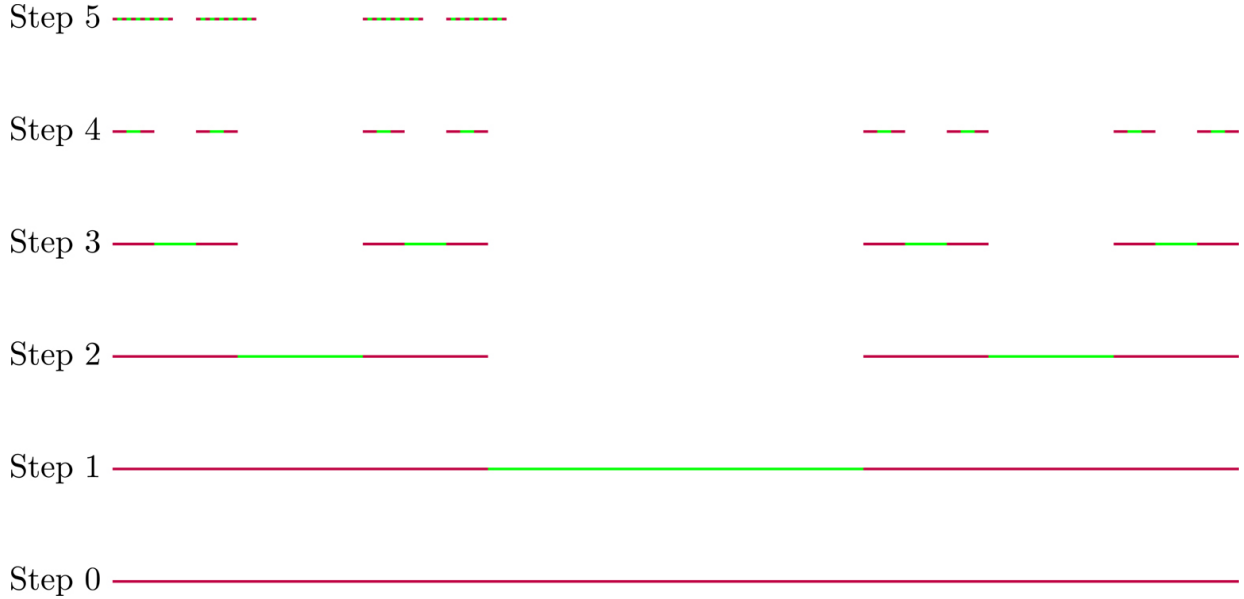


Figure 4.8 Cantor set construction up to Step 5. Purple segments remain; green segments are removed middle thirds. [↗](#)

Example 4.30 We show that any non-empty perfect subset $P \subseteq \mathbb{R}^n$ is uncountable.

Proof: Suppose, for contradiction, that P is countable. Then we can enumerate its points as

$$P = \{x_1, x_2, x_3, \dots\}.$$

Since P is perfect, it has no isolated points. In particular, x_1 is not isolated, so there exists a point $y_1 \in P$ such that

$$y_1 \neq x_1 \quad \text{and} \quad \|y_1 - x_1\| < 1.$$

Similarly, x_2 is not isolated, so there exists a point $y_2 \in P$ such that

$$y_2 \neq x_1, x_2 \quad \text{and} \quad \|y_2 - x_2\| < 1/2.$$

Continuing inductively, for each $n \in \mathbb{N}$, we can find a point $y_n \in P$ such that

$$y_n \neq x_1, \dots, x_n \quad \text{and} \quad \|y_n - x_n\| < 1/n.$$

By construction, the sequence (y_n) consists of points in P not in our original enumeration, which contradicts the assumption that P is countable.

Hence, P cannot be countable, and therefore P is uncountable.

In the following example, we establish several functional equations.

Example 4.31 *Let f be a real-valued function. Then the following characterizations hold.*

1. **Additive equation.** *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies*

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R},$$

then there exists a constant $a \in \mathbb{R}$ such that

$$f(x) = ax \quad \text{for all } x \in \mathbb{R}.$$

Proof. Let $a = f(1)$. We first show that $f(n) = an$ for all integers n . Indeed, by induction, for $n \geq 1$,

$$f(n) = f(1 + 1 + \cdots + 1) = f(1) + \cdots + f(1) = nf(1) = an.$$

For negative integers n , using $f(0) = f(n + (-n)) = f(n) + f(-n) = 0$, we get $f(-n) = -an$.

Next, for rational numbers of the form m/n , with $m \in \mathbb{Z}, n \in \mathbb{N}$, using the additivity and the previous step,

$$f\left(\frac{m}{n}\right) = \frac{1}{n}f(m) = \frac{1}{n}(am) = a\frac{m}{n}.$$

Finally, by continuity of f , for any real number x , choose a sequence of rationals $q_k \rightarrow x$. Then

$$f(x) = \lim_{k \rightarrow \infty} f(q_k) = \lim_{k \rightarrow \infty} aq_k = ax.$$

This completes the proof.

2. **Multiplicative equation.** *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies*

$$f(xy) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

then exactly one of the following holds:

$$f(x) \equiv 0, \quad f(x) \equiv 1, \quad \text{or} \quad f(x) = |x|^c \text{ for some } c \in \mathbb{R}.$$

Proof. First, if $f(0) \neq 0$, then $f(0) = f(0 \cdot 0) = f(0)^2$, so $f(0) = 1$. But then for any x ,

$$f(0) = f(0 \cdot x) = f(0)f(x) = 1 \cdot f(x) \implies f(x) = 1 \text{ for all } x.$$

If $f(0) = 0$, then $f(x) = f(x \cdot 0) = f(x)f(0) = f(x) \cdot 0 = 0$, so $f \equiv 0$.

Now assume f is not identically 0, and $f(0) = 1$ is excluded (so $f(0)$ cannot appear here). For $x > 0$, define $g(x) = \ln f(x)$, which is well-defined and continuous for $x > 0$ if $f(x) > 0$. Then

$$g(xy) = \ln f(xy) = \ln(f(x)f(y)) = \ln f(x) + \ln f(y) = g(x) + g(y),$$

so g satisfies the additive equation on $(0, \infty)$. By the previous result (additive equation), there exists $c \in \mathbb{R}$ such that

$$g(x) = c \ln x \implies f(x) = e^{g(x)} = x^c.$$

If f takes negative values, then $f(x) = |x|^c$. Finally, checking $x = 0$ and $x < 0$, the formula $f(x) = |x|^c$ extends consistently.

3. **Logarithmic equation.** If $f : (0, \infty) \rightarrow \mathbb{R}$ is continuous and satisfies

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y > 0,$$

then there exists a constant $c \in \mathbb{R}$ such that

$$f(x) = c \ln x \quad \text{for all } x > 0.$$

Proof. Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(t) = f(e^t)$ for $t \in \mathbb{R}$. Then for all $s, t \in \mathbb{R}$,

$$g(s + t) = f(e^{s+t}) = f(e^s e^t) = f(e^s) + f(e^t) = g(s) + g(t),$$

so g satisfies the additive equation on $\mathbb{R}$. By the first result, there exists $c \in \mathbb{R}$ such that $g(t) = ct$. Hence, for $x > 0$ (letting $t = \ln x$),

$$f(x) = g(\ln x) = c \ln x.$$

4.6 EXERCISES

Exercise 4.1

1. Let (X, d) be a metric space and $x, y, z, w \in X$. Show that

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w).$$

2. Show that in $X = \mathbb{R}$ with the Euclidean topology,

$$\overline{\mathbb{Q}} = \overline{\mathbb{Q}^c} = \mathbb{Q}' = (\mathbb{Q}^c)' = \mathbb{R}.$$

3. Show that in $X = \mathbb{R}$,

$$\mathbb{Q}^\circ = (\mathbb{Q}^c)^\circ = \emptyset.$$

4. Let $X = \mathbb{R}$ with the Euclidean metric. Show that

$$A = \{\sqrt{m} - \sqrt{n} : m, n \in \mathbb{N}\}$$

is dense in X .

5. Let $X = \mathbb{R}^2$ with the Euclidean topology and

$$A = \{(x, y) \in X : xy > 1\}.$$

Determine whether A is open or closed and justify.

6. Let $X = \mathbb{R}^2$ with the Euclidean topology and

$$A = \{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 < 1\}.$$

Compute A° and $\overline{A}$.

7. Let $X = \mathbb{R}$. Determine the cluster points, accumulation points, and interior points of the following sets:

(a) $A = \mathbb{N}$

(b) $B = \{\frac{1}{m} + e^{-k} : m, k \in \mathbb{N}\}$

(c) $C = \{\frac{m}{2^p}, \frac{n}{2^q} : p, q \in \mathbb{N}, 1 \leq m \leq 2^p, 1 \leq n \leq 2^q\}$

(d) $D = \{\sqrt{p} : p \text{ is prime}\}$

8. Show that $(0, 1]$ and $[0, 1)$ are both G_δ and F_σ in $\mathbb{R}$.

9. Determine G_δ and F_σ sets in a discrete space.

10. Let (X, d) be a metric space and $A \subseteq X$. Show that

$$\partial(\partial(\partial A)) = \partial(\partial A).$$

11. Show that $A \subseteq X$ is closed if and only if $\partial(A) \subseteq A$.

12. Find the frontier points of $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{R} \setminus \mathbb{N}$ in the Euclidean topology.

13. Let (X, d) be a discrete metric space and $A \subseteq X$. Show that

$$d(A) = \begin{cases} 0, & \text{if } A \text{ is singleton} \\ 1, & \text{otherwise} \end{cases}.$$

(a) Any finite union of closed sets is closed.

(b) Any arbitrary union of open sets is open.

(c) Any finite intersection of open sets is open.

(d) Any arbitrary intersection of closed sets is closed.

14. Define $d : P(X) \times P(X) \rightarrow \mathbb{R}$ via

$$d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}.$$

Show that d is decreasing in each component.

15. Let $Y \subset P(X)$ consist of all bounded subsets of X , and define

$$D(A, B) = \sup \{d(a, b) : a \in A, b \in B\}.$$

Show that D is increasing in each component.

16. Let $Y \subset P(X)$ consist of all bounded subsets of X , and define

$$\tau(A, B) = \sup \{d(a, B) : a \in A\}.$$

Show that τ is increasing in the first component and decreasing in the second component.

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Sequences: The Ambassadors of the Archipelago of Convergences

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5.1 IMPORTANCE OF SEQUENCES IN THE MATH OCEAN

Sequences: Professional Ambassadors and Competent representatives

The sequences, which are generally representatives and ambassadors of the convergence archipelago in other parts of the ocean of mathematics, can be organized and targeted to carry out various projects and manage them until they reach their objectives. In every corner of the ocean of mathematics, through arranging and selecting the best options, they can help projects and progress bear fruit. When a project comes to fruition (by arranging the sequences and using the upstream rules of the Topology realm), we consider that sequence to be a successful ambassador and grant it the valuable stamp of convergence (we will see later that this stamp also allows the beholder to enter the advanced and industrialized town of Compactness). This means that the residents of that region have well abided by the relevant norms and rules, and we grant them (along with their ambassador) the stamp of boundedness (the stamp granted to a normal citizen and resident). Undoubtedly, being successful in a project means having originality and being normal in the residents' work, but having originality and a norm (boundedness) does not necessarily mean that the project is fruitful (convergent). In other words, it is a necessary but not sufficient condition. For our originality and norms to lead to the project's success and the achievement of the objective, the residents must maintain the necessary perseverance and enthusiasm until the very end. In such a case, we will include a stamp indicating increasing enthusiasm (monotonicity) on the residents' method card. For example, in the very important realm of Real Numbers with Euclidean Topology, the convergence stamp is as valuable as the monotonicity and originality stamps. In this part, we are going to talk about an archipelago located near the topology territories, named the archipelago of convergences. This unique place has a very important role in communication in the math ocean. Different factories for construction of every boat and every submarine. Although most of the ownership of these factories belong to the topology, there were some important "structural" factories whose raw materials are not provided by the topologists. They were advanced factories but we do not talk about them in this book; it is a separate line for research. Anyway, we intend to talk about the topological parts. The factories produce oars, sails, motor engines for boats and submarines and many other materials. The inhabitants of these archipelagos are called: convergences. These inhabitants have many ambassadors in different islands and lands; they are called "sequences." Also, a nice and invaluable product from topologists, named "nets," was made during a combination of "sequences" and some "algae of ideas" and "fast planktons." In fact, some planktons were discovered by the topologists that make convergence speed very faster and

more accurate. Therefore, sequences and nets are ambassadors of the archipelago of convergences and play an important role in construction of transportation vehicles at the math ocean.

Issuing a visa for archipelago of convergences

As we said before, every inhabitant of a math place in our story, has a “method card” with that, could transfer between different places and it has the role of a visa, ticket, etc. However, just some inhabitants of the topology territory could go freely between the archipelago of convergences. Of course, the inhabitants of the province of metric spaces and the city of normed spaces had this fantastic stamp on their “method cards” to go freely to the archipelago of convergences. The required permission for entrance into the archipelago of convergences (visa) was issued by an office located in the heart of the archipelago, named Hausdorff. It was a fair office with a deep gratitude to math and very trusted; in the means that it does everything to make mathematics prosper. Now, we shall talk about the math language of this mysterious archipelago.

5.2 SEQUENCES AND METRIC SPACES

Every real valued function whose domain is the natural numbers is called a real sequence or briefly a sequence. Since the Euclidean topology inherited from $\mathbb{R}$ on $\mathbb{N}$ is just the discrete topology, we can consider every sequence as a continuous function, as well (we shall talk about this phenomenon later in details). Because of the proposed topology of the domain of a sequence, we cannot expect a limit at each point (another approach to the discrete topology in $\mathbb{N}$). Nevertheless, since the natural numbers are as powerful as the reals in large numbers as well as small ones (the Archimedean property and the related consequences in [Chapter 3](#)), we can expect the limit a sequence at infinity. This notion that we call it the limit of a sequence (since we do not have any other limits there), is the key of many applications in analysis. In fact, sequences are the ambassadors of the archipelago of convergences and control every commuting and hence the behavior of inhabitants of a topological object. We recall now the math notion of convergence for sequences. As the theme of the book demands, we focus our attention on the inhabitants of the metric spaces province.

Definition 5.1 Suppose that (X, d) is a metric space and $(x_n) \subseteq X$ is a sequence. In [Figures 5.1](#) and [5.2](#), we see two convergent and divergent sequences, respectively. We say that (x_n) converges in X if there exists $x \in X$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ with

$$d(x_n, x) < \varepsilon \quad \text{for all } n > n_0.$$

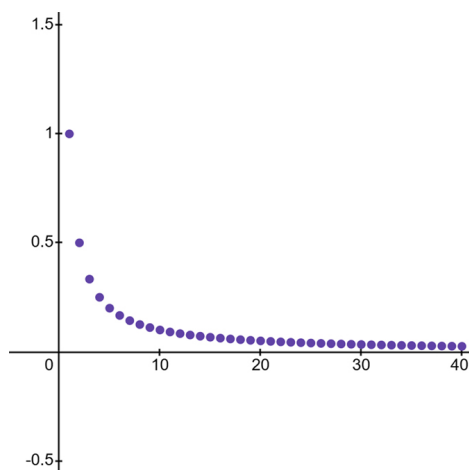


Figure 5.1 A convergent sequence. ↗

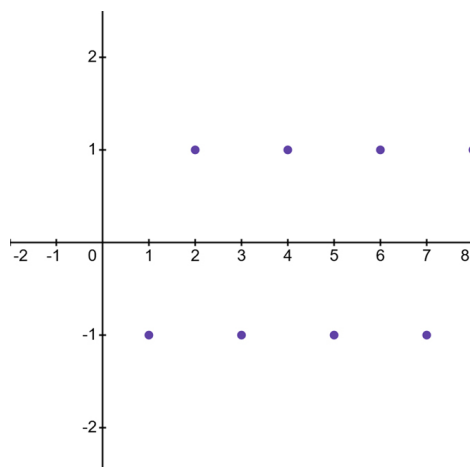


Figure 5.2 A divergent sequence. ↗

In this case, we write $x_n \rightarrow x$. If no such x exists, then the sequence (x_n) is said to be divergent.

Example 5.1 Let $X = \mathbb{R}$ equipped with the Euclidean metric. Then the sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ converges to 0.

Indeed, by the Archimedean property, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \varepsilon.$$

Hence, for all $n > n_0$, we have

$$\frac{1}{n} < \frac{1}{n_0} < \varepsilon,$$

which shows that $\frac{1}{n} \rightarrow 0$.

If we change the metric, the situation can be different.

Example 5.2 Let $X = \mathbb{R}$ be endowed with the discrete metric d . Then the sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ is divergent.

To see this, suppose to the contrary that $x_n = \frac{1}{n} \rightarrow x$ for some $x \in \mathbb{R}$. Take $\varepsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$d\left(\frac{1}{n}, x\right) < \frac{1}{2} \quad \text{for all } n > n_0.$$

Since d is the discrete metric, this implies

$$d\left(\frac{1}{n}, x\right) = 0 \quad \text{for all } n > n_0,$$

and hence $\frac{1}{n} = x$ for all $n > n_0$, which is impossible. Therefore, the sequence $\left(\frac{1}{n}\right)$ is divergent in $(\mathbb{R}, d)$.

The Hausdorff Authority

When we use a metric space, due to the honor of Hausdorff office in issuing the right visas for applications of the archipelago of convergences, we have two options. The former one is that they issue the visa and an inhabitant of the math places can go there, sell or buy required equipment and also use facilities provided by the archipelago of convergences for trip to the heaven of the analysis island: continuity; that is the entrance of the gate to a world of opportunities of applied islands and lands. In other words, by entrance to this heaven, you are given a “green star” on your “method card” and are allowed to transfer in a high volume in applied territories. However, in general topology parts, convergence may not be so well-behaved.

Example 5.3 $\hookleftarrow$ Consider $X = \mathbb{R}$ endowed with the indiscrete (or trivial) topology (you will learn more about this topology in an advanced course in topology; for now, note that the only open sets are $\emptyset$ and X).

Then the sequence $(\frac{1}{n})_{n \in \mathbb{N}}$ converges to every point of X .

Indeed, let $x \in \mathbb{R}$ be arbitrary. The only open neighborhood of x in this topology is X itself. Since

$$\frac{1}{n} \in X \quad \text{for all } n \in \mathbb{N},$$

it follows that for every open neighborhood U of x , we have $\frac{1}{n} \in U$ for all $n \in \mathbb{N}$. Hence,

$$\frac{1}{n} \rightarrow x$$

in the indiscrete topology.

Yet, Importance of Hausdorff Authority

This example shows that “convergence products” cannot be as strong as we expect for a purposeful, accurate and legal transportation if we use non-standard products. But there is no reason to be worried in the places related to mathematics since mathematics has an infinite power and sense to solve any problem. In fact, the situation as described in [Example 5.3](#) is very scarce in the topology territories and is caused by smuggling. In other words, some smugglers try to enter into the archipelago of convergences without a visa issued by the “Hausdorff” office and make a chaos situation for the archipelago. Of course, topology needs some discrete and no-discrete topologies on the frontiers of its territories for two reasons. First to protect its territories from being attacked or infiltrated and also for some products in counterexamples factories.

Example 5.4 $\hookleftarrow$ Suppose that $X = \mathbb{R}$ with the Euclidean topology (inherited from the modulus metric). We show that the sequence $((-1)^n)_{n \in \mathbb{N}}$ is not convergent. Assume, to the contrary, that there exists $a \in \mathbb{R}$ such that $(-1)^n \rightarrow a$. Therefore, for $\varepsilon = \frac{1}{3}$, there exists $N \in \mathbb{N}$ such that

$$|(-1)^n - a| < \frac{1}{3} \quad \text{for all } n \geq N.$$

In particular, for $n = 2N$, we have $|1 - a| < \frac{1}{3}$, and for $n = 2N + 1$, we have $|-1 - a| < \frac{1}{3}$. Therefore,

$$2 = |1 - a + a + 1| \leq |1 - a| + |1 + a| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$

which is a contradiction. Hence, the sequence $((-1)^n)$ is divergent.

5.3 SUBSEQUENCES: WHERE HOPE DEFIES DISAPPOINTMENT

Subsequences: a Well-deserved Reward for Relentless Effort

Many projects, in the ocean of mathematics, are very bulky, technical, and require a lot of time and concentration. This makes the task very difficult for the project management and the residents involved. On the other hand, most projects aim to reach the paradise of mathematics (the Topology realm) and the main port of export and import (the ocean of Applications). Due to upstream rules on topology and the discipline of the works, there is a very intense, high-level competition among residents of different regions to secure the stamp of applied trade (which signifies the importance and ability of a region's residents). Therefore, a project may fail to converge (even with effort, practice, and regular management), but some parts of it achieve convergence and contribute to applied trade (the stamp of limited applied trade). In fact, in this case, the volume and time of trade are limited. We call such a project management, together with the relevant members, a subsequence.

Definition 5.2 Assume that (X, d) is a metric space and (x_n) is a sequence in X . Let (n_k) be a strictly increasing sequence of natural numbers; that is,

$$n_1 < n_2 < \dots.$$

Then the sequence $(x_{n_k})_{k \in \mathbb{N}}$ is called a subsequence of the sequence (x_n) .

Proposition 5.1 Suppose (X, d) is a metric space and (x_n) is a sequence in X that is convergent. Then every subsequence of (x_n) is also convergent to the same limit.

Proof 5.1 Assume that $x_n \rightarrow x$ and (x_{n_k}) is a subsequence of (x_n) . For a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \text{for each } n > N.$$

Choose an index $n_{k_0} \in \mathbb{N}$ such that $n_{k_0} > N$. Then, for each $n_k > n_{k_0}$, we have

$$d(x_{n_k}, x) < \varepsilon,$$

which shows that $x_{n_k} \rightarrow x$ as well.

Cauchy Sequences: an Ideal Space for Shared Mathematical Growth

In the places related to mathematics, we have two special advantages: freedom and the best for all. The former case states that every inhabitant can freely go to every place provided that it possesses the required equipment (for example stars on its method card). However, there exists a connectivity between different inhabitants for a better life. In other words, with suitable cooperation, the desired inhabitants can get more stars on their method cards so that a better export and life etc. These two items are the key to the success of mathematics with respect to other sciences and show the real power of mathematics. In the archipelago of convergences, some elements have this opportunity to merge together and under their manager (metric) produce a powerful sequence that can be convergent to a particular target. In fact, in the world of mathematics, every inhabitant takes an oath to do its best for the whole of mathematics. The inhabitants are always ready to make every sacrifice for mathematics. So, they try to do their best for all of the mathematics. In the archipelago of convergences, when a new target is announced, (here by a target, we mean some parts of the math worlds need new equipment for their boats,

submarines or in general for better life), the inhabitants try to make this target swiftly and accurate. We call this character: Cauchy. In the other words, among the residents of metric spaces, Cauchy sequences of members represent the cohesion of a community under the control of a given metric to achieve a specific goal or carry out a project to develop the desired region. Some cases fail to realize because they lack a representative of the Banach street or the Convergence archipelago to manage and lead the project toward its intended objective. In such cases, residents only comprise a Cauchy sequence of efforts. The presence of a representative and a guide leads to achieving the desired outcome. It is noteworthy that being Cauchy or convergent (that is, successfully reaching a project or a specific objective) implies boundedness under the supervision of the chosen metric: persistent effort and teamwork.

Definition 5.3 Suppose (X, d) is a metric space and (x_n) is a sequence in X . We say that (x_n) is a Cauchy sequence provided that for each $\varepsilon > 0$, there exists an index n_0 such that for all $m > n > n_0$, we have

$$d(x_m, x_n) < \varepsilon.$$

It is obvious that when we reach a target, some sacrifices have already been made.

Theorem 5.1 $\hookrightarrow$ Suppose (X, d) is a metric space and (x_n) is a convergent sequence in X . Then (x_n) is a Cauchy sequence.

Proof 5.2 Assume that $x_n \rightarrow x$ and let $\varepsilon > 0$ be given. Then we can find n_0 such that for each $n > n_0$, we have

$$d(x_n, x) < \frac{\varepsilon}{2}.$$

Therefore, for each $m > n > n_0$, we obtain

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The converse of [Theorem 5.1](#) is not true in general. For example, consider $X = (0, 1]$ with the Euclidean topology induced from the real line. The sequence $(\frac{1}{n})_{n \in \mathbb{N}}$ in X is Cauchy, but it is not convergent. Indeed, if it were convergent, then it would converge to 0, but $0 \notin X$.

In the world of mathematics, in terms of sequences, managing to reach a target is a Cauchy sequence and reaching the target is convergence. These notions are very close together. Nevertheless, [Theorem 5.1](#) tells us to get a target, we must manage our power but having a management is not a guarantee to reach the target. We need to have a special insurance, named completeness, issued by an office in the analysis island owned by Banach and Cauchy simultaneously. In fact, without this insurance, we cannot reach the target even if we manage everything in an appropriate way. Of course, in general, completeness, is not a major problem and we can make complete every non-complete metric space with a suitable insurance (issued by the company with a higher price for non-accurate inhabitants who do not get an appropriate insurance on time and do not equip them self with it at first (before commence of managing to reach a target)).

Now, we turn our attention to mathematical language.

Definition 5.4 Assume that (X, d) is a metric space. We say that X is complete provided that every Cauchy sequence in X is convergent.

As an example, we have seen that $X = (0, 1]$ with the Euclidean metric is not complete. However, $\mathbb{R}$ is complete. Also, $\mathbb{Q}$ and $\mathbb{Q}^c$ are not complete when equipped with the Euclidean

metric. For a general intuition, note that we may have many rational numbers very close to each other, but they eventually converge to an irrational number. For example, put $x_1 = 1$, $x_2 = 1.4$, $x_3 = 1.41$, $x_4 = 1.414$, and so on. We can see that $x_n \rightarrow \sqrt{2}$, and clearly $\sqrt{2}$ is not rational.

The following example is simple but useful.

Example 5.5 $\hookrightarrow$ Assume that (X, d) is a metric space. Then every constant sequence (or a sequence that is constant for sufficiently large n) is convergent. To see this, suppose (x_n) is a sequence that is constant for sufficiently large n ; namely, there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0} = x$ for each $n \geq n_0$. Therefore, for a given $\varepsilon > 0$, we have

$$d(x_n, x) = d(x, x) = 0 < \varepsilon,$$

provided that $n \geq n_0$.

Now, we characterize convergent sequences as well as Cauchy sequences in discrete spaces. As usual, we do not expect any surprise.

Theorem 5.2 Assume that (X, d) is a discrete space. Then a sequence (x_n) is convergent (Cauchy) if and only if it is constant for sufficiently large $n \in \mathbb{N}$.

Proof 5.3 The proofs for convergent and Cauchy sequences follow the same line. We prove the statement for convergent sequences. The direct implication is trivial by [Example 5.5](#). For the converse, suppose that (x_n) converges to $x \in X$. Then, for $\varepsilon = \frac{1}{2}$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{1}{2} \quad \text{for each } n > n_0,$$

which implies that $d(x_n, x) = 0$. Hence, $x_n = x$ for sufficiently large n , as claimed.

Theorem 5.3 $\hookrightarrow$ Assume that (X, d) is a metric space and $A \subseteq X$. Then the following statements hold.

1. $x \in \overline{A}$ if and only if there exists a sequence (x_n) in A such that $x_n \rightarrow x$.
2. $x \in A'$ if and only if there exists a sequence (x_n) of distinct elements of A such that $x_n \rightarrow x$.

Proof 5.4 Both statements follow the same line of argument. We prove the first one and leave the second to the reader. Assume that $x \in \overline{A}$. For each $n \in \mathbb{N}$, we have

$$N_{\frac{1}{n}}^{(x)} \cap A \neq \emptyset.$$

Hence, we can choose $x_n \in A$ such that $d(x_n, x) < \frac{1}{n}$. This shows that $x_n \rightarrow x$.

Conversely, assume that $x \in X$ and there exists a sequence $(x_n) \subseteq A$ such that $x_n \rightarrow x$. For each $r > 0$, we can find $n_0 \in \mathbb{N}$ with

$$d(x_{n_0}, x) < r.$$

Thus, $x_{n_0} \in N_r^{(x)} \cap A$, which implies that $x \in \overline{A}$.

The following result is easy to verify. For a proof, just note that for convergence, we only need the elements of a sequence that are sufficiently large.

Theorem 5.4 $\hookrightarrow$ Assume that (X, d) is a metric space and (x_n) and (y_n) are sequences in X that are eventually the same. More precisely, there exists $n_0 \in \mathbb{N}$ such that $x_n = y_n$ for each $n > n_0$. Then both sequences are either convergent to the same limit or divergent.

To illustrate [Theorem 5.4](#), consider the following example.

Example 5.6 Assume that $X = \mathbb{R}$ with the Euclidean metric. Define the sequences (a_n) and (b_n) in X as follows:

$$a_n = \begin{cases} n^{n!}, & n \leq 10^9, \\ \frac{1}{n}, & n > 10^9, \end{cases}$$

and

$$b_n = \frac{1}{n^2}.$$

By [Theorem 5.4](#), we see that both (a_n) and (b_n) converge to zero. Although the initial elements of (a_n) are very large, the subsequent elements determine the behavior of the sequence. This is well captured by an Iranian proverb:

“Don't count your chickens before they hatch.”

Although not every Cauchy sequence is convergent in general, we have the following approach to convergence of a Cauchy sequence using subsequences.

Theorem 5.5 $\hookrightarrow$ Assume that (X, d) is a metric space and (x_n) is a Cauchy sequence in X . If (x_n) has a subsequence that is convergent, then the sequence (x_n) is also convergent to the same limit.

Proof 5.5 Let $\varepsilon > 0$ be arbitrary and assume that the subsequence (x_{n_k}) of (x_n) converges to $x \in X$. We show that $x_n \rightarrow x$. By the Cauchy property, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_n) < \frac{\varepsilon}{2} \quad \text{for all } m > n > n_0.$$

Moreover, since (x_{n_k}) converges to x , there exists an index $n_{k_0} > n_0$ such that

$$d(x_{n_k}, x) < \frac{\varepsilon}{2} \quad \text{for all } n_k > n_{k_0}.$$

Therefore, for each $n > n_{k_0}$, we have

$$d(x_n, x) \leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that $x_n \rightarrow x$.

Let (X, d) be a metric space. Recall that X is *complete* if every Cauchy sequence in X converges to a point of X . In other words, in a complete metric space, every Cauchy sequence is convergent; equivalently, the notions of being Cauchy and being convergent coincide. Before we proceed further, let us explain the notion of “completeness” with some illustrative examples.

Example 5.7 Suppose that $X = \mathbb{R}$ with the Euclidean metric. We show that $\mathbb{Q}$ is not complete.

Proof 5.6 It is enough to find a Cauchy sequence in $\mathbb{Q}$ that is not convergent. Consider the recurrent sequence (a_n) defined as follows:

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right).$$

It is easy to see that (a_n) is bounded below (indeed, $a_n \geq 0$ for each $n \in \mathbb{N}$). Moreover, we have

$$a_{n+1} \geq \sqrt{2} \iff \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \geq \sqrt{2} \iff (a_n - \sqrt{2})^2 \geq 0.$$

Therefore, we see that $a_n \geq \sqrt{2}$ for each $n \in \mathbb{N}$. Furthermore, we have

$$\begin{aligned} a_n \geq \sqrt{2} &\iff a_n^2 \geq 2 \iff a_n \geq \frac{2}{a_n} \iff a_n + \frac{2}{a_n} \leq 2a_n \\ &\iff \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) \leq a_n \iff a_{n+1} \leq a_n. \end{aligned}$$

Thus, we conclude that (a_n) is decreasing as well. Consequently, (a_n) is a Cauchy sequence. If (a_n) were convergent, say $\lim_{n \rightarrow \infty} a_n = a$, then we would have

$$a = \frac{1}{2} \left(a + \frac{2}{a} \right),$$

which implies $a^2 = 2$. This is not possible since $\sqrt{2} \notin \mathbb{Q}$.

Theorem 5.6 Assume that (X, d) is a complete metric space and $Y \subseteq X$. Then the metric space (Y, d) is complete if and only if Y is closed in X .

Proof 5.7 First, suppose that (Y, d) , equipped with the inherited metric, is complete. We claim that $\overline{Y} = Y$. Take any $x \in \overline{Y}$. By [Theorem 5.3](#), there exists a sequence (x_n) in Y such that $x_n \rightarrow x$. Then (x_n) is convergent in X , and hence it is a Cauchy sequence. Since the Cauchy property of a sequence in a subspace does not depend on the ambient space, we conclude that (x_n) is also Cauchy in Y . As Y is complete, (x_n) converges to some $y \in Y$. By the uniqueness of limits in metric spaces, we must have $x = y$, and hence $x \in Y$.

Conversely, suppose that Y is closed in X , that is, $Y = \overline{Y}$. We show that Y is complete. Let (x_n) be a Cauchy sequence in Y . Then it is also a Cauchy sequence in X . Since X is complete, there exists $x \in X$ such that $x_n \rightarrow x$. By [Theorem 5.3](#), we have $x \in \overline{Y} = Y$. Therefore, (x_n) converges in Y , which shows that Y is complete.

Definition 5.5 We say that a sequence (x_n) in a metric space (X, d) is bounded provided that the set

$$A = \{x_n : n \in \mathbb{N}\}$$

is bounded; in other words,

$$\text{diam}(A) = \sup \{d(x_n, x_m) : n, m \in \mathbb{N}\} < \infty.$$

Otherwise, we say that (x_n) is unbounded.

For example, the sequences $((-1)^n)$, $(\sin n)$, $(\cos n)$, and (e^{-n}) are bounded. On the other hand, the sequences $(\tan n)$, (e^n) , and $(\ln n)$ are unbounded. Now, we investigate further the importance of this notion and its relationship with other concepts related to sequences.

Proposition 5.2 $\blacktriangleleft$ Assume that (X, d) is a metric space and (x_n) is a convergent sequence in X . Then (x_n) is bounded.

Proof 5.8 Suppose that $x_n \rightarrow x$ for some $x \in X$. For $\varepsilon = 1$, we can find $n_0 \in \mathbb{N}$ such that

$$d(x_n, x) < 1 \quad \text{for each } n \geq n_0.$$

Put

$$M = \max \{d(x_1, x), \dots, d(x_{n_0-1}, x), 1\}.$$

It is easy to see that

$$\text{diam}(\{x_n\}) = \sup \{d(x_n, x_m) : n, m \in \mathbb{N}\} \leq 2M.$$

This proves the desired result.

Remark 5.1 The key point in the proof of [Proposition 5.2](#) is that for sufficiently large $n \in \mathbb{N}$, we obtain a uniform bound (namely, 1) for all the terms $d(x_n, x)$ with $n \geq n_0$; this follows from convergence. Therefore, only finitely many terms of the form $d(x_n, x)$, namely those with $n = 1, \dots, n_0 - 1$, remain, and hence the maximum exists.

Note that the converse of [Proposition 5.2](#) is not true in general; see [Example 5.4](#). Nevertheless, if we impose an additional condition on boundedness, we can recover convergence. Although in some situations monotonicity may be studied in partially ordered sets, due to the theme of this book, we return to the real numbers, the heart of the “analysis island.”

Definition 5.6 Assume that $X = \mathbb{R}$ with the usual order. We say that a sequence $(x_n) \subseteq X$ is increasing provided that $x_{n+1} \geq x_n$ for each $n \in \mathbb{N}$; (x_n) is called decreasing provided that $x_n \geq x_{n+1}$ for each $n \in \mathbb{N}$. The sequence is said to be strictly increasing (respectively, strictly decreasing) if the above inequalities hold strictly; that is, $x_{n+1} > x_n$ (respectively, $x_n > x_{n+1}$).

For example, the sequence $(\frac{1}{n})$ is strictly decreasing, (2^n) is strictly increasing, and the constant sequence (2) is both increasing and decreasing. We briefly call every increasing or decreasing sequence monotone.

Now, we are ready to prove the converse of [Proposition 5.2](#) in a special but popular case.

Theorem 5.7 Assume that $X = \mathbb{R}$ with the Euclidean topology. Then every monotone bounded sequence in X is convergent.

Proof 5.9 Let $(x_n) \subseteq \mathbb{R}$ be a bounded (say, by a constant M) increasing sequence. Since the set $\{x_n : n \in \mathbb{N}\}$ is bounded above by M , by the axiom of completeness it has a supremum, say α . We claim that $x_n \rightarrow \alpha$.

Let $\varepsilon > 0$ be given. By the definition of the supremum, there exists $n_0 \in \mathbb{N}$ such that

$$x_{n_0} > \alpha - \varepsilon,$$

or equivalently, $\alpha - x_{n_0} < \varepsilon$. Thus, for each $n \geq n_0$, we have

$$\alpha - x_n < \alpha - x_{n_0} < \varepsilon.$$

On the other hand, by monotonicity, we have $x_n - \alpha \leq 0 < \varepsilon$, and hence

$$|x_n - \alpha| < \varepsilon.$$

This proves the claim. The proof for the decreasing case is similar.

Proposition 5.3 [↗](#) Assume that $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above. Then the following statements hold.

1. If $\sup A = \alpha$, then there exists a sequence (a_n) in A such that $a_n \rightarrow \alpha$.
2. If there exists an increasing sequence $(a_n) \subseteq A$ such that $a_n \rightarrow \alpha$, then $\sup A = \alpha$.

Proof 5.10 The proof is straightforward and relies on the essential properties of the supremum. To prove the first statement, note that by the definition of the supremum, for each $n \in \mathbb{N}$ we can find $a_n \in A$ such that

$$a_n > \alpha - \frac{1}{n},$$

and hence $\alpha - a_n < \frac{1}{n}$. Moreover, since $a_n \leq \alpha$, we have

$$a_n - \alpha \leq 0 < \frac{1}{n} \quad \text{for each } n \in \mathbb{N}.$$

Therefore,

$$|a_n - \alpha| < \frac{1}{n},$$

which shows that the sequence (a_n) converges to α .

For the second statement, assume that (a_n) is an increasing sequence in A converging to α . Since A is bounded above, the axiom of completeness guarantees that $\sup A$ exists. We show that $\alpha = \sup A$. It is clear that $a_n \leq \alpha$ for each $n \in \mathbb{N}$. Moreover, if z is any upper bound of A , then $a_n \leq z$ for each $n \in \mathbb{N}$. Letting $n \rightarrow \infty$, we obtain $\alpha \leq z$, which shows that α is the least upper bound of A .

Corollary 5.1 Assume that $\emptyset \neq A \subseteq \mathbb{R}$ is bounded below. Then the following statements hold.

1. If $\inf A = \alpha$, then there exists a sequence (a_n) in A such that $a_n \rightarrow \alpha$.
2. If there exists a decreasing sequence $(a_n) \subseteq A$ such that $a_n \rightarrow \alpha$, then $\inf A = \alpha$.

Proof 5.11 Although we can present a direct proof, we prefer to use our previous results. For the first statement, note that $-A$ is bounded above, so $\sup(-A)$ exists and satisfies

$$\sup(-A) = -\inf(A).$$

By [Proposition 5.3](#), there exists a sequence $(-a_n)$ in $-A$ such that $-a_n \rightarrow \sup(-A)$. Consequently, $a_n \rightarrow \inf(A)$.

For the second statement, the given sequence (a_n) is decreasing, and hence $(-a_n)$ is increasing. Moreover, since $a_n \rightarrow \alpha$, we have $-a_n \rightarrow -\alpha$. By [Proposition 5.3](#), we obtain

$$-\alpha = \sup(-A) = -\inf(A),$$

which implies $\alpha = \inf(A)$, as claimed.

Corollary 5.2 Assume that $\emptyset \neq A \subseteq \mathbb{R}$ is bounded below. Then the following statements hold.

1. If $\inf A = \alpha$, then there exists a sequence (a_n) in A such that $a_n \rightarrow \alpha$.
2. If there exists a decreasing sequence $(a_n) \subseteq A$ such that $a_n \rightarrow \alpha$, then $\inf A = \alpha$.

Proof 5.12 Although we can present a direct proof, we prefer to use our previous results. For the first statement, note that $-A$ is bounded above, so $\sup(-A)$ exists and satisfies

$$\sup(-A) = -\inf(A).$$

By [Proposition 5.3](#), there exists a sequence $(-a_n)$ in $-A$ such that $-a_n \rightarrow \sup(-A)$. Consequently, $a_n \rightarrow \inf(A)$.

For the second statement, the given sequence (a_n) is decreasing, and hence $(-a_n)$ is increasing. Moreover, since $a_n \rightarrow \alpha$, we have $-a_n \rightarrow -\alpha$. By [Proposition 5.3](#), we obtain

$$-\alpha = \sup(-A) = -\inf(A),$$

which implies $\alpha = \inf(A)$, as claimed.

Nets: Ambassadors at every Moment and in every Place, without Boundaries

In general, the stamp of the Convergence archipelago has high credibility in the Topology realm. For example, the credibility of the convergence stamp of sequences in metric spaces is similar to that of the continuity stamp, but this is not the case everywhere across the Topology realm, and sequences are not essentially able to manage and represent the archipelago. In fact, the Topology realm covers a very large area of the ocean of mathematics, whose complete management and productivity is very difficult. In addition, there are remote, untouched areas full of ideas and creativity, with precious gems for export and for product development. These parts are located deep in some black holes, where it is not possible to manage a sequence with countable inhabitants (this management is suitable for the inhabitants of parts such as metric spaces, for which well-defined upstream rules have been formulated). Therefore, in the convergence factories located within the Convergence archipelago, topology forges nets from a secret combination of the Diamond of Hope and the Blue Diamond, competent representatives for entering and exiting all areas of the Topology realm. The management of these nets is the execution of all possible projects in the Topology realm. For example, some technically valuable planktons cannot be processed by the sequences, but nets can discover and process them. In general, the role of sequences and nets in the Convergence archipelago is the orderly management of projects, the discovery and process of ideas and techniques, and the completion of projects, achieving the stamp of effective commercial exchanges between different domains of mathematics and the ocean of Applications. In fact, the engines of all boats, trains, and submarines of continuity are formed with them and provide the basis for entering the paradise of mathematics.

We recall the following definition from set theory; see [Chapter 3](#) for its first introduction.

Definition 5.7 Assume that $\Lambda \neq \emptyset$ is a set and that $\preceq$ is a binary relation on Λ . We say that $(\Lambda, \preceq)$ is a partially ordered set provided that the relation $\preceq$ is reflexive, transitive, and anti-symmetric; that is, for every $x, y, z \in \Lambda$, we have

1. $x \preceq x$ (reflexivity),

2. $x \preceq y$ and $y \preceq x$ imply $x = y$ (anti-symmetry),
3. $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ (transitivity).

We now make the following observation.

Definition 5.8 Assume that $(\Lambda, \preceq)$ is a partially ordered set. We say that Λ is upward directed provided that for each $x, y \in \Lambda$, there exists $z \in \Lambda$ such that $x \preceq z$ and $y \preceq z$. We say that Λ is downward directed if for each $t, s \in \Lambda$, there exists $w \in \Lambda$ such that $w \preceq t$ and $w \preceq s$. The set Λ is called directed if it is both upward and downward directed.

We illustrate this definition with a few concise examples.

Example 5.8 The usual order on $\mathbb{R}$ is directed, since for every $x, y \in \mathbb{R}$, we have

$$x \wedge y \leq x, y \leq x \vee y.$$

Moreover, every subset of $\mathbb{R}$, such as $\mathbb{N}$, is also a directed set.

Example 5.9 Assume that $\Lambda \neq \emptyset$. Consider the partially ordered set $(P(\Lambda), \subseteq)$. This is a directed set, since for every $A, B \in P(\Lambda)$ we have

$$A \cap B \subseteq A, B \subseteq A \cup B.$$

Now, we have the definition of a net, which is an extension of the notion of a sequence; this definition allows us to interpret continuity freely in terms of nets in arbitrary topological spaces.

Definition 5.9 Assume that $X \neq \emptyset$ is a set and $(\Lambda, \preceq)$ is a directed set. Any family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of X is called a net in X directed by Λ . In this case, we say that the net $\{x_\lambda\}_{\lambda \in \Lambda}$ is indexed by Λ , and we call Λ the index set for the net.

In the special case when $\Lambda = \mathbb{N}$ with the usual order inherited from $\mathbb{R}$, the notion of a net reduces to the familiar notion of a sequence. Thus, nets are an extension of sequences. Now, we can extend the results regarding sequences to the net version.

Definition 5.10 Suppose (X, d) is a metric space and $(x_\lambda)_{\lambda \in \Lambda} \subseteq X$ is a net. We say that $(x_\lambda)_{\lambda \in \Lambda}$ is convergent in X if there exists $x \in X$ such that for each $\varepsilon > 0$, there exists an index $\lambda_\varepsilon \in \Lambda$ such that for each $\lambda \geq \lambda_\varepsilon$, we have

$$d(x_\lambda, x) < \varepsilon.$$

Otherwise, we say that the net $(x_\lambda)_{\lambda \in \Lambda}$ is divergent.

In other words, a net $(x_\lambda)_{\lambda \in \Lambda}$ is convergent to $x \in X$ if, for any open set U containing x , there exists a sufficiently large λ such that $x_\lambda \in U$. As we see, this definition is very similar to the one for sequences, so almost all results obtained for sequences also hold for nets; however, there are some elegant differences.

Example 5.10 Assume that $X = \mathbb{R}$ with the usual order. Consider $\Lambda = [0, 1]$. The net $\left\{ \frac{1}{r^2+1} \right\}_{r \in \Lambda}$ is convergent to $\frac{1}{2}$. Take any $\varepsilon > 0$; without loss of generality, assume $\varepsilon < \frac{1}{2}$. Put

$$r_\varepsilon = 2\sqrt{\frac{1-2\varepsilon}{1+2\varepsilon}} \in [0, 1].$$

Then, for any $r \geq r_\varepsilon$, we have

$$r^2 \geq r_\varepsilon^2 = 4 \frac{1 - 2\varepsilon}{1 + 2\varepsilon} \Rightarrow r^2 + 2\varepsilon r_\varepsilon^2 \geq 4(1 - 2\varepsilon),$$

so that

$$\varepsilon(2r^2 + 8) \geq 4 - r^2.$$

Therefore,

$$\left| \frac{1}{1 + r^2} - \frac{1}{2} \right| = \frac{|1 - r^2|}{2(r^2 + 1)} < \frac{4 - r^2}{2r^2 + 8} \leq \varepsilon.$$

Remark 5.2 Note that the type of order is very important for a net to either converge or diverge. We did not have this issue for sequences, as there is only one natural order (inherited from the reals).

For example, if we consider $\Lambda = \mathbb{R}$ with the usual order, the net

$$\left(\frac{1}{r^2 + 1} \right)_{r \in \mathbb{R}}$$

is convergent to 0.

Now, consider Λ with the inverse order, that is, $x \preceq y$ if and only if $x \geq y$ for each $x, y \in \Lambda$. Here, $\preceq$ is the inverse order (as we define), and $\leq$ is the usual order of $\mathbb{R}$. In this case, we also have $\lim_{\lambda \rightarrow -\infty} \frac{1}{r^2 + 1} = 0$.

However, if we put $\Lambda = [0, 1]$ with the usual order (inherited from $\mathbb{R}$), then

$$\lim_{\lambda \rightarrow 1} \frac{1}{r^2 + 1} = \frac{1}{2}.$$

But if we consider $\Lambda = [0, 1]$ with the inverse order, we have

$$\lim_{\lambda \rightarrow 0} \frac{1}{r^2 + 1} = 1.$$

Let us now establish some similarities between nets and sequences. The proofs are analogous to those for sequences, so we leave them to the reader.

Theorem 5.8 Assume that (X, d) is a metric space. Then the following statements hold:

1. The limit of any net is unique.
2. If two nets $(x_\lambda)_{\lambda \in \Lambda}$ and $(y_\lambda)_{\lambda \in \Lambda}$ are eventually equal, i.e., there exists an index λ_0 such that $x_\lambda = y_\lambda$ for each $\lambda \geq \lambda_0$, then

$$x_\lambda \rightarrow x \quad \text{if and only if} \quad y_\lambda \rightarrow x.$$

Theorem 5.9

1. $x \in \overline{A}$ if and only if there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in A such that $x_\lambda \rightarrow x$.

2. $x \in A'$ if and only if there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ of distinct elements of A such that $x_\lambda \rightarrow x$.

Now, we present an important difference between nets and sequences. Recall that for a metric space (X, d) , a net $(x_\lambda)_{\lambda \in \Lambda}$ is *bounded* provided that the set $\{x_\lambda : \lambda \in \Lambda\}$ is bounded in X ; in other words,

$$\sup \{d(x_\lambda, x_\mu) : \lambda, \mu \in \Lambda\} < \infty.$$

We know that a convergent sequence is bounded ([Proposition 5.2](#)). However, this does not hold for general nets, as the following example shows.

Example 5.11 Assume that $\Lambda = [0, 1]$ with the usual order. Define the net $(x_\lambda)_{\lambda \in \Lambda}$ by

$$x_\lambda = \begin{cases} \frac{1}{\lambda}, & 0 < \lambda < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, $(x_\lambda)_{\lambda \in \Lambda}$ is not bounded, but $x_\lambda \rightarrow 1$.

So, the following definition makes sense in the context of nets.

Definition 5.11 Assume that (X, d) is a metric space and $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X . We say that this net is *eventually bounded* provided that there exists an index λ_0 such that the set

$$\{x_\lambda : \lambda \geq \lambda_0\}$$

is bounded in X .

Clearly, every bounded net is eventually bounded; however, as we have seen, the converse does not hold for general nets. Furthermore, it is obvious that a sequence is bounded if and only if it is eventually bounded.

Now, we have the following result:

Proposition 5.4 Every convergent net in a metric space is eventually bounded.

Proof 5.13 Assume that (X, d) is a metric space and $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X that converges to $x \in X$. For $\varepsilon = 1$, there exists an index λ_ε such that

$$d(x_\lambda, x) < 1 \quad \text{for each } \lambda \geq \lambda_\varepsilon.$$

Thus, for every $\lambda_1, \lambda_2 \geq \lambda_\varepsilon$, we have

$$d(x_{\lambda_1}, x_{\lambda_2}) \leq d(x_{\lambda_1}, x) + d(x_{\lambda_2}, x) < 1 + 1 = 2,$$

so that the set

$$\{x_\lambda : \lambda \geq \lambda_\varepsilon\}$$

is bounded in X , showing that the net is eventually bounded.

Now, we present the definition of a *subnet* (which is a generalization of the notion of a subsequence); however, it is slightly more complicated.

Definition 5.12 Assume that (X, d) is a metric space and $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X . Moreover, assume that Γ is another directed set and there exists a function $N : \Gamma \rightarrow \Lambda$ with the property that for each $\lambda_0 \in \Lambda$, there exists $\gamma_0 \in \Gamma$ such that if $\gamma \geq \gamma_0$, then

$$N(\gamma) \geq \lambda_0.$$

In this case, the net $(x_{N(\gamma)})_{\gamma \in \Gamma}$ is called a subnet of $(x_\lambda)_{\lambda \in \Lambda}$.

Remark 5.3 First, let us explain that the notion of a subnet generalizes the notion of a subsequence. In the definition of a subnet, take $\Lambda = \mathbb{N}$ and put $\Gamma = \mathbb{N}$. Define $N : \mathbb{N} \rightarrow \mathbb{N}$ via $N(k) = n_k$, where $\{n_k\}$ is any strictly increasing sequence of natural numbers.

For each $n_0 \in \mathbb{N}$, we can find $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq n_0$ (possible since $\{n_k\}$ is strictly increasing). Thus, for each $k \geq k_0$, we have

$$N(k) = n_k \geq n_{k_0} \geq n_0.$$

Hence, when we reduce the notion of a subnet to sequences, we obtain a subsequence.

Intuitively, when dealing with general nets, to construct a subnet we need to pick elements of the index set such that, for sufficiently large indices, the subnet captures the behavior of the original net. In other words, since in a general net the index set can be very large (potentially with high cardinality), we need to ensure that the chosen indices are “representative” enough to evaluate the behavior of the net via the subnet.

Theorem 5.10 Assume that (X, d) is a metric space and $(x_\lambda)_{\lambda \in \Lambda}$ is a net that converges to $x \in X$. Then, every subnet of $(x_\lambda)_{\lambda \in \Lambda}$ also converges to x .

Proof 5.14 Assume that $(x_{N(\gamma)})_{\gamma \in \Gamma}$ is a subnet of $(x_\lambda)_{\lambda \in \Lambda}$, in the sense of the definition. For a given $\varepsilon > 0$, we can find λ_ε such that for each $\lambda \geq \lambda_\varepsilon$,

$$d(x_\lambda, x) < \varepsilon.$$

By the definition of a subnet, for this λ_ε , there exists an index γ_ε such that for each $\gamma \geq \gamma_\varepsilon$, we have $N(\gamma) \geq \lambda_\varepsilon$. Hence,

$$d(x_{N(\gamma)}, x) < \varepsilon.$$

Definition 5.13 Assume that (X, d) is a metric space and $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X . We say that this net is Cauchy provided that for each $\varepsilon > 0$, there exists an index λ_ε such that for each $\lambda, \mu \geq \lambda_\varepsilon$,

$$d(x_\lambda, x_\mu) < \varepsilon.$$

It is easy to verify that for sequences (considered as nets), this definition coincides with the usual notion of a Cauchy sequence. The following results can be proved similarly and are left to the reader.

Proposition 5.5 In every metric space, every convergent net is a Cauchy net.

Proposition 5.6 Every Cauchy net in a metric space is eventually bounded.

Now, we have the following important theorem:

Theorem 5.11  Assume that (X, d) is a metric space and $(x_\lambda)_{\lambda \in \Lambda}$ is a Cauchy net in X . If there exists a subnet $(x_{N(\gamma)})_{\gamma \in \Gamma}$ of $(x_\lambda)_{\lambda \in \Lambda}$ that converges to some $x \in X$, then the net $(x_\lambda)_{\lambda \in \Lambda}$ also converges to x .

Proof 5.15 Since $(x_\lambda)_{\lambda \in \Lambda}$ is Cauchy, for a given $\varepsilon > 0$, there exists an index λ_0 such that for each $\lambda, \mu \geq \lambda_0$,

$$d(x_\lambda, x_\mu) < \frac{\varepsilon}{2}.$$

On the other hand, since $x_{N(\gamma)} \rightarrow x$, there exists an index γ_0 such that for each $\gamma \geq \gamma_0$,

$$d(x_{N(\gamma)}, x) < \frac{\varepsilon}{2}.$$

By the definition of a subnet, for λ_0 , there exists γ_1 such that for each $\gamma \geq \gamma_1$, $N(\gamma) \geq \lambda_0$. Since Γ is a directed set, there exists γ_2 with $\gamma_2 \geq \gamma_0$ and $\gamma_2 \geq \gamma_1$. Put $\mu_0 = N(\gamma_2)$. Then, $\gamma_2 \geq \gamma_0$ and $N(\gamma_2) \geq \lambda_0$, so for each $\lambda \geq \lambda_0$, we have

$$d(x_\lambda, x) \leq d(x_\lambda, x_{\mu_0}) + d(x_{\mu_0}, x) = d(x_\lambda, x_{N(\gamma_2)}) + d(x_{N(\gamma_2)}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Furthermore, we have the following lemma:

Lemma 5.1 $\hookrightarrow$ In a metric space (X, d) , every subnet of a Cauchy net is also Cauchy.

Proof 5.16 Assume that $(x_\lambda)_{\lambda \in \Lambda}$ is a Cauchy net and $(x_{N(\gamma)})_{\gamma \in \Gamma}$ is a subnet of it. For a given $\varepsilon > 0$, there exists an index λ_ε such that for each $\lambda, \mu \geq \lambda_\varepsilon$,

$$d(x_\lambda, x_\mu) < \varepsilon.$$

By the definition of a subnet, for this λ_ε , there exists $\gamma_\varepsilon \in \Gamma$ such that for each $\gamma \geq \gamma_\varepsilon$, $N(\gamma) \geq \lambda_\varepsilon$. Hence, for each $\gamma, \delta \geq \gamma_\varepsilon$, we have $N(\gamma), N(\delta) \geq \lambda_\varepsilon$, so that

$$d(x_{N(\gamma)}, x_{N(\delta)}) < \varepsilon,$$

which completes the proof.

Recall that a metric space (X, d) is complete provided that every Cauchy sequence in X is convergent. Now, we characterize completeness of a metric space in terms of nets.

Theorem 5.12 A metric space (X, d) is complete if and only if every Cauchy net in X is convergent.

Proof 5.17 Let (X, d) be complete and $(x_\lambda)_{\lambda \in \Lambda}$ be a Cauchy net in X . We can choose a subsequence $(x_{N(n)})_{n \in \mathbb{N}}$ (by taking $\Gamma = \mathbb{N}$). By [Lemma 5.1](#), this subsequence is also Cauchy, hence it converges to some $x \in X$. Then, by [Theorem 5.11](#), the net $(x_\lambda)_{\lambda \in \Lambda}$ has a convergent subnet (the chosen subsequence) and therefore converges to $x \in X$.

Conversely, if every Cauchy net is convergent, then in particular every Cauchy sequence (a special case of a net) is convergent, so (X, d) is complete.

5.4 SEQUENCES IN THE REAL NUMBERS: A REMARKABLE INTERPLAY OF ORDER AND COMPLETENESS

In this section, we make use of the deep connection between the order structure and the topological structure of the real numbers to establish several surprising and useful results.

Lemma 5.2 $\hookrightarrow$ Assume that $(x_n) \subseteq \mathbb{R}$ is a sequence such that $x_n \rightarrow x$ and $x > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n > 0$ for each $n \geq n_0$.

Proof 5.18 Take $\varepsilon = \frac{x}{2} > 0$. Since $x_n \rightarrow x$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$,

$$|x_n - x| < \frac{x}{2} \Rightarrow \frac{x}{2} < x_n < \frac{3x}{2}.$$

Corollary 5.3 $\square$ Assume that (x_n) is a sequence in $\mathbb{R}$ that converges to $x > 0$. Then the sequence $\left(\frac{1}{x_n}\right)$ is bounded.

Proof 5.19 By [Lemma 5.2](#), the elements of (x_n) are positive for sufficiently large n , and moreover $x_n > \frac{x}{2}$ for $n \geq n_0$. Therefore,

$$\frac{1}{x_n} < \frac{2}{x},$$

showing that the sequence $\left(\frac{1}{x_n}\right)$ is bounded.

Theorem 5.13 Assume that $X = \mathbb{R}$ with the Euclidean metric, and let $(x_n), (y_n) \subseteq X$ be sequences such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Then:

1. $x_n \pm y_n \rightarrow x \pm y$;
2. $x_n y_n \rightarrow xy$;
3. $\frac{x_n}{y_n} \rightarrow \frac{x}{y}$, provided that $y \neq 0$ and $y_n \neq 0$ for sufficiently large n .

Proof 5.20 Let $\varepsilon > 0$ be arbitrary.

(1) There exist $K_1, K_2 \in \mathbb{N}$ such that for all $n > K_1$, $|x_n - x| < \frac{\varepsilon}{2}$ and for all $n > K_2$, $|y_n - y| < \frac{\varepsilon}{2}$. Hence, for $n > \max \{K_1, K_2\}$,

$$|(x_n \pm y_n) - (x \pm y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(2) Since (y_n) is convergent, it is bounded, so there exists $M > 0$ such that $|y_n| \leq M$ for all n . Since $x_n \rightarrow x$, there exists $K_3 \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2M}$ for $n \geq K_3$. Similarly, there exists $K_4 \in \mathbb{N}$ such that $|y_n - y| < \frac{\varepsilon}{2(|x|+1)}$ for $n \geq K_4$. Then, for $n \geq \max \{K_3, K_4\}$,

$$|x_n y_n - xy| \leq |x_n - x||y_n| + |x||y_n - y| < \frac{\varepsilon}{2M}(M) + \frac{\varepsilon}{2(|x|+1)}(|x|) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(3) Since $|y| > 0$, by [Corollary 5.3](#), there exists $K_5 \in \mathbb{N}$ such that $\frac{1}{|y_n|} < \frac{2}{|y|}$ for $n \geq K_5$. Also, choose K_6 such that $|x_n - x| < \frac{\varepsilon|y|}{4}$ for each $n > K_6$ and K_7 such that $|y_n - y| < \frac{\varepsilon|y|^2}{4(|x|+1)}$ for each $n > K_7$. Now, for n large enough,

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| = \left| \frac{x_n y - x y_n}{y_n y} \right| \leq \frac{1}{|y_n||y|} (|x_n - x||y| + |x||y_n - y|) < \varepsilon.$$

Definition 5.14 Assume that $X = \mathbb{R}$ with the Euclidean topology. A sequence $(x_n) \subseteq X$ is called

- increasing if $x_n \leq x_{n+1}$ for all n , and decreasing if $x_n \geq x_{n+1}$ for all n ;
- strictly increasing (resp. strictly decreasing) if the inequalities hold strictly;

- *monotone (resp. strictly monotone) if it is increasing or decreasing (resp. strictly increasing or strictly decreasing).*

In most results regarding monotone sequences, the required condition only needs to hold for sufficiently large n , since finitely many initial terms do not affect the limiting behavior of the sequence.

Theorem 5.14 $\hookrightarrow$ *Any increasing sequence (x_n) in $\mathbb{R}$ that is bounded above is convergent.*

Proof 5.21 *By hypothesis, there exists $y \in \mathbb{R}$ such that $x_n \leq y$ for all $n \in \mathbb{N}$. By the axiom of completeness in $\mathbb{R}$,*

$$x = \sup \{x_n : n \in \mathbb{N}\}$$

exists. We claim that $x_n \rightarrow x$.

Let $\varepsilon > 0$ be given. By the definition of the supremum, there exists $K_\varepsilon \in \mathbb{N}$ such that

$$x - \varepsilon < x_{K_\varepsilon}.$$

Since (x_n) is increasing, for all $n \geq K_\varepsilon$,

$$x - \varepsilon < x_{K_\varepsilon} \leq x_n \leq x < x + \varepsilon.$$

Hence,

$$|x_n - x| < \varepsilon,$$

showing that $x_n \rightarrow x$.

Corollary 5.4 *Every decreasing sequence (x_n) in $\mathbb{R}$ that is bounded below is convergent.*

Proof 5.22 *If (x_n) is decreasing and bounded below, then $(-x_n)$ is increasing and bounded above. By [Theorem 5.14](#), $-x_n \rightarrow x$ for some $x \in \mathbb{R}$. Hence, $x_n \rightarrow -x$.*

Theorem 5.15 (Squeeze Theorem) $\hookrightarrow$ *Let (x_n) , (y_n) , and (z_n) be sequences in $\mathbb{R}$ such that*

$$x_n \leq y_n \leq z_n \quad \text{for all } n \in \mathbb{N}.$$

If $x_n \rightarrow x$ and $z_n \rightarrow x$ for some $x \in \mathbb{R}$, then $y_n \rightarrow x$ as well.

Proof 5.23 *Let $\varepsilon > 0$ be given. Since $x_n \rightarrow x$, there exists $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$,*

$$x - \varepsilon < x_n.$$

Since $z_n \rightarrow x$, there exists $K_2 \in \mathbb{N}$ such that for all $n \geq K_2$,

$$z_n < x + \varepsilon.$$

Set $K_\varepsilon = \max \{K_1, K_2\}$. Then, for all $n \geq K_\varepsilon$,

$$x - \varepsilon < x_n \leq y_n \leq z_n < x + \varepsilon,$$

so that

$$|y_n - x| < \varepsilon.$$

Hence, $y_n \rightarrow x$.

Corollary 5.5 Let (x_n) be a null sequence (i.e., $x_n \rightarrow 0$) and (y_n) a bounded sequence in $\mathbb{R}$. Then the sequence $(x_n y_n)$ is also null.

Proof 5.24 Since (y_n) is bounded, there exists $M > 0$ such that for all $n \in \mathbb{N}$,

$$|y_n| \leq M \quad \Rightarrow \quad -M \leq y_n \leq M.$$

Then, for all n ,

$$-M|x_n| \leq x_n y_n \leq M|x_n|.$$

Since $x_n \rightarrow 0$, by [Theorem 5.15](#), it follows that $x_n y_n \rightarrow 0$.

Definition 5.15 Assume that $X = \mathbb{R}$ with the Euclidean metric and $(x_n) \subseteq X$. We call a term x_{n_0} a tip of the sequence if

$$x_n \leq x_{n_0} \quad \text{for all } n \geq n_0.$$

For example, in the sequence $(\frac{1}{n})$ (and more generally any decreasing sequence), every term is a tip. On the other hand, the sequence $((-1)^n)$ has no tip.

Theorem 5.16 $\spadesuit$ Every sequence (x_n) in $\mathbb{R}$ contains a monotone subsequence.

Proof 5.25 Consider the indices corresponding to tips. If there are infinitely many such indices, the corresponding terms form a decreasing subsequence. Indeed, if $n_1 < n_2$ are two tip indices, then by definition $x_{n_2} \leq x_{n_1}$.

Otherwise, if there are only finitely many tips, there exists $K \in \mathbb{N}$ such that for all $n \geq K$, x_n is not a tip. Since x_K is not a tip, there exists $n_1 > K$ with $x_{n_1} > x_K$. Similarly, since x_{n_1} is not a tip, there exists $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. Proceeding inductively, we obtain an increasing subsequence (x_{n_k}) .

Theorem 5.17 (Bolzano–Weierstrass Theorem) $\spadesuit$ Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.

Proof 5.26 Let (x_n) be bounded. By [Theorem 5.16](#), it contains a monotone subsequence, which is still bounded. By [Theorem 5.14](#) (or its decreasing counterpart), this monotone subsequence converges. Hence, (x_n) has a convergent subsequence.

Theorem 5.18 The Euclidean spaces $\mathbb{R}$ and $\mathbb{R}^n$ are complete.

Proof 5.27 Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Then (x_n) is bounded, so by [Theorem 5.17](#), it has a convergent subsequence. By [Theorem 5.5](#), the entire sequence (x_n) converges to the same limit.

Definition 5.16 We extend the real numbers to the extended reals:

$$\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}.$$

This allows every sequence (x_n) in $\mathbb{R}^*$ to have a supremum. If the sequence is bounded, its supremum exists in $\mathbb{R}$ by the axiom of completeness; otherwise, the supremum is $+\infty$. Similarly, the infimum exists or equals $-\infty$.

Definition 5.17 Let (x_n) be a sequence in $\mathbb{R}$. The set of all subsequential limits of (x_n) is denoted by

$$\overline{(x_n)} \subseteq \mathbb{R}^*.$$

If (x_n) is bounded, then $\overline{(x_n)} \subseteq \mathbb{R}$.

Here, $s \in \overline{(x_n)}$ if there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \rightarrow s$.

Lemma 5.3 Let (X, d) be a metric space and (x_n) a sequence in X . Then the set $\overline{(x_n)}$ of subsequential limits is closed.

Proof 5.28 Let $S = \overline{(x_n)}$. Take any $y \in \overline{S}$. Then there exists a sequence $(y_k) \subseteq S$ such that $y_k \rightarrow y$. For each k , since y_k is a subsequential limit, there exists an index n_k with

$$d(x_{n_k}, y_k) < \frac{1}{2^k}.$$

By choosing $\{n_k\}$ strictly increasing, we have

$$d(x_{n_k}, y) \leq d(x_{n_k}, y_k) + d(y_k, y) \rightarrow 0.$$

Hence, $y \in S$, showing that S is closed.

Theorem 5.19 Let (X, d) be a metric space, (x_n) a sequence in X , and set $A = \{x_n : n \in \mathbb{N}\}$. Then

$$\overline{(x_n)} \cup A = \overline{A}.$$

Proof 5.29 If $x \in \overline{(x_n)}$, there exists a subsequence (x_{n_k}) converging to x . For every $r > 0$, there exists K_r such that for all $k \geq K_r$, $x_{n_k} \in N_r^{(x)} \cap A$. Thus, $x \in \overline{A}$. Clearly, $A \subseteq \overline{A}$, so

$$\overline{(x_n)} \cup A \subseteq \overline{A}.$$

Conversely, if $x \in \overline{A} - A = A'$, there exists a sequence of distinct elements of A converging to x , i.e., a subsequence of (x_n) . Hence, $x \in \overline{(x_n)}$.

Definition 5.18 Let (x_n) be a sequence in $\mathbb{R}$. The limit superior of (x_n) , denoted $\limsup x_n$, is defined as the largest element of $\overline{(x_n)}$ in $\mathbb{R}^*$. Similarly, the limit inferior of (x_n) , denoted $\liminf x_n$, is the smallest element of $\overline{(x_n)}$.

Lemma 5.4 In $\mathbb{R}$, $\limsup x_n = +\infty$ if and only if (x_n) is not bounded above. Similarly, $\liminf x_n = -\infty$ if and only if (x_n) is not bounded below.

Proof 5.30 Assume $\limsup x_n = +\infty$, i.e., $\sup \overline{(x_n)} = +\infty$. Then $\overline{(x_n)}$ is not bounded above, so for every $M \in \mathbb{R}$, there exists $\alpha_M \in \overline{(x_n)}$ with $\alpha_M > M$. Since α_M is a subsequential limit, infinitely many x_n satisfy $x_n > M$. Hence (x_n) is unbounded above.

Conversely, if (x_n) is not bounded above, it contains a subsequence diverging to $+\infty$, so $+\infty \in \overline{(x_n)}$, giving $\limsup x_n = +\infty$.

The proof for $\liminf x_n$ is analogous.

Lemma 5.5 Let $A \subseteq \mathbb{R}$ be non-empty and bounded above. Then $\sup A \in \overline{A}$. Similarly, if A is bounded below, $\inf A \in \overline{A}$.

Proof 5.31 Let $\alpha = \sup A$. For any $r > 0$, there exists $a_0 \in A$ with $\alpha - r < a_0 \leq \alpha < \alpha + r$. Then $a_0 \in N_r^{(\alpha)} \cap A$, so $\alpha \in \overline{A}$. The argument for $\inf A$ is similar.

Definition 5.19 Let X be a set and $\phi : X \rightarrow X$ a function. A point $x_0 \in X$ is a fixed point if $\phi(x_0) = x_0$.

Definition 5.20 Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : (X, d_1) \rightarrow (Y, d_2)$ is strongly contractive with constant $r < 1$ if

$$d_2(f(x), f(y)) \leq r d_1(x, y), \quad \forall x, y \in X.$$

Example 5.12 Let $X = \mathbb{R}$ with the Euclidean metric. For $0 < r < 1$, define $f_r(x) = r \sin x$. Then f_r is strongly contractive with constant r , because by the Mean Value Theorem, $|\sin x - \sin y| \leq |x - y|$, so

$$|f_r(x) - f_r(y)| = r |\sin x - \sin y| \leq r |x - y|.$$

Example 5.13 Similarly, $g_r(x) = r \cos x$ is strongly contractive on $\mathbb{R}$ with constant $r \in (0, 1)$.

Now, we state a very useful fact.

Theorem 5.20 (Banach Fixed Point Theorem) $\hookrightarrow$ Let (X, d) be a complete metric space and $\phi : X \rightarrow X$ a strongly contractive function with constant $r < 1$. Then ϕ has exactly one fixed point.

Proof 5.32 Choose any $x_0 \in X$ and define the sequence (x_n) by

$$x_{n+1} = \phi(x_n), \quad n \geq 0.$$

We first show by induction that

$$d(x_{n+1}, x_n) \leq r^n d(x_1, x_0), \quad \forall n \geq 0.$$

The base case $n = 0$ is

$$d(x_1, x_0) = d(\phi(x_0), x_0) \leq d(x_1, x_0).$$

Assume the claim for n . Then

$$d(x_{n+2}, x_{n+1}) = d(\phi(x_{n+1}), \phi(x_n)) \leq r d(x_{n+1}, x_n) \leq r^{n+1} d(x_1, x_0).$$

Now, assume that $m > n$. Using the triangle inequality repeatedly:

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{m-1} r^i d(x_1, x_0) \leq \sum_{i=n}^{\infty} r^i d(x_1, x_0).$$

Since $0 < r < 1$, the geometric series converges, so (x_n) is Cauchy. Completeness of X implies $x_n \rightarrow x \in X$.

Finally, continuity of ϕ (by contractivity) gives

$$\phi(x) = \lim_{n \rightarrow \infty} \phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x,$$

so x is the fixed point (We shall talk about this in the next Chapter, thoroughly). Now, we show the uniqueness. Suppose $x, y \in X$ are distinct fixed points. Then

$$d(x, y) = d(\phi(x), \phi(y)) \leq rd(x, y) < d(x, y),$$

a contradiction. So the fixed point is unique.

Consider [Figure 5.3](#) for a geometric illustration related to the Fixed Point Theorem.

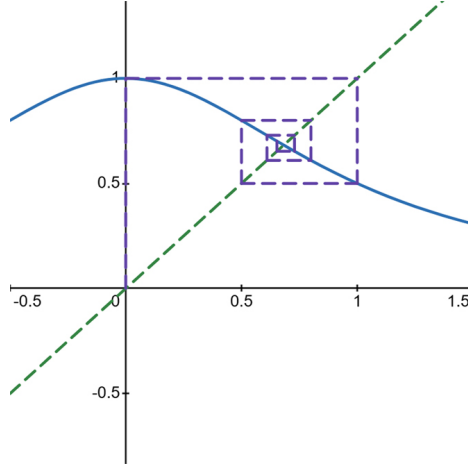


Figure 5.3 An intuitive construction of a fixed point. [↗](#)

Theorem 5.21 [↗](#) Let (x_n) be a bounded sequence in $\mathbb{R}$. Then,

$$\alpha = \limsup_{n \rightarrow \infty} x_n = \sup \overline{(x_n)},$$

if and only if the following statements hold simultaneously:

1. For every $\varepsilon > 0$, there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n > \alpha - \varepsilon;$$

2. For every $\varepsilon > 0$, there exist only finitely many $n \in \mathbb{N}$ such that

$$x_n > \alpha + \varepsilon.$$

Proof 5.33 For the direct implication, suppose $\alpha = \limsup_{n \rightarrow \infty} x_n = \sup \overline{(x_n)}$.

Since $\alpha = \max \overline{(x_n)}$, there exists a subsequence (x_{n_k}) of (x_n) such that

$$x_{n_k} \rightarrow \alpha.$$

Then, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$|x_{n_k} - \alpha| < \varepsilon \implies x_{n_k} > \alpha - \varepsilon,$$

so statement (1) holds.

For statement (2), suppose, for a contradiction, that there exists $\varepsilon_0 > 0$ such that $x_n > \alpha + \varepsilon_0$ for infinitely many n . Then we can extract a subsequence $(x_{n'_k})$ such that

$$x_{n'_k} > \alpha + \varepsilon_0 \quad \forall k.$$

Since (x_n) is bounded, $(x_{n'_k})$ is bounded and therefore has a convergent subsequence $(x_{n_{k'_l}}) \rightarrow x_0$. Clearly,

$$x_0 \geq \alpha + \varepsilon_0 > \alpha,$$

contradicting the fact that $\alpha = \sup \overline{(x_n)}$. Hence, statement (2) holds.

For the reverse implication, suppose statements (1) and (2) hold.

Statement (1) implies that for each $\varepsilon > 0$, there are infinitely many n with $x_n > \alpha - \varepsilon$, so we can extract a subsequence (x_{n_k}) satisfying

$$x_{n_k} > \alpha - \varepsilon \quad \forall k.$$

By taking $\varepsilon = \frac{1}{m}$ for $m \in \mathbb{N}$, we obtain a subsequence converging to some limit $x^* \geq \alpha - \frac{1}{m}$. Letting $m \rightarrow \infty$, we get

$$\limsup x_n \geq \alpha.$$

Statement (2) implies that for any $\varepsilon > 0$, there are only finitely many n with $x_n > \alpha + \varepsilon$. Thus, no subsequence of (x_n) can converge to a limit greater than α , so $\limsup x_n \leq \alpha$.

Combining both inequalities, we conclude

$$\limsup x_n = \alpha = \sup \overline{(x_n)}.$$

Similarly, we have an equivalent definition for the *limit inferior* of a sequence; the proof is left to the reader.

Theorem 5.22 Let (x_n) be a bounded sequence in $\mathbb{R}$. Then

$$\beta = \liminf_{n \rightarrow \infty} x_n = \inf \overline{(x_n)}$$

if and only if the following statements hold simultaneously:

1. For every $\varepsilon > 0$, there exist infinitely many n such that $x_n < \beta + \varepsilon$;
2. For every $\varepsilon > 0$, there are only finitely many n such that $x_n < \beta - \varepsilon$.

We now revisit the notions of $\limsup$ and $\liminf$ for a sequence, approaching them from a different perspective.

Definition 5.21 Let (x_n) be a sequence in $\mathbb{R}$ and $k \in \mathbb{N}$. The tail T_k of the sequence is defined by

$$T_k = \{x_n : n \geq k\}.$$

Remark 5.4  If (x_n) is bounded, then each tail T_k is also bounded, so $\sup T_k$ and $\inf T_k$ are well-defined. Moreover, $T_{k+1} \subseteq T_k$, so

$$\sup T_{k+1} \leq \sup T_k \quad \text{and} \quad \inf T_{k+1} \geq \inf T_k.$$

Theorem 5.23 Let (x_n) be a bounded sequence in $\mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} \sup x_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} x_n = \inf_k T_k.$$

Proof 5.34 Define $\alpha_k = \sup T_k$ and $\alpha = \inf_k \alpha_k$. We claim $\alpha = \lim \sup x_n$.

Statement 1 of [Theorem 5.21](#): Suppose it fails. Then there exists $\varepsilon > 0$ such that only finitely many n satisfy $x_n > \alpha - \varepsilon$. Hence, there exists $K \in \mathbb{N}$ with

$$x_n \leq \alpha - \varepsilon \quad \forall n \geq K.$$

This implies $\alpha_K = \sup_{n \geq K} x_n \leq \alpha - \varepsilon$. But $\alpha = \inf_k \alpha_k \leq \alpha_K$, giving

$$\alpha \leq \alpha_K \leq \alpha - \varepsilon < \alpha,$$

a contradiction. Hence, statement 1 holds.

Statement 2 of [Theorem 5.21](#): Given $\varepsilon > 0$, choose k_0 such that $\alpha_{k_0} < \alpha + \varepsilon$ (since $\alpha = \inf_k \alpha_k$). By [Remark 5.4](#), (α_k) is decreasing, so for all $n \geq k \geq k_0$,

$$x_n \leq \alpha_k \leq \alpha_{k_0} < \alpha + \varepsilon.$$

Thus, only finitely many n satisfy $x_n > \alpha + \varepsilon$, so statement 2 also holds. Therefore, $\alpha = \lim \sup x_n$.

An analogous statement holds for $\lim \inf$; its proof is left to the reader.

Theorem 5.24 Let (x_n) be a bounded sequence in $\mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} \inf x_n = \sup \inf_{k \in \mathbb{N}} x_n = \sup \inf_k T_k.$$

Proposition 5.7 Let $(x_n), (y_n) \subseteq \mathbb{R}$. Then

1. $\lim_{n \rightarrow \infty} \sup (x_n + y_n) \leq \lim_{n \rightarrow \infty} \sup x_n + \lim_{n \rightarrow \infty} \sup y_n$;
2. $\lim_{n \rightarrow \infty} \inf (x_n + y_n) \geq \lim_{n \rightarrow \infty} \inf x_n + \lim_{n \rightarrow \infty} \inf y_n$.

Proof 5.35

1. Using the tail characterization:

$$\sup_{n \geq k} (x_n + y_n) \leq \sup_{n \geq k} x_n + \sup_{n \geq k} y_n.$$

Taking the infimum over k , we get

$$\lim_n \sup (x_n + y_n) = \inf_k \sup_{n \geq k} (x_n + y_n) \leq \inf_k (\sup_{n \geq k} x_n + \sup_{n \geq k} y_n) \leq \lim_n \sup x_n + \lim_n \sup y_n.$$

2. Similarly,

$$\inf_{n \geq k} (x_n + y_n) \geq \inf_{n \geq k} x_n + \inf_{n \geq k} y_n.$$

Taking the supremum over k , we obtain

$$\lim_n \inf (x_n + y_n) = \sup_k \inf_{n \geq k} (x_n + y_n) \geq \sup_k \inf_{n \geq k} x_n + \sup_k \inf_{n \geq k} y_n = \lim_n \inf x_n + \lim_n \inf y_n.$$

Theorem 5.25 Baire Category Theorem) $\hookrightarrow$ If (X, d) is a complete metric space and $(U_n)_{n \in \mathbb{N}}$ is a sequence of open dense subsets of X , then

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X .

Proof 5.36 To prove density, let $V \subseteq X$ be a non-empty open set. We must show

$$V \cap \bigcap_{n=1}^{\infty} U_n \neq \emptyset.$$

Since U_1 is dense and V is open and non-empty,

$$V \cap U_1$$

is a non-empty open set.

Hence there exist $x_1 \in X$ and $r_1 > 0$ such that

$$N_{r_1}^{(x_1)} \subseteq V \cap U_1.$$

Choose $r_1 < 1$; also small enough (if necessary), and consider the closed ball

$$N_{r_1}^{[x_1]} \subseteq V \cap U_1.$$

Assume we have chosen $x_n \in X$ and $r_n > 0$ such that

$$N_{r_n}^{[x_n]} \subseteq V \cap U_1 \cap \cdots \cap U_n,$$

and $r_n < \frac{1}{n}$.

Since U_{n+1} is dense, the set

$$N_{r_n}^{(x_n)} \cap U_{n+1}$$

is non-empty and open.

Thus there exist $x_{n+1} \in X$ and $r_{n+1} > 0$ such that

$$N_{r_{n+1}}^{(x_{n+1})} \subseteq N_{r_n}^{(x_n)} \cap U_{n+1}.$$

Choose $r_{n+1} < \frac{1}{n+1}$ and take the closed ball

$$N_{r_{n+1}}^{[x_{n+1}]} \subseteq N_{r_n}^{(x_n)} \cap U_{n+1}.$$

Therefore we obtain a nested sequence of closed balls

$$N_{r_1}^{[x_1]} \supseteq N_{r_2}^{[x_2]} \supseteq N_{r_3}^{[x_3]} \supseteq \cdots$$

with $r_n \rightarrow 0$.

Now, we show that the sequence (x_n) is Cauchy. Let $\varepsilon > 0$. Choose N such that $r_N < \varepsilon$. If $m \geq n \geq N$, then

$$x_m \in N_{r_n}^{[x_n]},$$

so

$$d(x_m, x_n) \leq r_n < \varepsilon.$$

Thus (x_n) is a Cauchy sequence.

Since X is complete, there exists $x \in X$ such that

$$x_n \rightarrow x.$$

For each fixed n , all x_m with $m \geq n$ lie in the closed ball $N_{r_n}^{[x_n]}$. Because closed balls are closed sets and contain all their limit points,

$$x \in N_{r_n}^{[x_n]}.$$

Hence for every n ,

$$x \in U_n.$$

Also $x \in V$ because every ball was chosen inside V . Therefore

$$x \in V \cap \bigcap_{n=1}^{\infty} U_n.$$

Since V was arbitrary non-empty open,

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X .

There exists an interesting consequence of the Baire category theorem; it is in terms of nowhere dense subsets.

Corollary 5.6 Assume that (X, d) is a complete metric space, then X cannot be written as a countable union of nowhere dense sets.

Proof 5.37 Suppose

$$X = \bigcup_{n=1}^{\infty} A_n,$$

where each A_n is nowhere dense.

Since $(\overline{A_n})^\circ = \emptyset$, the set

$$X - \overline{A_n}$$

is open and dense in X .

Define

$$U_n := X - \overline{A_n}.$$

Each U_n is open and dense. By the Baire Category Theorem,

$$\bigcap_{n=1}^{\infty} U_n$$

is dense and in particular non-empty.

Let x belong to this intersection. Then for every n ,

$$x \notin \overline{A_n}.$$

In particular $x \notin A_n$ for all n , contradicting

$$X = \bigcup_{n=1}^{\infty} A_n.$$

Therefore X cannot be a countable union of nowhere dense sets.

5.5 EXAMPLES AROUND SEQUENTIAL SPACES

In this section, we consider several sequence spaces, that is, normed spaces whose elements are sequences.

Example 5.14 Let $1 \leq p < \infty$. Consider the class of sequences

$$\ell_p = \left\{ (x_n) : x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define the function $\|\cdot\|_p: \ell_p \rightarrow \mathbb{R}$ by

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

By Minkowski's inequality, it can be easily verified that $(\ell_p, \|\cdot\|_p)$ is a normed space. For $p = \infty$, the space $(\ell_\infty, \|\cdot\|_\infty)$ is known to be a Banach space. We will show that $(\ell_p, \|\cdot\|_p)$ is also a Banach space in the sequel.

Example 5.15 Let $1 \leq p < q < \infty$. Then $\ell_p \subseteq \ell_q$.

Proof. Let $(x_n) \in \ell_p$, i.e. $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Then $x_n \rightarrow 0$ as $n \rightarrow \infty$. Hence, for sufficiently large n , $|x_n| < 1$, which implies $|x_n|^q \leq |x_n|^p$. By the comparison test, $\sum_{n=1}^{\infty} |x_n|^q < \infty$.

The converse is not true in general. For example, $(\frac{1}{n}) \in \ell_2$ but $(\frac{1}{n}) \notin \ell_1$ since the harmonic series diverges.

Example 5.16 For any $1 \leq p < \infty$, we have $\ell_p \subseteq \ell_\infty$. To see this, take any sequence $(x_n) \in \ell_p$ so that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Then $\lim_{n \rightarrow \infty} |x_n|^p = 0$, so that $x_n \rightarrow 0$. Every convergent sequence is

bounded, so $\| (x_n) \|_\infty = \sup_n |x_n| < \infty$ by the completeness of the real numbers.

The converse is not true in general. For example, the sequence $((-1)^n)$ is bounded, but $\lim_{n \rightarrow \infty} (-1)^n$ does not exist. This means that $((-1)^n) \notin \ell_p$ for any $1 \leq p < \infty$.

Example 5.17 Assume that $1 \leq p < q < \infty$. Then we show that $\ell_p \subseteq \ell_q$. To see this, take any sequence $(x_n) \in \ell_p$ so that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Using an elementary fact about series in Calculus, we know that $\lim_{n \rightarrow \infty} |x_n|^p = 0$, so that $x_n \rightarrow 0$. Thus, for sufficiently large $n \in \mathbb{N}$, $|x_n| < 1$. Therefore, $|x_n|^q \leq |x_n|^p$. By the comparison test, we conclude that $\sum_{n=1}^{\infty} |x_n|^q < \infty$. Nevertheless, the converse is not true in general. For example, the sequence $(\frac{1}{n})$ belongs to ℓ_2 but not to ℓ_1 (the harmonic series is divergent).

Example 5.18 For any $1 \leq p < \infty$, we have $\ell_p \subseteq \ell_\infty$. To see this, take any sequence $(x_n) \in \ell_p$ so that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Therefore, $\lim_{n \rightarrow \infty} |x_n|^p = 0$, so that $x_n \rightarrow 0$. Every convergent sequence is bounded, so $\| (x_n) \|_\infty = \sup_n |x_n| < \infty$ by the completeness of the reals. The converse is not true in general. The sequence $((-1)^n)$ is bounded but $\lim_{n \rightarrow \infty} (-1)^n$ does not exist, which means that $((-1)^n) \notin \ell_p$ for every $1 \leq p$.

Example 5.19  Assume that $X = \mathbb{R}$ with the Euclidean metric. Then the sequences $(\sin n)$ and $(\cos n)$ are not convergent.

First, we show that $(\cos n)$ is divergent. Suppose, to the contrary, that it is convergent; more precisely, $\lim_{n \rightarrow \infty} \cos n = a$ for some real a . The subsequence $(\cos 2n)$ is also convergent to a . From the identity

$$1 + \cos 2n = 2 \cos^2 n$$

we conclude that $1 + a = 2a^2$, so that

$$2a^2 - a - 1 = 0.$$

By elementary Calculus, either $a = 1$ or $a = -\frac{1}{2}$.

First, assume that $a = 1$. From the identity $\sin^2 n + \cos^2 n = 1$, we conclude that $\lim_{n \rightarrow \infty} \sin n = 0$. On the other hand, we have

$$\sin(n+1) = \sin n \cos 1 + \cos n \sin 1.$$

Taking the limit on both sides gives

$$0 = 0 \times \cos 1 + a \sin 1,$$

showing a contradiction.

Now, assume that $a = -\frac{1}{2}$. Then $\lim_{n \rightarrow \infty} \sin n = \pm \frac{\sqrt{3}}{2}$. From the identity

$$\sin 2n = 2 \sin n \cos n$$

we again obtain a contradiction.

The proof for $(\sin n)$ is similar. Suppose, to the contrary, that $\lim_{n \rightarrow \infty} \sin n = b$ for some real b . Then

$$\lim_{n \rightarrow \infty} \cos n = \pm \sqrt{1 - b^2},$$

which contradicts our previous result.

Yet, we can have another approach to [Example 5.19](#).

Example 5.20 We show that

$$\lim_{x \rightarrow 0} \cos \frac{1}{x}$$

does not exist in $\mathbb{R}$, and hence

$$\lim_{x \rightarrow \infty} \cos x$$

also does not exist.

Intuitively, consider

$$x_1 = \frac{1}{2k\pi}, \quad x_2 = \frac{1}{2k\pi + \pi/2}, \quad x_3 = \frac{1}{2k\pi + \pi}.$$

Then for large k , x_1, x_2, x_3 are small but

$$\cos \frac{1}{x_1} = 1, \quad \cos \frac{1}{x_2} = 0, \quad \cos \frac{1}{x_3} = -1.$$

For a formal proof, suppose $\lim_{x \rightarrow 0} \cos \frac{1}{x} = \ell$ for some $\ell \in \mathbb{R}$. Let $\varepsilon = 1/2$. Then there exists $\delta > 0$ such that $|x| < \delta \implies |\cos(1/x) - \ell| < 1/2$. Pick k large enough that $x_1 = 1/(2k\pi) < \delta$ and $x_2 = 1/(2k\pi + \pi/2) < \delta$. Then

$$|1 - \ell| < 1/2, \quad |0 - \ell| = |\ell| < 1/2.$$

Hence,

$$1 = |1 - \ell + \ell| \leq |1 - \ell| + |\ell| < 1/2 + 1/2 = 1,$$

a contradiction. Therefore, the limit does not exist.

Theorem 5.26 For any $1 \leq p < \infty$, $(\ell_p, \|\cdot\|_p)$ is a Banach space.

Proof 5.38 Assume that (x^n) is a Cauchy sequence in ℓ_p . Note that for each $n \in \mathbb{N}$, we have $x^n = (x^n_k)_{k \in \mathbb{N}}$ and $\sum_{k=1}^{\infty} |x^n_k|^p < \infty$.

For a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $m > n > n_0$,

$$\|x^m - x^n\|_p < \varepsilon,$$

so that

$$\|x^m - x^n\|_p^p < \varepsilon^p.$$

For a fixed k , we have $|x^m_k - x^n_k| < \varepsilon$ for sufficiently large m, n . This means that $(x^n_k)_n$ is a Cauchy sequence in $\mathbb{R}$, so it converges. Assume $x^n_k \rightarrow x_k \in \mathbb{R}$. Put $x = (x_k)_{k \in \mathbb{N}}$. We show that $x \in \ell_p$ and that $\|x^n - x\|_p \rightarrow 0$.

$$\|x^m - x^n\|_p^p = \sum_{k=1}^{\infty} |x_k^m - x_k^n|^p < \varepsilon^p,$$

so that letting $m \rightarrow \infty$ and taking sufficiently large n , we have

$$\|x - x^n\|_p^p = \sum_{k=1}^{\infty} |x_k - x_k^n|^p \leq \varepsilon^p.$$

Now, for a sufficiently large n , using Minkowski's inequality, we have

$$\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} = \left(\sum_{k=1}^{\infty} |x_k + x_k^n - x_k^n|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_k - x_k^n|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |x_k^n|^p\right)^{1/p} < \varepsilon + M_n.$$

This shows that $x \in \ell_p$, as claimed.

Example 5.21 Recall that

$$\ell_{\infty} = \{(x_n) : (x_n) \text{ is a real sequence with } \sup_n |x_n| < \infty\},$$

and the norm is defined by

$$\|(x_n)\|_{\infty} = \sup_n |x_n|.$$

We claim that $(\ell_{\infty}, \|\cdot\|_{\infty})$ is a Banach space, i.e., it is complete with respect to the $\|\cdot\|_{\infty}$ norm.

Proof 5.39 Let $(x^k)_{k \in \mathbb{N}}$ be a Cauchy sequence in ℓ_{∞} with respect to the $\|\cdot\|_{\infty}$ norm. That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|x^m - x^n\|_{\infty} = \sup_{j \in \mathbb{N}} |x_j^m - x_j^n| < \varepsilon.$$

Fix a coordinate $j \in \mathbb{N}$. Then for $m, n \geq N$,

$$|x_j^m - x_j^n| \leq \sup_{i \in \mathbb{N}} |x_i^m - x_i^n| = \|x^m - x^n\|_{\infty} < \varepsilon.$$

This shows that for each fixed j , the sequence of real numbers $(x_j^k)_{k \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists a limit

$$x_j := \lim_{k \rightarrow \infty} x_j^k \in \mathbb{R}.$$

Define the sequence $x = (x_j)_{j \in \mathbb{N}}$. We now show that $x \in \ell_{\infty}$ and that $x^k \rightarrow x$ in the $\|\cdot\|_{\infty}$ norm.

Since (x^k) is Cauchy in ℓ_{∞} , it is bounded: there exists $M > 0$ such that $\|x^k\|_{\infty} \leq M$ for all k . Hence, for each j , $|x_j| = \lim_{k \rightarrow \infty} |x_j^k| \leq M$, showing that x is bounded. Therefore, $x \in \ell_{\infty}$.

Finally, we show convergence in norm. For any $\varepsilon > 0$, choose N such that $\|x^m - x^n\|_{\infty} < \varepsilon$ for all $m, n \geq N$. Fix $n \geq N$. Then for all j ,

$$|x_j - x_j^n| = \lim_{m \rightarrow \infty} |x_j^m - x_j^n| \leq \limsup_{m \rightarrow \infty} \|x^m - x^n\|_{\infty} < \varepsilon.$$

Taking the supremum over all j , we get

$$\|x - x^n\|_\infty = \sup_j |x_j - x_j^n| \leq \varepsilon.$$

This proves that $x^n \rightarrow x$ in the $\|\cdot\|_\infty$ norm.

Hence, $(\ell_\infty, \|\cdot\|_\infty)$ is complete, i.e., it is a Banach space.

Example 5.22 Let

$$c = \{(x_n) : \lim_{n \rightarrow \infty} x_n \text{ exists}\}.$$

Then c is a vector subspace of ℓ_∞ . Moreover, c is closed in ℓ_∞ with the uniform norm, hence a Banach space. Let $(x^k)_{k \in \mathbb{N}}$ be a sequence in c with $x^k \rightarrow x$ in the $\|\cdot\|_\infty$ norm. Each x^k converges to some limit $L_k \in \mathbb{R}$, i.e., $\lim_{n \rightarrow \infty} x_n^k = L_k$.

Since $x^k \rightarrow x$ uniformly (sup norm convergence), we can pass the limit under $n \rightarrow \infty$: for any $\varepsilon > 0$, choose k large enough so that $\|x^k - x\|_\infty < \varepsilon$. Then

$$|x_n - L_k| \leq |x_n - x_n^k| + |x_n^k - L_k| < \varepsilon + \varepsilon = 2\varepsilon$$

for all sufficiently large n . Hence $\lim_{n \rightarrow \infty} x_n = \lim_{k \rightarrow \infty} L_k$ exists. Therefore, $x \in c$, showing that c is closed. By completeness of ℓ_∞ , $(c, \|\cdot\|_\infty)$ is a Banach space.

Example 5.23 Let

$$c_0 = \{(x_n) : \lim_{n \rightarrow \infty} x_n = 0\}.$$

It is easy to see that c_0 is a vector subspace of c , hence a normed space with the infinity norm $\|\cdot\|_\infty$.

Moreover, c_0 is complete. Let $(x^k)_{k \in \mathbb{N}}$ be a sequence in c_0 such that $x^k \rightarrow x$ in the $\|\cdot\|_\infty$ norm. That is, for any $\varepsilon > 0$, there exists N such that for all $k \geq N$,

$$\|x^k - x\|_\infty = \sup_n |x_n^k - x_n| < \varepsilon.$$

Since each $x^k \in c_0$, for given $\varepsilon > 0$, there exists M such that for all $n \geq M$, $|x_n^k| < \varepsilon$ for all k . Then for $n \geq M$,

$$|x_n| \leq |x_n - x_n^k| + |x_n^k| < \varepsilon + \varepsilon = 2\varepsilon.$$

This shows $\lim_{n \rightarrow \infty} x_n = 0$, so $x \in c_0$. Therefore, c_0 is closed in ℓ_∞ . By completeness of ℓ_∞ , it follows that $(c_0, \|\cdot\|_\infty)$ is a Banach space.

Example 5.24 Define

$$c_{00} = \{(x_n) : \exists N \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n > N\}.$$

Then c_{00} is a vector subspace of c_0 , hence a normed space with $\|\cdot\|_\infty$.

However, c_{00} is not complete. Consider the sequence $(u^n) \subset c_{00}$ defined by

$$u^1 = (1, 0, 0, \dots), \quad u^2 = \left(1, \frac{1}{2}, 0, 0, \dots\right), \quad \dots, \quad u^n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right),$$

and let

$$u = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) \notin c_{00}.$$

Then

$$u^n - u = (0, \dots, 0, \frac{1}{n+1}, \frac{1}{n+2}, \dots),$$

where 0 appears n times. Clearly,

$$\|u^n - u\|_\infty = \sup_{k \in \mathbb{N}} \left\{ \frac{1}{n+1}, \frac{1}{n+2}, \dots \right\} = \frac{1}{n+1} \rightarrow 0.$$

Hence $u^n \rightarrow u$ in c_0 , but $u \notin c_{00}$. Therefore, c_{00} is not closed in c_0 and not a Banach space.

Consider [Figure 5.4](#) for an illustration of a sequence that eventually becomes zero (and thus belongs to c_{00}). Furthermore, see [Figure 5.5](#) to visualize a sequence that belongs to c_0 but not to c_{00} .

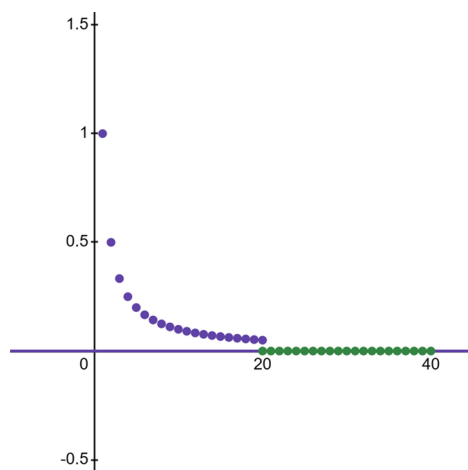


Figure 5.4 An element in c_{00} . ↗

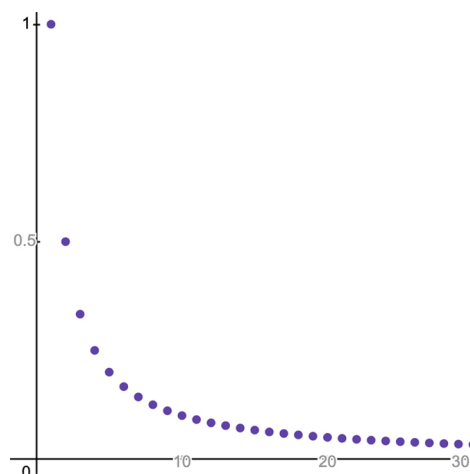


Figure 5.5 A sequence in c_0 but not in c_{00} . 

5.6 EXERCISES

Exercise 5.1

1. Let (X, d) be a metric space and let (x_n) be a sequence in X . Prove that (x_n) converges to a point $x \in X$ if and only if every subsequence of (x_n) has a further subsequence that converges to x .
2. Let (X, d) be a metric space, and let $(x_n), (y_n)$ be sequences in X such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Show that $(d(x_n, y_n)) \rightarrow d(x, y)$ in $\mathbb{R}$.

3. Let $x_n \rightarrow x$ in $\mathbb{R}$. Show that the Cesàro means

$$s_n = \frac{x_1 + \cdots + x_n}{n} \rightarrow x.$$

Is the converse true?

4. Let $(x_n) \subseteq X$ satisfy $d(x_{n+1}, x_n) \rightarrow 0$. Can we conclude that (x_n) is convergent or Cauchy?
5. Show that $\mathbb{Q}^c$ is not complete in $\mathbb{R}$ with the Euclidean metric.
6. Let $\{(X_n, d_n)\}_{n=1}^\infty$ be a sequence of metric spaces such that

$$\text{diam}(X_n) \leq 1 \quad \text{for all } n \in \mathbb{N}.$$

Define a function $d : X \times X \rightarrow \mathbb{R}$ by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n), \quad x = (x_n), y = (y_n) \in X := \prod_{n=1}^{\infty} X_n.$$

Show that d is a metric on X . Is d a product metric?

7. Let $\{(X_n, d_n)\}_{n=1}^\infty$ be a sequence of metric spaces, and let d be any product metric on

$$X := \prod_{n=1}^{\infty} X_n.$$

Show that for every sequence of subsets $A_n \subseteq X_n$, $n \in \mathbb{N}$, we have

$$\overline{\prod_{n=1}^{\infty} A_n} = \prod_{n=1}^{\infty} \overline{A_n},$$

where closures are taken in the corresponding spaces.

8. Let $(F_n)_{n=1}^\infty$ be a decreasing sequence of non-empty closed and bounded subsets of a complete metric space (X, d) such that

$$\text{diam}(F_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Show that

$$\bigcap_{n=1}^{\infty} F_n$$

consists of exactly one point.

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Continuity: It Knows All of the Applications

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6.1 WHY IS CONTINUITY USEFUL?

Continuity: The Hidden Paradise in the Ocean of Mathematics

Let us recall that the Topology realm, whose inhabitants this book is about, is a region in the ocean of mathematics. This ocean includes various islands, cities located on them, and cities at its depth, which actually comprise a very vast and impressive part of it. Some regions of the realm are visible, and some are hidden. In the world of mathematics, topology is known as the visible and hidden paradise, due to the effective connections and interactions between its different parts, which are carried out by trains, boats, and submarines, and are known as continuity (the external connections of members of one region with other regions). In fact, the power and impact of topology is due to its significant presence in various parts, such as the convergence archipelago (the gateway to the paradise of exports and imports) and, consequently, the ports of entry and exit to the ocean of applications. This is the secret of topology's permanence. The joy of continuity is the hidden paradise of the ocean of mathematics within the heart of its visible paradise (topology). In fact, topology uses its tools (which will be introduced in the following chapters) to provide the necessary equipment and means for a profitable and safe journey into the ocean of applications and the various areas of mathematics through Continuity. Note that continuity is not the only hidden paradise in the world of topology. Submarines and open and closed trains are other gifts of this beautiful realm to the world of mathematics and applications. For example, there are hidden mines in the heart of the ocean of mathematics (which belongs to the Topology realm), rich with pure diamonds of ideas and applications. Although Continuity submarines can reach there, they cannot return because of the mines' curvature (due to being untouched), and therefore open or closed submarines (which can return) will deliver the diamonds to the factories of the Topology realm. Hence, the more stamps on the method cards of residents of the Topology realm, the more opportunities they will have for profitable exchanges and trade (for example, stamps for various types of continuity, openness, and closure, etc.).

6.2 CONTINUITY AND METRIC SPACES

First, we recall the following notions from [Chapter 4](#).

Definition 6.1 Assume that (X, d) is a metric space and $x_0 \in X$. A neighborhood of x_0 with radius $r > 0$ (denoted by $N_r^{(x_0)}$) is defined by

$$N_r^{(x_0)} = \{x \in X \mid d(x, x_0) < r\}.$$

Moreover, the punctured open neighborhood of x_0 with radius r (denoted by $N_r^{(x_0)} - \{x_0\}$ or $N_r^{)x_0(}$) is defined as

$$N_r^{)x_0(} = N_r^{(x_0)} - \{x_0\} = \{x \in X \mid 0 < d(x, x_0) < r\}.$$

The motivation for introducing punctured open neighborhoods is that, in the definition of the limit of a function (to be introduced shortly), it is not necessary for the function to be defined at the point itself. Rather, it suffices that the function be defined on a neighborhood surrounding that point.

Definition 6.2 Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function defined on a punctured open neighborhood of $x_0 \in X$. We say that the limit of $f(x)$ as x approaches x_0 is equal to $l \in Y$, and we write

$$\lim_{x \rightarrow x_0} f(x) = l,$$

provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < d_1(x, x_0) < \delta \implies d_2(f(x), l) < \varepsilon.$$

Intuitively, this means that $f(x)$ can be made arbitrarily close to l whenever x is sufficiently close to x_0 .

In other words, the concept of limit describes the idea of approaching a point as closely as desired in a precise and logical manner. Having defined the limit of a function, a natural question arises: *Is the limit unique?* Uniqueness plays a fundamental role in many areas of mathematics. Indeed, it allows us to identify a single distinguished object among many possibilities. This leads to the following result.

Theorem 6.1 Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function defined on a punctured open neighborhood of $x_0 \in X$. If the limit of f at x_0 exists, then it is unique.

Proof 6.1 Assume that

$$\lim_{x \rightarrow x_0} f(x) = l_1 \quad \text{and} \quad \lim_{x \rightarrow x_0} f(x) = l_2.$$

Suppose, to the contrary, that $l_1 \neq l_2$. Then

$$\varepsilon = \frac{d_2(l_1, l_2)}{2} > 0.$$

By the definition of the limit, there exists $\delta_1 > 0$ such that

$$0 < d_1(x, x_0) < \delta_1 \implies d_2(f(x), l_1) < \frac{\varepsilon}{2}.$$

Similarly, there exists $\delta_2 > 0$ such that

$$0 < d_1(x, x_0) < \delta_2 \implies d_2(f(x), l_2) < \frac{\varepsilon}{2}.$$

Let $\delta = \min \{\delta_1, \delta_2\}$. Then for every x satisfying $0 < d_1(x, x_0) < \delta$, both inequalities hold. Hence,

$$d_2(l_1, l_2) \leq d_2(f(x), l_1) + d_2(f(x), l_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = \frac{d_2(l_1, l_2)}{2},$$

which is a contradiction. Therefore, the limit of f at x_0 , if it exists, must be unique.

One of the advantages of the notions of limit and continuity is that they provide us with alternative and equivalent ways to approach these concepts. We shall encounter this phenomenon several times throughout this chapter; the first instance appears now.

Proposition 6.1 Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function defined on a punctured open neighborhood of $x_0 \in X$. Then,

$$\lim_{x \rightarrow x_0} f(x) = l$$

if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$N_\delta^{(x_0)} \subseteq f^{-1}(N_\varepsilon^{(l)}).$$

Proof 6.2 Assume that $\lim_{x \rightarrow x_0} f(x) = l$ and let $\varepsilon > 0$ be given. By the definition of the limit, there exists $\delta > 0$ such that

$$0 < d_1(x, x_0) < \delta \implies d_2(f(x), l) < \varepsilon.$$

Equivalently, if $x \in N_\delta^{(x_0)}$, then $f(x) \in N_\varepsilon^{(l)}$, or $x \in f^{-1}(N_\varepsilon^{(l)})$. Hence,

$$N_\delta^{(x_0)} \subseteq f^{-1}(N_\varepsilon^{(l)}).$$

The proof of the converse implication follows in a similar manner.

The definition of continuity is essentially based on the notion of limit. Nevertheless, it is a powerful and indispensable tool in both pure and applied mathematics.

Definition 6.3 Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function defined on an open neighborhood of $x_0 \in X$. We say that f is continuous at x_0 provided that

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

In other words, f is continuous at x_0 if the limit of f at x_0 exists and is equal to the value of f at x_0 . If this condition fails, we say that f is discontinuous at x_0 .

The usual and standard equivalent definition of continuity (via limits) states that for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_1(x, x_0) < \delta \implies d_2(f(x), f(x_0)) < \varepsilon.$$

Equivalently, f is continuous at x_0 if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$N_\delta^{(x_0)} \subseteq f^{-1}\left(N_\varepsilon^{(f(x_0))}\right).$$

We now present another equivalent characterization of continuity using open sets.

Theorem 6.2 $\hookleftarrow$ Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function defined on an open neighborhood of $x_0 \in X$. Then f is continuous at x_0 if and only if for every open set $H \subseteq Y$ containing $f(x_0)$, we have

$$x_0 \in (f^{-1}(H))^\circ.$$

Proof 6.3 Assume that f is continuous at x_0 and let $H \subseteq Y$ be an open set containing $f(x_0)$. Since H is open, $f(x_0)$ is an interior point of H , and thus there exists $r_0 > 0$ such that

$$N_{r_0}^{(f(x_0))} \subseteq H.$$

By continuity of f at x_0 , for $\varepsilon = r_0$ there exists $\delta > 0$ such that

$$d_1(x, x_0) < \delta \implies d_2(f(x), f(x_0)) < r_0.$$

Equivalently,

$$x \in N_\delta^{(x_0)} \implies f(x) \in N_{r_0}^{(f(x_0))}.$$

Hence,

$$N_\delta^{(x_0)} \subseteq f^{-1}\left(N_{r_0}^{(f(x_0))}\right) \subseteq f^{-1}(H),$$

which shows that x_0 is an interior point of $f^{-1}(H)$.

Conversely, let $\varepsilon > 0$ be given and set $H = N_\varepsilon^{(f(x_0))}$, which is an open set in Y containing $f(x_0)$. By assumption, $x_0 \in (f^{-1}(H))^\circ$, so there exists $\delta > 0$ such that

$$N_\delta^{(x_0)} \subseteq f^{-1}(H).$$

This implies that if $x \in N_\delta^{(x_0)}$, then $f(x) \in H = N_\varepsilon^{(f(x_0))}$, which is precisely the ε – δ definition of continuity at x_0 .

By taking another look at the previous theorem, we observe that open sets are, in some sense, preserved under continuity. Consequently, the following result is immediate. Note that by

continuity of f (without mentioning a particular point), we mean that f is continuous at every point of its domain.

Corollary 6.1 $\hookrightarrow$ Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then f is continuous if and only if for every open set $H \subseteq Y$, the inverse image $f^{-1}(H)$ is open in X .

Proof 6.4 For the direct implication, assume that f is continuous (at every point) and let $H \subseteq Y$ be an arbitrary open set. We show that $f^{-1}(H)$ is open in X . Take $x_0 \in f^{-1}(H)$. Then $f(x_0) \in H$. Since H is open in Y , by [Theorem 6.2](#) we have $x_0 \in (f^{-1}(H))^\circ$. Therefore, there exists $r_0 > 0$ such that

$$N_{r_0}^{(x_0)} \subseteq f^{-1}(H),$$

which shows that x_0 is an interior point of $f^{-1}(H)$. Hence, $f^{-1}(H)$ is open in X .

For the converse direction, let $x_0 \in X$ be arbitrary and let $\varepsilon > 0$ be given. We show that f is continuous at x_0 . Set

$$H = N_\varepsilon^{(f(x_0))},$$

which is an open set in Y . By assumption, $f^{-1}(H)$ is open in X . Since $x_0 \in f^{-1}(H)$, there exists $\delta > 0$ such that

$$N_\delta^{(x_0)} \subseteq f^{-1}(H).$$

This is precisely the definition of continuity of f at x_0 .

Since closed sets are, in a sense, dual to open sets in a metric space, it is natural to expect a corresponding relationship between continuity and closed sets, as shown in the following result.

Corollary 6.2 Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then f is continuous if and only if for every closed set $F \subseteq Y$, the inverse image $f^{-1}(F)$ is closed in X .

Proof 6.5 For the direct implication, assume that f is continuous and let $F \subseteq Y$ be closed. We show that $f^{-1}(F)$ is closed in X . Since $F^c = Y - F$ is open in Y , by [Corollary 6.1](#), $f^{-1}(F^c)$ is open in X . But

$$f^{-1}(F^c) = f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F).$$

Hence, $f^{-1}(F)$ is closed in X .

Conversely, using [Corollary 6.1](#), it suffices to show that the inverse image of every open set in Y is open in X . Let $H \subseteq Y$ be open. Then $H^c = Y - H$ is closed in Y , and by assumption, $f^{-1}(H^c)$ is closed in X . Since

$$f^{-1}(H^c) = X - f^{-1}(H),$$

we conclude that $f^{-1}(H)$ is open in X .

As we have seen several times before, the discrete topology plays a rather limited role in analysis, mainly in the construction of counterexamples. Therefore, continuity in the discrete

setting is not surprising, as the next result shows.

Corollary 6.3 Assume that (X, d_1) is a discrete metric space and (Y, d_2) is an arbitrary metric space. Then every function $f : (X, d_1) \rightarrow (Y, d_2)$ is continuous.

Proof 6.6 By [Corollary 6.1](#), it suffices to show that the inverse image of every open set in Y is open in X . Recall that in a discrete metric space, every subset is open. Therefore, if $H \subseteq Y$ is open, then $f^{-1}(H)$ is automatically open in X .

Moreover, we recall from Calculus that constant functions are continuous (when considered as functions from $\mathbb{R}$ into $\mathbb{R}$ endowed with the Euclidean topology). The good news is that this fact can be generalized to arbitrary metric spaces, as the next result shows.

Proposition 6.2 Assume that (X, d_1) and (Y, d_2) are metric spaces and let $y_0 \in Y$. Consider the function $f : (X, d_1) \rightarrow (Y, d_2)$ defined by $f(x) = y_0$. Then f is continuous.

Proof 6.7 We use [Corollary 6.1](#). Let $H \subseteq Y$ be an arbitrary open set. Then

$$f^{-1}(H) = \begin{cases} X, & y_0 \in H, \\ \emptyset, & y_0 \notin H. \end{cases}$$

In either case, $f^{-1}(H)$ is open in X , as required.

Furthermore, the distance functions introduced in [Chapter 4](#) can be viewed as continuous functions.

Proposition 6.3  Assume that (X, d) is a metric space and $\emptyset \neq A \subseteq X$. Then the distance function $f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$ defined by

$$f(x) = d(x, A) = \inf \{d(x, a) : a \in A\}$$

is continuous.

Proof 6.8 We use [Exercise 1](#) from [Chapter 4](#). Take any $a, b \in A$. Then

$$d(x, a) - d(y, b) \leq |d(x, a) - d(y, b)| \leq d(x, y) + d(a, b).$$

Fix $b \in A$ and take the infimum over all $a \in A$ on both sides to obtain

$$d(x, A) - d(y, b) \leq d(x, y) + d(b, A) = d(x, y).$$

Now, taking the infimum over all $b \in A$, we get

$$d(x, A) - d(y, A) \leq d(x, y).$$

Similarly,

$$d(y, A) - d(x, A) \leq d(x, y),$$

and hence

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Therefore, for any $\varepsilon > 0$, choosing $\delta = \varepsilon$ yields the desired result.

Proposition 6.4 $\hookleftarrow$ Assume that (X, d) is a metric space and that $\emptyset \neq A \subseteq X$ is bounded. Then the function $f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$ defined by

$$f(x) = D(x, A) = \sup \{d(x, a) : a \in A\}$$

is continuous.

Proof 6.9 We again use [Exercise 1](#) from [Chapter 4](#). The proof is similar to the previous one. Take any $a, b \in A$. Then

$$d(x, a) - d(y, b) \leq |d(x, a) - d(y, b)| \leq d(x, y) + d(a, b).$$

Fix $b \in A$ and take the supremum over all $a \in A$ on both sides to obtain

$$D(x, A) - d(y, b) \leq d(x, y) + D(b, A) = d(x, y).$$

Now, taking the supremum over all $b \in A$, we get

$$D(x, A) - D(y, A) \leq d(x, y).$$

Similarly,

$$D(y, A) - D(x, A) \leq d(x, y),$$

and hence

$$|D(x, A) - D(y, A)| \leq d(x, y).$$

Therefore, for any $\varepsilon > 0$, choosing $\delta = \varepsilon$ completes the proof.

Proposition 6.5 $\hookleftarrow$ Assume that $\{(X_i, d_i)\}_{i=1}^n$ is a finite family of metric spaces and let

$$X = \prod_{i=1}^n X_i$$

be endowed with any product metric. Then the projections $\pi_i : X \rightarrow X_i$, defined by

$$\pi_i(x_1, \dots, x_n) = x_i, \quad 1 \leq i \leq n,$$

are continuous.

Proof 6.10 By the results of [Chapter 4](#), the collection

$$\{U_1 \times \dots \times U_n : U_i \subseteq X_i \text{ is open}\}$$

forms a basis for the product metric. For each i , we have

$$\pi_i^{-1}(U_i) = \{(x_1, \dots, x_n) \in X : x_i \in U_i\} = X_1 \times \dots \times X_{i-1} \times U_i \times X_{i+1} \times \dots \times X_n,$$

which is open in the product metric. By [Corollary 6.1](#), this completes the proof.

Theorem 6.3 Assume that (X, d) , (X_1, d_1) , and (X_2, d_2) are metric spaces. Let $f_1 : (X, d) \rightarrow (X_1, d_1)$ and $f_2 : (X, d) \rightarrow (X_2, d_2)$ be functions. Define $f : (X, d) \rightarrow X_1 \times X_2$ by

$$f(x) = (f_1(x), f_2(x)),$$

where $X_1 \times X_2$ is equipped with a product metric. Then f is continuous if and only if both f_1 and f_2 are continuous.

Proof 6.11 Assume first that f is continuous. By [Proposition 6.5](#), the projection maps π_1 and π_2 are continuous. Observe that

$$f_1 = \pi_1 \circ f \quad \text{and} \quad f_2 = \pi_2 \circ f.$$

Thus, each f_i is the composition of continuous functions, and hence both f_1 and f_2 are continuous.

Conversely, assume that f_1 and f_2 are continuous. Let $U \subseteq X_1 \times X_2$ be an open set. Since sets of the form $U_1 \times U_2$, where $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ are open, form a basis for the product topology, it suffices to consider the case $U = U_1 \times U_2$.

By continuity of f_1 and f_2 , the sets $f_1^{-1}(U_1)$ and $f_2^{-1}(U_2)$ are open in X . Moreover,

$$f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2),$$

which is open as an intersection of open sets. Hence f is continuous.

6.3 CONTINUITY AND SEQUENCES

In this section, we use sequences to uncover further elegant results in the realm of continuity.

Definition 6.4 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. We say that f is sequentially continuous at $x_0 \in X$ if, for every sequence $(x_n) \subseteq X$ converging to x_0 , we have $f(x_n) \rightarrow f(x_0)$. We say that f is sequentially continuous if it is sequentially continuous at every point of its domain.

Although this definition makes sense for functions between general topological spaces, an important fact is that, in metric spaces, sequential continuity is equivalent to the usual notion of continuity.

Theorem 6.4  Suppose (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then f is continuous at $x_0 \in X$ if and only if for every sequence $(x_n) \subseteq X$ with $x_n \rightarrow x_0$, we have $f(x_n) \rightarrow f(x_0)$.

Proof 6.12 Assume that f is continuous at x_0 and let $\varepsilon > 0$. By continuity, there exists $\delta > 0$ such that

$$d_1(x, x_0) < \delta \quad \Rightarrow \quad d_2(f(x), f(x_0)) < \varepsilon.$$

Let $(x_n) \subseteq X$ be a sequence converging to x_0 . Then there exists $n_0 \in \mathbb{N}$ such that $d_1(x_n, x_0) < \delta$ for all $n \geq n_0$. Consequently, $d_2(f(x_n), f(x_0)) < \varepsilon$ for all $n \geq n_0$, and hence $f(x_n) \rightarrow f(x_0)$.

Conversely, assume that f is sequentially continuous at x_0 but not continuous at x_0 . Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there is $x_\delta \in X$ with

$$d_1(x_\delta, x_0) < \delta \quad \text{and} \quad d_2(f(x_\delta), f(x_0)) \geq \varepsilon_0.$$

For each $n \in \mathbb{N}$, let $\delta_n = \frac{1}{n}$ and choose $x_n = x_{\delta_n}$ accordingly. Then $x_n \rightarrow x_0$, but $f(x_n) \not\rightarrow f(x_0)$, contradicting sequential continuity.

At first glance, it may seem (based on intuition from real analysis) that the identity map is always continuous. However, in the context of metric spaces, this need not be the case.

Example 6.1 Let X be a non-empty set, let d_1 be the discrete metric on X , and let d_2 be a non-discrete metric on X . Then the identity map

$$I : (X, d_2) \rightarrow (X, d_1)$$

is not continuous in general. Indeed, there exist open sets in (X, d_1) that are not open in (X, d_2) , and the inverse image of any set under the identity map is the set itself.

For example, let $X = \mathbb{R}$, let d_1 be the discrete metric, and let d_2 be the Euclidean metric. Then the singleton $\{0\}$ is open in (X, d_1) but not open in (X, d_2) , and hence $I : (X, d_2) \rightarrow (X, d_1)$ is not continuous.

Theorem 6.5 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f, g : (X, d_1) \rightarrow (Y, d_2)$ are continuous functions. Moreover, suppose $A \subseteq X$ is a dense set such that $f(a) = g(a)$ for all $a \in A$. Then, $f(x) = g(x)$ for each $x \in X$.

Proof 6.13 Let $x \in X$. Since $\overline{A} = X$, by the sequential characterization of closure in metric spaces, there exists a sequence $(a_n) \subseteq A$ such that $a_n \rightarrow x$. Since f and g are continuous, we conclude that $f(a_n) \rightarrow f(x)$ and $g(a_n) \rightarrow g(x)$. By the assumption, $f(a_n) = g(a_n)$ for each $n \in \mathbb{N}$. Since the limit of a sequence is unique, we conclude that $f(x) = g(x)$.

Definition 6.5 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. We say that f satisfies the Lipschitz condition with constant M provided that there exists a positive real number M such that for each $x, y \in X$, we have

$$d_2(f(x), f(y)) \leq M d_1(x, y).$$

If f satisfies the condition with constant less than 1, we say that f is a strongly contractive mapping.

The first simple observation about Lipschitz functions is continuity.

Proposition 6.6  Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a Lipschitz function with constant M . Then, f is continuous.

Proof 6.14 The proof is straightforward. Fix any point $x_0 \in X$ and let $\varepsilon > 0$ be given. It suffices to choose $\delta = \frac{\varepsilon}{M}$. Then the desired result follows immediately.

Remark 6.1 At first glance, it might seem that Lipschitz functions are rather scarce in the literature. However, there are many non-trivial examples. Consider the functions introduced in [Proposition 6.3](#) and [Proposition 6.4](#). Both functions are Lipschitz with constant 1. Moreover, we have more familiar examples.

The following results examine the interaction between continuity and boundedness for real-valued functions defined on a closed and bounded interval.

Example 6.2 Consider $f : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$ defined by $f(x) = \sin x$. Then f is Lipschitz with constant 1. To see this, observe that f is differentiable on $\mathbb{R}$. Hence, for any real numbers x, y , by the Mean Value Theorem, there exists a real number c between x and y such that

$$\cos c = \frac{\sin x - \sin y}{x - y},$$

which implies that $|\sin x - \sin y| \leq |x - y|$. The same argument applies to show that $f(x) = \cos x$ is also Lipschitz with constant 1.

Theorem 6.6 $\hookleftarrow$ Let $f : ([a, b], |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$ be a continuous function. Then $f([a, b])$ is a bounded subset of $\mathbb{R}$.

Proof 6.15 Suppose, on the contrary, that $f([a, b])$ is unbounded. Then for each $n \in \mathbb{N}$, there exists a point $x_n \in [a, b]$ such that

$$|f(x_n)| \geq n.$$

The sequence (x_n) is bounded since $x_n \in [a, b]$ for all n . By the Bolzano–Weierstrass Theorem ([Theorem 5.17](#)), there exists a convergent subsequence (x_{n_k}) of (x_n) such that

$$x_{n_k} \rightarrow x_0 \quad \text{for some } x_0 \in \mathbb{R}.$$

Since $[a, b]$ is closed, we have $x_0 \in [a, b]$.

Because f is continuous at x_0 and $x_{n_k} \rightarrow x_0$, it follows that

$$f(x_{n_k}) \rightarrow f(x_0).$$

Hence, the sequence $(f(x_{n_k}))$ is bounded, which contradicts the inequality

$$|f(x_{n_k})| \geq n_k \rightarrow \infty.$$

This contradiction shows that $f([a, b])$ must be bounded.

Theorem 6.7 (Extreme Value Theorem) Let $f : ([a, b], |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$ be a continuous function. Then f attains both its maximum and its minimum on the interval $[a, b]$. In other words, there exist points $x_1, x_2 \in [a, b]$ such that

$$f(x_1) = \max_{x \in [a, b]} f(x) \quad \text{and} \quad f(x_2) = \min_{x \in [a, b]} f(x).$$

Proof 6.16 Since $f([a, b])$ is bounded, by the axiom of completeness,

$$M = \sup_{x \in [a, b]} f(x), \quad m = \inf_{x \in [a, b]} f(x).$$

exist. Suppose, on the contrary, that f does not attain M . Then for all $x \in [a, b]$, $f(x) < M$. Define

$$g(x) = \frac{1}{M - f(x)}, \quad x \in [a, b].$$

Then g is well-defined and continuous. By [Theorem 6.6](#), $g([a, b])$ is bounded, so there exists $K > 0$ such that

$$g(x) \leq K \quad \forall x \in [a, b].$$

This implies

$$M - f(x) \geq \frac{1}{K} > 0 \quad \forall x \in [a, b],$$

so that

$$f(x) \leq M - \frac{1}{K} \quad \forall x \in [a, b].$$

This contradicts the definition of M as the supremum. Hence, f attains its maximum. The argument for the minimum is analogous.

If we take another look at the proof of [Proposition 6.6](#), we see that for proving continuity at a point, the chosen δ does not depend on the particular point. In other words, the same δ works for all points. This special property deserves a different name.

Definition 6.6 (Uniform Continuity) Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. We say that f is uniformly continuous provided that for each $\varepsilon > 0$, there exists $\delta > 0$ (depending only on ε) such that for any $x, y \in X$, if $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \varepsilon$.

Remark 6.2 The main difference between uniform continuity and usual continuity is that in the former, the choice of δ depends only on ε , whereas in the latter, δ may depend on both ε and the point under consideration. It is clear that every uniformly continuous function is continuous. However, the converse is not true in general.

Example 6.3  Consider $f : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$ defined by $f(x) = x^2$. Then f is uniformly continuous on every bounded interval, but it is not uniformly continuous on $\mathbb{R}$, although it is continuous.

To see this, assume that $A \subseteq \mathbb{R}$ is bounded. Then there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$. Take any $x, y \in A$ and let $\varepsilon > 0$ be given. Put $\delta = \frac{\varepsilon}{2M}$. We have

$$|x^2 - y^2| = |x - y||x + y| \leq |x - y|(|x| + |y|) \leq 2M|x - y| < 2M\delta = \varepsilon.$$

To see why f is not uniformly continuous on $\mathbb{R}$, we argue by contradiction. Suppose that f is uniformly continuous on $\mathbb{R}$. Then for $\varepsilon = 1$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|x^2 - y^2| < 1$. For each $n \in \mathbb{N}$, set $x = n$ and $y = n + \frac{\delta}{2}$. Clearly, $|x - y| = \frac{\delta}{2} < \delta$. However,

$$|x^2 - y^2| = |n^2 - (n + \frac{\delta}{2})^2| = n\delta + \frac{\delta^2}{4},$$

which becomes arbitrarily large as $n \rightarrow \infty$, a contradiction. Hence, f is not uniformly continuous on $\mathbb{R}$.

Remark 6.3 A significant number of continuous functions that we have studied so far are, in fact, uniformly continuous. For example, Lipschitz functions (and now we know that distance functions are among them) are uniformly continuous. Moreover, when the domain space is endowed with the discrete topology, every function between metric spaces is uniformly continuous.

In certain situations, continuity implies uniform continuity. We establish one such instance below. Although a simpler proof can be given using the material from [Chapter 7](#), with the knowledge available at this stage, the argument is necessarily more elaborate.

Theorem 6.8 Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then f is uniformly continuous on $[a, b]$.

Proof 6.17 Let $\varepsilon > 0$ be given. Since f is continuous at each point $y \in [a, b]$, there exists $\delta'_{\varepsilon, y} > 0$ such that

$$|x - y| < \delta'_{\varepsilon, y} \implies |f(x) - f(y)| < \varepsilon.$$

For each $y \in [a, b]$, define

$$\delta_{\varepsilon, y} = \sup \left\{ \delta'_{\varepsilon, y} > 0 : |x - y| < \delta'_{\varepsilon, y} \implies |f(x) - f(y)| < \varepsilon \right\}.$$

Clearly, $\delta_{\varepsilon, y} > 0$ for all $y \in [a, b]$.

Define the function $\varphi : [a, b] \rightarrow \mathbb{R}$ by

$$\varphi(y) = \delta_{\varepsilon, y}.$$

We claim that

$$\inf_{y \in [a, b]} \varphi(y) > 0.$$

Suppose, on the contrary, that

$$\inf_{y \in [a, b]} \varphi(y) = 0.$$

Then there exists a sequence $\{y_n\} \subset [a, b]$ such that

$$\varphi(y_n) = \delta_{\varepsilon, y_n} \longrightarrow 0.$$

Since $[a, b]$ is bounded, the sequence $\{y_n\}$ is bounded, hence it has a convergent subsequence (which we denote again by $\{y_n\}$) such that

$$y_n \longrightarrow y_0 \in [a, b].$$

By continuity of f at y_0 , there exists $\eta > 0$ such that

$$|x - y_0| < \eta \implies |f(x) - f(y_0)| < \frac{\varepsilon}{2}.$$

Choose n large enough so that

$$|y_n - y_0| < \frac{\eta}{2} \quad \text{and} \quad \delta_{\varepsilon, y_n} < \frac{\eta}{2}.$$

Now, if $|x - y_n| < \delta_{\varepsilon, y_n}$, then

$$|x - y_0| \leq |x - y_n| + |y_n - y_0| < \eta,$$

and hence

$$|f(x) - f(y_0)| < \frac{\varepsilon}{2}.$$

Also,

$$|f(y_n) - f(y_0)| < \frac{\varepsilon}{2}.$$

Therefore,

$$|f(x) - f(y_n)| \leq |f(x) - f(y_0)| + |f(y_0) - f(y_n)| < \varepsilon,$$

which shows that

$$\delta_{\varepsilon, y_n} \geq \frac{\eta}{2},$$

a contradiction.

Thus,

$$\inf_{y \in [a, b]} \delta_{\varepsilon, y} > 0.$$

Let

$$\delta = \inf_{y \in [a, b]} \delta_{\varepsilon, y}.$$

Finally, if $x, y \in [a, b]$ and $|x - y| < \delta$, then $\delta \leq \delta_{\varepsilon, y}$, and therefore

$$|f(x) - f(y)| < \varepsilon.$$

Hence f is uniformly continuous on $[a, b]$.

As we have seen in [Theorem 6.4](#), continuity can be expressed in terms of sequences. A simple but important question is whether uniform continuity can also be expressed in terms of sequences, in a similar way. The following result provides an affirmative answer.

Theorem 6.9 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then f is uniformly continuous if and only if for every two sequences (x_n) and (y_n)

in X such that $d_1(x_n, y_n) \rightarrow 0$, we have $d_2(f(x_n), f(y_n)) \rightarrow 0$.

Proof 6.18 The proof follows the same line as in the proof of [Theorem 6.4](#).

For the direct implication, let f be uniformly continuous and let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $d_1(x, y) < \delta$ for some $x, y \in X$, we have $d_2(f(x), f(y)) < \varepsilon$. Now, let (x_n) and (y_n) be sequences in X such that $d_1(x_n, y_n) \rightarrow 0$. By the definition of convergence, there exists $n_0 \in \mathbb{N}$ such that $d_1(x_n, y_n) < \delta$ for all $n \geq n_0$. Hence, $d_2(f(x_n), f(y_n)) < \varepsilon$ for all $n \geq n_0$, which shows that $d_2(f(x_n), f(y_n)) \rightarrow 0$.

To prove the converse, suppose on the contrary that f is not uniformly continuous. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, we can find $x_\delta, y_\delta \in X$ satisfying

$$d_1(x_\delta, y_\delta) < \delta \quad \text{and} \quad d_2(f(x_\delta), f(y_\delta)) \geq \varepsilon_0.$$

For each $n \in \mathbb{N}$, set $\delta_n = \frac{1}{n}$. Then there exist sequences (x_n) and (y_n) in X such that

$$d_1(x_n, y_n) < \frac{1}{n} \quad \text{and} \quad d_2(f(x_n), f(y_n)) \geq \varepsilon_0.$$

It follows that $d_1(x_n, y_n) \rightarrow 0$, while $d_2(f(x_n), f(y_n)) \not\rightarrow 0$, which completes the proof.

Remark 6.4 We are now able to present another proof showing that the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ (see [Example 6.3](#)). For each $n \in \mathbb{N}$, set $x_n = n$ and $y_n = n + \frac{1}{n}$. Clearly,

$$|x_n - y_n| = \frac{1}{n} \rightarrow 0.$$

However,

$$|x_n^2 - y_n^2| = |n^2 - (n + \frac{1}{n})^2| = 2 + \frac{1}{n^2} \rightarrow 2,$$

which shows that f is not uniformly continuous on $\mathbb{R}$.

Furthermore, we have another characterization of uniformly continuous functions in terms of the diameter of a set.

Proposition 6.7 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then f is uniformly continuous if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every set $A \subseteq X$ with $\text{diam}(A) < \delta$, we have $\text{diam}(f(A)) < \varepsilon$.

Proof 6.19 First, assume that f is uniformly continuous. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, if $d_1(x, y) < \delta$, then $d_2(f(x), f(y)) < \frac{\varepsilon}{2}$. Let $A \subseteq X$ satisfy

$$\text{diam}(A) = \sup \{d_1(x, y) : x, y \in A\} < \delta.$$

Then for all $x, y \in A$, we have $d_2(f(x), f(y)) < \frac{\varepsilon}{2}$. Consequently,

$$\text{diam}(f(A)) = \sup \{d_2(f(x), f(y)) : x, y \in A\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

Conversely, assume that the stated condition holds. For a given $\varepsilon > 0$, let $\delta > 0$ be as in the hypothesis. If $x, y \in X$ satisfy $d_1(x, y) < \delta$, then by taking $A = \{x, y\}$, we obtain $d_2(f(x), f(y)) < \varepsilon$. Hence, f is uniformly continuous.

Remark 6.5 Consider the function $f : ((0, 1], |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$ defined by $f(x) = \frac{1}{x}$. The function f is continuous, but it is not uniformly continuous. To see this, consider the sequences $(\frac{1}{n})_{n \in \mathbb{N}}$ and $(\frac{1}{n^2})_{n \in \mathbb{N}}$. We have

$$\left| \frac{1}{n} - \frac{1}{n^2} \right| \rightarrow 0,$$

but

$$|n^2 - n| \not\rightarrow 0.$$

We now investigate this function further. The sequence $(\frac{1}{n})$ is Cauchy in $(0, 1]$, but $f(\frac{1}{n}) = n$, and hence the sequence $(f(\frac{1}{n}))$ is not Cauchy in $\mathbb{R}$. This shows that continuous functions do not preserve Cauchy sequences in general. Furthermore, the interval $(0, 1]$ is bounded, while

$$f((0, 1]) = [1, \infty),$$

which is not bounded. Therefore, continuous functions do not preserve bounded sets in general.

Now, we aim to identify functions or conditions that preserve either the Cauchy property or boundedness. We begin with uniform continuity.

Proposition 6.8 Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is uniformly continuous. Then f maps Cauchy sequences to Cauchy sequences.

Proof 6.20 Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta = \delta_\varepsilon > 0$ such that for any $x, y \in X$,

$$d_1(x, y) < \delta \quad \Rightarrow \quad d_2(f(x), f(y)) < \varepsilon.$$

Now, assume that $(x_n) \subseteq X$ is a Cauchy sequence. Then, for the above δ , there exists $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$, we have

$$d_1(x_m, x_n) < \delta.$$

Hence, by the hypothesis,

$$d_2(f(x_m), f(x_n)) < \varepsilon,$$

which shows that the sequence $(f(x_n))$ is Cauchy in Y .

Therefore, uniform continuity guarantees that Cauchy sequences are mapped to Cauchy sequences. However, the situation is more delicate when it comes to bounded sets.

Example 6.4 Consider the identity map $I : (\mathbb{R}, d) \rightarrow (\mathbb{R}, |\cdot|)$, where d is the discrete metric. This map is uniformly continuous. However, it maps the bounded space $(\mathbb{R}, d)$ onto the

unbounded space $(\mathbb{R}, |\cdot|)$. Note that in a discrete metric space, every set is bounded.

This example shows that even uniform continuity is not sufficient to preserve boundedness. Therefore, we need a stronger notion.

Definition 6.7 (ε -net) Assume that (X, d) is a metric space and $A \subseteq X$. For any $\varepsilon > 0$, the family

$$\{N_\varepsilon^{(x)}\}_{x \in A}$$

is called an ε -net for A . Intuitively, the neighborhoods $N_\varepsilon^{(x)}$ are spread like a net over the set A .

Definition 6.8 (Totally boundedness) Assume that (X, d) is a metric space and $A \subseteq X$. We say that A is totally bounded if for every $\varepsilon > 0$, there exist finitely many points $\{x_1, \dots, x_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^n N_\varepsilon^{(x_i)}.$$

It may seem confusing whether, in the definition of total boundedness, the points $x_1, \dots, x_n$ must be chosen in A or can be taken anywhere in X . The following useful fact shows that either choice is equivalent.

Proposition 6.9 Let (X, d) be a metric space and $A \subseteq X$. In the definition of total boundedness, it is equivalent to require the centers of the covering balls to lie in A or in X . That is, A is totally bounded if and only if for every $\varepsilon > 0$ there exist points $x_1, \dots, x_m \in X$ such that

$$A \subseteq \bigcup_{k=1}^m N_\varepsilon^{(x_k)}.$$

Proof 6.21 Assume first the usual definition: for every $\varepsilon > 0$, there exist points $a_1, \dots, a_m \in A$ such that

$$A \subseteq \bigcup_{k=1}^m N_\varepsilon^{(a_k)}.$$

Clearly, since $A \subseteq X$, these points $a_1, \dots, a_m$ also belong to X . Therefore, the covering with centers in A is also a covering with centers in X .

Conversely, suppose that for every $\varepsilon > 0$ there exist points $x_1, \dots, x_m \in X$ such that

$$A \subseteq \bigcup_{k=1}^m N_{\varepsilon/2}^{(x_k)}.$$

For each k , if $N_{\varepsilon/2}^{(x_k)} \cap A \neq \emptyset$, choose a point

$$a_k \in N_{\varepsilon/2}^{(x_k)} \cap A.$$

Then for any $y \in A$ with $y \in N_{\varepsilon/2}^{(x_k)}$, we have

$$d(y, a_k) \leq d(y, x_k) + d(x_k, a_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence,

$$A \subseteq \bigcup_{k=1}^m N_{\varepsilon}^{(a_k)},$$

with centers $a_k \in A$. This shows that the definitions using centers in A or in X are equivalent.

To better understand this definition and to present some illustrative examples, we begin with a natural relationship between boundedness and total boundedness.

Proposition 6.10 $\Rightarrow$ Assume that (X, d) is a metric space and $A \subseteq X$ is totally bounded. Then A is bounded.

Proof 6.22 Take $\varepsilon = 1$. By total boundedness, there exist points $\{x_1, \dots, x_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^n N_1^{(x_i)}.$$

Each neighborhood $N_1^{(x_i)}$ is bounded, and a finite union of bounded sets is bounded. Hence, A is bounded.

Remark 6.6 The converse of [Proposition 6.10](#) is not true in general. Indeed, in a discrete metric space (X, d) , every set is bounded. However, if X is infinite, it cannot be totally bounded. To see this, consider a $\frac{1}{2}$ -net. If there existed finitely many points $\{x_1, \dots, x_n\} \subseteq X$ such that

$$X \subseteq \bigcup_{i=1}^n N_{\frac{1}{2}}^{(x_i)},$$

then, since $N_{\frac{1}{2}}^{(x_i)} = \{x_i\}$ for each i , we would have

$$X = \{x_1, \dots, x_n\},$$

which contradicts the assumption that X is infinite.

The converse of [Proposition 6.10](#) can hold in some situations. Consider the following result.

Proposition 6.11 Let $A \subseteq \mathbb{R}^n$ (with the Euclidean metric). If A is bounded, then A is totally bounded.

Proof 6.23 Assume that A is bounded. Then there exists $r > 0$ such that

$$A \subseteq N_r^{(0)}.$$

Hence

$$A \subseteq [-r, r] \times \cdots \times [-r, r],$$

which is a cube in $\mathbb{R}^n$ of side length $2r$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that

$$\sqrt{n} \delta < \varepsilon.$$

Partition the cube $[-r, r] \times \cdots \times [-r, r]$ into finitely many smaller cubes of side length δ . (The number of such cubes is finite since each coordinate interval $[-r, r]$ is divided into finitely many subintervals of length δ .)

Each small cube has diameter $\sqrt{n} \delta$, hence is contained in some open ball of radius ε . Since only finitely many such cubes cover $[-r, r] \times \cdots \times [-r, r]$, it follows that $[-r, r] \times \cdots \times [-r, r]$, and therefore A , can be covered by finitely many open balls of radius ε .

Because $\varepsilon > 0$ was arbitrary, A is totally bounded.

A remarkable theorem regarding totally bounded sets in a metric space is the following.

Theorem 6.10 (Bolzano–Weierstrass Theorem) *Let (X, d) be a metric space and let $A \subseteq X$ be an infinite set. If A is totally bounded (in particular, if A is bounded in $\mathbb{R}^n$), then A has a limit point in X .*

Proof 6.24 Since A is totally bounded, for each $n \in \mathbb{N}$ there exists a finite collection of open balls of radius $\frac{1}{n}$ whose union covers A . Hence, A can be covered by finitely many balls of radius $\frac{1}{n}$.

Because A is infinite, at least one of these balls must contain infinitely many points of A . Choose such a ball and denote it by N_0 . Thus, there exists a point $x_1 \in X$ such that

$$A \cap N_1^{(x_1)} \text{ is infinite.}$$

Proceed inductively. Suppose that for some $k \in \mathbb{N}$ we have found a ball $N_{\frac{1}{k}}^{(x_k)}$ such that

$$A \cap N_{\frac{1}{k}}^{(x_k)} \text{ is infinite.}$$

Cover $A \cap N_{\frac{1}{k}}^{(x_k)}$ by finitely many balls of radius $\frac{1}{k+1}$. Again, at least one of these balls must contain infinitely many points of A . Choose such a ball and denote it by $N_{\frac{1}{k+1}}^{(x_{k+1})}$.

In this way, we obtain a nested sequence of balls

$$N_1^{(x_1)} \supseteq N_{\frac{1}{2}}^{(x_2)} \supseteq \cdots \supseteq N_{\frac{1}{k}}^{(x_k)} \supseteq \cdots$$

each containing infinitely many points of A .

Choose a sequence (a_k) in A such that

$$a_k \in A \cap N_{\frac{1}{k}}^{(x_k)} \text{ and } a_k \neq a_j \text{ for } k \neq j.$$

We claim that (a_k) is a Cauchy sequence. Indeed, if $m, n \geq N$, then both a_m and a_n belong to $N_{\frac{1}{N}}^{(x_N)}$, and hence

$$d(a_m, a_n) \leq d(a_m, x_N) + d(x_N, a_n) < \frac{2}{N}.$$

This shows that (a_k) is Cauchy.

We claim that

$$\bigcap_{n=1}^{\infty} \overline{N_{\frac{1}{k}}^{(x_n)}} \neq \emptyset.$$

To see this, suppose on a contrary, $\bigcap_{n=1}^{\infty} \overline{N_{\frac{1}{k}}^{(x_n)}} = \emptyset$. Thus, for each $m \in \mathbb{N}$, we can find $n \in \mathbb{N}$ such that $x_m \notin \overline{N_{\frac{1}{n}}^{(x_n)}}$. Obviously, $n \neq m$. Now, we consider two cases. If $n < m$, then since the balls are nested, we have

$$x_m \in N_{\frac{1}{m}}^{(x_m)} \subseteq N_{\frac{1}{m}}^{(x_n)},$$

which is not possible.

Now, assume that $n > m$. Then there exists $\delta > 0$ such that

$$N_{\delta}^{(x_m)} \cap N_{\frac{1}{n}}^{(x_n)} = \emptyset.$$

Since $x_n \in N_{\frac{1}{n}}^{(x_n)}$, we conclude that $x_n \notin N_{\delta}^{(x_m)}$ showing that

$$d(x_m, x_n) \geq \delta.$$

Since m and n can take large enough, it is a contradiction. Thus,

$$\bigcap_{n=1}^{\infty} \overline{N_{\frac{1}{n}}^{(x_n)}} \neq \emptyset.$$

Take $x \in \overline{N_{\frac{1}{n}}^{(x_n)}}$. We show that x is a limit point of A . Let $\varepsilon > 0$. Choose N such that $\frac{1}{N} < \varepsilon$. Then for all $n \geq N$,

$$a_n \in N_{\frac{1}{N}}^{(x_N)} \subseteq N_{\varepsilon}^{(x)},$$

since x lies in the closure of $N_{\varepsilon}^{(x)}$.

Thus every neighborhood of x contains infinitely many points of A (namely a_n for $n \geq N$). Therefore x is a limit point of A .

Remark 6.7 In $\mathbb{R}^n$ with the Euclidean metric, every bounded set is totally bounded and the space is complete. Hence, every bounded infinite subset of $\mathbb{R}^n$ has a limit point.

Corollary 6.4 Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Consider [Figure 6.1](#) for a better understanding Bolzano–Weierstrass Theorem. Now, we investigate the behavior of uniformly continuous functions on totally bounded sets.

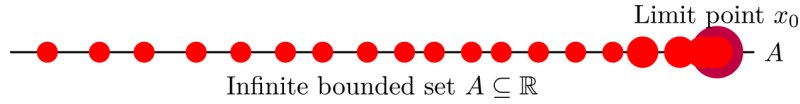


Figure 6.1 Illustration of the Bolzano–Weierstrass theorem: an infinite bounded set A contains a limit point x_0 (purple), approached by a convergent subsequence (red points). $\hookleftarrow$

Proposition 6.12 $\hookleftarrow$ Assume that $f : (X, d_1) \rightarrow (Y, d_2)$ is a uniformly continuous function and that $A \subseteq X$ is totally bounded. Then $f(A)$ is totally bounded in Y .

Proof 6.25 Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta = \delta_\varepsilon > 0$ such that for all $x, y \in X$,

$$d_1(x, y) < \delta \quad \Rightarrow \quad d_2(f(x), f(y)) < \varepsilon.$$

Since A is totally bounded, there exist finitely many points $\{y_1, \dots, y_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^n N_\delta^{(y_i)}.$$

We claim that

$$f(A) \subseteq \bigcup_{i=1}^n N_\varepsilon^{(f(y_i))}.$$

To see this, take any $f(x) \in f(A)$ with $x \in A$. Then there exists $1 \leq i_0 \leq n$ such that $x \in N_\delta^{(y_{i_0})}$, which implies $d_1(x, y_{i_0}) < \delta$. By the hypothesis,

$$d_2(f(x), f(y_{i_0})) < \varepsilon,$$

and hence

$$f(x) \in N_\varepsilon^{(f(y_{i_0}))} \subseteq \bigcup_{i=1}^n N_\varepsilon^{(f(y_i))}.$$

This shows that $f(A)$ is totally bounded.

Note that the converse of [Proposition 6.12](#) is not true in general. For example, the sign function (sgn) maps every totally bounded set into a finite set, which is certainly totally bounded; however, it is not even continuous.

Proposition 6.13 $\hookleftarrow$ Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is continuous. If X is separable, then $f(X)$ is also separable.

Proof 6.26 Since X is separable, there exists a countable subset $A \subseteq X$ such that $\overline{A} = X$. Clearly, $f(A) \subseteq f(X)$ is countable.

Since f is continuous, it preserves closures; that is,

$$f(\overline{A}) \subseteq \overline{f(A)}.$$

Therefore,

$$f(X) = f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{f(X)} = f(X).$$

Hence $\overline{f(A)} = f(X)$, which shows that $f(X)$ is separable.

Proposition 6.14 Assume that (X_1, d_1) and (X_2, d_2) are metric spaces. Then the product space $X_1 \times X_2$ is separable if and only if both X_1 and X_2 are separable.

Proof 6.27 Assume first that $X_1 \times X_2$ is separable (with respect to any product metric). Consider the projection maps

$$\pi_1 : X_1 \times X_2 \rightarrow X_1, \quad \pi_2 : X_1 \times X_2 \rightarrow X_2.$$

Both π_1 and π_2 are continuous. Hence, by [Proposition 6.13](#), their images $\pi_1(X_1 \times X_2) = X_1$ and $\pi_2(X_1 \times X_2) = X_2$ are separable.

Conversely, assume that X_1 and X_2 are separable. Then there exist countable subsets $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ such that

$$\overline{A_1} = X_1 \quad \text{and} \quad \overline{A_2} = X_2.$$

The set $A_1 \times A_2$ is countable, and moreover,

$$\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2} = X_1 \times X_2.$$

Thus $X_1 \times X_2$ is separable.

Definition 6.9 A topological space (X, τ) is called normal if for any two disjoint closed subsets $E, F \subseteq X$, there exist disjoint open sets $U, V \subseteq X$ such that $E \subseteq U$ and $F \subseteq V$.

We now show that every metric space is normal.

Proposition 6.15 $\hookrightarrow$ Every metric space (X, d) is a normal topological space.

Proof 6.28 Let E and F be disjoint closed subsets of X . If one of them is empty, the conclusion is trivial. Assume therefore that both E and F are non-empty.

Define a function $f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$ by

$$f(x) = d(x, E) - d(x, F).$$

The distance functions $x \mapsto d(x, E)$ and $x \mapsto d(x, F)$ are continuous, hence f is continuous.

Note that $d(x, E) = 0$ if and only if $x \in E$, and $d(x, F) = 0$ if and only if $x \in F$. Since E and F are disjoint, we have

$$f(x) < 0 \quad \text{for all } x \in E, \quad f(x) > 0 \quad \text{for all } x \in F.$$

Define

$$U = f^{-1}((-\infty, 0)), \quad V = f^{-1}((0, \infty)).$$

Since f is continuous, U and V are open subsets of X . Moreover, $E \subseteq U$, $F \subseteq V$, and clearly $U \cap V = \emptyset$. This proves that X is normal.

6.4 AROUND FUNCTIONS RELATED TO THE TOPOLOGICAL ATTITUDE

The Bullet Train of Uniform Continuity and the Special Train of Absolute Continuity

Among the Continuity trains capable of transporting residents and various products across regions of the Topology realm, some possess special features and facilities that set them apart from other trains. Although ordinary Continuity trains have a reasonable speed, a certain type, called “Uniform Continuity,” travels at a very high speed and stops only at a limited number of major stations. These trains exist in some parts of the Topology realm. Additionally, there are special locomotives that draw their engine power and fuel from the Compact town to upgrade Continuity trains into Uniform Continuity ones. Similarly, there are boats and submarines of Uniform Continuity that are used in certain regions. However, there is another primary type of Continuity train for transporting residents or products, known as the special train of Absolute Continuity. The special engine of this type of train allows any finite number of cars to be added to the train without causing any change in the speed mechanism of the train. Clearly, these trains are considered Continuity trains (provided the number of cars does not change), but the reverse is not true. It means that such trains are distinct and unique, and operate more powerfully than Continuity trains. Similarly, there are boats and submarines of Absolute Continuity. There are other important kinds of functions between metric spaces that are worthwhile to investigate. In fact, we aim to explore as many relationships as possible between different notions in metric spaces, such as openness, closedness, and continuity.

Definition 6.10 (Open and Closed Functions) *Assume that (X, d_1) and (Y, d_2) are metric spaces and that $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. We say that f is an open mapping if it maps open sets to open sets; that is, whenever $G \subseteq X$ is open, $f(G)$ is open in Y . Furthermore, we say that f is closed if it maps closed sets to closed sets; more precisely, if $F \subseteq X$ is closed, then $f(F)$ is closed in Y .*

It might seem that there are some direct relations between continuous functions and these new notions (open and closed mappings). However, we shall see that, in general, there is no such relation. We present several examples to illustrate these differences; this also helps us build a collection of functions with specific desired properties.

Example 6.5 *Consider the identity map $I : (\mathbb{R}, d) \rightarrow (\mathbb{R}, |\cdot|)$, where d is the discrete metric. Clearly, I is continuous since the domain is a discrete space. However, I is neither open nor closed. Indeed, the interval $(0, 1)$ is closed in $(\mathbb{R}, d)$ but not closed in $(\mathbb{R}, |\cdot|)$. Moreover, the singleton $\{0\}$ is open in $(\mathbb{R}, d)$ but not open in $(\mathbb{R}, |\cdot|)$.*

Now, consider the identity map from $(\mathbb{R}, |\cdot|)$ into $(\mathbb{R}, d)$ (that is, I^{-1}). It is easily seen that I^{-1} is not continuous. Nevertheless, it is both open and closed.

Example 6.6 *Consider the sign function. More precisely, define*

$$\text{sgn} : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$$

by

$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Then sgn is a closed mapping but is neither continuous nor open. To see this, observe that

$$\operatorname{sgn}^{-1}((0, +\infty)) = \{1\},$$

where $(0, +\infty)$ is open while $\{1\}$ is closed. Moreover,

$$\operatorname{sgn}((0, 1)) = \{1\},$$

where $(0, 1)$ is open but $\{1\}$ is closed.

To prove that sgn is a closed mapping, note that for every closed set $F \subseteq \mathbb{R}$, we have

$$\operatorname{sgn}(F) \subseteq \{-1, 0, 1\},$$

which is a closed set.

We have seen in [Proposition 6.5](#) that the projections in a product space are continuous. Now, we show that they are, in fact, open mappings as well.

Proposition 6.16 Assume that $\{(X_i, d_i)\}_{i=1}^n$ is a finite family of metric spaces and $X = \prod_{i=1}^n X_i$ is equipped with any product metric. Then the projections $\pi_i : X \rightarrow X_i$, defined by $\pi_i(x_1, \dots, x_n) = x_i$ for $1 \leq i \leq n$, are open mappings.

Proof 6.29 By the results of [Chapter 3](#), the collection

$$\{U_1 \times \cdots \times U_n : U_i \subseteq X_i \text{ is open}\}$$

forms a base for the product metric. Since

$$\pi_i(U_1 \times \cdots \times U_n) = U_i,$$

the result follows.

Nevertheless, projections need not be closed mappings, as the following example shows.

Example 6.7 Consider $\mathbb{R}$ with the Euclidean metric, and define $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $\pi_1(x_1, x_2) = x_1$. The map π_1 is not closed. Let

$$F = \left\{ \left(\frac{1}{n}, n \right) : n \in \mathbb{N} \right\}.$$

The set F is closed in $\mathbb{R}^2$ since it has no cluster point in $\mathbb{R}^2$. However,

$$\pi_1(F) = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\},$$

which is not closed in $\mathbb{R}$, since 0 is a cluster point that does not belong to the set.

It seems that there may exist a relation between open and closed mappings, at least in some special cases. First, we record the following simple observation.

Lemma 6.1  Assume that $f : X \rightarrow Y$ is a one-to-one function. Then for each $A \subseteq X$, we have

$$f(A^c) = f(X) - f(A).$$

Proof 6.30 Let $y \in f(A^c)$. Then there exists $x \in A^c$ such that $y = f(x)$. If $y \in f(A)$, then there exists $x' \in A$ with $y = f(x')$. Since f is one-to-one, we would have $x = x'$, which contradicts $x \in A^c$ and $x' \in A$. Hence, $y \in f(X) - f(A)$.

Conversely, let $y \in f(X) - f(A)$. Then there exists $x \in X$ such that $y = f(x)$ and $x \notin A$. Hence, $x \in A^c$ and $y \in f(A^c)$.

Now, we have the following.

Proposition 6.17 Assume that $f : (X, d_1) \rightarrow (Y, d_2)$ is a one-to-one function. Then, f is open if and only if it is closed.

Proof 6.31 Suppose that f is open. We show that it is also closed. Let $F \subseteq X$ be closed. Since F^c is open, $f(F^c)$ is open in Y by the hypothesis. By [Lemma 6.1](#),

$$f(F^c) = f(X) - f(F).$$

Since $f(X)$ is open in Y (because X is open in X and f is open), openness in $f(X)$ coincides with openness in Y for subsets of $f(X)$. Therefore, $f(F)$ is closed in Y , as claimed.

As we said before, “continuity” can be viewed from several equivalent perspectives. We now present some of them.

Theorem 6.11 Suppose (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then the following statements are equivalent:

1. f is continuous;
2. For each closed set $F \subseteq Y$, $f^{-1}(F)$ is closed in X ;
3. For each $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$;
4. For each $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof 6.32 $1 \rightarrow 2$. Let $F \subseteq Y$ be closed. Then F^c is open, so $f^{-1}(F^c)$ is open in X . This means that

$$f^{-1}(F^c) = f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$$

is open in X so that $f^{-1}(F)$ is closed.

$2 \rightarrow 3$. For $B \subseteq Y$, $B \subseteq \overline{B}$ implies $f^{-1}(B) \subseteq f^{-1}(\overline{B})$. Since $\overline{B}$ is closed, $f^{-1}(\overline{B})$ is closed by (2). Hence, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

$3 \rightarrow 4$. Let $A \subseteq X$ and put $B = f(A)$. By (3),

$$\overline{f^{-1}(f(A))} \subseteq f^{-1}(\overline{f(A)}).$$

Since $A \subseteq f^{-1}(f(A))$, we have $\overline{A} \subseteq \overline{f^{-1}(f(A))}$. Applying f , we get $f(\overline{A}) \subseteq f(\overline{f^{-1}(f(A))}) \subseteq \overline{f(A)}$.

$4 \rightarrow 1$. Let $V \subseteq Y$ be open. Then V^c is closed. Let $A = f^{-1}(V^c) = X - f^{-1}(V)$. By (4),

$$f(\overline{A}) \subseteq \overline{f(A)}.$$

This means that

$$f(\overline{f^{-1}(V^c)}) \subseteq \overline{f(f^{-1}(V^c))} \subseteq \overline{V^c} = V^c.$$

Therefore,

$$\overline{f^{-1}(V^c)} \subseteq f^{-1}(\overline{f(f^{-1}(V^c))}) \subseteq f^{-1}(V^c).$$

This shows that $f^{-1}(V^c)$ is closed so that $f^{-1}(V)$ is open; showing that f is continuous.

Furthermore, there are equivalent statements for open and closed functions as well, highlighting the power of topology (using open and closed sets).

Theorem 6.12 Suppose (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then the following are equivalent:

1. f is open;
2. For each $A \subseteq X$, $f(A^\circ) \subseteq (f(A))^\circ$;
3. For each $B \subseteq Y$, $(f^{-1}(B))^\circ \subseteq f^{-1}(B^\circ)$;
4. For each $D \subseteq Y$, $f^{-1}(\overline{D}) \subseteq \overline{f^{-1}(D)}$.

Proof 6.33 $1 \rightarrow 2$. Take any $A \subseteq X$. Note that $A^\circ \subseteq A$ so that $f(A^\circ) \subseteq f(A)$. By the hypothesis, $f(A^\circ)$ is open and $f(A)^\circ$ is the greatest open set involved in $f(A)$ so that $f(A^\circ) \subseteq (f(A))^\circ$.

To prove $2 \rightarrow 3$, for a set $B \subseteq Y$, we apply 2 with $A = f^{-1}(B)$. We have

$$f((f^{-1}(B))^\circ) \subseteq (f(f^{-1}(B)))^\circ \subseteq B^\circ.$$

Therefore, we have

$$(f^{-1}(B))^\circ \subseteq f^{-1}(f((f^{-1}(B))^\circ)) \subseteq f^{-1}(B^\circ).$$

$3 \rightarrow 4$. Take any $D \subseteq Y$. In 3, consider $B = Y - D = D^c$. We have

$$(f^{-1}(Y - D))^\circ = (f^{-1}(Y) - f^{-1}(D))^\circ = (X - f^{-1}(D))^\circ \subseteq f^{-1}((Y - D)^\circ).$$

On the other hand,

$$(X - f^{-1}(D))^\circ = ((f^{-1}(D))^c)^\circ = (\overline{(f^{-1}(D))}^c)^c,$$

and

$$f^{-1}((Y - D)^\circ) = f^{-1}((D^c)^\circ) = f^{-1}(\overline{D}^c) = f^{-1}(Y - \overline{D}) = X - f^{-1}(\overline{D}) = (f^{-1}(\overline{D}))^c.$$

We conclude that

$$(\overline{(f^{-1}(D))}^c)^c \subseteq (f^{-1}(\overline{D}))^c \Rightarrow f^{-1}(\overline{D}) \subseteq \overline{f^{-1}(D)}.$$

Finally, we need to prove that 4 implies 1. Assume that $G \subseteq X$ is open. We shall show that $f(G)$ is open in Y . In 4, put $D = Y - f(G) = (f(G))^c$. We have

$$f^{-1}(\overline{Y - f(G)}) \subseteq \overline{f^{-1}(Y - f(G))} = \overline{X - f^{-1}(f(G))}.$$

On the other hand, we have

$$f^{-1}(\overline{Y - f(G)}) = f^{-1}(\overline{(f(G))^c}) = f^{-1}(((f(G))^\circ)^c) = f^{-1}(Y - (f(G))^\circ) = X - f^{-1}((f(G))^\circ),$$

and

$$\overline{X - f^{-1}((f(G))^\circ)} = \overline{(f^{-1}((f(G))^\circ))^c} = ((f^{-1}((f(G))^\circ))^\circ)^c = X - (f^{-1}((f(G))^\circ))^\circ.$$

Therefore,

$$X - f^{-1}((f(G))^\circ) \subseteq X - (f^{-1}((f(G))^\circ))^\circ \Rightarrow (f^{-1}((f(G))^\circ))^\circ \subseteq (f^{-1}((f(G))^\circ)).$$

Note that $G \subseteq f^{-1}(f(G))$ so that $G \subseteq (f^{-1}(f(G)))^\circ$ since G is open. Thus, $G \subseteq (f^{-1}(f(G))^\circ)$ so that $f(G) \subseteq f((f^{-1}(f(G))^\circ)) \subseteq (f(G))^\circ$. This means that $f(G)$ is open in Y .

Proposition 6.18 Suppose (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a function. Then f is closed if and only if for each $A \subseteq X$,

$$\overline{f(A)} \subseteq f(\overline{A}).$$

Proof 6.34 Assume that f is closed and let $A \subseteq X$. Since $A \subseteq \overline{A}$, we have $f(A) \subseteq f(\overline{A})$. By hypothesis, $f(\overline{A})$ is closed and contains $f(A)$, so

$$\overline{f(A)} \subseteq f(\overline{A}).$$

Conversely, assume that for each $A \subseteq X$, $\overline{f(A)} \subseteq f(\overline{A})$. Let $F \subseteq X$ be closed. Then $\overline{F} = F$, and hence

$$\overline{f(F)} \subseteq f(\overline{F}) = f(F),$$

which implies $\overline{f(F)} = f(F)$. Therefore, f is closed.

Definition 6.11 Let (X, d_1) and (Y, d_2) be metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ a function. We say that f is strongly contractive with constant $0 < r < 1$ if for all $x, y \in X$,

$$d_2(f(x), f(y)) \leq r d_1(x, y).$$

Remark 6.8 Note that every strongly contractive function is Lipschitz with constant $M < 1$; in particular, it is continuous. As we have seen in the Banach Fixed Point Theorem ([Theorem 5.20](#)), a strongly contractive mapping defined on a complete metric space admits a unique fixed point. In what follows, we present another version of this result.

Theorem 6.13 Let (X, d) be a non-empty complete metric space and $f : (X, d) \rightarrow (X, d)$ a continuous function such that for all $x, y \in X$, the series

$$\sum_{n=1}^{\infty} d(f^n(x), f^n(y))$$

converges. Then f has a unique fixed point.

Proof 6.35 Let $\varepsilon > 0$ be given. Fix $x_0 \in X$ and set $x = x_0$ and $y = f(x_0)$. By assumption, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$,

$$\sum_{k=n}^m d(f^k(x_0), f^k(f(x_0))) < \varepsilon.$$

Observe that

$$d(f^m(x_0), f^n(x_0)) \leq d(f^m(x_0), f^{n+1}(x_0)) + d(f^{n+1}(x_0), f^n(x_0)),$$

and by iterating this estimate we obtain

$$d(f^m(x_0), f^n(x_0)) \leq \sum_{k=n}^m d(f^k(x_0), f^k(f(x_0))) < \varepsilon.$$

This shows that the sequence $(f^n(x_0))$ is Cauchy in X and hence convergent by completeness. Assume that

$$f^n(x_0) \rightarrow x \quad \text{for some } x \in X.$$

We claim that x is a fixed point of f . Indeed,

$$f(x) = f\left(\lim_{n \rightarrow \infty} f^n(x_0)\right) = \lim_{n \rightarrow \infty} f^{n+1}(x_0) = x.$$

To prove uniqueness, suppose that x and y are two fixed points of f . Then

$$f(x) = x, \quad f(y) = y,$$

and hence $f^k(x) = x$ and $f^k(y) = y$ for every $k \in \mathbb{N}$. Since, by assumption, the series

$$\sum_{n=1}^{\infty} d(f^n(x), f^n(y))$$

converges, it follows that

$$\lim_{k \rightarrow \infty} d(f^k(x), f^k(y)) = d(x, y) = 0.$$

Therefore $x = y$, and the fixed point is unique.

The following result is interesting in that it allows a uniformly continuous function to be extended to the closure (or completion) of its domain. We shall see in what follows that completeness plays an almost magical role in many fundamental theorems of analysis.

Definition 6.12 (Extension) Let (X, d_1) and (Y, d_2) be metric spaces and let $A \subseteq X$. Assume that $f : (A, d_1) \rightarrow (Y, d_2)$ is uniformly continuous. A function $\tilde{f} : (X, d_1) \rightarrow (Y, d_2)$ is called an extension of f if the restriction of $\tilde{f}$ to A coincides with f ; that is,

$$\tilde{f}|_A = f.$$

In general, the existence of an extension of a function is a subtle matter. However, as usual, completeness turns out to be an extremely powerful and useful property.

Theorem 6.14 Let (X, d_1) be a metric space, (Y, d_2) a complete metric space, and $A \subseteq X$. Suppose $f : A \rightarrow Y$ is uniformly continuous. Then there exists an extension $\tilde{f} : \bar{A} \rightarrow Y$ which is uniformly continuous.

Proof 6.36 Define $\tilde{f} : \bar{A} \rightarrow Y$ by

$$\tilde{f}(x) = \lim_{n \rightarrow \infty} f(a_n),$$

where $(a_n) \subseteq A$ is any sequence with $a_n \rightarrow x$.

We show that $\tilde{f}$ is well-defined. If $(a_n), (b_n) \subseteq A$ both converge to x , then $d_1(a_n, b_n) \rightarrow 0$. Continuity of f implies

$$d_2(f(a_n), f(b_n)) \rightarrow 0.$$

Since Y is complete, both $\lim f(a_n)$ and $\lim f(b_n)$ exist and are equal. Thus, $\tilde{f}$ is well-defined, and clearly $\tilde{f}|_A = f$.

Now, we show uniform continuity. Let $\varepsilon > 0$. By continuity of f on A , there exists $\delta > 0$ such that $d_1(a_1, a_2) < \delta$ implies $d_2(f(a_1), f(a_2)) < \varepsilon/2$ for $a_1, a_2 \in A$. For $x, y \in \bar{A}$, choose sequences $(a_n), (b_n) \subseteq A$ with $a_n \rightarrow x, b_n \rightarrow y$. For large n , $d_1(a_n, b_n) < \delta$, so

$$d_2(f(a_n), f(b_n)) < \varepsilon/2.$$

Passing to the limit, we obtain

$$d_2(\tilde{f}(x), \tilde{f}(y)) \leq \varepsilon,$$

showing that $\tilde{f}$ is uniformly continuous.

Because of importance of continuity, In this part, we present several elementary examples using different types of continuity.

Example 6.8 Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \sin(x^2)$. Then f is continuous but not uniformly continuous.

Indeed, let $x_n = \sqrt{n\pi}$ and $y_n = \sqrt{n\pi + \pi/2}$. Then

$$|x_n - y_n| = \frac{\pi/2}{\sqrt{n\pi} + \sqrt{n\pi + \pi/2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

but

$$|f(x_n) - f(y_n)| = |\sin(n\pi) - \sin(n\pi + \pi/2)| = 1 \not\rightarrow 0.$$

Hence f is not uniformly continuous.

Example 6.9 Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$. Then f is uniformly continuous.

Indeed, for any $\varepsilon > 0$, take $\delta = \varepsilon/2$. For $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have

$$\begin{aligned} \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| &= \frac{|x-y||x+y|}{(1+x^2)(1+y^2)} \leq |x-y| \left(\frac{|x|}{(1+x^2)(1+y^2)} + \frac{|y|}{(1+x^2)(1+y^2)} \right) \\ &< 2|x-y| < \varepsilon. \end{aligned}$$

Example 6.10 Let $\mathbb{Q}_+ = \{r_n\}_{n \in \mathbb{N}}$ be an enumeration of the positive rational numbers. Define the function $f : (0, \infty) \rightarrow \mathbb{R}$ by

$$f(x) = \sum_{r_n < x} \frac{x - r_n}{2^n}.$$

We show that f is well-defined. For each fixed $x > 0$, the sum is taken over infinitely many indices, but

$$0 \leq \frac{x - r_n}{2^n} \leq \frac{x}{2^n},$$

and the series $\sum_{n=1}^{\infty} \frac{x}{2^n}$ converges. Hence the defining series converges absolutely, and $f(x)$ is well defined for all $x > 0$.

Fix $x_0 > 0$ and let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$\sum_{n > N} \frac{x_0 + 1}{2^n} < \frac{\varepsilon}{2}.$$

Write

$$f(x) = \sum_{\substack{r_n < x_0 \\ n \leq N}} \frac{x - r_n}{2^n} + \sum_{n > N} \frac{x - r_n}{2^n} =: g(x) + h(x).$$

The function g is a finite sum of linear functions, hence continuous at x_0 . Therefore, there exists $\delta_1 > 0$ such that

$$|x - x_0| < \delta_1 \quad \Rightarrow \quad |g(x) - g(x_0)| < \frac{\varepsilon}{2}.$$

For the tail, note that if $|x - x_0| < 1$, then

$$|h(x) - h(x_0)| \leq \sum_{n>N} \frac{|x - x_0|}{2^n} < \sum_{n>N} \frac{1}{2^n} < \frac{\varepsilon}{2}.$$

Let $\delta = \min \{\delta_1, 1\}$. Then for all x with $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \leq |g(x) - g(x_0)| + |h(x) - h(x_0)| < \varepsilon.$$

Thus f is continuous at x_0 . Since x_0 was arbitrary, f is continuous on $(0, \infty)$.

Finally, we show that f is not uniformly continuous on $(0, \infty)$. Let $\delta > 0$ be given. By the density of the rationals, there exists $k \in \mathbb{N}$ such that

$$0 < r_k < \frac{\delta}{2}.$$

Set

$$x = r_k, \quad y = r_k + \frac{\delta}{4}.$$

Then $|x - y| < \delta$.

However, the k -th term contributes

$$|f(y) - f(x)| \geq \frac{y - r_k}{2^k} = \frac{\delta}{4 \cdot 2^k}.$$

Since such indices k can be chosen arbitrarily large while keeping $|x - y| < \delta$, there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ one can find x, y with

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon_0.$$

This violates the definition of uniform continuity.

There exists another interesting type of continuity.

Definition 6.13 (Absolute Continuity) Let $f : [a, b] \rightarrow \mathbb{R}$. We say that f is absolutely continuous on $[a, b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite collection of pairwise disjoint open intervals

$$\{(x_k, y_k)\}_{k=1}^n \subset [a, b]$$

satisfying

$$\sum_{k=1}^n (y_k - x_k) < \delta,$$

we have

$$\sum_{k=1}^n |f(y_k) - f(x_k)| < \varepsilon.$$

Obviously, every absolutely continuous function is uniformly continuous so that continuous. However, the converse is not, in general as the following example shows.

Example 6.11 Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

First, we show that f is continuous at 0. Using the inequality $|\sin x - x| \leq \frac{x^2}{2}$ for all $x \in \mathbb{R}$, we obtain

$$\left| \frac{\sin x}{x} - 1 \right| = \frac{|\sin x - x|}{|x|} \leq \frac{|x|}{2}, \quad x \neq 0.$$

Hence,

$$\lim_{x \rightarrow 0} f(x) = 1 = f(0),$$

so f is continuous at 0. Since $\sin x/x$ is continuous on $(0, 1]$, the function f is continuous on $[0, 1]$.

Because $[0, 1]$ is compact, the definition of uniform continuity implies that every continuous function on $[0, 1]$ is uniformly continuous. Therefore, f is uniformly continuous on $[0, 1]$.

Now, we show that f is in fact absolutely continuous. For $x > 0$, observe that

$$\frac{\sin x}{x} = \int_0^1 \cos(tx) dt.$$

This identity also holds at $x = 0$, since

$$\int_0^1 \cos(0) dt = 1 = f(0).$$

Now, we show that f is in fact absolutely continuous. Let $\varepsilon > 0$ be given. Put $\delta = \frac{\varepsilon}{2}$. for any $x, y \in [0, 1]$ with $|x - y| < \delta$, By the intermediate value theorem, we have Since $\cos(tx)$ is uniformly continuous on the compact set $[0, 1] \times [0, 1]$, there exists $\delta > 0$ such that

$$|\cos(tx) - \cos(ty)| \leq t|x - y| \leq |x - y| < \varepsilon \quad \text{for all } t \in [0, 1].$$

Now let $\{(a_k, b_k)\}$ be a finite collection of disjoint subintervals of $[0, 1]$ with

$$\sum_{k=1}^n (b_k - a_k) < \delta.$$

Then

$$\begin{aligned}
\sum_{k=1}^n |f(b_k) - f(a_k)| &= \sum_{k=1}^n \left| \int_0^1 (\cos(tb_k) - \cos(ta_k)) dt \right| \\
&\leq \sum_{k=1}^n \int_0^1 |\cos(tb_k) - \cos(ta_k)| dt \\
&< \sum_{k=1}^n \int_0^1 \varepsilon dt = \varepsilon \sum_{k=1}^n (b_k - a_k) < \varepsilon.
\end{aligned}$$

This verifies the definition of absolute continuity. Hence f is absolutely continuous on $[0, 1]$.

For an example of a uniformly continuous function on a compact set which is not absolutely continuous, we refer the reader to [11, Theorem 1].

6.5 FUNCTION SPACES WITH CONCENTRATE EXAMPLES

In this section, we study several well-known function spaces, along with their principal results and representative examples. Theorems and examples are developed simultaneously to facilitate a clearer and more illustrative understanding of the material.

First, we have the following notions.

Definition 6.14 (Function Space) Let (X, d) be a metric space. A function space on X (with real values) is a collection $\mathcal{F}$ of functions

$$f : X \rightarrow \mathbb{R}.$$

Typical examples include

$$\mathbb{R}^X = \{f : X \rightarrow \mathbb{R}\}, \quad C(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous}\},$$

and

$$B(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is bounded}\}.$$

Such collections become metric spaces once equipped with a suitable metric, for instance the supremum metric when defined.

Definition 6.15 (Pointwise Convergence) Let (f_n) be a sequence of real-valued functions on X . We say that (f_n) converges pointwise to a function $f : X \rightarrow \mathbb{R}$ if for every $x \in X$ and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (depending on x and ε) such that for all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Definition 6.16 (Uniform Convergence) Let (f_n) be a sequence of real-valued functions on X . We say that (f_n) converges uniformly to $f : X \rightarrow \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (depending only on ε) such that for all $n \geq N$ and for all $x \in X$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Remark 6.9 The essential difference between pointwise and uniform convergence lies in the dependence of N .

In pointwise convergence, the integer N may depend on both the point x and ε .

In uniform convergence, the integer N depends only on ε and works simultaneously for all $x \in X$.

Thus uniform convergence is strictly stronger than pointwise convergence.

Example 6.12 Define $f_n : [0, 1] \rightarrow \mathbb{R}$ by

$$f_n(x) = x^n.$$

For each fixed $x \in [0, 1)$, we have $x^n \rightarrow 0$, and clearly $f_n(1) = 1$ for all n . Hence the pointwise limit function f is

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

Thus (f_n) converges pointwise to f on $[0, 1]$.

However, the convergence is not uniform. Indeed,

$$\sup_{x \in [0, 1]} |f_n(x) - f(x)| = 1 \quad \text{for all } n,$$

since near $x = 1$ the values x^n remain close to 1 while $f(x) = 0$ for $x < 1$.

Therefore, pointwise convergence does not imply uniform convergence.

Uniform convergence allows one to interchange limits with both integration and differentiation. Also, it preserves continuity.

Theorem 6.15 $\Leftarrow$ *Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ such that $f_n \rightarrow f$ uniformly on $[a, b]$. Then f is continuous on $[a, b]$, and*

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx.$$

Proof 6.37 *Since each f_n is continuous and $f_n \rightarrow f$ uniformly, it follows that f is continuous on $[a, b]$. Let $\varepsilon > 0$. By uniform convergence, there exists N such that for all $n \geq N$ and all $x \in [a, b]$,*

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b - a}.$$

Then

$$\begin{aligned} \left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| &= \left| \int_a^b (f_n(x) - f(x)) dx \right| \leq \int_a^b |f_n(x) - f(x)| dx \\ &< \int_a^b \frac{\varepsilon}{b - a} dx = \varepsilon. \end{aligned}$$

Since ε was arbitrary, this proves

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx. \quad \square$$

Theorem 6.16 $\Leftarrow$ Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$ such that $\{f'_n\}$ converges uniformly to a function g on $[a, b]$, and assume there exists $x_0 \in [a, b]$ such that $f_n(x_0)$ converges. Then $\{f_n\}$ converges uniformly to a differentiable function f on $[a, b]$, and

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x) = g(x), \quad \forall x \in [a, b].$$

Proof 6.38 Fix $x \in [a, b]$. By the Fundamental Theorem of Calculus,

$$f_n(x) - f_n(x_0) = \int_{x_0}^x f'_n(t) dt.$$

Since $f'_n \rightarrow g$ uniformly on $[a, b]$, it follows from [Theorem 6.15](#) that

$$\int_{x_0}^x f'_n(t) dt \rightarrow \int_{x_0}^x g(t) dt \quad \text{as } n \rightarrow \infty.$$

By assumption, $f_n(x_0) \rightarrow f(x_0)$ for some $f(x_0)$. Define

$$f(x) := f(x_0) + \int_{x_0}^x g(t) dt.$$

Then $f_n(x) = f_n(x_0) + \int_{x_0}^x f'_n(t) dt \rightarrow f(x_0) + \int_{x_0}^x g(t) dt = f(x)$ uniformly on $[a, b]$. Moreover, f is differentiable and

$$f'(x) = g(x) = \lim_{n \rightarrow \infty} f'_n(x). \quad \square$$

As we have seen, integral and derivative are stable under uniform convergence. However, pointwise convergence does not behave well.

Example 6.13 Consider the sequence $(f_n) \subseteq C([0, 1])$ defined by

$$f_n(t) = t^n, \quad t \in [0, 1].$$

It can be easily seen that $f_n \rightarrow f$ pointwise, where

$$f(t) = \begin{cases} 0, & t = 0, \\ 1, & t \in (0, 1]. \end{cases}$$

However, $f \notin C([0, 1])$, since it is discontinuous at $t = 0$.

Remark 6.10 As we have seen in [Theorem 6.16](#), some additional conditions are needed in order to establish a relationship between uniform convergence of a sequence of functions and their derivatives. More precisely, we cannot expect a direct analogue of [Theorem 6.15](#) for differentiation.

For example, in $C^1([0, 1])$, consider the sequence

$$f_n(t) = \frac{\sin(nt)}{n},$$

which converges uniformly on $[0, 1]$ to the zero function. However, the derivatives

$$f'_n(t) = \cos(nt)$$

do not converge pointwise (they oscillate between -1 and 1). This shows that uniform convergence of a sequence of functions alone is not sufficient to guarantee convergence of their derivatives.

Example 6.14 $C^1([0, 1])$ is complete under the norm

$$\|f\| := \|f\|_\infty + \|f'\|_\infty, \quad \forall f \in C^1([0, 1]).$$

To see this, consider a Cauchy sequence $(f_n) \subseteq C^1([0, 1])$. For sufficiently large m, n , we have

$$\|f_m - f_n\| = \|f_m - f_n\|_\infty + \|f'_m - f'_n\|_\infty \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Since $(C([0, 1]), \|\cdot\|_\infty)$ is complete, there exist functions $f, g \in C([0, 1])$ such that

$$f_n \rightarrow f \quad \text{and} \quad f'_n \rightarrow g \quad \text{uniformly on } [0, 1].$$

By [Theorem 6.16](#), it follows that $f' = g$, so that $f \in C^1([0, 1])$. Hence,

$$\|f_n - f\| \rightarrow 0,$$

and this completes the proof.

In this part, we focus on some illustrative and fruitful examples in elementary functional analysis.

Example 6.15  Denote by $C([0, 1])$ the space of all continuous real-valued functions defined on $[0, 1]$. It is easy to see that $C([0, 1])$ is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication. By elementary calculus, the sum of two continuous functions and the scalar multiple of a continuous function are also continuous.

Moreover, elements of $C([0, 1])$ are bounded. Indeed, $[0, 1]$ is compact in $\mathbb{R}$ (closed and bounded), so for each $f \in C([0, 1])$, the image $f([0, 1])$ is compact in $\mathbb{R}$ and hence bounded. Define the norm $\|\cdot\|_\infty$ on $C([0, 1])$ via

$$\|f\|_\infty := \sup \{|f(t)| : t \in [0, 1]\}, \quad \forall f \in C([0, 1]).$$

It is straightforward to verify that $\|\cdot\|_\infty$ is a norm:

$$\|f\|_\infty \geq 0, \quad \|f\|_\infty = 0 \Leftrightarrow f = 0,$$

$$\|\alpha f\|_\infty = |\alpha| \|f\|_\infty, \quad \forall \alpha \in \mathbb{R},$$

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Moreover, $(C([0, 1]), \|\cdot\|_\infty)$ is complete, i.e., a Banach space. Indeed, let $(f_n) \subset C([0, 1])$ be Cauchy. For a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $m > n > n_0$, we have

$$\|f_m - f_n\|_\infty = \sup \{|f_m(t) - f_n(t)| : t \in [0, 1]\} < \frac{\varepsilon}{2}.$$

So, for each $t \in [0, 1]$, we have

$$|f_m(t) - f_n(t)| < \frac{\varepsilon}{2}.$$

This means that the sequence $(f_n(t))$ is Cauchy in $\mathbb{R}$ so that it is convergent to some real α_t . Define $f : \mathbb{R} \rightarrow \mathbb{R}$ via $f(t) = \alpha_t$. It is easy to see that f is well-defined. By fixing n and letting m to tend to infinity, we have

$$|f(t) - f_n(t)| \leq \frac{\varepsilon}{2} < \varepsilon,$$

provided that n is sufficiently large; this happens for any $t \in [0, 1]$. This means that

$$\|f_n - f\|_\infty \rightarrow 0.$$

By [Theorem 6.15](#), f is also continuous and we are done.

Example 6.16 Consider $C^1([0, 1])$, the space of all differentiable functions on $[0, 1]$ whose derivative is continuous. $C^1([0, 1])$ is a vector subspace of $C([0, 1])$ under pointwise operations, so it is a normed space with $\|f\|_\infty$.

However, it is not complete. For instance, let $f_n(t) = \sqrt{t + \frac{1}{n}}$ and $f(t) = \sqrt{t}$. Then

$$\|f_n - f\|_\infty = \sup_{t \in [0, 1]} \left| \sqrt{t + \frac{1}{n}} - \sqrt{t} \right| \leq \frac{1}{\sqrt{n}} \rightarrow 0,$$

but $f \notin C^1([0, 1])$ since f is not differentiable at $t = 0$.

Example 6.17 $C([0, 1])$ is not complete with respect to the following norm.

$$\|f\|_1 = \int_0^1 |f(t)| dt.$$

Indeed, consider $f_n(t) = t^{1/n}$. Then (f_n) is Cauchy in $\|\cdot\|_1$, but its limit $f(t) = \begin{cases} 1, & t \neq 0 \\ 0, & t = 0 \end{cases}$ is not continuous. Hence, (f_n) does not converge in $(C([0, 1]), \|\cdot\|_1)$.

Example 6.18 Let $X \neq \emptyset$ be a set, and denote by $B(X)$ the collection of all bounded real-valued functions on X . Then $B(X)$ is a vector space over $\mathbb{R}$. Define the uniform norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|, \quad \forall f \in B(X).$$

Then $(B(X), \|\cdot\|_\infty)$ is a Banach space. The norm axioms are verified easily using the properties of the supremum.

As we have seen so far, complete metric spaces enjoy certain advantages over general metric spaces (for example, compare Banach spaces with normed spaces). The good news is that it is possible to make any metric space complete by embedding it into a suitably constructed complete space. Before we explain this construction in more detail, let us recall a useful notion.

Definition 6.17 (Isometry and Embedding) Let (X, d_X) and (Y, d_Y) be metric spaces.

1. A map $f : X \rightarrow Y$ is called an isometry if for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

In other words, f preserves distances exactly.

2. A map $f : X \rightarrow Y$ is called a metric embedding (or simply an embedding) if f is injective and an isometry onto its image. That is, f preserves distances and $f(X) \subseteq Y$ is a metric subspace of Y .

Theorem 6.17 (Completion of a Metric Space) Let (X, d) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{d})$ such that X can be regarded as a subset of $\tilde{X}$. In other words, there exists a distance preserving mapping

$$\tilde{\alpha} : X \rightarrow \tilde{X}$$

such that $\tilde{\alpha}(X) = \tilde{X}$.

Moreover, if $(\hat{X}, \hat{d})$ is another complete metric space with the same property (i.e., there exists a distance preserving mapping

$$\hat{\alpha} : X \rightarrow \hat{X}$$

such that $\hat{\alpha}(X) = \hat{X}$), then there exists a distance preserving mapping

$$\varphi : \tilde{X} \rightarrow \hat{X}$$

such that

$$\varphi \circ \tilde{\alpha} = \hat{\alpha}.$$

Proof 6.39 *Proof.* The construction principle will be presented in several steps.

First step. Let C be the family of all Cauchy sequences in X . Define a relation $\sim$ on C by

$$\{x_n\} \sim \{y_n\} \iff d(x_n, y_n) \rightarrow 0.$$

First, $\sim$ is well-defined on C . Second, $\sim$ is an equivalence relation on C .

Indeed, let $\{x_n\}$ and $\{y_n\}$ be arbitrary elements of C . Since they are Cauchy sequences, the sequence $\{d(x_n, y_n)\}$ is Cauchy in $\mathbb{R}$ because

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n).$$

Since $\mathbb{R}$ is complete, $\{d(x_n, y_n)\}$ converges to a nonnegative number. Since the limit is unique, the relation $\sim$ is well-defined.

Clearly $\{x_n\} \sim \{x_n\}$ because $d(x_n, x_n) = 0 \rightarrow 0$. Symmetry of d implies symmetry of $\sim$. If $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$, then $d(x_n, y_n) \rightarrow 0$ and $d(y_n, z_n) \rightarrow 0$, and by the triangle inequality, $d(x_n, z_n) \rightarrow 0$. Thus $\sim$ is an equivalence relation on C .

Second step. Set

$$\tilde{X} = C / \sim = \{[\{x_n\}] \mid \{x_n\} \in C\}.$$

Define $\tilde{d}$ on $\tilde{X}$ by

$$\tilde{d}([\{x_n\}], [\{y_n\}]) = \lim_{n \rightarrow \infty} d(x_n, y_n).$$

Since $\{d(x_n, y_n)\}$ converges for every $\{x_n\}, \{y_n\} \in C$, $\tilde{d}$ is well-defined.

Third step. $\tilde{d}$ is a metric on $\tilde{X}$.

First, since $\{d(x_n, y_n)\}$ converges to a nonnegative number,

$$\tilde{d}([\{x_n\}], [\{y_n\}]) \geq 0.$$

Second, since $d(x_n, y_n) = d(y_n, x_n)$, $\tilde{d}$ is symmetric.

Third,

$$\tilde{d}([\{x_n\}], [\{y_n\}]) = 0$$

if and only if $d(x_n, y_n) \rightarrow 0$, which is equivalent to $[\{x_n\}] = [\{y_n\}]$.

Fourth, for $[\{x_n\}], [\{y_n\}], [\{z_n\}]$ we have

$$\begin{aligned} \tilde{d}([\{x_n\}], [\{y_n\}]) &= \lim_{n \rightarrow \infty} d(x_n, y_n) \leq \lim_{n \rightarrow \infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \tilde{d}([\{x_n\}], [\{z_n\}]) + \tilde{d}([\{z_n\}], [\{y_n\}]). \end{aligned}$$

Fourth step. $(\tilde{X}, \tilde{d})$ is complete.

Let $\{u_m\}$ be a Cauchy sequence in $\tilde{X}$. Each u_m can be written as

$$u_m = [\{x_{m,k}\}_{k \in \mathbb{N}}].$$

We claim that $\{x_{n,n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X and

$$[\{x_{n,n}\}] \rightarrow u_m.$$

Since $\{u_m\}$ is Cauchy, for every $\varepsilon > 0$ there exists $K_1 \in \mathbb{N}$ such that

$$\tilde{d}(u_m, u_n) < \varepsilon \quad \text{whenever } m, n \geq K_1.$$

Hence

$$\lim_{k \rightarrow \infty} d(x_{m,k}, x_{n,k}) < \varepsilon.$$

By definition of limit, there exists K_2 such that for all $k \geq K_2$,

$$|d(x_{m,k}, x_{n,k}) - 0| < \frac{\varepsilon}{2}.$$

Thus for $m \geq \max \{K_1, K_2\}$ and $n \geq K_1$,

$$d(x_{m,m}, x_{n,m}) < \frac{\varepsilon}{2}.$$

Since each $\{x_{n,k}\}_k$ is Cauchy, there exists K_3 such that

$$d(x_{n,m}, x_{n,n}) < \frac{\varepsilon}{2} \quad \text{for } m, n \geq K_3.$$

Let $K_\varepsilon = \max \{K_1, K_2, K_3\}$. Then for $m, n \geq K_\varepsilon$,

$$d(x_{m,m}, x_{n,n}) \leq d(x_{m,m}, x_{n,m}) + d(x_{n,m}, x_{n,n}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{x_{n,n}\}$ is Cauchy in X , hence $[\{x_{n,n}\}] \in \tilde{X}$.

One then shows that $u_m \rightarrow [\{x_{n,n}\}]$.

Fifth step. Define

$$\tilde{\alpha} : X \rightarrow \tilde{X}, \quad \tilde{\alpha}(x) = [\{x\}_{n \in \mathbb{N}}].$$

Then $\tilde{\alpha}$ is distance preserving, since

$$\tilde{d}(\tilde{\alpha}(x), \tilde{\alpha}(y)) = \lim_{n \rightarrow \infty} d(x, y) = d(x, y).$$

Sixth step. X is dense in $\tilde{X}$; that is, $\tilde{\alpha}(X) = \tilde{X}$.

Let $u \in \tilde{X}$ be arbitrary. Then $u = [\{x_n\}]$ where $\{x_n\}$ is a Cauchy sequence in X . Define

$$u_m = \tilde{\alpha}(x_m) \in \tilde{\alpha}(X) \subseteq \tilde{X}.$$

We show that $u_m \rightarrow u$.

Let $\varepsilon > 0$ be given. Since $\{x_n\}$ is Cauchy in X , there exists $K_\varepsilon \in \mathbb{N}$ such that

$$d(x_m, x_n) < \frac{\varepsilon}{2} \quad \text{whenever } m, n \geq K_\varepsilon.$$

For fixed $m \geq K_\varepsilon$ we have

$$\tilde{d}(u_m, u) = \tilde{d}(\tilde{\alpha}(x_m), [\{x_n\}]) = \tilde{d}([\{x_m\}_{n \in \mathbb{N}}], [\{x_n\}_{n \in \mathbb{N}}]) = \lim_{n \rightarrow \infty} d(x_m, x_n) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Therefore $u_m \rightarrow u$, and hence $\tilde{\alpha}(X)$ is dense in $\tilde{X}$.

Seventh step.

Let $(\hat{X}, \hat{d})$ be another complete metric space with a distance preserving mapping

$$\hat{\alpha} : X \rightarrow \hat{X}$$

such that $\hat{\alpha}(X)$ is dense in $\hat{X}$.

We define

$$\varphi : \tilde{X} \rightarrow \hat{X}.$$

Let $u \in \tilde{X}$. Since $\tilde{\alpha}(X) = \tilde{X}$, there exists a sequence

$$u_m = \tilde{\alpha}(x_m)$$

with $u_m \rightarrow u$.

Define

$$v_m = \hat{\alpha}(x_m) \in \hat{X}.$$

Since $\{u_m\}$ converges, it is Cauchy. Hence for every $\varepsilon > 0$ there exists $K_\varepsilon \in \mathbb{N}$ such that

$$\tilde{d}(u_m, u_n) < \varepsilon \quad \text{whenever } m, n \geq K_\varepsilon.$$

Because $\tilde{\alpha}$ and $\hat{\alpha}$ preserve distance, for $m, n \geq K_\varepsilon$,

$$\hat{d}(v_m, v_n) = \hat{d}(\hat{\alpha}(x_m), \hat{\alpha}(x_n)) = d(x_m, x_n) = \tilde{d}(\tilde{\alpha}(x_m), \tilde{\alpha}(x_n)) = \tilde{d}(u_m, u_n) < \varepsilon.$$

Thus $\{v_m\}$ is Cauchy in $\hat{X}$. Since $\hat{X}$ is complete, $\{v_m\}$ converges to some $\hat{u} \in \hat{X}$.

Define

$$\varphi(u) = \hat{u}.$$

First, φ is well-defined.

Indeed, suppose $u_m \rightarrow u$ and $t_m \rightarrow u$ with

$$u_m = \tilde{\alpha}(x_m), \quad t_m = \tilde{\alpha}(y_m).$$

Let

$$v_m = \hat{\alpha}(x_m), \quad s_m = \hat{\alpha}(y_m).$$

Then

$$\hat{d}(v_m, s_m) = \hat{d}(\hat{\alpha}(x_m), \hat{\alpha}(y_m)) = d(x_m, y_m) = \tilde{d}(\tilde{\alpha}(x_m), \tilde{\alpha}(y_m)) = \tilde{d}(u_m, t_m) \rightarrow 0.$$

Since limits are unique and

$$\hat{d}(v_m, s_m) \rightarrow \hat{d}(v, s),$$

we obtain $\hat{d}(v, s) = 0$, hence $v = s$. Thus φ is well-defined.

Second, for every $x \in X$,

$$(\varphi \circ \tilde{\alpha})(x) = \varphi(\tilde{\alpha}(x)) = \varphi([\{x\}_{m \in \mathbb{N}}]) = \lim_{m \rightarrow \infty} \hat{\alpha}(x) = \hat{\alpha}(x).$$

Therefore

$$\varphi \circ \tilde{\alpha} = \hat{\alpha}.$$

Third, φ preserves distance. Let $u, t \in \tilde{X}$ and choose sequences

$$u_m = \tilde{\alpha}(x_m) \rightarrow u, \quad t_m = \tilde{\alpha}(y_m) \rightarrow t.$$

Then

$$\begin{aligned} \hat{d}(\varphi(u), \varphi(t)) &= \hat{d}\left(\lim_{m \rightarrow \infty} \hat{\alpha}(x_m), \lim_{m \rightarrow \infty} \hat{\alpha}(y_m)\right) = \lim_{m \rightarrow \infty} \hat{d}(\hat{\alpha}(x_m), \hat{\alpha}(y_m)) \\ &= \lim_{m \rightarrow \infty} d(x_m, y_m) = \lim_{m \rightarrow \infty} \tilde{d}(\tilde{\alpha}(x_m), \tilde{\alpha}(y_m)) = \tilde{d}(u, t). \end{aligned}$$

Fourth, φ is onto.

Let $v \in \hat{X}$. Since $\hat{\alpha}(X)$ is dense in $\hat{X}$, there exists

$$v_m = \hat{\alpha}(x_m) \rightarrow v.$$

Set

$$u_m = \tilde{\alpha}(x_m).$$

As in the argument above, $\{u_m\}$ converges to some $u \in \tilde{X}$. Moreover,

$$\varphi(u) = \varphi\left(\lim_{m \rightarrow \infty} u_m\right) = \lim_{m \rightarrow \infty} \hat{\alpha}(x_m) = \lim_{m \rightarrow \infty} v_m = v.$$

Hence φ is onto.

6.6 BANACH STREET: THE ELITE OF THE TOPOLOGY TERRITORY

Banach Street: The Beating Heart and the Unrivaled Miracle of the Rich and Vibrant City of Normed Spaces in the Realm of Topology

Imagine a street where every resident is polished, robust, and complete. Here, in the heart of Topology Territory, only those who measure up in every sense are welcome. On Banach Street, where the residents are complete and uniquely distinguished, there exist features and structures that make something akin to a miracle occur within the realm of topology. The attainment of every goal, and the presence of the most refined and skillful individuals in every position, characterize this remarkable place. In truth, the completeness of its inhabitants symbolizes the highest level of intellectual, personal, and social capability—an excellence through which the seemingly impossible becomes possible by the power of mind and uniquely developed abilities.

For this reason, many individuals from across the domain of topology strive tirelessly to reside on this extraordinary street. They acquire the necessary equipment, strengthen their preparation, and continually refine themselves in the hope of earning the privilege of belonging there. Yet even the residents themselves must constantly maintain their excellence and preserve the

essential structures that qualify them to remain. A subtle shift in foundational principles, or the loss of certain essential tools, may result in the revocation of their residency, compelling them to leave Banach Street and return to the ordinary city of normed spaces.

Thus, perseverance and continuous refinement are indispensable both for entering and for remaining in this astonishing street.

We shall now examine this remarkable “miracle” from a mathematical perspective through a series of illustrative examples.

Throughout this section, we assume that $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are normed vector spaces over $\mathbb{R}$.

Definition 6.18 (Linear Operator) A map $T : X \rightarrow Y$ is called linear if for all $x_1, x_2 \in X$ and scalars α, β ,

$$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2).$$

Definition 6.19 (Bounded Linear Operator) A linear operator $T : X \rightarrow Y$ is called bounded if there exists a constant $C \geq 0$ such that

$$\| T(x) \|_Y \leq C \| x \|_X \quad \text{for all } x \in X \quad \Leftrightarrow \quad \| T(x) \|_Y \leq C \quad \text{for all } x \in X, \| x \| \leq 1$$

The smallest such constant C is called the operator norm of T , denoted by

$$\| T \| = \sup_{\| x \|_X \leq 1} \| T(x) \|_Y.$$

It can be shown easily that

$$\| T \| = \inf \{ C > 0 : \| T(x) \|_Y \leq C \| x \|_X, \quad \forall x \in X \}.$$

Remark 6.11 Note that every bounded linear operator is, in particular, a function. It is then natural to ask:

Question. Why do we not define a linear operator $T : X \rightarrow Y$ as bounded in the usual sense of functions; that is, why not simply require that there exists $M > 0$ such that

$$\| T(x) \|_Y \leq M \quad \text{for all } x \in X?$$

If we try to adopt the usual notion of bounded functions for linear operators, we quickly see that this would force every bounded linear operator to be the zero operator. Indeed, suppose $T : X \rightarrow Y$ is linear and satisfies $\| T(x) \|_Y \leq M$ for all $x \in X$.

- Choose $x_0 \in X$ such that $T(x_0) \neq 0$. - For each $n \in \mathbb{N}$, by linearity we have

$$T(nx_0) = n T(x_0) \quad \Rightarrow \quad \| T(nx_0) \|_Y = n \| T(x_0) \|_Y.$$

- But by the usual boundedness assumption, $\| T(nx_0) \|_Y \leq M$, so

$$n \| T(x_0) \|_Y \leq M \quad \Rightarrow \quad \| T(x_0) \|_Y \leq \frac{M}{n}.$$

- Letting $n \rightarrow \infty$, we obtain $\| T(x_0) \|_Y = 0$ so that $T(x_0) = 0$, which is a contradiction.

Therefore, a nonzero linear operator can never be bounded in the usual sense of functions. This justifies the operator norm definition of boundedness for linear operators:

T is bounded $\iff \exists C > 0$ such that $\|T(x)\|_Y \leq C \|x\|_X$ for all $x \in X$.

Proposition 6.19 (Equivalence of Boundedness and Continuity) *Let $T : X \rightarrow Y$ be linear. Then the following statements are equivalent:*

1. T is bounded;
2. T is continuous at 0;
3. T is continuous everywhere on X ;
4. T is uniformly continuous on X .

Proof 6.40 (1 $\implies$ 4): If T is bounded, there exists $C > 0$ such that $\|T(x)\|_Y \leq C \|x\|_X$ for all $x \in X$. Then for all $x, y \in X$,

$$\|T(x) - T(y)\|_Y = \|T(x - y)\|_Y \leq C \|x - y\|_X,$$

which shows that T is a Lipschitz function, hence uniformly continuous.

(4 $\implies$ 3 $\implies$ 2): Uniform continuity implies continuity everywhere, which obviously implies continuity at 0. **(2 $\implies$ 1):** Suppose T is continuous at 0. Then there exists $\delta > 0$ such that $\|x\|_X < \delta \implies \|T(x)\|_Y < 1$. For any $x \neq 0$, write $x = (\frac{\|x\|_X}{\delta}) \cdot (\frac{\delta x}{\|x\|_X})$. By linearity,

$$T(x) = \frac{\|x\|_X}{\delta} T\left(\frac{\delta x}{\|x\|_X}\right),$$

so

$$\|T(x)\|_Y = \frac{\|x\|_X}{\delta} \left\| T\left(\frac{\delta x}{\|x\|_X}\right) \right\|_Y \leq \frac{\|x\|_X}{\delta}.$$

Hence, T is bounded with $\|T\| \leq 1/\delta$.

Remark 6.12 - Boundedness is therefore equivalent to continuity and uniform continuity for linear operators between normed spaces. - This equivalence is special to linear operators; for nonlinear maps, continuity does not imply uniform continuity in general as we have seen in this Chapter, before.

Proposition 6.20 $\hookleftarrow$ Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a linear operator. Then there exists a real number a such that

$$T(x) = ax \quad \text{for all } x \in \mathbb{R}.$$

Proof 6.41 Since T is linear, it satisfies

$$T(x + y) = T(x) + T(y), \quad T(\lambda x) = \lambda T(x)$$

for all $x, y \in \mathbb{R}$ and all $\lambda \in \mathbb{R}$.

Set $a := T(1)$. For any integer $n \in \mathbb{Z}$, linearity implies

$$T(n) = nT(1) = na.$$

If $x = \frac{p}{q} \in \mathbb{Q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then

$$qT\left(\frac{p}{q}\right) = T(p) = pa,$$

and hence

$$T\left(\frac{p}{q}\right) = \frac{p}{q}a = ax.$$

Thus $T(x) = ax$ for all $x \in \mathbb{Q}$.

By Theorem 6.23, T extends to a function from $\mathbb{R}$ into $\mathbb{R}$, and we retain the notation T for this extension. The extension is uniformly continuous, and it is easily checked that it preserves linearity. Now let $x \in \mathbb{R}$ be arbitrary. Choose a sequence $(x_n) \subset \mathbb{Q}$ such that $x_n \rightarrow x$. Since T is linear, it is bounded on $\mathbb{R}$ and therefore continuous. Hence,

$$T(x) = \lim_{n \rightarrow \infty} T(x_n) = \lim_{n \rightarrow \infty} ax_n = ax.$$

This shows that $T(x) = ax$ for all $x \in \mathbb{R}$, completing the proof.

Corollary 6.5 Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a linear operator. Then T is absolutely continuous if and only if it is uniformly continuous.

Proof 6.42 Since T is linear, by [Proposition 6.20](#), there exists a real constant a such that

$$T(x) = ax \quad \text{for all } x \in \mathbb{R}.$$

If T is uniformly continuous, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |T(x) - T(y)| = |a||x - y| < \varepsilon.$$

In particular, for any finite collection of disjoint intervals $\{(x_k, y_k)\}$ with total length less than δ , we have

$$\sum_k |T(y_k) - T(x_k)| = |a| \sum_k |y_k - x_k| < \varepsilon,$$

so T is absolutely continuous.

The reverse implication is immediate from the definitions.

Before examining some fundamental results that demonstrate the importance of completeness in normed spaces and linear operators, we first introduce two important notions of convergence in function spaces. These concepts play a central role in analysis and will be essential in understanding the behavior of sequences of functions.

Definition 6.20 Let X and Y be normed spaces, and let $\{T_\alpha\}_{\alpha \in I}$ be a family (net) of bounded linear operators

$$T_\alpha : X \rightarrow Y,$$

indexed by a directed set I .

1. The net $\{T_\alpha\}_{\alpha \in I}$ is said to converge pointwise to a linear operator $T : X \rightarrow Y$ if, for each $x \in X$, the net $\{T_\alpha(x)\}_{\alpha \in I}$ converges to $T(x)$ in Y ; that is,

$$\lim_{\alpha} T_\alpha(x) = T(x) \quad \text{for every } x \in X.$$

2. The net $\{T_\alpha\}_{\alpha \in I}$ is said to converge uniformly to the linear operator $T : X \rightarrow Y$ if

$$\|T_\alpha - T\| \longrightarrow 0,$$

where the norm is the operator norm on $B(X, Y)$. Equivalently,

$$\sup_{\|x\| \leq 1} \|T_\alpha(x) - T(x)\| \longrightarrow 0.$$

Remark 6.13 It is clear that uniform convergence implies pointwise convergence. However, the converse is generally false.

Consider the sequence of functions (f_n) in $C([0, 1])$ defined by

$$f_n(t) = t^n, \quad t \in [0, 1].$$

For each fixed $t \in [0, 1)$, we have

$$\lim_{n \rightarrow \infty} t^n = 0,$$

and for $t = 1$,

$$\lim_{n \rightarrow \infty} 1^n = 1.$$

Therefore, (f_n) converges pointwise to the function

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 1, & t = 1. \end{cases}$$

To see that the convergence is not uniform, note that for any $n \in \mathbb{N}$,

$$\sup_{t \in [0, 1]} |f_n(t) - f(t)| = \sup_{t \in [0, 1)} |t^n - 0| = 1$$

Thus

$$\sup_{t \in [0, 1]} |f_n(t) - f(t)| \not\rightarrow 0,$$

so the convergence is not uniform.

Theorem 6.18 (Open Mapping Theorem) Let X and Y be Banach spaces and let $T : X \rightarrow Y$ be a bounded linear operator. If T is surjective, then T is an open mapping; that is, for every open set $U \subseteq X$, the image $T(U)$ is open in Y .

Proof 6.43 Since T is linear, it suffices to show that T maps a neighborhood of the origin in X onto a neighborhood of the origin in Y .

We shall prove that there exists $\delta > 0$ such that

$$N_\delta^{(0)} \subseteq T(N_1^{(0)}).$$

(★)

Since T is surjective,

$$Y = T(X).$$

Moreover,

$$X = \bigcup_{n=1}^{\infty} N_n^{(0)},$$

hence

$$Y = \bigcup_{n=1}^{\infty} T(N_n^{(0)}).$$

(1)

Note that Y is a Banach space. So, by using the Baire Category Theorem ([Theorem 5.25](#)), in (1), at least one of the sets $T(N_n^{(0)})$ has non-empty interior. Thus there exist $n_0 \in \mathbb{N}$, $y_0 \in Y$, and $\varepsilon > 0$ such that

$$N_\varepsilon^{(y_0)} \subseteq T(N_{n_0}^{(0)}).$$

(2)

Since $y_0 \in T(N_{n_0}^{(0)})$, there exists $x_0 \in N_{n_0}^{(0)}$ such that $T(x_0) = y_0$. Subtracting y_0 from both sides of (2) and using linearity of T , we obtain

$$N_\varepsilon^{(0)} \subseteq T(N_{n_0}^{(0)} - x_0).$$

Since

$$N_{n_0}^{(0)} - x_0 \subseteq N_{2n_0}^{(0)},$$

it follows that

$$N_\varepsilon^{(0)} \subseteq T(N_{2n_0}^{(0)}).$$

(3)

By linearity of T ,

$$T(N_{2n_0}^{(0)}) = 2n_0 T(N_1^{(0)}).$$

Hence from (3),

$$N_{\frac{\varepsilon}{2n_0}}^{(0)} \subseteq T(N_1^{(0)}).$$

This proves (*).

Let $U \subseteq X$ be open and let $x \in U$. Then there exists $r > 0$ such that

$$N_r^{(x)} = x + N_r^{(0)} \subseteq U.$$

Keep in mind that (using (*)), in Y , we have

$$N_\delta^{(0)} \subseteq T(N_1^{(0)}).$$

We claim that

$$N_{r\delta}^{(T(x))} \subseteq T(U).$$

Take any $z \in N_{r\delta}^{(T(x))}$ so that $\|z - T(x)\| < r\delta$ showing that $\|\frac{z - T(x)}{r}\| < \delta$. We conclude that there exists $y \in X$ with $\|y\| < 1$ such that $\frac{z - T(x)}{r} = T(y)$. On the other hand, note that $\|x + ry - x\| = r\|y\| < r$ showing that $x + ry \in N_r^{(x)}$ so that $x + ry \in U$. Thus, $T(x) + rT(y) \in T(U)$. Therefore, $z = T(x) + rT(y) \in T(U)$. This completes the proof.

Remark 6.14 Completeness is an essential assumption in the Open Mapping Theorem. The following examples illustrate that if either space is not complete, a bounded linear operator—even the identity—may fail to be open.

Example 6.19  Consider the space $C([0, 1])$ of continuous functions on $[0, 1]$, equipped with the supremum norm $\|\cdot\|_\infty$:

$$\|f\|_\infty := \sup_{t \in [0, 1]} |f(t)|.$$

We have seen in [Example 6.15](#) that $(C([0, 1]), \|\cdot\|_\infty)$ is a Banach space. Now, define the following norm on $C([0, 1])$:

$$\|f\|_1 := \int_0^1 |f(t)| dt.$$

It is easy to see that $(C([0, 1]), \|\cdot\|_1)$ is a normed space. However, this space is not complete. Consider the following sequence $(f_n) \subseteq C([0, 1])$:

$$f_n(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}t - \frac{n}{4} + \frac{1}{2}, & \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} + \frac{1}{n}, \\ 1, & \frac{1}{2} + \frac{1}{n} < t \leq 1. \end{cases}$$

The sequence (f_n) is a **Cauchy sequence** in $(C([0, 1]), \|\cdot\|_1)$. Indeed, for natural numbers $m > n$, we have

$$\|f_m - f_n\|_1 \leq \frac{1}{n} + \frac{1}{m} \longrightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Now, define the function f by

$$f(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < t \leq 1. \end{cases}$$

Clearly, $f \notin C([0, 1])$. Nevertheless,

$$\|f_n - f\|_1 \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This shows that (f_n) converges in the $\|\cdot\|_1$ -norm to a function $f \notin C([0, 1])$, hence $(C([0, 1]), \|\cdot\|_1)$ is not complete.

Define the identity operator

$$I : (C([0, 1]), \|\cdot\|_\infty) \rightarrow (C([0, 1]), \|\cdot\|_1), \quad I(f) = f.$$

First note that I is bounded: for any f ,

$$\|I(f)\|_1 = \int_0^1 |f(x)| dx \leq \|f\|_\infty.$$

However, I is not open. To see this, consider the open unit ball in $(C([0, 1]), \|\cdot\|_\infty)$:

$$(N_1^{(0)})_\infty := \{f \in C([0, 1]) : \|f\|_\infty < 1\}.$$

If I were open, $I((N_1^{(0)})_\infty)$ would contain some $\|\cdot\|_1$ -ball $(N_1^{(0)})_1$ for some $r > 0$. But this is impossible: for any $r > 0$, by the Archimedean property, there exists $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < r$. Now, define the sequence $(f_n) \subseteq C([0, 1])$ as follows. Define the function

$$f_n(t) = \begin{cases} -2nt + 2, & t \in [0, 1/n], \\ 0, & t \in (1/n, 1]. \end{cases}$$

Then for each $n > n_0$, $f_n \in (N_1^{(0)})_1$. But $\|f_n\|_\infty = 2$ which means that $f_n \notin (N_1^{(0)})_\infty$.

Example 6.20 Define the linear operator

$$T : c_{00} \longrightarrow c_{00}, \quad T((x_n)) = \left(\frac{x_n}{n}\right).$$

Then T is bounded and surjective, but it is not an open map.

First, note that if $x = (x_n) \in c_{00}$, then there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n > N$. Hence for $n > N$,

$$T(x) = (x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, 0, \dots).$$

So $T(x) \in c_{00}$.

Let $x \in c_{00}$. Then

$$\|Tx\|_{\infty} = \sup_{n \in \mathbb{N}} \left| \frac{x_n}{n} \right| \leq \sup_{n \in \mathbb{N}} |x_n| = \|x\|_{\infty}.$$

Hence $\|T\| \leq 1$, and therefore T is bounded (continuous).

Now, we show surjectivity.

Let $y = (y_n) \in c_{00}$. Then there exists N such that $y_n = 0$ for all $n > N$.

Define a sequence $x = (x_n)$ by

$$x_n = ny_n.$$

Then for $n > N$, we have $x_n = n \cdot 0 = 0$, so $x \in c_{00}$.

Moreover,

$$T(x) = \left(\frac{x_n}{n} \right) = \left(\frac{ny_n}{n} \right) = (y_n) = y.$$

Hence T is surjective.

Now, we show that T is not open.

Consider the open unit ball in c_{00} :

$$N_1^{(0)} = \{x \in c_{00} : \|x\|_{\infty} < 1\}.$$

If $x \in N_1^{(0)}$, then $|x_n| < 1$ for all n so that $\frac{|x_n|}{n} < 1$.

Hence

$$T(N_1^{(0)}) = \left\{ y \in c_{00} : |y_n| < \frac{1}{n} \text{ for all } n \right\}.$$

We show that this set is not open in c_{00} .

Assume, for contradiction, that T is open. Then $T(N_1^{(0)})$ must be an open neighborhood of 0. Therefore there exists $\delta > 0$ such that

$$N_{\delta}^{(0)} \subseteq T(N_1^{(0)}).$$

Let $\delta > 0$ be arbitrary. Choose $n \in \mathbb{N}$ large enough such that

$$\frac{1}{n} < \frac{\delta}{2}.$$

Define the sequence $y^{(n)} \in c_{00}$ by

$$y_k^{(n)} = \begin{cases} \frac{\delta}{2}, & k = n, \\ 0, & k \neq n. \end{cases}$$

Then

$$\|y^{(n)}\|_{\infty} = \frac{\delta}{2} < \delta,$$

so $y^{(n)} \in N_{\delta}^{(0)}$.

However,

$$|y_n^{(n)}| = \frac{\delta}{2} > \frac{1}{n},$$

so $y^{(n)} \notin T(N_1^{(0)})$.

This contradicts the inclusion $N_{\delta}^{(0)} \subseteq T(N_1^{(0)})$. Hence $T(N_1^{(0)})$ is not open, and therefore T is not an open map.

Corollary 6.6 $\hookrightarrow$ Let X be a vector space equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces. Suppose there exists a constant $\alpha > 0$ such that

$$\|x\|_1 \leq \alpha \|x\|_2 \quad \text{for all } x \in X.$$

Then the two norms are equivalent: there exists a constant $\beta > 0$ such that

$$\|x\|_2 \leq \beta \|x\|_1 \quad \text{for all } x \in X.$$

Proof 6.44 Consider the identity map

$$I : (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1), \quad I(x) = x.$$

By assumption, for all $x \in X$,

$$\|I(x)\|_1 = \|x\|_1 \leq \alpha \|x\|_2.$$

Hence, $I : (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1)$ is bounded.

Both $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces. The identity map is also surjective. Therefore, by the Open Mapping Theorem, a bounded, surjective linear map between Banach spaces is open. Therefore, I is an open map, which means there exists a constant $\beta > 0$ such that for all $x \in X$,

$$\|x\|_2 = \|I^{-1}(x)\|_2 \leq \beta \|x\|_1.$$

Hence, for all $x \in X$, we have

$$\|x\|_1 \leq \alpha \|x\|_2 \leq \alpha\beta \|x\|_1.$$

Thus, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.

Theorem 6.19 (Closed Graph Theorem) Let X and Y be Banach spaces, and let $T : X \rightarrow Y$ be a linear operator. Suppose that the graph of T ,

$$Gr(T) := \{(x, Tx) \in X \times Y : x \in X\},$$

is closed in $X \times Y$ (with the product norm $\| (x, y) \| := \| x \|_X + \| y \|_Y$). Then T is bounded, i.e., continuous.

Proof 6.45 Define

$$\| |x| \| := \| x \|_X + \| Tx \|_Y, \quad x \in X.$$

It can be easily seen that $\| | \cdot | \|$ is a norm on X , as well.

Now, we show that $(X, \| | \cdot | \|)$ is also a Banach space. Take a Cauchy sequence (x_n) in $\| | \cdot | \|$ norm. Then

$$\| |x_m - x_n| \| = \| x_m - x_n \|_X + \| T(x_m - x_n) \|_Y \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

In particular, (x_n) is Cauchy in $(X, \| \cdot \|)$ and (Tx_n) is Cauchy in $(Y, \| \cdot \|_Y)$. Since X and Y are Banach spaces, there exist

$$x \in X, \quad y \in Y \quad \text{such that } x_n \rightarrow x \text{ in } X, \quad Tx_n \rightarrow y \text{ in } Y.$$

But the graph of T is closed, so $(x_n, Tx_n) \rightarrow (x, y)$ implies $y = Tx$. Therefore $x_n \rightarrow x$ in $\| | \cdot | \|$ so that $(X, \| | \cdot | \|)$ is complete.

Observe that for all $x \in X$,

$$\| x \|_X \leq \| |x| \|.$$

Thus, the identity map

$$I : (X, \| | \cdot | \|) \rightarrow (X, \| \cdot \|_X), \quad I(x) = x$$

is bounded and surjective. Both $(X, \| | \cdot | \|)$ and $(X, \| \cdot \|_X)$ are Banach spaces.

So, by [Corollary 6.6](#), there exists a constant $C > 0$ such that for all $x \in X$,

$$\| |x| \| \leq C \| x \|_X.$$

Thus, for each $x \in X$, we have

$$\| Tx \|_Y \leq \| |x| \| \leq C \| x \|_X.$$

Hence, T is bounded (continuous).

Completeness of both the domain and codomain is essential in the Closed Graph Theorem. The following examples illustrate that if either space is not complete, a linear operator can have a closed graph but fail to be bounded.

Example 6.21 Define the linear operator

$$T : (C^1([0, 1]), \| \cdot \|_\infty) \rightarrow (C([0, 1]), \| \cdot \|_\infty), \quad T(f) = f'.$$

The graph of T is closed in $C^1([0, 1]) \times C([0, 1])$ with the product norm $\| (f, g) \| = \| f \|_\infty + \| g \|_\infty$; it is enough to use from [Theorem 6.16](#).

Nevertheless, T is not continuous (bounded). Consider the sequence $(f_n) \subseteq C^1([0, 1])$ defined via $f_n(t) = \frac{\sin(nt)}{n}$. Indeed, $\| f_n \|_\infty \rightarrow 0$ but $\| f_n' \|_\infty \cos(nt)$ which is not convergent

to zero. Note that we have seen that $(C^1([0, 1]), \|\cdot\|_\infty)$ is not complete.

Example 6.22 Consider the identity operator

$$I : (C^1([0, 1]), \|\cdot\|_1) \rightarrow (C([0, 1]), \|\cdot\|_\infty).$$

We have seen in [Example 6.19](#) that $(C([0, 1]), \|\cdot\|_1)$ is not complete. We show that $\text{Gr}(I)$ is closed.

Assume that $f_n \rightarrow f$ in $\|\cdot\|_1$ and $f_n \rightarrow g$ in $\|\cdot\|_\infty$. From the inequality

$$\|f_n - g\|_1 = \int_0^1 |f_n(t) - g(t)| dt \leq \|f_n - g\|_\infty,$$

we conclude that $\|f_n - g\|_1 \rightarrow 0$, as well. Since the limit is unique, we conclude that $f = g$. However, I is not bounded. For example, consider the sequence $(f_n) \subseteq C([0, 1])$ defined via $f_n(t) = t^n$. We have

$$\|f_n\|_1 = \int_0^1 t^n dt = \frac{1}{n+1} \rightarrow 0,$$

nevertheless,

$$\|f_n\|_\infty = 1 \not\rightarrow 0.$$

Now, we are going to discover the honor of completeness in bounded setting. First, we have the following definition.

Definition 6.21 Let X be a normed space, Y a normed space, and let $\{T_\alpha\}_{\alpha \in I}$ be a family (net) of bounded linear operators $T_\alpha : X \rightarrow Y$, indexed by a directed set I .

1. The net $\{T_\alpha\}_{\alpha \in I}$ is pointwise bounded if, for each $x \in X$, there exists a constant $C_x > 0$ such that

$$\|T_\alpha x\|_Y \leq C_x, \quad \forall \alpha \in I.$$

In other words, for each fixed $x \in X$, the set $\{\|T_\alpha x\|_Y : \alpha \in I\}$ is bounded in Y .

2. The net $\{T_\alpha\}_{\alpha \in I}$ is uniformly bounded if there exists a constant $C > 0$ such that

$$\|T_\alpha\| := \sup_{\|x\|_X \leq 1} \|T_\alpha x\|_Y \leq C, \quad \forall \alpha \in I.$$

That is, the operator norms of all T_α are bounded by a common constant.

It is clear that every uniformly bounded family (or net) is pointwise bounded. However, in general, the converse is false.

Example 6.23 $\triangleleft$ Define a family of linear operators $\{f_k\}_{k \geq 1}$ from c_{00} into $\mathbb{R}$ by

$$f_k((a_n)_{n \geq 1}) := \sum_{n=1}^k k a_n.$$

The sequence $(f_k)_{k \in \mathbb{N}}$ is pointwise bounded. For any fixed sequence $(a_n) \in c_{00}$, there exists N such that $a_n = 0$ for all $n > N$. Then for all $k > N$,

$$f_k((a_n)) = \sum_{n=1}^k k a_n = \sum_{n=1}^N k a_n = \text{finite.}$$

Hence, for each (a_n) , the set $\{|f_k((a_n))| : k \geq 1\}$ is bounded. So the family $\{f_k\}$ is pointwise bounded.

However, this sequence is not uniformly bounded.

$$\|f_k\| = \sup_{\|(a_n)\|_\infty \leq 1} |f_k((a_n))| \geq |f_k(e_k)| = k,$$

where $e_k = (0, 0, \dots, 1, 0, \dots)$ with 1 in the k -th position. Since $\|f_k\| \geq k \rightarrow \infty$, the family $\{f_k\}$ is not uniformly bounded.

We now present the following result, which employs completeness to obtain a rather surprising conclusion.

Theorem 6.20 (Uniform Boundedness Principle) *Let X be a Banach space and Y a normed space. Let $\{T_\alpha\}_{\alpha \in I} \subseteq \mathcal{B}(X, Y)$ be a net of bounded linear operators which is pointwise bounded. Then the net is uniformly bounded:*

Proof 6.46 For each $n \in \mathbb{N}$, define

$$A_n := \{x \in X : \|T_\alpha x\|_Y \leq n \text{ for all } \alpha \in I\}.$$

Each A_n is closed in X because each T_α is continuous. Pointwise boundedness implies

$$X = \bigcup_{n=1}^{\infty} A_n.$$

Since X is Banach, by Baire Category Theorem, there exists N such that A_N has non-empty interior. That is, there exist $x_0 \in X$ and $r > 0$ with

$$N_r^{(x_0)} \subseteq A_N.$$

For any $x \in N_r^{(0)}$, $x + x_0 \in N_r^{(x_0)} \subseteq A_N$, so

$$\|T_\alpha(x + x_0)\|_Y \leq N, \quad \forall \alpha \in I.$$

Linearity gives

$$\|T_\alpha x\|_Y = \|T_\alpha(x + x_0) - T_\alpha x_0\|_Y \leq \|T_\alpha(x + x_0)\|_Y + \|T_\alpha x_0\|_Y \leq 2N.$$

Now, for any $x \neq 0$, set $y = (r / \|x\|_X)x \in N_r^{(0)}$. Then

$$\|T_\alpha x\|_Y = \left(\frac{\|x\|_X}{r}\right) \|T_\alpha y\|_Y \leq \left(\frac{\|x\|_X}{r}\right) 2N.$$

Hence

$$\|T_\alpha\| \leq \frac{2N}{r}, \quad \forall \alpha \in I,$$

showing uniform boundedness of the net.

Consider [Example 6.23](#). It presents a pointwise bounded sequence which is not uniformly bounded. Observe that c_{00} is not complete.

6.7 EXERCISES

Exercise 6.1

1. Let

$$f : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$$

be a function. Show that, in order to prove that f is an open map, it suffices to show that f maps open intervals onto open intervals. On the other hand, show that to prove that f is a closed map, it is not enough to verify that f maps closed intervals onto closed intervals.

2. Give an example of an open (respectively, closed) map between metric spaces whose restriction to a subset of its domain is not open (respectively, not closed).
3. Show that the restriction of a continuous function to any subspace is again continuous.
4. Determine all continuous functions into a discrete metric space.
5. Suppose $f : (X = A \cup B, d_1) \rightarrow (Y, d_2)$ is a function whose restrictions to A and B are continuous. Prove that if A and B are both open or both closed, then f is continuous on X . Is the statement still true in general if we do not assume that A and B are both open or both closed?
6. Let (X, d_1) be a complete metric space and let $\varphi : A \subseteq X \rightarrow (Y, d_2)$ be a continuous function defined on an arbitrary subset A of X .

Define

$$d(x, y) = d_1(x, y) + d_2(\varphi(x), \varphi(y)), \quad x, y \in A.$$

Prove that d is a metric on A which is topologically equivalent to the metric induced on A by d_1 .

7. Suppose $f : (X, d) \rightarrow (X, d)$ satisfies a Lipschitz condition.

Is the function

$$g(x) = d(x, f(x))$$

from (X, d) into $(\mathbb{R}, |\cdot|)$ continuous?

8. Suppose $f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$ is continuous and define

$$P = \{x \in X \mid f(x) \geq 0\}, \quad S = \{x \in X \mid f(x) = 0\}.$$

Show that P and S are closed and that $S \subseteq \partial P$.

Give an example for which $S - \partial P \neq \emptyset$.

9. Describe all continuous functions from $[0, 1]$ into the Cantor set (with the Euclidean metric).

10. Suppose f is a one-to-one function from the Euclidean space $\mathbb{N}$ onto the Euclidean space $\mathbb{Q}$.

Prove that f is continuous at every point and that f^{-1} is discontinuous at every point.

11. Consider $C([0, 1])$ with $\|\cdot\|_\infty$. Prove that the set

$$F = \{f \in C([0, 1]) \mid f([0, \frac{1}{3}]) \cap f([\frac{2}{3}, 1]) \neq \emptyset\}$$

is closed in $C([0, 1])$.

12. Let

$$X = \{(t, 0) \mid 0 < t \leq 1\} \cup \left\{ \left(\frac{1}{n}, t \right) \mid n \in \mathbb{N}, 0 \leq t \leq 1 \right\}.$$

Define a function d on $X \times X$ by letting $d(x, y)$ be the length of the shortest path in X joining x to y .

(a) Prove that (X, d) is a metric space.

(b) Is d topologically equivalent to the Euclidean metric?

(c) Is X complete?

(d) Does X have any other interesting properties?

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Compactness: A Magic Sense

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7.1 COMPACTNESS: A GOVERNING PRINCIPLE IN THE REALM OF TOPOLOGY

Compactness: a Magic Industrial Town

In general, the ocean of mathematics is a miracle and hope for its inhabitants and those around them (like other sciences that are on its shores) to face the unknown world; the most practical science that can provide the basis for the peace and comfort of different sciences. But in the world of mathematics, we also have many unknowns and possibilities, the importance of discovering and processing which is very crucial and influential. For example, the infinite trenches in the depths of the ocean of mathematics or the active oceanic volcanoes in the vicinity and under the deep valleys all have unique gifts that mathematics, in its territory, must explore and process. One of the common tools is the management of sequences and nets (ambassadors and representatives of the convergence archipelago), along with other mathematical entities, which is used to organize and carry out progress projects. Like other mathematical pillars, they do their job pretty adequately. However, there is also another miraculous concept in between, which, on the one hand, aligns with the management and organization of sequences and nets, and on the other hand, due to its nature and extent, paves the way for advancement for all residents. In the vicinity of the convergence archipelago, there is an industrial town called Compactness, which produces high-quality, useful essences for many products in the Topology realm. Economic efficiency, convenient transportation, and a variety of high-quality products have given this industrial town a special place. For example, by using the products of this town, the inhabitants of the City of Real Functions can simultaneously have the value, continuity, and boundedness stamps on their Method Cards, which, in turn, greatly facilitate trade and help them achieve their development plans. Without using the products of this industrial town, the effective progression of bounded and continuous sets will be highly challenging, if not impossible. As another example, the spotlights produced by this company allow us to observe and exploit the depths of oceanic trenches and the activities of volcanoes very well. It is like possessing an infinite treasure in the depths of the trenches of Number cities. Furthermore, the powerful shrimps of Extremum, which, due to their unique, delicate taste, are popular among buyers in the ocean of applications, are processed and utilized from the management of upper and lower constraints in the city of Compactness. It should be noted that sequences and nets of the Convergence archipelago from the depths of marine trenches are the ones catching these shrimps.

7.2 COMPACTNESS IN ANALYSIS

Definition 7.1 Assume that (X, d) is a metric space and $A \subseteq X$. A family $\{G_\alpha\}_{\alpha \in I}$ of open subsets of X is called an open covering for A if

$$A \subseteq \bigcup_{\alpha \in I} G_\alpha.$$

Furthermore, for a subset $J \subseteq I$, we say that $\{G_\alpha\}_{\alpha \in J}$ is a subcovering for A if

$$A \subseteq \bigcup_{\alpha \in J} G_\alpha.$$

The subcovering is said to be finite if the index set J is finite.

Definition 7.2 Assume that (X, d) is a metric space and $K \subseteq X$. We say that K is compact if every open covering of K in X has a finite subcovering. In other words, if $\{G_\alpha\}_{\alpha \in I}$ is an open covering for K , then there exists a finite subset $\{\alpha_1, \dots, \alpha_n\} \subseteq I$ such that

$$K \subseteq \bigcup_{i=1}^n G_{\alpha_i}.$$

Unlike open or closed sets, compactness is stable under subspaces. That is, compactness does not depend on the choice of the ambient space. We illustrate this in the following theorem.

Theorem 7.1 $\hookleftarrow$ Assume that (X, d) is a metric space and $Y \subseteq X$. For a set $K \subseteq Y \subseteq X$, K is compact in (X, d) if and only if it is compact in (Y, d) .

Proof 7.1 First, assume that K is compact in (X, d) and let $\{H_\alpha\}_{\alpha \in I}$ be an arbitrary open covering of K in (Y, d) ; that is, each H_α is open in Y and $K \subseteq \bigcup_{\alpha \in I} H_\alpha$. We need to show that this covering has a finite subcovering.

By the definition of the subspace topology ([Theorem 4.15](#)), for each α , there exists an open set G_α in (X, d) such that

$$H_\alpha = G_\alpha \cap Y.$$

Moreover,

$$K \subseteq \bigcup_{\alpha \in I} H_\alpha = \bigcup_{\alpha \in I} (G_\alpha \cap Y) = \left(\bigcup_{\alpha \in I} G_\alpha \right) \cap Y \subseteq \bigcup_{\alpha \in I} G_\alpha.$$

Hence, $\{G_\alpha\}_{\alpha \in I}$ is an open covering of K in (X, d) , which has a finite subcovering since K is compact. Therefore, there exist finite indices $\{\alpha_1, \dots, \alpha_n\} \subseteq I$ such that

$$K \subseteq \bigcup_{i=1}^n G_{\alpha_i}.$$

Intersecting with Y gives

$$K \subseteq \left(\bigcup_{i=1}^n G_{\alpha_i} \right) \cap Y = \bigcup_{i=1}^n H_{\alpha_i},$$

showing that K is compact in (Y, d) .

Conversely, assume that K is compact in (Y, d) and let $\{G_\alpha\}_{\alpha \in I}$ be an arbitrary open covering of K in (X, d) . Define

$$H_\alpha = G_\alpha \cap Y.$$

Then each H_α is open in Y , and

$$K \subseteq \left(\bigcup_{\alpha \in I} G_\alpha \right) \cap Y = \bigcup_{\alpha \in I} (G_\alpha \cap Y) = \bigcup_{\alpha \in I} H_\alpha.$$

Hence, $\{H_\alpha\}_{\alpha \in I}$ is an open covering of K in (Y, d) , which has a finite subcovering by assumption. Therefore, there exist finite indices $\{\alpha_1, \dots, \alpha_n\} \subseteq I$ such that

$$K \subseteq \bigcup_{i=1}^n H_{\alpha_i} \subseteq \bigcup_{i=1}^n G_{\alpha_i}.$$

This shows that K is compact in (X, d) , as claimed.

Example 7.1 The real line $\mathbb{R}$ with the Euclidean topology is not compact. To see this, consider the open covering $\{G_n\}_{n \in \mathbb{N}}$ defined by

$$G_n = (-n, n).$$

Clearly, $\bigcup_{n=1}^{\infty} G_n = \mathbb{R}$. However, no finite sub-collection of $\{G_n\}$ can cover $\mathbb{R}$. Indeed, if $\{G_{n_1}, \dots, G_{n_N}\}$ were such a subcovering, then $\mathbb{R} \subseteq (-n_N, n_N)$ (assuming $n_1 < \dots < n_N$), which is impossible since $\mathbb{R}$ is unbounded.

Proposition 7.1 Assume that (X, d) is a metric space and $(x_n) \subseteq X$ is a convergent sequence with limit x . Then

$$A = \{x_n : n \in \mathbb{N}\} \cup \{x\}$$

is compact.

Proof 7.2 Let $\{G_\alpha\}_{\alpha \in I}$ be an arbitrary open covering of A . Since $x \in A$, there exists $\alpha_0 \in I$ such that $x \in G_{\alpha_0}$. By openness, there exists $r_0 > 0$ such that $N_r^{(x)} \subseteq G_{\alpha_0}$. Since $x_n \rightarrow x$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < r$ so that $x_n \in N_{r_0}^{(x)} \subseteq G_{\alpha_0}$ for all $n \geq n_0$.

For the remaining finitely many points $x_1, \dots, x_{n_0-1}$, choose indices $\alpha_1, \dots, \alpha_{n_0-1}$ such that $x_i \in G_{\alpha_i}$. Then

$$A \subseteq \bigcup_{i=0}^{n_0-1} G_{\alpha_i},$$

showing that A has a finite subcovering and so is compact.

Proposition 7.2 $\hookrightarrow$ Assume that (X, d) is a metric space and $K \subseteq X$ is finite. Then K is compact.

Proof 7.3 Suppose $K = \{x_1, \dots, x_n\}$ and let $\{G_\alpha\}_{\alpha \in I}$ be an open covering of K . For each $x_i \in K$, choose $\alpha_i \in I$ such that $x_i \in G_{\alpha_i}$. Then

$$K \subseteq \bigcup_{i=1}^n G_{\alpha_i},$$

so K has a finite subcovering and is compact.

In discrete spaces, the converse of [Proposition 7.2](#) also holds. In other words, in discrete spaces, only finite sets are compact.

Proposition 7.3 Assume that (X, d) is a discrete metric space. Then a subset $K \subseteq X$ is compact if and only if it is finite.

Proof 7.4 Assume that $K \subseteq X$ is compact. Consider the family

$$\{N_{\frac{1}{2}}^{(x)}\}_{x \in K}.$$

This is an open covering of K and, by compactness, it must have a finite subcovering. Hence, there exist $\{x_1, \dots, x_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=1}^n N_{\frac{1}{2}}^{(x_i)} = \bigcup_{i=1}^n \{x_i\} = \{x_1, \dots, x_n\}.$$

This shows that K is finite. The converse follows directly from [Proposition 7.2](#).

Theorem 7.2 $\hookrightarrow$ Assume that (X, d) is a metric space and $K \subseteq X$ is compact. Then K is closed.

Proof 7.5 To show that K is closed, it suffices to prove that K^c is open. Take any $x \in K^c$. For each $y \in K$, since $x \neq y$ and metric spaces are Hausdorff, there exists $r_y > 0$ such that

$$N_{r_y}^{(x)} \cap N_{r_y}^{(y)} = \emptyset.$$

The family $\{N_{r_y}^{(x)}\}_{y \in K}$ is an open covering of K , which has a finite subcovering by compactness. Hence, there exist $\{y_1, \dots, y_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=1}^n N_{r_{y_i}}^{(y_i)}.$$

Let

$$r_0 = \min \{r_{y_1}, \dots, r_{y_n}\}.$$

We claim that $N_{r_0}^{(x)} \subseteq K^c$. Suppose, for a contradiction, that there exists $y \in N_{r_0}^{(x)} \cap K$. Then there exists $1 \leq i_0 \leq n$ such that $y \in N_{r_{y_{i_0}}}^{(y_{i_0})}$. On the other hand, $y \in N_{r_0}^{(x)} \subseteq N_{r_{y_{i_0}}}^{(x)}$. This implies

$$y \in N_{r_{y_{i_0}}}^{(y_{i_0})} \cap N_{r_{y_{i_0}}}^{(x)},$$

which is a contradiction. Therefore, $N_{r_0}^{(x)} \subseteq K^c$, completing the proof.

Theorem 7.3 $\hookleftarrow$ Assume that (X, d) is a metric space and $F \subseteq K \subseteq X$, where K is compact and F is closed. Then F is also compact.

Proof 7.6 Let $\{G_\alpha\}_{\alpha \in I}$ be an arbitrary open covering of F . Since F is closed, F^c is open, and we have

$$K \subseteq X = F \cup F^c \subseteq \left(\bigcup_{\alpha \in I} G_\alpha \right) \cup F^c.$$

Hence, the family $\{G_\alpha\}_{\alpha \in I} \cup \{F^c\}$ is an open covering of K , which has a finite subcovering. Therefore, there exist indices $\{\alpha_1, \dots, \alpha_n\}$ such that

$$K \subseteq \bigcup_{i=1}^n G_{\alpha_i} \cup F^c.$$

Since $F \cap F^c = \emptyset$, we conclude

$$F \subseteq \bigcup_{i=1}^n G_{\alpha_i}.$$

This shows that F is compact.

Note that in [Theorem 7.3](#), it does not matter whether F is closed in X or in K . Indeed, by [Theorem 7.2](#), K is closed, so by [Corollary 4.3](#), closeness of F with respect to both X and K is equivalent. Thus, we have the following simple but important fact.

Corollary 7.1 $\hookleftarrow$ A subset F of a compact set K is compact if and only if it is closed.

Proposition 7.4 $\hookleftarrow$ Assume that (X, d) is a metric space. Then the following statements hold:

1. A finite union of compact sets is compact;
2. An arbitrary intersection of compact sets is compact.

Proof 7.7 For the first statement, let $\{K_i\}_{i=1}^n$ be a finite collection of compact sets, and put $K = \bigcup_{i=1}^n K_i$. Let $\{G_\alpha\}_{\alpha \in I}$ be an open covering of K . Then $K_i \subseteq \bigcup_{\alpha \in I} G_\alpha$ for each i . By compactness of K_i , we can choose finite indices $\{\alpha_{i_1}, \dots, \alpha_{i_{n_i}}\} \subseteq I$ such that

$$K_i \subseteq \bigcup_{j=1}^{n_i} G_{\alpha_{ij}}.$$

Thus,

$$K \subseteq \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} G_{\alpha_{ij}},$$

showing that K is compact.

For the second statement, let $\{K_\alpha\}_{\alpha \in I}$ be an arbitrary family of compact sets, and put $K = \bigcap_{\alpha \in I} K_\alpha$. By [Theorem 7.2](#), each K_α is closed, and since an intersection of closed sets is closed, K is closed. Moreover, for any fixed α_0 , $K \subseteq K_{\alpha_0}$, and by [Corollary 7.1](#), K is compact.

Remark 7.1 Observe that an arbitrary union of compact sets need not be compact, even when the index set is countable. Consider $X = \mathbb{R}$ with the Euclidean metric and let $A = \mathbb{Q}$. Since $\mathbb{Q}$ is countable, we can write

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}.$$

Each singleton $\{r_n\}$ is compact; however, $\mathbb{Q}$ itself is certainly not compact.

Now, we investigate the relationship between compactness and boundedness.

Theorem 7.4 $\hookrightarrow$ Every compact set in a metric space is bounded.

Proof 7.8 Let (X, d) be a metric space and $K \subseteq X$ be compact. Consider the family

$$\{N_1^{(x)}\}_{x \in K}.$$

This is an open covering of K , so by compactness, it has a finite subcovering. Hence, there exist $\{x_1, \dots, x_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=1}^n N_1^{(x_i)}.$$

Note that each neighborhood $N_1^{(x_i)}$ is bounded, with

$$\text{diam}(N_1^{(x_i)}) = \sup \{d(x, y) : x, y \in N_1^{(x_i)}\} \leq 2.$$

By [Proposition 7.4](#), the finite union $\bigcup_{i=1}^n N_1^{(x_i)}$ is also bounded, so K is bounded.

We have seen that every compact set in a metric space is closed and bounded. A natural question arises: is the converse true? That is, is every closed and bounded set in a metric space necessarily compact? The following examples show that this is not true in general.

Example 7.2 Let (X, d) be a discrete space that is infinite. Then X is closed and bounded, but it is not compact, since it is not finite.

Our examples are not limited to discrete spaces.

Example 7.3 Consider $X = \mathbb{Q}$ with the Euclidean metric induced from $\mathbb{R}$, and let

$$A = [0, \sqrt{2}] \cap \mathbb{Q}.$$

The set A is bounded; in fact, $\text{diam}(A) = \sqrt{2}$. Moreover, A is closed in $\mathbb{Q}$, since it is the intersection of the closed set $[0, \sqrt{2}] \subseteq \mathbb{R}$ with $\mathbb{Q}$.

We claim that A is not compact. Consider the open covering

$$\left\{ \left(-1, \sqrt{2} - \frac{1}{n} \right) \cap \mathbb{Q} \right\}_{n \in \mathbb{N}}.$$

Each element of this covering is open in $\mathbb{Q}$, being the intersection of an open set in $\mathbb{R}$ with $\mathbb{Q}$. Moreover,

$$A \subseteq \bigcup_{n=1}^{\infty} \left((-1, \sqrt{2} - \frac{1}{n}) \cap \mathbb{Q} \right).$$

However, this covering has no finite subcovering. Indeed, suppose for contradiction that there exists $N \in \mathbb{N}$ such that

$$A \subseteq \bigcup_{n=1}^N \left((-1, \sqrt{2} - \frac{1}{n}) \cap \mathbb{Q} \right).$$

Then

$$[0, \sqrt{2}] \cap \mathbb{Q} = A \subseteq \bigcup_{n=1}^N \left((-1, \sqrt{2} - \frac{1}{n}) \cap \mathbb{Q} \right) = (-1, \sqrt{2} - \frac{1}{N}),$$

which is impossible, since the interval $(\sqrt{2} - \frac{1}{N}, \sqrt{2})$ contains infinitely many rational numbers not covered by the finite union. This shows that A is not compact.

7.3 COMPACTNESS AND CONTINUITY: AN EFFICIENT COMBINATION

In this section, we investigate some relations between compactness and continuity via functions.

Theorem 7.5 $\hookrightarrow$ Let (X, d_1) and (Y, d_2) be metric spaces, and let $f : (X, d_1) \rightarrow (Y, d_2)$ be continuous. Then f maps compact sets in X to compact sets in Y .

Proof 7.9 Let $K \subseteq X$ be compact. We show that $f(K)$ is compact in Y . Let $\{H_\alpha\}_{\alpha \in I}$ be an open covering of $f(K)$. Since f is continuous, for each α the preimage

$$G_\alpha = f^{-1}(H_\alpha)$$

is open in X . Moreover,

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1}\left(\bigcup_{\alpha \in I} H_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(H_\alpha) = \bigcup_{\alpha \in I} G_\alpha.$$

Thus, $\{G_\alpha\}_{\alpha \in I}$ is an open covering of K in X . By compactness, it has a finite subcovering, so there exist indices $\{\alpha_1, \dots, \alpha_n\}$ such that

$$K \subseteq \bigcup_{i=1}^n G_{\alpha_i}.$$

Applying f , we obtain

$$f(K) \subseteq f\left(\bigcup_{i=1}^n G_{\alpha_i}\right) \subseteq \bigcup_{i=1}^n f(G_{\alpha_i}) \subseteq \bigcup_{i=1}^n H_{\alpha_i},$$

which shows that $f(K)$ is compact.

Corollary 7.2 Let (X, d_1) be a compact metric space and (Y, d_2) any metric space. Then every continuous function $f : (X, d_1) \rightarrow (Y, d_2)$ is closed.

Proof 7.10 Let $F \subseteq X$ be closed. Since X is compact, F is also compact by [Theorem 7.3](#). By [Theorem 7.5](#), $f(F)$ is compact in Y , and by [Theorem 7.2](#), every compact set in a metric space is closed. Hence, $f(F)$ is closed.

Remark 7.2 In [Theorem 7.5](#), we saw that a continuous function maps compact sets to compact sets. A natural question arises in this context:

Question. Does a continuous function map the inverse image of a compact set to a compact set?

In general, the answer is negative. Consider the identity map

$$I : (\mathbb{R}, d) \rightarrow (\mathbb{R}, |\cdot|),$$

where d is the discrete metric. The map I is continuous (in fact, uniformly continuous). Let $A = [0, 1]$. Clearly, A is compact in $(\mathbb{R}, |\cdot|)$. However, A is not compact in $(\mathbb{R}, d)$, since it is infinite.

Corollary 7.3  Assume that $f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$ is a continuous function and $K \subseteq X$ is compact. Then f attains its maximum and minimum on K .

Proof 7.11 Since $f(K)$ is compact in $\mathbb{R}$, it is both bounded and closed. Being bounded and non-empty, $f(K)$ has a supremum and infimum by the axiom of completeness. In the Euclidean space, the supremum and infimum are limit points, hence

$$\sup f(K), \inf f(K) \in \overline{f(K)} = f(K).$$

Therefore, the supremum and infimum are actually the maximum and minimum of f on K .

Corollary 7.4 $\Leftarrow$ Assume that (X, d) is a compact metric space. Then for any $x_0 \in X$, there exists $x_1 \in K$ such that

$$d(x_0, K) = d(x_0, x_1).$$

In other words, K attains its distance to x_0 .

Proof 7.12 Consider the function $d_{x_0} : K \rightarrow \mathbb{R}$ defined by

$$d_{x_0}(x) = d(x, x_0).$$

This function is continuous. By [Corollary 7.3](#), d_{x_0} attains its minimum on K , so there exists $x_1 \in K$ with

$$d_{x_0}(x_1) = \min_{x \in K} d_{x_0}(x) = \inf \{d(x, x_0) : x \in K\} = d(x_0, K).$$

Note that compactness is essential in [Corollary 7.4](#) and cannot be replaced by a weaker condition such as closeness. For example, in the Euclidean space $\mathbb{Q}$, the set $F = [0, \sqrt{2}] \cap \mathbb{Q}$ is closed, but for $x_0 = 2$, we have $d(x_0, F) = 2 - \sqrt{2}$, which is not attained. Compactness is a really magic tool. In what follows, we show that every continuous function is uniformly continuous provided that the domain is a compact space.

Theorem 7.6 $\Leftarrow$ Let $f : (X, d_1) \rightarrow (Y, d_2)$ be a continuous function and let K be a compact subset of X . Then f is uniformly continuous on K .

Proof 7.13 Let $\varepsilon > 0$ be given. For each fixed $y \in K$, since f is continuous at y , there exists $\delta_{\varepsilon, y} > 0$ such that

$$d_1(x, y) < \delta_{\varepsilon, y}$$

implies

$$d_2(f(x), f(y)) < \varepsilon.$$

(★)

Now the collection $\{N_{\delta_{\varepsilon, y}/2}(y)\}_{y \in K}$ is an open cover of K . Since K is compact, there exist $y_1, \dots, y_n$ in K such that

$$K \subset \bigcup_{i=1}^n N_{\frac{\delta_{\varepsilon, y_i}}{2}}(y_i).$$

Now set

$$\delta_\varepsilon = \frac{1}{2} \min \{\delta_{\varepsilon, y_1}, \dots, \delta_{\varepsilon, y_n}\} > 0.$$

If $x, y \in K$ and

$$d_1(x, y) < \delta_\varepsilon,$$

then since $y \in K$, there exists i_0 with $1 \leq i_0 \leq n$ such that

$$d_1(y, y_{i_0}) < \frac{\delta_{\varepsilon, y_{i_0}}}{2}.$$

Hence, by (*),

$$d_2(f(y), f(y_{i_0})) < \frac{\varepsilon}{2}.$$

(†)

Also,

$$d_1(x, y_{i_0}) \leq d_1(x, y) + d_1(y, y_{i_0}) < \delta_\varepsilon + \frac{\delta_{\varepsilon, y_{i_0}}}{2} \leq \frac{\delta_{\varepsilon, y_{i_0}}}{2} + \frac{\delta_{\varepsilon, y_{i_0}}}{2} = \delta_{\varepsilon, y_{i_0}}.$$

Thus, by (*),

$$d_2(f(x), f(y_{i_0})) < \frac{\varepsilon}{2}.$$

(‡)

Therefore, using (†) and (‡),

$$d_2(f(x), f(y)) \leq d_2(f(x), f(y_{i_0})) + d_2(f(y_{i_0}), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus δ_ε satisfies the condition for uniform continuity.

Next, we show that in $\mathbb{R}$ with the Euclidean topology, every closed and bounded set is compact. We first need the following result, known as the Chinese Box Theorem.

Theorem 7.7 $\square$ Assume that $\{I_n\}_{n \in \mathbb{N}}$ is a sequence of nested closed intervals in $\mathbb{R}$; that is,

$$I_1 \supseteq I_2 \supseteq \dots$$

Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof 7.14 Write $I_n = [a_n, b_n]$ with $a_n \leq b_n$. Since the sequence (I_n) is nested, each b_n is an upper bound for the set $\{a_n : n \in \mathbb{N}\}$, which is therefore bounded above. By the axiom of completeness, it has a supremum α . Moreover, $\alpha \leq b_n$ and $a_n \leq \alpha$ for all $n \in \mathbb{N}$. Hence, $\alpha \in I_n$ for each n , showing that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

We can generalize [Theorem 7.7](#) to $\mathbb{R}^k$ for $k \in \mathbb{N}$.

Theorem 7.8 Assume $k \in \mathbb{N}$ and let $(I_n)_{n \in \mathbb{N}}$ be a decreasing sequence of k -dimensional hypercubes in $\mathbb{R}^k$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof 7.15 Without loss of generality, assume

$$I_n = [a_n^1, b_n^1] \times \cdots \times [a_n^k, b_n^k]$$

for each n , where each $[a_n^i, b_n^i]$ is a closed interval in $\mathbb{R}$. By hypothesis, $I_{n+1} \subseteq I_n$, so that for each $1 \leq i \leq k$,

$$[a_{n+1}^i, b_{n+1}^i] \subseteq [a_n^i, b_n^i].$$

The result then follows from applying the one-dimensional case to each coordinate separately.

Now, we are ready to state and prove the Heine–Borel theorem, which asserts that in the Euclidean space $\mathbb{R}$, every closed and bounded set is compact.

Theorem 7.9 (Heine–Borel) $\Leftarrow$ A subset K of the Euclidean space $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof 7.16 The direct implication holds in every metric space by [Theorems 7.2](#) and [7.4](#).

Now, assume that $K \subseteq \mathbb{R}$ is closed and bounded. We shall show that K is compact. By boundedness, there exists an interval $[a, b] \subseteq \mathbb{R}$ such that $K \subseteq [a, b]$. If we show that $[a, b]$ is compact, then by the fact that K is closed ([Theorem 7.3](#)), it follows that K is also compact.

Suppose, for the sake of contradiction, that $[a, b]$ is not compact. Then there exists an open covering $\{G_\alpha\}_{\alpha \in I}$ of $[a, b]$ that does not admit a finite subcovering. Divide $[a, b]$ into two subintervals, $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. At least one of these subintervals cannot be covered by finitely many elements of the covering; denote it by $[a_1, b_1]$.

Next, divide $[a_1, b_1]$ into two subintervals, $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$. Again, at least one of these is not covered by finitely many elements of the covering; denote it by $[a_2, b_2]$. Proceeding inductively, we obtain a sequence of nested closed intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$$

such that none of them can be covered by finitely many elements of the open covering.

On the other hand, the lengths of these intervals shrink to zero:

$$\text{diam}([a_n, b_n]) = \frac{b-a}{2^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By the Chinese Box Theorem ([Theorem 7.7](#)), the intersection of these intervals is non-empty:

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset.$$

Let x_0 be a point in this intersection. Since $x_0 \in [a, b] \subseteq \bigcup_{\alpha \in I} G_{\alpha}$, there exists $\alpha_0 \in I$ such that $x_0 \in G_{\alpha_0}$. Because G_{α_0} is open, x_0 is an interior point, so there exists $r > 0$ such that

$$N_r^{(x_0)} \subseteq G_{\alpha_0}.$$

Since $x_0 \in [a_n, b_n]$ for all n and $d([a_n, b_n]) \rightarrow 0$, there exists $n_0 \in \mathbb{N}$ such that

$$[a_{n_0}, b_{n_0}] \subseteq N_r^{(x_0)} \subseteq G_{\alpha_0}.$$

This contradicts the construction of $[a_{n_0}, b_{n_0}]$, which was chosen so that it cannot be covered by finitely many elements of the covering. Therefore, $[a, b]$ must be compact, and hence K is compact as well.

Now, we illustrate these results with some classical examples.

Example 7.4

1. In the Euclidean space $\mathbb{R}^3$, the closed unit ball

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$$

and the set

$$B = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$$

are compact since both are closed and bounded.

2. The set

$$A = \{(x, \frac{1}{x}) : 0 < x \leq 1\} \subseteq \mathbb{R}^2$$

is closed but unbounded, so it is not compact.

3. The set

$$B = \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \subseteq \mathbb{R}^2$$

is bounded but not closed, hence not compact. To see that it is not closed, observe that $(0, 0)$ is a cluster point of B but $(0, 0) \notin B$.

4. The set

$$C = \{(0, \frac{1}{n}) : n \in \mathbb{N}\} \cup \{(0, 0)\} \subseteq \mathbb{R}^2$$

is both bounded and closed, and therefore compact.

In [Theorem 7.6](#), we have shown that every continuous function $f : ([a, b], |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$ is uniformly continuous; the main reason (we know now) is that the interval $[a, b]$ is compact. Now, we generalize this result to general compact metric spaces.

Theorem 7.10 Assume that (X, d_1) and (Y, d_2) are metric spaces, $f : (X, d_1) \rightarrow (Y, d_2)$ is continuous, and $K \subseteq X$ is compact. Then f is uniformly continuous on K .

Proof 7.17 Let $\varepsilon > 0$ be given. Since f is continuous at each $y \in K$, there exists $\delta_{\varepsilon, y} > 0$ such that for all $x \in X$,

$$d_1(x, y) < \delta_{\varepsilon, y} \implies d_2(f(x), f(y)) < \varepsilon.$$

By compactness of K , the family of neighborhoods

$$\{N_{\frac{\delta_{\varepsilon, y}}{2}}(y)\}_{y \in K}$$

forms an open cover of K , which admits a finite subcovering. In other words, there exist points $\{y_1, \dots, y_n\} \subset K$ such that

$$K \subseteq \bigcup_{i=1}^n N_{\frac{\delta_{\varepsilon, y_i}}{2}}(y_i).$$

Define

$$\delta_\varepsilon = \frac{1}{2} \min \{\delta_{\varepsilon, y_1}, \dots, \delta_{\varepsilon, y_n}\} > 0.$$

Now, for any $x, y \in K$ with $d_1(x, y) < \delta_\varepsilon$, there exists $1 \leq i_0 \leq n$ such that

$$d_1(y, y_{i_0}) < \frac{\delta_{\varepsilon, y_{i_0}}}{2} \implies d_2(f(y), f(y_{i_0})) < \frac{\varepsilon}{2}.$$

Also,

$$d_1(x, y_{i_0}) \leq d_1(x, y) + d_1(y, y_{i_0}) < \delta_\varepsilon + \frac{\delta_{\varepsilon, y_{i_0}}}{2} \leq \delta_{\varepsilon, y_{i_0}}.$$

Hence, $d_2(f(x), f(y_{i_0})) < \frac{\varepsilon}{2}$. Finally,

$$d_2(f(x), f(y)) \leq d_2(f(x), f(y_{i_0})) + d_2(f(y_{i_0}), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

showing that f is uniformly continuous on K .

7.4 COMPACTNESS IN MANY FORMS: ONE TIMELESS MATHEMATICAL MAGIC

In this section, we establish different forms of compactness; each of them, has many useful advantages. Nevertheless, they follow the same idea with a noble nature.

Theorem 7.11 $\Leftarrow$ Assume that (X, d) is a compact metric space. Then every infinite subset $A \subseteq X$ has a limit point in X .

Proof 7.18 Suppose, for the sake of contradiction, that A has no limit point. Then for each $x \in X$, there exists $r_x > 0$ such that

$$(N_{r_x}^{(x)} - \{x\}) \cap A = \emptyset,$$

which is equivalent to

$$N_{r_x}^{(x)} \cap A \subseteq \{x\}.$$

Consider the open cover $\{N_{r_x}^{(x)}\}_{x \in X}$ of X . By compactness, there exists a finite subcovering $\{N_{r_{x_i}}^{(x_i)}\}_{i=1}^n$ such that

$$X \subseteq \bigcup_{i=1}^n N_{r_{x_i}}^{(x_i)}.$$

Intersecting with A , we get

$$A \subseteq \bigcup_{i=1}^n (N_{r_{x_i}}^{(x_i)} \cap A) \subseteq \bigcup_{i=1}^n \{x_i\} = \{x_1, \dots, x_n\},$$

which is a contradiction since A is infinite. Therefore, A must have a limit point in X .

Remark 7.3 $\Leftarrow$ Because of the similarity between the definitions of compactness and total boundedness, it might seem at first glance that these notions coincide. It is easily seen that a compact set is totally bounded: for a compact set K in a metric space (X, d) and a given $\varepsilon > 0$, the family $\{N_\varepsilon^{(x)}\}_{x \in K}$ is an open cover, so it admits a finite subcovering, which is precisely the definition of total boundedness.

However, not every totally bounded set is compact. For example, in the Euclidean space $\mathbb{R}$, every open interval is totally bounded, but it is not compact since it is not closed. Nevertheless, totally bounded sets retain some useful topological properties.

Recall that a metric space (X, d) is *separable* if there exists a countable set $A \subseteq X$ with $\overline{A} = X$.

Theorem 7.12 If (X, d) is a totally bounded metric space, then X is separable.

Proof 7.19 For each $n \in \mathbb{N}$, consider a $\frac{1}{n}$ -net of X . Since X is totally bounded, there exists a finite set $\{x_{n,1}, \dots, x_{n,k_n}\} \subseteq X$ such that

$$X \subseteq \bigcup_{i=1}^{k_n} N_{\frac{1}{n}}(x_{n,i}).$$

Define the countable set

$$A = \{x_{n,i} : n \in \mathbb{N}, 1 \leq i \leq k_n\}.$$

We claim that $\overline{A} = X$. Indeed, let $x \in X$ and $r > 0$. By the Archimedean property, choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < r$. Then there exists i_0 with $1 \leq i_0 \leq k_{n_0}$ such that $x \in N_{\frac{1}{n_0}}(x_{n_0,i_0})$, so that $d(x, x_{n_0,i_0}) < r$. Hence, x is a limit point of A , completing the proof.

As an immediate consequence:

Corollary 7.5 $\Leftarrow$ Every totally bounded metric space satisfies the second axiom of countability; that is, it has a countable base.

Since every compact metric space is totally bounded ([Remark 7.3](#)), we also have:

Corollary 7.6 Every compact metric space is separable.

Definition 7.3 Let $f : X \rightarrow Y$ be a function. The graph of f is

$$Gr(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y.$$

Equipped with any metric on the Cartesian product $X \times Y$, $Gr(f)$ is itself a metric space.

Theorem 7.13 $\Leftarrow$ Let (X, d_1) and (Y, d_2) be metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ be continuous. Then $Gr(f)$ is closed in $(X \times Y, d)$, for any product metric d on $X \times Y$.

Proof 7.20 Let $z \in \overline{Gr(f)}$. Then there exists a sequence $(z_n) \subseteq Gr(f)$ with $z_n \rightarrow z$. Write $z_n = (x_n, f(x_n))$ and $z = (x, y)$. Since $z_n \rightarrow z$, we have $x_n \rightarrow x$ and $f(x_n) \rightarrow y$. By continuity of f , $f(x_n) \rightarrow f(x)$, so $y = f(x)$. Hence, $z = (x, f(x)) \in Gr(f)$, proving that $Gr(f)$ is closed.

Remark 7.4 The converse of [Theorem 7.13](#) is not true in general. For example, define

$$f : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|), \quad f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is not continuous, but $Gr(f)$ is closed in $\mathbb{R}^2$.

Now, we present a very useful fact using compactness and continuity of a real-valued function.

Theorem 7.14 (Intermediate Value Theorem) $\Leftarrow$ Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then for any k between $f(a)$ and $f(b)$, there exists $c \in (a, b)$ with $f(c) = k$.

In [Figure 7.1](#), we see the importance of continuity in the intermediate value theorem.

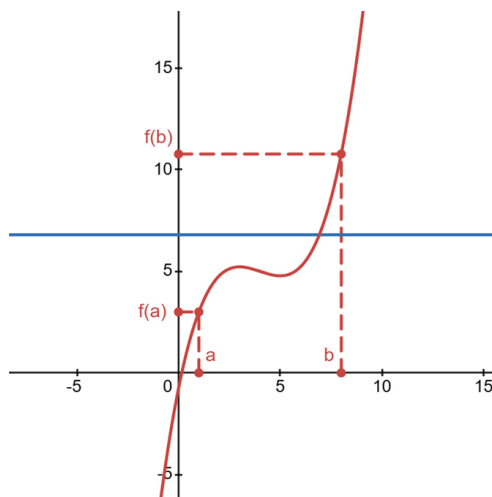


Figure 7.1 Illustration of the importance of continuity in the intermediate value theorem. ↩

Proof 7.21 Since $f([a, b])$ is compact ([Theorem 7.3](#)), it is closed and bounded. Without loss of generality, assume $f(a) < k < f(b)$. Define $g : [a, b] \rightarrow \mathbb{R}$ by $g(x) = f(x) - k$. Then $g(a) < 0$ and $g(b) > 0$.

Let

$$c = \sup \{x \in [a, b] : g(x) < 0\}.$$

By definition, $g(x) \geq 0$ for $x > c$, and there exists a sequence (x_n) in $[a, c)$ with $x_n \rightarrow c$ and $g(x_n) < 0$. By continuity of g , $\lim_{n \rightarrow \infty} g(x_n) = g(c) \leq 0$. Also, $\lim_{x \rightarrow c} g(x) \geq 0$, so $g(c) = 0$, hence $f(c) = k$.

Remark 7.5 Continuity is an essential assumption for the intermediate value theorem and cannot be omitted.

Consider

$$f : [0, 1] \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \\ 4, & \frac{1}{2} < x \leq 1. \end{cases}$$

The interval $[0, 1]$ is compact, but the discontinuity of f prevents f from taking all intermediate values. This motivates the classical intermediate value theorem.

The intermediate value theorem has several important consequences. We mention a few of them.

Corollary 7.7 Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then

$$f([a, b]) = [m, M],$$

where $m = \min_{x \in [a, b]} f(x)$ and $M = \max_{x \in [a, b]} f(x)$.

Proof 7.22 By [Corollary 7.3](#), f attains its maximum and minimum on $[a, b]$. By the intermediate value theorem ([Theorem 7.14](#)), f takes all values between m and M , so the result follows.

Corollary 7.8 (Existence of a root) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous with $f(a)f(b) < 0$. Then f has at least one root in (a, b) .

Proof 7.23 Since 0 lies between $f(a)$ and $f(b)$, the intermediate value theorem guarantees $c \in (a, b)$ such that $f(c) = 0$.

Corollary 7.9 (Existence of a fixed point) Let $f : [a, b] \rightarrow [a, b]$ be continuous. Then f has at least one fixed point in $[a, b]$.

Proof 7.24 Define $g : [a, b] \rightarrow \mathbb{R}$ by $g(x) = f(x) - x$. Then g is continuous, $g(a) = f(a) - a \geq 0$, and $g(b) = f(b) - b \leq 0$. By the intermediate value theorem, there exists $c \in [a, b]$ such that $g(c) = 0$, i.e., $f(c) = c$.

Theorem 7.15 $\Leftarrow$ Let (X, d) be a metric space and $A \subseteq X$. Then A is totally bounded if and only if every sequence in A has a Cauchy subsequence.

Proof 7.25 For the direct implication, assume that $A \subseteq X$ is totally bounded. We show that every sequence in A has a Cauchy subsequence. Let (x_n) be any sequence in A .

Fix $\varepsilon_1 = 1$. By total boundedness, there exist finitely many points $y_1^{(1)}, \dots, y_{k_1}^{(1)} \in A$ such that

$$A \subseteq \bigcup_{i=1}^{k_1} N_{y_{\varepsilon_1}^{(1)}}.$$

Since there are only finitely many balls, infinitely many x_n lie in some ball. Pick this ball and extract a subsequence $(x_n^{(1)})$ entirely contained in it.

Next, take $\varepsilon_2 = \frac{1}{2}$. Cover A by finitely many balls of radius ε_2 . Again, infinitely many terms of $(x_n^{(1)})$ lie in one such ball. Extract a subsequence $(x_n^{(2)})$ lying entirely in that ball.

Continue this process for $\varepsilon_k = \frac{1}{k}$. At step k , we get a subsequence $(x_n^{(k)})$ lying entirely in the assumed ball.

In this part, define the diagonal sequence:

$$y_n := x_n^{(n)}.$$

Then for any $\varepsilon > 0$, choose N such that $\frac{1}{N} < \varepsilon$. For $m, n \geq N$, both y_m and y_n lie in a ball of radius $\frac{1}{N}$, so

$$d(y_m, y_n) < 2 \cdot \frac{1}{N} < 2\varepsilon.$$

Hence (y_n) is Cauchy.

$\Leftarrow$. Now assume that every sequence in A has a Cauchy subsequence. We show that A is totally bounded.

Suppose A is not totally bounded. Then there exists $\varepsilon_0 > 0$ such that for any finite set $F \subseteq A$, there is $x \in A$ with

$$d(x, y) \geq \varepsilon_0 \quad \text{for all } y \in F.$$

Construct a sequence (x_n) inductively:

- Pick $x_1 \in A$.
- Given $x_1, \dots, x_n$, choose $x_{n+1} \in A$ such that $d(x_{n+1}, x_i) \geq \varepsilon_0$ for $i = 1, \dots, n$.

By construction, any two distinct terms satisfy $d(x_m, x_n) \geq \varepsilon_0$, so no subsequence can be Cauchy. Contradiction. Hence A must be totally bounded.

We have seen that every metric space has a topological base. It is natural to ask whether compactness can be verified using only elements of the base rather than all open sets. The following proposition confirms this.

Proposition 7.5 $\hookrightarrow$ A metric space (X, d) is compact if and only if every open covering of X consisting of elements of a base has a finite subcovering.

Proof 7.26 The forward implication is trivial since base elements are open.

Conversely, let $\{G_\alpha\}_{\alpha \in I}$ be an arbitrary open cover of X . For each $x \in X$, choose $\alpha_x \in I$ such that $x \in G_{\alpha_x}$, and pick a base element B_{α_x} with $x \in B_{\alpha_x} \subseteq G_{\alpha_x}$. Then $\{B_{\alpha_x}\}_{x \in X}$ is an open cover of X by base elements. By assumption, a finite subcovering $\{B_{\alpha_{x_1}}, \dots, B_{\alpha_{x_n}}\}$ exists. Hence,

$$X = \bigcup_{i=1}^n B_{\alpha_{x_i}} \subseteq \bigcup_{i=1}^n G_{\alpha_{x_i}},$$

so the original cover has a finite subcovering. This proves the proposition.

The following result is immediate and noteworthy.

Corollary 7.10 $\hookrightarrow$ Let (X, d) be a metric space satisfying the second axiom of countability (i.e., X has a countable base). Then X is compact if and only if every countable open covering $\{G_n\}_{n \in \mathbb{N}}$ has a finite subcovering.

Proof 7.27 This follows directly from [Proposition 7.5](#) and the fact that X has a countable base.

We now present another equivalent form of compactness. Recall that in a compact metric space, every infinite subset has a limit point. We now prove the converse under the assumption of the second axiom of countability; later, we shall see that this assumption can be removed.

Theorem 7.16 Let (X, d) be a metric space satisfying the second axiom of countability. If every infinite subset of X has a limit point, then X is compact.

Proof 7.28 Suppose X satisfies the second axiom of countability but is not compact. Then there exists a countable open covering $\{G_n\}_{n \in \mathbb{N}}$ with no finite subcovering.

Construct a sequence (x_n) inductively as follows: choose $x_1 \in X \setminus G_1$, then $x_2 \in X \setminus (G_1 \cup G_2)$ distinct from x_1 , and so on. Inductively, define $x_n \in X \setminus \bigcup_{i=1}^n G_i$. This is possible since otherwise, a finite union of G_i 's would cover X , contradicting the assumption.

Let $A = \{x_n : n \in \mathbb{N}\}$, which is infinite. By assumption, A has a limit point $x \in X$. Then $x \in G_{n_0}$ for some n_0 . Since x is a limit point of A , the set $A \cap G_{n_0}$ is infinite, so there exists $x_{n_1} \in A \cap G_{n_0}$ with $n_1 > n_0$. However, by construction, $x_{n_1} \notin \bigcup_{i=1}^{n_1} G_i$, a contradiction. Hence X must be compact.

Theorem 7.17 $\Leftarrow$ A metric space (X, d) is compact if and only if every sequence in X has a convergent subsequence.

Proof 7.29 Let (X, d) is a compact metric space and (x_n) be a sequence in X , and let $A = \{x_n : n \in \mathbb{N}\}$.

- If A is finite, the pigeonhole principle guarantees a constant (and hence convergent) subsequence. - If A is infinite, by [Theorem 7.11](#), A has a limit point x . By [Theorem 5.3](#), (x_n) has a subsequence converging to x .

Now, assume that (X, d) is sequentially compact; that is every sequence possesses a convergent subsequence. We show that X is compact.

If every sequence has a convergent subsequence, then every sequence has a Cauchy subsequence. By [Theorem 7.15](#), X is totally bounded, hence satisfies the second axiom of countability ([Corollary 7.5](#)). By [Corollary 7.10](#), it suffices to show that every countable open covering has a finite subcovering.

Suppose, for contradiction, that $\{G_n\}_{n \in \mathbb{N}}$ is a countable open cover with no finite subcovering. Construct (x_n) inductively as in the previous theorem: $x_n \notin \bigcup_{i=1}^n G_i$. Then (x_n) has no convergent subsequence, a contradiction. To see this exactly, assume that a subsequence (x_{n_k}) of (x_n) is convergent to some $x \in X$. Thus, there exists some $n_0 \in \mathbb{N}$ with $x \in G_{n_0}$. Furthermore, there is $k_0 \in \mathbb{N}$ with $x_{n_k} \in G_{n_0}$ for all $k > k_0$. Moreover, we may assume that $k_0 > n_0$. This means that $x_{n_k} \in \bigcup_{i=1}^{n_k} G_i$ that is a contradiction. Hence X is compact.

Theorem 7.18 A metric space (X, d) is compact if and only if it is complete and totally bounded.

Proof 7.30 First, note that compactness implies total boundedness ([Remark 7.3](#)). Let (x_n) be a Cauchy sequence in X . By [Theorem 7.17](#), it has a convergent subsequence. By [Theorem 5.5](#), a Cauchy sequence with a convergent subsequence converges to the same limit. Hence, X is complete.

Now, we consider the reverse. Let (x_n) be any sequence in X . By total boundedness ([Theorem 7.15](#)), (x_n) has a Cauchy subsequence. By completeness, this subsequence converges. Thus, every sequence has a convergent subsequence, and by [Theorem 7.17](#), X is compact.

We first extend [Corollary 7.10](#) by showing that the second axiom of countability can be omitted.

Theorem 7.19 A metric space (X, d) is compact if and only if every countable open covering has a finite subcovering.

Proof 7.31 *The direct implication is trivial. For the converse, assume that every countable open covering of X has a finite subcovering. We show that every sequence (x_n) in X has a convergent subsequence (see [Theorem 7.17](#)).*

Let

$$A = \{x_n : n \in \mathbb{N}\}, \quad G = \bigcup_{n=1}^{\infty} N_{\frac{1}{n}}^{(x_n)}, \quad H = \overline{G}^c.$$

Then H is open and

$$X = G \cup H = \left(\bigcup_{n=1}^{\infty} N_{\frac{1}{n}}^{(x_n)} \right) \cup H.$$

By assumption, this countable cover has a finite subcovering:

$$X \subseteq \bigcup_{i=1}^m N_{\frac{1}{n_i}}^{(x_{n_i})} \cup H.$$

If infinitely many elements of A lie in some $N_{1/n_i}^{(x_{n_i})}$, then we have a convergent subsequence converging to x_{n_i} . Otherwise, for sufficiently large n , $x_n \in H$. But then $x_n \in \overline{G}^c$ contradicts

$$x_n \in N_{\frac{1}{n}}^{(x_n)} \subseteq G \subseteq \overline{G}.$$

This contradiction implies that (x_n) has a convergent subsequence, completing the proof.

We now present another approach to compactness via closed sets.

Definition 7.4 *A family $\{A_\alpha\}_{\alpha \in I}$ of non-empty subsets of a set X has the finite intersection property if the intersection of every finite subfamily is non-empty. It has the total intersection property if*

$$\bigcap_{\alpha \in I} A_\alpha \neq \emptyset.$$

Clearly, the total intersection property implies the finite intersection property, but the converse is not generally true:

Example 7.5 *Let $X = \mathbb{R}$ and consider the family $\{(0, \frac{1}{n})\}_{n \in \mathbb{N}}$. This family has the finite intersection property, but*

$$\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset,$$

so it does not have the total intersection property.

Theorem 7.20 $\hookleftarrow$ *A metric space (X, d) is compact if and only if every family $\{F_\alpha\}_{\alpha \in I}$ of closed subsets of X with the finite intersection property also has the total intersection*

property.

Proof 7.32 Let (X, d) be compact, and let $\{F_\alpha\}_{\alpha \in I}$ be a family of closed subsets with the finite intersection property. Suppose, for contradiction, that

$$\bigcap_{\alpha \in I} F_\alpha = \emptyset.$$

Set $G_\alpha = F_\alpha^c$, which is open. Then

$$X = \bigcup_{\alpha \in I} G_\alpha.$$

By compactness, there exist indices $\alpha_1, \dots, \alpha_n$ such that

$$X \subseteq \bigcup_{i=1}^n G_{\alpha_i}.$$

Taking complements gives

$$\bigcap_{i=1}^n F_{\alpha_i} = \left(\bigcup_{i=1}^n G_{\alpha_i} \right)^c \subseteq X^c = \emptyset,$$

contradicting the finite intersection property.

Assume every family of closed subsets with the finite intersection property has non-empty total intersection. Let (x_n) be any sequence in X , and define the tails

$$T_n = \{x_k : k \geq n\}.$$

Then $\overline{T_1} \supseteq \overline{T_2} \supseteq \dots$ forms a decreasing family of closed sets. By assumption, the intersection

$$\bigcap_{n=1}^{\infty} \overline{T_n} \neq \emptyset.$$

Pick $x \in \bigcap_{n=1}^{\infty} \overline{T_n}$. For each k , since $x \in \overline{T_k}$, the neighborhood $N_{\frac{1}{k}}^{(x)} \cap T_k \neq \emptyset$, so there exists $n_k \geq k$ with

$$d(x_{n_k}, x) < 1/k.$$

Hence, the subsequence (x_{n_k}) converges to x , proving compactness.

We now discuss another equivalent form of compactness related to functions. Recall that any continuous real-valued function on a compact metric space is bounded.

Definition 7.5 A metric space (X, d) is called *pseudo compact* if every continuous real-valued function on X is bounded.

Clearly, every compact metric space is pseudo compact. In metric spaces, the converse also holds, although in general topological spaces these notions can differ. First, we have the following useful fact.

Lemma 7.1 $\hookrightarrow$ Let (X, d) be a metric space that is not totally bounded. Then there exists a continuous real-valued function on X which is unbounded.

Proof 7.33 Since X is not totally bounded, by [Theorem 7.15](#), there exists a sequence (x_n) in X with no Cauchy subsequence. Then there exists $r > 0$ such that, possibly after passing to a subsequence,

$$d(x_m, x_n) \geq r \quad \text{for all } m \neq n.$$

Define $f : X \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} n, & x \in N_r^{(x_n)}, \\ 0, & \text{otherwise.} \end{cases}$$

First, note that f is well-defined as the balls $\{N_r^{(x_i)}\}_{i=1}^{\infty}$ are pairwise disjoint. Clearly f is unbounded. So, it is enough to show that f is continuous.

To check continuity, let $x_0 \in X$ and $\varepsilon > 0$ be given. Suppose $x_0 \in N_r^{(x_{n_0})}$ for some $n_0 \in \mathbb{N}$ so that $d(x_0, x_{n_0}) < r$. Put $\delta = \frac{r - d(x_0, x_{n_0})}{2}$. clearly, $\delta > 0$. We claim that

$$N_{\delta}^{(x_0)} \subseteq N_r^{(x_{n_0})}.$$

Take any $z \in N_{\delta}^{(x_0)}$ so that $d(z, x_0) < \delta$. Therefore,

$$d(z, x_{n_0}) \leq d(z, x_0) + d(x_0, x_{n_0}) < r - d(x_0, x_{n_0}) + d(x_0, x_{n_0}) = r,$$

showing that $z \in N_r^{(x_{n_0})}$. Thus, for each $z \in N_{\delta}^{(x_0)}$, we have

$$|f(z) - f(x_0)| = |n - n| = 0 < \varepsilon.$$

Now, assume that $x_0 \notin N_r^{(x_n)}$ for any $n \in \mathbb{N}$ so that $d(x_0, x_n) \geq r$ for each $n \in \mathbb{N}$. Put

$$\gamma = \inf \{d(x_0, x_n) : n \in \mathbb{N}\}.$$

Clearly $\gamma \geq r$. We consider two cases. First, assume that $\gamma > r$. Take

$$\gamma_1 = \frac{\gamma - r}{2}.$$

Clearly, $\gamma_1 > 0$. We claim that

$$N_{\gamma_1}^{(x_0)} \cap N_r^{(x_n)} = \emptyset \quad \forall n \in \mathbb{N}.$$

Suppose on a contrary, $z \in N_{\gamma_1}^{(x_0)} \cap N_r^{(x_n)}$ so that $d(x_0, z) < \gamma_1$ and $d(z, x_n) < r$. So,

$$d(x_0, x_n) \leq d(x_0, z) + d(z, x_n) < \gamma_1 + r < \gamma - r + r = \gamma.$$

That is a contradiction with definition of γ .

Now, assume that $\gamma = r$. Thus, we can find $n_1 \in \mathbb{N}$ with

$$d(x_0, x_{n_1}) < r - \frac{r}{2} = \frac{r}{2} < r \Rightarrow x_0 \in N_r^{(x_{n_1})},$$

which is impossible.

Theorem 7.21 A metric space (X, d) is compact if and only if it is pseudo compact.

Proof 7.34 The direct implication is immediate from [Corollary 7.3](#). For the converse, if X were not compact, then it would not be totally bounded, and by [Lemma 7.1](#), there would exist an unbounded continuous function on X , contradicting pseudo compactness.

Proposition 7.6 Every compact metric space (X, d) is complete.

Proof 7.35 Let (x_n) be a Cauchy sequence in X . By [Theorem 7.17](#), (x_n) has a convergent subsequence $(x_{n_k}) \rightarrow x \in X$. Then by [Theorem 5.5](#), the full sequence (x_n) converges to x , so X is complete.

Finally, we present a fixed-point theorem for compact metric spaces.

Theorem 7.22 (Contraction-type Fixed Point) $\hookrightarrow$ Let (X, d) be a compact metric space, and let $f : X \rightarrow X$ satisfy

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \in X, \quad (x \neq y).$$

Then f has a unique fixed point.

Proof 7.36 Define $g : X \rightarrow \mathbb{R}$ by $g(x) = d(x, f(x))$. Then g is continuous, so by compactness, it attains a minimum at some $x_0 \in X$, i.e.,

$$g(x_0) \leq g(x) \quad \text{for all } x \in X.$$

We claim x_0 is a fixed point. If not, then

$$g(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = g(x_0),$$

contradicting the minimality of $g(x_0)$. Hence $f(x_0) = x_0$.

For uniqueness, suppose $x_1 \neq x_2$ are fixed points. Then

$$d(f(x_1), f(x_2)) = d(x_1, x_2) < d(x_1, x_2),$$

which is impossible. Thus, the fixed point is unique.

Remark 7.6 Note that the compactness hypothesis in [Theorem 7.22](#) is essential and cannot be replaced with completeness. Consider

$$f : ([1, \infty), |\cdot|) \rightarrow ([1, \infty), |\cdot|), \quad f(x) = x + \frac{1}{x}.$$

Indeed, f is continuous and the interval $[1, \infty)$ is complete (since it is closed). However, f does not have a fixed point; if $f(x_0) = x_0$ for some $x_0 \in [1, \infty)$, then

$$x_0 + \frac{1}{x_0} = x_0$$

so that $\frac{1}{x_0} = 0$, which is impossible. Now, we prove that f satisfies the condition of [Theorem 7.22](#). We have

$$|f(x) - f(y)| = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = \left| x - y + \frac{y - x}{xy} \right| = \left| \frac{xy - 1}{xy} \right| |x - y| < |x - y|.$$

Note that since $x, y \geq 1$, we conclude that $\left| \frac{xy-1}{xy} \right| < 1$.

Example 7.6 Assume that (X, d) is a compact metric space and $f : (X, d) \rightarrow (X, d)$ is a continuous function. Put

$$A = \bigcap_{n=1}^{\infty} f^n(X),$$

where f^n denotes the n -th composition of f with itself. Then A is non-empty and compact, and we have $f(A) = A$.

Proof 7.37 Note that $f(X)$ is compact in X by [Theorem 7.5](#); the same holds for $f^2(X) = f(f(X)) \subseteq f(X) \subseteq X$. In other words, we have a nested sequence of compact sets, and A is their total intersection. Note that since X is non-empty, the sequence $\{f^n(X)\}_{n \in \mathbb{N}}$ has the finite intersection property, so this family has the total intersection property by [Theorem 7.20](#). Thus, we conclude that A is non-empty and compact.

Furthermore, since the sequence $\{f^n(X)\}_{n \in \mathbb{N}}$ is nested, we have

$$f(A) = f\left(\bigcap_{n=1}^{\infty} f^n(X)\right) \subseteq \bigcap_{n=1}^{\infty} f^{n+1}(X) = \bigcap_{n=1}^{\infty} f^n(X) = A.$$

To see the converse, note that $A \subseteq f^{-1}(f(A))$, so that we have

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(f\left(\bigcap_{n=1}^{\infty} f^n(X)\right)\right) \subseteq f^{-1}\left(\bigcap_{n=1}^{\infty} f^n(X)\right) = \bigcap_{n=1}^{\infty} f^n(X) = A.$$

So, we have the desired result.

Example 7.7 Assume that (X, d) is a compact metric space and $f : (X, d) \rightarrow (X, d)$ is a distance-preserving function; that is,

$$d(f(x), f(y)) = d(x, y) \quad \text{for each } x, y \in X.$$

Then f is onto. Furthermore, compactness is necessary as a hypothesis; if we replace it with completeness, f is not onto in general.

Proof 7.38 It is easy to see that every distance-preserving function is continuous (even uniformly continuous) and one-to-one. Furthermore, f is open and therefore closed (since it is one-to-one), so we have a homeomorphism. We only need to show that f is onto. Note that $f(X)$ is closed; if we show that it is also dense in X , then $f(X) = X$ and we are done.

Take any $z \in X \setminus f(X)$. Define the sequence (z_n) in X via $z_n = f^n(z)$ for each $n \in \mathbb{N}$, where f^n is the n -th composition of f with itself. We have

$$\begin{aligned} d(z, z_1) &= d(z, f(z)), \\ d(z_1, z_2) &= d(f(z), f(z_1)) = d(z, z_1), \\ d(z_1, z_3) &= d(f(z), f(z_2)) = d(z, z_2). \end{aligned}$$

Furthermore,

$$\begin{aligned} d(z_1, z_4) &= d(f(z), f(z_3)) = d(z, z_3), \\ d(z_2, z_4) &= d(f(z_1), f(z_3)) = d(z_1, z_3) = d(f(z), f(z_2)) = d(z, z_2). \end{aligned}$$

So, we have $d(z_m, z_n) = d(z, z_{m-n})$ for naturals $m > n$. Therefore, if m and n tend to infinity, we see that (z_n) is a Cauchy sequence, so it is convergent since X is complete.

Now, for a given $\varepsilon > 0$, find $n_0 \in \mathbb{N}$ such that for each $r > s > n_0$, $d(z_r, z_s) < \varepsilon$. Take naturals m and n sufficiently large with $m > m - n > n_0$. We have

$$d(f^m(z), z) = d(f^m(z), f^{m-n}(z)) < \varepsilon,$$

which implies that $f^m(z) = z_m \rightarrow z$, as claimed.

7.5 LOCALLY COMPACT METRIC SPACES

Definition 7.6 Assume that (X, d) is a metric space. We say that X is locally compact provided that for each $x \in X$, there exist a compact set $K \subseteq X$ and an open set $U \subseteq X$ such that

$$x \in U \subseteq K.$$

Remark 7.7 This definition is crucial but somehow unclear, since we need a neighborhood contained in another compact set. Intuitively, when we say a metric space is locally compact, we mean that for each element, we can find a local compact set containing that element. On the other hand, “local” means that we are working with neighborhoods. By merging these notions, we may have a clearer understanding of “local compactness,” as the following lemma shows.

Lemma 7.2 A metric space (X, d) is locally compact if and only if for each $x \in X$, there is an open neighborhood $U \subseteq X$ containing x whose closure is compact.

Proof 7.39 For the direct statement, assume that (X, d) is locally compact and $x \in X$. By the definition, there exists a compact set $K \subseteq X$ and an open neighborhood $U \subseteq X$ with $x \in U \subseteq K$. So,

$$\overline{U} \subseteq \overline{K} = K,$$

since K is closed. Thus, we conclude that $\overline{U}$ is compact.

For the converse, assume that for each $x \in X$, we can find an open neighborhood $U \subseteq X$ containing x whose closure is compact. Then $\overline{U}$ plays exactly the role of K in the original definition of local compactness.

Let us illustrate this definition with some familiar examples. Note that every compact space is locally compact. However, the converse is not true in general.

Example 7.8 Consider $\mathbb{R}$ with the Euclidean metric. It is not compact, but it is locally compact. For each $x \in \mathbb{R}$, consider the interval $U = (x - 1, x + 1)$ whose closure is the compact interval $[x - 1, x + 1]$.

Example 7.9 Every discrete metric space (X, d) is locally compact. Indeed, for any $x \in X$, the singleton $\{x\}$ is both compact and an open neighborhood containing x .

Theorem 7.23 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a continuous open function. Then, if X is locally compact, $f(X)$ is also locally compact.

Proof 7.40 Take any $y \in f(X)$ and find $x \in X$ such that $f(x) = y$. Since X is locally compact, there exists an open neighborhood $U \subseteq X$ containing x such that $\overline{U}$ is compact. Since f is both continuous and open, we have

$$f(\overline{U}) = \overline{f(U)},$$

so that $\overline{f(U)}$ is compact. On the other hand, $f(U)$ is open and contains y . This shows that $f(X)$ is locally compact.

Proposition 7.7 Assume that (X, d) is a locally compact metric space and $E \subseteq X$ is closed. Then E is also locally compact.

Proof 7.41 Take any $x \in E$. By assumption, there is an open neighborhood $U \subseteq X$ containing x whose closure is compact. Now, we have

$$x \in E \cap U \subseteq E \cap \overline{U} = \overline{E \cap U} = \overline{E \cap U}.$$

Note that $E \cap U$ is an open neighborhood of x in E , whose closure is compact. Hence, E is locally compact.

7.6 COMPACTNESS AND SEPARATION IN METRIC SPACES

Now, we present another separation property regarding metric spaces with a compactness flavor.

Theorem 7.24 $\hookrightarrow$ Assume that (X, d) is a metric space and $Y \subseteq X$ is compact. Then for every $x_0 \in X - Y$, there exist open sets $U, V \subseteq X$ with $x_0 \in U$ and $Y \subseteq V$ such that $U \cap V = \emptyset$.

Proof 7.42 We know that every metric space is Hausdorff, so for every $y \in Y$, we can find open sets $U_y, V_y \subseteq X$ such that $x_0 \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. Note that

$$Y \subseteq \bigcup_{y \in Y} V_y,$$

so the collection $\{V_y\}_{y \in Y}$ forms an open covering of Y . By compactness, there exist $y_1, \dots, y_n \in Y$ such that

$$Y \subseteq \bigcup_{i=1}^n V_{y_i}.$$

Now, put

$$U = U_{y_1} \cap \dots \cap U_{y_n}, \quad V = V_{y_1} \cup \dots \cup V_{y_n}.$$

Indeed, both U and V are open, $x_0 \in U$, and $Y \subseteq V$. We claim that $U \cap V = \emptyset$. Suppose, to the contrary, that there exists $z \in U \cap V$. Then there is $1 \leq i_0 \leq n$ such that $z \in V_{y_{i_0}}$ and also $z \in U_{y_{i_0}}$. This implies that $U_{y_{i_0}} \cap V_{y_{i_0}} \neq \emptyset$, a contradiction.

For open subsets, we have a similar result.

Theorem 7.25 Assume that (X, d) is a locally compact metric space and $E \subseteq X$ is open. Then E is also locally compact.

Proof 7.43 Assume that $y \in E$. Since X is locally compact, there exists an open set $U \subseteq X$ containing y such that $\overline{U}$ is compact. Put

$$B = \overline{U} - E = \overline{U} \cap E^c = \overline{U} \cap (X - E).$$

Note that B is closed since E^c is closed and $B \subseteq \overline{U}$, so B is compact. Also, $y \notin B$. By [Theorem 7.24](#), we can find open disjoint sets $V_1, V_2 \subseteq X$ such that $y \in V_1$ and $B \subseteq V_2$. Put $W = V_1 \cap U$.

We claim that $\overline{W}$ is compact and W is open in E . First, $\overline{W}$ is compact because $W \subseteq \overline{U}$ and $\overline{U}$ is compact.

Next, note that $W \cap B = \emptyset$. Indeed, if $x \in W \cap B$, then $x \in B \subseteq V_2$ and $x \in W \subseteq V_1$, which contradicts $V_1 \cap V_2 = \emptyset$. Therefore,

$$W \subseteq B^c = (\overline{U} \cap E^c)^c = \overline{U} \cup E.$$

Since $W \subseteq \overline{U}$, it follows that $W \subseteq E$, so $W \cap E = W$. This implies

$$W = V_1 \cap U = V_1 \cap U \cap E,$$

showing that W is open in E . Hence, E is locally compact.

Before we present the next theorem, which has an illustrative and interesting result, we need an elementary but essential lemma.

Lemma 7.3 $\hookrightarrow$ Assume that (X, d) is a metric space and $Y \subseteq X$ is dense. Furthermore, suppose $V \subseteq Y$ is open. Then,

$$\overline{V} = (\overline{V})_Y,$$

where $(\overline{V})_Y$ denotes the closure of V in the induced metric space (Y, d) .

Proof 7.44 Assume that $x \in (\overline{V})_Y$ and let $U \subseteq X$ be an arbitrary open set containing x . Then

$$W = U \cap Y$$

is an open set in Y containing x . Since $x \in (\overline{V})_Y$, we have

$$V \cap W \neq \emptyset \implies V \cap U \supseteq V \cap U \cap Y = V \cap W \neq \emptyset,$$

showing that $x \in \overline{V}$.

Conversely, suppose $x \in \overline{V}$ and let $W \subseteq Y$ be an arbitrary open set containing x . Then there exists an open set $U \subseteq X$ such that $W = U \cap Y$. Also, since $V \subseteq Y$ is open in Y , there exists $U_1 \subseteq X$ open with $V = U_1 \cap Y$. By density,

$$U \cap U_1 \cap Y = W \cap V \neq \emptyset.$$

Thus, $x \in (\overline{V})_Y$, as claimed.

Theorem 7.26 $\hookrightarrow$ Assume that (X, d) is a metric space and $Y \subseteq X$ is dense. Moreover, assume that the metric space (Y, d) is locally compact. Then Y is open in X .

Proof 7.45 Take any $y \in Y$. We show that y is an interior point of Y . Since Y is locally compact, there exists an open set $V \subseteq Y$ containing y whose closure in Y is compact. By [Theorem 7.1](#), $\overline{V}$ is also compact in X .

By definition of the subspace topology, there exists an open set $U \subseteq X$ such that

$$V = U \cap Y.$$

Taking closures in X gives

$$\overline{V} = \overline{U \cap Y} \subseteq \overline{U} \cap \overline{Y} = \overline{U} \cap X = \overline{U}.$$

Thus, $\overline{U}$ is compact in X . Using [Lemma 7.3](#), we have

$$y \in V = U \cap Y \subseteq \overline{U} = \overline{V} = (\overline{V})_Y \subseteq Y.$$

This shows that y is an interior point of Y , so Y is open in X .

Example 7.10 Let $X = \mathbb{R}$ with the Euclidean metric. Then $Y_1 = \mathbb{Q}$ and $Y_2 = \mathbb{Q}^c$ are not locally compact. Indeed, if, for example, Y_1 were locally compact, then by [Theorem 7.26](#), Y_1 would be open in $\mathbb{R}$, which is impossible. Similarly, Y_2 is not locally compact.

As a routine result, we now consider product spaces.

Proposition 7.8 Assume that $(X_1, d_1), \dots, (X_n, d_n)$ are locally compact metric spaces. Then the Cartesian product space

$$X = \prod_{i=1}^n X_i$$

with any product metric is also locally compact.

Proof 7.46 Take any $(x_1, \dots, x_n) \in X$. Since each X_i is locally compact, there exists an open set $U_i \subseteq X_i$ containing x_i whose closure is compact in X_i . Consider the open set

$$U = \prod_{i=1}^n U_i \subseteq X.$$

By [Proposition 4.12](#),

$$\overline{U} = \overline{U_1 \times \dots \times U_n} = \overline{U_1} \times \dots \times \overline{U_n}.$$

The Cartesian product of compact metric spaces is compact, so $\overline{U}$ is compact. Hence X is locally compact.

We have seen in [Theorem 7.9](#) that $\mathbb{R}$ with the Euclidean metric is a Heine-Borel space; that is, every closed and bounded subset is compact. Now, we formally state and investigate this property in general metric spaces.

Definition 7.7 Assume that (X, d) is a metric space. We say that it satisfies the Heine-Borel property provided that compactness in this space is equivalent to closeness plus boundedness.

Proposition 7.9 $\Leftarrow$ Assume that (X, d) has the Heine-Borel property. For every $x_0 \in X$, every closed set $F \subseteq X$ attains its distance from x_0 ; in other words, there exists $x \in F$ such that

$$d(x_0, F) = \inf \{d(x_0, a) : a \in F\} = d(x_0, x).$$

Proof 7.47 If $x_0 \in F$, put $x = x_0$ so that $d(x_0, F) = d(x_0, x_0) = 0$. Otherwise, assume that $x_0 \in X - F$. Choose any $a \in F$ and put $r_0 = d(x_0, a)$; it is obvious that $r_0 > 0$. Define

$$K = N_{r_0}^{[x_0]} \cap F.$$

It is clear that K is closed since both $N_{r_0}^{[x_0]}$ and F are closed. Moreover, since $K \subseteq N_{r_0}^{[x_0]}$, K is bounded. By the Heine-Borel property, K is compact. Hence, by [Corollary 7.4](#), it attains its distance from x_0 . Thus, there exists $x \in K \subseteq F$ with $d(x_0, K) = d(x_0, x)$. Finally, note that $d(x_0, K) = d(x_0, F)$.

Remark 7.8 Although [Proposition 7.9](#) is proved using the Heine-Borel property, this assumption is not strictly necessary. For example, an infinite discrete metric space (X, d) does not have the Heine-Borel property; however, for every $x_0 \in X$ and every closed set $F \subseteq X$, we have either $d(x_0, F) = 0$ or $d(x_0, F) = 1$. In either case, this distance is attained by F .

Theorem 7.27 $\Leftarrow$ Assume that (X, d) is a metric space and $K_1, K_2 \subseteq X$ are compact sets. Then K_1 and K_2 attain their distance; in other words, there exist $x_0 \in K_1$ and $y_0 \in K_2$ such that

$$d(K_1, K_2) = d(x_0, y_0).$$

Proof 7.48 By definition,

$$d(K_1, K_2) = \inf \{d(x, y) : x \in K_1, y \in K_2\}.$$

Hence, $d(K_1, K_2)$ is a cluster point of the set $\{d(x, y) : x \in K_1, y \in K_2\}$. By [Theorem 5.3](#), there is a sequence $(d(x_n, y_n))$ in $\mathbb{R}$ such that $d(x_n, y_n) \rightarrow d(K_1, K_2)$, where $x_n \in K_1$ and $y_n \in K_2$.

Since K_1 is compact and $(x_n) \subseteq K_1$, there exists a subsequence (x_{n_k}) convergent to some $x_0 \in K_1$. Consider the corresponding subsequence (y_{n_k}) of (y_n) . Since K_2 is compact, there exists a further subsequence $(y_{n_{kl}})$ convergent to $y_0 \in K_2$. It is clear that $x_{n_{kl}} \rightarrow x_0$. Therefore,

$$d(x_{n_{kl}}, y_{n_{kl}}) \rightarrow d(x_0, y_0),$$

so that

$$d(K_1, K_2) = d(x_0, y_0),$$

completing the proof.

Remark 7.9 If even one of the sets K_1 or K_2 is only closed (instead of compact), the conclusion of [Theorem 7.27](#) is not true in general. For example, in the Euclidean space $\mathbb{Q}$, put $A = \{2\}$ and $B = [0, \sqrt{2}] \cap \mathbb{Q}$. Here, A is compact in $\mathbb{Q}$ but B is not (although it is closed). The distance between them cannot be attained.

Proposition 7.10 $\Leftarrow$ Assume that (X, d) is a metric space, $K \subseteq X$ is compact, and $F \subseteq X$ is closed. Moreover, assume that $K \cap F = \emptyset$. Then $d(K, F) > 0$.

Proof 7.49 Suppose, to the contrary, that $d(K, F) = 0$. Then there exists a sequence $\{d(x_n, y_n)\}$ in $\mathbb{R}$, with $(x_n) \subseteq K$ and $(y_n) \subseteq F$, such that $d(x_n, y_n) \rightarrow 0$. Since K is compact, there exists a subsequence (x_{n_k}) converging to some $x_0 \in K$. Then

$$d(x_0, y_{n_k}) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y_{n_k}) \rightarrow 0,$$

so that $y_{n_k} \rightarrow x_0$. This implies $x_0 \in \overline{F} = F$, contradicting $K \cap F = \emptyset$.

Being compact is a necessary condition for K in [Proposition 7.10](#). If $F_1, F_2 \subseteq X$ are merely closed and disjoint, it is possible that $d(F_1, F_2) = 0$, as the following example shows.

Example 7.11 Consider the Euclidean space $\mathbb{R}$. Put

$$F_1 = \{n : n \in \mathbb{N}, n \geq 2\}, \quad F_2 = \{n + \frac{1}{n} : n \in F_1\}.$$

Both sets are closed and $F_1 \cap F_2 = \emptyset$, yet $d(F_1, F_2) = 0$. Indeed, the sequence $(\frac{1}{n}) = (n + \frac{1}{n} - n)$ lies in the set $\{|a - b| : a \in F_1, b \in F_2\}$ and converges to 0.

Now we generalize the product metric to a countable family of metric spaces. Let $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ be a family of metric spaces and put

$$X = \prod_{n=1}^{\infty} X_n = \{(x_n) : x_n \in X_n\}.$$

To define a metric on X , set

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}, \quad x = (x_n), y = (y_n) \in X.$$

First, note that (X, d) is a metric space. The non-trivial part is the triangle inequality. Define $f : [0, +\infty) \rightarrow [0, +\infty)$ via $f(t) = \frac{t}{1+t}$. Then f is increasing since $f'(t) = \frac{1}{(1+t)^2}$. Furthermore,

$$f(t + s) = \frac{t + s}{1 + t + s} = \frac{t}{t + s + 1} + \frac{s}{t + s + 1} \leq f(t) + f(s).$$

Hence, for any $x_n, y_n, z_n \in X_n$,

$$d_n(x_n, y_n) \leq d_n(x_n, z_n) + d_n(z_n, y_n) \implies f(d_n(x_n, y_n)) \leq f(d_n(x_n, z_n)) + f(d_n(z_n, y_n)),$$

and summing over n gives the triangle inequality for d .

Definition 7.8 Assume that (X, d) is a metric space. a set $A \subseteq X$ is called relatively compact provided that $\overline{A}$ is compact.

Remark 7.10 Indeed, every relatively compact subset is totally bounded. Assume that (X, d) is a complete metric space. Then the notions of relatively compactness and totally boundedness agree. More precisely, assume that A is totally bounded so that $\overline{A}$ is also totally bounded. Moreover, $\overline{A}$ is also complete. Therefore, $\overline{A}$ is compact showing that it is relatively compact.

Now, assume that (X, d) is a compact metric space. Define

$$C(X) = \{f : (X, d) \rightarrow (\mathbb{R}, \|\cdot\|) : f \text{ is continuous}\}.$$

It is easy to see that $C(X)$ is a real vector space. Since for each $f \in C(X)$, $f(X)$ is compact in $\mathbb{R}$, we conclude that it is bounded. Define

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\}.$$

It is easy to see that $(C(X), \|\cdot\|_\infty)$ is a normed space. Now, we investigate a relatively compact subset in this space; it is known as the Arzelà–Ascoli Theorem for Real-Valued Functions.

Theorem 7.28 (Arzelà–Ascoli Theorem for Real-Valued Functions) *Let (X, d) be a compact metric space and $\mathcal{F} \subseteq C(X)$. Then $\mathcal{F}$ is relatively compact in $(C(X), \|\cdot\|_\infty)$ (where $\|f\|_\infty = \sup_{x \in X} |f(x)|$) if and only if:*

1. $\mathcal{F}$ is equicontinuous: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$

2. $\mathcal{F}$ is pointwise bounded: for each $x \in X$, the set

$$\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$$

is bounded in $\mathbb{R}$.

Proof 7.50 *First, we prove the direct implication. Assume $\mathcal{F}$ is relatively compact in $(C(X), \|\cdot\|_\infty)$.*

For fixed $x \in X$, consider the evaluation map

$$E_x : C(X) \rightarrow \mathbb{R}, \quad E_x(f) = f(x).$$

Since E_x is continuous with respect to the uniform norm and $\overline{\mathcal{F}}$ is compact, $E_x(\overline{\mathcal{F}})$ is compact in $\mathbb{R}$. Compact subsets of $\mathbb{R}$ are bounded, so $\mathcal{F}(x)$ is bounded.

Now, suppose on a contrary that $\mathcal{F}$ is not equicontinuous. Then there exists $\varepsilon_0 > 0$, sequences $(x_n), (y_n)$ in X with $d(x_n, y_n) \rightarrow 0$, and $f_n \in \mathcal{F}$ such that

$$|f_n(x_n) - f_n(y_n)| \geq \varepsilon_0.$$

By compactness of X , pass to a subsequence so $x_n \rightarrow x$ and $y_n \rightarrow x$. Since $\mathcal{F}$ is relatively compact, there exists a subsequence f_{n_k} converging uniformly to some $f \in C(X)$. Then

$$|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| \leq |f_{n_k}(x_{n_k}) - f(x_{n_k})| + |f(x_{n_k}) - f(y_{n_k})| + |f(y_{n_k}) - f_{n_k}(y_{n_k})| \rightarrow 0,$$

contradicting $|f_{n_k}(x_{n_k}) - f_{n_k}(y_{n_k})| \geq \varepsilon_0$. Hence $\mathcal{F}$ is equicontinuous.

Now, we prove the indirect implication. Assume $\mathcal{F}$ is equicontinuous and pointwise bounded.

Since X is compact metric, it is separable. Let $\{x_1, x_2, \dots\}$ be a countable dense subset of X .

For x_1 , $(f_n(x_1))$ is bounded, so it has a convergent subsequence $(f_n^{(1)})$. From $(f_n^{(1)})$, extract a subsequence $(f_n^{(2)})$ such that $(f_n^{(2)}(x_2))$ converges. Continue inductively. Let

$g_n = f_n^{(n)}$ be the diagonal sequence. Then $g_n(x_m)$ converges for all m .

Let $\varepsilon > 0$. By equicontinuity, choose $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{3}$ for all $f \in \mathcal{F}$. Cover X by finitely many balls of radius δ centered at $x_{i_1}, \dots, x_{i_k}$. For n, m large, $|g_n(x_{i_j}) - g_m(x_{i_j})| < \frac{\varepsilon}{3}$. Then for any $x \in X$, choose x_{i_j} in the same ball:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_{i_j})| + |g_n(x_{i_j}) - g_m(x_{i_j})| + |g_m(x_{i_j}) - g_m(x)| < \varepsilon.$$

Hence (g_n) is uniformly Cauchy and converges uniformly to some $g \in C(X)$. This shows $\mathcal{F}$ is relatively compact.

Example 7.12 Let $X = \mathbb{R}$ with the usual metric, and define a sequence

$$f_n(x) := \sin(x + n), \quad x \in \mathbb{R}.$$

Then the sequence $(f_n) \subseteq C(\mathbb{R})$ is uniformly bounded and equicontinuous, but it has no uniformly convergent subsequence on $\mathbb{R}$.

Proof 7.51 First, each function f_n is continuous on $\mathbb{R}$.

We show that the sequence (f_n) is uniformly bounded. Since $|\sin t| \leq 1$ for all $t \in \mathbb{R}$, we have

$$|f_n(x)| = |\sin(x + n)| \leq 1 \quad \text{for all } x \in \mathbb{R} \text{ and all } n.$$

Hence (f_n) is uniformly bounded.

Now, we show equicontinuity. The derivative of f_n satisfies

$$|f'_n(x)| = |\cos(x + n)| \leq 1 \quad \text{for all } x \in \mathbb{R} \text{ and all } n.$$

Thus, for any $x, y \in \mathbb{R}$,

$$|f_n(x) - f_n(y)| \leq |x - y| \quad \text{for all } n,$$

which implies that (f_n) is equicontinuous.

Finally, we show that no subsequences of the sequence (f_n) converge uniformly. Suppose on a contrary that (f_{n_k}) converges uniformly on $\mathbb{R}$ to some function f . Uniform convergence implies pointwise convergence, so for every $x \in \mathbb{R}$,

$$f_{n_k}(x) = \sin(x + n_k) \longrightarrow f(x).$$

Fix $x = 0$. Then $\sin(n_k)$ converges. Fix $x = \frac{\pi}{2}$. Then $\sin(\frac{\pi}{2} + n_k) = \cos(n_k)$ converges. But we have seen in [Chapter 4](#) that is not possible.

Therefore, no subsequence of (f_n) can converge uniformly on $\mathbb{R}$.

We now present another proof of [Proposition 6.15](#) in the compact case.

Corollary 7.11 Every compact metric space (X, d) is a normal topological space. More precisely, for any two disjoint closed subsets $F_1, F_2 \subseteq X$, there exist disjoint open sets $U_1, U_2 \subseteq X$ such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

Proof 7.52 Since X is compact and F_1, F_2 are closed subsets of X , both F_1 and F_2 are compact. Because F_1 and F_2 are disjoint compact sets, we have

$$d(F_1, F_2) := \inf \{d(x, y) : x \in F_1, y \in F_2\} > 0.$$

Set $\varepsilon = \frac{1}{2}d(F_1, F_2)$. Define

$$U_1 = \{x \in X : d(x, F_1) < \varepsilon\}, \quad U_2 = \{x \in X : d(x, F_2) < \varepsilon\}.$$

Then U_1 and U_2 are open, $F_1 \subseteq U_1$, $F_2 \subseteq U_2$, and $U_1 \cap U_2 = \emptyset$. This completes the proof.

7.7 EXAMPLES

Although we have established several examples in the course of this chapter, in this section, following our previous custom, we present further examples concerning compact metric spaces.

Example 7.13 $\hookrightarrow$ assume that $(X, \|\cdot\|)$ is a normed space, $F \subseteq X$ is closed and $K \subseteq X$ is compact. Then $F + K$ is a closed in X , in which

$$F + K = \{x + y : x \in F, y \in K\}.$$

Proof. Let $(z_n) \subseteq F + K$ be a sequence converging to some $z \in X$. Then for each n there exist $x_n \in F$ and $y_n \in K$ such that

$$z_n = x_n + y_n.$$

Since K is compact, there exists a subsequence (y_{n_k}) of (y_n) that converges to some $y \in K$ (compact subsets of any metric space are closed).

Because

$$z_{n_k} = x_{n_k} + y_{n_k} \longrightarrow z \quad \text{and} \quad y_{n_k} \longrightarrow y,$$

it follows that

$$x_{n_k} = z_{n_k} - y_{n_k} \longrightarrow z - y.$$

Since F is closed and $x_{n_k} \in F$, we conclude that $z - y \in F$. Hence

$$z = (z - y) + y \in F + K.$$

Therefore $F + K$ is closed.

Remark 7.11 Compactness is an essential assumption in [Example 7.13](#), in general, and cannot be replaced by closeness. Consider the Euclidean normed space $(\mathbb{R}, |\cdot|)$ and put

$$F_1 = \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\} \quad \text{and} \quad F_2 = \{-n : n \in \mathbb{N}\}.$$

Both sets are closed in $\mathbb{R}$.

Observe that

$$\left(n + \frac{1}{n}\right) + (-n) = \frac{1}{n},$$

so the sequence $\left(\frac{1}{n}\right)$ is contained in $F_1 + F_2$. However,

$$\frac{1}{n} \longrightarrow 0,$$

and $0 \notin F_1 + F_2$. Hence $F_1 + F_2$ is not closed.

Example 7.14 Let (X, d) be a metric space and $A \subseteq X$. Prove that A is totally bounded if and only if its closure $\overline{A}$ is totally bounded.

Proof.($\Rightarrow$) Assume that A is totally bounded. Then for every $\varepsilon > 0$ there exist $x_1, \dots, x_n \in A$ such that

$$A \subseteq \bigcup_{k=1}^n N_{\frac{\varepsilon}{2}}^{(x_k)}.$$

Taking closures, we obtain

$$\overline{A} \subseteq \bigcup_{k=1}^n \overline{N_{\frac{\varepsilon}{2}}^{(x_k)}} = \bigcup_{k=1}^n \overline{N_{\frac{\varepsilon}{2}}^{(x_k)}}.$$

Since

$$\overline{N_{\frac{\varepsilon}{2}}^{(x_k)}} \subseteq N_{\frac{\varepsilon}{2}}^{[x_k]} \subseteq N_{\varepsilon}^{(x_k)},$$

it follows that

$$\overline{A} \subseteq \bigcup_{k=1}^n N_{\varepsilon}^{(x_k)}.$$

Thus $\overline{A}$ is totally bounded.

($\Leftarrow$) Assume that $\overline{A}$ is totally bounded. Then for every $\varepsilon > 0$ there exist $x_1, \dots, x_n \in \overline{A}$ such that

$$\overline{A} \subseteq \bigcup_{k=1}^n N_{\frac{\varepsilon}{2}}^{(x_k)}.$$

For each k , choose $a_k \in A$ such that $d(a_k, x_k) < \varepsilon/2$ (possible since $x_k \in \overline{A}$). Then

$$N_{\frac{\varepsilon}{2}}^{(x_k)} \subset N_{\varepsilon}^{(a_k)}.$$

Hence

$$A \subseteq \overline{A} \subseteq \bigcup_{k=1}^n N_{\varepsilon}^{(a_k)},$$

with $a_k \in A$. Therefore A is totally bounded.

Example 7.15 Let

$$A = \{x \in [0, 1] : \text{every digit in the decimal expansion of } x \text{ belongs to } \{0, 2, 3\}\}.$$

Then A is uncountable and compact.

Proof. First, we show that A is uncountable. Consider the subset

$$B = \{x \in [0, 1] : \text{every digit is either 2 or 3}\}.$$

Clearly $B \subseteq A$.

Each element of B corresponds uniquely to a sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in \{2, 3\}$ via

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

Now associate to each such sequence a binary sequence (b_n) by defining

$$b_n = \begin{cases} 0 & \text{if } a_n = 2, \\ 1 & \text{if } a_n = 3. \end{cases}$$

This gives a one-to-one correspondence between B and the set $\{0, 1\}^{\mathbb{N}}$ of all infinite binary sequences.

Since the set of all infinite binary sequences is uncountable, it follows that B is uncountable. Therefore A is uncountable.

To show that A is compact, it suffices to prove that A is closed. Indeed, since $A \subseteq [0, 1]$ and $[0, 1]$ is compact in $\mathbb{R}$ by the Heine–Borel Theorem, any closed subset of $[0, 1]$ is compact. Therefore, once we show that A is closed, its compactness follows immediately.

Let $x \notin A$. Then in the decimal expansion of x there exists a first digit position m such that the digit is not in $\{0, 2, 3\}$.

All numbers sufficiently close to x share the same first m digits. Hence every number in a small neighborhood of x also has a digit outside $\{0, 2, 3\}$. Thus the complement of A is open, and A is closed.

Example 7.16 Let (X, d) be compact and $f : X \rightarrow X$ continuous. Define

$$S = \{x \in X : x_n = f(x_{n+1}), \forall n \in \mathbb{N}\}.$$

Show that S is compact.

Proof. The condition defining S is closed under limits since f is continuous.

Thus S is closed in X . Since X is compact and closed subsets of compact spaces are compact, S is compact.

7.8 EXERCISES

Exercise 7.1

1. Assume that (X, d) is a metric space and $A \subseteq X$. Prove that A is totally bounded if and only if $\overline{A}$ is totally bounded.
2. Let $X = \mathbb{R}^2$ with the Euclidean metric. Show that:
 - (a) $A = \{(x, y) \in X : x^2 + y^2 \leq 1\}$ is compact.
 - (b) $B = \{(x, y) \in X : x^2 + y^2 = 1\}$ is compact.
 - (c) $C = \{(x, y) \in X : x^2 + y^2 < 1\}$ is not compact.
 - (d) $D = B \setminus \{(0, 1)\}$ is not compact.
3. Show that the interior of every compact set in $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) is empty.
4. Determine whether the following subsets of $\mathbb{R}^2$ are compact:
 - (a) $A = \{(\frac{n-1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$.
 - (b) $B = (\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}) \times [0, 1]$.
 - (c) $C = \{(x, y) \in \mathbb{R}^2 : x = 0\}$.
5. Let (X, d) be a compact metric space. Prove that for every $\varepsilon > 0$ there exists a finite family of subsets

$$X = A_1 \cup \cdots \cup A_n$$

such that

$$\text{diam}(A_i) < \varepsilon \quad \text{for all } i = 1, \dots, n.$$

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \rightarrow Y$ be continuous. Assume that X is compact and that f is injective. Show that the inverse mapping

$$f^{-1} : f(X) \rightarrow X$$

is continuous.

7. Let X be a compact metric space and let d_1, d_2 be two metrics on X which generate the same topology. Prove that the metrics d_1 and d_2 are uniformly equivalent, that is, for any sequences (x_n) and (y_n) in X ,

$$d_1(x_n, y_n) \rightarrow 0 \iff d_2(x_n, y_n) \rightarrow 0.$$

8. Let (X, d_X) and (Y, d_Y) be compact metric spaces. Define a metric d on $X \times Y$ by

$$d((x, y), (x', y')) = \max \{d_X(x, x'), d_Y(y, y')\}.$$

Prove that $(X \times Y, d)$ is a compact metric space.

9. *Let (X, d) be a metric space.*

- (a) *Let $(x_\alpha)_{\alpha \in A}$ be a net in X . Show that if (x_α) converges to $x \in X$, then every subnet $(x_\beta)_{\beta \in B}$ also converges to x .*
- (b) *Prove that X is compact if and only if every net in X has a convergent subnet.*
- (c) *Construct a net (x_α) in a metric space X which converges, but for which there exists no sequence extracted from the net that converges to the same limit (outside a compact subset of X).*

10. *Let (X, d) be a metric space, and let $A, B \subseteq X$ with A compact and B closed. Suppose that $A \cap B = \emptyset$. Show that $d(A, B) > 0$.*

11. *Let (X, d) be a metric space and let $A \subseteq X$ be compact. Show that the set of all limit points of A , denoted by A' , is also compact.*

12. *Consider the Banach space $(C([0, 1]), \|\cdot\|_\infty)$. Let*

$$A = \{f \in C([0, 1]) : f([0, 1]) \subseteq [0, 1]\}.$$

Then A is bounded and closed but not compact.

13. *Let (X_1, d_1) and (X_2, d_2) be metric spaces, and suppose that X_2 is compact. Show that the projection map*

$$\phi_1 : X_1 \times X_2 \rightarrow X_1$$

is a closed map. item Let

$$X = \{(t, 0) \mid 0 < t \leq 1\} \cup \left\{ \left(\frac{1}{n}, t \right) \mid n \in \mathbb{N}, 0 \leq t \leq 1 \right\}.$$

Define a function d on $X \times X$ by letting $d(x, y)$ be the length of the shortest path in X joining x to y .

- (a) *Is X compact?*
- (b) *Is X totally bounded?*
- (c) *Does X have any other interesting properties?*

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Connectedness: A Fruitful Bridge

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8.1 WHY DO WE NEED BRIDGES?

Connectedness: Bridges Fostering Security, Peace, and Advancement

Connecting bridges between different islands on the ocean surface or different places below it are built by connectedness engineers. They create most of the routes to facilitate communication between islands and cities in the Topology realm. Some of these routes are local and designed for specific areas, while others are general and cover a large area. In addition to connected routes, there are also connected areas that serve as corridors between subsets of a region of the Topological realm. Another group among the inhabitants of connected mathematics is the hoteliers of an interdisciplinary hotel chain called “Hotel Connectedness”: luxurious hotels with unique facilities scattered throughout the Topology realm. Every day, they host different delegations of ambassadors and representatives of other sciences, as well as residents of the ocean of applications, who come there for important business deals and to benefit from the latest discoveries of the ocean of mathematics. For example, Hotel Connectedness of the Real Number district is one of the most popular and crowded hotels. Another name for this hotel is Hotel Convex, which hosts famous people from the world of mathematics every year, including open balls in normed spaces. Let us remember that normed spaces have a unique status in the ocean of mathematics and the ocean of applications due to their ability to become “Banach,” and enter the Banach Street. Furthermore, if you recall, we previously stated that the areas related to Real Numbers and the algae of ideas make a great contribution to the advancement of mathematics. The interesting thing about these hotels is that the union of any number of hotels can also form a hotel, provided that every pair of hotels has common rooms or are connected like a chain. These unique features, not found at any other hotel, attract numerous tourists from around the world to visit the Topology realm. In general, the intrinsic management of topology itself, the management of metrics on certain sets, cities of normed inhabitants, connected regions and paths, the industrial town of Compactness, and the transportation systems of continuity, openness, and closeness are among the most influential tools of the Topological realm, whose importance has been discussed in this book. All of these tools lay the groundwork for further interaction and the advancement of mathematics with the internal and external worlds.

8.2 DIFFERENT KINDS OF CONNECTEDNESS

Recall that for subsets A, B in a set X , we say that A and B are *disjoint* provided that $A \cap B = \emptyset$. Now, we have the following definition.

Definition 8.1  Assume that (X, d) is a metric space and $A, B \subseteq X$. We say that A and B are separated provided that

$$A \cap \overline{B} = \emptyset \quad \text{and} \quad \overline{A} \cap B = \emptyset.$$

Indeed, every separated pair of subsets of a metric space is disjoint, but the converse is not true in general, as the following simple example shows.

Example 8.1 Consider $X = \mathbb{R}$ with the Euclidean metric. Put

$$A = (0, 1), \quad B = (1, 2).$$

Then A and B are disjoint but not separated.

If we change the metric space, the situation may differ.

Example 8.2 Consider $X = (0, 2) - \{1\}$ with the induced metric from $\mathbb{R}$. Put

$$A = (0, 1), \quad B = (1, 2).$$

Then A and B are separated. To see this, note that A and B are closed in X :

$$A = [0, 1] \cap X, \quad B = [1, 2] \cap X.$$

Hence, being separated and being disjoint agree in this case.

A closer look at [Definition 8.1](#) might suggest that the conditions are symmetric; that is, one might think $\overline{A} \cap B = \emptyset$ is equivalent to $A \cap \overline{B} = \emptyset$. But this is not true even in Euclidean spaces.

Example 8.3 Consider $X = \mathbb{R}$ with the Euclidean metric. Put

$$A = (0, 1], \quad B = (1, 2).$$

Then

$$\overline{A} \cap B = [0, 1] \cap (1, 2) = \emptyset,$$

but

$$A \cap \overline{B} = (0, 1] \cap [1, 2] = \{1\}.$$

So, A and B are disjoint but not separated. This shows that the two conditions in [Definition 8.1](#) are not equivalent.

As expected, in discrete metric spaces the situation is trivial.

Proposition 8.1 Assume that (X, d) is a discrete metric space and $A, B \subseteq X$. Then A and B are separated if and only if they are disjoint.

Furthermore, in an arbitrary metric space, we have the following.

Proposition 8.2 Assume that (X, d) is a metric space. Then every two closed disjoint subsets are separated.

Moreover, a surprising fact holds for open sets as well.

Proposition 8.3 Assume that (X, d) is a metric space. Then every two open disjoint subsets are separated.

Proof 8.1 Suppose $G, H \subseteq X$ are open and disjoint. We show that they are separated. Take any $x \in G$. Since G is open, there exists $r_0 > 0$ such that $N_{r_0}^{(x)} \subseteq G$.

Now, if $x \in \overline{H}$, then for any $r > 0$, e.g., $r = r_0$, we would have $N_{r_0}^{(x)} \cap H \neq \emptyset$, which contradicts $G \cap H = \emptyset$. Hence, $G \cap \overline{H} = \emptyset$. Similarly, $\overline{G} \cap H = \emptyset$.

Definition 8.2 Assume that (X, d) is a metric space. A subset $D \subseteq X$ is called disconnected if there exist two non-empty separated subsets A, B of X such that $D = A \cup B$. The pair $\{A, B\}$ is called a disconnection of D .

A set that is not disconnected is called connected.

Remark 8.1 $\Leftarrow$ As we have seen, a subset C of a metric space (X, d) is connected if we cannot write C as the union of two non-empty separated sets.

Now, assume that X is disconnected, so that there exists a disconnection $\{A, B\}$ for it. Then

$$A \cap \overline{B} = \emptyset \implies \overline{A} \subseteq B^c.$$

On the other hand, $X = A \cup B$ so that $B^c = A$. Hence,

$$\overline{A} \subseteq A,$$

showing that A is closed. Similarly, B is also closed. But since $A = B^c$ and $B = A^c$, we conclude that A and B are also open.

Thus, when we have a disconnection $\{A, B\}$ for a metric space (X, d) , both A and B are clopen (simultaneously open and closed).

Proposition 8.4 A metric space (X, d) is disconnected if and only if it contains a non-trivial (non-empty and distinct from X) clopen subset.

Proof 8.2 Assume that M is a non-trivial clopen subset, i.e., $\emptyset \neq M \neq X$. Then $N = M^c$ is also clopen and non-trivial. Thus, $\{M, N\}$ is a disconnection for X .

Conversely, if X is disconnected, there exists a disconnection $\{A, B\}$ for X . By [Remark 8.1](#), A (and B) are non-trivial clopen sets, so the statement holds.

Corollary 8.1 $\Leftarrow$ A metric space (X, d) is connected if and only if the only clopen subsets of X are the trivial ones: X and $\emptyset$.

We illustrate [Corollary 8.1](#) with an example.

Example 8.4 Consider $X = (0, 1) \cup (1, 2)$ with the induced metric from $\mathbb{R}$. Then $(0, 1)$ is clopen in X , so X is not connected. Indeed,

$$(0, 1) = [0, 1] \cap X = (0, 1) \cap X,$$

showing that it is simultaneously open and closed in X .

Furthermore, connectedness can be characterized using the boundary of a set.

Proposition 8.5 Assume that (X, d) is a metric space. Then X is connected if and only if for each non-empty proper subset $A \subseteq X$, we have

$$\partial(A) \neq \emptyset.$$

Proof 8.3 Assume X is connected. By [Corollary 8.1](#), the only clopen subsets are X and $\emptyset$. Suppose, for contradiction, that there exists a non-empty proper subset $A \subset X$ with $\partial(A) = \emptyset$. Then

$$\partial(A) = \overline{A} \cap \overline{A^c} = \overline{A} \cap (A^\circ)^c = \overline{A} - A^\circ = \emptyset.$$

Hence, $\overline{A} = A^\circ$, so A is clopen, a contradiction.

Conversely, assume that $\partial(A) \neq \emptyset$ for each non-empty proper subset $A \subset X$. If X were disconnected, by [Corollary 8.1](#), there would exist a non-trivial clopen set $M \subset X$. But then

$$\partial(M) = \overline{M} - M^\circ = \emptyset,$$

a contradiction. Hence X is connected.

Proposition 8.6 Every discrete space with more than one point is disconnected.

Proof 8.4 Let (X, d) be a discrete space with more than one point. Take $x \in X$. Then $X - \{x\}$ is non-empty, and

$$X = \{x\} \cup (X - \{x\}).$$

In a discrete space, every set is closed, so this is a disconnection.

Proposition 8.7 Every singleton in any metric space is connected; we call these the trivial connected sets.

Proof 8.5 Let (X, d) be a metric space and $x \in X$. Suppose, for contradiction, that $\{x\}$ is disconnected, so there is a disconnection $\{A, B\}$:

$$\{x\} = A \cup B.$$

Since A and B are separated, they are disjoint, so x belongs to exactly one of them. This implies either $A = \emptyset$ or $B = \emptyset$, a contradiction. Hence, $\{x\}$ is connected.

Proposition 8.8 In a metric space (X, d) , every finite set with more than one point, such as $D = \{x_1, \dots, x_n\}$, is disconnected. We call these the trivial disconnected subsets.

Proof 8.6 Put

$$A = \{x_1\}, \quad B = \{x_2, \dots, x_n\}.$$

By assumption, both A and B are non-empty. Moreover, A and B are closed since they are finite, and they are separated because they are disjoint. Hence, $\{A, B\}$ is a disconnection for D , as claimed.

Definition 8.3 A metric space (X, d) is called totally disconnected if it contains no non-trivial connected subset; in other words, the only connected subsets of X are singletons.

Example 8.5 Every discrete metric space (X, d) with more than one point is totally disconnected. To see this, assume that $D \subseteq X$ has more than one point and take $x \in D$. Put

$$A = \{x\}, \quad B = D - \{x\}.$$

Then $\{A, B\}$ is a disconnection for D , showing that D is disconnected.

Totally disconnected sets are not limited to discrete spaces.

Example 8.6 In the Euclidean space $\mathbb{R}$, the rationals $\mathbb{Q}$ are totally disconnected. To see this, assume that $A \subseteq \mathbb{Q}$ has at least two points. Take $r, s \in A$ and choose an irrational number α between them. Put

$$T = [\alpha, \infty) \cap A.$$

Then T is clopen in A (we could also write $T = (\alpha, \infty) \cap A$). Moreover, $T \neq \emptyset$ and $T \neq A$, showing that A is not connected. Hence, $\mathbb{Q}$ is totally disconnected. Similarly, $\mathbb{Q}^c$ (the irrationals) is also totally disconnected.

Before giving another example, we need a useful lemma.

Lemma 8.1 $\hookrightarrow$ Assume that (X, d) is a metric space and $A \subseteq X$ is a non-empty clopen subset. Moreover, assume that $Y \subseteq X$ is connected. Then either

$$Y \subseteq A \quad \text{or} \quad Y \subseteq X - A.$$

Proof 8.7 Note that A is open and closed in X so that the set $A \cap Y$ is open and closed in Y . Since Y is connected, by [Corollary 8.1](#), we have

$$A \cap Y = Y \quad \text{or} \quad A \cap Y = \emptyset.$$

This shows that either $Y \subseteq A$ or $Y \subseteq X - A$.

Example 8.7 $\hookrightarrow$ Assume that $X = \mathbb{R}$ with the Euclidean metric, and let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

Then A is totally disconnected.

To see this, note first that for each $n \in \mathbb{N}$, the singleton $\{\frac{1}{n}\}$ is both open and closed in A . Indeed,

$$\frac{1}{n} = \left(\frac{1}{n+1}, \frac{1}{n-1} \right) \cap A \quad \text{for } n > 1,$$

and

$$\{1\} = \left(\frac{1}{2}, \frac{3}{2} \right) \cap A.$$

Now, assume that $Y \subseteq A$ is connected and has more than one point. Then Y contains an element of the form $\frac{1}{k}$. By [Lemma 8.1](#), either

$$Y \subseteq \left\{ \frac{1}{k} \right\} \quad \text{or} \quad Y \subseteq A - \left\{ \frac{1}{k} \right\},$$

which is impossible since Y has more than one point. Therefore, A is totally disconnected.

Now, assume that C_1 and C_2 are connected sets in a metric space. A natural question arises about the connectivity of their possible relations. More precisely:

Question. Is the union or the intersection of two connected sets again connected?

Question. Is the closure, interior, or boundary of a connected set again connected?

We start by showing some counterexamples.

Example 8.8 Assume that (X, d) is a metric space and $x, y \in X$ with $x \neq y$. Put

$$C_1 = \{x\}, \quad C_2 = \{y\}.$$

Both C_1 and C_2 are connected (singletons). However, $C_1 \cup C_2 = \{x, y\}$ is not connected.

Example 8.9 Consider a square C_1 in the Euclidean space $\mathbb{R}^2$, and let C_2 be a diameter of the square. Both C_1 and C_2 are connected. Nevertheless, $C_1 \cap C_2$ consists of just two separated points in $\mathbb{R}^2$, which is disconnected.

We now consider situations in which the union of connected sets is again connected. First, we need an auxiliary result.

Lemma 8.2 $\hookrightarrow$ Assume that (X, d) is a metric space and A, B are two non-empty separated subsets of X . Let C be a connected subset of X such that $C \subseteq A \cup B$. Then

$$C \subseteq A \quad \text{or} \quad C \subseteq B.$$

Proof 8.8 Suppose, on the contrary, that $C \not\subseteq A$ and $C \not\subseteq B$. Then

$$A_1 = C \cap A \neq \emptyset, \quad B_1 = C \cap B \neq \emptyset.$$

We claim that $\{A_1, B_1\}$ is a disconnection for C . Indeed,

$$A_1 \cup B_1 = (C \cap A) \cup (C \cap B) = C \cap (A \cup B) = C.$$

Furthermore,

$$\overline{A_1} \cap B_1 \subseteq \overline{A} \cap B = \emptyset, \quad A_1 \cap \overline{B_1} \subseteq A \cap \overline{B} = \emptyset.$$

This contradicts the connectedness of C .

Proposition 8.9 Assume that (X, d) is a metric space and $C_1, C_2 \subseteq X$ are connected sets with $C_1 \cap C_2 \neq \emptyset$. Then $C_1 \cup C_2$ is connected.

Proof 8.9 Let $C = C_1 \cup C_2$ and suppose, on the contrary, that C is disconnected. Then $C = A \cup B$, where A and B are non-empty separated sets. By [Lemma 8.2](#), we have

$$C_1 \subseteq A \text{ or } C_1 \subseteq B, \quad C_2 \subseteq A \text{ or } C_2 \subseteq B.$$

There are four cases:

1. $C_1 \subseteq A, C_2 \subseteq A$: then $C = C_1 \cup C_2 \subseteq A$, contradicting $B \neq \emptyset$.
2. $C_1 \subseteq B, C_2 \subseteq B$: similar contradiction.
3. $C_1 \subseteq A, C_2 \subseteq B$: then

$$C_1 \cap C_2 \subseteq \overline{A} \cap B = \emptyset,$$

contradicting $C_1 \cap C_2 \neq \emptyset$.

4. $C_1 \subseteq B, C_2 \subseteq A$: similar contradiction.

Hence, C must be connected.

Corollary 8.2 $\Leftarrow$ Assume that (X, d) is a metric space and $\{C_i\}_{i=1}^n$ is a finite family of connected sets in X such that $C_i \cap C_j \neq \emptyset$ for all $1 \leq i, j \leq n$. Then

$$C = \bigcup_{i=1}^n C_i$$

is connected.

Proof 8.10 We proceed by induction on n . For $n = 2$, the result follows from the previous proposition. Assume the result holds for $n - 1$. For n , note that

$$\bigcup_{i=1}^n C_i = C_n \cap \bigcup_{i=1}^{n-1} C_i = \bigcup_{i=1}^{n-1} (C_n \cap C_i) \neq \emptyset,$$

so by the induction hypothesis, $\bigcup_{i=1}^n C_i$ is connected.

Corollary 8.3 $\Leftarrow$ Assume that (X, d) is a metric space and $\{C_i\}_{i=1}^n$ is a finite family of connected sets in X such that

$$\bigcap_{i=1}^n C_i \neq \emptyset.$$

Then

$$C = \bigcup_{i=1}^n C_i$$

is connected.

Proof 8.11 If $\bigcap_{i=1}^n C_i \neq \emptyset$, then $C_i \cap C_j \neq \emptyset$ for all i, j . Hence, the conclusion follows from [Corollary 8.2](#).

As usual, we are interested in conditions under which an arbitrary union of connected sets is again connected. Before that, we prove the statement for a chain of connected sets. First, we recall the notion of a chain.

Definition 8.4 Let (S, r) be a partially ordered set. A subset $T \subseteq S$ is called a chain if the restriction of r to T makes T totally ordered; that is, every two elements of T are comparable.

Example. Let $X \neq \emptyset$ be a set and consider $(P(X), \subseteq)$. A subset $T \subseteq P(X)$ is a chain if for each $A, B \in T$, either $A \subseteq B$ or $B \subseteq A$.

Proposition 8.10 Assume that (X, d) is a metric space and $\{C_\alpha\}_{\alpha \in I}$ is a chain of connected subsets of X . Then

$$C = \bigcup_{\alpha \in I} C_\alpha$$

is connected.

Proof 8.12 Suppose, on the contrary, that C is disconnected with a disconnection $\{A, B\}$, i.e., $C = A \cup B$ with A and B separated. By [Lemma 8.2](#), for each $\alpha \in I$, we have

$$C_\alpha \subseteq A \quad \text{or} \quad C_\alpha \subseteq B.$$

Fix $\alpha_0 \in I$ and assume $C_{\alpha_0} \subseteq A$. For any $\alpha \in I$, either $C_\alpha \subseteq C_{\alpha_0}$ or $C_{\alpha_0} \subseteq C_\alpha$, since $\{C_\alpha\}$ is a chain.

- If $C_\alpha \subseteq C_{\alpha_0}$, then $C_\alpha \subseteq A$. - If $C_{\alpha_0} \subseteq C_\alpha$, then we have $C_\alpha \subseteq A$; otherwise, if $C_\alpha \subseteq B$, then $C_{\alpha_0} \subseteq B$, which contradicts $C_{\alpha_0} \subseteq A$ and the fact that A and B are disjoint.

Hence, for all α , $C_\alpha \subseteq A$, so that

$$C = \bigcup_{\alpha \in I} C_\alpha \subseteq A,$$

implying $B \subseteq A$, a contradiction. If $C_{\alpha_0} \subseteq B$, we get a similar contradiction. Therefore, C is connected.

In [Figure 8.5](#), we see the geometric view of a chain of connected sets.

Being a chain is a strong condition. We now investigate milder conditions. First, we extend [Corollary 8.3](#).

Theorem 8.1 $\Leftarrow$ Assume that (X, d) is a metric space and $\{C_\alpha\}_{\alpha \in I}$ is an arbitrary family of connected sets in X such that

$$\bigcap_{\alpha \in I} C_\alpha \neq \emptyset.$$

Then

$$C = \bigcup_{\alpha \in I} C_\alpha$$

is connected.

Proof 8.13 Choose $x_0 \in \bigcap_{\alpha \in I} C_\alpha$. Suppose, on the contrary, that C is disconnected with disconnection $\{A, B\}$. Since $x_0 \in C$, we have either $x_0 \in A$ or $x_0 \in B$. W.l.o.g., assume $x_0 \in A$.

For any $\alpha \in I$, C_α is connected, and $\{A, B\}$ is a disconnection for C . By [Lemma 8.2](#), $C_\alpha \subseteq A$ or $C_\alpha \subseteq B$. The latter cannot occur because $x_0 \in C_\alpha \cap A$, so $C_\alpha \subseteq A$.

Hence, $C = \bigcup_{\alpha \in I} C_\alpha \subseteq A$, contradicting $B \neq \emptyset$. Therefore, C is connected.

In [Figure 8.5](#), we see the geometric view of a chain of connected sets.

Corollary 8.4 $\Leftarrow$ Assume that (X, d) is a metric space and $C_1, C_2 \subseteq X$ are connected with $C_1 \cap C_2 \neq \emptyset$. Then

$$C_1 \cup C_2$$

is connected.

The following observation can be considered as a consequence of [Theorem 8.1](#).

Proposition 8.11 $\hookrightarrow$ Assume that (X, d) is a metric space and $\{C_\alpha\}_{\alpha \in I}$ is a family of connected sets. Let $C \subseteq X$ be another connected set such that

$$C \cap C_\alpha \neq \emptyset \quad \text{for all } \alpha \in I.$$

Then

$$C \cup \bigcup_{\alpha \in I} C_\alpha$$

is connected.

Proof 8.14 For each $\alpha \in I$, set $D_\alpha = C \cup C_\alpha$, which is connected by [Corollary 8.4](#). Observe that

$$\bigcap_{\alpha \in I} D_\alpha = \bigcap_{\alpha \in I} (C \cup C_\alpha) \supseteq C \neq \emptyset.$$

By [Theorem 8.1](#), $\bigcup_{\alpha \in I} D_\alpha = C \cup \bigcup_{\alpha \in I} C_\alpha$ is connected.

Remark 8.2 As we look closer at [Corollary 8.2](#), we see that a sufficient condition for the union of two connected sets to be connected is that their intersection is non-empty. However, this condition is not necessary.

Example. Consider $X = \mathbb{R}^2$ with the Euclidean metric. Let

$$C_1 = N_{\frac{1}{2}}^{(\frac{1}{2}, 0)}, \quad C_2 = N_{\frac{1}{2}}^{(-\frac{1}{2}, 0)},$$

the open balls of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0)$, respectively. Both C_1 and C_2 are connected, and so is $C_1 \cup C_2$, but $C_1 \cap C_2 = \emptyset$.

This shows that non-empty intersection is sufficient but not necessary. We now proceed to more general conditions.

Proposition 8.12 $\hookrightarrow$ Assume that (X, d) is a metric space and $C_1, C_2 \subseteq X$ are connected sets such that either

$$\overline{C_1} \cap C_2 \neq \emptyset \quad \text{or} \quad C_1 \cap \overline{C_2} \neq \emptyset.$$

Then $C_1 \cup C_2$ is connected.

Proof 8.15 We prove the case $\overline{C_1} \cap C_2 \neq \emptyset$; the other case is similar. Suppose, on the contrary, that $C = C_1 \cup C_2$ is disconnected with disconnection $\{A, B\}$. Let $x_0 \in \overline{C_1} \cap C_2$. Then $x_0 \in C$, so either $x_0 \in A$ or $x_0 \in B$. W.l.o.g., assume $x_0 \in A$.

Since C_1 is connected in C and $\{A, B\}$ is a disconnection, by [Lemma 8.2](#), either $C_1 \subseteq A$ or $C_1 \subseteq B$. If $C_1 \subseteq B$, then $x_0 \in \overline{C_1} \subseteq \overline{B}$, which contradicts $x_0 \in A$ because A and B are separated. Hence, $C_1 \subseteq A$, and similarly, $C_2 \subseteq A$ or $C_2 \subseteq B$. By the choice of $x_0 \in C_2 \cap A$, we get $C_2 \subseteq A$.

Thus, $C = C_1 \cup C_2 \subseteq A$, contradicting $B \neq \emptyset$. Therefore, C is connected.

Theorem 8.2 Assume that (X, d) is a metric space and $\{C_\alpha\}_{\alpha \in I}$ is a family of connected subsets of X such that for each $\alpha, \beta \in I$, either

$$\overline{C_\alpha} \cap C_\beta \neq \emptyset \quad \text{or} \quad C_\alpha \cap \overline{C_\beta} \neq \emptyset.$$

Then

$$\bigcup_{\alpha \in I} C_\alpha$$

is connected.

Proof 8.16 If $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$, the result follows from [Theorem 8.1](#). Otherwise, fix any $\alpha_0 \in I$ and define

$$D_\alpha = C_\alpha \cup C_{\alpha_0} \quad \text{for each } \alpha \in I.$$

By [Proposition 8.12](#), each D_α is connected. Moreover,

$$\bigcap_{\alpha \in I} D_\alpha = \bigcap_{\alpha \in I} (C_\alpha \cup C_{\alpha_0}) = C_{\alpha_0} \neq \emptyset.$$

Applying [Theorem 8.1](#) to $\{D_\alpha\}_{\alpha \in I}$, we conclude that

$$\bigcup_{\alpha \in I} D_\alpha = \bigcup_{\alpha \in I} C_\alpha$$

is connected.

Proposition 8.13 Assume that (X, d) is a metric space and $\{C_n\}_{n \in \mathbb{N}}$ is a sequence of connected sets such that for each $n \in \mathbb{N}$, either

$$C_n \cap \overline{C_{n+1}} \neq \emptyset \quad \text{or} \quad \overline{C_n} \cap C_{n+1} \neq \emptyset.$$

Then

$$\bigcup_{n \in \mathbb{N}} C_n$$

is connected.

Proof 8.17 Define

$$D_n = C_1 \cup \dots \cup C_n, \quad n \in \mathbb{N}.$$

We show by induction that each D_n is connected.

For $n = 1$, it is trivial. Assume D_n is connected. Now consider $D_{n+1} = D_n \cup C_{n+1}$.

If $C_n \cap \overline{C_{n+1}} \neq \emptyset$, then

$$D_n \cap \overline{C_{n+1}} \supseteq C_n \cap \overline{C_{n+1}} \neq \emptyset.$$

If $C_n \cap \overline{C_{n+1}} = \emptyset$, then by hypothesis $\overline{C_n} \cap C_{n+1} \neq \emptyset$, and

$$\overline{D_n} \cap C_{n+1} \supseteq \overline{C_n} \cap C_{n+1} \neq \emptyset.$$

In either case, by [Proposition 8.12](#), D_{n+1} is connected.
 Since $D_1 \subseteq D_2 \subseteq \dots$, we have

$$\bigcup_{n \in \mathbb{N}} D_n = \bigcup_{n \in \mathbb{N}} C_n,$$

which is connected by [Theorem 8.1](#).

Theorem 8.3 Assume that (X, d) is a metric space and $Y \subseteq X$ is connected. Then every subset $Z \subseteq X$ such that

$$Y \subseteq Z \subseteq \overline{Y}$$

is also connected.

Proof 8.18 Suppose, on the contrary, that Z is disconnected with disconnection $\{A, B\}$. Since Y is connected, by [Lemma 8.2](#), either $Y \subseteq A$ or $Y \subseteq B$.

Assume $Y \subseteq A$; then $\overline{Y} \subseteq \overline{A}$, so

$$Z \subseteq \overline{Y} \subseteq \overline{A}.$$

The closure of A in Z is $(\overline{A})_Z = \overline{A} \cap Z = Z$. Since A is closed in Z , we have $(\overline{A})_Z = A$. Hence,

$$A \cup B = Z \subseteq A \implies B = \emptyset,$$

a contradiction. The case $Y \subseteq B$ is analogous.

Corollary 8.5 $\hookrightarrow$ If (X, d) is a metric space and $Y \subseteq X$ is connected, then $\overline{Y}$ is connected.

However, the situation for the interior is different.

Example 8.10 Consider the set in $\mathbb{R}^2$ defined by the union of two closed unit balls centered at $(1, 0)$ and $(-1, 0)$:

$$C = N_1^{[(1,0)]} \cup N_1^{[(-1,0)]},$$

Note that

$$(0, 0) \in N_1^{[(1,0)]} \cap N_1^{[(-1,0)]}.$$

Therefore, C is connected. However, the interior of C is

$$C^\circ = N_1^{((1,0))} \cup N_1^{((-1,0))},$$

which is the union of two disjoint open balls. Hence, C° is not connected.

More precisely, we see how closure or interior of connected sets can be connected or disconnected, in general. See [Figures 8.1](#) and [8.2](#) for illustrative depictions.

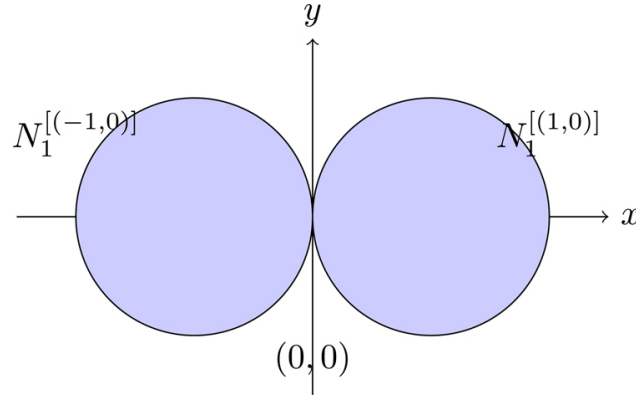


Figure 8.1 Union of closed balls $C = N_1^{[(1,0)]} \cup N_1^{[(-1,0)]}$, which is connected. [↗](#)

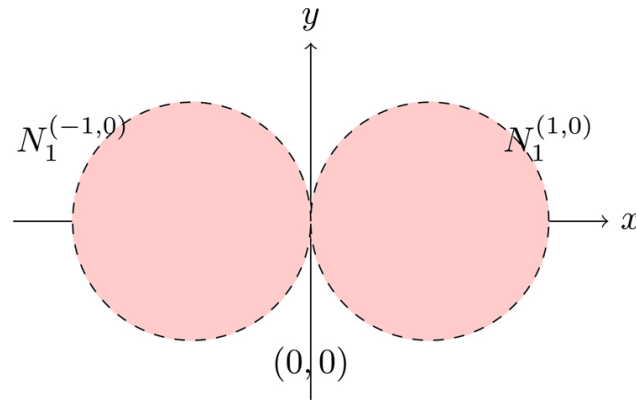


Figure 8.2 Interior of C : $C^\circ = N_1^{(1,0)} \cup N_1^{(-1,0)}$, which is disconnected. [↗](#)

Theorem 8.4 Assume that (X, d_1) and (Y, d_2) are connected metric spaces. Then the Cartesian product $X \times Y$ (with any standard product metric) is connected.

Proof 8.19 Fix $(a, b) \in X \times Y$. Note that $X \times \{b\}$ is homeomorphic to X and hence connected, and similarly $\{x\} \times Y$ is connected for each $x \in X$.

For any $x \in X$, define

$$T_x = (\{x\} \times Y) \cup (X \times \{b\}).$$

Each T_x is connected, being the union of two connected sets with common point (x, b) . Applying [Theorem 8.1](#) to the family $\{T_x : x \in X\}$, we conclude that

$$\bigcup_{x \in X} T_x = X \times Y$$

is connected.

Theorem 8.5 [↗](#) Let (X, d) be a metric space. Then X is connected if and only if every continuous function $f : (X, d) \rightarrow \{0, 1\}$ is constant.

Proof 8.20 First, note that $\{0, 1\}$ with the Euclidean metric is discrete.

($\Rightarrow$) If X is connected and $f : X \rightarrow \{0, 1\}$ is continuous, then $M = f^{-1}(\{0\})$ and $N = f^{-1}(\{1\})$ are clopen, $M \cap N = \emptyset$, and $M \cup N = X$. Since X is connected, one of M or N

must be empty, so f is constant.

($\Leftarrow$) Conversely, suppose every continuous $f : X \rightarrow \{0, 1\}$ is constant. If X were disconnected with $\{A, B\}$, define

$$f(x) = \begin{cases} 0, & x \in A, \\ 1, & x \in B. \end{cases}$$

Then f is continuous (preimages of all subsets are clopen), but f is not constant, a contradiction.

Definition 8.5 Let (X, d) be a metric space. A connected set $A \subseteq X$ is a *connected component* if it is maximal among connected sets, i.e., if $B \subseteq X$ is connected and $A \subseteq B$, then $A = B$. Connected components are the maximal connected subsets of X .

Example 8.11 If (X, d) is connected, the only connected component is X itself.

Example 8.12 If (X, d) is a discrete space, the connected components are singletons.

Theorem 8.6 $\Leftarrow$ Let (X, d) be a metric space. Then:

1. The connected components of X are closed;
2. Distinct components are disjoint;
3. Every non-empty connected subset of X is contained in exactly one component;
4. X is the union of its (pairwise disjoint) connected components.

Proof 8.21

(1) Let A be a component. Then $\overline{A}$ is connected ([Corollary 8.5](#)) and contains A , so maximality implies $A = \overline{A}$, i.e., A is closed.

(2) If A_1 and A_2 are distinct components and $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cup A_2$ is connected ([Proposition 8.12](#)), contradicting maximality.

(3) Let Y be connected. Consider the collection $\{F_\alpha\}_{\alpha \in I}$ of connected sets containing Y , and put $A = \bigcup_{\alpha \in I} F_\alpha$. By [Theorem 8.1](#), A is connected and maximal, so A is the component containing Y . Uniqueness follows by definition.

(4) Every $x \in X$ belongs to some component A_x (by (3) applied to $\{x\}$). Then $X = \bigcup_{x \in X} A_x$. Distinct components are disjoint by (2).

Proposition 8.14 $\Leftarrow$ Assume that (X, d) is a metric space and $A \subseteq X$ is a component. Then for every connected subset $C \subseteq X$, we have either $C \subseteq A$ or $C \subseteq X - A$.

Proof 8.22 It is enough to show that either $C \cap A \neq \emptyset$ or $C \cap (X - A) \neq \emptyset$. From the former one, we conclude that $C \cup A$ is also connected. From the latter one, it can be easily seen that $C \cup A \neq A$. So, since $A \subseteq C \cup A$, this is a contradiction with the fact that A is a component.

8.3 CONNECTEDNESS AND CONTINUITY, A BRIDGE FOR NOVEL IDEAS

Theorem 8.7 $\Leftarrow$ Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a continuous function. If $C \subseteq X$ is connected, then $f(C) \subseteq Y$ is also connected.

Proof 8.23 Consider the restriction of f from C onto $f(C)$ (denoted by f , again). The only difference is that f is onto in this case. Suppose on a contrary that $f(C)$ is not connected, so that it has a disconnection $\{M, N\}$. Put $M_1 = f^{-1}(M)$ and $N_1 = f^{-1}(N)$. Since f is continuous,

we conclude that M_1 and N_1 are clopen in X . Moreover, $M_1 = f^{-1}(M) \neq f^{-1}(f(C)) = C$. Also, $M_1 \neq \emptyset$; otherwise, $f^{-1}(M) = \{x \in C : f(x) \in M\} = \emptyset$, showing that $f(C) \subseteq N$, which is a contradiction. Similarly, $\emptyset \neq N_1 \neq C$. Thus, $\{M_1, N_1\}$ is a disconnection for C , a contradiction.

Motivated by [Theorem 8.6](#) and [Theorem 8.7](#), we consider the following question.

Question. Does a continuous function preserve connected components? How about the inverse image of a component?

In general, the answer is not affirmative, as the following example shows.

Example 8.13 Assume that $X = \mathbb{R}$ and d is the discrete metric. Consider the identity map $I : (\mathbb{R}, d) \rightarrow (\mathbb{R}, |\cdot|)$. I is continuous. Indeed, every singleton is connected in $(\mathbb{R}, d)$; in fact, every singleton is a component. However, every singleton is not a component in $(\mathbb{R}, |\cdot|)$, certainly.

For the inverse image, there is relatively good news.

Proposition 8.15 $\hookleftarrow$ Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is continuous. If $B \subseteq Y$ is a component, then the inverse image $f^{-1}(B)$ is a union of components of X .

Proof 8.24 Put $A = f^{-1}(B)$. For any $a \in A$, by [Theorem 8.6](#), $\{a\}$ is contained in just one component of X , namely C_a . Now, since f is continuous, we conclude that $f(C_a)$ is also connected in Y by [Theorem 8.7](#). On the other hand, $f(a) \in f(C_a)$ and $f(a) \in B$, so that $B \cap f(C_a) \neq \emptyset$. By [Proposition 8.14](#), $f(C_a) \subseteq B$. Now, we have

$$C_a \subseteq f^{-1}(f(C_a)) \subseteq f^{-1}(B) = A.$$

Therefore, $A = \bigcup_{a \in A} C_a$. This completes the proof.

Theorem 8.8 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a homeomorphism. Then, if A and B are components in X and Y , respectively, we have that $f(A)$ and $f^{-1}(B)$ are components in Y and X , respectively.

Proof 8.25 Assume that $A \subseteq X$ is a component. Since f is continuous, $f(A)$ is a connected subset of Y by [Theorem 8.7](#). Now, assume that $C \subseteq Y$ is a connected set such that $f(A) \subseteq C$, so that $A \subseteq f^{-1}(C)$. Since $f^{-1} : (Y, d_2) \rightarrow (X, d_1)$ is also continuous and $C \subseteq Y$ is connected, we conclude that $f^{-1}(C)$ is connected in X . Since A is a component, from $A \subseteq f^{-1}(C)$, we have $A = f^{-1}(C)$, so that $C = f(A)$. This shows that $f(A)$ is also a component in Y . Similarly, we can show that $f^{-1}(B)$ is also a component in X provided that $B \subseteq Y$ is a component.

In this part, we establish some results for connected sets in the reals. First, we have the following useful fact.

Lemma 8.3 $\hookleftarrow$ A non-empty set $I \subseteq \mathbb{R}$ is an interval if and only if for each $x, y \in I$ ($x < y$) and for each z with $x < z < y$, we have $z \in I$.

Proof 8.26 The direct implication is straightforward. Since I is an interval, we conclude that $[x, y] \subseteq I$, so that $z \in I$.

For the other side, assume that the mentioned condition holds. Put

$$\alpha = \inf I, \quad \beta = \sup I.$$

Note that if I is a bounded interval, by the axiom of completeness, α and β exist and are real numbers; otherwise, either $\alpha = -\infty$ or $\beta = +\infty$.

We proceed the proof with the bounded case; the unbounded case can be treated similarly. We claim that $(\alpha, \beta) \subseteq I \subseteq [\alpha, \beta]$.

Assume that $\alpha < z < \beta$. Therefore, we can find the reals α' and β' with $\alpha < \alpha' < z < \beta' < \beta$. By the definition of the supremum and the infimum, there are $x, y \in I$ with $x < \alpha'$ and $\beta' < y$, so that z lies between x and y , and thus $z \in I$ by the assumption. Moreover, for every $r \in I$, we have $\alpha \leq r \leq \beta$, so that $r \in [\alpha, \beta]$. We conclude that I is of the form one of the following four cases:

$$(\alpha, \beta), \quad [\alpha, \beta), \quad (\alpha, \beta], \quad [\alpha, \beta].$$

Recall that a subset C of a vector space V is called *convex* provided that for each $x, y \in C$ and for each $0 \leq t \leq 1$, we have $tx + (1 - t)y \in C$. In other words, for any distinct points in C , the segment between them also lies in C . Now, we have the following surprising fact.

Corollary 8.6 *A non-empty set $C \subseteq \mathbb{R}$ is convex if and only if it is an interval.*

Proof 8.27 Assume that C is convex, $x, y \in C$ and z lies between them. We conclude that z lies, in fact, in the segment between x and y , so that $z \in C$ by the convexity. By [Lemma 8.3](#), we see that C is an interval.

For the other direction, assume that C is an interval and $x, y \in C$. For each $0 \leq t \leq 1$, the point $tx + (1 - t)y$ lies between x and y , so that $tx + (1 - t)y \in C$ by [Lemma 8.3](#), again. Therefore, C is convex.

Theorem 8.9 $\Leftrightarrow$ *A subset C of the Euclidean metric space $\mathbb{R}$ is connected if and only if it is an interval.*

Proof 8.28 Assume that C is connected but it is not an interval. We use [Lemma 8.3](#). Thus, there are $x, y \in C$ with $x < y$ and $x < z < y$ such that $z \notin C$. Put

$$A = (-\infty, z) \cap C, \quad B = (z, +\infty) \cap C.$$

We claim that $\{A, B\}$ is a disconnection for C , which is impossible. First, note that $x < z$ and $x \in C$, so that $A \neq \emptyset$. Also, $z < y$ and $y \in C$, so that $B \neq \emptyset$. Moreover,

$$A \cup B = ((-\infty, z) \cap C) \cup ((z, +\infty) \cap C) = ((-\infty, z) \cup (z, +\infty)) \cap C = (\mathbb{R} - \{z\}) \cap C = C,$$

since $z \notin C$. On the other hand,

$$\overline{A} \cap B = \overline{(-\infty, z) \cap C} \cap ((z, +\infty) \cap C) \subseteq (-\infty, z] \cap (z, +\infty) = \emptyset.$$

Similarly, $A \cap \overline{B} = \emptyset$. This shows that $\{A, B\}$ is a disconnection for C , a contradiction.

For the other side, assume that C is an interval but not connected, so that there exists a disconnection $\{A, B\}$ for C . Since A and B are non-empty, we can choose $x_0 \in A$ and $y_0 \in B$. On the other hand, A and B are disjoint, so $x_0 \neq y_0$. W.L.O.G., assume that $x_0 < y_0$. Put

$$A_1 = C \cap [x_0, y_0] \cap A = [x_0, y_0] \cap A.$$

Note that $x_0 \in A_1$, so $A_1 \neq \emptyset$. Moreover, $A_1 \subseteq [x_0, y_0]$, so it is bounded. By the axiom of completeness of $\mathbb{R}$, $\alpha = \sup A_1$ exists. Also, $\alpha \in \overline{A_1} \subseteq \overline{[x_0, y_0]} = [x_0, y_0]$. So α is a point between x_0 and y_0 . Furthermore,

$$x_0 \in A \subseteq A \cup B = C, \quad y_0 \in B \subseteq A \cup B = C.$$

Since C is an interval, we conclude that $\alpha \in C = A \cup B$. But $\alpha \in \overline{A_1} \subseteq \overline{A}$, showing that $\alpha \notin B$ since $\overline{A} \cap B = \emptyset$. On the other hand, for each real $x > \alpha$, $x \notin A_1$ since α is an upper bound for A_1 . So for each $\alpha \leq x \leq y_0$, $x \in [x_0, y_0] \subseteq C$ but $x \notin A_1 = [x_0, y_0] \cap A$. Therefore,

$$[\alpha, y_0] \subseteq C - A = (A \cup B) - A = B.$$

Since A and B are disjoint, we conclude that

$$\alpha \in [\alpha, y_0] \subseteq \overline{[\alpha, y_0]} \subseteq \overline{B}.$$

Since $A \cap \overline{B} = \emptyset$, we get $\alpha \notin A$, so that $\alpha \notin A \cup B = C$, a contradiction.

Remark 8.3 Although connectedness, convexity, and being an interval are equivalent notions in $\mathbb{R}$, this does not hold in higher-dimensional Euclidean spaces. For example, the infinity symbol in $\mathbb{R}^2$ is connected but not convex. Moreover, there are several examples waiting for us in the next section!

Remark 8.4 A natural and crucial question comes to our mind as follows.

Question. Assume that a metric space is connected. Can we conclude that a subspace is also connected? How about the converse?

In general, the answer is negative. We know that the only connected sets in the reals with the Euclidean metric are the intervals. Note that the natural numbers, $\mathbb{N}$, form a subspace that is totally disconnected. Moreover, the singletons in a discrete space are connected (in fact, the only connected sets), but any discrete space with more than one point is disconnected. So, we conclude that if a subspace is connected, we cannot expect that the surrounding space is also connected.

Now, we present the following useful fact (as a version of the mean value theorem).

Theorem 8.10 Assume that (X, d) is a connected metric space and $f : (X, d) \rightarrow (\mathbb{R}, | \cdot |)$ is continuous. Moreover, assume that r is a real number between $f(a)$ and $f(b)$. Then, there exists $c \in X$ with $f(c) = r$.

Proof 8.29 W.L.O.G., we may assume that $f(a) < f(b)$, so that $f(a) < r < f(b)$. Put

$$A = f(X) \cap (-\infty, r), \quad B = f(X) \cap (r, +\infty).$$

Indeed, $f(a) \in A$ and $f(b) \in B$, so $A, B \neq \emptyset$. Note that both $(-\infty, r)$ and $(r, +\infty)$ are open in $\mathbb{R}$, so A and B are open in $f(X)$.

Suppose, on the contrary, that there is no $c \in X$ with $f(c) = r$. Then $f(X) = A \cup B$. Moreover, A and B are closed in $f(X)$; to see this, note that $A = f(X) \cap (-\infty, r]$ and $B = f(X) \cap [r, +\infty)$. Since $A \cap B = \emptyset$, we conclude that A and B form a disconnection of $f(X)$, which is impossible since X is connected and f is continuous. By [Theorem 8.7](#), $f(X)$ is connected. This shows that there exists $c \in X$ with $f(c) = r$.

8.4 LOCAL CONNECTEDNESS

Definition 8.6 Assume that (X, d) is a metric space and $x \in X$. We say that X is locally connected at x provided that for each open set $U \subseteq X$ containing x , there exists an open set $V \subseteq U$ containing x that is connected.

Remark 8.5 In spite of the relation between compactness and local compactness, there is no specific relation between connected metric spaces and locally connected ones. For example, every discrete space with more than one point is not connected but it is locally connected. In this case, for every point x , the singleton $\{x\}$ plays the role of a connected open set.

To illustrate further, note that by [Theorem 8.9](#), intervals are the only connected sets in the reals. But since the restriction of the Euclidean metric to either $\mathbb{N}$ or $\mathbb{Z}$ is the discrete metric, we see that $\mathbb{N}$ and $\mathbb{Z}$ are locally connected, although they are not connected. We shall present a locally connected metric space which is not connected, soon.

Lemma 8.4 Assume that (X, d) is a locally connected metric space and $Y \subseteq X$ is open. Then Y is also locally connected.

Proof 8.30 Assume that $y \in Y$ and $U \subseteq Y$ is an open set containing y . Then there exists an open set $V \subseteq X$ such that $U = V \cap Y$. We see that $y \in V$ and conclude that U is also open in X . Since X is locally connected, we can find an open set $W \subseteq X$ containing y that is connected and moreover, $W \subseteq U$, as claimed.

Theorem 8.11 $\Leftarrow$ Assume that (X, d) is a metric space. Then the following statements are equivalent.

1. X is locally connected.
2. For every open set $U \subseteq X$, each component of U is also open in X .
3. There exists a base of open sets for (X, d) whose elements are connected.

Proof 8.31 (1) $\rightarrow$ (2). Assume that X is locally connected and $U \subseteq X$ is open. Let $C \subseteq U$ be a component. Take any $x \in C$. By the assumption, there exists an open set $V \subseteq U$ such that $x \in V$ and V is connected. By [Theorem 8.6](#), V is contained in the unique component of U that intersects it. Since $V \cap C \neq \emptyset$, we have $V \subseteq C$, showing that x is an interior point of C .

(2) $\rightarrow$ (3). Assume that $U \subseteq X$ is arbitrary open. By (2), each component of U is also open in X . By [Theorem 8.5](#), U can be written as a union of its components. Let $\mathbf{B}_U$ be the collection of components of U , and put

$$\mathbf{B} = \{\mathbf{B}_U : U \text{ is an open set in } X\}.$$

It is easy to see that $\mathbf{B}$ forms a basis of open sets consisting of connected sets.

(3) $\rightarrow$ (1). Assume that (X, d) has a topological basis consisting of connected sets. For every $x \in X$ and every open set $U \subseteq X$ containing x , there exists a connected set V from the basis such that $x \in V \subseteq U$. This shows that X is locally connected.

Corollary 8.7 Assume that (X, d) is a locally connected metric space. Then every component is open, hence clopen.

Theorem 8.12 Assume that (X, d_1) and (Y, d_2) are metric spaces and $f : (X, d_1) \rightarrow (Y, d_2)$ is a continuous open function. Then, if X is locally connected, $f(X)$ is also locally connected.

Proof 8.32 It suffices to use the equivalent forms from [Theorem 8.11](#). Take any open set $V \subseteq f(X)$ and let B be a component of V . Then $f^{-1}(V)$ is open in X since f is continuous, and by [Proposition 8.15](#), $f^{-1}(B)$ is a union of components of $f^{-1}(V)$. Since X is locally connected, each component is open, so $f^{-1}(B)$ is open in X . Since f is also an open map, $B = f(f^{-1}(B))$ is open in $f(X)$.

Proposition 8.16 $\hookrightarrow$ Assume that (X, d) is a totally disconnected metric space and also locally connected. Then (X, d) is a discrete space.

Proof 8.33 Since X is totally disconnected, every singleton $\{x\}$ is a component. Since X is also locally connected, each $\{x\}$ is open. This shows that the metric is discrete.

We illustrate these results with a few examples.

Example 8.14

1. The induced Euclidean metric space $\mathbb{Q}$ is not locally connected. If it were, since it is also totally disconnected, by [Proposition 8.16](#), the induced metric should be discrete, which is impossible.
2. In the same way, the induced Euclidean metric space $\mathbb{Q}^c$ is not locally connected.
3. Consider the Euclidean space $X = \mathbb{R}$. As seen in [Example 8.7](#), the set $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is totally disconnected. Now, we claim it is not locally connected either. If it were, by [Proposition 8.16](#), the induced metric would be discrete, which is impossible since $\{0\}$ is not open in A . Indeed, if $\{0\}$ were open, there would exist $\varepsilon > 0$ such that $\{0\} = (-\varepsilon, \varepsilon) \cap A$, which is impossible by the Archimedean property, as there exists $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$.

8.5 PATHWISE CONNECTEDNESS

Definition 8.7 Assume that (X, d) is a metric space. For any $a, b \in X$, we say that there is a path from a to b if there exists a continuous function $f : [0, 1] \rightarrow X$ such that $f(0) = a$ and $f(1) = b$. We say that X is pathwise connected if for any $a, b \in X$, there exists a path between them.

Theorem 8.13 Assume that (X, d) is a pathwise connected metric space. Then X is also connected.

Proof 8.34 Suppose, on the contrary, that X is not connected, so it has a disconnection $\{A, B\}$. Choose $a \in A$ and $b \in B$. Since X is pathwise connected, there exists a path $f : [0, 1] \rightarrow X$ from a to b . By [Theorem 8.9](#), $[0, 1]$ is connected, so $f([0, 1])$ is connected. By [Lemma 8.2](#), either $f([0, 1]) \subseteq A$ or $f([0, 1]) \subseteq B$, implying that a and b lie in the same component, a contradiction.

Theorem 8.14 $\hookrightarrow$ Assume that (X, d_1) and (Y, d_2) are metric spaces and $g : (X, d_1) \rightarrow (Y, d_2)$ is continuous. If X is pathwise connected, then so is $g(X)$.

Proof 8.35 Take any $c, d \in g(X)$. Find $a, b \in X$ with $g(a) = c$ and $g(b) = d$. Since X is pathwise connected, there exists a continuous function $f : [0, 1] \rightarrow X$ with $f(0) = a$ and $f(1) = b$. Then $g \circ f : [0, 1] \rightarrow Y$ is continuous, with $(g \circ f)(0) = c$ and $(g \circ f)(1) = d$. Thus, $g(X)$ is pathwise connected.

Corollary 8.8 Assume that (X_1, d_1) and (X_2, d_2) are metric spaces. Then the Cartesian product $X_1 \times X_2$ is pathwise connected if and only if both X_1 and X_2 are pathwise connected.

Proof 8.36 Suppose $X_1 \times X_2$ is pathwise connected and let $\pi_1 : X_1 \times X_2 \rightarrow X_1$ be the projection. By [Theorem 8.14](#), $\pi_1(X_1 \times X_2) = X_1$ is pathwise connected, and similarly X_2 is pathwise connected.

Conversely, assume X_1 and X_2 are pathwise connected. For $(a_1, a_2), (b_1, b_2) \in X_1 \times X_2$, let $\alpha : [0, 1] \rightarrow X_1$ be a path from a_1 to b_1 , and $\beta : [0, 1] \rightarrow X_2$ be a path from a_2 to b_2 . Then $t \mapsto (\alpha(t), \beta(t))$ is a continuous path in $X_1 \times X_2$ connecting (a_1, a_2) and (b_1, b_2) .

Lemma 8.5 $\hookrightarrow$ In a normed space $(X, \|\cdot\|)$, every convex set is pathwise connected.

Proof 8.37 Let $C \subseteq X$ be convex and $x, y \in C$. Define $\alpha : [0, 1] \rightarrow X$ by $\alpha(t) = tx + (1 - t)y$. Then α is continuous with $\alpha(0) = y$ and $\alpha(1) = x$.

Lemma 8.6 $\hookrightarrow$ In a normed space $(X, \|\cdot\|)$, every open ball is convex and thus pathwise connected.

Proof 8.38 Let $x \in X$ and $r > 0$. For $y, z \in N_r^{(x)}$ and $0 \leq t \leq 1$, we have

$$\|ty + (1 - t)z - x\| \leq t\|y - x\| + (1 - t)\|z - x\| < tr + (1 - t)r = r,$$

so $ty + (1 - t)z \in N_r^{(x)}$. By [Lemma 8.5](#), $N_r^{(x)}$ is pathwise connected.

Theorem 8.15 Assume $U \subseteq \mathbb{R}^n$ is open. Then U is pathwise connected if and only if it is connected.

Proof 8.39 The forward direction follows from the general fact that pathwise connected implies connected.

For the converse, assume U is connected. Fix $t \in U$ and let

$$A = \{a \in U : \text{there exists a path from } t \text{ to } a\}, \quad B = U - A.$$

Then $A \cap B = \emptyset$. For any $a \in A$, there exists a path from t to a . Since U is open, there exists $\delta > 0$ with $N_\delta^{(a)} \subseteq U$, and by [Lemma 8.6](#), $N_\delta^{(a)}$ is pathwise connected, so every point in $N_\delta^{(a)}$ has a path to t , showing $N_\delta^{(a)} \subseteq A$. Thus, A is open. Similarly, B is open. Since U is connected, B must be empty, so $U = A$, showing U is pathwise connected.

Example 8.15 Assume (X, d) is a metric space and $a, b, c \in X$. If there is a path from a to b and a path from b to c , then there is a path from a to c .

To see this, let $\alpha : [0, 1] \rightarrow X$ be a path from a to b and $\beta : [0, 1] \rightarrow X$ a path from b to c . Define

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then γ is continuous, with $\gamma(0) = a$ and $\gamma(1) = c$.

Example 8.16 $\hookrightarrow$ Assume that $F \subseteq \mathbb{R}^2$ is a countable set and we equip $\mathbb{R}^2$ with the Euclidean metric. Then $X = \mathbb{R}^2 - F$ is pathwise connected. To see this, using our assumption, put

$$F = \{(p_n, q_n) : n \in \mathbb{N}, p_n, q_n \in \mathbb{R}\}.$$

Take any $(a, b), (c, d) \in X$. First, assume that $a < c$ and $b < d$. Moreover, in each ordered pair, there exists at least one component that is not either of the form p_n or q_n for each $n \in \mathbb{N}$. W. L. O. G, assume that $a \neq p_n$ and $c \neq q_n$ for each $n \in \mathbb{N}$; the other cases can be proved in a similar way.

Now, consider $\alpha_1 : [0, 1] \rightarrow X$ defined via

$$\alpha_1(t) = (a, td + (1 - t)b);$$

it is a continuous functions that maps (a, b) into (a, d) . Also, define continuous function $\alpha_2 : [0, 1] \rightarrow X$ via

$$\alpha_2(t) = (ct + (1 - t)a, d),$$

that maps (a, d) into (c, d) . By example, we have a path from (a, b) to (c, d) .

If $a = c$, the function $\alpha_3 : [0, 1] \rightarrow X$ defined via $\alpha_3(t) = (a, td + (1 - t)b)$ has the desired properties. Finally, if $b = d$, Choose any irrational s such that $a < s < c$ and $s \neq p_n$ for each $n \in \mathbb{N}$; note that this is possible since F is countable so that in each interval, we can find elements that does not belong to F . Now, consider the functions $\alpha_4 : [0, 1] \rightarrow X$ with $\alpha_4(t) = (a, ts + (1 - t)b)$, $\alpha_5 : [0, 1] \rightarrow X$ with $\alpha_5(t) = (tc + (1 - t)a, s)$ and $\alpha_6 : [0, 1] \rightarrow X$ defined via $\alpha_6(t) = (c, bt + (1 - t)s)$. This means that we have a continuous path from (a, b) into (c, d) .

Corollary 8.9 Assume that $F \subseteq \mathbb{R}^2$ is a finite set and we equip $\mathbb{R}^2$ with the Euclidean metric. Then $X = \mathbb{R}^2 - F$ is pathwise connected.

Example 8.17 The unit circle S^1 in $\mathbb{R}^2$ (Euclidean metric) is pathwise connected. To see this, consider the continuous function $g : \mathbb{R}^2 - \{0\} \rightarrow S^1$ defined by

$$g(x) = \frac{x}{\|x\|}, \quad x \in \mathbb{R}^2 - \{0\}.$$

By having a glance at [Example 8.16](#), it might seem that if we omit an uncountable set from $\mathbb{R}^2$, we may have not a pathwise connected set. However, the following example shows that this conjecture does not hold, in general.

Example 8.18 $X = \mathbb{R}^2 - \mathbb{Q}^c \times \mathbb{Q}^c$ with the Euclidean metric is pathwise connected. The proof of this fact is also similar. Take any $(a, b), (c, d) \in X$. First, assume that $a < c$ and $b < d$. Moreover, assume that a and c are rational; the other cases, can be proved in a similar way.

Now, consider $\alpha_1 : [0, 1] \rightarrow X$ defined via

$$\alpha_1(t) = (a, td + (1 - t)b);$$

it is a continuous functions that maps (a, b) into (a, d) . Also, define continuous function $\alpha_2 : [0, 1] \rightarrow X$ via

$$\alpha_2(t) = (ct + (1 - t)a, d),$$

that maps (a, d) into (c, d) . By example, we have a path from (a, b) to (c, d) . If $a = c$, the function $\alpha_3 : [0, 1] \rightarrow X$ defined via $\alpha_3(t) = (a, td + (1 - t)b)$ has the desired properties. Finally, if $b = d$, Choose any rational s such that $a < s < c$. Now, consider the functions

$\alpha_4 : [0, 1] \rightarrow X$ with $\alpha_4(t) = (a, ts + (1 - t)b)$, $\alpha_5 : [0, 1] \rightarrow X$ with $\alpha_5(t) = (tc + (1 - t)a, s)$ and $\alpha_6 : [0, 1] \rightarrow X$ defined via $\alpha_6(t) = (c, bt + (1 - t)s)$. This means that we have a continuous path from (a, b) into (c, d) .

The next example is known as “the topologist's sin curve.” It is a connected subset of the Euclidean space $\mathbb{R}^2$ that is not pathwise connected.

Example 8.19 Put

$$S = \{(0, 0)\} \cup \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}.$$

We show that S is connected but not pathwise connected. To show that S is connected, first, note that the singleton $A = \{(0, 0)\}$ is connected. Now, define $f : (0, 1] \rightarrow \mathbb{R}^2$ via $f(x) = (x, \sin \frac{1}{x})$. f is continuous and since $(0, 1]$ is connected, we conclude that $B = f((0, 1]) = \{(x, \sin \frac{1}{x}) : x \in (0, 1]\}$ is also connected. Furthermore, it is easy to see that $(0, 0) \in \overline{B}$ so that $A \cap \overline{B} \neq \emptyset$. By..., $S = A \cup B$ is connected.

Now, we show that S is not pathwise connected. Although, intuitively, it is obvious that there is no way from $\{(0, 0)\}$ to the set B . But the proof is somehow complicated. Suppose on a contrary, there is a way from $\{(0, 0)\}$ to $\{(1, \sin 1)\}$; in other words, there exists continuous function $\alpha : [0, 1] \rightarrow S$ such that $\alpha(0) = (0, 0)$ and $\alpha(1) = (1, \sin 1)$. Assume that $\alpha(t) = (f(t), g(t))$ for any $t \in [0, 1]$. Since for each $0 \leq t \leq 1$, $\alpha(t) \in S$, we have $0 \leq f(t) \leq 1$ and

$$g(t) = \begin{cases} 0, & f(t) = 0 \\ \sin \frac{1}{f(t)}, & 0 < f(t) \leq 1 \end{cases}$$

Since $\alpha(0) = (0, 0)$, there exists $0 \leq t \leq 1$ such that $f(t) = 0$. Put

$$t_0 = \sup \{t : 0 \leq t \leq 1, f(t) = 0\}.$$

First, we show that $t_0 \neq 1$ and then, we show that $f(t_0) = 0$. Suppose on a contrary, $t_0 = 1$. Since f is continuous at t_0 from the right side, we conclude that there exists $0 < \delta_0 < 1$ such that if $1 - \delta_0 < t < 1$ then $-\frac{1}{2} < f(t) - f(1) < \frac{1}{2}$. Note that since $\alpha(1) = (1, \sin 1)$, we conclude that $f(t_0) = f(1) = 1$ so that $f(t) > \frac{1}{2}$ for each $t \in (1 - \delta_0, 1]$. Now, since

$$t_0 = \sup \{t : 0 \leq t \leq 1, f(t) = 0\} = 1,$$

there exists $1 - \delta_0 < t' \leq t_0$ with $f(t') = 0$ so that $0 > \frac{1}{2}$ that is a contradiction. So, we see that $t_0 \neq 1$ so that $0 \leq t_0 < 1$. Now, we show that $f(t_0) = 0$. Otherwise (i.e. $f(t_0) \neq 0$), we see that $0 < t_0 < 1$; since $\alpha(0) = (0, 0)$, $f(0) = 0$. Since $f(t_0) > 0$ (by using definition of g), we conclude that there exists $\delta_1 > 0$ such that f is strictly positive on the interval $(t_0 - \delta_1, t_0 + \delta_1) \subseteq (0, 1)$. On the other hand, by the definition of t_0 (in fact, using the definition of the supremum), we can find $t_0 - \delta_1 < t'' \leq t_0$ with $f(t'') = 0$ that is a contradiction so that $f(t_0) = 0$. Thus, $g(t_0) = 0$, as well.

Now, since g is continuous at t_0 (from right side), there exists $\delta_2 > 0$ with $\delta_2 < 1 - t_0$ and also if $t_0 \leq t < t_0 + \delta_2$, then, $|g(t) - g(t_0)| < \frac{1}{2}$ so that $|g(t)| < \frac{1}{2}$ for each $t_0 \leq t < t_0 + \delta_2$. On

the other hand, by definition of t_0 , it is obvious that $f(t_0 + \delta_2) > 0$. By using a somehow Archimedean property of the reals, we can find $n \in \mathbb{N}$ with $\frac{1}{(2n+1)\frac{\pi}{2}} < f(t_0 + \delta_2)$. Therefore,

$$f(t_0) = 0 < \frac{1}{(2n+1)\frac{\pi}{2}} < f(t_0 + \delta_2).$$

Now, by the mean value theorem, there exists $t_0 < c < t_0 + c$ with $f(c) = \frac{1}{(2n+1)\frac{\pi}{2}}$. Now, using definition of g , convinces us that $g(c) = 1$ that is a contradiction with this fact that $|g(c)| < \frac{1}{2}$. So, there is not a continuous path between $(0, 0)$ and $(1, \sin 1)$.

In [Figure 8.3](#), we see the geometric behavior of the topologist's sin curve.

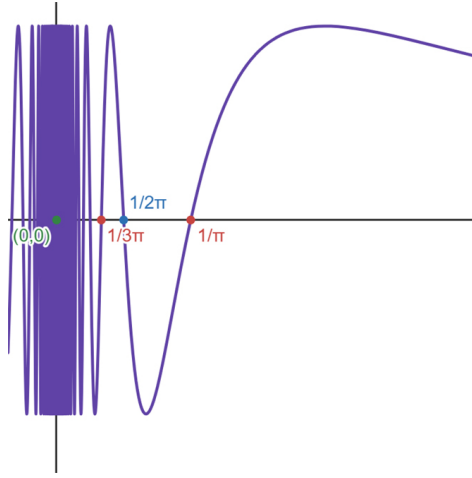


Figure 8.3 Topologist's sine curve: connected but not pathwise connected. ↴

Example 8.20 The function

$$f : [0, 1) \rightarrow S^1, \quad f(t) = (\cos 2\pi t, \sin 2\pi t)$$

is continuous, where S^1 is the unit circle. Since $[0, 1)$ is connected, $f([0, 1)) = S^1$ is connected.

Example 8.21 (Comb space) Define

$$S = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}, \quad X = ([0, 1] \times \{0\}) \cup (S \times [0, 1]) \subseteq \mathbb{R}^2.$$

Let $C = [0, 1] \times \{0\}$, $C_0 = \{0\} \times [0, 1]$, and $C_n = \{\frac{1}{n}\} \times [0, 1]$ for $n \in \mathbb{N}$. Each C and C_i is connected. To see this, consider this point that each of them is homeomorphic (topologically the same) with the interval $[0, 1]$; that is connected, so that all of them are connected. Furthermore, $C \cap C_i \neq \emptyset$. By [Proposition 8.11](#), $X = C \cup \bigcup_{i \in \mathbb{N}} C_i$ is connected.

X is not locally connected. Consider

$$Y = \{(x, y) \in X : 0 < y < \frac{1}{2}\} = X \cap \{(x, y) \in \mathbb{R}^2 : 0 < y < \frac{1}{2}\}.$$

Y is open in X . But $Y = S \times (0, \frac{1}{2})$, and S is not locally connected; hence Y cannot be locally connected, a contradiction.

In [Figure 8.4](#), we see the geometric behavior of the comb space.

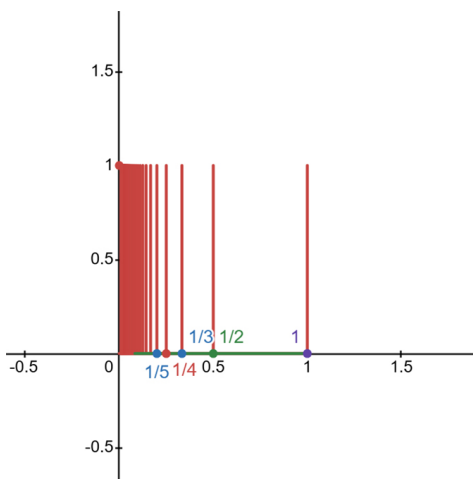


Figure 8.4 The comb space: connected but not locally connected. ↗



Figure 8.5 The union of a chain of connected sets is connected. ↗

8.6 EXERCISES

Exercise 8.1

1. Let (X, d) be a metric space and $f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$ a non-constant continuous function. Show that X is uncountable.
2. Let (X, d) be a metric space in which every continuous function $f : (X, d) \rightarrow (X, d)$ has a fixed point. Show that X is connected.
3. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Show that $X_1 \times X_2$ (with any product metric) is totally disconnected if and only if X_1 and X_2 are totally disconnected.
4. Let (X_1, d_1) and (X_2, d_2) be metric spaces, and let $C_1 \subseteq X_1$, $C_2 \subseteq X_2$ be connected components. Show that $C_1 \times C_2$ is a connected component of $X_1 \times X_2$.
5. Let (X, d) be a metric space and $E, F \subseteq X$ be closed (or open) subsets such that $E \cup F$ and $E \cap F$ are connected. Show that E and F are connected.
6. Let (X, d) be compact and locally connected. Show that X has finitely many connected components.
7. Let (X_1, d_1) and (X_2, d_2) be metric spaces, $A \subseteq X_1$ connected, and $f : A \rightarrow X_2$ continuous. Show that the graph of f , $\{(x, f(x)) : x \in A\}$, is connected in $X_1 \times X_2$.

8. Consider $\mathbb{R}^2$ endowed with the Euclidean metric. Give an example of a closed and connected subset $C \subset \mathbb{R}^2$ such that $\mathbb{R}^2 \setminus C$ has infinitely many connected components.

9. Let (X, d) be a connected metric space and let

$$f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$$

be a non-constant continuous function. Show that X is uncountable.

10. Let (X, d) be a compact connected metric space. Show that if

$$f : (X, d) \rightarrow (\mathbb{R}, |\cdot|)$$

is continuous, then $f(X) \subseteq \mathbb{R}$ is a closed interval. Deduce that every continuous function

$$f : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, |\cdot|)$$

maps closed intervals to closed intervals. Can we conclude that f is a closed map? Why or why not?

11. Let (X, d) be a metric space and let $x_0 \in X$. Suppose there exists a constant $c > 0$ such that

$$d(x, x_0) \neq c, \quad \text{for all } x \in X.$$

Prove that X is disconnected.

12. Let (X, d) be a metric space and let x_0 be a limit point of X . Let $\delta > 0$. Prove that the sets

$$A = \{x \in X \mid d(x, x_0) > \delta\}, \quad B = \{x \in X \mid d(x, x_0) < \delta\}$$

are separated. item Let

$$X = \{(t, 0) \mid 0 < t \leq 1\} \cup \left\{ \left(\frac{1}{n}, t \right) \mid n \in \mathbb{N}, 0 \leq t \leq 1 \right\}.$$

Define a function d on $X \times X$ by letting $d(x, y)$ be the length of the shortest path in X joining x to y .

(a) Is X connected?

(b) Is X path connected?

(c) Is X locally connected?

(d) Does X have any other interesting properties?

13. Assume that $Y = \{(x, 0) : x \in \mathbb{R}\}$. For each rational r , assume that $Y_r = \{(r, y) : y \in \mathbb{R}\}$. Put

$$X = Y \cup \left(\bigcup_{r \in \mathbb{Q}} Y_r \right).$$

Show that X (as a subset of the Euclidean space $\mathbb{R}^2$) is connected but it is not locally connected.

FURTHER READING

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