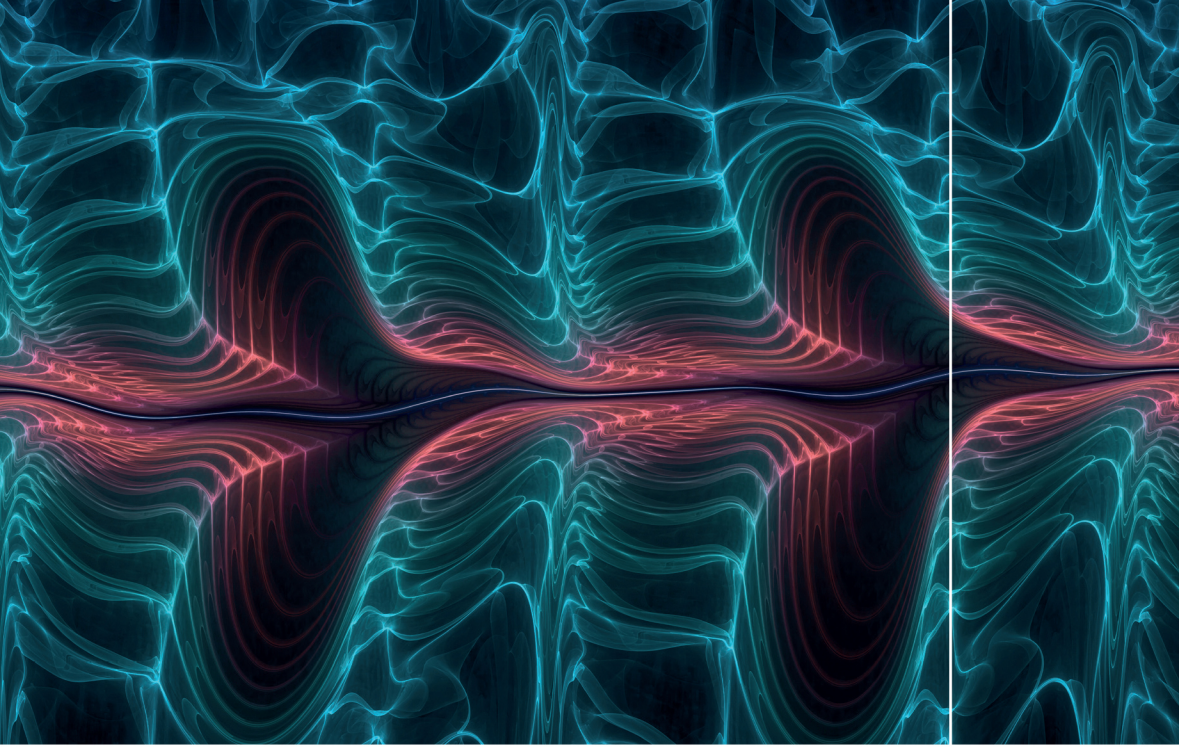




A VISUAL APPROACH TO INTRODUCTORY FUNCTIONAL ANALYSIS

Paul Dayao

A **Chapman & Hall** Book



CRC Press
Taylor & Francis Group

A Visual Approach to Introductory Functional Analysis

A Visual Approach to Introductory Functional Analysis is designed to offer a visual, geometric approach to functional analysis that bridges the gap between elementary analysis and advanced Banach Space Theory.

The material is organized to build progressively from foundational topology to the core structural theorems of the discipline. Commencing with the rigorous formalization of metric spaces, topological neighborhoods, and the critical analytical property of completeness, the text subsequently transitions to normed and Banach spaces, formalizing the evaluation of linear operators and dual spaces. This structural foundation culminates in the exposition of the “pillars” of functional analysis: the Hahn-Banach Theorem, the Uniform Boundedness Principle, the Open Mapping Theorem, the Closed Graph Theorem, and the Banach-Alaoglu Theorem.

This book can serve as a concise textbook for an introductory functional analysis course, or as a geometrically focused supplement to a more advanced treatment of the subject.

Features

- Curated selections of exercises ending every chapter, designed to build topological reasoning and proof writing skills
- Numerous color figures and illustrative examples to aid comprehension.



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Preface

Functional analysis represents a profound synthesis of linear algebra, topology, and classical real analysis. By abstracting the finite-dimensional concepts of vectors and matrices into infinite-dimensional spaces of functions and operators, it provides the essential mathematical machinery required to solve complex differential equations, analyze quantum mechanical systems, and formalize approximation theory.

This textbook is constructed as a rigorous introduction to the subject. The primary objective is to bridge the conceptual gap between elementary analysis and advanced operator theory. Students frequently encounter functional analysis as a daunting landscape of abstract definitions and dense topological proofs. To mitigate this, the text is structured to continuously anchor abstract axiomatic concepts to their geometric intuitions. Throughout the chapters, rigorous analytical proofs are paired directly with visual geometric interpretations—from the strict topological boundaries of metric spaces to the separating hyperplanes of the Hahn-Banach Theorem—ensuring that the underlying spatial logic of the theorems remains central to the learning process.

The material is organized to build progressively from foundational topology to the core structural theorems of the discipline. We commence with the rigorous formalization of metric spaces, topological neighborhoods, and the critical analytical property of completeness. The text subsequently transitions to normed and Banach spaces, formalizing the evaluation of linear operators and dual spaces. This structural foundation culminates in the exposition of the ‘pillars’ of functional analysis: the Hahn-Banach Theorem, the Uniform Boundedness Principle, the Open Mapping Theorem, the Closed Graph Theorem, and the Banach-Alaoglu Theorem.

Each chapter is accompanied by a curated selection of exercises designed not merely to test mechanical computation, but to challenge and refine the reader’s topological reasoning and proof-writing capabilities.

It is my hope that this text serves as a clear, precise, and illuminating guide through the elegant structural truths of infinite-dimensional spaces, equipping the reader with both the analytical rigor and the geometric intuition necessary for advanced mathematical applications.

Paul L. Dayao



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Author Biography

Paul Dayao is a mathematician affiliated with the Department of Mathematics at Ateneo de Manila University. His research centers on real analysis and functional analysis. Beyond this, Paul is driven by a deep passion for teaching and a commitment to make high-level mathematics intuitive by grounding abstract theory in visual clarity.



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Metric Spaces

[1.1 Definition of Metric Spaces](#)

[1.2 Metric Topology](#)

[1.3 Metric Subspaces](#)

[1.4 Continuous Mappings](#)

THE transition from classical real analysis to functional analysis necessitates a fundamental departure from finite-dimensional Euclidean geometry. In this broader framework, our primary mathematical objects are no longer simple vectors, but complex, infinite-dimensional entities such as continuous functions, infinite sequences, and linear operators. In this chapter, we establish the rigorous architecture for quantifying proximity and convergence between these abstract objects. By generalizing the axiomatic properties of physical distance, we define the concept of a metric space. We will survey the standard sequence and function spaces that serve as the primary domains of functional analysis, construct the essential metric topology of open and closed sets, and ultimately reframe the classical concept of continuous mappings through a purely topological perspective.

1.1 DEFINITION OF METRIC SPACES

In classical analysis and geometry, the concepts of limits, convergence, and continuity rely upon the Euclidean distance between points in spaces such as $\mathbb{R}^2$ or $\mathbb{R}^3$. For any two given points, the standard Euclidean metric quantifies their spatial separation.

Functional analysis broadens this topological framework. Our primary objects of study are not restricted to finite-dimensional coordinate vectors; rather, we investigate the structural properties of spaces whose elements are infinite-dimensional entities. Consequently, we must formulate strict notions of proximity for these abstract objects:

Functions: How is the deviation of a polynomial approximation $P(t)$ from a target continuous function $f(t)$ quantified?

Sequences: By what measure do two infinite streams of data, (x_n) and (y_n) , become mathematically indistinguishable?

Sets: How is the separation between two distinct geometric regions or subsets formally defined?

Abstract Elements: Can a robust theory of convergence be established for abstract spaces lacking inherent visual representation?

To address these questions, the axiomatic properties of Euclidean distance are extended to arbitrary sets. By treating a complex entity—such as a continuous function or an infinite sequence—as a single *point* within an abstract space, geometric intuition can resolve analytical problems. This abstraction culminates in the definition of a *metric space*: a pair (X, d) , where X is a set and $d : X \times X \rightarrow \mathbb{R}$ is the *metric* (or distance function). To constitute a valid metric, d must satisfy four axioms that capture the algebraic behavior of physical distance.

Definition 1.1 Metric Space

A **metric space** is a pair (X, d) consisting of a non-empty set X and a mapping $d : X \times X \rightarrow \mathbb{R}$, called a **metric** or distance function, which satisfies the following axioms for all $x, y, z \in X$:

(M1) Non-negativity The distance between any two points is a non-negative real number:

$$d(x, y) \geq 0$$

(M2) Identity of Indiscernibles The distance between two points is zero if and only if the points are identical:

$$d(x, y) = 0 \iff x = y$$

(M3) Symmetry The metric does not depend on the direction of measurement:

$$d(x, y) = d(y, x)$$

(M4) Triangle Inequality The distance between any two points is bounded above by the sum of the distances through any intermediate point z :

$$d(x, y) \leq d(x, z) + d(z, y)$$

Regardless of their underlying mathematical structure, the elements of X are conventionally referred to as *points* within the context of the metric space (X, d) .

Among the axioms, the triangle inequality (M4) is highly consequential for analytical proofs. It derives its nomenclature from Euclidean geometry, wherein the length of any side of a triangle is bounded by the sum of the lengths of the remaining two sides as visualized in Figure 1.1.

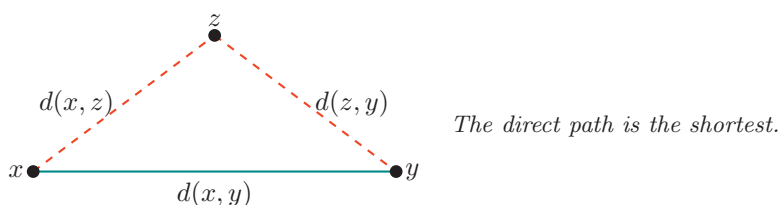


FIGURE 1.1 The triangle inequality. The metric distance $d(x, y)$ cannot exceed the sum of the distances through an intermediate point z .

Example 1.2 The Standard Examples

The following examples establish the standard metric spaces utilized throughout functional analysis. These spaces serve as prototypes for geometric intuition and test cases for evaluating theorems in infinite-dimensional settings.

The Real Line $\mathbb{R}$

The set of real numbers $\mathbb{R}$ forms a metric space where the points are scalars. The metric evaluates the absolute value of their difference:

$$d(x, y) = |x - y|$$

This metric satisfies the axioms, with the triangle inequality taking the form $|x - y| \leq |x - z| + |z - y|$.

The Euclidean Plane $\mathbb{R}^2$

The space $\mathbb{R}^2$ consists of ordered pairs of real numbers. Denoting points as vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the metric is defined by the Pythagorean

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formula:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

This space establishes geometric frameworks for open balls and neighborhoods.

Euclidean Space $\mathbb{R}^n$

Generalizing to arbitrary finite dimensions, $\mathbb{R}^n$ consists of ordered n -tuples of real numbers. The metric is the standard Euclidean distance:

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

Unitary Space $\mathbb{C}^n$

The complex analogue of Euclidean space is the Unitary Space $\mathbb{C}^n$, consisting of ordered n -tuples of complex numbers. To ensure the distance evaluates to a non-negative real number, the metric squares the modulus:

$$d(z, w) = \sqrt{\sum_{k=1}^n |z_k - w_k|^2}$$

The Space of All Sequences (s)

The space s consists of all sequences of complex numbers. Because unbounded sequences exist in s , standard distance formulas diverge. A valid metric is constructed by bounding the difference at each coordinate. For sequences $u = (u_k)$ and $v = (v_k)$, the metric utilizes a weighted sum:

$$d(u, v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|u_k - v_k|}{1 + |u_k - v_k|}$$

Since the rational term $\frac{|a|}{1+|a|}$ is strictly bounded by 1, the factor $\frac{1}{2^k}$ guarantees the convergence of the infinite series. As visualized in [Figure 1.2](#), distance in s is primarily determined by the initial terms of the sequences.

The Space of Bounded Sequences (ℓ^∞)

The space ℓ^∞ is a subspace of s consisting of all bounded sequences of complex numbers (u_k) ; a real constant M exists such that $|u_k| \leq M$ for all indices k . The distance between sequences u and v is evaluated using the supremum over all coordinates (see [Figure 1.3](#)):

$$d(u, v) = \sup_{k \in \mathbb{N}} |u_k - v_k|$$

This formulation is termed the *supremum metric*, where convergence corresponds to uniform convergence.

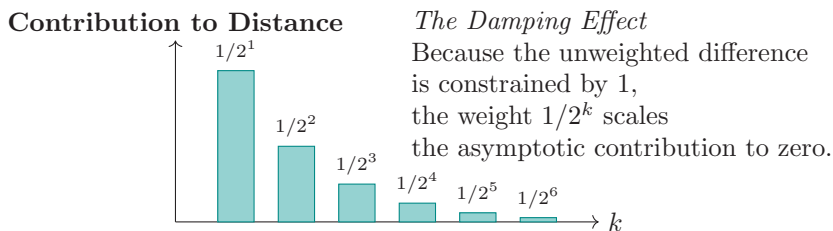
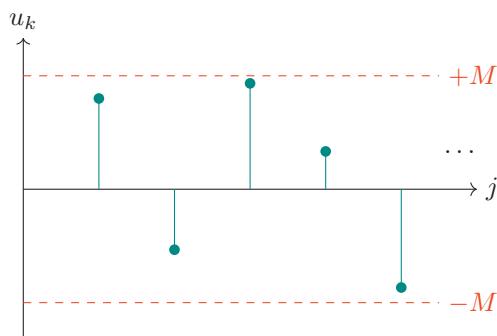


FIGURE 1.2 The metric on s . The sequence terms are weighted by $1/2^k$, defining the distance based on the initial coordinates.



Visualizing ℓ^∞

The terms u_k are constrained within the interval $[-M, M]$.

FIGURE 1.3 A bounded sequence evaluated in ℓ^∞ .

The Space of Convergent Sequences (c)

The subspace $c \subset \ell^\infty$ consists of all sequences converging to a limit L . Because every convergent sequence is bounded, c resides within ℓ^∞ . It is equipped with the supremum metric.

The Space of Summable Sequences (ℓ^p)

For a fixed real number $p \geq 1$, the space ℓ^p consists of sequences $u = (u_k)$ satisfying $\sum_{k=1}^{\infty} |u_k|^p < \infty$. The metric generalizes the Euclidean distance formula to infinite sums:

$$d(u, v) = \left(\sum_{k=1}^{\infty} |u_k - v_k|^p \right)^{1/p}$$

The specific case where $p = 2$ designates the *Hilbert sequence space* ℓ^2 .

The Function Space $C[a, b]$

The space $C[a, b]$ consists of all continuous, real-valued functions $f : [a, b] \rightarrow \mathbb{R}$.

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The distance between functions f and g evaluates the maximum vertical separation between their graphs over the interval (see Figure 1.4):

$$d(f, g) = \max_{t \in [a, b]} |f(t) - g(t)|$$

This is termed the **uniform metric**. Convergence in this metric guarantees that the limit function is continuous.

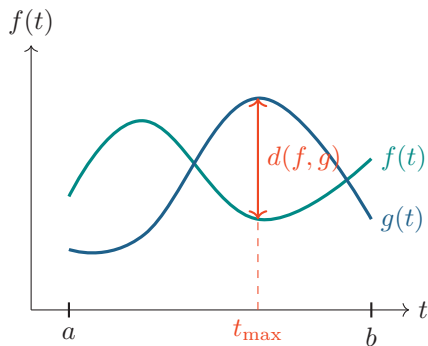


FIGURE 1.4 The metric in $C[a, b]$ evaluates the maximum vertical separation between the functions.

The L^p Spaces

For a fixed $p \geq 1$, the space $L^p[a, b]$ consists of all Lebesgue-measurable functions f on the interval $[a, b]$ such that the integral of $|f|^p$ is finite. The metric evaluates the accumulation of differences across the interval (see Figure 1.5):

$$d(f, g) = \left(\int_a^b |f(t) - g(t)|^p dt \right)^{1/p}$$

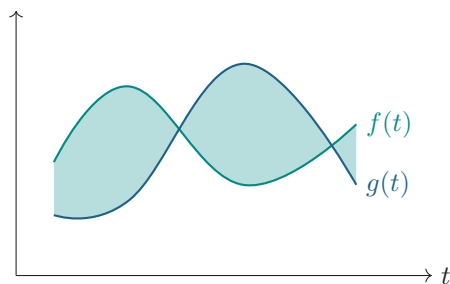
The elements of L^p are formulated as equivalence classes of functions. If two functions f and g differ on a set of measure zero, the integral of their difference evaluates to zero ($d(f, g) = 0$). To satisfy the Identity Axiom, such functions constitute the same element in the space.

The Discrete Metric Space

On any non-empty set X , a metric is defined independent of coordinate geometry:

$$d(u, v) = \begin{cases} 0 & \text{if } u = v \\ 1 & \text{if } u \neq v \end{cases}$$

This function satisfies the metric axioms, inducing the discrete topology where every subset is both open and closed, and convergent sequences are constant.



In L^p , distance evaluates the accumulation of the shaded area between the graphs.

FIGURE 1.5 The L^p metric integrates the differences over the interval domain.

EXERCISES

1. *Testing the Axioms.* Let $X = \mathbb{R}$. Determine whether the following functions $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ define a metric on $\mathbb{R}$. If not, state which axiom (M1–M4) is violated.

- (a) $d(x, y) = |x - y|^2$
- (b) $d(x, y) = |x^2 - y^2|$
- (c) $d(x, y) = \max\{|x|, |y|\}$
- (d) $d(x, y) = \min\{|x - y|, 1\}$

2. *The Reverse Triangle Inequality.* Applying the Triangle Inequality (M4), prove the following lower bound for the distance in any metric space (X, d) :

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

(Hint: Formulate $d(x, z) \leq d(x, y) + d(y, z)$, rearrange the terms, and evaluate the symmetry with x and y .)

3. *Bounded Metrics.* Let (X, d) be a metric space. Prove that the function ρ defined by:

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

constitutes a metric on X . (Hint: To establish the triangle inequality for ρ , apply the property that the function $f(t) = \frac{t}{1+t}$ is monotonically increasing for $t \geq 0$.)

4. *Product Metrics.* Let (X, d_X) and (Y, d_Y) be metric spaces. Define the Cartesian product $Z = X \times Y$, where points are structured as pairs $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Verify that the following function defines a metric on Z :

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

5. *Metrics on Sets.* Let A and B be non-empty subsets of a metric space (X, d) . Define the distance between sets as $D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$. Determine if $D(A, B)$ satisfies the axioms of a metric on the power set of X . (*Hint: Evaluate the condition where A and B overlap, or when $A = (0, 1)$ and $B = (1, 2)$ in $\mathbb{R}$. Check Axiom M2.*)
6. *Comparing Metrics on $\mathbb{R}^2$.* Consider the vectors $u = (1, 2)$ and $v = (4, -2)$ in $\mathbb{R}^2$. Calculate the distance $d(u, v)$ using three distinct metrics:
 - (a) The Euclidean metric (d_2).
 - (b) The taxicab metric ($d_1(u, v) = \sum |u_i - v_i|$).
 - (c) The supremum metric ($d_\infty(u, v) = \max |u_i - v_i|$).
7. *Sequence Spaces (ℓ^p vs ℓ^∞).* Consider the harmonic sequence $u = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots)$.
 - (a) Is u an element of ℓ^∞ (bounded sequences)?
 - (b) Is u an element of ℓ^1 (summable sequences)?
 - (c) Is u an element of ℓ^2 (square-summable sequences)?

(*Hint: Apply the convergence properties of the harmonic series $\sum \frac{1}{k}$ versus the series $\sum \frac{1}{k^2}$.*)
8. *Function Space Metrics.* Let $X = C[0, 1]$ be the space of continuous functions on $[0, 1]$. Evaluate the functions $f(t) = t$ and $g(t) = t^2$.
 - (a) Calculate the distance $d_\infty(f, g)$ using the uniform metric. At what coordinate t is the maximum separation achieved?
 - (b) Calculate the distance using the integral metric $d_1(f, g) = \int_0^1 |f(t) - g(t)| dt$.
9. *The Shape of the Unit Ball.* Sketch the unit ball $B(0, 1) = \{u \in \mathbb{R}^2 \mid d(u, 0) < 1\}$ in $\mathbb{R}^2$ for the following metrics:
 - (a) The Euclidean metric (ℓ^2).
 - (b) The Taxicab metric (ℓ^1).
 - (c) The Supremum metric (ℓ^∞).
10. *The Discrete Metric on Functions.* Let X be the space of all real-valued functions on $[0, 1]$. If X is equipped with the discrete metric, describe the open ball $B(f, \frac{1}{2})$ centered at the function $f(t) = \sin t$. Identify the specific functions it contains.

1.2 METRIC TOPOLOGY

Having established the axiomatic foundation of metric spaces, this section examines the topological structure induced by the metric. To analyze concepts such as convergence, continuity, and compactness, a formal framework classifies

the spatial properties of subsets. Functional analysis formalizes geometric notions through the abstraction of *metric topology*.

Topology evaluates spatial properties that remain invariant under continuous transformations. Rather than relying on quantitative distance measurements, analysis focuses on qualitative structural relationships: Is a given point enveloped by a set? Does a sequence of elements within a set approach a limit point outside of it?

The geometric construct utilized to evaluate these structural relationships is the ball. Serving as the abstract generalization of the open interval (a, b) on the real line $\mathbb{R}$, the open ball acts as the primary analytical probe for evaluating local properties. Constructing a neighborhood of radius $r > 0$ centered at a given point determines whether that point resides within the interior of a subset or upon its boundary.

Definition 1.3 Balls and Topological Sets

Let (X, d) be a metric space. For any point $x_0 \in X$ and any real number $r > 0$, the geometric neighborhoods are defined as follows:

1. The Geometry (Balls)

- The open ball $B(x_0, r)$ is the set of all points at a distance strictly less than r from the center x_0 :

$$B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$$

- The closed ball $\overline{B}(x_0, r)$ is the set of all points at a distance less than or equal to r from x_0 :

$$\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$$

2. The Topology (Sets) Let $S \subseteq X$ be an arbitrary subset.

- S is an open set if, for every point $x \in S$, there exists a radius $r > 0$ such that the open ball $B(x, r)$ is contained in S .
- S is a closed set if its complement in X , denoted $X \setminus S$ (or S^c), is an open set.

As visualized in [Figure 1.6](#), the open ball serves as the primary building block for the following topological definitions.

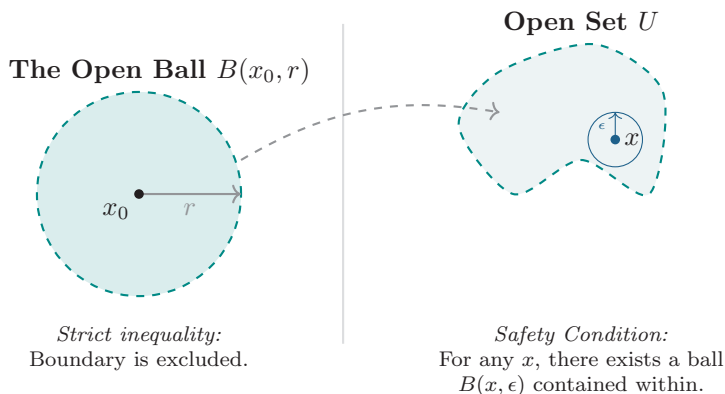


FIGURE 1.6 The geometry of openness. Left: The open ball $B(x_0, r)$. Right: An Open Set U requires that every constituent point is surrounded by an open ball contained in U .

Definition 1.4 Classification of Points and Sets

Let (X, d) be a metric space and $S \subseteq X$ be a subset. For any point $x \in X$, its relationship to the subset S is classified based on the intersection properties of its open neighborhoods $B(x, r)$:

1. Classification of Points

- **Interior Point:** There exists a radius $r > 0$ such that the open ball is contained in S .

$$B(x, r) \subseteq S$$

- **Boundary Point:** Every open ball centered at x contains at least one point in S and at least one point in its complement, $X \setminus S$.

$$\forall r > 0, \quad B(x, r) \cap S \neq \emptyset \quad \text{and} \quad B(x, r) \cap (X \setminus S) \neq \emptyset$$

- **Limit Point (Accumulation Point):** Every open ball centered at x contains a point of S distinct from x .

$$\forall r > 0, \quad (B(x, r) \setminus \{x\}) \cap S \neq \emptyset$$

2. Derived Sets Aggregating these classes of points yields the foundational derived sets:

- **The Interior (S° or $\text{int}(S)$):** The set of all interior points of S . This constitutes the largest open set contained in S .
- **The Boundary (∂S):** The set of all boundary points of S .

- The Closure ($\bar{S}$): The union of S and all its limit points. Equivalently, we define the closure as the intersection of all closed sets K containing S :

$$\bar{S} = \bigcap \{K \subseteq X \mid K \supseteq S \text{ and } K \text{ is closed}\}$$

It is the smallest closed set containing S .

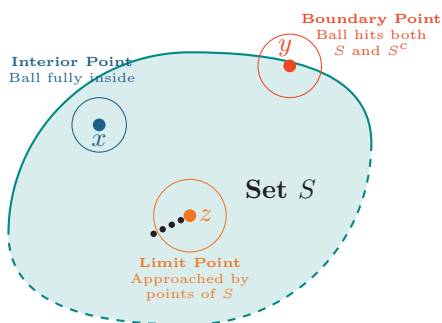


FIGURE 1.7 Classification of points. The interior point x is contained in the set, the boundary point y straddles the topological threshold, and the limit point z is the destination of a convergent sequence of elements in S .

Consider the half-open interval $M = (0, 1]$ equipped with the standard Euclidean metric in $\mathbb{R}$. The interior is $M^\circ = (0, 1)$ (the maximal open interval contained within M), its closure is $\bar{M} = [0, 1]$ (incorporating the limit point 0), and its boundary is $\partial M = \{0, 1\}$ (the topological extremities of the subset).

Because the closure operation completes a set by appending all its limit points, a set is closed if and only if it remains invariant under the closure operation.

Theorem 1.5 Closure and Set Equality

A subset M of a metric space is closed if and only if $M = \bar{M}$.

PROOF: The proof establishes mutual implication by using the definition of closure as the intersection of all closed sets containing the subset.

($\Leftarrow$) Assume that the set M equals its closure, $M = \bar{M}$. By definition, the closure of any subset is a closed set. Therefore, M itself must be closed.

($\Rightarrow$) Assume M is a closed set. By definition, every set is contained within its own closure, establishing the first inclusion: $M \subseteq \bar{M}$.

To establish equality, the reverse inclusion, $\overline{M} \subseteq M$, must be proven. The closure $\overline{M}$ is defined as the intersection of all closed sets containing M . Because M is a closed set, M is a member of this collection.

Conclusion Because the intersection of sets is contained in each set, it follows that $\overline{M} \subseteq M$. Thus, $M = \overline{M}$. ■

EXERCISES

1. *Calculating Topology.* Consider the metric space $X = \mathbb{R}$ equipped with the standard Euclidean metric. For each of the following subsets M , determine its interior M° , its closure $\overline{M}$, and its boundary ∂M .
 - (a) $M = (0, 1]$
 - (b) $M = \mathbb{Z}$ (The set of integers)
 - (c) $M = \mathbb{Q}$ (The set of rational numbers)
 - (d) $M = \{\frac{1}{n} \mid n \in \mathbb{N}\}$

(Hint: For $\mathbb{Q}$, evaluate the property that every real number can be approximated by rational numbers.)

2. *Topology of the Discrete Metric.* Let X be a non-empty set equipped with the discrete metric ($d(x, y) = 1$ if $x \neq y$).
 - (a) Describe the open ball $B(x, 0.5)$.
 - (b) Describe the open ball $B(x, 1.5)$.
 - (c) Prove that every subset of X is an open set in this topology.
 - (d) Consequently, prove that every subset is a closed set.
3. *Finite Sets Are Closed.* Applying the definition of closed sets or open sets, prove that any finite subset of a metric space is closed. (Hint: If $F = \{x_1, \dots, x_n\}$, establish that its complement F^c is open by formulating a radius r around an arbitrary point $y \in F^c$ that excludes all points in F .)
4. *Unions and Intersections.* The union of any collection of open sets is open, and the intersection of any finite collection of open sets is open. Construct a counter-example demonstrating that the infinite intersection of open sets is not open. (Hint: Evaluate the intersection of the open intervals $(-\frac{1}{n}, \frac{1}{n})$ in $\mathbb{R}$ as $n \rightarrow \infty$.)
5. *Balls and Closures.* Let $B(x_0, r)$ be an open ball in a metric space.
 - (a) Prove that the closed ball $\overline{B}(x_0, r) = \{x \mid d(x, x_0) \leq r\}$ is a closed set.
 - (b) Formulate an example of a metric space where the closure of the open ball, $\overline{B(x_0, r)}$, is unequal to the closed ball $\overline{B}(x_0, r)$.

(Hint: Evaluate the discrete metric space with radius $r = 1$. Determine the open ball, determine its closure, and compare it to the closed ball.)

1.3 METRIC SUBSPACES

In functional analysis, it is frequently necessary to focus on a specific subset of a larger metric space (for instance, evaluating the closed interval $[0, 1]$ within the real line $\mathbb{R}$). Upon restriction, the subset inherits the topological and geometric structure of the ambient (original) space, constituting a *metric subspace*. Within a subspace, the distance between any two elements is evaluated using the metric defined for the ambient space.

Definition 1.6 Metric Subspace

Let (X, d) be a metric space and $Y \subseteq X$ be a non-empty subset. The **induced metric** d_Y on Y is defined by restricting the domain of the original metric d to the Cartesian product $Y \times Y$:

$$d_Y(u, v) = d(u, v) \quad \text{for all } u, v \in Y$$

The resulting pair (Y, d_Y) is termed a **subspace** of (X, d) .

As visualized in Figure 1.8, the subspace Y operates strictly within the confines of its inherited metric d_Y . Distances involving elements exterior to Y are undefined within the subspace topology.

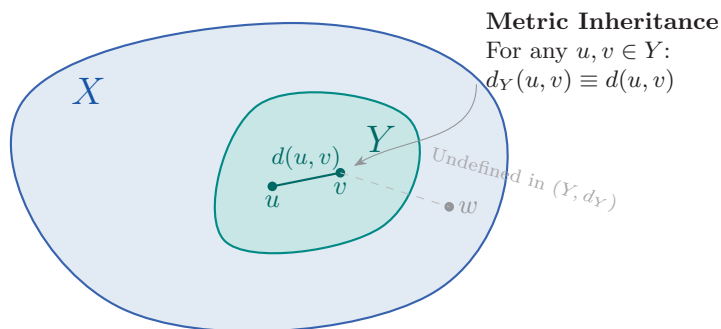


FIGURE 1.8 Visualization of a metric subspace. The subspace $Y \subseteq X$ inherently adopts the geometric structure of X . Distances incorporating points external to Y (such as w) fall outside the domain of the subspace metric d_Y .

The induced mapping d_Y satisfies the metric axioms because the original metric d guarantees these properties for all elements in the space X . A consequence of this restriction is that openness is relative to the space under consideration. A subset $A \subseteq Y$ is open relative to X if, for every element

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$u \in A$, there exists an open ball $B_X(u, r)$ contained entirely in A . It is *open relative* to Y if there exists an open ball $B_Y(u, r)$ contained entirely in A .

An open ball in the subspace Y is the intersection of an open ball from X with the subset Y :

$$B_Y(u, r) = \{v \in Y \mid d(u, v) < r\} = B_X(u, r) \cap Y$$

This relationship establishes the characterization of relative topology.

Theorem 1.7 Relative Open Sets

A subset $A \subseteq Y$ is open within the metric subspace (Y, d_Y) if and only if there exists a subset G , which is open in the space (X, d) such that:

$$A = Y \cap G$$

PROOF:

Subspace Openness Implies Intersection Assume that A is an open subset of the metric subspace (Y, d_Y) . By definition, for every point $y \in A$, there exists $r_y > 0$ such that $B_Y(y, r_y)$ is in A . Consequently,

$$A = \bigcup_{y \in A} B_Y(y, r_y)$$

Substituting the intersection identity:

$$A = \bigcup_{y \in A} (B_X(y, r_y) \cap Y)$$

Applying the distributive property of set operations:

$$A = Y \cap \left(\bigcup_{y \in A} B_X(y, r_y) \right)$$

Define the set $G = \bigcup_{y \in A} B_X(y, r_y)$. Because G is a union of open balls in X , G is an open set in X . Therefore, an open set $G \subseteq X$ is constructed satisfying $A = Y \cap G$.

Intersection Implies Subspace Openness Assume there exists a subset G that is open in (X, d) such that $A = Y \cap G$. It must be shown that A is in (Y, d_Y) .

Let y be an arbitrary point in A . By the assumption, $y \in Y$ and $y \in G$. Because G is an open set in X , there exists $r > 0$ such that $B_X(y, r) \subseteq G$. Taking the intersection of both sides with Y yields

$$B_X(y, r) \cap Y \subseteq G \cap Y$$

The left side defines a subspace open ball, and the right side equates to the subset A , yielding

$$B_Y(y, r) \subseteq A$$

Thus, the subset A is open in (Y, d_Y) . ■

EXERCISES

1. *Relative Open Sets.* Let $X = \mathbb{R}$ be equipped with the standard metric. Let $Y = [0, 2]$ be a subspace of X . Determine whether the set $A = [0, 1)$ is open, closed, or neither:

- (a) When considered as a subset of X .
- (b) When considered as a subset of Y .

(Hint: For part (b), express A as the intersection of Y with an open interval of $\mathbb{R}$.)

2. *Relative Closed Sets.* A set $A \subset Y$ is open in Y if $A = Y \cap G$ for some open $G \subset X$. Prove the analogous theorem for closed sets: A set $K \subset Y$ is closed in the subspace Y if and only if there exists a set F which is closed in X such that:

$$K = Y \cap F$$

3. *The Discrete Subspace.* Let $X = \mathbb{R}$ and let the subspace be the set of integers $Y = \mathbb{Z}$. Prove that in the subspace topology of Y , the singleton set $\{0\}$ is an open set. Consequently, prove that every subset of Y is open in Y (i.e., Y induces the discrete topology), although Y inherits its distance from the continuous line $\mathbb{R}$.
4. *Subspace of a Subspace.* Let $Z \subset Y \subset X$ be metric spaces. Prove the transitivity of open sets: If A is open in the subspace Z , and Z is open in the subspace Y , and Y is open in the full space X , then A is open in X .
5. *Boundedness Is Absolute.* A subset M is bounded if it is contained in an open ball $B(x_0, r)$. Let Y be a subspace of X and let $M \subset Y$. Prove that M is bounded in Y if and only if M is bounded in X . (Note: Unlike the topological property of openness, geometric boundedness does not depend on the ambient space.)

1.4 CONTINUOUS MAPPINGS

In classical analysis, the continuity of a function is associated with the uninterrupted trajectory of its graph. In the framework of metric spaces, this concept is formalized through the topological lenses of limits and open sets. The classical ϵ - δ formulation adapts to arbitrary metric spaces by replacing the one-dimensional absolute value with the metric functions of the abstract spaces.

Definition 1.8 Continuous Mapping

Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $T : X \rightarrow Y$ is continuous at a point $x_0 \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such

that

$$d_X(x, x_0) < \delta \implies d_Y(Tx, Tx_0) < \epsilon$$

The mapping T is continuous on X if it is continuous at every point in X .

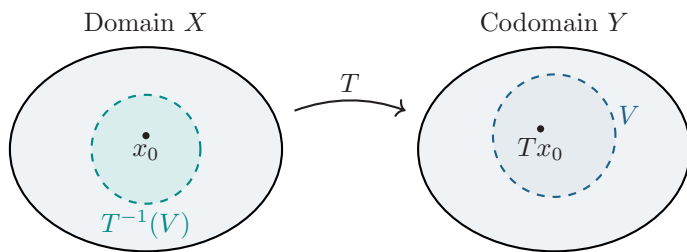
While the ϵ - δ definition provides a computational criterion for verifying continuity at a localized point, its application is pointwise. Verifying this condition across an infinite-dimensional space is frequently complex. Furthermore, relying exclusively on metric inequalities can obscure the topological relationship between the spaces.

In functional analysis, a global perspective characterizes continuity directly through the topology of the space, specifically via the behavior of its open sets. The subsequent theorem establishes this connection. As visualized in Figure 1.9, continuous functions are characterized by how they pull back the topological structure of the codomain into the domain.

Theorem 1.9 Topological Continuity

A mapping $T : X \rightarrow Y$ is continuous on X if and only if the inverse image of every open set in Y is an open set in X :

$$V \text{ open in } Y \implies T^{-1}(V) \text{ open in } X$$



Topological Continuity

If an arbitrary target V is open,
its pre-image $T^{-1}(V)$ must be open.

FIGURE 1.9 Visualization of topological continuity. The continuous mapping T pulls open sets from the codomain back to open sets in the domain.

PROOF: The proof establishes mutual implication by bridging the local ϵ - δ metric definition with the global open-set topological definition.

Continuity Implies Topological Pullback ($\Rightarrow$) Assume T is continuous on X , and let V be an arbitrary open set in the codomain Y . It must be shown that the pre-image $T^{-1}(V)$ is open in X . Let x_0 be any point in $T^{-1}(V)$ which means that $Tx_0 \in V$. Because V is open, there exists $\epsilon > 0$ such that $B_Y(Tx_0, \epsilon)$ is contained in V :

$$B_Y(Tx_0, \epsilon) \subseteq V$$

By continuity at x_0 , there exists $\delta > 0$ such that for all $x \in X$

$$d_X(x, x_0) < \delta \implies d_Y(Tx, Tx_0) < \epsilon$$

This implies that $T(B_X(x_0, \delta)) \subseteq B_Y(Tx_0, \epsilon) \subseteq V$. Taking the inverse image of this relation yields

$$B_X(x_0, \delta) \subseteq T^{-1}(V)$$

This proves x_0 is an interior point of $T^{-1}(V)$. Thus, $T^{-1}(V)$ is an open set.

Topological Pullback Implies Continuity ($\Leftarrow$) Assume that $T^{-1}(V)$ is open in X for every open subset $V \subseteq Y$. It must be proven that T is continuous at an arbitrary point $x_0 \in X$. Let $\epsilon > 0$ be given. Consider the open ball $V = B_Y(Tx_0, \epsilon)$ in the codomain Y .

By assumption, the pre-image $U = T^{-1}(V)$ is open in X . Also $x_0 \in U$ since Tx_0 is the center of V . Because U is an open set, there exists $\delta > 0$ such that the open ball $B_X(x_0, \delta)$ is contained in U . Thus:

$$x \in T^{-1}(V) \implies Tx \in V \implies d_Y(Tx, Tx_0) < \epsilon$$

This satisfies the criteria of the ϵ - δ definition of continuity at x_0 . ■

This topological characterization possesses a dual formulation involving closed sets. Because open and closed sets operate as exact complements, and the set-theoretic inverse image operation preserves complements (i.e., $T^{-1}(Y \setminus S) = X \setminus T^{-1}(S)$), the theorem can be restated as follows: a mapping T is continuous if and only if the inverse image of every closed set in Y constitutes a closed set in X .

EXERCISES

1. *Closed Set Characterization.* Prove: A mapping $T : X \rightarrow Y$ is continuous on X if and only if the inverse image $T^{-1}(F)$ is a closed set in X for every closed set F in Y . (*Hint: Utilize the set-theoretic identity $T^{-1}(Y \setminus F) = X \setminus T^{-1}(F)$ and the definition of closed sets as complements of open sets.*)
2. *Continuity of the Distance Function.* Let (X, d) be a metric space and fix a point $a \in X$. Define the mapping $f : X \rightarrow \mathbb{R}$ by $f(x) = d(x, a)$. Prove that f is a continuous mapping. (*Hint: Apply the Reverse Triangle Inequality to establish an upper bound for $|f(x) - f(y)|$ and use the ϵ - δ definition of continuity.*)

3. *Lipschitz Continuity.* A mapping $T : X \rightarrow Y$ is called Lipschitz continuous if there exists a constant $L > 0$ such that for all $x_1, x_2 \in X$:

$$d_Y(Tx_1, Tx_2) \leq L \cdot d_X(x_1, x_2)$$

- (a) Prove that every Lipschitz continuous mapping is continuous on X .
 - (b) Provide an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is continuous but fails to be Lipschitz continuous. (*Hint: Consider the function $f(x) = \sqrt{x}$ on the interval $[0, 1]$.*)
4. *Composition of Continuous Mappings.* Let X, Y , and Z be metric spaces. Let $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ be continuous mappings. Prove that the composition $ST : X \rightarrow Z$ is a continuous mapping.
5. *Isometries and Topological Equivalence.* Let $T : X \rightarrow Y$ be an isometry between two metric spaces.
- (a) Prove that T is uniformly continuous on X .
 - (b) If T is bijective, prove that the inverse mapping $T^{-1} : Y \rightarrow X$ is also continuous.
 - (c) Conclude that a bijective isometry is a homeomorphism (a continuous bijection with a continuous inverse), ensuring that X and Y are topologically indistinguishable.

Convergence and Completeness

2.1 Convergence

2.2 Separability

2.3 Cauchy Sequences

2.4 Completeness

2.5 Completeness Proofs

ANALYSIS is the study of limits. While the previous chapter defined the static structure of metric spaces, this chapter explores the dynamic behavior of sequences within them. We begin by formalizing convergence, extending our geometric intuition of "getting closer" to abstract environments where points may represent continuous functions or infinite data streams. We will investigate the fundamental concept of **completeness**—the structural guarantee that a space possesses no "topological gaps" and that every sequence whose terms cluster together (a Cauchy sequence) actually reaches a destination within the space. Along the way, we introduce **separability**, identifying spaces that can be efficiently approximated by countable subsets. These concepts form the rigorous backbone required to ensure that mathematical approximations lead to actual, existing solutions.

2.1 CONVERGENCE

Functional analysis relies upon the theory of approximation. Because exact solutions to complex differential equations or continuous optimization problems are frequently unattainable, a sequence of approximate solutions, denoted $(x_n)_{n=1}^{\infty}$, is constructed to converge toward the exact solution. This necessitates generalizing the classical definition of limits from real analysis to the abstract framework of metric spaces.

Definition 2.1 Convergence

A sequence (x_n) in a metric space (X, d) is said to **converge** to a limit $x \in X$ if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0$$

Symbolically, this convergence is denoted as $x_n \rightarrow x$ or $\lim x_n = x$.

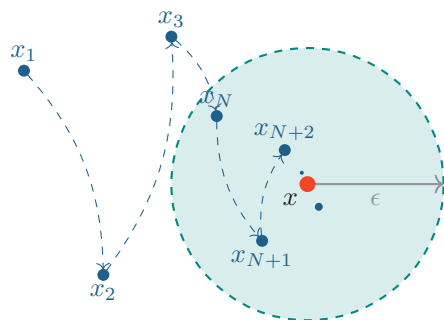


FIGURE 2.1 Geometric visualization of metric convergence. For any prescribed error tolerance $\epsilon > 0$, an index N exists such that all subsequent terms x_n (where $n \geq N$) are contained within the open ball $B(x, \epsilon)$.

As visualized in Figure 2.1, this definition states that for any small error tolerance $\epsilon > 0$, an index N exists in the sequence beyond which all subsequent terms reside in an open ball of radius ϵ centered at the limit x . A fundamental prerequisite for convergence theory is that a sequence cannot simultaneously converge to multiple distinct destinations.

Theorem 2.2 Uniqueness of Limits

If a sequence (x_n) converges in a metric space, its limit must be unique.

PROOF: The proof proceeds by contradiction, leveraging the metric axioms—specifically the triangle inequality—to demonstrate that assuming distinct limits yields a logical impossibility.

Assumption of Distinct Limits Assume for contradiction that a sequence (x_n) converges to two distinct limits, x and y . Because $x \neq y$, the distance between them must be positive. Let $d(x, y) = \delta > 0$.

Application of Convergence Criteria By the definition of convergence, for any $\epsilon > 0$, there exist indices N_1 and N_2 such that $d(x_n, x) < \epsilon$ for all $n > N_1$ and $d(x_n, y) < \epsilon$ for all $n > N_2$. By selecting $N = \max(N_1, N_2)$, both conditions hold for all $n > N$. Choose $\epsilon = \frac{\delta}{2}$.

The Triangle Inequality Contradiction Applying the triangle inequality:

$$d(x, y) \leq d(x, x_n) + d(x_n, y)$$

Substituting the established bounds yields

$$d(x, y) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

which is a contradiction. Therefore, the limit must be unique. ■

EXERCISES

1. *Convergence in the Discrete Metric.* Let (X, d) be a metric space equipped with the discrete metric. Prove that a sequence (x_n) in X converges to x if and only if the sequence is *eventually constant*; demonstrate that there exists an integer N such that $x_n = x$ for all $n \geq N$. (*Hint: Evaluate the definition of convergence for the error tolerance $\epsilon = 0.5$.*)
2. *Coordinate-wise Convergence.* Let $X = \mathbb{R}^2$ equipped with the standard Euclidean metric. Let $u_n = (\xi_n, \eta_n)$ be a sequence of vectors. Prove that $u_n \rightarrow u = (\xi, \eta)$ in X if and only if the respective scalar sequences converge: $\xi_n \rightarrow \xi$ and $\eta_n \rightarrow \eta$ in $\mathbb{R}$.
3. *Uniqueness of Limits in Pseudo-metrics.* Theorem 2.2 establishes the uniqueness of limits in a valid metric space. Consider a pseudo-metric space where the identity of indiscernibles axiom ($d(x, y) = 0 \iff x = y$) is relaxed to require only $d(x, x) = 0$. Construct a counterexample demonstrating that in a pseudo-metric space, a sequence can converge to multiple distinct limits simultaneously. (*Hint: Consider $X = \mathbb{R}^2$ equipped with the pseudo-metric $d(u, v) = |\xi_1 - \eta_1|$, which measures horizontal deviation.*)
4. *Boundedness of Convergent Sequences.* A subset M of a metric space is defined as *bounded* if it is contained in an open ball $B(x_0, r)$. Prove that if a sequence (x_n) converges to a limit x , then the underlying set of its terms $\{x_n \mid n \in \mathbb{N}\}$ is a bounded set. (*Hint: Apply the formal definition of convergence for $\epsilon = 1$ to bound the infinite tail of the sequence, and evaluate the finite collection of preceding terms.*)

5. *Topological Dependence of Convergence.* Consider the sequence $x_n = \frac{1}{n}$.
- Determine if this sequence converges in the metric space $X = \mathbb{R}$.
 - Determine if this sequence converges in the metric space $Y = (0, 1)$ equipped with the standard subspace metric.

Explain why this distinction requires explicitly specifying the ambient topological space when evaluating sequence convergence.

6. *Convergence of Functions: Pointwise vs. Uniform.* Let $X = C[0, 1]$ denote the space of continuous real-valued functions on the unit interval. Consider the sequence of functions $f_n(t) = t^n$.
- For each fixed $t \in [0, 1]$, determine the pointwise limit $\lim_{n \rightarrow \infty} f_n(t)$. Let this pointwise limit be denoted as the function $f(t)$.
 - Evaluate if the pointwise limit function $f(t)$ is continuous on $[0, 1]$.
 - Determine if the sequence (f_n) converges to f within the metric space (X, d_∞) , where $d_\infty(g, h) = \max_{t \in [0, 1]} |g(t) - h(t)|$.
 - Determine if the sequence (f_n) converges to f within the metric space (X, d_1) , where $d_1(g, h) = \int_0^1 |g(t) - h(t)| dt$.

(This exercise underscores how the limit of a functional sequence is contingent upon the specific metric imposed upon the space.)

7. *Coordinate-wise Convergence in Sequence Spaces.* Let s denote the space of all real sequences. Let d be the Fréchet metric defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

Consider a sequence of sequences denoted $x^{(n)}$, where $x^{(n)} = (\xi_1^{(n)}, \xi_2^{(n)}, \dots)$. Prove that $x^{(n)} \rightarrow x$ in (s, d) if and only if for every fixed coordinate index j , the scalar sequence $\xi_j^{(n)} \rightarrow \xi_j$ as $n \rightarrow \infty$.

8. *Distance to a Set.* Let M be a non-empty subset of a metric space (X, d) . For any point $x \in X$, the distance from x to M is defined as $\text{dist}(x, M) = \inf_{y \in M} d(x, y)$.
- Prove that if a sequence (x_n) in M converges to a limit $x \in X$, then $\text{dist}(x, M) = 0$.
 - Prove the converse: if $\text{dist}(x, M) = 0$, there exists a sequence (x_n) in M such that $x_n \rightarrow x$.
9. *Continuity of the Metric Mapping.* Let (X, d) be a metric space.
- Let (x_n) and (y_n) be two convergent sequences such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Prove that the sequence $d(x_n, y_n)$ converges to $d(x, y)$.

- (b) Provide a counterexample in $\mathbb{R}$ demonstrating that strict inequalities are not preserved under metric limits. Identify sequences where $x_n < y_n$ for all $n \in \mathbb{N}$, yet the respective limits satisfy $x = y$.
10. *Alternating Sequences and Metric Transformation.* Let $X = \mathbb{R}$ equipped with the standard Euclidean metric. Consider the alternating sequence $x_n = (-1)^n$.
- (a) Prove that (x_n) fails to converge under the standard metric.
- (b) Construct an alternative metric d' on $\mathbb{R}$ under which this alternating sequence converges to 1.

(Hint: Construct a metric identifying 1 and -1 geometrically, such as mapping via a quadratic function and evaluating their separation. Verify that the proposed function satisfies the metric axioms.)

2.2 SEPARABILITY

In applied mathematics and numerical analysis, spaces such as $\mathbb{R}$ or $C[a, b]$ are uncountable. Computational frameworks are restricted to finite or countable discrete data structures, such as rational numbers or algebraic polynomials. This operational constraint necessitates the concept of density. A subset M is dense in X if element in X can be approximated to any precision using elements from M .

Definition 2.3 Dense Set

A subset M of a metric space X is said to be dense in X if its closure is the entire space:

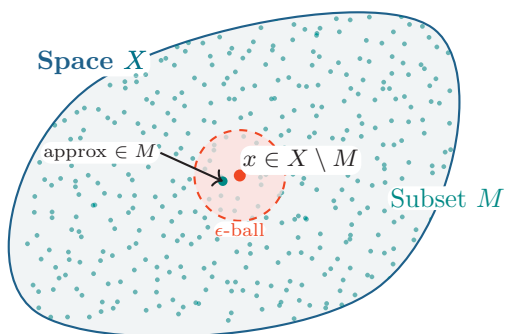
$$\overline{M} = X$$

Equivalently, M is dense if for every $x \in X$ and every $\epsilon > 0$, there exists a point $y \in M$ distinct from x such that $d(x, y) < \epsilon$.

As illustrated in Figure 2.2, a dense set permeates the whole space, ensuring that every open neighborhood contains at least one of its elements. For instance, the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ since any real number can be approximated using a sequence of rational numbers. Similarly, the set of polynomial functions is dense in $C[a, b]$ since any continuous function can be uniformly approximated by a polynomial sequence.¹

While density ensures that a subset approximates the entire space, analysis requires that this approximating set be countable. The synthesis of density and countability is formalized by the concept of separability.

¹This is guaranteed by the *Weierstrass Approximation Theorem*



Visualizing Density

The set M (teal dots) is dense because any arbitrary point x in the space falls within the ϵ -neighborhood of an element in M .

FIGURE 2.2 A dense set M approximates any element of the full space X with arbitrary metric precision.

Definition 2.4 Separable Space

A metric space X is said to be **separable** if it contains a countable subset that is dense in X .

The real line $\mathbb{R}$ is separable because the countable set of rational numbers $\mathbb{Q}$ is dense within it. Similarly, the sequence spaces ℓ^p (for $1 \leq p < \infty$) are separable because the set of sequences with finitely many non-zero rational terms is both countable and dense. The space of bounded sequences ℓ^∞ , however, is non-separable as will be proven in the next theorem.

Theorem 2.5 Inseparability of ℓ^∞

The space ℓ^∞ of bounded sequences is not separable. It contains no countable subset that is dense within the supremum metric topology.

PROOF: We proceed by contradiction. The methodology constructs an uncountable discrete subset within ℓ^∞ where all elements are separated by a constant distance.

Constructing an Uncountable Discrete Subset Consider the subset $M \subset \ell^\infty$ consisting of all sequences wherein every component is 0 or 1:

$$M = \{u = (\xi_1, \xi_2, \dots) \mid \xi_k \in \{0, 1\} \text{ for all } k \in \mathbb{N}\}$$

Every sequence in M is bounded by 1, confirming $M \subset \ell^\infty$. A bijection exists between the elements of M and the binary representations of real numbers in the interval $[0, 1]$. Because the interval $[0, 1]$ is uncountable, the subset M is uncountable.

Evaluating the Metric Distance Let u and v be two distinct elements of M . By definition, they must differ in at least one coordinate index k . Because the components are binary, the absolute difference at this coordinate is exactly $|u_k - v_k| = 1$. The distance under the supremum metric evaluates to

$$d(u, v) = \sup_{k \in \mathbb{N}} |u_k - v_k| = 1 \quad \text{for all } u \neq v \in M$$

This implies that open balls of radius $1/2$ centered at every point in M are mutually disjoint.

Contradiction with the Separability Hypothesis Assume for contradiction that ℓ^∞ is separable. Under this assumption, there exists a countable dense subset $D = \{w_1, w_2, \dots\} \subset \ell^\infty$. Because D is dense in ℓ^∞ , every open ball $B(u, 1/2)$ centered at $u \in M$ must contain at least one element $w_n \in D$. Let a mapping be defined that assigns each $u \in M$ to a corresponding $w_n \in D$ contained in this $1/2$ -neighborhood.

Conclusion This mapping is injective. Suppose distinct elements $u, v \in M$ map to the identical sequence $w_n \in D$. Applying the triangle inequality

$$d(u, v) \leq d(u, w_n) + d(w_n, v) < \frac{1}{2} + \frac{1}{2} = 1$$

which is a contradiction. Thus, the mapping is injective. An injective mapping from an uncountable set M into a countable set D is an impossibility. We conclude that ℓ^∞ is not separable. ■

EXERCISES

1. *Density of Rationals in $\mathbb{R}^2$.* The set $\mathbb{Q}$ is dense in $\mathbb{R}$. Formally prove that the Cartesian product $\mathbb{Q}^2 = \{(q_1, q_2) \mid q_1, q_2 \in \mathbb{Q}\}$ is dense in the Euclidean plane $\mathbb{R}^2$. Demonstrate that for any vector $u = (x, y) \in \mathbb{R}^2$ and any $\epsilon > 0$, there exists $v \in \mathbb{Q}^2$ such that $d(u, v) < \epsilon$.
2. *Separability of Discrete Spaces.* Let X be an uncountable set equipped with the discrete metric. Prove that X is not separable. (*Hint: Evaluate the density requirement for subsets under the discrete topology.*)
3. *Separability of Subspaces.* Any subspace of a separable metric space is separable. Let $X = \mathbb{R}$ and consider its discrete subspace $Y = \mathbb{Z}$. Identify a countable dense subset of Y to verify its separability.
4. *Continuous Maps Preserve Separability.* Let X be a separable metric space and $T : X \rightarrow Y$ be a continuous surjective mapping onto a metric

space Y . Prove that the image space Y is separable. (*Hint: If D is a countable dense subset of X , analyze the topological properties of the image set $T(D)$ within Y .*)

5. *Nowhere Dense Sets.* A subset $A \subset X$ is defined as *nowhere dense* if its topological closure $\bar{A}$ possesses an empty interior ($(\bar{A})^\circ = \emptyset$).
 - (a) Prove that the set of integers $\mathbb{Z}$ is nowhere dense within $\mathbb{R}$.
 - (b) Prove that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.
6. *Separability of $C[a, b]$.* Let $X = C[a, b]$ be equipped with the uniform metric. The Weierstrass Approximation Theorem establishes that the set of all polynomials with rational coefficients, denoted $\mathcal{P}_{\mathbb{Q}}$, is dense within $C[a, b]$.
 - (a) Demonstrate that the set $\mathcal{P}_{\mathbb{Q}}$ is countable.
 - (b) Deduce that $C[a, b]$ constitutes a separable metric space.
 - (c) Determine whether $C[a, b]$ retains separability if equipped with the integral metric $d(f, g) = \int_a^b |f(t) - g(t)| dt$.
7. *The Space of Bounded Functions.* Let $B[0, 1]$ denote the space of all bounded functions $f : [0, 1] \rightarrow \mathbb{R}$, equipped with the supremum metric $d(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|$. Prove that $B[0, 1]$ is not separable. (*Hint: Construct an uncountable family of characteristic functions $f_s(t) = 1$ if $t = s$ and 0 otherwise, for $s \in [0, 1]$. Evaluate the uniform metric distance $d(f_s, f_r)$ for distinct parameters $s \neq r$.*)
8. *Separability and Bases.* Let X be a normed vector space. A sequence of vectors (e_n) is a Schauder Basis for X if every vector $x \in X$ admits a unique representation as a convergent infinite series $x = \sum_{n=1}^{\infty} \alpha_n e_n$. Prove that the existence of a Schauder Basis guarantees the space X is separable. (*Hint: Construct a dense subset by considering the collection of all finite linear combinations of the basis vectors restricted to rational scalar coefficients.*)
9. *Separability of Product Spaces.* Let (X, d_X) and (Y, d_Y) be separable metric spaces. Prove that the Cartesian product space $Z = X \times Y$, equipped with the sum metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$, is a separable metric space. (*Hint: Evaluate the density of the product set $D_X \times D_Y$, where D_X and D_Y are countable dense subsets.*)
10. *Isometry and Topological Invariance.* Let X and Y be metric spaces. Assume there exists a surjective isometry $T : X \rightarrow Y$. Prove that X is separable if and only if Y is separable.

2.3 CAUCHY SEQUENCES

Having formalized the metric definition of limits, the analytical criteria required to evaluate sequence convergence intrinsically are introduced. This investigation

culminates in the topological property of **completeness**. Before formalizing completeness, a bridge must be established between the static geometry of closed sets and the dynamic behavior of sequences within a metric space.

Theorem 2.6 Sequences and Closure

Let M be a non-empty subset of a metric space (X, d) . A point x belongs to the closure $\overline{M}$ if and only if there exists a sequence (x_n) in M that converges to x .

PROOF: The proof establishes mutual implication by linking the static definition of accumulation points to the sequential construction of limits.

Closure Implies Sequential Convergence Assume $x \in \overline{M}$. A sequence in M converging to x must be constructed. If $x \in M$, the proposition is satisfied by utilizing the limit constant sequence $x_n = x$ for all $n \in \mathbb{N}$. If $x \notin M$, then x must be an accumulation point of M . Every open ball centered at x contains at least one point from M distinct from x . For each integer $n \geq 1$, consider the open ball $B(x, \frac{1}{n})$. Select a point $x_n \in M$ from this neighborhood. Since $d(x_n, x) < \frac{1}{n}$, the distance approaches zero as $n \rightarrow \infty$. Thus, $x_n \rightarrow x$.

Sequential Convergence Implies Closure Conversely, assume there exists a sequence (x_n) in M such that $x_n \rightarrow x$. It must be shown that $x \in \overline{M}$. If $x \in M$, limit the conclusion is immediate. If $x \notin M$, it suffices to show that x is an accumulation point. Consider any $r > 0$. By the definition of convergence, there exists an index N such that for all $n > N$, $d(x_n, x) < r$. Consequently, every neighborhood of x contains points from M and therefore $x \in \overline{M}$. ■

In applied analysis, the limit of a sequence is frequently unknown *a priori*. Consequently, it is necessary to establish a criterion for convergence that relies on the distances between terms, independent of the limit point.

Definition 2.7 Cauchy Sequence

A sequence (x_n) in a metric space (X, d) is a **Cauchy sequence** if, for every $\epsilon > 0$, there exists an integer N such that

$$d(x_n, x_m) < \epsilon \quad \text{for all } n, m > N$$

As differentiated in [Figure 2.3](#), a convergent sequence requires elements to accumulate around a limit, whereas in a Cauchy sequence the elements eventually cluster close to one another.

The necessary condition linking convergence to the Cauchy property is established next. If a sequence approaches a defined limit within a metric space, its terms must satisfy this internal clustering criterion.

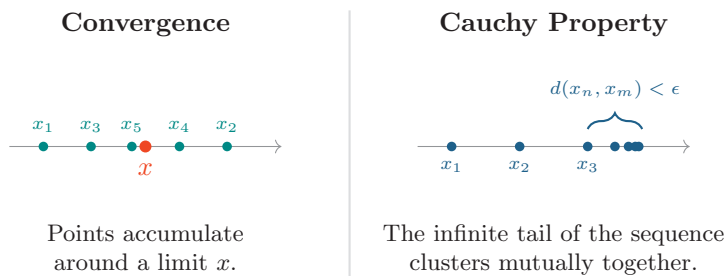


FIGURE 2.3 Geometric distinction between convergent and Cauchy sequences. A convergent sequence approaches a specific point, whereas a Cauchy sequence requires mutual proximity.

Theorem 2.8 Convergence Implies Cauchy

Every convergent sequence within a metric space is necessarily a Cauchy sequence.

PROOF: The proof leverages the triangle inequality to bound the mutual distance between sequence terms.

Applying the Convergence Criterion Let (x_n) be a sequence in a metric space (X, d) that converges to a limit $x \in X$. Fix $\epsilon > 0$. Since $x_n \rightarrow x$, there exists an integer N such that for all $n > N$

$$d(x_n, x) < \frac{\epsilon}{2}$$

Bounding the Mutual Distance Consider any two arbitrary indices $n, m > N$. By applying the triangle inequality:

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$

Conclusion Substituting the convergence bound yields:

$$d(x_n, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since this holds for all $n, m > N$, the sequence (x_n) must be a Cauchy sequence. ■

EXERCISES

1. *Boundedness of Cauchy Sequences.* Every convergent sequence is bounded. Prove that every Cauchy sequence in a metric space (X, d) is bounded,

independent of the completeness of the underlying space. (*Hint: Apply the definition of a Cauchy sequence for $\epsilon = 1$, to bound the tail of the sequence, then evaluate the finite collection of preceding terms alongside the bounded tail.*)

2. *Cauchy without Convergence.* Consider the metric space $X = (0, 1]$ equipped with the standard Euclidean metric $d(x, y) = |x - y|$. Let the sequence be defined by $x_n = \frac{1}{n}$.
 - (a) Verify that (x_n) constitutes a Cauchy sequence within X .
 - (b) Demonstrate why (x_n) fails to converge to any valid limit $x \in X$.
3. *Cauchy Sequences in the Discrete Metric.* Let (X, d) be a metric space equipped with the discrete metric. Prove that a sequence (x_n) is Cauchy if and only if it is eventually constant; demonstrate there exists an integer N such that $x_n = x_m$ for all indices $n, m \geq N$.
4. *Convergence of Metric Distances.* Let (x_n) and (y_n) be two independent Cauchy sequences within a metric space (X, d) . Prove that the sequence of real numbers defined by $a_n = d(x_n, y_n)$ converges in $\mathbb{R}$. (*Hint: To establish that (a_n) is a Cauchy sequence in $\mathbb{R}$, utilize repeated applications of the triangle inequality to demonstrate that $|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$.*)
5. *The Integers as a Metric Space.* Let $X = \mathbb{Z}$ denote the set of integers equipped with the standard metric $d(m, n) = |m - n|$. Determine the classification of all Cauchy sequences within X . Based upon this classification, deduce whether the metric space $\mathbb{Z}$ is complete.
6. *Product of Cauchy Sequences.* Let $X = \mathbb{R}$ be equipped with the standard metric. Let (x_n) and (y_n) be two Cauchy sequences in X . Prove that their term-wise product sequence $(x_n y_n)$ is also a Cauchy sequence. (*Hint: Express the difference using the algebraic identity $|x_n y_n - x_m y_m| = |x_n y_n - x_n y_m + x_n y_m - x_m y_m|$ and apply the triangle inequality.*)
7. *Equivalent Cauchy Sequences.* Two Cauchy sequences (x_n) and (y_n) in a metric space (X, d) are defined as equivalent (denoted $(x_n) \sim (y_n)$) if $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.
 - (a) Prove that $\sim$ is a formal equivalence relation.
 - (b) Prove that if $x_n \rightarrow x$ and the sequences are equivalent $((x_n) \sim (y_n))$, then $y_n \rightarrow x$.
8. *Subsequences of Cauchy Sequences.* Let (x_n) be a Cauchy sequence in a metric space X . Suppose that a specific subsequence (x_{n_k}) converges to a limit $x \in X$. Prove that the entirety of the sequence (x_n) must also converge to x .
9. *Contractive Sequences.* A sequence (x_n) is defined as contractive if there exists a constant $k \in (0, 1)$ such that the metric distance satisfies

$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$ for all $n \geq 2$. Prove that every contractive sequence is a Cauchy sequence. (*Hint: Utilize the sum of a geometric series to establish a strict upper bound for $d(x_n, x_m)$.*)

10. *Topological Preservation of the Cauchy Property.* Let X and Y be metric spaces, and let $f : X \rightarrow Y$ be a mapping between them.
- (a) Prove that if f is uniformly continuous, it maps Cauchy sequences in X to Cauchy sequences in Y .
 - (b) Construct a counterexample demonstrating that a mapping f which is continuous (but not uniformly continuous) can map a Cauchy sequence to a non-Cauchy sequence. (*Hint: Consider the mapping $f(x) = \frac{1}{x}$ defined on the open interval $(0, 1)$.*)

2.4 COMPLETENESS

In the set of real numbers $\mathbb{R}$, every Cauchy sequence converges. However, this property is not shared among all metric spaces. Consider the space of rational numbers $\mathbb{Q}$ equipped with the standard metric. The sequence $(1, 1.4, 1.41, 1.414, \dots)$ consists entirely of rational elements. It satisfies the Cauchy criterion but it converges to the irrational limit $\sqrt{2}$ exterior to $\mathbb{Q}$. This sequence diverges because it lacks a limit point in $\mathbb{Q}$.

Definition 2.9 Complete Metric Space

A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges to a limit in X .

Completeness is an essential prerequisite for solving functional equations. It guarantees that if an approximation sequence converges, a valid mathematical solution exists in the prescribed topological space.

However, even when operating in a complete space, specific sub-domains may fail to inherit this property. For instance, the open interval $(0, 1)$ is incomplete because a Cauchy sequence can approach the boundary endpoints.

Before addressing the systematic completion of spaces, the precise condition determining which subspaces preserve completeness must be established. In a complete metric space, the property of subspace completeness is equivalent to being a closed set.

Theorem 2.10 Subspace Completeness

Let X be a complete metric space and let $Y \subset X$ be a subspace. The subspace Y is complete if and only if Y is a closed set in X .

PROOF: The proof establishes both necessity and sufficiency by linking Cauchy convergence to the topological properties of closed sets.

Complete Implies Closed Assume that Y is a complete subspace. It must be shown that $Y = \bar{Y}$. Let y be a limit point in $\bar{Y}$. By Theorem 2.6, there exists a sequence (y_n) in Y such that $y_n \rightarrow y$ in X . This implies (y_n) is a Cauchy sequence in Y . Since Y is complete, this sequence must converge to a limit in Y . By uniqueness of limits, this limit must be also y . Therefore, $y \in Y$. Thus, $\bar{Y} \subset Y$, proving that Y is closed.

Closed Implies Complete This direction is trivial and is left to the reader as an exercise. ■

Because spaces such as $\mathbb{Q}$ or $P[a, b]$ are incomplete, the objective is to embed such an incomplete space X into a complete space $\hat{X}$. This topological process is called *completion*. This necessitates the concept of isometry which is a bijection between spaces that preserves distances.

Definition 2.11 Isometry

Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $T : X \rightarrow Y$ is called an **isometry** if it preserves metric distances:

$$d_Y(Tx, Ty) = d_X(x, y) \quad \text{for all } x, y \in X$$

The spaces X and Y are termed **isometric** if there exists a bijective isometry between them.

Isometric spaces are indistinguishable from the perspective of metric geometry—they differ exclusively in the set-theoretic nature of their elements, not in their topological structure. Every metric space admits a completion, and this completion is unique up to isometry.

Theorem 2.12 Completion Theorem

For any metric space (X, d) , there exists a complete metric space $(\hat{X}, \hat{d})$ which contains a subspace W that is:

1. Isometric to X .
2. Dense in $\hat{X}$.

Furthermore, this complete space $\hat{X}$ is unique up to isometry.

PROOF: The construction of $\hat{X}$ relies on the property that the missing limit points of an incomplete metric space can be represented by Cauchy sequences.

Construction of $\hat{X}$ via Equivalence Classes Let $\mathcal{C}$ denote the set of all Cauchy sequences in X . Distinct sequences may converge to the same limit; therefore, $\mathcal{C}$ is partitioned into equivalence classes. An equivalence relation $\sim$ is defined on $\mathcal{C}$: two Cauchy sequences $x = (x_n)$ and $y = (y_n)$ are equivalent ($x \sim y$) if

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$$

The space $\hat{X}$ is defined as the set of all such equivalence classes. Let $\hat{x} \in \hat{X}$ denote the class containing the representative sequence (x_n) .

Defining the Metric on $\hat{X}$ The metric function $\hat{d}$ between two equivalence classes $\hat{x}, \hat{y} \in \hat{X}$ is defined as the limit of the distances between their respective sequence elements:

$$\hat{d}(\hat{x}, \hat{y}) = \lim_{n \rightarrow \infty} d(x_n, y_n)$$

To ensure this is well-defined:

- *Existence of the Limit:* Because (x_n) and (y_n) are Cauchy sequences, the sequence of real numbers $a_n = d(x_n, y_n)$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, this limit exists.
- *Independence of Representatives:* If alternative representatives $x' \sim x$ and $y' \sim y$ are selected, the calculated limit remains invariant because $d(x_n, x'_n) \rightarrow 0$ and $d(y_n, y'_n) \rightarrow 0$.

Isometry of the Original Space The original space X is embedded into $\hat{X}$. For any element $b \in X$, consider the constant sequence $(b, b, b, \dots)$, which inherently satisfies the Cauchy criterion. Let $T: X \rightarrow \hat{X}$ be the mapping that assigns b to the equivalence class $[(b, b, \dots)]_{\sim}$.

$$\hat{d}(T(b), T(c)) = \lim_{n \rightarrow \infty} d(b, c) = d(b, c)$$

Thus, T preserves metric distances. The corresponding image $W = T(X)$ forms a subspace of $\hat{X}$ that is isometric to X .

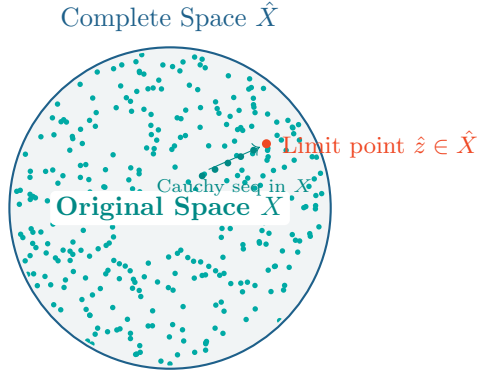
Density of the Embedded Subspace It must be shown that W is dense in $\hat{X}$, as visualized in [Figure 2.4](#). Let $\hat{x} \in \hat{X}$ be an arbitrary point represented by a Cauchy sequence (x_n) in X . For any $\epsilon > 0$, the Cauchy property guarantees an index N such that $d(x_n, x_N) < \epsilon/2$ for all $n > N$. The corresponding element $T(x_N)$ is contained in W . Evaluating the distance yields

$$\hat{d}(\hat{x}, T(x_N)) = \lim_{n \rightarrow \infty} d(x_n, x_N) \leq \frac{\epsilon}{2} < \epsilon$$

This implies that $\overline{W} = \hat{X}$.

Completeness of $\hat{X}$ It must be shown that an arbitrary Cauchy sequence in $\hat{X}$ converges to a limit in $\hat{X}$. Let $(\hat{y}_k)$ be a Cauchy sequence in $\hat{X}$. Because W is a dense subspace, for each term $\hat{y}_k$ there exists an element $z_k \in X$ such that

$$\hat{d}(\hat{y}_k, T(z_k)) < \frac{1}{k}$$



X is embedded densely into $\hat{X}$.
 The topological completion formally constructs and
 incorporates the limits of divergent Cauchy sequences.

FIGURE 2.4 Visualizing metric completion. The original space X permeates $\hat{X}$ as a dense subspace. The formal completion isolates and integrates the requisite limit points to secure structural completeness.

As illustrated in Figure 2.5, because $(\hat{y}_k)$ satisfies the Cauchy criterion, the derived sequence (z_k) is intrinsically Cauchy in X . Let $\hat{z}$ denote the equivalence class of this sequence. By the triangle inequality

$$\hat{d}(\hat{y}_k, \hat{z}) \leq \hat{d}(\hat{y}_k, T(z_k)) + \hat{d}(T(z_k), \hat{z})$$

Since $\hat{y}_k \rightarrow \hat{z}$, both terms on the right side of the inequality approach zero as $k \rightarrow \infty$. Thus, the Cauchy sequence $(\hat{y}_k)$ converges to $\hat{z} \in \hat{X}$. Therefore, $\hat{X}$ is a complete metric space.

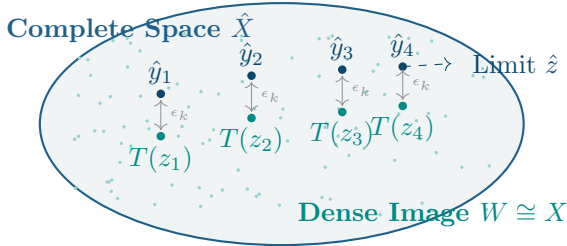


FIGURE 2.5 Visualizing the convergence in $\hat{X}$. To prove $\hat{X}$ is complete, the Cauchy sequence in $\hat{X}$ is bounded by a correlated sequence in the dense subspace W .

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Uniqueness up to Isometry Assume $(\tilde{X}, \tilde{d})$ is another complete metric space containing a dense subspace $\tilde{W}$ that is isometric to X . It follows that the subspaces $\tilde{W}$ and W are isometric to one another. It remains to show that this isometry extends to the spaces $\tilde{X}$ and $\hat{X}$.

For any arbitrary points $\tilde{x}, \tilde{y} \in \tilde{X}$, the density of $\tilde{W}$ guarantees the existence of sequences $(\tilde{x}_n)$ and $(\tilde{y}_n)$ in $\tilde{W}$ such that $\tilde{x}_n \rightarrow \tilde{x}$ and $\tilde{y}_n \rightarrow \tilde{y}$. By the reverse triangle inequality

$$|\tilde{d}(\tilde{x}, \tilde{y}) - \tilde{d}(\tilde{x}_n, \tilde{y}_n)| \leq \tilde{d}(\tilde{x}, \tilde{x}_n) + \tilde{d}(\tilde{y}, \tilde{y}_n)$$

As $n \rightarrow \infty$, both terms on the right approach zero so that $\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \rightarrow \infty} \tilde{d}(\tilde{x}_n, \tilde{y}_n)$. Let $J : \tilde{W} \rightarrow W$ denote the isometry. For any Cauchy sequences $(\tilde{x}_n)$ and $(\tilde{y}_n)$ in $\tilde{W}$, define their respective image sequences in W as $(x_n) = (J(\tilde{x}_n))$ and $(y_n) = (J(\tilde{y}_n))$. The isometric property of J guarantees metric equivalence:

$$d_{\tilde{X}}(\tilde{x}_n, \tilde{y}_n) = d_{\hat{X}}(x_n, y_n) \quad \text{for all } n \in \mathbb{N}$$

Because the spaces are completely defined by their closures ($\overline{\tilde{W}} = \tilde{X}$ and $\overline{W} = \hat{X}$), these sequences converge to limits $\tilde{x}, \tilde{y} \in \tilde{X}$ and $\hat{x}, \hat{y} \in \hat{X}$. The continuity of the metric preserves the sequence of distances in the limit:

$$d_{\tilde{X}}(\tilde{x}, \tilde{y}) = \lim_{n \rightarrow \infty} d_{\tilde{X}}(\tilde{x}_n, \tilde{y}_n) = \lim_{n \rightarrow \infty} d_{\hat{X}}(x_n, y_n) = d_{\hat{X}}(\hat{x}, \hat{y})$$

By defining the extension mapping $\tilde{J} : \tilde{X} \rightarrow \hat{X}$ via the limit $\tilde{J}(\tilde{x}) = \hat{x}$, the derived equality becomes:

$$d_{\tilde{X}}(\tilde{x}, \tilde{y}) = d_{\hat{X}}(\tilde{J}(\tilde{x}), \tilde{J}(\tilde{y}))$$

This confirms $\tilde{J}$ is a global isometry mapping $\tilde{X}$ onto $\hat{X}$. ■

EXERCISES

1. *Incomplete Subspaces.* The real line $\mathbb{R}$ is a complete metric space.
 - (a) Prove that the interval $X = (0, 1)$ equipped with the standard subspace metric is not complete.
 - (b) Construct an example of a Cauchy sequence in X that fails to converge to any limit in X .
2. *Completeness of Subspaces.* Let X be a complete metric space and $Y \subset X$ be a subspace. Prove that Y is complete if and only if Y is a closed set in X . (*Hint: Utilize the sequential characterization of closed sets established in Theorem 2.6.*)
3. *Cantor's Intersection Theorem.* Let (X, d) be a complete metric space. Let (F_n) be a nested sequence of non-empty closed sets $(F_1 \supset F_2 \supset \dots)$

...) such that the diameter of F_n approaches zero ($\text{diam}(F_n) \rightarrow 0$ as $n \rightarrow \infty$). Prove that intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point. (*Hint: Construct a sequence by selecting an arbitrary point x_n from each respective set F_n , and demonstrate that (x_n) satisfies the Cauchy criterion.*)

4. *Completion of Standard Spaces.* Describe the topological completion of the following metric spaces:
 - (a) The space of rational numbers $\mathbb{Q}$ equipped with the standard Euclidean metric $d(x, y) = |x - y|$.
 - (b) The space $\mathbb{Q}$ equipped with the discrete metric.
 - (c) The open interval $X = (0, 1)$ equipped with the standard Euclidean metric.
5. *Isometry of Trivial Completions.* The Completion Theorem establishes that the completion $\hat{X}$ is unique up to isometry. Assume the original metric space X is complete. Prove that its formal completion $\hat{X}$ is isometric to X itself.
6. *Completeness of $C^1[a, b]$.* Let $X = C^1[a, b]$ denote the space of continuously differentiable real-valued functions defined on the closed interval $[a, b]$. Consider the composite norm $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$. Prove that $(X, \|\cdot\|)$ is a complete metric space. (*Hint: Apply the theorem stating that if a sequence of differentiable functions $f_n \rightarrow f$ uniformly and their derivatives $f'_n \rightarrow g$ uniformly, then the limit function f is continuously differentiable and $f' = g$.*)
7. *Incompleteness of Functions with Compact Support.* Let $X = C_c(\mathbb{R})$ denote the space of continuous functions possessing compact support.² Equip X with the uniform supremum metric. Construct a Cauchy sequence in X that uniformly converges to a continuous function in $C(\mathbb{R})$ that lacks compact support. Conclude that X is not a complete metric space.
8. *Intersection of Nested Balls (Open vs. Closed).* Cantor's Intersection Theorem states that in a complete space, a nested sequence of closed sets with diameters tending to zero possesses a non-empty intersection. Construct a nested sequence of open intervals $I_n = (a_n, b_n)$ in $\mathbb{R}$ such that their respective lengths approach zero, yet the infinite intersection is empty.
9. *Density of Open Sets (Baire Category Lemma).* Let X be a complete metric space. Let U_1 and U_2 be open and dense subsets of X . Prove that their intersection $U_1 \cap U_2$ is both open and dense in X .

²The **support** of a function f , denoted $\text{supp}(f)$, is the closure of the set of points where the function is non-zero: $\text{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}$.

10. *Completeness of Closed Balls.* Let X be a complete metric space. Let $B = \overline{B}(x_0, r)$ denote a closed ball in X . Prove that B itself, when equipped with the inherited subspace metric, is a complete metric space.

2.5 COMPLETENESS PROOFS

In this section, the completeness of the standard functional spaces is established. The analytical methodology for establishing completeness involves three sequential stages:

1. *Construct a Candidate:* Utilize the inherent completeness of the underlying scalar field ($\mathbb{R}$ or $\mathbb{C}$) to isolate a potential limit point.
2. *Verify Convergence:* Prove the sequence analytically approaches this candidate under the specific metric topology of the space.
3. *Verify Membership:* Prove the constructed candidate satisfies all defining properties of the space (e.g., boundedness, continuity).

The completeness of the real line is directly leveraged to prove the completeness of finite-dimensional spaces:

Theorem 2.13 Completeness of $\mathbb{R}^n$

The Euclidean space $\mathbb{R}^n$ (and the unitary space $\mathbb{C}^n$), equipped with the standard metric, is a complete metric space.

PROOF: Let $(u_m)_{m=1}^\infty$ be a Cauchy sequence in $\mathbb{R}^n$. Each vector is expressed in terms of its n coordinate components:

$$u_m = (\alpha_{m,1}, \alpha_{m,2}, \dots, \alpha_{m,n})$$

Analyzing Components Because the sequence is Cauchy, the distance between terms becomes arbitrarily small. For any $\epsilon > 0$, there exists an integer N such that for all indices $m, j > N$

$$d(u_m, u_j) = \sqrt{\sum_{k=1}^n |\alpha_{m,k} - \alpha_{j,k}|^2} < \epsilon$$

Thus, for any coordinate k (where $1 \leq k \leq n$)

$$|\alpha_{m,k} - \alpha_{j,k}| < \epsilon$$

This implies that the sequence of scalars $(\alpha_{1,k}, \alpha_{2,k}, \dots)$ is a Cauchy sequence in $\mathbb{R}$ for each k .

Constructing the Limit Because $\mathbb{R}$ is complete, these scalar sequences converge. The limit components are defined:

$$\alpha_k = \lim_{m \rightarrow \infty} \alpha_{m,k}$$

The candidate limit vector $u = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is constructed. Since it comprises an n -tuple of real numbers, it trivially holds that $u \in \mathbb{R}^n$.

Metric Convergence Finally, it must be verified that $u_m \rightarrow u$ in the Euclidean metric. Applying standard limit theorems for finite sums yields

$$\lim_{m \rightarrow \infty} d(u_m, u)^2 = \lim_{m \rightarrow \infty} \sum_{k=1}^n |\alpha_{m,k} - \alpha_k|^2 = \sum_{k=1}^n 0 = 0$$

Thus, the sequence u_m converges to u . ■

Infinite-dimensional spaces equipped with the supremum metric share a common property: pointwise Cauchy bounds translate to uniform topological bounds.

Theorem 2.14 Completeness of ℓ^∞

The space ℓ^∞ is a complete metric space.

PROOF: Let (u_m) be a Cauchy sequence in ℓ^∞ . Each term represents an infinite sequence of scalars:

$$u_m = (\alpha_{m,1}, \alpha_{m,2}, \alpha_{m,3}, \dots)$$

Pointwise Convergence Given an arbitrary $\epsilon > 0$, an index N exists such that for all $m, j > N$:

$$d(u_m, u_j) = \sup_{k \in \mathbb{N}} |\alpha_{m,k} - \alpha_{j,k}| < \epsilon$$

If the supremum is bounded by ϵ , then the absolute difference at any index k is identically bounded:

$$|\alpha_{m,k} - \alpha_{j,k}| < \epsilon \quad \text{for every fixed } k$$

Thus, for each k , the scalar sequence $(\alpha_{m,k})_m$ is Cauchy in $\mathbb{C}$ and must converge. The limit is defined component-wise:

$$\alpha_k = \lim_{m \rightarrow \infty} \alpha_{m,k}$$

Let $u = (\alpha_1, \alpha_2, \dots)$ denote the candidate limit sequence.

Uniform Convergence It must be shown that $u_m \rightarrow u$ in the supremum metric. From the Cauchy condition, $|\alpha_{m,k} - \alpha_{j,k}| < \epsilon$ for all k . Keeping m fixed and

letting the index $j \rightarrow \infty$, the term $\alpha_{j,k} \rightarrow \alpha_k$. The strict inequality relaxes to a weak inequality:

$$|\alpha_{m,k} - \alpha_k| \leq \epsilon$$

This bound holds for all indices k . Taking the supremum over all coordinates:

$$\sup_{k \in \mathbb{N}} |\alpha_{m,k} - \alpha_k| \leq \epsilon$$

This confirms that $d(u_m, u) \leq \epsilon$ proving uniform convergence.

Membership (Boundedness) It must be proven that the limit sequence u is bounded. From the previous analysis, fix a sufficiently large index M such that $d(u_M, u) \leq 1$. Applying the triangle inequality:

$$\sup_k |\alpha_k| \leq \sup_k |\alpha_k - \alpha_{M,k}| + \sup_k |\alpha_{M,k}|$$

The first term is bounded by 1. The second term is a finite constant (defined as K) because $u_M \in \ell^\infty$. Thus, $\sup |\alpha_k| \leq 1 + K < \infty$, confirming $u \in \ell^\infty$. ■

The subspace of convergent sequences inherits this completeness directly from the original space.

Theorem 2.15 Completeness of c

The space c is a complete metric space.

PROOF: Let (u_m) be a Cauchy sequence in c .

The Candidate Limit Because $c \subset \ell^\infty$, the sequence (u_m) is also Cauchy in ℓ^∞ . Theorem 2.14 guarantees the existence of a limit sequence $u = (\alpha_k)$ such that $u_m \rightarrow u$ uniformly.

Membership It must be established that the limit sequence u belongs to c . This requires proving that the scalar sequence (α_k) converges as $k \rightarrow \infty$. By the triangle inequality

$$|\alpha_j - \alpha_k| \leq |\alpha_j - \alpha_{m,j}| + |\alpha_{m,j} - \alpha_{m,k}| + |\alpha_{m,k} - \alpha_k|$$

For any $\epsilon > 0$: Fix m sufficiently large such that $d(u_m, u) < \epsilon/3$. This bounds the first and third terms. Fix j, k . Because $u_m \in c$, its scalar components form a convergent (and therefore Cauchy) sequence. For sufficiently large indices j and k , the bound $|\alpha_{m,j} - \alpha_{m,k}| < \epsilon/3$ is obtained.

Summing these bounds yields $|\alpha_j - \alpha_k| < \epsilon$. Thus, (α_k) is Cauchy in $\mathbb{C}$ and must converge, verifying $u \in c$. ■

The space of continuous functions can be shown to be complete by a similar proof by replacing the discrete index k with a continuous parameter t .

Theorem 2.16 Completeness of $C[a, b]$

The space $C[a, b]$ is a complete metric space.

PROOF: Let (x_m) be a Cauchy sequence of functions in $C[a, b]$. Let $x_m(t)$ denote the evaluation of the m -th function at point t .

Pointwise Limit Because $\mathbb{R}$ is complete, the candidate limit function $x(t)$ is defined pointwise:

$$x(t) = \lim_{m \rightarrow \infty} x_m(t)$$

Uniform Convergence Let $\epsilon > 0$. Because the sequence is Cauchy under the uniform norm, for every fixed $t_0 \in [a, b]$, the sequence of scalars $(x_m(t_0))_{m=1}^{\infty}$ is Cauchy. There exists an index N such that $d(x_m, x_j) < \epsilon$ for all $m, j > N$. Consequently

$$|x_m(t_0) - x_j(t_0)| \leq \max_{t \in [a, b]} |x_m(t) - x_j(t)| = d(x_m, x_j) < \epsilon$$

Letting $j \rightarrow \infty$ yields:

$$|x_m(t) - x(t)| \leq \epsilon$$

Because this inequality holds for all t independently of N , the convergence is uniform: $d(x_m, x) \leq \epsilon$.

Membership (Continuity) A fundamental theorem of real analysis states that the uniform limit of a sequence of continuous functions is continuous. Because each x_m is continuous and the convergence $x_m \rightarrow x$ is uniform, $x(t)$ is continuous. Thus, $x \in C[a, b]$. ■

Concluding with the spaces ℓ^p , these diverge from the supremum metric paradigm. Because the metric involves infinite series, the analytical technique requires truncating the sums and utilizing Minkowski's inequality.

Theorem 2.17 Completeness of ℓ^p

The space ℓ^p is a complete metric space for $1 \leq p < \infty$.

PROOF: Let $(u_m)_{m=1}^{\infty}$ be a Cauchy sequence in ℓ^p , where $u_m = (\alpha_{m,1}, \alpha_{m,2}, \dots)$.

Constructing the Limit Candidate For any $\epsilon > 0$, an index N exists such that for all $m, j > N$:

$$\|u_m - u_j\|_p^p = \sum_{k=1}^{\infty} |\alpha_{m,k} - \alpha_{j,k}|^p < \epsilon^p$$

For any index k , the individual term is bounded by the total sum. Thus, $|\alpha_{m,k} - \alpha_{j,k}| < \epsilon$, verifying pointwise Cauchy behavior. The component-wise limit is defined:

$$\alpha_k = \lim_{m \rightarrow \infty} \alpha_{m,k}$$

Let $u = (\alpha_1, \alpha_2, \dots)$.

Proving Convergence To manage the infinite sum in the Cauchy inequality, it is restricted to an integer K :

$$\sum_{k=1}^K |\alpha_{m,k} - \alpha_{j,k}|^p < \epsilon^p$$

Allowing $j \rightarrow \infty$, the term $\alpha_{j,k} \rightarrow \alpha_k$:

$$\sum_{k=1}^K |\alpha_{m,k} - \alpha_k|^p \leq \epsilon^p$$

Since this bound holds independently of the limit K , letting $K \rightarrow \infty$ yields:

$$\sum_{k=1}^{\infty} |\alpha_{m,k} - \alpha_k|^p \leq \epsilon^p$$

Thus, $\|u_m - u\|_p \leq \epsilon$, confirming that $u_m \rightarrow u$ in the p -norm.

Membership Applying Minkowski's inequality verifies $u \in \ell^p$:

$$\|u\|_p \leq \|u - u_N\|_p + \|u_N\|_p$$

The first term is bounded by ϵ , and the second is finite. Consequently, $\|u\|_p < \infty$. ■

EXERCISES

1. *Completeness of the Space c_0 .* Let c_0 be the set of all sequences of complex numbers that converge to zero. This is a subspace of the bounded sequences space ℓ^∞ .
 - (a) Prove that c_0 is a closed subspace of ℓ^∞ .
 - (b) Conclude from the Subspace Completeness Theorem that c_0 is complete.
2. *Incompleteness of $C[0, 1]$ with Integral Norm.* Let X be the vector space of continuous functions $C[0, 1]$. Equip X with the L^1 metric defined by the integral from 0 to 1 of the absolute difference between $f(t)$ and $g(t)$. Consider the sequence of functions $f_n(t)$ defined as 0 for $t \in [0, 0.5)$, a linear segment with slope n for $t \in [0.5, 0.5 + 1/n]$, and 1 for $t > 0.5 + 1/n$.

- (a) Sketch the graph of $f_n(t)$.
 - (b) Prove that (f_n) is a Cauchy sequence in this metric.
 - (c) Prove that (f_n) does not converge to any continuous function in X . (*Hint: Evaluate its approach toward a step function, which is discontinuous and therefore exterior to the space.*)
3. *Incompleteness of $C^1[-1, 1]$ with Uniform Norm.* Let X be the vector space of continuously differentiable functions on the closed interval $[-1, 1]$. Equip X with the uniform norm. Consider the sequence of functions $f_n(t) = \sqrt{t^2 + 1/n}$.
- (a) Prove that (f_n) converges uniformly to the absolute value function $|t|$.
 - (b) Establish why the absolute value function is not contained in the space X .
 - (c) Conclude that X under the uniform norm is not complete.
4. *Completeness of Bounded Functions.* Let S be any non-empty set. Let $\mathcal{B}(S)$ denote the set of all bounded real-valued functions defined on S , equipped with the supremum metric over S . Prove that $\mathcal{B}(S)$ is a complete metric space. (*Hint: Analyze the structure of the proof for the space of bounded sequences, noting that sequences are functions mapping the natural numbers to the real numbers.*)
5. *Incompleteness of Finite Support Sequences.* Let c_{00} denote the space of all sequences possessing finitely many non-zero terms. Equip c_{00} with the supremum norm from ℓ^∞ .
- (a) Consider the sequence of vectors $x^{(n)}$ defined by terms $1/k$ up to index n , followed by zeros. Demonstrate that this sequence is in c_{00} .
 - (b) Prove that $(x^{(n)})$ is a Cauchy sequence.
 - (c) Identify the limit in the space of bounded sequences and determine if it possesses finitely many non-zero terms.
 - (d) Conclude that c_{00} is not a complete space.
6. *Completeness of Matrices.* Let $M_{n \times n}$ be the space of all real square matrices of dimension n . Define a metric based on the maximum absolute difference between corresponding scalar entries. Prove that this space is a complete metric space. (*Hint: Isometrically identify the matrix space with Euclidean space of dimension n^2 , or construct a direct proof verifying component-wise sequence convergence.*)
7. *Space of Convergent Series.* Let X be the subspace of absolutely summable sequences ℓ^1 consisting of all sequences whose infinite sum evaluates to zero.
- (a) Prove that X is a closed set in ℓ^1 .
 - (b) Using the theorem that a closed subset of a complete space is itself complete, prove that X is a Banach space.

Normed Spaces and Banach Spaces

3.1 Vector Spaces

3.2 Normed Spaces

3.3 Banach Spaces

3.4 Hilbert Spaces

3.5 Completion of Normed Spaces

3.6 Finite Dimensional Spaces

3.7 Direct Sums and Projections

WHILE metric spaces provided the geometric framework for analysis, the study of differential equations and operator theory requires the power of linear algebra. In this chapter, we synthesize these two disciplines by introducing **Normed Spaces**—vector spaces equipped with a measure of length.

We will investigate the structural stability of **Banach spaces** and the geometric elegance of **Hilbert spaces**, where orthogonality and projections allow us to decompose complex spaces into independent, manageable parts. Furthermore, we will establish why finite-dimensional spaces exhibit a unique topological rigidity compared to the “spacious” and often counter-intuitive nature of infinite-dimensional settings. This union of algebra and topology provides the essential arena for the modern study of functional analysis.

3.1 VECTOR SPACES

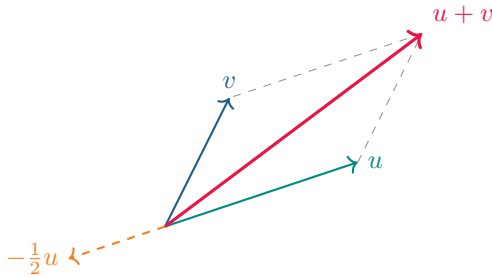
Before defining a norm, the underlying structure must be established. The object of study is the *Vector Space*. In functional analysis, the elements of a vector space are often complex objects, such as infinite sequences or families of continuous functions, rather than finite-dimensional coordinate tuples. A vector space provides a framework where elements can be added together and scaled by a field.

A **Vector Space** X over a field $\mathbb{F}$ (where $\mathbb{F}$ is usually $\mathbb{R}$ or $\mathbb{C}$) is a non-empty set of elements, called **vectors**, equipped with two operations:

1. **Vector Addition:** A mapping $(u, v) \mapsto u + v$ from $X \times X$ into X .
2. **Scalar Multiplication:** A mapping $(\alpha, u) \mapsto \alpha u$ from $\mathbb{F} \times X$ into X .

These operations satisfy the standard axioms: commutativity of addition, associativity of addition and scalar multiplication, the existence of a unique zero vector $\mathbf{0}$, the existence of unique additive inverses $(-u)$, and the distributivity of scalars over vector addition.

In this text, u, v, w denote vectors and Greek letters α, β denote scalars. The geometric interpretation of these operations is shown in [Figure 3.1](#).



Vector Operations

Vector addition satisfies the parallelogram identity.

Scalar multiplication scales the vector length.

FIGURE 3.1 The basic operations within a vector space.

We illustrate these definitions by examining the standard spaces used in this text.

1. **Euclidean Space $\mathbb{R}^n$:** The space of all ordered n -tuples of real numbers $u = (u_1, u_2, \dots, u_n)$.
 - *Addition:* Component-wise, $u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$.
 - *Scalar Multiplication:* $\alpha u = (\alpha u_1, \alpha u_2, \dots, \alpha u_n)$.
 - *Zero Vector:* $\mathbf{0} = (0, 0, \dots, 0)$.

2. **The Space of Continuous Functions** $C[a, b]$: The space of continuous real-valued or complex-valued functions $u(t)$ defined on the closed interval $[a, b]$.

- *Addition*: $(u + v)(t) = u(t) + v(t)$ for all $t \in [a, b]$.
- *Scalar Multiplication*: $(\alpha u)(t) = \alpha u(t)$.

The vector space axioms are satisfied pointwise.

3. **The Space of All Sequences** s : This space includes all infinite sequences of complex numbers $u = (u_j)_{j=1}^{\infty}$, regardless of convergence or boundedness.

- *Addition*: $u + v = (u_1 + v_1, u_2 + v_2, \dots)$.
- *Scalar Multiplication*: $\alpha u = (\alpha u_1, \alpha u_2, \dots)$.

4. **The Space of Bounded Sequences** ℓ^∞ : A subset of s . If u and v are bounded sequences (satisfying $|u_j| \leq M_u$ and $|v_j| \leq M_v$ for all j), their sum $u + v$ is bounded by $M_u + M_v$, and any scalar multiple αu is bounded by $|\alpha|M_u$. This closure confirms that ℓ^∞ is a vector space.

Applications frequently restrict focus to specific subsets. A subset that is closed under the operations of the space is a subspace.

A non-empty subset Y of a vector space X is a **subspace** if, for all vectors $u, v \in Y$ and scalars $\alpha, \beta \in \mathbb{F}$, the linear combination is in Y :

$$\alpha u + \beta v \in Y$$

- In $\mathbb{R}^3$: Any plane intersecting the origin is a subspace. The sum of any two vectors on the plane is also on the plane.
- In $C[a, b]$: The set of all polynomials is a subspace of the space of continuous functions. The sum of two polynomials is a polynomial.

Vectors can be constructed using linear combinations.

Definition 3.1 Linear Combination

A vector $w \in X$ is a **linear combination** of a finite set of vectors $\{u_1, u_2, \dots, u_m\}$ if there exist scalars $\alpha_1, \dots, \alpha_m$ such that:

$$w = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_m u_m$$

The set of all vectors constructed from a given subset is the span.

Definition 3.2 Span

The **span** of a set $M \subseteq X$, denoted by $\text{span}(M)$, is the set of all finite linear combinations of vectors in M :

$$\text{span } M = \left\{ \sum_{i=1}^k \alpha_i u_i \mid u_i \in M, \alpha_i \in \mathbb{F}, k \in \mathbb{N} \right\}$$

The span of any set M is a subspace of X . Furthermore, it is the smallest subspace containing M .

The dimension of a subspace is determined by finding a minimal spanning set of vectors, which requires the concept of linear independence. A set is independent if no vector in the set can be derived from the others.

Definition 3.3 Linear Independence

A finite set of vectors $\{u_1, u_2, \dots, u_m\}$ is linearly independent if the equation

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_m u_m = \mathbf{0}$$

is satisfied only by the trivial solution $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$. An infinite set of vectors M is linearly independent if every finite subset of M is linearly independent.

A subset that is linearly independent and spans the space is a basis. In infinite-dimensional spaces, this algebraic basis is called a Hamel basis.

Definition 3.4 Hamel Basis and Dimension

A set $\mathcal{B} \subset X$ is a Hamel Basis for X if:

1. $\mathcal{B}$ is linearly independent.
2. Every vector $u \in X$ can be expressed as a linear combination of a finite number of elements from $\mathcal{B}$.

If $\mathcal{B}$ has a finite number of elements n , X is finite dimensional and denoted $\dim X = n$. If $\mathcal{B}$ is infinite, X is infinite dimensional.

The Basis Extension Theorem establishes that any linearly independent set can be expanded to form a basis for the space.

Theorem 3.5 Basis Extension Theorem

Let X be a vector space.

1. If S is a linearly independent subset of X , then S can be extended to form a basis for X .
2. If X is finite-dimensional and Y is a subspace of X , any basis for Y can be extended to a basis for X .

PROOF:

Finite Dimensions The proof is constructive. If S does not span X , select a vector $v_1 \in X \setminus \text{span}(S)$. The union $S \cup \{v_1\}$ remains independent. This

iterative process is repeated. Because $\dim X$ is finite, the process terminates, yielding a basis.

Infinite Dimensions In infinite-dimensional spaces, the addition of vectors may not terminate. Proving the existence of a maximal linearly independent set requires Zorn's Lemma, an axiom explored in [Chapter 5](#). ■

To illustrate a Hamel basis, consider the canonical basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ in $\mathbb{R}^3$. In the space of all polynomials $\mathcal{P}$, the infinite set $\{1, t, t^2, t^3, \dots\}$ is a Hamel basis. By definition, any polynomial has a finite degree and therefore is a finite sum of these basis elements. The requirement of finite summation is visualized in [Figure 3.2](#).

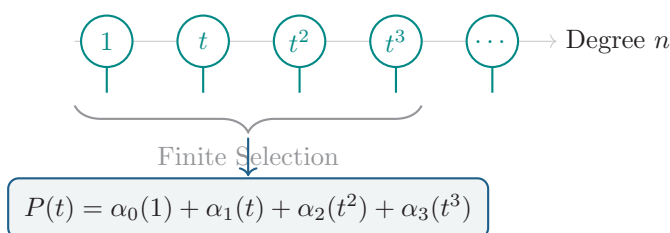


FIGURE 3.2 Visualizing a Hamel Basis. Within the infinite basis $\{1, t, t^2, \dots\}$, any single polynomial uses only a finite sequence of basis vectors.

The requirement of finite summation restricts the usefulness of the Hamel basis in complete infinite-dimensional spaces. For instance, in $C[a, b]$, the sequence of polynomials $\{1, t, t^2, \dots\}$ is linearly independent but fails to span the space, as transcendental functions (e.g., e^t) require infinite series representations. A Hamel basis for $C[a, b]$ is guaranteed by Zorn's Lemma, but it is uncountable and non-constructive. Consequently, functional analysis prioritizes the Schauder basis, which uses countably infinite bases.

It is a property of vector spaces that the dimension of a subspace is bounded by the dimension of the space.

Theorem 3.6 Dimension of Subspaces

Let X be a vector space with a finite dimension n . Any subspace $Y \subset X$ has a finite dimension $m \leq n$. If $\dim Y = \dim X$, then $Y = X$.

EXERCISES

1. *Identifying Subspaces.* Determine whether the following subsets Y are subspaces of $X = \mathbb{R}^3$. If a subset is not a subspace, identify the axiom that is violated.

- (a) $Y = \{u = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \mid \eta_1 + \eta_2 + \eta_3 = 0\}$
 (b) $Y = \{u = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \mid \eta_1 \cdot \eta_2 = 0\}$
 (c) $Y = \{u = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 \mid \eta_1 = 1\}$
2. *Intersection of Subspaces.* Let Y and Z be subspaces of a vector space X . Prove that their intersection $W = Y \cap Z$ is a subspace of X . (*Hint: Show that if $u, v \in W$, then any linear combination $\alpha u + \beta v$ is in W .*)
3. *Linear Independence in Function Spaces.* Consider the vector space $C[0, 1]$. Prove that the set of functions $\{1, t, t^2\}$ is linearly independent. (*Hint: Consider the linear combination $\alpha_1(1) + \alpha_2(t) + \alpha_3(t^2) = 0$. Since this equation must hold for all $t \in [0, 1]$, evaluate the expression at distinct values such as $t = 0, 0.5, 1$, or use differentiation.*)
4. *Space of Matrices.* Let X be the set of 2×2 real matrices.
- (a) Verify that X is a vector space under matrix addition and scalar multiplication.
 (b) Construct a basis for X and find the dimension of the space.
5. *Union of Subspaces.* Give a counterexample to show that the union of two subspaces $Y \cup Z$ is not always a subspace. (*Hint: In $X = \mathbb{R}^2$, let Y be the x -axis and Z be the y -axis. Consider the sum of a vector from Y and a vector from Z .*)
6. *Span and Linear Dependence.* Let $S = \{v_1, v_2, \dots, v_n\}$ be a set of vectors in a vector space X . If v_1 is a linear combination of $\{v_2, \dots, v_n\}$, prove that:
- $$\text{span}(S) = \text{span}(S \setminus \{v_1\})$$
- (*This establishes the principle for reducing a spanning set to a basis.*)
7. *Intersection of Infinite Subspaces.* Let X be an infinite-dimensional vector space. Let Y and Z be subspaces such that $X = Y + Z$ (every $x \in X$ can be written as $x = y + z$). Prove that if Y is finite-dimensional, then Z is infinite-dimensional.
8. *Quotient Spaces.* Let Y be a subspace of a vector space X . The quotient space X/Y is the set of cosets $[x] = \{x + y \mid y \in Y\}$. Addition is defined by $[u] + [v] = [u + v]$ and scalar multiplication by $\alpha[u] = [\alpha u]$.
- (a) Verify that these operations are well-defined (independent of the choice of representative).
 (b) Let $X = \mathbb{R}^3$ and Y be the z -axis. Describe the geometry of X/Y and find its dimension.
9. *Hamel Basis for Polynomials.* Let X be the space of real polynomials.
- (a) Prove that $\mathcal{B}_1 = \{1, t, t^2, \dots\}$ is a Hamel basis for X .
 (b) Determine whether the set $\mathcal{B}_2 = \{1, 1 + t, 1 + t + t^2, \dots\}$ is a Hamel basis.

- (c) Determine whether the set $\mathcal{B}_3 = \{1 + t, t + t^2, t^2 + t^3, \dots\}$ is a basis by verifying linear independence and spanning.

3.2 NORMED SPACES

We have established two mathematical structures: metric spaces, which measure the distance $d(u, v)$ between elements, and vector spaces, which provide the operations of vector addition and scalar multiplication.

A *normed space* combines these concepts. It is a vector space equipped with a function that assigns a non-negative length to each vector, called a *norm*. By measuring the length of individual vectors, a norm induces a metric to measure the distance between any two elements, allowing the simultaneous use of limits and linear algebra. The norm is a generalization of Euclidean length in $\mathbb{R}^n$ or the absolute value in $\mathbb{R}$.

Definition 3.7 Norm

Let X be a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). A *norm* on X is a real-valued function, denoted by $\|\cdot\|$, satisfying the following axioms for all $u, v \in X$ and scalars $\alpha \in \mathbb{F}$:

(N1) Non-negativity: $\|u\| \geq 0$

(N2) Definiteness: $\|u\| = 0 \iff u = \mathbf{0}$

(N3) Homogeneity: $\|\alpha u\| = |\alpha| \|u\|$

(N4) Triangle Inequality: $\|u + v\| \leq \|u\| + \|v\|$

The pair $(X, \|\cdot\|)$ is called a *normed space*.

The Triangle Inequality (N4) is the defining geometric constraint of a normed space, requiring that the length of a vector sum cannot exceed the sum of the individual vector lengths. This principle is shown in [Figure 3.3](#).

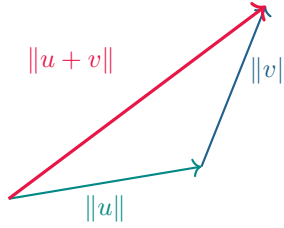
The definition of a norm measures the length of an isolated vector. To evaluate convergence, we need a metric to measure the distance between vectors. Every norm generates a metric.

Theorem 3.8 The Induced Metric

Every norm $\|\cdot\|$ on a vector space X induces a metric d on X , defined by the distance function:

$$d(u, v) = \|u - v\|$$

This mapping is called the *induced metric*. Consequently, every normed space is a metric space.



The Triangle Inequality

$$\|u + v\| \leq \|u\| + \|v\|$$

FIGURE 3.3 The norm axioms. The length of the resultant vector (purple) cannot exceed the combined lengths of its components.

VERIFICATION OF METRIC AXIOMS: We confirm that the function $d(u, v) = \|u - v\|$ satisfies the metric axioms by applying the norm axioms (N1)–(N4). For all vectors $u, v, w \in X$:

- (M1) Non-negativity By (N1), the distance is non-negative:

$$d(u, v) = \|u - v\| \geq 0$$

- (M2) Identity By (N2), the distance is zero if and only if the vectors are identical:

$$\begin{aligned} d(u, v) = 0 &\iff \|u - v\| = 0 \\ &\iff u - v = \mathbf{0} \\ &\iff u = v \end{aligned}$$

- (M3) Symmetry By (N3), extracting the scalar $\alpha = -1$ preserves the length:

$$\begin{aligned} d(u, v) &= \|u - v\| \\ &= \|(-1)(v - u)\| \\ &= |-1| \cdot \|v - u\| \\ &= d(v, u) \end{aligned}$$

- (M4) Triangle Inequality By (N4), introducing an intermediate vector v establishes the bound:

$$\begin{aligned} d(u, w) &= \|u - w\| \\ &= \|(u - v) + (v - w)\| \\ &\leq \|u - v\| + \|v - w\| \\ &= d(u, v) + d(v, w) \end{aligned}$$

The relationship between the norm and the induced metric establishes a lower bound for vector differences. Just as the Triangle Inequality establishes an upper bound for a sum, the Reverse Triangle Inequality establishes a lower bound for a difference. ■

Theorem 3.9 Reverse Triangle Inequality

For all vectors u, v in a normed space X :

$$|\|u\| - \|v\|| \leq \|u - v\|$$

PROOF: We use the identity $u = (u - v) + v$. Applying the Triangle Inequality yields

$$\|u\| \leq \|u - v\| + \|v\| \implies \|u\| - \|v\| \leq \|u - v\|$$

Exchanging the roles of u and v :

$$\|v\| - \|u\| \leq \|v - u\| = \|u - v\|$$

Combining these two inequalities implies

$$-\|u - v\| \leq \|u\| - \|v\| \leq \|u - v\|$$

This compound inequality is equivalent to $|\|u\| - \|v\|| \leq \|u - v\|$. ■

A requirement for analysis is that the norm function preserves limits.

Theorem 3.10 Continuity of the Norm

The norm function $\|\cdot\| : X \rightarrow \mathbb{R}$ is continuous. That is, if a sequence $u_n \rightarrow u$ in the normed space (meaning $\|u_n - u\| \rightarrow 0$), then the scalar sequence $\|u_n\| \rightarrow \|u\|$ in $\mathbb{R}$.

PROOF: Let (u_n) be a sequence such that $u_n \rightarrow u$. Applying the Reverse Triangle Inequality:

$$|\|u_n\| - \|u\|| \leq \|u_n - u\|$$

As $n \rightarrow \infty$, the right side of the inequality approaches 0. Consequently, the left side must approach 0, proving that $\lim_{n \rightarrow \infty} \|u_n\| = \|u\|$. ■

While every normed space is a metric space, the converse is false. For example, the discrete metric ($d(u, v) = 1$ if $u \neq v$) cannot be induced by a norm. If such a norm existed, evaluating the scalar multiple would yield $d(2u, 2v) = \|2u - 2v\| = 2\|u - v\| = 2$. However, the definition of the discrete

metric requires that $d(2u, 2v) = 1$. This contradiction violates the homogeneity axiom.

The axioms of a normed space determine the geometry of its unit ball. This geometry is formalized through convexity.

Definition 3.11 Convex Set

A subset K of a vector space is **convex** if, for any two points $x, y \in K$, the line segment connecting them is in K :

$$(1 - t)x + ty \in K \quad \text{for all } t \in [0, 1]$$

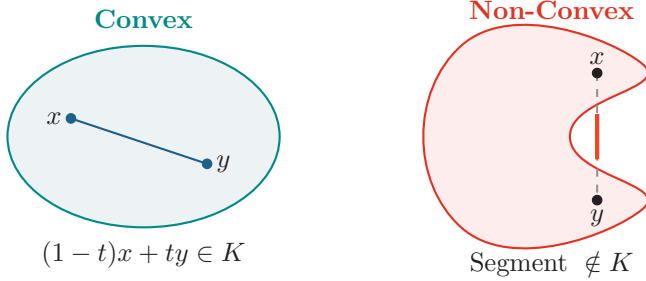


FIGURE 3.4 The geometric interpretation of convexity. In a convex set (left), any chord connecting two points lies entirely in the set. In a non-convex set (right), at least one chord exists passes through the exterior of the set.

Theorem 3.12 Convexity of the Unit Ball

In any normed space X , the closed unit ball $\bar{B} = \{x : \|x\| \leq 1\}$ is a convex set.

PROOF: Let $x, y \in \bar{B}$ (meaning $\|x\| \leq 1$ and $\|y\| \leq 1$) and let $t \in [0, 1]$. We evaluate the norm of the linear combination:

$$\|(1 - t)x + ty\| \leq (1 - t)\|x\| + t\|y\| \leq (1 - t)(1) + t(1) = 1$$

Thus, the segment remains within the unit ball. ■

EXERCISES

1. *The Discrete Metric vs. Norms.* Let X be a vector space where $X \neq \{\mathbf{0}\}$. Prove that the discrete metric d , defined by

$$d(u, v) = \begin{cases} 0 & \text{if } u = v \\ 1 & \text{if } u \neq v \end{cases}$$

cannot be generated by a norm. (*Hint: If d were induced by a norm, then $d(u, v) = \|u - v\|$. Evaluate the homogeneity axiom (N3) using $\alpha = 2$.)*

2. *Continuity of Vector Addition.* Using the Triangle Inequality, prove that vector addition is continuous. Show that if (u_n) and (v_n) are sequences in a normed space X such that $u_n \rightarrow u$ and $v_n \rightarrow v$, then:

$$(u_n + v_n) \rightarrow (u + v)$$

3. *The Taxicab Norm (ℓ^1 on $\mathbb{R}^2$).* Consider the vector space $\mathbb{R}^2$. Verify that the function defined by $\|u\|_1 = |\xi_1| + |\xi_2|$ (where $u = (\xi_1, \xi_2)$) satisfies the axioms of a norm. Sketch the boundary of the unit ball for this norm, defined by $S = \{u \in \mathbb{R}^2 : \|u\|_1 = 1\}$, and compare it with the Euclidean unit ball.
4. *Boundedness of Convergent Sequences.* Apply the Reverse Triangle Inequality to prove that every convergent sequence in a normed space is bounded. (*Hint: If $u_n \rightarrow u$, there exists an index N such that $\|u_n - u\| < 1$ for all $n > N$. Use the inequality $|\|u_n\| - \|u\|| \leq \|u_n - u\|$ to find an upper bound M for the scalar sequence $\|u_n\|$.)*
5. *Translation Invariance.* Let d be the metric induced by a norm on X . Prove that distance is preserved under translation:

$$d(u + a, v + a) = d(u, v)$$

for any vectors $u, v, a \in X$. Describe the geometric implications of this invariance for an open ball $B_r(a)$ relative to $B_r(\mathbf{0})$.

6. *Norm of Scalar Multiplication.* In a normed space X , prove that scalar multiplication is a continuous operation. If $\alpha_n \rightarrow \alpha$ in $\mathbb{F}$ and $u_n \rightarrow u$ in X , show that $\alpha_n u_n \rightarrow \alpha u$. (*Hint: Use the identity $\alpha_n u_n - \alpha u = \alpha_n(u_n - u) + (\alpha_n - \alpha)u$ and the theorem that convergent sequences are bounded.*)
7. *The Max Norm on $\mathbb{R}^n$.* Prove that $\|u\|_\infty = \max_{1 \leq j \leq n} |\xi_j|$ is a norm on $\mathbb{R}^n$. Sketch the unit ball for this norm in $\mathbb{R}^2$ and compare its boundaries with those of the Euclidean and Taxicab norms.
8. *Convexity of the Open Unit Ball.* Let $(X, \|\cdot\|)$ be a normed space. Prove that the open unit ball $B = \{u \in X : \|u\| < 1\}$ is a convex set.

9. *Strict Inequality in the Triangle Inequality.* In the Euclidean plane $\mathbb{R}^2$, the equality $\|u + v\| = \|u\| + \|v\|$ holds if and only if one vector is a non-negative scalar multiple of the other. Provide a counterexample in another normed space using two vectors u, v such that u is not a scalar multiple of v , yet the equality holds. (*Hint: Use the max norm on $\mathbb{R}^2$ with vectors $u = (1, 0)$ and $v = (1, 1)$.*)
10. *Distance between a Point and a Subspace.* Let Y be a subspace of a normed space X . For any vector $x \in X$, define the distance to the subspace as $d(x, Y) = \inf_{y \in Y} \|x - y\|$. Prove that for any scalar $\alpha \neq 0$, the distance scales according to the magnitude: $d(\alpha x, Y) = |\alpha|d(x, Y)$.

3.3 BANACH SPACES

The preceding sections established completeness in metric spaces—the guarantee that Cauchy sequences converge—and algebraic structure in normed spaces. A *Banach Space* integrates these properties. It is a normed vector space that is complete with respect to the metric induced by its norm.

This stability is essential for functional analysis, ensuring that limits, derivatives, and integrals of vectors evaluate to elements in the space. The concept is named after Stefan Banach, who formalized these spaces in his 1932 treatise, *Théorie des Opérations Linéaires*.

Definition 3.13 Banach Space

A **Banach Space** is a normed space $(X, \|\cdot\|)$ that is complete in the metric induced by its norm. Equivalently, X is a Banach space if every Cauchy sequence in X converges to a limit $u \in X$.

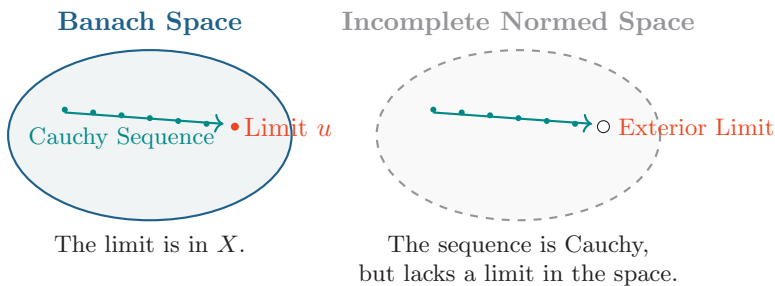


FIGURE 3.5 The topological distinction between a general normed space and a Banach space.

The completeness proofs from [Chapter 2](#) classify the following functional spaces as Banach spaces:

1. **Euclidean Space $\mathbb{R}^n$ (and $\mathbb{C}^n$):** Equipped with the Euclidean norm $\|u\|_2 = (\sum |u_i|^2)^{1/2}$, these are finite-dimensional Banach spaces.
2. **Sequence Spaces ℓ^p ($1 \leq p < \infty$):** The space of sequences $u = (\xi_j)$ satisfying $\sum |\xi_j|^p < \infty$ is a Banach space under the norm $\|u\|_p = (\sum |\xi_j|^p)^{1/p}$.
3. **Bounded Sequences ℓ^∞ :** The space of bounded sequences is a Banach space under the supremum norm $\|u\|_\infty = \sup_j |\xi_j|$.
4. **Continuous Functions $C[a, b]$:** The space of continuous functions on a closed interval $[a, b]$ is a Banach space when equipped with the uniform norm $\|u\|_\infty = \max_{t \in [a, b]} |u(t)|$.

Not every normed space is a Banach space. Consider the vector space of all polynomials, $\mathcal{P}[0, 1]$, equipped with the uniform norm.

- A sequence of polynomials defined by $p_n(t) = \sum_{k=0}^n \frac{t^k}{k!}$ satisfies the Cauchy criterion because it converges uniformly to the exponential function e^t .
- However, the limit function e^t is not a polynomial.

Consequently, the limit is not in $\mathcal{P}[0, 1]$. This space has a norm but lacks completeness.

In applications, solutions are frequently sought within a subspace Y of a Banach space X . A subspace inherits the operations of the space but does not always inherit completeness. Because a Banach space is a complete metric space, the condition as depicted in [Figure 3.6](#), for subspace completeness is identical to the condition established in [Chapter 2](#).

Theorem 3.14 Subspace Completeness

Let X be a Banach space and let Y be a subspace of X . The subspace Y is a Banach space if and only if Y is a closed set in X .

PROOF: By Theorem 2.10 (Subspace Completeness), a subset of a complete metric space is complete if and only if it is closed. Because the metric on X is induced by its norm, and Y is a subspace, Y is a complete normed space (a Banach space) if and only if it is closed in X . ■

Consider the space ℓ^∞ of bounded sequences.

- The subspace c , consisting of convergent sequences, is closed in ℓ^∞ because the uniform limit of convergent sequences is convergent. Consequently, c is a Banach space.
- The subspace c_{00} , consisting of sequences with finitely many non-zero terms, is not closed. A sequence in c_{00} can uniformly approximate the harmonic sequence $1/n$. Therefore, c_{00} is not a Banach space. ┘



Case 1: Closed Subspace
Closure guarantees
the limit is in Y .

Case 2: Not Closed
Cauchy sequences converge
to limits outside Y .

FIGURE 3.6 Completeness within a subspace requires a closed boundary.

Operations frequently require evaluating infinite series. An infinite series $\sum u_k$ in a normed space is evaluated as the limit of its partial sums.

Definition 3.15 Infinite Series

Let (u_k) be a sequence of vectors in a normed space X . The sequence of partial sums (s_n) is defined by:

$$s_n = u_1 + u_2 + \cdots + u_n = \sum_{k=1}^n u_k$$

The infinite series $\sum_{k=1}^{\infty} u_k$ **converges** to a sum $s \in X$ if the sequence of partial sums converges to s :

$$\lim_{n \rightarrow \infty} \|s_n - s\| = 0$$

Evaluating vector series limits directly is often difficult. A stronger condition uses non-negative scalars. A series is **absolutely convergent** if the sum of the vector norms converges in $\mathbb{R}$:

$$\sum_{k=1}^{\infty} \|u_k\| < \infty$$

In $\mathbb{R}$, absolute convergence guarantees convergence. This property extends to all Banach spaces and serves as a characterization of completeness.

Theorem 3.16

Absolute Convergence and Completeness: A normed space X is a Banach space if and only if every absolutely convergent series in X converges to an element in X .

PROOF:

Absolute Convergence Implies Convergence Assume X is a Banach space. Let $\sum u_k$ be an absolutely convergent series. We must prove that $\sum u_k$ has a limit in X . By the Triangle Inequality, the distance between two partial sums $s_n = \sum_{k=1}^n u_k$ and $s_m = \sum_{k=1}^m u_k$ (where $m > n$) is bounded by the partial sums of the norms:

$$\|s_m - s_n\| = \left\| \sum_{k=n+1}^m u_k \right\| \leq \sum_{k=n+1}^m \|u_k\|$$

Because $\sum \|u_k\|$ converges, it satisfies the Cauchy criterion for series in $\mathbb{R}$. Thus, for any $\epsilon > 0$, there exists an integer N such that for all $m > n \geq N$, we have:

$$\sum_{k=n+1}^m \|u_k\| < \epsilon$$

Thus, $\|s_m - s_n\| < \epsilon$ for all $m > n \geq N$ and so (s_n) is a Cauchy sequence in X . Because X is complete, (s_n) converges to a limit s in X , therefore, the original series converges.

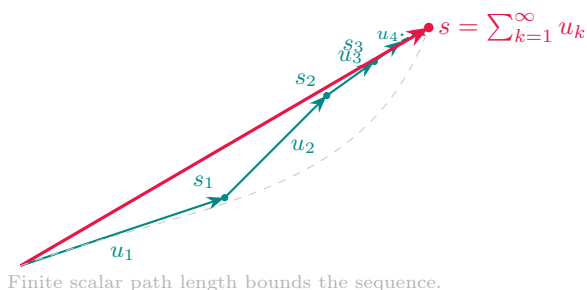


FIGURE 3.7 Visualizing absolute convergence. If the sum of individual vector lengths is finite, the sequence of partial sums forms a Cauchy sequence.

Convergence Implies Completeness Assume every absolutely convergent series in X converges to an element in X . We must show X is complete. Let (x_n)

be a Cauchy sequence. For each $k \in \mathbb{N}$, select an index n_k such that for all $n, m \geq n_k$:

$$\|x_n - x_m\| < \frac{1}{2^k}$$

Require the indices to be increasing ($n_1 < n_2 < \dots$). This extracts a subsequence (x_{n_k}) satisfying $\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}$. Define a telescoping series $\sum u_k$ by setting $u_1 = x_{n_1}$ and $u_k = x_{n_k} - x_{n_{k-1}}$ for $k \geq 2$. Evaluating absolute convergence:

$$\sum_{k=1}^{\infty} \|u_k\| = \|x_{n_1}\| + \sum_{k=2}^{\infty} \|x_{n_k} - x_{n_{k-1}}\| < \|x_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

Because the geometric series converges, $\sum u_k$ is absolutely convergent. By hypothesis, the series converges to a limit $x \in X$. Because the k -th partial sum of this series equals x_{n_k} , the subsequence (x_{n_k}) converges to x . If a Cauchy sequence has a convergent subsequence, the entire sequence must converge to the same limit. Thus $x_n \rightarrow x$, establishing that X is complete. ■

If a vector series converges but does not converge absolutely ($\sum \|u_k\| = \infty$), it is classified as *conditionally convergent*. In finite dimensions ($\mathbb{R}^n$), the Riemann Rearrangement Theorem proves that the terms of a conditionally convergent series can be reordered to yield convergence to any vector. This phenomenon persists in infinite-dimensional Banach spaces.

EXERCISES

1. *Subspaces and Completeness (c_0 vs. c_{00})*. Let $X = \ell^\infty$ be the Banach space of bounded sequences under the supremum norm. Consider the following subspaces:
 - c_0 : The subspace of sequences converging to zero.
 - c_{00} : The subspace of sequences with finitely many non-zero terms.
 - (a) Prove that c_0 is a closed set in ℓ^∞ . Conclude that c_0 is a Banach space.
 - (b) Prove that c_{00} is not closed in ℓ^∞ by constructing a sequence of vectors in c_{00} that converges to an element in $c_0 \setminus c_{00}$. Conclude that c_{00} is not complete.
2. *Absolute Convergence Test in $C[0, 1]$* . Let $X = C[0, 1]$ be equipped with the uniform norm $\|f\|_\infty = \max_{t \in [0, 1]} |f(t)|$. Use the Absolute Convergence Theorem to prove that the following series converges to a continuous function:

$$s(t) = \sum_{n=1}^{\infty} \frac{\sin(n^2 t)}{n^2}$$

(Hint: Compute the supremum norm $\|u_n\|_\infty$ for the term $u_n(t) = \frac{\sin(n^2 t)}{n^2}$. Evaluate the absolute convergence of the resulting scalar series.)

3. *Incompleteness of Polynomials.* Let X be the space of polynomials on $[0, 1]$, equipped with the uniform norm.

- (a) Verify that X satisfies the axioms of a normed vector space.
- (b) Construct a sequence of polynomials (p_n) that is Cauchy in X but fails to converge to a polynomial.
- (c) Identify the completion $\hat{X}$ of this space.

(Hint: For part (b), use the Taylor series partial sums for the function e^t .)

4. *Equivalent Norms and Completeness.* Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are equivalent if there exist positive constants $a, b > 0$ such that for all $x \in X$:

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

Prove that if X is a Banach space with respect to $\|\cdot\|_1$, it is also a Banach space with respect to any equivalent norm $\|\cdot\|_2$. (Hint: Prove that a sequence is Cauchy under the first norm if and only if it is Cauchy under the second norm.)

5. *Schauder Basis Coefficients.* Let (e_n) be a Schauder basis for a Banach space X . Prove that the coordinate functionals $f_k(x) = \alpha_k$ (where the expansion is $x = \sum_{n=1}^{\infty} \alpha_n e_n$) are linear mappings. Specifically, if $x = \sum_{n=1}^{\infty} \alpha_n e_n$ and $y = \sum_{n=1}^{\infty} \beta_n e_n$, prove that the coordinates of the sum $x + y$ are $\alpha_n + \beta_n$.

3.4 HILBERT SPACES

While a normed space quantifies vector magnitude and distance, the geometric structure of a general normed space lacks the orthogonal properties of Euclidean spaces. In $\mathbb{R}^n$, angles and orthogonal projections are defined via the dot product. To extend this geometric structure to infinite-dimensional spaces, we introduce the *Inner Product Space*.

Definition 3.17 Inner Product Space

Let X be a vector space over a scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). An inner product on X is a mapping $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{K}$ satisfying the following axioms for all vectors $x, y, z \in X$ and scalars $\alpha, \beta \in \mathbb{K}$:

- (IP1) **Linearity in the First Argument:** The inner product distributes over vector addition and scalar multiplication in the first coordinate.

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$$

- (IP2) **Conjugate Symmetry:** Reversing the order of the vector arguments

complex-conjugates the result.

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

(Note: If the field is $\mathbb{R}$, this reduces to standard symmetry $\langle x, y \rangle = \langle y, x \rangle$.)

(IP3) **Positive Definiteness:** The inner product of a vector with itself is a real, non-negative scalar, equaling zero if and only if the vector is the zero vector.

$$\langle x, x \rangle \geq 0 \quad \text{and} \quad \langle x, x \rangle = 0 \iff x = \mathbf{0}$$

The pair $(X, \langle \cdot, \cdot \rangle)$ is an **inner product space**. Note that by combining (IP1) and (IP2), it follows that the inner product is conjugate-linear in the second argument. Consequently, the inner product distributes over vector addition in both coordinates.

An inner product induces a norm via the relation $\|x\| = \sqrt{\langle x, x \rangle}$. However, showing that this function satisfies the Triangle Inequality requires the following bounding inequality.

Theorem 3.18 Cauchy-Schwarz Inequality

For all vectors x, y in an inner product space:

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Equality holds if and only if x and y are linearly dependent.

PROOF: The inequality holds if $y = \mathbf{0}$. Assume $y \neq \mathbf{0}$, which means $\|y\|^2 = \langle y, y \rangle > 0$.

Algebraic Expansion By positive definiteness, the inner product of any vector with itself is non-negative. For any scalar $\lambda \in \mathbb{K}$, we evaluate the squared norm of $x - \lambda y$:

$$0 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y \rangle$$

Expanding this using linearity and conjugate symmetry yields:

$$0 \leq \langle x, x \rangle - \lambda \langle y, x \rangle - \bar{\lambda} \langle x, y \rangle + \lambda \bar{\lambda} \langle y, y \rangle$$

Optimal Scalar Substitution We select a scalar λ that minimizes this expression.

Let $\lambda = \frac{\langle x, y \rangle}{\langle y, y \rangle}$. Consequently, the complex conjugate is $\bar{\lambda} = \frac{\overline{\langle x, y \rangle}}{\langle y, y \rangle} = \frac{\langle y, x \rangle}{\|y\|^2}$.

Substituting λ and $\bar{\lambda}$ produces:

$$0 \leq \|x\|^2 - \left(\frac{\langle x, y \rangle}{\|y\|^2} \right) \langle y, x \rangle - \left(\frac{\langle y, x \rangle}{\|y\|^2} \right) \langle x, y \rangle + \left(\frac{\langle x, y \rangle \langle y, x \rangle}{\|y\|^4} \right) \|y\|^2$$

Algebraic Resolution Because $\langle x, y \rangle \langle y, x \rangle = |\langle x, y \rangle|^2$, the expression simplifies:

$$0 \leq \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} - \frac{|\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^2}$$

$$0 \leq \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$$

Rearranging the terms gives:

$$\frac{|\langle x, y \rangle|^2}{\|y\|^2} \leq \|x\|^2$$

$$|\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2$$

Taking the square root establishes the inequality:

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Condition for Equality Equality holds if and only if $\|x - \lambda y\|^2 = 0$. By definiteness, this means $x - \lambda y = \mathbf{0}$, or $x = \lambda y$, which implies x and y are linearly dependent. ■

A consequence of this inequality is the continuity of the inner product: if $x_n \rightarrow x$ and $y_n \rightarrow y$, then the scalar sequence $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$. This guarantees that limit operations are preserved by the inner product.

Just as a Banach space is a complete normed space, a Hilbert space is a complete inner product space.

Definition 3.19 Hilbert Space

A Hilbert Space H is an inner product space that is complete with respect to the norm induced by its inner product.

Since every Hilbert space has an induced norm and is complete, every Hilbert space is a Banach space. The converse is false.

Example 3.20

The following canonical spaces are complete under their induced norms.

Euclidean Space $\mathbb{R}^n$: Equipped with the standard inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, this space is a finite-dimensional Hilbert space.

Sequence Space ℓ^2 : The space of square-summable sequences evaluated with the inner product $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$ is the standard example of a separable Hilbert space.

Function Space $L^2[a, b]$: The space of square-integrable functions evaluated with the inner product $\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$ provides the framework for quantum mechanics. $\lrcorner$

The utility of the inner product lies in its ability to formalize geometric orthogonality using algebraic identities.

Definition 3.21 Orthogonality

Two vectors $x, y \in X$ are **orthogonal** (denoted $x \perp y$) if their inner product is zero:

$$\langle x, y \rangle = 0$$

This definition formalizes the geometric property of perpendicularity. Orthogonal vectors satisfy the generalized Pythagorean identity.

Theorem 3.22 Pythagorean Theorem

Let X be an inner product space. If two vectors $x, y \in X$ are orthogonal, then:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

PROOF: We evaluate the squared norm by expanding its definition in terms of the inner product, $\|u\|^2 = \langle u, u \rangle$.

Algebraic Expansion By definition:

$$\|x + y\|^2 = \langle x + y, x + y \rangle$$

Applying linearity to the first argument yields:

$$\|x + y\|^2 = \langle x, x + y \rangle + \langle y, x + y \rangle$$

Expanding via additivity in the second argument produces:

$$\|x + y\|^2 = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

Application of Orthogonality By definition, $\langle x, x \rangle = \|x\|^2$ and $\langle y, y \rangle = \|y\|^2$. Furthermore, orthogonality implies $\langle x, y \rangle = 0$. By conjugate symmetry, $\langle y, x \rangle = \overline{0} = 0$. Substituting these values yields:

$$\|x + y\|^2 = \|x\|^2 + 0 + 0 + \|y\|^2$$

Consequently, $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. $\blacksquare$

Whereas the Pythagorean identity requires orthogonality, the Parallelogram Law describes the geometry of any two vectors in an inner product space. This identity states that the sum of the squared diagonal lengths equals the sum of the squared side lengths, as shown in [Figure 3.8](#).

Theorem 3.23 The Parallelogram Law

A normed space is an inner product space if and only if its norm satisfies the identity:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

PROOF: We expand both norm terms using the properties of the inner product.

Evaluating the Sum Diagonal

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2\end{aligned}$$

Evaluating the Difference Diagonal Expanding the terms and preserving the subtraction signs yields:

$$\begin{aligned}\|x - y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle -y, -y \rangle \\ &= \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2\end{aligned}$$

Resolution Summing the equations eliminates the inner product terms:

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= (\|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2) \\ &\quad + (\|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2) \\ &= 2\|x\|^2 + 2\|y\|^2\end{aligned}$$

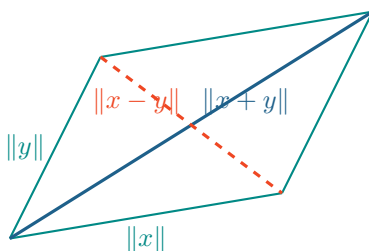


FIGURE 3.8 The parallelogram law. In a Hilbert space, the geometry of vectors x and y satisfies this identity.

EXERCISES

1. *Violating the Parallelogram Law.* Consider the Banach space $C[0, 1]$ equipped with the uniform norm $\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|$. Show that this space is not a Hilbert space by constructing two functions $x(t)$ and $y(t)$ that violate the Parallelogram Law. (*Hint: Evaluate the polynomial functions $x(t) = 1$ and $y(t) = t$.*)
2. *Continuity of the Inner Product.* Let X be an inner product space. Assume that the sequence (x_n) converges to x , and (y_n) converges to y . Prove that the sequence of inner products converges: $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$. (*Hint: Apply the identity $\langle x_n, y_n \rangle - \langle x, y \rangle = \langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle$, followed by the Cauchy-Schwarz inequality and the principle that convergent sequences are bounded.*)
3. *The Polarization Identity.* Let X be a complex inner product space. Prove that the inner product can be reconstructed from the norm via the Polarization Identity:

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$$

(*Hint: Expand each of the four squared norm terms using the definition $\|u\|^2 = \langle u, u \rangle$ and isolate $\langle x, y \rangle$.*)

4. *Orthogonal Complements.* Let M be a non-empty subset of a Hilbert space H . The orthogonal complement of M , denoted $M^\perp$, is the set of all vectors orthogonal to every element in M :

$$M^\perp = \{x \in H \mid \langle x, y \rangle = 0 \text{ for all } y \in M\}$$

Prove that $M^\perp$ is a closed subspace of H , regardless of whether M is a subspace or closed. (*Hint: To verify closure, assume a sequence $x_n \in M^\perp$ converges to an element $x \in H$, and use the continuity of the inner product to evaluate the limit of $\langle x_n, y \rangle$ for an arbitrary vector $y \in M$.*)

5. *Generalized Pythagorean Theorem.* Let $\{x_1, x_2, \dots, x_n\}$ be a finite set of mutually orthogonal vectors in a Hilbert space H (satisfying $\langle x_i, x_j \rangle = 0$ for all $i \neq j$). Prove by induction that:

$$\left\| \sum_{k=1}^n x_k \right\|^2 = \sum_{k=1}^n \|x_k\|^2$$

6. *Cauchy-Schwarz in Sequence Spaces.* Define the sequence space ℓ^2 . For sequences $x = (\xi_j)$ and $y = (\eta_j)$ in ℓ^2 , state the Cauchy-Schwarz inequality in its infinite series representation. Deduce from this inequality that the series $\sum_{j=1}^\infty \frac{1}{j} \xi_j$ converges for any sequence $x \in \ell^2$. (*Hint: Define y as the harmonic sequence $\eta_j = \frac{1}{j}$ and verify its membership in ℓ^2 before applying the inequality.*)

3.5 COMPLETION OF NORMED SPACES

Chapter 2 established that every metric space has a topological completion, constructed using equivalence classes of Cauchy sequences. A question arises: if the initial space is a *normed* space (possessing an algebraic structure), does its completion preserve this normed structure?

The answer is affirmative. The vector operations—vector addition and scalar multiplication—along with the norm, can be extended to the completed space, structuring it as a Banach space. This process is visualized in Figure 3.9.

Theorem 3.24 Completion of Normed Spaces

Let $X = (X, \|\cdot\|)$ be a normed space. There exists a Banach space $\hat{X}$ and an isometry T mapping X onto a subspace W of $\hat{X}$ such that W is dense in $\hat{X}$.

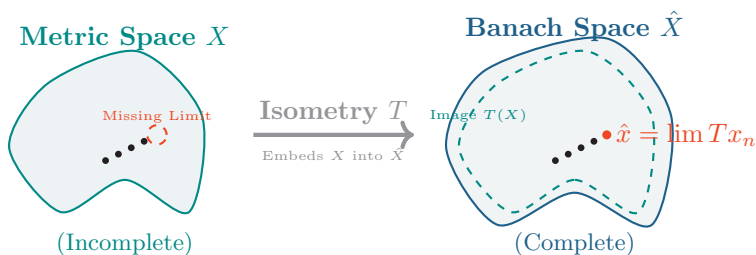


FIGURE 3.9 The completion theorem. The incomplete space X lacks limits for specific Cauchy sequences. The isometric mapping T embeds X into a complete space $\hat{X}$ where all topological limits ($\hat{x}$) are defined. The resulting image $T(X)$ is dense in $\hat{X}$.

CONSTRUCTION OF THE BANACH COMPLETION: Metric space theory guarantees the existence of a complete metric space $(\hat{X}, \hat{d})$ constructed from equivalence classes of Cauchy sequences in X . To establish $\hat{X}$ as a Banach space, vector addition, scalar multiplication, and a corresponding norm must be defined on these equivalence classes.

Defining Vector Operations Let $\hat{x}, \hat{y} \in \hat{X}$ denote equivalence classes. Let (x_n) be a Cauchy sequence representing $\hat{x}$, and let (y_n) represent $\hat{y}$. We define the sum $\hat{x} + \hat{y}$ as the equivalence class containing the sequence $(x_n + y_n)$:

$$\hat{x} + \hat{y} := [(x_n + y_n)]$$

This definition is well-defined because the sum of two Cauchy sequences is a Cauchy sequence. By the Triangle Inequality:

$$\|(x_n + y_n) - (x_m + y_m)\| \leq \|x_n - x_m\| + \|y_n - y_m\|$$

As the indices $m, n \rightarrow \infty$, the right side of the inequality approaches 0, implying that the sequence $(x_n + y_n)$ is Cauchy. Similar logic applies to define scalar multiplication, where $\alpha\hat{x} := [(\alpha x_n)]$.

Defining the Norm The norm on the complete space $\hat{X}$ is defined by taking the limit of the norms of the representative sequence in X :

$$\|\hat{x}\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x_n\|_X$$

The existence of this limit is guaranteed. Because (x_n) is a Cauchy sequence, the corresponding sequence of real numbers $(\|x_n\|)$ forms a Cauchy sequence in $\mathbb{R}$, guaranteed by the Reverse Triangle Inequality $(\|x_n\| - \|x_m\| \leq \|x_n - x_m\|)$. Because $\mathbb{R}$ is complete, the limit exists.

Verification It is straightforward to verify that this defined norm induces the metric generated during the initial completion process:

$$\hat{d}(\hat{x}, \hat{y}) = \|\hat{x} - \hat{y}\|_{\hat{X}}$$

Consequently, the metric completion of any normed space is a complete Banach space. ■

EXERCISES

1. *Constructing the Completion (Abstract).* Let X be an incomplete normed space. Let $\hat{X}$ represent the set of equivalence classes of Cauchy sequences derived from X .
 - (a) Verify that the vector addition operation, defined by $[(x_n)] + [(y_n)] = [(x_n + y_n)]$, is well-defined. (*Prove that if $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, then the summation preserves equivalence: $(x_n + y_n) \sim (x'_n + y'_n)$).*)
 - (b) Verify that the norm defined by $\|\hat{x}\| = \lim_{n \rightarrow \infty} \|x_n\|$ is well-defined and satisfies the triangle inequality in $\hat{X}$.
2. *Concrete Example: Polynomials.* Let X be the space of polynomials on $[0, 1]$, equipped with the uniform norm. It is established that X lacks completeness.
 - (a) Describe the completion $\hat{X}$. (Identify the resulting function space).
 - (b) Describe the mapping $T : X \rightarrow \hat{X}$. Is this linear operator surjective?
 - (c) Let $p_n(t) = \sum_{k=0}^n \frac{t^k}{k!}$. This constitutes a Cauchy sequence in X . Which element in $\hat{X}$ does the equivalence class $[(p_n)]$ represent?

3. *Completion of Subspaces.* Let X be a normed space and Y be a dense subspace of X . Prove that the completion of Y is isometrically isomorphic to the completion of X . (*This theorem dictates that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ share the same completion: $\mathbb{R}$.*)
4. *Preservation of Identity.* Assume X is a Banach space. Prove that its completion $\hat{X}$ is isometric to X . (*Hint: Construct the map $T : X \rightarrow \hat{X}$ that maps x to the constant sequence $(x, x, \dots)$. Demonstrate that T is surjective.*)
5. *Extension of Continuous Maps.* Let X and Y be normed spaces, where Y is a Banach space. Let $T : D \rightarrow Y$ be a bounded linear operator defined on a dense subspace $D \subset X$. Prove that T possesses a unique continuous extension $\hat{T} : X \rightarrow Y$ such that $\hat{T}(x) = T(x)$ for all $x \in D$. (*This theorem, the BLT Theorem, serves as the functional equivalent of topological completion: we complete the operator rather than the space.*)

3.6 FINITE DIMENSIONAL SPACES

In elementary analysis and linear algebra, vector spaces exhibit topological stability. Several properties presumed to be universal depend on the space having a finite dimension.

Consider the finite-dimensional Euclidean space $\mathbb{R}^n$:

- **Equivalence of Norms:** Irrespective of the metric applied (e.g., the Euclidean norm versus the taxicab norm), the topology remains invariant. A sequence converging under one norm converges under all norms on the space.
- **Compactness:** Any subset that is both closed and bounded is a compact set. This property guarantees the existence of convergent subsequences and ensures continuous operators attain extrema.
- **Completeness:** The space is complete; every Cauchy sequence converges to a limit in the space.

In contrast, consider infinite-dimensional spaces such as the function space $C[a, b]$ or the sequence space ℓ^2 :

- **Divergence of Topologies:** Distinct norms generate different topologies. A sequence may converge under an integral norm while diverging under a uniform norm.
- **Failure of Compactness:** A set may be closed and bounded (such as the closed unit ball) yet fail to be compact, precluding the extraction of convergent subsequences.
- **Incompleteness of Subspaces:** Linear subspaces may exhibit algebraic closure without topological closure. For example, the subspace of polynomials is topologically incomplete in $C[a, b]$.

These structural disparities are visualized in [Figure 3.10](#).

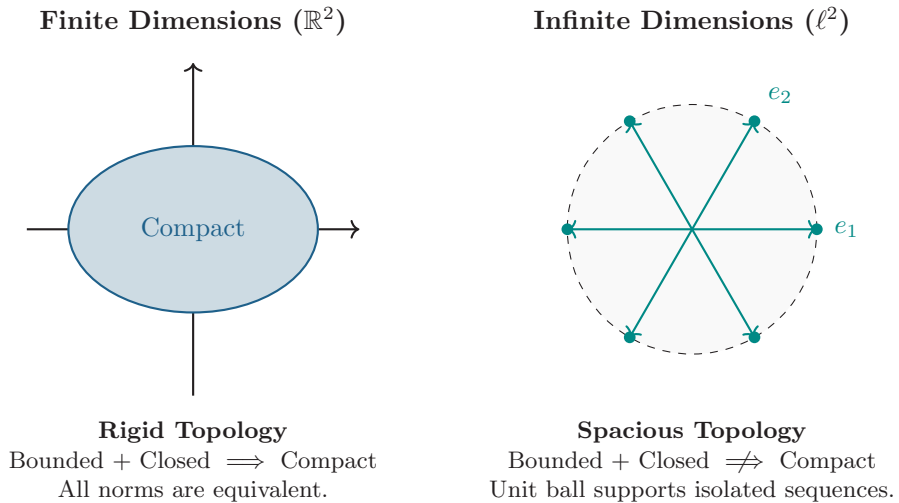


FIGURE 3.10 The topological divergence between finite and infinite dimensions.

In the subsequent sections, these properties are established. We prove that finite-dimensional normed spaces inherit the stability of Euclidean space $\mathbb{R}^n$. This establishes the baseline against which infinite-dimensional phenomena are evaluated.

Completeness of Finite Dimensional Spaces

Chapter 2 established that the Euclidean space $\mathbb{R}^n$ is complete. This property is not exclusive to the Euclidean norm; it is intrinsic to the finite dimension of the space.

To establish this completeness a bounding lemma is required. This lemma links the norm of a vector to the magnitude of its coordinate scalars, permitting the transfer of the Cauchy sequence property from the vector space X into the complete scalar field $\mathbb{K}$.

Lemma 3.25 Lower Bound Lemma

Let $\{e_1, \dots, e_n\}$ be a linearly independent set of vectors in a normed space X . There exists a constant $c > 0$ such that for any set of scalars $\alpha_1, \dots, \alpha_n$:

$$\left\| \sum_{j=1}^n \alpha_j e_j \right\| \geq c \left(\sum_{j=1}^n |\alpha_j| \right)$$

PROOF: Let $s = \sum_{j=1}^n |\alpha_j|$. If $s = 0$, the inequality is satisfied. For $s > 0$, normalize the coefficients by defining $\beta_j = \alpha_j/s$. Consequently, $\sum |\beta_j| = 1$. The bounding inequality simplifies to:

$$\left\| \sum_{j=1}^n \beta_j e_j \right\| \geq c$$

Assume, for contradiction, that no positive constant c exists. This implies the existence of a sequence of vectors (y_m) defined by $y_m = \sum_{j=1}^n \beta_j^{(m)} e_j$ satisfying two conditions:

1. The coefficients maintain unit magnitude: $\sum_{j=1}^n |\beta_j^{(m)}| = 1$ for all m .
2. The sequence norm vanishes: $\|y_m\| \rightarrow 0$ as $m \rightarrow \infty$.

Extracting Convergent Coefficients Consider the first coordinate coefficient $\beta_1^{(m)}$. Because $\sum |\beta_j^{(m)}| = 1$, the sequence $|\beta_1^{(m)}|$ is bounded by 1. By the Bolzano-Weierstrass Theorem, this sequence has a convergent subsequence. Let this limit be denoted by β_1 . This extraction process is iterated for each coefficient $j = 2, \dots, n$.

Following n applications of the theorem, we derive a subsequence (retaining the notation y_m) where every coordinate converges:

$$\beta_j^{(m)} \rightarrow \beta_j \quad \text{as } m \rightarrow \infty \quad \text{for each } j = 1, \dots, n$$

Evaluating the Limit Vector Because the sum $\sum_{j=1}^n |\beta_j^{(m)}| = 1$ for all m , the limit scalars must satisfy $\sum_{j=1}^n |\beta_j| = 1$. This guarantees that the limit scalars are not all zero. Construct the limit vector $y = \sum_{j=1}^n \beta_j e_j$.

The Contradiction Evaluating the norm of the difference between y_m and y :

$$\begin{aligned} \|y_m - y\| &= \left\| \sum_{j=1}^n (\beta_j^{(m)} - \beta_j) e_j \right\| \\ &\leq \sum_{j=1}^n |\beta_j^{(m)} - \beta_j| \|e_j\| \end{aligned}$$

As $m \rightarrow \infty$, the right side approaches 0 due to scalar convergence, establishing $y_m \rightarrow y$. However, by hypothesis, $\|y_m\| \rightarrow 0$, which implies $y_m \rightarrow \mathbf{0}$. By the uniqueness of limits, $y = \mathbf{0}$, thus:

$$\sum_{j=1}^n \beta_j e_j = \mathbf{0}$$

Since the basis vectors $\{e_j\}$ are linearly independent, this sum is zero if and

only if every coefficient $\beta_j = 0$. This contradicts the condition $\sum |\beta_j| = 1$. Therefore, the constant $c > 0$ exists. ■

This lemma permits the proof of finite-dimensional completeness, visualized in Figure 3.11.

Theorem 3.26 Completeness of Finite Dimensional Spaces

Every finite-dimensional normed space $(X, \|\cdot\|)$ is complete. Consequently, every finite-dimensional normed space is a Banach space.

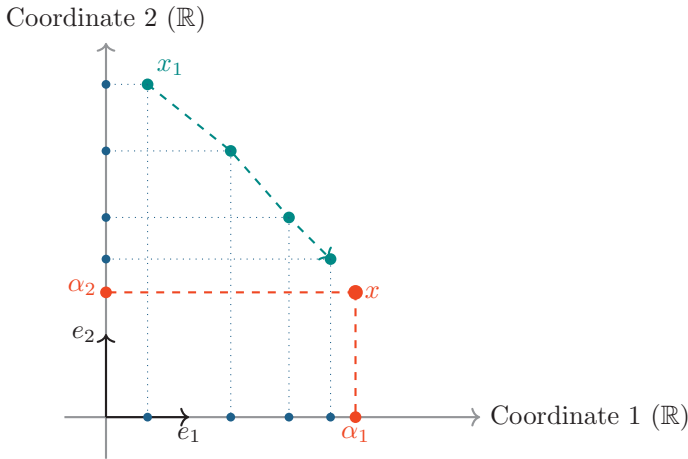


FIGURE 3.11 Theorem 3.30. The finite-dimensional vector sequence converges because its coordinate shadows converge within a complete field.

PROOF: Let X be a normed space characterized by dimension n . Let (x_m) denote a Cauchy sequence in X .

Coordinate Representation Let $\{e_1, \dots, e_n\}$ be a basis for X . Every vector x_m admits a unique representation:

$$x_m = \alpha_{m,1}e_1 + \alpha_{m,2}e_2 + \cdots + \alpha_{m,n}e_n$$

Coordinate-wise Cauchy Property Applying the Lower Bound Lemma, there exists a constant $c > 0$ such that for any indices $m, p \in \mathbb{N}$:

$$\|x_m - x_p\| = \left\| \sum_{j=1}^n (\alpha_{m,j} - \alpha_{p,j})e_j \right\| \geq c \sum_{j=1}^n |\alpha_{m,j} - \alpha_{p,j}|$$

Because (x_m) satisfies the Cauchy criterion, for any $\epsilon > 0$, there exists an

index N such that $\|x_m - x_p\| < c\epsilon$ for all $m, p > N$. Substituting this upper bound:

$$c \sum_{j=1}^n |\alpha_{m,j} - \alpha_{p,j}| \leq \|x_m - x_p\| < c\epsilon$$

Dividing by c yields $\sum |\alpha_{m,j} - \alpha_{p,j}| < \epsilon$. This implies that for each coordinate index j :

$$|\alpha_{m,j} - \alpha_{p,j}| < \epsilon$$

Consequently, every sequence of coordinate scalars $(\alpha_{1,j}, \alpha_{2,j}, \dots)$ independently forms a Cauchy sequence in the scalar field $\mathbb{K}$.

Convergence of Coordinates Because the scalar fields $\mathbb{R}$ and $\mathbb{C}$ are complete, every coordinate Cauchy sequence converges to a limit α_j :

$$\lim_{m \rightarrow \infty} \alpha_{m,j} = \alpha_j$$

Define the candidate vector $x \in X$ using these scalar limits:

$$x = \sum_{j=1}^n \alpha_j e_j$$

Verification of Norm Convergence Evaluating the limit of the vector sequence (x_m) toward x :

$$\begin{aligned} \|x_m - x\| &= \left\| \sum_{j=1}^n (\alpha_{m,j} - \alpha_j) e_j \right\| \\ &\leq \sum_{j=1}^n |\alpha_{m,j} - \alpha_j| \|e_j\| \end{aligned}$$

Let $M = \max_{1 \leq j \leq n} \|e_j\|$. Substituting this upper bound yields:

$$\|x_m - x\| \leq M \sum_{j=1}^n |\alpha_{m,j} - \alpha_j|$$

As $m \rightarrow \infty$, the scalar difference $|\alpha_{m,j} - \alpha_j| \rightarrow 0$ for each j . Because the summation is finite, the overall norm converges to 0, establishing that $x_m \rightarrow x$. ■

A critical corollary applies to finite-dimensional subspaces embedded within arbitrary normed spaces.

Theorem 3.27 Closure of Finite Dimensional Subspaces

Every finite-dimensional subspace Y contained in a normed space X is closed in X .

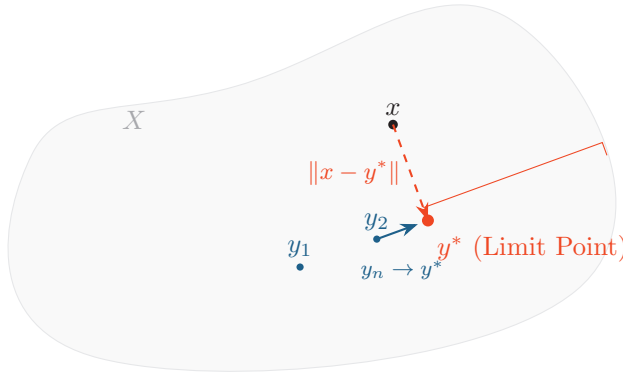


FIGURE 3.12 Theorem 3.27. Because a finite-dimensional subspace Y is complete, it contains all of its limit points, establishing closure.

PROOF: Consider the subspace Y equipped with the norm restricted from X . Because $\dim Y$ is finite, Theorem 3.26 implies that Y is a complete normed space. Let y be a limit point of Y . By definition, there exists a sequence (y_n) in Y such that $y_n \rightarrow y$. Because this sequence converges in X , it satisfies the Cauchy criterion. Because Y is complete, every Cauchy sequence in Y must converge to a limit in Y . Because limits are unique, the limit point y must belong to Y . Thus, Y contains all its limit points and is closed. ■

Example 3.28

Theorem 3.27 is restricted to finite dimensions. Infinite-dimensional linear subspaces frequently exhibit algebraic closure without satisfying topological closure.

- Evaluate the space $X = C[0, 1]$ equipped with the uniform norm.
- Let the subspace $Y \subset X$ consist of all polynomials of arbitrary degree.
- The partial sums of the Taylor series form a sequence of polynomials converging to the function e^t .
- Because $e^t \notin Y$, the limit lies outside the subspace boundary,

proving that the infinite-dimensional subspace Y is not closed in $C[0, 1]$.

The preceding section established the completeness of finite-dimensional spaces. This stability originates from a geometric principle: in finite dimensions, the choice of norm does not influence the convergence of sequences or the topology of open sets.

In $\mathbb{R}^2$, distance can be evaluated using the Euclidean norm ($\|x\|_2$) or the taxicab norm ($\|x\|_1$). While the numerical magnitudes differ, the definitions of convergence remain identical. A sequence converging under one norm must converge under the other. We formalize this invariance through equivalent norms.

Definition 3.29 Equivalent Norms

Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a vector space X are **equivalent** if there exist positive constants $c, C > 0$ such that for every vector $x \in X$:

$$c\|x\|_a \leq \|x\|_b \leq C\|x\|_a$$

Geometrically, this condition implies that the unit ball corresponding to one norm is bounded internally and externally by scaled evaluations of the unit ball of the alternative norm, as visualized in Figure 3.13. Equivalent norms generate the same topology. A set is open or bounded under $\|\cdot\|_a$ if and only if it is open or bounded under $\|\cdot\|_b$.

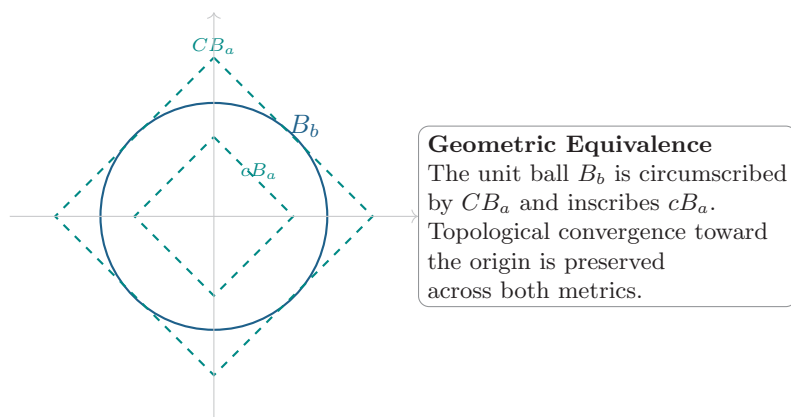


FIGURE 3.13 Equivalent norms on $\mathbb{R}^2$. The existence of bounding constants ensures the topologies are indistinguishable.

In infinite-dimensional spaces, norms frequently fail to be equivalent. For instance, in $C[0, 1]$, a sequence of functions may converge to zero under the integral norm (L^1) while diverging under the uniform norm (L^∞). However, such divergence is impossible in finite dimensions.

Theorem 3.30 Equivalence in Finite Dimensions

On a finite-dimensional vector space X , any two norms are equivalent.

PROOF: This proof relies upon compactness. Let $\dim X = n$ and let $\{e_1, \dots, e_n\}$ be a basis for X . We introduce a comparative *sum norm* defined by:

$$\|x\|_1 = \sum_{j=1}^n |\alpha_j| \quad \text{where } x = \sum_{j=1}^n \alpha_j e_j$$

To establish the theorem, it is sufficient to prove that any norm $\|\cdot\|$ is equivalent to this sum norm $\|\cdot\|_1$. General equivalence follows via transitivity.

Upper Bound (Continuity) Applying the Triangle Inequality (N4) and homogeneity (N3):

$$\|x\| = \left\| \sum_{j=1}^n \alpha_j e_j \right\| \leq \sum_{j=1}^n |\alpha_j| \|e_j\|$$

Define the constant $K = \max_j \|e_j\|$. This maximum exists because the basis vectors form a finite set. Substituting this upper bound yields

$$\|x\| \leq K \sum_{j=1}^n |\alpha_j| = K \|x\|_1$$

This inequality $\|x\| \leq K \|x\|_1$ guarantees that the norm mapping $x \mapsto \|x\|$ is continuous with respect to the topology induced by $\|\cdot\|_1$.

Lower Bound (Compactness) We must derive a constant $c > 0$ satisfying $\|x\| \geq c \|x\|_1$. Consider the unit sphere restricted to the sum norm:

$$S_1 = \{x \in X \mid \|x\|_1 = 1\}$$

Since the dimension is finite, S_1 is closed and bounded. By the Heine-Borel Theorem, S_1 is a compact set.

Consider the continuous mapping $f : X \rightarrow \mathbb{R}$ defined by $f(x) = \|x\|$. Because f evaluates a continuous function across a compact domain S_1 , the Extreme Value Theorem implies that f attains a minimum value c at a coordinate $x_0 \in S_1$:

$$c = \min_{x \in S_1} \|x\| = \|x_0\|$$

Since $x_0 \in S_1$, its sum norm equals 1, ensuring $x_0 \neq \mathbf{0}$. By the norm axiom

of definiteness (N2), $\|x_0\| > 0$. Consequently, $c > 0$. Therefore, for all $x \in S_1$, the inequality $\|x\| \geq c$ is satisfied.

To evaluate an arbitrary non-zero vector $y \in X$, scale it to intersect the unit sphere S_1 :

$$\frac{y}{\|y\|_1} \in S_1 \implies \left\| \frac{y}{\|y\|_1} \right\| \geq c$$

Extracting the scalar via homogeneity:

$$\frac{1}{\|y\|_1} \|y\| \geq c \implies \|y\| \geq c\|y\|_1$$

Synthesizing the upper and lower bounds establishes $c\|x\|_1 \leq \|x\| \leq K\|x\|_1$. ■

In elementary analysis, compactness guarantees the convergence of subsequences and the existence of extrema for continuous functions. For subsets in Euclidean space $\mathbb{R}^n$, the Heine-Borel Theorem establishes an equivalence: a set is compact if and only if it is closed and bounded.

The extension of this theorem to abstract normed spaces depends on the dimension of the space. To formalize this relationship, the sequential definition of compactness must be established.

Definition 3.31 Compactness

A subset M of a normed space X is **compact** if every sequence in M contains a convergent subsequence whose limit belongs to M . A subset M is **relatively compact** if its closure $\overline{M}$ is compact.

The geometric distinction between finite and infinite dimensions is that finite dimensions impose sufficient rigidity such that boundedness guarantees sequential clustering, as visualized in [Figure 3.14](#).

Theorem 3.32 Compactness in Finite Dimensions

Let X be a finite-dimensional normed space. A subset $M \subset X$ is a compact set if and only if M is closed and bounded.

PROOF:

Necessity (Compact implies Closed and Bounded) Assume M is compact. This condition applies to all metric spaces, irrespective of dimension.

1. **Boundedness:** Assume M is unbounded. For any $n \in \mathbb{N}$, a vector $x_n \in M$ can be selected satisfying $\|x_n\| > n$. The sequence (x_n) diverges in norm to infinity. Because convergent sequences are bounded, no subsequence can converge. This contradiction implies that M must be bounded.

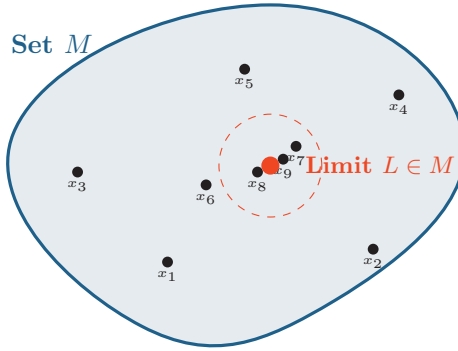


FIGURE 3.14 Compactness in finite dimensions. A closed and bounded boundary forces any infinite sequence to cluster around a limit point in the set.

2. **Closedness:** Let x be a limit point of M . By definition, there exists a sequence (x_n) in M converging to x . Because M is compact, this sequence must possess a subsequence converging to a limit in M . Because the original sequence converges to x , the uniqueness of limits forces the subsequence to converge to x . Thus, $x \in M$, verifying closure.

Sufficiency (Closed and Bounded implies Compact) Assume M is closed and bounded. This direction requires the space to have a finite dimension. Let $\dim X = n$ and designate $\{e_1, \dots, e_n\}$ as a basis. Let (u_m) denote a sequence in M .

1. **Coordinate Representation:** Because X is finite-dimensional, each vector u_m admits a unique expansion:

$$u_m = \alpha_{m,1}e_1 + \alpha_{m,2}e_2 + \cdots + \alpha_{m,n}e_n$$

Since M is bounded, there exists a constant k satisfying $\|u_m\| \leq k$ for all m .

2. **Boundedness of Coordinates:** Applying the Lower Bound Lemma, there exists a constant $c > 0$ such that:

$$k \geq \|u_m\| = \left\| \sum_{j=1}^n \alpha_{m,j}e_j \right\| \geq c \sum_{j=1}^n |\alpha_{m,j}|$$

This implies that for each j , the corresponding sequence $(\alpha_{m,j})$ is bounded in $\mathbb{K}$:

$$|\alpha_{m,j}| \leq \sum_{i=1}^n |\alpha_{m,i}| \leq \frac{k}{c}$$

3. **Extracting Convergent Subsequences:** By repeatedly applying the Bolzano-Weierstrass Theorem to each bounded coordinate sequence, we extract nested subsequences until arriving at a final vector subsequence (v_m) where every coordinate converges:

$$v_{m,j} \rightarrow v_j \quad \text{as } m \rightarrow \infty, \quad (j = 1, \dots, n)$$

4. **Norm Convergence and Closure:** Define the limit vector $u = \sum_{j=1}^n v_j e_j$. Evaluating convergence in the metric space:

$$\|v_m - u\| = \left\| \sum_{j=1}^n (v_{m,j} - v_j) e_j \right\| \leq \sum_{j=1}^n |v_{m,j} - v_j| \|e_j\|$$

As $m \rightarrow \infty$, the coordinate differences converge to zero, forcing $\|v_m - u\| \rightarrow 0$. Because M is closed and the subsequence (v_m) resides in M , the limit u belongs to M . Thus, M is compact. ■

To evaluate compactness in infinite-dimensional spaces, a geometric mechanism for constructing separated vectors is required. Riesz's Lemma formalizes the existence of a unit vector that maintains a minimum distance from any proper closed subspace, as visualized in [Figure 3.15](#).

Theorem 3.33 Riesz's Lemma

Let W be a subspace of a normed space X , and Y a proper closed subspace of W . For every real scalar $t \in (0, 1)$, there exists a vector $w \in W$ satisfying $\|w\| = 1$ such that:

$$\|w - y\| \geq t \quad \text{for all } y \in Y$$

PROOF:

Existence of a Topological Gap Because Y is a proper subspace, there exists a vector $v \in W \setminus Y$. Because Y is closed, the distance from v to the subspace Y must be positive. Let d denote this distance:

$$d = \text{dist}(v, Y) = \inf_{y \in Y} \|v - y\| > 0$$

Approximating the Infimum If $t \in (0, 1)$, then $d < d/t$. By the definition of the infimum, there must exist a vector $y_0 \in Y$ satisfying:

$$d \leq \|v - y_0\| < \frac{d}{t}$$

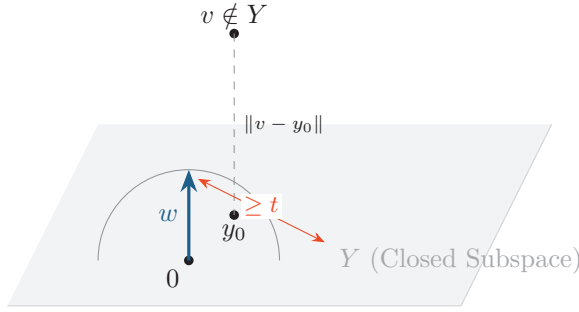


FIGURE 3.15 Riesz's lemma. While infinite-dimensional spaces lack the guarantee of orthogonal projections, one can construct a unit vector w that approximates orthogonality, maintaining a minimum distance $t < 1$ from the subspace Y .

Constructing the Candidate Vector Define w as the normalized vector connecting y_0 to v . Because $v \in W$ and $y_0 \in Y \subset W$, the difference remains in W :

$$w = \frac{v - y_0}{\|v - y_0\|} \in W$$

Note that by definition, $\|w\| = 1$.

Verifying the Distance Constraint To confirm that w maintains the required distance t from all vectors $y \in Y$, evaluate the norm:

$$\|w - y\| = \left\| \frac{v - y_0}{\|v - y_0\|} - y \right\| = \frac{1}{\|v - y_0\|} \|v - y_0 - \|v - y_0\|y\|$$

Isolate the internal vector grouping:

$$v - \underbrace{(y_0 + \|v - y_0\|y)}_{\in Y}$$

Since Y is a linear subspace, the grouped term is equivalent to an alternative vector $y' \in Y$. By the initial definition of the infimum distance d , the inequality $\|v - y'\| \geq d$ is forced. Substituting this bound yields

$$\|w - y\| \geq \frac{d}{\|v - y_0\|}$$

From the infimum approximation, we established $\|v - y_0\| < d/t$, which algebraically inverts to $\frac{1}{\|v - y_0\|} > \frac{t}{d}$. Synthesizing these inequalities:

$$\|w - y\| > d \cdot \frac{t}{d} = t$$

Thus, the condition $\|w - y\| \geq t$ is satisfied. ■

This lemma is used to prove the theorem establishing that geometric compactness is exclusively a finite-dimensional property.

Theorem 3.34 Compactness and Dimension

The closed unit ball $\overline{B}(\mathbf{0}; 1)$ of a normed space X is a compact set if and only if X is finite-dimensional.

PROOF:

($\Leftarrow$) If X is finite-dimensional, the closed unit ball $\overline{B}(\mathbf{0}; 1)$ is closed and bounded. By Theorem 3.32, it is compact.

($\Rightarrow$) We prove the contrapositive: If $\dim X = \infty$, the unit ball fails the criteria for compactness. We construct a sequence (x_n) situated on the unit sphere, from which no convergent subsequence can be extracted.

1. Select an initial vector x_1 satisfying $\|x_1\| = 1$. Define the subspace $Y_1 = \text{span}\{x_1\}$. Because Y_1 has a finite dimension, it is closed. Because X is infinite-dimensional, Y_1 is a proper subspace ($Y_1 \neq X$).
2. Applying Riesz's Lemma for $t = 1/2$, there exists a unit vector x_2 such that $\|x_2 - x_1\| \geq 1/2$.
3. Define $Y_2 = \text{span}\{x_1, x_2\}$. Applying Riesz's Lemma to this closed, proper subspace, there exists a unit vector x_3 such that $\|x_3 - y\| \geq 1/2$ for all $y \in Y_2$. This guarantees $\|x_3 - x_1\| \geq 1/2$ and $\|x_3 - x_2\| \geq 1/2$.
4. Iterating this geometric construction indefinitely yields an infinite sequence (x_n) satisfying $\|x_n - x_m\| \geq 1/2$ for all $n \neq m$.

This sequence violates the Cauchy criterion, guaranteeing that no extracted subsequence can converge. Consequently, the unit ball in an infinite-dimensional space cannot be compact. ■

The topological stability of compact sets is preserved when mapped through continuous operators. This preservation is essential for establishing the existence of extrema in optimization theory.

Theorem 3.35 Preservation of Compactness

Let X and Y represent metric or normed spaces, and let $T : X \rightarrow Y$ denote a continuous mapping. If M is a compact subset in X , its image $T(M)$ forms a compact subset in Y .

PROOF: We execute the proof utilizing the sequential definition of compactness. Let (y_n) be a sequence contained in the image $T(M)$. By definition, there exists a corresponding pre-image sequence (x_n) in M such that $y_n = T(x_n)$.

Since the domain M is compact, the sequence (x_n) possesses a subsequence (x_{n_k}) converging to a limit $x \in M$:

$$x_{n_k} \rightarrow x \quad \text{as } k \rightarrow \infty$$

Since the operator T is continuous, limits are preserved across the mapping:

$$y_{n_k} = T(x_{n_k}) \rightarrow T(x)$$

Define the mapped limit $y = T(x)$. Because x is in M , the limit y is in $T(M)$. Consequently, the sequence (y_n) yields a subsequence (y_{n_k}) converging to an element of $T(M)$, confirming that $T(M)$ is compact. ■

Corollary 3.36 Extreme Value Theorem

A continuous mapping T from a compact domain $M \subset X$ into the scalar field $\mathbb{R}$ must attain a maximum and a minimum in M .

PROOF: By Theorem 3.35, the scalar image $T(M)$ is a compact subset of $\mathbb{R}$. By the Heine-Borel theorem for real numbers, $T(M)$ must be both closed and bounded. Since the set is bounded, its supremum and infimum exist as finite real numbers. Because the set is closed, it must contain these limit points. ■

EXERCISES

1. *A Concrete Closed Subspace.* Consider the space $X = \mathbb{R}^2$ equipped with the standard Euclidean norm. Let Y represent the subspace mapping the relation $y = x$:

$$Y = \{(\xi, \xi) \in \mathbb{R}^2 \mid \xi \in \mathbb{R}\}$$

- (a) Let (y_n) represent a sequence in Y converging to a vector $z \in \mathbb{R}^2$. Define $y_n = (\xi_n, \xi_n)$ and $z = (\eta_1, \eta_2)$. Utilizing the definition of convergence in $\mathbb{R}^2$, prove that $\eta_1 = \eta_2$.
 - (b) Deduce that $z \in Y$, proving that Y is closed.
2. *The Polynomial Contrast (P_n vs. $\mathcal{P}$).* Let $X = C[0, 1]$ equipped with the uniform norm.
 - (a) Let Y_n denote the subspace of polynomials restricted to a maximum degree n (possessing algebraic dimension $n + 1$). Is Y_n complete? Is it a closed set in $C[0, 1]$? Justify the result utilizing Theorem 3.27.
 - (b) Let Y represent the subspace of all polynomials of arbitrary degree. The Weierstrass Approximation Theorem states that Y is dense in $C[0, 1]$, that is, $\overline{Y} = X$. Prove that because $Y \neq X$, Y is not closed.

- (c) Explain why the conditions of Theorem 3.27 fail to apply to the subspace Y in part (b).
3. *Ranges of Linear Operators.* Let X and Y be normed spaces, and let $T : X \rightarrow Y$ denote a bounded linear operator. Prove that if the domain X is finite-dimensional, the range $\mathcal{R}(T) = \{Tx \mid x \in X\}$ is a closed subspace of Y . (*Hint: Utilize linear algebraic principles to verify the algebraic dimension of the image, then apply Theorem 3.27.*)
4. *Completeness of Matrix Spaces.* Let $\mathcal{M}_{n \times n}$ denote the vector space of all $n \times n$ matrices over $\mathbb{R}$. Identify the dimension of this vector space to prove that $\mathcal{M}_{n \times n}$ constitutes a complete Banach space under any matrix norm (e.g., the Frobenius norm).
5. *Finite vs. Infinite Sums.* Let Y be a subspace of a Banach space X .
- Show that if Y is finite-dimensional, it is closed, implying that any convergent sequence $y_n \rightarrow y$ (where $y_n \in Y$) forces $y \in Y$.
 - Consider the space $X = \ell^2$. Let Y be the subspace of sequences possessing finitely many non-zero terms (the infinite linear span of the basis vectors e_n). Prove that the sequence $z = (1, 1/2, 1/4, \dots)$ belongs to the closure $\overline{Y}$ but $z \notin Y$. Explain why this exclusion does not contradict Theorem 3.27.
6. *Concrete Equivalence on $\mathbb{R}^2$.* Let $X = \mathbb{R}^2$ containing the vector $x = (\xi_1, \xi_2)$. Consider the Euclidean norm $\|x\|_2 = (\xi_1^2 + \xi_2^2)^{1/2}$ and the Max norm $\|x\|_\infty = \max(|\xi_1|, |\xi_2|)$. Derive the bounding constants c and C required to satisfy:

$$c\|x\|_\infty \leq \|x\|_2 \leq C\|x\|_\infty$$

(*Hint: Geometrically sketch the unit square and the unit circle. Determine the radius of the circle inscribed within the square, and the circle circumscribing the square.*)

7. *Failure in Infinite Dimensions ($C[0, 1]$).* Consider the infinite-dimensional space $X = C[0, 1]$. Define $\|\cdot\|_1$ as the integral norm: $\|f\|_1 = \int_0^1 |f(t)| dt$. Define $\|\cdot\|_\infty$ as the uniform norm: $\|f\|_\infty = \max_{t \in [0, 1]} |f(t)|$.
- Prove that $\|f\|_1 \leq \|f\|_\infty$ for all $f \in X$. (This establishes the upper bound of the equivalence relation).
 - Prove that no positive constant $c > 0$ exists satisfying $c\|f\|_\infty \leq \|f\|_1$ for all f . (*Hint: Construct a sequence of functions f_n exhibiting geometric properties akin to triangles possessing a height of n and a base width of $1/n^2$. Demonstrate that the resulting integral area converges to 0 while the maximum magnitude diverges.*)
8. *Topological Equivalence.* Assume $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms on a vector space X . Prove that a subset $U \subset X$ is an open set with respect to $\|\cdot\|_a$ if and only if it is an open set with respect to $\|\cdot\|_b$.

9. *Matrix Norms.* Let X denote the vector space of all $n \times n$ matrices, which maps to the finite-dimensional space $\mathbb{R}^{n^2}$. Utilizing the Equivalence Theorem, justify why a sequence of matrices converging under the Frobenius norm ($\|A\|_F = \sqrt{\sum |a_{ij}|^2}$) must converge under the induced Operator norm ($\|A\|_{op} = \sup_{\|x\|=1} \|Ax\|$).

3.7 ROTUNDITY AND SMOOTHNESS

The topological properties of a normed space are determined by the shape of its unit ball. While the triangle inequality guarantees that the closed unit ball of every normed space forms a convex set, it does not prevent the boundary—the unit sphere—from possessing flat facets.

To preclude these flat edges, analysis requires a stronger condition known as strict convexity, historically referred to as rotundity.

Definition 3.37 Strict Convexity (Rotundity)

A normed space X is **strictly convex** (or **rotund**) if for any two distinct vectors $x, y \in X$ residing on the unit sphere ($\|x\| = \|y\| = 1$ and $x \neq y$), the strict inequality holds:

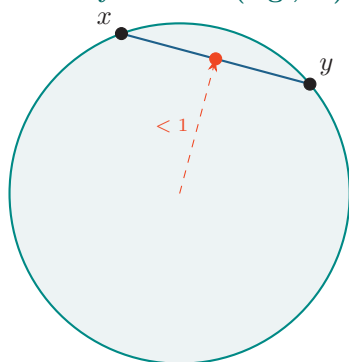
$$\left\| \frac{x + y}{2} \right\| < 1$$

Geometrically this means that the midpoint of any chord connecting two distinct points on the unit sphere must lie strictly in the interior of the open unit ball. If the space is not strictly convex, distinct unit vectors can share a midpoint that remains exactly on the boundary, indicating a flat edge.

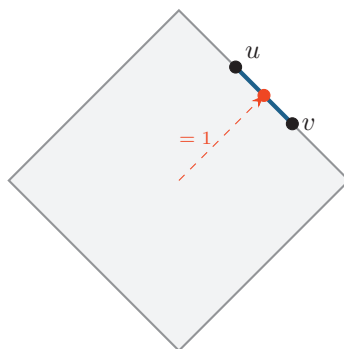
The property of strict convexity is instrumental for optimization theory. In a strictly convex space, every point outside a closed convex set possesses exactly one unique closest point within that set.

Example 3.38 Convexity of Sequence Spaces

- **Hilbert Spaces:** The space ℓ^2 is strictly convex. The parallelogram law forces the strict inequality for midpoints.
- **The L^p Spaces:** For any $1 < p < \infty$, the spaces ℓ^p and $L^p[a, b]$ are strictly convex.
- **Non-Rotund Spaces:** The space ℓ^1 (the Taxicab norm) and ℓ^∞ (the supremum norm) fail to be strictly convex. Their unit balls possess flat "facets" that permit infinite sets of vectors to share identical distances to external targets.

Strictly Convex (e.g., ℓ^2)

The midpoint falls strictly inside the unit ball.

Not Strictly Convex (e.g., ℓ^1)

The midpoint remains on the unit sphere.

FIGURE 3.16 Geometric comparison of rotundity. In a strictly convex space, any convex combination of distinct unit vectors yields a vector with a length strictly less than 1.

The geometric dual to strict convexity is smoothness. Where strict convexity prohibits flat edges on the unit sphere, smoothness prohibits sharp corners.

In finite-dimensional calculus, a sharp corner on a curve implies that the derivative is undefined at that specific location, permitting multiple distinct tangent lines to touch the curve simultaneously. In functional analysis, a “tangent line” generalizes to a supporting hyperplane, formalized by bounded linear functionals.

Definition 3.39 Smoothness

A normed space X is **smooth** if at every point x on the unit sphere ($\|x\| = 1$), there exists a unique supporting hyperplane. That is, there exists a *unique* bounded linear functional $f \in X^*$ satisfying:

$$\|f\| = 1 \quad \text{and} \quad f(x) = 1$$

The Hahn-Banach Theorem (explored thoroughly in [Chapter 5](#)) guarantees that *at least one* such norming functional exists for every point on the unit sphere. Smoothness is the precise structural property dictating whether this functional is unique.

The relationship between rotundity and smoothness highlights the elegant duality of Banach spaces. If a Banach space X is strictly convex, its dual space X^* is smooth. Conversely, if X is smooth, its dual space X^* is strictly convex.

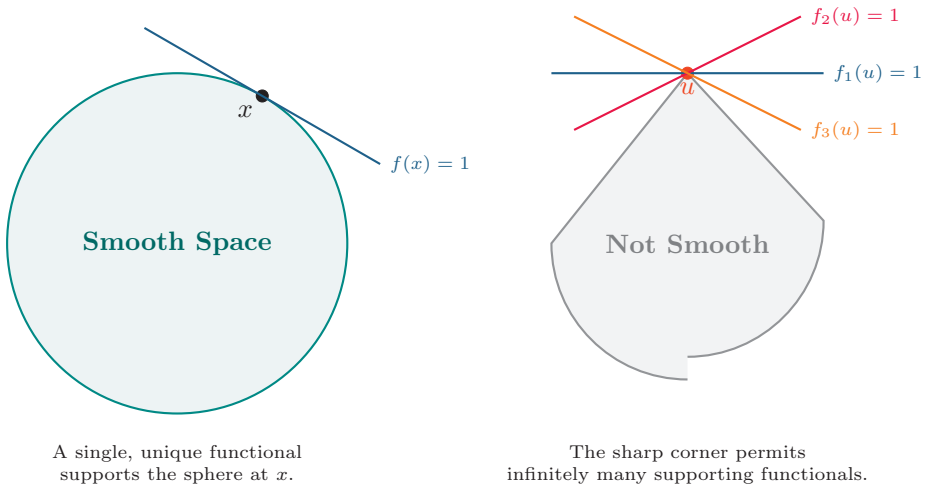


FIGURE 3.17 Geometric comparison of smoothness. The lack of smoothness corresponds to points of non-differentiability on the norm's surface, allowing the Hahn-Banach extension to branch into multiple valid mappings.

Hilbert spaces (such as ℓ^2) hold a privileged position in functional analysis because they are the only spaces whose geometry is perfectly balanced: they are both strictly convex and smooth.

The geometric definition of smoothness—the existence of a unique supporting hyperplane at every point on the unit sphere—requires a rigorous analytical test. In finite-dimensional calculus, a unique tangent line corresponds to the existence of a derivative. To apply this principle to infinite-dimensional Banach spaces, the concept of differentiation must be generalized to directional derivatives along abstract vectors.

This generalization is achieved through the Gâteaux derivative, named after René Gâteaux.

Definition 3.40 The Gâteaux Derivative

Let X be a normed space and let $F : X \rightarrow \mathbb{R}$ be a continuous functional. The Gâteaux differential of F at a point $x \in X$ in the direction of a vector $h \in X$ is defined as the limit:

$$dF(x; h) = \lim_{t \rightarrow 0} \frac{F(x + th) - F(x)}{t}$$

where t is a real scalar.

If this limit exists for all $h \in X$, and if the mapping $h \mapsto dF(x; h)$

is a bounded linear functional on X , then F is said to be **Gâteaux differentiable** at x . The corresponding linear functional is denoted as $F'(x)$, such that $dF(x; h) = F'(x)h$.

The Gâteaux derivative measures the instantaneous rate of change of the functional F as one moves away from x strictly along the straight-line path determined by h . By defining $g(t) = F(x + th)$, the Gâteaux derivative simplifies to the standard scalar derivative $g'(0)$.

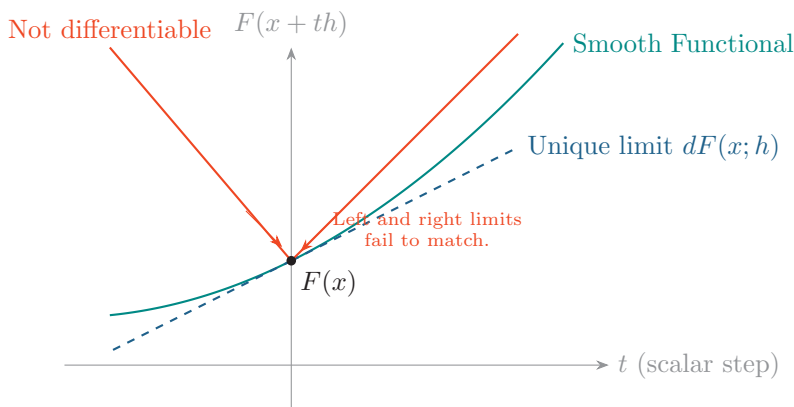


FIGURE 3.18 Visualizing the Gâteaux derivative. The dashed line provides the unique directional limit for the smooth functional, while the red corner represents a divergence in directional limits.

The profound connection between calculus and Banach space geometry emerges when the Gâteaux derivative is applied directly to the norm function, $F(x) = \|x\|$. The existence of this derivative dictates the smoothness of the unit sphere.

EXERCISES

1. *Proving Failure of Rotundity.* Let $X = \ell^\infty$ evaluated in $\mathbb{R}^2$. Consider the vectors $x = (1, 1)$ and $y = (1, -1)$.
 - (a) Verify that both x and y reside on the unit sphere ($\|x\|_\infty = 1$ and $\|y\|_\infty = 1$).
 - (b) Calculate the midpoint $z = \frac{x+y}{2}$.
 - (c) Show that $\|z\|_\infty = 1$, and conclude that ℓ^∞ is not strictly convex.

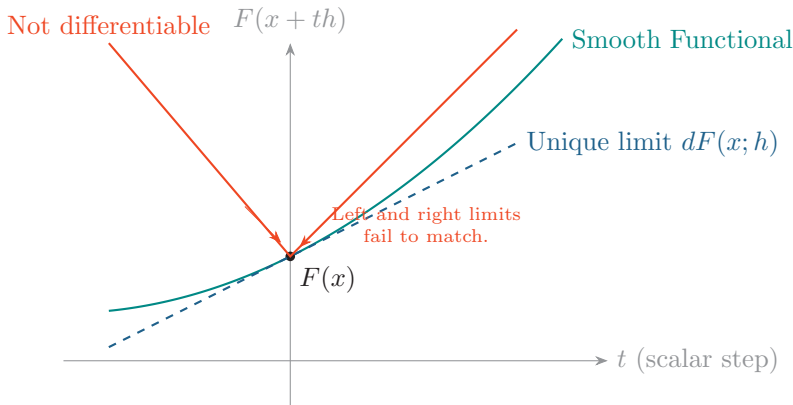


FIGURE 3.19 Visualizing the Gâteaux derivative. The dashed line provides the unique directional limit for the smooth functional, while the red corner represents a divergence in directional limits.

2. *Rotundity of Euclidean Space.* Using the strict condition for equality in the Cauchy-Schwarz inequality, prove that $\mathbb{R}^n$ equipped with the Euclidean metric (ℓ^2 norm) is strictly convex.
3. *Failure of Smoothness.* Let $X = \ell^1$ evaluated in $\mathbb{R}^2$. The unit sphere forms a diamond.
 - (a) Locate the "corner" at the unit vector $u = (1, 0)$.
 - (b) Construct two distinct linear functionals $f_1, f_2 \in (\mathbb{R}^2)^*$ such that $\|f_1\| = \|f_2\| = 1$ and both functionals yield $f_1(u) = 1$ and $f_2(u) = 1$. (Hint: Recall that the dual of ℓ^1 is ℓ^∞ . Select two distinct vectors from the ℓ^∞ unit square that yield a dot product of 1 with u .)
4. *Gâteaux Differentiability of Euclidean Space.* Let $X = \mathbb{R}^2$ equipped with the standard Euclidean norm $\|x\|_2 = \sqrt{\xi_1^2 + \xi_2^2}$. Let $x = (1, 0)$ and select an arbitrary direction vector $h = (h_1, h_2)$.
 - (a) Formulate the expression for $\|x + th\|_2$.
 - (b) Calculate the Gâteaux derivative $d\|\cdot\|_2(x; h) = \lim_{t \rightarrow 0} \frac{\|x + th\|_2 - \|x\|_2}{t}$ using standard calculus limit techniques.
 - (c) Verify that the resulting expression is a linear functional with respect to h , confirming that ℓ^2 is a smooth space.

Linear Operators and Functionals

4.1 Linear Operators

4.2 Bounded Operators

4.3 Linear Functionals

4.4 Dual Spaces

4.5 Adjoint Operators

4.6 Operators on Finite-Dimensional Spaces

4.7 Normed Spaces of Linear Operators

FUNCTIONAL analysis is not merely the study of abstract spaces, but the study of how these spaces interact through mappings. Having established the “where” (metric and Banach spaces), we now turn our attention to the “how”—the **Linear Operators** that transform one space into another. These operators serve as the high-dimensional generalizations of matrices, representing fundamental analytical processes such as differentiation and integration.

In this chapter, we bridge the gap between algebra and topology by exploring the equivalence between **boundedness** and **continuity**. We will investigate the structure of the **Dual Space**, where the measurements we perform on vectors (functionals) themselves become vectors in a new space. Finally, we establish the theory of **Adjoint Operators** and **Reflexivity**, providing the tools necessary to evaluate operator equations and the convergence of operator sequences in a rigorous framework.

4.1 LINEAR OPERATORS

In linear algebra, matrices map $\mathbb{R}^n$ to $\mathbb{R}^m$. Functional analysis generalizes this to linear operators, which map one normed space to another. Because the points in these spaces are frequently functions, these operators represent transformations such as differentiation, integration, or signal filtering. The primary focus will be bounded operators, as boundedness is equivalent to continuity for linear mappings.

Whereas calculus examines functions mapping real numbers to real numbers, functional analysis evaluates mappings that preserve the algebraic structure of the vector spaces they connect.

Definition 4.1 Linear Operator

A linear operator T is a mapping defined such that:

1. The domain $\mathcal{D}(T)$ is a vector space X , and the range $\mathcal{R}(T)$ lies in a vector space Y defined over the identical scalar field.
2. For all vectors $u, v \in \mathcal{D}(T)$ and all scalars α, β :

$$T(\alpha u + \beta v) = \alpha Tu + \beta Tv$$

The linearity property encapsulates additivity ($T(u + v) = Tu + Tv$) and homogeneity ($T(\alpha u) = \alpha Tu$). These fundamental axioms imply that a linear operator maps the zero vector of the domain to the zero vector of the codomain: $T(\mathbf{0}) = \mathbf{0}$.

Associated with every linear operator is the *null space* (or kernel) defined as $\mathcal{N}(T) = \{x \in \mathcal{D}(T) \mid Tx = \mathbf{0}\}$. As a consequence of operator linearity, the null space is inherently a vector subspace of the domain $\mathcal{D}(T)$ as well as the range $\mathcal{R}(T)$. The proofs of these are left to the reader.

A linear operator $T : X \rightarrow Y$ is a *homomorphism* because it preserves vector addition and scalar multiplication. If T is bijective (injective and surjective), the inverse T^{-1} exists and is linear. In this case, T is an *isomorphism*, and the spaces X and Y are *isomorphic*. Algebraically, isomorphic spaces operate identically under vector addition and scalar multiplication. Any n -dimensional real vector space is isomorphic to $\mathbb{R}^n$.

Example 4.2

The following operators demonstrate the linearity property.

1. **Identity Operator:** The operator $I_X : X \rightarrow X$ defined by $I_X u = u$ for all $u \in X$ is linear.
2. **Zero Operator:** The operator $0 : X \rightarrow Y$ defined by $0u = \mathbf{0}$ for all $u \in X$ is linear.

3. **Differentiation Operator:** Let X be the space of polynomials on $[a, b]$. Define T as:

$$(Tu)(t) = \frac{du}{dt}$$

Since the derivative of a linear combination is the linear combination of the derivatives, T is linear.

4. **Integration Operator:** Let $X = C[a, b]$. Define T by:

$$(Tu)(t) = \int_a^t u(\tau) d\tau$$

By the linearity of integration, $T(\alpha u + \beta v) = \alpha Tu + \beta Tv$.

5. **Multiplication by t :** In $C[a, b]$, let T be defined by $(Tu)(t) = tu(t)$. This is linear because $t(\alpha u(t) + \beta v(t)) = \alpha tu(t) + \beta tv(t)$.
6. **Elementary Matrix Operators:** Let A be a real $m \times n$ matrix. Define $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ as the standard matrix-vector product. If u is a vector in $\mathbb{R}^n$, then:

$$Tu = Au$$

Linearity follows from the distributive and associative properties of matrix algebra: $A(\alpha u + \beta v) = \alpha(Au) + \beta(Av)$. $\square$

The dimension of a vector space quantifies its degrees of freedom. When a linear operator acts on a finite-dimensional space, the distribution of these dimensions is established by the Rank-Nullity Theorem, visualized in [Figure 4.1](#).

Theorem 4.3 Dimension of the Range

Let $T : \mathcal{D}(T) \rightarrow Y$ be a linear operator with a finite-dimensional domain $\mathcal{D}(T)$. Then the range $\mathcal{R}(T)$ is finite-dimensional, and its dimension satisfies the identity:

$$\dim \mathcal{D}(T) = \dim \mathcal{N}(T) + \dim \mathcal{R}(T)$$

PROOF: Let $\dim \mathcal{D}(T) = n$ and let the null space $\mathcal{N}(T)$ have dimension k . We construct a basis for the domain that separates the vectors mapped to zero by T from those preserved by T .

Construction of the Basis Let $\{e_1, e_2, \dots, e_k\}$ be a basis for the null space $\mathcal{N}(T)$. Because $\mathcal{N}(T)$ is a subspace of $\mathcal{D}(T)$, the Basis Extension Theorem permits adding $n - k$ vectors to form a complete basis for the domain $\mathcal{D}(T)$:

$$\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$$

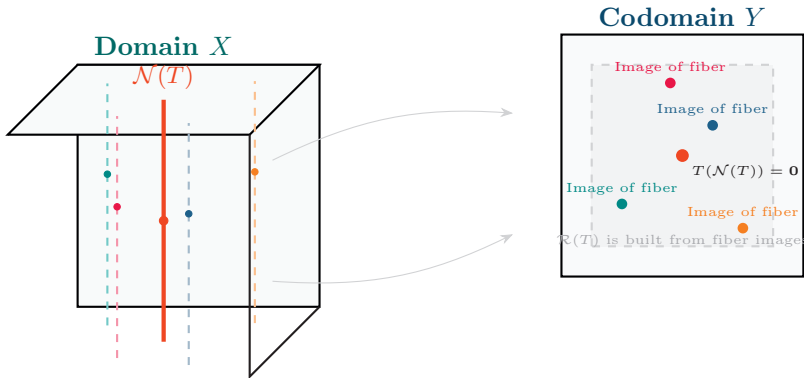


FIGURE 4.1 Rank-Nullity Visualization. The domain X decomposes into fibers parallel to the Null Space. Each fiber in the domain corresponds to a one point in the codomain. The Null Space (red) is the unique fiber that collapses to the zero vector.

Mapping to the Range Let $w_j = Te_j$ for the indices $j = k + 1, \dots, n$. We will establish that the set $\{w_{k+1}, \dots, w_n\}$ forms a basis for the range $\mathcal{R}(T)$.

Spanning Property Take any vector $w \in \mathcal{R}(T)$. By definition, $w = Tu$ for some $u \in \mathcal{D}(T)$. We represent u in the extended basis: $u = \sum_{i=1}^n \alpha_i e_i$. By the linearity of T :

$$w = T \left(\sum_{i=1}^n \alpha_i e_i \right) = \sum_{i=1}^k \alpha_i T e_i + \sum_{i=k+1}^n \alpha_i T e_i$$

Since $e_1, \dots, e_k$ are in the null space, $T e_i = \mathbf{0}$ for $i \leq k$. Thus:

$$w = \sum_{i=k+1}^n \alpha_i w_i$$

proving that $\{w_{k+1}, \dots, w_n\}$ spans $\mathcal{R}(T)$.

Linear Independence Assume $\sum_{i=k+1}^n \beta_i w_i = \mathbf{0}$. By linearity, $T \left(\sum_{i=k+1}^n \beta_i e_i \right) = \mathbf{0}$. This implies the vector $v = \sum_{i=k+1}^n \beta_i e_i$ belongs to $\mathcal{N}(T)$. If $v \in \mathcal{N}(T)$, it must be representable as a combination of the first k basis vectors $\{e_1, \dots, e_k\}$. Because the set $\{e_1, \dots, e_n\}$ is linearly independent, the only vector that can be represented by both the first k vectors and the remaining $n - k$ vectors is the zero vector. Thus, $\beta_i = 0$ for all i .

The set $\{w_{k+1}, \dots, w_n\}$ is therefore a basis for $\mathcal{R}(T)$. Its dimension is $n - k$. We conclude:

$$\dim \mathcal{R}(T) = n - k = \dim \mathcal{D}(T) - \dim \mathcal{N}(T)$$

Rearranging yields the identity. ■

The following theorem establishes the algebraic rule for inverting a composition of operators, formalizing the reversal of sequential operations, as visualized in Figure 4.2.

Theorem 4.4 Inverse of a Product

Let X, Y , and Z be vector spaces. Let $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ be bijective linear operators. The product $ST : X \rightarrow Z$ is invertible, and its inverse equals the product of the inverses in reverse order:

$$(ST)^{-1} = T^{-1}S^{-1}$$

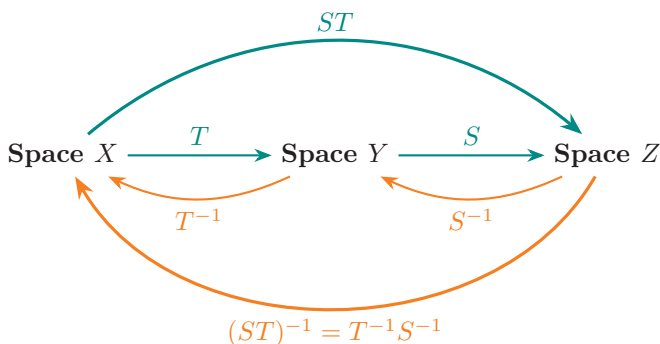


FIGURE 4.2 The composition of operators and their inverses. The inverse of the sequence maps backward through the intermediate space Y .

PROOF: To prove that $T^{-1}S^{-1}$ is the inverse of ST , it must be shown that their composition yields the identity operator on the respective domains.

Existence Because T and S are bijective (injective and surjective), their composition ST is a bijection mapping X onto Z . Therefore, the inverse mapping $(ST)^{-1}$ exists.

Left Inverse Property Apply the proposed inverse to the product ST . For any vector $u \in X$:

$$(T^{-1}S^{-1})(ST)u = T^{-1}(S^{-1}(S(Tu)))$$

By the associativity of operator multiplication, regroup the terms. Because S^{-1} is the inverse of S , $S^{-1}S$ is the identity operator I_Y on space Y :

$$T^{-1}(S^{-1}S)(Tu) = T^{-1}(I_Y)(Tu) = T^{-1}(Tu)$$

Because $T^{-1}T$ is the identity operator I_X on space X :

$$T^{-1}(Tu) = (T^{-1}T)u = I_X u = u$$

Thus, $(T^{-1}S^{-1})(ST) = I_X$.

Right Inverse Property The proof for the right inverse is analogous, establishing $(ST)(T^{-1}S^{-1}) = I_Z$.

Because the operator $T^{-1}S^{-1}$ acts as both a left and right inverse to ST , it is the unique inverse $(ST)^{-1}$. ■

Since operator multiplication is generally non-commutative, the order of factors must be preserved. When inverting transformations, the sequence of inverted operators is reversed.

EXERCISES

1. *Verifying Linearity.* Determine whether the following mappings T are linear operators. If not, identify which property (additivity or homogeneity) fails.
 - (a) *Euclidean Translation:* $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(u) = u + v_0$ where $v_0 \neq \mathbf{0}$ is a fixed vector.
 - (b) *Modulus:* $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $T(u) = \|u\|_2$.
 - (c) *Evaluation:* $T : C[a, b] \rightarrow \mathbb{R}$ defined by $T(u) = u(c)$ for a fixed $c \in [a, b]$.
 - (d) *Squaring:* $T : C[a, b] \rightarrow C[a, b]$ defined by $(Tu)(t) = (u(t))^2$.
 - (e) *Right Shift:* $T : \ell^2 \rightarrow \ell^2$ defined by $T(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$.
2. *Null Space and Range.* Let X be the space of all polynomials. Consider the differentiation operator $D : X \rightarrow X$ defined by $Du(t) = u'(t)$ and the integration operator $J : X \rightarrow X$ defined by $Ju(t) = \int_0^t u(\tau) d\tau$.
 - (a) Describe the null space $\mathcal{N}(D)$ and the range $\mathcal{R}(D)$.
 - (b) Describe the null space $\mathcal{N}(J)$ and the range $\mathcal{R}(J)$.
 - (c) Is D injective? Is J injective?
3. *Commutativity of Operators.* Let X be the space of polynomials. Let D be the differentiation operator and M be the multiplication operator defined by $(Mu)(t) = tu(t)$.
 - (a) Calculate $(DM)u$ and $(MD)u$ for an arbitrary polynomial $u(t)$.
 - (b) Show that $DM - MD = I$ (the identity operator).
4. *Rank-Nullity Application.* Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be a linear operator represented by a matrix A .
 - (a) What is the maximum possible dimension of the range $\mathcal{R}(T)$?

- (b) If the dimension of the range is 3, what is the dimension of the null space $\mathcal{N}(T)$?
- (c) Can such an operator T ever be injective? Use Theorem 4.3 to explain why or why not.
5. *Inverse of a Translation.* Consider the operator $T : \ell^\infty \rightarrow \ell^\infty$ defined by $T(\xi_1, \xi_2, \dots) = (\xi_2, \xi_3, \dots)$ (the Left Shift).
- (a) Find a non-zero sequence u such that $Tu = \mathbf{0}$.
- (b) Does T^{-1} exist? Explain using the relationship between inverses and null spaces.
6. *Properties of the Null Space.* Let $S, T : X \rightarrow Y$ be linear operators.
- (a) Prove that $\mathcal{N}(T) \subseteq \mathcal{N}(ST)$ is false in general. (Hint: Consider the domains).
- (b) Let $T : X \rightarrow Y$ and $S : Y \rightarrow Z$. Prove that $\mathcal{N}(T) \subseteq \mathcal{N}(ST)$.
- (c) Prove that if S is injective, then $\mathcal{N}(ST) = \mathcal{N}(T)$.
7. *Algebraic Reflexivity.* Let $T : X \rightarrow Y$ be an invertible linear operator. Prove the following algebraic identities:
- (a) $(T^{-1})^{-1} = T$.
- (b) $(cT)^{-1} = \frac{1}{c}T^{-1}$ for any scalar $c \neq 0$.
8. *The Cayley-Hamilton Concept.* Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.
- (a) Calculate the operator $T^2 - 4T + 3I$.
- (b) Verify that for any vector $u \in \mathbb{R}^2$, $(T^2 - 4T + 3I)u = \mathbf{0}$.
9. *Finite vs. Infinite Dimensions.* Let $T : X \rightarrow X$ be a linear operator.
- (a) If X is finite-dimensional, prove that T is injective if and only if it is surjective.
- (b) Let X be the space of polynomials. Let D be the differentiation operator. Is D surjective? Is it injective? Does this contradict part (a)?

4.2 BOUNDED AND CONTINUOUS OPERATORS

Section 4.1 analyzed linear operators from an algebraic standpoint, treating vector spaces as structures defined by addition and scalar multiplication. Functional analysis unifies algebra and topology. The analysis extends beyond whether a mapping preserves vector operations to whether it respects the geometry of the space.

In finite-dimensional spaces (e.g., $\mathbb{R}^n$), geometry and algebra are linked: every linear operator is continuous. If a sequence of vectors u_n approaches a limit u , the transformed sequence Tu_n approaches Tu .

In infinite-dimensional spaces, this stability is not guaranteed. A linear operator can map a convergent sequence to a divergent one, or map a bounded vector to one of infinite length. To classify stable operators, we define boundedness.

Definition 4.5 Bounded Linear Operator

Let X and Y be normed spaces. A linear operator $T : \mathcal{D}(T) \rightarrow Y$ is **bounded** if there exists a real number $c > 0$ such that for all vectors $u \in \mathcal{D}(T)$:

$$\|Tu\| \leq c\|u\|$$

If such a constant exists, the operator is bounded. Otherwise, we say the operator is **unbounded**.

The inequality $\|Tu\| \leq c\|u\|$ provides a geometric constraint: T maps bounded sets in the domain to bounded sets in the codomain. If a set $M \subset \mathcal{D}(T)$ is contained in a ball of radius r , its image $T(M)$ must be contained in a ball of radius cr .

If an operator is bounded, numerous constants c satisfy the boundedness inequality. The norm is the infimum of these constants.

Definition 4.6 Norm of an Operator

The norm of a bounded linear operator T , denoted by $\|T\|$, is defined as:

$$\|T\| = \sup_{\substack{u \in \mathcal{D}(T) \\ u \neq 0}} \frac{\|Tu\|}{\|u\|}$$

This definition establishes the bounding inequality:

$$\|Tu\| \leq \|T\|\|u\|$$

Applying the homogeneity of the norm ($\|\alpha u\| = |\alpha|\|u\|$), the definition is reformulated to evaluate vectors exclusively on the unit sphere, as visualized in [Figure 4.3](#).

$$\|T\| = \sup_{\substack{u \in \mathcal{D}(T) \\ \|u\|=1}} \|Tu\|$$

This formulation means that the norm of T is the radius of the minimal ball in Y containing the image of the unit sphere from X .

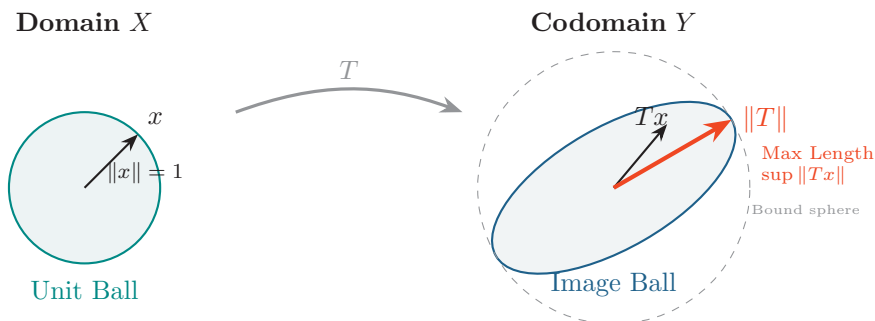


FIGURE 4.3 The operator norm. The operator T maps the unit ball of X into an ellipsoid in Y . The norm $\|T\|$ is the length of the longest vector in this image.

Example 4.7

To evaluate boundedness, we examine operators using the uniform norm $\|u\| = \max |u(t)|$ for function spaces.

1. The Identity Operator (I): Let $I : X \rightarrow X$ be defined by $Iu = u$.

$$\|Iu\| = \|u\|$$

Comparing this to the inequality $\|Tu\| \leq c\|u\|$, the constant $c = 1$ satisfies the bound. Because a vector exists where equality holds, the norm is equal to 1.

$$\|I\| = 1$$

2. The Zero Operator (0): Let $0 : X \rightarrow Y$ be defined by $0u = \mathbf{0}$.

$$\|0u\| = \|\mathbf{0}\| = 0 \leq 0 \cdot \|u\|$$

Thus, the zero operator is bounded with norm $\|0\| = 0$.

3. The Differentiation Operator (Unbounded): Let X be the space of polynomials on $[0, 1]$ equipped with the uniform norm. Let T be defined by $Tu(t) = u'(t)$.

Evaluate the sequence of polynomials $u_n(t) = t^n$.

- *Input Size:* The maximum of $|t^n|$ on $[0, 1]$ is 1. Thus, $\|u_n\| = 1$.
- *Output Action:* The derivative is $u'_n(t) = nt^{n-1}$.
- *Output Size:* The maximum of $|nt^{n-1}|$ on $[0, 1]$ occurs at $t = 1$, simplifying to n . Thus, $\|Tu_n\| = n$.

The ratio of output to input is:

$$\frac{\|Tu_n\|}{\|u_n\|} = \frac{n}{1} = n$$

Since n can be arbitrarily large, no finite constant c bounds this ratio. Consequently, differentiation is an unbounded operator on this space.

4. **The Integration Operator (Bounded):** Let $X = C[0, 1]$ with the uniform norm. Define T by $Tu(t) = \int_0^t u(\tau) d\tau$. We evaluate the bound of the output:

$$|Tu(t)| = \left| \int_0^t u(\tau) d\tau \right| \leq \int_0^t |u(\tau)| d\tau$$

Since $|u(\tau)| \leq \max |u(t)| = \|u\|$, substituting the bound yields

$$|Tu(t)| \leq \int_0^t \|u\| d\tau = \|u\| \cdot t$$

Since $t \in [0, 1]$, the maximum value is bounded by $\|u\|$:

$$\|Tu\| = \max_{t \in [0, 1]} |Tu(t)| \leq \|u\|$$

This establishes $\|Tu\| \leq 1 \cdot \|u\|$. Thus T is bounded and $\|T\| \leq 1$. ($u(t) = 1$ implies $\|T\| = 1$). $\square$

In general topology, determining continuity requires evaluating limits at every point in the domain. However, because linear operators preserve vector addition and scalar multiplication, their local behavior determines their global behavior. The following theorem establishes that for linear operators, boundedness and continuity are equivalent, a geometric relationship visualized in Figure 4.4.

Theorem 4.8 Equivalence of Boundedness and Continuity

Let X and Y be normed vector spaces, and let $T : X \rightarrow Y$ be a linear operator. The following statements are equivalent:

1. T is continuous at the origin (0).
2. T is bounded ($\|T\| < \infty$).
3. T is uniformly continuous on the space X .

PROOF: We prove the implications in the circular order (1) $\implies$ (2) $\implies$ (3) $\implies$ (1).

Continuity at Origin implies Boundedness Assume T is continuous at 0. By definition, for $\epsilon = 1$, there exists a $\delta > 0$ such that for all $u \in X$:

$$\|u\| \leq \delta \implies \|Tu\| \leq 1$$

Restrict the operator on the unit sphere. Let v be a vector with $\|v\| = 1$. Define the scaled vector $u = \delta v$. Because $\|u\| = \delta$, the bound is satisfied:

$$\|T(\delta v)\| \leq 1$$

By the linearity of T and the homogeneity of the norm, the scalar δ is extracted:

$$\delta \|Tv\| \leq 1 \implies \|Tv\| \leq \frac{1}{\delta}$$

Since this holds for any unit vector v , the operator norm is finite: $\|T\| = \sup_{\|v\|=1} \|Tv\| \leq \frac{1}{\delta} < \infty$.

Boundedness Implies Uniform Continuity Assume T is bounded with $\|T\| < \infty$. For any two vectors $u, w \in X$:

$$\|Tu - Tw\| = \|T(u - w)\| \leq \|T\| \|u - w\|$$

Given any $\epsilon > 0$, define $\delta = \epsilon/\|T\|$ (assuming $\|T\| > 0$). Substituting this limit:

$$\|u - w\| < \delta \implies \|Tu - Tw\| < \|T\| \cdot \left(\frac{\epsilon}{\|T\|} \right) = \epsilon$$

Since δ depends exclusively on the constant $\|T\|$ and ϵ (independent of the position of u), T is uniformly continuous.

Uniform Continuity Implies Continuity at Origin If T is uniformly continuous on X , it is continuous at every specific point, including the origin $u = \mathbf{0}$. ■

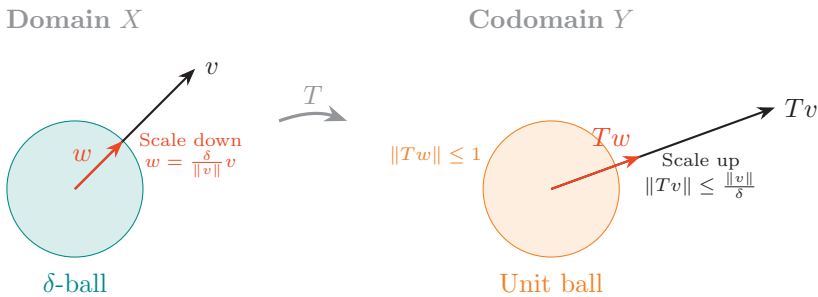


FIGURE 4.4 Continuity at the origin guarantees that small vectors (w) map to bounded vectors (Tw). By linearity, this bound scales up to arbitrary vectors (v), proving global boundedness.

The null space $\mathcal{N}(T)$ of a linear operator is a vector subspace. When the operator is bounded, this subspace is automatically closed.

Corollary 4.9

Let $T : X \rightarrow Y$ be a bounded linear operator defined on a normed space X . The null space $\mathcal{N}(T)$ is a closed subspace of X .

PROOF: The proof uses the definition of continuity regarding preimages.

Identify the Preimage The null space is the set of vectors mapping to the zero vector:

$$\mathcal{N}(T) = \{u \in X \mid Tu = \mathbf{0}\}$$

In set-theoretic terms, this is the inverse image (preimage) of the singleton set $\{\mathbf{0}\}$:

$$\mathcal{N}(T) = T^{-1}(\{\mathbf{0}\})$$

Evaluate the Target Set In any normed space, a singleton set consisting of a single point (such as $\{\mathbf{0}\}$) is a closed set.

Apply Continuity Because T is a bounded linear operator, Theorem 4.8 establishes that T is continuous. The inverse image of a closed set under a continuous mapping is closed. Thus, the preimage $T^{-1}(\{\mathbf{0}\}) = \mathcal{N}(T)$ is closed. ■

This corollary provides a method to establish that an operator is unbounded. If a sequence of vectors (u_n) satisfies the following conditions:

1. Each u_n is in the null space ($Tu_n = \mathbf{0}$),
2. The sequence converges to a limit u ($u_n \rightarrow u$),
3. The limit u is not in the null space ($Tu \neq \mathbf{0}$),

then the null space is not closed. By the contrapositive of Corollary 4.9, the operator T is not bounded. ┘

Differentiation acts on the infinite-dimensional space of polynomials and is an unbounded operator. In a finite-dimensional domain, the geometric structure prevents unbounded behavior. The existence of a finite basis forces every linear operator to be bounded, as visualized in [Figure 4.5](#).

Theorem 4.10 Boundedness in Finite Dimensions

Let X be a finite-dimensional normed space and Y be any normed space. Every linear operator $T : X \rightarrow Y$ is bounded.

PROOF: Let $\dim X = n$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis for X .

Bound the Basis Images Because the basis is finite, the set of norms of the

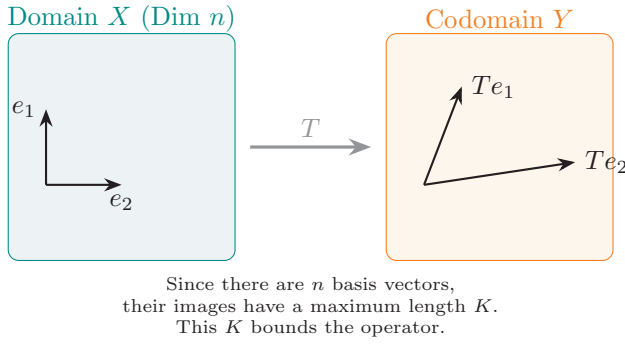


FIGURE 4.5 In finite dimensions, the operator is determined by a finite set of images Te_j , preventing unbounded scaling.

image vectors $\{\|Te_1\|, \|Te_2\|, \dots, \|Te_n\|\}$ is a finite set of real numbers. Let K denote the maximum of these lengths:

$$K = \max_{1 \leq j \leq n} \|Te_j\|$$

Basis Expansion and Linearity Consider an arbitrary vector $u \in X$. Expand u uniquely in terms of the basis:

$$u = \sum_{j=1}^n \alpha_j e_j$$

Calculate the norm of the image $\|Tu\|$ by applying linearity:

$$\|Tu\| = \left\| T \left(\sum_{j=1}^n \alpha_j e_j \right) \right\| = \left\| \sum_{j=1}^n \alpha_j (Te_j) \right\|$$

Apply the Triangle Inequality Applying the Triangle Inequality isolates the scalar coefficients:

$$\left\| \sum_{j=1}^n \alpha_j (Te_j) \right\| \leq \sum_{j=1}^n |\alpha_j| \|Te_j\|$$

Substitute the maximum bound K :

$$\|Tu\| \leq K \sum_{j=1}^n |\alpha_j|$$

Relate Coefficients to the Norm We relate the sum of coefficients $\sum |\alpha_j|$ to the

norm of the vector $\|u\|$. On a finite-dimensional space, all norms are equivalent. Consider the ℓ^1 -style norm defined by $\|u\|_1 = \sum |\alpha_j|$. Because $\|\cdot\|_1$ and the given norm $\|\cdot\|$ are equivalent on X , a constant $m > 0$ exists such that:

$$\sum_{j=1}^n |\alpha_j| \leq \frac{1}{m} \|u\|$$

Final Inequality Substituting this equivalence into the estimate yields

$$\|Tu\| \leq \frac{K}{m} \|u\|$$

Defining $c = K/m$ satisfies the condition $\|Tu\| \leq c\|u\|$ for all u . Thus, T is bounded. ■

In analysis, defining an operator on a dense subset (such as polynomials or smooth functions) and extending it to the entire space is a standard procedure. The following theorem, visualized in [Figure 4.6](#), guarantees that if the operator is bounded, it can be uniquely extended without increasing its norm.

Theorem 4.11 Bounded Linear Extension

Let $T : \mathcal{D}(T) \rightarrow Y$ be a bounded linear operator where $\mathcal{D}(T)$ is a dense subset of a normed space X , and Y is a Banach space. There exists a unique bounded linear extension $\tilde{T} : X \rightarrow Y$ such that:

1. $\tilde{T}u = Tu$ for all $u \in \mathcal{D}(T)$.
2. The norm is preserved: $\|\tilde{T}\| = \|T\|$.

PROOF: The proof is constructive, defining $\tilde{T}$ explicitly using limits.

Construction of $\tilde{T}$ Let x be a vector in X . Because $\mathcal{D}(T)$ is dense, there exists a sequence (u_n) in $\mathcal{D}(T)$ such that $u_n \rightarrow x$. Consider the sequence of images (Tu_n) in Y :

$$\|Tu_n - Tu_m\| = \|T(u_n - u_m)\| \leq \|T\| \|u_n - u_m\|$$

Since (u_n) converges, it forms a Cauchy sequence in X . The bounding inequality establishes that (Tu_n) forms a Cauchy sequence in Y .

Use Completeness of Y Because Y is a Banach space, every Cauchy sequence converges. Therefore, (Tu_n) possesses a limit $y \in Y$. Define the extension by this limit:

$$\tilde{T}x = y = \lim_{n \rightarrow \infty} Tu_n$$

Well-definedness Assume (v_n) is an alternative sequence in $\mathcal{D}(T)$ converging to x . It must be verified that $\lim Tv_n$ equals $\lim Tu_n$.

$$\|Tu_n - Tv_n\| = \|T(u_n - v_n)\| \leq \|T\| \|u_n - v_n\|$$

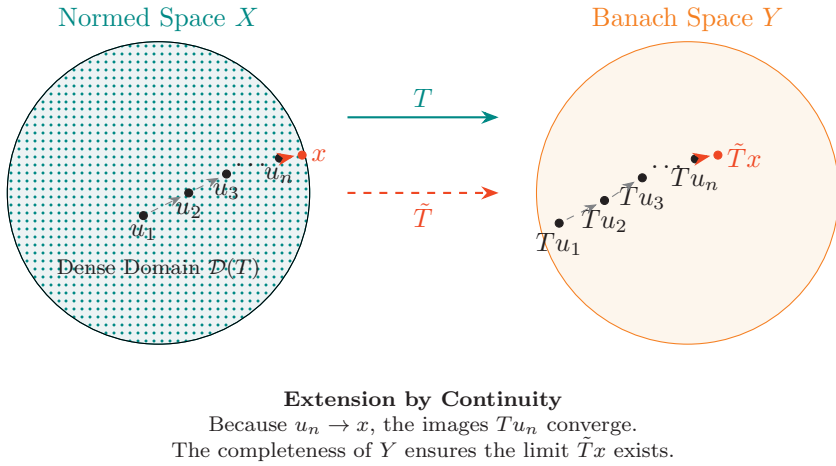


FIGURE 4.6 The bounded linear extension theorem defines the operator on the topological gaps (red dot) by evaluating the limit of the dense subset sequence (black dots).

Since $u_n \rightarrow x$ and $v_n \rightarrow x$, the difference $\|u_n - v_n\| \rightarrow 0$. The limits are identical, confirming $\tilde{T}$ is well-defined.

Linearity of $\tilde{T}$ Let $x, z \in X$. Select sequences $u_n \rightarrow x$ and $w_n \rightarrow z$. The linear combination converges: $(\alpha u_n + \beta w_n) \rightarrow (\alpha x + \beta z)$.

$$\tilde{T}(\alpha x + \beta z) = \lim_{n \rightarrow \infty} T(\alpha u_n + \beta w_n)$$

Applying the linearity of T :

$$= \lim_{n \rightarrow \infty} (\alpha T u_n + \beta T w_n) = \alpha \lim T u_n + \beta \lim T w_n = \alpha \tilde{T} x + \beta \tilde{T} z$$

Preservation of Norm Evaluating the limit as $n \rightarrow \infty$ and applying the continuity of the norm:

$$\|\tilde{T}x\| = \lim \|Tu_n\| \leq \lim (\|T\| \|u_n\|) = \|T\| \|x\|$$

This establishes $\|\tilde{T}\| \leq \|T\|$. Conversely, because $\tilde{T}$ is an extension of T acting on an expanded domain, its supremum cannot decrease. Thus, $\|\tilde{T}\| = \|T\|$.

Uniqueness Assume S is an alternative continuous extension. For any $x \in X$, construct $u_n \rightarrow x$ where $u_n \in \mathcal{D}(T)$. Because S is continuous:

$$Sx = \lim_{n \rightarrow \infty} S u_n$$

Since u_n lies in the original domain, $S u_n = T u_n$. Thus:

$$Sx = \lim_{n \rightarrow \infty} T u_n = \tilde{T}x$$

Therefore, $S = \tilde{T}$. ■

EXERCISES

1. *Matrix Norms on $\mathbb{R}^2$.* Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.
 - (a) Calculate $\|Tu\|$ where $u = (1, 0)$ and $u = (0, 1)$ using the Euclidean norm.
 - (b) Show that $\|T\| = 3$.
 - (c) If the norm on $\mathbb{R}^2$ is changed to the sum norm $\|u\|_1 = |\xi_1| + |\xi_2|$, what is the new operator norm $\|T\|_1$?
2. *The Integration Operator Norm.* Let $T : C[0, 1] \rightarrow C[0, 1]$ be the integration operator defined by $(Tu)(t) = \int_0^t u(\tau) d\tau$.
 - (a) It was established that $\|T\| \leq 1$. Consider the constant function $u(t) = 1$. Calculate $\|Tu\|$ and $\|u\|$.
 - (b) Does this prove that $\|T\| = 1$? Explain why.
 - (c) Consider the operator S defined by $(Su)(t) = \int_0^1 u(\tau) d\tau$. Show that $\|S\| = 1$.
3. *Unboundedness of Differentiation.* Consider the space of polynomials $P[0, 1]$ equipped with the uniform norm. Let D be the differentiation operator.
 - (a) Find a sequence of polynomials (u_n) such that $\|u_n\| = 1$ for all n , but $\|Du_n\| \rightarrow \infty$.
 - (b) Does this imply D is discontinuous everywhere, or exclusively at specific points? (Refer to Theorem 4.8).
4. *Multiplication Operator.* Let $X = C[0, 1]$. Define $T : X \rightarrow X$ by $(Tu)(t) = t^2 u(t)$.
 - (a) Show that T is linear.
 - (b) Prove that T is bounded and calculate the norm $\|T\|$. (Hint: Evaluate the maximum of t^2 on $[0, 1]$).
5. *Shift Operators on ℓ^∞ .* Let $X = \ell^\infty$ (the space of bounded sequences).
 - (a) **Right Shift:** $R(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$. Show that $\|R\| = 1$.
 - (b) **Left Shift:** $L(\xi_1, \xi_2, \dots) = (\xi_2, \xi_3, \dots)$. Show that $\|L\| = 1$.
 - (c) Is the Left Shift operator injective? Is the Right Shift operator surjective?
6. *Norm of a Composition.* Let $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ be bounded linear operators.
 - (a) Prove that the composition ST is bounded.

(b) Prove the sub-multiplicative inequality:

$$\|ST\| \leq \|S\|\|T\|$$

(c) Provide an example where strict inequality holds ($\|ST\| < \|S\|\|T\|$).
(Hint: Use projection matrices on $\mathbb{R}^2$).

7. *Space Dependence of Boundedness.* Let X be the space of sequences possessing finitely many non-zero terms. Let $T : X \rightarrow X$ be defined by $T(\xi_k) = (k\xi_k)$.

- (a) If X is equipped with the sup-norm $\|\cdot\|_\infty$, show that T is unbounded.
- (b) Can a weighted norm be defined on X such that T becomes bounded?

8. *Closedness of the Kernel.* Let $T : C[a, b] \rightarrow \mathbb{R}$ be the functional defined by $T(u) = u(a)$.

- (a) Show that T is a bounded linear operator.
- (b) Describe the null space $\mathcal{N}(T)$.
- (c) Explain why the set of functions vanishing at $x = a$ is a closed subspace of $C[a, b]$.

9. *Extension of Functionals.* Let X be the space of polynomials on $[0, 1]$ (a dense subspace of $C[0, 1]$). Let $T : X \rightarrow \mathbb{R}$ be defined by $T(u) = \int_0^1 u(t)dt$.

- (a) Show that T is bounded on the subspace of polynomials.
- (b) According to the Bounded Linear Extension Theorem, what is the unique extension $\tilde{T}$ of this operator to the full space $C[0, 1]$?

10. *Operator Limit.* Let $T_n : \ell^2 \rightarrow \ell^2$ be defined by $T_n(\xi_1, \xi_2, \dots) = (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)$ (the truncation operator).

- (a) Show that each T_n is a bounded linear operator.
- (b) Does the sequence of operators (T_n) converge to the identity operator I in the operator norm? (i.e., does $\|T_n - I\| \rightarrow 0$?)

11. *Fixed Point Concept.* Let $T : X \rightarrow X$ be a bounded linear operator satisfying $\|T\| < 1$.

- (a) Prove that for any $u \in X$, the sequence defined by $x_{n+1} = Tx_n$ converges to $\mathbf{0}$.
- (b) Why is the condition $\|T\| < 1$ required? What occurs if $T = I$ (where $\|T\| = 1$)?

4.3 DIRECT SUMS AND PROJECTIONS

In functional analysis, the decomposition of a vector space into independent, complementary subspaces is a core structural operation. Unlike elementary

linear algebra, where algebraic independence is sufficient, decompositions in infinite-dimensional normed spaces must preserve the topology to ensure that limits remain well-defined across the subspaces.

Topological Direct Sums

A normed space admits a direct sum decomposition if it can be partitioned into two complementary subspaces such that the projection between them is continuous.

Definition 4.12 Topological Direct Sum

Let X be a normed space, and let Y and Z be subspaces of X . The space X is the **topological direct sum** of Y and Z , denoted by $X = Y \oplus Z$, if it satisfies two criteria:

1. **Algebraic Decomposition:** Every vector $x \in X$ admits a unique representation as a sum:

$$x = y + z \quad \text{where } y \in Y \text{ and } z \in Z$$

(This uniqueness implies that the intersection is trivial: $Y \cap Z = \{0\}$).

2. **Topological Stability:** The linear projection operator $P : X \rightarrow Y$, defined by $P(y + z) = y$, is a continuous (bounded) mapping.

The operator P characterizes the projection of X onto the subspace Y parallel to the subspace Z . A property of any projection is *idempotence*, meaning iterative applications of the operator yield no further alteration ($P^2 = P$).

Evaluating a vector $x = y + z$, the initial projection yields:

$$Px = P(y + z) = y$$

Since the vector y lies in Y , its subsequent decomposition is $y = y + 0$. Consequently, evaluating the operator a second time resolves to

$$P(Px) = P(y) = P(y + 0) = y$$

Since $P(Px) = Px$ holds for all vectors in the space, the operator is idempotent.

While the definition of a topological direct sum requires the continuity of the projection P , direct verification is frequently difficult. In Banach spaces, the algebraic decomposition of the space into closed subspaces is sufficient to guarantee the continuity of the projection. We only state this result. The proof uses the Bounded Inverse Theorem and so is reserved in the Appendix.

Theorem 4.13 The Decomposition Theorem

Let X be a complete Banach space. If X is the algebraic direct sum of two closed subspaces Y and Z ($X = Y \oplus Z$), then the projection operator P mapping X onto Y along Z is continuous.

A one-to-one correspondence exists between algebraic direct sum decompositions and idempotent projection operators.

Theorem 4.14 Projections and Direct Sums

Let X denote a vector space.

1. If $P : X \rightarrow X$ is a linear operator satisfying idempotence ($P^2 = P$), then X decomposes into the algebraic direct sum of the range and the null space of P :

$$X = \mathcal{R}(P) \oplus \mathcal{N}(P)$$

2. Conversely, if $X = Y \oplus Z$, there exists a unique idempotent projection P satisfying $\mathcal{R}(P) = Y$ and $\mathcal{N}(P) = Z$.

PROOF OF (1):

Existence of Decomposition For an arbitrary vector $x \in X$, construct the identity:

$$x = Px + (x - Px)$$

Define the components $y = Px$ and $z = x - Px$. By definition, $y \in \mathcal{R}(P)$. To verify the placement of z , apply the operator P :

$$Pz = P(x - Px) = Px - P^2x$$

Since P is idempotent ($P^2 = P$), this evaluates to $Pz = \mathbf{0}$, guaranteeing $z \in \mathcal{N}(P)$. Thus, x admits a valid decomposition.

Uniqueness of Decomposition Assume x permits two distinct decompositions:

$$x = y_1 + z_1 = y_2 + z_2$$

where $y_1, y_2 \in \mathcal{R}(P)$ and $z_1, z_2 \in \mathcal{N}(P)$. Algebraic rearrangement yields $y_1 - y_2 = z_2 - z_1$. Define the vector $v = y_1 - y_2 = z_2 - z_1$. Because $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are linear subspaces, v belongs to both the range and the null space. Since $v \in \mathcal{R}(P)$, there exists a vector u such that $Pu = v$. Consequently, $Pv = P(Pu) = P^2u = Pu = v$. Because $v \in \mathcal{N}(P)$, the operator dictates $Pv = \mathbf{0}$. Synthesizing these evaluations forces $v = \mathbf{0}$, ensuring $y_1 = y_2$ and $z_1 = z_2$. The decomposition is unique. ■

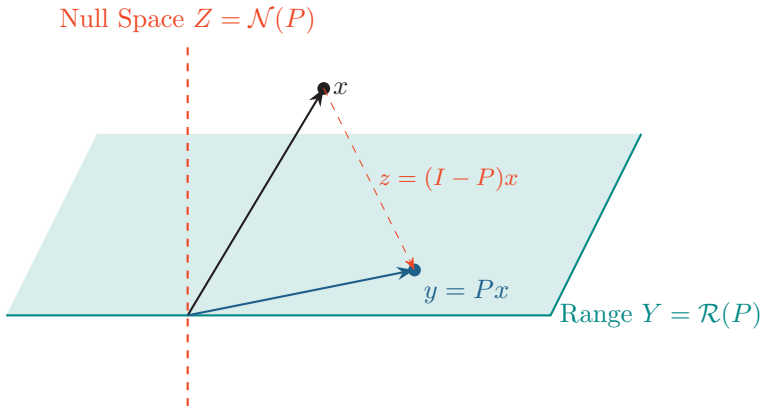


FIGURE 4.7 The mapping of a direct sum $X = Y \oplus Z$. The idempotent operator P projects onto the range Y along the trajectory of the null space Z .

Complemented Subspaces

A subspace Y is complemented in X if it can be isolated from the space via a decomposition that preserves the topology.

Definition 4.15 Complemented Subspace

Let X denote a normed space. A subspace $Y \subset X$ is **complemented** if there exists a closed subspace $Z \subset X$, defined as the **topological complement**, such that X forms the direct sum of Y and Z :

$$X = Y \oplus Z$$

The existence of topological complements highlights a geometric variance between finite and infinite-dimensional spaces:

- **Finite Dimensions:** Every subspace is complemented. A basis for the subspace can be extended to span the space.
- **Hilbert Spaces:** Every closed subspace is complemented. The geometry ensures the existence of an orthogonal complement ($Z = Y^\perp$).
- **Banach Spaces:** In general complete normed spaces, closed subspaces frequently fail to be complemented. For example, the subspace c_0 is closed in ℓ^∞ , yet theory dictates that no continuous projection maps ℓ^∞ onto c_0 .

EXERCISES

1. *The Complementary Projection.* Let X be a vector space and let $P : X \rightarrow X$ be an idempotent projection operator ($P^2 = P$). Define the complementary operator $Q = I - P$, where I is the identity operator.
 - (a) Prove that Q is also an idempotent projection operator.
 - (b) Prove that the range of Q equals the null space of P ($\mathcal{R}(Q) = \mathcal{N}(P)$), and the null space of Q equals the range of P ($\mathcal{N}(Q) = \mathcal{R}(P)$).
 - (c) Verify that $PQ = QP = \mathbf{0}$.
2. *Bounded Projections and Closed Subspaces.* Let X be a Banach space and $P : X \rightarrow X$ be a bounded projection operator. Prove that both the range $\mathcal{R}(P)$ and the null space $\mathcal{N}(P)$ are closed subspaces of X . (*Hint: For the null space, evaluate the continuity of P acting on a convergent sequence $x_n \rightarrow x$ where $Px_n = \mathbf{0}$. For the range, apply the same principle to the bounded complementary projection $I - P$.)*
3. *Decomposition of Continuous Functions.* Let $X = C[-1, 1]$ be the space of continuous functions on $[-1, 1]$ equipped with the uniform norm. Define Y as the subspace of even functions ($f(-t) = f(t)$) and Z as the subspace of odd functions ($f(-t) = -f(t)$).
 - (a) Prove that X is the algebraic direct sum of Y and Z ($X = Y \oplus Z$) by constructing the unique decomposition of an arbitrary function $f \in X$.
 - (b) Define the projection operator $P : X \rightarrow Y$. Prove that P is a continuous linear operator.
 - (c) Conclude that X is the topological direct sum of Y and Z .
4. *Product Space Norms.* Let Y and Z be complete Banach spaces. Construct the product space $W = Y \times Z$. Prove that W is a Banach space when equipped with the maximum norm:

$$\|(y, z)\|_W = \max(\|y\|_Y, \|z\|_Z)$$

(This establishes that the choice between the sum norm $\|y\| + \|z\|$ utilized in the text and the maximum norm yields equivalent topological spaces.)

5. *Orthogonal Complements in Hilbert Spaces.* Let H be a Hilbert space and Y a closed subspace of H . The Projection Theorem for Hilbert spaces dictates that any vector $x \in H$ possesses a unique closest point $y \in Y$, and the difference vector $z = x - y$ is orthogonal to Y ($z \in Y^\perp$). Utilize this theorem to prove that H is the topological direct sum of Y and $Y^\perp$:

$$H = Y \oplus Y^\perp$$

6. *Finite-Dimensional Complements.* Let X be a normed space and Y a

closed subspace. Assume Z is a finite-dimensional subspace of X such that $Y \cap Z = \{\mathbf{0}\}$. Prove that the algebraic direct sum $Y \oplus Z$ is a closed subspace of X . (*Hint: Utilize the Lower Bound Lemma to establish the continuity of projections on the finite-dimensional component.*)

4.4 LINEAR FUNCTIONALS

Sections 4.1 and 4.2 considered linear operators $T : X \rightarrow Y$ where Y is an arbitrary vector space. In this section, Y is restricted to a scalar field (either $\mathbb{R}$ or $\mathbb{C}$).

These operators function as measurement tools, compressing an object—a function, a sequence, or a vector—into a scalar. These mappings are called linear functionals. They form the foundation of the dual space.

Definition 4.16 Linear Functional

A linear functional f is a linear operator with domain $\mathcal{D}(f)$ in a vector space X and range in the scalar field $\mathbb{K}$ of X .

$$f : \mathcal{D}(f) \rightarrow \mathbb{K}$$

Thus, for all vectors $u, v \in \mathcal{D}(f)$ and scalars α, β :

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$$

As with general operators, linear functionals are classified as bounded or unbounded.

Definition 4.17 Bounded Linear Functional

A linear functional f is **bounded** if there exists a real number c such that for all $u \in \mathcal{D}(f)$:

$$|f(u)| \leq c\|u\|$$

The norm of a bounded linear functional is defined identically to the operator norm, evaluating the norm in the range as the absolute value:

$$\|f\| = \sup_{\substack{u \in \mathcal{D}(f) \\ u \neq \mathbf{0}}} \frac{|f(u)|}{\|u\|} = \sup_{\substack{u \in \mathcal{D}(f) \\ \|u\|=1}} |f(u)|$$

For every vector $u \in X$, the bounding inequality applies:

$$|f(u)| \leq \|f\|\|u\|$$

Since a linear functional is a specific type of linear operator, the equivalence of boundedness and continuity applies.

Example 4.18

The following examples evaluate linear functionals across different spaces to determine the norm $\|f\|$, representing the maximum scaling factor of the functional.

1. **The Dot Product Functional.** Let $X = \mathbb{R}^n$ equipped with the Euclidean norm $\|u\| = (\sum u_j^2)^{1/2}$. Fix a vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Define the linear functional $f : X \rightarrow \mathbb{R}$ by the dot product:

$$f(u) = u \cdot a = \sum_{j=1}^n u_j a_j$$

Applying the Cauchy-Schwarz Inequality estimates the absolute value of the output:

$$|f(u)| = \left| \sum_{j=1}^n u_j a_j \right| \leq \left(\sum_{j=1}^n u_j^2 \right)^{1/2} \left(\sum_{j=1}^n a_j^2 \right)^{1/2}$$

In vector notation, this evaluates to $|f(u)| \leq \|u\| \|a\|$, establishing the bound $\|f\| \leq \|a\|$. To evaluate if the supremum achieves this bound, let $a \neq \mathbf{0}$ and set $u = a$:

$$f(a) = a \cdot a = \|a\|^2$$

The ratio of output to input evaluates to

$$\frac{|f(a)|}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|$$

Since the supremum cannot be less than this value, and the upper bound is $\|a\|$, it follows that $\|f\| = \|a\|$. In Euclidean space, the norm of the linear functional equals the length of its defining vector.

2. **The Sequence Functional (Space ℓ^2).** Let $X = \ell^2$ (the space of square-summable sequences). Fix a sequence $a = (a_j) \in \ell^2$. Define $f : \ell^2 \rightarrow \mathbb{R}$ by:

$$f(u) = \sum_{j=1}^{\infty} u_j a_j$$

Applying the Hölder Inequality (the sequence generalization of Cauchy-Schwarz):

$$|f(u)| = \left| \sum_{j=1}^{\infty} u_j a_j \right| \leq \left(\sum_{j=1}^{\infty} |u_j|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{1/2}$$

This establishes $|f(u)| \leq \|u\|\|a\|$, yielding the bound $\|f\| \leq \|a\|$. Choosing the input $u = a$ evaluates to $f(a) = \|a\|^2$, establishing the ratio as $\|a\|$. Thus, $\|f\| = \|a\|$.

3. **The Definite Integral (Space $C[a, b]$).** Let $X = C[a, b]$ be the space of continuous functions on $[a, b]$, equipped with the uniform norm $\|u\| = \max_{t \in [a, b]} |u(t)|$. Define f by:

$$f(u) = \int_a^b u(t) dt$$

Evaluate the bound of the integral:

$$|f(u)| = \left| \int_a^b u(t) dt \right| \leq \int_a^b |u(t)| dt$$

Because $|u(t)| \leq \max |u(t)| = \|u\|$ for all t :

$$\int_a^b |u(t)| dt \leq \int_a^b \|u\| dt = \|u\|(b - a)$$

This establishes $|f(u)| \leq (b - a)\|u\|$, yielding the bound $\|f\| \leq b - a$. To maximize the integral, evaluate the constant function $u_0(t) = 1$. The norm is $\|u_0\| = 1$, and the functional evaluates to:

$$f(u_0) = \int_a^b 1 dt = b - a$$

The ratio is $(b - a)/1$. Therefore, $\|f\| = b - a$.

4. **The Evaluation Functional/Dirac Delta (Space $C[a, b]$).** Let $X = C[a, b]$ with the uniform norm. Fix a coordinate $t_0 \in [a, b]$. Define f by evaluating the function at that specific coordinate:

$$f(u) = u(t_0)$$

For any function u , the value at a single coordinate is bounded by the maximum value across the interval:

$$|f(u)| = |u(t_0)| \leq \max_{t \in [a, b]} |u(t)| = \|u\|$$

This establishes $|f(u)| \leq 1 \cdot \|u\|$, yielding $\|f\| \leq 1$. Evaluating the constant function $u_0(t) = 1$ yields $\|u_0\| = 1$ and $f(u_0) = 1$. Thus, $\|f\| = 1$.

5. **An Unbounded Functional (Space of Polynomials).** Let X be the space of all polynomials on $[0, 1]$ equipped with the uniform norm $\|u\| = \max |u(t)|$. Define f as the derivative evaluated at $t = 1$:

$$f(u) = u'(1)$$

Evaluate the sequence of polynomials $u_n(t) = t^n$.

- *Input Size*: $\|u_n\| = \max_{t \in [0,1]} |t^n| = 1$.
- *Action*: The derivative is $u'_n(t) = nt^{n-1}$.
- *Output*: $f(u_n) = u'_n(1) = n(1)^{n-1} = n$.

The ratio $|f(u_n)|/\|u_n\| = n/1 = n$. Because n diverges, no constant c satisfies $|f(u)| \leq c\|u\|$. Thus, this functional is unbounded. $\lrcorner$

EXERCISES FOR SECTION 4.3

1. *Verifying Linearity*. Determine whether the following mappings $f : X \rightarrow \mathbb{R}$ are linear functionals.
 - (a) On $X = \mathbb{R}^3$: $f(u) = \xi_1 + 2\xi_2 - 3\xi_3$.
 - (b) On $X = \mathbb{R}^3$: $f(u) = \|\xi\|$.
 - (c) On $X = C[0, 1]$: $f(u) = \int_0^1 u^2(t) dt$.
 - (d) On $X = C[0, 1]$: $f(u) = \int_0^1 u(t) \sin t dt$.
 - (e) On $X = \mathbb{R}^{n \times n}$ (matrices): $f(A) = \text{tr}(A)$ (the trace).
2. *Calculating Norms in Finite Dimensions*. Let $X = \mathbb{R}^2$. Consider the functional $f(u) = 2\xi_1 - 3\xi_2$. Find the norm $\|f\|$ if X is equipped with:
 - (a) The Euclidean norm $\|u\|_2 = (\xi_1^2 + \xi_2^2)^{1/2}$.
 - (b) The sum norm $\|u\|_1 = |\xi_1| + |\xi_2|$.
 - (c) The max norm $\|u\|_\infty = \max(|\xi_1|, |\xi_2|)$.
3. *Weighted Integral Functional*. Let $X = C[0, 1]$ with the uniform norm. Fix a continuous function $g \in C[0, 1]$. Define f by:

$$f(u) = \int_0^1 u(t)g(t) dt$$

Show that $\|f\| = \int_0^1 |g(t)| dt$. (Hint: To maximize the integral, define $u(t)$ to align with the sign of $g(t)$).

4. *Evaluation on Different Spaces*. Consider the evaluation functional $f(u) = u(0)$.
 - (a) It is established that f is bounded on $C[-1, 1]$ equipped with the uniform norm.
 - (b) Consider the subspace of continuous functions inside $L^2[-1, 1]$ (equipped with the L^2 norm). Show that f is unbounded in this setting. (Hint: Construct a sequence of functions that spike at 0 while the integral area converges to 0).
5. *The Sum Functional*. Let $X = \ell^1$ (absolutely summable sequences). Define $f(u) = \sum_{j=1}^{\infty} \xi_j$.

- (a) Show that f is a bounded linear functional.
 - (b) Calculate $\|f\|$.
 - (c) Does this functional remain bounded if defined on ℓ^2 ?
6. *The Geometric Series Functional.* Let $X = \ell^\infty$ (bounded sequences). Define $f(u) = \sum_{j=1}^{\infty} \frac{\xi_j}{2^j}$.
- (a) Prove that $|f(u)| \leq \|u\|_\infty$ for all u .
 - (b) Find the norm $\|f\|$.

4.5 DUAL SPACES

Previous examples established two classes of linear functionals: bounded (e.g., the definite integral) and unbounded (e.g., differentiation). This distinction requires specifying the functional collection under evaluation.

Evaluating a vector space X from an algebraic perspective—excluding distance, convergence, or continuity—yields the set of all linear functionals defined on X , as visualized in Figure 4.8.

Definition 4.19 Algebraic Dual Space X'

The Algebraic Dual Space, denoted by X' , is the vector space consisting of all linear functionals $f : X \rightarrow \mathbb{K}$.

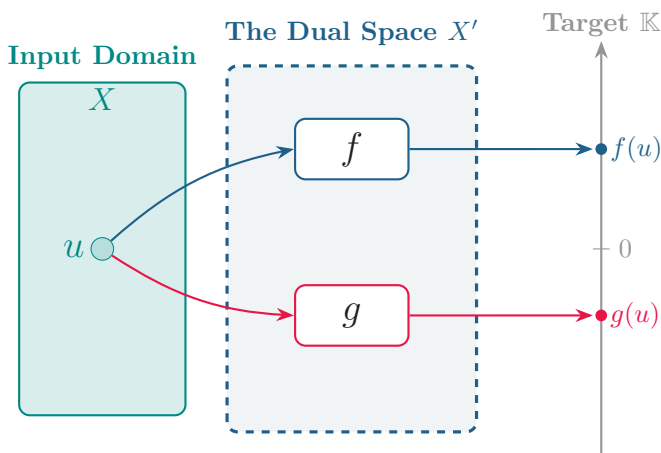


FIGURE 4.8 The algebraic dual X' . The dual space contains the functionals (f and g) that map vectors into scalars. The same vector u yields distinct scalar limits depending on the functional evaluating it.

The space X' is a vector space because functionals satisfy pointwise addition and scalar multiplication:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

In algebra, X' is the exclusive dual space. In analysis, X' frequently contains discontinuous functionals that violate topological structure. Consequently, analysis restricts evaluation to bounded (continuous) functionals.

Definition 4.20 Topological Dual Space X^*

The Topological Dual Space (or the Dual Space), denoted by X^* , is the vector space consisting of all bounded linear functionals defined on X .

In functional analysis and its applications, the term “dual space” defaults to X^* . The relationship between X' and X^* depends on the dimension of the domain X .

On a finite-dimensional space, every linear operator is bounded. Consequently, the algebraic and topological duals are identical ($X' = X^*$). In infinite-dimensional spaces, unbounded functionals exist. These functionals belong to the algebraic dual X' but are excluded from the topological dual X^* . The topological dual X^* forms a proper subspace of the algebraic dual X' .

In finite-dimensional linear algebra, an operator A possesses a transpose A^T . In abstract Banach spaces, this concept is generalized as the adjoint operator, visualized in [Figure 4.9](#).

Definition 4.21 The Adjoint Operator

Let X and Y be normed spaces and let $T : X \rightarrow Y$ be a bounded linear operator. The adjoint operator $T^* : Y^* \rightarrow X^*$ is defined by the identity:

$$(T^*y^*)(x) = y^*(Tx)$$

for all functionals $y^* \in Y^*$ and all vectors $x \in X$.

Elements of the dual space Y^* are denoted as y^* . Because y^* is a functional:

- $y^*(Tx)$ denotes the functional y^* evaluated at the vector Tx .
- $(T^*y^*)(x)$ denotes the new functional (T^*y^*) evaluated at the vector x .

While T maps vectors forward from X to Y , the adjoint T^* maps functionals backward from Y^* to X^* .

The adjoint operator preserves the geometry of the original operator.

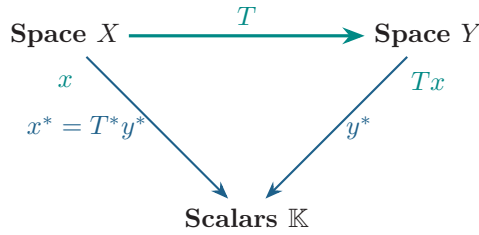


FIGURE 4.9 The adjoint operator T^* makes the diagram commute. The evaluation can proceed by mapping the vector forward (T) and measuring (y^*), or mapping the measurement backward (T^*) to evaluate the original vector directly.

Theorem 4.22 Isometry of the Adjoint

The adjoint operator T^* is bounded, linear, and preserves the norm of the original operator:

$$\|T^*\| = \|T\|$$

The proof requires a corollary of the Hahn-Banach Theorem and so is reserved in the Appendix. ■

Because X^* is a Banach space, it possesses a dual space, denoted $X^{**} = (X^*)^*$. This is the space of functionals that evaluate functionals. Vectors in X can be structured as elements of this double dual.

If a vector $x \in X$ is fixed, it defines a function J_x that acts on elements of the dual space:

$$J_x(x^*) = x^*(x)$$

The perspective shifts from x^* evaluating x to x evaluating x^* . This mapping is visualized in Figure 4.10.

Definition 4.23 Canonical Embedding

The canonical mapping (or evaluation map) $J : X \rightarrow X^{**}$ is defined by mapping a vector x to the functional $J_x \in X^{**}$ defined by:

$$(J_x)(x^*) = x^*(x) \quad \text{for all } x^* \in X^*.$$

Because J is a linear isometry, X is structurally a subspace embedded within X^{**} as shown in Figure 4.11. If the image $J(X)$ is the entirety of X^{**} , we say that X is reflexive.

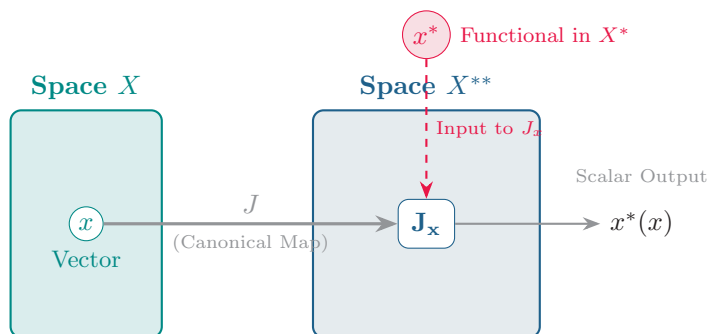


FIGURE 4.10 The canonical embedding. The embedding J assigns each vector x to a functional J_x in the double dual. This functional acts by evaluating elements of the dual space X^* at the point x .

Definition 4.24 Reflexive Space

A normed space X is **reflexive** if the canonical mapping J is surjective.

$$J(X) = X^{**}$$

Examples of reflexivity include:

- Every finite-dimensional space is reflexive.
- ℓ^p spaces ($1 < p < \infty$) are reflexive.
- ℓ^1 and ℓ^∞ are not reflexive.

Theorem 4.25 Reflexivity implies Completeness

If a normed space X is reflexive, it must be a Banach space (complete).

PROOF: The double dual X^{**} is the dual of the normed space X^* . Dual spaces are complete. The space X is isometric to X^{**} in a reflexive space (see [Appendix](#)). Therefore, X is complete. ■

EXERCISES

1. *Null Space Geometry.* Let f be a non-zero bounded linear functional on a normed space X .
 - (a) Prove that the null space $\mathcal{N}(f)$ is a subspace of codimension 1 (a hyperplane).

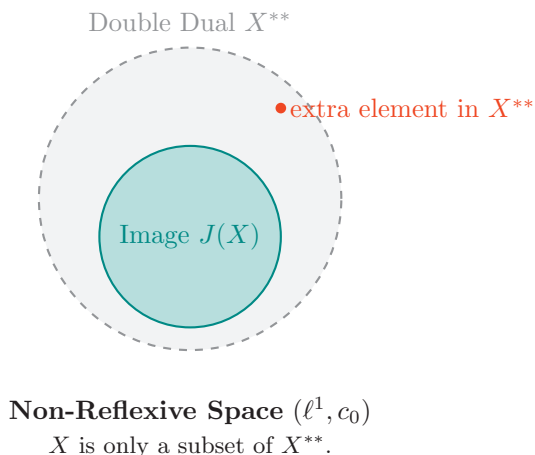


FIGURE 4.11 Reflexivity. In a reflexive space, X^{**} is identically mapped to X . In non-reflexive spaces, the double dual contains functionals that do not correspond to any original vector.

- (b) If $f(u) = c$ defines a level set parallel to the null space, calculate the distance from the origin to this hyperplane in terms of c and $\|f\|$.
2. *Continuity of Coefficients.* Let X be a finite-dimensional space with basis $\{e_1, \dots, e_n\}$. Any vector can be written $u = \sum \alpha_j e_j$. Define the coordinate functionals $f_j(u) = \alpha_j$. Prove that each f_j is a continuous linear functional. (This establishes that coordinates depend continuously on the vector).
3. *Annihilators.* Let S be a subset of a normed space X . The annihilator $S^\perp$ is defined as the set of all bounded linear functionals f such that $f(u) = 0$ for all $u \in S$.
 - (a) Prove that $S^\perp$ is a vector subspace of the dual space.
 - (b) If S is dense in X , prove that $S^\perp = \{0\}$.
4. *Composition.* Let $T : X \rightarrow Y$ be a bounded linear operator and $g : Y \rightarrow \mathbb{K}$ be a bounded linear functional on Y . Define $f(u) = g(Tu)$.
 - (a) Show that f is a bounded linear functional on X .
 - (b) Prove that $\|f\| \leq \|g\| \|T\|$.
5. *Matrix Transpose.* Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $T : X \rightarrow Y$ be represented by the matrix A . Show that the matrix representing the adjoint $T^* : Y^* \rightarrow X^*$ (with respect to the dual bases) is the transpose A^T .

6. *The Shift Adjoint.* Let $X = \ell^2$. Let R be the Right Shift operator: $R(\xi_1, \xi_2, \dots) = (0, \xi_1, \xi_2, \dots)$. Identify the adjoint operator R^* . (Hint: Evaluate a functional satisfying $(R^*y^*)(x) = y^*(Rx)$).
7. *Properties of the Adjoint.* Prove the following algebraic properties:
- (a) $(S + T)^* = S^* + T^*$.
 - (b) $(\alpha T)^* = \alpha T^*$.
 - (c) $(ST)^* = T^*S^*$.
8. *Annihilators and Ranges.* Let $T : X \rightarrow Y$ be a bounded operator. Prove that the null space of the adjoint is the annihilator of the range:

$$\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$$

9. *Subspaces of Reflexive Spaces.* Prove that a closed subspace of a reflexive space is reflexive.
10. *Separability and Reflexivity.* Assume the theorem: If X^* is separable, then X is separable.
- (a) Utilize this theorem to prove that ℓ^1 is not reflexive. (Hint: ℓ^1 is separable. Its dual is ℓ^∞ , which is not separable. If ℓ^1 were reflexive, X^{**} would be isomorphic to X , generating a contradiction).
11. *Adjoint of a Dense Operator.* Let $T : X \rightarrow Y$ be a bounded linear operator. Prove that if the range $\mathcal{R}(T)$ is dense in Y ($\overline{\mathcal{R}(T)} = Y$), then the adjoint operator $T^* : Y^* \rightarrow X^*$ is injective. (Hint: If $T^*y^* = 0$, then $y^*(Tx) = 0$ for all x . Apply the density of Tx to establish that y^* is the zero functional.)
12. *Adjoint of a Surjection.* Prove the condition: If $T : X \rightarrow Y$ is surjective (onto), then T^* is injective and bounded below ($\|T^*y^*\| \geq c\|y^*\|$ for a constant $c > 0$).
13. *The Double Adjoint Identity.* Consider the second adjoint $T^{**} : X^{**} \rightarrow Y^{**}$. Let $J_X : X \rightarrow X^{**}$ and $J_Y : Y \rightarrow Y^{**}$ be the canonical embeddings. Prove the commutative relation:

$$T^{**}J_X = J_YT$$

14. *Reflexivity is an Invariant.* Let X and Y be isomorphic Banach spaces (establishing a linear homeomorphism exists between them). Prove that X is reflexive if and only if Y is reflexive.
15. *The James Space.* The James Space (J) is a Banach space isometric to its double dual J^{**} , yet the canonical map $J_X : J \rightarrow J^{**}$ is not surjective. Explain why Definition 4.24 requires the canonical map, rather than an arbitrary isometry, to establish reflexivity.

4.6 OPERATORS ON FINITE-DIMENSIONAL SPACES

This section considers the application of general operator theory to finite-dimensional spaces, establishing matrix representations and defining the dual basis to compute coordinates.

Matrix Representation of Linear Operators

While operators are denoted algebraically as $y = Tx$, finite-dimensional analysis evaluates these mappings via matrices.

Let X and Y be finite-dimensional vector spaces over the field $\mathbb{K}$, with $\dim X = n$ and $\dim Y = m$. Establishing numeric representation requires defining bases:

- Basis for X : $E = \{e_1, e_2, \dots, e_n\}$
- Basis for Y : $B = \{b_1, b_2, \dots, b_m\}$

Because $\{e_j\}$ forms a basis, the operator T is determined by its action on these n vectors. The image Te_j is a vector in the codomain Y , formulated as a unique linear combination of the basis vectors of Y :

$$Te_j = \sum_{i=1}^m \tau_{ij} b_i$$

The scalar coefficients τ_{ij} are collected to form a rectangular array, yielding the matrix of the operator.

Definition 4.26 Matrix Representation

The matrix representation T_E defining the operator T with respect to bases E and B is the $m \times n$ matrix with entries τ_{ij} :

$$T_E = \begin{pmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1n} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{m1} & \tau_{m2} & \cdots & \tau_{mn} \end{pmatrix}$$

The j -th column of this matrix consists of the coordinates of the image vector Te_j .

The standard algorithm for matrix multiplication is derived from the linearity property. Let $x \in X$ be a vector formulated with coordinates ξ_j : $x = \sum_{j=1}^n \xi_j e_j$. Let $y = Tx$ denote the image vector formulated with coordinates η_i : $y = \sum_{i=1}^m \eta_i b_i$.

Evaluate y by applying T to the expansion of x :

$$\begin{aligned} y = T(x) &= T \left(\sum_{j=1}^n \xi_j e_j \right) \\ &= \sum_{j=1}^n \xi_j (T e_j) \\ &= \sum_{j=1}^n \xi_j \left(\sum_{i=1}^m \tau_{ij} b_i \right) \end{aligned}$$

Interchanging the summations (permitted for finite sums):

$$y = \sum_{i=1}^m \left(\sum_{j=1}^n \tau_{ij} \xi_j \right) b_i$$

Equating this result to the coordinate expression $y = \sum \eta_i b_i$ identifies the coefficients:

$$\eta_i = \sum_{j=1}^n \tau_{ij} \xi_j$$

This establishes that the i -th coordinate of the output equals the dot product of the i -th row of the matrix and the coordinate vector of x . The linear mapping $y = Tx$ aligns directly with the matrix product $y = T_E x$.

Linear Functionals and the Dual Basis

On an n -dimensional space X , every linear functional is bounded. The algebraic dual X' and topological dual X^* are equivalent. For every basis of X , a corresponding basis exists for the dual space.

Theorem 4.27 The Dual Basis

Let $\{e_1, \dots, e_n\}$ be a basis for X . There exists a unique basis $\{f_1, \dots, f_n\}$ for the dual space X^* satisfying:

$$f_j(e_k) = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

The set $\{f_j\}$ is the Dual Basis of X^* relative to the basis $\{e_j\}$.

PROOF:

Construction For any vector $x \in X$, construct the expansion $x = \sum_{k=1}^n \xi_k e_k$. Define the functional f_j as the mapping extracting the j -th coefficient:

$$f_j(x) = \xi_j$$

This mapping is linear. Evaluating the defining condition, the expansion of the basis vector e_k is $1 \cdot e_k$ (all alternative coefficients equal 0). Therefore:

$$f_j(e_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Linear Independence Assume a linear combination yields the zero functional:

$$\sum_{j=1}^n \alpha_j f_j = \mathbf{0}$$

Apply this functional to the basis vector e_k :

$$\left(\sum_{j=1}^n \alpha_j f_j \right) (e_k) = \mathbf{0}(e_k)$$

$$\sum_{j=1}^n \alpha_j f_j(e_k) = 0$$

Applying the property $f_j(e_k) = \delta_{jk}$, the single non-zero term occurs where $j = k$:

$$\alpha_k \cdot 1 = 0 \implies \alpha_k = 0$$

Since this holds for all $k = 1 \dots n$, all coefficients α_k equal zero. The set is linearly independent.

Spanning Since $\dim X^* = \dim X = n$, the n linearly independent vectors form a basis for X^* . ■

Any linear functional $f \in X^*$ is formulated as an expansion of the dual basis:

$$f = \sum_{j=1}^n \gamma_j f_j$$

Applying f to the primal basis vector e_k isolates the coefficient γ_k :

$$f(e_k) = \sum_{j=1}^n \gamma_j f_j(e_k) = \sum_{j=1}^n \gamma_j \delta_{jk} = \gamma_k$$

Thus, the expansion evaluates to

$$f = \sum_{j=1}^n f(e_j) f_j$$

EXERCISES

1. *Matrix Representation Calculation.* Let $X = \mathbb{R}^2$. Let $E = \{e_1, e_2\}$ be the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let $B = \{b_1, b_2\}$ be an alternative basis where $b_1 = (1, 1)$ and $b_2 = (1, -1)$. Let $T : X \rightarrow X$ be the operator defined by $T(\xi_1, \xi_2) = (2\xi_1, \xi_2)$.
 - (a) Determine the matrix T_E representing T relative to the standard basis E .
 - (b) Determine the matrix T_B representing T relative to the basis B .
2. *Differentiation Matrix.* Let X be the space of polynomials of degree at most 3 (P_3). Let the basis be $E = \{1, t, t^2, t^3\}$. Let $D : X \rightarrow X$ be the differentiation operator $Du = u'$.
 - (a) Determine the matrix representing D in the basis E .
 - (b) Using this matrix, evaluate the derivative of $u(t) = 3 + 2t - t^3$.
 - (c) Determine if D is invertible by calculating the determinant of the matrix.
3. *Dual Basis Calculation.* Let $X = \mathbb{R}^2$. Consider the basis $E = \{e_1, e_2\}$ where $e_1 = (1, 0)$ and $e_2 = (1, 1)$. Let $\{f_1, f_2\}$ be the dual basis of X' .
 - (a) Formulate explicit equations for $f_1(x)$ and $f_2(x)$ where $x = (\xi_1, \xi_2)$. (Hint: Apply the conditions $f_i(e_j) = \delta_{ij}$).
 - (b) Verify that if $x = (3, 4)$, the expansion is $x = f_1(x)e_1 + f_2(x)e_2$.
4. *The Trace Functional.* Let $X = \mathbb{R}^{n \times n}$ be the space of square matrices. Define the trace functional $f(A) = \text{tr}(A) = \sum_{i=1}^n a_{ii}$.
 - (a) Prove that f is a linear functional.
 - (b) If the standard basis for matrices is $\{E_{ij}\}$ (where E_{ij} has a 1 at position (i, j) and 0 elsewhere), determine the coefficients of f in the corresponding dual basis.
5. *Annihilator Dimension.* Let X be an n -dimensional vector space. Let M be a subspace of X with $\dim M = m$. The annihilator M^a is the set of all functionals $f \in X'$ such that $f(x) = 0$ for all $x \in M$. Prove that $\dim M^a = n - m$. (Hint: Extend a basis of M to a basis of X and determine the dual basis).
6. *Change of Dual Basis.* Let $\{e_1, e_2\}$ be a basis for X and $\{f_1, f_2\}$ be the corresponding dual basis. Let an alternative basis be defined by $u_1 = e_1 + e_2$ and $u_2 = e_1 - e_2$. Determine the dual basis $\{g_1, g_2\}$ corresponding to $\{u_1, u_2\}$ formulated in terms of f_1 and f_2 .
7. *Double Dual Evaluation.* Let $X = \mathbb{R}^3$. Let $x = (1, 2, 3)$. Let $J : X \rightarrow X^{**}$ be the canonical mapping. Let $F = J(x)$. Evaluate the functional F on the dual vector $f \in X'$ defined by $f(\xi_1, \xi_2, \xi_3) = \xi_1 - \xi_2 + 2\xi_3$.

8. *Non-Reflexive Example Constraints.* Establish why a non-reflexive finite-dimensional space cannot exist, referencing Definition 4.24.
9. *Operator Transpose.* Let $T : X \rightarrow Y$ be a linear operator. The **dual operator** (or transpose) $T^\times : Y' \rightarrow X'$ is defined by $(T^\times g)(x) = g(Tx)$ for all $g \in Y'$ and $x \in X$. Let $X = Y = \mathbb{R}^2$ with the standard basis. Let T have the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Prove that the matrix of the dual operator $T^\times$ relative to the dual basis is the transpose A^T .
10. *Evaluating the Norm of a Functional.* Let $X = \mathbb{R}^2$ with the Euclidean norm. Consider the functional f formulated with coordinates $(3, -4)$ in the standard dual basis.
 - (a) Formulate the explicit equation for $f(x)$.
 - (b) Calculate $\|f\|$.
 - (c) Find a vector u satisfying $\|u\| = 1$ such that $f(u) = \|f\|$.

4.7 NORMED SPACES OF OPERATORS

Previous sections evaluated a linear operator T as a mapping between spaces. This section considers operators as elements within a geometric space. This framework enables the analytical evaluation of the operators:

- Measuring the distance between two operators.
- Evaluating the convergence of a sequence of operators $T_1, T_2, \dots$ to a limit.
- Evaluating the existence of an infinite series of operators $\sum T_n$.

Let X and Y be normed spaces. The set of all bounded linear operators mapping X into Y is denoted by $B(X, Y)$.

Theorem 4.28 The Vector Space $B(X, Y)$

The set $B(X, Y)$ forms a normed vector space subject to the operations:

1. **Vector Addition:** $(T_1 + T_2)x = T_1x + T_2x$
2. **Scalar Multiplication:** $(\alpha T)x = \alpha(Tx)$
3. **Norm:** The operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

To establish $B(X, Y)$ as a normed space, the operator norm must satisfy the four axioms (N1–N4).

(N1) Positivity: $\|T\| \geq 0$ is established because norms of vectors $\|Tx\|$ are non-negative.

(N2) Definiteness: Assume $\|T\| = 0$. By definition, $\sup_{\|x\|=1} \|Tx\| = 0$. This implies $\|Tx\| = 0$ for every vector on the unit sphere. By homogeneity, $Tx = 0$ for all $x \in X$. Thus, T simplifies to the zero operator ($T = 0$).

(N3) Homogeneity:

$$\|\alpha T\| = \sup_{\|x\|=1} \|(\alpha T)x\| = \sup_{\|x\|=1} \|\alpha(Tx)\| = \sup_{\|x\|=1} |\alpha| \|Tx\| = |\alpha| \|T\|$$

(N4) Triangle Inequality: Applying the Triangle Inequality of the target space Y , for any unit vector x :

$$\|(T_1 + T_2)x\| = \|T_1x + T_2x\| \leq \|T_1x\| + \|T_2x\|$$

By definition, $\|T_1x\| \leq \|T_1\|$ and $\|T_2x\| \leq \|T_2\|$. Thus:

$$\|(T_1 + T_2)x\| \leq \|T_1\| + \|T_2\|$$

Since this bound holds for all unit vectors x , the supremum satisfies the inequality

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$$

■

In analysis, exact operators are frequently approximated by a sequence. Establishing that approximations converge to a valid operator requires completeness. The completeness of the operator space $B(X, Y)$ is independent of the domain X ; it depends exclusively on the completeness of the target space Y , a convergence behavior visualized in [Figure 4.12](#).

Theorem 4.29 Completeness of $B(X, Y)$

If Y is a Banach space, then $B(X, Y)$ is a Banach space.

PROOF: Assume Y is complete. Let (T_n) be a Cauchy sequence in $B(X, Y)$. The proof constructs a limit operator T and establishes it belongs to $B(X, Y)$.

Construct the Candidate Limit (Pointwise) Because (T_n) is Cauchy in the operator norm, for every $\epsilon > 0$, an N exists such that for $n, m > N$:

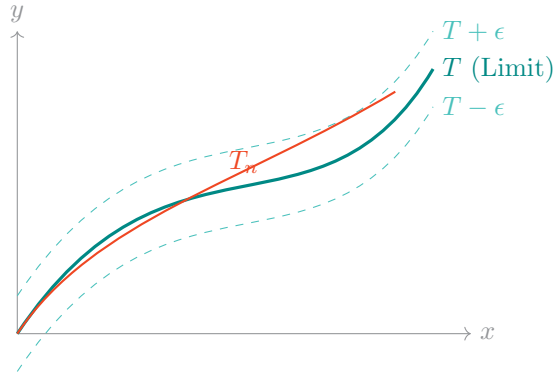
$$\|T_n - T_m\| < \epsilon$$

Fix a vector $x \in X$. Consider the sequence of image vectors (T_nx) in Y :

$$\|T_nx - T_mx\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\|$$

Since $\|T_n - T_m\| \rightarrow 0$, the scalar coefficient converges to zero. Thus, (T_nx) is a Cauchy sequence of vectors in Y . Because Y is a Banach space, this sequence converges. Define the limit operator T pointwise:

$$Tx = \lim_{n \rightarrow \infty} T_nx$$



Visualizing Operator Convergence

For large n , the operator T_n must reside entirely within the ϵ -tube around T for all unit vectors x .

FIGURE 4.12 Uniform operator convergence. Evaluating convergence requires the entire mapping function T_n to approach T uniformly across the domain.

Verify Linearity The limit of linear maps preserves linearity. For scalars α, β and vectors u, v :

$$T(\alpha u + \beta v) = \lim_{n \rightarrow \infty} T_n(\alpha u + \beta v)$$

Applying the linearity of T_n :

$$= \lim_{n \rightarrow \infty} (\alpha T_n u + \beta T_n v)$$

Applying the continuity of vector operations in Y :

$$= \alpha \lim T_n u + \beta \lim T_n v = \alpha T u + \beta T v$$

Verify Boundedness Because (T_n) is a Cauchy sequence, the sequence of real numbers $\|T_n\|$ is bounded. Let $K = \sup_n \|T_n\|$. For any x , the bound evaluates to $\|T_n x\| \leq \|T_n\| \|x\| \leq K \|x\|$. Evaluating the limit as $n \rightarrow \infty$:

$$\|Tx\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq K \|x\|$$

This establishes that T is bounded ($\|T\| \leq K$), concluding $T \in B(X, Y)$.

Verify Convergence in Operator Norm The operator T was constructed such that $T_n x \rightarrow T x$ for each x . Uniform convergence ($\|T_n - T\| \rightarrow 0$) must be established. Returning to the Cauchy condition: given $\epsilon > 0$, an N exists such that for all $n, m > N$:

$$\|T_n - T_m\| < \epsilon$$

For any vector x satisfying $\|x\| = 1$:

$$\|T_n x - T_m x\| < \epsilon$$

Evaluating the limit as $m \rightarrow \infty$, where $T_m x \rightarrow Tx$, the strict inequality transitions to a non-strict bound:

$$\|T_n x - Tx\| \leq \epsilon$$

This inequality is satisfied for every unit vector x simultaneously (for a fixed $n > N$). Therefore, the supremum is bounded:

$$\sup_{\|x\|=1} \|T_n x - Tx\| \leq \epsilon \implies \|T_n - T\| \leq \epsilon$$

Thus, $T_n \rightarrow T$ in the operator norm. ■

This theorem applies to linear functionals. The dual space is formulated as $X^* = B(X, \mathbb{K})$, where the scalar field $\mathbb{K}$ evaluates to either $\mathbb{R}$ or $\mathbb{C}$. Because the real line $\mathbb{R}$ and complex plane $\mathbb{C}$ act as complete metric spaces, the target space $Y = \mathbb{K}$ is complete. Theorem 4.29 guarantees $B(X, \mathbb{K})$ is complete.

Corollary 4.30 Completeness of the Dual Space

The dual space X^* of a normed space X is a Banach space.

This property holds independently of the completeness of the domain X . The stability of the dual space is derived entirely from the stability of the scalar field.

EXERCISES

1. *Completeness Logic.* Theorem 4.29 establishes that if Y is a Banach space, $B(X, Y)$ is a Banach space.
 - (a) Assume X is not complete (e.g., the space of polynomials). Is $B(X, \mathbb{R})$ complete? Detail the conditions.
 - (b) **Converse:** Prove that if $B(X, Y)$ is a Banach space and $X \neq \{0\}$, then Y must be a Banach space. (*Hint: Construct an operator T mapping a specific vector x_0 to an arbitrary $y \in Y$, and apply the completeness of the operator space to evaluate limits in Y .*)
2. *Pointwise vs. Uniform Convergence.* Let $X = \ell^2$. Let P_n be the projection operator targeting the first n coordinates:

$$P_n(\xi_1, \xi_2, \dots) = (\xi_1, \dots, \xi_n, 0, 0, \dots)$$

Let I be the identity operator.

- (a) Prove that for any fixed vector $x \in \ell^2$, $P_n x \rightarrow Ix$ as $n \rightarrow \infty$.
- (b) Calculate the operator norm $\|P_n - I\|$.
- (c) Determine if the sequence (P_n) converges to I uniformly in the operator norm.
3. *Series of Operators.* Let X be a Banach space. Let $T \in B(X, X)$ be an operator satisfying $\|T\| < 1$. Consider the series of operators $\sum_{n=0}^{\infty} T^n$ (where $T^0 = I$).
- (a) Prove that the sequence of partial sums $S_k = \sum_{n=0}^k T^n$ forms a Cauchy sequence in $B(X, X)$.
- (b) Conclude that the limit sum exists (i.e., the operator $(I - T)^{-1}$ exists). This establishes the **Neumann Series**.
4. *Closed Subspaces of Operators.* Let X be a Banach space. Fix a bounded operator $S \in B(X, X)$. Let M be the set of all operators commuting with S :

$$M = \{T \in B(X, X) \mid TS = ST\}$$

Prove that M forms a closed subspace of $B(X, X)$. (*Hint: Evaluate sequential limits. If $T_n \rightarrow T$ and $T_n S = ST_n$, does $TS = ST$?*).

5. *Finite Dimensions.* Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.
- (a) Explain why $B(X, Y)$ is isomorphic to the space of $m \times n$ matrices.
- (b) Because finite-dimensional spaces are complete, $B(X, Y)$ is a Banach space. Explain the implications for matrix convergence. (If a sequence of matrices converges entry-wise, does it converge in the operator norm?).
6. *Multiplication Operator Norm.* Let $X = C[0, 1]$ equipped with the uniform norm. Let $g \in C[0, 1]$ act as a fixed function. Define the multiplication operator $M_g : X \rightarrow X$ by $(M_g u)(t) = g(t)u(t)$.
- (a) Prove that M_g is a bounded linear operator.
- (b) Prove that $\|M_g\| = \|g\|_{\infty} = \max_{t \in [0, 1]} |g(t)|$.
7. *Double Dual Embeddings.* Let X be a normed space.
- (a) X' is established as complete. Evaluate if $X'' = (X')'$ is inherently complete.
- (b) If X is not complete, determine if it can be reflexive. (Reference the canonical mapping $J : X \rightarrow X^{**}$).
8. *Continuity of Composition.* Let X, Y, Z be normed spaces. Let $T_n \rightarrow T$ in $B(X, Y)$ and $S_n \rightarrow S$ in $B(Y, Z)$. Prove that the composition $S_n T_n$ converges to ST in $B(X, Z)$. (*Hint: Apply the substitution $S_n T_n - ST = S_n(T_n - T) + (S_n - S)T$.*)

9. *Unbounded Codomain.* The completeness of $B(X, Y)$ relies on Y . Let $X = \mathbb{R}$ (complete). Let Y be the space of polynomials equipped with the uniform norm (incomplete). Prove that $B(X, Y)$ is not complete. (*Hint: Show that $B(\mathbb{R}, Y)$ is isomorphic to Y .*)
10. *Equivalent Norms.* Let $\|\cdot\|_1$ and $\|\cdot\|_2$ act as equivalent norms on a vector space X (satisfying $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$). Let $T : X \rightarrow X$ be a linear operator. Prove that T is bounded relative to $\|\cdot\|_1$ if and only if it is bounded relative to $\|\cdot\|_2$.

Fundamental Theorems for Normed and Banach Spaces

5.1 Zorn's Lemma

5.2 Hahn-Banach Theorem

5.3 Uniform Boundedness Principle

5.4 Open Mapping Theorem

5.5 Closed Graph Theorem

FUNCTIONAL analysis rests upon four foundational results often referred to as the “Four Pillars”: the Hahn-Banach Theorem, the Uniform Boundedness Principle, the Open Mapping Theorem, and the Closed Graph Theorem. Together, these theorems provide the essential bridge between algebraic manipulation and topological stability in infinite-dimensional spaces.

In this chapter, we first equip ourselves with Zorn's Lemma to handle existence proofs where geometric intuition alone fails. We then establish the rigorous extension of linear functionals, the structural thickness of complete spaces via the Baire Category Theorem, and the automatic continuity of inverse mappings. These results form the rigorous backbone of the subject, ensuring that our mathematical models remain stable and predictable in the realm of Banach spaces.

5.1 ZORN'S LEMMA

To establish the Hahn-Banach theorem and several existence proofs in infinite-dimensional spaces, a set-theoretic tool known as *Zorn's Lemma* is needed. While logically equivalent to the Axiom of Choice, it is formulated to identify *maximal* configurations in analysis.

Definition 5.1 Partially Ordered Set

A partially ordered set (or *poset*) is a set $\mathcal{P}$ equipped with a binary relation $\preceq$ such that for all $p, q, r \in \mathcal{P}$:

1. Reflexivity: $p \preceq p$.
2. Antisymmetry: If $p \preceq q$ and $q \preceq p$, then $p = q$.
3. Transitivity: If $p \preceq q$ and $q \preceq r$, then $p \preceq r$.

In a poset, it is not required that every pair of elements be comparable. If neither $p \preceq q$ nor $q \preceq p$ holds, the elements are classified as incomparable. This partial structure is visualized via a Hasse diagram in Figure 5.1.

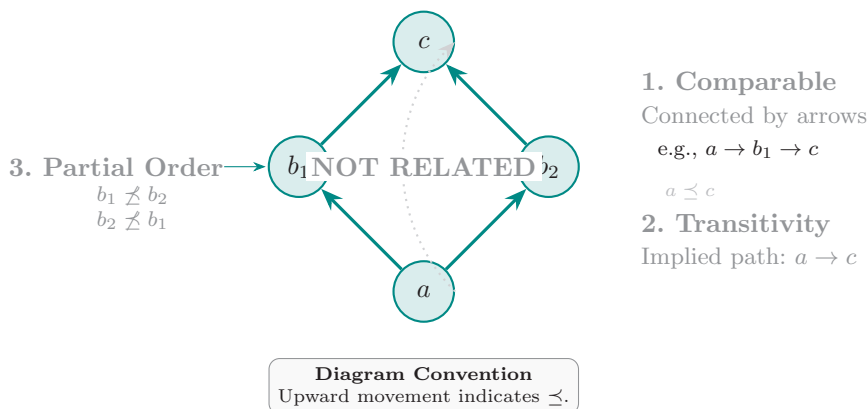


FIGURE 5.1 Visualizing a Partially Ordered Set (Hasse Diagram). The elements b_1 and b_2 are situated on separate branches, rendering them incomparable. The order is "partial" because not every pair of elements can be compared.

Definition 5.2 Chains and Bounds

1. A chain $\mathcal{C} \subset \mathcal{P}$ is a subset in which every pair of elements is comparable (a totally ordered subset).

2. An upper bound for a subset $\mathcal{S} \subset \mathcal{P}$ is an element $s \in \mathcal{P}$ such that $x \preccurlyeq s$ for all $x \in \mathcal{S}$.
3. A maximal element of $\mathcal{P}$ is an element $m \in \mathcal{P}$ such that if $m \preccurlyeq x$ for some $x \in \mathcal{P}$, then $x = m$.

The structural relationship between chains, upper bounds, and maximal elements is illustrated in Figure 5.2.

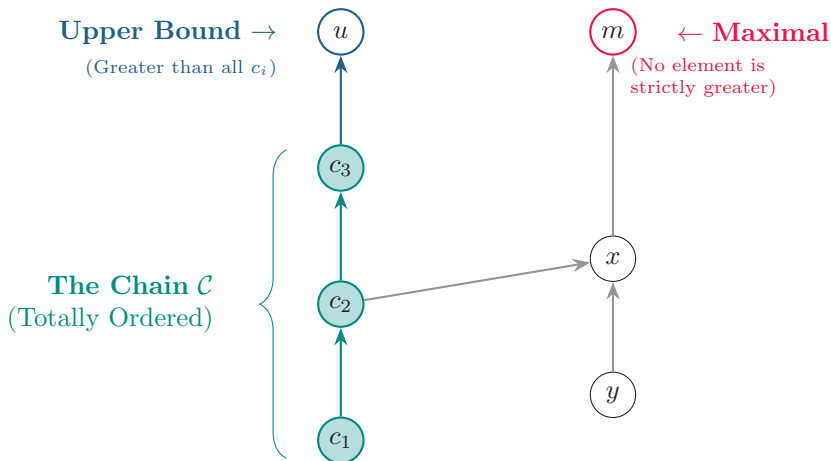


FIGURE 5.2 Visualizing chains and bounds. The highlighted nodes form a chain. The node u serves as an upper bound for the chain. The node m is maximal because no subsequent elements strictly succeed it.

To conceptualize the utility of Zorn's Lemma, the following standard examples distinguish between different topological orderings.

Example 5.3 The Real Line (Total Ordering)

Let $\mathcal{P} = \mathbb{R}$ with the relation $\preccurlyeq$ defined as the standard inequality $\leq$.

Comparability Any two elements are comparable (e.g., $2 \preccurlyeq 5$). Therefore, the set $\mathcal{P}$ is a chain.

Maximality This set lacks a maximal element, as for any $x \in \mathbb{R}$, there exists an element $x + 1$ satisfying $x \preccurlyeq x + 1$.

Example 5.4 Power Sets (Partial Ordering)

Let $\mathcal{X}$ be a set containing at least two elements, and $\mathcal{P} = 2^{\mathcal{X}}$ denote its power set. Let the relation be $A \preceq B$ if $A \subseteq B$.

Incomparability This constitutes a partial order because if $\mathcal{X} = \{1, 2\}$, the specific subsets $\{1\}$ and $\{2\}$ remain incomparable.

Maximality The set $\mathcal{X}$ itself acts as the maximal element (and in this specific formulation, the unique maximum).

Example 5.5 Proper Subsets

Let $\mathcal{M}$ be the collection of all *proper* subsets of $\mathcal{X} = \{a, b, c\}$, ordered by inclusion.

Maximal Elements The subsets $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$ are identically classified as maximal elements.

Upper Bounds If the chain is defined as $\mathcal{C} = \{\emptyset, \{a\}, \{a, b\}\}$, an upper bound for this chain within $\mathcal{P}$ is the element $\{a, b\}$. ┘

Zorn's Lemma is an axiom equivalent to the Axiom of Choice. In functional analysis, it is utilized to facilitate the construction of bases and functional extensions across infinite-dimensional spaces.

Theorem 5.6 Zorn's Lemma

Let $\mathcal{P} \neq \emptyset$ be a partially ordered set. If every chain $\mathcal{C} \subset \mathcal{P}$ possesses an upper bound in $\mathcal{P}$, then $\mathcal{P}$ contains at least one maximal element.

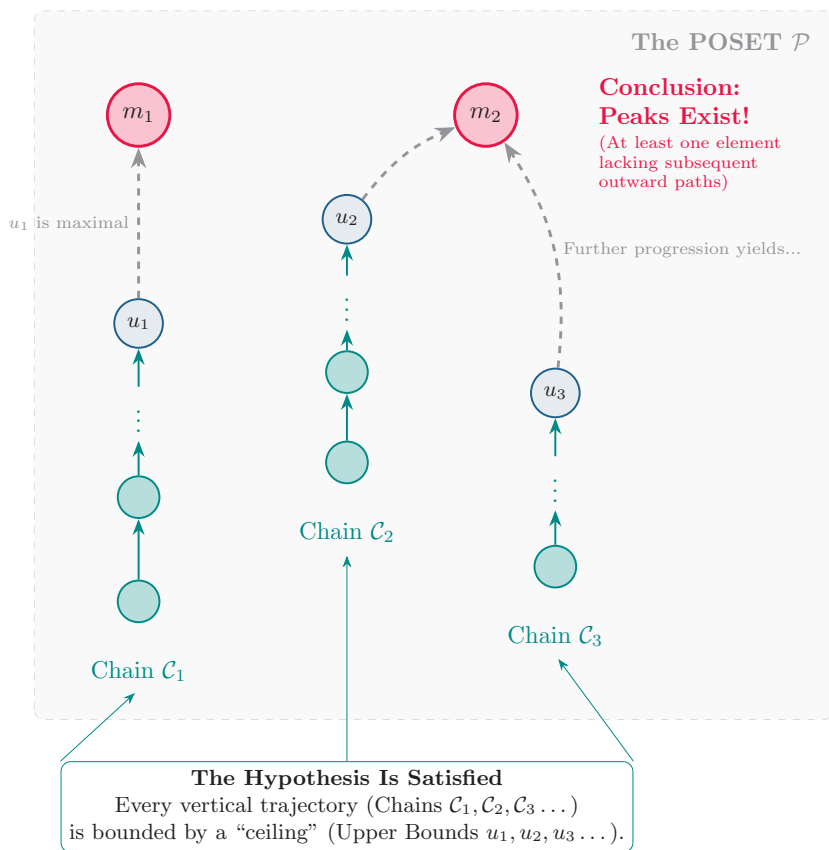


FIGURE 5.3 Visualizing Zorn’s lemma with multiple chains. The hypothesis dictates that every possible chain (teal paths) possesses an upper bound (blue nodes). Because every upward trajectory is locally bounded, the lemma ensures the overarching structure contains at least one ultimate maximal element (purple nodes m_1, m_2).

Example 5.7

Let V be a vector space. Let $\mathcal{P}$ be the collection of all linearly independent subsets of V , ordered by inclusion ($\subseteq$).

Chains This structure forms a poset. A chain $\mathcal{C}$ within this poset constitutes a nested collection of independent sets.

Upper Bounds The union of all sets within such a chain is itself an independent set, serving as the upper bound.

Conclusion By Zorn's Lemma, $\mathcal{P}$ contains a maximal element. This maximal independent set defines a Hamel Basis.

Zorn's lemma is invoked to establish that a mathematical process can be advanced to its absolute limit across an infinite set. For example, verifying whether every vector space possesses a basis requires Zorn's Lemma when evaluating infinite-dimensional spaces.

Theorem 5.8 Existence of Hamel Basis

Every vector space $V \neq \{0\}$ possesses a Hamel basis.

PROOF: Let V be a non-zero vector space. A Hamel basis is defined as a maximal linearly independent subset of V . The existence of such a set is proven by constructing a poset of independent sets and applying Zorn's Lemma.

Construct the Poset Let $\mathcal{I}$ denote the collection of all linearly independent subsets of V . Because $V \neq \{0\}$, a non-zero vector $v \in V$ exists, and the isolated set $\{v\}$ is linearly independent. Consequently, $\mathcal{I}$ is non-empty. $\mathcal{I}$ is ordered by set inclusion. For $A, B \in \mathcal{I}$, the relation is defined as $A \preceq B$ if $A \subseteq B$.

Verify the Chain Condition Let $\mathcal{C}$ be an arbitrary chain in $\mathcal{I}$ (a totally ordered collection of independent sets). The union of these sets is an upper bound for $\mathcal{C}$, and it belongs to $\mathcal{I}$ because the union of any chain of linearly independent sets is linearly independent.

Apply Zorn's Lemma Because every chain in $\mathcal{I}$ possesses an upper bound, Zorn's Lemma guarantees the existence of a maximal element M .

Conclusion The set M is linearly independent by definition. It must be shown that M spans V . Assume, for contradiction, that $\text{span } M \neq V$. A vector $v \in V$ must exist such that $v \notin \text{span } M$. Evaluate the set $M' = M \cup \{v\}$. Because v is not a linear combination of vectors in M , the set M' remains linearly independent. However, $M \subset M'$ strictly, which contradicts the established fact that M is a maximal element of $\mathcal{I}$. Therefore, $\text{span } M = V$, confirming M is a Hamel basis. ■

EXERCISES

1. *Basis of $\mathbb{R}$.* Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$. Prove that a Hamel basis for this space must be uncountable.
2. *Total Ordering.* A poset is defined as totally ordered if it consists of a single chain. Determine whether the poset $(\mathbb{R}, \leq)$ is totally ordered. Determine whether the power set $(\mathcal{P}(\mathbb{R}), \subseteq)$ is totally ordered.
3. *Visualizing Chains.* Within the power set of $\{a, b, c, d\}$, construct a chain of maximum possible length.

4. *Maximality.* Construct a poset $(\mathcal{P}, \preceq)$ possessing exactly two distinct maximal elements.
5. *The Real Line.* Consider the set of all bounded closed intervals $[a, b] \subset \mathbb{R}$ ordered by the inclusion relation $\supseteq$. Determine if every chain within this poset possesses an upper bound.
6. *Validity of Partial Orders.* Determine whether the following relations define a partial order on the complex numbers $\mathbb{C}$. If a relation fails to define a partial order, identify the violated axiom (Reflexivity, Antisymmetry, Transitivity).
 - (a) $z_1 \preceq z_2$ if $|z_1| \leq |z_2|$.
 - (b) $z_1 \preceq z_2$ if $\operatorname{Re}(z_1) \leq \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) \leq \operatorname{Im}(z_2)$.
7. *The Extension Principle.* Zorn's Lemma facilitates the extension of a fundamental structure into a maximal configuration. Let V be a vector space and S be a linearly independent subset of V . Apply Zorn's Lemma to prove the existence of a Hamel basis B for V such that $S \subseteq B$. (*Hint: Modify the poset utilized in Theorem 5.8 to consist exclusively of independent sets containing S .*)
8. *Basis of $\mathbb{R}$ over $\mathbb{Q}$.* Consider the real numbers $\mathbb{R}$ evaluated as a vector space over the field of rational numbers $\mathbb{Q}$.
 - (a) Prove that $\{1, \sqrt{2}\}$ constitutes a linearly independent set in this space.
 - (b) Analytically demonstrate the existence of a Hamel basis for this space using Zorn's Lemma.
 - (c) Prove that any such basis must be uncountable. (*Hint: Evaluate the cardinality of $\mathbb{R}$ under the assumption of a countable basis.*)
9. *Minimal Elements.* Zorn's Lemma possesses a dual formulation. Prove that if $\mathcal{P}$ is a poset where every chain possesses a lower bound, then $\mathcal{P}$ contains at least one minimal element. (*Hint: Define a reversed order $x \sqsubseteq y \iff y \preceq x$ and apply the standard formulation of Zorn's Lemma.*)
10. *Maximal Ideals in Function Rings.* Let $C[0, 1]$ denote the ring of continuous real-valued functions defined on $[0, 1]$. An ideal is a subspace I such that if $f \in I$ and $g \in C[0, 1]$, then $fg \in I$. A proper ideal satisfies $I \neq C[0, 1]$. Apply Zorn's Lemma to prove that every proper ideal is contained strictly within a maximal ideal. (*Note: This algebraic application structurally mirrors the Hahn-Banach extension logic.*)

5.2 THE HAHN-BANACH THEOREM

The *Hahn-Banach Theorem* constitutes a foundational theorem in the theory of Banach spaces. While Zorn's Lemma furnishes the set-theoretic machinery for existence, the theorem guarantees that linear functionals can be extended

from a subspace to the entire space without losing their essential bounding properties. This theorem establishes that sufficient continuous linear functionals exist on a vector space to distinguish between different vectors, enabling the structural study of the space via its dual.

Before analyzing norms and continuity, the theorem is established in a purely algebraic setting. A generalized notion of length, termed a sublinear functional, is required.

Definition 5.9 Sublinear Functional

Let X be a real vector space. A functional $f : X \rightarrow \mathbb{R}$ is called **sublinear** if it satisfies:

1. **Subadditivity:** $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$.
2. **Positive Homogeneity:** $f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$.

A norm $\|\cdot\|$ constitutes a sublinear functional, but a sublinear functional need not be symmetric ($p(-x)$ need not equal $p(x)$) nor strictly non-negative.

Theorem 5.10 Hahn-Banach Theorem (Real Vector Spaces)

Let X be a real vector space and p be a sublinear functional on X . Let f be a linear functional defined on a subspace $Z \subset X$ such that:

$$f(z) \leq p(z) \quad \text{for all } z \in Z$$

Then, there exists a linear extension $\tilde{f}$ defined on the entire space X such that:

1. $\tilde{f}(x) = f(x)$ for all $x \in Z$ (Extension).
2. $\tilde{f}(x) \leq p(x)$ for all $x \in X$ (Domination).

PROOF: The proof proceeds in two stages: first, it is proven that the functional can be extended by a single dimension; second, Zorn's Lemma is utilized to extend the functional across the entire infinite-dimensional space. The geometric intuition for this extension is visualized in [Figure 5.4](#).

The One-Step Extension Assume Z is a proper subspace of X and $y_1 \in X \setminus Z$. Consider the subspace Z_1 spanned by Z and y_1 :

$$Z_1 = \{z + \alpha y_1 \mid z \in Z, \alpha \in \mathbb{R}\}$$

Because $y_1 \notin Z$, the representation of any vector in Z_1 as $z + \alpha y_1$ is unique. Any linear extension $\tilde{f}$ on Z_1 is determined entirely by the value assigned to y_1 . Let $\tilde{f}(y_1) = c$. By linearity, for any $x = z + \alpha y_1$, it must hold that

$$\tilde{f}(x) = f(z) + \alpha c$$

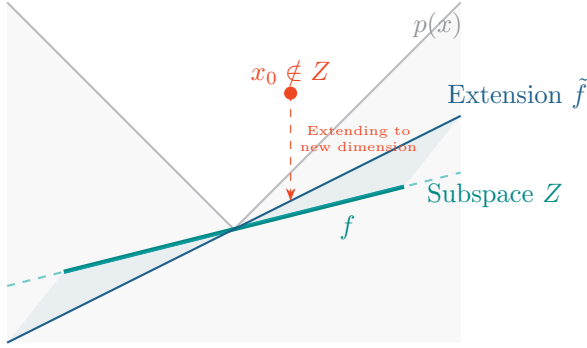


FIGURE 5.4 The Hahn-Banach extension. A linear functional defined on a subspace Z is extended to the entire space X while remaining strictly dominated by the sublinear functional p .

A real number c must be selected such that the domination condition $\tilde{f}(x) \leq p(x)$ holds for all $x \in Z_1$. That is,

$$f(z) + \alpha c \leq p(z + \alpha y_1) \quad \text{for all } z \in Z, \alpha \in \mathbb{R}$$

This inequality is analyzed for positive and negative scalar values of α :

- Case $\alpha > 0$: Dividing by α the inequality becomes

$$f(z/\alpha) + c \leq p(z/\alpha + y_1)$$

Letting $v = z/\alpha$ (which represents an arbitrary element of Z), an upper bound for c is derived:

$$c \leq p(v + y_1) - f(v) \quad \text{for all } v \in Z \quad (\text{A})$$

- Case $\alpha < 0$: Let $\alpha = -\beta$ where $\beta > 0$. The inequality becomes

$$f(z) - \beta c \leq p(z - \beta y_1)$$

Dividing by β yields

$$f(z/\beta) - c \leq p(z/\beta - y_1)$$

Letting $w = z/\beta$ (an arbitrary element of Z), a lower bound for c is derived:

$$c \geq f(w) - p(w - y_1) \quad \text{for all } w \in Z \quad (\text{B})$$

For a valid scalar c to exist, the lower bound (B) must remain less than or equal to the upper bound (A) for any arbitrary choice of $w, v \in Z$:

$$f(w) - p(w - y_1) \stackrel{?}{\leq} p(v + y_1) - f(v)$$

Rearranging the algebraic terms yields

$$f(w) + f(v) \leq p(v + y_1) + p(w - y_1)$$

This is verified by applying the linearity of the functionals and the domination of f by p on Z :

$$f(w) + f(v) = f(w + v) \leq p(w + v) = p(w - y_1 + v + y_1)$$

By the subadditivity property of p

$$p(w - y_1 + v + y_1) \leq p(w - y_1) + p(v + y_1)$$

This confirms the inequality holds. Therefore, the supremum and infimum are defined as

$$m = \sup_{w \in Z} [f(w) - p(w - y_1)], \quad M = \inf_{v \in Z} [p(v + y_1) - f(v)]$$

Since $m \leq M$, any $c \in [m, M]$ can be selected so that the extension $\tilde{f}$ satisfies $\tilde{f}(x) \leq p(x)$ across Z_1 .

Maximal Extension via Zorn's Lemma Let $\mathcal{E}$ define the set of all pairs (g, D_g) , where D_g is a subspace containing Z , and g is a linear extension of f defined on D_g satisfying $g \leq p$. The set $\mathcal{E}$ is ordered by extension: $(g, D_g) \preceq (h, D_h)$ if $D_g \subseteq D_h$ and $h|_{D_g} = g$.

- **Upper Bound.** Let $\mathcal{C} = \{(g_\alpha, D_\alpha)\}$ be a chain in $\mathcal{E}$. The union of the respective domains, $D = \bigcup D_\alpha$, constitutes a subspace. A functional g is defined on this union by setting $g(x) = g_\alpha(x)$ for any $x \in D_\alpha$. Because $\mathcal{C}$ is a chain, any two domains within it are comparable, ensuring g is well-defined. Furthermore, the domination condition $g \leq p$ holds on D . Thus, the pair $(g, D) \in \mathcal{E}$ serves as an upper bound for the chain $\mathcal{C}$.
- **Maximality:** By Zorn's Lemma, $\mathcal{E}$ possesses a maximal element $(\tilde{f}, D_{max})$.
- **Conclusion:** Claim that $D_{max} = X$. If not, a vector $y \in X \setminus D_{max}$ exists. Utilizing the one-step extension, $\tilde{f}$ could be extended to $D_{max} \oplus \text{span } y$, thereby creating a proper extension. This contradicts the maximality of $\tilde{f}$. Thus, $\tilde{f}$ must be defined on all of X . ■

A fundamental application of the Hahn-Banach theorem occurs in normed spaces, where the sublinear functional constitutes the norm scaled by a constant.

Theorem 5.11 Hahn-Banach Theorem (Normed Spaces)

Let X be a normed space (real or complex) and f be a bounded linear functional defined on a subspace $Z \subset X$. Then there exists a bounded linear functional $\tilde{f}$ defined on X such that:

1. $\tilde{f}(z) = f(z)$ for all $z \in Z$.
2. $\|\tilde{f}\|_X = \|f\|_Z$.

PROOF:

Construct the Sublinear Functional The functional $p(x) = \|f\|_Z \cdot \|x\|$ is defined. This is a valid sublinear functional because the norm is subadditive and positively homogeneous.

Verify Domination on Z For any arbitrary $z \in Z$, linearity and geometric boundedness dictate:

$$f(z) \leq |f(z)| \leq \|f\|_Z \|z\| = p(z)$$

Thus, the fundamental condition of Theorem 5.10 is satisfied.

Apply Extension By the real Hahn-Banach theorem, there exists a linear extension $\tilde{f}$ defined on X such that $\tilde{f}(x) \leq p(x) = \|f\|_Z \|x\|$ for all $x \in X$.

Verify Norm Preservation It must be verified that the norm of $\tilde{f}$ has not increased during the extension process. It is established that $\tilde{f}(x) \leq \|f\|_Z \|x\|$. To establish the corresponding lower bound, the linearity of $\tilde{f}$ is applied:

$$-\tilde{f}(x) = \tilde{f}(-x) \leq p(-x) = \|f\|_Z \|-x\| = \|f\|_Z \|x\|$$

Thus, $\tilde{f}(x) \geq -\|f\|_Z \|x\|$. Combining these dual inequalities yields the absolute bound:

$$|\tilde{f}(x)| \leq \|f\|_Z \|x\|$$

This inequality implies that $\|\tilde{f}\|_X \leq \|f\|_Z$. Because $\tilde{f}$ is an extension of f , the supremum $\|\tilde{f}\|_X$ over the entire space cannot be less than the supremum $\|f\|_Z$ over the subspace. Therefore, the norms are equivalent: $\|\tilde{f}\|_X = \|f\|_Z$. ■

The Hahn-Banach theorem serves as the guarantee that the dual space X^* is sufficiently robust to detect every topological nuance of the original space X .

The Norming Functional

For every non-zero vector $x_0 \in X$, there exists a bounded linear functional $\tilde{f} \in X^*$ such that:

$$\|\tilde{f}\| = 1 \quad \text{and} \quad \tilde{f}(x_0) = \|x_0\|$$

PROOF: Construct a one-dimensional subspace $Z = \text{span}\{x_0\}$. Define a functional f on Z by specifying its action on the basis vector: $f(x_0) = \|x_0\|$. For any vector $z = \alpha x_0 \in Z$:

$$|f(z)| = |f(\alpha x_0)| = |\alpha| \|x_0\| = \|\alpha x_0\| = \|z\|$$

Thus, restricted to the subspace Z , the norm is $\|f\| = 1$. By the Hahn-Banach Theorem, f can be extended to a functional $\tilde{f}$ defined on X such that $\|\tilde{f}\| = \|f\| = 1$ and $\tilde{f}(x_0) = \|x_0\|$. ■

Separation of Points

If x and y are distinct vectors in X , then there exists a bounded linear functional $f \in X^*$ that separates them:

$$f(x) \neq f(y)$$

PROOF: Because $x \neq y$, the difference vector $x_0 = x - y$ is non-zero. By the preceding corollary, a functional f exists such that $f(x_0) = \|x - y\| \neq 0$. By linearity:

$$f(x - y) = f(x) - f(y) \neq 0 \implies f(x) \neq f(y)$$

■

Geometric Separation by Hyperplanes

Let K be a closed convex set in a normed space X , and let x_0 be a point outside K . Then, there exists a non-zero functional $f \in X^*$ and a real scalar α such that the hyperplane $H = \{x : f(x) = \alpha\}$ strictly separates x_0 from K .

As visualized in [Figure 5.5](#), this separation relies on utilizing the kernel of the functional to act as an impenetrable topological boundary.

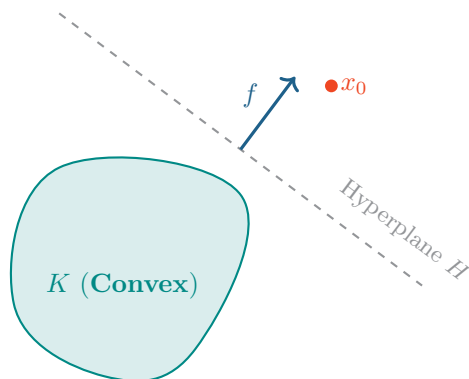


FIGURE 5.5 Visualization of the strict separation of an external point from a closed convex set. The kernel of the functional establishes the separating hyperplane.

PROOF: The proof converts the convex structure into a sublinear functional in order to apply the Hahn-Banach theorem.

Create an Open Convex Neighborhood Because K is closed and $x_0 \notin K$, the distance $d = \text{dist}(x_0, K) > 0$. An open convex set U is constructed K :

$$U = \{y \in X : \text{dist}(y, K) < d/2\}$$

The point x_0 remains outside the set U .

Shift to the Origin An internal point $u_0 \in U$ is fixed. The coordinate space is translated such that u_{int} establishes the new origin. Let $C = U - u_0$. Consequently, $0 \in C$, and $z_0 = x_0 - u_0 \notin C$.

The Minkowski Functional The Minkowski Functional (or gauge) $p : X \rightarrow \mathbb{R}$ is defined for the set C :

$$p(x) = \inf\{\lambda > 0 : x \in \lambda C\}$$

Because C is both convex and open, p satisfies all criteria for a sublinear functional. Since $z_0 \notin C$, $p(z_0) \geq 1$.

Apply Hahn-Banach Define g on $Z = \text{span}\{z_0\}$ by $g(tz_0) = tp(z_0)$.

- For $t > 0$: $g(tz_0) = p(tz_0)$ (dominated by p).
- For $t \leq 0$: $g(tz_0) \leq p(tz_0)$ (satisfied trivially, because $p \geq 0$).

By the real Hahn-Banach Theorem, g can be extended to a functional f on X such that $f(x) \leq p(x)$. Since C contains an open ball around the origin, p is bounded. The condition $f(x) \leq p(x)$ thus ensures f is also bounded, that is, $f \in X^*$ (see [Appendix](#)). The condition $f(x) \leq p(x)$ thus ensures f is also bounded, that is, $f(x) \in X^*$.

Now any coordinate $x \in C$, $p(x) < 1$, guaranteeing $f(x) < 1$. Also z_0 , $f(z_0) = p(z_0) \geq 1$. Thus, f separates z_0 from C . Translating back to the original coordinates yields the hyperplane separating x_0 from K . ■

While the real formulation of the Hahn-Banach theorem is fundamental, numerous spaces in functional analysis (such as Hilbert spaces and $\mathbb{C}^n$) are complex. The theorem is now extended to complex vector spaces.

The strategy is to treat the complex vector space as a real vector space, applying the real Hahn-Banach theorem to the real component of the functional, and utilizing the complex algebraic structure to reconstruct the complete complex functional.

Theorem 5.15 Hahn-Banach Theorem (Complex)

Let X be a complex vector space and p be a seminorm on X . Let f be a linear functional defined on a subspace $Z \subset X$ such that:

$$|f(z)| \leq p(z) \quad \text{for all } z \in Z$$

Then there exists a linear extension $\tilde{f}$ defined on X such that:

1. $\tilde{f}(z) = f(z)$ for all $z \in Z$.
2. $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

PROOF: The complex functional can be decomposed into real components, the real part extended, and the full complex functional reconstructed.

Decomposition into Real and Imaginary Parts Let $f(z) = u(z) + iv(z)$ where $u(z) = \operatorname{Re} f(z)$ and $v(z) = \operatorname{Im} f(z)$ which are real-valued functionals. Because f is linear, u and v are linear with respect to real scalars. However, u and v are not independent. Because f is complex-linear, the identity $f(iz) = if(z)$ must hold:

- $f(iz) = u(iz) + iv(iz)$
- $if(z) = i(u(z) + iv(z)) = -v(z) + iu(z)$

Equating the real components yields the identity $u(iz) = -v(z)$, or $v(z) = -u(iz)$. Thus, the imaginary part v is entirely determined by the real part u and we can write

$$f(z) = u(z) - iu(iz)$$

Extension of the Real Part From the hypothesis $|f(z)| \leq p(z)$, and since the real component cannot exceed the modulus:

$$u(z) = \operatorname{Re} f(z) \leq |f(z)| \leq p(z) \quad \text{for all } z \in Z$$

Thus, u acts as a linear functional on the real subspace Z that is dominated by the sublinear functional p (a seminorm is inherently sublinear). By the real Hahn-Banach Theorem, there exists a real-linear extension $\tilde{u}$ defined on X such that $\tilde{u}(x) \leq p(x)$ for all $x \in X$.

Reconstruction and Verification The candidate extension $\tilde{f}$ is defined on X by substituting u with its extension $\tilde{u}$:

$$\tilde{f}(x) := \tilde{u}(x) - i\tilde{u}(ix)$$

VERIFICATION:

Extension Property For $z \in Z$, because $\tilde{u}$ extends u , the evaluation yields:

$$\tilde{f}(z) = \tilde{u}(z) - i\tilde{u}(iz) = u(z) - iu(iz) = f(z)$$

Complex Linearity Because $\tilde{u}$ is real-linear, $\tilde{f}$ is inherently additive. It remains to verify homogeneity for complex scalars $\alpha = a + ib$. First, $\tilde{f}(ix)$ is computed:

$$\begin{aligned} \tilde{f}(ix) &= \tilde{u}(ix) - i\tilde{u}(i(ix)) = \tilde{u}(ix) - i\tilde{u}(-x) \\ &= \tilde{u}(ix) + i\tilde{u}(x) \quad (\text{because } \tilde{u} \text{ is real-linear}) \\ &= i(\tilde{u}(x) - i\tilde{u}(ix)) = i\tilde{f}(x) \end{aligned}$$

Subsequently, $\tilde{f}((a + ib)x) = a\tilde{f}(x) + bi\tilde{f}(x) = (a + ib)\tilde{f}(x)$. Thus, $\tilde{f}$ is complex-linear.

Norm Condition (Domination) It must be shown that $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$. If $\tilde{f}(x) = 0$, the inequality holds trivially because $p(x) \geq 0$. Assume $\tilde{f}(x) \neq 0$. The complex number $\tilde{f}(x)$ is expressed in polar form:

$$\tilde{f}(x) = |\tilde{f}(x)|e^{i\theta}$$

where θ represents the argument of $\tilde{f}(x)$. From this formulation:

$$|\tilde{f}(x)| = e^{-i\theta} \tilde{f}(x) = \tilde{f}(e^{-i\theta}x)$$

Define the vector $y = e^{-i\theta}x$. Two properties of y are noted:

1. $p(y) = p(e^{-i\theta}x) = |e^{-i\theta}|p(x) = p(x)$ (because p constitutes a seminorm).
2. $\tilde{f}(y)$ is real-valued (it is equal to $|\tilde{f}(x)|$).

Because $\tilde{f}(y)$ is real, it equals its real part:

$$\tilde{f}(y) = \operatorname{Re} \tilde{f}(y) = \tilde{u}(y)$$

Applying the domination property yields $\tilde{u}(y) \leq p(y)$. Chaining these equalities produces the final bound:

$$|\tilde{f}(x)| = \tilde{f}(y) = \tilde{u}(y) \leq p(y) = p(x)$$

Thus, the extension is valid and norm-preserving. ■

The normed space version is a direct corollary of the preceding theorem.

Corollary 5.16 Hahn-Banach (Complex Normed Spaces)

Let X be a complex normed space and f be a bounded linear functional on a subspace $Z \subseteq X$. There exists a bounded linear extension $\tilde{f}$ defined on X such that:

1. $\tilde{f}(z) = f(z)$ for all $z \in Z$.
2. $\|\tilde{f}\|_X = \|f\|_Z$.

PROOF: The proof applies the complex vector space formulation of the theorem by constructing a suitable seminorm capturing the boundedness of f .

Construct the Seminorm Define the constant $K = \|f\|_Z$. The functional $p : X \rightarrow \mathbb{R}$ is defined by $p(x) = K\|x\|$. This functional p fulfills the requisite properties of a seminorm:

- **Subadditivity:** $p(x + y) = K\|x + y\| \leq K(\|x\| + \|y\|) = p(x) + p(y)$.
- **Homogeneity:** $p(\alpha x) = K\|\alpha x\| = K|\alpha|\|x\| = |\alpha|p(x)$.

Verify the Domination Condition on Z For any $z \in Z$:

$$|f(z)| \leq \|f\|_Z \|z\| = K\|z\| = p(z)$$

Apply Extension Theorem There exists a complex-linear extension $\tilde{f}$ defined on X such that $|\tilde{f}(x)| \leq p(x) = K\|x\|$ for all $x \in X$.

Verify Norm Preservation It must be established that the extension preserves the magnitude of the functional.

- **Upper Bound:** The inequality $|\tilde{f}(x)| \leq K\|x\|$ dictates that $\tilde{f}$ is bounded, and:

$$\|\tilde{f}\|_X = \sup_{x \neq 0} \frac{|\tilde{f}(x)|}{\|x\|} \leq K = \|f\|_Z$$

- **Lower Bound:** Because $\tilde{f}$ is an extension of f : $\|\tilde{f}\|_X \geq \|f\|_Z$

These two inequalities confirm that $\|\tilde{f}\|_X = \|f\|_Z$. ■

A fundamental inquiry in functional analysis is whether the dual space X' is sufficiently robust to distinguish between distinct vectors within X . The subsequent result guarantees that if a vector remains undetected by the dual space, it must be the zero vector.

Norm and the Zero Vector

Let X be a normed space and let $x \in X$. If $f(x) = 0$ for every bounded linear functional $f \in X^*$, then $x = 0$.

PROOF: The proof proceeds by contradiction to establish that the dual space contains a functional that can detect any non-zero vector.

Assume Non-Zero Vector Assume that $x \neq 0$ so that $\|x\| > 0$.

Invoke Hahn-Banach Extension The corollary of the Hahn-Banach Theorem regarding the existence of functionals at a point is utilized. For any non-zero vector x , a bounded linear functional $\hat{f} \in X^*$ exists satisfying:

$$\|\hat{f}\| = 1 \quad \text{and} \quad \hat{f}(x) = \|x\|$$

Establish Contradiction Because $\|x\| > 0$, a functional $\hat{f}$ can be constructed such that $\hat{f}(x) \neq 0$. This contradicts the assumption that $f(x) = 0$ for all $f \in X^*$. ■

EXERCISES FOR SECTION 6.2

1. *Properties of Sublinear Functionals.* Let p be a sublinear functional on a real vector space X . Prove the following elementary properties:

- (a) $p(0) = 0$.
- (b) $p(-x) \geq -p(x)$ for all $x \in X$.
- (c) If p constitutes a seminorm (i.e., symmetric $p(-x) = p(x)$), then $p(x) \geq 0$ universally for all x .

2. *Concrete Extension in $\mathbb{R}^2$.* Let $X = \mathbb{R}^2$ be equipped with the Euclidean norm. Let Z be the subspace spanned by $u = (1, 1)$. Define f on Z by the formula $f(\alpha(1, 1)) = \alpha$.
 - (a) Calculate the norm $\|f\|_Z$.
 - (b) Apply the Hahn-Banach theorem to formally extend f to a functional $\tilde{f}$ defined on $\mathbb{R}^2$ such that $\|\tilde{f}\|_X = \|f\|_Z$.
 - (c) Determine whether this extension is uniquely defined. (*Hint: Evaluate the strict convexity of the unit ball.*)
3. *Non-Uniqueness of Extensions.* Let $X = \mathbb{R}^2$ be equipped with the max-norm: $\|(x_1, x_2)\|_\infty = \max(|x_1|, |x_2|)$. Let Z denote the subspace spanned by $(1, 0)$ (the x-axis). Define f on Z by $f(\alpha, 0) = \alpha$.
 - (a) Verify analytically that $\|f\|_Z = 1$.
 - (b) Prove that $\tilde{f}_1(x_1, x_2) = x_1$ constitutes a valid norm-preserving extension.
 - (c) Prove that $\tilde{f}_2(x_1, x_2) = x_1 + x_2$ constitutes an alternative valid norm-preserving extension. (*Hint: Verify that the condition $|\tilde{f}_2(x)| \leq \|x\|_\infty$ is satisfied.*)
 - (d) Conclude that Hahn-Banach extensions inherently lack uniqueness in spaces that fail to qualify as Hilbert spaces.
4. *Distance to a Subspace.* Let Y be a closed subspace of a normed space X , and let $x_0 \in X \setminus Y$. Let $d = \inf_{y \in Y} \|x_0 - y\|$ denote the topological distance from x_0 to Y . Prove there exists a bounded linear functional f defined on X satisfying:
 - (a) $\|f\| = 1$.
 - (b) $f(y) = 0$ for all $y \in Y$ (i.e., $Y \subset \ker(f)$).
 - (c) $f(x_0) = d$.

(*Hint: Define f on the subspace spanned by Y and x_0 utilizing the formula $f(y + \alpha x_0) = \alpha d$, verify the operator norm, and subsequently apply the extension theorem.*)
5. *Banach Limits.* Let ℓ^∞ denote the space of bounded real sequences. The Hahn-Banach theorem facilitates the definition of a generalized limit for divergent sequences. Prove that there exists a linear functional $L : \ell^\infty \rightarrow \mathbb{R}$ satisfying:
 - (a) If $x \in c$ (the space of convergent sequences), then $L(x) = \lim_{n \rightarrow \infty} x_n$.
 - (b) L operates as shift-invariant: $L(x_1, x_2, \dots) = L(x_2, x_3, \dots)$.
 - (c) $\liminf x_n \leq L(x) \leq \limsup x_n$.
6. *Separation of Convex Sets (Geometric Hahn-Banach).* Let K be a closed convex subset of a real normed space X , and let $x_0 \in X \setminus K$. Prove

that there exists a bounded linear functional f defined on X and a real constant c satisfying:

$$f(x_0) > c \geq f(k) \quad \text{for all } k \in K$$

(Note: This establishes that the geometric hyperplane $H = \{x \mid f(x) = c\}$ strictly separates the point x_0 from the convex set K .)

7. *Supporting Hyperplanes.* Let x_0 be a point residing exactly on the boundary of the closed unit ball $S = \{x \in X \mid \|x\| \leq 1\}$. Prove that there exists a bounded linear functional f satisfying:

- (a) $\|f\| = 1$.
- (b) $f(x_0) = 1$.
- (c) $f(x) \leq 1$ for all $x \in S$.

8. *Biorthogonal Systems.* Let $\{x_1, x_2, \dots, x_n\}$ constitute a linearly independent set of vectors within a normed space X . Prove the existence of specific linear functionals $f_1, f_2, \dots, f_n \in X'$ satisfying:

$$f_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(Hint: Define f_1 on the specific subspace spanned by $\{x_1, \dots, x_n\}$ by establishing $f_1(x_1) = 1$ and $f_1(x_j) = 0$ for $j \neq 1$. Subsequently, extend the functional employing the Hahn-Banach Theorem.)

9. *Uniqueness of Extension from Dense Subspaces.* The Hahn-Banach theorem guarantees abstract existence, excluding uniqueness; however, the specified domain dictates constraints. Let Z be a dense subspace of a normed space X . Let f be a bounded linear functional defined on Z . Prove that the Hahn-Banach extension of f evaluated on X is strictly unique. (Hint: Utilize the uniform continuity of f in conjunction with the topological reality that any element $x \in X$ operates as the limit of a sequence defined in Z .)
10. *The Canonical Embedding (Reflexivity Intro).* For any fixed vector $x \in X$, a corresponding functional g_x is defined on the dual space X' through evaluation: $g_x(f) = f(x)$.
- (a) Prove that g_x operates as a linear functional defined on X' .
 - (b) Applying the Hahn-Banach theorem, rigorously prove that the operator norm equates identically: $\|g_x\| = \|x\|$.
 - (c) Conclude that the geometric map $x \mapsto g_x$ establishes an isometric embedding of X directly into its bidual X'' .
11. *Real vs Complex.* Let $X = \mathbb{C}$ be evaluated strictly as a real vector space. Define the functional $f(z) = \operatorname{Re}(z)$. This constitutes a real-linear functional. Attempt to extend it to a complex-linear functional. Determine the analytical result of applying the explicit formula $f(z) - if(iz)$.

12. *Extension from Real Line.* Let $X = \mathbb{C}$ evaluated as a complex space. Let the subspace be $Z = \mathbb{R}$ (the real axis). Let $f(x) = x$ exclusively for $x \in \mathbb{R}$. Determine if f qualifies as linear over the field $\mathbb{C}$. Evaluate whether the Hahn-Banach theorem can be applied directly without real-part decomposition.
13. *The Conjugate Linear Case.* Within physics, anti-linear (conjugate linear) functionals are frequently utilized. Formulate a rigorous version of the Hahn-Banach theorem specifically adapted for anti-linear functionals satisfying $f(\alpha x) = \bar{\alpha} f(x)$.
14. *Norm of the Real Part.* Let f be a bounded linear functional defined on a complex normed space X , and let $u(x) = \operatorname{Re} f(x)$ define its real component.
 - (a) Prove that $\|u\| \leq \|f\|$.
 - (b) Utilizing the polar decomposition technique ($f(x) = |f(x)|e^{i\theta}$), demonstrate that for any x , a complex scalar α exists with $|\alpha| = 1$ such that the relation $|f(x)| = u(\alpha x)$ is satisfied.
 - (c) Conclude that the specific norms are strictly identical: $\|u\| = \|f\|$.
15. *Concrete Extension in $\mathbb{C}^2$.* Let $X = \mathbb{C}^2$ be equipped with the standard Euclidean norm $\|z\|_2 = \sqrt{|z_1|^2 + |z_2|^2}$. Let Z define the specific subspace spanned by the complex vector $v = (1, 1)$. Define a functional f defined on Z via the mapping $f(\lambda(1, 1)) = \lambda$.
 - (a) Calculate the explicit operator norm $\|f\|_Z$.
 - (b) Formulate an extension $\tilde{f}$ defined on all of $\mathbb{C}^2$ such that the geometric relation $\|\tilde{f}\|_X = \|f\|_Z$ is strictly maintained.
 - (c) Determine if this specific complex extension evaluates uniquely.
16. *Real vs. Complex Hyperplanes.* Within a real vector space, the kernel of a non-zero linear functional inherently possesses a codimension of 1 (a hyperplane).
 - (a) Let f be a linear functional defined on a complex space X . Prove that the kernel $\ker(f) = \{x \mid f(x) = 0\}$ establishes a complex subspace possessing exactly codimension 1.
 - (b) Prove that the specific set $\ker(\operatorname{Re} f)$ constitutes a real subspace possessing codimension 1.
 - (c) Explain geometrically why a complex functional f fundamentally defines a subspace possessing a real codimension of 2.
17. *Boundedness of the Imaginary Part.* Let $f(x) = u(x) + iv(x)$ constitute a linear functional defined on a complex normed space. Prove that if the real component u is demonstrably bounded, the imaginary component v must concurrently be bounded, establishing explicitly that $\|v\| = \|u\|$. (Hint: Apply the complex identity $v(x) = -u(ix)$.)

18. *Extension from c_0 to ℓ^∞ (Complex)*. Let the domain $X = \ell^\infty$ represent complex bounded sequences and the subspace $Z = c_0$ represent complex sequences converging asymptotically to 0. Consider the functional defined strictly on Z by the infinite sum:

$$f(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

- Prove that f evaluates as bounded on the subspace Z and calculate its precise norm.
- Apply the Hahn-Banach theorem to mathematically guarantee an explicit extension $\tilde{f}$ spanning ℓ^∞ .
- Explicitly construct such an extension if analytically possible, or formally explain how the extended functional relates strictly to the dual space $(\ell^\infty)^*$.

5.3 THE UNIFORM BOUNDEDNESS PRINCIPLE

The proof of uniform boundedness relies upon a topological framework established by the **Baire Category Theorem**. This theorem states that a complete metric space possesses too much topological "thickness" to be completely covered by a countable collection of topologically "thin" sets.

Definition 5.18 Rare and Meager Sets

Let X be a metric space and $M \subset X$.

- M is **nowhere dense** (or **rare**) if its closure has an empty interior: $(\overline{M})^\circ = \emptyset$.
- M is **meager** (or of the *first category*) if it is a countable union of nowhere dense sets: $M = \bigcup_{n=1}^{\infty} M_n$.
- If M is not meager, it is designated as a space of the **second category**.

Theorem 5.19 Baire Category Theorem

A complete metric space X is a non-meager set.

Equivalent Formulation: If a complete metric space X is the union of a sequence of closed sets $X = \bigcup_{k=1}^{\infty} A_k$, then at least one component set A_k must possess a non-empty interior.

PROOF: The equivalent formulation is proven: if $X = \bigcup A_k$ where each A_k is closed, at least one A_k contains an open ball. The proof proceeds by contradiction. Assume $X = \bigcup_{k=1}^{\infty} A_k$, where each A_k is closed and possesses an empty interior (i.e., is nowhere dense). A point $x \in X$ will be constructed that avoids every set A_k , contradicting the assumption that the sets cover X .

Escaping A_1 Because A_1 is closed and possesses an empty interior, it cannot contain any open ball. Therefore, its complement A_1^c is open and dense in X . A center point x_1 and a radius $r_1 < 1/2$ can be selected such that the resulting closed ball lies exterior to A_1 :

$$\overline{B}(x_1; r_1) \cap A_1 = \emptyset$$

Escaping A_2 Consider the open ball $B(x_1; r_1)$. Because A_2 possesses an empty interior, it cannot contain this ball. Thus, the open set $B(x_1; r_1) \setminus A_2$ is non-empty. A subsequent point x_2 and a smaller radius $r_2 < 1/4$ can be selected such that

$$\overline{B}(x_2; r_2) \subset B(x_1; r_1) \quad \text{and} \quad \overline{B}(x_2; r_2) \cap A_2 = \emptyset$$

This geometric nesting is visualized in Figure 5.6.

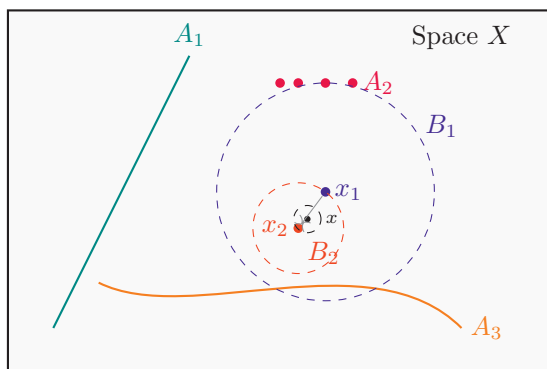


FIGURE 5.6 Visualization of the Baire category theorem proof. A nested sequence of centers $x_1, x_2, \dots$ and shrinking radii is constructed. The limit point x must exist because the space is complete, and lies outside every nowhere dense set A_n .

The Inductive Construction An inductive construction is utilized. Assume x_n and r_n have been selected. Because A_{n+1} is nowhere dense, the open ball $B(x_n; r_n)$ contains points not located in A_{n+1} . A subsequent center x_{n+1} and radius $r_{n+1} < 1/2^{n+1}$ can be chosen satisfying:

$$\overline{B}(x_{n+1}; r_{n+1}) \subset B(x_n; r_n) \quad \text{and} \quad \overline{B}(x_{n+1}; r_{n+1}) \cap A_{n+1} = \emptyset$$

This generates a nested sequence of closed balls:

$$\overline{B}(x_1; r_1) \supset \overline{B}(x_2; r_2) \supset \overline{B}(x_3; r_3) \dots$$

Completeness The sequence of centers (x_n) is Cauchy. For any index values $n, m > N$:

$$d(x_n, x_m) < r_N < \frac{1}{2^N} \rightarrow 0$$

Because X is a complete metric space, $\lim_{n \rightarrow \infty} x_n$ converges to a point $x \in X$.

Contradiction Because the balls are closed and nested, the limit x is contained in $\overline{B}(x_n; r_n)$ for all n . However, the n -th ball is disjoint from A_n :

$$x \in \overline{B}(x_n; r_n) \implies x \notin A_n$$

Since this holds for all n :

$$x \notin \bigcup_{n=1}^{\infty} A_n$$

This contradicts the assumption that $X = \bigcup A_n$. Consequently, at least one subset A_n must have a non-empty interior. ■

The second foundational pillar of functional analysis is the *Uniform Boundedness Principle* (or the Banach-Steinhaus Theorem). It addresses a subtle question regarding the convergence of functional operators:

If a sequence of operators is bounded at every single point individually, is it necessarily bounded uniformly across the entire space?

In finite dimensions, the answer is trivially affirmative. In infinite dimensions, without the property of completeness, the answer is negative. The following theorem guarantees that in a Banach space pointwise boundedness necessitates uniform boundedness. This result is indispensable for proving the rigorous convergence of numerical schemes (e.g., quadrature formulas) and infinite Fourier series.

Theorem 5.20 Uniform Boundedness Theorem

Let X be a Banach space and Y be a normed space. Let $\{T_n\}_{n \in I}$ be a family of bounded linear operators from X to Y . Assume the family is pointwise bounded, meaning for every arbitrary $x \in X$, there exists a finite real constant c_x satisfying:

$$\|T_n x\| \leq c_x \quad \text{for all } n \in I$$

Then the family is uniformly bounded, that is, there exists a constant C such that

$$\|T_n\| \leq C \quad \text{for all } n \in I$$

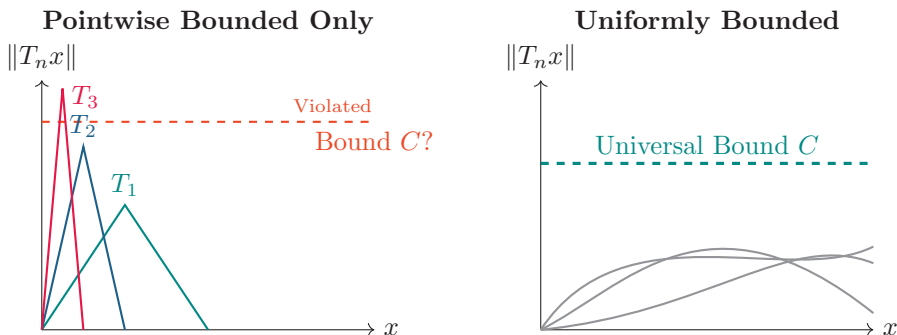


FIGURE 5.7 Visualization detailing the uniform boundedness theorem. If the evaluated domain is complete, functional "spikes" that increase indefinitely (left) cannot manifest without violating boundedness at a specific geometric point; the operators must uniformly stabilize (right).

PROOF:

Constructing the Closed Sets For each integer $k \geq 1$ let

$$A_k = \{x \in X \mid \|T_n x\| \leq k \text{ for all } n \in I\}$$

Since the map $x \mapsto \|T_n x\|$ is continuous, the set $\{x \mid \|T_n x\| \leq k\}$ is closed since it is the inverse image of the closed interval $[0, k]$. Thus, A_k must also be closed since it is the intersection of closed sets.

Applying the Baire Category Theorem By assumption, the functional values $\|T_n x\|$ are bounded by a scalar c_x for every $x \in X$. Choosing $k \geq c_x$ guarantees that $x \in A_k$. This implies that every vector in X belongs to some set A_k . Consequently,

$$X = \bigcup_{k=1}^{\infty} A_k$$

Because X is a Banach space, by the Baire Category Theorem an index k_0 exists such that A_{k_0} has a non-empty interior.

Geometry of the Interior (Shifting the Ball) Because A_{k_0} has an interior, there is an open ball $B(x_0; r) \subset A_{k_0}$ (refer to [Figure 5.8](#)). Thus, for any vector z with $\|z\| < r$, the translated point $x_0 + z$ lies inside A_{k_0} . By the definition of A_{k_0} :

$$\|T_n(x_0 + z)\| \leq k_0 \quad \text{for all } n$$

Applying linearity:

$$T_n(z) = T_n(x_0 + z) - T_n(x_0)$$

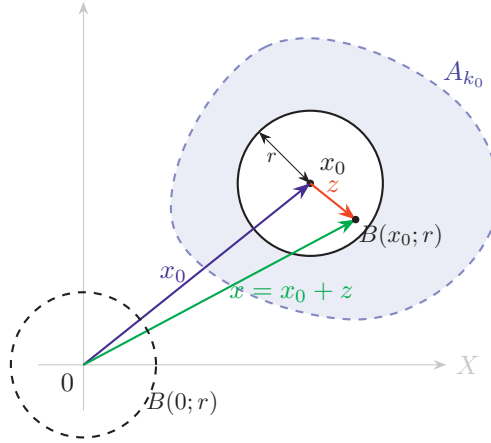


FIGURE 5.8 Geometric interpretation of the translation vector z . Establishing an explicit bound on the ball $B(x_0; r) \subset A_{k_0}$ translates the problem to a centralized ball of equivalent radius r anchored at the geometric origin.

Applying the Triangle Inequality yields

$$\|T_n z\| \leq \|T_n(x_0 + z)\| + \|T_n x_0\|$$

The initial term is bounded by k_0 (because $x_0 + z \in A_{k_0}$). The subsequent term is bounded by k_0 (because the center $x_0 \in A_{k_0}$). Hence,

$$\|T_n z\| \leq k_0 + k_0 = 2k_0 \quad \text{for all } \|z\| < r$$

The bound has been successfully shifted from an arbitrary location to a ball centered at the origin (detailed in [Figure 5.8](#)).

Establishing the Operator Norm Bound

Let u be any non-zero in X and $z = \frac{r}{2\|u\|}u$. It is evident that $\|z\| = r/2 < r$. Substitution yields

$$\left\| T_n \left(\frac{r}{2\|u\|} u \right) \right\| \leq 2k_0$$

By homogeneity of the linear operator

$$\frac{r}{2\|u\|} \|T_n u\| \leq 2k_0$$

Solving this inequality to isolate $\|T_n u\|$ produces

$$\|T_n u\| \leq \frac{4k_0}{r} \|u\|$$

This inequality holds for all vectors $u \in X$ and all indices $n \in I$. By the definition of the operator norm, $\|T_n\| \leq \frac{4k_0}{r}$. By choosing $C = \frac{4k_0}{r}$, it is proven that $\|T_n\| \leq C$ for all n . ■

EXERCISES

1. *Incompleteness Counter-Example.* Let X denote the space of polynomials $P[0, 1]$ evaluated with the uniform norm. (This specified space is demonstrably not complete). Let $T_n : X \rightarrow \mathbb{R}$ be functionally defined by $T_n(p) = np(1/n)$.
 - (a) Prove that for any specific polynomial p , the generated sequence $|T_n(p)|$ evaluates as bounded.
 - (b) Prove mathematically that $\|T_n\| = n$.
 - (c) Provide a rigorous explanation detailing why this result contradicts the conclusion of the Uniform Boundedness Theorem, specifying the precise hypothesis that is violated.
2. *Resonance Theorem.* An applied variation of the theorem is termed the "Resonance Theorem." Let X be a complete Banach space and let T_n represent bounded operators. If $\sup_n \|T_n\| = \infty$, prove that a dense topological set of vectors $x \in X$ must exist such that $\sup_n \|T_n x\| = \infty$. (Note: This establishes that if the operators fundamentally diverge, they diverge practically almost everywhere across the space).
3. *Weak Convergence and Boundedness.* A specific sequence (x_n) in X is defined as weakly convergent to x if $f(x_n) \rightarrow f(x)$ evaluates true for every bounded linear functional $f \in X'$.
 - (a) Evaluate the specific operators $T_n : X' \rightarrow \mathbb{F}$ defined functionally by $T_n(f) = f(x_n)$.
 - (b) Prove that for each rigidly fixed functional f , the sequence $|T_n(f)|$ evaluates as structurally bounded.
 - (c) Apply the Uniform Boundedness Theorem to conclusively prove that the specific sequence of norms $\|x_n\|$ must inherently evaluate as bounded.
4. *The Limit Operator.* Let X be a complete Banach space and Y a designated normed space. Let (T_n) define a sequence of bounded linear operators mapping from X to Y . Assume the defined sequence converges strictly pointwise; i.e., for every vector $x \in X$, the sequence $(T_n x)$ mathematically converges to a specific limit $y \in Y$. Define the target operator $T : X \rightarrow Y$ functionally by $Tx = \lim_{n \rightarrow \infty} T_n x$.
 - (a) Prove that the derived T constitutes a mathematically linear operator.
 - (b) Applying the Uniform Boundedness Theorem, prove that T structurally evaluates as a bounded operator.
 - (c) Prove analytically that $\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$.
5. *Separate vs. Joint Continuity.* Let X and Y define complete Banach spaces. Let $b : X \times Y \rightarrow \mathbb{R}$ dictate a bilinear functional (strictly linear

evaluated in each variable independently). Assume the functional b evaluates as separately continuous, defined strictly as:

- For every rigidly fixed $x \in X$, the mapping $y \mapsto b(x, y)$ evaluates continuous.
- For every rigidly fixed $y \in Y$, the mapping $x \mapsto b(x, y)$ evaluates continuous.

Apply the Uniform Boundedness Principle to mathematically prove that b is jointly continuous (topologically bounded). Specifically, prove the existence of a real scalar $M > 0$ such that the geometric inequality $|b(x, y)| \leq M\|x\|\|y\|$ evaluates true for all elements x, y . (*Hint: Analytically evaluate the family of specific functionals $f_x(y) = b(x, y)$ defined strictly for x constrained within the geometric unit ball.*)

6. *Convergence of Quadrature Formulas.* Let $X = C[0, 1]$ be strictly equipped with the uniform norm. Applied numerical integration utilizes discrete quadrature formulas defined mathematically as:

$$Q_n(x) = \sum_{k=1}^n c_{nk} x(t_{nk})$$

where $t_{nk} \in [0, 1]$ represent specified coordinate nodes and c_{nk} dictate applied scalar weights. Assume that for every evaluated polynomial $p(t)$, the limit evaluates identically: $\lim_{n \rightarrow \infty} Q_n(p) = \int_0^1 p(t) dt$. Prove rigorously that the identity $\lim_{n \rightarrow \infty} Q_n(x) = \int_0^1 x(t) dt$ evaluates true for all continuous functions $x \in X$ if and only if a specific scalar constant M exists such that:

$$\sum_{k=1}^n |c_{nk}| \leq M \quad \text{for all indices } n$$

7. *Pointwise Divergence.* Let the specific space evaluate to $X = \ell^1$. Evaluate the particular sequence of linear functionals $f_n(x) = nx_n$ (where x_n dictates the strictly n -th coordinate component of the evaluated sequence x).

- (a) Prove analytically that $f_n(x) \rightarrow 0$ for every specific vector $x \in X$ possessing strictly finitely many non-zero structural terms.
- (b) Prove that the specific geometric set of elements x where $f_n(x)$ evaluates bounded constitutes a dense subset in X .
- (c) Prove conclusively that the operator sequence physically diverges: $\|f_n\| \rightarrow \infty$.
- (d) Evaluate whether the identity $f_n(x) \rightarrow 0$ holds universally for all vectors $x \in X$. (*Apply the Uniform Boundedness Theorem directly to conclusively dictate the analytical answer without manually constructing a specific mathematical counter-example.*)

5.4 THE OPEN MAPPING THEOREM

The third pillar of functional analysis is the *Open Mapping Theorem*. It addresses the analytical stability of solutions to linear equations $Tx = y$. If an operator T is continuous and bijective, a unique solution $x = T^{-1}y$ is established algebraically. The Open Mapping Theorem provides the topological guarantee that this inverse map T^{-1} is continuous,

Definition 5.21 Open Mapping

Let X and Y be metric spaces. A map $T : X \rightarrow Y$ is an **open mapping** if for every open set $U \subset X$, the corresponding image $T(U)$ is an open set in Y .

Not every bounded linear operator $T : X \rightarrow Y$ is an open mapping. The following theorem establishes the guarantee that for complete Banach spaces, surjectivity alone is sufficient to ensure this topological property.

Theorem 5.22 Open Mapping Theorem

Let X and Y be Banach spaces. If $T : X \rightarrow Y$ is a surjective bounded linear operator, then T is an **open mapping**. That is, for every open set $U \subset X$, the image $T(U)$ is open in Y .

PROOF: The proof combines topological mechanisms (the Baire Category Theorem) with analytical sequences (geometric series). It must be proven that if $T : X \rightarrow Y$ is a surjective bounded linear operator between Banach spaces, then T maps open sets to open sets.

The Baire Category Argument The space X is systematically covered with expanding geometric balls:

$$X = \bigcup_{k=1}^{\infty} B_k \quad \text{where } B_k = B(0; k).$$

Because the operator T is surjective, the images of these geometric balls must cover the target space Y :

$$Y = T(X) = \bigcup_{k=1}^{\infty} T(B_k).$$

Because Y is a Banach space, it is complete. The Baire Category Theorem states that a complete metric space cannot be a countable union of nowhere dense sets (subsets whose topological closure has empty interior). Therefore, at

least one of the sets, say $\overline{T(B_{k_0})}$, must have non-empty interior. Furthermore, by the linearity of T , the set $T(B_{k_0})$ is just the set $T(B_1)$ scaled by k_0 :

$$\overline{T(B_{k_0})} = k_0 \cdot \overline{T(B_1)}$$

Scaling a set does not eliminate its interior. Thus, $\overline{T(B_1)}$ has a non-empty interior.

Shifting to the Origin Thus, $\overline{T(B_1)}$ contains an open ball $V(y_0; \epsilon)$. It must be shown that it contains an open ball centered at 0. Since B_1 is symmetric and convex, so is $\overline{T(B_1)}$. Since $V(y_0; \epsilon) \subset \overline{T(B_1)}$, symmetry gives $V(-y_0; \epsilon) \subset \overline{T(B_1)}$ (see [Appendix](#)). For $y \in V(0; \epsilon)$, the vectors $y \pm y_0$ lie in these respective balls. By convexity, their midpoint $\frac{1}{2}(y + y_0) + \frac{1}{2}(y - y_0) = y$ is in $\overline{T(B_1)}$, implying $V(0; \epsilon) \subset \overline{T(B_1)}$ (visualized in [Figure 5.9](#)).

By scaling, for any constant $k > 0$

$$[V(0, k\delta) \subset \overline{T(B_k)}]$$

where the scalar $\delta = \epsilon/2$.

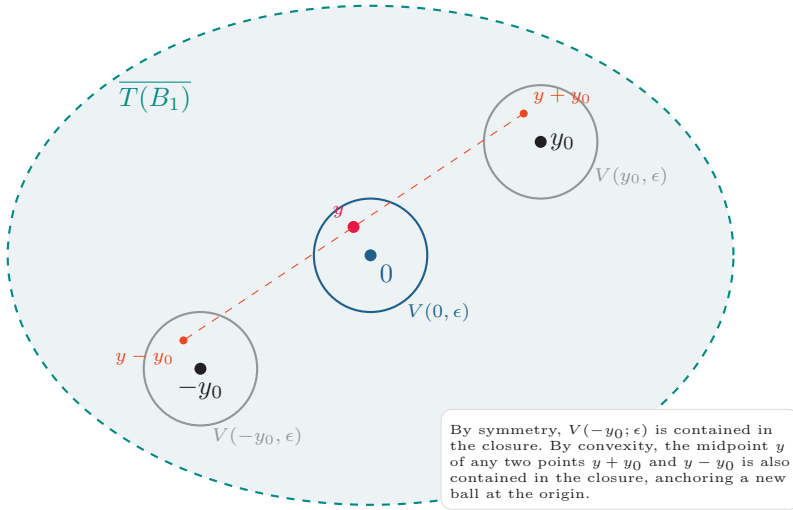


FIGURE 5.9 Translation of the topological interior to the origin. By leveraging the symmetry and convexity of the set, taking the midpoint of elements in the mirrored neighborhoods proves that a ball centered at the origin is contained within the closure of the image.

The Successive Approximation It will now be proven that $V(0; \delta/2) \subset T(B_1)$. Let $y \in Y$ define an arbitrary target vector satisfying $\|y\| < \delta/2$. A rigorous sequence of approximations will be constructed.

1. Since $y \in V_{\delta/2}$, by linearity, $V(0; \delta/2) \subset \overline{T(B_{1/2})}$. Because y is in the closure of the image, there exists a vector x_1 in the ball $B_{1/2}$ such that

$$\|y - Tx_1\| < \frac{\delta}{4}$$

Let $y_1 = y - Tx_1$ define the error (visualized in Figure 5.10).

2. Since $V_{\delta/4} \subset \overline{T(B_{1/4})}$, there exists $x_2 \in B_{1/4}$ such that

$$\|y_1 - Tx_2\| < \frac{\delta}{8}$$

Let $y_2 = y_1 - Tx_2 = y - (Tx_1 + Tx_2)$.

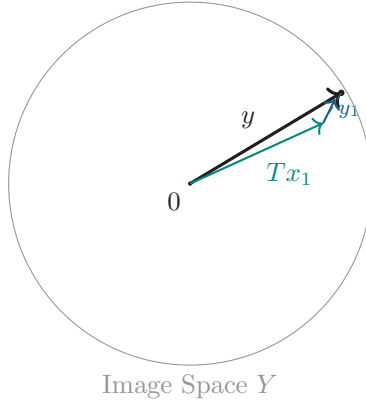


FIGURE 5.10 Visualization of the successive approximation stage. Because the image evaluates as dense, any target vector y can be approached via a sequence of mapped images. Because the ambient space is complete (Banach), the sum of the pre-images x_n converges, eliminating the error gap completely.

3. By induction, there exists a vector $x_n \in B_{1/2^n}$ satisfying

$$\left\| y - \sum_{k=1}^n Tx_k \right\| < \frac{\delta}{2^{n+1}}$$

Convergence Thus, the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent since $\sum \|x_n\| < \sum \frac{1}{2^n} = 1$. Because X is a Banach space, this series converges to a limit $x \in X$. Additionally, the bound $\|x\| < 1$ holds, confirming $x \in B_1$. Because T is continuous:

$$Tx = T \left(\lim_{N \rightarrow \infty} \sum_{n=1}^N x_n \right) = \lim_{N \rightarrow \infty} \sum_{n=1}^N Tx_n = y$$

Thus, an element $x \in B_1$ is found such that $Tx = y$. This conclusively proves that $V(0; \delta/2) \subset T(B_1)$.

Conclusion The primary theorem is now established. It must be shown that for any arbitrary open set $A \subset X$, the corresponding image $T(A)$ is open in Y . Let $y \in T(A)$. This implies $y = Tx$ for some vector $x \in A$. Because A is an open

set, a radius $r > 0$ exists such that $B(x; r) \subset A$. This implies $x + B(0; r) \subset A$, or equivalently

$$B(0; r) \subset A - x.$$

By linearity, the relation $T(B(0; r)) \subset T(A) - Tx$ holds. This relation can be scaled. Because $T(B(0; r)) = rT(B_1)$, the condition becomes

$$rT(B_1) \subset T(A) - y.$$

By the successive approximation argument established previously, $T(B_1)$ contains an open ball V centered at 0. Therefore, the scaled set rV is an open ball centered at 0 contained in $T(A) - y$. Shifting the coordinate frame back by y , it is established that $T(A)$ contains an open ball centered at y . Thus, the set $T(A)$ is open.

Consequently, if T is bijective, T^{-1} must exist. The continuity of T^{-1} is equivalent to the requirement that the inverse image of any open set under T^{-1} is open. Because the identity $(T^{-1})^{-1}(U) = T(U)$, which is open, the inverse mapping T^{-1} must be continuous. ■

The Bounded Inverse Theorem

The most practical corollary of the Open Mapping Theorem is the guaranteed continuity of the inverse operator.

Theorem 5.23 Bounded Inverse Theorem

Let X and Y be Banach spaces. If $T : X \rightarrow Y$ acts as a bounded linear operator that is bijective, then the algebraic inverse operator $T^{-1} : Y \rightarrow X$ is bounded (continuous).

PROOF: The proof translates the definition of general continuity into a structural mapping problem that can be solved applying the Open Mapping Theorem.

Existence of the Inverse Because $T : X \rightarrow Y$ is bijective, the operator $T^{-1} : Y \rightarrow X$ exists. The objective is to prove that this inverse is bounded, which, for linear functional operators, is equivalent to proving continuity.

The Topological Criterion The topological characterization of continuity is utilized: a mapping is continuous if and only if the inverse image of every open set in the codomain is open in the domain.

Mapping Analysis Let $U \subset X$ represent an open set in the domain of T . By definition of an inverse mapping,

$$(T^{-1})^{-1}(U) = T(U)$$

Invoking the Open Mapping Theorem Because T is a continuous and surjective

linear operator between the complete Banach spaces X and Y , the Open Mapping Theorem applies. The theorem guarantees that the image of the open set U is an open in Y .

Conclusion Because $T(U)$ is open in Y for every open set $U \subset X$, the inverse image of every open set under T^{-1} is open. Therefore, T^{-1} is continuous and, consequently, bounded. ■

EXERCISES

1. *Incomplete Spaces.* Let $X = C[0, 1]$ be equipped with the $\|\cdot\|_\infty$ norm (Banach). Let $Y = C[0, 1]$ be equipped with the $\|\cdot\|_1$ norm (Not Banach). Let $I : X \rightarrow Y$ define the identity map $Ix = x$.
 - (a) Prove that I is bounded (continuous).
 - (b) Prove that I is bijective.
 - (c) Prove that the inverse operator I^{-1} is unbounded (discontinuous).
 - (d) Provide an explanation detailing why this result does not contradict the Bounded Inverse Theorem.
2. *Equivalent Norms.* Let X denote a vector space equipped with two distinct norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume X is complete with respect to both norms. Assume there exists a scalar constant C such that $\|x\|_1 \leq C\|x\|_2$ holds for all x . Prove that the norms are equivalent (implying the existence of an alternative constant c such that the geometric inequality $\|x\|_2 \leq c\|x\|_1$ holds). (*Hint: Apply the Bounded Inverse Theorem directly to the identity mapping $I : (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1)$.*)
3. *Closed Graph Preview.* Let $T : X \rightarrow Y$ define a general linear operator. The formal graph of T represents the geometric subspace $G(T) = \{(x, Tx) \mid x \in X\} \subset X \times Y$. If X and Y are Banach spaces, prove that if T is bounded, the graph $G(T)$ is a closed subspace of $X \times Y$.

5.5 THE CLOSED GRAPH THEOREM

The final pillar of the fundamental theory is the *Closed Graph Theorem*. It provides a practical criterion for determining the continuity of an operator. Typically, verifying continuity requires proving that if $x_n \rightarrow x$, then $Tx_n \rightarrow Tx$. The Closed Graph Theorem permits a stronger assumption: it is assumed that Tx_n converges to a specific limit y , reducing the task to verifying that y converges to Tx .

Definition 5.24 Closed Linear Operator

Let X and Y be normed spaces. A linear operator $T : \mathcal{D}(T) \subset X \rightarrow Y$ is closed if its graph

$$G(T) = \{(x, Tx) \mid x \in \mathcal{D}(T)\}$$

is a closed subspace of $X \times Y$.

Sequential Criterion: T is closed if and only if for any sequence (x_n) in $\mathcal{D}(T)$ satisfying

1. $x_n \rightarrow x$ (in X), and
2. $Tx_n \rightarrow y$ (in Y),

it follows that $x \in \mathcal{D}(T)$ and $Tx = y$.

Theorem 5.25 Closed Graph Theorem

Let X and Y be Banach spaces. If $T : X \rightarrow Y$ is a closed linear operator, then T is bounded.

The geometric interpretation of this theorem is visualized in [Figure 5.11](#).

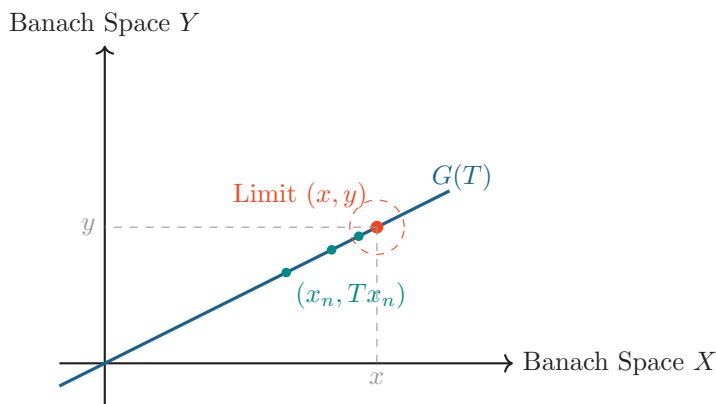


FIGURE 5.11 Visualization of the closed graph. The graph is a subspace within the product space $X \times Y$. If a sequence of points on the line converges to a limit point in the plane, that point must lie on the line.

PROOF: The proof relies on the Bounded Inverse Theorem. The problem is shifted from evaluating the operator T directly to evaluating the projection map on its graph.

Setup The product space $X \times Y$ is equipped with the norm

$$\|(x, y)\| = \|x\|_X + \|y\|_Y$$

Since X and Y are Banach spaces, the product $X \times Y$ is complete. Because T is a closed operator, $G(T)$ is a closed subspace of a Banach space. Therefore, $G(T)$ is a Banach space under the norm inherited from $X \times Y$.

Define the Projection Map The mapping $P : G(T) \rightarrow X$ is defined by:

$$P(x, Tx) = x$$

This maps a point located on the graph directly to its respective x -coordinate. The operator P is also linear.

Verify Continuity of P For any element $z = (x, Tx) \in G(T)$:

$$\|P(z)\|_X = \|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|(x, Tx)\|_{X \times Y} = \|z\|$$

Since $\|P(z)\| \leq 1 \cdot \|z\|$ holds, the operator P is bounded and thus continuous.

Verify Bijectivity of P It must be shown that P is a bijection from $G(T)$ onto X . Since the domain of T is the entire space X , for every $x \in X$, a corresponding pair $(x, Tx) \in G(T)$ exists. Thus, the range of P is all of X , confirming surjectivity.

Assume $P(x, Tx) = 0$. This implies $x = 0$. Since T is linear, $T(0) = 0$. Thus, the designated element is uniquely $(0, 0)$ and so the kernel of P is trivial, confirming injectivity. Therefore, P is a continuous bijection between two Banach spaces, $G(T)$ and X .

Apply the Bounded Inverse Theorem By the Bounded Inverse Theorem, the map $P^{-1} : X \rightarrow G(T)$ is bounded. The inverse mapping is

$$P^{-1}(x) = (x, Tx)$$

Deduce the Boundedness of T Since P^{-1} is bounded, a scalar $C > 0$ exists such that for all $x \in X$

$$\|P^{-1}(x)\|_{X \times Y} \leq C\|x\|_X$$

Thus,

$$\|x\|_X + \|Tx\|_Y \leq C\|x\|_X$$

This inequality is rearranged to isolate the norm involving T :

$$\|Tx\|_Y \leq C\|x\|_X - \|x\|_X = (C - 1)\|x\|_X$$

This proves that T is a bounded linear operator. ■

Example 5.26

The differentiation operator is the standard counter-example that an operator can be closed without being continuous, provided the domain is not a Banach space.

Let $X = C[0, 1]$ define the space of continuous functions equipped with the uniform norm $\|f\|_\infty = \sup_{t \in [0, 1]} |f(t)|$. The differentiation operator $T : \mathcal{D}(T) \subset X \rightarrow X$ is defined by

$$T(f) = f'$$

The domain is $\mathcal{D}(T) = C^1[0, 1]$ (functions possessing continuous derivatives).

T is Unbounded. The norm of the function is $\|f_n\|_\infty = \sup |t^n| = 1$ and its derivative, $Tf_n(t) = nt^{n-1}$. Thus, the norm of the image is equal to $\|Tf_n\|_\infty = n$. Thus, T is unbounded (discontinuous).

T Is Closed. A sequence (f_n) in the domain is considered such that $f_n \rightarrow f$ uniformly in X and $f'_n \rightarrow g$ uniformly in X . A foundational theorem of classical analysis states that if functional derivatives converge uniformly, the limit of the functions must be differentiable, and the derivative of the limit equals the limit of the derivatives:

$$\left(\lim_{n \rightarrow \infty} f_n \right)' = \lim_{n \rightarrow \infty} (f'_n) \implies f' = g$$

Thus, the graph is closed. This does not contradict the Closed Graph Theorem because the theorem requires the domain $\mathcal{D}(T)$ to be a Banach space. The space $\mathcal{D}(T) = C^1[0, 1]$ equipped with the uniform norm $\|\cdot\|_\infty$ fails to be complete. This geometric leakage is visualized in [Figure 5.12](#).

The analytical norm defined on the domain can be altered to the **graph norm**:

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$$

Under this metric, the functional space is complete (a Banach space). Subject to this norm:

$$\|Tf\|_\infty = \|f'\|_\infty \leq \|f\|_\infty + \|f'\|_\infty = \|f\|_{C^1}$$

The designated operator is trivially bounded ($C = 1$), satisfying the prerequisites of the Closed Graph Theorem. $\lrcorner$

EXERCISES

1. *Differentiation Example.* Let $X = C[0, 1]$ be defined with the uniform norm. Let T be the differentiation operator on $\mathcal{D}(T) = C^1[0, 1]$.
 - (a) Prove that T is unbounded by analyzing the sequence $x_n(t) = t^n$.

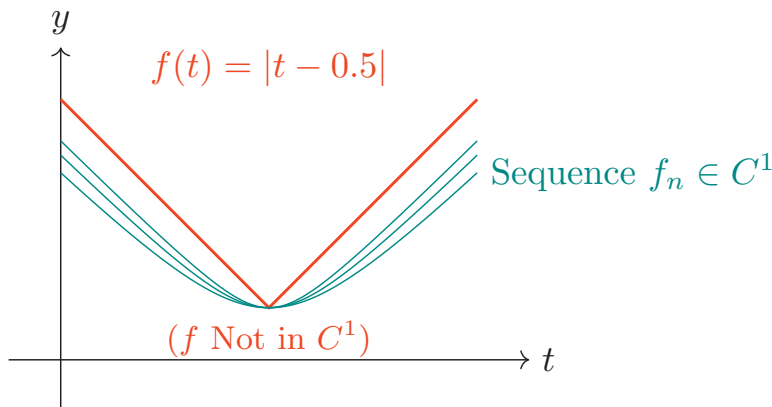


FIGURE 5.12 The formal differentiation operator is defined on smooth functions. However, smooth functions possess the capacity to converge (evaluated in the uniform norm) to non-smooth functions. Because the topological domain "leaks" specific limits, the Closed Graph Theorem is inapplicable.

- (b) Prove that T is closed. (*Hint: Apply the theorem stating that if $f_n \rightarrow f$ uniformly and $f'_n \rightarrow g$ uniformly, then f is differentiable and $f' = g$.*)
 - (c) Explain why this does not contradict the Closed Graph Theorem.
2. *Inverse of Closed Operators.* Let $T : \mathcal{D}(T) \subset X \rightarrow Y$ be a closed linear operator. Assume T is injective (one-to-one). Prove that the inverse operator $T^{-1} : \mathcal{R}(T) \rightarrow X$ is closed. (*Hint: Analyze the graph of T^{-1} in the product space $Y \times X$, noting it functions as the transposition of $G(T)$.*)
3. *The Direct Sum Decomposition.* Let X be a Banach space. Let M and N be closed subspaces satisfying $X = M \oplus N$ (so that every x possesses a mathematically unique representation $x = m + n$). Let P be the projection onto M along N , defined by $P(m + n) = m$.
 - (a) Prove that P is a linear operator in X .
 - (b) Prove that P is a closed operator. (*Hint: Establish $x_k \rightarrow x$ and $Px_k \rightarrow y$. Exploit the closedness of M and N to determine the limits.*)
 - (c) Conclude that P must be continuous.
4. *The Graph Norm.* Let $T : \mathcal{D}(T) \subset X \rightarrow Y$ denote a closed linear operator between Banach spaces. While T may be unbounded with respect to the norm defined on X , an alternative norm can be used on the domain

$\mathcal{D}(T)$, designated as the **Graph Norm**:

$$\|x\|_T := \|x\|_X + \|Tx\|_Y$$

- (a) Prove that $\mathcal{D}(T)$ equipped with the graph norm $\|\cdot\|_T$ is a Banach space.
 - (b) Prove that T is continuous with respect to this norm (specifically, analyzing it as a mapping from $(\mathcal{D}(T), \|\cdot\|_T)$ to $(Y, \|\cdot\|_Y)$).
5. *Hellinger-Toeplitz Theorem.* Let H be a Hilbert space and let $T : H \rightarrow H$ be a linear operator. Assume T is symmetric, defined by

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \text{for all vectors } x, y \in H$$

- (a) Apply the definition of symmetry to prove that T is a closed operator. (*Hint: Show $x_n \rightarrow x$ and $Tx_n \rightarrow z$. Compute the inner product $\langle z, y \rangle$ using limits.*)
 - (b) Apply the Closed Graph Theorem to conclude that T is bounded.
6. *Product of Closed Operators.* Let S and T be closed linear functionals on a Banach space X .
- (a) Determine if the algebraic sum $S + T$ is a closed operator. (*Hint: Consider the specific operators T and $S = -T$.*)
 - (b) Determine if the composition TS is closed.
 - (c) Assuming T is closed and S is bounded, prove that the sum $T + S$ must be closed.

The Weak and Weak* Topologies

2.1 Topology and Nets

2.2 The Weak Topology

2.3 The Weak* Topology

2.4 The Banach-Alaoglu Theorem

IN the study of Banach spaces, the standard norm topology often contains “too many” open sets, making compactness a rare and restrictive property. To bypass the limitations of the norm topology in infinite-dimensional spaces, functional analysis introduces weakened topological structures. By defining convergence through the lens of bounded linear functionals rather than distance, we construct the weak and weak* topologies.

This chapter establishes the topological foundations necessary to work with these concepts. It begins by reviewing general topological spaces and basis generation, ensuring a rigorous framework before extending these ideas to vector spaces, dual spaces, and the powerful theorems that govern their geometry.

6.1 TOPOLOGY AND NETS

Throughout the previous chapter, reliance has been placed extensively on the norm topology of Banach spaces. However, in infinite-dimensional normed spaces, the closed unit ball is intrinsically never compact under the standard norm topology. This poses a significant limitation for mathematical optimization, existence proofs, and establishing limits for sequences of operators.

The *Banach-Alaoglu Theorem* circumvents this fundamental limitation. By shifting the analytical perspective to the dual space and equipping it with a “weaker” topology—one defined by fewer open sets—it becomes feasible for sequences to converge and for specific sets to attain compactness. To establish this framework, the concept of a topological space must first be defined, generalizing the notion of open sets independent of metric distance functions.

This section reviews the fundamental axioms that govern general topological spaces, local bases, and subbases.

Definition 6.1 Topological Space

A topology on a given set X is defined as a family $\mathcal{T}$ of subsets of X that satisfies three foundational axioms:

- (1) The trivial subsets, namely the empty set $\emptyset$ and the entire space X , are elements of $\mathcal{T}$.
- (2) The collection is closed under arbitrary unions: if $\{U_\alpha\}$ is any subcollection of sets in $\mathcal{T}$, their union $\bigcup U_\alpha$ must also be an element of $\mathcal{T}$.
- (3) The collection is closed under finite intersections: if $U_1, U_2, \dots, U_n$ are sets in $\mathcal{T}$, their intersection $\bigcap_{i=1}^n U_i$ must also be an element of $\mathcal{T}$.

The paired structure $(X, \mathcal{T})$ constitutes a **topological space**. Whenever the topology $\mathcal{T}$ is unambiguously understood from context, we simply refer to the space as X . The sets within $\mathcal{T}$ are designated as *open sets*.

Technically, the first axiom is a natural consequence of the latter two. By standard set-theoretic convention, the union of an empty collection of subsets of X is the empty set $\emptyset$, and the intersection of an empty collection of subsets of X is the entire space X .

Specifying every open set in a complex topology is practically impossible. Instead, topologies are defined via smaller, generating collections of sets. The most direct method of generating a topology is through a local or global basis.

Definition 6.2 Local Basis

Let x be an arbitrary point within a topological space $(X, \mathcal{T})$. A **local basis** (or neighborhood basis) at x is a subcollection $\mathcal{B}_x \subseteq \mathcal{T}$ of open sets, each containing x , characterized by the following absorption property: for any arbitrary open set $U \in \mathcal{T}$ containing x , there exists at least one basis element $B \in \mathcal{B}_x$ such that

$$x \in B \subseteq U.$$

This strict nesting property is visualized in [Figure 6.1](#), which illustrates how the local basis elements provide a system of neighborhoods that fit inside any arbitrarily complex open set containing the point.

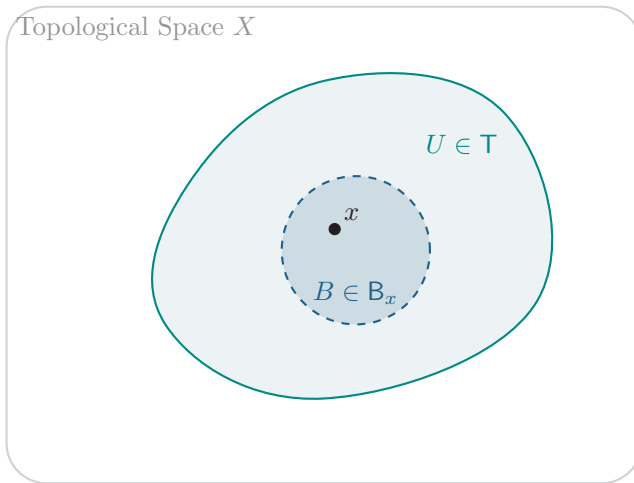


FIGURE 6.1 *Geometry of a local basis:* The local basis $\mathcal{B}_x$ provides a localized “coordinate system” of neighborhoods. No matter how the target open set U is deformed, a basic open set B can always be found nested entirely within it, capturing the point x .

Definition 6.3 Topological Basis

A collection $\mathcal{B}$ of subsets of a set X qualifies as a **basis** for a topology on X provided it meets two geometric criteria:

- (1) The collection covers the space: $\bigcup\{B : B \in \mathcal{B}\} = X$.
- (2) Given any two sets $B_1, B_2 \in \mathcal{B}$ and a point x situated in their intersection ($x \in B_1 \cap B_2$), there must exist a third basic set

$B_3 \in \mathcal{B}$ nested entirely within that intersection:

$$x \in B_3 \subseteq B_1 \cap B_2.$$

If these conditions are satisfied, the topology generated by the basis $\mathcal{B}$, denoted $\mathcal{T}$, is constructed by taking all possible arbitrary unions of the sets in $\mathcal{B}$.

In certain scenarios, even verifying the intersection property of a basis is cumbersome. In such cases, we can drop all structural requirements and generate a topology from an arbitrary collection of sets, known as a subbasis.

Definition 6.4 Subbasis

Let $\mathcal{S}$ be an arbitrary collection of subsets of X . The topology generated by the subbasis $\mathcal{S}$ is constructed in a two-step process:

1. First, form a basis $\mathcal{B}_{\mathcal{S}}$ by taking all possible intersections of finitely many sets from $\mathcal{S}$.
2. Second, generate the topology $\mathcal{T}$ by taking all possible arbitrary unions of the sets in $\mathcal{B}_{\mathcal{S}}$.

Because the empty intersection of sets in $\mathcal{S}$ is the entire space X , the intermediate collection $\mathcal{B}_{\mathcal{S}}$ is guaranteed to cover X , ensuring it functions as a valid basis.

Several fundamental properties emerge directly from the preceding definitions of bases and subbases. Most evidently, any collection that qualifies as a basis trivially serves as a subbasis for the same topology, and any member of a basis or subbasis is, by definition, an open set within the topology it generates.

Furthermore, we can characterize a basis entirely through its relationship with open sets: a collection $\mathcal{B}$ of subsets of X acts as a basis for a topology $\mathcal{T}$ if and only if every element of $\mathcal{B}$ is in $\mathcal{T}$, and every set in $\mathcal{T}$ can be expressed as a union of elements from $\mathcal{B}$. Equivalently, localized to individual points, this holds true if and only if for every point $x \in X$, the subcollection $\{B \in \mathcal{B} : x \in B\}$ is a local basis for $\mathcal{T}$ at x .

Additionally, the topology $\mathcal{T}_{\mathcal{S}}$ generated by a subbasis $\mathcal{S}$ is characterized by an important minimality condition. If $\mathcal{T}$ is any other valid topology on X that contains the subbasis ($\mathcal{S} \subseteq \mathcal{T}$), it must necessarily contain the generated topology as well ($\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}$). In other words, the topology generated by a subbasis is the absolute coarsest (or smallest) topology that includes every set in that subbasis.

Definition 6.5 The Product Topology

Let $\{X_\alpha : \alpha \in I\}$ represent an arbitrary family of topological spaces indexed by a set I . Consider the Cartesian product $\prod_{\alpha \in I} X_\alpha$. We define a subbasis $\mathbf{S}$ comprised of subsets of the Cartesian product that take the form $\prod_{\alpha \in I} U_\alpha$, subject to two conditions:

1. Each U_α is an open set in its respective component space X_α .
2. For all but at most one index $\alpha \in I$, the set U_α is the entire space X_α .

The topology generated by this specific subbasis $\mathbf{S}$ is defined as the **product topology** on $\prod_{\alpha \in I} X_\alpha$. Consequently, the Cartesian product endowed with this topology is referred to as the **topological product** of the given family of spaces.

It is crucial to examine the structure of the basis $\mathbf{B}$ that is constructed from finite intersections of the subbasis $\mathbf{S}$ defined above. Taking finite intersections of these subbasic elements yields basic open sets of the form $\prod_{\alpha \in I} U_\alpha$, where each U_α is open in X_α , and the condition $U_\alpha \neq X_\alpha$ occurs for only a finite number of indices $\alpha \in I$. Because this topology behaves so well with respect to continuity and limits, it is the standard convention in functional analysis that whenever a Cartesian product of topological spaces is introduced without an explicitly defined topology, the product topology is automatically assumed.

Finally, we consider the degenerate, albeit theoretically necessary, case where the index set I is entirely empty. In this scenario, the Cartesian product $\prod_{\alpha \in I} X_\alpha$ reduces to a set containing a single element: the empty set, interpreted functionally as the empty mapping from I into $\bigcup_{\alpha \in I} X_\alpha$. If $\mathbf{S}$ and $\mathbf{B}$ represent the subbasis and basis for the product topology in this empty-index case, they trivially collapse into a single collection:

$$\mathbf{S} = \mathbf{B} = \left\{ \prod_{\alpha \in I} X_\alpha \right\} = \{\{\emptyset\}\}$$

To establish a lexicon for our study of general and weak topologies, we formalize several classes of topological properties. We categorize them into: fundamental operations, separation axioms, variations of compactness, and sequential convergence.

Definition 6.6 Subspaces and Boundary Operators

Let X be a topological space. The operations on open sets generate the following structures for any subset $S \subseteq X$:

- **Closed Sets and Neighborhoods:** A subset is closed if its complement

$X \setminus S$ is open. A neighborhood of a point $x \in X$ is an open set U that contains x .

- **Relative Topology:** The relative topology (or induced/inherited topology) on S is the collection of all intersections $S \cap U$, where U ranges over all open sets in X .
- **Interior (S°):** The interior is the largest open set contained in S , formed by the union of all open subsets of S .
- **Closure ($\overline{S}$):** The closure is the smallest closed set containing S , formed by the intersection of all closed sets containing S .
- **Boundary (∂S):** The boundary consists of points that are close to both S and its complement, defined as $\partial S = \overline{S} \setminus S^\circ$, or equivalently, $\overline{S} \cap \overline{X \setminus S}$.
- **Limit Points:** A point $x \in X$ is a limit point (or cluster/accumulation point) of S if every neighborhood of x intersects S at a point distinct from x .
- **Density:** A subset $D \subseteq X$ is dense in S if S is contained in the closure of D , meaning $D \subseteq S \subseteq \overline{D}$.

The ability to distinguish points or disjoint closed sets via open neighborhoods characterizes a topological space. These constraints are formalized via the separation axioms.

Definition 6.7 T_0 Space

A topological space X is a T_0 space if, for any two distinct points $x, y \in X$, there exists an open set that contains exactly one of these points.

This minimal separation ensures distinct points can be topologically distinguished, though they need not possess disjoint neighborhoods. As illustrated in Figure 6.2, an open set U captures x while excluding y .

Definition 6.8 T_1 Space

A topological space X is a T_1 space if, for any two distinct points $x, y \in X$, there exists an open set containing x but not y , and another open set containing y but not x .

This condition is equivalent to requiring that every singleton set $\{x\}$ is closed. In a T_1 space, each point possesses a neighborhood excluding the other,

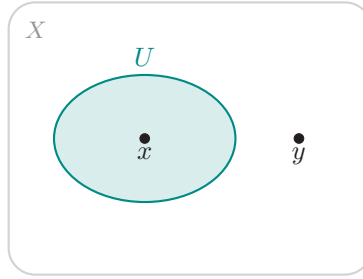


FIGURE 6.2 T_0 separation: The open set U contains x but not y , providing minimal distinguishability.

as shown in Figure 6.3. However, these separating open sets U and V may still intersect.

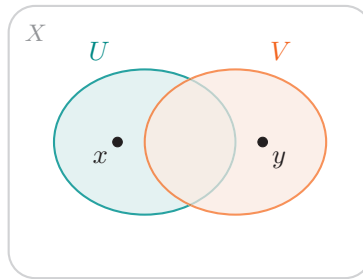


FIGURE 6.3 T_1 separation: Each point is contained in an open set that excludes the other, though the sets U and V are not required to be disjoint.

Definition 6.9 T_2 Space (Hausdorff)

A topological space X is a T_2 space, or a Hausdorff space, if any two distinct points $x, y \in X$ can be separated by disjoint neighborhoods. There exist open sets U containing x and V containing y such that $U \cap V = \emptyset$.

Figure 6.4 illustrates this strong separation, which is a standard feature of metric spaces, guaranteeing that convergent sequences have unique limits.

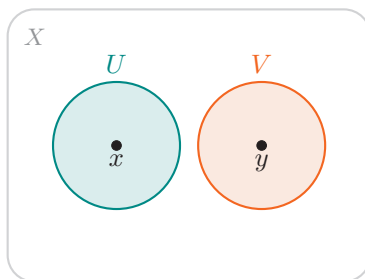


FIGURE 6.4 T_2 (Hausdorff) separation: The points x and y are isolated within completely disjoint open neighborhoods U and V .

Definition 6.10 T_3 Space (Regular)

A topological space X is a T_3 space, or a **regular space**, if it is a T_1 space and any point $x \in X$ and closed set $F \subseteq X$ not containing x can be separated by disjoint open sets. There exist open sets U containing x and V containing F such that $U \cap V = \emptyset$.

This elevates the separation capability from point-point to point-set, as visualized in Figure 6.5.

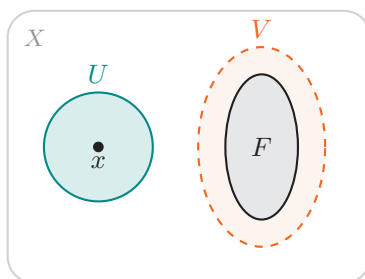


FIGURE 6.5 T_3 (Regular) separation: The point x and the disjoint closed set F are separated by non-overlapping open sets U and V .

Definition 6.11 $T_{3.5}$ Space (Tychonoff)

A topological space X is a $T_{3.5}$ space, or a **completely regular space**, if it is a T_1 space and any point $x \in X$ and closed set $F \subseteq X$ not containing x can be separated by a continuous function. There exists a continuous map $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in F$.

Figure 6.6 demonstrates this functional separation, ensuring the topology supports non-trivial continuous real-valued functions.

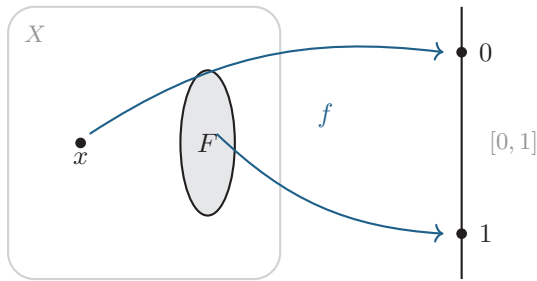


FIGURE 6.6 $T_{3.5}$ (Tychonoff) Separation: The point x and the closed set F are separated by a continuous function f mapping them to distinct endpoints of the unit interval.

Definition 6.12 T_4 Space (Normal)

A topological space X is a T_4 space, or a normal space, if it is a T_1 space and any two disjoint closed sets $E, F \subseteq X$ can be separated by disjoint open sets. There exist open sets U containing E and V containing F such that $U \cap V = \emptyset$.

Through Urysohn's Lemma, normal spaces also guarantee functional separation between disjoint closed sets. Figure 6.7 visualizes this set-set isolation.

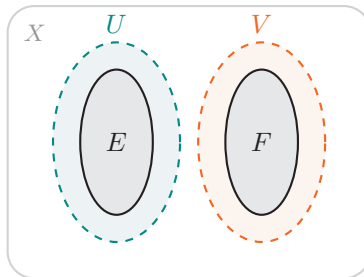


FIGURE 6.7 T_4 (Normal) separation: Any two completely disjoint closed sets E and F can be contained within mutually disjoint open sets U and V .

It is a standard exercise to demonstrate that a space satisfies the T_1 axiom if and only if every single-element subset (singleton) is closed. Also these

separation axioms form a logical hierarchy:

$$T_4 \implies T_{3\frac{1}{2}} \implies T_3 \implies T_2 \implies T_1 \implies T_0$$

In metric spaces, compactness can be characterized via sequences or open covers. However, in abstract topological spaces, compactness divides into several non-equivalent concepts.

Definition 6.13 Varieties of Compactness

Let S be a subset of a topological space X . We define the following stability properties:

- **Compact:** Every open cover of S admits a finite subcover.
- **Locally Compact:** Every point $x \in X$ possesses a compact neighborhood (i.e., there exists a compact set K_x such that $x \in K_x^\circ$).
- **Countably Compact:** Every countable open cover of S admits a finite subcover.
- **Limit Point Compact (Fréchet Compact):** Every infinite subset of S possesses at least one limit point in S .
- **Sequentially Compact:** Every sequence contained in S possesses a convergent subsequence whose limit is in S .

For each of the compactness variations defined above, a set S is termed **relatively compact** (e.g., relatively sequentially compact) if its closure $\bar{S}$ satisfies the respective compactness condition. In these relative cases, the limit points or subsequential limits are only required to be in $\bar{S}$, not necessarily in S .

The equivalencies between these compactness conditions depend on the underlying topology. Compactness implies countable compactness, which in turn implies limit point compactness. If the space is Hausdorff (T_2), countable compactness and limit point compactness are equivalent. However, even in Hausdorff spaces, sequential compactness is independent of these other forms. Only in a metric space do these definitions—compactness, limit point compactness, and sequential compactness—become equivalent.

Finally, while sequences are sufficient for analyzing metric topologies, we must formalize their behavior before demonstrating their inadequacies in weak topologies.

Definition 6.14 Sequential Convergence and Closure

Let (x_n) be a sequence in a topological space X .

- **Convergence:** The sequence converges to a limit $x \in X$ (denoted $x_n \rightarrow x$) if the sequence is eventually captured by every

neighborhood of x . Formally, for every open set U containing x , there exists an integer N such that $x_n \in U$ for all $n \geq N$.

- **Sequential Closure:** The sequential closure of a subset S is the collection of all limit points generated exclusively by sequences whose terms lie in S .
- **Sequentially Closed/Dense:** A set is sequentially closed if it contains all its sequential limits (i.e., it equals its sequential closure). A set D is sequentially dense in S if S is contained in the sequential closure of D .

Unless a topological space is known to be Hausdorff (T_2), limits of sequences are not guaranteed to be unique. In a non-Hausdorff space, a single sequence might converge simultaneously to multiple distinct points. For this reason, the notation $x_n \rightarrow x$ is preferred over $\lim_{n \rightarrow \infty} x_n = x$, as the latter implies equality between potentially distinct limit points.

Definition 6.15 Continuity and Mappings

Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a function. We define the following topological properties for f :

- **Local Continuity:** The function f is continuous at a point $x_0 \in X$ if, for every neighborhood V of $f(x_0)$ in Y , there exists a neighborhood U of x_0 in X such that $f(U) \subseteq V$.
- **Global Continuity:** The function f is continuous on X if the preimage $f^{-1}(V)$ of every open set $V \subseteq Y$ is open in X .
- **Sequential Continuity:** The function f is sequentially continuous at x_0 if for every sequence (x_n) converging to x_0 in X , the image sequence $(f(x_n))$ converges to $f(x_0)$ in Y . It is globally sequentially continuous if this condition holds for all points in X .
- **Open Mapping:** The function f is open if the image $f(U)$ of every open set $U \subseteq X$ is open in Y .
- **Homeomorphism:** The function f is a homeomorphism if it is a bijection of X onto Y where both f and its inverse f^{-1} are continuous. Equivalently, f is a continuous, open bijection.

A function from one topological space to another is globally continuous if and only if it is continuous at every point in its domain. The same pointwise-to-global relationship applies to sequential continuity. Furthermore, continuity at a point always implies sequential continuity at that point.

The converse, however, requires specific topological conditions. To understand when sequences are sufficient to determine continuity, we recall the first

countability axiom. A topological space satisfies this axiom if every point in the space possesses a countable local basis. Metric spaces rational satisfy this condition; the collection of open balls centered at a point with radii decreasing to 0 forms a countable local basis.

In any topological space satisfying the first countability axiom, sequences possess enough resolution to capture the topology. In these spaces, a function is continuous at a point if and only if it is sequentially continuous at that point, making global continuity and global sequential continuity equivalent. Similarly, a subset of such a space is closed if and only if it is sequentially closed. The methods for proving these equivalences mirror standard metric space arguments.

In broader topological contexts, straightforward sequential testing fails. It is possible to construct spaces—including Hausdorff spaces—containing subsets that are sequentially closed but not closed. Because sequences are indexed by the natural numbers, they cannot always navigate the neighborhood structures of more complex topologies. To recover the analytical power of limit operations in these general spaces, the sequential methods useful in metric spaces must be replaced by nets.

Definition 6.16 Directed Sets and Nets

A directed set is a non-empty set I equipped with a binary relation $\preceq$ satisfying the following conditions for all $\alpha, \beta, \gamma \in I$:

- (1) Reflexivity: $\alpha \preceq \alpha$.
- (2) Transitivity: If $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.
- (3) Upper Bound: There exists an element $\delta \in I$ such that $\alpha \preceq \delta$ and $\beta \preceq \delta$.

A non-empty set satisfying conditions (1) and (2) is a preordered set. A directed set is a preordered set satisfying condition (3). A net in a set X is a function from a directed set I into X . The set I serves as the index set for the net.

The geometric intuition for net convergence mirrors that of sequences, but replaces the linear number line with a branching directed set as shown in Figure 6.8.

Certain developments of net theory require the relation $\preceq$ to be a partial order, imposing the condition of anti-symmetry: if $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then $\alpha = \beta$. The theory of topological convergence functions consistently with or without this axiom.

If $f : I \rightarrow X$ is a net, the image $f(\alpha)$ for an index $\alpha \in I$ is conventionally denoted by x_α . The entire net is denoted by $(x_\alpha)_{\alpha \in I}$ or simply (x_α) . Borrowing terminology from sequences, we state that x_α precedes x_β when $\alpha \preceq \beta$. This extends the language of sequences to generalized index sets.

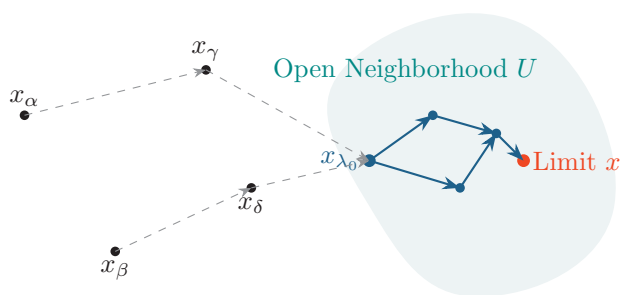


FIGURE 6.8 Visualization of net convergence in a general topological space. The net may branch across an uncountable directed set, but it is eventually captured completely within any open neighborhood of the limit x .

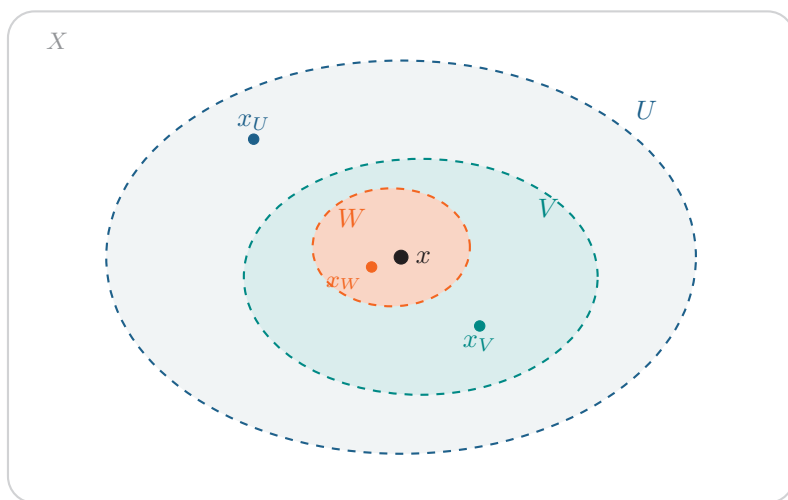
Example 6.17 Sequences and Reals as Nets

Every sequence is a net. The directed set is the set of natural numbers $\mathbb{N}$, equipped with its standard numerical order $\leq$. Similarly, the set of real numbers $\mathbb{R}$, equipped with its standard order $\leq$, forms a directed set. Consequently, any function mapping $\mathbb{R}$ into a space X is a net. However, unlike sequences indexed by $\mathbb{N}$, a term in an $\mathbb{R}$ -indexed net can be preceded by infinitely many terms, and the net lacks a first term.

Example 6.18 The Neighborhood Net

A specific net construction is fundamental to topological arguments. Let X be a topological space and $x \in X$. Let the index set I be the collection of all neighborhoods of x . We direct I by reverse inclusion: $U \preceq V$ if $U \supseteq V$. Under this relation, I is a directed set (the intersection of any two neighborhoods provides the required upper bound). By selecting a point $x_U \in U$ for each neighborhood $U \in I$, the resulting function (x_U) forms a net in X (see Figure 6.9).

A sequence in a topological space converges to a point if, for every neighborhood of that point, all terms of the sequence eventually lie in that neighborhood. This concept translates directly to nets.

**Directed by Reverse Inclusion:**

Since $U \supseteq V \supseteq W$, the indices follow $U \preceq V \preceq W$.
As the index “increases,” the neighborhood shrinks,
forcing the choice points x_U, x_V, x_W closer to x .

FIGURE 6.9 *The Neighborhood net:* The collection of neighborhoods of x forms a directed set under reverse inclusion. By choosing a point inside each neighborhood, the resulting net (x_U) is topologically trapped and must converge to x .

Definition 6.19 Net Convergence

Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological space X , and let $x \in X$. The net **converges** to x (and x is a limit of the net) if for every neighborhood U of x , there exists an index $\alpha_U \in I$ such that $x_\alpha \in U$ whenever $\alpha_U \preceq \alpha$. We denote this convergence by writing $x_\alpha \rightarrow x$ or $\lim_\alpha x_\alpha = x$.

When verifying net convergence to a point x , it is sufficient to check the convergence condition against a basis (or subbasis) for the local neighborhood system of x .

Theorem 6.20 Subbasic Convergence

Let $\mathcal{S}$ be a subbasis for the topology on a space X . A net $(x_\alpha)_{\alpha \in I}$ converges to a point $x \in X$ if and only if every subbasic open set $U \in \mathcal{S}$

containing x eventually captures the net. That is, for each $U \in \mathcal{S}$ with $x \in U$, there exists an index $\alpha_U \in I$ such that $x_\alpha \in U$ for all $\alpha \succeq \alpha_U$.

PROOF: If the net converges to x , the convergence condition holds for all open neighborhoods of x , which includes all sets in the subbasis.

For the converse, assume the net is eventually captured by every subbasic set containing x . A basic open neighborhood of x is formed by a finite intersection of subbasic sets $U_1, U_2, \dots, U_n$, each containing x . By assumption, for each index $k \in \{1, \dots, n\}$, there exists an index $\alpha_k \in I$ such that $x_\alpha \in U_k$ whenever $\alpha \succeq \alpha_k$. Because the index set I is directed, it contains an upper bound γ such that $\gamma \succeq \alpha_k$ for all k . Consequently, for all $\alpha \succeq \gamma$, the term $x_\alpha \in \bigcap_{k=1}^n U_k$. Since these finite intersections form a local basis at x , the net converges to x . $\lrcorner$

The preceding proof highlights a fundamental utility of directed sets: if a finite collection of conditions each hold eventually, they must all hold simultaneously eventually. This principle simplifies the analysis of convergence in product spaces.

Theorem 6.21 Convergence in Product Spaces

Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of topological spaces, and let $X = \prod_{\gamma \in \Gamma} X_\gamma$ be their topological product. A net $(x_\alpha)_{\alpha \in I}$ in X converges to a point $x \in X$ if and only if it converges coordinate-wise. That is, for every $\gamma \in \Gamma$, the projection net $(x_{\alpha, \gamma})_{\alpha \in I}$ converges to the coordinate x_γ in the component space X_γ .

PROOF: Let $\mathcal{S}$ denote the standard subbasis for the product topology. The members of $\mathcal{S}$ are sets where a single coordinate is restricted to an open set $U_\gamma \subseteq X_\gamma$, while all other coordinates remain unconstrained.

Applying the previous theorem, the net (x_α) converges to x in X if and only if it is eventually captured by every such subbasic set containing x . This requires that for any $\gamma \in \Gamma$ and any open neighborhood U_γ of x_γ , the coordinate $x_{\alpha, \gamma}$ is eventually in U_γ . This condition is the definition of coordinate-wise convergence for each projection of the net. $\lrcorner$

Recall that in a Hausdorff space, a convergent sequence has a unique limit. For nets, this property provides a complete characterization of the Hausdorff separation axiom among all topological spaces.

Theorem 6.22 Uniqueness of Net Limits

A topological space X is a Hausdorff (T_2) space if and only if every convergent net in X possesses exactly one limit.

PROOF: The proof establishes mutual implication by linking the existence of disjoint neighborhoods to the behavior of the directed index set.

Hausdorff Implies Unique Limits Assume X is a Hausdorff space. Let $(x_\alpha)_{\alpha \in I}$ be a net in X converging to a limit x . Assume, for the sake of contradiction, that the net also converges to a distinct limit y ($x \neq y$).

Because X is Hausdorff, there exist disjoint open neighborhoods U containing x and V containing y ($U \cap V = \emptyset$). By the definition of net convergence, the net must eventually be in U . Thus, there exists an index $\alpha_U \in I$ such that $x_\alpha \in U$ for all $\alpha \succeq \alpha_U$. Simultaneously, the net must eventually be in V , meaning there exists an index $\alpha_V \in I$ such that $x_\alpha \in V$ for all $\alpha \succeq \alpha_V$.

Because I is a directed set, it possesses an upper bound for any pair of indices. Therefore, there exists an index $\gamma \in I$ satisfying both $\gamma \succeq \alpha_U$ and $\gamma \succeq \alpha_V$. Consequently, the element x_γ must be in both U and V , implying $x_\gamma \in U \cap V$. This contradicts U and V being disjoint. Thus, the limit must be unique.

Unique Limits Implies Hausdorff We proceed by contraposition. Assume X is not a Hausdorff space. There must exist two distinct points $x, y \in X$ ($x \neq y$) that cannot be separated by disjoint neighborhoods. Consequently, for every open neighborhood U of x and every open neighborhood V of y , the intersection is non-empty ($U \cap V \neq \emptyset$).

We construct a directed set to form a net. Let $\mathcal{N}_x$ and $\mathcal{N}_y$ denote the collections of all open neighborhoods of x and y , respectively. Define the index set I as the Cartesian product $\mathcal{N}_x \times \mathcal{N}_y$. We direct I by reverse inclusion: $(U_1, V_1) \preceq (U_2, V_2)$ if and only if $U_2 \subseteq U_1$ and $V_2 \subseteq V_1$.

For each index pair $(U, V) \in I$, select a point $z_{(U,V)}$ from the non-empty intersection $U \cap V$. This selection defines a net $(z_{(U,V)})$ indexed by I .

We now verify convergence. Let U_0 be an arbitrary neighborhood of x , and let V_0 be an arbitrary neighborhood of y . For any index $(U, V) \succeq (U_0, V_0)$, the construction guarantees $U \subseteq U_0$. Because $z_{(U,V)} \in U \cap V \subseteq U \subseteq U_0$, the net is eventually contained in U_0 , proving $z_{(U,V)} \rightarrow x$. By identical logic applied to the second coordinate, $z_{(U,V)} \rightarrow y$.

Because a single net converges simultaneously to two distinct limits, the condition of unique limits is violated. Therefore, if every net possesses a unique limit, the space must be Hausdorff. ■

In metric spaces, sequential continuity provides a complete test for global continuity. Because sequences fail to capture the neighborhood structure of arbitrary topological spaces, this operational test must be upgraded to utilize nets.

Theorem 6.23 Net Characterization of Continuity

Let X and Y be topological spaces. A mapping $f : X \rightarrow Y$ is continuous at a point $x \in X$ if and only if, for every net $(x_\alpha)_{\alpha \in I}$ converging to x in X , the image net $(f(x_\alpha))_{\alpha \in I}$ converges to $f(x)$ in Y .

PROOF: We establish mutual implication by linking the convergence of the directed index set to the topological pullback of open neighborhoods.

Necessity (Continuity Implies Net Convergence) Assume f is continuous at x , and let (x_α) be a net converging to x . We must show that $f(x_\alpha) \rightarrow f(x)$. Let V be an arbitrary open neighborhood of $f(x)$ in Y . Because f is continuous at x , there exists an open neighborhood U of x in X such that $f(U) \subseteq V$. Because the net (x_α) converges to x , it must eventually be in U . Thus, there exists an index $\alpha_U \in I$ such that $x_\alpha \in U$ for all $\alpha \succeq \alpha_U$. Mapping these points through f guarantees that $f(x_\alpha) \in f(U) \subseteq V$ for all $\alpha \succeq \alpha_U$. Because the image net is eventually captured by the arbitrary neighborhood V , it converges to $f(x)$.

Sufficiency (Net Convergence Implies Continuity) We proceed by contraposition. Assume f is *not* continuous at x . Consequently, there exists an open neighborhood V of $f(x)$ in Y such that no neighborhood U of x satisfies $f(U) \subseteq V$. This dictates that for every open neighborhood U containing x , the set $U \setminus f^{-1}(V)$ is non-empty.

We construct a directed set I consisting of all open neighborhoods of x , directed by reverse inclusion ($U_1 \preceq U_2 \iff U_2 \subseteq U_1$). For each neighborhood $U \in I$, select a point $x_U \in U \setminus f^{-1}(V)$. This selection generates a net $(x_U)_{U \in I}$. By the exact logic of the standard neighborhood net, (x_U) converges to x . However, the construction guarantees that $f(x_U) \notin V$ for all $U \in I$. Because the image net never enters the neighborhood V , it cannot converge to $f(x)$. Thus, a failure of continuity guarantees a failure of net convergence. ■

To extend the concept of sequential compactness to arbitrary topological spaces, we require a generalization of a subsequence. A subsequence extracts an infinite subset of terms while preserving their original chronological order. For nets, this extraction is formalized via a mapping between directed sets.

Definition 6.24 Subnet

Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological space X . A net $(y_\beta)_{\beta \in J}$ is a **subnet** of (x_α) if there exists a function $h : J \rightarrow I$ mapping the directed set J into the directed set I such that:

1. **Composition:** $y_\beta = x_{h(\beta)}$ for all $\beta \in J$.
2. **Cofinality:** For every index $\alpha_0 \in I$, there exists an index $\beta_0 \in J$ such that $h(\beta) \succeq \alpha_0$ whenever $\beta \succeq \beta_0$.

The cofinality condition ensures that the subnet does not become “trapped” in a bounded region of the original index set I ; it must push arbitrarily deep into the topological tail of the original net. This mechanism is visualized in [Figure 6.10](#).

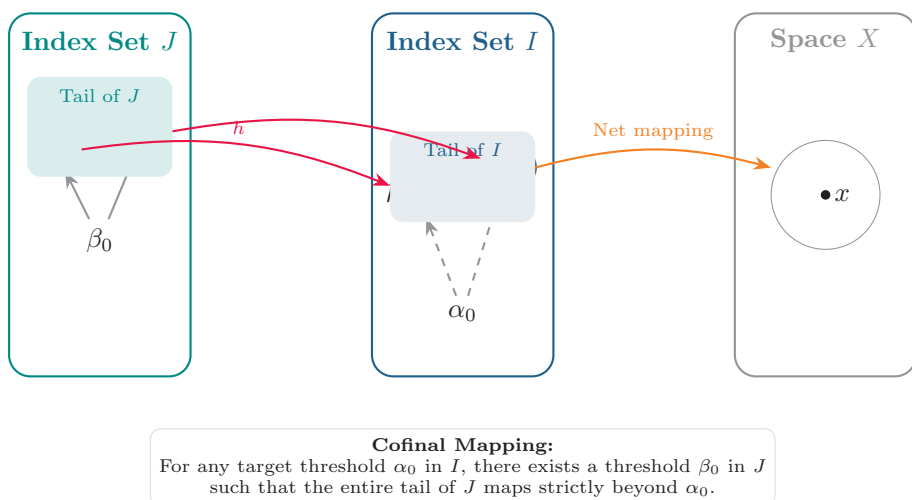


FIGURE 6.10 Visualizing the subnet mapping. The function h extracts a generalized subsequence by ensuring the new directed set J successfully drives the indices deep into the original directed set I .

The utility of subnets lies in their ability to completely characterize compactness in abstract topological spaces, generalizing the Bolzano-Weierstrass theorem beyond metric boundaries.

Theorem 6.25 Compactness via Subnets

A topological space X is **compact** if and only if every net in X possesses a convergent subnet whose limit is in X .

This theorem establishes the analytical machinery required to prove existence theorems in the weak topology. Rather than verifying open covers, one constructs a bounded net and extracts a convergent subnet.

The capstone of this topological foundation is the behavior of compact spaces under Cartesian products. If we take an infinite product of compact intervals (such as $[0, 1]$), does the resulting multi-dimensional cube remain compact? In the metric-induced box topology, the answer is definitively no. However, under the coarse constraints of the product topology, compactness is preserved. This result, proven by Andrey Tychonoff in 1930, relies heavily on the Axiom of Choice (often via Zorn's Lemma or ultrafilters) and serves as the linchpin for the Banach-Alaoglu theorem.

Theorem 6.26 Tychonoff's Theorem

Let $\{X_\gamma\}_{\gamma \in \Gamma}$ be an arbitrary family of compact topological spaces. The Cartesian product

$$X = \prod_{\gamma \in \Gamma} X_\gamma$$

is strictly compact when equipped with the product topology.

Because the weak* topology on the dual space X^* is formally equivalent to the subspace topology inherited from the Cartesian product of scalar fields, Tychonoff's Theorem provides the exact topological guarantee required to prove that the closed unit ball of the dual space is weak* compact.

EXERCISES

1. *Subbasis of the Real Line.* Let $X = \mathbb{R}$. Consider the collection of sets $\mathcal{S} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$. Prove that $\mathcal{S}$ forms a subbasis for the standard topology on $\mathbb{R}$. (*Hint: Detail the finite intersections required to construct arbitrary open intervals (a, b) .*)
2. *Singletons in T_1 Spaces.* Prove that a topological space X is a T_1 space if and only if for every point $x \in X$, the singleton set $\{x\}$ is a closed set.
3. *Finite Subsets as a Directed Set.* Let X be an infinite set. Let I be the collection of all finite subsets of X . Define a relation on I by $A \preceq B$ if and only if $A \subseteq B$. Verify that I satisfies the three axioms of a directed set.
4. *Subnet Convergence.* Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological space X that converges to a limit x . Prove that every subnet $(y_\beta)_{\beta \in J}$ must also converge to x .
5. *Compactness of Closed Subsets.* Utilizing the net characterization of compactness (Theorem 6.25), prove that any closed subset F of a compact topological space X is itself compact.
6. *Product Topology and Pointwise Convergence.* Let $X = \mathbb{R}^{[0,1]}$ denote the space of all functions $f(t)$ mapping the interval $[0, 1]$ to $\mathbb{R}$. Equip X with the product topology. Let $(f_\alpha)_{\alpha \in I}$ be a net of functions in X . Prove that $f_\alpha \rightarrow f$ in the product topology if and only if $f_\alpha(t) \rightarrow f(t)$ in $\mathbb{R}$ for every fixed $t \in [0, 1]$.
7. *Hausdorff Spaces and Compact Sets.* Prove that a compact subset K of a Hausdorff space X is always a closed set. (*Hint: Let a net (x_α) in K converge to a limit $x \in X$. Apply the subnet characterization of compactness to show that x must belong to K .*)

8. *Failure of Sequential Convergence.* Let X be an uncountable set equipped with the cocountable topology, where a subset $U \subseteq X$ is open if and only if $U = \emptyset$ or its complement $X \setminus U$ is countable.
- (a) Prove that the only convergent sequences in X are those that are eventually constant.
 - (b) Deduce why sequences are insufficient to characterize the closure of subsets in this specific topology, demonstrating the necessity of nets.

6.2 THE WEAK TOPOLOGY

In a normed space X , the standard norm induces a refined topological structure, commonly referred to as the *strong topology*. Because the open balls generated by the norm restrict movement in every dimension simultaneously, strong open sets are confined. Consequently, bounded sets in infinite-dimensional spaces stretch outward without ever achieving the density required for compactness.

To recover compactness, we must construct a topology with fewer open sets—a *coarser* topology. We achieve this by redefining proximity. Rather than measuring the geometric distance between vectors, we measure their separation through the lens of bounded linear functionals.

Defining the Weak Topology

Let X be a normed space and X^* be its continuous dual space. Every functional $f \in X^*$ is continuous with respect to the strong (norm) topology. Our objective is to strip away superfluous open sets from X , retaining only the minimum required to ensure that every functional in X^* remains continuous.

A function is continuous if and only if the preimage of every open set is open. For a linear functional $f : X \rightarrow \mathbb{K}$, the basic open sets in the scalar field $\mathbb{K}$ are open balls (or intervals) $B(c, \epsilon) = \{\lambda \in \mathbb{K} : |\lambda - c| < \epsilon\}$. The preimage of such an interval under f forms a *strip* or *slab* in the vector space X .

Definition 6.27 The Weak Topology $\sigma(X, X^*)$

Let X be a normed space. For any point $x_0 \in X$, any functional $f \in X^*$, and any real scalar $\epsilon > 0$, we define the subbasic neighborhood:

$$U(x_0, f, \epsilon) = \{x \in X : |f(x) - f(x_0)| < \epsilon\}$$

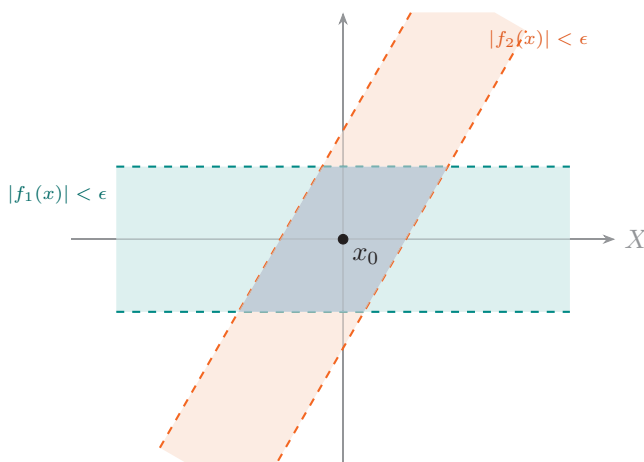
The weak topology on X , denoted by $\sigma(X, X^*)$, is the topology generated by taking the collection of all such sets for all $x_0 \in X$, $f \in X^*$, and $\epsilon > 0$ as a subbasis.

Because it is generated directly from this subbasis, the weak topology is the *coarsest* topology on X such that every bounded linear functional $f \in X^*$ remains continuous.

To construct a *local basis* at x_0 for the weak topology, we take finite intersections of these subbasic strips. Thus, a standard basic weak neighborhood of x_0 takes the form:

$$W(x_0, f_1, \dots, f_n, \epsilon) = \{x \in X : |f_i(x) - f_i(x_0)| < \epsilon \text{ for all } i = 1, \dots, n\}$$

where $\{f_1, \dots, f_n\}$ is a finite subset of X^* .



A Basic Weak Neighborhood

The intersection of finitely many unbounded strips forms the neighborhood W . In infinite dimensions, this intersection always contains an infinite-dimensional subspace, confirming that weak open sets are geometrically “massive” compared to norm open balls.

FIGURE 6.11 Visualizing the weak topology. A basic weak neighborhood is formed by the finite intersection of subbasic functional strips.

Weak Convergence

Because the weak topology is constructed from a specific subbasis, we can leverage the net convergence theorems established in [Section 6.1](#) to analyze sequential limits. A net converges in a topology if and only if it eventually enters every subbasic neighborhood of the limit.

Theorem 6.28 Characterization of Weak Convergence

Let $(x_\alpha)_{\alpha \in I}$ be a net in a normed space X . The net converges weakly to a limit $x \in X$, denoted by $x_\alpha \rightharpoonup x$, if and only if for every bounded

linear functional $f \in X^*$:

$$\lim_{\alpha} f(x_{\alpha}) = f(x) \quad \text{in } \mathbb{K}$$

PROOF: The net $x_{\alpha} \rightarrow x$ in the weak topology if and only if for every subbasic set $U(x, f, \epsilon)$ containing x , the net is eventually in U . This requires that for any $f \in X^*$ and any $\epsilon > 0$, there exists an index α_0 such that for all $\alpha \succeq \alpha_0$:

$$x_{\alpha} \in U(x, f, \epsilon) \implies |f(x_{\alpha}) - f(x)| < \epsilon$$

This condition is the definition of the scalar net $f(x_{\alpha})$ converging to $f(x)$ in the field $\mathbb{K}$. ■

Because strong (norm) convergence ($\|x_{\alpha} - x\| \rightarrow 0$) implies that $f(x_{\alpha}) \rightarrow f(x)$ via the continuity bound $|f(x_{\alpha}) - f(x)| \leq \|f\| \|x_{\alpha} - x\|$, strong convergence always implies weak convergence. The converse, however, fails in infinite-dimensional spaces.

The weak topology possesses fewer open sets than the strong topology. By the definition of closed sets as the complements of open sets, the weak topology must also possess fewer closed sets. Every weakly closed set is strongly closed, but a strongly closed set is not guaranteed to be weakly closed.

However, when geometry is constrained by convexity, this topological gap vanishes. Stanisław Mazur proved that for convex sets, the two topologies are indistinguishable regarding closure.

Theorem 6.29 Mazur's Theorem

Let X be a normed space and let $C \subseteq X$ be a convex subset. The set C is closed in the weak topology if and only if it is closed in the strong (norm) topology.

PROOF:

Weak Closure Implies Strong Closure This direction is trivial and independent of convexity. Because the weak topology is coarser than the strong topology ($\sigma(X, X^*) \subseteq \tau_{\text{norm}}$), any set that is open in the weak topology is open in the strong topology. By taking complements, any weakly closed set is strongly closed.

Strong Closure Implies Weak Closure (requires Convexity) Assume C is convex and strongly closed. We must prove it is weakly closed. We proceed by showing its complement $X \setminus C$ is weakly open. Let $x_0 \notin C$. Because C is strongly closed and convex, we can apply the strict separation form of the Hahn-Banach

Theorem (established in Chapter 5). There exists a bounded linear functional $f \in X^*$ and a real scalar $c \in \mathbb{R}$ such that:

$$\operatorname{Re}(f(x_0)) < c < \inf_{x \in C} \operatorname{Re}(f(x))$$

Define the open half-space $V = \{y \in X : \operatorname{Re}(f(y)) < c\}$. Because f is a continuous functional in the weak topology, the set V (being the continuous preimage of the open scalar interval $(-\infty, c)$) is a weakly open set in X . By construction, $x_0 \in V$, and because all elements $x \in C$ satisfy $\operatorname{Re}(f(x)) > c$, the intersection $V \cap C = \emptyset$. Thus, V is a weakly open neighborhood of x_0 contained in the complement $X \setminus C$. Because every point outside C possesses such a weak neighborhood, $X \setminus C$ is weakly open, confirming C is weakly closed. ■

Corollary 6.30

If a sequence (x_n) converges weakly to x ($x_n \rightharpoonup x$), then there exists a sequence of convex combinations of the elements (x_n) that converges strongly (in norm) to x .

EXERCISES FOR SECTION 6.2

1. *Uniqueness of Weak Limits.* Prove that the weak topology $\sigma(X, X^*)$ on a normed space X is a Hausdorff (T_2) space. (*Hint: If $x \neq y$, apply a corollary of the Hahn-Banach Theorem to find a bounded linear functional $f \in X^*$ that separates x and y , then construct disjoint weak neighborhoods.*)
2. *Weak vs. Strong Convergence.* Let (x_n) be a sequence in a normed space X . Prove that if $x_n \rightarrow x$ in the strong (norm) topology, then $x_n \rightharpoonup x$ in the weak topology.
3. *Weak Convergence of Operators.* Let X and Y be normed spaces, and let $T : X \rightarrow Y$ be a bounded linear operator. Prove that if a sequence (x_n) converges weakly to x in X , then the image sequence (Tx_n) converges weakly to Tx in Y . (*Hint: For any functional $g \in Y^*$, consider the composition $g \circ T$ and verify it belongs to X^* .*)
4. *Constructing Strong Convergence from Weak Convergence.* Recall the example demonstrating that the standard basis vectors (e_n) in ℓ^2 converge weakly to 0 but fail to converge in norm. Mazur's Theorem (Corollary 6.30) guarantees the existence of a sequence of convex combinations of (e_n) that converges to 0 in the strong topology. Define the sequence of arithmetic means:

$$y_n = \frac{1}{n} \sum_{k=1}^n e_k$$

- (a) Verify that y_n is a convex combination of the basis vectors.

- (b) Calculate the norm $\|y_n\|_2$ and show that $y_n \rightarrow 0$ in the strong topology.
5. *Weak Convergence in finite dimensions.* Prove that if X is a finite-dimensional normed space, weak convergence is equivalent to strong convergence. (*Hint: Let $\dim(X) = k$ with basis $\{b_1, \dots, b_k\}$. Use the dual basis $\{f_1, \dots, f_k\}$ in X^* to show that weak convergence implies coordinate-wise convergence, which ensures convergence in norm.*)
6. *Lower Semicontinuity of the Norm.* Prove that the norm function $x \mapsto \|x\|$ is weakly sequentially lower semicontinuous. Specifically, show that if $x_n \rightharpoonup x$ in X , then:

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$$

(*Hint: By the Hahn-Banach Theorem, choose a functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Apply the functional to the weakly convergent sequence.*)

7. *Boundedness of Weakly Convergent Sequences.* Prove that if a sequence (x_n) converges weakly to x in a Banach space X , then the sequence of norms $(\|x_n\|)$ is bounded in $\mathbb{R}$. (*Hint: Treat each x_n as an evaluation functional J_{x_n} operating on X^* . Apply the Uniform Boundedness Principle.*)
8. *Schur's Property in ℓ^1 .* While infinite-dimensional spaces generally contain sequences that converge weakly but not strongly, the space ℓ^1 is a rare exception. It possesses *Schur's property*, meaning weak sequential convergence implies strong sequential convergence. Let (x_n) be a sequence in ℓ^1 such that $x_n \rightharpoonup 0$. Explain why testing this weak convergence against the specific functionals in c_0 (a subspace of ℓ^∞) forces coordinate-wise convergence.

6.3 THE WEAK* TOPOLOGY

Because the continuous dual space X^* is a vector space equipped with the operator norm, it constitutes a Banach space in its own right. Consequently, we can equip X^* with its own weak topology. Following the construction established in the previous section, the standard weak topology on the dual space, denoted $\sigma(X^*, X^{**})$, is generated by all bounded linear functionals operating on X^* . These functionals reside in the bidual space, X^{**} .

However, unless the space is reflexive, the bidual X^{**} is larger than the original space X . Utilizing all functionals in X^{**} generates a topology that often possesses too many open sets to permit compactness. To construct a coarser topology, we must restrict the generating functionals. Rather than using the entire bidual X^{**} , we use only the original space X .

To utilize elements of X as functionals acting on X^* , we must establish a mechanism that reverses their roles. Normally, a functional $f \in X^*$ operates

on a vector $x \in X$ to produce a scalar $f(x)$. We can invert this perspective by fixing the vector x and allowing the functional f to vary.

Definition 6.31 The Canonical Evaluation Map

For every fixed vector $x \in X$, we define a mapping $J_x : X^* \rightarrow \mathbb{K}$ by the evaluation rule:

$$J_x(f) = f(x) \quad \text{for all } f \in X^*$$

The mapping J_x is a bounded linear functional on X^* , meaning $J_x \in X^{**}$. The transformation $J : X \rightarrow X^{**}$ defined by $x \mapsto J_x$ is called the **canonical embedding** of X into its bidual.

By the Hahn-Banach Theorem, the operator norm of J_x equals the norm of x ($\|J_x\| = \|x\|$). Therefore, the canonical embedding is an isometric isomorphism onto its image $J(X) \subseteq X^{**}$. By identifying X with its image $J(X)$, we treat X as a subspace of X^{**} .

Defining the Weak* Topology

We are now positioned to define the weak* topology. Instead of requiring continuity for every functional in the vast bidual X^{**} , we require continuity only for the specific evaluation functionals J_x generated by the original space X .

Definition 6.32 The Weak* Topology $\sigma(X^*, X)$

Let X be a normed space and X^* be its dual. For any functional $f_0 \in X^*$, any vector $x \in X$, and any real scalar $\epsilon > 0$, we define the subbasic neighborhood:

$$U(f_0, x, \epsilon) = \{f \in X^* : |f(x) - f_0(x)| < \epsilon\}$$

The **weak* topology** on X^* , denoted by $\sigma(X^*, X)$, is the topology generated by taking the collection of all such sets for all $f_0 \in X^*$, $x \in X$, and $\epsilon > 0$ as a **subbasis**.

The weak* topology is the coarsest topology on X^* such that for every $x \in X$, the evaluation map $f \mapsto f(x)$ is continuous.

A standard basic weak* neighborhood of f_0 is formed by the finite intersection of these subbasic sets. For a finite collection of vectors $\{x_1, \dots, x_n\}$ in X and a scalar $\epsilon > 0$, the basic neighborhood is:

$$W(f_0, x_1, \dots, x_n, \epsilon) = \{f \in X^* : |f(x_i) - f_0(x_i)| < \epsilon \text{ for all } i = 1, \dots, n\}$$

Topological Hierarchy and Convergence

Because $J(X) \subseteq X^{**}$, the collection of subbasic sets generating the weak* topology is a subcollection of the sets generating the weak topology on X^* . Consequently, the weak* topology contains fewer open sets than the weak topology. This establishes a strict hierarchy of topologies on the dual space.

Because the weak* topology is coarse, it is the easiest topology in which to achieve convergence. A net of functionals must only converge pointwise on the original space X .

Theorem 6.33 Characterization of Weak* Convergence

Let $(f_\alpha)_{\alpha \in I}$ be a net in the dual space X^* . The net converges weak* to a limit $f \in X^*$, denoted by $f_\alpha \rightharpoonup^* f$, if and only if for every vector $x \in X$:

$$\lim_{\alpha} f_\alpha(x) = f(x) \quad \text{in } \mathbb{K}$$

PROOF: The net $f_\alpha \rightarrow f$ in the weak* topology if and only if for every subbasic set $U(f, x, \epsilon)$ containing f , the net eventually is in U . This requires that for any $x \in X$ and any $\epsilon > 0$, there exists an index α_0 such that for all $\alpha \succeq \alpha_0$:

$$f_\alpha \in U(f, x, \epsilon) \implies |f_\alpha(x) - f(x)| < \epsilon$$

This condition is the precise definition of the scalar net $f_\alpha(x)$ converging to $f(x)$ in $\mathbb{K}$. ■

This pointwise convergence reveals why the weak* topology is so prevalent in applications: to verify that a sequence of measurement tools (functionals) converges, one must only verify that their measurements align on every test object (vectors in X).

The distinction between weak and weak* convergence is profound in non-reflexive spaces. A sequence of functionals can easily converge in the weak* topology without converging in the weak topology.

EXERCISES FOR SECTION 6.3

1. *Isometry of the Canonical Embedding.* Let $J : X \rightarrow X^{**}$ be the canonical embedding defined by $J_x(f) = f(x)$ for all $f \in X^*$.
 - (a) Prove that J_x is a bounded linear functional on X^* with operator norm $\|J_x\| \leq \|x\|$.
 - (b) Using the Hahn-Banach Theorem, prove that there exists a functional $f_0 \in X^*$ with $\|f_0\| = 1$ such that $f_0(x) = \|x\|$. Use this to conclude that $\|J_x\| = \|x\|$, proving J is an isometry.

2. *Hausdorff Property of the Weak* Topology.* Prove that the weak* topology $\sigma(X^*, X)$ on the dual space X^* is a Hausdorff (T_2) space. (*Hint: If $f, g \in X^*$ are distinct, there exists a vector $x \in X$ such that $f(x) \neq g(x)$. Use this vector to construct disjoint subbasic open sets in the weak* topology.*)
3. *Boundedness of Weak* Convergent Sequences.* Let X be a Banach space. Assume (f_n) is a sequence in X^* that converges in the weak* topology to a limit $f \in X^*$. Prove that the sequence of norms $(\|f_n\|)$ is bounded. (*Hint: Weak* convergence implies that for every $x \in X$, the scalar sequence $f_n(x)$ converges and is therefore bounded. Apply the Uniform Boundedness Principle.*)
4. *Weak* Convergence of Dirac Functionals.* Let $X = C[0, 1]$ be the space of continuous functions on the interval $[0, 1]$ equipped with the supremum norm. For any point $t \in [0, 1]$, define the evaluation functional (Dirac measure) $\delta_t \in X^*$ by $\delta_t(g) = g(t)$. Let (t_n) be a sequence of points in $[0, 1]$ converging to a limit t in the standard topology of $\mathbb{R}$.
 - (a) Prove that $\delta_{t_n} \rightharpoonup^* \delta_t$ in the weak* topology of X^* .
 - (b) Prove that $\|\delta_{t_n} - \delta_t\| = 2$ for any $t_n \neq t$. Deduce that the sequence of functionals does not converge in the strong (norm) topology.
5. *Coincidence of Topologies in Reflexive Spaces.* A Banach space X is reflexive if the canonical embedding $J(X)$ equals the entire bidual X^{**} . Prove that if X is reflexive, the weak topology $\sigma(X^*, X^{**})$ and the weak* topology $\sigma(X^*, X)$ on the dual space X^* are the same topology.
6. *Weak* Closed Hyperplanes.* Let X be a Banach space and $F : X^* \rightarrow \mathbb{K}$ be a linear functional defined on the dual space (meaning $F \in (X^*)^* = X^{**}$). Prove that the kernel (null space) of F is closed in the weak* topology if and only if F belongs to the canonical image $J(X)$. (*This establishes that the only weak* continuous linear functionals on X^* are the evaluation maps generated by elements in X .*)
7. *The Failure of Completeness.* Let X be an infinite-dimensional Banach space. The dual space X^* is a complete metric space under the strong (norm) topology. However, when equipped with the weak* topology, X^* is not complete. Explain why Cauchy nets are required instead of Cauchy sequences to evaluate completeness in this topology, referring to the failure of the first countability axiom.

6.4 THE BANACH-ALAOGLU THEOREM

In an infinite-dimensional normed space X , the closed unit ball B_X is never compact in the norm topology. This lack of compactness poses a severe obstacle for optimization and existence proofs, as bounded sequences are not guaranteed to possess convergent subsequences.

The weak* topology was constructed specifically to solve this problem. By coarsening the topology on the dual space X^* , we reduce the number of open sets required to cover a bounded region. Leonidas Alaoglu and Stefan Banach proved that this topological relaxation is sufficient to render the closed unit ball of the dual space compact.

Theorem 6.34 The Banach-Alaoglu Theorem

Let X be a normed space. The closed unit ball of the continuous dual space, defined as

$$B_{X^*} = \{f \in X^* : \|f\| \leq 1\}$$

is a compact set in the weak* topology $\sigma(X^*, X)$.

PROOF: The proof relies upon embedding the unit ball into a Cartesian product of compact scalar intervals, allowing us to apply Tychonoff's Theorem.

Embedding in a Product Space For any functional $f \in B_{X^*}$ and any vector $x \in X$, the definition of the operator norm imposes the bound:

$$|f(x)| \leq \|f\| \|x\| \leq 1 \cdot \|x\| = \|x\|$$

It follows that $f(x) \in D_x = \{\lambda \in \mathbb{K} : |\lambda| \leq \|x\|\}$.

Because D_x is a closed and bounded subset of $\mathbb{K}$, the Heine-Borel theorem ensures D_x is compact. We construct the Cartesian product of these compact disks for every $x \in X$:

$$P = \prod_{x \in X} D_x$$

By Tychonoff's Theorem (Theorem 6.26), the infinite product space P is compact when equipped with the product topology.

Every functional $f \in B_{X^*}$ is uniquely identified by its values on all $x \in X$. Thus, we can embed B_{X^*} directly into P via the mapping $f \mapsto (f(x))_{x \in X}$. The product topology on P is defined by pointwise convergence. Because the weak* topology on B_{X^*} is also defined by pointwise convergence on X , the weak* topology on B_{X^*} is identical to the subspace topology inherited from P .

Proving the Image Is Closed To prove B_{X^*} is compact, it suffices to prove that its embedded image is a closed set in the compact space P . Let $(f_\alpha)_{\alpha \in I}$ be a net in B_{X^*} that converges to an element $g \in P$ in the product topology. We must show that g corresponds to a bounded linear functional in B_{X^*} .

Because convergence in P is pointwise, $f_\alpha(x) \rightarrow g(x)$ for all $x \in X$. First, we verify that g is linear. For any vectors $x, y \in X$ and scalars $a, b \in \mathbb{K}$:

$$g(ax + by) = \lim_{\alpha} f_\alpha(ax + by)$$

Because each f_α is linear, this expands to:

$$\lim_{\alpha} [af_\alpha(x) + bf_\alpha(y)] = a \lim_{\alpha} f_\alpha(x) + b \lim_{\alpha} f_\alpha(y) = ag(x) + bg(y)$$

Thus, g is a linear mapping.

Second, we verify that g is bounded by 1. For any $x \in X$, the net $f_\alpha(x)$ resides in D_x , meaning $|f_\alpha(x)| \leq \|x\|$. Because the absolute value is a continuous function, the limit preserves this bound:

$$|g(x)| = \lim_{\alpha} |f_\alpha(x)| \leq \|x\|$$

This ensures that the operator norm $\|g\|$ is less than or equal to 1, confirming $g \in B_{X^*}$.

Because B_{X^*} contains all its limit points in P , it is a closed subset of the compact space P . A closed subset of a compact space is always compact. Therefore, B_{X^*} is compact in the weak* topology. ■

EXERCISES

1. *Weak Compactness vs. Weak* Compactness.* Let X be an infinite-dimensional Banach space. The Banach-Alaoglu theorem establishes that the closed unit ball B_{X^*} is compact in the weak* topology $\sigma(X^*, X)$. Prove that B_{X^*} is compact in the weak topology $\sigma(X^*, X^{**})$ if and only if X is a reflexive space. (*Hint: Recall that in a reflexive space, the canonical embedding $J(X)$ equals X^{**} , making the weak and weak* topologies identical.*)
2. *Compactness of Annihilators.* Let Y be a closed subspace of a Banach space X . The annihilator of Y is defined as $Y^\perp = \{f \in X^* : f(y) = 0 \text{ for all } y \in Y\}$.
 - (a) Prove that $Y^\perp$ is a closed set in the weak* topology.
 - (b) Prove that the intersection of $Y^\perp$ and the closed unit ball B_{X^*} is a compact set in the weak* topology.
3. *Pointwise Boundedness and Weak* Limits.* Let (f_α) be a net of bounded linear functionals in X^* . Assume that for every vector $x \in X$, the net of scalars $(f_\alpha(x))$ is bounded in the field $\mathbb{K}$.
 - (a) Apply the Uniform Boundedness Principle to conclude that the net (f_α) is bounded in the operator norm (there exists a constant $M > 0$ such that $\|f_\alpha\| \leq M$ for all indices α).
 - (b) Use the Banach-Alaoglu theorem to prove that this net possesses a weak* convergent subnet.
4. *Weak* Closure of the Unit Sphere.* In finite-dimensional spaces, the unit sphere $S_{X^*} = \{f \in X^* : \|f\| = 1\}$ is a closed set. Let X be an infinite-dimensional Banach space. Prove that the closure of the unit sphere S_{X^*} in the weak* topology is the entire closed unit ball B_{X^*} . (*This fact demonstrates that the weak* topology fails to separate the boundary of the ball from its interior.*)

5. *Sequential Compactness in Separable Spaces.* Assume X is a separable Banach space. Let (x_n) be a countable dense subset contained in the unit ball of X . We define a metric on B_{X^*} by:

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f(x_n) - g(x_n)|$$

- (a) Verify that $d(f, g)$ satisfies the standard metric axioms on B_{X^*} .
 - (b) Prove that convergence in this metric is equivalent to weak* convergence on the bounded set B_{X^*} .
 - (c) Conclude that every bounded sequence of functionals in a separable space possesses a weak* convergent subsequence.
6. *Failure of Norm Density.* Let X be a non-reflexive Banach space, and let $J : X \rightarrow X^{**}$ be the canonical embedding.
- (a) Prove that the image $J(X)$ is a closed subspace of X^{**} in the strong (norm) topology.
 - (b) Use this fact to explain why the image of the unit ball $J(B_X)$ can never be dense in $B_{X^{**}}$ under the strong topology.

Appendix

Proof: The Sequence Space ℓ^p is a Metric Space

For any fixed real number $p \geq 1$, we define the distance between two sequences $x = (\xi_j)$ and $y = (\eta_j)$ in ℓ^p as:

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p}$$

To prove this defines a metric, we verify the four axioms. Non-negativity, identity of indiscernibles, and symmetry follow directly from the properties of the absolute value. The *triangle inequality* is established through the following structural derivation.

PART (A): YOUNG'S INEQUALITY: Let $p > 1$ and define q such that $\frac{1}{p} + \frac{1}{q} = 1$. These are called **conjugate exponents**. For any positive numbers α and β , we have *Young's Inequality*:

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Geometric Intuition Consider the function $u = t^{p-1}$. The area under this curve from 0 to α is $\int_0^\alpha t^{p-1} dt = \frac{\alpha^p}{p}$. The inverse function is $t = u^{1/(p-1)} = u^{q-1}$. The area between this curve and the u -axis from 0 to β is $\int_0^\beta u^{q-1} du = \frac{\beta^q}{q}$. By comparing these areas to the area of the rectangle with sides α and β , the inequality becomes geometrically evident.

PART (B): HÖLDER INEQUALITY FOR SUMS: For any sequences $x \in \ell^p$ and $y \in \ell^q$, we have:

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q \right)^{1/q}$$

Normalization If $\|x\|_p = 0$ or $\|y\|_q = 0$, the inequality is trivially $0 \leq 0$. Assume $\|x\|_p > 0$ and $\|y\|_q > 0$. We define the normalized terms $\tilde{\xi}_j = \frac{|\xi_j|}{\|x\|_p}$ and $\tilde{\eta}_j = \frac{|\eta_j|}{\|y\|_q}$.

Summation Applying Young's Inequality to each term $\tilde{\xi}_j \tilde{\eta}_j$:

$$\tilde{\xi}_j \tilde{\eta}_j \leq \frac{\tilde{\xi}_j^p}{p} + \frac{\tilde{\eta}_j^q}{q}$$

Summing over all j :

$$\sum_{j=1}^{\infty} \tilde{\xi}_j \tilde{\eta}_j \leq \frac{1}{p} \sum_{j=1}^{\infty} \tilde{\xi}_j^p + \frac{1}{q} \sum_{j=1}^{\infty} \tilde{\eta}_j^q = \frac{1}{p}(1) + \frac{1}{q}(1) = 1$$

Multiplying both sides by the scalar $\|x\|_p \|y\|_q$ yields the desired Hölder inequality.

PART (C): MINKOWSKI INEQUALITY FOR SUMS: The Minkowski inequality provides the functional backbone for the triangle inequality. For $x, y \in \ell^p$:

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p} + \left(\sum_{m=1}^{\infty} |\eta_m|^p \right)^{1/p}$$

Decomposition For $p > 1$, let $\omega_j = |\xi_j + \eta_j|$. Using the triangle inequality for real numbers, we write:

$$\sum_{j=1}^{\infty} \omega_j^p = \sum_{j=1}^{\infty} |\xi_j + \eta_j| \omega_j^{p-1} \leq \sum_{j=1}^{\infty} |\xi_j| \omega_j^{p-1} + \sum_{j=1}^{\infty} |\eta_j| \omega_j^{p-1}$$

Hölder Application We apply Hölder's inequality to the sums on the right side. Note that the conjugate of p is q , which implies $(p-1)q = p$:

$$\sum_{j=1}^{\infty} |\xi_j| \omega_j^{p-1} \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} (\omega_j^{p-1})^q \right)^{1/q} = \|x\|_p \left(\sum_{j=1}^{\infty} \omega_j^p \right)^{1/q}$$

Applying the same logic to the η_j terms and substituting back into the decomposition:

$$\sum_{j=1}^{\infty} \omega_j^p \leq (\|x\|_p + \|y\|_p) \left(\sum_{j=1}^{\infty} \omega_j^p \right)^{1/q}$$

Assuming the sum is non-zero, dividing by $(\sum \omega_j^p)^{1/q}$ and utilizing the exponent relation $1 - \frac{1}{q} = \frac{1}{p}$ yields $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

PART (D): VERIFICATION OF THE TRIANGLE INEQUALITY: Finally, we apply the Minkowski inequality to the distance function $d(x, y)$.

Substitution For any $x, y, z \in \ell^p$, we observe by adding and subtracting intermediate terms that $\xi_j - \eta_j = (\xi_j - \zeta_j) + (\zeta_j - \eta_j)$. Let $a_j = \xi_j - \zeta_j$ and $b_j = \zeta_j - \eta_j$.

Final Result By applying Minkowski's Inequality to sequences a and b :

$$d(x, y) = \left(\sum_{j=1}^{\infty} |a_j + b_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |b_j|^p \right)^{1/p}$$

Substituting the original sequence terms a_j and b_j back into the inequality:

$$d(x, y) \leq \left(\sum_{j=1}^{\infty} |\xi_j - \zeta_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |\zeta_j - \eta_j|^p \right)^{1/p}$$

$$d(x, y) \leq d(x, z) + d(z, y)$$

This completes the proof that (ℓ^p, d) satisfies the metric axioms rigorously.

The Decomposition Theorem

Let X be a complete Banach space. If X is the algebraic direct sum of two closed subspaces Y and Z ($X = Y \oplus Z$), then the projection operator P mapping X onto Y along Z is continuous.

PROOF: This proof uses the Bounded Inverse Theorem to evaluate the projection via a product space, a mapping visualized in [Figure A.1](#).

Constructing the Product Space Define the Cartesian product space $W = Y \times Z$. Define the norm evaluating W as $\|(y, z)\|_W = \|y\|_X + \|z\|_X$. Because Y and Z are closed subspaces in a complete Banach space X , they are complete Banach spaces. The topological product of complete Banach spaces is complete; thus, W is a Banach space.

The Summation Mapping Define the summation operator $T : W \rightarrow X$ by:

$$T(y, z) = y + z$$

Because $X = Y \oplus Z$ algebraically, every vector $x \in X$ possesses a unique decomposition. Thus, T is a bijection (injective and surjective). Evaluating continuity via the Triangle Inequality:

$$\|T(y, z)\|_X = \|y + z\|_X \leq \|y\|_X + \|z\|_X = \|(y, z)\|_W$$

This bounding inequality guarantees $\|T\| \leq 1$, confirming T is a bounded continuous operator.

The Inverse Mapping Because T acts as a bijective, continuous linear operator between complete Banach spaces (W and X), the Bounded Inverse Theorem dictates that the inverse mapping $T^{-1} : X \rightarrow W$ is bounded and continuous. Given $x = y + z$, the inverse evaluates to:

$$T^{-1}(x) = (y, z)$$

The Projection Composition The projection $P : X \rightarrow Y$ defined by $P(x) = y$ can be formalized as the composition of the inverse mapping T^{-1} and the coordinate projection $\pi_1 : W \rightarrow Y$ defined by $\pi_1(y, z) = y$:

$$P = \pi_1 \circ T^{-1}$$

The inverse T^{-1} is continuous. The coordinate mapping π_1 is continuous because $\|\pi_1(y, z)\| = \|y\| \leq \|y\| + \|z\| = \|(y, z)\|_W$. Because the composition of continuous mappings yields a continuous mapping, the projection P is continuous. ■

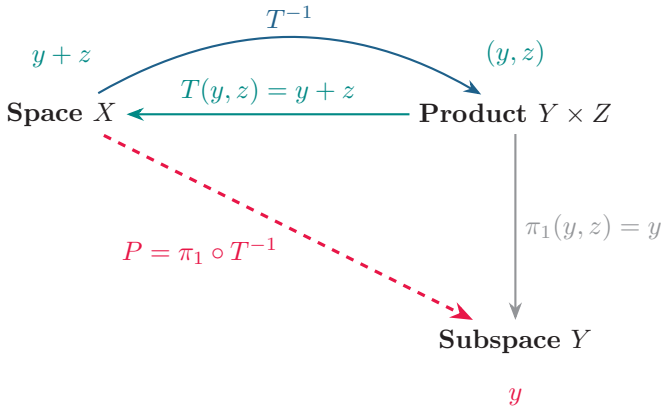


FIGURE A.1 The Decomposition Theorem. The continuity of the projection P is proven by factoring the operator through the complete product space $Y \times Z$.

Isometry of the Adjoint

The adjoint operator T^* is bounded, linear, and preserves the norm of the original operator:

$$\|T^*\| = \|T\|$$

PROOF: The proof verifies that T^* is linear, bounded, and bounded from below by $\|T\|$, establishing isometry.

Linearity of the Adjoint Let $y_1^*, y_2^* \in Y^*$ be bounded linear functionals, and let α, β be scalars. For any vector $x \in X$:

$$\begin{aligned} (T^*(\alpha y_1^* + \beta y_2^*))(x) &= (\alpha y_1^* + \beta y_2^*)(Tx) \\ &= \alpha y_1^*(Tx) + \beta y_2^*(Tx) \\ &= \alpha(T^*y_1^*)(x) + \beta(T^*y_2^*)(x) \end{aligned}$$

Because this equality holds for all $x \in X$, the operators are equivalent: $T^*(\alpha y_1^* + \beta y_2^*) = \alpha T^*y_1^* + \beta T^*y_2^*$. Thus, T^* is linear.

Expanding the Norm Definition The norm of the operator T^* mapping Y^* to X^* evaluates to the supremum of its action on the unit sphere of Y^* :

$$\|T^*\| = \sup_{\|y^*\|_{Y^*}=1} \|T^*y^*\|_{X^*}$$

The internal term, $\|T^*y^*\|_{X^*}$, is the norm of a functional in X^* . Expanding this internal norm using the supremum over the unit sphere of X :

$$\|T^*y^*\|_{X^*} = \sup_{\|x\|_X=1} |(T^*y^*)(x)|$$

Applying the definition of the adjoint, $(T^*y^*)(x) = y^*(Tx)$, formulates the operator norm as a double supremum:

$$\|T^*\| = \sup_{\|y^*\|=1} \left(\sup_{\|x\|=1} |y^*(Tx)| \right)$$

The Upper Bound ($\|T^*\| \leq \|T\|$) Applying the fundamental inequality for bounded functionals ($|f(u)| \leq \|f\|\|u\|$):

$$|y^*(Tx)| \leq \|y^*\|_{Y^*} \|Tx\|_Y$$

Because T is bounded, $\|Tx\|_Y \leq \|T\|\|x\|_X$. Substituting this yields:

$$|y^*(Tx)| \leq \|y^*\|_{Y^*} \|T\| \|x\|_X$$

Evaluating this on the unit spheres, where $\|y^*\|_{Y^*} = 1$ and $\|x\|_X = 1$:

$$|y^*(Tx)| \leq 1 \cdot \|T\| \cdot 1 = \|T\|$$

Because this upper bound holds for all unit vectors x and all unit functionals y^* , the supremum cannot exceed this bound. Therefore:

$$\|T^*\| \leq \|T\|$$

This establishes that T^* is bounded.

The Lower Bound ($\|T^*\| \geq \|T\|$) Let $x \in X$ be a vector satisfying $\|x\|_X = 1$.

Consider its image $Tx \in Y$. If $Tx = \mathbf{0}$, the lower bound is satisfied. If $Tx \neq \mathbf{0}$, we borrow Corollary 5.12 again to guarantee the existence of a bounded linear functional $y_0^* \in Y^*$ such that:

$$\|y_0^*\|_{Y^*} = 1 \quad \text{and} \quad y_0^*(Tx) = \|Tx\|_Y$$

Evaluating the norm of the adjoint functional $T^*y_0^*$ at x :

$$\|T^*y_0^*\|_{X^*} = \sup_{\|w\|=1} |(T^*y_0^*)(w)| \geq |(T^*y_0^*)(x)|$$

Substituting the definition of the adjoint and the Hahn-Banach functional:

$$|(T^*y_0^*)(x)| = |y_0^*(Tx)| = \|Tx\|_Y$$

Thus, $\|T^*y_0^*\|_{X^*} \geq \|Tx\|_Y$.

Returning to the evaluation of the operator norm $\|T^*\|$, because y_0^* is a unit functional, the supremum over all unit functionals must be at least the value achieved by y_0^* :

$$\|T^*\| = \sup_{\|y^*\|=1} \|T^*y^*\|_{X^*} \geq \|T^*y_0^*\|_{X^*} \geq \|Tx\|_Y$$

Because $\|T^*\| \geq \|Tx\|_Y$ for any unit vector x , evaluating the supremum over all $x \in X$ yields:

$$\|T^*\| \geq \sup_{\|x\|=1} \|Tx\|_Y = \|T\|$$

Because both $\|T^*\| \leq \|T\|$ and $\|T^*\| \geq \|T\|$ are established, it concludes that $\|T^*\| = \|T\|$. ■

Completeness of the Operator Space

Let X and Y be normed spaces, with $X \neq \{\mathbf{0}\}$. The space of all bounded linear operators $B(X, Y)$ constitutes a complete Banach space if and only if the codomain Y evaluates as a Banach space.

PROOF: The proof considers the forward and reverse implications independently.

If Y is complete, then $B(X, Y)$ is complete: Assume Y is a Banach space. Let (T_n) represent a Cauchy sequence of bounded linear operators in $B(X, Y)$. For any vector $x \in X$, the definition of the operator norm establishes the inequality

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\| \|x\|_X$$

Because (T_n) is a Cauchy sequence, as $n, m \rightarrow \infty$, $\|T_n - T_m\|_{B(X, Y)} \rightarrow 0$. Consequently, $(T_n x)$ is a Cauchy sequence in Y . Because Y is complete, this

sequence must converge to a defined limit in Y . The target operator $T : X \rightarrow Y$ is defined pointwise by this limit:

$$Tx = \lim_{n \rightarrow \infty} T_n x$$

Verification of Linearity and Boundedness Because limits preserve linear operations, the operator T is inherently linear. Furthermore, because Cauchy sequences are bounded, a constant M exists such that the bound $\|T_n\| \leq M$ holds for all indices n . Thus, $\|Tx\|_Y \leq M\|x\|_X$ and so that T is a bounded operator ($T \in B(X, Y)$).

Verification of Operator Norm Convergence It must be demonstrated that $T_n \rightarrow T$ with respect to the operator norm. Because (T_n) is Cauchy, for any $\epsilon > 0$, an index N exists such that for all $n, m \geq N$:

$$\|T_n - T_m\| < \epsilon$$

Applying this specific bound to an arbitrary vector x :

$$\|T_n x - T_m x\|_Y \leq \epsilon \|x\|_X$$

Fixing the index n and evaluating the limit as $m \rightarrow \infty$ yields:

$$\|T_n x - Tx\|_Y \leq \epsilon \|x\|_X \quad \text{for all } n \geq N$$

Extracting the supremum over all unit vectors $x \in X$ produces the strict bound $\|T_n - T\| \leq \epsilon$. Thus, $T_n \rightarrow T$ strictly in the operator norm, proving $B(X, Y)$ is complete.

If $B(X, Y)$ is complete, then Y is complete We will need to borrow a result in [Chapter 5](#) to prove this. Assume $B(X, Y)$ is a Banach space. Let (y_n) be a Cauchy sequence in Y . Because $X \neq \{0\}$, a non-zero vector $x_0 \in X$ can be selected. By Corollary 5.12 of the Hahn-Banach Theorem, a bounded linear functional $f \in X^*$ exists satisfying the condition $f(x_0) = 1$. A specific sequence of linear operators $T_n : X \rightarrow Y$ is defined explicitly by the mapping:

$$T_n(x) = f(x)y_n$$

We calculate the operator norm corresponding to the difference:

$$\|T_n - T_m\|_{B(X, Y)} = \sup_{\|x\|_X=1} \|f(x)(y_n - y_m)\|_Y = \|f\|_{X^*} \|y_n - y_m\|_Y$$

Because (y_n) is a Cauchy sequence in Y , the sequence (T_n) is a Cauchy sequence in $B(X, Y)$. Because $B(X, Y)$ is complete, (T_n) converges to a limit operator $T \in B(X, Y)$. Evaluating this convergence at x_0 :

$$y_n = f(x_0)y_n = T_n(x_0) \rightarrow T(x_0)$$

Thus, (y_n) converges to $T(x_0) \in Y$ which proves Y is complete. ■

Isometry of the Canonical Embedding

Let X be a normed space and X^{**} be its double dual. The canonical embedding $J : X \rightarrow X^{**}$ defined by $J_x(x^*) = x^*(x)$ for all $x^* \in X^*$ is an isometry.

PROOF: To demonstrate that J is an isometry, we must prove that the operator norm of the functional J_x in the double dual equals the norm of the vector x in X , establishing $\|J_x\|_{X^{**}} = \|x\|_X$ for all $x \in X$.

The Upper Bound ($\|J_x\| \leq \|x\|$) By the definition of the operator norm in the double dual X^{**} :

$$\|J_x\| = \sup_{\|x^*\|_{X^*} \leq 1} |J_x(x^*)| = \sup_{\|x^*\|_{X^*} \leq 1} |x^*(x)|$$

Applying the fundamental inequality for bounded linear functionals, we know that $|x^*(x)| \leq \|x^*\|_{X^*} \|x\|_X$. For any functional situated in the closed unit ball ($\|x^*\|_{X^*} \leq 1$), this inequality simplifies to:

$$|x^*(x)| \leq (1)\|x\|_X = \|x\|_X$$

Because this bound holds for all such functionals, the supremum cannot exceed $\|x\|_X$. Thus, we establish the upper bound:

$$\|J_x\| \leq \|x\|_X$$

The Lower Bound ($\|J_x\| \geq \|x\|$) If $x = \mathbf{0}$, the equality $\|J_x\| = \|x\|_X$ holds trivially. For any non-zero vector $x \in X$, we rely on Corollary 5.12 of the Hahn-Banach Theorem. This corollary guarantees the existence of a bounded linear functional $f \in X^*$ such that:

$$\|f\|_{X^*} = 1 \quad \text{and} \quad f(x) = \|x\|_X$$

Evaluating the canonical functional J_x at this specific norming functional f yields:

$$|J_x(f)| = |f(x)| = \|x\|_X$$

Returning to the supremum definition of the norm $\|J_x\|$, because f is a unit functional, the supremum over all unit functionals must be at least the specific value achieved by f :

$$\|J_x\| = \sup_{\|x^*\|_{X^*} \leq 1} |J_x(x^*)| \geq |J_x(f)| = \|x\|_X$$

This establishes the lower bound:

$$\|J_x\| \geq \|x\|_X$$

Conclusion Because both the upper bound ($\|J_x\| \leq \|x\|_X$) and the lower bound ($\|J_x\| \geq \|x\|_X$) are established, it follows that $\|J_x\| = \|x\|_X$. This confirms that the canonical embedding J preserves distances and thus an isometry. ■

Boundedness of the Minkowski Functional

Let X be a normed space and $C \subset X$ be an open convex set containing the origin. The Minkowski functional $p : X \rightarrow \mathbb{R}$ defined by

$$p(x) = \inf\{\lambda > 0 : x \in \lambda C\}$$

is bounded. There exists a constant $M > 0$ such that $p(x) \leq M\|x\|$ for all $x \in X$.

PROOF: Because C is an open set containing the origin, there exists a radius $r > 0$ such that the open ball $B(0; r)$ is contained in C . For any non-zero vector $x \in X$, the scaled vector $\frac{r}{2\|x\|}x$ has a norm of $r/2$. This places the vector in $B(0; r)$, and therefore in C . By the definition of the Minkowski functional, if a vector $y \in C$, then $p(y) \leq 1$. Applying this to the scaled vector:

$$p\left(\frac{r}{2\|x\|}x\right) \leq 1$$

The Minkowski functional is positively homogeneous. Factoring out the positive scalar $\frac{r}{2\|x\|}$ yields:

$$\frac{r}{2\|x\|}p(x) \leq 1$$

Rearranging the inequality provides the explicit bound for $p(x)$:

$$p(x) \leq \frac{2}{r}\|x\|$$

Defining $M = \frac{2}{r}$, the inequality $p(x) \leq M\|x\|$ holds for all non-zero $x \in X$. For $x = 0$, $p(0) = 0 \leq M\|0\|$ holds trivially. Thus, the functional is bounded by a constant multiple of the norm. ■

Preservation of Convexity

Let X and Y be vector spaces and $T : X \rightarrow Y$ be a linear operator. If $C \subseteq X$ is a convex set, then the image $T(C)$ is a convex set in Y .

PROOF: To establish that $T(C)$ is convex, it must be shown that any convex combination of two points in $T(C)$ remains in $T(C)$.

Identify Pre-images Let $y_1, y_2 \in T(C)$. By the definition of the image, there exist vectors $x_1, x_2 \in C$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Apply Convexity of the Domain Let $t \in [0, 1]$ be a scalar. Since C is a convex set, the convex combination of x_1 and x_2 is contained in C :

$$tx_1 + (1 - t)x_2 \in C$$

Apply Linearity Evaluate the image of this convex combination under the operator T . Since T is linear, it preserves vector addition and scalar multiplication:

$$T(tx_1 + (1-t)x_2) = tT(x_1) + (1-t)T(x_2)$$

Substituting the established values y_1 and y_2 yields:

$$T(tx_1 + (1-t)x_2) = ty_1 + (1-t)y_2$$

Since $tx_1 + (1-t)x_2 \in C$, its mapped image $ty_1 + (1-t)y_2$ lies in $T(C)$. Thus, $T(C)$ is convex. ■

Preservation of Symmetry

Let X and Y be vector spaces and $T : X \rightarrow Y$ be a linear operator. If $S \subseteq X$ is a symmetric set with respect to the origin, then the image $T(S)$ is a symmetric set in Y .

PROOF: To establish that $T(S)$ is symmetric, it must be shown that for any point $y \in T(S)$, its negation $-y$ is also in $T(S)$.

Identify Pre-image Let $y \in T(S)$. By definition, there exists a vector $x \in S$ such that $T(x) = y$.

Apply Symmetry of the Domain Because S is a symmetric set, the negation of any vector in S is also contained in S . Thus, $-x \in S$.

Apply Linearity Evaluate the operator T applied to this negated vector. Because T is linear, it preserves scalar multiplication, including multiplication by -1 :

$$T(-x) = -T(x) = -y$$

Since $-x \in S$, its mapped image $-y$ lies in $T(S)$. Thus, $T(S)$ is symmetric. ■

Epilogue

The foundational exploration of functional analysis presented within this text reaches its analytical conclusion. Throughout these chapters, the rigorous transition from the finite-dimensional structures of classical linear algebra to the topological complexities of infinite-dimensional spaces has been systematically established.

The mathematical journey commenced with the formulation of fundamental metric topologies and culminated in the rigorous proof of the Four Pillars of functional analysis. The Hahn-Banach Theorem, the Uniform Boundedness Principle, the Open Mapping Theorem, and the Closed Graph Theorem have been formalized not merely as abstract analytical properties, but as the indispensable structural laws governing the behavior of continuous linear operators. Furthermore, the Banach-Alaoglu Theorem provided the necessary topological machinery to recover mathematical compactness within these vast spaces via the weak* topology.

However, this text serves strictly as an introductory foundation. The analytical landscape of functional analysis extends far beyond these preliminary bounds. The rigorous study of spectral theory, the analysis of unbounded operators essential to the formulation of quantum mechanics, the structural classification of Banach algebras, and the broader generalizations found within non-linear functional analysis constitute the natural mathematical continuation of this study. The infinite-dimensional framework established within these pages provides the exact topological tools required to navigate those advanced domains.

It is intended that the synthesis of algebraic structure and geometric intuition developed throughout this text will serve as a permanent, rigorous analytical foundation for the reader. The structural elegance of functional analysis demonstrates a profound mathematical unity, where discrete algebraic operations and continuous topological spaces converge flawlessly.

Paul L. Dayao

Soli Deo Gloria



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