

TEXTBOOK

Jun-Yi Guo

# Physical Geodesy

Global Elastic Deformation  
of the Earth



Springer

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Global Elastic Deformation of the Earth



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*In memory of my wife Xiaolan  
and  
To my children Yuanyuan and Manman*

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## Preface

During the last half-century, with the advancement of radio, satellite, and information technologies, geodesy has transitioned from a discipline of science mainly relying on ground-to-ground and ground-to-space optical observations (of next importance are ground distance and gravity measurements) to a discipline of science mainly relying on ground-to-space, space-to-ground, and space-to-space radio observations (of next importance are ground and airborne gravity measurements). A large amount of space geodetic data has been collected, and the amount is growing more and more rapidly. These data are extremely accurate that monitoring changes of the Earth's shape, gravity, and rotation became a routine task. Therefore, studying the relevant Earth system properties and processes using these data became an integrated and significant part of geodesy. The contents of this book are expected to serve as a framework of fundamental theories for this kind of studies, and belong to the essential background physics of geodesy nowadays.

The main focus of this book is on the theory of elastic deformation and rotation of the solid Earth in response to external forcings, with an emphasis on the present-day mass transport in the hydrosphere-cryosphere-atmosphere system, which is closely related to climate changes. This is the underlying theory to invert data of changes of Earth's shape, gravity, and rotation for mass transport in the atmosphere-hydrosphere-cryosphere system, which could be in turn used as data in climate studies. Gravity and shape of the Earth are essential for the deformation theory, and are included as a chapter. Elastic deformation of the Earth is also the main concern of seismology. Particularly, the structure, density, and elastic parameters of the Earth are determined using seismic data. Therefore, a chapter on seismology and its application to determine the structure, density, and elastic parameters of the Earth is included, with a particular emphasis on the free oscillation, which is also a kind of global elastic deformation of the Earth. Central to the main focus of this book is the formulation of the Love-Shida numbers of the spherical elastic Earth model not found elsewhere in both comprehensiveness and detail, including the method of computation. Major applications of the theory to interpret data are addressed.

It is my hope that this book serve as a self-sufficient textbook, which is understandable stand-alone. Particular care is taken for the need of students who desire to understand the theories thoroughly starting from the first principles. Hence, the formulations are presented in extensive detail starting from very fundamental mathematics and physics. Results without the detail of derivation are cited only in a

few instances, which are less relevant to the scope of this book, and the readers are referred to other references for derivations.

The readers are required to have a good understanding of single and multivariable calculus and some elementary knowledge of ordinary differential equations and matrix algebra. This is a mathematical background similar to that acquired by most physical science or engineering undergraduates during their freshman and sophomore years. Mathematics beyond are included in the book in detail, most notably, vector and tensor analysis, spherical harmonics, as well as the methods and implementations of numerical computations concerned. The readers are also required to have taken a course of calculus-based mechanics, preferably a course of *Classical Mechanics*. Relevant knowledge of physics including elasticity and solid body rotation is explained from first principles without jump of logic or derivation. The readers who are familiar with the mathematics and physics mentioned can use the texts for refreshing the knowledge and as references for subsequent texts of the book. The readers are not required to have any background knowledge in Earth sciences; relevant Earth science knowledge is included in the brief first chapter of the book to the depth required.

I also expect this book to serve as a reference for researchers. Hence, a compromise is made in various subjects between the need for the self-containdness of this book and that required in understanding the relevant literature, particularly in Chaps. 2 to 4 dedicated largely to mathematics and physics. To the author's knowledge, this is the first book in this discipline that brings mathematics, physics, and geophysics together in such a detail and depth starting from very fundamental undergraduate mathematics and physics.

I wrote programs in C/C++ for verifying formulations and algorithms and for producing numerical values, tables, and figures in the book. These programs, together with some programs written for my research, are arranged into two packages, PREM-PLUS and GRACE-L2-MASS, the former for computing additional physical quantities of PREM-like Earth models (i.e., spherically symmetrical elastic Earth models like the Preliminary Reference Earth Model), and the later for studying mass changes on the Earth's surface using GRACE/GRACE-FO Level 2 data (i.e., the monthly gravitational potential in the form of spherical harmonic series of the Gravity Recovery and Climate Experiment satellite mission and its follow-on). These programs are available through GitHub:

- <https://github.com/jun-yi-guo/PREM-PLUS>
- <https://github.com/jun-yi-guo/GRACE-L2-MASS>

The readers choosing to run the programs can easily relate the relevant ones to the texts in the book. These programs are unparalleled resources for readers to understand the relevant texts to the exact details and to write their own programs.

The origin of this book traces back to a course of geophysics I taught to students of geodesy at the Wuhan Technical University of Surveying and Mapping (now a part of Wuhan University) during the 1990s. The lecture notes, after several revisions, led to the publication of a book in 2001 in Chinese. In an occasion in 2004,

I have shown this book to Professor Michael Rochester, Memorial University of Newfoundland, and Professor David Crossley, Saint Louis University. Both of them applauded the idea of including the relevant mathematics, physics, and seismology to the detail and depth in the book whose main subject is the Love number theory, and suggested me to put this book into English. I have been an enthusiastic reader of their publications since I started graduate study, and share the common background knowledge of global geophysics with them. I have certainly kept their suggestion in mind through the years. At present, the subjects covered become much more important in geodesy, but there is no book in English that presents these subjects systematically. Therefore, I decided to rewrite and comprehensively update it, resulting in the finalization of this book.

I am grateful to many who assisted me in the learning of the knowledge involved and in the preparation of this book. I am most grateful to my doctoral advisor, Professor Paul Melchior, Royal Observatory of Belgium, who is acknowledged as “the Father of Earth tides” in the geodetic community. As his student, mastering the theory of the Love numbers, which is the primary concern of this book, is certainly a must. Professor Paul Melchior’s enthusiasm for science and his attractive personality made the Royal Observatory of Belgium a very welcoming international center, where many people helped me in all aspects of my study, research, and personal life. Particularly, Professor Paul Pâquet has virtually taken my dad’s role. These experiences were the most important that shaped me as a researcher and teacher. I have learned the fundamentals of gravity, spherical harmonics, Earth tide, and Love numbers from the undergraduate course *Earth’s shape and external gravity field* taught by Professors Zeling Guan and Jinsheng Ning at the Wuhan Technical University of Surveying and Mapping, which were the most important background knowledge for my graduate study. During my early career, I received immense help from Professor Houze Xu, Wuhan Institute of Geodesy and Geophysics, who is a world-renowned expert in Earth tide. He was undoubtedly the one who knew best what I knew among my teacher’s generation in China. I have always felt free to ask questions to him, and occasionally argue with him; he was a well-known friend of the youngs. During the last few years, I received constant support and encouragement from my geodesy colleagues at The Ohio State University, Professors Michael Bevis and C.K. Shum, and from the Director of the School of Earth Sciences, Professor Steven Lower. I received corrections from several colleagues and students, most notably, Professor Chengli Huang, Shanghai Astronomical Observatory, who provided a long list of corrections covering the entire original book in Chinese. I thank all these generous people, as well as many others, who have provided me with corrections, comments, suggestions, challenges, supports, and encouragements over the years.

Columbus, OH, USA  
July 2025

Jun-Yi Guo

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## Declarations

**Competing Interests** The author has no competing interests to declare that are relevant to the content of this manuscript.

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# Background Knowledge

# 1

## Abstract

This chapter provides some elementary knowledge of geophysics, geodesy and astronomy for the readers of diverse background to understand the rest of the book. The focus is to provide a brief description of the necessary terminologies concerning the internal structure of the Earth, plate tectonics, geodetic coordinates, celestial coordinates, precession-nutation, polar motion, time systems, and spherical trigonometry.

## 1.1 The Restless Earth

The knowledge of the Earth's deep interior is attributed to the development of theory and observation of seismic waves. In the sense of spherically symmetrical structure, the Earth's interior is very well known. Preliminarily, the Earth is composed of, from the surface to the center, the ocean, the crust, the mantle, the outer core and the inner core. The ocean and outer core are fluid, and all the other layers are solid. The core-mantle boundary (CMB) is about 2890 km deep; the inner core boundary (ICB) is about 5150 km deep; and the Earth center is about 6371 km deep, which is the radius of the Earth. There is a major discontinuity within the mantle at the depth 670 km, which is the largest focal depth of Earthquakes observed. This discontinuity separates the mantle into an upper mantle and a lower mantle. Other discontinuities also exist, such as those with depths of 220 and 400 km. Furthermore, some thin transition layers exist, such as the  $D''$  layer just above the CMB. In spherical models, the Earth is divided into concentric layers. The physical parameters such as the density and elastic parameters (elasticity will be defined in the paragraph after the next, the elastic parameters describe how strong the body resists deformation in response to a forcing) are functions of the distance to the center, which are continuous in each layer, and may have a jump across the boundaries between adjacent layers. The principle of the determination of the

spherically symmetrical model of the Earth's interior using seismic data will be the subject of Chap. 6. An example of a such model is the Preliminary Reference Earth Model (PREM) (Dziewonski & Anderson, 1981). Studying the deformation of this kind of Earth model is the main topic of this book.

Besides the ocean that covers only a part of the Earth, the crust is the most irregular layer with a thickness varying from 0 up to 100 km. The crustal thickness in continents is mostly from 30 to 65 km with an average of about 35 km, and that in open oceans is mostly in the range from 5 to 8 km with an average of about 7 km. The crust-mantle boundary is referred to as Mohorovičić discontinuity, or briefly Moho, to credit the seismologist Mohorovičić who discovered it. In spherical Earth models, all layers are averaged globally.

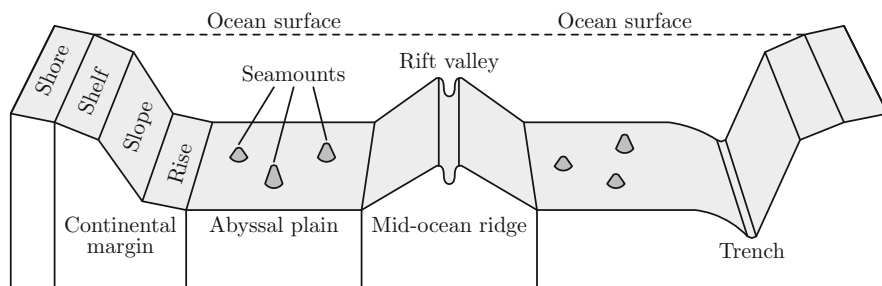
The outermost cold and hard layer of the Earth considered to be rigid is referred to as lithosphere, which includes the crust and a part of the mantle, and the layer below considered to be able to slowly flow is referred to asthenosphere. The term *rigid* is used as an antonym of the term *to be able to slowly flow*. The lithosphere is also often defined as an elastic solid layer. An elastic body is defined such that, when a force is applied to it, it deforms and stays as is in the deformed state without further motion, and after the force is removed, it returns to its initial shape and size.<sup>1</sup> In the sense used for defining the lithosphere, the terms *rigid* and *elastic* have the same meaning. Hence, we can say alternatively that the lithosphere is the outermost cold and hard layer of the Earth that can purely elastically support forcing without further flowing. As whether a forcing can be supported purely elastically depends on its distribution, magnitude and time span of action, the lithospheric thicknesses—also referred to as elastic thickness of the lithosphere to be more specific—for different geophysical phenomena are different. For the action of the weight of seamount and topography, the lithospheric thickness is mostly in the range of 10 to 35 km, meaning that the lithosphere cannot support huge mountains elastically without slowly flowing. For the motion of tectonic plates to be discussed later in this section, the oceanic lithosphere is 0 (mostly 50) to 140 km thick, and the continental lithosphere is 40 to 280 km thick. The bottom of the asthenosphere is not well defined.

About 71% of the Earth's surface is covered by oceans, and only 29% is land higher than the ocean surface. There are marshes and mountains over the continents, which are easily visible. The highest mountain peak reaches 8.8 km. The oceanic bottom is composed of mainly continental margins, abyssal plains (also referred to as ocean basins) and mid-ocean ridges, as demonstrated conceptually in Fig. 1.1.

The region between the shoreline and the flat ocean bottom is called continental margin. A typical continental margin is composed of three components, as shown at the left side of Fig. 1.1. The shallow water region adjacent to the shoreline is called continental shelf. Its width varies from place to place, ranging from 0 km to 1300

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<sup>1</sup> Strictly speaking, the application and relief of the force should be extremely slow, so that the elastic body remains in quasi-equilibrium state. A sudden forcing such as a shock would cause the elastic body to vibrate.



**Fig. 1.1** A conceptual view of an ocean bottom from one continent to another

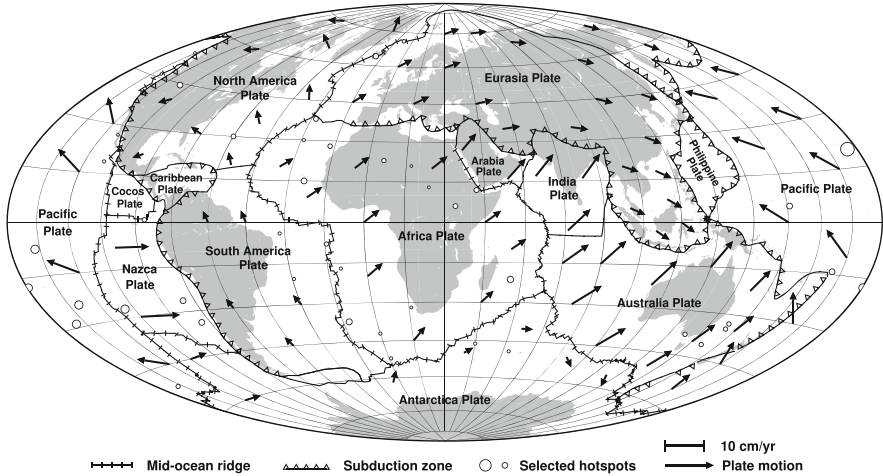
km with an average of about 75 km; its slope is on the order of  $1/1000$ ; and the water depth at the outer edge is about 130 m. Beyond the continental shelf is a very steep continental slope of about 3 km high, typically with a slope ranging from  $1/40$  to  $1/6$ , and a width from tens to hundreds of kilometers. Below the continental slope is a slightly slope region of sediments called continental rise, typically with a slope on the order of  $1/700$  to  $1/50$ , and a width on the order of hundreds to even thousands of kilometers. The continental margins around the Atlantic ocean, the Indian ocean, and the Arctic oceans are all in this form. Around the Pacific ocean, the continental margin does not include a continental rise, and right beyond the continental slope is a steep and deep trench, as shown at the right side of Fig. 1.1. The deepest trench reaches a depth of about 11 km. For geophysicists, besides the land higher than the ocean surface, a continent includes also the continental shelf extended into below the water. In this way, the oceans occupy about 65% of the Earth's surface, and the continents about 35%.

Beyond continental margins are abyssal plains and mid-ocean ridges, also shown in Fig. 1.1. Abyssal plains are very flat, with a slope normally smaller than  $1/1000$ , and a water depth of about 4 km. Seamounts of heights ranging from 0.5 to 4 km exist over abyssal plains. Mid-ocean ridges are extremely large scale mountains of very mild slope across the middle of oceans, with widths reaching the scale of 1500 km. The typical water depth on top of mid-ocean ridges is on the order of 2 to 3 km. In the middle of some mid-ocean ridges, such as the one across the Atlantic ocean, there exists a rift valley of 1 to 2 km deep and varying width from several to over a hundred kilometers.

By comparing the heights of mountains reaching 8.8 km, the depths of trenches reaching 11 km, and the radius of the Earth of 6371 km, we realize that the surface of the Earth is in fact very even.

The most important process of the Earth's interior associated with the displacement of the crust is the plate tectonics. The discovery of the plate tectonic motion is a very important event in Earth science in the 20th century. It was only after the advent of the plate tectonics when Earth scientists have begun to really correctly explain the various phenomena of the large scale motion of the outermost layer of the solid Earth.



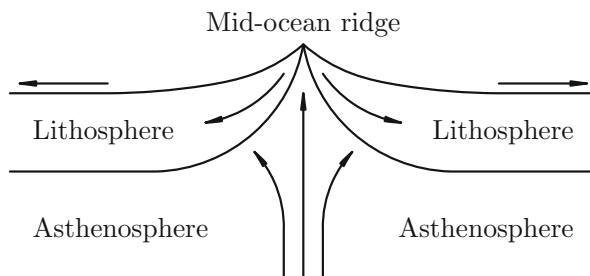
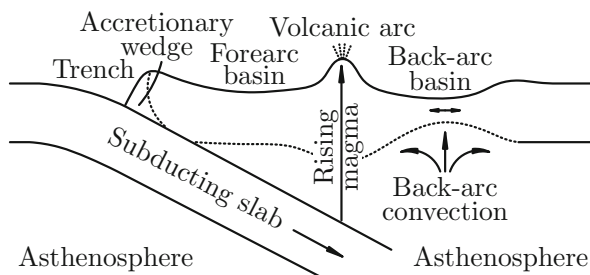


**Fig. 1.2** Major plates and their relative motion

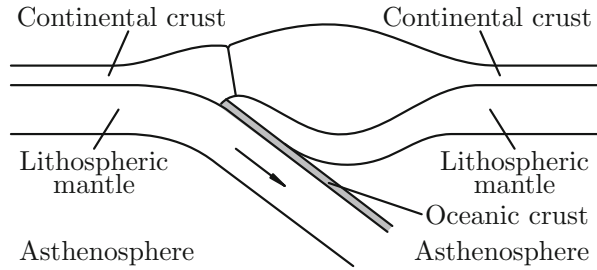
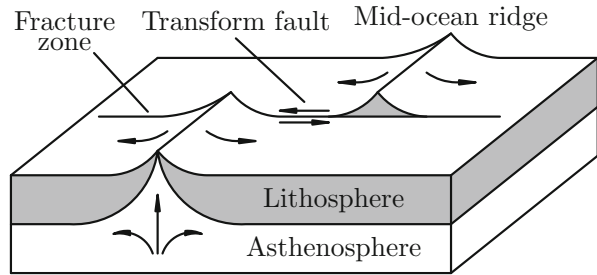
We have already mentioned that the outermost layer of the Earth, which is cold and is considered to be rigid in long time scale, is the lithosphere, and the layer below is the asthenosphere. Although the material in the asthenosphere is still considered to be solid, it is soft enough with a temperature not far from the solidus temperature, and can flow slowly. The plate tectonics states that the hard lithosphere is broken into several plates, and are moving one with respect to another, just like broken ices moving one piece with respect to another over the water of a lake. A global model of the major plates as well as their relative motion is shown in Fig. 1.2 for demonstrative purpose, which is based on the no-net-rotation (NNR) model NUVEL-1A (DeMets et al., 1994) computed using the Plate Motion Calculator provided over the internet by Okino.<sup>2</sup> The meaning of NNR is to be explained in the next section. Three concepts, mid-ocean ridge, subduction zone and hotspot are concerned in this figure, which are to be explained in the next paragraphs. The mid-ocean ridge has already been briefly mentioned previously.

The boundaries between plates are classified as divergent, convergent and transform based on how the two plates move one with respect to another. The mid-ocean ridges are divergent boundaries, as shown in Fig. 1.3, where the two plates at both sides move away from the ridge, and the magma upwells from the asthenosphere to form new lithosphere while cooling down, causing lithosphere to grow. This phenomenon is called seafloor spreading. We realize immediately that the thicknesses of the crust and lithosphere in the middle of the ridges are nearly 0. There are two types of convergent boundaries. One is called subduction zone, as shown in Fig. 1.4, where one oceanic plate submerges into below another oceanic

<sup>2</sup> [http://ofgs.aori.u-tokyo.ac.jp/~okino/platecalc\\_new.html](http://ofgs.aori.u-tokyo.ac.jp/~okino/platecalc_new.html)

**Fig. 1.3** Mid-ocean ridge**Fig. 1.4** Subduction zone

or (mostly) continental plate, causing the lithosphere to diminish. The subducting plate is often referred to as subducting slab. Subduction zones are Earthquake-frequent, and are found by analyzing seismic wave data, thus also being referred to as Wadati-Benioff zone to credit the seismologists who discovered them. Besides the subducting slab and the trench, a subduction zone is often associated with a volcanic arc (when an oceanic slab subducts down to a continental slab) or island arc (when an oceanic slab subducts down to another oceanic slab), a forearc basin, a back-arc basin, and an accretionary wedge of sediment scrapped from the oceanic plate. The back-arc basin is also referred to as marginal basin. The ridge at the edge of the forearc basin next to the trench is referred to as outer-arc ridge, which could be the accretionary wedge, the uplifted plate edge, or a combination of both. The model shown in Fig. 1.4 is illustrative. In reality, some components, such as the forearc basin and back-arc basin, are not always present, and some less common features not shown in Fig. 1.4 may be present. The dotted lines indicate that the reality there may be very complicated. The other converging boundary is the continent-continent or continent-island arc collision zone, as demonstrated in Fig. 1.5. This kind of collision happens during the subduction when a continent meets another continent or an island arc. Associated with such a collision is a process called orogeny, in which a section of the crust is folded and deformed by lateral compression to form a mountain range. The Himalaya is the product of the orogeny in the still ongoing collision between the Indian plate and the Eurasian plate, and the Alps, the result of collision between the African plate and the Eurasian plate. The transform boundary is also called transform fault, where two plates slide one with respect to another. The most common are those at the offsets in the mid-ocean ridges, as demonstrated in Fig. 1.6.

**Fig. 1.5** Continental collision**Fig. 1.6** Transform fault

The motion of a continent associated with the plate tectonic motion is referred to as continental drift. The shelf-slope-trench type continental margin is associated with a subduction zone, and is referred to as active continental margin (also called Pacific-type or Andean-type continental margin). The shelf-slope-rise type continental margin is not associated with a subduction zone, and is referred to as passive continental margin (also called Atlantic-type continental margin).

Due to the plate tectonic motion, oceanic lithosphere has been generated at mid-ocean ridges and recycled at subduction zones. The oldest oceanic lithosphere is on the order of 0.17 billion years, while the oldest continental lithosphere is on the order of 3.8 billion years. The speed of seafloor spreading is on the order of 3 cm per year, and about 2 km<sup>2</sup> lithosphere is generated and recycled every year.

Besides the plate tectonic motion that is the most known, the Earth is undergoing other deformations. The first to mention is the Earth tide. As the name implies, Earth tide is also caused by the lunisolar gravitational attraction as the ocean tide (other celestial bodies in the solar system also contribute a little). More specifically, Earth tide and ocean tide are caused by the difference of the strength and direction of the lunisolar gravitational attraction at different locations on the Earth; while observed on the Earth, this difference acts like a stretching or compressing force that causes motion or deformation. The motions in the atmosphere-hydrosphere-cryosphere system, including the ocean tide, induce a change of the system's gravitational attraction to the interior of the Earth and its pressure on the surface of the Earth, thus deforming the Earth. This phenomenon is referred to as loading. When the present-day variation in a time scale up to decades is concerned, the Earth behaves like an elastic body and the term elastic loading is used. In much longer time scale such as the glacial-interglacial cycles on the order of 120 thousand years, the lithosphere

still behaves like an elastic body, but the mantle below behaves like a Maxwell viscoelastic body. When a force is applied to and remain on a Maxwell viscoelastic body, the body first deforms like an elastic body and then flows like a viscous fluid; a parameter called viscosity is used to describe how strong the body is against the flow in response to the forcing. The last glacial cycle attained its maximum about 20 thousand years ago, and ended about six thousand years ago. The procedure of Earth deformation associated with the glacial-interglacial cycles is referred to as glacial isostatic adjustment (GIA), which mainly consists of an elastic deformation of the lithosphere and a viscous flow of the mantle below to achieve an equilibrium. The GIA signal observed at present is mainly the on-going rebound from the very large deformation caused by the last glacial cycle, and is referred to as post-glacial rebound (PGR). It mainly consists of an elastic bouncing-back of the lithosphere accompanied by a viscous flow in the mantle below. At present, to a time scale of up to decades, the PGR is a motion with constant speed, or the deformation is linear in time. Earth tide and elastic loading will be the subject of Chap. 7. More about GIA will be provided in Sect. 9.2.6.

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## 1.2 Terrestrial Reference System and Reference Earth Ellipsoid

The Earth is rotating. An apparent demonstration of the rotation of the Earth that we see every day is the rising from east and setting to the west of the Sun, the Moon and other stars. This is the rotation of the Earth with respect to the inertial space, or the celestial sphere represented by the directions of distant celestial objects to be defined in the next section. The physical quantity describing this rotation is the angular velocity of the Earth with respect to the inertial space. Two concepts are defined for describing the rotation of the Earth: the rotation axis and rotation rate. The rotation axis is parallel to the angular velocity and pass through the center of mass of the Earth system bounded by the top of the atmosphere. The rotation rate is the angle that the Earth rotates per unit time around the rotation axis. Evidently, the rotation axis is the direction of the angular velocity tied to the center of mass, and the rotation rate is the magnitude of the angular velocity. It is found based on long term observations that the rotation of the Earth is very stable; the variation of the rotation rate and that of rotation axis with respect to the Earth itself is very small.

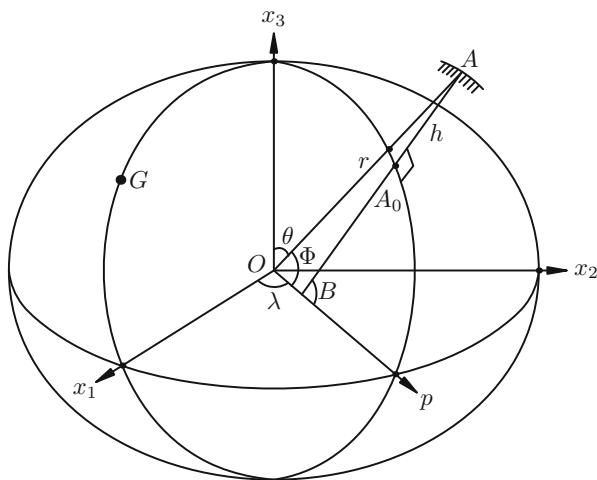
The Earth is approximately spherical. A visible phenomenon of the sphericity of the Earth is that the height of the Polaris (i.e., the North Star, which appears to be not moving, while other stars are doing circular motions around it as a result of the rotation of the Earth) above the horizon is different at different places in the South-North direction. When the shape of the Earth is concerned, a very important concept is the geoid, which is defined as a closed surface perpendicular to the plumb line everywhere, and is customarily chosen to be the nearest possible to the mean surface of the oceans. The height above sea level familiar to the public is in fact the height above the geoid. Therefore, if the surface of the Earth is understood as a geometrical figure of the Earth, the geoid is understood as a physical figure of the Earth. Although the shape of the geoid is not as complicated as that of the

geometrical surface of the Earth, it is still very complicated. However, the geoid is very close to an oblate ellipsoid of revolution, whose minor axis is the symmetry axis, and is extremely close to the rotation axis of the Earth.

The terrestrial reference system (TRS) used to express positions on the Earth is defined based on the facts described in the previous two paragraphs. The positions can be expressed using rectangular coordinates, spherical coordinates or geodetic coordinates, which are to be detailed below.

We first assume that the Earth is rigid, i.e., its shape does not vary with time, and that the rotation axis does not vary with respect to the Earth itself. In this simplified situation, we define the TRS as the Earth-fixed rectangular coordinate system  $Ox_1x_2x_3$ , such that the origin is on the center of mass of the Earth system, the  $x_3$ -axis coincides with the rotation axis and points to the North, the  $x_1$ -axis is conventionally chosen, and the  $x_2$ -axis is defined according to the right-hand rule. The spherical coordinate system  $Or\theta\lambda$  is defined by the rectangular coordinate system  $Ox_1x_2x_3$  in the usual way, as shown in Fig. 1.7.

The geodetic coordinates are defined based on a reference Earth ellipsoid chosen as a first approximation of the Earth. The reference Earth ellipsoid is defined to be an oblate ellipsoid of revolution. Its center is on the center of mass of the Earth system. Its symmetry axis is its minor axis and coincides with the rotation axis, i.e., the  $x_3$ -axis. Its size and shape are determined using observations based on the criterion that the surface of the reference Earth ellipsoid is best fit to the geoid. The intersections of the symmetry axis with the surface are called terrestrial poles. The one at the North is called north terrestrial pole, and the one at the South is called south terrestrial pole. The plane perpendicular to the symmetry axis and passing through the center is called terrestrial equatorial plane. Its intersection with the surface of the reference Earth ellipsoid is called terrestrial equator. A half plane with



**Fig. 1.7** Rectangular, geocentric, and geodetic coordinates

the symmetry axis as the edge is called a terrestrial meridional plane. Its intersection with the surface of the reference Earth ellipsoid is called a terrestrial meridian. We denote the semi-major axis of the reference Earth ellipsoid, i.e., the distance from its center to a point on the equator, using  $a$ , and its semi-minor axis, i.e., the distance from its center to a pole, using  $b$ . Evidently, the equator is a circle of radius  $a$ , and a meridian is a half ellipse of semi-major and semi-minor axes  $a$  and  $b$ . Another line of importance on the surface of the reference Earth ellipsoid is the prime vertical. For a point on the reference Earth ellipsoid, the prime vertical is defined as the intersection line between the reference Earth ellipsoid and the plane perpendicular to the terrestrial meridian.

The geodetic coordinates as shown in Fig. 1.7, the geodetic longitude  $\lambda$ , the geodetic latitude  $B$  and the geodetic height  $h$ , are used to express positions around the surface of the reference Earth ellipsoid. The geodetic longitude of a point is the angle between the meridional plane through this point and that through the  $x_1$ -axis. The geodetic latitude of a point is the angle between the normal to the surface of the reference Earth ellipsoid and the terrestrial equatorial plane. The geodetic longitude is positive to the East, and the geodetic latitude is positive to the North. The meridian through the  $x_1$ -axis is referred to as prime meridian, which is conventionally chosen to be the one through Greenwich, called Greenwich meridian, and is shown in Fig. 1.7 as the point  $G$ . This should be understood as the definition of the  $x_1$ -axis. The geodetic height is the height above the surface of the reference Earth ellipsoid measured along the line perpendicular to this surface. The complementary angle of  $\theta$ ,  $\Phi$ , is called geocentric latitude. The line on the reference Earth ellipsoid connecting all points with the same latitude is a circle, called parallel. It is evident that the parallel and prime vertical through a point are tangential to each other.

In the above, we assumed that the Earth is rigid, and that the rotation axis does not vary with respect to the Earth itself. However, none of the two assumptions holds true in reality. The Earth's surface is ceaselessly moving one point with respect to another, and the direction of the rotation axis is also moving with respect to the Earth itself. Therefore, the definition of the TRS is much more complicated; the absolute criteria—the system is fixed on the Earth, the origin is on the center of mass of the Earth system, and the  $x_3$ -axis coincides with the rotation axis—must be replaced by relative criteria. Hence, there is some room of arbitrariness in the definition of the TRS. Nevertheless, the TRS must be reasonably fixed on the Earth, although the meaning of the term *Earth-fixed* can only be understood in a relative sense.

Conceptually, a relative Earth-fixity can be realized in the following way. We choose a globally distributed network of stations. Based on long term observations, the relative motion among all the stations are determined, and can be predicted. After removing the relative motion among the stations, we obtain a network of relatively fixed virtual stations. Naturally, we attach the TRS on this network of virtual stations. The coordinates of a real station are then the sum of the coordinates of the virtual station and the time-dependent component describing the relative motion with respect to the network of virtual stations. The  $x_3$ -axis is conventionally chosen to be the average of the Earth rotation axis during a certain period of time, and is referred to as conventional terrestrial pole (CTP). The position of the center

of mass of the Earth system and the parameters of the reference Earth ellipsoid are determined based on satellite observations. Once the reference Earth ellipsoid is chosen, the geodetic coordinates are defined.

In summary, a realization of the TRS is a conventional choice, and it is attached on the Earth via a network of stations. The rectangular, spherical and geodetic coordinate systems are mathematical tools to represent positions. We cannot get hold of them directly. What are accessible to us are the objects these coordinate systems are tied to, which are the stations in the network. These stations form a materialized frame, called terrestrial reference frame (TRF). The coordinates of an unknown point can be determined by measuring the relative position of this point with respect to the stations in the frame.

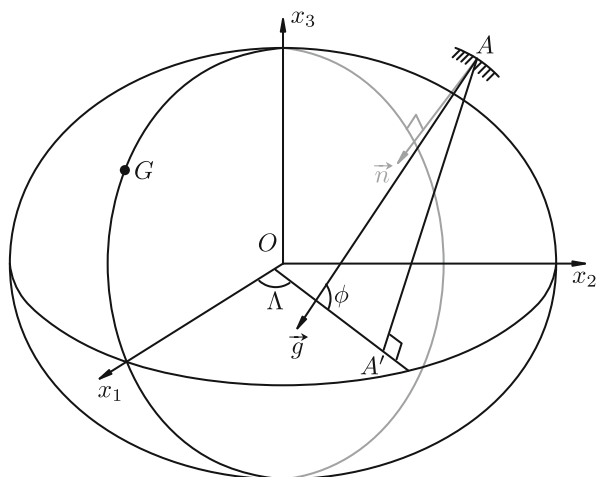
Some other terms related to TRS and TRF are often used in the literature. The TRS and TRF adopted by the International Earth Rotation and Reference Systems Service (IERS) are referred to as international terrestrial reference system (ITRS) and frame (ITRF), the pole as IERS (or international) reference pole (IRP), and the prime meridian as IERS (or international) reference meridian (IRM). A more accurate term for the TRS with origin on the center of mass of the Earth system is geocentric terrestrial reference system (GTRS), which is used interchangeably with ITRS. The IERS terminologies of reference systems became standard nowadays.

A reference system is understood to be three mutually orthogonal coordinate axes defined to satisfy a set of roles, and a reference frame is understood to be a network of stations with known coordinates. Therefore, a reference frame is a realization of a reference system, and can also be used to indicate the three orthogonal axes tied to the network.

The TRF can be understood as a fictitious rigid network of stations after removing the relative motion. However, removing relative motion does not uniquely determine the frame representing the rectangular coordinate system, as a rotation of the frame as a whole can be attributed to the velocity of the stations. Hence, an extra constraint is necessary. The most apparent choice is to minimize the squared sum of the velocity of all stations with respect to the TRF. However, as we do already know the relative motion of the Earth's surface to a reasonable extent, we should take this knowledge into consideration. Hence, what we actually minimize is the squared sum of the velocity of the stations after removing the known velocity, the largest component being the plate tectonic motion. The plate tectonic motion model used for this purpose is the no-net-rotation (NNR) model. As indicated by the name, NNR implies that the different plates do move with respect to one another, but they do not rotate as a whole. Hence, this model serves as a neat separation of the relative motion among the plates from an overall rotation.

The direction of the gravity at any point can also be expressed with respect to the reference Earth ellipsoid, as shown in Fig. 1.8. The angle between the direction of gravity and the equatorial plane,  $\phi$ , is called astronomical latitude, and that between the local astronomical meridional plane and the Greenwich meridional plane,  $\Lambda$ , is called astronomical longitude. The local astronomical meridional plane is the plane perpendicular to the equatorial plane and containing the direction of gravity. More specifically, the local astronomical meridional plane can be understood as the

**Fig. 1.8** Definition of the astronomical latitude  $\phi$  and longitude  $\Lambda$ , where  $AA'$  is perpendicular to the equatorial plane. Compare with the geodetic latitude and longitude defined by the normal of the reference Earth ellipsoid,  $\vec{n} = \overrightarrow{AA_0}$ , as shown in Fig. 1.7



plane defined by two straight lines through the station or point of interest, one being along the direction of gravity, and the other being perpendicular to the equatorial plane. The signs of the astronomical coordinates, i.e., the astronomical latitude and longitude, are defined in the same way as the respective geodetic coordinates.

### 1.3 Celestial Reference System and Celestial Coordinate Systems

In the vast universe, there are innumerable amounts of galaxies, and in each galaxy, there are millions to even trillions of stars. In our galaxy, there are about a hundred billion stars, and our Sun is only an ordinary one of them. The closest star to our Sun is about 4.22 light years away, i.e., it takes the light to travel 4.22 years to reach it. Hence, the solar system can be considered as a closed system as far as the dynamics of the objects within it is studied.

Our solar system is mainly composed of the Sun and eight planets orbiting it in virtually the same plane, which are, from inward to outward, Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and Neptune. In between the Mars and Jupiter is the asteroid belt, where a lot of asteroids composed of rock are orbiting the Sun. Beyond the Neptune is the Kuiper belt, where many objects of frozen volatiles, such as methane, ammonia and water, are orbiting the Sun. The Pluto, once known as a planets, is now considered to be one of the Kuiper belt objects. The planets can be classified in two categories: the four inner planets are smaller with larger density, and are referred to as terrestrial planets; the four outer planets are larger with smaller density, and are referred to as gaseous planets. The mass of the Sun is about 99.9% of the total mass of whole the solar system.

In order to describe the directions of celestial objects, we introduce the concept of celestial sphere, which is a fictitious sphere with an arbitrarily (or infinitely) large

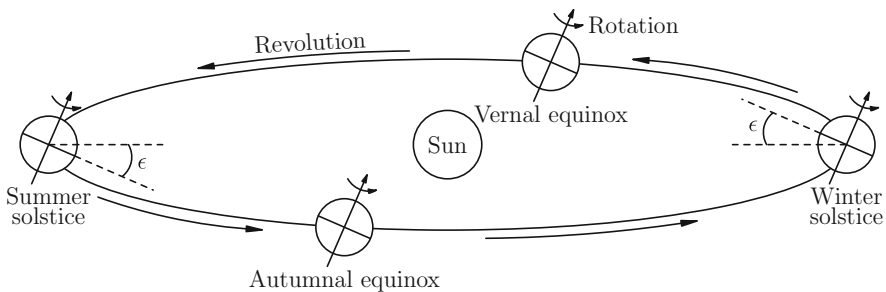


radius, so that only the direction matters for observers on the Earth, or within the solar system, depending on the problem studied. The center of the celestial sphere can be chosen as the observation station, the Earth center, or the center of the solar system, depending again on the problem studied. The position of a celestial object on the celestial sphere tells only the direction, but not the distance: any two parallel lines with a finite distance apart intersect the celestial sphere at the same point. Another property of the celestial sphere is that it is selected to be not rotating with respect to the inertial space, which is represented by three mutually orthogonal coordinate axes, called celestial reference system (CRS). The realization of the CRS by the directions of a set of extremely distant celestial objects assumed to be not moving one relative to another (or after removing relative motions in extreme high accuracy) is referred to as a celestial reference frame (CRF).

The Earth, or more exactly the Earth-Moon system, is revolving around the Sun. To a reasonable accuracy, the orbit of the Earth-Moon barycenter, i.e., the center of mass of the Earth-Moon system, can be considered to be within a plane, and is referred to as ecliptic plane. Figure 1.9 serves as an illustration of the Earth's rotation and revolution. The variation of the orientation of the ecliptic plane with respect to the celestial sphere is extremely small. It has been found based on observations that the variation of the direction of the Earth rotation axis with respect to the ecliptic plane is very slow and quite regular. We call the plane perpendicular to the Earth rotation axis as celestial equatorial plane. The angle between the ecliptic and celestial equatorial planes, called obliquity of the ecliptic, remains almost a constant,  $\epsilon = 23^\circ 27' = 23.45^\circ$ . The above observed facts serve as the background for the definitions of circles and points on the celestial sphere, based on which the celestial coordinate systems are defined.

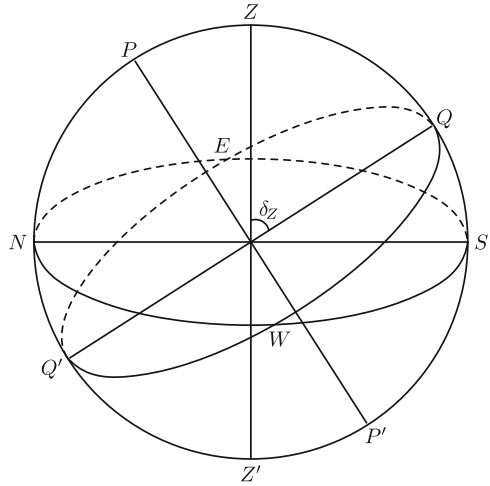
The circles and points defined on the celestial sphere, as shown in Figs. 1.10 and 1.11, serve as the basis for establishing celestial coordinate systems to describe the directions of celestial objects.

The direction of the gravity at the observation station intersects the celestial sphere at two points. The one above is called zenith, and is denoted as  $Z$ . The one below is called nadir, and is denoted as  $Z'$ . The plane perpendicular to  $ZZ'$  is called

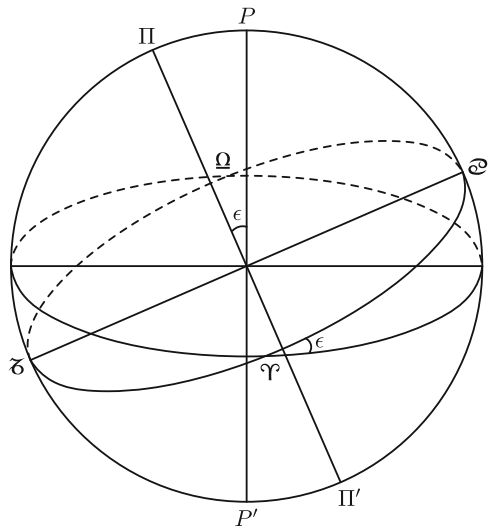


**Fig. 1.9** Earth's rotation and revolution

**Fig. 1.10** Relative positions of zenith, nadir, circle of the horizon, north celestial pole, south celestial pole and celestial equator



**Fig. 1.11** Relative positions of north celestial pole, south celestial pole, celestial equator, north ecliptic pole, south ecliptic pole and ecliptic



plane of the horizon, and its intersection with the celestial sphere is called circle of the horizon.

The Earth rotation axis intersects the celestial sphere at two points, called celestial poles. The one in the North is called north celestial pole, and is denoted as  $P$ . The one in the South is called south celestial pole, and is denoted as  $P'$ . The plane perpendicular to the rotation axis  $PP'$  is called celestial equatorial plane, which has already been mentioned. The intersection of the celestial equatorial plane with the celestial sphere is called celestial equator. The half-circle  $PZP'$  is called upper celestial meridian, and  $PZ'P'$  lower celestial meridian. The half-planes through the celestial meridians are called celestial meridional planes.

The intersection of the ecliptic plane with the celestial sphere is called ecliptic. The line perpendicular to the ecliptic plane intersects the celestial sphere at two points, called ecliptic poles. The one in the north is called north ecliptic pole, and is denoted as  $\Pi$ . The one in the south is called south ecliptic pole, and is denoted as  $\Pi'$ .

We show in Fig. 1.10 the relative positions of the zenith, the nadir, the circle of the horizon, the north and south celestial poles, and the celestial equator, and in Fig. 1.11 the relative positions of the north and south celestial poles, the celestial equator, the north and south ecliptic poles, and the ecliptic. By rotating Fig. 1.11 an angle  $\epsilon$  clockwise, the point  $\Pi$  is turned to the top, and we obtain a figure in correspondence with Fig. 1.9. We have chosen to draw  $P$  upright in Fig. 1.11 for the name of the angle  $\epsilon$ , i.e., obliquity of the ecliptic, indicating that the ecliptic plane is oblique.

We denote the elevation (or altitude) of the zenith  $Z$  above the celestial equatorial plane as  $\delta_Z$ , which is in fact the angle between the directions of the points  $Q$  and  $Z$ . It is the declination of the zenith as will be defined below.

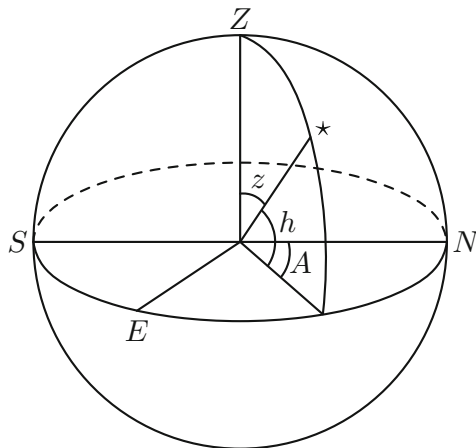
The obliquity of the ecliptic, i.e., the angle between the celestial equatorial and ecliptic planes, is also the angle between the celestial and ecliptic poles, and is denoted as  $\epsilon$  in Fig. 1.11. The celestial equator does move with respect to the ecliptic, or in other words, the celestial pole  $P$  does move with respect to the ecliptic pole  $\Pi$ . However, the obliquity of the ecliptic remains almost constant, which we understand here as a fact observed, implying that  $P$  moves around  $\Pi$  almost on a circle on the celestial sphere parallel to the ecliptic. This motion is westward, and is referred to as precession, which is the topic of the next section.

The center of the celestial sphere in both Figs. 1.10 and 1.11 could be understood as any point on the Earth. However, it is more convenient to understand the center of the celestial sphere in Fig. 1.10 as the observation station, and that in Fig. 1.11 as the Earth center. The direction  $ZZ'$  is fixed with respect to the Earth, thus it varies with time with respect to the celestial sphere following the Earth rotation.

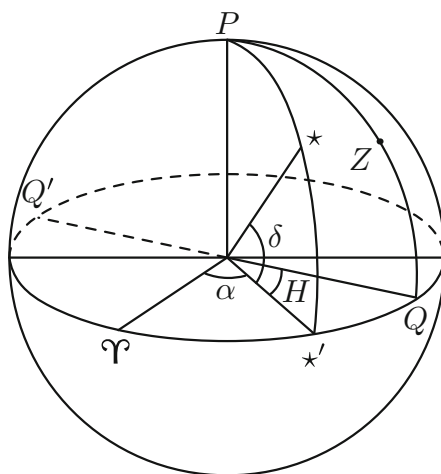
In Fig. 1.10, the circle of the horizon intersects the celestial equator at two points,  $E$  and  $W$ , which are called east point and west point, respectively. The two points  $90^\circ$  apart from  $E$  and  $W$  on the circle of the horizon,  $S$  and  $N$ , are called south point and north point, respectively. The intersection of the celestial equator with the upper and lower meridians,  $Q$  and  $Q'$ , are called upper and lower culminations of the equator, respectively.

In Fig. 1.11, the celestial equator and the ecliptic intersect at two points,  $\Omega$  and  $\Upsilon$ , which are the positions of the Sun on the celestial sphere at the autumnal (fall) equinox and vernal (spring) equinox, respectively. The two points  $90^\circ$  apart from  $\Omega$  and  $\Upsilon$  on the ecliptic,  $\varrho$  and  $\zeta$ , are the positions of the Sun on the celestial sphere at the summer solstice and the winter solstice, respectively. As seen from the north of the ecliptic, the motion of the Sun with respect to the Earth is counterclockwise in the cycle  $\Upsilon \rightarrow \varrho \rightarrow \Omega \rightarrow \zeta \rightarrow \Upsilon$ . The Sun crosses the equator from south to north through the vernal equinox, and from north to south through the autumnal equinox.

**Fig. 1.12** Horizon celestial coordinates



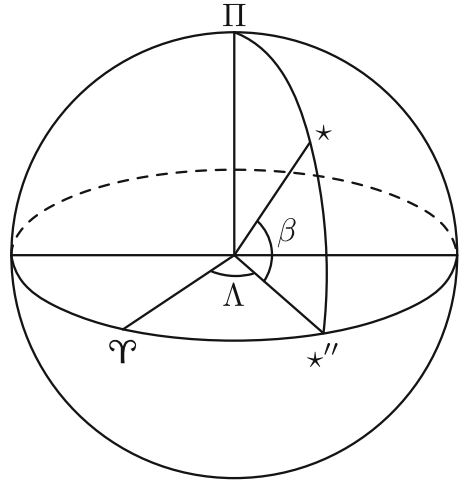
**Fig. 1.13** Equatorial celestial coordinates



Based on the points and circles defined above, several celestial coordinate systems can be defined to describe positions of celestial objects, such as a star denoted as  $\star$  in Figs. 1.12, 1.13, and 1.14.

Shown in Fig. 1.12 is an observation station-dependent celestial coordinate system with the circle of the horizon as reference circle, and the zenith as reference point, called horizon celestial coordinate system. The two coordinates are the azimuth  $A$  and the elevation (or altitude)  $h$ , the latter is also substituted by the zenith angle (or zenith distance)  $z$ . The azimuth is an angle measured in the plane of the horizon from the north toward the east. The altitude  $h$  is the height of the celestial object above the horizon measured as an angle (the angle between the direction of the celestial object and the plane of the horizon), and its complementary angle  $z$  is the angle between the celestial object and the zenith with respect to the center of the celestial sphere.

**Fig. 1.14** Ecliptic celestial coordinates



Shown in Fig. 1.13 are two coordinate systems with the celestial equator as the reference circle and the north celestial pole as the reference point, which are referred to as equatorial celestial coordinate systems. One coordinate system consists of the right ascension (or celestial equatorial longitude)  $\alpha$  and the declination (or celestial equatorial latitude)  $\delta$ , and the other the hour angle  $H$  and the declination  $\delta$ . The right ascension is the angle from the celestial meridional plane through the vernal equinox to that through the celestial object, positive toward the east, i.e., it is measured counterclockwise as seen from the north. The declination is the angle between the direction of the celestial object and the equatorial plane, positive if the celestial object is at the northern side of the equator. The hour angle is the angle from the celestial meridional plane through the observation station to that through the celestial object, positive toward west, i.e., it is measured clockwise as seen from the north. Evidently, the hour angle depends on the location of the observation station. In practice, the hour angle is also expressed using time, with  $360^\circ$  corresponding to 24 hours. By definition, the angle  $\delta_Z$  in Fig. 1.10 is the declination of the zenith.

Figure 1.14 shows a celestial coordinate system with the ecliptic as reference circle and the north ecliptic pole as reference point, which is referred to as ecliptic celestial coordinate system. The coordinates are the ecliptic longitude  $\Lambda$  and the ecliptic latitude  $\beta$ . The ecliptic longitude  $\Lambda$  is measured from the vernal equinox toward the east, i.e., counterclockwise as seen from the north, and the ecliptic latitude  $\beta$  is measured from the ecliptic plane to the north, i.e., positive at the north side, and negative at the south side.

We mention again that the azimuth, the zenith angle, and the hour angle depend on the location of the observation station.

According to Fig. 1.13, it can be realized that, for any celestial body, the sum of its right ascension  $\alpha$  and hour angle  $H$  is equal to the hour angle of the vernal equinox, which we denote as  $\Theta$ , and is referred to as local sidereal time. Hence we

have the relation

$$\Theta = \alpha + H. \quad (1.1)$$

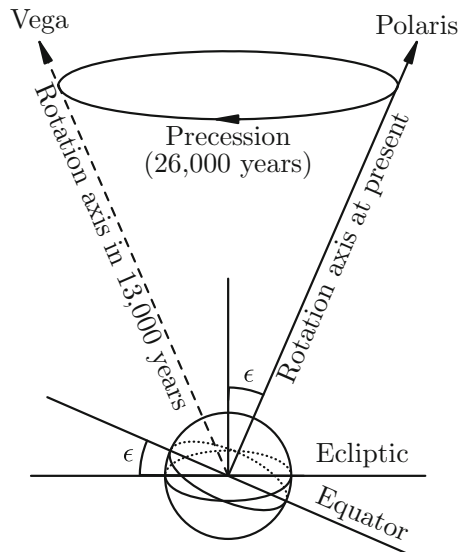
Although the local sidereal time  $\Theta$  is the sum of the right ascension and the hour angle of the celestial object, it actually is independent of the celestial object; it is the same for all celestial objects. As far as the hour angle is concerned, the location of the station must be known. Hence, we can understand that the station position is involved through the local sidereal time  $\Theta$ .

The ecliptic and the celestial pole can be defined differently for convenience of practical applications to reach higher accuracy, as will be elaborated in the next section. Nevertheless, whatever the alternative definitions are, the ecliptic should be very close to the orbital plane of the Earth-Moon barycenter around the Sun, and the celestial pole should be very close to the rotation axis of the Earth.

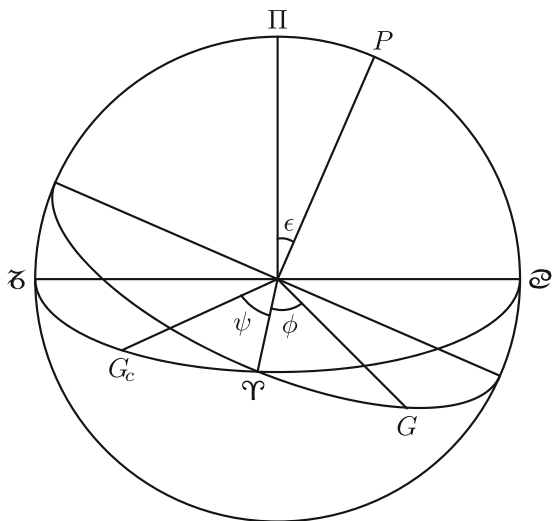
## 1.4 Precession-Nutation and Polar Motion

The variation of the orientation of the ecliptic with respect to the celestial sphere is extremely small. We first assume the ecliptic to be fixed on the celestial sphere, and describe the variation of the direction of the Earth rotation axis with the ecliptic as reference. The major component of the variation of the Earth rotation axis with respect to the ecliptic plane is a precession with a period of about 26,000 years, while the obliquity of the ecliptic remains practically fixed at about  $23.45^\circ$ , as shown in Fig. 1.15. The north celestial pole is currently pointing to the Polaris. Due to the precession, in 13,000 years, the north celestial pole will approximately be farthest

**Fig. 1.15** A graphical representation of the Earth's precession



**Fig. 1.16** Euler angles representing the Earth's precession-nutation



from its current direction, and will approximately point to the star Vega. Due to the precession, the vernal equinox moves about  $50''$  westwards every year on the ecliptic. As the Earth is rotating eastwards, which is defined to be prograde, we say that the precession of the vernal equinox is retrograde. A minor component called nutation, which is composed of periodical elliptic motions with very small amplitudes, is superimposed on the precession.

Figure 1.16 shows a representation of the precession-nutation with 3 angles,  $\psi$ ,  $\epsilon$  and  $\phi$ , called Euler angles, where  $G_c$  is a point on the ecliptic not moving with respect to the celestial sphere, and  $G$  is the intersection of the Greenwich meridian with the celestial equator. The precession-nutation of the vernal equinox  $\Upsilon$  is the change of the angle  $\psi$ , and the nutation of the obliquity of the ecliptic is the change of the angle  $\epsilon$ . The rotation of the Earth with respect to the vernal equinox is the change of the angle  $\phi$ . The precession causes the angle  $\psi$  to decrease about  $50''$  per year with the angle  $\epsilon$  fixed. In Chap. 8, the variations of  $\psi$  and  $\epsilon$  will be formulated as functions of the lunisolar gravitational attraction and Earth parameters.

To facilitate the description of precession-nutation, we define a mean celestial pole, which is a fictitious point on the celestial sphere moving according the precession alone, and the celestial pole (or the true celestial pole in contrast to the mean celestial pole) moves with respect to the mean celestial pole according to the nutation. We show in Fig. 1.17 the precession superimposed with one nutation ellipse, where  $P_0$  is the mean celestial pole. There are numerous nutation ellipses above the accuracy of observation at present. The celestial equator defined by the mean celestial pole is called mean celestial equator, and the intersection point of the mean celestial equator with the ecliptic is called mean vernal equinox. Evidently, the nutation renders the vernal equinox (or the true equinox in contrast to the mean equinox) to oscillates around the mean equinox.

**Fig. 1.17** A nutation ellipse superimposed on the precession

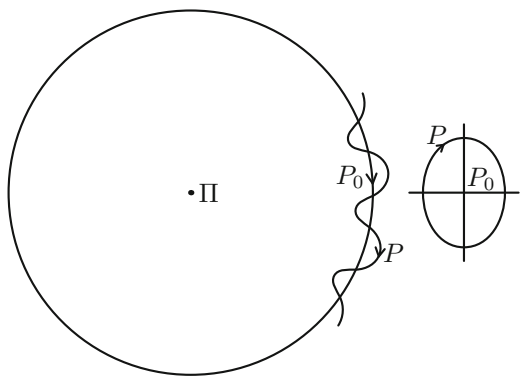


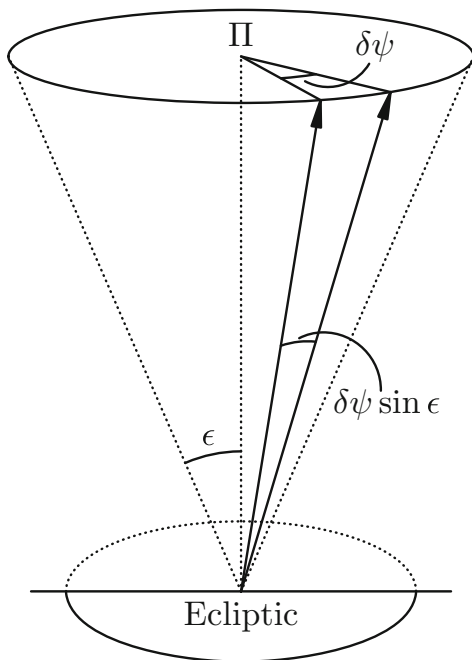
Figure 1.17 displays the trace of the celestial pole in a plane parallel to the ecliptic, such as the one in Fig. 1.15 demonstrating the precession. The precession-nutation is usually represented using the Euler angles  $\psi$  and  $\epsilon$  defined in Fig. 1.16. The angle  $\epsilon$  is the angle between the direction of the celestial pole and that of the ecliptic pole, thus being a direct measure the direction of the celestial pole; a change of  $\epsilon$ ,  $\delta\epsilon$ , is the angle of change of the direction of the celestial pole. However, this is not the case for the angle  $\psi$ , which expresses the direction of the celestial pole indirectly; it is not an angle between the direction of the celestial pole and another direction. In Fig. 1.18, we show how a change of the angle  $\psi$ ,  $\delta\psi$ , is related to the angle of change of direction of the celestial pole,  $\delta\psi \sin \epsilon$ . For this reason, the nutation is often expressed using  $\delta\epsilon$  and  $\delta\psi \sin \epsilon$ .

We have defined the celestial pole as the direction of the rotation axis of the Earth, which is the most natural choice, and is more strictly speaking the direction of the angular velocity of the Earth rotation with respect to the celestial sphere. We can also define the celestial pole as the direction of the angular momentum of the Earth with respect to its center of mass, or otherwise. When confusion is possible, we explicitly call the pole as pole of the angular velocity (or rotation pole), pole of the angular momentum, or otherwise. The celestial pole currently defined by the International Astronomical Union (IAU) to represent precession-nutation is neither the pole of the angular velocity nor the pole of the angular momentum; it is called celestial intermediate pole, briefly referred to as CIP, which is chosen for the convenience to represent the orientation of the Earth with respect to the celestial sphere as to be explained briefly in the next two paragraphs.

Current accuracy of observations requires the problem to be formulated in the framework of general relativity, where the origin of the CRS can be defined on the center of mass of the Earth system or on the barycenter of the solar system. For observers on the Earth, the origin is chosen to be on the center of mass of the Earth system, and the geocentric CRS and CRF thus defined are called GCRS and GCRF, respectively. The directions of the 3 axes of the GCRS are defined to be fixed to the celestial sphere, and the GCRF is tied to the directions of distant celestial objects representing the celestial sphere. The GCRS is also said to be an Earth-fixed inertial



**Fig. 1.18** The change of the Euler angle  $\psi$ ,  $\delta\psi$ , corresponds to an angle of the change of the direction of the celestial pole,  $\delta\psi \sin \epsilon$



reference system or geocentric inertial reference system, as it is non-rotating with respect to the celestial sphere, which is assumed to represent the inertial space. In this regard, the ITRS is an Earth-centered Earth-fixed (ECEF) or Earth-centered rotating (ECR) reference system.

In this book, the terms *inertial space* and *inertial reference system* are also used to indicate the *geocentric inertial reference system*, i.e., the GCRS, in describing the rotation of the Earth. This should not cause any confusion within the contexts.

The CIP is defined based on the precession-nutation of the IRP with respect to the GCRS, which includes a polynomial of time and a set of sine functions of different periods. The CIP is a fictitious pole whose precession-nutation includes only the polynomial of time and the sine functions of period longer than two days. More precisely speaking, this *day* is the *sidereal day* to be defined in the next section. A mean celestial pole can also be defined for the CIP, which includes only the polynomial of time. A mean equinox can be defined correspondingly. The statements on precession-nutation in the previous paragraphs remain valid. In reality, the CIP is extremely close to the pole of angular velocity.

The precession-nutation is mainly caused by the torque of the gravitational attraction of the Sun and the Moon on the Earth (there is also an observed small component of nutation originated from the dynamics of the Earth system, called free core nutation). The torque of the gravitational attraction of other planets also have observable effects, although much smaller. The precession-nutation due to the torque of gravitational attraction is modeled physically as the variation of the CIP, or

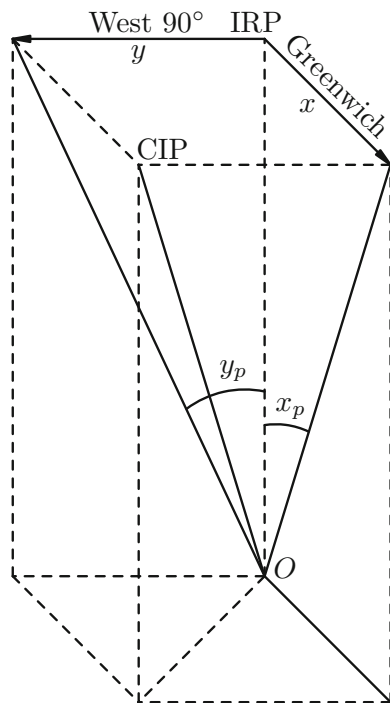
the CIP equator, with respect to the celestial sphere, and is referred to as precession-nutation of the equator. The rate of precession of the equator is proportional to the dynamic flattening of the Earth,  $H = (C - A)/C$ , where  $C$  and  $A$  are the moments of inertia of the Earth around the polar axis and an axis in the equator assuming the Earth to be axially symmetrical. Actually,  $H$  is determined by the rate of precession of the equator. The amplitude of nutation is also proportional to  $H$ . These are studied analytically in Chap. 8 where Earth rotation is formulate.

The orientation of the ecliptic also varies with respect to the celestial sphere, although the variation is much smaller and slower than that of the celestial equator. Currently, the ecliptic is defined as a kind of mean by the IAU: the ecliptic pole is defined as the direction of the mean orbital angular momentum vector of the Earth-Moon barycenter with respect to the center of mass of the entire Solar System, and the ecliptic plane is defined to be perpendicular to the direction of the ecliptic pole. Here the term *mean* indicates the polynomial part alone of the variation of the angular momentum vector with time. Therefore, the variation of the direction of the ecliptic pole with respect to the celestial sphere is a polynomial of time. The effect of the variation of the direction of the ecliptic pole with respect to the celestial sphere on precession is called precession of the ecliptic, and it is a polynomial of time. The effect of this variation on the obliquity of the ecliptic with respect to the mean celestial equator is also a polynomial of time.

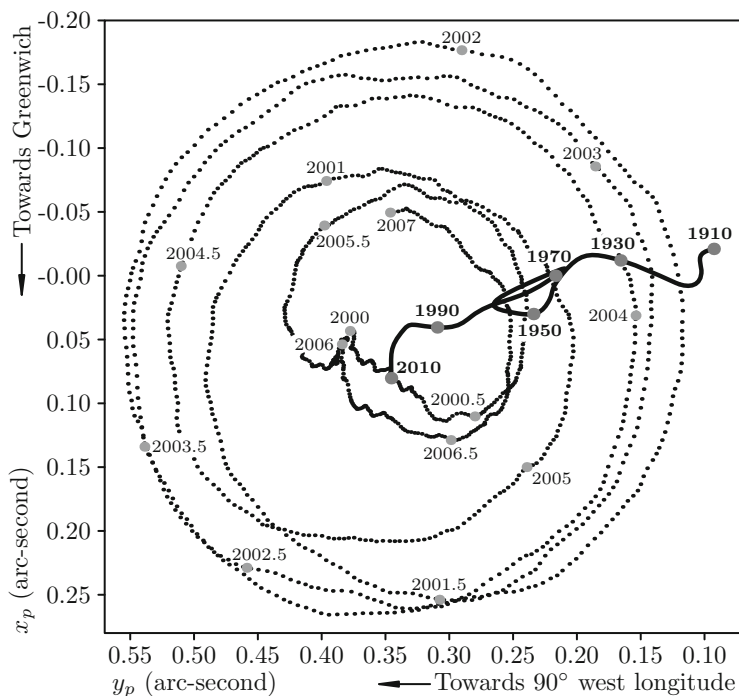
The definition of the ecliptic in the sense of a mean is sufficient for the definition of the tropical year, which is closely related to our daily life. Inclusion of short term variations in the definition of the ecliptic is of little concern. Therefore, the nutation is implicitly understood as the nutation of the equator, which is a variation of the Earth's orientation with respect to the celestial sphere. We mention that the term *mean ecliptic* is also used in the literature. As the ecliptic is currently defined in the sense of a mean, the meaning of the terminologies *ecliptic* and *mean ecliptic* in the literature must be understood within the specific context.

We have described the motion of the CIP with respect to the ecliptic as precession-nutation. The precession is a long term motion of the vernal equinox along the ecliptic with the obliquity of the ecliptic to be  $23^\circ 26' 21.448'' = 23.439291^\circ$  with an extremely small variation, and is composed of two components: the dominant retrograde (i.e., westward) precession of the equator and a small prograde (i.e., eastward) precession of the ecliptic. The combination of the two components is referred to as general precession, which is retrograde. The rate of the precession of the equator is  $50.384''$  per year, and the rate of the general precession is  $50.288''$  per year. We emphasize that the retrograde precession of the equator is the motion of the mean equator with respect to the celestial sphere. The nutation is composed of periodical terms in the form of ellipses. The largest term has a period of 18.6 years with the vernal equinox varying in between  $-17''$  and  $17''$ , and the obliquity of the ecliptic in between  $-9.2''$  and  $9.2''$ . Other nutation components are much smaller, with the next largest component being only on the order of 1/10 of the largest one. We mention that the term precession is used very generally, but the terms general precession, precession of the equator and precession of the ecliptic have well defined specific meanings.

**Fig. 1.19** Coordinates of the CIP along the Greenwich and west 90° meridians



The direction of the CIP also moves with respect to the Earth itself, called polar motion. Here we also use the term CIP to denote the intersection of the CIP-axis (the axis through the center of mass of the Earth system pointing to the CIP) with the reference Earth ellipsoid. The polar motion is described by the time-dependent position of the CIP in the ITRS. The coordinates of the CIP are expressed in the rectangular coordinate system  $Oxy$ , where the origin  $O$  is on the north pole of the reference Earth ellipsoid, i.e., on the IERS reference pole (IRP), and the axes are tangent to the reference Earth ellipsoid. The  $x$ -axis points to the Greenwich meridian, and the  $y$ -axis to the meridian of 90° west, as shown in Fig. 1.19. The coordinates of the CIP could be the projections of the angle between the CIP-axis and the IRP-axis, or that of the distance between the CIP and the IRP, along the  $x$ - and  $y$ -axes, and is denoted as  $(x_p, y_p)$ . The two main components of the polar motion during the last 100 years are a linear drift of 0.3'' per century toward the west and a periodical motion reaching a magnitude on the order of 0.5''  $\times$  0.5'' or 15  $\times$  15 meters. The main cause of the linear drift is the glacial isostatic adjustment (GIA). The periodical component includes mainly a prograde annual wobble with a period of 1 year and a prograde Chandler wobble with a period of 1.19 years, and is mainly caused by the motions in the atmosphere and oceans. We show in Fig. 1.20 the position of the CIP as a function of time, called polhody, over the period 2000–2007 based on the IERS solution C04, as well as that of a filtered pole, which is defined here as the position of the pole after removing periodical and short



**Fig. 1.20** Position of the CIP as a function of time, where the CIP is plotted as a dot every other day, and the filtered pole as a thick solid line

term variations of the CIP by filtering,<sup>3</sup> from 1910 to 2010. Both the trend and the periodical components are not steady. The rate of the trend and the amplitudes and phases of the annual and Chandler wobbles are varying with time. Besides, the polar motion has other irregularities, such as the knot of the filtered pole around 1970.

The theory of precession-nutation and polar motion is the subject of Chap. 8.

## 1.5 Time Systems

When we think of time, we would immediately think of dividing a day. The most natural way to define a day is to use the position of the Sun as seen from the Earth's surface: a day is the time-span for the Sun to go through the cycle *lower meridian-sunrise-higher meridian-sunset-lower meridian*. The day thus defined based on the real Sun is called true solar day. Within a true solar day, the natural way to define time is still to use the position of the Sun: the hour angle of the Sun plus 12 hours is called true solar time. The true solar time is most natural for the daily life—

<sup>3</sup> <https://hpiers.obspm.fr/eop-pc/earthor/polmot/pm.html>

the moment when the Sun is on the lower meridian is the end of one day and the beginning of another day, and it is 12 o'clock in the noon when the Sun is on the upper meridian.

The true solar time is not homogeneous. Two causes are dominant. (1) As the orbit of the Earth's revolution around the Sun is an ellipse with the Sun at one of the foci, the rate of visual motion of the Sun in the ecliptic plane as seen from the Earth is not homogeneous, causing the length of day to be different. (2) Even if the Earth orbits the Sun strictly homogeneously on a circle, the angle between the equator and the ecliptic, i.e., the obliquity of the ecliptic, renders the rate of change of the hour angle of the Sun non-homogeneous. Minor causes include the influences of nutation, polar motion, motion of the Earth around the Earth-Moon barycenter, and the inhomogeneity in Earth rotation rate, which we assume to be negligible in the discussion of this paragraph. First, we realize that the true solar time would be homogeneous if (1) the Earth orbits the Sun strictly on a circle with constant velocity, which can be understood reversely as that the Sun orbits the Earth in the ecliptic plane strictly on a circle with constant rate of visual motion, and (2) the ecliptic and the equator coincide. In order to eliminate the two types of inhomogeneity mentioned above, we introduce the concepts of mean solar day and mean solar time. We first define a fictitious ecliptic mean Sun having the following properties: (1) it goes through the mean vernal equinox at the same time as the real Sun; (2) it orbits the Earth in the ecliptic plane on a strict circle with a constant rate of visual motion equal to the average rate of visual motion of the real Sun. We also define a fictitious equatorial mean Sun that goes through the mean vernal equinox at the same time as the ecliptic mean Sun, and orbits the Earth with the same constant rate of visual motion as that of the ecliptic mean Sun. This is to define the right ascension of the equatorial mean Sun equals to the ecliptic longitude of the ecliptic mean Sun. In this way, the rate of change of the hour angle of the equatorial mean Sun is homogeneous. We thus define the mean solar day as the time-span for the equatorial mean Sun to go through the cycle *lower meridian-sunrise-higher meridian-sunset-lower meridian*, and define the mean solar time as the hour angle of the equatorial mean Sun plus 12 hours. The length of a mean solar day is in fact the yearly average of the length of the true solar day. We mention that in the literature, the term *solar day* is often used to indicate *mean solar day*.

The mean solar time is location-dependent. The mean solar time at Greenwich, called Greenwich mean time, or GMT, is also referred to as universal time, and is denoted as UT0. A more homogeneous version is obtained by applying the correction due to the polar motion, and is denoted as UT1. By further applying the correction due to the seasonal variation of the Earth rotation rate, it is denoted as UT2. The inhomogeneity in Earth rotation rate is completely reflected in UT1, but a part of it is removed from UT2. We will use the term UT to denote the universal time in general. At present, the UT2 is obsolete, and the UT1 is used as one of the parameters describing the orientation of the TRS with respect to the CRS. As the rotation rate of the Earth slows down, the UT1 cannot be used as a long term continuous measure of time.

The variation of the Earth rotation is very complicated. With the improvement of observation accuracy, even the UT2 becomes inhomogeneous. A more homogeneous time, called ephemeris time, and briefly referred to as ET, was defined based on the dynamic theory of the orbits of the Earth and the Moon, and was set to equal UT2 at 12h, January 0, 1900 (which actually refers to noon, December 31, 1899). The length of an ET second is defined based on the period of the revolution of the Earth around the Sun, i.e., the length of a tropical year, as the length of a year is much stable than the length of a day, and the ET can be determined according to the dynamic theory of the orbits of the Sun and the Moon using their positions observed. The ET is a continuous measure of time independent of Earth rotation.

With the advent and advancement of the atomic clock technology, a much more precise time system based on atomic clocks came into practice, which is referred to as international atomic time, and is denoted briefly as TAI (following its French name *Temps Atomique International*). The TAI is set to equal UT2 at 0h, January 1, 1958, and then used as a continuous measure of time. In 1967, the length of a TAI second was redefined to be the same as that of ET, and is now used as the second of the international system of units, briefly referred to as SI unit (following its French name *Système Internationale d'Unités*) of the second. Since January 1, 1977, ET was determined by TAI, and the relation between TAI and ET was determined to be

$$ET = TAI + 32.184s. \quad (1.2)$$

The lag of 32.184s is mainly due to the slowing down of Earth rotation from 1990 to 1958.

By definition, the UT is the basis of the time system for societal applications. Due to the long term variation of Earth rotation, the difference between the UT and the TAI would increase progressively with time. In order to take advantage of the homogeneity of the TAI and the convenience of the UT, a compromise called coordinated universal time, briefly referred to as UTC, is introduced in 1972. The SI unit of second, which is in fact the length of a TAI second, is adopted in UTC, and the UTC is adjusted so that UT1–UTC to be within  $\pm 0.9''$  by jumping a second forth or back when necessary. Base on past observations, there is a need of jump at most two seconds each year, and, as the Earth rotation rate has been decreasing, the jump has always been to add a second, called leap second. It is the responsibility of the International Earth Rotation and Reference Systems Service (IERS). The time of the jump is arranged to happen by the end of June or December. When a leap second happens, the next time after 23:59:59 is 23:59:60, and then 00:00:00 on the next day, i.e., the leap second causes the last day of June or December to have one more second. At present, UT1–UTC with the leap seconds corrected is used to represent the variation of the rotation rate of the Earth.<sup>4</sup>

The most accurate time scales are defined by the International Astronomical Union (IAU) in the framework of general relativity, called dynamical time. This

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<sup>4</sup> A resolution to eliminate the leap second was made by the General Conference on Weights and Measures (CGPM) on November 18, 2022, and it is expected to be implemented in 2035.

was done in recognition to the fact that atomic clocks tick in different rates at different locations, which is a consequence of gravitational time dilation. Here we just provide the terminologies. Originally, the IAU defined two dynamical time scales, the Barycentric Dynamical Time (TDB) and the Terrestrial Dynamical Time (TDT). These were later replaced by the Barycentric Coordinate Time (TCB) and Geocentric Coordinate Time (TCG). The time scale currently used to measure time in astronomical observations made from the surface of Earth is called Terrestrial Time (TT), which is a replacement of TDT, and a continuation of ET. The TT is in agreement of ET, and the old almanacs can be used without any correction by replacing ET by TT. In reality, TAI is a particular realization of TT. In the framework of general relativity, the rate of an atomic clock is different at different height above the sea level. The TT is obtained by adding general relativity corrections to the readings of atomic clocks.

In astronomical ephemerides, the unit of time is often chosen as Julian day and century, which is a continuous measure of time. The number of mean solar days elapsed since 12h, January 1, 4713 BC is defined as Julian day, and is briefly referred to as JD. A Julian century consists of 36525 mean solar days. The JD at 12h, January 1, 2000, which is often denoted as January 1.5d, 2000, is 2451545. This epoch is defined as the initial epoch of the *J2000.0* system used at present. The time used in the *J2000.0* system is the number of Julian century counted starting from the initial epoch, where the day is defined as  $24 \times 60 \times 60 = 86400$  TT seconds. A modified version of the Julian day denoted as MJD is also often used; its relation to the JD is  $\text{MJD} = \text{JD} - 2400000.5$ . The initial epoch of the MJD is 24h, November 16, 1858.

In astronomy, the hour angle of the vernal equinox is called sidereal time, the Greenwich sidereal time being the angle  $\phi$  in Fig. 1.16. The time-span between two successive passings of the vernal equinox through the upper meridian is called a sidereal day. A mean sidereal time and a mean sidereal day can be defined based on the mean vernal equinox. In the literature, the terms *sidereal time* and *sidereal day* are often used to indicate *mean sidereal time* and *mean sidereal day*, respectively. The time-span for the Earth to revolve the Sun from the vernal equinox back to the vernal equinox is called a tropical year, which is a whole cycle of the seasons spring, summer, fall and winter. The sidereal year is defined with respect to the celestial sphere, or fixed stars, i.e., a sidereal year is the time-span between two passings of the Earth through a point on the ecliptic fixed on the celestial sphere. Note particularly the difference between the reference points in the definitions of a sidereal day and a sidereal year—it is the vernal equinox for the former, and a celestial-sphere-fixed point in the ecliptic for the latter.

In a tropical year, there are 365.2422 mean solar days, but there are 366.2422 mean sidereal days, where the exact 1 day difference is due to the visual motion of the Sun related to the revolution of the Earth around the Sun. To understand this, we imagine the extreme case where the Earth revolves around the Sun as usual, but it does not rotate visually with respect to the vernal equinox, i.e., the vernal equinox remains in the same direction for any observer fixed on the Earth. In this case, we examine the whole cycle between the successive epochs when the Sun is in the direction of the vernal equinox. This cycle is exactly a tropical year, during

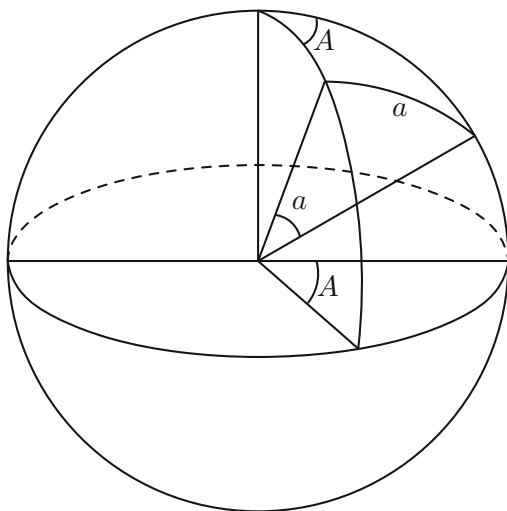
which the Earth revolves the sun a whole cycle, and visually, the Sun goes through a whole cycle around the Earth eastward, i.e., there is  $-1$  mean solar day. However, as the vernal equinox (which is on the celestial sphere infinitely far away) does not move at all visually, there is 0 sidereal day. Taking reference of the discussion of this imaginary scenario, if the Earth is rotating, causing the mean vernal equinox to have visually moved  $n$  cycles with respect to the Earth westward during a tropical year, the mean Sun has visually moved  $n - 1$  cycles around the Earth westward. In a tropical year, we know that there are  $n - 1 = 365.2422$  mean solar days, and therefore, there are  $n = 366.2422$  mean sidereal days.

## 1.6 Spherical Trigonometry

Spherical trigonometry is the mathematical tool for handling the relations among the quantities of a spherical triangle, which is defined by three points on a sphere, each pair being connected by a great arc, i.e., the arc with center on the center of the sphere. The angle between the planes holding two arcs is called an angle, and the angle between two points with respect to the center of the sphere is called a side. Definitions of an angle  $A$  and a side  $a$  is illustrated in Fig. 1.21. Conventionally, as shown in Fig. 1.22,  $A$ ,  $B$  and  $C$  are used to represent the angles at the respective vertex, and  $a$ ,  $b$  and  $c$  are used to represent the side opposite the respective vertex.

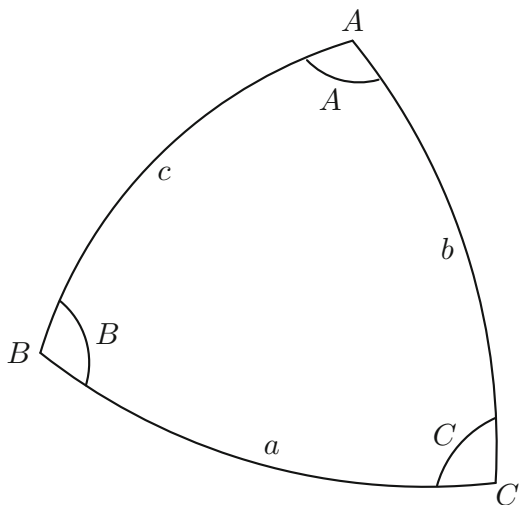
As shown in Fig. 1.23, for a spherical triangle  $ABC$ , define a point  $D$  on the great arc through  $A$  and  $B$  such that  $\widehat{AD} = 90^\circ$ , and a point  $E$  on the great arc through  $A$  and  $C$  such that  $\widehat{AE} = 90^\circ$ . We then draw a great arc to connect  $D$  and  $E$ . Another two great arcs can be similarly drawn. The three great arcs drawn form a triangle  $A'B'C'$ . The vertex  $A$  of the triangle  $ABC$  is the pole of the side  $B'C'$  of the triangle  $A'B'C'$ , i.e., if  $A$  is defined as the pole of a spherical coordinate

**Fig. 1.21** Definition of angle and side of a spherical triangle

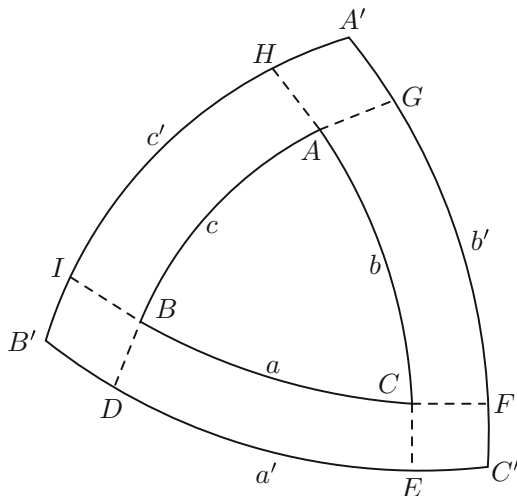




**Fig. 1.22** Notation of angles and sides of a spherical triangle



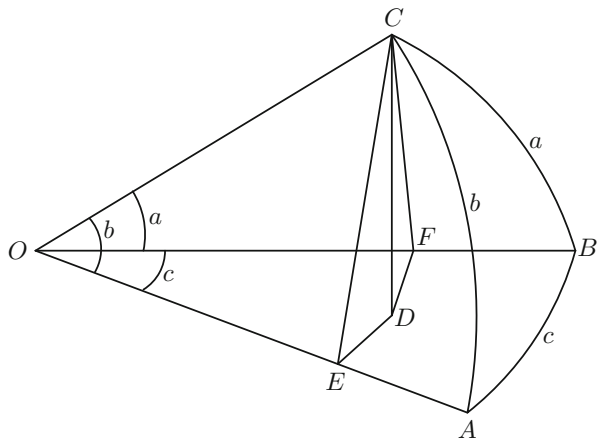
**Fig. 1.23** Polar spherical triangles



system, the arc  $B'C'$  is in the equatorial plane. The same is for the other two vertex-arc combinations. Hence, the triangle  $ABC$  is referred to as the polar triangle of  $A'B'C'$ . It can be readily shown that  $\widehat{A'I} = \widehat{A'F} = 90^\circ$  (hint: the angles at vertices  $I$  and  $F$  are all right angles). The same is for the other two pairs of sides. Therefore,  $A'B'C'$  is also a polar triangle of  $ABC$ , implying that the relation of a triangle to its polar triangle is reciprocal. We can see that  $A = \widehat{DE}$ ,  $\widehat{EB'} = \widehat{DC'} = 90^\circ$  and  $A' = \widehat{IF}$ ,  $\widehat{BF} = \widehat{CI} = 90^\circ$ . Consequently, we have

$$a' + A = 180^\circ, \quad A' + a = 180^\circ. \quad (1.3)$$

**Fig. 1.24** Proof of the sine formula of spherical trigonometry



In the following, we derive the fundamental relations among the angles and sides of a spherical triangle.

As shown in Fig. 1.24,  $CD$  is perpendicular to the plane  $OAB$ ,  $DE$  is perpendicular to the line  $OA$ , and  $DF$  is perpendicular to the line  $OB$ . We see that  $A = \angle CED$  and  $B = \angle CFD$ , and  $\angle CEO$  and  $\angle CFO$  are right triangles. Therefore,

$$CD = CE \sin A = OC \sin b \sin A, \quad (1.4)$$

$$CD = CF \sin B = OC \sin a \sin B. \quad (1.5)$$

By dividing one of them by the other, we obtain

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}, \quad (1.6)$$

which is called sine formula.

As shown in Fig. 1.25,  $AM$  is tangential to the side  $b$  at  $A$ , and  $AN$  is tangential to the side  $c$  at  $A$ . We have

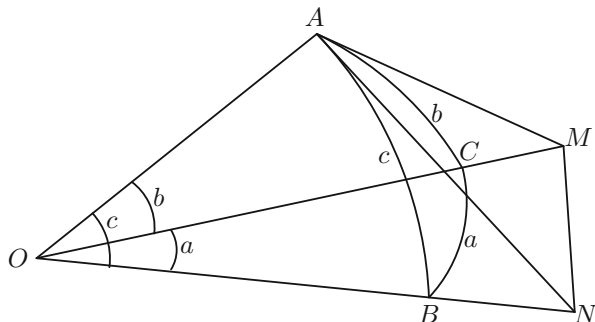
$$MN^2 = OM^2 + ON^2 - 2OM \cdot ON \cos a, \quad (1.7)$$

$$MN^2 = AM^2 + AN^2 - 2AM \cdot AN \cos A. \quad (1.8)$$

These equations can be solved to obtain

$$\begin{aligned} 2OM \cdot ON \cos a &= OM^2 + ON^2 - MN^2 \\ &= OM^2 + ON^2 - AM^2 - AN^2 + 2AM \cdot AN \cos A \\ &= 2OA^2 + 2AM \cdot AN \cos A, \end{aligned} \quad (1.9)$$

**Fig. 1.25** Proof of the cosine formula of spherical trigonometry



which leads to

$$\cos a = \frac{OA}{OM} \cdot \frac{OA}{ON} + \frac{AM}{OM} \cdot \frac{AN}{ON} \cos A. \quad (1.10)$$

We finally obtain

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \quad (1.11)$$

which is called cosine formula for the sides. Apply the above formula to the polar triangle  $A'B'C'$ . We obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'. \quad (1.12)$$

This formula becomes, by substituting (1.3) into it,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \quad (1.13)$$

which is called cosine formula for the angles.

The sine and cosine formulae are most fundamental. We have given only one instance for each of them. Other instances can be obtained by applying the cyclic permutation  $(A, a) \rightarrow (B, b) \rightarrow (C, c) \rightarrow (A, a)$  in the formulae.

Many other formulae can be derived from the sine and cosine formulae, and some are listed below. The following are called five-part formulae:

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \quad (1.14)$$

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A, \quad (1.15)$$

$$\sin A \cos b = \cos B \sin C + \sin B \cos C \cos a, \quad (1.16)$$

$$\sin A \cos c = \cos C \sin B + \sin C \cos B \cos a. \quad (1.17)$$

The following are called four-part formulae:

$$\cot A \sin C = -\cos C \cos b + \sin b \cot a, \quad (1.18)$$

$$\cot A \sin B = -\cos B \cos c + \sin c \cot a, \quad (1.19)$$

$$\cot a \sin c = \cos c \cos B + \sin B \cot A, \quad (1.20)$$

$$\cot a \sin b = \cos b \cos C + \sin C \cot A. \quad (1.21)$$

Other instances of the five-part and four-part formulae can be obtained by applying the cyclic permutation  $(A, a) \rightarrow (B, b) \rightarrow (C, c) \rightarrow (A, a)$  in the formulae.

As an exercise, the readers are suggested derive the relations among the different celestial coordinate systems using the spherical trigonometric formulae.

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## Bibliographic Notes

Seismology and its application to study the Earth's interior will be the topic of Chap. 6. A book in moderate depth in these subjects is Bullen and Bolt (1985). A recent book in plate tectonics is Frisch et al. (2022), which includes plenty of impressive color figures, and is quite easy to follow. Earth tide and elastic loading will be the topic of Chap. 7 with a lot of detail not found elsewhere. The GIA will be briefly discussed in Chap. 9. A comprehensive book on its theory is Sabadini et al. (2004). The detail of establishing a ITRF can be found in Altamimi et al. (2016). The physical background of defining the reference Earth ellipsoid and elementary ellipsoidal geometry will be the topic of Chap. 5. More in-depth detail can be found in any book of *Physical Geodesy* dedicated to the static gravity field of the Earth, such as Guo (2023). Precession-nutation and polar motion will be the topic of Chap. 8. A recent comprehensive book in this topic is Dehant and Mathews (2015). A comprehensive book in the definitions of time throughout the history is McCarthy and Seidelmann (2018). Some figures in the book, e.g., Fig. 1.2, are drawn using the Generic Mapping Tools (Wessel et al., 2019). Other figure are drawn using the METAPOST that comes in the  $\text{\LaTeX}$  package  $\text{\texttt{MiKTeX}}$  that the author uses.

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# Three-Dimensional Vector and Tensor Analysis 2

## Abstract

This chapter is a concise yet rigorous presentation of three-dimensional vector and tensor analysis. Subjects are specifically chosen for the need of continuum mechanics and geophysics. Besides the definitions, the algebraic operations, the operations of the vector differential operator *nabla*, and the basic field theory, also included are the fourth order isotropic tensor, the reduction of a symmetrical second order tensor to the diagonal form, Green's identities, and Helmholtz's theorem. The curvilinear coordinate system is treated to some length. The general curvilinear coordinate system is formulated first, and then the orthogonal curvilinear coordinate system is treated as a special case. The spherical and cylindrical coordinate systems are considered as instantiations. Formulae of the *nabla* operator in the curvilinear coordinate system are systematically derived. Throughout the chapter, the formulations are rigorous, i.e., there is no jump of logic or derivation. However, the learning curve is steep, as the text is concise. Less experienced learners of mathematics are suggested to read a book providing a less steep learning curve in parallel, such as those recommended by the end of the chapter.

## 2.1 Vectors and Their Algebraic Operations

### 2.1.1 Vectors and Their Algebraic Operations: Definitions

The quantities which are most familiar to us are scalars, which are fully represented by a magnitude. Our focus in this subsection is vectors. By definition, a vector is fully represented by both a magnitude and a direction, and furthermore, the sum of two vectors obeys the parallelogram law. In physics, the most common scalars include mass, time, length, density, temperature, etc., and the most common vectors include force, displacement, velocity, acceleration, torque, electric field intensity,

magnetic induction, etc. A vector is customarily graphically represented using a directed line segment, whose length and direction represent the magnitude and direction, respectively, of the vector. A vector represented by the directed line segment from the starting point  $A$  to the end point  $B$  is denoted as  $\vec{AB}$ . A vector is also represented by adding an arrow on top of a letter, such as  $\vec{A}$ ,  $\vec{B}$ ,  $\vec{a}$ , and  $\vec{b}$ . In practice, we may come across vectors which are dependent on or independent from the starting point, the former being referred to as bound vector (the starting point is fixed) or sliding vector (the starting point is on the line through the vector), and the latter free vector. In this book, we are limited to the free vectors. Therefore, as the translational displacement of a vector does not alter its magnitude and direction, it is naturally defined that two vectors are equal if their magnitudes are equal and their directions point to the same direction in parallel. The magnitude is also referred to as norm, which is customarily denoted as  $|\vec{AB}|$ ,  $|\vec{A}|$ , and  $|\vec{a}|$  for the vectors  $\vec{AB}$ ,  $\vec{A}$ , and  $\vec{a}$ , respectively. If no confusion may arise,  $AB$ ,  $A$ , and  $a$  are also used. A vector with norm equal to zero is referred to as a zero vector, and is denoted as  $\vec{0}$ , or simply  $0$  as the direction is meaningless. A vector with norm equal to 1 is referred to as a unit vector, and is denoted with a hat over a letter, e.g.,  $\hat{e}$ .

By definition, the sum of two vectors obey the parallelogram law. It can be seen from Fig. 2.1 that the sum of vectors obey the commutative and associative laws, i.e.,

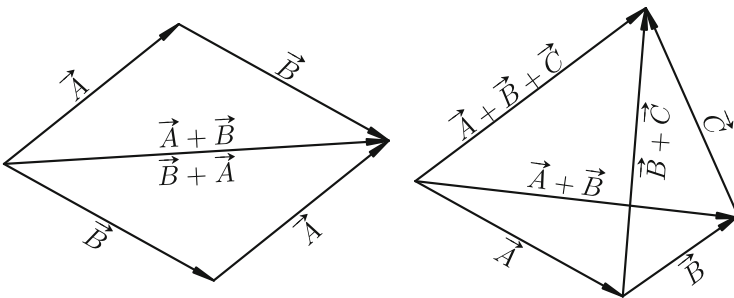
$$\vec{A} + \vec{B} = \vec{B} + \vec{A}, \quad (2.1)$$

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) = \vec{A} + \vec{B} + \vec{C}. \quad (2.2)$$

We can also obtain the triangle inequality based on the parallelogram law,

$$|\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|. \quad (2.3)$$

The negative of a vector,  $\vec{B} = -\vec{A}$ , is defined such that  $\vec{A} + \vec{B} = \vec{0}$ . Therefore,  $\vec{A}$  and  $-\vec{A}$  have the same norm, but with opposite directions. The difference between two vectors is defined as



**Fig. 2.1** The commutative (left) and associative (right) laws of the sum of vectors

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}). \quad (2.4)$$

The product between a vector  $\vec{A}$  and a scalar  $\lambda$ , denoted as  $\lambda\vec{A}$  or  $\vec{A}\lambda$ , is defined to be a vector, such that its norm is  $|\lambda||\vec{A}|$ , and its direction is the same as  $\vec{A}$  when  $\lambda > 0$ , and opposite to  $\vec{A}$  when  $\lambda < 0$ . When  $\lambda = 0$ , the norm is zero, and the product is a zero vector. Evidently, we have  $1\vec{A} = \vec{A}$  and  $-1\vec{A} = -\vec{A}$ . It is easy to verify that the product between a vector and a scalar has the following properties:

$$\lambda(\mu\vec{A}) = (\lambda\mu)\vec{A} = \mu(\lambda\vec{A}), \quad (2.5)$$

$$(\lambda + \mu)\vec{A} = \lambda\vec{A} + \mu\vec{A}, \quad (2.6)$$

$$\lambda(\vec{A} + \vec{B}) = \lambda\vec{A} + \lambda\vec{B}. \quad (2.7)$$

It is also easy to verify that, if  $\vec{A}$  and  $\vec{B}$  are parallel each other, denoted as  $\vec{A} // \vec{B}$ , there must be a scalar  $\lambda$ , such that

$$\vec{A} = \lambda\vec{B}. \quad (2.8)$$

Particularly, if we use  $\hat{e}_A$  to denote the unit vector in the same direction as  $\vec{A}$ , we have

$$\vec{A} = A\hat{e}_A, \quad (2.9)$$

where  $A$  is the norm of  $\vec{A}$ .

The scalar product of two vectors  $\vec{A}$  and  $\vec{B}$ , denoted as  $\vec{A} \cdot \vec{B}$ , is defined to be a scalar,

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(\vec{A}, \vec{B}), \quad (2.10)$$

where  $(\vec{A}, \vec{B})$  is the angle between the directions of  $\vec{A}$  and  $\vec{B}$ . This product has the following properties:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}, \quad (2.11)$$

$$(\lambda\vec{A}) \cdot \vec{B} = \lambda(\vec{A} \cdot \vec{B}) = \vec{A} \cdot (\lambda\vec{B}), \quad (2.12)$$

$$(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C}. \quad (2.13)$$

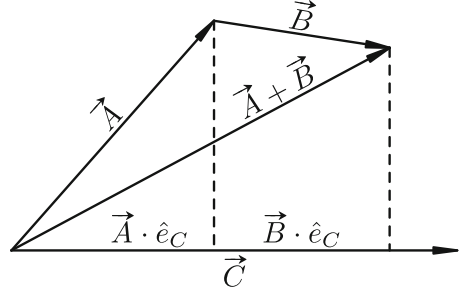
The first two formulae can be verified directly based on the definition. The last one can be obtained by multiplying  $|\vec{C}|$  to both sides of  $(\vec{A} + \vec{B}) \cdot \hat{e}_C = \vec{A} \cdot \hat{e}_C + \vec{B} \cdot \hat{e}_C$ , which can be readily inferred from Fig. 2.2.

The projection of a vector along a direction is referred to as the component of the vector along the direction. The component of  $\vec{A}$  along  $\hat{e}$  is

$$A_e = |\vec{A}|\cos(\vec{A}, \hat{e}) = \vec{A} \cdot \hat{e}. \quad (2.14)$$



**Fig. 2.2** Proof of the distributive law of the scalar product between vectors



The vector product of two vectors  $\vec{A}$  and  $\vec{B}$ , denoted as  $\vec{A} \times \vec{B}$ , is defined to be a vector; its norm is defined as

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin(\vec{A}, \vec{B}), \quad (2.15)$$

and its direction is defined to be perpendicular to both  $\vec{A}$  and  $\vec{B}$  obeying the right-hand rule—it is the direction of the thumb while turning the four fingers from  $\vec{A}$  to  $\vec{B}$  counterclockwise using the right hand. An evident geometrical property of  $\vec{A} \times \vec{B}$  is that, its norm is the area of the parallelogram with  $\vec{A}$  and  $\vec{B}$  as sides. We can also understand  $\vec{A} \times \vec{B}$  as the vector area of the parallelogram, which is a vector pointing to the direction of  $\vec{A} \times \vec{B}$  with the norm equal to the area of the parallelogram. Another geometrical property of  $\vec{A} \times \vec{B}$  is that, if  $\vec{A}$  and  $\vec{B}$  are non-zero vectors,  $\vec{A} \times \vec{B} = 0$  implies that  $\vec{A}$  and  $\vec{B}$  are parallel each other, i.e.,  $\vec{A} // \vec{B}$ . The vector product of two vectors has the following properties:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}, \quad (2.16)$$

$$(\lambda \vec{A}) \times \vec{B} = \lambda(\vec{A} \times \vec{B}) = \vec{A} \times (\lambda \vec{B}), \quad (2.17)$$

$$(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}. \quad (2.18)$$

The first two identities can be easily verified based on the definition, and the last one will be proven later.

We call  $(\vec{A} \times \vec{B}) \cdot \vec{C}$  the scalar triple product of  $\vec{A}$ ,  $\vec{B}$ , and  $\vec{C}$ , which is also denoted as  $(\vec{A}, \vec{B}, \vec{C})$  some times. It can be easily verified that  $(\vec{A} \times \vec{B}) \cdot \vec{C}$  is the signed volume of the parallelepiped with  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$  as sides, which is positive when the vectors  $\vec{A}$ ,  $\vec{B}$  and  $\vec{C}$  approximately obey the right-hand rule (the angle between the direction of the thumb and  $\vec{C}$  is less than  $90^\circ$ ), and negative otherwise. Based on this geometrical interpretation, we have

$$\begin{aligned} (\vec{A} \times \vec{B}) \cdot \vec{C} &= (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B} = -(\vec{B} \times \vec{A}) \cdot \vec{C} \\ &= -(\vec{C} \times \vec{B}) \cdot \vec{A} = -(\vec{A} \times \vec{C}) \cdot \vec{B}. \end{aligned} \quad (2.19)$$

Finally, we provide a proof of (2.18) using (2.19). Let  $\vec{y}$  be an arbitrary vector. We have

$$\begin{aligned} [(\vec{A} + \vec{B}) \times \vec{C}] \cdot \vec{y} &= (\vec{C} \times \vec{y}) \cdot (\vec{A} + \vec{B}) = (\vec{C} \times \vec{y}) \cdot \vec{A} + (\vec{C} \times \vec{y}) \cdot \vec{B} \\ &= (\vec{A} \times \vec{C}) \cdot \vec{y} + (\vec{B} \times \vec{C}) \cdot \vec{y} = (\vec{A} \times \vec{C} + \vec{B} \times \vec{C}) \cdot \vec{y}. \end{aligned} \quad (2.20)$$

This equation can be written as

$$[(\vec{A} + \vec{B}) \times \vec{C} - (\vec{A} \times \vec{C} + \vec{B} \times \vec{C})] \cdot \vec{y} = 0. \quad (2.21)$$

As  $\vec{y}$  is arbitrary, in order for the above equation to hold, the following equation must hold:

$$(\vec{A} + \vec{B}) \times \vec{C} - (\vec{A} \times \vec{C} + \vec{B} \times \vec{C}) = 0, \quad (2.22)$$

which is identical to (2.18). We mention that this proof can be valid only if (2.19) is independent of (2.18), which is indeed the case.

### 2.1.2 Summation Index, Free Index, Kronecker Delta $\delta_{ij}$ and Levi-Civita Epsilon $\epsilon_{ijk}$

We are limited to study problems in the three-dimensional space, and will frequently come across of similar equations such as

$$\begin{cases} y_1 = a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3, \\ y_2 = a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3, \\ y_3 = a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3. \end{cases} \quad (2.23)$$

In these equations, the comma in the subscript of  $a$  is for preventing understanding the two numbers together, e.g., 1, 1 implies 1 and 1, but not 11. In order to write these equations briefly, we define a system of notation: When an index, i.e., a letter used as subscript of superscript, appears twice in a term, it means that this term is indeed the sum of the three results obtained by setting the index to be 1, 2, and 3 in the term. When an index appears only once in each term in an equation, it means that this equation represents the three equations by setting the index to be 1, 2, and 3. The former is referred to as summation index or dummy index, and the latter free index. With the convention of indices thus defined, (2.23) can be written as

$$y_i = a_{ij}x_j, \quad (2.24)$$

where  $j$  is a summation index, and  $i$  a free index. When the convention of summation is used, it is strictly prohibited to have an index appearing three times

or more in a term. Evidently, changing a summation index to a different letter does not affect the value of the term, e.g., in (2.24), the index  $j$  can be changed to  $k$  without any effect. While reading this book, the readers are suggested to expand the summation and free indices to 1, 2, and 3 till adequately familiar with these notations.

The Kronecker Delta  $\delta_{ij}$  is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \quad (2.25)$$

It has the following properties:

$$\delta_{ij} = \delta_{ji}, \quad (2.26)$$

$$\delta_{ii} = 3, \quad (2.27)$$

$$\delta_{ij}a_j = a_i. \quad (2.28)$$

We mention that the subscript  $i$  in (2.27) is an summation index. These formulae can be proven by just expanding the summation and free indices to 1, 2, and 3. Some other formulae can be readily inferred from the above formulae, e.g.,  $\delta_{ij}a_i = a_j$ ,  $\delta_{ij}a_{jk} = a_{ik}$ ,  $\delta_{ij}a_{ik} = a_{jk}$ , etc., which will be used frequently hereafter.

The Levi-Civita Epsilon  $\epsilon_{ijk}$  is defined as

$$\epsilon_{ijk} = \begin{cases} 1, & ijk = 123, 231, 312; \\ -1, & ijk = 213, 321, 132; \\ 0, & \text{Two of the indices are equal.} \end{cases} \quad (2.29)$$

It has the following properties:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}, \quad (2.30)$$

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}, \quad (2.31)$$

which can also be proven by expanding the summation and free indices to 1, 2, and 3.

Later on, the Kronecker Delta  $\delta_{ij}$  will also be used in a broader sense by extending the dimension from 3 to any number, including infinity. This will not cause confusion within the respective context. Furthermore, another Kronecker Delta  $\delta_k$  defined by

$$\delta_k = \begin{cases} 0, & k \neq 0; \\ 1, & k = 0; \end{cases} \quad (2.32)$$

will be used. It is in fact a simpler form of  $\delta_{k0}$ .

### 2.1.3 Algebraic Operations of Vectors in Rectangular Coordinate Systems

Unless specified otherwise, the rectangular coordinate system  $Ox_1x_2x_3$  is always implicitly assumed to be a right-handed system, i.e., the thumb points to the direction of the  $x_3$ -axis while the four fingers turn from the direction of the  $x_1$ -axis to that of the  $x_2$ -axis counterclockwise using the right hand.

According to the definitions of the sum of vectors and the product of a vector with a scalar, any vector  $\vec{A}$  can be expressed as the sum of three terms in the rectangular coordinate system  $Ox_1x_2x_3$ ,

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = A_i\hat{e}_i, \quad (2.33)$$

as shown in Fig. 2.3, where  $\hat{e}_i$  is the unit vector along the  $x_i$ -axis, and is referred to as base vector. By definition,

$$|\vec{A}| = (A_1^2 + A_2^2 + A_3^2)^{1/2} = (A_i A_i)^{1/2}, \quad (2.34)$$

and it can be verified that

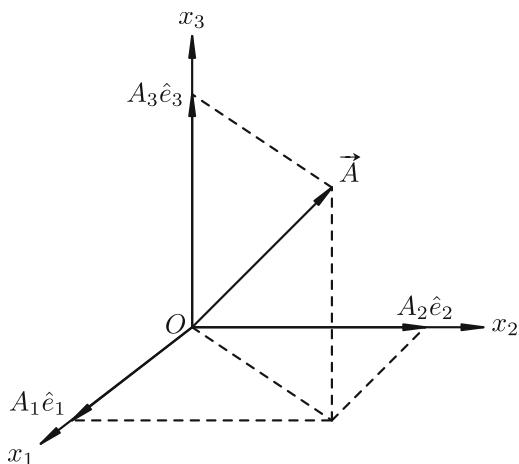
$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \quad (2.35)$$

$$\hat{e}_i \times \hat{e}_j = \epsilon_{ijk} \hat{e}_k. \quad (2.36)$$

Define  $n_i = \cos(\vec{A}, \hat{e}_i)$ , called directional cosine. We have, according to Fig. 2.3,

$$n_i = \cos(\vec{A}, \hat{e}_i) = A_i/|\vec{A}|. \quad (2.37)$$

**Fig. 2.3** Expression of a vector in the rectangular coordinate system



It is evident that

$$n_1^2 + n_2^2 + n_3^2 = n_i n_i = 1. \quad (2.38)$$

The unit vector in the same direction as  $\vec{A}$  can be expressed as

$$\hat{e}_A = \vec{A}/|\vec{A}| = n_i \hat{e}_i. \quad (2.39)$$

According to Fig. 2.3, we have

$$A_i = |\vec{A}| n_i = \vec{A} \cdot \hat{e}_i, \quad (2.40)$$

implying that  $A_i$  is the component or projection of  $\vec{A}$  along the direction of  $\hat{e}_i$ . We mention that the last formula can also be obtained by changing the summation index  $i$  to  $j$  in (2.33), performing a scalar product of both sides with  $\hat{e}_i$ , and then using (2.35). The detail is left to the readers as an exercise.

The algebraic operations of vectors in terms of their components in the rectangular coordinate system  $Ox_1x_2x_3$  can be straightforwardly proven to be

$$\vec{A} \pm \vec{B} = A_i \hat{e}_i \pm B_i \hat{e}_i = (A_i \pm B_i) \hat{e}_i, \quad (2.41)$$

$$\vec{A} \cdot \vec{B} = A_i \hat{e}_i \cdot B_j \hat{e}_j = A_i B_j \delta_{ij} = A_i B_i, \quad (2.42)$$

$$\vec{A} \times \vec{B} = A_i \hat{e}_i \times B_j \hat{e}_j = A_i B_j \epsilon_{ijk} \hat{e}_k, \quad (2.43)$$

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = A_i B_j \epsilon_{ijk} \hat{e}_k \cdot C_l \hat{e}_l = A_i B_j C_l \epsilon_{ijk} \delta_{kl} = A_i B_j C_k \epsilon_{ijk}. \quad (2.44)$$

The identities  $\hat{e}_k \cdot \hat{e}_l = \delta_{kl}$  and  $C_l \delta_{kl} = C_k$  were used in the derivation of the last formula. We emphasize again that changing a summation index from one letter to another does not affect the value of the expression. It can be also shown that

$$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{B} \cdot \vec{C}) \vec{A}, \quad (2.45)$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}). \quad (2.46)$$

To familiarize with the notations, we provide the proof of (2.46):

$$\begin{aligned} (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) &= A_i B_j \epsilon_{ijk} \hat{e}_k \cdot C_m D_n \epsilon_{mnl} \hat{e}_l \quad [\text{using (2.43)}] \\ &= A_i B_j \epsilon_{ijk} C_m D_n \epsilon_{mnl} \delta_{kl} \quad [\text{using (2.35)}] \\ &= A_i B_j C_m D_n \epsilon_{ijk} \epsilon_{mnk} \quad [\epsilon_{mnl} \delta_{kl} = \epsilon_{mnk}] \\ &= A_i B_j C_m D_n (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \quad [\text{using (2.31)}] \\ &= A_i B_j C_m \delta_{im} D_n \delta_{jn} - A_i B_j C_m \delta_{jm} D_n \delta_{in} \\ &= A_i B_j C_i D_j - A_i B_j C_j D_i \quad [\text{using (2.28)}] \end{aligned}$$

$$\begin{aligned}
&= (A_i C_i)(B_j D_j) - (A_i D_i)(B_j C_j) \\
&= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}). \quad [\text{using (2.42)}] \quad (2.47)
\end{aligned}$$

The proof of (2.45) is left to the readers as an exercise.

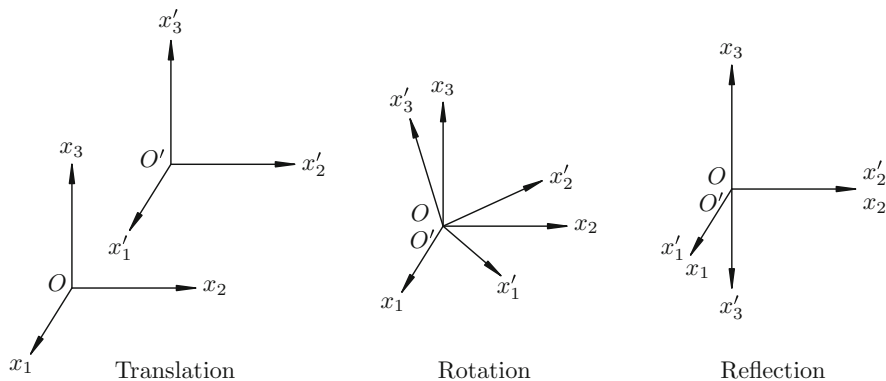
## 2.2 Coordinate Transformations and Tensors

### 2.2.1 Coordinate Transformations

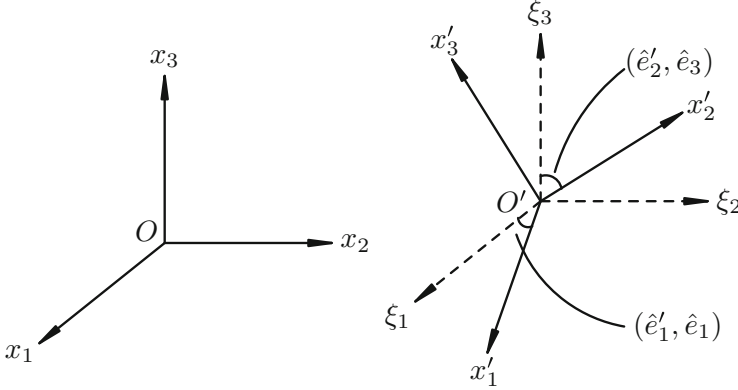
A coordinate transformation concerns the computation of scalars, vectors or tensors in a new coordinate system from their values in an old coordinate system. We consider only the coordinate transformation from one rectangular coordinate system to another rectangular coordinate system. A new coordinate system  $O'x'_1x'_2x'_3$  can be obtained from an old coordinate system  $Ox_1x_2x_3$  by a combination of three basic types of coordinate transformation as shown in Fig. 2.4: (1) Translation, which is a translation of the origin while maintaining the directions of the three axes unchanged. (2) Rotation, which is a rigid rotation of the triad composed of the three axes around the origin; the direction of any axis with respect to those of the other two axes remains unchanged. (3) Reflection, which is a flipping of the direction of one axis while maintaining the directions of the other two axes unchanged. Another coordinate transformation, called dilatation, concerns a change in the scales of the axes, and is not involved in this book.

As shown in Fig. 2.5, let  $O'x'_1x'_2x'_3$  be the new coordinate system, and  $Ox_1x_2x_3$  the old coordinate system, and their base vectors are denoted as  $\hat{e}'_i$  and  $\hat{e}_i$ , respectively. As both coordinate systems are rectangular, we have

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \quad \hat{e}'_i \cdot \hat{e}'_j = \delta_{ij}. \quad (2.48)$$



**Fig. 2.4** Three basic types of coordinate transformation



**Fig. 2.5** Angles among the axes of a new coordinate system  $O'x'_1x'_2x'_3$  and an old coordinate system  $Ox_1x_2x_3$

For evident reason, these relations hold for both Left- and right-handed coordinate systems.

We define an auxiliary coordinate system  $O'\xi_1\xi_2\xi_3$  with axes parallel to those of  $Ox_1x_2x_3$  one by one, as shown in Fig. 2.5. The base vector along the  $\xi_i$ -axis is then the same as that along the  $x_i$ -axis, and is denoted as  $\hat{e}_i$ . Hence, the angle between  $x'_i$  and  $x_j$  is identical to that between  $x'_i$  and  $\xi_j$ , and is denoted as  $(\hat{e}'_i, \hat{e}_j)$ . Define the cosines of the angles between the axes of  $O'\xi_1\xi_2\xi_3$  and those of  $Ox_1x_2x_3$  as

$$n_{ij} = \cos(\hat{e}'_i, \hat{e}_j) = \hat{e}'_i \cdot \hat{e}_j, \quad (2.49)$$

the second identity being a consequence that both  $\hat{e}'_i$  and  $\hat{e}_j$  are unit vectors. Evidently,  $n_{ij}$  is the component of  $\hat{e}'_i$  along  $\hat{e}_j$ , we have

$$\hat{e}'_i = n_{ij} \hat{e}_j. \quad (2.50)$$

Furthermore,  $n_{ij}$  can also be understood as the component of  $\hat{e}_j$  along  $\hat{e}'_i$ , resulting in

$$\hat{e}_j = n_{ij} \hat{e}'_i. \quad (2.51)$$

These last two formulae can be understood as the coordinate transformation between the base vectors.

By performing the scalar product of both sides of (2.50) with  $\hat{e}'_k$ , we obtain

$$n_{ij}n_{kj} = \delta_{ik}. \quad (2.52)$$

Similarly, by performing the scalar product of both sides of (2.51) with  $\hat{e}_k$ , we obtain

$$n_{ij}n_{ik} = \delta_{jk} . \quad (2.53)$$

These last two formulae are consequences of the fact that both  $Ox_1x_2x_3$  and  $O'x'_1x'_2x'_3$  are rectangular coordinate systems.

### 2.2.2 Coordinate Transformation of Coordinate-System-Independent Scalars and Vectors

When a quantity is coordinate-system-independent, it can be represented in different coordinate systems without affecting its value.

For a scalar, we use  $\phi$  and  $\phi'$  to represent its values in  $Ox_1x_2x_3$  and  $O'x'_1x'_2x'_3$ , respectively. Its coordinate-system-independence means, for the same spatial location,

$$\phi = \phi' . \quad (2.54)$$

A vector  $\vec{A}$  can be represented using its three components in a coordinate system. We use  $A_i$  and  $A'_i$  to denote its components in  $Ox_1x_2x_3$  and  $O'x'_1x'_2x'_3$ , respectively. Its coordinate-system-independence means

$$A_j \hat{e}_j = A'_j \hat{e}'_j . \quad (2.55)$$

Perform the scalar products of this equation by  $\hat{e}'_i$ . We obtain

$$A'_i = n_{ij} A_j . \quad (2.56)$$

This is the coordinate transformation of the coordinate-system-independent vector  $\vec{A}$  from  $Ox_1x_2x_3$  to  $O'x'_1x'_2x'_3$ . We can obtain (2.55) by multiplying both sides of (2.56) by  $\hat{e}'_i$  and then using (2.51). Hence, (2.56) is the sufficient and necessary condition for  $\vec{A}$  to be coordinate-system-independent. Therefore, we can also understand (2.56) as the analytic definition of the coordinate-system-independence of  $\vec{A}$ .

Change the summation index  $j$  to  $i$  in (2.55), and then perform the scalar product with  $\hat{e}_j$ . We obtain

$$A_j = n_{ij} A'_i . \quad (2.57)$$

This is the coordinate transformation of the coordinate-system-independent vector  $\vec{A}$  from  $O'x'_1x'_2x'_3$  to  $Ox_1x_2x_3$ . We can similarly show that this is the sufficient and necessary condition for  $\vec{A}$  to be coordinate-system-independent.



Finally, we further demonstrate that the coordinate transformations does not change the value of the vector:

$$\vec{A} = A_j \hat{e}_j = A_j n_{ij} \hat{e}'_i = A'_i \hat{e}'_i = \vec{A}, \quad (2.58)$$

$$\vec{A} = A'_i \hat{e}'_i = A'_i n_{ij} \hat{e}_j = A_j \hat{e}_j = \vec{A}. \quad (2.59)$$

These implies that the three components of a vector in a coordinate system can not be regarded as individual quantities.

### 2.2.3 Coordinate Transformation of the Coordinates of a Point

The transformation of the coordinates of a point from one coordinate system to another is essential for the transformation of the expression of a function from one coordinate system to another.

Let  $A$  be a point in the space, its coordinates in  $Ox_1x_2x_3$  and  $O'x'_1x'_2x'_3$  are denoted as  $x_i$  and  $x'_i$ , respectively. Its position vector with respect to  $O$ ,  $\vec{OA}$ , expressed in  $Ox_1x_2x_3$  is

$$\vec{r} = x_i \hat{e}_i, \quad (2.60)$$

and its position vector with respect to  $O'$ ,  $\vec{O'A}$ , expressed in  $O'x'_1x'_2x'_3$  is

$$\vec{r}' = x'_i \hat{e}'_i. \quad (2.61)$$

Let

$$\vec{r}^{O'} = x_i^{O'} \hat{e}_i \quad (2.62)$$

be the position vector of  $O'$  with respect to  $O$ ,  $\vec{OO'}$ , where  $x_i^{O'}$  are the coordinates of  $O'$  in  $Ox_1x_2x_3$ . We have

$$\vec{r} = \vec{r}^{O'} + \vec{r}', \quad (2.63)$$

whence

$$(x_j - x_j^{O'}) \hat{e}_j = \vec{r} - \vec{r}^{O'} = \vec{r}' = x'_i \hat{e}'_i, \quad (2.64)$$

where we have intentionally used different summation indices in the most left-hand side and the most right-hand side for convenience in further derivations. By substituting (2.51) and (2.50) into this equation, we obtain

$$n_{ij}(x_j - x_j^{O'}) \hat{e}'_i = x'_i \hat{e}'_i, \quad (2.65)$$

and

$$(x_j - x_j^{O'})\hat{e}_j = n_{ij}x_i'\hat{e}_j, \quad (2.66)$$

respectively. These are vector equations equivalent to the following equations of the components,

$$x_i' = n_{ij}(x_j - x_j^{O'}), \quad (2.67)$$

$$x_j - x_j^{O'} = n_{ij}x_i', \quad (2.68)$$

which are the coordinate transformations of the coordinates of the point  $A$  between  $Ox_1x_2x_3$  and  $O'x_1'x_2'x_3'$ .

Replace the summation index  $j$  by  $k$  in (2.67), and then take the derivative with respect to  $x_j$ . As  $\partial x_k / \partial x_j = \delta_{kj}$ , we obtain a property of the coordinate transformation,

$$\frac{\partial x_i'}{\partial x_j} = n_{ik} \frac{\partial x_k}{\partial x_j} = n_{ik} \delta_{kj} = n_{ij}. \quad (2.69)$$

Similarly, we can obtain, from (2.68),

$$\frac{\partial x_j}{\partial x_i'} = n_{kj} \frac{\partial x_k}{\partial x_i'} = n_{kj} \delta_{ki} = n_{ij}. \quad (2.70)$$

The readers are suggested to derive the alternative formulae when the coordinates of  $O$  in  $O'x_1'x_2'x_3'$  are known, and further examine the special case when  $O'$  and  $O$  coincide.

### 2.2.4 Definition of Tensors

A coordinate-system-independent scalar has  $3^0 = 1$  component, which is independent of the directions of the coordinate axes. A coordinate-system-independent vector has  $3^1 = 3$  components, where each component is associated to the direction of one coordinate axis. We now generalize the definition of coordinate-system-independent scalars and vectors to introduce a new quantity with  $3^2 = 9$  components, where each component is associated to the directions of two axes. We write this quantity in  $Ox_1x_2x_3$  in the form of

$$\overline{T} = T_{ij}\hat{e}_i\hat{e}_j, \quad (2.71)$$

where  $\hat{e}_i\hat{e}_j$  is used as a symbol, called dyadic, indicating that the component  $T_{ij}$  is first associated to  $\hat{e}_i$ , and then to  $\hat{e}_j$ . Naturally, by definition,  $\hat{e}_i$  and  $\hat{e}_j$  in the dyadic

cannot be swapped. Similarly, we write the expression of  $\overline{T}$  in the coordinate system  $O'x'_1x'_2x'_3$  as

$$\overline{T} = T'_{mn} \hat{e}'_m \hat{e}'_n. \quad (2.72)$$

In accordance with the coordinate transformation of the components of a coordinate-system-independent vector, (2.56), if the coordinate transformation from the components  $T_{ij}$  to the components  $T'_{mn}$  is

$$T'_{mn} = n_{mi} n_{nj} T_{ij}, \quad (2.73)$$

we call  $\overline{T}$  a second order tensor, and  $T_{ij}$  the component of the tensor. In this book, we use an overline to denote tensors.

The coordinate transformation from the components  $T'_{mn}$  to the components  $T_{ij}$  can be obtained by multiplying both sides of (2.73) by  $n_{mk} n_{nl}$ ,

$$n_{mk} n_{nl} T'_{mn} = n_{mi} n_{mk} n_{nj} n_{nl} T_{ij}. \quad (2.74)$$

According to (2.53), we obtain

$$n_{mk} n_{nl} T'_{mn} = \delta_{ik} \delta_{jl} T_{ij} = T_{kl}. \quad (2.75)$$

After replacing the summation indices  $k$  and  $l$  by  $i$  and  $j$ , respectively, we obtain

$$T_{ij} = n_{mi} n_{nj} T'_{mn}. \quad (2.76)$$

This is evidently a kind of generalization of the coordinate transformation of a coordinate-system-independent vector, (2.57).

Coordinate transformations for the dyadics must also be defined in conformance with those for the components:

$$\hat{e}'_m \hat{e}'_n = n_{mi} \hat{e}_i n_{nj} \hat{e}_j = n_{mi} n_{nj} \hat{e}_i \hat{e}_j, \quad (2.77)$$

$$\hat{e}_i \hat{e}_j = n_{mi} \hat{e}'_m n_{nj} \hat{e}'_n = n_{mi} n_{nj} \hat{e}'_m \hat{e}'_n. \quad (2.78)$$

These mean that the coordinate transformations of the two base vectors in a dyadic are done independently without swapping the order.

With the coordination transformations of the dyadics defined in (2.77) and (2.78), we can perform coordinate transformations for a second order tensor in the natural way:

$$\overline{T} = T'_{mn} \hat{e}'_m \hat{e}'_n = T'_{mn} n_{mi} n_{nj} \hat{e}_i \hat{e}_j = T_{ij} \hat{e}_i \hat{e}_j = \overline{T}, \quad (2.79)$$

$$\overline{T} = T_{ij} \hat{e}_i \hat{e}_j = T_{ij} n_{mi} n_{nj} \hat{e}'_m \hat{e}'_n = T'_{mn} \hat{e}'_m \hat{e}'_n = \overline{T}. \quad (2.80)$$

These confirm that, with the coordinate transformations defined for the components and dyadics, the second order tensor is coordinate-system-independent, and its components in a coordinate system cannot be treated as individual quantities.

When  $T_{ij} = \delta_{ij}$ ,  $\bar{T}$  is referred to as a unit tensor, and is denoted as  $\bar{I}$ .

When  $T_{ij} = T_{ji}$ ,  $\bar{T}$  is referred to as symmetric, and when  $T_{ij} = -T_{ji}$ ,  $\bar{T}$  is referred to as antisymmetric. If  $T_{ij} = T_{ji}$ , we have

$$T'_{mn} = n_{mi}n_{nj}T_{ij} = n_{nj}n_{mi}T_{ji} = T'_{nm}. \quad (2.81)$$

If  $T_{ij} = -T_{ji}$ , we have

$$T'_{mn} = n_{mi}n_{nj}T_{ij} = -n_{nj}n_{mi}T_{ji} = -T'_{nm}. \quad (2.82)$$

Hence, a symmetric or antisymmetric tensor remains symmetric or antisymmetric after a coordinate transformation.

The concept of second order tensor can be generalized to any order. A coordinate-system-independent scalar is a zero order tensor. A coordinate-system-independent vector is a first order tensor. A third order tensor is defined as

$$\bar{T} = T_{ijk}\hat{e}_i\hat{e}_j\hat{e}_k, \quad (2.83)$$

$$\bar{T} = T'_{lmn}\hat{e}'_l\hat{e}'_m\hat{e}'_n, \quad (2.84)$$

with the coordinate transformations

$$T'_{lmn} = n_{li}n_{mj}n_{nk}T_{ijk}, \quad (2.85)$$

$$T_{ijk} = n_{li}n_{mj}n_{nk}T'_{lmn}, \quad (2.86)$$

and

$$\hat{e}'_l\hat{e}'_m\hat{e}'_n = n_{li}n_{mj}n_{nk}\hat{e}_i\hat{e}_j\hat{e}_k, \quad (2.87)$$

$$\hat{e}_i\hat{e}_j\hat{e}_k = n_{li}n_{mj}n_{nk}\hat{e}'_l\hat{e}'_m\hat{e}'_n. \quad (2.88)$$

Higher order tensors can be defined similarly. The most used in mechanics are second and forth order tensors.

### 2.2.5 Algebraic Operations of Tensors

As second order tensors are to be concerned mostly, we limit ourselves to second order tensors here.

The addition and subtraction of two tensors are defined as

$$\bar{T} \pm \bar{P} = (T_{ij} \pm P_{ij})\hat{e}_i\hat{e}_j. \quad (2.89)$$

The tensor product of two first order tensors is defined as

$$\vec{A}\vec{B} = A_i B_j \hat{e}_i \hat{e}_j . \quad (2.90)$$

The product of a scalar and a tensor is defined as

$$\lambda \vec{T} = (\lambda T_{ij}) \hat{e}_i \hat{e}_j . \quad (2.91)$$

It can be easily shown that the results of all these operation are second order tensors, which is left to the readers.

Two forms of inner product of a first order tensor and a second one are defined:

$$\vec{A} \cdot \vec{T} = A_k \hat{e}_k \cdot T_{ij} \hat{e}_i \hat{e}_j = A_k T_{ij} (\hat{e}_k \cdot \hat{e}_i) \hat{e}_j = A_k T_{ij} \delta_{ik} \hat{e}_j = A_i T_{ij} \hat{e}_j , \quad (2.92)$$

$$\vec{T} \cdot \vec{A} = T_{ij} \hat{e}_i \hat{e}_j \cdot A_k \hat{e}_k = T_{ij} A_k \hat{e}_i (\hat{e}_j \cdot \hat{e}_k) = T_{ij} A_k \hat{e}_i \delta_{jk} = T_{ij} A_j \hat{e}_i . \quad (2.93)$$

The results of both of them are first order tensors. It can be seen that, in  $\vec{A} \cdot \vec{T}$ , the base vector of  $\vec{A}$  operates with the first base vector in  $\vec{T}$ , and in  $\vec{T} \cdot \vec{A}$ , the last base vector in  $\vec{T}$  operates with the base vector in  $\vec{A}$ , i.e., the operations of base vectors are performed between adjacent ones, which is the rule of operations of base vectors in tensor algebra.

A type of inner product of two tensors is defined as

$$\begin{aligned} \vec{T} \cdot \vec{P} &= T_{ij} \hat{e}_i \hat{e}_j \cdot P_{mn} \hat{e}_m \hat{e}_n = T_{ij} P_{mn} \hat{e}_i (\hat{e}_j \cdot \hat{e}_m) \hat{e}_n \\ &= T_{ij} P_{mn} \hat{e}_i \delta_{jm} \hat{e}_n = T_{ij} P_{jn} \hat{e}_i \hat{e}_n . \end{aligned} \quad (2.94)$$

We see again that the operations of the base vectors are performed between adjacent ones, and the orders of the base vectors do not swap. It can be shown that the result is a first order tensor.

Two other types of inner product of two tensors are defined as

$$\vec{T} : \vec{P} = T_{ij} \hat{e}_i \hat{e}_j : P_{mn} \hat{e}_m \hat{e}_n = T_{ij} P_{mn} (\hat{e}_i \cdot \hat{e}_m) (\hat{e}_j \cdot \hat{e}_n) = T_{ij} P_{ij} . \quad (2.95)$$

$$\vec{T} \cdot \cdot \vec{P} = T_{ij} \hat{e}_i \hat{e}_j \cdot \cdot P_{mn} \hat{e}_m \hat{e}_n = T_{ij} P_{mn} (\hat{e}_j \cdot \hat{e}_m) (\hat{e}_i \cdot \hat{e}_n) = T_{ij} P_{ji} . \quad (2.96)$$

It can be shown that the results of these operations are zero order tensors. Another operation leading to a zero order tensor is the tensor contraction defined as

$$\text{tr} \vec{T} = T_{ii} . \quad (2.97)$$

The result is referred to as the trace of the tensor.

It can be seen that the components of a second order tensor is related to the tensor itself in the form of

$$T_{ij} = (\hat{e}_i \cdot \vec{T}) \cdot \hat{e}_j = \hat{e}_i \cdot (\vec{T} \cdot \hat{e}_j) = \vec{T} : \hat{e}_i \hat{e}_j . \quad (2.98)$$

The transpose of a tensor consists of swapping the indices of the components. The transpose of the tensor  $\bar{T}$  is denoted as  $\bar{T}^T$ , which is, by definition,

$$\bar{T}^T = T_{ji} \hat{e}_i \hat{e}_j. \quad (2.99)$$

It is also a second order tensor.

Any tensor  $\bar{T}$  can be decomposed into the sum of a symmetric tensor  $\bar{S}$  and an antisymmetric tensor  $\bar{A}$ ,

$$\bar{T} = \bar{S} + \bar{A}, \quad (2.100)$$

where

$$\bar{S} = \frac{1}{2}(\bar{T} + \bar{T}^T), \quad \bar{A} = \frac{1}{2}(\bar{T} - \bar{T}^T), \quad (2.101)$$

or, in components,

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ji}), \quad A_{ij} = \frac{1}{2}(T_{ij} - T_{ji}). \quad (2.102)$$

It is left to the readers to proof that both  $\bar{S}$  and  $\bar{A}$  are second order tensors.

## 2.3 Some Properties of Second Order Tensors

### 2.3.1 Principal Values, Principal Axes and Numerical Invariants

Second order tensors have extensive applications in mechanics. The most common examples include the moment of inertia tensor, stress tensor and strain tensor.

We start by solving the following problem: For any second order tensor  $\bar{T} = T_{ij} \hat{e}_i \hat{e}_j$ , define a vector

$$\vec{t} = \hat{l} \cdot \bar{T} = l_i T_{ij} \hat{e}_j, \quad (2.103)$$

where  $\hat{l} = l_i \hat{e}_i$  is a unit vector. We seek such an  $\hat{l}$  that  $\vec{t}$  and  $\hat{l}$  be in the same direction, i.e.,  $\vec{t} = \sigma \hat{l}$ , where  $\sigma$  is the norm of  $\vec{t}$ .

To find such an  $\hat{l}$ , we express  $\vec{t}$  in the form of

$$\vec{t} = \sigma \hat{l} = l_j \sigma \hat{e}_j = l_i \delta_{ij} \sigma \hat{e}_j. \quad (2.104)$$

We then have, according to (2.103),

$$l_i T_{ij} \hat{e}_j - l_i \delta_{ij} \sigma \hat{e}_j = l_i (T_{ij} - \delta_{ij} \sigma) \hat{e}_j = 0. \quad (2.105)$$

This is a vector equation, which can be written as

$$l_i(T_{ij} - \delta_{ij}\sigma) = 0. \quad (2.106)$$

In matrix form, it is

$$\begin{bmatrix} T_{11} - \sigma & T_{12} & T_{13} \\ T_{21} & T_{22} - \sigma & T_{23} \\ T_{31} & T_{32} & T_{33} - \sigma \end{bmatrix}^T \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = 0. \quad (2.107)$$

This is a system of homogeneous linear algebraic equation. In order for it to have a non-vanishing solution, the determinant of the coefficient matrix must vanish, i.e.,

$$\begin{vmatrix} T_{11} - \sigma & T_{12} & T_{13} \\ T_{21} & T_{22} - \sigma & T_{23} \\ T_{31} & T_{32} & T_{33} - \sigma \end{vmatrix} = 0. \quad (2.108)$$

By expanding the determinant, we can write it as

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0, \quad (2.109)$$

where

$$I_1 = T_{11} + T_{22} + T_{33} = T_{ii} = \text{tr} \bar{T}, \quad (2.110)$$

$$\begin{aligned} I_2 &= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} \\ &= \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji}) = \frac{1}{2}[(\text{tr} \bar{T})^2 - \bar{T} : \bar{T}], \end{aligned} \quad (2.111)$$

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \det \bar{T}, \quad (2.112)$$

where  $\det \bar{T}$  denotes the determinant of  $\bar{T}$ .

Since (2.109) is a third-degree algebraic equation, it has three solutions. Let  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  be the solutions. We can then write (2.109) as

$$(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3) = 0. \quad (2.113)$$

This leads to

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad (2.114)$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1, \quad (2.115)$$

$$I_3 = \sigma_1 \sigma_2 \sigma_3 . \quad (2.116)$$

We define the solutions of (2.109),  $\sigma$ , as the principal values of  $\bar{T}$ , also referred to as eigenvalues, and the corresponding non-vanishing solutions of (2.107),  $\hat{l} = l_i \hat{e}_i$ , as the principal axes of  $\bar{T}$ , also referred to as eigendirections.

Next, we show that the principal values  $\sigma$  remain the same after a coordinate transformation. In the new coordinate system  $O'x'_1x'_2x'_3$ , (2.106) has the form

$$l'_m (T'_{mn} - \delta_{mn} \sigma') = 0 . \quad (2.117)$$

By definition,  $\hat{l} = l'_m \hat{e}'_m = l_i \hat{e}_i$ , and hence,  $l'_m = n_{mi} l_i$ . We can thus write the above equation as

$$l_i (n_{mi} T'_{mn} - n_{ni} \sigma') = 0 . \quad (2.118)$$

By multiplying both sides by  $n_{nj}$ , we obtain, according to the relation between  $T'_{mn}$  and  $T_{ij}$  as well as the relation  $n_{ni} n_{nj} = \delta_{ij}$ ,

$$l_i (T_{ij} - \delta_{ij} \sigma') = 0 . \quad (2.119)$$

By comparing this result with (2.106), we obtain

$$\sigma' = \sigma . \quad (2.120)$$

This implies that the principal values are coordinate-system-independent. Consequently,  $I_1$ ,  $I_2$  and  $I_3$  are also coordinate-system-independent, and are referred to as invariants of coordinate transformations.

### 2.3.2 A Special Case: Second Order Symmetric Tensors

First, the principal values of a symmetric second order tensor are all real numbers. Write the equation for principal values, (2.106), as

$$l_i T_{ij} = \sigma l_j . \quad (2.121)$$

We assume that  $l_i$ ,  $l_j$  and  $\sigma$  are all complex. We use the superscript '\*' to denote the complex conjugate. We obtain, by multiplying both sides of (2.121) by  $l_j^*$ ,

$$l_i T_{ij} l_j^* = \sigma l_j l_j^* . \quad (2.122)$$

Take the conjugate at both sides. As  $T_{ij}$  is real, we obtain

$$l_i^* T_{ij} l_j = \sigma^* l_j^* l_j . \quad (2.123)$$



According to the symmetry of  $\bar{T}$ ,  $T_{ij} = T_{ji}$ , we obtain

$$l_i T_{ij} l_j^* = l_j^* T_{ji} l_i = l_i^* T_{ij} l_j. \quad (2.124)$$

Hence, by subtracting (2.123) from (2.122), we obtain

$$(\sigma - \sigma^*) l_j l_j^* = 0. \quad (2.125)$$

As  $l_j l_j^* \neq 0$ ,  $\sigma - \sigma^* = 0$  must hold. Therefore,

$$\sigma = \sigma^*, \quad (2.126)$$

which implies that  $\sigma$  is a real number.

Second, the principal axes of a symmetric second order tensor are orthogonal each other. As the principal values  $\sigma$  of a symmetric second order tensor are real numbers, the components of the principal axes  $\hat{l}$  are also real numbers according to (2.106) or (2.107). Denote the three principal axes as  $\hat{l}_1$ ,  $\hat{l}_2$  and  $\hat{l}_3$ . We have

$$\hat{l}_1 \cdot \bar{T} = \sigma_1 \hat{l}_1, \quad \hat{l}_2 \cdot \bar{T} = \sigma_2 \hat{l}_2, \quad \hat{l}_3 \cdot \bar{T} = \sigma_3 \hat{l}_3. \quad (2.127)$$

Perform the scalar products of the first and second identities with  $\hat{l}_2$  and  $\hat{l}_1$ , respectively. We obtain

$$(\hat{l}_1 \cdot \bar{T}) \cdot \hat{l}_2 = \sigma_1 \hat{l}_1 \cdot \hat{l}_2, \quad (\hat{l}_2 \cdot \bar{T}) \cdot \hat{l}_1 = \sigma_2 \hat{l}_2 \cdot \hat{l}_1. \quad (2.128)$$

According to the symmetry of  $\bar{T}$ , i.e.,  $T_{ij} = T_{ji}$ , we have

$$(\hat{l}_1 \cdot \bar{T}) \cdot \hat{l}_2 = l_{1i} T_{ij} l_{2j} = l_{2j} T_{ji} l_{1i} = (\hat{l}_2 \cdot \bar{T}) \cdot \hat{l}_1, \quad (2.129)$$

where  $l_{ij}$  is the component of  $\hat{l}_i$  along  $\hat{e}_j$ , i.e.,  $l_{ij} = \hat{l}_i \cdot \hat{e}_j$ . It can then be readily obtained from (2.128) by taking reference to (2.129) that  $\sigma_1 \hat{l}_1 \cdot \hat{l}_2 = \sigma_2 \hat{l}_2 \cdot \hat{l}_1$ , i.e.,

$$(\sigma_1 - \sigma_2) \hat{l}_1 \cdot \hat{l}_2 = 0. \quad (2.130)$$

In the general case, as  $\sigma_1 - \sigma_2 \neq 0$ , we have  $\hat{l}_1 \cdot \hat{l}_2 = 0$ , i.e.,  $\hat{l}_1 \perp \hat{l}_2$ . Similarly,  $\hat{l}_2 \perp \hat{l}_3$  and  $\hat{l}_3 \perp \hat{l}_1$ . We have thus proven the orthogonality among the principal axes. We remark that, in the case of a multiple root of  $\sigma$ , the solution of  $\hat{l}$  corresponding to this root is not unique; in fact, the number of solution is equal to the number of order of the root, which can be made orthogonal based on an orthogonalization procedure.

An important result obtained from the orthogonality of the principal axes is that a symmetric second order tensor can be reduced to a diagonal form by a coordinate transformation. Taking into account of the orthogonality of the principal axes, we obtain from the first equation of (2.127),

$$\hat{l}_1 \cdot \bar{T} \cdot \hat{l}_j = \sigma_1 \hat{l}_1 \cdot \hat{l}_j = \delta_{1j} \sigma_1. \quad (2.131)$$

If we take  $x'_1, x'_2$  and  $x'_3$  to be along the principal axes  $\hat{l}_1, \hat{l}_2$  and  $\hat{l}_3$ , respectively, i.e., let  $\hat{e}'_1 = \hat{l}_1, \hat{e}'_2 = \hat{l}_2$  and  $\hat{e}'_3 = \hat{l}_3$ , we can obtain from the equation above,

$$T'_{1j} = \hat{e}'_1 \cdot \bar{T} \cdot \hat{e}'_j = \delta_{1j} \sigma_1, \quad (2.132)$$

which means  $T'_{11} = \sigma_1$  and  $T'_{12} = T'_{13} = 0$ . Similarly, we can prove  $T'_{22} = \sigma_2$  and  $T'_{21} = T'_{23} = 0$ , and  $T'_{33} = \sigma_3$  and  $T'_{31} = T'_{32} = 0$ . Hence,  $\bar{T}$  is diagonal in the coordinate system  $O'x'_1x'_2x'_3$ . This kind of coordinate system is referred to as principal axes coordinate system.

### 2.3.3 Decomposition into the Sum of a Spherical and a Deviatoric Tensors

In continuum mechanics, the stress and strain tensors are often expressed as the sum of a spherical tensor and a deviatoric tensor. For a second order tensor  $\bar{T}$ , define

$$T^S_{ij} = T_0 \delta_{ij}, \quad (2.133)$$

where

$$T_0 = \frac{1}{3} T_{kk}. \quad (2.134)$$

As  $T_{kk}$  is coordinate-system-independent, the components  $T^S_{ij}$  do not vary with coordinate transformation, and hence, they form a tensor,  $\bar{T}^S$ , which is referred to as the spherical tensor of  $\bar{T}$ . Evidently, it is diagonal, and all components on the diagonal are equal, which is the average of the components of  $T$  on the diagonal. The difference between  $\bar{T}$  and  $\bar{T}^S$ ,

$$\bar{T}^D = \bar{T} - \bar{T}^S, \quad (2.135)$$

is referred to as the deviatoric tensor of  $\bar{T}$ . In this way,  $\bar{T}$  is decomposed into the sum of the spherical tensor  $\bar{T}^S$  and the deviatoric tensor  $\bar{T}^D$ ,

$$\bar{T} = \bar{T}^S + \bar{T}^D. \quad (2.136)$$

## 2.4 Quotient Rule and Isotropic Tensors

### 2.4.1 Quotient Rule

In physics, there is often a need to relate two tensors. The simplest is to express each component of one tensor as a linear combination of all components of another

tensor, i.e.,

$$\sigma_{i_1 i_2 \dots i_p} = C_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q} \epsilon_{j_1 j_2 \dots j_q}, \quad (2.137)$$

where  $\sigma_{i_1 i_2 \dots i_p}$  are the components of a tensor  $\bar{\sigma}$  of order  $p$ , and  $\epsilon_{j_1 j_2 \dots j_q}$  those of a tensor  $\bar{\epsilon}$  of order  $q$ .

The quotient rule tells that, if the result of  $\bar{\sigma}$  computed according to (2.137) is always a tensor of order  $p$  for any tensor  $\bar{\epsilon}$  of order  $q$  arbitrarily chosen, the coefficients  $C_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q}$  are the components of a tensor  $\bar{C}$  of order  $p + q$ .

Here we provide the proof for the practical case when both  $\bar{\sigma}$  and  $\bar{\epsilon}$  are second order tensors, which appear in elasticity. In this case, (2.137) degenerates to

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}. \quad (2.138)$$

Its form after the coordinate transformation is

$$\sigma'_{mn} = C'_{mnpq} \epsilon'_{pq}. \quad (2.139)$$

By transforming  $\sigma_{ij}$  and  $\epsilon_{kl}$  in (2.138) to  $\sigma'_{rs}$  and  $\epsilon'_{pq}$ , respectively, we obtain

$$n_{ri} n_{sj} \sigma'_{rs} = C_{ijkl} n_{pk} n_{ql} \epsilon'_{pq}. \quad (2.140)$$

By multiplying both sides by  $n_{mi} n_{nj}$ , we obtain

$$n_{mi} n_{ri} n_{nj} n_{sj} \sigma'_{rs} = n_{mi} n_{nj} n_{pk} n_{ql} C_{ijkl} \epsilon'_{pq}, \quad (2.141)$$

which can be simplified to, according to (2.52),

$$\delta_{mr} \delta_{ns} \sigma'_{rs} = n_{mi} n_{nj} n_{pk} n_{ql} C_{ijkl} \epsilon'_{pq}. \quad (2.142)$$

Finally,

$$\sigma'_{mn} = n_{mi} n_{nj} n_{pk} n_{ql} C_{ijkl} \epsilon'_{pq}. \quad (2.143)$$

By subtracting this formula from (2.139), we obtain

$$(C'_{mnpq} - n_{mi} n_{nj} n_{pk} n_{ql} C_{ijkl}) \epsilon'_{pq} = 0. \quad (2.144)$$

Considering that  $\epsilon'_{pq}$  is arbitrary based on the assumption that  $\epsilon_{kl}$  is arbitrary, we finally obtain

$$C'_{mnpq} = n_{mi} n_{nj} n_{pk} n_{ql} C_{ijkl}. \quad (2.145)$$

This relation means that  $\bar{C}$  is a fourth order tensor.

As an exercise, the readers are suggested to prove for the case when both  $\bar{\sigma}$  and  $\bar{\epsilon}$  are first order tensors.

### 2.4.2 Fourth Order Isotropic Tensor

In physical formulae in the form of (2.137),  $\bar{C}$  is mostly used to describe a property of the material studied. The special form (2.138) is of particular importance in elasticity, where  $\bar{\sigma}$  is the stress tensor,  $\bar{\epsilon}$  the strain tensor, and  $\bar{C}$  the elastic coefficient tensor. The simplest case is for a material that is isotropic, i.e., the property is independent of direction, which is defined mathematically as that the relation (2.138) remains the same in all coordinate systems. This implies that the components of  $\bar{C}$ ,  $C_{ijkl}$ , is the same in all coordinate systems. This kind of tensor is referred to as isotropic tensor. In this book, we provide the formulation for the fourth order isotropic tensor. The readers are suggested to formulate for the second order isotropic tensor in parallel.

By definition, the components are identical in all coordinate systems, i.e.,

$$C'_{ijkl} = C_{ijkl} . \quad (2.146)$$

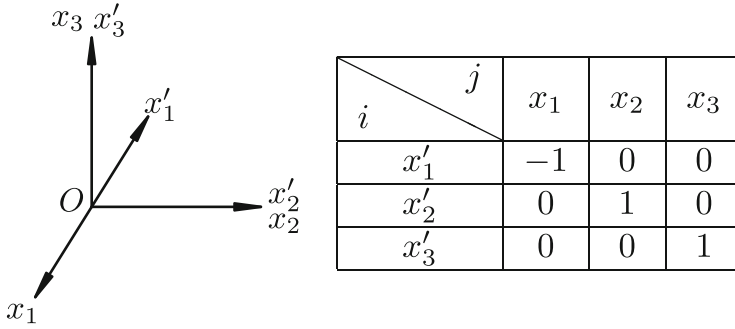
All of them are listed in Fig. 2.6, where the meanings of other symbols are to be explained later. Our purpose of this subsection is to derive a general expression of  $\bar{C}$  that obeys (2.146).

#### Reflection of One Coordinate Axis

As shown in Fig. 2.7, we define a new coordinate system  $O'x'_1x'_2x'_3$  by reflecting the  $x_1$ -axis while keeping the  $x_2$ - and  $x_3$ -axes unchanged. The values of  $n_{ij}$  are

|                              |                              |                              |                              |                              |                              |                              |                              |                              |
|------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|
| <u><math>C_{1111}</math></u> | $C_{1112}$                   | $C_{1113}$                   | $C_{1121}$                   | <u><math>C_{1122}</math></u> | $C_{1123}$                   | $C_{1131}$                   | $C_{1132}$                   | <u><math>C_{1133}</math></u> |
| $C_{1211}$                   | <u><math>C_{1212}</math></u> | $C_{1213}$                   | <u><math>C_{1221}</math></u> | $C_{1222}$                   | $C_{1223}$                   | $C_{1231}$                   | $C_{1232}$                   | $C_{1233}$                   |
| $C_{1311}$                   | $C_{1312}$                   | <u><math>C_{1313}</math></u> | $C_{1321}$                   | $C_{1322}$                   | $C_{1323}$                   | <u><math>C_{1331}</math></u> | $C_{1332}$                   | $C_{1333}$                   |
| $C_{2111}$                   | <u><math>C_{2112}</math></u> | $C_{2113}$                   | <u><math>C_{2121}</math></u> | $C_{2122}$                   | $C_{2123}$                   | $C_{2131}$                   | $C_{2132}$                   | $C_{2133}$                   |
| <u><math>C_{2211}</math></u> | $C_{2212}$                   | $C_{2213}$                   | <u><math>C_{2221}</math></u> | <u><math>C_{2222}</math></u> | $C_{2223}$                   | $C_{2231}$                   | $C_{2232}$                   | <u><math>C_{2233}</math></u> |
| $C_{2311}$                   | $C_{2312}$                   | $C_{2313}$                   | $C_{2321}$                   | $C_{2322}$                   | <u><math>C_{2323}</math></u> | $C_{2331}$                   | <u><math>C_{2332}</math></u> | $C_{2333}$                   |
| $C_{3111}$                   | $C_{3112}$                   | <u><math>C_{3113}</math></u> | $C_{3121}$                   | $C_{3122}$                   | $C_{3123}$                   | <u><math>C_{3131}</math></u> | $C_{3132}$                   | $C_{3133}$                   |
| $C_{3211}$                   | $C_{3212}$                   | $C_{3213}$                   | $C_{3221}$                   | $C_{3222}$                   | <u><math>C_{3223}</math></u> | $C_{3231}$                   | <u><math>C_{3232}</math></u> | $C_{3233}$                   |
| <u><math>C_{3311}</math></u> | $C_{3312}$                   | $C_{3313}$                   | $C_{3321}$                   | <u><math>C_{3322}</math></u> | $C_{3323}$                   | $C_{3331}$                   | $C_{3332}$                   | <u><math>C_{3333}</math></u> |

**Fig. 2.6** All components of a fourth order isotropic tensor

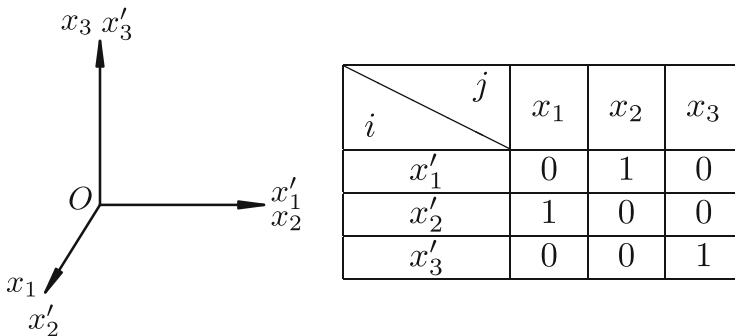


**Fig. 2.7** Reflection of  $x_1$  (left) and the values of  $n_{ij}$  (right)

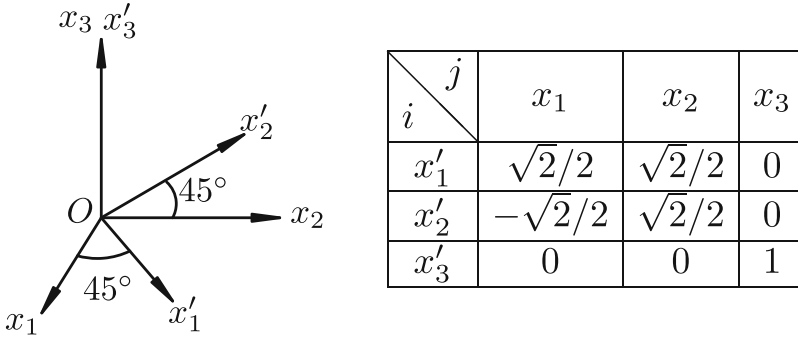
also listed. According to (2.145), we have  $C'_{1222} = n_{1i}n_{2j}n_{2k}n_{2l}C_{ijkl} = -C_{1222}$ . According to (2.146), we also have  $C'_{1222} = C_{1222}$ . Hence,  $C_{1222} = 0$ . This logic can be applied to all components with one index equal to 1, and all other indices different from 1, and consequently, all these components are also equal to zero. Similarly, all components with one index equal to 2, and all other indices different from 2, are equal to zero; and all components with one index equal to 3, and all other indices different from 3, are equal to zero. In Fig 2.6, we have crossed out all these zero components using a slant line, and we can see that only 21 non-zero components are left.

### Swapping of Two Coordinate Axes

As shown in Fig. 2.8, we define a new coordinate system  $O'x'_1x'_2x'_3$  by swapping the  $x_1$ - and  $x_2$ -axes while keeping the  $x_3$ -axis unchanged. The values of  $n_{ij}$  are also listed. According to (2.145), we have  $C'_{3311} = n_{3i}n_{3j}n_{1k}n_{1l}C_{ijkl} = C_{3322}$ . As  $C'_{3311} = C_{3311}$  according to (2.146), we obtain  $C_{3311} = C_{3322}$ . Many similar identities can be found based on this approach. By inspecting Fig. 2.8, we can infer that, for any  $C_{ijkl}$ , changing the index 1 to 2, 2 to 1, while keeping 3 unchanged lead



**Fig. 2.8** Swapping of  $x_1$  and  $x_2$  (left) and the values of  $n_{ij}$  (right)



**Fig. 2.9** Rotation of  $45^\circ$  around  $x_3$  (left) and the values of  $n_{ij}$  (right)

to a component of the same value. Similarly, changing the index 1 to 3, 3 to 1, while keeping 2 unchanged lead to a component of the same value, and changing the index 2 to 3, 3 to 2, while keeping 1 unchanged lead to a component of the same value. We have marked in Fig. 2.6 the equal components with the same symbols, including underline, double underline, tilde underline and double tilde underline. We see that there are only 4 independent components left.

### Rotation of the Coordinate System

As shown in Fig. 2.9, we define a new coordinate system  $O'x'_1x'_2x'_3$  by rotating the original coordinate system  $45^\circ$  around the  $x_3$ -axis. The values of  $n_{ij}$  are also listed. According to (2.145), we have, after omitting the zero-value components crossed out in Fig. 2.6,

$$\begin{aligned}
 C'_{1111} &= n_{1i}n_{1j}n_{1k}n_{1l}C_{ijkl} \\
 &= n_{11}n_{11}n_{11}n_{11}C_{1111} + n_{11}n_{11}n_{12}n_{12}C_{1122} + n_{11}n_{12}n_{11}n_{12}C_{1212} \\
 &\quad + n_{11}n_{12}n_{12}n_{11}C_{1221} + n_{12}n_{11}n_{11}n_{12}C_{2112} + n_{12}n_{11}n_{12}n_{11}C_{2121} \\
 &\quad + n_{12}n_{12}n_{11}n_{11}C_{2211} + n_{12}n_{12}n_{12}n_{12}C_{2222} \\
 &= \frac{1}{4}(C_{1111} + C_{1122} + C_{1212} + C_{1221} + C_{2112} + C_{2121} + C_{2211} + C_{2222}).
 \end{aligned} \tag{2.147}$$

We have proven, as shown in Fig. 2.6, that  $C_{1111} = C_{2222}$ ,  $C_{1122} = C_{2211}$ ,  $C_{1212} = C_{2121}$  and  $C_{1221} = C_{2112}$ . According to (2.146), we also have  $C'_{1111} = C_{1111}$ . Hence, the above formula is reduced to

$$C_{1111} = \frac{1}{2}(C_{1111} + C_{1122} + C_{1212} + C_{1221}). \tag{2.148}$$

Finally, we obtain

$$C_{1111} = C_{1122} + C_{1212} + C_{1221} . \quad (2.149)$$

This relation reduce the number of independent components to 3.

### The General Expression of a Fourth Order Isotropic Tensor

Let  $\lambda = C_{1122}$ ,  $\mu = C_{1212}$  and  $\gamma = C_{1221}$ . We can write  $C_{ijkl}$  as, taking Fig. 2.6 and (2.149) into account,

$$C_{ijkl} = \begin{cases} \lambda, & i = j, k = l, \text{ and } i \neq k; \\ \mu, & i = k, j = l, \text{ and } i \neq j; \\ \gamma, & i = l, j = k, \text{ and } i \neq j; \\ \lambda + \mu + \gamma, & i = j = k = l; \\ 0, & \text{Otherwise;} \end{cases} \quad (2.150)$$

which is equivalent to the expression

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} . \quad (2.151)$$

What we have shown is that (2.151) holds if  $\bar{C}$  is an isotropic tensor. We must also verify that (2.151) is a sufficient condition for  $\bar{C}$  to be an isotropic tensor. This is straightforward based on the coordinate transformation (2.145):

$$\begin{aligned} C'_{mnpq} &= \lambda n_{mi} n_{nj} n_{pk} n_{ql} \delta_{ij} \delta_{kl} + \mu n_{mi} n_{nj} n_{pk} n_{ql} \delta_{ik} \delta_{jl} + \gamma n_{mi} n_{nj} n_{pk} n_{ql} \delta_{il} \delta_{jk} \\ &= \lambda n_{mi} n_{ni} n_{pl} n_{ql} + \mu n_{mk} n_{nl} n_{pk} n_{ql} + \gamma n_{ml} n_{nk} n_{pk} n_{ql} \\ &= \lambda \delta_{mn} \delta_{pq} + \mu \delta_{mp} \delta_{nq} + \gamma \delta_{mq} \delta_{np} , \end{aligned} \quad (2.152)$$

where (2.52) was used to derive the last identity. Based on (2.151) and (2.152), we conclude that (2.146) holds for the  $C_{ijkl}$  given by (2.151), meaning that (2.151) is a sufficient condition for  $\bar{C}$  to be an isotropic tensor.

The readers are suggested to show that the general expression of second order isotropic tensors is

$$C_{ij} = \lambda \delta_{ij} . \quad (2.153)$$

It is used to relate two first order tensors in physics. For example, we can understand the force-acceleration relation of a point mass,  $\vec{F} = m \vec{a}$ , as the degenerate form of the more general relation  $\vec{F} = \bar{C} \cdot \vec{a}$  by assuming the space to be isotropic, where  $\lambda = m$  is referred to as the mass.

## 2.5 Vector- and Tensor-Valued Functions

### 2.5.1 Vector-Valued Functions and Their Derivatives

If the a vector  $\vec{A}$  depends on a scalar variable  $t$ , i.e., its value varies with  $t$ , we call  $\vec{A}$  a vector-valued function of the scalar variable  $t$ . We will use the brief term vector function hereafter. The limit of a vector function can be defined based on the limit of a scalar function. If

$$\lim_{t \rightarrow t_0} |\vec{A}(t) - \vec{A}_0| = 0, \quad (2.154)$$

we call  $\vec{A}_0$  the limit of  $\vec{A}(t)$  when  $t$  approaches  $t_0$ , and denote it is

$$\lim_{t \rightarrow t_0} \vec{A}(t) = \vec{A}_0. \quad (2.155)$$

Similar to the definition of derivative of a scalar function, the derivative of a vector function is defined as

$$\frac{d\vec{A}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\vec{A}(t + \delta t) - \vec{A}(t)}{\delta t}, \quad (2.156)$$

if the limit exists. Evidently, the derivative of a vector function is also a vector function.

Similar to the case of a multivariable scalar function, partial derivatives are defined for a multivariable vector function. Take for example the vector function of two variables,  $\vec{A}(\xi, \eta)$ . It degenerates to a single variable vector function depending on the scalar  $\xi$  when  $\eta$  is fixed. The derivative of this single variable vector function is then defined as the partial derivative of  $\vec{A}$  with respect to  $\xi$ , and is denoted as  $\partial \vec{A} / \partial \xi$ . The partial derivative of  $\vec{A}$  with respect to  $\eta$ ,  $\partial \vec{A} / \partial \eta$ , can be similarly defined.

Let the vector functions  $\vec{A}$  and  $\vec{B}$  and the scalar function  $u$  be all functions of the scalar variable  $t$ . It can be verified based on definition that the derivative of vector functions has the following properties:

$$\frac{d}{dt}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}, \quad (2.157)$$

$$\frac{d}{dt}(u\vec{A}) = \frac{du}{dt}\vec{A} + u\frac{d\vec{A}}{dt}, \quad (2.158)$$

$$\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt}, \quad (2.159)$$

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}. \quad (2.160)$$



Based on the above rules, if the base vectors of the coordinate system  $Ox_1x_2x_3$  do not vary with  $t$ , we have

$$\frac{d\vec{A}}{dt} = \frac{dA_i}{dt} \hat{e}_i. \quad (2.161)$$

### 2.5.2 Application to Curves and Surfaces

Single variable vector functions can be used to study curves in the space. The parametric equation of a curve is

$$x_i = x_i(t). \quad (2.162)$$

The position vector of a point on the curve is

$$\vec{r}(t) = x_i(t) \hat{e}_i. \quad (2.163)$$

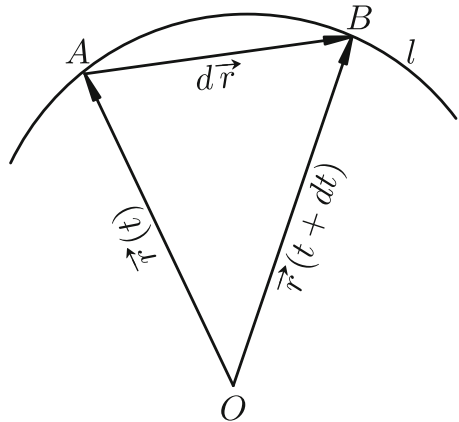
As  $t$  varies, the trace of  $\vec{r}(t)$  forms a curve  $l$ .

Taking reference to Fig. 2.10, we use  $d\vec{r}$  to denote the difference of the position vectors of the points  $A$  and  $B$ . We assume that the curve is smooth, and that  $|d\vec{r}|$  is infinitesimally small. In this case, the curve  $AB$  can be approximated by the straight line  $AB$ . Hence, we also use  $d\vec{l}$  to denote  $d\vec{r}$  for specifying that the infinitesimal vector is an approximation of a line segment on the curve. By definition, we have

$$d\vec{l} = d\vec{r} = \frac{d\vec{r}}{dt} dt. \quad (2.164)$$

It can be seen in Fig. 2.10 that, when  $dt$  tends to 0 (but not equal to 0), the limit of  $d\vec{l} = d\vec{r}$  is tangential to the curve. Hence,  $d\vec{r}/dt$  is tangential to the curve. The

**Fig. 2.10** The tangent of a curve



unit tangential vector is

$$\hat{l} = \frac{d\vec{r}}{dt} \left| \frac{d\vec{r}}{dt} \right|^{-1}. \quad (2.165)$$

Furthermore, according to (2.164), the length of arc on the curve corresponding to  $dt$  is

$$dl = |d\vec{r}| = \left| \frac{d\vec{r}}{dt} \right| dt, \quad (2.166)$$

which can be further written as

$$dl = |d\vec{r}| = \left( \frac{dx_i}{dt} \frac{dx_i}{dt} \right)^{1/2} dt. \quad (2.167)$$

Vector functions of two variables are suitable for studying surfaces in the space. The parametric equation of a surface in space is

$$x_i = x_i(\xi, \eta). \quad (2.168)$$

The position vector of a point on the surface is

$$\vec{r}(\xi, \eta) = x_i(\xi, \eta) \hat{e}_i. \quad (2.169)$$

As  $\xi$  and  $\eta$  vary, the trace of the point forms a surface.

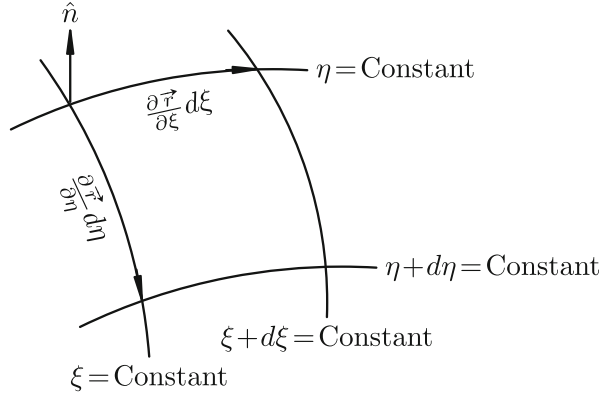
The result in the study of curves using the derivative of single variable vector functions can be used to study surfaces using the partial derivatives of vector functions of two variable. When  $\xi$  varies while keeping  $\eta$  fixed, (2.169) becomes the parametric equation of a curve on the surface with  $\xi$  as the parameter, and the differential of the position vector on this curve is  $(\partial \vec{r} / \partial \xi) d\xi$ . When  $\eta$  varies while keeping  $\xi$  fixed, (2.169) becomes the parametric equation of another curve on the surface with  $\eta$  as the parameter, and the differential of the position vector on this curve is  $(\partial \vec{r} / \partial \eta) d\eta$ . See Fig. 2.11. It can be seen that the area element corresponding to  $d\xi$  and  $d\eta$  is

$$dS = \left| \frac{\partial \vec{r}}{\partial \xi} d\xi \times \frac{\partial \vec{r}}{\partial \eta} d\eta \right| = \left| \frac{\partial \vec{r}}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta} \right| d\xi d\eta. \quad (2.170)$$

Besides, as the two curves obtained by setting  $\xi$  or  $\eta$  fixed are both on the surface, their tangents,  $\partial \vec{r} / \partial \xi$  and  $\partial \vec{r} / \partial \eta$ , are thus both tangential to the surface. Therefore, the normal to the surface is perpendicular to both these tangents, and the unit normal vector is

$$\hat{n} = \pm \frac{\partial \vec{r}}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta} \left| \frac{\partial \vec{r}}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta} \right|^{-1}. \quad (2.171)$$

**Fig. 2.11** Area and normal of a curved surface



As the surface has two sides, the unit normal vector  $\hat{n}$  may point to two opposite directions. Hence, a ‘ $\pm$ ’ sign appear in the above formula.

Evidently, the above formulations require the surface to be smooth, so that the infinitesimal surface element in Fig. 2.11 can be approximated as a plane, and the curves  $\xi = \text{Constant}$  and  $\eta = \text{Constant}$  there as straight lines.

### 2.5.3 Integrals of Vector-Valued Functions

In this section, we understand the vector  $\vec{A}$  as a function of position, which can be expressed as  $\vec{A}(x_1, x_2, x_3)$ , or briefly  $\vec{A}(\vec{r})$ .

The line, surface and volume integrals of a scalar function can be understood as the operation to compute a quantity from its density. The integrals of a vector function can be understood similarly.

Two commonly used line integrals are defined for vectors. We express the curve  $l$  in terms of the parameter  $t$  in the form of (2.163). These integrals are

$$\int_l \vec{A} dl = \left( \int_l A_i dl \right) \hat{e}_i = \left[ \int_{t_1}^{t_2} A_i \left( \frac{dx_j}{dt} \frac{dx_j}{dt} \right)^{1/2} dt \right] \hat{e}_i, \quad (2.172)$$

$$\int_l \vec{A} \cdot d\vec{l} = \int_l \vec{A} \cdot d\vec{r} = \int_{t_1}^{t_2} A_i dx_i = \int_{t_1}^{t_2} A_i \frac{dx_i}{dt} dt. \quad (2.173)$$

where  $t_1$  and  $t_2$  are the values of the parameter  $t$  of the line  $l$  at the starting and end points, respectively. Both integrals require that the line be piece-wise smooth. The second one can also be written as  $\int_l \vec{A} \cdot \hat{l} dl$ .

Two commonly used surface integrals defined for a vector are, with the surface expressed by (2.169) in terms of the parameters  $\xi$  and  $\eta$ ,

$$\int_S \vec{A} dS = \left( \int_S A_i dS \right) \hat{e}_i = \left( \int_{\eta_1}^{\eta_2} \int_{\xi_1}^{\xi_2} A_i \left| \frac{\partial \vec{r}}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta} \right| d\xi d\eta \right) \hat{e}_i, \quad (2.174)$$

$$\int_S \vec{A} \cdot d\vec{S} = \int_S \vec{A} \cdot \hat{n} dS = \pm \int_{\eta_1}^{\eta_2} \int_{\xi_1}^{\xi_2} \vec{A} \cdot \left( \frac{\partial \vec{r}}{\partial \xi} \times \frac{\partial \vec{r}}{\partial \eta} \right) d\xi d\eta. \quad (2.175)$$

In the second integral, the positive side of the surface should be specified to clarify whether ‘+’ or ‘-’ should be assigned to  $\pm$ . Both integrals require that the surface be piece-wise smooth, in which case, any infinitesimal element of the surface can be treated as a plane. The second integral also requires that the surface be two-sided, i.e., a point cannot slide from one side to the other side without crossing the boundary. We can also write (2.175) as

$$\begin{aligned} \int_S \vec{A} \cdot d\vec{S} &= \int_S \vec{A} \cdot \hat{n} dS = \int_S A_i n_i dS \\ &= \int_S (\pm A_1 dx_2 dx_3 \pm A_2 dx_3 dx_1 \pm A_3 dx_1 dx_2). \end{aligned} \quad (2.176)$$

This is due to the fact that  $n_i dS$  is the projection of the surface element  $dS$  on the coordinate plane  $x_2 x_3$ , and so forth. The ‘ $\pm$ ’ in front of  $A_i$  takes the same sign as that of  $n_i$ ,  $\text{sign}(n_i)$ , i.e., it takes ‘+’ when  $n_i > 0$ , and ‘-’ when  $n_i < 0$ .

The most commonly used volume integral of a vector function is

$$\int_V \vec{A} dV = \left( \int_V A_i dV \right) \hat{e}_i. \quad (2.177)$$

We remark that integrals of vector functions can always be reduced to integrals of scalar functions.

### 2.5.4 Tensor-Valued Functions and Their Derivatives and Integrals

The definition of a tensor-valued function, or a tensor function, is similar to that of a vector function. The limit of a tensor function can also be defined similarly. If

$$\lim_{t \rightarrow t_0} [\bar{T}(t) - \bar{T}_0] : [\bar{T}(t) - \bar{T}_0] = 0, \quad (2.178)$$

we call  $\bar{T}_0$  the limit of  $\bar{T}(t)$  when  $t$  approaches  $t_0$ ,

$$\lim_{t \rightarrow t_0} \bar{T}(t) = \bar{T}_0. \quad (2.179)$$

The definitions of derivative and partial derivatives of a tensor function are similar to the case of a vector function, and are not to be reproduced here. The properties of the derivative of tensor functions are also similar to the case of vector

functions. Particularly, if the base vectors of the coordinate system  $Ox_1x_2x_3$  do not vary with  $t$ , we have

$$\frac{d\bar{T}}{dt} = \frac{dT_{ij}}{dt} \hat{e}_i \hat{e}_j. \quad (2.180)$$

Let  $\bar{T}$  and  $\bar{P}$  be second order tensors, and  $\vec{A}$  a vector. All of them are functions of the spatial position  $\vec{r} = x_i \hat{e}_i$ . Some most commonly used integrals including a tensor are

$$\int_{(l,S,V)} \bar{T}(dl, dS, dV), \quad (2.181)$$

$$\int_{(l,S,V)} \vec{A} \cdot \bar{T}(dl, dS, dV), \quad (2.182)$$

$$\int_{(l,S,V)} \bar{T} \cdot \vec{A}(dl, dS, dV), \quad (2.183)$$

$$\int_{(l,S,V)} \bar{T} : \bar{P}(dl, dS, dV) \quad (2.184)$$

etc., where the symbols in the parentheses are alternatives, with  $dl$ ,  $dS$  and  $dV$  corresponding to  $l$ ,  $S$  and  $V$  respectively, representing a line, surface or volume integral. As all integrals we are going to meet do have very clear meanings, and all of them can be reduced to integrals of scalar functions, we are not going to further discuss the integrals defined above.

## 2.6 Field Theory

### 2.6.1 The Concept of Field, the nabla Operator and the Laplacian

A field is understood as a function of spatial position. If the value of a scalar, vector, or tensor function is defined at every point within a spatial domain  $V$ , we say that a scalar, vector, or tensor field is defined in  $V$ .

Most differential operations in field theory can be expressed very briefly using the vector differential operator

$$\nabla = \hat{e}_i \frac{\partial}{\partial x_i}, \quad (2.185)$$

also called nabla operator based on its shape. Its algebraic operations are defined to be indifferent from those of a usual vector, but the derivative operator is applied to the quantity after it. Let  $u$  be a scalar,  $\vec{A}$  and  $\vec{B}$  vectors, and  $\bar{T}$  a tensor. The rules of the operations of the nabla operator in the rectangular coordinate system  $Ox_1x_1x_3$

are analytically expressed as

$$\nabla u = \hat{e}_i \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x_i} \hat{e}_i, \quad (2.186)$$

$$\nabla \vec{A} = \hat{e}_i \frac{\partial}{\partial x_i} (A_j \hat{e}_j) = \frac{\partial A_j}{\partial x_i} \hat{e}_i \hat{e}_j, \quad (2.187)$$

$$\nabla \bar{T} = \hat{e}_i \frac{\partial}{\partial x_i} (T_{jk} \hat{e}_j \hat{e}_k) = \frac{\partial T_{jk}}{\partial x_i} \hat{e}_i \hat{e}_j \hat{e}_k, \quad (2.188)$$

$$\nabla \cdot \vec{A} = \left( \hat{e}_i \frac{\partial}{\partial x_i} \right) \cdot (A_j \hat{e}_j) = \frac{\partial A_j}{\partial x_i} \hat{e}_i \cdot \hat{e}_j = \frac{\partial A_j}{\partial x_i} \delta_{ij} = \frac{\partial A_i}{\partial x_i}, \quad (2.189)$$

$$\nabla \cdot \bar{T} = \left( \hat{e}_k \frac{\partial}{\partial x_k} \right) \cdot (T_{ij} \hat{e}_i \hat{e}_j) = \frac{\partial T_{ij}}{\partial x_k} (\hat{e}_k \cdot \hat{e}_i) \hat{e}_j = \frac{\partial T_{ij}}{\partial x_k} \delta_{ki} \hat{e}_j = \frac{\partial T_{ij}}{\partial x_i} \hat{e}_j, \quad (2.190)$$

$$\nabla \times \vec{A} = \left( \hat{e}_i \frac{\partial}{\partial x_i} \right) \times (A_j \hat{e}_j) = \frac{\partial A_j}{\partial x_i} \hat{e}_i \times \hat{e}_j = \frac{\partial A_j}{\partial x_i} \epsilon_{ijk} \hat{e}_k, \quad (2.191)$$

$$\vec{A} \cdot \nabla = (A_i \hat{e}_i) \cdot \left( \hat{e}_j \frac{\partial}{\partial x_j} \right) = A_i (\hat{e}_i \cdot \hat{e}_j) \frac{\partial}{\partial x_j} = A_i \delta_{ij} \frac{\partial}{\partial x_j} = A_i \frac{\partial}{\partial x_i}. \quad (2.192)$$

In (2.192),  $\vec{A} \cdot \nabla$  is in fact a differential operator, called advection operator, which appears in the governing equation of fluid flow. It applies to scalars and vectors in the following ways:

$$(\vec{A} \cdot \nabla)u = A_i \frac{\partial u}{\partial x_i}, \quad (2.193)$$

$$(\vec{A} \cdot \nabla)\vec{B} = A_i \frac{\partial}{\partial x_i} (B_j \hat{e}_j) = A_i \frac{\partial B_j}{\partial x_i} \hat{e}_j. \quad (2.194)$$

We also define

$$\begin{aligned} \nabla^2 &= \nabla \cdot \nabla = \left( \hat{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left( \hat{e}_j \frac{\partial}{\partial x_j} \right) = (\hat{e}_i \cdot \hat{e}_j) \frac{\partial^2}{\partial x_i \partial x_j} \\ &= \delta_{ij} \frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \end{aligned} \quad (2.195)$$

called Laplacian.

### 2.6.2 Gauss' and Stokes' Theorems

Both Gauss' and Stokes' theorems provide a relation between an integral over a domain and an integral over the boundary of the domain. As their proofs can be found in most textbooks of multivariable calculus, here we just cite them.

Gauss' theorem provides a relation between a volume integral over a domain  $V$  and a surface integral over the boundary of  $V$  denoted as  $S$ . Let the closed domain  $V$  be connected, i.e., any two points within it can be connected with a broken line completely within it, its boundary  $S$  is piece-wise smooth, and the functions  $A_i$  and their first order derivatives are all continuous over  $V$ . The following Gauss' theorem holds:

$$\oint_S n_i A_i dS = \int_V \frac{\partial A_i}{\partial x_i} dV, \quad (2.196)$$

where  $\hat{n} = n_i \hat{e}_i$  is the unit outward normal vector of  $S$ . The integral sign with a circle indicates that  $S$  is a closed surface. This theorem can be expressed in the vector form

$$\oint_S \hat{n} \cdot \vec{A} dS = \int_V \nabla \cdot \vec{A} dV. \quad (2.197)$$

Several other forms of Gauss' theorem can be derived. Let  $\vec{A} = u \hat{e}_i$  in (2.197). We obtain

$$\oint_S u n_i dS = \int_V \frac{\partial u}{\partial x_i} dV, \quad (2.198)$$

which is the equation of the components of the vector equation

$$\oint_S \hat{n} u dS = \int_V \nabla u dV. \quad (2.199)$$

Let  $u = A_j$  in (2.198), we obtain

$$\oint_S A_j n_i dS = \int_V \frac{\partial A_j}{\partial x_i} dV. \quad (2.200)$$

Multiply both its sides by  $\epsilon_{ijk}$ . We obtain

$$\oint_S n_i A_j \epsilon_{ijk} dS = \int_V \frac{\partial A_j}{\partial x_i} \epsilon_{ijk} dV, \quad (2.201)$$

which is the equation of the components of the vector equation

$$\oint_S \hat{n} \times \vec{A} dS = \int_V \nabla \times \vec{A} dV. \quad (2.202)$$

Let  $A_i = T_{ij}$  in (2.196). We obtain

$$\oint_S T_{ij} n_i dS = \int_V \frac{\partial T_{ij}}{\partial x_i} dV, \quad (2.203)$$

which is the equation of the components of the vector equation

$$\oint_S \hat{n} \cdot \vec{T} dS = \int_V \nabla \cdot \vec{T} dV. \quad (2.204)$$

By summarizing (2.197), (2.199), (2.202), and (2.204), we can write all forms of Gauss' theorem in the following general form,

$$\oint_S \hat{n}(\quad) dS = \int_V \nabla(\quad) dV, \quad (2.205)$$

where the spaces within the parentheses can be filled with  $u$ ,  $\vec{A}$ ,  $\nabla \times \vec{A}$  and  $\vec{T}$ , or even  $\vec{A}$ ,  $\vec{T}$ , etc.

Stokes' theorem provides a relation between a surface integral over  $S$  and a line integral over the boundary of  $S$  denoted as  $l$ . Let the functions  $A_i$  and their first order derivatives be all continuous over  $V$ , the two-sided smooth surface  $S$  is completely enclosed within  $V$ , and the boundary of  $S$ ,  $l$ , is piece-wise smooth. The following Stokes' theorem holds:

$$\oint_l A_i dx_i = \int_S \frac{\partial A_j}{\partial x_i} n_k \epsilon_{ijk} dS. \quad (2.206)$$

The integral sign with a circle indicates that  $l$  is a closed line. This theorem can be written in the vector form

$$\oint_l \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot \hat{n} dS, \quad (2.207)$$

where the relation between the directions of  $d\vec{l}$  and  $\hat{n}$  approximately obeys the right-hand rule (the angle between the direction of the thumb and  $\hat{n}$  is less than  $90^\circ$ ).

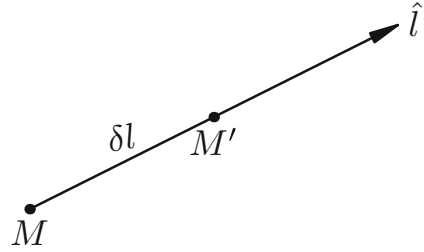
### 2.6.3 Gradient, Divergence and Curl

We first defined the gradient based on the concept of directional derivative. As shown in Fig. 2.12, let  $M$  be a point in the space,  $\hat{l}$  the unit vector along a direction, and  $M'$  another point  $\delta l$  apart from  $M$  with  $\vec{MM'}$  in the same direction as that of  $\hat{l}$ . Let  $u$  be a scalar field, i.e., it is a function of position in the space. Denote the increment of its value from  $M$  to  $M'$  as

$$\delta u = u(M') - u(M). \quad (2.208)$$



**Fig. 2.12** The concept of directional derivative



The directional derivative of  $u$  along  $\hat{l}$  is defined as

$$\frac{\partial u}{\partial l} = \lim_{\delta l \rightarrow 0} \frac{\delta u}{\delta l}, \quad (2.209)$$

if the limit exists.

The concept of gradient can be introduced by analyzing (2.208) and (2.209). According to Taylor's theorem, we can express  $\delta u$  as

$$\delta u = \delta x_i \frac{\partial u}{\partial x_i} + \text{higher order terms of } \delta x_i, \quad (2.210)$$

where  $\delta x_i$  are the increments of the coordinates of  $M'$  with respect to those of  $M$ . As  $\delta x_i$  is evidently the projection of  $\delta \vec{l} = (\delta l)\hat{l}$  along  $\hat{e}_i$ , we have

$$\delta x_i = l_i \delta l, \quad (2.211)$$

where  $l_i$  is the components of  $\hat{l}$ , i.e.,  $\hat{l} = l_i \hat{e}_i$ . By substituting (2.211) into (2.210), we obtain

$$\delta u = l_i \delta l \frac{\partial u}{\partial x_i} + \text{higher order terms of } \delta l. \quad (2.212)$$

Divide both sides by  $\delta l$ , and then take the limit  $\delta l \rightarrow 0$ . We obtain

$$\frac{\partial u}{\partial l} = l_i \frac{\partial u}{\partial x_i} = \hat{l} \cdot \nabla u. \quad (2.213)$$

It can be seen from this equation that  $\nabla u$  does have a unique physical meaning: its magnitude is the maximum that  $\partial u / \partial l$  may attain as the direction of  $\hat{l}$  varies, and its direction is the direction of  $\hat{l}$  when  $\partial u / \partial l$  attains the maximum. We thus define  $\nabla u$  as the gradient of the scalar field  $u$ . For this reason,  $\nabla u$  is also often written as  $\text{grad } u$ . It can be readily verified that, if  $u$  is coordinate-system-independent,  $\nabla u$  is also coordinate-system-independent.

The gradient of a vector can be introduced similarly. We have

$$\begin{aligned}
 \delta \vec{A} &= \vec{A}(M') - \vec{A}(M) \\
 &= \delta x_i \frac{\partial \vec{A}}{\partial x_i} + \text{higher order terms of } \delta x_i \\
 &= l_i \frac{\partial \vec{A}}{\partial x_i} \delta l + \text{higher order terms of } \delta l \\
 &= \hat{l} \cdot \nabla \vec{A} \delta l + \text{higher order terms of } \delta l.
 \end{aligned} \tag{2.214}$$

We define the directional derivative  $\partial \vec{A} / \partial l$  in analogy to that of a scalar field as

$$\frac{\partial \vec{A}}{\partial l} = \hat{l} \cdot \nabla \vec{A}. \tag{2.215}$$

We further define  $\nabla \vec{A}$  as the gradient of the vector  $\vec{A}$ . It can also be easily verified that, if  $\vec{A}$  is coordinate-system-independent,  $\nabla \vec{A}$  is also coordinate-system-independent, i.e., it is a second order tensor.

It can be inferred from the definition that the gradient is independent of the type of coordinate system, either rectangular, spherical, cylindrical or otherwise. This can also be shown based on Gauss' theorem, here written generally as

$$\oint_S \hat{n} (u, \vec{A}, \vec{T}) dS = \int_V \nabla (u, \vec{A}, \vec{T}) dV. \tag{2.216}$$

It represents three formulae for the three quantities in the parentheses, respectively. Now we let  $V$  shrinks to a point  $M$ . The integrand of the volume integral can then be considered as constant, and can be moved out of the integral. We thus obtain

$$\nabla (u, \vec{A}, \vec{T}) = \lim_{V \rightarrow M} \frac{1}{V} \oint_S \hat{n} (u, \vec{A}, \vec{T}) dS, \tag{2.217}$$

where  $V$  is also used to denote the volume of the domain  $V$ , and  $V \rightarrow M$  indicates the shrinking of the domain  $V$  to the point  $M$ , which also implies  $V \rightarrow 0$ . If the limit in (2.217) exists, it is independent of the type of coordinate system. Although we can also understand (2.217) as the definition of the gradient, its physical meaning is not as clear as (2.213) and (2.215).

Another two forms of Gauss' theorem similar to (2.216) are

$$\oint_S \hat{n} \cdot (\vec{A}, \vec{T}) dS = \int_V \nabla \cdot (\vec{A}, \vec{T}) dV, \tag{2.218}$$

$$\oint_S \hat{n} \times \vec{A} dS = \int_V \nabla \times \vec{A} dV. \tag{2.219}$$

We can obtain from them,

$$\nabla \cdot (\vec{A}, \vec{T}) = \lim_{V \rightarrow M} \frac{1}{V} \oint_S \hat{n} \cdot (\vec{A}, \vec{T}) dS, \quad (2.220)$$

$$\nabla \times \vec{A} = \lim_{V \rightarrow M} \frac{1}{V} \oint_S \hat{n} \times \vec{A} dS. \quad (2.221)$$

Evidently, if the limits at the right-hand sides of these two equations exist, they are independent of the type of the coordinate system. We thus define  $\nabla \cdot \vec{A}$  and  $\nabla \cdot \vec{T}$  as the divergences of  $\vec{A}$  and  $\vec{T}$ , respectively, and define  $\nabla \times \vec{A}$  as the curl of  $\vec{A}$ . Evidently,  $\nabla \cdot \vec{A}$  is a scalar, and  $\nabla \cdot \vec{T}$  and  $\nabla \times \vec{A}$  are vectors. We can also define  $\nabla \times \vec{T}$  similarly. The expressions  $\nabla \cdot \vec{A}$  and  $\nabla \cdot \vec{T}$  are also often written as  $\text{div } \vec{A}$  and  $\text{div } \vec{T}$ , respectively, and  $\nabla \times \vec{A}$  as  $\text{rot } \vec{A}$  or  $\text{curl } \vec{A}$ .

The physical meanings of the divergence and curl are not as evident as that of the gradient. The physical meaning of divergence will be discussed in the next section, and that of curl will be discussed in the next chapter in the description of deformation of a solid body. We mention that Gauss' theorem (2.197) is also referred to as divergence theorem.

Let  $u$  and  $v$  be scalar fields, and  $\vec{A}$  and  $\vec{B}$  vector fields. The operations gradient, divergence and curl have the following properties:

$$\nabla(u + v) = \nabla u + \nabla v, \quad (2.222)$$

$$\nabla(uv) = (\nabla u)v + u(\nabla v), \quad (2.223)$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \times (\nabla \times \vec{B}) + (\vec{B} \cdot \nabla)\vec{A} + (\vec{A} \cdot \nabla)\vec{B}, \quad (2.224)$$

$$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}, \quad (2.225)$$

$$\nabla \cdot (u\vec{A}) = (\nabla u) \cdot \vec{A} + u(\nabla \cdot \vec{A}), \quad (2.226)$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\nabla \times \vec{B}), \quad (2.227)$$

$$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}, \quad (2.228)$$

$$\nabla \times (\nabla u) = 0, \quad (2.229)$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0, \quad (2.230)$$

$$\nabla \times (u\vec{A}) = (\nabla u) \times \vec{A} + u(\nabla \times \vec{A}), \quad (2.231)$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}, \quad (2.232)$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}, \quad (2.233)$$

$$(\vec{A} \cdot \nabla)u = \vec{A} \cdot \nabla u, \quad (2.234)$$

$$(\vec{A} \cdot \nabla)\vec{B} = \vec{A} \cdot \nabla \vec{B}, \quad (2.235)$$

where the symbol  $\nabla^2 \vec{A}$  in (2.233) is defined as

$$\nabla^2 \vec{A} = \nabla \cdot (\nabla \vec{A}) = \frac{\partial^2 A_i}{\partial x_j \partial x_j} \hat{e}_i = \nabla^2 A_i \hat{e}_i. \quad (2.236)$$

In order for the readers to familiarize with the operations of  $\nabla$ , we provide the proof for a few relations.

- The proof of (2.229):

$$\nabla \times (\nabla u) = \left( \frac{\partial}{\partial x_i} \hat{e}_i \right) \times \left( \frac{\partial u}{\partial x_j} \hat{e}_j \right) = \frac{\partial^2 u}{\partial x_i \partial x_j} \epsilon_{ijk} \hat{e}_k, \quad (2.237)$$

$$\nabla \times (\nabla u) = \left( \frac{\partial}{\partial x_j} \hat{e}_j \right) \times \left( \frac{\partial u}{\partial x_i} \hat{e}_i \right) = \frac{\partial^2 u}{\partial x_j \partial x_i} \epsilon_{jik} \hat{e}_k. \quad (2.238)$$

As  $\epsilon_{ijk}$  and  $\epsilon_{jik}$  have opposite signs, (2.229) can be readily obtained by comparing the above two formulae.

- The proof of (2.231)

$$\begin{aligned} \nabla \times (u \vec{A}) &= \left( \frac{\partial}{\partial x_i} \hat{e}_i \right) \times (u A_j \hat{e}_j) \\ &= \frac{\partial}{\partial x_i} (u A_j) \epsilon_{ijk} \hat{e}_k \\ &= \frac{\partial u}{\partial x_i} A_j \epsilon_{ijk} \hat{e}_k + u \frac{\partial A_j}{\partial x_i} \epsilon_{ijk} \hat{e}_k \\ &= (\nabla u) \times \vec{A} + u (\nabla \times \vec{A}). \end{aligned} \quad (2.239)$$

- The proof of (2.233):

$$\begin{aligned} \nabla \times (\nabla \times \vec{A}) &= \left( \frac{\partial}{\partial x_i} \hat{e}_i \right) \times \left( \frac{\partial A_k}{\partial x_j} \epsilon_{jkm} \hat{e}_m \right) \\ &= \frac{\partial^2 A_k}{\partial x_i \partial x_j} \epsilon_{jkm} \epsilon_{imn} \hat{e}_n \\ &= \frac{\partial^2 A_k}{\partial x_i \partial x_j} \epsilon_{jkm} \epsilon_{nim} \hat{e}_n \\ &= \frac{\partial^2 A_k}{\partial x_i \partial x_j} \delta_{jn} \delta_{ki} \hat{e}_n - \frac{\partial^2 A_k}{\partial x_i \partial x_j} \delta_{ji} \delta_{kn} \hat{e}_n \\ &= \frac{\partial}{\partial x_j} \left( \frac{\partial A_k}{\partial x_k} \right) \hat{e}_j - \frac{\partial^2 A_k}{\partial x_i \partial x_i} \hat{e}_k \\ &= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}. \end{aligned} \quad (2.240)$$

As a supplement, we derive an identity used in the formulation of the magnetic field (e.g., Wangsness, 1986). By integrating (2.226) over a volume and then using (2.205) and (2.234), we get

$$\oint_S u \vec{A} \cdot \hat{n} dS = \int_V [(\vec{A} \cdot \nabla)u + u(\nabla \cdot \vec{A})] dV. \quad (2.241)$$

Consider a vector  $\vec{B} = B_i \hat{e}_i$ . By setting  $u = B_i$ , we obtain

$$\oint_S B_i \vec{A} \cdot \hat{n} dS = \int_V [(\vec{A} \cdot \nabla)B_i + B_i(\nabla \cdot \vec{A})] dV. \quad (2.242)$$

This is the equation for the  $i$ th component of the vector equation

$$\oint_S \vec{B}(\vec{A} \cdot \hat{n}) dS = \int_V [(\vec{A} \cdot \nabla)\vec{B} + \vec{B}(\nabla \cdot \vec{A})] dV. \quad (2.243)$$

As a field is a function of the position vector, a scalar field can be expressed as  $u = u(\vec{r})$ . Hence, the equation  $u = C$ , where  $C$  is a constant, represents a surface in the space, called equi-value surface. Scalar fields are often geometrically represented using equi-value surfaces, e.g., isothermal surfaces for a temperature field and isobar surfaces for a pressure field. The gradient of a scalar is a vector perpendicular to its equi-value surface, which can be readily inferred from (2.213) by assuming  $\hat{l}$  to be tangential or perpendicular the equi-value surface. Particularly, if  $\hat{l}$  is in the same direction as  $\nabla u$ , we have  $\partial u / \partial l = |\nabla u|$ , and the distance from the equi-value surface  $u = C$  to a neighboring equi-value surface  $u = C + \delta C$  is  $\delta C / |\nabla u|$  measured positive along the direction of  $\nabla u$ .

In order to represent a vector field  $\vec{A} = A_i \hat{e}_i$  geometrically, we introduce the concept of line of force, which is a curve, whose tangent is always in the same direction as that of  $\vec{A}$ . Let  $\vec{r}(t) = x_i(t) \hat{e}_i$  be the parametric equation of a line of force, we then have  $(dx_i/dt)/A_1 = (dx_2/dt)/A_2 = (dx_3/dt)/A_3$ . While each line of force represents only the direction of the vector along the line, the magnitude of the vector, which is referred to as the intensity of the field, is represented by the density of the line of force. Magnetic flux lines and plumb lines are examples of line of force.

The case for a tensor is more complicated, and is not to be further discussed.

### 2.6.4 Conservative, Solenoidal and Harmonic Vector Fields

A vector field  $\vec{A}$  is called conservative if the integral  $\int_l \vec{A} \cdot d\vec{l}$  depends only on the starting and end points of the path  $l$ , and independent of the choice of the path  $l$  between the two points. A concrete example is that, if  $\vec{A}$  is a field of force, the integral is the work done by the force to a unit point mass that moved through the path  $l$ . The field of force is conservative if the work done by the force to a point

mass that moved from one point to another is independent of the path between the two points. If  $\nabla \times \vec{A} = 0$ , the field  $\vec{A}$  is called irrotational. A scalar field  $u$  is called the potential of the vector field  $\vec{A}$ , if  $\vec{A} = \nabla u$ .<sup>1</sup> We will show in the following paragraphs that the three statements— $\vec{A}$  is conservative,  $\vec{A}$  is irrotational, and  $\vec{A}$  has a potential—are equivalent.

First, we show that, if  $\vec{A}$  is an irrotational field, it is also a conservative field. According to Stokes' theorem (2.207), we have

$$\oint_{l_C} \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot \hat{n} dS = 0, \quad (2.244)$$

where the closed line  $l_C$  is the boundary of  $S$ . This means that, over any closed line  $l_C$ , the integral  $\oint_{l_C} \vec{A} \cdot d\vec{l}$  always vanishes. With this property, it can be readily inferred based on the proof by contradiction that the integral  $\int_l \vec{A} \cdot d\vec{l}$  depends only on the starting and end points of the path  $l$ , and independent of the choice of the path  $l$  between the two points.

Second, we show that a conservative field has a potential. By definition, as  $\int_l \vec{A} \cdot d\vec{l}$  depends only on the starting and end points of the path  $l$ , and independent of the choice of the path  $l$  between the two points, we can define a single-valued function of  $x_1, x_2, x_3$  by

$$u = \int_{(x_1^0, x_2^0, x_3^0)}^{(x_1, x_2, x_3)} A_i(y_1, y_2, y_3) dy_i, \quad (2.245)$$

where  $(x_1^0, x_2^0, x_3^0)$  and  $(x_1, x_2, x_3)$  are the starting and end points of the path. This implies that  $u$  is considered as a function of the end point of the integral. As the integral is independent of how the path from  $(x_1^0, x_2^0, x_3^0)$  and  $(x_1, x_2, x_3)$  is chosen, we defined the path to be composed of three line segments parallel to the coordinate axes as shown in Fig. 2.13. The integral is then the sum of the integrals over the three line segments,

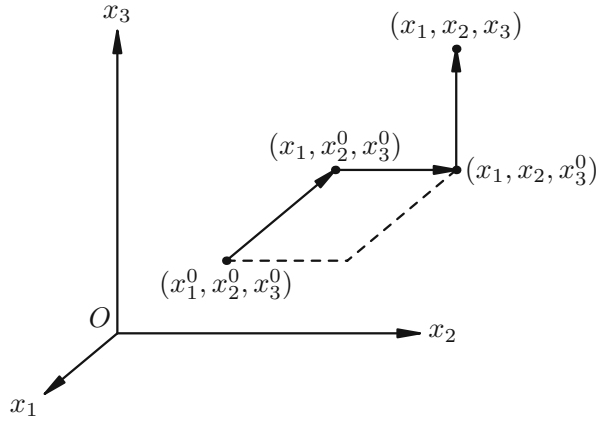
$$u = \int_{x_1^0}^{x_1} A_1(y_1, x_2^0, x_3^0) dy_1 + \int_{x_2^0}^{x_2} A_2(x_1, y_2, x_3^0) dy_2 + \int_{x_3^0}^{x_3} A_3(x_1, x_2, y_3) dy_3. \quad (2.246)$$

By taking the partial derivative of both sides with respect to  $x_1, x_2$  and  $x_3$ , we obtain, according to the Newton-Leibniz formula,

$$\frac{\partial u}{\partial x_1} = A_1, \quad \frac{\partial u}{\partial x_2} = A_2, \quad \frac{\partial u}{\partial x_3} = A_3. \quad (2.247)$$

<sup>1</sup> The relation  $\vec{A} = \nabla u$  is the more common definition of the potential in the study of the gravity field of the Earth in geodesy, which we follow in this book. In physics, the potential  $u$  of  $\vec{A}$  is usually defined through the relation  $\vec{A} = -\nabla u$ .

**Fig. 2.13** Division of the integration path into three line segments



These are equivalent to

$$\vec{A} = \nabla u, \quad (2.248)$$

meaning that  $u$  is a potential of  $\vec{A}$ .

Last, we shown that if  $\vec{A}$  has a potential, it is also irrotational. The proof is straightforward:

$$\nabla \times \vec{A} = \nabla \times (\nabla u) = 0. \quad (2.249)$$

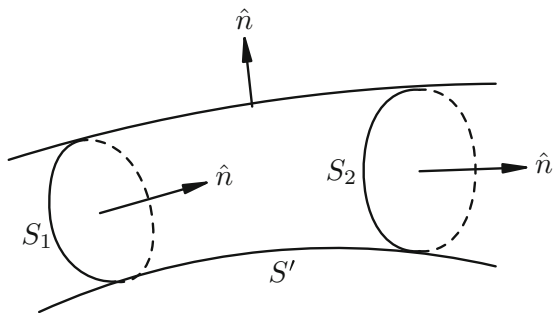
The equivalence of the three statements can be inferred from the conclusions of the previous three paragraphs. The term *conservative field* is most referred.

A vector field  $\vec{A}$  is referred to as source-free or solenoidal if  $\nabla \cdot \vec{A} = 0$ , with each name representing an aspect of the property of the field.

We first examine what source-free implies. Assume that  $\vec{A}$  is a velocity field of a homogeneous fluid, and  $S$  is a closed surface enclosing the three-dimensional domain  $V$ . We also use  $V$  to denote the volume of the domain  $V$ . The integral  $\oint_S \hat{n} \cdot \vec{A} dS$  can be understood as the volume flux, i.e., the flow rate, of the fluid from inside  $S$  to the outside. If no gap is allowed, the same amount of fluid should be produced within  $V$ , and  $(1/V) \oint_S \hat{n} \cdot \vec{A} dS$  can be understood as the average density of fluid production rate (the volume of fluid produced per unit time within a unit volume). At the limit that  $V$  shrinks to a point,  $(1/V) \oint_S \hat{n} \cdot \vec{A} dS$  becomes  $\nabla \cdot \vec{A}$ . Hence,  $\nabla \cdot \vec{A}$  is considered as the density of fluid production rate, or source of  $\vec{A}$ , which is the reason a vector field is referred to as source-free when  $\nabla \cdot \vec{A} = 0$ . Furthermore,  $\nabla \cdot \vec{A} > 0$  implies that the velocity diverges to the outside of  $S$ , hence the name divergence for  $\nabla \cdot \vec{A}$ .

The meaning of the term solenoidal is tube-shaped. Draw a closed line, which is not a line of force of  $\vec{A}$ , and then draw the line of force of  $\vec{A}$  through every point on the closed line. The lines of force then form a tube, called vector tube. From a

**Fig. 2.14** The property of a solenoidal field



vector tube as shown in Fig. 2.14, we take a portion  $V$  with cross sections  $S_1$  and  $S_2$  and side wall  $S'$ . If  $\nabla \cdot \vec{A} = 0$ , we obtain, according to Gauss' theorem (2.197),

$$-\int_{S_1} \hat{n} \cdot \vec{A} dS + \int_{S'} \hat{n} \cdot \vec{A} dS + \int_{S_2} \hat{n} \cdot \vec{A} dS = \int_V \nabla \cdot \vec{A} dV = 0, \quad (2.250)$$

where the unit normal vectors are shown in 2.14. Based on how the vector tube is drawn,  $\vec{A}$  and  $\hat{n}$  are perpendicular on  $S'$ , implying that  $\hat{n} \cdot \vec{A} = 0$  on  $S'$ . Hence, (2.250) is simplified to

$$\int_{S_1} \hat{n} \cdot \vec{A} dS = \int_{S_2} \hat{n} \cdot \vec{A} dS. \quad (2.251)$$

This implies that the flux of the vector through different cross sections of the same vector tube are all equal, which is what the term solenoidal implies. Two properties can be further inferred. First, a vector tube cannot start or end inside the domain of the field; it can only extend to infinity, terminate on the boundary of the domain, or be closed inside the domain of the field. Second, the fluxes through all surfaces with the same closed line as boundary are equal.

A vector field is called harmonic if it is both conservative and solenoidal. If  $\vec{A}$  is conservative, there exists a scalar field  $u$ , such that

$$\vec{A} = \nabla u. \quad (2.252)$$

If it is also solenoidal, we have  $\nabla \cdot \vec{A} = 0$ , leading to

$$\nabla \cdot (\nabla u) = \nabla^2 u = 0, \quad (2.253)$$

which is referred to as Laplace's equation. A function with continuous derivatives up to second order satisfying Laplace's equation is called harmonic function.



### 2.6.5 Green's Identities

Green's identities are broadly used in the theory of the Earth's gravity field.

Let  $\vec{A} = u\nabla v$  in Gauss' theorem (2.197). We obtain, according to (2.226), (2.195), and (2.213),

$$\int_V (u\nabla^2 v + \nabla u \cdot \nabla v) dV = \oint_S u \frac{\partial v}{\partial n} dS, \quad (2.254)$$

which is referred to as Green's first identity.

Consider Green's first identity (2.254). We first obtain a new formula by swapping  $u$  and  $v$ , and then subtract the new formula from the original one. We obtain

$$\int_V (u\nabla^2 v - v\nabla^2 u) dV = \oint_S \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS, \quad (2.255)$$

which is referred to as Green's second identity.

Let  $l$  be the distance between the volume element  $dV$  and a fixed point  $A$  in the space. By understanding  $l$  as a function of the position of  $dV$ , it can be verified that  $1/l$  is a harmonic function everywhere except at  $A$ , i.e., when  $dV$  is at  $A$ . Hence,  $1/l$  satisfies Laplace's equation  $\nabla^2(1/l) = 0$  everywhere except at  $A$ . By setting  $u = 1/l$  in Green's second identity (2.255), we obtain Green's third identity,

$$\int_V \frac{1}{l} \nabla^2 v dV = \oint_S \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS - p v(A), \quad (2.256)$$

where  $v(A)$  is the value of  $v$  at  $A$ , and  $p$  is defined as

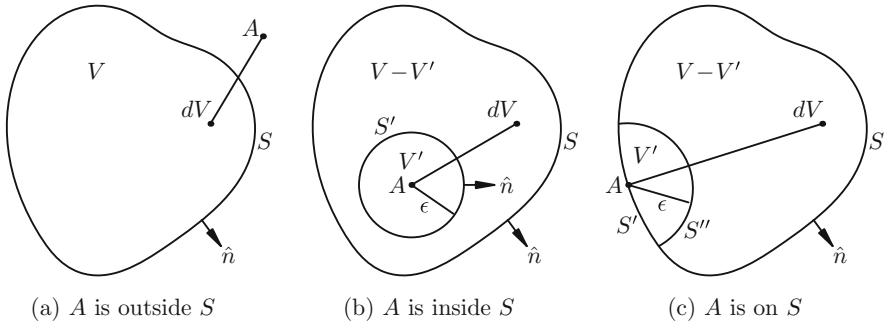
$$p = \begin{cases} 4\pi, & A \text{ is inside } S; \\ 2\pi, & A \text{ is on } S; \\ 0, & A \text{ is outside } S. \end{cases} \quad (2.257)$$

The proof of this identity is lengthy, which will be the objective of the following paragraphs. We first present a degenerate case of Green's third identity (2.256) by setting  $v = 1$ , and then using (2.257),

$$\oint_S \frac{\partial}{\partial n} \left( \frac{1}{l} \right) dS = -p = \begin{cases} -4\pi, & A \text{ is inside } S; \\ -2\pi, & A \text{ is on } S; \\ 0, & A \text{ is outside } S; \end{cases} \quad (2.258)$$

which is a very useful formula.

We now proceed to prove Green's third identity.



**Fig. 2.15** Proof of Green's third theorem

When  $A$  is outside  $S$ , as shown in Fig. 2.15a, since  $1/l$  is a harmonic function inside  $V$ , we obtain, by setting  $u = 1/l$  in (2.255),

$$\int_V \frac{1}{l} \nabla^2 v dV = \oint_S \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS, \quad (2.259)$$

which is the part of (2.256) when  $A$  is outside  $S$ .

When  $A$  is inside  $S$ , as shown in Fig. 2.15b, since both  $1/l$  and  $\nabla^2(1/l)$  tend to infinitely large at  $A$ , we cannot set  $u = 1/l$  in (2.255) as is. To overcome this complication, we draw a small sphere of radius  $\epsilon$  centered at  $A$  and completely within  $V$ . We denote the surface of the sphere as  $S'$ , and its inside as  $V'$ . By applying Green's second theorem (2.255) to the domain  $V - V'$  and its surface  $S$  and  $S'$ , and then setting  $u = 1/l$ , we obtain

$$\int_{V-V'} \frac{1}{l} \nabla^2 v dV = \oint_S \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS - \oint_{S'} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS, \quad (2.260)$$

where the negative sign of the second term at the right-hand side is due to the fact that the unit normal vector  $\hat{n}$  of the surface  $S'$  points to the inside of  $V - V'$ .

On the sphere  $S'$ ,  $l = \epsilon$  and  $dS = \epsilon^2 \sin \theta d\theta d\lambda$ . Furthermore, as  $\hat{n}$  is in the radial direction, and  $l$  is the radial distance from the center  $A$ , we have  $(\partial/\partial n)(1/l)_{S'} = (\partial/\partial l)(1/l)_{l=\epsilon} = -1/\epsilon^2$ . Hence,

$$\oint_{S'} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS = \int_0^{2\pi} d\lambda \int_0^\pi \left( \epsilon \frac{\partial v}{\partial n} + v \right) \sin \theta d\theta. \quad (2.261)$$

We assume that both  $v$  and its first order derivatives are finite within  $V$ , which includes certainly the point  $A$ . The latter ensures  $\partial v/\partial n$  to be finite. We can then

obtain, by setting  $\epsilon \rightarrow 0$ ,

$$\lim_{\epsilon \rightarrow 0} \oint_{S'} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS = \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} d\lambda \int_0^\pi \left( \epsilon \frac{\partial v}{\partial n} + v \right) \sin \theta d\theta = 4\pi v(A), \quad (2.262)$$

where  $\epsilon(\partial v / \partial n) \rightarrow 0$  when  $\epsilon \rightarrow 0$  is evident, and the second identity is due to the fact that the value of  $v$  on  $S'$  when  $\epsilon \rightarrow 0$  can be considered as its value at  $A$ , and can be taken out of the integral sign.

We also assume that  $\nabla^2 v$  is finite within  $V$ , which includes the point  $A$ . We then have

$$\lim_{\epsilon \rightarrow 0} \int_{V'} \frac{1}{l} \nabla^2 v dV = \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} d\lambda \int_0^\pi \int_0^\epsilon l \nabla^2 v \sin \theta dl = 0. \quad (2.263)$$

Hence, we further have

$$\lim_{\epsilon \rightarrow 0} \int_{V-V'} \frac{1}{l} \nabla^2 v dV = \int_V \frac{1}{l} \nabla^2 v dV. \quad (2.264)$$

Finally, by setting  $\epsilon \rightarrow 0$  at both sides of (2.260), we obtain, according to (2.262) and (2.264),

$$\int_V \frac{1}{l} \nabla^2 v dV = \oint_S \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS - 4\pi v(A). \quad (2.265)$$

which is the part of (2.256) when  $A$  is inside  $S$ .

When  $A$  is on  $S$ , as shown in Fig. 2.15c, we assume that  $S$  is smooth around  $A$ , so that a small disk  $S'$  of radius  $\epsilon$  centered at  $A$  can be drawn on  $S$ , and can be considered as a plane disk. We further draw a half sphere of radius  $\epsilon$  centered at  $A$  within  $V$ , and denote its surface as  $S''$ , and its inside as  $V'$ . We can use Green's second theorem (2.255) to obtain

$$\int_{V-V'} \frac{1}{l} \nabla^2 v dV = \int_{S-S'} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS - \int_{S''} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS. \quad (2.266)$$

Similar to the case when  $A$  is inside  $S$ , we can show that

$$\lim_{\epsilon \rightarrow 0} \int_{S''} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS = 2\pi v(A), \quad (2.267)$$

$$\lim_{\epsilon \rightarrow 0} \int_{V-V'} \frac{1}{l} \nabla^2 v dV = \int_V \frac{1}{l} \nabla^2 v dV. \quad (2.268)$$

On  $S'$ ,  $\hat{n}$  is perpendicular to  $S'$ , and  $l$  is the radial distance from the center  $A$ . Hence, we have  $(\partial/\partial n)(1/l) = 0$  and  $dS = ldl d\alpha$  (by understanding  $l$  and  $\alpha$  as two-dimensional polar coordinates). We obtain, assuming  $\partial v/\partial n$  to be finite,

$$\lim_{\epsilon \rightarrow 0} \int_{S'} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS = \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} d\alpha \int_0^\epsilon \frac{\partial v}{\partial n} dl = 0. \quad (2.269)$$

Consequently,

$$\lim_{\epsilon \rightarrow 0} \int_{S-S'} \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS = \oint_S \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS. \quad (2.270)$$

Finally, by setting  $\epsilon \rightarrow 0$  at both sides of (2.266), we obtain, according to (2.267), (2.268), and (2.270),

$$\int_V \frac{1}{l} \nabla^2 v dV = \oint_S \left[ \frac{1}{l} \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} \left( \frac{1}{l} \right) \right] dS - 2\pi v(A), \quad (2.271)$$

which is the part of (2.256) when  $A$  is on  $S$ . Remember that  $S$  is assumed to be smooth around  $A$  in the proof. This requires that  $S$  to be piecewise smooth.

Now, the proof of Green's third theorem (2.256) and (2.257) is complete.

### 2.6.6 Helmholtz's Theorem

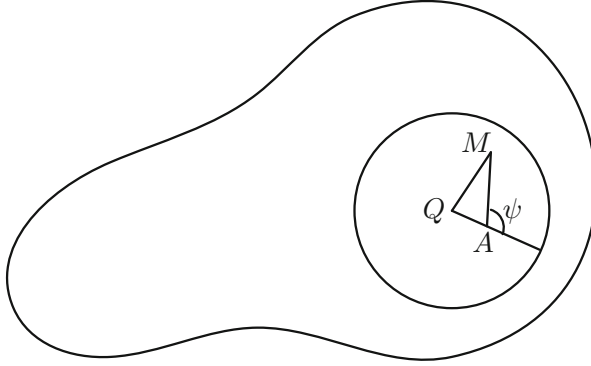
Helmholtz's theorem expresses a vector field using its divergence and curl within its domain as well as its value on the surface of its domain. In this subsection, we progressively prove this theorem.

We first derive an auxiliary formula. Denote the position vectors of two points  $A$  and  $M$  as  $\vec{r} = x_i \hat{e}_i$  and  $\vec{r}' = x'_i \hat{e}_i$ , respectively, and the distance between them as  $l(\vec{r}, \vec{r}')$ . As we will consider functions depending on the positions of both  $A$  and  $M$ , we denote the nabla operator and Laplacian corresponding to the two points as  $\nabla$ ,  $\nabla^2$  and  $\nabla'$ ,  $\nabla'^2$ , i.e.,  $\nabla = \hat{e}_i (\partial/\partial x_i)$ ,  $\nabla^2 = \partial^2/\partial x_i \partial x_i$  and  $\nabla' = \hat{e}_i (\partial/\partial x'_i)$ ,  $\nabla'^2 = \partial^2/\partial x'_i \partial x'_i$ . For any continuous function  $u$ , we have the following formula:

$$u(\vec{r}) = \int_V u(\vec{r}') \nabla'^2 \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}')} \right] dV', \quad (2.272)$$

where  $u(\vec{r})$  and  $u(\vec{r}')$  are the values of  $u$  at  $A$  and  $M$ , respectively,  $dV'$  is a volume element at  $M$ , implying that the integration variables are  $x'_1, x'_2$ , and  $x'_3$ , with  $dV' = dx'_1 dx'_2 dx'_3$ . The above formula means that  $\nabla'^2 \{-1/[4\pi l(\vec{r}, \vec{r}')] \}$  is a  $\delta$  function. The proof is the subject of the subsequent paragraphs.

It can be easily verified that  $\nabla'^2 \{-1/[4\pi l(\vec{r}, \vec{r}')] \} = 0$  when  $\vec{r} \neq \vec{r}'$ . Hence, only a very small domain around  $A$  contributes to the integral in (2.272). As shown



**Fig. 2.16** Definition of a small sphere within the domain

in Fig. 2.16, we draw a sphere of radius  $\epsilon$  centered at a point  $Q$  very close to  $A$ , and enclosing  $A$  inside. We denote the domain inside the sphere as  $V_1$ . The domain  $V$  is then divided into two parts,  $V - V_1$  and  $V_1$ , where  $V - V_1$  does not contribute to the integral in (2.272). Hence,

$$\int_V u(\vec{r}') \nabla^2 \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}') } \right] dV' = \int_{V_1} u(\vec{r}') \nabla^2 \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}') } \right] dV'. \quad (2.273)$$

As  $u$  is assumed to be continuous, by setting the distance between  $A$  and  $Q$ ,  $\rho$ , to approach 0, and then the radius of  $V_1$ ,  $\epsilon$ , to approach 0, we can understand the value of  $u(\vec{r}')$  in the integral at the right-hand side of (2.273) as  $u(\vec{r})$ , and can be moved out of the integral sign. Hence, we obtain

$$\int_V u(\vec{r}') \nabla^2 \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}') } \right] dV' = u(\vec{r}) \lim_{\epsilon \rightarrow 0} \lim_{\rho \rightarrow 0} \int_{V_1} \nabla^2 \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}') } \right] dV'. \quad (2.274)$$

As  $\nabla^2$  concerns derivatives with respect to  $x_i$ , it can also be moved out of the integral sign, leading to

$$\int_V u(\vec{r}') \nabla^2 \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}') } \right] dV' = u(\vec{r}) \lim_{\epsilon \rightarrow 0} \lim_{\rho \rightarrow 0} \nabla^2 \int_{V_1} \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}') } \right] dV'. \quad (2.275)$$

What remains is to compute the integral at the right-hand side of (2.275). Write  $l(\vec{r}, \vec{r}')$  briefly as  $l$ , and consider the spherical coordinate system  $Al\psi\gamma$  with  $A$  as the origin,  $l$  as the radial distance,  $\psi$  as the polar angle, and  $\gamma$  as the longitude, as shown in Fig. 2.16, where  $\gamma$  is not shown explicitly, but its meaning is evident. We have  $dV' = l^2 \sin \psi dl d\psi d\gamma$ . The equation of the sphere is

$$\epsilon^2 = \rho^2 + l_0^2 + 2\rho l_0 \cos \psi, \quad (2.276)$$

where  $l_0$  is the distance from  $A$  to the surface of the sphere, i.e., the distance  $AM$  when  $M$  is on the surface of the sphere. The above equation has two solutions, and the positive one we need is (as we assumed  $\rho < \epsilon$ )

$$l_0 = -\rho \cos \psi + \sqrt{\epsilon^2 - \rho^2 \sin^2 \psi}. \quad (2.277)$$

Hence, we obtain

$$\begin{aligned} \int_{V_1} \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}')} \right] dV' &= -\frac{1}{4\pi} \int_0^{2\pi} d\gamma \int_0^\pi d\psi \int_0^{l_0} l \sin \psi dl \\ &= -\frac{1}{4} \int_0^\pi \left( -\rho \cos \psi + \sqrt{\epsilon^2 - \rho^2 \sin^2 \psi} \right)^2 \sin \psi d\psi \\ &= -\frac{1}{4} \int_0^\pi \left( \epsilon^2 - \rho^2 + 2\rho^2 \cos^2 \psi - 2\rho \cos \psi \sqrt{\epsilon^2 - \rho^2 \sin^2 \psi} \right) \sin \psi d\psi \\ &= -\frac{1}{4} \left[ (\epsilon^2 - \rho^2) \int_0^\pi \sin \psi d\psi + 2\rho^2 \int_0^\pi \cos^2 \psi \sin \psi d\psi \right. \\ &\quad \left. - 2\rho \int_0^\pi \sin \psi \cos \psi \sqrt{\epsilon^2 - \rho^2 \sin^2 \psi} d\psi \right]. \end{aligned} \quad (2.278)$$

The last integral should vanish, as its integrand is anti-symmetric with respect to the center of the integration domain, i.e.,  $\psi = \pi/2$ . Hence, we obtain

$$\begin{aligned} \int_{V_1} \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}')} \right] dV' &= -\frac{1}{4} \left[ (\epsilon^2 - \rho^2)(-\cos \psi)_0^\pi + 2\rho^2 \left( -\frac{1}{3} \cos^3 \psi \right)_0^\pi \right] \\ &= -\frac{1}{4} \left[ 2(\epsilon^2 - \rho^2) + \frac{4}{3} \rho^2 \right] \\ &= -\frac{1}{6} (3\epsilon^2 - \rho^2). \end{aligned} \quad (2.279)$$

Denote the coordinates of the point  $Q$  as  $x_i^0$ . We have

$$\rho^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2. \quad (2.280)$$

After substituting this formula into (2.279), we can readily obtain

$$\nabla^2 \int_{V_1} \left[ -\frac{1}{4\pi l(\vec{r}, \vec{r}')} \right] dV' = 1. \quad (2.281)$$

Finally (2.272) can be obtained by substituting this formula into (2.275).

We now progressively introduce Helmholtz's theorem base on (2.272).

In (2.272), as the Laplacian  $\nabla^2$  concerns the computation of derivatives with respect to  $x_i$ , it can be moved out of the integral sign. Hence, we have

$$u(\vec{r}) = -\frac{1}{4\pi} \nabla^2 \int_V \frac{u(\vec{r}')}{l(\vec{r}, \vec{r}')} dV'. \quad (2.282)$$

Let  $\vec{A}$  be a continuous vector function. By replacing  $u$  by each component of  $\vec{A}$  in a rectangular coordinate system, we obtain an equation for each of the three components of  $\vec{A}$ . These three equations can be written together in vector form as

$$\vec{A}(\vec{r}) = -\frac{1}{4\pi} \nabla^2 \int_V \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV'. \quad (2.283)$$

The right-hand side can be expressed in two terms according to (2.233),

$$\vec{A}(\vec{r}) = -\frac{1}{4\pi} \nabla \left[ \nabla \cdot \int_V \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' \right] + \frac{1}{4\pi} \nabla \times \left[ \nabla \times \int_V \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' \right]. \quad (2.284)$$

Let

$$\phi(\vec{r}) = -\frac{1}{4\pi} \nabla \cdot \int_V \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV', \quad (2.285)$$

$$\vec{\psi}(\vec{r}) = \frac{1}{4\pi} \nabla \times \int_V \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV'. \quad (2.286)$$

We can write (2.284) as

$$\vec{A} = \nabla \phi + \nabla \times \vec{\psi}, \quad (2.287)$$

where it can be easily verified based on (2.286) that

$$\nabla \cdot \vec{\psi} = 0. \quad (2.288)$$

The above two formulae are referred to as Helmholtz's theorem of decomposition. As  $\vec{A}$  has three components, while  $\phi$  is a scalar and  $\vec{\psi}$  has three components, there is an extra component at the right-hand side of (2.287), and hence, (2.288) can be understood as a constraint to reduce the three components of  $\psi$  to two independent components.

The scalar function  $\phi$  and vector function  $\vec{\psi}$  can be expressed using the divergence and curl of  $\vec{A}$  inside  $V$  and the value of  $\vec{A}$  on the surface of  $V$  denoted as  $S$ . By definition,

$$\frac{1}{l(\vec{r}, \vec{r}')} = [(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2]^{-1/2}. \quad (2.289)$$

We can readily verify that

$$\nabla \left[ \frac{1}{l(\vec{r}, \vec{r}')} \right] = -\nabla' \left[ \frac{1}{l(\vec{r}, \vec{r}')} \right]. \quad (2.290)$$

According to this relation, (2.285) and (2.286) can be further developed as

$$\begin{aligned} \phi(\vec{r}) &= -\frac{1}{4\pi} \int_V \vec{A}(\vec{r}') \nabla \cdot \left[ \frac{1}{l(\vec{r}, \vec{r}')} \right] dV' \\ &= \frac{1}{4\pi} \int_V \vec{A}(\vec{r}') \nabla' \cdot \left[ \frac{1}{l(\vec{r}, \vec{r}')} \right] dV' \\ &= \frac{1}{4\pi} \left[ -\int_V \frac{\nabla' \cdot \vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' + \int_V \nabla' \cdot \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' \right] \\ &= \frac{1}{4\pi} \left[ -\int_V \frac{\nabla' \cdot \vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' + \oint_S \frac{\hat{n} \cdot \vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dS' \right], \end{aligned} \quad (2.291)$$

$$\begin{aligned} \vec{\psi}(\vec{r}) &= \frac{1}{4\pi} \int_V \nabla \times \left[ \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} \right] dV' \\ &= \frac{1}{4\pi} \int_V \nabla \left[ \frac{1}{l(\vec{r}, \vec{r}')} \right] \times \vec{A}(\vec{r}') dV' \\ &= -\frac{1}{4\pi} \int_V \nabla' \left[ \frac{1}{l(\vec{r}, \vec{r}')} \right] \times \vec{A}(\vec{r}') dV' \\ &= \frac{1}{4\pi} \left[ -\int_V \frac{\nabla' \times \vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' - \int_V \nabla' \times \frac{\vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' \right] \\ &= \frac{1}{4\pi} \left[ \int_V \frac{\nabla' \times \vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dV' - \oint_S \frac{\hat{n} \times \vec{A}(\vec{r}')}{l(\vec{r}, \vec{r}')} dS' \right]. \end{aligned} \quad (2.292)$$

These complete the proof of Helmholtz's theorem.

On  $S$ , we can decompose  $\vec{A}$  into two terms according to (2.45),

$$\vec{A} = (\hat{n} \cdot \vec{A})\hat{n} - \hat{n} \times (\hat{n} \times \vec{A}). \quad (2.293)$$

As the first term is evidently the component along  $\hat{n}$ , the second term is the component perpendicular to  $\hat{n}$ , or tangential to  $S$ . It can be seen from (2.291) and (2.292) how  $\phi$  and  $\vec{\psi}$  depend on  $\vec{A}$ :  $\phi$  depends on the divergence of  $\vec{A}$  within  $V$  and the normal component of  $\vec{A}$  on  $S$ ;  $\vec{\psi}$  depends on the curl of  $\vec{A}$  within  $V$  and the tangential component of  $\vec{A}$  on  $S$ , where  $\hat{n} \times \vec{A}$  can be understood as the result of rotating  $-\hat{n} \times (\hat{n} \times \vec{A})$  by  $\pi/2$  around the normal  $\hat{n}$ .



## 2.7 Curvilinear Coordinate Systems

### 2.7.1 Definition of Curvilinear Coordinate Systems

We have been using the rectangular coordinates system in the preceding sections. However, based on the properties of the problems studied, other types of coordinate systems may be more convenient. In geophysics, the spherical and cylindrical coordinate systems are frequently used, which are actually the most common non-rectangular coordinate systems. Both the spherical and cylindrical coordinate systems are orthogonal curvilinear coordinate systems. However, the formulation for general curvilinear coordinate systems is not much lengthier than that for orthogonal curvilinear coordinate systems. Hence, we provide the formulation for general curvilinear coordinate systems, which is indeed used in geophysics, where, e.g., the Clairaut's coordinates are non-orthogonal curvilinear coordinates (e.g., Rochester et al., 2014). Orthogonal curvilinear coordinate systems will be treated as a special case of general curvilinear coordinate systems, and the spherical and cylindrical coordinate systems will be treated as instantiations.

The purpose is for application in geophysics, particularly in problems related to continuum mechanics. As the governing equations are mostly easier to be formulated in the rectangular coordinate system, our focus is to provide the mathematical tools to transform the governing equations from the rectangular coordinate system to a curvilinear coordinate system.

#### Definition of Curvilinear Coordinate Systems

In the rectangular coordinate system, the position of a point in the space is represented by three coordinates,  $x_1$ ,  $x_2$ , and  $x_3$ , and the position vector  $\vec{r}$  is expressed as

$$\vec{r} = x_i \hat{e}_i. \quad (2.294)$$

In a curvilinear coordinate system, the position of a point in the space is represented by another three coordinates,  $q^1$ ,  $q^2$ , and  $q^3$ ; the use of superscripts to denote the three curvilinear coordinates will be explained later. We express the relation between the rectangular coordinates  $x_1$ ,  $x_2$ , and  $x_3$  and the curvilinear coordinates  $q^1$ ,  $q^2$ , and  $q^3$  as

$$x_i = x_i(q^1, q^2, q^3). \quad (2.295)$$

The position vector of a point in the space, (2.294), can then be expressed in the form of

$$\vec{r} = x_i(q^1, q^2, q^3) \hat{e}_i. \quad (2.296)$$

We now assume that, among  $q^1$ ,  $q^2$ , and  $q^3$ , only one of them, say  $q^i$ , varies, while the other two remain fixed. In this circumstance, (2.296) becomes the parametric

equation of a curve, called  $q^i$ -curve. The  $q^1$ -,  $q^2$ -, and  $q^3$ -curves are referred to as coordinate curves of the curvilinear coordinate system. The tangential vector of the  $q^i$ -curve is

$$\vec{i}_i = \frac{\partial \vec{r}}{\partial q^i}. \quad (2.297)$$

The vectors  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$  are referred to as the covariant base vectors of the curvilinear coordinate system, which are demonstrated in Fig. 2.17. Another set of base vectors,  $\vec{i}^1$ ,  $\vec{i}^2$ , and  $\vec{i}^3$ , are defined according to

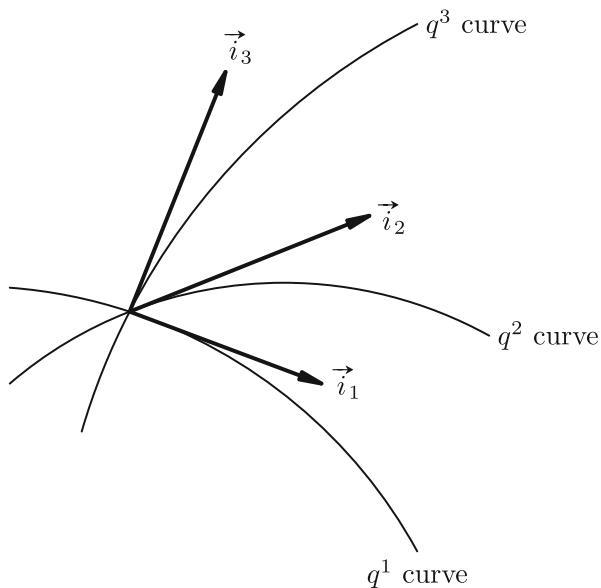
$$\vec{i}_i \cdot \vec{i}^j = \delta_{ij}, \quad (2.298)$$

called contravariant base vectors. The rectangular coordinate system can be understood as a special case with identical covariant and contravariant base vectors, both denoted as  $\hat{e}_i$ ,

$$\hat{e}_i = \frac{\partial \vec{r}}{\partial x_i}. \quad (2.299)$$

The meaning of the terms *covariant* and *contravariant* will be explained later. Here we just understand them as a naming convention. We note that quantities numbered with a subscript are covariant, and those numbered with a superscript are contravariant. The coordinates  $q^1$ ,  $q^2$ , and  $q^3$  are contravariant. Any set of

**Fig. 2.17** Covariant base vectors of a curvilinear coordinate system



base vectors, either covariant or contravariant, are not necessarily unit vectors, or orthogonal each other.

Among  $q^1$ ,  $q^2$ , and  $q^3$  if one of them is assumed fixed, and the other two vary, (2.296) becomes the parametric equation of a surface, called coordinate surface. For  $i \neq j$ , the coordinate surface corresponding to varying  $q^i$  and  $q^j$  is referred to as  $q^i q^j$ -surface.

Not any three quantities  $q^1$ ,  $q^2$ , and  $q^3$  defined by (2.295) can form a curvilinear coordinate system. The requirements for them to form a curvilinear coordinate system are: (a) the function defined in (2.295) is single-valued and continuous; (b) the covariant base vectors  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$  are not in the same plane, i.e.,

$$(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3 = \left( \frac{\partial \vec{r}}{\partial q^1} \times \frac{\partial \vec{r}}{\partial q^2} \right) \cdot \frac{\partial \vec{r}}{\partial q^3} \neq 0; \quad (2.300)$$

(c) the covariant base vectors  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$  as well as their first order partial derivatives are continuous. The requirement (a) has already been implied by (2.296)—in order for a set of values of  $q^1$ ,  $q^2$ , and  $q^3$  corresponds to a unique point in the space, the functions defined by (2.295) must be single valued; in order for the values of  $q^1$ ,  $q^2$ , and  $q^3$  to vary continuously when the point in the space moves continuously, the functions defined by (2.295) must be continuous. The implication of the other two requirements will be made clear later in this section and Sects. 2.7.2 and 2.8.2.

### Transformation Between the Base Vectors of Rectangular and Curvilinear Coordinate Systems

In order to express vectors and tensors in a curvilinear coordinate system, we must first derive the relation between the base vectors of the rectangular coordinate system and the base vectors of the curvilinear coordinate system, which is considered as a fundamental property of the curvilinear coordinate system.

The relation between  $\hat{e}_j$  and  $\vec{i}_i$  can be obtained readily based on the definitions (2.297) and (2.299),

$$\vec{i}_i = \frac{\partial \vec{r}}{\partial q^i} = \frac{\partial \vec{r}}{\partial x_j} \frac{\partial x_j}{\partial q^i} = \frac{\partial x_j}{\partial q^i} \hat{e}_j, \quad (2.301)$$

$$\hat{e}_j = \frac{\partial \vec{r}}{\partial x_j} = \frac{\partial \vec{r}}{\partial q^i} \frac{\partial q^i}{\partial x_j} = \frac{\partial q^i}{\partial x_j} \vec{i}_i, \quad (2.302)$$

which can be used to transform from  $\hat{e}_j$  to  $\vec{i}_i$ , and from  $\vec{i}_i$  to  $\hat{e}_j$ .

The derivation of the relation between  $\hat{e}_j$  and  $\vec{i}^i$  is more complicated. According to (2.302), we have

$$\vec{i}^i \cdot \hat{e}_j = \frac{\partial q^k}{\partial x_j} \vec{i}^i \cdot \vec{i}_k = \frac{\partial q^k}{\partial x_j} \delta_{ki} = \frac{\partial q^i}{\partial x_j}, \quad (2.303)$$

which is the component of  $\vec{i}^i$  along  $\hat{e}_j$ . Hence,

$$\vec{i}^i = \frac{\partial q^i}{\partial x_j} \hat{e}_j. \quad (2.304)$$

According to (2.301), we have  $\vec{i}_i \cdot \hat{e}_j = \partial x_j / \partial q^i$ . Changing the free index  $j$  to  $k$  in (2.302), and then perform the scalar products of both sides with  $\hat{e}_j$ , we obtain

$$\delta_{kj} = \hat{e}_k \cdot \hat{e}_j = \frac{\partial q^i}{\partial x_k} \vec{i}_i \cdot \hat{e}_j = \frac{\partial q^i}{\partial x_k} \frac{\partial x_j}{\partial q^i}. \quad (2.305)$$

Swapping the two sides of (2.304), changing the summation index  $j$  to  $k$ , and then multiplying both sides by  $\partial x_j / \partial q^i$ , we obtain, by referencing further (2.305),

$$\hat{e}_j = \delta_{kj} \hat{e}_k = \frac{\partial q^i}{\partial x_k} \frac{\partial x_j}{\partial q^i} \hat{e}_k = \frac{\partial x_j}{\partial q^i} \vec{i}^i. \quad (2.306)$$

We see that (2.304) is the transformation from  $\hat{e}_j$  to  $\vec{i}^i$ , and (2.306) from  $\vec{i}^i$  to  $\hat{e}_j$ .

### The Metric Tensor

Define

$$g_{ij} = \vec{i}_i \cdot \vec{i}_j = \left( \frac{\partial x_k}{\partial q^i} \hat{e}_k \right) \cdot \left( \frac{\partial x_l}{\partial q^j} \hat{e}_l \right) = \frac{\partial x_k}{\partial q^i} \frac{\partial x_k}{\partial q^j}, \quad (2.307)$$

$$g^{ij} = \vec{i}^i \cdot \vec{i}^j = \left( \frac{\partial q^i}{\partial x_k} \hat{e}_k \right) \cdot \left( \frac{\partial q^j}{\partial x_l} \hat{e}_l \right) = \frac{\partial q^i}{\partial x_k} \frac{\partial q^j}{\partial x_k}. \quad (2.308)$$

Each of  $g_{ij}$  and  $g^{ij}$  is the component of a second order tensor, the former is called metric tensor, and the latter, conjugate metric tensor. Their meanings will be made clear later.

By substituting (2.302) into (2.304), and (2.306) into (2.301), we obtain

$$\vec{i}^i = \frac{\partial q^i}{\partial x_k} \hat{e}_k = \frac{\partial q^i}{\partial x_k} \frac{\partial q^j}{\partial x_k} \vec{i}_j = g^{ij} \vec{i}_j, \quad (2.309)$$

$$\vec{i}_j = \frac{\partial x_k}{\partial q^j} \hat{e}_k = \frac{\partial x_k}{\partial q^j} \frac{\partial x_k}{\partial q^i} \vec{i}^i = g_{ij} \vec{i}^i. \quad (2.310)$$

These are the transformation between the covariant and contravariant base vectors.

In the above pairs of transformations between base vectors, (2.301) and (2.302), (2.304) and (2.306), and (2.309) and (2.310), two kinds of quantities,  $\partial x_j / \partial q^i$  and  $\partial q^i / \partial x_j$ , are included. We now derive the relation between them. According to

(2.301) and (2.302),

$$\vec{i}_i \cdot \hat{e}_j = \frac{\partial x_k}{\partial q^i} \hat{e}_k \cdot \hat{e}_j = \frac{\partial x_j}{\partial q^i}, \quad (2.311)$$

$$\vec{i}_i \cdot \hat{e}_j = \vec{i}_i \cdot \left( \frac{\partial q^k}{\partial x_j} \vec{i}_k \right) = \frac{\partial q^k}{\partial x_j} \vec{i}_i \cdot \vec{i}_k = g_{ik} \frac{\partial q^k}{\partial x_j}. \quad (2.312)$$

Hence, we obtain

$$\frac{\partial x_j}{\partial q^i} = g_{ik} \frac{\partial q^k}{\partial x_j}, \quad (2.313)$$

which is the relation for computing  $\partial x_j / \partial q^i$  from  $\partial q^i / \partial x_j$ . According to (2.304) and (2.306),

$$\vec{i}^i \cdot \hat{e}_j = \frac{\partial q^i}{\partial x_k} \hat{e}_k \cdot \hat{e}_j = \frac{\partial q^i}{\partial x_j}, \quad (2.314)$$

$$\vec{i}^i \cdot \hat{e}_j = \vec{i}^i \cdot \left( \frac{\partial x_j}{\partial q^k} \vec{i}^k \right) = \frac{\partial x_j}{\partial q^k} \vec{i}^i \cdot \vec{i}^k = g^{ik} \frac{\partial x_j}{\partial q^k}. \quad (2.315)$$

Hence, we obtain

$$\frac{\partial q^i}{\partial x_j} = g^{ik} \frac{\partial x_j}{\partial q^k}. \quad (2.316)$$

which is the relation to compute  $\partial q^i / \partial x_j$  from  $\partial x_j / \partial q^i$ .

The relations between  $\partial x_j / \partial q^i$  and  $\partial q^i / \partial x_j$ , i.e., (2.313) and (2.316), should be complemented with the relations between  $g_{ij}$  and  $g^{ij}$ . Both  $g_{ij}$  and  $g^{ij}$  are symmetric:

$$g_{ij} = g_{ji}, \quad g^{ij} = g^{ji}. \quad (2.317)$$

According to (2.309) and (2.310), we have

$$g_{ik} g^{kj} = \delta_{ij}, \quad (2.318)$$

which can be written in matrix form as

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \bar{I}. \quad (2.319)$$

where  $\bar{I}$  is the unit tensor. Hence,

$$\begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}^{-1}, \quad (2.320)$$

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix}^{-1}. \quad (2.321)$$

According to the formula of the inverse of rank 3 matrices, the formula for computing  $g^{ij}$  from  $g_{ij}$  can be obtained from (2.320) as

$$g^{ij} = G_{ij}/g, \quad (2.322)$$

where  $g$  is the determinant of the metric tensor,

$$\begin{aligned} g &= \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_k}{\partial q^1} \frac{\partial x_k}{\partial q^1} & \frac{\partial x_k}{\partial q^1} \frac{\partial x_k}{\partial q^2} & \frac{\partial x_k}{\partial q^1} \frac{\partial x_k}{\partial q^3} \\ \frac{\partial x_k}{\partial q^2} \frac{\partial x_k}{\partial q^1} & \frac{\partial x_k}{\partial q^2} \frac{\partial x_k}{\partial q^2} & \frac{\partial x_k}{\partial q^2} \frac{\partial x_k}{\partial q^3} \\ \frac{\partial x_k}{\partial q^3} \frac{\partial x_k}{\partial q^1} & \frac{\partial x_k}{\partial q^3} \frac{\partial x_k}{\partial q^2} & \frac{\partial x_k}{\partial q^3} \frac{\partial x_k}{\partial q^3} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial q^1} \frac{\partial x_2}{\partial q^1} \frac{\partial x_3}{\partial q^1} \\ \frac{\partial x_1}{\partial q^1} \frac{\partial x_2}{\partial q^2} \frac{\partial x_3}{\partial q^2} \\ \frac{\partial x_1}{\partial q^2} \frac{\partial x_2}{\partial q^2} \frac{\partial x_3}{\partial q^2} \\ \frac{\partial x_1}{\partial q^3} \frac{\partial x_2}{\partial q^3} \frac{\partial x_3}{\partial q^3} \end{vmatrix} \begin{vmatrix} \frac{\partial x_1}{\partial q^1} \frac{\partial x_1}{\partial q^2} \frac{\partial x_1}{\partial q^3} \\ \frac{\partial x_2}{\partial q^1} \frac{\partial x_2}{\partial q^2} \frac{\partial x_2}{\partial q^3} \\ \frac{\partial x_3}{\partial q^1} \frac{\partial x_3}{\partial q^2} \frac{\partial x_3}{\partial q^3} \\ \frac{\partial q^1}{\partial q^1} \frac{\partial q^2}{\partial q^2} \frac{\partial q^3}{\partial q^3} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_1}{\partial q^1} \frac{\partial x_2}{\partial q^1} \frac{\partial x_3}{\partial q^1} \\ \frac{\partial x_1}{\partial q^1} \frac{\partial x_2}{\partial q^2} \frac{\partial x_3}{\partial q^2} \\ \frac{\partial x_1}{\partial q^2} \frac{\partial x_2}{\partial q^2} \frac{\partial x_3}{\partial q^2} \\ \frac{\partial x_1}{\partial q^3} \frac{\partial x_2}{\partial q^3} \frac{\partial x_3}{\partial q^3} \end{vmatrix}^2 \\ &= [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3]^2, \end{aligned} \quad (2.323)$$

and  $G_{ij}$  is an element of the adjugate matrix of  $g$ , which can be explicitly written as

$$G_{ij} = g_{mr}g_{ns} - g_{nr}g_{ms}, \quad (2.324)$$

where the values of  $m, n, r, s$  being related to those of  $i$  and  $j$  such that  $jmn$  and  $irs$  must be in the order of 123, 231 or 312. The formula for computing  $g_{ij}$  from  $g^{ij}$  can be obtained similarly from (2.321)

Evidently, in order to express  $\hat{e}_1, \hat{e}_2$ , and  $\hat{e}_3$  using  $\vec{i}_1, \vec{i}_2$ , and  $\vec{i}_3$  according to (2.302),  $\vec{i}_1, \vec{i}_2$ , and  $\vec{i}_3$  must not be in the same plane, which is the requirement (b) for  $q^1, q^2$ , and  $q^3$  defined in (2.295) to form a curvilinear coordinate system. The same is for expressing  $\hat{e}_1, \hat{e}_2$ , and  $\hat{e}_3$  using  $\vec{i}^1, \vec{i}^2$ , and  $\vec{i}^3$  according to (2.306). It can also be shown according to (2.320) and (2.323), as well as the corresponding

formulae for the conjugate metric tensor, that, if the covariant base vectors  $\vec{i}_1, \vec{i}_2$ , and  $\vec{i}_3$  are not in the same plane, the contravariant base vectors  $\vec{i}^1, \vec{i}^2$ , and  $\vec{i}^3$  are not in the same plane either, which can also be inferred later from (2.362), (2.363), and (2.364).

### Unit Base Vectors

For expressing physical equations in a curvilinear coordinate system, unit vectors in the same directions as those of the covariant and contravariant base vectors are also used, called unit covariant and contravariant base vectors, and are denoted as  $\hat{i}_1, \hat{i}_2, \hat{i}_3$  and  $\hat{i}^1, \hat{i}^2, \hat{i}^3$ , respectively. We have, according to (2.307) and (2.308),

$$\hat{i}_i = \frac{1}{\sqrt{\vec{i}_i \cdot \vec{i}_i}} \vec{i}_i = \frac{1}{\sqrt{g_{ii}}} \vec{i}_i, \quad i \text{ is not a summation index}; \quad (2.325)$$

$$\hat{i}^i = \frac{1}{\sqrt{\vec{i}^i \cdot \vec{i}^i}} \vec{i}^i = \frac{1}{\sqrt{g^{ii}}} \vec{i}^i, \quad i \text{ is not a summation index}. \quad (2.326)$$

### Spherical and Cylindrical Coordinate Systems

As instantiations, we derive the formula of the metric and conjugate metric tensors for the spherical and cylindrical coordinate systems.

For the spherical coordinate system, the coordinate transformation is defined by

$$x_1 = r \sin \theta \cos \lambda, \quad x_2 = r \sin \theta \sin \lambda, \quad x_3 = r \cos \theta. \quad (2.327)$$

The curvilinear coordinates are

$$q^1 = r, \quad q^2 = \theta, \quad q^3 = \lambda. \quad (2.328)$$

The components of the metric tensor are

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta, \quad g_{ij} = 0 \text{ when } i \neq j. \quad (2.329)$$

The diagonality of the metric tensor implies that  $g^{ii} = 1/g_{ii}$  (where  $i$  is not a summation index), and  $g^{ij} = 0$  when  $i \neq j$ . Hence, the components of the conjugate metric tensor are

$$g^{11} = 1, \quad g^{22} = 1/r^2, \quad g^{33} = 1/(r^2 \sin^2 \theta), \quad g^{ij} = 0 \text{ when } i \neq j. \quad (2.330)$$

For the cylindrical coordinate system, the coordinate transformation is defined by

$$x_1 = r \cos \lambda, \quad x_2 = r \sin \lambda, \quad x_3 = z. \quad (2.331)$$

The curvilinear coordinates are

$$q^1 = r, \quad q^2 = \lambda, \quad q^3 = z. \quad (2.332)$$

The components of the metric tensor are

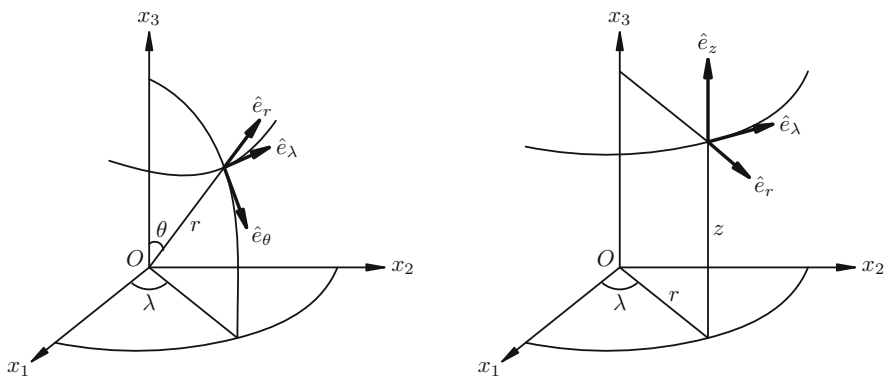
$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1, \quad g_{ij} = 0 \text{ when } i \neq j. \quad (2.333)$$

Again, the diagonality of the metric tensor implies that  $g^{ii} = 1/g_{ii}$  (where  $i$  is not a summation index), and  $g^{ij} = 0$  when  $i \neq j$ . Hence, the components of the conjugate metric tensor are

$$g^{11} = 1, \quad g^{22} = 1/r^2, \quad g^{33} = 1, \quad g^{ij} = 0 \text{ when } i \neq j. \quad (2.334)$$

It is left to the readers to further derive the formulae of the transformations between the base vectors of these two coordinate systems and those of the rectangular coordinate system. In both these coordinate systems, the covariant base vectors are orthogonal each other, and the metric tensor and the conjugate metric tensor are diagonal, implying that the contravariant base vectors are also diagonal each other. Therefore, the unit covariant and contravariant base vectors are identical, and are both referred to as unit base vectors, which are denoted as  $\hat{e}_r, \hat{e}_\theta$ , and  $\hat{e}_\lambda$  for the spherical coordinate system, and  $\hat{e}_r, \hat{e}_\lambda$ , and  $\hat{e}_z$  for the cylindrical coordinate system, as shown in Fig. 2.18. As reference we provide the transformation from the base vectors of the rectangular coordinate system to the unit base vectors of these curvilinear coordinate systems. For the spherical coordinate system,

$$\begin{cases} \hat{e}_r = \sin \theta \cos \lambda \hat{e}_1 + \sin \theta \sin \lambda \hat{e}_2 + \cos \theta \hat{e}_3, \\ \hat{e}_\theta = \cos \theta \cos \lambda \hat{e}_1 + \cos \theta \sin \lambda \hat{e}_2 - \sin \theta \hat{e}_3, \\ \hat{e}_\lambda = -\sin \lambda \hat{e}_1 + \cos \lambda \hat{e}_2. \end{cases} \quad (2.335)$$



**Fig. 2.18** The unit base vectors of the spherical (left) and cylindrical (right) coordinate systems



For the cylindrical coordinate system,

$$\begin{cases} \hat{e}_r = \cos \lambda \hat{e}_1 + \sin \lambda \hat{e}_2, \\ \hat{e}_\lambda = -\sin \lambda \hat{e}_1 + \cos \lambda \hat{e}_2, \\ \hat{e}_z = \hat{e}_3. \end{cases} \quad (2.336)$$

## 2.7.2 Vectors and Tensors in Curvilinear Coordinate Systems

### Expressions of Vectors and Tensors in Curvilinear Coordinate Systems

Expressing a scalar function in a curvilinear coordinate system is simple. For a scalar function expressed in the rectangular coordinate system as  $u(x_1, x_2, x_3)$ , we just have to substitute  $x_1, x_2$ , and  $x_3$  by  $q^1, q^2$ , and  $q^3$  according to (2.295) to obtain  $u(q^1, q^2, q^3)$ .

Expressing a vector function in a curvilinear coordinate system is not so straightforward. In the rectangular coordinate system, a vector can be expressed graphically as a directed line segment starting from the origin, and its projections on the axes are the components. The basis for this kind of representation is that the base vectors  $\hat{e}_1, \hat{e}_2$ , and  $\hat{e}_3$  are independent of the coordinates, and are orthogonal each other. In a curvilinear coordinate system, complications are brought by the facts that (a) the base vectors, either the covariant ones  $\vec{i}_1, \vec{i}_2$ , and  $\vec{i}_3$ , or the contravariant ones  $\vec{i}^1, \vec{i}^2$ , and  $\vec{i}^3$ , may vary with the coordinates  $q^1, q^2$ , and  $q^3$ ; and (b) neither of the two sets of base vectors are necessarily orthogonal each other. Hence, the convention in the rectangular coordinate system of using a directed line segment starting from the origin can no longer be adopted. However, for any specific position in the space, both sets of base vectors of a curvilinear coordinate system are unique. Hence, the value of a vector at a position in the space is chosen to be expressed using a set of base vectors at this position; either the covariant ones  $\vec{i}_1, \vec{i}_2$ , and  $\vec{i}_3$ , or the contravariant ones  $\vec{i}^1, \vec{i}^2$ , and  $\vec{i}^3$ , serve the purpose. We have

$$\vec{A}(q^1, q^2, q^3) = \tilde{A}^i(q^1, q^2, q^3) \vec{i}_i(q^1, q^2, q^3), \quad (2.337)$$

$$\vec{A}(q^1, q^2, q^3) = \tilde{A}_i(q^1, q^2, q^3) \vec{i}^i(q^1, q^2, q^3). \quad (2.338)$$

For brevity, we write them as

$$\vec{A} = \tilde{A}^i \vec{i}_i, \quad (2.339)$$

$$\vec{A} = \tilde{A}_i \vec{i}^i, \quad (2.340)$$

where we must keep in mind that the base vectors are functions of the coordinates. In this way, we have defined two sets of components as well for a vector. The components  $\tilde{A}^i$  are referred to as contravariant, and the components  $\tilde{A}_i$  are referred to as covariant. Either expressed using covariant or contravariant base vectors, the

vector is the same. Hence, the above two expressions can be written together as

$$\vec{A} = \tilde{A}^i \vec{i}_i = \tilde{A}_i \vec{i}^i. \quad (2.341)$$

The expression for a tensor is similar,

$$\vec{T} = \tilde{T}^{ij} \vec{i}_i \vec{i}_j = \tilde{T}_{ij} \vec{i}^i \vec{i}^j = \tilde{T}^i_{\cdot j} \vec{i}_i \vec{i}^j = \tilde{T}^{\cdot j}_i \vec{i}^i \vec{i}_j, \quad (2.342)$$

where  $\tilde{T}^{ij}$  is referred to as contravariant component,  $\tilde{T}_{ij}$ , covariant component, and  $\tilde{T}^i_{\cdot j}$  and  $\tilde{T}^{\cdot j}_i$ , mixed components.

Naturally, the value of a vector or tensor at a point in the space is unique, whatever base vectors are used to express it. Therefore, the transformations among the components corresponding to different base vectors can be determined based on the transformations among base vectors. By comparing (2.341) with the expressing of  $\vec{A}$  in the rectangular coordinate system,  $\vec{A} = A_j \hat{e}_j$ , we can readily use the transformations among the base vectors to obtain the transformations among the different components of a vector,

$$\tilde{A}_i = g_{ij} \tilde{A}^j = \frac{\partial x_j}{\partial q^i} A_j, \quad (2.343)$$

$$\tilde{A}^i = g^{ij} \tilde{A}_j = \frac{\partial q^i}{\partial x_j} A_j, \quad (2.344)$$

$$A_j = \frac{\partial x_j}{\partial q^i} \tilde{A}^i = \frac{\partial q^i}{\partial x_j} \tilde{A}_i. \quad (2.345)$$

We realize that, in the curvilinear coordinate system, the components of the vector  $\vec{A}$  are related to the vector itself in the form of

$$\tilde{A}^i = \vec{A} \cdot \vec{i}^i, \quad \tilde{A}_i = \vec{A} \cdot \vec{i}_i, \quad (2.346)$$

which, together with  $A_i = \vec{A} \cdot \hat{e}_i$ , can be conveniently used to derive (2.343)–(2.345) using the relations among  $\vec{i}_i$ ,  $\vec{i}^i$ , and  $\hat{e}_i$ .

For the components of a tensor, we must first define how the dyadics  $\vec{i}_i \vec{i}_j$ ,  $\vec{i}^i \vec{i}^j$ ,  $\vec{i}_i \vec{i}^j$ , and  $\vec{i}^i \vec{i}_j$  are related to the dyadics in the rectangular coordinate system, which also implicitly define the transformations among them. We define that each base vector in a dyadic is transformed to the base vectors of the rectangular coordinate system without swapping their order, such as  $\vec{i}_i \vec{i}_j = [(\partial x_m / \partial q^i) \hat{e}_m][(\partial x_n / \partial q^j) \hat{e}_n] = (\partial x_m / \partial q^i)(\partial x_n / \partial q^j) \hat{e}_m \hat{e}_n$ . In this way, all the dyadics can be transformed from one to another similarly, e.g.,  $\vec{i}_i \vec{i}_j = (g^{im} \vec{i}^m)(g^{jn} \vec{i}^n) = g^{im} g^{jn} \vec{i}^m \vec{i}^n$ . In summary,

$$\vec{i}_i \vec{i}_j = g_{im} g_{jn} \vec{i}^m \vec{i}^n = g_{jn} \vec{i}_i \vec{i}^n = g_{im} \vec{i}^m \vec{i}_j = \frac{\partial x_m}{\partial q^i} \frac{\partial x_n}{\partial q^j} \hat{e}_m \hat{e}_n, \quad (2.347)$$

$$\vec{i}^i \vec{i}^j = g^{im} g^{jn} \vec{i}_i \vec{i}_j = g^{im} \vec{i}_m \vec{i}^j = g^{jn} \vec{i}^i \vec{i}_n = \frac{\partial q^i}{\partial x_m} \frac{\partial q^j}{\partial x_n} \hat{e}_m \hat{e}_n, \quad (2.348)$$

$$\vec{i}_i \vec{i}^j = g^{jn} \vec{i}_i \vec{i}_n = g_{im} \vec{i}^m \vec{i}^j = g_{im} g^{jn} \vec{i}^m \vec{i}_n = \frac{\partial x_m}{\partial q^i} \frac{\partial q^j}{\partial x_n} \hat{e}_m \hat{e}_n, \quad (2.349)$$

$$\vec{i}^i \vec{i}_j = g^{im} \vec{i}_m \vec{i}_j = g_{jn} \vec{i}^i \vec{i}^n = g^{im} g_{jn} \vec{i}_m \vec{i}^n = \frac{\partial q^i}{\partial x_m} \frac{\partial x_n}{\partial q^j} \hat{e}_m \hat{e}_n. \quad (2.350)$$

By comparing (2.342) with the expression of  $\bar{T}$  in the rectangular coordinate system,  $\bar{T} = T_{ij} \hat{e}_i \hat{e}_j$ , we obtain, according to the transformations among the dyadics,

$$\tilde{T}_{ij} = g_{im} g_{jn} \tilde{T}^{mn} = g_{im} \tilde{T}^m_{\cdot j} = g_{jn} \tilde{T}_i^{\cdot n} = \frac{\partial x_m}{\partial q^i} \frac{\partial x_n}{\partial q^j} T_{mn}, \quad (2.351)$$

$$\tilde{T}^{ij} = g^{im} g^{jn} \tilde{T}_{mn} = g^{jn} \tilde{T}^i_{\cdot n} = g^{im} \tilde{T}_m^{\cdot j} = \frac{\partial q^i}{\partial x_m} \frac{\partial q^j}{\partial x_n} T_{mn}, \quad (2.352)$$

$$\tilde{T}_i^{\cdot j} = g_{jn} \tilde{T}_{in} = g^{im} \tilde{T}^{mj} = g^{im} g_{jn} \tilde{T}_m^{\cdot n} = \frac{\partial x_m}{\partial q^i} \frac{\partial q^j}{\partial x_n} T_{mn}, \quad (2.353)$$

$$\tilde{T}^i_{\cdot j} = g^{im} \tilde{T}_{mj} = g_{jn} \tilde{T}^{in} = g^{im} g_{jn} \tilde{T}_m^{\cdot n} = \frac{\partial q^i}{\partial x_m} \frac{\partial x_n}{\partial q^j} T_{mn}. \quad (2.354)$$

In the curvilinear coordinate system, the components of the tensor  $\bar{T}$  are related to the tensor itself in the form of

$$\tilde{T}_{ij} = \bar{T} : \vec{i}_i \vec{i}_j, \quad \tilde{T}^{ij} = \bar{T} : \vec{i}^i \vec{i}^j, \quad \tilde{T}_i^{\cdot j} = \bar{T} : \vec{i}_i \vec{i}^j, \quad \tilde{T}^i_{\cdot j} = \bar{T} : \vec{i}^i \vec{i}_j, \quad (2.355)$$

which, together with  $T_{ij} = \bar{T} : \hat{e}_i \hat{e}_j$ , can be conveniently used to derive (2.351)–(2.354) using (2.347)–(2.350). The expression of the dyadics and components in the rectangular coordinate system in terms of those in the curvilinear coordinate system are

$$\hat{e}_m \hat{e}_n = \frac{\partial q^i}{\partial x_m} \frac{\partial q^j}{\partial x_n} \vec{i}_i \vec{i}_j = \frac{\partial x_m}{\partial q^i} \frac{\partial x_n}{\partial q^j} \vec{i}^i \vec{i}^j = \frac{\partial q^i}{\partial x_m} \frac{\partial x_n}{\partial q^j} \vec{i}_i \vec{i}^j = \frac{\partial x_m}{\partial q^i} \frac{\partial q^j}{\partial x_n} \vec{i}^i \vec{i}_j, \quad (2.356)$$

$$T_{mn} = \frac{\partial q^i}{\partial x_m} \frac{\partial q^j}{\partial x_n} \tilde{T}_{ij} = \frac{\partial x_m}{\partial q^i} \frac{\partial x_n}{\partial q^j} \tilde{T}^{ij} = \frac{\partial q^i}{\partial x_m} \frac{\partial x_n}{\partial q^j} \tilde{T}_i^{\cdot j} = \frac{\partial x_m}{\partial q^i} \frac{\partial q^j}{\partial x_n} \tilde{T}^i_{\cdot j}. \quad (2.357)$$

Evidently, in order to express a vector or tensor in a curvilinear coordinate system, the base vectors  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$ , or equivalently,  $\vec{i}^1$ ,  $\vec{i}^2$ , and  $\vec{i}^3$ , must not be in the same plane, which can be inferred from (2.302) and (2.306). Furthermore, for a vector or tensor that varies continuously in space, in order for its expression in a curvilinear coordinate system to vary continuously, the base vectors of the

curvilinear coordinate system must vary continuously in space, i.e., they must be continuous functions of the rectangular or curvilinear coordinates. These have already been mentioned in Sect. 2.7.1. It can be readily verified that, if the covariant base vectors are continuous, the contravariant base vectors are also continuous, and vice versa.

We have been using the terms *covariant* and *contravariant*. Here we explain how the covariant and contravariant base vectors and the components of a vector or tensor are defined based on their coordinate transformations. We take the case of a vector as an example. The coordinate transformations of covariant base vectors and components are  $\{\vec{i}_i, \tilde{A}_i\} = (\partial x_j / \partial q^i) \{\hat{e}_j, A_j\}$  and  $\{\hat{e}_j, A_j\} = (\partial q^i / \partial x_j) \{\vec{i}_i, \tilde{A}_i\}$ , where  $\hat{e}_j$  and  $A_j$  are regarded as the covariant base vectors and components. The coordinate transformations of contravariant base vectors and components are  $\{\vec{i}^i, \tilde{A}^i\} = (\partial q^i / \partial x_j) \{\hat{e}_j, A_j\}$  and  $\{\hat{e}_j, A_j\} = (\partial x_j / \partial q^i) \{\vec{i}^i, \tilde{A}^i\}$ , where  $\hat{e}_j$  and  $A_j$  are regarded as the contravariant base vectors and components.

### Algebraic Operations of Vectors and Tensors in Curvilinear Coordinate Systems

The sum and subtraction are similar to the case in the rectangular coordinate system:

$$\vec{A} \pm \vec{B} = (\tilde{A}^i \pm \tilde{B}^i) \vec{i}_i = (\tilde{A}_i \pm \tilde{B}_i) \vec{i}^i, \quad (2.358)$$

$$\begin{aligned} \vec{T} \pm \vec{P} &= (\tilde{T}^{ij} \pm \tilde{P}^{ij}) \vec{i}_i \vec{i}_j = (\tilde{T}_{ij} \pm \tilde{P}_{ij}) \vec{i}^i \vec{i}^j \\ &= (\tilde{T}^{\cdot j} \pm \tilde{P}^{\cdot j}) \vec{i}_i \vec{i}^j = (\tilde{T}_i^{\cdot j} \pm \tilde{P}_i^{\cdot j}) \vec{i}^i \vec{i}_j. \end{aligned} \quad (2.359)$$

The products depend on the products among the base vectors. The scalar products among the base vectors,  $\vec{i}_i \cdot \vec{i}_j$ ,  $\vec{i}^i \cdot \vec{i}^j$ , and  $\vec{i}_i \cdot \vec{i}^j$ , have already been defined in (2.307), (2.308), and (2.298). The vector products can be obtained based on the coordinate transformations among the base vectors. We have

$$\begin{aligned} \vec{i}_i \times \vec{i}_j &= \left( \frac{\partial x_m}{\partial q^i} \hat{e}_m \right) \times \left( \frac{\partial x_n}{\partial q^j} \hat{e}_n \right) = \frac{\partial x_m}{\partial q^i} \frac{\partial x_n}{\partial q^j} \epsilon_{mnl} \hat{e}_l \\ &= \frac{\partial x_m}{\partial q^i} \frac{\partial x_n}{\partial q^j} \frac{\partial x_l}{\partial q^k} \epsilon_{mnl} \vec{i}^k = \frac{\partial x_m}{\partial q^i} \frac{\partial x_n}{\partial q^j} \frac{\partial x_l}{\partial q^k} [(\hat{e}_n \times \hat{e}_m) \cdot \hat{e}_l] \vec{i}^k \\ &= [(\vec{i}_i \times \vec{i}_j) \cdot \vec{i}^k] \vec{i}^k = V \epsilon_{ijk} \vec{i}^k, \end{aligned} \quad (2.360)$$

$$\begin{aligned} \vec{i}^i \times \vec{i}^j &= \left( \frac{\partial q^i}{\partial x_m} \hat{e}_m \right) \times \left( \frac{\partial q^j}{\partial x_n} \hat{e}_n \right) = \frac{\partial q^i}{\partial x_m} \frac{\partial q^j}{\partial x_n} \epsilon_{mnl} \hat{e}_l \\ &= \frac{\partial q^i}{\partial x_m} \frac{\partial q^j}{\partial x_n} \frac{\partial q^k}{\partial x_l} \epsilon_{mnl} \vec{i}_k = \frac{\partial q^i}{\partial x_m} \frac{\partial q^j}{\partial x_n} \frac{\partial q^k}{\partial x_l} [(\hat{e}_n \times \hat{e}_m) \cdot \hat{e}_l] \vec{i}_k \\ &= [(\vec{i}^i \times \vec{i}^j) \cdot \vec{i}^k] \vec{i}_k = V' \epsilon_{ijk} \vec{i}_k, \end{aligned} \quad (2.361)$$

where  $V$  and  $V'$  are defined as

$$V = (\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3 = \pm\sqrt{g}, \quad (2.362)$$

$$V' = (\vec{i}^1 \times \vec{i}^2) \cdot \vec{i}^3 = \pm\sqrt{g'}, \quad (2.363)$$

with  $g$  and  $g'$  being the determinants of the metric and conjugate metric tensors, respectively. The ‘ $\pm$ ’ signs in the two formulae take the signs of  $V$  and  $V'$ , respectively. Evidently,  $V$  and  $V'$  obey the relation

$$V V' = 1, \quad (2.364)$$

whose proof is left to the readers as an excise. Hence, the ‘ $\pm$ ’ signs in (2.362) and (2.363) are both either ‘+’ or ‘-’. Evidently, The ‘+’ or ‘-’ depends on how  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$ , or equivalently,  $q^1$ ,  $q^2$ , and  $q^3$ , are ordered. We always assume that the order of  $q^1$ ,  $q^2$ , and  $q^3$  is chosen so that the ‘ $\pm$ ’ sign take ‘+’, which requires the angle between  $\vec{i}_1 \times \vec{i}_2$ , and  $\vec{i}_3$  be smaller than  $\pi/2$  (it is 0 in the case of rectangular coordinates). In this case, we say approximately that  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$  obey the right-hand rule, or in other words, the coordinates  $q^1$ ,  $q^2$ , and  $q^3$  constitute a right-handed coordinate system. We thus write (2.360) and (2.361) as

$$\vec{i}_i \times \vec{i}_j = \sqrt{g} \epsilon_{ijk} \vec{i}^k = \sqrt{g} \epsilon_{ijk} g^{kl} \vec{i}_l, \quad (2.365)$$

(in right-handed coordinate systems)

$$\vec{i}^i \times \vec{i}^j = (1/\sqrt{g}) \epsilon_{ijk} \vec{i}^k = (1/\sqrt{g}) \epsilon_{ijk} g_{kl} \vec{i}^l. \quad (2.366)$$

(in right-handed coordinate systems)

Hereafter, we always understand the coordinate system to be right handed if no explicit remark is made. The derivation of the products between a covariant and a contravariant base vector,  $\vec{i}_i \times \vec{i}^j$  and  $\vec{i}^i \times \vec{i}_j$ , are left to the readers. The following are some examples of products of vectors and tensors in curvilinear coordinate systems:

$$\vec{A} \cdot \vec{B} = (\tilde{A}_i \vec{i}^i) \cdot (\tilde{B}_j \vec{i}^j) = \tilde{A}_i \tilde{B}_j \vec{i}^i \cdot \vec{i}^j = \tilde{A}_i \tilde{B}_j g^{ij}, \quad (2.367)$$

$$\vec{A} \times \vec{B} = (\tilde{A}_i \vec{i}^i) \times (\tilde{B}_j \vec{i}^j) = \tilde{A}_i \tilde{B}_j \vec{i}^i \times \vec{i}^j = \tilde{A}_i \tilde{B}_j (1/\sqrt{g}) \epsilon_{ijk} g_{kl} \vec{i}^l, \quad (2.368)$$

$$\vec{A} \cdot \vec{T} = (\tilde{A}_i \vec{i}^i) \cdot (\tilde{T}_{jk} \vec{i}^j \vec{i}^k) = \tilde{A}_i \tilde{T}_{jk} (\vec{i}^i \cdot \vec{i}^j) \vec{i}^k = \tilde{A}_i \tilde{T}_{jk} g^{ij} \vec{i}^k, \quad (2.369)$$

$$\begin{aligned} \vec{T} \cdot \vec{A} &= (\tilde{T}_{ij} \vec{i}^i \vec{i}^j) \cdot (\tilde{A}_k \vec{i}^k) = \tilde{T}_{ij} \tilde{A}_k \vec{i}^i (\vec{i}^j \cdot \vec{i}^k) \\ &= \tilde{T}_{ij} \tilde{A}_k g^{jk} \vec{i}^i = \tilde{T}_{ij} \tilde{A}_k g^{jk} \vec{i}^i, \end{aligned} \quad (2.370)$$

$$\begin{aligned}\bar{T} : \bar{P} &= (\tilde{T}_{ij} \vec{i}^i \vec{i}^j) : (\tilde{T}_{mn} \vec{i}^m \vec{i}^n) \\ &= \tilde{T}_{ij} \tilde{T}_{mn} (\vec{i}^i \cdot \vec{i}^m) (\vec{i}^j \cdot \vec{i}^n) = \tilde{T}_{ij} \tilde{T}_{mn} g^{im} g^{jn}.\end{aligned}\quad (2.371)$$

In all these formulae, covariant components and contravariant base vectors are adopted. The readers are suggested to write out the formulae using contravariant components and covariant base vectors. For tensors, mixed components and dyadics can also be adopted. According to (2.298), i.e.,  $\vec{i}_i \cdot \vec{i}^j = \delta_{ij}$ , the relation between the components and the base vectors are

$$\tilde{A}_i = \vec{A} \cdot \vec{i}_i, \quad \tilde{A}^i = \vec{A} \cdot \vec{i}^i; \quad (2.372)$$

$$\tilde{T}_{ij} = \bar{T} : \vec{i}_i \vec{i}_j, \quad \tilde{T}^{ij} = \bar{T} : \vec{i}^i \vec{i}^j, \quad \tilde{T}_i^j = \bar{T} : \vec{i}_i \vec{i}^j, \quad \tilde{T}_{\cdot j}^i = \bar{T} : \vec{i}^i \vec{i}_{\cdot j}. \quad (2.373)$$

### Physical Components

Physical components are the quantities assumed to be measured by physical instruments. Therefore, the physical component of a vector along a direction is the projection of the vector along this direction. The physical components of a vector along the directions of the base vectors are then the scalar products of the vector with the unit base vectors,

$$\tilde{A}_{(i)} = \vec{A} \cdot \hat{i}_i, \quad \tilde{A}^{(i)} = \vec{A} \cdot \hat{i}^i, \quad (2.374)$$

where  $\tilde{A}_{(i)}$  is referred to as covariant physical component, and  $\tilde{A}^{(i)}$ , contravariant physical component. The covariant and contravariant physical components of a tensor are similarly defined as

$$\tilde{T}_{(i)(j)} = \bar{T} : \hat{i}_i \hat{i}_j, \quad \tilde{T}^{(i)(j)} = \bar{T} : \hat{i}^i \hat{i}^j. \quad (2.375)$$

Base on the definitions in the above two formulae, we can use (2.325), (2.326), (2.372), and (2.373) to derive the relation between the covariant physical components and the covariant components as well as that between the contravariant physical components and the contravariant components.

$$\begin{cases} \tilde{A}_{(i)} = (1/\sqrt{g_{ii}}) \tilde{A}_i, \\ \tilde{A}^{(i)} = (1/\sqrt{g^{ii}}) \tilde{A}^i, \end{cases} \quad i \text{ is not a summation index}; \quad (2.376)$$

$$\begin{cases} \tilde{T}_{(i)(j)} = (1/\sqrt{g_{ii}g_{jj}}) \tilde{T}_{ij}, \\ \tilde{T}^{(i)(j)} = (1/\sqrt{g^{ii}g^{jj}}) \tilde{T}^{ij}, \end{cases} \quad i \text{ and } j \text{ are not summation indices.} \quad (2.377)$$

Vectors and tensors can be expressed using physical components as

$$\vec{A} = \sum_{i=1}^3 \tilde{A}_{(i)} \sqrt{g_{ii} g^{ii}} \hat{i}^i = \sum_{i=1}^3 \tilde{A}^{(i)} \sqrt{g^{ii} g_{ii}} \hat{i}_i, \quad (2.378)$$

$$\bar{T} = \sum_{i=1}^3 \sum_{j=1}^3 \tilde{T}_{(i)(j)} \sqrt{g_{ii} g_{jj} g^{ii} g^{jj}} \hat{i}^i \hat{j}^j = \sum_{i=1}^3 \sum_{j=1}^3 \tilde{T}^{(i)(j)} \sqrt{g^{ii} g^{jj} g_{ii} g_{jj}} \hat{i}_i \hat{j}_j. \quad (2.379)$$

It can be seen that, as summation indices can no longer be used, it is lengthier to formulate using physical components.

### Spherical and Cylindrical Coordinate Systems

As instantiations, we consider again the spherical and cylindrical coordinate systems, where, as the directions of covariant base vectors are the same as those of the corresponding contravariant base vectors, covariant physical components are equal to the corresponding contravariant physical components. For a vector  $\vec{A}$ , we denote its physical components as  $A_r$ ,  $A_\theta$ , and  $A_\lambda$  in the spherical coordinate system, and as  $A_r$ ,  $A_\lambda$ , and  $A_z$  in the cylindrical coordinate system. For a tensor  $\bar{T}$ , we denote its physical components as  $T_{rr}$ ,  $T_{r\theta}$ ,  $T_{r\lambda}$ ,  $T_{\theta r}$ ,  $T_{\theta\theta}$ ,  $T_{\theta\lambda}$ ,  $T_{\lambda r}$ ,  $T_{\lambda\theta}$ , and  $T_{\lambda\lambda}$  in the spherical coordinate system, and as  $T_{rr}$ ,  $T_{r\lambda}$ ,  $T_{rz}$ ,  $T_{\lambda r}$ ,  $T_{\lambda\lambda}$ ,  $T_{\lambda z}$ ,  $T_{zr}$ ,  $T_{z\lambda}$ , and  $T_{zz}$  in the cylindrical coordinate system. As already been mentioned, unit covariant and contravariant base vectors are identical in these coordinate systems, and  $g_{ii} g^{ii} = 1$  (where  $i$  is not a summation index). For the spherical coordinate system, we obtain, based on the definition (2.327), and according to (2.343), (2.376), and (2.378),

$$\begin{cases} A_r = \sin \theta \cos \lambda A_1 + \sin \theta \sin \lambda A_2 + \cos \theta A_3, \\ A_\theta = \cos \theta \cos \lambda A_1 + \cos \theta \sin \lambda A_2 - \sin \theta A_3, \\ A_\lambda = -\sin \lambda A_1 + \cos \lambda A_2. \end{cases} \quad (2.380)$$

For the cylindrical coordinate system, we obtain similarly

$$\begin{cases} A_r = \cos \lambda A_1 + \sin \lambda A_2, \\ A_\lambda = -\sin \lambda A_1 + \cos \lambda A_2, \\ A_z = A_3. \end{cases} \quad (2.381)$$

Transformations of the components given in (2.380) and (2.381) and those of the unit base vectors given in (2.335) and (2.336) constitute a transformation of a vector from the rectangular coordinate system to the spherical and cylindrical coordinate systems. It is left to the readers to derive the formulae for the transformation from the spherical and cylindrical coordinate systems to the rectangular coordinate system.

The spherical and cylindrical coordinate systems are orthogonal coordinate systems like the rectangular coordinate system. The same as in the rectangular

coordinate system, in the spherical and cylindrical coordinate systems, the unit base vectors (it has been proven that covariant and contravariant unit base vectors are identical) are orthogonal each other. Hence, the scalar and vector products among the unit base vectors in the spherical and cylindrical coordinate systems are similar to those in the rectangular coordinate system. Therefore, the formulae of scalar and vector products based on the physical components in the spherical and cylindrical coordinate systems are similar to those in the rectangular coordinate system.

## 2.8 Some Properties of Curvilinear Coordinate Systems

### 2.8.1 Infinitesimal Line, Surface and Volume Elements

Integration and differentiation are fundamental operations in infinitesimal analysis. The formulation of infinitesimal line, surface and volume elements in this subsection is for the computation of integrals using curvilinear coordinates.

#### Line Elements

According to (2.166) and Fig. 2.10, the infinitesimal line element on the coordinate curve  $q^i$  corresponding to an coordinate increment of  $dq^i$  is

$$dl_i = \left| \frac{\partial \vec{r}}{\partial q^i} \right| dq^i, \quad i \text{ is not a summation index.} \quad (2.382)$$

It can also be written as, according to (2.301) and (2.307),

$$dl_i = \sqrt{\vec{i}_i \cdot \vec{i}_i} dq^i = \sqrt{g_{ii}} dq^i, \quad i \text{ is not a summation index.} \quad (2.383)$$

We now consider a general infinitesimal line element in the space, which is naturally an increment of the position vector, and we denote it as  $d\vec{l} = d\vec{r}$ . Denote the increments of coordinates associated with it as  $dq^1, dq^2$ , and  $dq^3$ . We then have

$$d\vec{l} = d\vec{r} = \frac{\partial \vec{r}}{\partial q^i} dq^i = \vec{i}_i dq^i, \quad (2.384)$$

where  $i$  is understood as a summation index by default. The length of a general infinitesimal line element in the space can be obtained as

$$dl = \sqrt{d\vec{r} \cdot d\vec{r}} = \sqrt{(\vec{i}_i dq^i) \cdot (\vec{i}_j dq^j)} = \sqrt{g_{ij} dq^i dq^j}. \quad (2.385)$$

If the line is given in the form of a parametric equation,

$$q^i = q^i(t), \quad (2.386)$$



the length of the infinitesimal increment of the position vector, (2.385), becomes

$$dl = \sqrt{g_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt}} dt. \quad (2.387)$$

We now see from (2.384) why the three coordinates are denoted using superscripts— $dq^1$ ,  $dq^2$  and  $dq^3$  are the contravariant components of the infinitesimal position vector element  $d\vec{r}$ .

In the spherical coordinate system  $Or\theta\lambda$ , the infinitesimal line elements along the coordinate curves are

$$dl_r = dr, \quad dl_\theta = r d\theta, \quad dl_\lambda = r \sin \theta d\lambda; \quad (2.388)$$

and the length of a general infinitesimal line element is

$$\begin{aligned} dl &= \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\lambda^2} \\ &= \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 + r^2 \sin^2 \theta \left(\frac{d\lambda}{dt}\right)^2} dt, \end{aligned} \quad (2.389)$$

where, in the last expression, the line is assumed to be written in the form of parametric equations  $r = r(t)$ ,  $\theta = \theta(t)$ , and  $\lambda = \lambda(t)$ .

In the cylindrical coordinate system  $Or\lambda z$ , the infinitesimal line elements along the coordinate curves are

$$dl_r = dr, \quad dl_\lambda = r d\lambda, \quad dl_z = dz; \quad (2.390)$$

and the length of a general infinitesimal line element is

$$dl = \sqrt{dr^2 + r^2 d\lambda^2 + dz^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt, \quad (2.391)$$

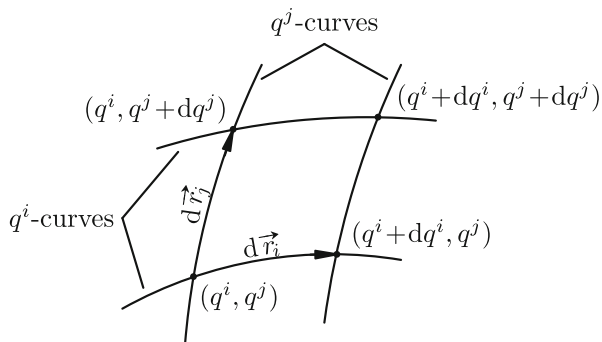
where, in the last expression, the line is assumed to be written in the form of parametric equations  $r = r(t)$ ,  $\lambda = \lambda(t)$ , and  $z = z(t)$ .

### Surface Elements

We compute the area of an infinitesimal parallelogram within the coordinate surface  $q^i q^j$  bounded by two infinitesimally close  $q^i$ -curves, and two infinitesimally close  $q^j$ -curves, as shown in Fig. 2.19. We consider the area to be directed, and denote it as  $d\vec{S}_{ij}$ . The meaning of the direction will be made clear in the derivation. This infinitesimal parallelogram is an infinitesimal surface element in a curvilinear coordinate system.

On the coordinate surface  $q^i q^j$ , only  $q^i$  and  $q^j$  vary. Hence, a  $q^i$ -curve is fully determined by the value of  $q^j$ , and a  $q^j$ -curve is fully determined by the value of

**Fig. 2.19** An infinitesimal surface element bounded by coordinate curves



$q^i$ . As shown in Fig. 2.19, we use  $q^j$  and  $q^j + dq^j$  to denote the values of  $q^j$  on the two  $q^i$ -curves, and use  $q^i$  and  $q^i + dq^i$  to denote the values of  $q^i$  on the two  $q^j$ -curves. The directed sides of the parallelogram along the  $q^i$ - and  $q^j$ -curves are

$$d\vec{r}_i = \vec{i}_i dq^i \quad \text{and} \quad d\vec{r}_j = \vec{i}_j dq^j, \quad i \text{ and } j \text{ are not summation indices.} \quad (2.392)$$

The area of the parallelogram is

$$\begin{aligned} d\vec{S}_{ij} &= d\vec{r}_i \times d\vec{r}_j = \vec{i}_i \times \vec{i}_j dq^i dq^j \\ &= \sqrt{g} \epsilon_{ijk} dq^i dq^j \vec{i}^k, \quad i \text{ and } j \text{ are not summation indices.} \end{aligned} \quad (2.393)$$

It can also be written as, using the unit contravariant base vectors,

$$\begin{aligned} d\vec{S}_{ij} &= \sqrt{g g^{kk}} dq^i dq^j \hat{i}^k, \\ i, j, \text{ and } k \text{ are not summation indices, } ijk &= 123, 231 \text{ or } 312. \end{aligned} \quad (2.394)$$

In the spherical coordinate system,

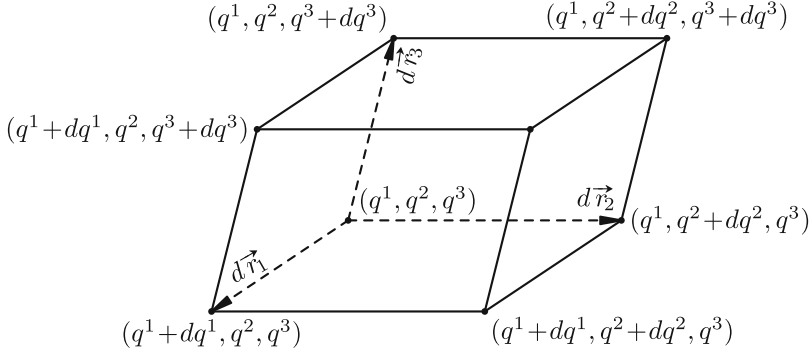
$$d\vec{S}_{r\theta} = r dr d\theta \hat{e}_\lambda, \quad d\vec{S}_{\theta\lambda} = r^2 \sin \theta d\theta d\lambda \hat{e}_r, \quad d\vec{S}_{\lambda r} = r \sin \theta d\lambda dr \hat{e}_\theta. \quad (2.395)$$

In the cylindrical coordinate system,

$$d\vec{S}_{r\lambda} = r dr d\lambda \hat{e}_z, \quad d\vec{S}_{\lambda z} = r d\lambda dz \hat{e}_r, \quad d\vec{S}_{zr} = dz dr \hat{e}_\lambda. \quad (2.396)$$

### Volume Elements

Finally, we seek the expression of the volume of the parallelepiped by two  $q^1 q^2$  surfaces, two  $q^2 q^3$  surfaces and two  $q^3 q^1$  surfaces. This is the infinitesimal volume element.



**Fig. 2.20** An infinitesimal volume element bounded by coordinate surfaces

We know that a  $q^1 q^2$ -surface is fully determined by the value of  $q^3$ , a  $q^2 q^3$ -surface is fully determined by the value of  $q^1$ , and a  $q^3 q^1$ -surface is fully determined by the value of  $q^2$ . As shown in Fig. 2.20, we use  $q^3$  and  $q^3 + dq^3$  to denote the values of  $q^3$  on the two  $q^1 q^2$ -surfaces, use  $q^1$  and  $q^1 + dq^1$  to denote the values of  $q^1$  on the two  $q^2 q^3$ -surfaces, and use  $q^2$  and  $q^2 + dq^2$  to denote the values of  $q^2$  on the two  $q^3 q^1$ -surfaces. The directed sides of the parallelepiped along the  $q^1$ -,  $q^2$ -, and  $q^3$ -curves are

$$d\vec{r}_1 = \vec{i}_1 dq^1, \quad d\vec{r}_2 = \vec{i}_2 dq^2, \quad d\vec{r}_3 = \vec{i}_3 dq^3, \quad (2.397)$$

respectively. The volume of the parallelepiped is

$$dV = (d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3 = [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3] dq^1 dq^2 dq^3 = \sqrt{g} dq^1 dq^2 dq^3. \quad (2.398)$$

In the spherical coordinate system,

$$dV = r^2 \sin \theta dr d\theta d\lambda. \quad (2.399)$$

In the cylindrical coordinate system,

$$dV = r dr d\lambda dz. \quad (2.400)$$

## 2.8.2 Derivative of Base Vectors with Respect to Coordinates

The fundamental difference between a curvilinear coordinate system and the rectangular coordinate system is that, in a curvilinear coordinate system, the base vectors vary with coordinates, which is expressed by the partial derivatives of the base vectors with respect to the coordinates. These partial derivatives are

fundamental properties of a curvilinear coordinate system. Here we discuss only the covariant base vectors.

The partial derivatives of the covariant base vectors are represented by the Christoffel symbols of the first kind,  $\Gamma_{ij,m}$ , and those of the second kind,  $\Gamma_{ij}^k$ ,

$$\frac{\partial \vec{i}_i}{\partial q^j} = \Gamma_{ij,m} \vec{i}^m = \Gamma_{ij}^k \vec{i}_k. \quad (2.401)$$

This implies that  $\Gamma_{ij,m}$  and  $\Gamma_{ij}^k$  are defined to be the covariant and contravariant components of  $\partial \vec{i}_i / \partial q^j$ , respectively. According to (2.301) and (2.372), we can express the two kinds of Christoffel symbols as

$$\Gamma_{ij,m} = \frac{\partial^2 \vec{r}}{\partial q^i \partial q^j} \cdot \vec{i}_m = \frac{\partial^2 x_l}{\partial q^i \partial q^j} \frac{\partial x_l}{\partial q^m}, \quad (2.402)$$

$$\Gamma_{ij}^k = \frac{\partial^2 \vec{r}}{\partial q^i \partial q^j} \cdot \vec{i}^k = \frac{\partial^2 x_l}{\partial q^i \partial q^j} \frac{\partial q^k}{\partial x_l}. \quad (2.403)$$

Evidently, they are both symmetrical in  $i$  and  $j$ . Hence,

$$\Gamma_{ij,m} = \Gamma_{ji,m}, \quad (2.404)$$

$$\Gamma_{ij}^k = \Gamma_{ji}^k. \quad (2.405)$$

Furthermore, according to the relation between the covariant and contravariant components of vectors,  $\Gamma_{ij,m}$  and  $\Gamma_{ij}^k$  have the relation

$$\Gamma_{ij}^k = g^{mk} \Gamma_{ij,m}. \quad (2.406)$$

In the following we first compute  $\Gamma_{ij,m}$ , and then compute  $\Gamma_{ij}^k$  using (2.406). According to (2.307), we have

$$\frac{\partial x_l}{\partial q^i} \frac{\partial x_l}{\partial q^j} = g_{ij}. \quad (2.407)$$

Take the partial derivative of both sides with respect to  $q^m$ . We obtain

$$\frac{\partial^2 x_l}{\partial q^i \partial q^m} \frac{\partial x_l}{\partial q^j} + \frac{\partial^2 x_l}{\partial q^j \partial q^m} \frac{\partial x_l}{\partial q^i} = \frac{\partial g_{ij}}{\partial q^m}. \quad (2.408)$$

By comparing with (2.402), we obtain

$$\Gamma_{im,j} + \Gamma_{jm,i} = \frac{\partial g_{ij}}{\partial q^m}. \quad (2.409)$$

By swapping  $i$  and  $j$  with  $m$  one at a time, and then using (2.404), we obtain

$$\Gamma_{im,j} + \Gamma_{ij,m} = \frac{\partial g_{mj}}{\partial q^i}, \quad (2.410)$$

$$\Gamma_{ij,m} + \Gamma_{jm,i} = \frac{\partial g_{im}}{\partial q^j}. \quad (2.411)$$

By adding (2.410) and (2.411), and then subtracting (2.409), we obtain

$$\Gamma_{ij,m} = \frac{1}{2} \left( \frac{\partial g_{im}}{\partial q^j} + \frac{\partial g_{mj}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^m} \right), \quad (2.412)$$

which is the result for  $\Gamma_{ij,m}$ . The result for  $\Gamma_{ij}^k$  can be obtained using (2.406),

$$\Gamma_{ij}^k = \frac{1}{2} g^{mk} \left( \frac{\partial g_{im}}{\partial q^j} + \frac{\partial g_{mj}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^m} \right). \quad (2.413)$$

Both sets of base vectors, either the covariant ones  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$ , or the contravariant ones  $\vec{i}^1$ ,  $\vec{i}^2$ , and  $\vec{i}^3$ , form a frame. These are referred to as moving frames, as both of them vary with position.

Finally, we derive a relation between the Christoffel symbols of the second kind and the determinant of the metric tensor. As we always assume that  $q^1$ ,  $q^2$  and  $q^3$  form a right-handed system, the ‘ $\pm$ ’ sign in (2.362) takes ‘+’. We obtain, according to (2.401),

$$\begin{aligned} \frac{\partial \sqrt{g}}{\partial q^j} &= \frac{\partial}{\partial q^j} [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3] \\ &= \left[ \left( \frac{\partial \vec{i}_1}{\partial q^j} \times \vec{i}_2 \right) \cdot \vec{i}_3 \right] + \left[ \left( \vec{i}_1 \times \frac{\partial \vec{i}_2}{\partial q^j} \right) \cdot \vec{i}_3 \right] + \left[ \left( \vec{i}_1 \times \vec{i}_2 \right) \cdot \frac{\partial \vec{i}_3}{\partial q^j} \right] \\ &= \Gamma_{1j}^m [(\vec{i}_m \times \vec{i}_2) \cdot \vec{i}_3] + \Gamma_{2j}^m [(\vec{i}_1 \times \vec{i}_m) \cdot \vec{i}_3] + \Gamma_{3j}^m [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_m] \\ &= \Gamma_{1j}^1 [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3] + \Gamma_{2j}^2 [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3] + \Gamma_{3j}^3 [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3] \\ &= (\Gamma_{1j}^1 + \Gamma_{2j}^2 + \Gamma_{3j}^3) [(\vec{i}_1 \times \vec{i}_2) \cdot \vec{i}_3] \\ &= \Gamma_{ij}^i \sqrt{g}. \end{aligned} \quad (2.414)$$

Hence, we have

$$\Gamma_{ij}^i = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial q^j} = \frac{\partial \ln \sqrt{g}}{\partial q^j} = \frac{1}{2} \frac{\partial \ln g}{\partial q^j}. \quad (2.415)$$

At this point, we are ready to fully understand the last requirement of a curvilinear coordinate system specified in Sect. 2.7.1, i.e., the base vectors  $\vec{i}_1$ ,  $\vec{i}_2$ , and  $\vec{i}_3$  as well as their first order partial derivatives are continuous. The need for the continuity of the base vectors is evident; it ensures the components of a continuously varying vector or tensor in the curvilinear coordinate system to be continuous, which has already been mentioned in Sect. 2.7.2. The continuity of the first order partial derivatives of the base vectors ensures the continuity of the first order partial derivatives of the components of a vector or tensor in the curvilinear coordinate system, if the vector or tensor does have such properties in the rectangular coordinate system. Based on the formulations, it is straightforward to infer that the continuity of the base vectors is ensured by the continuity of the first order partial derivatives of the functions defined in (2.295). The continuity of the first order partial derivatives of the base vectors is ensured by the continuity of  $\Gamma_{ij,m}$  or  $\Gamma_{ij}^i$ , which require continuity of the second order partial derivatives of the functions defined in (2.295). It is certainly more advantageous to ensure the continuity of the partial derivatives up to any order, at least to the highest order that appears in the formulation of the problem being studied.

It is left to the readers to convert (2.401) to the formula for unit base vectors, and then apply the result to the spherical and cylindrical coordinate systems.

### 2.8.3 Orthogonal Curvilinear Coordinate Systems

Orthogonal curvilinear coordinate systems are a very important class of curvilinear coordinate systems. As the name implies, orthogonal means that the coordinate curves are perpendicular each other.

Naturally, we start by deriving the formula for the angles among the coordinate curves of a curvilinear coordinate System. By definition, the angle between the  $q^i$ - and  $q^j$ -curves,  $\theta_{ij}$ , are determined by the relation

$$\vec{i}_i \cdot \vec{i}_j = |\vec{i}_i| |\vec{i}_j| \cos \theta_{ij}, \quad i \text{ and } j \text{ are not summation indices.} \quad (2.416)$$

According to the definition of  $g_{ij}$ , we obtain

$$\cos \theta_{ij} = \frac{\vec{i}_i \cdot \vec{i}_j}{|\vec{i}_i| |\vec{i}_j|} = \frac{g_{ij}}{\sqrt{g_{ii} g_{jj}}}, \quad i \text{ and } j \text{ are not summation indices.} \quad (2.417)$$

The sufficient and necessary condition for a curvilinear coordinate system to be orthogonal, i.e., for the coordinate curves to be perpendicular each other, is

$$\cos \theta_{ij} = 0, \quad i \neq j. \quad (2.418)$$

Due to this property, as compared to the general curvilinear coordinate systems, formulae in the orthogonal curvilinear coordinate systems can be significantly

simplified. An orthogonal curvilinear coordinate system is characterized by the Lamé parameters defined by

$$H_i = \left| \frac{\partial \vec{r}}{\partial q^i} \right| = \left[ \left( \frac{\partial x_1}{\partial q^i} \right)^2 + \left( \frac{\partial x_2}{\partial q^i} \right)^2 + \left( \frac{\partial x_3}{\partial q^i} \right)^2 \right]^{1/2}, \quad i \text{ is not a summation index.} \quad (2.419)$$

It can be readily verified that the metric and conjugate metric tensors are related to the Lamé parameters as

$$g_{ij} = \begin{cases} H_i^2, & i = j, \quad i \text{ is not a summation index;} \\ 0, & i \neq j; \end{cases} \quad (2.420)$$

$$g^{ij} = \begin{cases} 1/H_i^2, & i = j, \quad i \text{ is not a summation index;} \\ 0, & i \neq j. \end{cases} \quad (2.421)$$

The readers should have already noticed that the spherical and cylindrical coordinate systems are both orthogonal curvilinear coordinate systems.

In the following, we simplify some formulae of the general curvilinear coordinate systems to the special case of orthogonal curvilinear coordinate systems.

The relation between the base vectors and the corresponding unit base vectors, (2.325) and (2.326), are simplified to

$$\hat{i}_i = (1/H_i) \vec{i}_i, \quad i \text{ is not a summation index;} \quad (2.422)$$

$$\hat{i}^i = H_i \vec{i}^i, \quad i \text{ is not a summation index.} \quad (2.423)$$

According to the definition of  $\vec{i}^i$  given in (2.298), it can be readily shown that

$$\hat{i}_i = \hat{i}^i, \quad (2.424)$$

i.e., the corresponding unit covariant and contravariant base vectors are the same, and hence, we simply call them unit base vectors. The product operations between the unit base vectors are

$$\hat{i}_i \cdot \hat{i}_j = \delta_{ij}, \quad (2.425)$$

$$\hat{i}_i \times \hat{i}_j = \epsilon_{ijk} \hat{i}_k, \quad (2.426)$$

where the right-handed assumption is made. The relations between the covariant and contravariant components and the corresponding physical components, (2.376) and (2.377), are simplified to

$$\tilde{A}_{(i)} = (1/H_i) \tilde{A}_i, \quad \tilde{A}^{(i)} = H_i \tilde{A}^i, \quad i \text{ is not a summation index;} \quad (2.427)$$

$$\tilde{T}_{(i)(j)} = [1/(H_i H_j)] \tilde{T}_{ij}, \quad \tilde{T}^{(i)(j)} = H_i H_j \tilde{T}^{ij}, \quad i \text{ and } j \text{ are not a summation indices.} \quad (2.428)$$

It can also be readily verified that the covariant and contravariant physical components are equal,

$$\tilde{A}_{(i)} = \tilde{A}^{(i)}, \quad (2.429)$$

$$\tilde{T}_{(i)(j)} = \tilde{T}^{(i)(j)}. \quad (2.430)$$

Hence, we simply call them physical components. As there is no need to distinguish whether a physical component or a unit base vector is covariant or contravariant, we can express vectors and tensors as

$$\vec{A} = \tilde{A}_{(i)} \hat{i}_i, \quad (2.431)$$

$$\overline{T} = \tilde{T}_{(i)(j)} \hat{i}_i \hat{i}_j. \quad (2.432)$$

The transformation of a vector between an orthogonal curvilinear coordinate system and the rectangular coordinate system requires the relations between  $\{\hat{i}_i, \tilde{A}_{(i)}\}$  and  $\{\hat{e}_j, A_j\}$ . According to (2.422), (2.427), (2.301), and (2.343), we obtain

$$\{\hat{i}_i, \tilde{A}_{(i)}\} = \sum_{j=1}^3 \frac{1}{H_i} \frac{\partial x_j}{\partial q^i} \{\hat{e}_j, A_j\}, \quad i \text{ is not a summation index;} \quad (2.433)$$

According to (2.306), (2.345), (2.423), (2.427), together with (2.424), we obtain

$$\{\hat{e}_j, A_j\} = \sum_{i=1}^3 \frac{1}{H_i} \frac{\partial x_j}{\partial q^i} \{\hat{i}_i, \tilde{A}_{(i)}\}, \quad j \text{ is not a summation index.} \quad (2.434)$$

By observing the transformation between the two unit vectors  $\hat{i}_i$  and  $\hat{e}_j$ , we realize that

$$n_{ij} = \cos(\hat{i}_i, x_j) = \hat{i}_i \cdot \hat{e}_j = \frac{1}{H_i} \frac{\partial x_j}{\partial q^i}, \quad i \text{ is not a summation index.} \quad (2.435)$$

As the three unit base vectors of the orthogonal curvilinear coordinate system are perpendicular each other, the relations (2.52) and (2.53) also hold here, and the transformations of coordinates for vectors are just like those between rectangular coordinate systems with  $n_{ij}$  defined in (2.435). It can be readily inferred that the same are for tensors, which is not reproduced here.



The infinitesimal line element (2.385), surface element (2.394) and the volume element (2.398) are simplified to

$$dl = \left[ H_1^2 (dq^1)^2 + H_2^2 (dq^2)^2 + H_3^2 (dq^3)^2 \right]^{1/2}; \quad (2.436)$$

$$d\vec{S}_{ij} = H_i H_j dq^i dq^j \hat{i}_k, \quad ijk = 123, 231 \text{ or } 312; \quad (2.437)$$

( $i$  and  $j$  are not summation indices)

$$dV = H_1 H_2 H_3 dq^1 dq^2 dq^3. \quad (2.438)$$

Furthermore, in orthogonal coordinate systems, as  $g^{ij} = g_{ij} = 0$  when  $i \neq j$ , the Christoffel symbols of the first kind  $\Gamma_{ij,m}$  given in (2.412) are non-zero only when two of the indices  $i, j$  and  $m$  are identical, and that of the second kind  $\Gamma_{ij}^k$  given in (2.413) are non-zero only when two of the indices  $i, j$  and  $k$  are identical. We have

$$\left\{ \begin{array}{ll} \Gamma_{ij,m} = 0, & i, j \text{ and } m \text{ are all different;} \\ \Gamma_{ii,m} = -\frac{1}{2} \frac{\partial g_{ii}}{\partial q^m} = -H_i \frac{\partial H_i}{\partial q^m}, & m \neq i, i \text{ and } m \text{ are not summation indices;} \\ \Gamma_{ij,i} = \Gamma_{ji,i} = \frac{1}{2} \frac{\partial g_{ii}}{\partial q^j} = H_i \frac{\partial H_i}{\partial q^j}, & j \neq i, i \text{ and } j \text{ are not summation indices;} \\ \Gamma_{ii,i} = \frac{1}{2} \frac{\partial g_{ii}}{\partial q^i} = H_i \frac{\partial H_i}{\partial q^i}, & i \text{ is not a summation index;} \end{array} \right. \quad (2.439)$$

and

$$\left\{ \begin{array}{ll} \Gamma_{ij}^k = 0, & i, j \text{ and } k \text{ are all different;} \\ \Gamma_{ii}^k = -\frac{1}{2} g^{kk} \frac{\partial g_{ii}}{\partial q^k} = -\frac{H_i}{H_k^2} \frac{\partial H_i}{\partial q^k}, & k \neq i, i \text{ and } k \text{ are not summation indices;} \\ \Gamma_{ij}^i = \Gamma_{ji}^i = \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial q^j} = \frac{1}{H_i} \frac{\partial H_i}{\partial q^j}, & j \neq i, i \text{ and } j \text{ are not summation indices;} \\ \Gamma_{ii}^i = \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial q^i} = \frac{1}{H_i} \frac{\partial H_i}{\partial q^i}, & i \text{ is not a summation index.} \end{array} \right. \quad (2.440)$$

To reduce the formula for the partial derivative of the base vectors, (2.401), to the simpler case of orthogonal curvilinear coordinate systems, we write it as

$$\frac{\partial}{\partial q^j} (H_i \hat{i}_i) = \sum_{k=1}^3 \Gamma_{ij}^k H_k \hat{i}_k, \quad i \text{ is not a summation index.} \quad (2.441)$$

We further obtain

$$\frac{\partial \hat{i}_i}{\partial q^j} = -\frac{1}{H_i} \frac{\partial H_i}{\partial q^i} \hat{i}_i + \frac{1}{H_i} \sum_{k=1}^3 \Gamma_{ij}^k H_k \hat{i}_k, \quad i \text{ is not a summation index.} \quad (2.442)$$

This formula holds for both  $i \neq j$  and  $i = j$ . When  $i \neq j$ , according to (2.440),  $\Gamma_{ij}^k$  is non-zero only when  $k = i$  or  $k = j$ . Hence, (2.442) can be written as

$$\begin{aligned}\frac{\partial \hat{i}_i}{\partial q^j} &= -\frac{1}{H_i} \frac{\partial H_i}{\partial q^i} \hat{i}_i + \frac{1}{H_i} \left( \frac{1}{H_i} \frac{\partial H_i}{\partial q^j} H_i \hat{i}_i + \frac{1}{H_j} \frac{\partial H_j}{\partial q^i} H_j \hat{i}_j \right) \\ &= \frac{1}{H_i} \frac{\partial H_j}{\partial q^i} \hat{i}_j, \quad i \neq j, i \text{ and } j \text{ are not summation indices.}\end{aligned}\quad (2.443)$$

When  $i = j$ , (2.442) is more complicated. We treat its special case  $i = j = 1$ ,

$$\frac{\partial \hat{i}_1}{\partial q^1} = -\frac{1}{H_1} \frac{\partial H_1}{\partial q^1} \hat{i}_1 + \frac{1}{H_1} \sum_{k=1}^3 \Gamma_{1,1}^k H_k \hat{i}_k, \quad (2.444)$$

where the subscript 1, 1 is for avoiding understanding 1 and 1 as 11. According to (2.440),

$$\begin{aligned}\frac{\partial \hat{i}_1}{\partial q^1} &= -\frac{1}{H_1} \frac{\partial H_1}{\partial q^1} \hat{i}_1 + \frac{1}{H_1} \left( \frac{1}{H_1} \frac{\partial H_1}{\partial q^1} H_1 \hat{i}_1 - \frac{H_1}{H_2^2} \frac{\partial H_1}{\partial q^2} H_2 \hat{i}_2 - \frac{H_1}{H_3^2} \frac{\partial H_1}{\partial q^3} H_3 \hat{i}_3 \right) \\ &= -\frac{1}{H_2} \frac{\partial H_1}{\partial q^2} \hat{i}_2 - \frac{1}{H_3} \frac{\partial H_1}{\partial q^3} \hat{i}_3.\end{aligned}\quad (2.445)$$

The cases  $i = j = 2$  and 3 are similar. We can write them together as

$$\frac{\partial \hat{i}_i}{\partial q^i} = -\frac{1}{H_j} \frac{\partial H_i}{\partial q^j} \hat{i}_j - \frac{1}{H_k} \frac{\partial H_i}{\partial q^k} \hat{i}_k, \quad ijk = 123, 231 \text{ or } 312, \quad (2.446)$$

where  $i, j$  and  $k$  are not summation indices. In summary, the partial derivative of the unit base vectors are given in (2.443) and (2.446).

We now consider the spherical and cylindrical coordinate systems. For both of them, we have already shown that  $g_{ij} = 0$  when  $i \neq j$ . Hence, according to (2.417) and (2.418), they are both orthogonal.

For the spherical coordinate system, we have

$$H_1 = 1, \quad H_2 = r, \quad H_3 = r \sin \theta. \quad (2.447)$$

For vectors, the instantiation of the transformation (2.433) has already been given in (2.335) and (2.380) based on the formulation of the general orthogonal coordinate systems. Here we provide the instantiation of the transformation (2.434),

$$\begin{cases} \hat{e}_1 = \sin \theta \cos \lambda \hat{e}_r + \cos \theta \cos \lambda \hat{e}_\theta - \sin \lambda \hat{e}_\lambda, \\ \hat{e}_2 = \sin \theta \sin \lambda \hat{e}_r + \cos \theta \sin \lambda \hat{e}_\theta + \cos \lambda \hat{e}_\lambda, \\ \hat{e}_3 = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta; \end{cases} \quad (2.448)$$

$$\begin{cases} A_1 = \sin \theta \cos \lambda A_r + \cos \theta \cos \lambda A_\theta - \sin \lambda A_\lambda, \\ A_2 = \sin \theta \sin \lambda A_r + \cos \theta \sin \lambda A_\theta + \cos \lambda A_\lambda, \\ A_3 = \cos \theta A_r - \sin \theta A_\theta. \end{cases} \quad (2.449)$$

The partial derivative of the unit base vectors with respect to the coordinates are obtained according to (2.443) and (2.446). We write them in matrix form as

$$\frac{\partial(\hat{e}_r, \hat{e}_\theta, \hat{e}_\lambda)}{\partial(r, \theta, \lambda)} = \begin{bmatrix} \frac{\partial \hat{e}_r}{\partial r} & \frac{\partial \hat{e}_\theta}{\partial r} & \frac{\partial \hat{e}_\lambda}{\partial r} \\ \frac{\partial \hat{e}_r}{\partial \theta} & \frac{\partial \hat{e}_\theta}{\partial \theta} & \frac{\partial \hat{e}_\lambda}{\partial \theta} \\ \frac{\partial \hat{e}_r}{\partial \lambda} & \frac{\partial \hat{e}_\theta}{\partial \lambda} & \frac{\partial \hat{e}_\lambda}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \hat{e}_\theta & -\hat{e}_r & 0 \\ \sin \theta \hat{e}_\lambda & \cos \theta \hat{e}_\lambda & -\sin \theta \hat{e}_r - \cos \theta \hat{e}_\theta \end{bmatrix}. \quad (2.450)$$

For the cylindrical coordinate system, we have

$$H_1 = 1, \quad H_2 = r, \quad H_3 = 1. \quad (2.451)$$

Again, the instantiation of the transformation (2.433) has already been given in (2.336) and (2.381) based on the formulation of the general orthogonal coordinate systems. Here we provide the instantiation of the transformation (2.434),

$$\begin{cases} \hat{e}_1 = \cos \lambda \hat{e}_r - \sin \lambda \hat{e}_\lambda, \\ \hat{e}_2 = \sin \lambda \hat{e}_r + \cos \lambda \hat{e}_\lambda, \\ \hat{e}_3 = \hat{e}_r; \end{cases} \quad (2.452)$$

$$\begin{cases} A_1 = \cos \lambda A_r - \sin \lambda A_\lambda, \\ A_2 = \sin \lambda A_r + \cos \lambda A_\lambda, \\ A_3 = A_r. \end{cases} \quad (2.453)$$

Again, the partial derivative of the unit base vectors with respect to the coordinates are obtained according to (2.443) and (2.446). We write them in matrix form as

$$\frac{\partial(\hat{e}_r, \hat{e}_\lambda, \hat{e}_z)}{\partial(r, \lambda, z)} = \begin{bmatrix} \frac{\partial \hat{e}_r}{\partial r} & \frac{\partial \hat{e}_\lambda}{\partial r} & \frac{\partial \hat{e}_z}{\partial r} \\ \frac{\partial \hat{e}_r}{\partial \lambda} & \frac{\partial \hat{e}_\lambda}{\partial \lambda} & \frac{\partial \hat{e}_z}{\partial \lambda} \\ \frac{\partial \hat{e}_r}{\partial z} & \frac{\partial \hat{e}_\lambda}{\partial z} & \frac{\partial \hat{e}_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \hat{e}_\lambda & -\hat{e}_r & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.454)$$

## 2.9 Field Theory Formulae in Curvilinear Coordinate Systems

Our purpose is to convert physical equations from the rectangular coordinate system to curvilinear coordinate systems. Physical equations are used to express natural phenomenon, which are coordinate-system-independent, and hence, can be written in coordinate-system-independent forms. The more practical approach is to (1) derive the equations of general physical principles using the coordinate system that is simplest for the purpose, which is the rectangular coordinate system in elasticity concerned in this book; (2) write the equations in coordinate-system-independent forms; (3) convert the coordinate-system-independent equations to the curvilinear coordinate system that is most appropriate based on the geometry of the specific problem to be solved. In this book, we will be concerned with the global elastic deformation of the Earth, where spherical symmetry is a first approximation of the Earth's internal structure. In physical equations in coordinate-system-independent forms, what appear must be coordinate-system-independent quantities and operators—the most common physical quantities being tensors, including coordinate-system-independent scalars and vectors considered as zero and first order tensors, and the most common operators being field theory operators, including gradient, divergence and curl.

We derive the formulae involving field theory operators in general curvilinear coordinate systems by expressing vectors and tensors using their contravariant components and covariant base vectors. If required, the result can also be converted to forms expressed using covariant components and contravariant base vectors, which is straightforward using the formulae derived in the preceding two sections.

Considering the importance of orthogonal curvilinear coordinate systems, we also further express the formulae using physical components and unit base vectors in this particular class of curvilinear coordinate system. We emphasize again that, in orthogonal curvilinear coordinate systems, the corresponding covariant and contravariant physical components are identical, thus being simply referred to as physical components, and the corresponding covariant and contravariant unit base vectors are identical, thus being simply referred to as unit base vectors.

In the formulations in this section, as formulae derived in the preceding two sections will be very frequently used, equation numbers will not be cited for not to be wordy. The readers are assumed to be familiar with the notations, and hence, can trace to the formulae used.

The nabla operator  $\nabla$  is central to the derivations in this section. By expressing  $\partial/\partial x_i = (\partial q^j/\partial x_i)\partial/\partial q^j$  and  $e_i = (\partial q^k/\partial x_i)\vec{i}_k$ , we obtain its expression in curvilinear coordinate systems,

$$\nabla = \hat{e}_i \frac{\partial}{\partial x_i} = \left( \vec{i}_k \frac{\partial q^k}{\partial x_i} \right) \left( \frac{\partial q^j}{\partial x_i} \frac{\partial}{\partial q^j} \right) = \vec{i}_k \left( \frac{\partial q^k}{\partial x_i} \frac{\partial q^j}{\partial x_i} \right) \frac{\partial}{\partial q^j} = \vec{i}_k g^{jk} \frac{\partial}{\partial q^j}. \quad (2.455)$$

In orthogonal curvilinear coordinate systems, it can be readily obtained to be

$$\nabla = \frac{\hat{i}_1}{H_1} \frac{\partial}{\partial q^1} + \frac{\hat{i}_2}{H_2} \frac{\partial}{\partial q^2} + \frac{\hat{i}_3}{H_3} \frac{\partial}{\partial q^3}. \quad (2.456)$$

With (2.447) or (2.451), we obtain

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\lambda \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda} \quad (2.457)$$

for the spherical coordinate system, and

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\lambda \frac{1}{r} \frac{\partial}{\partial \lambda} + \hat{e}_z \frac{\partial}{\partial z} \quad (2.458)$$

for the cylindrical coordinate system.

Derivations of formulae involving  $\nabla$  can be done in two ways. One way is to purely use coordinate transformations starting from the formula in the rectangular coordinate system, like in the derivation of (2.455). We take the divergence of a vector as an example:

$$\begin{aligned} \nabla \cdot \vec{A} &= \frac{\partial A_i}{\partial x_i} = \frac{\partial}{\partial q^j} \left( \frac{\partial x_i}{\partial q^k} \tilde{A}^k \right) \frac{\partial q^j}{\partial x_i} \\ &= g^{jl} \frac{\partial x_i}{\partial q^l} \frac{\partial^2 x_i}{\partial q^j \partial q^k} \tilde{A}^k + g^{jl} \frac{\partial x_i}{\partial q^l} \frac{\partial x_i}{\partial q^k} \frac{\partial \tilde{A}^k}{\partial q^j} \\ &= \Gamma_{jk}^j \tilde{A}^k + g^{jl} g_{lk} \frac{\partial \tilde{A}^k}{\partial q^j} \\ &= \Gamma_{jk}^j \tilde{A}^k + \delta_{jk} \frac{\partial \tilde{A}^k}{\partial q^j} \\ &= \Gamma_{jk}^j \tilde{A}^k + \frac{\partial \tilde{A}^j}{\partial q^j}. \end{aligned} \quad (2.459)$$

The other way is to apply the nabla operator  $\nabla$  in curvilinear coordinate systems given in (2.455) directly to a vector or tensor expressed in curvilinear coordinate systems. For the divergence of a vector, we have

$$\begin{aligned} \nabla \cdot \vec{A} &= \vec{i}_k g^{jk} \frac{\partial}{\partial q^j} \cdot (\tilde{A}^i \vec{i}_i) \\ &= g^{jk} \frac{\partial \tilde{A}^i}{\partial q^j} \vec{i}_k \cdot \vec{i}_i + g^{jk} \tilde{A}^i \vec{i}_k \cdot \frac{\partial \vec{i}_i}{\partial q^j} \end{aligned}$$

$$\begin{aligned}
&= g^{jk} \frac{\partial \tilde{A}^i}{\partial q^j} \vec{i}_k \cdot \vec{i}_i + g^{jk} \tilde{A}^i \Gamma_{ij}^l \vec{i}_k \cdot \vec{i}_l \\
&= g^{jk} \frac{\partial \tilde{A}^i}{\partial q^j} g_{ki} + g^{jk} A^i \Gamma_{ij}^l g_{kl} \\
&= \delta_{ij} \frac{\partial \tilde{A}^i}{\partial q^j} + \delta_{jl} \Gamma_{ij}^l A^i \\
&= \frac{\partial \tilde{A}^i}{\partial q^i} + \Gamma_{ij}^j \tilde{A}^i.
\end{aligned} \tag{2.460}$$

We see that both approaches lead to the same result.

In introductory books of vector analysis, formulations for the spherical and cylindrical coordinate systems are often provided individually following (2.460) with the nabla operator given by (2.457) or (2.458), and the partial derivatives of the unit base vectors given by (2.450) or (2.454).

In this section, we formulate the general case using coordinate transformations as in (2.459), and provide the result in the spherical and cylindrical coordinate systems as special cases. The nabla operator in the spherical coordinate system will be further discussed in Sect. 4.7.

### 2.9.1 Formulae for Gradients

#### Gradient of Scalars

In general curvilinear coordinate systems,

$$\nabla u = \frac{\partial u}{\partial x_i} \hat{e}_i = \frac{\partial u}{\partial q^j} \frac{\partial q^j}{\partial x_i} \frac{\partial q^k}{\partial x_i} \vec{i}_k = \frac{\partial u}{\partial q^j} g^{jk} \vec{i}_k = g^{jk} \frac{\partial u}{\partial q^j} \vec{i}_k. \tag{2.461}$$

In orthogonal curvilinear coordinate systems,

$$\nabla u = \sum_{i=1}^3 \frac{1}{H_i^2} \frac{\partial u}{\partial q^i} \vec{i}_i = \sum_{i=1}^3 \frac{1}{H_i} \frac{\partial u}{\partial q^i} \hat{i}_i. \tag{2.462}$$

In the spherical coordinate system,

$$\nabla u = \frac{\partial u}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial u}{\partial \lambda} \hat{e}_\lambda. \tag{2.463}$$

In the cylindrical coordinate system,

$$\nabla u = \frac{\partial u}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial u}{\partial \lambda} \hat{e}_\lambda + \frac{\partial u}{\partial z} \hat{e}_z. \tag{2.464}$$

### Gradient of Vectors

In general curvilinear coordinate systems,

$$\begin{aligned}
 \nabla \vec{A} &= \frac{\partial A_j}{\partial x_i} \hat{e}_i \hat{e}_j = \frac{\partial}{\partial q^k} \left( \frac{\partial x_j}{\partial q^l} \tilde{A}^l \right) \frac{\partial q^k}{\partial x_i} \frac{\partial q^m}{\partial x_i} \vec{i}_m \frac{\partial q^n}{\partial x_j} \vec{i}_n \\
 &= \frac{\partial q^k}{\partial x_i} \frac{\partial q^m}{\partial x_i} \left( \frac{\partial^2 x_j}{\partial q^l \partial q^k} \tilde{A}^l + \frac{\partial x_j}{\partial q^l} \frac{\partial \tilde{A}^l}{\partial q^k} \right) g^{np} \frac{\partial x_j}{\partial q^p} \vec{i}_m \vec{i}_n \\
 &= \left( g^{km} g^{np} \frac{\partial^2 x_j}{\partial q^l \partial q^k} \frac{\partial x_j}{\partial q^p} \tilde{A}^l + g^{km} g^{np} \frac{\partial x_j}{\partial q^l} \frac{\partial x_j}{\partial q^p} \frac{\partial \tilde{A}^l}{\partial q^k} \right) \vec{i}_m \vec{i}_n \\
 &= \left( g^{km} \Gamma_{lk}^n \tilde{A}^l + g^{km} g^{np} g^{lp} \frac{\partial \tilde{A}^l}{\partial q^k} \right) \vec{i}_m \vec{i}_n \\
 &= \left( g^{km} \Gamma_{lk}^n \tilde{A}^l + g^{km} \delta_{nl} \frac{\partial A_l}{\partial q^k} \right) \vec{i}_m \vec{i}_n \\
 &= g^{km} \left( \Gamma_{lk}^n \tilde{A}^l + \frac{\partial \tilde{A}^n}{\partial q^k} \right) \vec{i}_m \vec{i}_n. \tag{2.465}
 \end{aligned}$$

In orthogonal curvilinear coordinate systems (where the subscripts  $m = n$  or  $m \neq n$  implies that the term exists only when  $m = n$  or  $m \neq n$ , respectively),

$$\begin{aligned}
 \nabla \vec{A} &= \sum_{m=1}^3 \sum_{n=1}^3 \frac{1}{H_m^2} \left( \sum_{l=1}^3 \Gamma_{lm}^n \tilde{A}^l + \frac{\partial \tilde{A}^n}{\partial q^m} \right) \vec{i}_m \vec{i}_n \\
 &= \sum_{m=1}^3 \sum_{n=1}^3 \frac{1}{H_m^2} \left[ \left( \sum_{l=1}^3 \frac{1}{H_m} \frac{\partial H_m}{\partial q^l} \tilde{A}^l \right)_{m=n} \right. \\
 &\quad \left. + \left( -\frac{H_m}{H_n^2} \frac{\partial H_m}{\partial q^n} \tilde{A}^m + \frac{1}{H_n} \frac{\partial H_n}{\partial q^m} \tilde{A}^n \right)_{m \neq n} + \frac{\partial \tilde{A}^n}{\partial q^m} \right] \vec{i}_m \vec{i}_n \\
 &= \sum_{m=1}^3 \sum_{n=1}^3 \frac{1}{H_m^2} \left[ \left( \sum_{l=1}^3 \frac{1}{H_m} \frac{\partial H_m}{\partial q^l} \frac{A^{(l)}}{H_l} \right)_{m=n} + \left( -\frac{H_m}{H_n^2} \frac{\partial H_m}{\partial q^n} \frac{A^{(m)}}{H_m} \right. \right. \\
 &\quad \left. \left. + \frac{1}{H_n} \frac{\partial H_n}{\partial q^m} \frac{A^{(n)}}{H_n} \right)_{m \neq n} + \frac{\partial}{\partial q^m} \left( \frac{A^{(n)}}{H_n} \right) \right] H_m H_n \hat{i}_m \hat{i}_n \\
 &= \sum_{m=1}^3 \sum_{n=1}^3 \frac{H_n}{H_m} \left[ \left( \sum_{l=1}^3 \frac{1}{H_m H_l} \frac{\partial H_m}{\partial q^l} A^{(l)} \right)_{m=n} + \left( -\frac{1}{H_n^2} \frac{\partial H_m}{\partial q^n} A^{(m)} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{H_n^2} \frac{\partial H_n}{\partial q^m} A^{(n)} \Big)_{m \neq n} + \frac{1}{H_n} \frac{\partial A^{(n)}}{\partial q^m} - \frac{1}{H_n^2} \frac{\partial H_n}{\partial q^m} A^{(n)} \Big] \hat{i}_m \hat{i}_n \\
& = \sum_{m=1}^3 \sum_{n=1}^3 \left[ \left( \frac{1}{H_m} \frac{\partial A^{(m)}}{\partial q^m} + \sum_{\substack{l=1 \\ l \neq m}}^3 \frac{1}{H_m H_l} \frac{\partial H_m}{\partial q^l} A^{(l)} \right)_{m=n} \right. \\
& \quad \left. + \left( \frac{1}{H_m} \frac{\partial A^{(n)}}{\partial q^m} - \frac{1}{H_m H_n} \frac{\partial H_m}{\partial q^n} A^{(m)} \right)_{m \neq n} \right] \hat{i}_m \hat{i}_n. \tag{2.466}
\end{aligned}$$

In the spherical coordinate system,

$$\begin{aligned}
\nabla \vec{A} &= \frac{\partial A_r}{\partial r} \hat{e}_r \hat{e}_r + \frac{\partial A_\theta}{\partial r} \hat{e}_r \hat{e}_\theta + \frac{\partial A_\lambda}{\partial r} \hat{e}_r \hat{e}_\lambda + \left( \frac{1}{r} \frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r} \right) \hat{e}_\theta \hat{e}_r \\
&+ \left( \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r} \right) \hat{e}_\theta \hat{e}_\theta + \frac{1}{r} \frac{\partial A_\lambda}{\partial \theta} \hat{e}_\theta \hat{e}_\lambda \\
&+ \left( \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \lambda} - \frac{A_\lambda}{r} \right) \hat{e}_\lambda \hat{e}_r + \left( \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \lambda} - \frac{\cot \theta A_\lambda}{r} \right) \hat{e}_\lambda \hat{e}_\theta \\
&+ \left( \frac{1}{r \sin \theta} \frac{\partial A_\lambda}{\partial \lambda} + \frac{A_r}{r} + \frac{\cot \theta A_\theta}{r} \right) \hat{e}_\lambda \hat{e}_\lambda. \tag{2.467}
\end{aligned}$$

In the cylindrical coordinate system,

$$\begin{aligned}
\nabla \vec{A} &= \frac{\partial A_r}{\partial r} \hat{e}_r \hat{e}_r + \frac{\partial A_\lambda}{\partial r} \hat{e}_r \hat{e}_\lambda + \frac{\partial A_z}{\partial r} \hat{e}_r \hat{e}_z + \left( \frac{1}{r} \frac{\partial A_r}{\partial \lambda} - \frac{A_\lambda}{r} \right) \hat{e}_\lambda \hat{e}_r \\
&+ \left( \frac{1}{r} \frac{\partial A_\lambda}{\partial \lambda} + \frac{A_r}{r} \right) \hat{e}_\lambda \hat{e}_\lambda + \frac{1}{r} \frac{\partial A_z}{\partial \lambda} \hat{e}_\lambda \hat{e}_z + \frac{\partial A_r}{\partial z} \hat{e}_z \hat{e}_r \\
&+ \frac{\partial A_\lambda}{\partial z} \hat{e}_z \hat{e}_\lambda + \frac{\partial A_z}{\partial z} \hat{e}_z \hat{e}_z. \tag{2.468}
\end{aligned}$$

## 2.9.2 Formulae for Divergences

### Divergence of Vectors

In general curvilinear coordinate systems,

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{\partial A_i}{\partial x_i} = \frac{\partial}{\partial q^j} \left( \frac{\partial x_i}{\partial q^k} \tilde{A}^k \right) \frac{\partial q^j}{\partial x_i} \\
&= g^{jl} \frac{\partial x_i}{\partial q^l} \frac{\partial^2 x_i}{\partial q^j \partial q^k} \tilde{A}^k + g^{jl} \frac{\partial x_i}{\partial q^l} \frac{\partial x_i}{\partial q^k} \frac{\partial \tilde{A}^k}{\partial q^j}
\end{aligned}$$



$$\begin{aligned}
&= \Gamma_{jk}^j A^k + g^{jl} g_{lk} \frac{\partial A^k}{\partial q^j} \\
&= \Gamma_{jk}^j A^k + \delta_{jk} \frac{\partial A^k}{\partial q^j} \\
&= \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial q^k} A^k + \frac{\partial A^k}{\partial q^k} \\
&= \frac{1}{\sqrt{g}} \left( \frac{\partial \sqrt{g}}{\partial q^k} \tilde{A}^k + \sqrt{g} \frac{\partial \tilde{A}^k}{\partial q^k} \right) \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^k} (\sqrt{g} \tilde{A}^k). \tag{2.469}
\end{aligned}$$

In orthogonal curvilinear coordinate systems,

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{1}{H_1 H_2 H_3} \sum_{k=1}^3 \frac{\partial}{\partial q^k} (H_1 H_2 H_3 \tilde{A}^k) \\
&= \frac{1}{H_1 H_2 H_3} \left\{ \frac{\partial}{\partial q^1} [H_2 H_3 A^{(1)}] + \frac{\partial}{\partial q^2} [H_3 H_1 A^{(2)}] + \frac{\partial}{\partial q^3} [H_1 H_2 A^{(3)}] \right\}. \tag{2.470}
\end{aligned}$$

In the spherical coordinate system,

$$\nabla \cdot \vec{A} = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (r^2 \sin \theta A_r) + \frac{\partial}{\partial \theta} (r \sin \theta A_\theta) + \frac{\partial}{\partial \lambda} (r A_\lambda) \right]. \tag{2.471}$$

In the cylindrical coordinate system,

$$\nabla \cdot \vec{A} = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_r) + \frac{\partial A_\lambda}{\partial \lambda} + \frac{\partial}{\partial z} (r A_z) \right]. \tag{2.472}$$

### Divergence of Tensors

In general curvilinear coordinate systems,

$$\begin{aligned}
\nabla \cdot \bar{T} &= \frac{\partial T_{ij}}{\partial x_i} \hat{e}_j = \frac{\partial}{\partial q^k} \left( \frac{\partial x_i}{\partial q^m} \frac{\partial x_j}{\partial q^n} \tilde{T}^{mn} \right) \frac{\partial q^k}{\partial x_i} \frac{\partial q^l}{\partial x_j} \vec{i}_l \\
&= \left( g^{kp} \frac{\partial x_i}{\partial q^p} \frac{\partial^2 x_i}{\partial q^k \partial q^m} \frac{\partial x_j}{\partial q^n} \tilde{T}^{mn} g^{lq} \frac{\partial x_j}{\partial q^q} + g^{kp} \frac{\partial x_i}{\partial q^p} \frac{\partial x_i}{\partial q^m} \frac{\partial^2 x_j}{\partial q^k \partial q^n} \tilde{T}^{mn} g^{lq} \frac{\partial x_j}{\partial q^q} \right. \\
&\quad \left. + g^{kp} \frac{\partial x_i}{\partial q^p} \frac{\partial x_i}{\partial q^m} \frac{\partial x_j}{\partial q^n} \frac{\partial \tilde{T}^{mn}}{\partial q^k} g^{lq} \frac{\partial x_j}{\partial q^q} \right) \vec{i}_l
\end{aligned}$$

$$\begin{aligned}
&= \left( \Gamma_{km}^k g_{nq} g^{lq} \tilde{T}^{mn} + g^{kp} g_{pm} \Gamma_{kn}^l \tilde{T}^{mn} + g^{kp} g_{pm} g_{nq} g^{lq} \frac{\partial \tilde{T}^{mn}}{\partial q^k} \right) \vec{i}_l \\
&= \left( \Gamma_{km}^k \delta_{nl} \tilde{T}^{mn} + \delta_{km} \Gamma_{kn}^l \tilde{T}^{mn} + \delta_{km} \delta_{nl} \frac{\partial \tilde{T}^{mn}}{\partial q^k} \right) \vec{i}_l \\
&= \left( \Gamma_{km}^k \tilde{T}^{ml} + \Gamma_{kn}^l \tilde{T}^{kn} + \frac{\partial \tilde{T}^{ml}}{\partial q^m} \right) \vec{i}_l \\
&= \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial q^m} \tilde{T}^{ml} + \Gamma_{kn}^l \tilde{T}^{kn} + \frac{\partial \tilde{T}^{ml}}{\partial q^m} \right) \vec{i}_l \\
&= \left[ \frac{1}{\sqrt{g}} \left( \frac{\partial \sqrt{g}}{\partial q^m} \tilde{T}^{ml} + \sqrt{g} \frac{\partial \tilde{T}^{ml}}{\partial q^m} \right) + \Gamma_{kn}^l \tilde{T}^{kn} \right] \vec{i}_l \\
&= \left[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} (\sqrt{g} \tilde{T}^{ml}) + \Gamma_{kn}^l \tilde{T}^{kn} \right] \vec{i}_l. \tag{2.473}
\end{aligned}$$

In orthogonal curvilinear coordinate systems,  $\Gamma_{kn}^l$  is not zero only when at least two of  $k$ ,  $n$ , and  $l$  are equal. As  $\Gamma_{ll}^l$  should be considered as a special case of  $\Gamma_{ln}^l$ , we have

$$\begin{aligned}
\nabla \cdot \vec{T} &= \sum_{l=1}^3 \left\{ \sum_{m=1}^3 \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} (\sqrt{g} \tilde{T}^{ml}) + \sum_{n=1}^3 \Gamma_{ln}^l \tilde{T}^{ln} + \sum_{\substack{k=1 \\ k \neq l}}^3 \Gamma_{kl}^l \tilde{T}^{kl} + \sum_{\substack{k=1 \\ k \neq l}}^3 \Gamma_{kk}^l \tilde{T}^{kk} \right\} \vec{i}_l \\
&= \sum_{l=1}^3 \left\{ \sum_{m=1}^3 \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^m} (H_1 H_2 H_3 \tilde{T}^{ml}) + \sum_{n=1}^3 \frac{1}{H_l} \frac{\partial H_l}{\partial q^n} \tilde{T}^{ln} \right. \\
&\quad \left. + \sum_{\substack{k=1 \\ k \neq l}}^3 \frac{1}{H_l} \frac{\partial H_l}{\partial q^k} \tilde{T}^{kl} - \sum_{\substack{k=1 \\ k \neq l}}^3 \frac{H_k}{H_l^2} \frac{\partial H_k}{\partial q^l} \tilde{T}^{kk} \right\} \vec{i}_l \\
&= \sum_{l=1}^3 \left\{ \sum_{m=1}^3 \frac{H_l}{H_1 H_2 H_3} \frac{\partial}{\partial q^m} \left[ \frac{H_1 H_2 H_3}{H_m H_l} T^{(m)(n)} \right] + \sum_{n=1}^3 \frac{1}{H_l H_n} \frac{\partial H_l}{\partial q^n} T^{(l)(n)} \right. \\
&\quad \left. + \sum_{\substack{k=1 \\ k \neq l}}^3 \frac{1}{H_l H_k} \frac{\partial H_l}{\partial q^k} T^{(k)(l)} - \sum_{\substack{k=1 \\ k \neq l}}^3 \frac{1}{H_l H_k} \frac{\partial H_k}{\partial q^l} T^{(k)(k)} \right\} \hat{i}_l. \tag{2.474}
\end{aligned}$$

The second and third terms at the right-hand side can be rearranged as follows: change the summation sign  $n$  to  $k$  in the second term, write the instance  $k = l$  separately, and then write the instances  $k \neq l$  together with the third term. We

obtain

$$\begin{aligned} \nabla \cdot \bar{T} = & \sum_{l=1}^3 \left\{ \sum_{m=1}^3 \frac{H_l}{H_1 H_2 H_3} \frac{\partial}{\partial q^m} \left[ \frac{H_1 H_2 H_3}{H_m H_l} T^{(m)(n)} \right] + \frac{1}{H_l^2} \frac{\partial H_l}{\partial q^l} T^{(l)(l)} \right. \\ & \left. + \sum_{\substack{k=1 \\ k \neq l}}^3 \frac{1}{H_l H_k} \frac{\partial H_l}{\partial q^k} [T^{(l)(k)} + T^{(k)(l)}] - \sum_{\substack{k=1 \\ k \neq l}}^3 \frac{1}{H_l H_k} \frac{\partial H_k}{\partial q^l} T^{(k)(k)} \right\} \hat{i}_l. \end{aligned} \quad (2.475)$$

In the spherical coordinate system,

$$\begin{aligned} \nabla \cdot \bar{T} = & \left\{ \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (r^2 \sin \theta T_{rr}) + \frac{\partial}{\partial \theta} (r \sin \theta T_{\theta r}) + \frac{\partial}{\partial \lambda} (r T_{\lambda r}) \right] \right. \\ & \left. - \frac{1}{r} T_{\theta \theta} - \frac{1}{r} T_{\lambda \lambda} \right\} \hat{e}_r + \left\{ \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial r} (r \sin \theta T_{r\theta}) + \frac{\partial}{\partial \theta} (\sin \theta T_{\theta \theta}) \right. \right. \\ & \left. \left. + \frac{\partial}{\partial \lambda} (T_{\lambda \theta}) \right] - \frac{\cot \theta}{r} T_{\lambda \lambda} + \frac{1}{r} (T_{\theta r} + T_{r\theta}) \right\} \hat{e}_\theta + \left\{ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r T_{r\lambda}) + \frac{\partial}{\partial \theta} (T_{\theta \lambda}) \right. \right. \\ & \left. \left. + \frac{\partial}{\partial \lambda} \left( \frac{1}{\sin \theta} T_{\lambda \lambda} \right) \right] + \frac{1}{r} (T_{\lambda r} + T_{r\lambda}) + \frac{\cot \theta}{r} (T_{\lambda \theta} + T_{\theta \lambda}) \right\} \hat{e}_\lambda \\ = & \left\{ \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta r}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\lambda r}}{\partial \lambda} + \frac{1}{r} (2T_{rr} - T_{\theta \theta} - T_{\lambda \lambda} + \cot \theta T_{\theta r}) \right\} \hat{e}_r \\ & + \left\{ \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta \theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\lambda \theta}}{\partial \lambda} + \frac{1}{r} [2T_{r\theta} + T_{\theta r} + \cot \theta (T_{\theta \theta} - T_{\lambda \lambda})] \right\} \hat{e}_\theta \\ & + \left\{ \frac{\partial T_{r\lambda}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta \lambda}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\lambda \lambda}}{\partial \lambda} + \frac{1}{r} [2T_{r\lambda} + T_{\lambda r} + \cot \theta (T_{\lambda \theta} + T_{\theta \lambda})] \right\} \hat{e}_\lambda. \end{aligned} \quad (2.476)$$

In the cylindrical coordinate system,

$$\begin{aligned} \nabla \cdot \bar{T} = & \left\{ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r T_{rr}) + \frac{\partial T_{\lambda r}}{\partial \lambda} + \frac{\partial}{\partial z} (r T_{zr}) \right] - \frac{1}{r} T_{\lambda \lambda} \right\} \hat{e}_r \\ & + \left\{ \frac{\partial T_{r\lambda}}{\partial r} + \frac{\partial}{\partial \lambda} \left( \frac{1}{r} T_{\lambda \lambda} \right) + \frac{\partial T_{\lambda z}}{\partial z} + \frac{1}{r} (T_{\lambda r} + T_{r\lambda}) \right\} \hat{e}_\lambda \\ & + \left\{ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r T_{rz}) + \frac{\partial T_{\lambda z}}{\partial \lambda} + \frac{\partial}{\partial z} (r T_{zz}) \right] \right\} \hat{e}_z \\ = & \left[ \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{\lambda r}}{\partial \lambda} + \frac{\partial T_{zr}}{\partial z} + \frac{1}{r} (T_{rr} - T_{\lambda \lambda}) \right] \hat{e}_r \end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{\partial T_{r\lambda}}{\partial r} + \frac{1}{r} \frac{\partial T_{\lambda\lambda}}{\partial \lambda} + \frac{\partial T_{z\lambda}}{\partial z} + \frac{1}{r} (T_{\lambda r} + T_{r\lambda}) \right] \hat{e}_\lambda \\
& + \left( \frac{\partial T_{rz}}{\partial r} + \frac{1}{r} \frac{\partial T_{\lambda z}}{\partial \lambda} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r} T_{rz} \right) \hat{e}_z.
\end{aligned} \tag{2.477}$$

### 2.9.3 Curl of Vectors

We formulate only the curl for vectors. In general curvilinear coordinate systems,

$$\begin{aligned}
\nabla \times \vec{A} &= \frac{\partial A_j}{\partial x_i} \hat{e}_i \times \hat{e}_j = \frac{\partial}{\partial q^k} \left( \frac{\partial x_j}{\partial q^l} \tilde{A}^l \right) \frac{\partial q^k}{\partial x_i} \frac{\partial q^m}{\partial x_i} \vec{i}_m \times \frac{\partial q^n}{\partial x_j} \vec{i}_n \\
&= g^{km} \left( g^{np} \frac{\partial x_j}{\partial q^p} \frac{\partial^2 x_j}{\partial q^k \partial q^l} \tilde{A}^l + g^{np} \frac{\partial x_j}{\partial q^p} \frac{\partial x_j}{\partial q^l} \frac{\partial \tilde{A}^l}{\partial q^k} \right) \vec{i}_m \times \vec{i}_n \\
&= g^{km} \left( \Gamma_{kl}^n \tilde{A}^l + g^{np} g_{pl} \frac{\partial \tilde{A}^l}{\partial q^k} \right) \vec{i}_m \times \vec{i}_n \\
&= g^{km} \left( \Gamma_{kl}^n \tilde{A}^l + \delta_{nl} \frac{\partial \tilde{A}^l}{\partial q^k} \right) \vec{i}_m \times \vec{i}_n \\
&= g^{km} \left( \Gamma_{kl}^n A^l + \frac{\partial A^n}{\partial q^k} \right) \vec{i}_m \times \vec{i}_n \\
&= \sqrt{g} g^{ij} \epsilon_{mnj} g^{km} \left( \Gamma_{kl}^n A^l + \frac{\partial A^n}{\partial q^k} \right) \vec{i}_i.
\end{aligned} \tag{2.478}$$

Further derivations for orthogonal curvilinear coordinate systems is lengthy. We provide the detail only for the term  $i = 1$ ,

$$\begin{aligned}
& \sqrt{g} g^{1j} \epsilon_{mnj} g^{km} \left( \Gamma_{kl}^n \tilde{A}^l + \frac{\partial \tilde{A}^n}{\partial q^k} \right) \vec{i}_1 \\
&= \sqrt{g} g^{11} \left[ g^{22} \left( \Gamma_{22}^3 \tilde{A}^2 + \Gamma_{23}^3 \tilde{A}^3 + \frac{\partial \tilde{A}^3}{\partial q^3} \right) - g^{33} \left( \Gamma_{32}^2 \tilde{A}^2 + \Gamma_{33}^2 \tilde{A}^3 + \frac{\partial \tilde{A}^2}{\partial q^3} \right) \right] \vec{i}_1 \\
&= \frac{H_1 H_2 H_3}{H_1^2} \left[ \frac{1}{H_2^2} \left( -\frac{H_2}{H_3^2} \frac{\partial H_2}{\partial q^3} \tilde{A}^2 + \frac{1}{H_3} \frac{\partial H_3}{\partial q^2} \tilde{A}^3 + \frac{\partial \tilde{A}^3}{\partial q^2} \right) \right. \\
&\quad \left. - \frac{1}{H_3^2} \left( \frac{1}{H_2} \frac{\partial H_2}{\partial q^3} \tilde{A}^2 - \frac{H_3}{H_2^2} \frac{\partial H_3}{\partial q^2} \tilde{A}^3 + \frac{\partial \tilde{A}^2}{\partial q^3} \right) \right] \vec{i}_1
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \frac{H_3}{H_1 H_2} \left[ -\frac{1}{H_3^2} \frac{\partial H_2}{\partial q^3} A^{(2)} + \frac{1}{H_3^2} \frac{\partial H_3}{\partial q^2} A^{(3)} + \frac{\partial}{\partial q^2} \left( \frac{A^{(3)}}{H_3} \right) \right] \right. \\
&\quad \left. - \frac{H_2}{H_1 H_3} \left[ \frac{1}{H_2^2} \frac{\partial H_2}{\partial q^3} A^{(2)} - \frac{1}{H_2^2} \frac{\partial H_3}{\partial q^2} A^{(3)} + \frac{\partial}{\partial q^3} \left( \frac{A^{(2)}}{H_2} \right) \right] \right\} H_1 \hat{i}_1 \\
&= \left[ \frac{H_3}{H_2} \left( -\frac{1}{H_3^2} \frac{\partial H_2}{\partial q^3} A^{(2)} + \frac{1}{H_3} \frac{\partial A^{(3)}}{\partial q^2} \right) - \frac{H_2}{H_3} \left( -\frac{1}{H_2^2} \frac{\partial H_3}{\partial q^2} A^{(3)} + \frac{1}{H_2} \frac{\partial A^{(2)}}{\partial q^3} \right) \right] \hat{i}_1 \\
&= \frac{1}{H_2 H_3} \left( H_3 \frac{\partial A^{(3)}}{\partial q^2} + \frac{\partial H_3}{\partial q^2} A^{(3)} - \frac{\partial H_2}{\partial q^3} A^{(2)} - H_2 \frac{\partial A^{(2)}}{\partial q^3} \right) \hat{i}_1 \\
&= \frac{1}{H_2 H_3} \left[ \frac{\partial}{\partial q^2} [H_3 A^{(3)}] - \frac{\partial}{\partial q^3} [H_2 A^{(2)}] \right] \hat{i}_1. \tag{2.479}
\end{aligned}$$

The terms for  $i = 2$  and  $i = 3$  can be obtained by cyclic permutation of indices. Finally, the expression of  $\nabla \times \vec{A}$  in orthogonal curvilinear coordinate systems is obtained by combining the three terms together,

$$\begin{aligned}
\nabla \times \vec{A} &= \frac{1}{H_2 H_3} \left[ \frac{\partial}{\partial q^2} (H_3 A^{(3)}) - \frac{\partial}{\partial q^3} (H_2 A^{(2)}) \right] \hat{i}_1 \\
&\quad + \frac{1}{H_3 H_1} \left[ \frac{\partial}{\partial q^3} (H_1 A^{(1)}) - \frac{\partial}{\partial q^1} (H_3 A^{(3)}) \right] \hat{i}_2 \\
&\quad + \frac{1}{H_1 H_2} \left[ \frac{\partial}{\partial q^1} (H_2 A^{(2)}) - \frac{\partial}{\partial q^2} (H_1 A^{(1)}) \right] \hat{i}_3. \tag{2.480}
\end{aligned}$$

In the spherical coordinate system,

$$\begin{aligned}
\nabla \times \vec{A} &= \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta A_\lambda) - \frac{\partial A_\theta}{\partial \lambda} \right] \hat{e}_r + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \lambda} - \frac{\partial}{\partial r} (r A_\lambda) \right] \hat{e}_\theta \\
&\quad + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{e}_\lambda. \tag{2.481}
\end{aligned}$$

In the cylindrical coordinate system,

$$\nabla \times \vec{A} = \left( \frac{1}{r} \frac{\partial A_z}{\partial \lambda} - \frac{\partial A_\lambda}{\partial z} \right) \hat{e}_r + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{e}_\lambda + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\lambda) - \frac{\partial A_r}{\partial \lambda} \right] \hat{e}_z. \tag{2.482}$$

## 2.9.4 The Advection Operator

### The Advection Operator Applied to Scalars

The operator  $\mathbf{A} \cdot \nabla$  is called advection operator, as it appears in a term of the governing equation of fluid flow, called advection. In general curvilinear coordinate

systems,

$$\begin{aligned}
 (\vec{A} \cdot \nabla)u &= A_i \frac{\partial u}{\partial x_i} = \frac{\partial x_i}{\partial q^j} \tilde{A}^j \frac{\partial u}{\partial q^k} \frac{\partial q^k}{\partial x_i} \\
 &= \tilde{A}^j g^{kl} \frac{\partial x_i}{\partial q^l} \frac{\partial x_i}{\partial q^j} \frac{\partial u}{\partial q^k} = \tilde{A}^j g^{kl} g_{lj} \frac{\partial u}{\partial q^k} = \tilde{A}^k \frac{\partial u}{\partial q^k}. \quad (2.483)
 \end{aligned}$$

In orthogonal curvilinear coordinate systems,

$$(\vec{A} \cdot \nabla)u = \sum_{k=1}^3 \tilde{A}^k \frac{\partial u}{\partial q^k} = \sum_{k=1}^3 \frac{1}{H_k} A^{(k)} \frac{\partial u}{\partial q^k}. \quad (2.484)$$

In the spherical coordinate system,

$$(\vec{A} \cdot \nabla)u = A_r \frac{\partial u}{\partial r} + \frac{1}{r} A_\theta \frac{\partial u}{\partial \theta} + \frac{1}{r \sin \theta} A_\lambda \frac{\partial u}{\partial \lambda}. \quad (2.485)$$

In the cylindrical coordinate system,

$$(\vec{A} \cdot \nabla)u = A_r \frac{\partial u}{\partial r} + \frac{1}{r} A_\lambda \frac{\partial u}{\partial \lambda} + A_z \frac{\partial u}{\partial z}. \quad (2.486)$$

### The Advection Operator Applied to Vectors

In general curvilinear coordinate systems,

$$\begin{aligned}
 (\vec{A} \cdot \nabla)\vec{B} &= A_i \frac{\partial B_j}{\partial x_i} \hat{e}_j = \frac{\partial x_i}{\partial q^k} \tilde{A}^k \frac{\partial}{\partial q^l} \left( \frac{\partial x_j}{\partial q^m} \tilde{B}^m \right) \frac{\partial q^l}{\partial x_i} \frac{\partial q^n}{\partial x_j} \vec{i}_n \\
 &= \frac{\partial x_i}{\partial q^k} \tilde{A}^k \frac{\partial}{\partial q^l} \left( \frac{\partial x_j}{\partial q^m} \tilde{B}^m \right) g^{lp} \frac{\partial x_i}{\partial q^p} g^{nq} \frac{\partial x_j}{\partial q^q} \vec{i}_n \\
 &= \tilde{A}^k g^{lp} g_{kp} \left( g^{nq} \frac{\partial x_j}{\partial q^q} \frac{\partial^2 x_j}{\partial q^l \partial q^m} \tilde{B}^m + g^{nq} \frac{\partial x_j}{\partial q^q} \frac{\partial x_j}{\partial q^m} \frac{\partial \tilde{B}^m}{\partial q^l} \right) \vec{i}_n \\
 &= \tilde{A}^k \left( \delta_{lk} \Gamma_{lm}^n \tilde{B}^m + \delta_{lk} g^{nq} g^{qm} \frac{\partial \tilde{B}^m}{\partial q^l} \right) \vec{i}_n \\
 &= \tilde{A}^k \left( \Gamma_{km}^n \tilde{B}^m + \delta_{mn} \frac{\partial \tilde{B}^m}{\partial q^k} \right) \vec{i}_n \\
 &= \left( \Gamma_{km}^n A^k \tilde{B}^m + A^k \frac{\partial \tilde{B}^n}{\partial q^k} \right) \vec{i}_n. \quad (2.487)
 \end{aligned}$$

In orthogonal curvilinear coordinate systems, as  $\Gamma_{nn}^n$  should be considered as a special case of  $\Gamma_{kn}^n$ , we have

$$\begin{aligned}
 (\vec{A} \cdot \nabla) \vec{B} &= \sum_{n=1}^3 \left\{ \sum_{k=1}^3 \Gamma_{kn}^n \tilde{A}^k \tilde{B}^n + \sum_{\substack{m=1 \\ m \neq n}}^3 \Gamma_{nm}^n \tilde{A}^n \tilde{B}^m + \sum_{\substack{k=1 \\ k \neq n}}^3 \Gamma_{kk}^n \tilde{A}^k \tilde{B}^k + \sum_{k=1}^3 \tilde{A}^k \frac{\partial \tilde{B}^n}{\partial q^k} \right\} \vec{i}_n \\
 &= \sum_{n=1}^3 \left\{ \sum_{k=1}^3 \frac{1}{H_n} \frac{\partial H_n}{\partial q^k} \frac{A^{(k)}}{H_k} \frac{B^{(n)}}{H_n} + \sum_{\substack{m=1 \\ m \neq n}}^3 \frac{1}{H_n} \frac{\partial H_n}{\partial q^m} \frac{A^{(n)}}{H_n} \frac{B^{(m)}}{H_m} \right. \\
 &\quad \left. - \sum_{\substack{k=1 \\ k \neq n}}^3 \frac{H_k}{H_n^2} \frac{\partial H_k}{\partial q^n} \frac{A^{(k)}}{H_k} \frac{B^{(k)}}{H_k} + \sum_{k=1}^3 \frac{A^{(k)}}{H_k} \frac{\partial}{\partial q^k} \left[ \frac{B^{(n)}}{H_n} \right] \right\} H_n \hat{i}_n \\
 &= \sum_{n=1}^3 \left\{ \sum_{k=1}^3 \frac{1}{H_k H_n} \frac{\partial H_n}{\partial q^k} A^{(k)} B^{(n)} + \sum_{\substack{m=1 \\ m \neq n}}^3 \frac{1}{H_m H_n} \frac{\partial H_n}{\partial q^m} A^{(n)} B^{(m)} \right. \\
 &\quad \left. - \sum_{\substack{k=1 \\ k \neq n}}^3 \frac{1}{H_k H_n} \frac{\partial H_k}{\partial q^n} A^{(k)} B^{(k)} + \sum_{k=1}^3 \frac{H_n}{H_k} A^{(k)} \frac{\partial}{\partial q^k} \left[ \frac{B^{(n)}}{H_n} \right] \right\} \hat{i}_n \\
 &= \sum_{n=1}^3 \left\{ \sum_{\substack{m=1 \\ m \neq n}}^3 \frac{1}{H_m H_n} \frac{\partial H_n}{\partial q^m} A^{(n)} B^{(m)} - \sum_{\substack{k=1 \\ k \neq n}}^3 \frac{1}{H_k H_n} \frac{\partial H_k}{\partial q^n} A^{(k)} B^{(k)} \right. \\
 &\quad \left. + \sum_{k=1}^3 \frac{1}{H_k} A^{(k)} \frac{\partial B^{(n)}}{\partial q^k} \right\} \hat{i}_n. \tag{2.488}
 \end{aligned}$$

The first two terms at the right-hand side can be written together by changing the index  $m$  to  $k$  in the first term. We obtain

$$(\vec{A} \cdot \nabla) \vec{B} = \sum_{n=1}^3 \left\{ \sum_{\substack{k=1 \\ k \neq n}}^3 \frac{B^{(k)}}{H_k H_n} \left( \frac{\partial H_n}{\partial q^k} A^{(n)} - \frac{\partial H_k}{\partial q^n} A^{(k)} \right) + \sum_{k=1}^3 \frac{1}{H_k} A^{(k)} \frac{\partial B^{(n)}}{\partial q^k} \right\} \hat{i}_n. \tag{2.489}$$

In the spherical coordinate system,

$$(\vec{A} \cdot \nabla) \vec{B} = \left( A_r \frac{\partial B_r}{\partial r} + \frac{1}{r} A_\theta \frac{\partial B_r}{\partial \theta} + \frac{1}{r \sin \theta} A_\lambda \frac{\partial B_r}{\partial \lambda} - \frac{1}{r} A_\theta B_\theta - \frac{1}{r} A_\lambda B_\lambda \right) \hat{e}_r$$

$$\begin{aligned}
& + \left( A_r \frac{\partial B_\theta}{\partial r} + \frac{1}{r} A_\theta \frac{\partial B_\theta}{\partial \theta} + \frac{1}{r \sin \theta} A_\lambda \frac{\partial B_\theta}{\partial \lambda} + \frac{1}{r} A_\theta B_r - \frac{\cot \theta}{r} A_\lambda B_\lambda \right) \hat{e}_\theta \\
& + \left( A_r \frac{\partial B_\lambda}{\partial r} + \frac{1}{r} A_\theta \frac{\partial B_\lambda}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial B_\lambda}{\partial \lambda} + \frac{1}{r} A_\lambda B_\lambda + \frac{\cot \theta}{r} A_\lambda B_\theta \right) \hat{e}_\lambda .
\end{aligned} \tag{2.490}$$

In the cylindrical coordinate system,

$$\begin{aligned}
(\vec{A} \cdot \nabla) \vec{B} &= \left( A_r \frac{\partial B_r}{\partial r} + \frac{1}{r} A_\lambda \frac{\partial B_r}{\partial \lambda} + A_z \frac{\partial B_r}{\partial z} - \frac{1}{r} A_\lambda B_\lambda \right) \hat{e}_r \\
&+ \left( A_r \frac{\partial B_\lambda}{\partial r} + \frac{1}{r} A_\lambda \frac{\partial B_\lambda}{\partial \lambda} + A_z \frac{\partial B_\lambda}{\partial z} + \frac{1}{r} A_\lambda B_\lambda \right) \hat{e}_\lambda \\
&+ \left( A_r \frac{\partial B_z}{\partial r} + \frac{1}{r} A_\lambda \frac{\partial B_z}{\partial \lambda} + A_z \frac{\partial B_z}{\partial z} \right) \hat{e}_z .
\end{aligned} \tag{2.491}$$

## 2.9.5 The Laplacian

### The Laplacian Applied to Scalars

The formula of  $\nabla^2 u$  can be derived based on the results of  $\nabla u$  and  $\nabla \cdot \vec{A}$ . In general curvilinear coordinate systems,

$$\nabla^2 u = \nabla \cdot (\nabla u) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^k} \left( \sqrt{g} g^{jk} \frac{\partial u}{\partial q^j} \right) . \tag{2.492}$$

In orthogonal curvilinear coordinate systems,

$$\begin{aligned}
\nabla^2 u &= \sum_{k=1}^3 \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^k} \left( \sqrt{g} g^{kk} \frac{\partial u}{\partial q^k} \right) \\
&= \frac{1}{H_1 H_2 H_3} \left[ \frac{\partial}{\partial q^1} \left( \frac{H_2 H_3}{H_1} \frac{\partial u}{\partial q^1} \right) + \frac{\partial}{\partial q^2} \left( \frac{H_3 H_1}{H_2} \frac{\partial u}{\partial q^2} \right) + \frac{\partial}{\partial q^3} \left( \frac{H_1 H_2}{H_3} \frac{\partial u}{\partial q^3} \right) \right] .
\end{aligned} \tag{2.493}$$

In the spherical coordinate system,

$$\nabla^2 u = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{\partial}{\partial \lambda} \left( \frac{1}{\sin \theta} \frac{\partial u}{\partial \lambda} \right) \right] . \tag{2.494}$$



In the cylindrical coordinate system,

$$\nabla^2 u = \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \lambda} \left( \frac{1}{r} \frac{\partial u}{\partial \lambda} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial u}{\partial z} \right) \right]. \quad (2.495)$$

### The Laplacian Applied to Vectors

The formula of  $\nabla^2 \vec{A}$  can be derived based on the results of  $\nabla \cdot \vec{T}$  and  $\nabla \vec{A}$ ,

$$\begin{aligned} \nabla^2 \vec{A} &= \nabla \cdot (\nabla \vec{A}) \\ &= \left\{ \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} \left[ \sqrt{g} g^{km} \left( \Gamma_{ik}^l \tilde{A}^i + \frac{\partial \tilde{A}^l}{\partial q^k} \right) \right] + \Gamma_{ki}^l g^{mk} \left( \Gamma_{nm}^i \tilde{A}^n + \frac{\partial \tilde{A}^i}{\partial q^m} \right) \right\} \vec{i}_l \\ &= \left\{ \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} \left( \sqrt{g} g^{km} \frac{\partial \tilde{A}^l}{\partial q^k} \right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} (\sqrt{g} g^{km} \Gamma_{ik}^l \tilde{A}^i) \right. \\ &\quad \left. + \Gamma_{ki}^l g^{mk} \left( \Gamma_{nm}^i \tilde{A}^n + \frac{\partial \tilde{A}^i}{\partial q^m} \right) \right\} \vec{i}_l \\ &= \left\{ \nabla^2 \tilde{A}^l + g^{km} \Gamma_{ik}^l \frac{\partial \tilde{A}^i}{\partial q^m} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} (\sqrt{g} g^{km} \Gamma_{ik}^l) \tilde{A}^i \right. \\ &\quad \left. + \Gamma_{ki}^l g^{mk} \left( \Gamma_{nm}^i \tilde{A}^n + \frac{\partial \tilde{A}^i}{\partial q^m} \right) \right\} \vec{i}_l \\ &= \left\{ \nabla^2 \tilde{A}^l + 2g^{km} \Gamma_{ik}^l \frac{\partial \tilde{A}^i}{\partial q^m} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} (\sqrt{g} g^{km} \Gamma_{nk}^l) \tilde{A}^n + g^{mk} \Gamma_{ki}^l \Gamma_{nm}^i \tilde{A}^n \right\} \vec{i}_l \\ &= \left\{ \nabla^2 \tilde{A}^l + 2g^{km} \Gamma_{ik}^l \frac{\partial \tilde{A}^i}{\partial q^m} + \left[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^m} (\sqrt{g} g^{km} \Gamma_{nk}^l) + g^{mk} \Gamma_{ki}^l \Gamma_{nm}^i \right] \tilde{A}^n \right\} \vec{i}_l \\ &= \left\{ \nabla^2 \tilde{A}^l + 2g^{km} \Gamma_{ik}^l \frac{\partial \tilde{A}^i}{\partial q^m} + \left[ \frac{\partial}{\partial q^m} (g^{km} \Gamma_{nk}^l) + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial q^m} g^{km} \Gamma_{nk}^l \right. \right. \\ &\quad \left. \left. + g^{mk} \Gamma_{ki}^l \Gamma_{nm}^i \right] \tilde{A}^n \right\} \vec{i}_l \\ &= \left\{ \nabla^2 \tilde{A}^l + 2g^{km} \Gamma_{ik}^l \frac{\partial \tilde{A}^i}{\partial q^m} + \left[ \frac{\partial}{\partial q^m} (g^{km} \Gamma_{nk}^l) + \Gamma_{im}^i g^{km} \Gamma_{nk}^l + g^{km} \Gamma_{ki}^l \Gamma_{nm}^i \right] \tilde{A}^n \right\} \vec{i}_l \\ &= \left\{ \nabla^2 \tilde{A}^l + 2g^{km} \Gamma_{ik}^l \frac{\partial \tilde{A}^i}{\partial q^m} + \left[ \frac{\partial}{\partial q^m} (g^{km} \Gamma_{nk}^l) + g^{km} (\Gamma_{im}^i \Gamma_{nk}^l + \Gamma_{ki}^l \Gamma_{nm}^i) \right] \tilde{A}^n \right\} \vec{i}_l. \end{aligned} \quad (2.496)$$

We skip the tedious derivation to directly write out the result in orthogonal curvilinear coordinate systems (Wu, 2021),

$$\begin{aligned}
\nabla^2 \vec{A} = & \left[ \nabla^2 A^{(1)} + \frac{2}{H_1^2 H_2} \frac{\partial H_1}{\partial q^2} \frac{\partial A^{(2)}}{\partial q^1} - \frac{2}{H_1 H_2^2} \frac{\partial H_2}{\partial q^1} \frac{\partial A^{(2)}}{\partial q^2} + \frac{2}{H_1^2 H_3} \frac{\partial H_1}{\partial q^3} \frac{\partial A^{(3)}}{\partial q^1} \right. \\
& - \frac{2}{H_1 H_3^2} \frac{\partial H_3}{\partial q^1} \frac{\partial A^{(3)}}{\partial q^3} + \left\{ \frac{1}{H_1} \frac{\partial}{\partial q^1} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^1} (H_2 H_3) \right] \right. \\
& + \frac{1}{H_2 H_3} \frac{\partial}{\partial q^2} \left( \frac{H_3}{H_1 H_2} \frac{\partial H_1}{\partial q^2} \right) + \frac{1}{H_2 H_3} \frac{\partial}{\partial q^3} \left( \frac{H_2}{H_1 H_3} \frac{\partial H_1}{\partial q^3} \right) \left. \right\} A^{(1)} \\
& + \left\{ \frac{1}{H_1} \frac{\partial}{\partial q^1} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^2} (H_3 H_1) \right] - \frac{1}{H_2 H_3} \frac{\partial}{\partial q^2} \left( \frac{H_3}{H_1 H_2} \frac{\partial H_2}{\partial q^1} \right) \right\} A^{(2)} \\
& + \left\{ \frac{1}{H_1} \frac{\partial}{\partial q^1} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^3} (H_1 H_2) \right] - \frac{1}{H_2 H_3} \frac{\partial}{\partial q^3} \left( \frac{H_2}{H_3 H_1} \frac{\partial H_3}{\partial q^1} \right) \right\} A^{(3)} \Big] \hat{i}_1 \\
& + \left[ \nabla^2 A^{(2)} + \frac{2}{H_2^2 H_3} \frac{\partial H_2}{\partial q^3} \frac{\partial A^{(3)}}{\partial q^2} - \frac{2}{H_2 H_3^2} \frac{\partial H_3}{\partial q^2} \frac{\partial A^{(3)}}{\partial q^3} + \frac{2}{H_2^2 H_1} \frac{\partial H_2}{\partial q^1} \frac{\partial A^{(1)}}{\partial q^2} \right. \\
& - \frac{2}{H_2 H_1^2} \frac{\partial H_1}{\partial q^2} \frac{\partial A^{(1)}}{\partial q^1} + \left\{ \frac{1}{H_2} \frac{\partial}{\partial q^2} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^2} (H_3 H_1) \right] \right. \\
& + \frac{1}{H_3 H_1} \frac{\partial}{\partial q^3} \left( \frac{H_1}{H_2 H_3} \frac{\partial H_2}{\partial q^3} \right) + \frac{1}{H_3 H_1} \frac{\partial}{\partial q^1} \left( \frac{H_3}{H_2 H_1} \frac{\partial H_2}{\partial q^1} \right) \left. \right\} A^{(2)} \\
& + \left\{ \frac{1}{H_2} \frac{\partial}{\partial q^2} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^3} (H_1 H_2) \right] - \frac{1}{H_3 H_1} \frac{\partial}{\partial q^3} \left( \frac{H_1}{H_2 H_3} \frac{\partial H_3}{\partial q^2} \right) \right\} A^{(3)} \\
& + \left\{ \frac{1}{H_2} \frac{\partial}{\partial q^2} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^1} (H_2 H_3) \right] - \frac{1}{H_3 H_1} \frac{\partial}{\partial q^1} \left( \frac{H_3}{H_1 H_2} \frac{\partial H_1}{\partial q^2} \right) \right\} A^{(1)} \Big] \hat{i}_2 \\
& + \left[ \nabla^2 A^{(3)} + \frac{2}{H_3^2 H_1} \frac{\partial H_3}{\partial q^1} \frac{\partial A^{(1)}}{\partial q^3} - \frac{2}{H_3 H_1^2} \frac{\partial H_1}{\partial q^3} \frac{\partial A^{(1)}}{\partial q^1} + \frac{2}{H_3^2 H_2} \frac{\partial H_3}{\partial q^2} \frac{\partial A^{(2)}}{\partial q^3} \right. \\
& - \frac{2}{H_3 H_2^2} \frac{\partial H_2}{\partial q^3} \frac{\partial A^{(2)}}{\partial q^2} + \left\{ \frac{1}{H_3} \frac{\partial}{\partial q^3} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^3} (H_1 H_2) \right] \right. \\
& + \frac{1}{H_1 H_2} \frac{\partial}{\partial q^1} \left( \frac{H_2}{H_3 H_1} \frac{\partial H_3}{\partial q^1} \right) + \frac{1}{H_1 H_2} \frac{\partial}{\partial q^2} \left( \frac{H_1}{H_3 H_2} \frac{\partial H_3}{\partial q^2} \right) \left. \right\} A^{(3)} \\
& + \left\{ \frac{1}{H_3} \frac{\partial}{\partial q^3} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^1} (H_2 H_3) \right] - \frac{1}{H_1 H_2} \frac{\partial}{\partial q^1} \left( \frac{H_2}{H_3 H_1} \frac{\partial H_1}{\partial q^3} \right) \right\} A^{(1)} \\
& + \left\{ \frac{1}{H_3} \frac{\partial}{\partial q^3} \left[ \frac{1}{H_1 H_2 H_3} \frac{\partial}{\partial q^2} (H_3 H_1) \right] - \frac{1}{H_1 H_2} \frac{\partial}{\partial q^2} \left( \frac{H_1}{H_2 H_3} \frac{\partial H_2}{\partial q^3} \right) \right\} A^{(2)} \Big] \hat{i}_3 .
\end{aligned}
\tag{2.497}$$

In the spherical coordinate system,

$$\begin{aligned}\nabla^2 \vec{A} = & \left[ \nabla^2 A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) - \frac{2}{r^2 \sin \theta} \frac{\partial A_\lambda}{\partial \lambda} \right] \hat{e}_r \\ & + \left[ \nabla^2 A_\theta - \frac{1}{r^2 \sin^2 \theta} A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\lambda}{\partial \lambda} \right] \hat{e}_\theta \\ & + \left[ \nabla^2 A_\lambda - \frac{1}{r^2 \sin^2 \theta} A_\lambda + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \lambda} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \lambda} \right] \hat{e}_\lambda .\end{aligned}\quad (2.498)$$

In the cylindrical coordinate system,

$$\nabla^2 \vec{A} = \left( \nabla^2 A_r - \frac{1}{r^2} A_r - \frac{2}{r^2} \frac{\partial A_\lambda}{\partial \lambda} \right) \hat{e}_r + \left( \nabla^2 A_\lambda - \frac{1}{r^2} A_\lambda + \frac{2}{r^2} \frac{\partial A_r}{\partial \lambda} \right) \hat{e}_\lambda + \nabla^2 A_z \hat{e}_z .\quad (2.499)$$

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## Bibliographic Notes

Our favorite book in vector analysis is Davis and Snider (1995). This book exposes the subjects in a sequence that agree with our preference, and it is one of the rare books with a paragraph indicating that not all quantities with a direction and a magnitude obeys the parallelogram law of vector addition. The author also benefited from the Chap. 1 of Wangsness (1986) and the Chap. 1 of Wu (2021). We suggest Young (1993) as an alternative text in tensor analysis. These books provide a less steep learning curve, and include ample exercise problems for the readers to gain familiarity with the subjects.

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# Physical Equations of Rotation and Elastic Deformation of the Earth

## 3

### Abstract

This chapter starts by providing the background physics covering reference system and coordinate system, angular velocity and its combination, generalization of Newton's second law of motion into a translating and rotating reference system, gravity and gravity potential, Lagrangian and Eulerian descriptions of position and velocity of a continuum, dynamic equations of motion of a continuum, deformation of a continuum, and elasticity. Particular care is taken for the formulation to be general enough for studying the dynamics of the earth. The separation of the general motion of the Earth into three components—translation, rotation and deformation—is rigorously formulated. The equations derived include the equation of the translational motion of the center of mass, the equation of rotation of the Tisserand frame around the center of mass, and the elastic deformation with respect to the Tisserand frame. The elastic deformation of the translating and rotating prestressed general Earth model is formulated in extreme detail, most notably, the derivation of the constitutive equation and the boundary conditions. The hydrostatic equilibrium state considered as the undeformed configuration, which is mostly the case in the literature, is treated as a degenerate instance.

## 3.1 Rotating Coordinate Systems and Rotating Reference Systems

### 3.1.1 Coordinate System and Reference System

Coordinate system and reference system are two completely different concepts.

A reference system is tied to the object used as the reference, with respect to which physical phenomena are described. For example, when a velocity is concerned, we are implicitly assumed to know the answers to the following two

questions: (1) which object's velocity is it? (2) which object is the velocity with respect to? In this example, the motion of the former object is the physical phenomenon described, where the velocity is a physical quantity used as a quantitative measure of the phenomenon, and the latter object serves as the reference system. Hence, the role of a reference system is essentially indifferent from that of the object it is tied to.

A coordinate system is used to quantitatively represent a position, thus being a mathematical concept. In the three-dimensional space, the three axes of the rectangular coordinate system is actually the abstraction of three graduated rods. As the rods are objects, we would certainly ask whether the coordinate system is also so. Actually, a position must also be represented with respect to an object, and therefore, the frame consisting of the three coordinate axes must be tied to an object, so that the readings on the axes could be materialize.

Hence, when a physical phenomenon is described, there must be an object where the reference system is tied to, and another object where the coordinate system is tied to. In most applications, these two objects are the same, and the coordinate system also represents the reference system, which is the easiest way to perceive. However, there are instances where the coordinate system moves with respect to the reference system. A typical example in geophysics is the rotation of the Earth with respect to the inertial space. The variation of the direction of the rotation axis can be represented either in a coordinate system tied to the inertial space, or in a coordinate system tied to the Earth. In the former, what we obtain is the variation of the direction of the rotation axis with respect to the inertial space, called precession-nutation; in the latter, what we obtain is the variation of the direction of the rotation axis with respect to the Earth, called polar motion. In both precession-nutation and polar motion, the direction is that of the rotation axis with respect to the inertial space.

Here we give a simpler example, where the coordinate system and reference system move one with respect to another. We assume that a ship is moving in a zigzag river, and a person is walking on the ship. By definition, the velocity of the person with respect to the ship  $\vec{v}$  is related to only the person and the ship. When this velocity is expressed in a coordinate system  $Ox_1x_2x_3$  tied to the ship, its component along the  $x_i$ -axis is evidently  $\vec{v} \cdot \hat{e}_i$ . Our question here is, if this velocity is to be expressed in a coordinate system  $Ox'_1x'_2x'_3$  tied to the river bank, what is its component along the  $x'_i$ -axis? The answer is  $\vec{v} \cdot \hat{e}'_i$ , which is independent of the velocity of the ship with respect to the river bank, and which should be the case! In the above two coordinate systems, the expressions of the velocity  $\vec{v}$  are  $\vec{v} = (\vec{v} \cdot \hat{e}_i)\hat{e}_i$  and  $\vec{v} = (\vec{v} \cdot \hat{e}'_i)\hat{e}'_i$ , which are equal. We now further assume that a coordinate system  $Ox''_1x''_2x''_3$  is tied to the body of the person. What is the expression of the person's velocity with respect to the ship in  $Ox''_1x''_2x''_3$ ? The answer is certainly  $\vec{v} = (\vec{v} \cdot \hat{e}''_i)\hat{e}''_i$ , which is equal to those in the previous two cases.

### 3.1.2 Angular Velocity

It is well known that the Earth is rotating with respect to the inertial space, or, more exactly, with respect to the stars assumed to be at standstill. Many geophysical

phenomena are observed and studied on the Earth. Hence, rotating reference and coordinate systems are very important in geophysics.

We assume that the right-handed coordinate system  $Ox'_1x'_2x'_3$  is rotating with respect to the right-handed coordinate system  $Ox_1x_2x_3$  around their common origin  $O$ . Their relation is still described by the cosines of the angles between each pair of base vectors, one in each coordinate systems, i.e.,  $n_{ij} = \cos(\hat{e}'_i, \hat{e}_j)$ . However, here  $\hat{e}'_i$  and  $n_{ij}$  are assumed to be varying with time, and  $\hat{e}_j$  is regarded as time-independent. In order to clearly distinguish time-dependent and time-independent quantities, we express time-dependence explicitly. For the quantities defined above, we have

$$n_{ij}(t) = \hat{e}'_i(t) \cdot \hat{e}_j, \quad (3.1)$$

$$\hat{e}'_i(t) = n_{ij}(t)\hat{e}_j, \quad (3.2)$$

$$\hat{e}_j = n_{ij}(t)\hat{e}'_i(t). \quad (3.3)$$

In order to demonstrate how  $Ox'_1x'_2x'_3$  rotates with respect to  $Ox_1x_2x_3$ , we study how a point fixed in  $Ox'_1x'_2x'_3$  moves in  $Ox_1x_2x_3$ . Saying a point is fixed in  $Ox'_1x'_2x'_3$  is equivalent to say that the coordinates of this point in  $Ox'_1x'_2x'_3$  are time-independent, i.e.,  $x'_i$  is time-independent. However, as  $Ox'_1x'_2x'_3$  is rotating with respect to  $Ox_1x_2x_3$ , a point fixed in  $Ox'_1x'_2x'_3$  moves with respect to  $Ox_1x_2x_3$ , and the position vector of this point,  $\vec{r}$ , also varies with time. The expression of this position vector in the coordinate system  $Ox'_1x'_2x'_3$  is

$$\vec{r}(t) = x'_i\hat{e}'_i(t). \quad (3.4)$$

This implies that the variation of  $\vec{r}$  is due to the rotation of  $Ox'_1x'_2x'_3$  with respect to  $Ox_1x_2x_3$ , i.e., the time-dependence of  $\hat{e}'_i(t)$ . By substituting (3.2) into (3.4), we obtain

$$\vec{r}(t) = n_{ij}(t)x'_i\hat{e}_j. \quad (3.5)$$

Hence, the velocity of this point with respect to  $Ox_1x_2x_3$  is

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{dn_{ij}(t)}{dt}x'_i\hat{e}_j. \quad (3.6)$$

We used (3.5) instead of (3.4) to derive  $\vec{v}(t)$  for avoiding to take the derivative of the base vector  $\hat{e}'_i(t)$  with respect to time. By converting (3.6) into  $Ox'_1x'_2x'_3$  using (3.3), we obtain

$$\vec{v}(t) = \frac{dn_{ij}(t)}{dt}n_{kj}(t)x'_i\hat{e}'_k(t). \quad (3.7)$$

This formula implies several things: (1) This is the velocity of a point with respect to  $Ox_1x_2x_3$ ; (2) The coordinates of this point in  $Ox'_1x'_2x'_3$  are time-independent, i.e.,

the point is fixed in  $Ox'_1x'_2x'_3$ ; (3) The velocity is given in terms of its components in  $Ox'_1x'_2x'_3$ , i.e., it is expressed in  $Ox'_1x'_2x'_3$ .

We now introduce the definition of angular velocity based on (3.7). Let

$$\omega_{ik}(t) = \frac{dn_{ij}(t)}{dt}n_{kj}(t). \quad (3.8)$$

We first show that  $\omega_{ik}(t)$  is anti-symmetric. By definition as given in (3.1)–(3.3), we have

$$n_{ij}(t)n_{kj}(t) = \delta_{ik}. \quad (3.9)$$

By taking the derivative of both sides with respect to  $t$ , we obtain

$$\frac{dn_{ij}(t)}{dt}n_{kj}(t) + n_{ij}(t)\frac{dn_{kj}(t)}{dt} = 0. \quad (3.10)$$

The anti-symmetry of  $\omega_{ik}(t)$  can be readily inferred by comparing this equation with (3.8),

$$\omega_{ik} = -\omega_{ki}. \quad (3.11)$$

Hence, only three out of its nine components are independent. Define a vector (we will show that its combination, i.e., sum, obeys the parallelogram law)

$$\vec{\omega}(t) = \omega'_j(t)\hat{e}'_j(t), \quad (3.12)$$

where the components are defined by

$$\omega_{ik}(t) = \epsilon_{jik}\omega'_j(t), \quad (3.13)$$

which are, written explicitly,

$$\omega'_1(t) = \omega_{23}(t), \quad \omega'_2(t) = \omega_{31}(t), \quad \omega'_3(t) = \omega_{12}(t). \quad (3.14)$$

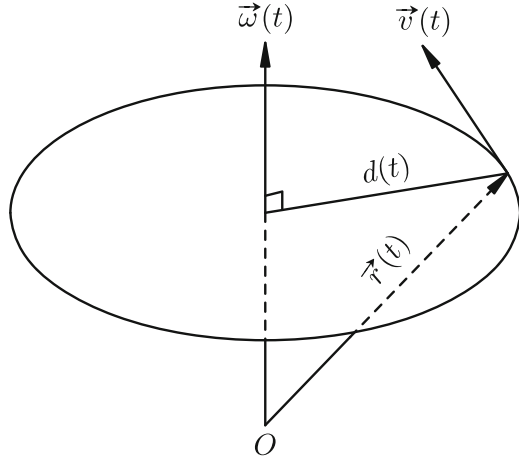
Evidently, the three components of  $\vec{\omega}(t)$  are the three independent components of  $\omega_{ik}(t)$ . The three formulae in (3.14) can be written in a single formula as

$$\omega'_j(t) = \frac{1}{2}\epsilon_{jik}\omega_{ik}(t), \quad (3.15)$$

which can be easily verified according to the anti-symmetry of  $\omega_{ik}(t)$ . Finally, according to (3.8) and (3.13), we can write (3.7) as

$$\vec{v}(t) = \omega'_j(t)x'_i\epsilon_{jik}\hat{e}'_k(t) = \vec{\omega}(t) \times \vec{r}(t). \quad (3.16)$$

**Fig. 3.1** The definition of angular velocity



The relation among  $\vec{v}(t)$ ,  $\vec{\omega}(t)$  and  $\vec{r}(t)$  is shown in Fig. 3.1. We define  $\vec{\omega}(t)$  as the angular velocity of  $Ox'_1x'_2x'_3$  with respect to  $Ox_1x_2x_3$ . Denote the distance of the point studied to the rotation axis as  $d(t)$ , as shown in 3.1. We obtain, according to (3.16),

$$|\vec{v}(t)| = |\vec{\omega}(t)|d(t). \quad (3.17)$$

This implies that the velocity is equal to the product between the magnitude of the angular velocity and the distance to the rotation axis. It can thus be inferred that the magnitude of the angular velocity is in fact the rate of change of the angle of rotation, and the direction is defined according to the right-hand rule.

### 3.1.3 Combination of Angular Velocities

In this subsection, we define the combination, i.e., the sum, of angular velocities, and show that the sum obeys the parallelogram law.

Let  $Ox'_1x'_2x'_3$  be rotating with respect to  $Ox_1x_2x_3$  with the angular velocity  $\vec{\omega}(t)$  around  $O$ , and  $Ox''_1x''_2x''_3$  be rotating with respect to  $Ox'_1x'_2x'_3$  with the angular  $\vec{\omega}'(t)$  around  $O$ . We defined the angular velocity of  $Ox''_1x''_2x''_3$  with respect to  $Ox_1x_2x_3$  around  $O$ ,  $\vec{\Omega}(t)$ , as the combination, i.e., the sum, of  $\vec{\omega}(t)$  and  $\vec{\omega}'(t)$ .

In the preceding subsection, we assumed  $Ox_1x_2x_3$  to be time-independent when studying the rotation of  $Ox'_1x'_2x'_3$  with respect to  $Ox_1x_2x_3$ . This is actually not really necessary, as the rotation of  $Ox'_1x'_2x'_3$  with respect to  $Ox_1x_2x_3$  is completely determined by  $n_{ij}(t)$ . However, assuming  $Ox_1x_2x_3$  to be time-independent helped to understand the derivations more apparently.

Now we assume that  $Ox'_1x'_2x'_3$  is rotating with respect to  $Ox_1x_2x_3$ , and  $Ox''_1x''_2x''_3$  is rotating with respect to  $Ox'_1x'_2x'_3$ . In this circumstance, both  $\hat{e}''_i$  and  $\hat{e}'_j$  are varying



with time. Hence, the relation between  $Ox_1''x_2''x_3''$  and  $Ox_1'x_2'x_3'$  is, according to (3.1)–(3.3),

$$n'_{ij}(t) = \hat{e}''_i(t) \cdot \hat{e}'_j(t), \quad (3.18)$$

$$\hat{e}''_i(t) = n'_{ij}(t)\hat{e}'_j(t), \quad (3.19)$$

$$\hat{e}'_j(t) = n'_{ij}(t)\hat{e}''_i(t). \quad (3.20)$$

The angular velocity of  $Ox_1''x_2''x_3''$  with respect to  $Ox_1'x_2'x_3'$  is defined according to

$$\omega'_{ik}(t) = \frac{dn'_{ij}(t)}{dt}n'_{kj}(t), \quad (3.21)$$

$$\omega'_{ik}(t) = \epsilon_{jik}\omega'_j(t), \quad (3.22)$$

$$\vec{\omega}'(t) = \omega'_j(t)\hat{e}'_j(t). \quad (3.23)$$

In order to compute the angular velocity of  $Ox_1''x_2''x_3''$  with respect to  $Ox_1x_2x_3$ ,  $\vec{\Omega}(t)$ , we derive the expression of the velocity of a point fixed in  $Ox_1''x_2''x_3''$  with respect to  $Ox_1x_2x_3$ . For this purpose, we have to first derive the relation between these two coordinate systems, which is straightforward,

$$N_{ij}(t) = \hat{e}''_i(t) \cdot \hat{e}_j = n'_{im}(t)\hat{e}'_m(t) \cdot \hat{e}_j = n'_{im}(t)n_{mj}(t), \quad (3.24)$$

$$\hat{e}''_i(t) = N_{ij}(t)\hat{e}_j = n'_{im}(t)n_{mj}(t)\hat{e}_j, \quad (3.25)$$

$$\hat{e}_j = N_{ij}(t)\hat{e}''_i(t) = n'_{im}(t)n_{mj}(t)\hat{e}''_i(t). \quad (3.26)$$

The position vector of a point fixed in  $Ox_1''x_2''x_3''$  is

$$\vec{r}(t) = x''_i\hat{e}''_i(t), \quad (3.27)$$

where  $x''_i$  is time-independent. By substituting (3.25) into (3.27), we obtain the position vector of the point fixed in  $Ox_1''x_2''x_3''$  expressed in  $Ox_1x_2x_3$ ,

$$\vec{r}(t) = n'_{im}(t)n_{mj}(t)x''_i\hat{e}_j. \quad (3.28)$$

The velocity of this point with respect to  $Ox_1x_2x_3$  is then

$$\vec{V}(t) = \frac{d}{dt}[n'_{im}(t)n_{mj}(t)]x''_i\hat{e}_j = \frac{dn'_{im}(t)}{dt}n_{mj}(t)x''_i\hat{e}_j + n'_{im}(t)\frac{dn_{mj}(t)}{dt}x''_i\hat{e}_j. \quad (3.29)$$

We need to further express  $\vec{V}(t)$  in terms of  $\vec{\omega}(t)$  and  $\vec{\omega}'(t)$ . By substituting  $\hat{e}_j$  by  $\hat{e}''_k$  according to (3.26) in the first term, and  $\hat{e}_j$  by  $\hat{e}'_k$  according to (3.3) in the second

term, we obtain

$$\vec{V}(t) = \frac{dn'_{im}(t)}{dt} n_{mj}(t) n'_{kn}(t) n_{nj}(t) x''_i \hat{e}''_k(t) + n'_{im}(t) \frac{dn_{mj}(t)}{dt} n_{kj}(t) x''_i \hat{e}'_k(t). \quad (3.30)$$

With the relations

$$n_{mj}(t) n_{nj}(t) = \delta_{mn}, \quad (3.31)$$

$$n'_{im}(t) x''_i = x'_m(t), \quad (3.32)$$

we can write (3.30) as

$$\vec{V}(t) = \frac{dn'_{im}(t)}{dt} n'_{km}(t) x''_i \hat{e}''_k(t) + \frac{dn_{mj}(t)}{dt} n_{kj}(t) x'_m(t) \hat{e}'_k(t). \quad (3.33)$$

By substituting (3.8) and (3.21) into this formula, we obtain

$$\vec{V}(t) = \omega'_{ik}(t) x''_i \hat{e}''_k(t) + \omega_{mk}(t) x'_m(t) \hat{e}'_k(t). \quad (3.34)$$

By further substituting (3.13) and (3.22) into it, we obtain

$$\vec{V}(t) = \omega''_j(t) x''_i \epsilon_{jik} \hat{e}''_k(t) + \omega'_j(t) x'_m(t) \epsilon_{jmk} \hat{e}'_k(t). \quad (3.35)$$

Finally, by substituting (3.12) and (3.23) into it, we obtain

$$\vec{V}(t) = \vec{\omega}'(t) \times \vec{r}(t) + \vec{\omega}(t) \times \vec{r}(t) = [\vec{\omega}'(t) + \vec{\omega}(t)] \times \vec{r}(t). \quad (3.36)$$

By definition, the relation between  $\vec{V}(t)$  and  $\vec{\Omega}(t)$  is

$$\vec{V}(t) = \vec{\Omega}(t) \times \vec{r}(t). \quad (3.37)$$

It can be obtained from the last two equations that

$$\vec{\Omega}(t) \times \vec{r}(t) = [\vec{\omega}'(t) + \vec{\omega}(t)] \times \vec{r}(t). \quad (3.38)$$

As the point fixed in  $Ox''_1x''_2x''_3$  can be arbitrarily chosen, i.e., the value of  $x''_i$  is arbitrary, the vector  $\vec{r}(t)$  has an infinite number of choices according to (3.28). Therefore, in order for this equation to hold, we must have

$$\vec{\Omega}(t) = \vec{\omega}'(t) + \vec{\omega}(t). \quad (3.39)$$

This is the conclusion we are seeking: The combination, i.e., the sum of angular velocities, obeys the parallelogram law.

### 3.1.4 The Derivative of Vectors with Respect to Time in Rotating Coordinate Systems

The purpose is to express, in a rotating coordinate system, a physical law that holds in the inertial reference system.

Let  $Ox_1x_2x_3$  be a coordinate system representing the inertial reference system assumed to be time-independent, and  $Ox'_1x'_2x'_3$ , a coordinate system rotating with respect to  $Ox_1x_2x_3$  with the angular velocity  $\vec{\omega}(t)$ . Let  $\vec{A}$  be a coordinate-independent vector known in  $Ox_1x_2x_3$ , i.e., its components in  $Ox_1x_2x_3$  are known. We can express this vector in both  $Ox_1x_2x_3$  and  $Ox'_1x'_2x'_3$  as follows:

$$\vec{A}(t) = A_i(t)\hat{e}_i = A'_i(t)\hat{e}'_i(t). \quad (3.40)$$

Its derivative with respect to time is

$$\frac{d\vec{A}(t)}{dt} = \frac{dA_i(t)}{dt}\hat{e}_i = \frac{dA'_i(t)}{dt}\hat{e}'_i(t) + A'_i(t)\frac{d\hat{e}'_i(t)}{dt}. \quad (3.41)$$

Evidently,  $d\hat{e}'_i(t)/dt$  is the value of  $\vec{v}(t)$  in (3.6) by setting  $\vec{r}(t) = \hat{e}'_i(t)$  in (3.4). Hence,

$$\frac{d\hat{e}'_i(t)}{dt} = \vec{\omega}(t) \times \hat{e}'_i(t). \quad (3.42)$$

By substituting this formula into (3.41), we obtain

$$\frac{d\vec{A}(t)}{dt} = \frac{dA_i(t)}{dt}\hat{e}_i = \frac{dA'_i(t)}{dt}\hat{e}'_i(t) + \vec{\omega}(t) \times [A'_i(t)\hat{e}'_i(t)]. \quad (3.43)$$

Define

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{in}} = \frac{dA_i(t)}{dt}\hat{e}_i, \quad (3.44)$$

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{rot}} = \frac{dA'_i(t)}{dt}\hat{e}'_i(t). \quad (3.45)$$

We can write (3.43) as

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{in}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{A}. \quad (3.46)$$

We must keep in mind that, in  $(d\vec{A}/dt)_{\text{in}}$ , the directions of the axes of the coordinate system  $Ox_1x_2x_3$  are assumed time-independent, which is straightforward, and in  $(d\vec{A}/dt)_{\text{rot}}$ , the directions of the axes of the coordinate system  $Ox'_1x'_2x'_3$  are

assumed time-independent, i.e., it includes only the contribution of the derivative of the components with respect to time. These are understood as the definitions of the symbols  $(d\vec{A}/dt)_{\text{in}}$  and  $(d\vec{A}/dt)_{\text{rot}}$  in (3.44) and (3.45), respectively. If we set  $\vec{A} = \vec{r}$  to be the position vector of a point with respect to the origin,  $(d\vec{r}/dt)_{\text{in}}$  is the velocity of the point with respect to the  $Ox_1x_2x_3$ , and  $(d\vec{r}/dt)_{\text{rot}}$ , the velocity of the point with respect to  $Ox'_1x'_2x'_3$ .

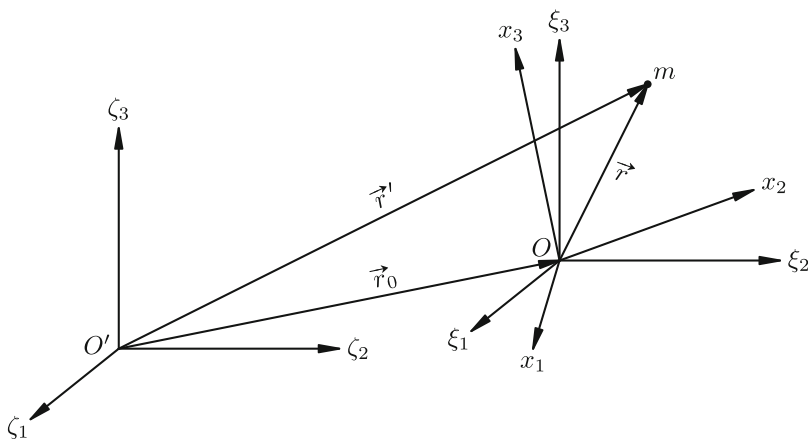
### 3.1.5 Equation of the Motion of Particles in Translating and Rotating Reference Systems

As an application of the results derived in the preceding subsections, we study the relations between the velocity and acceleration of a particle in the inertial reference system and those in a translating and rotating reference system. These relations are fundamental to convert dynamic equations from the inertial reference system to the Earth-fixed reference system.

As shown in Fig. 3.2, let  $O'\zeta_1\zeta_2\zeta_3$  represent the inertial reference system, and  $Ox_1x_2x_3$  is a translating and rotating reference system. The motion of  $Ox_1x_2x_3$  with respect to  $O'\zeta_1\zeta_2\zeta_3$  is composed of a translation and a rotation. Hence, their origins are not coincident. We use  $\vec{r}_0$  to denote the position vector of  $O$  with respect to  $O'$ , and  $\vec{r}'$  and  $\vec{r}$ , the position vectors of a particle of mass  $m$  with respect to  $O'$  and  $O$ , respectively. We then evidently have

$$\vec{r}' = \vec{r} + \vec{r}_0. \quad (3.47)$$

Define an auxiliary coordinate system  $O\xi_1\xi_2\xi_3$ , with origin coincident with that of  $Ox_1x_2x_3$ , and axes parallel to those of  $O'\zeta_1\zeta_2\zeta_3$ . The translation of  $Ox_1x_2x_3$  with



**Fig. 3.2** A translating and rotating reference systems

respect to  $O'\zeta_1\zeta_2\zeta_3$  is then represented by the translation of  $O\xi_1\xi_2\xi_3$  with respect to  $O'\zeta_1\zeta_2\zeta_3$ , and the rotation of  $Ox_1x_2x_3$  with respect to  $O'\zeta_1\zeta_2\zeta_3$  is represented by the rotation of  $O\xi_1\xi_2\xi_3$ . We denote the angular velocity as  $\vec{\omega}$ . By definition, the velocity of the particle with respect to the inertial reference system  $O'\zeta_1\zeta_2\zeta_3$  is

$$\vec{v}_{\text{in}} = \left( \frac{d\vec{r}'}{dt} \right)_{\text{in}} = \left( \frac{d\vec{r}}{dt} \right)_{\text{in}} + \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}}, \quad (3.48)$$

where the subscript 'in' indicates that the derivative is defined by assuming the directions of the axes of the coordinate system  $O'\zeta_1\zeta_2\zeta_3$  or  $O\xi_1\xi_2\xi_3$ , i.e., the inertial reference system, time-independent. According to (3.46), we can write the velocity of the particle with respect to  $O'\zeta_1\zeta_2\zeta_3$  as

$$\vec{v}_{\text{in}} = \left( \frac{d\vec{r}'}{dt} \right)_{\text{in}} = \left( \frac{d\vec{r}}{dt} \right)_{\text{rot}} + \vec{\omega} \times \vec{r} + \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}}, \quad (3.49)$$

where the subscript 'rot' indicates that the directions of the axes of the rotating coordinate system  $Ox_1x_2x_3$  are assumed time-independent. Evidently,  $[d\vec{r}(t)/dt]_{\text{rot}}$  is the velocity of the particle with respect to  $Ox_1x_2x_3$ , and  $\vec{\omega} \times \vec{r} + (d\vec{r}_0/dt)_{\text{in}}$ , the velocity of a point fixed in  $Ox_1x_2x_3$  with a position vector  $\vec{r}$  with respect to  $O'\zeta_1\zeta_2\zeta_3$ , called implicated velocity. The acceleration of the particle with respect to  $O'\zeta_1\zeta_2\zeta_3$  is

$$\begin{aligned} \vec{a}_{\text{in}} &= \left( \frac{d^2\vec{r}'}{dt^2} \right)_{\text{in}} = \left[ \frac{d}{dt} \left( \frac{d\vec{r}}{dt} \right)_{\text{rot}} \right]_{\text{in}} + \left[ \frac{d}{dt} (\vec{\omega} \times \vec{r}) \right]_{\text{in}} + \left( \frac{d^2\vec{r}_0}{dt^2} \right)_{\text{in}} \\ &= \left( \frac{d^2\vec{r}}{dt^2} \right)_{\text{rot}} + \vec{\omega} \times \left( \frac{d\vec{r}}{dt} \right)_{\text{rot}} + \left( \frac{d\vec{\omega}}{dt} \right)_{\text{in}} \times \vec{r} + \vec{\omega} \times \left( \frac{d\vec{r}}{dt} \right)_{\text{in}} + \left( \frac{d^2\vec{r}_0}{dt^2} \right)_{\text{in}} \\ &= \left( \frac{d^2\vec{r}}{dt^2} \right)_{\text{rot}} + \vec{\omega} \times \left( \frac{d\vec{r}}{dt} \right)_{\text{rot}} + \left( \frac{d\vec{\omega}}{dt} \right)_{\text{in}} \times \vec{r} + \vec{\omega} \times \left( \frac{d\vec{r}}{dt} \right)_{\text{rot}} \\ &\quad + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \left( \frac{d^2\vec{r}_0}{dt^2} \right)_{\text{in}} \\ &= \left( \frac{d^2\vec{r}}{dt^2} \right)_{\text{rot}} + 2\vec{\omega} \times \left( \frac{d\vec{r}}{dt} \right)_{\text{rot}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \left( \frac{d\vec{\omega}}{dt} \right)_{\text{in}} \times \vec{r} \\ &\quad + \left( \frac{d^2\vec{r}_0}{dt^2} \right)_{\text{in}}. \end{aligned} \quad (3.50)$$

In the last expression of this formula: the first term is the acceleration of the particle with respect to  $Ox_1x_2x_3$ ; the second term is called Coriolis acceleration; the sum of the last three terms is called implicated acceleration, which is the acceleration of a point fixed in  $Ox_1x_2x_3$  with a position vector  $\vec{r}$  with respect to  $O'\zeta_1\zeta_2\zeta_3$ ; the third

term is called centripetal acceleration, which is the acceleration toward the center if the particle is undertaking a uniform circular motion; the fourth term is called Poincaré or Euler acceleration, which is the influence of the variation of the angular velocity. It is to be noted that, as  $\vec{\omega} \times \vec{\omega} = 0$ ,  $(d\vec{\omega}/dt)_{\text{in}}$  and  $(d\vec{\omega}/dt)_{\text{rot}}$  are equal.

As we assumed  $O'\zeta_1\zeta_2\zeta_3$  to be an inertial reference system, Newton's second law of motion holds,

$$m\vec{a}_{\text{in}} = m\left(\frac{d^2\vec{r}'}{dt^2}\right)_{\text{in}} = \vec{F}, \quad (3.51)$$

where  $\vec{F}$  is the force exerted on the particle. We can use (3.50) to write the above equation as

$$\begin{aligned} m\left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{rot}} = & \vec{F} - 2m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ & - m\left(\frac{d\vec{\omega}}{dt}\right)_{\text{in}} \times \vec{r} - m\left(\frac{d^2\vec{r}_0}{dt^2}\right)_{\text{in}}. \end{aligned} \quad (3.52)$$

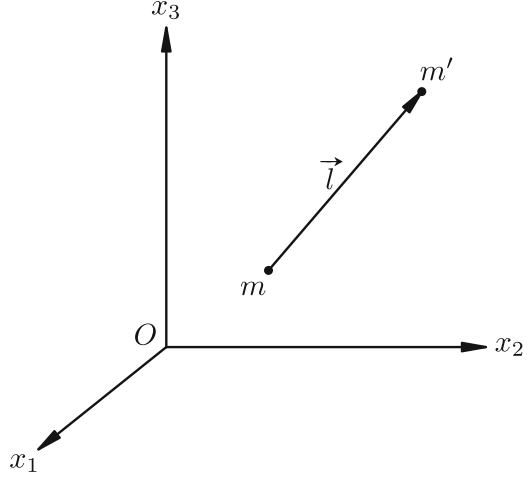
This is the equation of motion of a particle in the non-inertial reference system  $Ox_1x_2x_3$ . As compared to the equation of motion in the inertial reference system (3.51), we see that some extra terms are present at the right-hand side of this equation. If we understand these extra terms as force, the equation of motion in the non-inertial reference system appears identical to that in the inertial reference system. These extra terms are referred to as inertial forces; the term  $-2m\vec{\omega} \times (d\vec{r}/dt)_{\text{rot}}$  is called Coriolis force; the term  $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$  is called centrifugal force; the term  $-m(d\vec{\omega}/dt)_{\text{in}} \times \vec{r}$  is called Poincaré or Euler force; the sum  $-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m(d\vec{\omega}/dt)_{\text{in}} \times \vec{r} - m(d^2\vec{r}_0/dt^2)_{\text{in}}$  is also referred to as implicated inertial force. Evidently, inertial forces have the same dimension as that of the force. After dividing the inertial forces by the mass  $m$ , we obtain the inertial accelerations. The nomenclatures for the inertial forces also apply to the inertial accelerations, e.g., Coriolis acceleration.

## 3.2 Gravity Field

### 3.2.1 Gravitational Attraction, Centrifugal Acceleration and Gravity

In the theory of the Earth's gravity, two kinds of forces are concerned, one is the gravitational force, and the other is the centrifugal force. We know that the former is an attraction between objects with a mass, which is to be briefly referred to as an attraction between masses, and the latter is an inertial force arisen from the Earth rotation.

**Fig. 3.3** Gravitational attraction between point masses



As shown in Fig. 3.3, let  $m$  and  $m'$  be two point masses, and we use  $m$  and  $m'$  to denote both the masses and their positions. We denote the coordinates of  $m$  as  $\xi_i$ , and those of  $m'$  as  $x_i$ . The relative position vector from  $m$  to  $m'$  is

$$\vec{l} = (x_i - \xi_i)\hat{e}_i. \quad (3.53)$$

Its norm is denoted as  $l$ , which is

$$l = [(x_i - \xi_i)(x_i - \xi_i)]^{1/2}. \quad (3.54)$$

The gravitational force exerted on  $m'$  by  $m$  is then

$$\vec{F} = -G \frac{mm'}{l^2} \hat{l} = -G \frac{mm'}{l^3} \vec{l} = -G \frac{mm'}{l^3} (x_i - \xi_i) \hat{e}_i, \quad (3.55)$$

where  $\hat{l} = \vec{l}/l$  is the unit vector along  $\vec{l}$ . The negative sign in this expression indicates that  $\vec{F}$  and  $\vec{l}$  are in opposite directions. The universal gravitational constant  $G$  is determined by experiments, its most up to date determination is  $G = 6.6742 \times 10^{-11} \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} = 6.6742 \times 10^{-8} \text{cm}^3 \cdot \text{g}^{-1} \cdot \text{s}^{-2}$ . As this force was discovered by Newton, (3.55) is called Newton's law of gravitation.

In (3.55), we refer  $m$  to as the attracting mass, and  $m'$  the attracted mass. When the attracted mass is a unit mass, i.e.,  $m' = 1$ , we have

$$\vec{F} = -G \frac{m}{l^3} \vec{l} = -G \frac{m}{l^3} (x_i - \xi_i) \hat{e}_i. \quad (3.56)$$

We will always assume the attracted mass to be a unit mass hereafter, and refer the gravitational force exerted on the unit point mass as gravitational attraction, or

briefly, attraction. The gravitational attraction will be considered as a function of the position of the attracted mass. Evidently, the gravitational attraction is equal to the gravitational acceleration in value; therefore, the unit of acceleration,  $\text{m} \cdot \text{s}^{-2}$ , is also used as the unit of the gravitational attraction. This is equivalent to redefine the symbol  $\vec{F}$  as  $[-G(mm'/l^3)\vec{l}]/m'$ .

In geodesy,  $\text{cm} \cdot \text{s}^{-2}$  is also used as a standard unit, called Gal. The subdivisions  $10^{-3}\text{Gal}$ ,  $10^{-6}\text{Gal}$ , and  $10^{-9}\text{Gal}$  are referred to as miliGal, microGal, and nanoGal, and are denoted as mGal,  $\mu\text{Gal}$ , and nGal, respectively.

Experiments showed that the combination of gravitational attraction obeys the parallelogram law of vector sum. Hence, the total gravitational attraction of a system of point mass is the vector sum of the gravitational attraction of all the point masses.

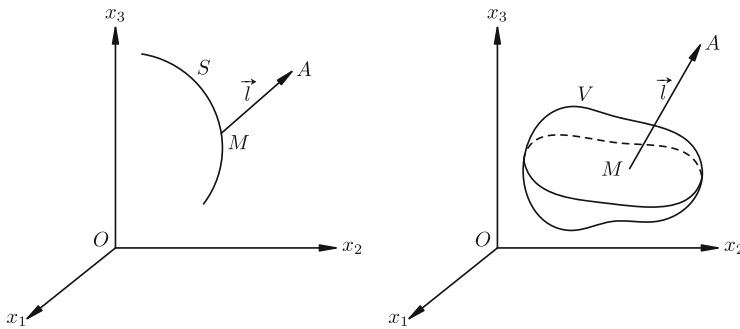
If the mass is concentrated on a thin shell, whose thickness is negligible, we call the shell a material surface. As shown in Fig. 3.4 (left), on the material surface  $S$ , a surface element  $dS$  is located at the point  $M$ , where the coordinates are denoted as  $\xi_i$ , and the surface mass density  $\mu$  is understood as a function of  $\xi_i$ . The gravitational attraction of this material surface at the point  $A$  of coordinates  $x_i$  is

$$\vec{F} = -G \int_S \frac{\mu}{l^3} \vec{l} dS = -G \int_S \frac{\mu}{l^3} (x_i - \xi_i) dS \hat{e}_i, \quad (3.57)$$

where the variables of integration are  $\xi_i$ , and  $x_i$  are considered as constants under the integral sign.

In general, the mass is not concentrated at a point or on a surface, it occupies a space domain, and is referred as a solid body (here solid means three-dimensional). As shown in Fig. 3.4 (right), within a solid body  $V$ , a volume element  $dV$  is located at the point  $M$ , where the coordinates are denoted as  $\xi_i$ , and the density  $\rho$  is understood as a function of  $\xi_i$ . The gravitational attraction of this solid body at the point  $A$  of coordinates  $x_i$  is

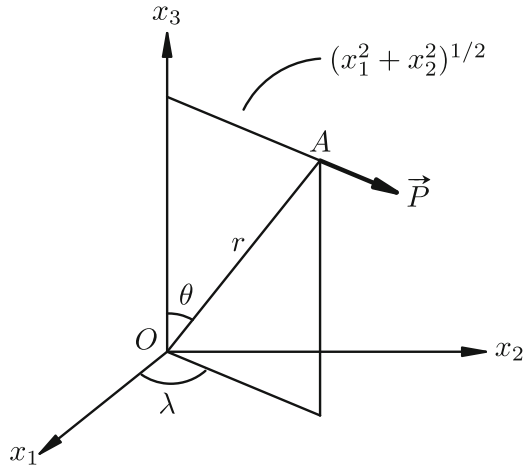
$$\vec{F} = -G \int_V \frac{\rho}{l^3} \vec{l} dV = -G \int_V \frac{\rho}{l^3} (x_i - \xi_i) dV \hat{e}_i, \quad (3.58)$$



**Fig. 3.4** Gravitational attraction of a material surface (left) and a solid body (right)



**Fig. 3.5** The centrifugal acceleration



where, the same as (3.57), the variables of integration are  $\xi_i$ , and  $x_i$  are considered as constants under the integral sign.

As shown in Fig. 3.5, we assume that the reference system  $Ox_1x_2x_3$  rotates with respect to the inertial reference system around the  $x_3$ -axis with an angular velocity  $\omega \hat{e}_3$ . The centrifugal acceleration at the point A of coordinate  $x_i$  fixed in the reference system  $Ox_1x_2x_3$  is

$$P = \omega^2 \left( x_1^2 + x_2^2 \right)^{1/2} = \omega^2 r \sin \theta. \quad (3.59)$$

Its direction points outward from the axis of rotation. It can be written in vector form as

$$\vec{P} = \omega^2 (x_1 \hat{e}_1 + x_2 \hat{e}_2). \quad (3.60)$$

In general, we assume the reference system  $Ox_1x_2x_3$  rotates with respect to the inertial reference system around O with the angular velocity  $\vec{\omega}$ . The expression of the centrifugal acceleration is

$$\vec{P} = -\vec{\omega} \times (\vec{\omega} \times \vec{r}). \quad (3.61)$$

which is obtained by setting  $m = 1$  in the centrifugal force term in (3.52). Therefore, just as the gravitational attraction is defined as the gravitational force exerted on a unit point mass, the centrifugal acceleration is the centrifugal force exerted on a unit point mass. Evidently, the centrifugal acceleration increases as the distance to the axis of rotation increases.

The gravity of the Earth is defined to be the sum of the Earth's (all the masses from the Earth center to the top of the atmosphere) gravitational attraction and the centrifugal acceleration due to the rotation of the Earth,

$$\vec{g} = \vec{F} + \vec{P}. \quad (3.62)$$

If we neglect the effects of the gravitational attraction of extraterrestrial bodies, the gravity of the Earth is the free fall acceleration of a mass not subject to electromagnetic force. Strictly speaking, the free fall acceleration is the right-hand side of (3.52) by understanding  $\vec{F}$  as the gravitational force and setting  $m = 1$ .

### 3.2.2 Gravitational, Centrifugal and Gravity Potentials

The gravitational field is a conservative field, and it has a potential. The gravitational potential of the point mass, material surface, and solid body are

$$\phi = G \frac{m}{l}, \quad (3.63)$$

$$\phi = G \int_S \frac{\mu}{l} dS, \quad (3.64)$$

$$\phi = G \int_V \frac{\rho}{l} dV, \quad (3.65)$$

respectively. By definition, we just have to verify that<sup>1</sup>

$$\vec{F} = \nabla \phi. \quad (3.66)$$

For the case of a point mass, the verification is straightforward,

$$\nabla \phi = Gm \frac{\partial}{\partial x_i} \left( \frac{1}{l} \right) \hat{e}_i = -G \frac{m}{l^3} (x_i - \xi_i) \hat{e}_i = \vec{F}. \quad (3.67)$$

For the cases of material surface and solid body, we just have to notice that the partial derivative with respect to  $x_i$  can be moved to under the integral signs.

The centrifugal acceleration also has a potential. When the angular velocity is assumed to be in the direction of  $x_3$ , for which the centrifugal acceleration is given by (3.60), the centrifugal potential is

$$\psi = \frac{1}{2} \omega^2 (x_1^2 + x_2^2) = \frac{1}{2} \omega^2 r^2 \sin^2 \theta. \quad (3.68)$$

The verification of

$$\vec{P} = \nabla \psi \quad (3.69)$$

<sup>1</sup> The definition  $\vec{F} = -\nabla \phi$  is also used. Hence, there is always a need to verify the definition in any literature.

is straightforward. The expression of the centrifugal potential corresponding to the general expression of centrifugal acceleration (3.61) is

$$\psi = \frac{1}{2}(\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r}) = \frac{1}{2}|\vec{\omega} \times \vec{r}|^2. \quad (3.70)$$

These general expressions can be written in the coordinate system  $Ox_1x_2x_3$  as

$$\vec{P} = (\vec{\omega} \cdot \vec{\omega}) \vec{r} - (\vec{r} \cdot \vec{\omega}) \vec{\omega} = (\omega_i \omega_i x_k - x_i \omega_i \omega_k) \hat{e}_k, \quad (3.71)$$

$$\psi = \frac{1}{2}[(\vec{\omega} \cdot \vec{\omega})(\vec{r} \cdot \vec{r}) - (\vec{\omega} \cdot \vec{r})(\vec{\omega} \cdot \vec{r})] = \frac{1}{2}(\omega_i \omega_i x_j x_j - \omega_i x_i \omega_j x_j). \quad (3.72)$$

It is easy to verify that (3.69) holds for the general expressions of  $\vec{P}$  and  $\psi$  using these last two formulae.

We emphasize that (3.60) and (3.68) require the coordinate system to be chosen such that  $x_3$  is in the direction of  $\vec{\omega}$ , but (3.61), (3.70), (3.71), and (3.72) do not have such a requirement.

By definition, the gravity also has a potential, called gravity potential, which is evidently the sum of the gravitational and centrifugal potentials,

$$W = \phi + \psi, \quad (3.73)$$

which is related to the gravity in the form of

$$\vec{g} = \nabla W. \quad (3.74)$$

In the coordinate system defined in Fig. 3.5, the gravity potential of the Earth is

$$W = G \int_V \frac{\rho}{l} dV + \frac{1}{2} \omega^2 r^2 \sin^2 \theta, \quad (3.75)$$

where the volume integral is understood as over whole the Earth, i.e., from the Earth center to the top of the atmosphere.

### 3.2.3 Gauss' Law of Gravity, Laplace's and Poisson's Equations of the Gravitational Potential

In this subsection, we first derive a formula called Gauss' law of gravity, and then use it to derive the differential equation satisfied by the gravitational potential.

Let a point mass  $m$  be located at  $A$ , and  $S$  is a smooth surface. Denote the gravitational potential of  $m$  as  $\phi$ , and the distance from  $A$  to the surface element  $dS$  as  $l$ . We obtain, by multiplying both sides of (2.258) by  $Gm$ ,

$$\oint_S \frac{\partial \phi}{\partial n} dS = \begin{cases} -4\pi Gm, & m \text{ is inside } S; \\ -2\pi Gm, & m \text{ is on } S; \\ 0, & m \text{ is outside } S; \end{cases} \quad (3.76)$$

where  $\hat{n}$  is the outward unit normal of  $S$ . Evidently, this formula can be generalized to a system of point masses as

$$\oint_S \frac{\partial \phi}{\partial n} dS = -4\pi Gm_i - 2\pi Gm_S, \quad (3.77)$$

where  $m_i$  and  $m_S$  are the total masses inside and on  $S$ , respectively. The gravitational potential of a system of point masses is the sum of that of each of them. As an integral is essentially a summation, (3.77) can be applied to a system of masses composed of point masses, material surfaces and solid bodies without any modification, and is called Gauss' law of gravity.

In order to use (3.77) to derive the differential equation satisfied by the gravitational potential, we first derive a formula according to Gauss' theorem (2.197),

$$\int_V \nabla^2 \phi dV = \int_V \nabla \cdot (\nabla \phi) dV = \oint_S \hat{n} \cdot \nabla \phi dS = \oint_S \frac{\partial \phi}{\partial n} dS. \quad (3.78)$$

Let  $V$  be completely outside the attracting masses, i.e., no attracting mass is inside it. We obtain from (3.78) and (3.77),

$$\int_V \nabla^2 \phi dV = \oint_S \frac{\partial \phi}{\partial n} dS = 0. \quad (3.79)$$

As the choice of  $V$  is arbitrary, in order for this equation to hold for any  $V$ , we must have

$$\nabla^2 \phi = 0. \quad (3.80)$$

Hence, outside the attracting masses, the gravitational potential satisfies Laplace's equation. In other words, the gravitational field is harmonic outside the attracting masses.

For a solid body, let  $V$  be completely inside the body, and the density is continuous within  $V$  and on its surface. We obtain from (3.78) and (3.77),

$$\int_V \nabla^2 \phi dV = \oint_S \frac{\partial \phi}{\partial n} dS = -4\pi G \int_V \rho dV. \quad (3.81)$$

Let  $V$  shrink to a point  $M$ , and denote the volume of  $V$  also as  $V$ . We obtain

$$\nabla^2 \phi = \lim_{V \rightarrow M} \frac{1}{V} \int_V \nabla^2 \phi dV = -4\pi G \lim_{V \rightarrow M} \frac{1}{V} \int_V \rho dV$$

$$= -4\pi G \lim_{V \rightarrow M} \frac{1}{V} (\rho V) = -4\pi G \rho, \quad (3.82)$$

which is referred to as Poisson's equation.<sup>2</sup> This equation is identical to (2.282) with  $u = G\rho$ ; therefore, its derivation can be understood as an alternative derivation of (2.282). Furthermore, by definition, we already know that (3.65) is its solution, as the quantity itself is the solution of its equation. However, the solution is not unique, for which the reason is left to the readers to figure out.

It can be easily verified that the centrifugal potential satisfies the following Poisson's equation:

$$\nabla^2 \psi = 2\omega^2. \quad (3.83)$$

It is left to the readers to write out the differential equation satisfied by the gravity potential  $W = \phi + \psi$  by combining (3.80) or (3.82) with (3.83).

### 3.2.4 Gravitational Potential and Attraction of Some Simple Shapes

In this subsection, we derive out the analytic formulae of the gravitational attraction and potential for some simple shapes, which often serve as approximations for practical problems.

#### Masses in Spherical Symmetry

The gravitational attraction and potential of masses in spherical symmetry can be most briefly derived using Gauss' law of gravity.

We suppose to have a homogeneous spherical material surface with radius  $R$ , centered at  $O$ , and surface mass density  $\mu$ . Its mass is then  $M = 4\pi R^2 \mu$ . Draw a sphere  $S$  of radius  $r$  and centered at  $O$ , and denote its unit normal to the exterior as  $\hat{n}$ . We know that  $\hat{n} = \hat{e}_r$ . According to the symmetry, the gravitational attraction on  $S$  is along the radial direction, and we write it as  $\vec{F}(r) = F(r)\hat{e}_r$ , with  $F = d\phi/dr = \partial\phi/\partial n$ . These leads to

$$\oint_S \frac{\partial\phi}{\partial n} dS = \oint_S F(r) dS = 4\pi r^2 F(r). \quad (3.84)$$

We can then obtain, using (3.77),

<sup>2</sup> If the potential is defined such that  $F = -\nabla\phi$ , the negative sign at the right-hand side of Poisson's equation is missing. Hence, this sign can be used to infer how the potential is defined in the literature if not explicitly provided.

$$F(r) = \frac{1}{4\pi r^2} \oint_S \frac{\partial \phi}{\partial n} dS = \begin{cases} -\frac{GM}{r^2} = -4\pi G \frac{R^2}{r^2} \mu, & r > R; \\ -\frac{1}{2} \frac{GM}{R^2} = -2\pi G \mu, & r = R; \\ 0, & r < R; \end{cases} \quad (3.85)$$

where the negative sign implies that  $\vec{F}$  is in the opposite direction of  $\hat{e}_r$ . As both  $\phi$  and  $F$  are functions of  $r$  alone,  $F = d\phi/dr$ , and  $\phi \rightarrow 0$  when  $r \rightarrow \infty$ . Hence, we can obtain  $\phi$  by integrating (3.85) over  $r$  from  $\infty$  to  $r_0$ ,

$$\phi(r_0) = \int_{\infty}^{r_0} F(r) dr = \begin{cases} \frac{GM}{r_0} = 4\pi G \frac{R^2}{r_0} \mu, & r_0 > R; \\ \frac{GM}{R} = 4\pi G \mu R, & r_0 = R; \\ \frac{GM}{R} = 4\pi G \mu R, & r_0 < R. \end{cases} \quad (3.86)$$

We see that, inside the homogeneous spherical material surface, the gravitational attraction vanishes, and the gravitational potential is a constant. In the outside, the gravitational attraction and potential are identical to those of a particle with the same mass located at the center  $O$ .

Now we suppose to have a homogeneous spherical body of radius  $R$ , centered at  $O$ , and density  $\rho$ . Its mass is  $M = (4/3)\pi R^3 \rho$ . Draw a sphere  $S$  of radius  $r$  centered at  $O$ . As the mass within  $S$  is  $M_r = (4/3)\pi r^3 \rho$  when  $r < R$ , we have, with the same notations and approach as in the case of the homogeneous spherical material surface,

$$F = \begin{cases} -\frac{GM}{r^2} = -\frac{4}{3}\pi G \frac{R^3}{r^2} \rho, & r > R; \\ -\frac{GM}{r^2} = -\frac{4}{3}\pi G r \rho, & r = R; \\ -\frac{GM_r}{r^2} = -\frac{4}{3}\pi G r \rho, & r < R; \end{cases} \quad (3.87)$$

$$\phi = \begin{cases} \frac{GM}{r} = \frac{4}{3}\pi G \frac{R^3}{r} \rho, & r > R; \\ \frac{GM}{R} = \frac{4}{3}\pi G R^2 \rho, & r = R; \\ \frac{2}{3}\pi G \rho (3R^2 - r^2), & r < R. \end{cases} \quad (3.88)$$

Evidently, when  $S$  is within the spherical body, the mass of the body outside  $S$  does not contribute to the gravitational attraction on  $S$ . When  $S$  is on or outside the spherical body, the gravitational attraction and potential are identical to those of a particle with the same mass located at the center  $O$ .

Finally, we suppose to have a spherical body with radius  $R$ , center on  $O$ , and the density is a function of the radial distance alone, i.e.,  $\rho = \rho(r)$ . Evidently, Gauss' law of gravity can still be applied. Here we derive the result by taking reference to Poisson's equation that the gravitational potential satisfies within the sphere. According to the symmetry,  $\phi$  and  $F$  are functions of  $r$  alone. Taking reference to the Laplacian in spherical coordinate system, (2.494), Poisson's equation (3.82) is simplified to

$$\nabla^2 \phi = \frac{1}{r^2} \left[ \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) \right] = \frac{1}{r^2} \left[ \frac{d}{dr} (r^2 F) \right] = -4\pi G \rho, \quad (3.89)$$

where  $F = d\phi/dr$  as in the previous cases. By multiplying both sides of this equation by  $r^2$ , and then integrating it over  $r$  from 0 to  $r_0$ , we obtain the gravitational attraction in the interior of the sphere,

$$F(r_0) = -\frac{4\pi G}{r_0^2} \int_0^{r_0} \rho r^2 dr = -\frac{GM_{r_0}}{r_0^2}, \quad (3.90)$$

where we have taken into account the fact  $F = 0$  at the center  $O$ , which can be inferred straightforwardly based on the symmetry, and  $M_{r_0} = 4\pi \int_0^{r_0} \rho r^2 dr$  is the mass enclosed within the sphere of radius  $r_0$  centered at  $O$ .

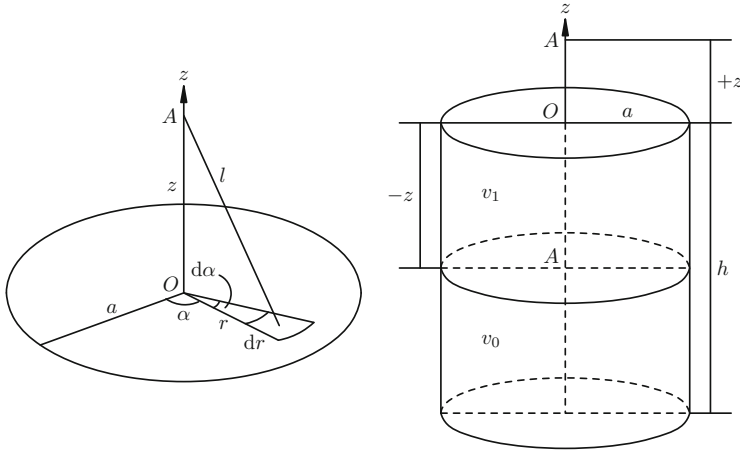
It is evident that on and outside the spheres, i.e., when  $r \geq R$ , (3.87) and (3.88) hold, i.e., the gravitational attraction and potential are identical to those of a particle with the same mass located at the center  $O$ . It is left to the readers to derive the formula for  $\phi$  inside the sphere.

### Homogeneous Disk and Cylinder

In this subsection, we derive the formulae of the gravitational potential and attraction for homogeneous disks and cylinders on their symmetry axes.

First, we consider a homogeneous disk of radius  $a$  and density  $\mu$ . As shown in Fig. 3.6 (left), we use the cylindrical coordinate system  $Or\lambda z$ , where  $O$  is the center of the disk. The gravitational potential at the point  $A$  on the symmetry axis can be obtained based on definition as

$$\begin{aligned} \phi &= G \int_0^{2\pi} \int_0^a \frac{\mu}{(r^2 + z^2)^{1/2}} r dr d\lambda \\ &= 2\pi G \mu \int_0^a \frac{r dr}{(r^2 + z^2)^{1/2}} \\ &= 2\pi G \mu [(r^2 + z^2)^{1/2}]_0^a \\ &= 2\pi G \mu [(a^2 + z^2)^{1/2} - |z|] \\ &= 2\pi G \mu [(a^2 + z^2)^{1/2} - z \operatorname{sign} z], \end{aligned} \quad (3.91)$$



**Fig. 3.6** A homogeneous disk (left) and a homogeneous cylinder (right)

where the sign  $z$  function is defined as 1, 0 and  $-1$  when  $z > 0$ ,  $z = 0$  and  $z < 0$ , respectively. According to the symmetry, the gravitational attraction can be written as  $\vec{F} = F\hat{e}_z$ , with

$$F = \frac{d\phi}{dz} = 2\pi G\mu \left[ \frac{z}{(a^2 + z^2)^{1/2}} - \text{sign } z \right]. \quad (3.92)$$

It can be readily verified that it points to the center  $O$ .

For the homogeneous cylinder shown in Fig. 3.6 (right), the gravitational potential can be obtained by integrating that of the homogeneous disk given in (3.91). We first derive the formula of the gravitational potential at the point  $A$  above the top denoted as  $\phi_E$ , where the subscript ‘ $E$ ’ indicates that the point  $A$  is in the exterior of the cylinder. In (3.91),  $z$  is actually the distance from  $A$  to the center of the disk when  $z \geq 0$ . We divide the cylinder to parallel thin discs, and use  $z'$  to denote the distance from  $A$  to the thin discs. The gravitational potential  $\phi_E$  is then the integral

$$\begin{aligned} \phi_E &= 2\pi G\rho \int_z^{z+h} \left[ (a^2 + z'^2)^{1/2} - z' \right] dz' \\ &= 2\pi G\rho \left\{ \int_z^{z+h} (a^2 + z'^2)^{1/2} dz' - \frac{1}{2} \left[ (z+h)^2 - z^2 \right] \right\}. \end{aligned} \quad (3.93)$$

With the indefinite integral formula

$$\int (a^2 + x^2)^{1/2} dx = \frac{1}{2} \left\{ x(a^2 + x^2)^{1/2} + a^2 \ln \left[ x + (a^2 + x^2)^{1/2} \right] \right\} + C, \quad (3.94)$$



where  $C$  is an integral constant, we obtain the integral in the braces at the right-hand side of (3.93) to be

$$\begin{aligned}
 & \frac{1}{2} \left\{ (z+h) \left[ a^2 + (z+h)^2 \right]^{1/2} + a^2 \ln \left\{ (z+h) + \left[ a^2 + (z+h)^2 \right]^{1/2} \right\} \right. \\
 & \quad \left. - z(a^2 + z^2)^{1/2} - a^2 \ln \left[ z + (a^2 + z^2)^{1/2} \right] \right\} \\
 &= \frac{1}{2} \left\{ (z+h) \left[ a^2 + (z+h)^2 \right]^{1/2} - z(a^2 + z^2)^{1/2} \right. \\
 & \quad \left. + a^2 \ln \frac{(z+h) + \left[ a^2 + (z+h)^2 \right]^{1/2}}{z + (a^2 + z^2)^{1/2}} \right\}. \tag{3.95}
 \end{aligned}$$

Finally, we can write (3.93) as

$$\begin{aligned}
 \phi_E = \pi G \rho \left\{ (z+h) \left\{ \left[ a^2 + (z+h)^2 \right]^{1/2} - (z+h) \right\} - z \left[ (a^2 + z^2)^{1/2} - z \right] \right. \\
 \left. + a^2 \ln \frac{(z+h) + \left[ a^2 + (z+h)^2 \right]^{1/2}}{z + (a^2 + z^2)^{1/2}} \right\}. \tag{3.96}
 \end{aligned}$$

Besides taking the derivative of the gravitational potential, the gravitational attraction of the homogeneous cylinder can also be obtained by integrating that of a homogeneous disk given in (3.92), which we follow. We denote the gravitational attraction at the point  $A$  above the top as  $\vec{F}_E = F_E \hat{e}_z$ . Similar to the case of the gravitational potential, the gravitational attraction  $F_E$  is the integral

$$\begin{aligned}
 F_E &= 2\pi G \rho \left\{ \int_z^{z+h} \left[ \frac{z'}{(a^2 + z'^2)^{1/2}} - 1 \right] dz' \right\} \\
 &= 2\pi G \rho \left\{ \left[ (a^2 + z'^2)^{1/2} - z' \right]_z^{z+h} \right\} \\
 &= -2\pi G \rho \left\{ (a^2 + z^2)^{1/2} + h - \left[ a^2 + (z+h)^2 \right]^{1/2} \right\}. \tag{3.97}
 \end{aligned}$$

If the point  $A$  is inside the cylinder, we have  $z < 0$ , also as shown in Fig. 3.6 (right), and the gravitational potential and attraction can be inferred from (3.96) and (3.97). We divide the cylinder into two, one below the point  $A$  denoted as  $v_0$ , and one above denoted as  $v_1$ . The gravitational potential and attraction of  $v_0$  can be inferred from (3.96) and (3.97) by replacing  $z$  by 0, which is the distance of  $A$  to the center of the top, and  $h$  by  $z+h$ , which is the height of the cylinder,

$$\phi_0 = \pi G\rho \left\{ (z+h) \left\{ \left[ a^2 + (z+h)^2 \right]^{1/2} - (z+h) \right\} + a^2 \ln \frac{(z+h) + \left[ a^2 + (z+h)^2 \right]^{1/2}}{a} \right\}, \quad (3.98)$$

$$F_0 = -2\pi G\rho \left\{ (a + (z+h)) - \left[ a^2 + (z+h)^2 \right]^{1/2} \right\}. \quad (3.99)$$

The gravitational potential and attraction of  $v_1$  at  $A$  can be obtained by replacing  $z+h$  by  $-z$  in the above formulae, with the sign of the gravitational attraction reversed, as  $A$  is below  $v_1$ ,

$$\phi_1 = \pi G\rho \left\{ -z \left[ (a^2 + z^2)^{1/2} + z \right] + a^2 \ln \frac{-z + (a^2 + z^2)^{1/2}}{a} \right\}, \quad (3.100)$$

$$F_1 = -2\pi G\rho \left\{ (a^2 + z^2)^{1/2} + z - a \right\}. \quad (3.101)$$

The total gravitational potential and attraction at the point  $A$  inside the cylinder are the sums,

$$\begin{aligned} \phi_I = \pi G\rho \left\{ (z+h) \left\{ \left[ a^2 + (z+h)^2 \right]^{1/2} - (z+h) \right\} - z \left[ (a^2 + z^2)^{1/2} + z \right] \right. \\ \left. + a^2 \ln \frac{\left\{ (z+h) + \left[ a^2 + (z+h)^2 \right]^{1/2} \right\} \left[ -z + (a^2 + z^2)^{1/2} \right]}{a^2} \right\}, \end{aligned} \quad (3.102)$$

$$F_I = -2\pi G\rho \left\{ (a^2 + z^2)^{1/2} + (h+2z) - \left[ a^2 + (z+h)^2 \right]^{1/2} \right\}, \quad (3.103)$$

noting that the vector form of the gravitational attraction is implicitly assumed to be  $\vec{F}_I = F_I \hat{e}_z$ .

### 3.2.5 Some Properties of the Gravitational Potential

Outside the attracting mass, it is evident by definition that the gravitational potential and its derivatives up to any order are finite and continuous. Hence, our focus will be in the interior of solid bodies, across boundaries of discontinuity of density in solid bodies, or across material surfaces.

### Solid Body

We already know that the gravitational potential satisfies Poisson's equation (3.82), i.e.,  $\nabla^2\phi = -4\pi G\rho$ , within solid bodies. This implies that its second order derivatives within solid bodies are discontinuous across boundaries of discontinuity in density. Actually, in deriving (3.82) using Gauss' theorem (3.78), it has been implicitly assumed that the gravitational potential has continuous first and second order derivatives where the density is continuous. In the formulation of the Earth's deformation, derivatives of the gravitational potential up to order two are involved, which are to be discussed below.

For any point  $A$  inside a solid body, we divide the body into two parts, part one being an infinitesimal sphere containing  $A$  inside centered at a point  $O$  very close to  $A$ , and part two being the rest. By definition, the gravitational potential is the sum of those of the two parts. We know that the gravitational potential of part two as well as its derivatives up to any order is finite and continuous at  $A$ . Hence, the continuities of the gravitational potential and its derivatives are completely determined by those of the part one, i.e., the infinitesimal sphere. If the density is continuous at  $A$ , we can treat part one as a homogeneous sphere. According to (3.88) and (3.87), its gravitational potential, as well as the first and second order derivatives (the first order derivative is the gravitational attraction), is continuous at  $A$ . If the point  $A$  is on the surface of the body, the density has a jump from 0 outside the body to a value  $\rho$  assumed non-vanishing inside the body. Assuming the surface of the body to be smooth, we can cut out an infinitesimal cylinder from the body with the top surface coincident with the surface of the body and centered by  $A$ . The continuities of the gravitational potential and its derivatives are all completely defined by those of the cylinder. According to the formulae derived in the previous subsection, we do have  $\phi_E = \phi_I$  and  $F_E = F_I$  when  $z = 0$ . We thus conclude that the gravitational potential and its first order derivative, which is the gravitational attraction, are continuous across the surface (assumed smooth in the proof) of a solid body. An immediate inference is that the gravitational potential and its first order derivative are both continuous across a surface (assumed smooth) of discontinuity of density inside a solid body.

The jump of the second order derivative of the gravitational potential can be formulated using the formula of gravitational attraction of a homogeneous cylinder on the symmetry axis derived in the previous subsection. We first consider the jump from the inside to the outside of a body, at a place on the surface,  $A$ , where the surface is smooth, and the density inside is continuous. In this case, we can cut out an infinitesimal cylinder, which can be considered as homogeneous, with the top surface of the cylinder on the surface of the body centered at  $A$ . As compared to Fig. 3.6 (right), the point  $A$  here corresponds to the point  $O$  in the figure. We define  $z$  as the distance from  $A$  along the normal of the surface, positive outward. By definition,  $F_E = d\phi_E/dz$  and  $F_I = d\phi_I/dz$  are given in (3.97) and (3.103), respectively, for this infinitesimal cylinder. We have

$$\left(\frac{d^2\phi_E}{dz^2}\right)_{z=0} - \left(\frac{d^2\phi_I}{dz^2}\right)_{z=0} = \left(\frac{dF_E}{dz}\right)_{z=0} - \left(\frac{dF_I}{dz}\right)_{z=0} = 4\pi G\rho. \quad (3.104)$$

As compared to Fig. 3.6 (right),  $\hat{e}_z$  is the unit outward normal  $\hat{n}$ . Furthermore, the derivatives of the gravitational potential of the mass outside the infinitesimal cylinder are continuous. Hence, we have the general formula for the solid body,

$$\left(\frac{\partial^2 \phi_E}{\partial n^2}\right)_S - \left(\frac{\partial^2 \phi_I}{\partial n^2}\right)_S = 4\pi G\rho, \quad (3.105)$$

where  $\phi_E$  and  $\phi_I$  are understood as the gravitational potentials of the solid body in the exterior and interior, respectively, and  $S$ , the surface of the body. This formula can be generalized to surfaces of discontinuity of density inside a solid body. We use ‘+’ and ‘−’ to indicate the two sides of a discontinuity surface, and define  $\hat{n}$  to point to the side ‘+’. The discontinuity is then

$$\left(\frac{\partial^2 \phi}{\partial n^2}\right)_+ - \left(\frac{\partial^2 \phi}{\partial n^2}\right)_- = -4\pi G(\rho_+ - \rho_-). \quad (3.106)$$

Since the gravitational potential is continuous across a surface (assumed smooth) of discontinuity, i.e., its values at the two sides of the surface are exactly the same, we can infer that the directional derivatives of the gravitational potential up to any order along a direction tangential to the surface must also be continuous. The readers are suggested to further consider why the boundary condition (3.106) do not contradict to Poisson’s equation  $\nabla^2 \phi = -4\pi G\rho$ .

### Material Surface

Some properties of the gravitational potential of material surfaces can be inferred from the formulae of the gravitational potential and attraction of homogeneous thin disks. Let  $A$  be a point on a material surface where the latter is smooth. We can then cut out an infinitesimal disk centered at  $A$ , which can be considered as homogeneous. The continuities of the gravitational potential and its derivatives of the material surface across the surface through  $A$  is completely determined by those of the disk. According to (3.91), the gravitational potential is finite and continuous. In (3.92), when  $z$  approaches 0 from the positive and negative sides, we obtain, for the disk,

$$\left(\frac{d\phi}{dz}\right)_{z=\pm 0} = \mp 2\pi G\mu. \quad (3.107)$$

As the gravitational attraction of the disk at the center vanishes, we have

$$\left(\frac{d\phi}{dz}\right)_{z=0} = 0. \quad (3.108)$$

By combining the above two formulae together, we have

$$\left(\frac{\partial \phi}{\partial z}\right)_{+0} - \left(\frac{\partial \phi}{\partial z}\right)_{0-} = \left(\frac{\partial \phi}{\partial z}\right)_0 - \left(\frac{\partial \phi}{\partial z}\right)_{-0} = -2\pi G\mu. \quad (3.109)$$

Considering that  $\hat{e}_z$  is actually the unit normal of the surface,  $\hat{n}$ , and that the gravitational potential and its derivatives of the mass outside the disk are all continuous, we have, for the whole material surface,

$$\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_0 = \left(\frac{\partial\phi}{\partial n}\right)_0 - \left(\frac{\partial\phi}{\partial n}\right)_- = -2\pi G\mu, \quad (3.110)$$

where we used  $+$ ,  $0$  and  $-$  to indicate the side that  $\hat{n}$  points to, on the surface, and the side  $\hat{n}$  points from, respectively. Particularly, the relation

$$\left(\frac{\partial\phi}{\partial n}\right)_+ - \left(\frac{\partial\phi}{\partial n}\right)_- = -4\pi G\mu \quad (3.111)$$

is used very often. The results (3.110) and (3.111) can in fact be derived using Gauss' law of gravity (3.77), which is left to the readers.

Finally, we generalize (3.110) and (3.111) to the directional derivative along any direction  $\hat{l}$ , as shown in Fig. 3.7. For the disk considered to be homogeneous, as  $(d\phi/dz)\hat{e}_z$  is the gravitational attraction,  $\partial\phi/\partial l$  is therefore the projection of the gravitational attraction along  $\hat{l}$ . Hence,

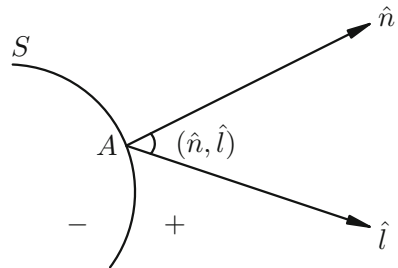
$$\frac{\partial\phi}{\partial l} = \frac{\partial\phi}{\partial n} \cos(\hat{n}, \hat{l}), \quad (3.112)$$

where  $(\hat{n}, \hat{l})$  is the angle between the normal  $\hat{n}$  and the direction  $\hat{l}$ . This formula can be straightforwardly used to generalize the formulae for whole the material surface, (3.110) and (3.111), to

$$\left(\frac{\partial\phi}{\partial l}\right)_+ - \left(\frac{\partial\phi}{\partial l}\right)_0 = \left(\frac{\partial\phi}{\partial l}\right)_0 - \left(\frac{\partial\phi}{\partial l}\right)_- = -2\pi G\mu \cos(\hat{n}, \hat{l}), \quad (3.113)$$

$$\left(\frac{\partial\phi}{\partial l}\right)_+ - \left(\frac{\partial\phi}{\partial l}\right)_- = -4\pi G\mu \cos(\hat{n}, \hat{l}). \quad (3.114)$$

**Fig. 3.7** Discontinuity of the first order derivative of the gravitational potential of a material surface



### 3.3 Dynamical Equations of the Motion of Continua

#### 3.3.1 Lagrangian and Eulerian Descriptions of the Motion

Two methods are used to describe the motion of continua. One is the Lagrangian description, where an observer follows an individual material element to determine its position and physical parameters at different time. The other one is the Eulerian description, where an observer focuses at a fixed position in the space to determine the physical parameters of different material elements passing through the point at different time. We take for example the observation of falling snow flakes outside the window (Melchior, 1986). In the Lagrangian description, the observer follows each snow flake to tell what are its position, velocity, acceleration, etc., at different time. In the Eulerian description, the observer's sight is fixed at each fixed point in the space to tell the velocity, acceleration, etc., of different snow flakes passing through the points at different time. Both descriptions are necessary in continuum mechanics. First, the velocity and acceleration are with respect to fixed material elements, just like the velocity and acceleration of a particle are with respect to the same particle. Hence, the velocity and acceleration should be defined based on the Lagrangian description. Second, in continuum mechanics, the physical parameters of the material studied are mostly function of both position and time, and it is more convenient to use the Eulerian description.

The key of the Lagrangian description is to create a way to distinguish all material elements one from another within the whole body of the material studied throughout the time, like the numbering of particles in the study of a system of particles. Numbering is impossible in a continuum. The practical way is to define a time-independent reference state for the material studied, e.g., the undeformed state or the initial state, where different material elements occupy different positions. In this way, different positions at the reference state represent different material elements, and reciprocally, different elements are specified by their different coordinates at the reference state. The coordinates of a material element at the reference state are called Lagrangian coordinates. We use  $X_i$  to denote the Lagrangian coordinate, and express the Lagrangian position vector as

$$\vec{R} = X_i \hat{e}_i. \quad (3.115)$$

By definition, the instantaneous coordinates of a material element at the time  $t$  are called Eulerian Coordinates. We use  $x_i$  to denote the Eulerian coordinate, and express the Eulerian position vector as

$$\vec{r} = x_i \hat{e}_i. \quad (3.116)$$

For the same material element, its Lagrangian coordinates are time-independent, while its Eulerian Coordinates vary with time. The relation between the Eulerian

and Lagrangian position vectors of the material element can then be written as

$$\vec{r} = \vec{r}(\vec{R}, t), \quad (3.117)$$

where  $\vec{r}$  is the Eulerian position vector, at the time  $t$ , of the material element with Lagrangian position vector  $\vec{R}$ . By inverting (3.117), we can obtain the Lagrangian position vector of the material element with Eulerian position vector  $\vec{r}$  at the time  $t$ ,

$$\vec{R} = \vec{R}(\vec{r}, t). \quad (3.118)$$

For a specific material element, its Lagrangian position vector  $R$  is time-independent, and the variation of its Eulerian position vector  $\vec{r}$  draws the trace of its motion with time.

When we describe a physical quantity as a function of position, we must first specify whether the Lagrangian or Eulerian description is used, and then specify the position vector. As the Eulerian description corresponds to the actual position and time, thus being more obvious than the Lagrangian description, we will more frequently use the Eulerian description. Hence, we assume the Eulerian description as the default, and use a superscript ‘ $L$ ’ to indicate the Lagrangian description. Take the density as example. We understand  $\rho$  as the density of the material element located at the position vector  $\vec{r}$  at the time  $t$ , and understand  $\rho^L$  as the density of the material element with a Lagrangian position vector  $\vec{R}$  at the time  $t$ . As the density of the same element is unique, we have

$$\rho(\vec{r}, t) = \rho^L[\vec{r}(\vec{R}, t), t], \quad (3.119)$$

where  $\vec{r}(\vec{R}, t)$  is understood to substitute  $\vec{r}$  by  $\vec{R}$  and  $t$  according to  $\vec{r}(\vec{R}, t)$ . Evidently, this relation holds for any physical parameter.

Based on the discussion in the previous paragraph, the velocity and acceleration of the material element with Lagrangian position vector  $\vec{R}$  at the time  $t$  are defined as

$$\vec{v}^L = \frac{\partial \vec{r}^L}{\partial t}, \quad \vec{a}^L = \frac{\partial \vec{v}^L}{\partial t} = \frac{\partial^2 \vec{r}^L}{\partial t^2}, \quad (3.120)$$

where the partial derivative with respect to  $t$  is defined such that the Lagrangian position vector  $\vec{R}$  is a fixed parameter, i.e., for the same material element. In order to elucidate the importance of the partial derivative with respect to time for the same material element, we call it material derivative, and denote it as  $D/Dt$  in Eulerian description. In mathematical language, when  $D/Dt$  is applied to a physical quantity, this quantity should be first converted to a function of  $\vec{R}$  and  $t$  according to (3.117), then the partial derivative with respect to  $t$  is taken, and finally, the result is converted to a function of  $\vec{r}$  and  $t$  according to (3.118). Based on the definition of the material derivative, we can write the velocity and acceleration of the material element at the

position vector  $\vec{r}$  at the time  $t$  as

$$\vec{v} = \frac{D\vec{r}}{Dt}, \quad \vec{a} = \frac{D\vec{v}}{Dt} = \frac{D^2\vec{r}}{Dt^2}. \quad (3.121)$$

We should keep in mind that, when the partial derivative with respect to  $t$ ,  $\partial/\partial t$ , is taken in Eulerian description, the Eulerian position vector  $\vec{r}$  is considered as fixed parameter.

$D/Dt$  and  $\partial/\partial t$  are evidently different. We derive out their relation. We still take the density  $\rho$  as example by expressing its dependence on  $\vec{r}$  as on  $x_1, x_2$  and  $x_3$ ,

$$\rho = \rho(x_1, x_2, x_3, t). \quad (3.122)$$

For a particular material element, according to (3.117), we have

$$x_1 = x_1(\vec{R}, t), \quad x_2 = x_2(\vec{R}, t), \quad x_3 = x_3(\vec{R}, t). \quad (3.123)$$

Hence,

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \frac{\partial x_1}{\partial t} \frac{\partial\rho}{\partial x_1} + \frac{\partial x_2}{\partial t} \frac{\partial\rho}{\partial x_2} + \frac{\partial x_3}{\partial t} \frac{\partial\rho}{\partial x_3} = \frac{\partial\rho}{\partial t} + \frac{\partial x_i}{\partial t} \frac{\partial\rho}{\partial x_i}. \quad (3.124)$$

As  $\partial x_i/\partial t = \partial x_i(\vec{R}, t)/\partial t$  is the component of the velocity of the material element along the  $x_i$ -axis, which we denote as  $v_i$ , we can write the above formula as

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + v_i \frac{\partial\rho}{\partial x_i} = \frac{\partial\rho}{\partial t} + (\vec{v} \cdot \nabla)\rho. \quad (3.125)$$

This formula holds for other physical quantities as well. Particularly, the acceleration is the material derivative of the velocity,

$$a_j = \frac{Dv_j}{Dt} = \frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i}, \quad (3.126)$$

whose vector form is

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}. \quad (3.127)$$

We mention that the term  $(\vec{v} \cdot \nabla)\vec{v}$  is called advection in fluid mechanics.

In solid mechanics, the reference state is often chosen as the undeformed state, and a displacement vector is defined as the difference between the Eulerian and Lagrangian vectors,

$$\vec{u} = \vec{r} - \vec{R}. \quad (3.128)$$



As  $R$  does not vary with time, the velocity and acceleration can be expressed using  $\vec{u}$  as

$$\vec{v}^L = \frac{\partial \vec{u}^L}{\partial t}, \quad \vec{a}^L = \frac{\partial \vec{v}^L}{\partial t} = \frac{\partial^2 \vec{u}^L}{\partial t^2}; \quad (3.129)$$

$$\vec{v} = \frac{D\vec{u}}{Dt}, \quad \vec{a} = \frac{D\vec{v}}{Dt} = \frac{D^2\vec{u}}{Dt^2}. \quad (3.130)$$

Generally speaking,  $\vec{v}$  is normally chosen as the fundamental physical quantity in fluid mechanics, and  $\vec{u}$  in solid mechanics. Mostly,  $\vec{u}$  is assumed to be very small in solid mechanics, thus the Lagrangian and Eulerian position vectors can be interchangeably used to represent position. However, they must be distinguished in larger deformation. In the elastic deformation equation of the Earth, their difference has to be considered for taking into account of the effect of the change of the gravitational field accompanying the deformation of the Earth.

### 3.3.2 Generalization of Newton's Laws of Motion of Particles to Continua

The dynamic equations of motion are first established based on experiments for the simplest model of material, i.e., the Newton's three laws of motion for particles, whose masses are implicitly assumed to be time-independent. Here we write the laws as three equations—the equation of mass conservation, the equation of linear momentum, and the equation of angular momentum—for to be generalized to the case of continua,

$$\frac{dm}{dt} = 0, \quad (3.131)$$

$$\frac{d}{dt}(m\vec{v}) = \vec{F}, \quad (3.132)$$

$$\frac{d}{dt}(m\vec{r} \times \vec{v}) = \vec{r} \times \vec{F} = \vec{N}, \quad (3.133)$$

where  $m$ ,  $\vec{r}$ , and  $\vec{v}$  are the mass, position vector, and velocity, respectively, of the point mass, and  $\vec{F}$  and  $\vec{N}$  are the force and torque exerted on it. It is to be mentioned that the last two formulae hold only in the inertial reference system, and both of them can be derived out based on Newton's second law of motion,  $\vec{F} = m\vec{a}$ , where  $\vec{a}$  is the acceleration.

The object studied by (3.131)–(3.133) is a particular particle understood as a point mass, and the physical parameters  $m$ ,  $\vec{r}$ ,  $\vec{v}$ ,  $\vec{F}$ , and  $\vec{N}$  are all instantaneous values at the time  $t$ . Hence, when these formulae are generalized to a continuum, the material studied should also be specifically defined, i.e., the same material throughout the time, and all physical parameters should also be the instantaneous

values at the time  $t$ . Evidently, it is more straightforward to describe the physical parameters using Eulerian description. The generalized forms of these dynamic equations of motion to a continuum is stated as follows: At any time  $t$ , for the specific material within the space domain  $V$ , its mass is conserved, the rate of change of its linear momentum is equal to the total force exerted on it, and the rate of change of its angular momentum is equal to the torque exerted on it. As these are for the specific material, the space domain  $V$  also changes with time, and the mathematical expression of these generalized dynamic equations of motion are

$$\frac{D}{Dt} \int_V \rho dV = 0, \quad (3.134)$$

$$\frac{D}{Dt} \int_V \rho \vec{v} dV = \vec{F}_V(t), \quad (3.135)$$

$$\frac{D}{Dt} \int_V \rho \vec{r} \times \vec{v} dV = \vec{N}_V(t), \quad (3.136)$$

where the material derivative is with respect to the same material within  $V$  at the time  $t$ , and  $\vec{F}_V(t)$  and  $\vec{N}_V(t)$  are the total force and torque exerted on the material at the time  $t$ . As already mentioned, the space domain  $V$  changes with time, and hence, the material derivative  $D/Dt$  cannot be moved to within the integral signs. In the rest of this subsection, we write  $\vec{F}_V(t)$  and  $\vec{N}_V(t)$  in more concrete forms, and leave the problem of how to move the material derivative to within the integral signs to the next subsection.

Two kinds of force are exerted on the material within  $V$ . The first is a body force distributed within the body of the material, i.e., within  $V$ , and the second is a surface force distributed on the surface of the material,  $S$ , which is the boundary of  $V$ . The body force is expressed using the body force density, i.e., the force per unit volume. Hence, at a point  $M$  within  $V$ , the body force density is

$$\vec{f} = \lim_{\delta V \rightarrow M} \frac{\delta \vec{F}}{\delta V}, \quad (3.137)$$

where  $\delta V$  is an infinitesimal space domain including  $M$  in its inside, with its volume also denoted as  $\delta V$ , the limit  $\delta V \rightarrow M$  indicates the shrinking of  $\delta V$  to the point  $M$ , and  $\delta \vec{F}$  is the total force exerted on the material within  $\delta V$ .

The surface force is expressed using the surface force density, i.e., the force per unit area, which is referred to as traction. Evidently, the traction depends not only on the position, but also the orientation of the surface. The orientation of the surface is best represented by its unit normal vector  $\hat{n}$  pointing toward the outside of  $V$ . Hence, for a point  $M$  on  $S$ , the analytical definition of traction is

$$\vec{t}(\hat{n}) = \lim_{\delta S \rightarrow M} \frac{\delta \vec{T}}{\delta S}. \quad (3.138)$$

where the meanings of  $\delta S$  and  $\delta \vec{T}$  are similar to those of  $\delta V$  and  $\delta \vec{F}$ , thus being not explained further.

In terms of the body and surface force densities, we can express  $\vec{F}_V(t)$  and  $\vec{N}_V(t)$  as

$$\vec{F}_V(t) = \oint_S \vec{t}(\hat{n}) dS + \int_V \vec{f} dV, \quad (3.139)$$

$$\vec{N}_V(t) = \oint_S \vec{r} \times \vec{t}(\hat{n}) dS + \int_V \vec{r} \times \vec{f} dV. \quad (3.140)$$

In these expressions, we implicitly assumed that the body and surface forces expressed through their densities defined in (3.137) and (3.138) are exclusive, i.e., no point force is present. Furthermore, no point couple (pure twist at a point) is assumed to be present.

Finally we can write (3.135) and (3.136) as

$$\frac{D}{Dt} \int_V \rho \vec{v} dV = \oint_S \vec{t}(\hat{n}) dS + \int_V \vec{f} dV, \quad (3.141)$$

$$\frac{D}{Dt} \int_V \rho \vec{r} \times \vec{v} dV = \oint_S \vec{r} \times \vec{t}(\hat{n}) dS + \int_V \vec{r} \times \vec{f} dV. \quad (3.142)$$

### 3.3.3 Reynolds Transport Theorem and Its Applications

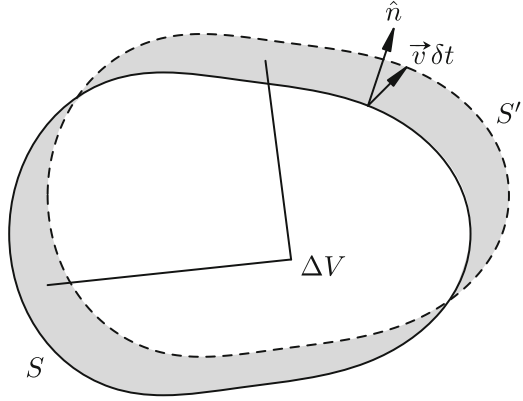
Before discussing how to move the material derivative  $D/Dt$  to under the integral signs in (3.134), (3.141), and (3.142), we first formulate the general expression  $(D/Dt) \int_V u dV$ , where  $u$  is a function of spatial position and time, which is, as well as its first order derivatives, finite and continuous. This expression implicitly involves the definition that, the derivative  $D/Dt$  applies to the material occupying  $V$  at the time  $t$ . We use  $V'$  to denote the space occupied by the same material at the time  $t + \delta t$ . We further denote the surface of the material at the times  $t$  and  $t + \delta t$  as  $S$  and  $S'$ , respectively. By definition, we have

$$\frac{D}{Dt} \int_V u dV = \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \left[ \int_{V'} u(t + \delta t) dV - \int_V u(t) dV \right]. \quad (3.143)$$

As shown in Fig. 3.8, we decompose  $V'$  into  $V$  and  $\delta V$ , i.e.,  $V' = V + \delta V$ , where  $\delta V$  is the spatial domain swept by the surface during the infinitesimal time interval  $\delta t$ . With the definition of  $\delta V$ , we can write (3.143) as

$$\begin{aligned} \frac{D}{Dt} \int_V u dV &= \lim_{\delta t \rightarrow 0} \left\{ \frac{1}{\delta t} \left[ \int_V u(t + \delta t) dV - \int_V u(t) dV \right] \right. \\ &\quad \left. + \frac{1}{\delta t} \int_{\delta V} u(t + \delta t) dV \right\}. \end{aligned} \quad (3.144)$$

**Fig. 3.8** Derivation of Reynolds transport theorem



Since

$$\begin{aligned} \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \left[ \int_V u(t + \delta t) dV - \int_V u(t) dV \right] &= \lim_{\delta t \rightarrow 0} \int_V \frac{1}{\delta t} [u(t + \delta t) - u(t)] dV \\ &= \int_V \frac{\partial u}{\partial t} dV, \end{aligned} \quad (3.145)$$

$$\begin{aligned} \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \int_{\delta V} u(t + \delta t) dV &= \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \int_S u(t + \delta t) (\vec{v} \delta t) \cdot \hat{n} dS \\ &= \int_S u \vec{v} \cdot \hat{n} dS = \int_V \nabla \cdot (u \vec{v}) dV, \end{aligned} \quad (3.146)$$

we obtain

$$\frac{D}{Dt} \int_V u dV = \int_V \left[ \frac{\partial u}{\partial t} + \nabla \cdot (u \vec{v}) \right] dV. \quad (3.147)$$

The term  $\partial u / \partial t$  is the contribution of temporal variation from the point of view of Eulerian description, and  $\nabla \cdot (u \vec{v})$  is the contribution of transportation due to the motion of the material, as can be seen based on (3.146). If  $u$  describes a material property, such as the density, it satisfies (3.125). We can thus write the above equation as

$$\frac{D}{Dt} \int_V u dV = \int_V \left[ \frac{Du}{Dt} + u \nabla \cdot \vec{v} \right] dV. \quad (3.148)$$

Both (3.147) and (3.148) are referred to as Reynolds transport theorem.

By applying (3.147) to the conservation equation of mass (3.134), we obtain

$$\int_V \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] dV = 0. \quad (3.149)$$

As the spatial domain  $V$  can be arbitrarily chosen within the bulk of the material studied, this equation implies that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \quad (3.150)$$

Another form of this equation is

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0. \quad (3.151)$$

These are the differential form of the equation of mass conservation, which are also referred to as continuity equation.

By replacing  $u$  by  $\rho u$  in (3.148), we obtain

$$\frac{D}{Dt} \int_V \rho u dV = \int_V \left[ \rho \frac{Du}{Dt} + u \frac{D\rho}{Dt} + \rho u \nabla \cdot \vec{v} \right] dV. \quad (3.152)$$

According to (3.151), the sum of the last two terms in the integrand at the right-hand side vanishes. Hence,

$$\frac{D}{Dt} \int_V \rho u dV = \int_V \rho \frac{Du}{Dt} dV. \quad (3.153)$$

The physical meaning of this formula is clear: Although both  $\rho$  and  $dV$  vary with time, the mass within the volume element  $dV$ ,  $\rho dV$ , does not vary, and hence,  $D(\rho dV)/Dt = 0$ . Therefore,  $D/Dt$  can be moved to within the integral sign, but it should be applied to the quantity other than  $\rho dV$ .

It can be readily inferred that (3.153) still holds if the scalar  $u$  is replaced by a vector  $\vec{A}$ . Hence,

$$\frac{D}{Dt} \int_V \rho \vec{A} dV = \int_V \rho \frac{D\vec{A}}{Dt} dV. \quad (3.154)$$

In continuum mechanics, quantities in the form of (3.153) and (3.154) are common. It can be obtained from (3.154) that

$$\frac{D}{Dt} \int_V \rho \vec{v} dV = \int_V \rho \frac{D\vec{v}}{Dt} dV = \int_V \rho \vec{a} dV, \quad (3.155)$$

$$\begin{aligned} \frac{D}{Dt} \int_V \rho \vec{r} \times \vec{v} dV &= \int_V \rho \frac{D}{Dt} (\vec{r} \times \vec{v}) dV \\ &= \int_V \rho \vec{r} \times \frac{D\vec{v}}{Dt} dV = \int_V \rho \vec{r} \times \vec{a} dV, \end{aligned} \quad (3.156)$$

where  $\vec{a}$  is the acceleration. These formulae can be used to reduce the linear momentum and angular momentum equations to

$$\int_V \rho \vec{a} dV = \oint_S \vec{t}(\hat{n}) dS + \int_V \vec{f} dV, \quad (3.157)$$

$$\int_V \rho \vec{r} \times \vec{a} dV = \oint_S \vec{r} \times \vec{t}(\hat{n}) dS + \int_V \vec{r} \times \vec{f} dV, \quad (3.158)$$

where the material derivative is no longer present.

### 3.3.4 Description of Interaction Within Continua: the Stress Tensor

In (3.157) and (3.158), the dependence of the traction  $\vec{t}(\hat{n})$  on the position and the direction of the normal of the surface,  $\hat{n}$ , is in need of further simplification. In this subsection, we introduce a second order tensor, called Cauchy stress tensor, to separate the dependence of  $\vec{t}(\hat{n})$  on position and on  $\hat{n}$ .

The traction  $\vec{t}(\hat{n})$  is not necessarily along the normal of the surface  $\hat{n}$ , its component along  $\hat{n}$  being called normal traction, and the component perpendicular to  $\hat{n}$ , shear traction. Evidently,  $\hat{n}$  and  $-\hat{n}$  represent the two sides of a surface, and  $\vec{t}(\hat{n})$  and  $\vec{t}(-\hat{n})$  are mutual action and reaction. According to Newton's third law of motion,

$$\vec{t}(-\hat{n}) = -\vec{t}(\hat{n}). \quad (3.159)$$

In the simplest case, let  $\hat{n} = \hat{e}_1$ , i.e., we are concerned with the traction on the surface with normal along the direction of the  $x_1$ -axis,  $\vec{t}(\hat{n}) = \vec{t}(\hat{e}_1)$ . We use  $\sigma_{11}$ ,  $\sigma_{12}$ , and  $\sigma_{13}$  to express the components of  $\vec{t}(\hat{e}_1)$  along  $x_1$ ,  $x_2$ , and  $x_3$ , respectively. The traction  $\vec{t}(\hat{e}_1)$  can then be expressed as

$$\vec{t}(\hat{e}_1) = \sigma_{11}\hat{e}_1 + \sigma_{12}\hat{e}_2 + \sigma_{13}\hat{e}_3. \quad (3.160)$$

Similarly, we use  $\sigma_{21}$ ,  $\sigma_{22}$ , and  $\sigma_{23}$  to express the components of the traction  $\vec{t}(\hat{e}_2)$  along  $x_1$ ,  $x_2$ , and  $x_3$ , respectively, and use  $\sigma_{31}$ ,  $\sigma_{32}$ , and  $\sigma_{33}$  to express those of the traction  $\vec{t}(\hat{e}_3)$ . We have

$$\vec{t}(\hat{e}_2) = \sigma_{21}\hat{e}_1 + \sigma_{22}\hat{e}_2 + \sigma_{23}\hat{e}_3, \quad (3.161)$$

$$\vec{t}(\hat{e}_3) = \sigma_{31}\hat{e}_1 + \sigma_{32}\hat{e}_2 + \sigma_{33}\hat{e}_3. \quad (3.162)$$

The above three formulae can be written together briefly as

$$\vec{t}(\hat{e}_i) = \sigma_{ij}\hat{e}_j. \quad (3.163)$$

We emphasize that  $\vec{t}(\hat{e}_i)$  is the traction on the surface with  $\hat{e}_i$  as normal vector, and  $\sigma_{ij}$  is the component of  $\vec{t}(\hat{e}_i)$  along the  $x_j$ -axis. It can be obtained according to (3.159) that

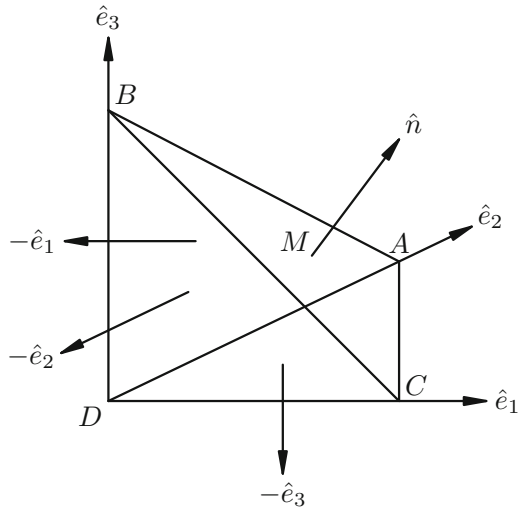
$$\vec{t}(-\hat{e}_i) = -\vec{t}(\hat{e}_i). \quad (3.164)$$

In the following, we first prove that  $\vec{t}(\hat{n})$  can be expressed using  $\sigma_{ij}$  and the components of the unit normal vector,  $n_i$ , and then prove that  $\sigma_{ij}$  are the components of a second order tensor.

In order to study the traction  $\vec{t}(\hat{n})$  at a point  $M$  on the surface with a unit normal vector  $\hat{n}$ , we draw a triangle  $ABC$  with  $M$  on it, and then find a point  $D$ , so that the triangles  $ABD$ ,  $BCD$ , and  $CAD$  are perpendicular to  $x_1$ ,  $x_2$ , and  $x_3$  respectively, as shown in Fig. 3.9. In this way, we can express  $\vec{t}(\hat{n})$  in terms of  $\sigma_{ij}$  and the components of  $\hat{n}$ ,  $n_i$ , by analyzing the forces exerted on the material within the tetrahedron  $ABCD$ . Evidently, the unit normals of the triangles  $ABD$ ,  $BCD$ , and  $CAD$  as surfaces of the tetrahedron  $ABCD$  are  $-\hat{e}_1$ ,  $-\hat{e}_2$ , and  $-\hat{e}_3$ , respectively. We assume that the density of the continuum,  $\rho$ , the body force density  $\vec{f}$ , and the traction  $\vec{t}$  on all surfaces are finite and continuous. The limits of their values are then those at the point  $M$  when the triangle shrinks to the point  $M$ . Therefore, when the tetrahedron  $ABCD$  is infinitesimally small, we obtain, by applying the linear momentum equation (3.157) to it,

$$\begin{aligned} \rho V_{ABCD} \vec{a} = & S_{ABC} \vec{t}(\hat{n}) + S_{ABD} \vec{t}(-\hat{e}_1) + S_{BCD} \vec{t}(-\hat{e}_2) \\ & + S_{CAD} \vec{t}(-\hat{e}_3) + V_{ABCD} \vec{f}, \end{aligned} \quad (3.165)$$

**Fig. 3.9** The forces exerted on a tetrahedron



where  $V$  and  $S$  are the respective volume and areas. By substituting (3.164) into this equation, we obtain

$$\vec{t}(\hat{n}) = \frac{S_{ABD}}{S_{ABC}} \vec{t}(\hat{e}_1) + \frac{S_{BCD}}{S_{ABC}} \vec{t}(\hat{e}_2) + \frac{S_{CAD}}{S_{ABC}} \vec{t}(\hat{e}_3) + \frac{V_{ABCD}}{S_{ABC}} (\rho \vec{a} - \vec{f}). \quad (3.166)$$

The triangle  $ABD$  is the projection of the triangle  $ABC$  on the coordinate surface  $x_2x_3$ , implying  $S_{ABD} = S_{ABC} \cos \alpha_1$ , where  $\alpha_1$  is the angle between the surfaces  $ABC$  and  $x_2x_3$ . As  $\alpha_1$  is also the angle between  $\hat{n}$  and  $\hat{e}_1$ ,  $\cos \alpha_1$  is in fact the component of  $\hat{n}$  along  $\hat{e}_1$  denoted as  $n_1$ . We thus obtain  $S_{ABD} = S_{ABC} n_1$ . Similarly, we can also obtain  $S_{BCD} = S_{ABC} n_2$  and  $S_{CAD} = S_{ABC} n_3$ . Substitute these relations into (3.166), and then let  $D$  approach  $M$ , which is equivalent to the shrinking of the triangle to  $M$ . Note that when  $D$  approaches  $M$ ,  $V_{ABCD}/S_{ABC} = (1/3)h$  approaches zero, where  $h$  is the distance between  $D$  and the surface  $ABC$ . We thus obtain

$$\vec{t}(\hat{n}) = n_1 \vec{t}(\hat{e}_1) + n_2 \vec{t}(\hat{e}_2) + n_3 \vec{t}(\hat{e}_3) = n_i \vec{t}(\hat{e}_i). \quad (3.167)$$

Finally, by substituting (3.163) into it, we obtain

$$\vec{t}(\hat{n}) = n_i \sigma_{ij} \hat{e}_j, \quad (3.168)$$

which is the expression of  $\vec{t}(\hat{n})$  in terms of  $\sigma_{ij}$  and  $n_i$ . This means that the traction within a continuum is completely represented by  $\sigma_{ij}$ , and the puzzling quantity  $\vec{t}(\hat{n})$  can be substituted by  $\sigma_{ij}$  and  $n_i$ .

We can use (3.168) to prove that  $\sigma_{ij}$  are the components of a second order tensor. As a coordinate transformation is required for the proof, we define a new rectangular coordinate system  $Ox'_1x'_2x'_3$  with base vectors denoted as  $\hat{e}'_i$ . We further denote the components of  $\hat{e}'_m$  along  $\hat{e}_i$  as  $n_{mi}$ . According to (3.168), the traction on the surface with  $\hat{e}'_m$  as normal is

$$\vec{t}(\hat{e}'_m) = n_{mi} \sigma_{ij} \hat{e}_j. \quad (3.169)$$

By definition, in the new coordinate system  $Ox'_1x'_2x'_3$ , the stress component  $\sigma'_{mn}$  is the component of  $\vec{t}(\hat{e}'_m)$  along  $\hat{e}'_n$ . Hence,

$$\sigma'_{mn} = \vec{t}(\hat{e}'_m) \cdot \hat{e}'_n. \quad (3.170)$$

The substitution of (3.169) into (3.170), together with  $\hat{e}'_n = n_{nk} \hat{e}_k$ , yields

$$\sigma'_{mn} = n_{mi} \sigma_{ij} \hat{e}_j \cdot (n_{nk} \hat{e}_k) = n_{mi} n_{nj} \sigma_{ij}. \quad (3.171)$$

This completes the proof that  $\sigma_{ij}$  are the components of a second order tensor, which is denoted as  $\bar{\sigma}$ , and is referred to as Cauchy stress tensor, or stress tensor, whose components with  $i = j$  are referred to as normal stress components, and those with



$i \neq j$  as shear stress components. We can thus use algebraic operations of tensors to write (3.168) as

$$\vec{t}(\hat{n}) = \hat{n} \cdot \bar{\sigma}. \quad (3.172)$$

Its component form is

$$t_j(\hat{n}) = n_i \sigma_{ij}. \quad (3.173)$$

We will briefly refer the Cauchy stress tensor as stress tensor.

### 3.3.5 Dynamical Equation of the Motion of Continua

After expressing  $\vec{t}(\hat{n})$  in terms of  $\bar{\sigma}$  and  $\hat{n}$  according to (3.172), we can convert the surface integrals in the linear momentum and angular momentum equations, i.e., (3.157) and (3.158), into volume integrals, thus eliminating the integral signs to obtain differential equations.

By substituting (3.172) into the linear momentum equation (3.157), we obtain

$$\int_V \rho \vec{a} dV = \oint_S \hat{n} \cdot \bar{\sigma} dS + \int_V \vec{f} dV. \quad (3.174)$$

After converting the surface integral to a volume integral according to Gauss' theorem (2.204), we obtain

$$\int_V \rho \vec{a} dV = \int_V \nabla \cdot \bar{\sigma} dV + \int_V \vec{f} dV. \quad (3.175)$$

Considering that  $V$  can be arbitrarily chosen within the body of the material studied, in order for the above equation to hold, we must have

$$\rho \vec{a} = \nabla \cdot \bar{\sigma} + \vec{f}. \quad (3.176)$$

This is the dynamic equation of motion of continua in Eulerian description. Its component form is

$$\rho a_j = \frac{\partial \sigma_{ij}}{\partial x_i} + f_j. \quad (3.177)$$

By substituting (3.172) and (3.176) into the angular momentum equation (3.158), we obtain

$$\int_V \vec{r} \times (\nabla \cdot \bar{\sigma}) dV = \oint_S \vec{r} \times (\hat{n} \cdot \bar{\sigma}) dS, \quad (3.178)$$

which can be written more concretely as

$$\int_V (x_i \hat{e}_i) \times \left( \frac{\partial \sigma_{kj}}{\partial x_k} \hat{e}_j \right) dV = \oint_S (x_i \hat{e}_i) \times (n_k \sigma_{kj} \hat{e}_j) dS. \quad (3.179)$$

After evaluating the vector products, we obtain

$$\int_V x_i \frac{\partial \sigma_{kj}}{\partial x_k} \epsilon_{ijl} \hat{e}_l dV = \oint_S x_i n_k \sigma_{kj} \epsilon_{ijl} \hat{e}_l dS. \quad (3.180)$$

In this vector equation, the corresponding components are equal. Hence,

$$\int_V x_i \frac{\partial \sigma_{kj}}{\partial x_k} \epsilon_{ijl} dV = \oint_S x_i n_k \sigma_{kj} \epsilon_{ijl} dS. \quad (3.181)$$

The right-hand side can be converted into a volume integral:

$$\begin{aligned} \oint_S x_i n_k \sigma_{kj} \epsilon_{ijl} dS &= \int_V \frac{\partial}{\partial x_k} (x_i \sigma_{kj} \epsilon_{ijl}) dV \\ &= \int_V \left( \frac{\partial x_i}{\partial x_k} \sigma_{kj} + x_i \frac{\partial \sigma_{kj}}{\partial x_k} \right) \epsilon_{ijl} dV \\ &= \int_V \left( \delta_{ik} \sigma_{kj} + x_i \frac{\partial \sigma_{kj}}{\partial x_k} \right) \epsilon_{ijl} dV \\ &= \int_V \left( \sigma_{ij} + x_i \frac{\partial \sigma_{kj}}{\partial x_k} \right) \epsilon_{ijl} dV. \end{aligned} \quad (3.182)$$

By substituting it back into (3.181), we obtain

$$\int_V \sigma_{ij} \epsilon_{ijk} dV = 0. \quad (3.183)$$

Again, as  $V$  is arbitrary, we must have

$$\sigma_{ij} \epsilon_{ijk} = 0. \quad (3.184)$$

The term at the left-hand side is non-vanishing only when  $i \neq j$ . Writing the summation term by term leads to

$$\sigma_{ij} = \sigma_{ji}. \quad (3.185)$$

This implies that the stress tensor is symmetric, which can be written using tensor notation as

$$\bar{\sigma} = \bar{\sigma}^T. \quad (3.186)$$

The differential equations (3.150) or (3.151) derived from the mass conservation equation, (3.176) or (3.177) from the linear momentum equation, and (3.185) or (3.186) from the angular momentum equation are the general dynamic equations of motion of continua. These equations do not depend on the property of the material studied, and thus hold for solids as well as fluids. Evidently, the responses of different materials to the same forcing are different, particularly, the deformation of solids or flow of fluids in response to the stress tensor. In order to completely determine the motion of a continuum, equations representing its response to the stress tensor must be formulated, which are referred to as constitutive equations. The continua are categorized as elastic body, fluid, viscoelastic body, plastic body, etc., based on the different forms of their constitutive equations.

### 3.4 Deformation of Solid Bodies and the Elastic Constitutive Equation

#### 3.4.1 Deformation and Strain Tensor of Solid Bodies

An elastic body is defined such that when a force is exerted on it, it deforms, and when the force is released, it goes back to its original form. Strictly speaking, the exertion and release of the force should be extremely slow, which will be realized later. Therefore, we have to first formulate the deformation before establishing the relation between the forcing and the deformation, i.e., the constitutive equation. We mention that the deformation formulated in this subsection and the next is not restricted to elastic bodies, but applies to any solid body.

We know that a rigid body is undeformable. In mathematical language, undeformability is equivalent to the statement that the distance between any two material elements within the body is invariant. Therefore, we start by analyzing the change of the distance between two material elements for introducing the quantities to describe deformations.

We study the distance between two infinitesimally close material elements situated at the points  $P$  and  $Q$  with Lagrangian position vectors

$$\vec{R}_P = X_i \hat{e}_i \quad \text{and} \quad \vec{R}_Q = (X_i + dX_i) \hat{e}_i, \quad (3.187)$$

respectively. The distance between them at the undeformed state,  $ds_0$ , is

$$ds_0^2 = (\vec{R}_Q - \vec{R}_P) \cdot (\vec{R}_Q - \vec{R}_P) = dX_i dX_i. \quad (3.188)$$

We denote their positions at the deformed state at the time  $t$  as  $P'$  and  $Q'$ , and their Eulerian position vectors as

$$\vec{r}_{P'} = x_i \hat{e}_i \quad \text{and} \quad \vec{r}_{Q'} = (x_i + dx_i) \hat{e}_i, \quad (3.189)$$

Respectively. The distance between them at the time  $t$ ,  $ds$ , is then given by

$$ds^2 = (\vec{r}_{Q'} - \vec{r}_{P'}) \cdot (\vec{r}_{Q'} - \vec{r}_{P'}) = dx_i dx_i . \quad (3.190)$$

We use  $ds^2 - ds_0^2$  to represent the change of distance between the two material elements from the undeformed state to their actual position at the deformed state at the time  $t$ ,

$$ds^2 - ds_0^2 = dx_i dx_i - dX_i dX_i . \quad (3.191)$$

Evidently,  $ds^2 - ds_0^2 = 0$  when there is no deformation.

In solid mechanics, the displacement from the undeformed state to the deformed state at the time  $t$ ,  $\vec{u}$ , is chosen as a fundamental function to be studied. We use  $\vec{u}$  to denote the displacement of the point  $P$ , which is defined as  $\overrightarrow{PP'}$ :

$$\vec{u} = \vec{r}_{P'} - \vec{R}_P = u_i \hat{e}_i . \quad (3.192)$$

We use  $\vec{u} + d\vec{u}$  to denote the displacement of the point  $Q$ , which is defined as  $\overrightarrow{QQ'}$ :

$$\vec{r}_{Q'} - \vec{R}_Q = \vec{u} + d\vec{u} = (u_i + du_i) \hat{e}_i . \quad (3.193)$$

According to (3.187) and (3.189), we have

$$u_i = x_i - X_i , \quad u_i + du_i = x_i + dx_i - (X_i + dX_i) . \quad (3.194)$$

Hence,

$$du_i = dx_i - dX_i . \quad (3.195)$$

As  $dx_j$  represents the difference between the Eulerian position vectors of the points  $Q'$  and  $P'$ , and  $dX_j$  the difference between the Lagrangian position vectors of the points  $Q$  and  $P$ ,  $du_i$  can be expressed based on Eulerian and Lagrangian descriptions as

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \quad \text{and} \quad du_i = \frac{\partial u_i^L}{\partial X_j} dX_j , \quad (3.196)$$

respectively. We can use these relations to express  $dX_i$  and  $dx_i$  one in terms of another according to (3.195),

$$dX_i = dx_i - du_i = dx_i - \frac{\partial u_i}{\partial x_j} dx_j , \quad (3.197)$$

$$dx_i = dX_i + du_i = dX_i + \frac{\partial u_i^L}{\partial X_j} dX_j . \quad (3.198)$$

According to (3.197), we obtain

$$\begin{aligned}
 dX_i dX_i &= \left( dx_i - \frac{\partial u_i}{\partial x_j} dx_j \right) \left( dx_i - \frac{\partial u_i}{\partial x_k} dx_k \right) \\
 &= dx_i dx_i - \frac{\partial u_i}{\partial x_j} dx_j dx_i - \frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} dx_j dx_k \\
 &= dx_i dx_i - \frac{\partial u_j}{\partial x_i} dx_i dx_j - \frac{\partial u_i}{\partial x_j} dx_i dx_j + \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_i} dx_i dx_j \\
 &= dx_i dx_i - \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right) dx_i dx_j. \tag{3.199}
 \end{aligned}$$

Similarly, we can use (3.198) to obtain

$$dx_i dx_i = dX_i dX_i + \left( \frac{\partial u_j^L}{\partial X_i} + \frac{\partial u_i^L}{\partial X_j} + \frac{\partial u_k^L}{\partial X_j} \frac{\partial u_k^L}{\partial X_i} \right) dX_i dX_j. \tag{3.200}$$

By substituting the above two formulae into (3.191), we obtain the expression of  $ds^2 - ds_0^2$  based on Eulerian and Lagrangian descriptions as

$$ds^2 - ds_0^2 = 2e_{ij} dx_i dx_j, \tag{3.201}$$

$$ds^2 - ds_0^2 = 2E_{ij} dX_i dX_j, \tag{3.202}$$

where  $e_{ij}$  and  $E_{ij}$  are defined as

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right), \tag{3.203}$$

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_j^L}{\partial X_i} + \frac{\partial u_i^L}{\partial X_j} + \frac{\partial u_k^L}{\partial X_j} \frac{\partial u_k^L}{\partial X_i} \right), \tag{3.204}$$

respectively. It can be readily verified that,  $e_{ij}$  are the components of a second order tensor in the coordinate system  $Ox_1x_2x_3$ , and  $E_{ij}$  those of a second order tensor in the coordinate system  $OX_1X_2X_3$ , where it is just in need to recognize the facts that the coordinate transformation of  $u_i$  is the same as that of  $x_i$ , and that of  $u_i^L$  is the same as that of  $X_i$ . We write these two tensors as

$$\bar{e} = \frac{1}{2} [\nabla \vec{u} + (\nabla \vec{u})^T - \nabla \vec{u} \cdot (\nabla \vec{u})^T], \tag{3.205}$$

$$\bar{E} = \frac{1}{2} [\nabla^L \vec{u}^L + (\nabla^L \vec{u}^L)^T + \nabla^L \vec{u}^L \cdot (\nabla^L \vec{u}^L)^T], \tag{3.206}$$

where  $\nabla^L$  is the nabla operator in the Lagrangian coordinates. The tensor  $\bar{e}$  is called Almansi strain tensor, and  $\bar{E}$  Green strain tensor. Both of them are symmetrical,

$$e_{ij} = e_{ji}, \quad E_{ij} = E_{ji}. \quad (3.207)$$

This symmetry is particularly arranged in deriving (3.201) and (3.202). While the tensor  $\nabla \vec{u}$  or  $\nabla^L \vec{u}^L$  has 9 components, the tensor  $\bar{e}$  or  $\bar{E}$  has only 6 independent components. This implies that the deformation can be completely described using 6 components.

A direct inference from (3.201) and (3.202) is that  $ds^2 - ds_0^2 = 0$  for any  $dx_i$  or  $dX_i$  implies  $\bar{e} = \bar{E} = 0$  and vice versa. Hence,  $\bar{e} = 0$  or  $\bar{E} = 0$  is a sufficient and necessary condition of rigid motion.

### 3.4.2 Infinitesimal Deformation of Solid Bodies: Strain Sensor, Rotation Angle and Volume Dilatation

We know that the deformation is completely described by  $\bar{e}$  and  $\bar{E}$ . In this subsection, we discuss their geometrical properties assuming the deformation to be infinitesimal, i.e.,

$$e_{ij} \ll 1 \text{ or } E_{ij} \ll 1 \quad (3.208)$$

for all  $i$  and  $j$ . This implies that, according to (3.203) and (3.204),

$$\frac{\partial u_i}{\partial x_j} \ll 1 \text{ or } \frac{\partial u_i^L}{\partial X_j} \ll 1. \quad (3.209)$$

Therefore, in infinitesimal deformation, the second order terms of  $\partial u_i / \partial x_j$  can be neglected, and we use  $\epsilon_{ij}$  to denote the strain components  $e_{ij}$  in (3.203). We have

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \quad (3.210)$$

These are also the components of a second order tensor,

$$\bar{\epsilon} = \frac{1}{2} [\nabla \vec{u} + (\nabla \vec{u})^T], \quad (3.211)$$

called Cauchy strain tensor, which depends on the displacement linearly, thus rendering the solution of problems of solid mechanics much easier. Another fact is that, the difference between the Almansi and Green strain tensors is of second order of  $\partial u_i / \partial x_j$ . Hence, they are identical under first order approximation, and both of them are simplified to the Cauchy strain tensor. Therefore, there is no need to distinguish  $\partial u_i / \partial x_j$  and  $\partial u_i^L / \partial X_j$  in infinitesimal deformation.

In the following, we interpret the geometrical meaning of the Cauchy strain tensor by analyzing the relative displacement of two infinitesimally close material elements.

We go back to (3.187) and (3.189). Denote

$$d\vec{R} = \overrightarrow{PQ} = dX_i \hat{e}_i, \quad d\vec{r} = \overrightarrow{P'Q'} = dx_i \hat{e}_i. \quad (3.212)$$

The change of the relative position vector is

$$d\vec{r} - d\vec{R} = (dx_i - dX_i) \hat{e}_i. \quad (3.213)$$

By substituting (3.197) into it, we obtain

$$d\vec{r} - d\vec{R} = dx_j \frac{\partial u_i}{\partial x_j} \hat{e}_i = d\vec{r} \cdot (\nabla \vec{u}). \quad (3.214)$$

This implies that the change of the relative position vector is fully described by  $\nabla \vec{u}$ , which is referred to as deformation tensor. According to (3.211), we can decompose the deformation tensor as the sum of a Cauchy strain tensor  $\bar{\epsilon}$  and an antisymmetric tensor  $\bar{\xi}$ ,

$$\nabla \vec{u} = \bar{\epsilon} + \bar{\xi}, \quad (3.215)$$

where the antisymmetric tensor is

$$\bar{\xi} = \frac{1}{2} [\nabla \vec{u} - (\nabla \vec{u})^T], \quad (3.216)$$

and its component form is

$$\xi_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right). \quad (3.217)$$

In this way, we can write (3.214) in the form of

$$d\vec{r} - d\vec{R} = d\vec{r} \cdot \bar{\epsilon} + d\vec{r} \cdot \bar{\xi}. \quad (3.218)$$

The term  $d\vec{r} \cdot \bar{\xi}$  can be further simplified. As  $\bar{\xi}$  is antisymmetric, only three components are independent among the nine components  $\xi_{ij}$ . By choosing the three independent terms as

$$\xi_1 = \xi_{23}, \quad \xi_2 = \xi_{31}, \quad \xi_3 = \xi_{12}, \quad (3.219)$$

we can express  $\xi_{ij}$  in terms of  $\xi_k$  as

$$\xi_{ij} = \epsilon_{ijk} \xi_k. \quad (3.220)$$

Hence,

$$d\vec{r} \cdot \vec{\bar{\xi}} = dx_i \xi_{ij} \hat{e}_j = dx_i \epsilon_{ijk} \xi_k \hat{e}_j = \xi_k dx_i \epsilon_{kij} \hat{e}_j = \vec{\bar{\xi}} \times d\vec{r}, \quad (3.221)$$

where the vector  $\vec{\bar{\xi}}$  is defined as

$$\vec{\bar{\xi}} = \xi_k \hat{e}_k. \quad (3.222)$$

By substituting (3.221) into (3.218), we finally obtain

$$d\vec{r} - d\vec{R} = d\vec{r} \cdot \vec{\bar{\epsilon}} + \vec{\bar{\xi}} \times d\vec{r}. \quad (3.223)$$

This is the change of a line segment following the displacement, with  $d\vec{R}$  representing the line segment at the undeformed state, and  $d\vec{r}$  at the deformed state. According to (3.201) and (3.205) neglecting the second order terms, the deformation, i.e., the change of shape, is completely described by the Cauchy strain tensor  $\vec{\bar{\epsilon}}$ , which is to be simply referred to as strain tensor. Hence, in the above formula, the term  $d\vec{r} \cdot \vec{\bar{\epsilon}}$  represents the deformation. As will be explained in detail in the rest of this subsection, the term  $\vec{\bar{\xi}} \times d\vec{r}$  represents a rotation of an infinitesimal material element enclosing  $d\vec{R}$  at the undeformed state, and  $d\vec{r}$  at the deformed state.

We can interpret  $\vec{\bar{\xi}}$  as a rotation angle. This can be best understood by comparing the contribution of  $\vec{\bar{\xi}}$  to the relative displacement,  $\vec{\bar{\xi}} \times d\vec{r}$ , with the contribution of the angular velocity  $\vec{\omega}$  to the velocity,  $\vec{\omega} \times \vec{r}$ . The difference is that, in the case of rotation of a coordinate system with respect to another, the same angular velocity is used to compute the velocity throughout the rotating system, while in the case of deformation here, the contribution of  $\vec{\bar{\xi}}$  to the relative displacement  $d\vec{r} - d\vec{R}$  is local, i.e., the values of  $\vec{\bar{\xi}}$  at different locations are different.

The rotation angle  $\vec{\bar{\xi}}$  can be expressed more explicitly in terms of  $\vec{u}$ . According to (3.219), we can express  $\xi_k$  as

$$\xi_k = \frac{1}{2} \epsilon_{ijk} \xi_{ij}, \quad (3.224)$$

where the antisymmetry of  $\xi_{ij}$  is used. We obtain, by substituting this formula into (3.222), and then using (3.217),

$$\begin{aligned} \vec{\bar{\xi}} &= \frac{1}{2} \epsilon_{ijk} \xi_{ij} \hat{e}_k \\ &= \frac{1}{4} \epsilon_{ijk} \left( \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) \hat{e}_k \\ &= \frac{1}{4} \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \hat{e}_k - \frac{1}{4} \epsilon_{ijk} \frac{\partial u_i}{\partial x_j} \hat{e}_k \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{4} \frac{\partial u_j}{\partial x_i} \epsilon_{ijk} \hat{e}_k + \frac{1}{4} \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \hat{e}_k \\
&= \frac{1}{2} \frac{\partial u_j}{\partial x_i} \epsilon_{ijk} \hat{e}_k \\
&= \frac{1}{2} \nabla \times \vec{u}.
\end{aligned} \tag{3.225}$$

This means that the rotation angle is half of the curl of the displacement, which can be understood as a geometrical property of the curl operator  $\nabla \times$ .

The geometrical interpretations of the strain components and the rotation angle are to be discussed in the following paragraphs.

We first discuss  $\epsilon_{11}$ . In infinitesimal deformation, (3.202) degenerates to

$$ds^2 - ds_0^2 = 2\epsilon_{ij} dX_i dX_j. \tag{3.226}$$

In particular, setting  $d\vec{R} = \overrightarrow{PQ} = dX_1 \hat{e}_1$  leads to

$$ds^2 - ds_0^2 = 2\epsilon_{11} dX_1^2. \tag{3.227}$$

As  $ds_0^2 = dX_1^2$ , and  $ds - ds_0$  is of smaller order of magnitude as compared to  $ds_0$ , we obtain

$$\epsilon_{11} = \frac{ds^2 - ds_0^2}{2ds_0^2} = \frac{ds + ds_0}{2ds_0} \frac{ds - ds_0}{ds_0} = \frac{ds - ds_0}{ds_0}. \tag{3.228}$$

This implies that  $\epsilon_{11}$  is the relative elongation of a line segment along the  $x_1$ -axis. The properties of  $\epsilon_{22}$  and  $\epsilon_{33}$  are similar. The components  $\epsilon_{11}$ ,  $\epsilon_{22}$  and  $\epsilon_{33}$  are referred to as normal strain components.

In order to discuss  $\epsilon_{12}$  and  $\xi_{12}$ , we write (3.214) as

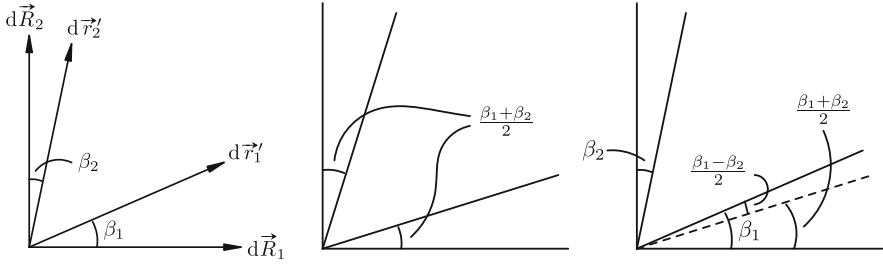
$$d\vec{r} = d\vec{R} + \frac{\partial u_i^L}{\partial X_j} dX_j \hat{e}_i. \tag{3.229}$$

which is permitted in infinitesimal deformation. Setting  $d\vec{R}$  as

$$d\vec{R}_1 = dX_1 \hat{e}_1 \quad \text{and} \quad d\vec{R}_2 = dX_2 \hat{e}_2, \tag{3.230}$$

the corresponding  $d\vec{r}$  are

$$\begin{cases} d\vec{r}_1 = \left( dX_1 + \frac{\partial u_1^L}{\partial X_1} dX_1 \right) \hat{e}_1 + \frac{\partial u_2^L}{\partial X_1} dX_1 \hat{e}_2 + \frac{\partial u_3^L}{\partial X_1} dX_1 \hat{e}_3, \\ d\vec{r}_2 = \frac{\partial u_1^L}{\partial X_2} dX_2 \hat{e}_1 + \left( dX_2 + \frac{\partial u_2^L}{\partial X_2} dX_2 \right) \hat{e}_2 + \frac{\partial u_3^L}{\partial X_2} dX_2 \hat{e}_3. \end{cases} \tag{3.231}$$



**Fig. 3.10** Change of angle between two infinitesimal line segments (left) and its decomposition into a strain component (middle) and a rotation component (right)

Their projection on the coordinate plane  $x_1x_2$  are

$$\begin{cases} d\vec{r}'_1 = \left( dX_1 + \frac{\partial u_1^L}{\partial X_1} dX_1 \right) \hat{e}_1 + \frac{\partial u_2^L}{\partial X_1} dX_1 \hat{e}_2, \\ d\vec{r}'_2 = \frac{\partial u_1^L}{\partial X_2} dX_2 \hat{e}_1 + \left( dX_2 + \frac{\partial u_2^L}{\partial X_2} dX_2 \right) \hat{e}_2. \end{cases} \quad (3.232)$$

The relative positions of  $d\vec{R}_1$ ,  $d\vec{R}_2$  and  $d\vec{r}'_1$ ,  $d\vec{r}'_2$  are shown in Fig. 3.10. We obtain

$$\beta_1 = \sin \beta_1 = \cos \left( \frac{\pi}{2} - \beta_1 \right) = \frac{d\vec{r}'_1 \cdot d\vec{R}_2}{|d\vec{r}'_1| |d\vec{R}_2|}. \quad (3.233)$$

By substituting (3.230) and (3.232) into this equation, we obtain, after neglecting second order terms,

$$\beta_1 = \frac{\partial u_2^L}{\partial X_1} = \frac{\partial u_2}{\partial x_1}. \quad (3.234)$$

Similarly, we have

$$\beta_2 = \frac{\partial u_1^L}{\partial X_2} = \frac{\partial u_1}{\partial x_2}. \quad (3.235)$$

In general,  $\beta_1$  and  $\beta_2$  are not equal, i.e., the deformation is not symmetrical with respect to the  $x_1$ - and  $x_2$ -axes. We separate the deformation into two parts. One part is a change of the angle maintaining symmetry with respect to the  $x_1$ - and  $x_2$ -axes, and the other part is a rotation angle, as shown in Fig. 3.10. The symmetrical change of the angle is in fact  $\epsilon_{12}$ ,

$$\epsilon_{12} = \frac{1}{2}(\beta_1 + \beta_2) = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right). \quad (3.236)$$

and the rotation angle is  $\xi_{12}$ ,

$$\xi_{12} = \frac{1}{2}(\beta_1 - \beta_2) = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right), \quad (3.237)$$

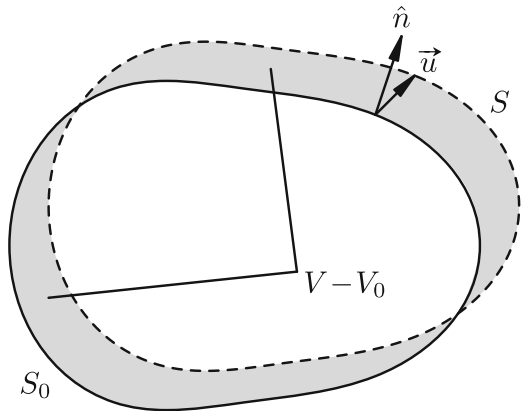
as defined in (3.217).

The geometrical property of  $\epsilon_{12}$  is made clear in (3.236): it is half of the decrease of the angle between two line segments due to the deformation, one line segment being along the  $x_1$ -axis, and the other the  $x_2$ -axis. The properties of  $\epsilon_{23}$  and  $\epsilon_{31}$  are similar (noting that the strain tensor is symmetrical). The components  $\epsilon_{12}$ ,  $\epsilon_{23}$  and  $\epsilon_{31}$  are referred to as shear strain components. The rotation angle  $\xi_{12}$  is a rotation around the  $x_3$ -axis following the right-hand rule, and it is the component of  $\vec{\xi}$  along the  $x_3$ -axis. According to (3.225), the vector  $\vec{\xi}$  is independent of the direction of  $d\vec{r}$ , i.e., it remains the same whatever the direction of  $d\vec{r}$  is. The displacement associated with the rotation is the second term at the right-hand side of (3.223), i.e.,  $\vec{\xi} \times d\vec{r}$ . This implies that this rotation behaves as a rigid rotation of an infinitesimal material element where (3.223) is applicable (the element encloses the infinitesimal vector  $d\vec{R}$  in the initial state, and the vector  $d\vec{r}$  after the displacement).

The last quantity describing the deformation is the volume dilatation, which can be formulated based on Gauss' theorem. To compute the volume dilatation at the point  $M$ , we consider a spatial domain that encloses  $M$  inside. We denote the domain and its surface at the undeformed state as  $V_0$  and  $S_0$ , respectively, and at the deformed state as  $V$  and  $S$ , respectively, as shown in Fig. 3.11. The volume dilatation at  $M$  is defined and formulated as

$$\begin{aligned} \theta &= \lim_{V \rightarrow M} \frac{V - V_0}{V} = \lim_{V \rightarrow M} \frac{1}{V} \int_S \hat{n} \cdot \vec{u} dS \\ &= \lim_{V \rightarrow M} \frac{1}{V} \int_V \nabla \cdot \vec{u} dV = \nabla \cdot \vec{u} = \frac{\partial u_i}{\partial x_i} = \epsilon_{ii}. \end{aligned} \quad (3.238)$$

**Fig. 3.11** Change of volume in the deformation



The change of density can be computed using  $\theta$ . Normally, as the change is small, it is not considered in most elastic problems. However, it will be considered in the elastic deformation equation of the Earth to take into account its effect on gravity.

### 3.4.3 Constitutive Equation of Elastic Bodies: The Stress-Strain Relation

The constitutive equation describes the intrinsic property of a material in response to forcings. The simplest example of elastic bodies is a spring, and its constitutive equation is Hooke's law: the extension of the spring, i.e., the difference between the distended and the relaxed lengths, is proportional to the force pulling on it at its ends. In order for the constitutive equation to be independent of the length of the spring, the extension should be replaced by the rate of extension, i.e., the ratio between the extension and the relaxed length, where the rate of extension can indeed be regarded as the strain. Our purpose at present is to establish a general relation like Hooke's law for elastic bodies, which is material specific, and is independent of the shape and size of the body studied. It can be conceptualized that such a relation for a three dimensional elastic body is between the stress and the strain, which are both local parameters, unlike the extension of a spring, which is the accumulated effect. A generalization of the Hooke's law for springs to general three dimensional bodies is that each component of the stress is a linear combination of all the strain components,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} , \quad (3.239)$$

where according to the quotient theorem,  $C_{ijkl}$  are the components of a fourth order tensor. We mention that this relation is an inference, and it does satisfy the definition of elasticity: When a force is exerted on the body, the body deforms, and when the force is released, the body returns back to its undeformed figure.

The relation (3.239) must obey all laws of physics as constraint on the elastic parameters  $C_{ijkl}$ .

The process of elastic deformation is also a thermodynamic process. Therefore, the relation (3.239) must obey the laws of thermodynamics. Here we discuss only the simpler case of adiabatic deformation, i.e., no heat transfer is involved in the process. Most elastic deformation can be approximately considered as an adiabatic process, where the deformation is so quick and the effect of heat transfer can be neglected. In this case, only the first law of thermodynamics, i.e., the energy conservation, has to be considered. The readers are suggested to consult a more comprehensive book of continuum mechanics for a thorough discussion of thermodynamics.

The first law of thermodynamics for an adiabatic elastic deformation process can be stated as follows: For the material within the spatial domain  $V$  with  $S$  as surface at the time  $t$ , the rate of change of the sum of the internal energy and kinetic energy is equal to the sum of the powers supplied, i.e., the rate of work done by the volume

force exerted within  $V$  and that done by the surface force on  $S$ . This implies that all the work done by the forces is converted to either internal energy or kinetic energy, which can be expressed mathematically as

$$\frac{D}{Dt}(E + K) = W, \quad (3.240)$$

where  $E$  is the internal energy,  $K$ , the kinetic energy, and  $W$ , the power of external forcing. We use the internal energy per unit mass,  $e$ , to describe the distribution of the internal energy. The internal energy  $E$  can be expressed in terms of  $e$  as

$$E = \int_V \rho e dV. \quad (3.241)$$

According to Reynolds transport theorem, its material derivative is

$$\frac{DE}{Dt} = \int_V \rho \frac{De}{Dt} dV. \quad (3.242)$$

The kinematic energy of the material within  $V$  is

$$K = \int_V \frac{1}{2} \rho |\vec{v}|^2 dV = \int_V \frac{1}{2} \rho \vec{v} \cdot \vec{v} dV. \quad (3.243)$$

Its material derivative is

$$\frac{DK}{Dt} = \int_V \frac{1}{2} \rho \frac{D}{Dt} (\vec{v} \cdot \vec{v}) dV = \int_V \rho \vec{a} \cdot \vec{v} dV. \quad (3.244)$$

The power of a force is the scalar product between the force and the velocity. Hence, the total power of the body and surfaces forces is

$$W = \oint_S \vec{\tau}(\hat{n}) \cdot \vec{v} dS + \int_V \vec{f} \cdot \vec{v} dV. \quad (3.245)$$

The surface integral can be transformed to a volume integral according to Gauss' theorem,

$$\begin{aligned} \oint_S \vec{\tau}(\hat{n}) \cdot \vec{v} dS &= \oint_S (\hat{n} \cdot \vec{\sigma}) \cdot \vec{v} dS = \oint_S n_i \sigma_{ij} v_j dS \\ &= \int_V \frac{\partial}{\partial x_i} (\sigma_{ij} v_j) dV = \int_V \left( \sigma_{ij} \frac{\partial v_j}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_i} v_j \right) dV \\ &= \int_V [\vec{\sigma} : \nabla \vec{v} + (\nabla \cdot \vec{\sigma}) \cdot \vec{v}] dV. \end{aligned} \quad (3.246)$$

Hence, we can write the expression of  $W$  as

$$W = \int_V [\bar{\sigma} : \nabla \vec{v} + (\nabla \cdot \bar{\sigma} + \vec{f}) \cdot \vec{v}] dV. \quad (3.247)$$

Finally, by substituting (3.242), (3.244), and (3.247) into (3.240), we obtain

$$\int_V \rho \frac{De}{Dt} dV = \int_V [\bar{\sigma} : \nabla \vec{v} + (-\rho \vec{a} + \nabla \cdot \bar{\sigma} + \vec{f}) \cdot \vec{v}] dV. \quad (3.248)$$

which can be further simplified to, by using the dynamic equation of motion, (3.176),

$$\int_V \rho \frac{De}{Dt} dV = \int_V \bar{\sigma} : \nabla \vec{v} dV. \quad (3.249)$$

As  $V$  arbitrary, we must have

$$\rho \frac{De}{Dt} = \bar{\sigma} : \nabla \vec{v} = \sigma_{ij} \frac{\partial v_j}{\partial x_i}. \quad (3.250)$$

By expression  $v_j$  in terms of  $u_j$ , we obtain

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial}{\partial x_i} \left( \frac{Du_j}{Dt} \right). \quad (3.251)$$

This is the exact form of energy conservation equation assuming the deformation to be adiabatic.

In infinitesimal deformation, as second order terms of infinitesimal quantities can be neglected, we do not need to distinguish between material derivative and partial derivative in infinitesimal terms. Hence,

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial t} \right) = \sigma_{ij} \frac{\partial}{\partial t} \left( \frac{\partial u_j}{\partial x_i} \right) = \sigma_{ij} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right). \quad (3.252)$$

According to the symmetry of  $\sigma_{ij}$  and the definition of the stress, we can write this formula as

$$\rho \frac{De}{Dt} = \frac{1}{2} \sigma_{ij} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = \sigma_{ij} \frac{D\epsilon_{ij}}{Dt}. \quad (3.253)$$

By multiplying  $dt$  at both sides, this formula can be written in an incremental form as

$$de = \frac{1}{\rho} \sigma_{ij} d\epsilon_{ij}. \quad (3.254)$$

As the energy per unit mass is a state parameter, its increment must be the total differential of all the independent variables it depends on. Hence, we can infer from (3.254) that

$$\frac{\partial e}{\partial \epsilon_{ij}} = \frac{1}{\rho} \sigma_{ij} . \quad (3.255)$$

By neglecting the effect of volume dilatation, which is of second order, we obtain

$$\frac{\partial(\rho e)}{\partial \epsilon_{ij}} = \sigma_{ij} , \quad (3.256)$$

where  $\rho e$  is the internal energy per unit volume. By substituting (3.239) into this formula, we obtain

$$\frac{\partial(\rho e)}{\partial \epsilon_{ij}} = C_{ijpq} \epsilon_{pq} . \quad (3.257)$$

By taking the partial derivative with respect to  $\epsilon_{kl}$  at both sides, we obtain

$$C_{ijkl} = \frac{\partial^2(\rho e)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} . \quad (3.258)$$

As switching the order of the partial derivatives with respect to  $\epsilon_{ij}$  and  $\epsilon_{kl}$  does not alter the result, we obtain

$$C_{ijkl} = C_{klij} . \quad (3.259)$$

i.e.,  $C_{ijkl}$  is symmetric with respect to  $ij$  and  $kl$ . This is the thermodynamic constraint to  $C_{ijkl}$ .

The symmetries of  $\sigma_{ij}$  and  $\epsilon_{kl}$  also impose constraints on  $C_{ijkl}$ . According to the symmetry of  $\sigma_{ij}$ , we have

$$\sigma_{ij} = \sigma_{ji} = C_{jikl} \epsilon_{kl} . \quad (3.260)$$

Hence, we can also write the generalized Hooke's law (3.239) as

$$\sigma_{ij} = \frac{1}{2} (C_{ijkl} + C_{jikl}) \epsilon_{kl} . \quad (3.261)$$

Therefore, the elastic parameters  $C_{ijkl}$  can be replaced by  $(1/2)(C_{ijkl} + C_{jikl})$ , the later being symmetrical with respect to  $i$  and  $j$ . We have

$$C_{ijkl} = C_{jikl} . \quad (3.262)$$

Furthermore, according to the symmetry of  $\epsilon_{kl}$ , we have

$$\sigma_{ij} = C_{ijkl}\epsilon_{lk} = C_{ijlk}\epsilon_{kl} . \quad (3.263)$$

Hence,

$$\sigma_{ij} = \frac{1}{2}(C_{ijkl} + C_{jilk})\epsilon_{kl} , \quad (3.264)$$

whence we can infer

$$C_{ijkl} = C_{ijlk} . \quad (3.265)$$

In fact, this symmetry can be derived from the symmetries (3.259) and (3.262).

In summary, among the 81 elastic parameters  $C_{ijkl}$ , only 21 are independent with the symmetries (3.259), (3.262), and (3.265) taken into account. This implies that the most general elastic body has 21 elastic parameters.

Finally, we assume the elastic body to be isotropic, i.e., assuming  $C_{ijkl}$  to be the components of a fourth order isotropic tensor, which is, according to (2.151),

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} . \quad (3.266)$$

Each of (3.259), (3.262), and (3.265) requires

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = \lambda\delta_{ji}\delta_{kl} + \mu\delta_{jk}\delta_{il} + \gamma\delta_{jl}\delta_{ik} , \quad (3.267)$$

which can be simplified to

$$\mu(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) = \gamma(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) . \quad (3.268)$$

We thus obtain

$$\gamma = \mu . \quad (3.269)$$

By substituting this formula back into (3.266), we obtain the elastic parameters  $C_{ijkl}$  for isotropic elastic bodies,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) . \quad (3.270)$$

By substituting (3.270) into (3.239), we obtain the stress-strain relation for isotropic elastic bodies,

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\epsilon_{kl} + \mu(\delta_{ik}\delta_{jl}\epsilon_{kl} + \delta_{il}\delta_{jk}\epsilon_{kl}) = \lambda\delta_{ij}\epsilon_{kk} + \mu(\epsilon_{ij} + \epsilon_{ji}) , \quad (3.271)$$



which can be further simplified to, by taking into account of the symmetry of  $\epsilon_{ij}$ ,

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} . \quad (3.272)$$

The expressions in terms of the components are

$$\begin{cases} \sigma_{11} = \lambda \theta + 2\mu \epsilon_{11} , & \sigma_{12} = 2\mu \epsilon_{12} , \\ \sigma_{22} = \lambda \theta + 2\mu \epsilon_{22} , & \sigma_{23} = 2\mu \epsilon_{23} , \\ \sigma_{33} = \lambda \theta + 2\mu \epsilon_{33} , & \sigma_{31} = 2\mu \epsilon_{31} , \end{cases} \quad (3.273)$$

where  $\theta = \epsilon_{ii}$  is the volume dilatation. We mention again that  $\sigma_{ij}$  and  $\epsilon_{ij}$  are both symmetrical. The parameters  $\lambda$  and  $\mu$  are referred to as Lamé's parameters (coefficients, constants or moduli), and  $\mu$  is also referred to as shear modulus or modulus of rigidity.

The Lamé's first parameter  $\lambda$  is often substituted by another parameter  $k$ . By adding  $\sigma_{11}$ ,  $\sigma_{22}$ , and  $\sigma_{33}$  in (3.273) together, we obtain

$$-3p = \sigma_{11} + \sigma_{22} + \sigma_{33} = (3\lambda + 2\mu)\theta , \quad (3.274)$$

where the first identity is understood as the definition of  $p$ , i.e.,  $p$  is the average of the three normal strain components with sign reversed. We define  $k$  as

$$k = -\frac{p}{\theta} = \lambda + \frac{2}{3}\mu . \quad (3.275)$$

When the three normal strain components are equal,  $p$  is equivalent to a pressure. Hence,  $k$  can be understood as the ratio between the pressure and the consequent volume contraction, i.e., the negative of volume dilatation. The larger  $k$  is, the smaller the volume contraction is. Hence,  $k$  is referred to as incompressibility or bulk modulus.

The stress-strain relation (3.272) can be decomposed into the relations between the spherical part and the deviatoric part of the two tensors using  $k$  and  $\mu$ :

$$\sigma_{ij}^S = 3k\epsilon_{ij}^S , \quad \sigma_{ij}^D = 2\mu\epsilon_{ij}^D ; \quad (3.276)$$

where the spherical and deviatoric parts of the stress and strain are defined as

$$\sigma_{ij}^S = \frac{1}{3}\sigma_{kk}\delta_{ij} , \quad \epsilon_{ij}^S = \frac{1}{3}\epsilon_{kk}\delta_{ij} ; \quad (3.277)$$

$$\sigma_{ij}^D = \sigma_{ij} - \sigma_{ij}^S , \quad \epsilon_{ij}^D = \epsilon_{ij} - \epsilon_{ij}^S . \quad (3.278)$$

Finally, we substitute (3.210) into (3.272) for a stress-displacement relation,

$$\sigma_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) , \quad (3.279)$$

which can be written in tensor form as

$$\bar{\sigma} = \lambda \nabla \cdot \vec{u} \bar{I} + \mu [\nabla \vec{u} + (\nabla \vec{u})^T]. \quad (3.280)$$

It is left to the readers to derive the formulae of spherical and deviatoric stresses expressed in terms of displacement.

In fluids, we have  $\mu = 0$ . Stated differently,  $\mu = 0$  could be understood as the definition of fluids: In a continuous medium, if the shear stress always vanishes in standstill state, the medium is a fluid.

### 3.4.4 A General Form of the Mathematical Equations of Elasticity Problems

Up to present, we derived all equations necessary to study the deformation of elastic bodies. The dynamic equation independent of the property of the body is (3.176), or (3.177) in component form. Here we write them as

$$\rho \frac{D^2 \vec{u}}{Dt^2} = \nabla \cdot \bar{\sigma} + \vec{f}, \quad (3.281)$$

$$\rho \frac{D^2 u_j}{Dt^2} = \frac{\partial \sigma_{ij}}{\partial x_i} + f_j. \quad (3.282)$$

The constitutive equation describing the property of the body is (3.280), or (3.279) in component form. Here we write them as

$$\bar{\sigma} = \lambda \nabla \cdot \vec{u} \bar{I} + \mu [\nabla \vec{u} + (\nabla \vec{u})^T], \quad (3.283)$$

$$\sigma_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \quad (3.284)$$

The symmetry of  $\bar{\sigma}$ , or  $\sigma_{ij}$  in component form, is implicitly included in the last two equation.

When the above differential equations are solved for studying the deformation of an elastic body, boundary conditions have to be imposed. In elasticity, two kinds of boundary conditions, displacement and stress, are involved mostly. As a general description, we divide the boundary into two parts,

$$S = S_u + S_\sigma. \quad (3.285)$$

On  $S_u$ , displacement condition is specified,

$$\vec{u}_{S_u} = \vec{u}^0, \quad (3.286)$$

where  $\vec{u}^0 = u_j^0 \hat{e}_j$  is the known displacement of the boundary. The component form of the above formula is

$$(u_j)_{S_u} = u_j^0. \quad (3.287)$$

On  $S_\sigma$ , stress condition is imposed,

$$(\hat{n} \cdot \bar{\sigma})_{S_\sigma} = \vec{t}^0, \quad (3.288)$$

where  $\vec{t}^0 = t_j^0 \hat{e}_j$  is the known traction exerted on the surface. The component form of the above formula is

$$(n_i \sigma_{ij})_{S_\sigma} = t_j^0. \quad (3.289)$$

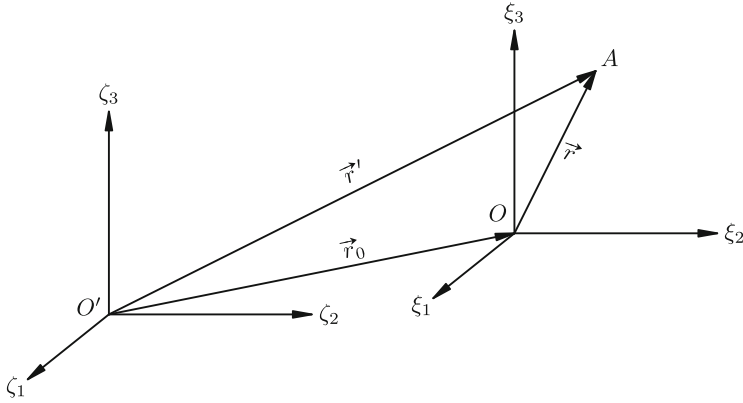
We mention that all the above formulae are derived based on inferences. The generalization of the dynamic equations of particles inferred from results of experiments to the case of continua is an inference; and the generalization of Hooke's law of springs inferred from results of experiments to the case of continua is also an inference, and is made in agreement with dynamic and thermodynamic equations. These inferences ensured that the formulae derived out are in agreement with known physical laws. However, their correctness should still be verified based on experiments done with real materials. Naturally, experiments show that they do apply to a wide variety of materials.

### 3.5 Governing Equations of the Rotation of the Earth

#### 3.5.1 Angular Momentum Equation with Respect to the Center of Mass

The motion of the Earth can be separated into three parts: the translation of its center of mass, the rotation around the center of mass, and a deformation superimposed on the rotation. The first step of the formulation is the separation between the dynamic equation of motion of the center of mass and the angular momentum equation with respect to the center of mass. Further study of the former is mostly a topic of celestial mechanics, and that of the latter, celestial mechanics, astronomy, geophysics and geodesy, which are largely overlapping disciplines. Therefore, the latter is to be elaborated further.

As shown in Fig. 3.12, let  $O'\zeta_1\zeta_2\zeta_3$  be an inertial reference system,  $O$ , the center of mass of the Earth, and  $O\xi_1\xi_2\xi_3$ , a reference system purely translating with respect to  $O'\zeta_1\zeta_2\zeta_3$  while keeping its axes parallel to those of  $O'\zeta_1\zeta_2\zeta_3$ . The system  $O\xi_1\xi_2\xi_3$  is in fact the geocentric celestial reference system (GCRS), which is also referred to as geocentric inertial reference system, as its axes are always kept parallel to those of the inertial reference system  $O'\zeta_1\zeta_2\zeta_3$ , and its origin, on the center of mass of the Earth. Strictly speaking,  $O\xi_1\xi_2\xi_3$  is not really an inertial



**Fig. 3.12** Translational motion of the Earth

reference system, as the motion of its origin with respect to the inertial reference system includes an acceleration.

The velocity of a material element located at the point  $A$  at the time  $t$  with respect to the inertial reference system  $O'\xi_1\xi_2\xi_3$  can be written as

$$\vec{v}_{\text{in}} = \left( \frac{D\vec{r}'}{Dt} \right)_{\text{in}} = \left( \frac{D\vec{r}}{Dt} \right)_{\text{in}} + \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}} = \vec{v}_{\text{gi}} + \vec{v}_0. \quad (3.290)$$

where  $\vec{v}_{\text{gi}} = (D\vec{r}/Dt)_{\text{in}}$  is the velocity of the material element with respect to the geocentric inertial system  $O\xi_1\xi_2\xi_3$ , and  $\vec{v}_0 = (d\vec{r}_0/dt)_{\text{in}}$ , the velocity of the center of mass of the Earth with respect to the inertial reference system  $O'\xi_1\xi_2\xi_3$ . By applying the linear momentum and angular momentum equations (3.141) and (3.142) to the whole Earth, we obtain

$$\frac{D}{Dt} \int_V \rho \left[ \left( \frac{D\vec{r}'}{Dt} \right)_{\text{in}} + \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}} \right] dV = \oint_S \vec{t} dS + \int_V \vec{f} dV, \quad (3.291)$$

$$\frac{D}{Dt} \int_V \rho \vec{r}' \times \left[ \left( \frac{D\vec{r}'}{Dt} \right)_{\text{in}} + \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}} \right] dV = \oint_S \vec{r}' \times \vec{t} dS + \int_V \vec{r}' \times \vec{f} dV. \quad (3.292)$$

We should keep in mind that (1) as  $V$  is the whole Earth,  $D/Dt$  can be replaced by  $d/dt$ , as both derivatives have the same meaning in this circumstance; (2) both equations hold in the inertial reference system.

As  $O$  is the center of mass of the Earth, we have

$$\int_V \rho \vec{r} dV = 0. \quad (3.293)$$

Hence,

$$\int_V \rho \left( \frac{D\vec{r}}{Dt} \right)_{\text{in}} dV = \left( \frac{D}{Dt} \int_V \rho \vec{r} dV \right)_{\text{in}} = 0. \quad (3.294)$$

By substituting this formula into (3.291), we obtain

$$\frac{D}{Dt} \int_V \rho \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}} dV = \oint_S \vec{r} dS + \int_V \vec{f} dV. \quad (3.295)$$

As  $\vec{v}_0 = (D\vec{r}_0/Dt)_{\text{in}}$  is independent of the coordinates of integration, and  $M = \int_V \rho dV$  is the total mass of the Earth, this equation can be simplified to

$$\frac{d}{dt} (M \vec{v}_0) = \oint_S \vec{r} dS + \int_V \vec{f} dV = \vec{F}, \quad (3.296)$$

where we have replaced  $D/Dt$  by  $d/dt$  at the left-hand side, as they have the same meaning for the present case, and  $\vec{F}$  is the total force exerted on the whole Earth. This equation is the dynamic equation of motion of the center of mass of the Earth. It includes only three parameters: the total mass of the Earth  $M$ , the velocity of the center of mass of the Earth with respect to the inertial space  $\vec{v}_0$ , and the total force exerted on the whole Earth  $\vec{F}$ . Its form is identical to Newton's second law of motion for particles.

By performing the vector product of  $\vec{r}_0$  with both sides of (3.291), and then subtracting the result from (3.292), we obtain, with  $\vec{r}' = \vec{r} + \vec{r}_0$ ,

$$\frac{D}{Dt} \int_V \rho \vec{r} \times \left[ \left( \frac{D\vec{r}}{Dt} \right)_{\text{in}} + \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}} \right] dV = \oint_S \vec{r} \times \vec{r} dS + \int_V \vec{r} \times \vec{f} dV. \quad (3.297)$$

As  $O$  is the center of mass of the Earth, we have, according to (3.293),

$$\int_V \rho \vec{r} \times \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}} dV = \left( \int_V \rho \vec{r} dV \right) \times \left( \frac{d\vec{r}_0}{dt} \right)_{\text{in}} = 0. \quad (3.298)$$

Hence, (3.297) can be simplified to

$$\frac{d}{dt} \int_V \rho \vec{r} \times \vec{v}_{\text{gi}} dV = \oint_S \vec{r} \times \vec{r} dS + \int_V \vec{r} \times \vec{f} dV, \quad (3.299)$$

where we have replaced  $D/Dt$  by  $d/dt$  at the left-hand side as in (3.296), and  $\vec{v}_{\text{gi}} = (D\vec{r}/Dt)_{\text{in}}$  is the velocity of the material element with respect to the geocentric inertial reference system  $O\xi_1\xi_2\xi_3$ .

We use  $\vec{L}$  to denote the angular momentum of the Earth around the center of mass of the Earth (the position vector in the equation below is defined with respect to this point) with respect to the geocentric inertial reference system  $O\xi_1\xi_2\xi_3$  (the

velocity  $\vec{v}_{gi}$  in the following equation is defined to be with respect to this system),

$$\vec{L} = \int_V \rho \vec{r} \times \vec{v}_{gi} dV, \quad (3.300)$$

and use  $\vec{N}$  to denote the total torque exerted on the whole Earth,

$$\vec{N} = \oint_S \vec{r} \times \vec{\tau} dS + \int_V \vec{r} \times \vec{f} dV, \quad (3.301)$$

we can write (3.299) as

$$\left( \frac{d\vec{L}}{dt} \right)_{in} = \vec{N}, \quad (3.302)$$

where we added the subscript ‘in’ to indicate that the directions of the axes of the geocentric inertial reference system  $O\xi_1\xi_2\xi_3$  are treated as time-independent in the derivative, i.e.,  $d\vec{L}$  is expressed in  $O\xi_1\xi_2\xi_3$ , and the derivative  $d/dt$  applies to the components alone, which follows most apparently from (3.299).

If  $d\vec{L}$  is expressed in a coordinate system  $Ox_1x_2x_3$  rotating with respect to  $O\xi_1\xi_2\xi_3$  with an angular velocity  $\vec{\omega}$ , we obtain

$$\left( \frac{d\vec{L}}{dt} \right)_{rot} + \vec{\omega} \times \vec{L} = \vec{N}. \quad (3.303)$$

where the subscript ‘rot’ indicates that the directions of the axes of the rotating coordinate system  $Ox_1x_2x_3$  are treated as time-independent in the derivative, i.e.,  $L$  is expressed in  $Ox_1x_2x_3$ , and the derivative  $d/dt$  applies to the components alone.

It must be kept in mind that, in (3.303),  $\vec{L}$  and  $\vec{N}$  are defined in the geocentric inertial reference system  $O\xi_1\xi_2\xi_3$  according to (3.300) and (3.301), i.e.,  $O\xi_1\xi_2\xi_3$  is the reference system, but they are expressed in the rotating coordinate system  $Ox_1x_2x_3$ , i.e., their components are given as projections along the axes of  $Ox_1x_2x_3$ .

In summary, (3.296) is the dynamic equation for studying the motion of the center of mass of the Earth, i.e., the translational motion of the Earth, and (3.303) is the dynamic equation for studying the rotation of the Earth with respect to the inertial space. They can be treated independently.

### 3.5.2 Governing Equations of the Rotation of Rigid Earth Models

As the simplest circumstance, we assume the Earth to be rigid, and the coordinate system  $Ox_1x_2x_3$  is fixed on the Earth. We then know that  $\vec{\omega}$  is the angular velocity of the Earth with respect to the inertial space. The velocity of any point within the

Earth with respect to the geocentric inertial reference system is

$$\vec{v}_{gi} = \vec{\omega} \times \vec{r} . \quad (3.304)$$

By substituting this formula into (3.300), and then expressing the angular velocity  $\vec{\omega}$  in the Earth-fixed coordinate system  $Ox_1x_2x_3$ ,  $\vec{\omega} = \omega_i \hat{e}_i$ , where  $\omega_i$  is the projection of  $\vec{\omega}$  along  $x_i$ , we obtain

$$\begin{aligned} L &= \int_V \rho \vec{r} \times (\vec{\omega} \times \vec{r}) dV \\ &= \int_V \rho [(\vec{r} \cdot \vec{r}) \vec{\omega} - (\vec{r} \cdot \vec{\omega}) \vec{r}] dV \\ &= \int_V \rho (x_k x_k \omega_j \hat{e}_j - x_j \omega_j x_i \hat{e}_i) dV \\ &= \int_V \rho (\delta_{ij} x_k x_k \omega_j \hat{e}_i - x_i x_j \omega_j \hat{e}_i) dV \\ &= \left[ \int_V \rho (\delta_{ij} x_k x_k - x_i x_j) dV \right] \omega_j \hat{e}_i . \end{aligned} \quad (3.305)$$

Define

$$C_{ij} = \int_V \rho (\delta_{ij} x_k x_k - x_i x_j) dV . \quad (3.306)$$

We can write  $\vec{L}$  as

$$\vec{L} = C_{ij} \omega_j \hat{e}_i . \quad (3.307)$$

It can be readily verified that  $C_{ij}$  are the components of a second order tensor, called moment of inertia tensor, which is symmetrical. The components with  $i = j$  are called moments of inertia, and those with  $i \neq j$ , products of inertia. We can also write (3.306) and (3.307) in tensor and vector forms as

$$\bar{C} = \int_V \rho [(\vec{r} \cdot \vec{r}) \bar{I} - \vec{r} \vec{r}] dV , \quad (3.308)$$

$$\vec{L} = \bar{C} \cdot \vec{\omega} . \quad (3.309)$$

The angular momentum equation (3.303) can then be written as

$$\bar{C} \cdot \frac{d\vec{\omega}}{dt} + \vec{\omega} \times (\bar{C} \cdot \vec{\omega}) = \vec{N} . \quad (3.310)$$

We remark that the subscript ‘rot’ is dropped, but the meaning of the derivative remains to be the same as in (3.303), i.e., the directions of the axes of the coordinate system  $Ox_1x_2x_3$  are treated as time-independent. This will be understood as default in the rest of this subsection. The component form of (3.310) is

$$C_{ij} \frac{d\omega_j}{dt} + \epsilon_{ijk} C_{kl} \omega_j \omega_l = N_i. \quad (3.311)$$

Equation (3.310) or (3.311) is referred to as Euler’s dynamic equation, which the rotation of rigid bodies obeys.

As the moment of inertia tensor is symmetric, we choose its principal axes as  $x_1$ ,  $x_2$ , and  $x_3$ , and denote the three principal values as  $A$ ,  $B$ , and  $C$ . We have  $C_{11} = A$ ,  $C_{22} = B$ ,  $C_{33} = C$ , and  $C_{ij} = 0$  when  $i \neq j$ . In this simpler case, Euler’s dynamic equation (3.311) is simplified to

$$\begin{cases} A \frac{d\omega_1}{dt} + (C - B)\omega_2\omega_3 = N_1, \\ B \frac{d\omega_2}{dt} + (A - C)\omega_3\omega_1 = N_2, \\ C \frac{d\omega_3}{dt} + (B - A)\omega_1\omega_2 = N_3. \end{cases} \quad (3.312)$$

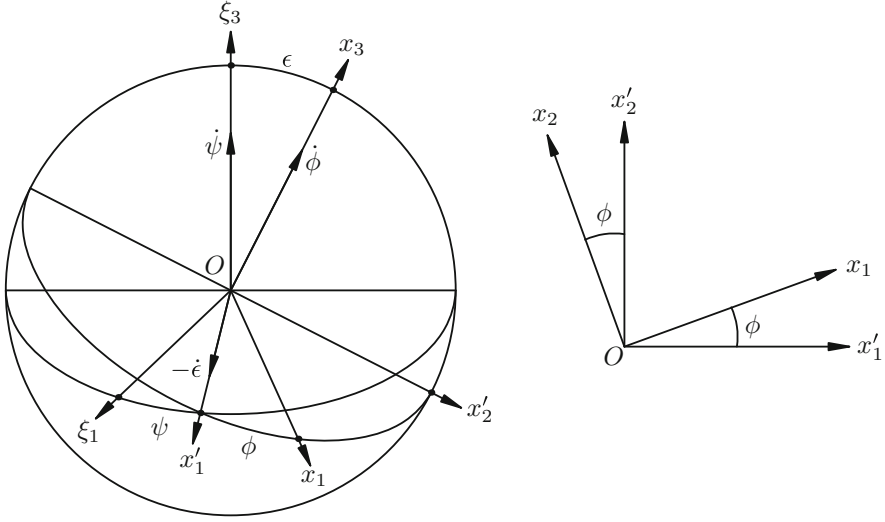
It must be noted that  $\omega_i$  is the component of the angular velocity of the Earth fixed system  $Ox_1x_2x_3$  with respect to the geocentric inertial reference system  $O\xi_1\xi_2\xi_3$  along the  $x_i$ -axis.

In the study of Earth rotation,  $A = B$  is a very good approximation, and Euler’s dynamic equations can be further simplified to

$$\begin{cases} A \frac{d\omega_1}{dt} + (C - A)\omega_2\omega_3 = N_1, \\ A \frac{d\omega_2}{dt} - (C - A)\omega_3\omega_1 = N_2, \\ C \frac{d\omega_3}{dt} = N_3. \end{cases} \quad (3.313)$$

Since the coordinate system  $Ox_1x_2x_3$  is fixed on the Earth, as far as the directions of its three axes with respect to  $O\xi_1\xi_2\xi_3$  are known, the orientation of the Earth in the space is known. Three angles,  $\phi$ ,  $\psi$  and  $\epsilon$ , as shown in Fig. 3.13, are used to describe the orientation of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$ , called Euler angles. In order to interpret the three Euler angles, we define an auxiliary rectangular coordinate system  $Ox'_1x'_2x'_3$ , with  $x'_1$  along the intersection line between the coordinate planes  $x_1x_2$  and  $\xi_1\xi_2$ . The orientation of  $Ox_1x_2x_3$  with respect to  $Ox'_1x'_2x'_3$  is then completely determined by the angle between the  $x_1$ - and  $x'_1$ -axes, which is denoted as  $\phi$ . The orientation of  $Ox'_1x'_2x'_3$  with respect to  $O\xi_1\xi_2\xi_3$  is determined by the angle between the  $x'_1$ - and  $\xi_1$ -axes, which is denoted as  $\psi$ , and the angle between the  $x_3$ - and  $\xi_3$ -axes, which is denoted as  $\epsilon$ . These show that the three





**Fig. 3.13** Definition of Euler angles

Euler angles completely define the orientation of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$ , and consequently, the components of the angular velocity of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$  can be expressed in terms of the three Euler angles.

We use an over-dot to denote the derivative with respect to time for the Euler angles. The angular velocity of  $Ox_1x_2x_3$  with respect to  $Ox'_1x'_2x_3$  can be inferred from Fig. 3.13 to be  $\dot{\phi}\hat{e}_3$ , and the angular velocity of  $Ox'_1x'_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$ ,  $\dot{\psi}\hat{e}_{\xi_3} - \dot{\epsilon}\hat{e}_{x'_1}$ , where  $\hat{e}_{\xi_3}$  and  $\hat{e}_{x'_1}$  are the base vectors along  $\xi_3$  and  $x'_1$ , respectively. Finally, we obtain the angular velocity of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$ , which is the sum of the two above,

$$\vec{\omega} = \dot{\psi}\hat{e}_{\xi_3} - \dot{\epsilon}\hat{e}_{x'_1} + \dot{\phi}\hat{e}_3. \quad (3.314)$$

The formula for each component can be inferred from the relations among the coordinate systems based on Fig. 3.13,

$$\begin{cases} \omega_1 = -\dot{\psi} \sin \epsilon \sin \phi - \dot{\epsilon} \cos \phi, \\ \omega_2 = -\dot{\psi} \sin \epsilon \cos \phi + \dot{\epsilon} \sin \phi, \\ \omega_3 = \dot{\psi} \cos \epsilon + \dot{\phi}. \end{cases} \quad (3.315)$$

This system of equations is referred to as Euler's kinematic equations.

The two systems of equations, Euler's dynamic equations (3.312) or (3.313) and Euler's kinematic equations (3.315) are the basic equations governing the rotation of rigid bodies.

### 3.5.3 Governing Equations of Rotation of Deformable Earth Models

The problem becomes much more complicated when the Earth deforms. In this circumstance, the coordinate system  $Ox_1x_2x_3$  can no longer be fixed on the deformable Earth; the motion of the material within the Earth with respect to  $Ox_1x_2x_3$  is unavoidable. Therefore, there is no longer any absolute criteria for the choice of  $Ox_1x_2x_3$ . We start by assuming that  $Ox_1x_2x_3$  is an arbitrarily chosen coordinate system rotating with respect to  $O\xi_1\xi_2\xi_3$  with an angular velocity  $\vec{\omega}$ . In such a case, according to (3.49), the velocity of a material element within the Earth with respect to the geocentric inertial reference system  $O\xi_1\xi_2\xi_3$  can be expressed as

$$\vec{v}_{gi} = \vec{\omega} \times \vec{r} + \vec{v}, \quad (3.316)$$

where  $\vec{v}$  is the velocity of the material element with respect to  $Ox_1x_2x_3$ . The substitution of this formula into (3.300) yields an expression of the angular momentum of the Earth,

$$\vec{L} = \int_V \rho \vec{r} \times (\vec{\omega} \times \vec{r}) dV + \int_V \rho \vec{r} \times \vec{v} dV. \quad (3.317)$$

Evidently, the first term has the same form as the angular momentum of a rigid body, (3.305), and can be reduced to the form of (3.307) or (3.309). Hence, (3.317) can be written as

$$\vec{L} = \bar{C} \cdot \vec{\omega} + \vec{h}, \quad (3.318)$$

where  $\vec{h}$  is the second term in (3.317),

$$\vec{h} = \int_V \rho \vec{r} \times \vec{v} dV. \quad (3.319)$$

The substitution of (3.318) into the angular momentum equation (3.303) yields the dynamic equation of rotation of deformable bodies,

$$\frac{d}{dt}(\bar{C} \cdot \vec{\omega} + \vec{h}) + \vec{\omega} \times (\bar{C} \cdot \vec{\omega} + \vec{h}) = \vec{N}, \quad (3.320)$$

called Euler-Liouville equation. It is mentioned that  $\bar{C}$ ,  $\vec{\omega}$ , and  $\vec{h}$  are all time-dependent. The meaning of the derivative of  $\bar{C} \cdot \vec{\omega} + \vec{h}$  with respect to  $t$  is the same as that in (3.303) with the subscript ‘rot’ dropped here. With

$$\bar{C} \cdot \vec{\omega} = C_{ij} \omega_j \hat{e}_i, \quad (3.321)$$

$$\vec{h} = h_i \hat{e}_i = \left( \int_V \rho x_j v_k \epsilon_{ijk} dV \right) \hat{e}_i, \quad (3.322)$$

the Euler-Liouville equation can be written in terms of the components as

$$\frac{d}{dt}(C_{ij}\omega_j + h_i) + \epsilon_{ijk}(C_{kl}\omega_l + h_k)\omega_j = N_i, \quad (3.323)$$

and further

$$C_{ij}\frac{d\omega_j}{dt} + \frac{dC_{ij}}{dt}\omega_j + \frac{dh_i}{dt} + \epsilon_{ijk}C_{kl}\omega_j\omega_l + \epsilon_{ijk}h_k\omega_j = N_i. \quad (3.324)$$

Even if we can chose the principal axes in the undeformed state, the moment of inertia tensor would still vary with time. Therefore, the principal axes also vary with time.

As already mentioned, due to the deformation of the Earth with time, the rotating system  $Ox_1x_2x_3$  cannot be fixed on the Earth. This allows some room of flexibility for its choice. We now define a vector  $\vec{\beta}$ , such that

$$\vec{h} = \vec{C} \cdot \vec{\beta}. \quad (3.325)$$

In this way, (3.318) can be written as  $\vec{L} = \vec{C} \cdot (\vec{\omega} + \vec{\beta})$ . Hence, while defining  $Ox_1x_2x_3$ , we can attribute  $\vec{\beta}$  into the angular velocity of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$ ,  $\vec{\omega}$ , such that  $\vec{\beta} = 0$  for the  $Ox_1x_2x_3$  chosen. In the coordinate system  $Ox_1x_2x_3$  defined in this way, the Euler-Liouville equation no longer has the term  $\vec{h}$  or  $h_i$ , and (3.320) and (3.324) are simplified to

$$\frac{d}{dt}(\vec{C} \cdot \vec{\omega}) + \vec{\omega} \times (\vec{C} \cdot \vec{\omega}) = \vec{N}, \quad (3.326)$$

$$C_{ij}\frac{d\omega_j}{dt} + \frac{dC_{ij}}{dt}\omega_j + \epsilon_{ijk}C_{kl}\omega_j\omega_l = N_i. \quad (3.327)$$

Although these equations no longer explicitly include velocity, they still explicitly include the variation of  $C_{ij}$  related to the time-dependent deformation of the Earth.

The coordinate system  $Ox_1x_2x_3$  rendering  $\vec{\beta} = 0$  is referred to as Tisserand frame, and its axes, Tisserand axes. By comparing (3.319) and (3.325), and then taking reference to (3.309) and (3.305), we obtain

$$\vec{h} = \int_V \rho \vec{r} \times \vec{v} dV = \vec{C} \cdot \vec{\beta} = \int_V \rho \vec{r} \times (\vec{\beta} \times \vec{r}) dV. \quad (3.328)$$

By comparing with (3.305), we see that  $\vec{\beta}$  behaves like an angular velocity that represents an average of the velocity field  $\vec{v}$ . In the Tisserand frame,  $\vec{\beta} = 0$  implies that the velocity field  $\vec{v}$  is a kind of pure deformation without any rotation of the Earth as a whole, which can be understood as a physical interpretation of the Tisserand frame.

Finally, we mention that, if the Tisserand frame is chosen as the Earth-fixed coordinate system  $Ox_1x_2x_3$ , the velocity representing the deformation no longer appear explicitly in the dynamic equation of rotation, it influences Earth rotation

only implicitly through its influence on the moment of inertia tensor. This is equivalent to have separated deformation and rotation in the formulation. We also mention that Euler's kinematic equations (3.315) are also required in the study of Earth rotation using the Euler-Liouville equation.

We have thus separated the motion of the material inside the Earth into three components: the motion of the Earth as a whole represented by the motion of center of mass of the Earth in the inertial space, the rotation of the Earth as a whole represented by the rotation of the Tisserand frame around the center of mass of the Earth, and the deformation of the Earth with respect to the Tisserand frame. The former two have been formulated in this section, and the last is to be formulated in the rest of this chapter assuming the deformation to be elastic.

### 3.6 Governing Equations of Infinitesimal Deformation of Elastic Earth Models

#### 3.6.1 Inclusion of Rotation and Prestress in Continuum Mechanics

The dynamic equation of motion of continua derived in Sect. 3.3 has two limitations for application to the Earth: (1) it holds only in the inertial reference system; (2) the stress is assumed to vanish at the undeformed state. These two factors must be considered for the Earth. Hence, the result formulated in Sect. 3.3 must be expressed in a coordinate system that is translating and rotating with respect to the inertial reference system, and the prestress must also be considered.

We first consider the effect of translation and rotation. For clarity in subsequent derivation, we write the dynamic equation of elastic deformation in the inertial reference system as

$$\rho \vec{a}_{\text{in}} = \nabla \cdot \vec{\sigma} + \vec{f}_{\text{bf}}, \quad (3.329)$$

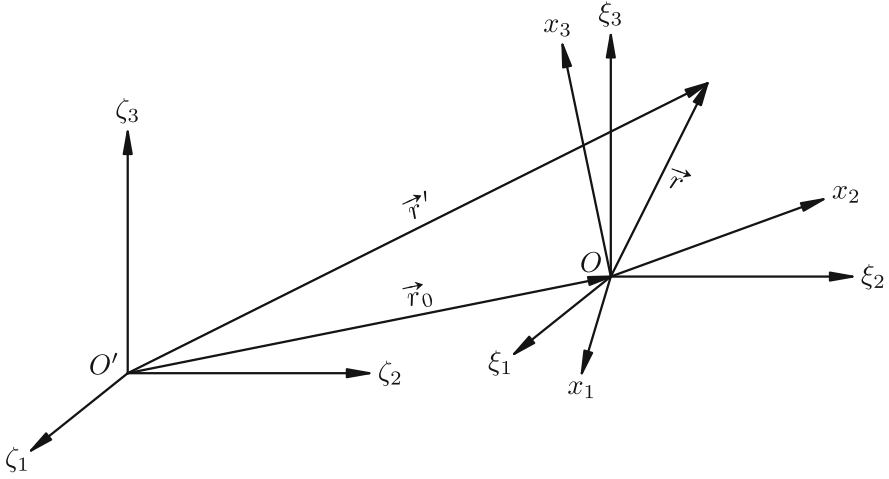
where the symbol  $\vec{f}_{\text{bf}}$  represents the total body force density.

As shown in Fig. 3.14, let  $O'\zeta_1\zeta_2\zeta_3$  be an inertial reference system,  $O$ , the center of mass of the Earth,  $O\xi_1\xi_2\xi_3$ , the geocentric inertial reference system translating with respect to  $O'\zeta_1\zeta_2\zeta_3$ , and  $Ox_1x_2x_3$ , a terrestrial reference system rotating with respect to  $O\xi_1\xi_2\xi_3$  with an angular velocity  $\vec{\omega}$ . Our purpose is to generalize the dynamic equation of elastic deformation established in  $O'\zeta_1\zeta_2\zeta_3$ , (3.329), into  $Ox_1x_2x_3$ . Evidently, we have

$$\vec{r}' = \vec{r} + \vec{r}_0. \quad (3.330)$$

By taking reference to (3.47) and (3.50), we can express  $\vec{a}_{\text{in}} = (D^2\vec{r}'/Dt^2)_{\text{in}}$  as

$$\vec{a}_{\text{in}} = \left( \frac{D^2\vec{r}}{Dt^2} \right)_{\text{rot}} + 2\vec{\omega} \times \left( \frac{D\vec{r}}{Dt} \right)_{\text{rot}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \left( \frac{D\vec{\omega}}{Dt} \right)_{\text{in}} \times \vec{r} + \left( \frac{D^2\vec{r}_0}{Dt^2} \right)_{\text{in}}, \quad (3.331)$$



**Fig. 3.14** Relation between the inertial and Earth-fixed reference systems

where the subscript ‘rot’ indicates that the unit base vectors of  $Ox_1x_2x_3$  are treated time-independent in the derivative. Neglecting the temporal variation of  $\vec{\omega}$ , the substitution of (3.331) into (3.329) leads to

$$\rho \left( \frac{D^2 \vec{r}}{Dt^2} + 2\vec{\omega} \times \frac{D\vec{r}}{Dt} \right) = \nabla \cdot \vec{\sigma} + \vec{f}_{\text{bf}} - \rho \vec{\omega} \times (\vec{\omega} \times \vec{r}) - \rho \vec{a}_0, \quad (3.332)$$

where  $\vec{a}_0 = (d^2 \vec{r}_0 / dt^2)_{\text{in}}$  is the acceleration of the center of mass of the Earth,  $O$ , with respect to the inertia reference system. We mention that in the material derivatives, the subscript ‘rot’ has been omitted, but their meanings remain the same, i.e., the unit base vectors of  $Ox_1x_2x_3$  are treated time-independent, which is understood as the default hereafter.

Equation (3.332) is the dynamic equation of elastic deformation in  $Ox_1x_2x_3$ , which is a generalization of that in  $O'\xi_1\xi_2\xi_3$ , (3.329); the former degenerates to the latter if  $\vec{\omega} = 0$  and  $\vec{a}_0 = 0$ . We emphasize again that, in the derivative with respect to time in (3.332), the direction of the axes of  $Ox_1x_2x_3$  is assumed time-independent.

We now apply (3.332) to the Earth. As the Earth is self-gravitating, we express  $\vec{f}_{\text{bf}}$  as the sum of the Earth’s self-gravitational attraction and an external force density. We denote the gravitational potential of the Earth as  $\phi$  and the external body force density as  $\vec{f}_e$ . We then have  $\vec{f}_{\text{bf}} = \rho \nabla \phi + \vec{f}_e$ . We further denote the centrifugal potential as  $\psi$ . The centrifugal acceleration  $-\vec{\omega} \times (\vec{\omega} \times \vec{r})$  can then be expressed as  $\nabla \psi$ . In this way, we can write (3.332) as

$$\rho \left( \frac{D^2 \vec{r}}{Dt^2} + 2\vec{\omega} \times \frac{D\vec{r}}{Dt} \right) = \nabla \cdot \vec{\sigma} + \rho \nabla (\phi + \psi) + \vec{f}, \quad (3.333)$$

where  $\vec{f}$  is the sum of the external body force density and the inertial force density associated with the acceleration of the center of mass of the Earth with respect to the inertial reference system,

$$\vec{f} = \vec{f}_e - \rho \vec{a}_0. \quad (3.334)$$

We now consider the prestress. As can be seen from (3.333), the stress  $\bar{\sigma}$  does not vanish when there is no deformation, i.e., when  $D^2\vec{r}/Dt^2 = 0$  and  $D\vec{r}/Dt = 0$ , and meanwhile, the external force  $\vec{f}_e$  and the inertial force  $-\rho\vec{a}_0$  are exactly balanced, i.e.,  $\vec{f} = 0$ . This stress is called prestress, and we denote it as  $\bar{T}_0$ , which, according to (3.333), satisfies the equation

$$\nabla \cdot \bar{T}_0 = -\rho_0 \nabla(\phi_0 + \psi), \quad (3.335)$$

where  $\rho_0$  and  $\phi_0$  are the density and gravitational potential at the undeformed state. Naturally, the deformation is superimposed over the undeformed state. We still use  $\vec{R}$  and  $X_i$  to denote the position vector and coordinates, respectively, in the Lagrangian description, i.e., in the undeformed state. The displacement is then

$$\vec{u} = \vec{r} - \vec{R}. \quad (3.336)$$

The substitution of  $\vec{r}$  by  $\vec{u}$  in (3.333) yields

$$\rho \left( \frac{D^2\vec{u}}{Dt^2} + 2\vec{\omega} \times \frac{D\vec{u}}{Dt} \right) = \nabla \cdot \bar{\sigma} + \rho \nabla(\phi + \psi) + \vec{f}. \quad (3.337)$$

where  $\rho$  and  $\phi$  have the same meanings as in (3.333), i.e., the instantaneous density and gravitational potential in the deformed state.

The equation for the prestress, (3.335), and that for the displacement, (3.337), are the governing equations for the elastic deformation of the Earth with translation, rotation and prestress considered. Poisson's equation relating the gravitational potential to the density should also be solved together. In the next subsection, we will linearize these equations to obtain a system of equations for the displacement as well as the increments of stress, density and gravitational potential as compared to the undeformed state.

### 3.6.2 Linearization of the Equations to Practical Forms

The context of this subsection is to linearize the elastic equation of the Earth, or to reduce it to a system of equations for the displacement as well as the increments of stress, density and gravitational potential as compared to the undeformed state.

In subsequent derivations, we will need the relation between the values of a function at the position vectors  $\vec{R}$  and  $\vec{r}$ . For brevity, we make the following

convention: the value at  $\vec{R}$  is explicitly expressed, while the value at  $\vec{r}$  is implicitly understood as default. Taking for example a function  $Y$ , its value at  $\vec{R}$  is denoted as  $Y(\vec{R})$ , and  $Y$  is implicitly understood as its value at  $\vec{r}$ . The difference between  $Y$  and  $Y(\vec{R})$  can be expressed using the displacement  $u_k = x_k - X_k$  as

$$Y - Y(\vec{R}) = \frac{\partial Y}{\partial x_k} u_k . \quad (3.338)$$

This is a relation based on the Eulerian description accurate to the first order of displacement, and will be used multiple times.

In the study of the Earth's elastic deformation, the Lagrangian increment of the stress denoted as  $\bar{T}$  and the Eulerian increment of the gravitational potential denoted as  $\phi_1$  are used. The Lagrangian increment of the stress is the increment of the instantaneous stress of a material element at the instantaneous position vector  $\vec{r}$  with respect to that of the same material element at its position vector  $\vec{R}$  in the undeformed state,

$$\bar{\sigma} = \bar{T}_0(\vec{R}) + \bar{T} . \quad (3.339)$$

Its component form is

$$\sigma_{ij} = T_{0ij}(\vec{R}) + T_{ij} . \quad (3.340)$$

The use of the Lagrangian increment of the stress for the prestressed Earth may probably be made easier to understand by comparing with a spring already distended: When an extra pulling force is applied at its ends, an extra extension happens with the total pulling force acting at its ends being the sum of the original one and the extra one. The Eulerian increment of the gravitational potential is the difference between the instantaneous value and that in the undeformed state at the same position vector  $\vec{r}$ ,

$$\phi = \phi_0 + \phi_1 . \quad (3.341)$$

The change of density should also be considered. We denote its Lagrangian and Eulerian increments as  $\delta\rho$  and  $\rho_1$ , respectively. We have, by definition,

$$\rho = \rho_0(\vec{R}) + \delta\rho , \quad (3.342)$$

$$\rho = \rho_0 + \rho_1 . \quad (3.343)$$

In subsequent formulations, we will always accurate to the first order of the increments.

We first derive the relation between the Lagrangian and Eulerian increments of the density for later reference. As the volume dilatation is  $\nabla \cdot \vec{u}$ , the conservation

of mass yields  $\rho_0(\vec{R})V_0 = \rho V = \rho V_0(1 + \nabla \cdot \vec{u})$ . Hence,

$$\delta\rho = -\rho\nabla \cdot \vec{u} = -\rho_0(\vec{R})\nabla \cdot \vec{u} = -\rho_0(\vec{R})\frac{\partial u_k}{\partial x_k}, \quad (3.344)$$

where the replacement of  $\rho$  by  $\rho_0(\vec{R})$  introduces only an error of second order of displacement. The Eulerian increment  $\rho_1$  can be obtained from (3.342), (3.344), and (3.338),

$$\begin{aligned} \rho_1 &= \rho - \rho_0 = \left[ \rho - \rho_0(\vec{R}) \right] - \left[ \rho_0 - \rho_0(\vec{R}) \right] \\ &= -\rho_0(\vec{R})\frac{\partial u_k}{\partial x_k} - u_k \frac{\partial \rho_0}{\partial x_k} = -\frac{\partial}{\partial x_k}(\rho_0 u_k) = -\nabla \cdot (\rho_0 \vec{u}), \end{aligned} \quad (3.345)$$

where the replacement of  $\rho_0(\vec{R})$  by  $\rho_0$  introduces again only an error of second order of displacement.

The elastic equation of the Earth with the displacement and the increments defined above can be derived from (3.337) and (3.335). The component form of (3.337) is

$$\rho \left( \frac{D^2 u_j}{Dt^2} + 2\omega_i \frac{Du_k}{Dt} \epsilon_{ikj} \right) = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho \frac{\partial}{\partial x_j} (\phi + \psi) + f_j. \quad (3.346)$$

At the position vector  $\vec{R}$ , Eq. (3.335) in component form can be written as

$$\left( \frac{\partial T_{0ij}}{\partial x_i} \right)_{\vec{R}} = -\rho_0(\vec{R}) \left[ \frac{\partial}{\partial x_j} (\phi_0 + \psi) \right]_{\vec{R}}, \quad (3.347)$$

where  $T_{0ij}$  are the components of  $\bar{T}_0$ . By adding (3.347) and (3.346) together, we obtain

$$\begin{aligned} \rho \left( \frac{D^2 u_j}{Dt^2} + 2\omega_i \frac{Du_k}{Dt} \epsilon_{ikj} \right) &= \left\{ \frac{\partial \sigma_{ij}}{\partial x_i} - \left( \frac{\partial T_{0ij}}{\partial x_i} \right)_{\vec{R}} \right\} \\ &\quad + \left\{ \rho \frac{\partial}{\partial x_j} (\phi + \psi) - \rho_0(\vec{R}) \left[ \frac{\partial}{\partial x_j} (\phi_0 + \psi) \right]_{\vec{R}} \right\} + f_j. \end{aligned} \quad (3.348)$$

In this equation, the contents of the two pairs of braces are both an Lagrangian increment, i.e., the difference between the instantaneous value of a parameter of a material element at the instantaneous position in the deformed state and that for the same material element in the undeformed state. Hence, they are both infinitesimal quantities on the order of the displacement, and our main task left is to express them in terms of displacement.



For the difference in the first pair of braces in (3.348), in order to compute  $\partial\sigma_{ij}/\partial x_i$  according to (3.340), we need to express  $T_{0ij}(\vec{R})$  in terms of  $T_{0ij}$ , which is

$$T_{0ij}(\vec{R}) = T_{0ij} - u_k \frac{\partial T_{0ij}}{\partial x_k}. \quad (3.349)$$

With this formula, we can write (3.340) as

$$\sigma_{ij} = T_{0ij} - u_k \frac{\partial T_{0ij}}{\partial x_k} + T_{ij}. \quad (3.350)$$

This formula is purely based on Eulerian description, and we can then take derivatives of it to obtain

$$\frac{\partial\sigma_{ij}}{\partial x_i} = \frac{\partial T_{0ij}}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial T_{0ij}}{\partial x_k} - u_k \frac{\partial^2 T_{0ij}}{\partial x_i \partial x_k} + \frac{\partial T_{ij}}{\partial x_i}. \quad (3.351)$$

The term  $\partial T_{0ij}/\partial x_i$  can be expressed using  $(\partial T_{0ij}/\partial x_i)_{\vec{R}}$  as

$$\frac{\partial T_{0ij}}{\partial x_i} = \left( \frac{\partial T_{0ij}}{\partial x_i} \right)_{\vec{R}} + u_k \frac{\partial^2 T_{0ij}}{\partial x_i \partial x_k}. \quad (3.352)$$

We finally obtain

$$\frac{\partial\sigma_{ij}}{\partial x_i} - \left( \frac{\partial T_{0ij}}{\partial x_i} \right)_{\vec{R}} = - \frac{\partial u_k}{\partial x_i} \frac{\partial T_{0ij}}{\partial x_k} + \frac{\partial T_{ij}}{\partial x_i}. \quad (3.353)$$

This is the expression we need for the difference in the first pairs of braces in (3.348).

For the difference in the second pair of braces in (3.348), we write the term  $\rho\partial(\phi + \psi)/\partial x_j$  as, according to (3.342) and (3.341),

$$\rho \frac{\partial}{\partial x_j}(\phi + \psi) = \rho_0(\vec{R}) \frac{\partial}{\partial x_j}(\phi_0 + \psi) + \delta\rho \frac{\partial}{\partial x_j}(\phi_0 + \psi) + \rho_0(\vec{R}) \frac{\partial\phi_1}{\partial x_j}. \quad (3.354)$$

By expressing  $\partial(\phi_0 + \psi)/\partial x_j$  using  $[\partial(\phi_0 + \psi)/\partial x_j]_{\vec{R}}$  as

$$\frac{\partial}{\partial x_j}(\phi_0 + \psi) = \left[ \frac{\partial}{\partial x_j}(\phi_0 + \psi) \right]_{\vec{R}} + u_k \frac{\partial^2}{\partial x_j \partial x_k}(\phi_0 + \psi), \quad (3.355)$$

we obtain, by using (3.344),

$$\begin{aligned} \rho \frac{\partial}{\partial x_j}(\phi + \psi) &= \rho_0(\vec{R}) \left[ \frac{\partial}{\partial x_j}(\phi_0 + \psi) \right]_{\vec{R}} + \rho_0(\vec{R}) u_k \frac{\partial^2}{\partial x_j \partial x_k}(\phi_0 + \psi) \\ &\quad - \rho_0(\vec{R}) \frac{\partial u_k}{\partial x_k} \frac{\partial}{\partial x_j}(\phi_0 + \psi) + \rho_0(\vec{R}) \frac{\partial\phi_1}{\partial x_j}. \end{aligned} \quad (3.356)$$

It can be readily inferred that, at the right-hand side, except the first term, replacing  $\rho_0(\vec{R})$  by  $\rho_0$  in all other terms would introduce only an error of second order of displacement. Hence,

$$\begin{aligned} & \rho \frac{\partial}{\partial x_j} (\phi + \psi) - \rho_0(\vec{R}) \left[ \frac{\partial}{\partial x_j} (\phi_0 + \psi) \right]_{\vec{R}} \\ &= \rho_0 u_k \frac{\partial^2}{\partial x_j \partial x_k} (\phi_0 + \psi) - \rho_0 \frac{\partial u_k}{\partial x_k} \frac{\partial}{\partial x_j} (\phi_0 + \psi) + \rho_0 \frac{\partial \phi_1}{\partial x_j}. \end{aligned} \quad (3.357)$$

The is the expression we need for the difference in the second pairs of braces in (3.348).

The equation of elastic deformation of the Earth is obtained by substituting (3.353) and (3.357) into (3.348). With some rearrangement of the terms, we can write it as

$$\begin{aligned} \rho_0 \left( \frac{D^2 u_j}{Dt^2} + 2\omega_i \frac{Du_k}{Dt} \epsilon_{ikj} \right) &= \frac{\partial T_{ij}}{\partial x_i} - \rho_0 \frac{\partial u_k}{\partial x_k} \frac{\partial}{\partial x_j} (\phi_0 + \psi) - \frac{\partial u_k}{\partial x_i} \frac{\partial T_{0ij}}{\partial x_k} \\ &+ \rho_0 u_k \frac{\partial^2}{\partial x_j \partial x_k} (\phi_0 + \psi) + \rho_0 \frac{\partial \phi_1}{\partial x_j} + f_j. \end{aligned} \quad (3.358)$$

where we replaced  $\rho$  by  $\rho_0$  at the left-hand side, which introduces only an error of second order of displacement.

Equation (3.358) can be further deduced to another form. By substituting the component form of (3.335) written as

$$\frac{\partial T_{0ij}}{\partial x_i} = -\rho_0 \frac{\partial}{\partial x_j} (\phi_0 + \psi) \quad (3.359)$$

into the second term at the right-hand side of (3.358), we obtain

$$\begin{aligned} \rho_0 \left( \frac{D^2 u_j}{Dt^2} + 2\omega_i \frac{Du_k}{Dt} \epsilon_{ikj} \right) &= \frac{\partial T_{ij}}{\partial x_i} + \frac{\partial u_k}{\partial x_k} \frac{\partial T_{0ij}}{\partial x_i} - \frac{\partial u_i}{\partial x_k} \frac{\partial T_{0kj}}{\partial x_i} \\ &+ \rho_0 u_k \frac{\partial^2}{\partial x_j \partial x_k} (\phi_0 + \psi) + \rho_0 \frac{\partial \phi_1}{\partial x_j} + f_j. \end{aligned} \quad (3.360)$$

The term  $(\partial u_i / \partial x_k)(\partial T_{0kj} / \partial x_i)$  is in fact the term  $(\partial u_k / \partial x_i)(\partial T_{0ij} / \partial x_k)$  in (3.358) with the indices  $i$  and  $k$  switched. Such an rearrangement is for convenience in later derivation. With the two relations

$$\frac{\partial u_k}{\partial x_k} \frac{\partial T_{0ij}}{\partial x_i} = \frac{\partial}{\partial x_i} \left( \frac{\partial u_k}{\partial x_k} T_{0ij} \right) - \frac{\partial^2 u_k}{\partial x_i \partial x_k} T_{0ij}, \quad (3.361)$$

$$\frac{\partial u_i}{\partial x_k} \frac{\partial T_{0kj}}{\partial x_i} = \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_k} T_{0kj} \right) - \frac{\partial^2 u_i}{\partial x_i \partial x_k} T_{0kj}, \quad (3.362)$$

where their second terms at the right-hand side are identical, we can write (3.360) as

$$\begin{aligned} \rho_0 \left( \frac{D^2 u_j}{Dt^2} + 2\omega_i \frac{Du_k}{Dt} \epsilon_{ikj} \right) &= \frac{\partial T_{ij}}{\partial x_i} + \frac{\partial}{\partial x_i} \left( \frac{\partial u_k}{\partial x_k} T_{0ij} \right) - \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_k} T_{0kj} \right) \\ &\quad + \rho_0 u_k \frac{\partial^2}{\partial x_j \partial x_k} (\phi_0 + \psi) + \rho_0 \frac{\partial \phi_1}{\partial x_j} + f_j. \end{aligned} \quad (3.363)$$

Define

$$\tilde{T}_{ij} = T_{ij} + \frac{\partial u_k}{\partial x_k} T_{0ij} - \frac{\partial u_i}{\partial x_k} T_{0kj} = T_{ij} + (T_{0ij} \delta_{kl} - T_{0kj} \delta_{li}) \frac{\partial u_l}{\partial x_k}. \quad (3.364)$$

We can rewrite (3.363) in the simpler form

$$\rho_0 \left( \frac{D^2 u_j}{Dt^2} + 2\omega_i \frac{Du_k}{Dt} \epsilon_{ikj} \right) = \frac{\partial \tilde{T}_{ij}}{\partial x_i} + \rho_0 u_k \frac{\partial^2}{\partial x_j \partial x_k} (\phi_0 + \psi) + \rho_0 \frac{\partial \phi_1}{\partial x_j} + f_j. \quad (3.365)$$

This is another form of the equation of elastic deformation of the Earth.

It can be shown that  $\tilde{T}_{ij}$  are the components of the second order tensor

$$\bar{\tilde{T}} = \bar{T} + (\nabla \cdot \vec{u}) \bar{T}_0 - (\nabla \vec{u})^T \cdot \bar{T}_0, \quad (3.366)$$

which is referred to as Lagrangian increment of the Piola-Kirchhoff stress tensor. Here we just understand  $\bar{\tilde{T}}$  as a symbol to represent the quantity at the right-hand side of the above formula.

We thus derived the equation of elastic deformation of the Earth in two forms, (3.358) and (3.365), both in component form. Their vector forms are

$$\begin{aligned} \rho_0 \left( \frac{D^2 \vec{u}}{Dt^2} + 2\vec{\omega} \times \frac{D\vec{u}}{Dt} \right) &= \nabla \cdot \bar{T} - \rho_0 (\nabla \cdot \vec{u}) \nabla (\phi_0 + \psi) - (\nabla \vec{u})^T : (\nabla \bar{T}_0) \\ &\quad + \rho_0 \vec{u} \cdot \nabla \nabla (\phi_0 + \psi) + \rho_0 \nabla \phi_1 + \vec{f}, \end{aligned} \quad (3.367)$$

$$\rho_0 \left( \frac{D^2 \vec{u}}{Dt^2} + 2\vec{\omega} \times \frac{D\vec{u}}{Dt} \right) = \nabla \cdot \bar{\tilde{T}} + \rho_0 \vec{u} \cdot \nabla \nabla (\phi_0 + \psi) + \rho_0 \nabla \phi_1 + \vec{f}. \quad (3.368)$$

The material derivatives in these equations can be replaced by partial derivatives,

$$\rho_0 \left( \frac{\partial^2 \vec{u}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{u}}{\partial t} \right) = \nabla \cdot \bar{\mathbf{T}} - \rho_0 (\nabla \cdot \vec{u}) \nabla (\phi_0 + \psi) - (\nabla \vec{u})^T : (\nabla \bar{\mathbf{T}}_0) + \rho_0 \vec{u} \cdot \nabla \nabla (\phi_0 + \psi) + \rho_0 \nabla \phi_1 + \vec{f}, \quad (3.369)$$

$$\rho_0 \left( \frac{\partial^2 \vec{u}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{u}}{\partial t} \right) = \nabla \cdot \bar{\bar{\mathbf{T}}} + \rho_0 \vec{u} \cdot \nabla \nabla (\phi_0 + \psi) + \rho_0 \nabla \phi_1 + \vec{f}, \quad (3.370)$$

which introduces only errors on second order of the displacement.

Finally, we derive the formula for  $\phi_1$ . Poisson's equations satisfied by  $\phi_0$  and  $\phi$  are

$$\nabla^2 \phi_0 = -4\pi G \rho_0, \quad (3.371)$$

$$\nabla^2 \phi = -4\pi G \rho. \quad (3.372)$$

The equation for  $\phi_1$  can be readily obtained by subtracting the former from the latter, and then using (3.345),

$$\nabla^2 \phi_1 = -4\pi G \rho_1 = 4\pi G \nabla \cdot (\rho_0 \vec{u}). \quad (3.373)$$

In summary, the system of equations to be solved consists of (3.369) or (3.370) and (3.373), together with a stress-strain relation to be formulated in the next subsection. Although they are derived based on Eulerian description, they can be assumed to hold only in the domain occupied by the Earth in the undeformed state. Therefore, the boundary conditions in the deformed instantaneous state should also be projected to the boundaries in the undeformed state. For this reason, these equations are often understood as based on Lagrangian description.

We mention that the prestress, i.e., the stress in the undeformed state, has not been solved. We only know that it obeys Eq. (3.335). However, the solution of (3.335) is not unique, and some kind of assumption is required to uniquely determine the initial stress. The simplest instance, i.e., when hydrostatic equilibrium is assumed, will be formulated in the last subsection.

### 3.6.3 The Constitutive Equation with Prestress

The elastic constitutive equation for the Earth must include the effect of the prestress, as the strain is not only affected by the increment of stress, but also by the prestress.

We assume the elastic constitutive equation to be in the form of

$$T_{ij} = \Gamma_{ijkl} \frac{\partial u_l}{\partial x_k}. \quad (3.374)$$

This is a more general form of the strain-stress relation (3.239) with the train expressed in terms of  $\partial u_l / \partial x_k$ , where the elastic parameter  $C_{ijkl}$  for the unstressed case is replaced by  $\Gamma_{ijkl}$  for the prestressed case. This relation is an assumption, which is to be further constrained based on laws of physics. As both  $\bar{T}_0$  and  $\bar{\sigma}$  are symmetrical,  $\bar{T}$  must be symmetrical according to (3.339). Hence, we can infer that  $\Gamma_{ijkl}$  is symmetrical with respect to  $i$  and  $j$ ,

$$\Gamma_{ijkl} = \Gamma_{jikl}. \quad (3.375)$$

When this symmetry is considered,  $\Gamma_{ijkl}$  has 54 independent components.

The elastic constitutive equation (3.374) must satisfy the energy conservation equation (3.250). Here we write it in the form of

$$\rho \frac{De}{Dt} = \bar{\sigma} : \nabla \vec{v}_{\text{in}}, \quad (3.376)$$

where  $e$  is the internal energy per unit mass, and  $\vec{v}_{\text{in}}$  the velocity in the inertial reference system. By definition, the velocity in the terrestrial reference system is  $D\vec{u}/Dt$ . We have

$$\vec{v}_{\text{in}} = \vec{v}_0 + \vec{\omega} \times \vec{r} + \frac{D\vec{u}}{Dt}, \quad (3.377)$$

where  $\vec{v}_0$  is the velocity of the center of mass of the Earth, and  $\vec{\omega}$  the angular velocity of the Earth-fixed coordinate system. By substituting (3.377) into (3.376), we obtain, written in component form,

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial}{\partial x_i} \left( v_{0j} + \epsilon_{klj} \omega_k x_l + \frac{Du_j}{Dt} \right), \quad (3.378)$$

where  $v_{0j}$  is the component of  $\vec{v}_0$ . As  $v_{0j}$  is position independent, we have

$$\frac{\partial v_{0j}}{\partial x_i} = 0. \quad (3.379)$$

Furthermore, as

$$\frac{\partial}{\partial x_i} (\epsilon_{klj} \omega_k x_l) = \epsilon_{klj} \omega_k \delta_{il} = \epsilon_{kij} \omega_k = -\epsilon_{kji} \omega_k = -\frac{\partial}{\partial x_j} (\epsilon_{kli} \omega_k x_l), \quad (3.380)$$

we obtain, according to the symmetry of  $\sigma_{ij}$ ,

$$\sigma_{ij} \frac{\partial}{\partial x_i} (\epsilon_{klj} \omega_k x_l) = 0. \quad (3.381)$$

Hence, (3.378) is simplified to

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial}{\partial x_i} \left( \frac{Du_j}{Dt} \right). \quad (3.382)$$

By substituting (3.340) into this formula, we obtain

$$\rho \frac{De}{Dt} = \left[ T_{0ij}(\vec{R}) + T_{ij} \right] \frac{\partial}{\partial x_i} \left( \frac{Du_j}{Dt} \right). \quad (3.383)$$

In the next, we develop  $De/Dt$  to the form of a total differential, for which we first further develop  $(\partial/\partial x_i)(Du_j/Dt)$ , so that  $(D/Dt)(\partial u_j/\partial x_i)$  appears, as it is  $\partial u_j/\partial x_i$  that appears in the elastic constitutive equation (3.374). For evident reason, we will accurate to the second order terms of the displacement in the derivations.

As  $x_k = X_k + u_k$ , we have

$$\begin{aligned} \frac{\partial}{\partial x_i} \left( \frac{Du_j}{Dt} \right) &= \frac{\partial}{\partial X_l} \left( \frac{\partial u_j^L}{\partial t} \right) \frac{\partial X_l}{\partial x_i} = \frac{D}{Dt} \left( \frac{\partial u_j^L}{\partial X_l} \right) \left( \delta_{li} - \frac{\partial u_l}{\partial x_i} \right) \\ &= \frac{D}{Dt} \left( \frac{\partial u_j^L}{\partial X_i} \right) - \frac{\partial u_l}{\partial x_i} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_l} \right), \end{aligned} \quad (3.384)$$

where we have directly replaced the derivative with respect to Lagrangian coordinates to that with respect to Eulerian coordinates in the second order term, which introduces an error of higher order. This formula can be further developed as (accurate to second order terms)

$$\begin{aligned} \frac{\partial}{\partial x_i} \left( \frac{Du_j}{Dt} \right) &= \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_l} \frac{\partial x_l}{\partial X_i} \right) - \frac{\partial u_l}{\partial x_i} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_l} \right) \\ &= \frac{D}{Dt} \left[ \frac{\partial u_j}{\partial x_l} \left( \delta_{li} + \frac{\partial u_l}{\partial X_i} \right) \right] - \frac{\partial u_l}{\partial x_i} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_l} \right) \\ &= \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial x_l} \frac{\partial u_l}{\partial x_i} \right) - \frac{\partial u_l}{\partial x_i} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_l} \right) \\ &= \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) + \frac{\partial u_j}{\partial x_l} \frac{D}{Dt} \left( \frac{\partial u_l}{\partial x_i} \right). \end{aligned} \quad (3.385)$$

By substituting this formula into (3.383), we obtain, accurate to second order terms,

$$\begin{aligned} \rho \frac{De}{Dt} &= T_{0ij}(\vec{R}) \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) + T_{0ij}(\vec{R}) \frac{\partial u_j}{\partial x_l} \frac{D}{Dt} \left( \frac{\partial u_l}{\partial x_i} \right) + T_{ij} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) \\ &= T_{0ij}(\vec{R}) \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) + T_{0il}(\vec{R}) \frac{\partial u_l}{\partial x_j} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) + T_{ij} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) \end{aligned}$$

$$= \left[ T_{0ij}(\vec{R}) + T_{0il}(\vec{R}) \frac{\partial u_l}{\partial x_j} + T_{ij} \right] \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right). \quad (3.386)$$

In this formula, the density  $\rho$  still depends on  $\vec{u}$ . According to (3.342) and (3.344),

$$\rho = \rho_0(\vec{R}) \left( 1 - \frac{\partial u_l}{\partial x_l} \right), \quad (3.387)$$

and hence, (3.386) can be developed as

$$\begin{aligned} \frac{De}{Dt} &= \frac{1}{\rho_0(\vec{R})} \left( 1 + \frac{\partial u_l}{\partial x_l} \right) \left[ T_{0ij}(\vec{R}) + T_{0il}(\vec{R}) \frac{\partial u_l}{\partial x_j} + T_{ij} \right] \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) \\ &= \frac{1}{\rho_0(\vec{R})} \left[ T_{0ij}(\vec{R}) + T_{0ij}(\vec{R}) \frac{\partial u_l}{\partial x_l} + T_{0il}(\vec{R}) \frac{\partial u_l}{\partial x_j} + T_{ij} \right] \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right). \end{aligned} \quad (3.388)$$

By substituting (3.374) into this formula, we obtain

$$\begin{aligned} \frac{De}{Dt} &= \frac{1}{\rho_0(\vec{R})} \left[ T_{0ij}(\vec{R}) + T_{0ij}(\vec{R}) \frac{\partial u_l}{\partial x_l} + T_{0il}(\vec{R}) \frac{\partial u_l}{\partial x_j} + \Gamma_{ijkl} \frac{\partial u_l}{\partial x_k} \right] \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right) \\ &= \left\{ \frac{T_{0ij}(\vec{R})}{\rho_0(\vec{R})} + \frac{1}{\rho_0(\vec{R})} \left[ T_{0ij}(\vec{R}) \delta_{kl} + T_{0il}(\vec{R}) \delta_{jk} + \Gamma_{ijkl} \right] \frac{\partial u_l}{\partial x_k} \right\} \frac{D}{Dt} \left( \frac{\partial u_j}{\partial x_i} \right). \end{aligned} \quad (3.389)$$

This formula can be written in a total differential form as

$$de = \left\{ \frac{T_{0ij}(\vec{R})}{\rho_0(\vec{R})} + \frac{1}{\rho_0(\vec{R})} \left[ T_{0ij}(\vec{R}) \delta_{kl} + T_{0il}(\vec{R}) \delta_{jk} + \Gamma_{ijkl} \right] \frac{\partial u_l}{\partial x_k} \right\} d \left( \frac{\partial u_j}{\partial x_i} \right). \quad (3.390)$$

It is to be noted that, as this formula is derived based on (3.376), the deformation is implicitly assumed to be adiabatic.

The constraint of energy conservation on the stress-displacement relation (3.374) can be derived based on (3.390). As  $e$  is a total differential, (3.390) implies

$$\frac{\partial e}{\partial (\partial u_j / \partial x_i)} = \frac{T_{0ij}(\vec{R})}{\rho_0(\vec{R})} + \frac{1}{\rho_0(\vec{R})} \left[ T_{0ij}(\vec{R}) \delta_{kl} + T_{0il}(\vec{R}) \delta_{jk} + \Gamma_{ijkl} \right] \frac{\partial u_l}{\partial x_k}. \quad (3.391)$$

We can further obtain

$$\frac{\partial^2 e}{\partial (\partial u_j / \partial x_i) \partial (\partial u_l / \partial x_k)} = \frac{1}{\rho_0(\vec{R})} \left[ T_{0ij}(\vec{R}) \delta_{kl} + T_{0il}(\vec{R}) \delta_{jk} + \Gamma_{ijkl} \right]. \quad (3.392)$$

As the left-hand side is symmetrical with respect to  $ij$  and  $kl$ , the right-hand side must also be so, i.e.,

$$T_{0ij}(\vec{R})\delta_{kl} + T_{0il}(\vec{R})\delta_{jk} + \Gamma_{ijkl} = T_{0kl}(\vec{R})\delta_{ij} + T_{0kj}(\vec{R})\delta_{li} + \Gamma_{klij}, \quad (3.393)$$

which can also be written as

$$\Gamma_{ijkl} - \Gamma_{klij} = T_{0kl}(\vec{R})\delta_{ij} + T_{0kj}(\vec{R})\delta_{li} - T_{0ij}(\vec{R})\delta_{kl} - T_{0il}(\vec{R})\delta_{jk}. \quad (3.394)$$

We choose the expression of  $\Gamma_{ijkl}$  satisfying this relation to be

$$\Gamma_{ijkl} = \Gamma'_{ijkl} + \frac{1}{2} \left[ T_{0kl}(\vec{R})\delta_{ij} + T_{0kj}(\vec{R})\delta_{li} - T_{0ij}(\vec{R})\delta_{kl} - T_{0il}(\vec{R})\delta_{jk} \right], \quad (3.395)$$

The coefficients  $\Gamma'_{ijkl}$  are assumed to be symmetrical with respect to  $ij$  and  $kl$ ,

$$\Gamma'_{ijkl} = \Gamma'_{klij}, \quad (3.396)$$

so that (3.395) satisfies (3.394). The expressions (3.374), (3.395), and (3.396) are the stress-displacement relation obeying the law of energy conservation under adiabatic displacement assumption.

Finally, we examine the constraint of the symmetry (3.375) on  $\Gamma_{ijkl}$ . According to (3.395), we have

$$\begin{aligned} \Gamma_{ijkl} - \Gamma_{jikl} &= \Gamma'_{ijkl} - \Gamma'_{jikl} \\ &+ \frac{1}{2} \left[ T_{0kj}(\vec{R})\delta_{li} + T_{0jl}(\vec{R})\delta_{ik} - T_{0il}(\vec{R})\delta_{jk} - T_{0ki}\delta_{lj} \right]. \end{aligned} \quad (3.397)$$

The symmetry (3.375) implies  $\Gamma_{ijkl} - \Gamma_{jikl} = 0$ . Hence, we obtain from the above formula,

$$\Gamma'_{ijkl} - \Gamma'_{jikl} = \frac{1}{2} \left[ T_{0il}(\vec{R})\delta_{jk} + T_{0ki}(\vec{R})\delta_{lj} - T_{0kj}(\vec{R})\delta_{li} - T_{0jl}(\vec{R})\delta_{ik} \right]. \quad (3.398)$$

We further choose the expression of  $\Gamma'_{ijkl}$  obeying (3.396) and (3.398) as

$$\Gamma'_{ijkl} = C_{ijkl} + \frac{1}{2} \left[ T_{0ki}(\vec{R})\delta_{lj} - T_{0jl}(\vec{R})\delta_{ik} \right]. \quad (3.399)$$

The coefficients  $C_{ijkl}$  are assumed to satisfy the symmetries

$$C_{ijkl} = C_{klij}, \quad \text{and} \quad C_{ijkl} = C_{jikl}, \quad (3.400)$$



so that (3.399) satisfies both the requirements (3.398) and (3.396). Based on these two symmetries, we can derive another symmetry of  $C_{ijkl}$ ,

$$C_{ijkl} = C_{ijlk} . \quad (3.401)$$

We see that  $C_{ijkl}$  satisfy all the symmetries of the elastic parameters of the unstressed state. Hence, there is no longer any constraint from prestress to be imposed.

Finally, by substituting (3.399) into (3.395), we obtain

$$\begin{aligned} \Gamma_{ijkl} = C_{ijkl} + \frac{1}{2} \Big[ & T_{0kl}(\vec{R})\delta_{ij} + T_{0kj}(\vec{R})\delta_{li} + T_{0ki}(\vec{R})\delta_{lj} \\ & - T_{0ij}(\vec{R})\delta_{kl} - T_{0il}(\vec{R})\delta_{jk} - T_{0jl}(\vec{R})\delta_{ik} \Big] . \end{aligned} \quad (3.402)$$

In the equation of motion,  $\Gamma_{ijkl}$  is to be multiplied with  $\partial u_l / \partial x_k$ . As the equation of motion is accurate to the first order of  $\vec{u}$ , we can replace  $\vec{T}_0(\vec{R})$  in (3.402) by  $\vec{T}_0$  to obtain

$$\Gamma_{ijkl} = C_{ijkl} + \frac{1}{2} (T_{0kl}\delta_{ij} + T_{0kj}\delta_{li} + T_{0ki}\delta_{lj} - T_{0ij}\delta_{kl} - T_{0il}\delta_{jk} - T_{0jl}\delta_{ik}) . \quad (3.403)$$

The stress-displacement relation satisfying all physical requirements we are seeking is then (3.374) with the coefficients given by (3.403). The coefficients  $C_{ijkl}$  satisfying (3.400) and (3.401) are assumed to be the material property, i.e., the elastic parameters of the material. The second term at the right-hand side of (3.403) is the contribution of the prestress to the incremental stress. We must mention that, although the coefficients  $\Gamma_{ijkl}$  given in (3.403) are proven to satisfy all physical requirements, but it has not been proven that it is unique—both (3.395) and (3.399) in deriving (3.403) are chosen based on simplicity, but they are not unique. To the depth of this book, we understand (3.403) as the definition of the elastic parameters  $C_{ijkl}$ , as  $\Gamma_{ijkl} = C_{ijkl}$  in the unstressed case. It can also be easily verified that, in the unstressed case, the stress-displacement relation given by (3.374) and (3.403) is equivalent to that formulated in Sect. 3.4.3. It can be further verified that the degenerate form  $\Gamma_{ijkl} = C_{ijkl}$  also holds in the particular prestressed case where the prestress tensor  $\vec{T}_0$  is a spherical tensor, which is assumed to be the case in the Earth's interior as a first order approximation, and will be the subject of the next subsection.

Finally, by substituting (3.374) and (3.403) into (3.364), we obtain the relation between the Lagrangian increment of the Piola-Kirchhoff stress tensor,  $\tilde{T}_{ij}$ , and the displacement,

$$\tilde{T}_{ij} = \Lambda_{ijkl} \frac{\partial u_l}{\partial x_k} , \quad (3.404)$$

where

$$\Lambda_{ijkl} = C_{ijkl} + \frac{1}{2} (T_{0ij}\delta_{kl} + T_{0kl}\delta_{ij} + T_{0ki}\delta_{lj} - T_{0kj}\delta_{li} - T_{0il}\delta_{jk} - T_{0jl}\delta_{ik}) . \quad (3.405)$$

In the stress-displacement relations derived in this subsection, we have made no assumption on the property of the material, such as isotropy, to further constrain the elastic coefficients  $C_{ijkl}$ . If isotropy is imposed, the property of  $C_{ijkl}$  formulated in Sect. 3.4.3 applies.

Although we have derived both the equation of motion and the elastic constitutive equation, in order to solve for the elastic deformation of the Earth, the prestress must be known. We have only assumed that the prestress is in balance with gravity, as given by (3.335), whose solution is not unique. In order to uniquely determine the prestress, we must know all kinds of interactions within the Earth's interior, which is not practical. Therefore, assumptions have to be made to obtain a unique prestress according to (3.335).

### 3.6.4 The Hydrostatic Equilibrium as Undeformed State

The simplest assumption on the undeformed state within the Earth is hydrostatic equilibrium, i.e., the Earth is assumed to behave like a fluid, and there is no motion one part with respect to another. At a first glance, assuming the Earth's interior to be fluid appears irrational. However, considering that the forcing of gravity and rotation on the Earth can practically be considered perpetual, the yielding of the apparently solid Earth to such forcing like fluid is then natural. Although it appears evident to imagine what a fluid is, its analytical definition must be provided in order to formulate its physical properties: A fluid is such a material that, in any standstill state, the shear stress within it vanishes. It can be shown that, if all the off-diagonal components of a second order symmetric tensor vanish in all coordinate systems, the diagonal components are equal. By applying this property to a fluid in standstill, we obtain the conclusion that the three normal components of the stress tensor are equal. Therefore, only one quantity is required to describe the stress tensor  $\bar{T}_0$ . We chose this quantity to be the pressure  $p_0$ , which is the negative of the normal stress. Hence,

$$T_{0ij} = -p_0\delta_{ij} , \quad (3.406)$$

which are the components of a spherical tensor. By substituting this formula into the equation of the Earth's undeformed state, (3.335), we obtain

$$\rho_0 \frac{\partial}{\partial x_j} (\phi_0 + \psi) = \frac{\partial}{\partial x_i} (p_0\delta_{ij}) = \frac{\partial p_0}{\partial x_j} , \quad (3.407)$$

which is the equation of the state of hydrostatic equilibrium.

The formula (3.407) can be derived out directly based on more fundamental physics. Let  $S$  be an arbitrary smooth closed surface within the Earth, and  $V$  is its interior. In hydrostatic equilibrium, the gravity acting in the interior and the pressure acting on the surface of the material in  $V$  is in balance, i.e.,

$$\int_V \rho_0 \nabla(\phi_0 + \psi) dV - \int_S p_0 \hat{n} dS = 0, \quad (3.408)$$

where  $\hat{n}$  is the unit outward normal of  $S$ . According to Gauss' theorem, the surface integral can be converted to a volume integral, yielding

$$\int_V \rho_0 \nabla(\phi_0 + \psi) dV - \int_V \nabla p_0 dV = 0. \quad (3.409)$$

As  $V$  is arbitrary, in order for this equation to hold for any  $V$ , we must have

$$\nabla p_0 = \rho_0 \nabla(\phi_0 + \psi). \quad (3.410)$$

which is the vector form of (3.407). This relation implies that the gradients of pressure and gravity potential are in the same direction, and hence, equi-value surfaces of pressure and gravity potential coincide. By taking the curl of the above equation, and then using (2.231) and (2.229), we obtain

$$\nabla \rho_0 \times \nabla(\phi_0 + \psi) = 0, \quad (3.411)$$

meaning that the gradients of density and gravity potential are parallel, and hence, equi-value surfaces of density and gravity potential coincide. Normally, the above properties are generalized such that the equi-value surfaces of the elastic parameters  $\lambda$  and  $\mu$  also coincide with those mentioned above. The coincidence of equi-value surfaces of density and gravity potential are used to formulate the shape of the equi-value surface in the Earth's interior, which will be the subject of Sect. 5.5.

We now apply (3.406) to the elastic equation of the Earth. First, the term  $(\nabla \vec{u})^T : (\nabla \vec{T}_0)$  in (3.369) can be simplified as

$$(\nabla \vec{u})^T : (\nabla \vec{T}_0) = \frac{\partial u_k}{\partial x_i} \frac{\partial T_{0ij}}{\partial x_k} \hat{e}_j = -\delta_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial p_0}{\partial x_k} \hat{e}_j = -\hat{e}_i \frac{\partial u_k}{\partial x_i} \frac{\partial p_0}{\partial x_k} = -\nabla \vec{u} \cdot \nabla p_0. \quad (3.412)$$

By substituting this formula back into (3.369), and then using (3.410), we obtain

$$\begin{aligned} \rho_0 \left( \frac{\partial^2 \vec{u}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{u}}{\partial t} \right) &= \nabla \cdot \vec{T} - \rho_0 (\nabla \cdot \vec{u}) \nabla(\phi_0 + \psi) + \rho_0 \nabla \vec{u} \cdot \nabla(\phi_0 + \psi) \\ &\quad + \rho_0 \vec{u} \cdot \nabla \nabla(\phi_0 + \psi) + \rho_0 \nabla \phi_1 + \vec{f}. \end{aligned} \quad (3.413)$$

The substitution of (3.406) into (3.403) yields

$$\Gamma_{ijkl} = C_{ijkl}. \quad (3.414)$$

The constitutive equation for a general elastic body is thus

$$T_{ij} = C_{ijkl} \frac{\partial u_l}{\partial x_k}. \quad (3.415)$$

For isotropic elastic bodies, we have

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (3.416)$$

and hence,

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \quad (3.417)$$

Its tensor form is

$$\bar{T} = \lambda \nabla \cdot \vec{u} \bar{I} + \mu [\nabla \vec{u} + (\nabla \vec{u})^T]. \quad (3.418)$$

This is the constitutive equation for isotropic elastic bodies, which is the most used. We remark that it is in the same form as that in the unstressed case, which has already been mentioned in the preceding subsection.

The elastic equation using the Lagrangian increment of the Piola-Kirchhoff stress tensor, (3.370), remains the same. However, the Lagrangian increment of the Piola-Kirchhoff stress tensor defined in (3.366) can be further developed as (switching the last two terms at the right-hand side)

$$\bar{\tilde{T}} = \bar{T} + p_0 (\nabla \vec{u})^T - p_0 (\nabla \cdot \vec{u}) \bar{I}. \quad (3.419)$$

The elastic coefficients defined in (3.405) can be simplified to

$$\Lambda_{ijkl} = C_{ijkl} + p_0 (\delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl}). \quad (3.420)$$

Thus the constitutive equation (3.404) is simplified to

$$\tilde{T}_{ij} = C_{ijkl} \frac{\partial u_l}{\partial x_k} + p_0 \left( \frac{\partial u_i}{\partial x_j} - \delta_{ij} \frac{\partial u_k}{\partial x_k} \right). \quad (3.421)$$

This formula agrees with (3.419) with  $\bar{T}$  expressed in terms of elastic parameters and displacement given by (3.415). For isotropic elastic bodies,  $\bar{T}$  is given by (3.418).

Finally, we mention that the elastic and constitutive equations must be combined with the equation of  $\phi_1$ , i.e., (3.373), for a solution.

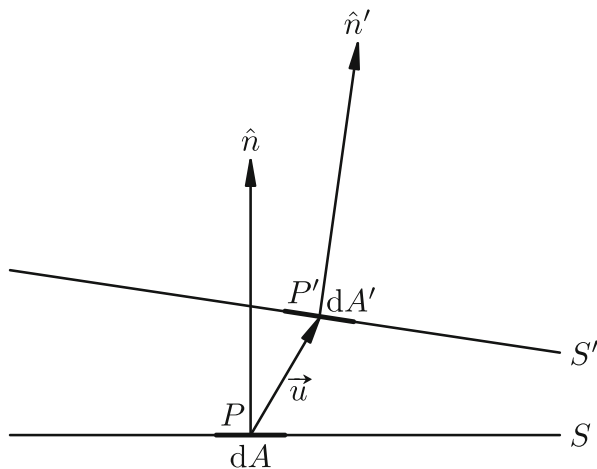
### 3.7 Boundary Conditions for the Elastic Deformation of the Earth

When solving the Earth's elastic equation, boundary conditions must be considered. The internal structure of the Earth is very complicated. Besides the surface of the Earth, there are internal boundaries where the parameters  $\rho_0$ ,  $\lambda$  and  $\mu$  are discontinuous. The conditions that the solution should satisfy across all these boundaries are within the scope of our consideration. Furthermore, these boundaries change their positions orientations following the deformation of the Earth, which is not negligible, thus adding complexity to the problem. In the previous section, the equations of deformation are based on Eulerian description, i.e., the equations hold in the instantaneous deformed state of the Earth. Therefore, the boundary conditions hold also in the instantaneous deformed state. It is much more complicated to solve a problem of varying boundaries than that of fixed boundaries. Hence, as the deformation of the Earth is small, for solving the Earth's elastic equation, we project the boundary conditions from the varying boundaries to the boundaries in the undeformed state, which reduces the problem with varying boundaries to a problem with fixed boundaries.

#### 3.7.1 Displacement and Stress Across Welded Boundaries

No slip happens at welded boundaries. As shown in Fig. 3.15, let  $S$  be a boundary in the undeformed state, and  $S'$  its instantaneous deformed state. Consider a surface element at  $P$  with an area  $dA$  and a unit normal  $\hat{n}$  in the undeformed state. It moves to  $P'$  following the deformation, where the area becomes  $dA'$ , and the unit normal,  $\hat{n}'$ . For convenience, we use a super script '+' and '-' to indicate the value of a function at the side that  $\hat{n}$  and  $\hat{n}'$  point to or from.

**Fig. 3.15** A welded boundary in undeformed and deformed states



### Displacement Boundary Conditions

The displacement is continuous at the deformed state, which is what a welded boundary implies,

$$(\vec{u}^+ - \vec{u}^-)_{S'} = 0, \quad (3.422)$$

as shown in Fig. 3.15. In order to reduce the condition on  $S'$  to  $S$ , we must continuously project the value of  $\vec{u}^\pm$  from  $S'$  to  $S$ , which is done by

$$(u_j)_{S'}^\pm = (u_j)_S^\pm + \left( \frac{\partial u_j^\pm}{\partial x_k} \right)_{S'} u_k^\pm. \quad (3.423)$$

The second term is of second order of displacement, and can be neglected. Hence, the boundary condition (3.422) is reduced to

$$(\vec{u}^+ - \vec{u}^-)_S = 0. \quad (3.424)$$

This is the displacement boundary condition reduced to the boundary in the undeformed state. We mention that  $(\vec{u}^\pm)_S$  is a continuous extension of  $(\vec{u}^\pm)_{S'}$ , which meanwhile extend the domain of the elastic equations with  $S'$  as boundary to the domain with  $S$  as boundary. As such, the elastic equations could be solved in the domain of the undeformed state.

### Stress Boundary Conditions

The traction is continuous in both the undeformed and deformed states. According to Fig. 3.15, we have

$$(\hat{n}' \cdot \sigma^+ - \hat{n}' \cdot \sigma^-)_{S'} = 0, \quad (3.425)$$

$$(\hat{n} \cdot \bar{T}_0^+ - \hat{n} \cdot \bar{T}_0^-)_S = 0. \quad (3.426)$$

Evidently, in order to reduce the condition on  $S'$  to  $S$ , we must express  $\hat{n}'$  in terms of  $\hat{n}$ . For convenience, we look for the relation between  $\hat{n}dA$  and  $\hat{n}'dA'$ . We express the equation of  $S$  as

$$\vec{r}_S(\xi, \eta) = x_i(\xi, \eta)\hat{e}_i. \quad (3.427)$$

According to (2.169), (2.170), and (2.171), we have

$$\hat{n}dA = \frac{\partial \vec{r}_S}{\partial \xi} \times \frac{\partial \vec{r}_S}{\partial \eta} d\xi d\eta, \quad (3.428)$$

where we assume that the parameters  $\xi$  and  $\eta$  are chosen such that the  $\pm$  sign in (2.171) takes '+'. As  $S'$  is the deformed form of  $S$ , the curves on  $S$  with a constant  $\xi$  or  $\eta$  become curves on  $S'$  with the same constant  $\xi$  or  $\eta$ . Hence, the equation of  $S'$  is

$$\vec{r}_{S'} = \vec{r}_S(\xi, \eta) + \vec{u}(\xi, \eta) = [x_i(\xi, \eta) + u_i(\xi, \eta)]\hat{e}_i. \quad (3.429)$$

Hence, we have

$$\hat{n}' dA' = \left( \frac{\partial \vec{r}_S}{\partial \xi} + \frac{\partial \vec{u}}{\partial \xi} \right) \times \left( \frac{\partial \vec{r}_S}{\partial \eta} + \frac{\partial \vec{u}}{\partial \eta} \right) d\xi d\eta. \quad (3.430)$$

For brevity, we use the component forms of (3.428) and (3.430) for further derivation,

$$n_j dA = \epsilon_{jkl} \frac{\partial x_k}{\partial \xi} \frac{\partial x_l}{\partial \eta} d\xi d\eta, \quad (3.431)$$

$$n'_i dA' = \epsilon_{ikl} \left( \frac{\partial x_k}{\partial \xi} + \frac{\partial u_k}{\partial \xi} \right) \left( \frac{\partial x_l}{\partial \eta} + \frac{\partial u_l}{\partial \eta} \right) d\xi d\eta. \quad (3.432)$$

Accurate to the first order of displacement, we can express  $n'_i dA'$  in terms of  $n_j dA$  with a correction as

$$\begin{aligned} n'_i dA' &= \delta_{ij} \epsilon_{jkl} \frac{\partial x_k}{\partial \xi} \frac{\partial x_l}{\partial \eta} d\xi d\eta + \epsilon_{ikl} \left( \frac{\partial x_k}{\partial \xi} \frac{\partial u_l}{\partial \eta} + \frac{\partial u_k}{\partial \xi} \frac{\partial x_l}{\partial \eta} \right) d\xi d\eta \\ &= \delta_{ij} n_j dA + \left( \epsilon_{ikl} \frac{\partial x_k}{\partial \xi} \frac{\partial x_m}{\partial \eta} \frac{\partial u_l}{\partial x_m} + \epsilon_{ikl} \frac{\partial x_m}{\partial \xi} \frac{\partial x_l}{\partial \eta} \frac{\partial u_k}{\partial x_m} \right) d\xi d\eta. \end{aligned} \quad (3.433)$$

By exchanging the summation indices  $k$  and  $l$  in the second term, we obtain

$$\begin{aligned} n'_i dA' &= \delta_{ij} n_j dA + \left( \epsilon_{ikl} \frac{\partial x_k}{\partial \xi} \frac{\partial x_m}{\partial \eta} \frac{\partial u_l}{\partial x_m} + \epsilon_{ilk} \frac{\partial x_m}{\partial \xi} \frac{\partial x_k}{\partial \eta} \frac{\partial u_l}{\partial x_m} \right) d\xi d\eta \\ &= \delta_{ij} n_j dA + \epsilon_{ikl} \left( \frac{\partial x_k}{\partial \xi} \frac{\partial x_m}{\partial \eta} - \frac{\partial x_m}{\partial \xi} \frac{\partial x_k}{\partial \eta} \right) \frac{\partial u_l}{\partial x_m} d\xi d\eta \\ &= \delta_{ij} n_j dA + \epsilon_{ikl} (\delta_{kp} \delta_{mq} - \delta_{mp} \delta_{kq}) \frac{\partial x_p}{\partial \xi} \frac{\partial x_q}{\partial \eta} \frac{\partial u_l}{\partial x_m} d\xi d\eta \\ &= \delta_{ij} n_j dA + \epsilon_{ikl} \epsilon_{jkm} \epsilon_{lpq} \frac{\partial x_p}{\partial \xi} \frac{\partial x_q}{\partial \eta} \frac{\partial u_l}{\partial x_m} d\xi d\eta \\ &= \delta_{ij} n_j dA + \epsilon_{kil} \epsilon_{kjm} \frac{\partial u_l}{\partial x_m} n_j dA \end{aligned}$$

$$\begin{aligned}
&= \delta_{ij} n_j dA + \left( \delta_{ij} \delta_{lm} \frac{\partial u_l}{\partial x_m} - \delta_{im} \delta_{lj} \frac{\partial u_l}{\partial x_m} \right) n_j dA \\
&= \left[ \left( 1 + \frac{\partial u_m}{\partial x_m} \right) \delta_{ij} - \frac{\partial u_j}{\partial x_i} \right] n_j dA.
\end{aligned} \tag{3.434}$$

In order to use (3.434), we express (3.425) and (3.426) in the following component forms,

$$(n'_i \sigma_{ik}^+ dA' - n'_i \sigma_{ik}^- dA')_{S'} = 0, \tag{3.435}$$

$$(n_i T_{0ik}^+ dA - n_i T_{0ik}^- dA)_S = 0. \tag{3.436}$$

As the displacements at both sides are the same, we have

$$(\sigma_{ik}^\pm)_{S'} = (T_{0ik}^\pm)_S + (T_{ik}^\pm)_{S'} = (T_{0ik}^\pm)_S + (T_{ik}^\pm)_S + \left( \frac{\partial T_{ik}^\pm}{\partial x_l} u_l \right)_{S'} = (T_{0ik}^\pm)_S + (T_{ik}^\pm)_S, \tag{3.437}$$

where a second order term was neglected to obtain the last identity.

According to (3.434) and (3.437), we obtain

$$\begin{aligned}
(\sigma_{ik}^\pm n'_i dA')_{S'} &= \left\{ (T_{0ik}^\pm + T_{ik}^\pm) \left[ \left( 1 + \frac{\partial u_m}{\partial x_m} \right) \delta_{ij} - \frac{\partial u_j}{\partial x_i} \right] n_j dA \right\}_S \\
&= (T_{0ik}^\pm n_i dA)_S + \left[ n_j \left( T_{jk}^\pm + \frac{\partial u_m}{\partial x_m} T_{0jk}^\pm - \frac{\partial u_j}{\partial x_i} T_{0ik}^\pm \right) dA \right]_S \\
&= (T_{0ik}^\pm n_i dA)_S + n_j \tilde{T}_{jk}^\pm dA,
\end{aligned} \tag{3.438}$$

where the definition of  $\tilde{T}_{jk}$ , (3.364), was used. By substituting the above formula into (3.435), and then using (3.436), we obtain the stress boundary condition

$$(n_j \tilde{T}_{jk}^+ - n_j \tilde{T}_{jk}^-)_S = 0. \tag{3.439}$$

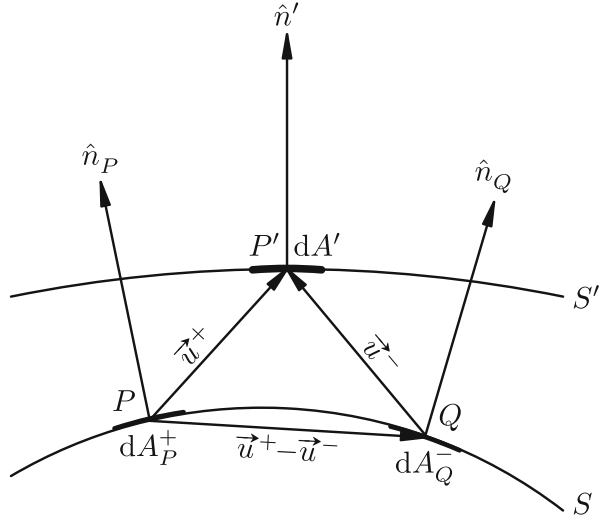
This means that the Lagrangian increment of the Piola-Kirchhoff stress tensor on the surface is continuous. Its vector form is

$$(\hat{n} \cdot \tilde{\bar{T}}^+ - \hat{n} \cdot \tilde{\bar{T}}^-)_S = 0. \tag{3.440}$$

It is left to the readers to infer the boundary conditions for the stress tensor, where displacement and prestress are inevitably involved.



**Fig. 3.16** A slip boundary in undeformed and deformed states



### 3.7.2 Displacement and Stress Across Slip Boundaries

#### Displacement Boundary Conditions

As shown in Fig. 3.16, let  $S$  be the undeformed surface of discontinuity, and  $S'$  its instantaneous deformed state. We use again the superscripts '+' and '-' to represent the two sides of the surface. We consider a surface element with an area of  $dA'$  at both sides at the point  $P'$  on the deformed surface  $S'$ . As slip happens at slip boundaries, the surface elements at the two sides are from different places on the undeformed surface  $S$ , say, from the points  $P$  at the '+' side, and  $Q$  at the '-' side, and the surface elements are  $dA_P^+$  and  $dA_Q^-$  on  $S$ . As pure slip is assumed, no vacancy or penetration can happen, we have  $\vec{u}^+ - \vec{u}^- = \vec{PQ}$ , where, as shown in Fig. 3.16,  $\vec{u}^+$  and  $\vec{u}^-$  are the displacements of the material elements at the '+' and '-' sides of the deformed surface  $S'$  at the point  $P'$ , which are originally at the points  $P$  and  $Q$  at the '+' and '-' sides of the undeformed surface  $S$ . Hence,

$$\hat{n}' \cdot \vec{u}^+ - \hat{n}' \cdot \vec{u}^- = \hat{n}' \cdot \vec{PQ}. \quad (3.441)$$

The differences among the unit vectors  $\hat{n}_P$ ,  $\hat{n}_Q$  and  $\hat{n}'$  are of the first order of displacement. We therefore do not need to distinguish these vector in (3.441), as ignoring their differences would introduce errors of the second order of displacement. In this respect,  $\vec{PQ}$  is perpendicular to  $\hat{n}_P$  and  $\hat{n}_Q$ , and the term at the right-hand side of (3.441) can be neglected. In order to project the boundary condition from the surface  $S'$  to  $S$ , we must continuously extend the value of  $\vec{u}^\pm$  on  $S'$  to  $S$ . Here we choose to project the point  $P'$  to the point  $P$ ,

$$u_j^\pm = (u_j^\pm)_P + \left( \frac{\partial u_j^\pm}{\partial x_k} \right)_{P'} u_k^+. \quad (3.442)$$

By substituting this relation into (3.441), and neglecting second order terms of displacement, we obtain the displacement boundary condition

$$\hat{n}_P \cdot \vec{u}_P^+ - \hat{n}_P \cdot \vec{u}_P^- = 0. \quad (3.443)$$

This is the boundary condition we are seeking. As a general form, we replace the point  $P$  by the surface  $S$  as subscript,

$$(\hat{n} \cdot \vec{u}^+ - \hat{n} \cdot \vec{u}^-)_S = 0. \quad (3.444)$$

### Stress Boundary Conditions

For stress, the boundary condition in the deformed state is

$$(\hat{n}' \cdot \vec{\sigma}^+ - \hat{n}' \cdot \vec{\sigma}^-)_{P'} = 0, \quad (3.445)$$

and that at the undeformed state is

$$(\hat{n} \cdot \vec{T}_0^+ - \hat{n} \cdot \vec{T}_0^-)_P = 0. \quad (3.446)$$

Both of them are the continuation of the traction. Due to the complexity of the problem, we have indicated the points where the values are taken. In subsequent derivations, we will use their component forms

$$(\sigma_{ik}^+ n'_i dA' - \sigma_{ik}^- n'_i dA')_{P'} = 0, \quad (3.447)$$

$$(T_{0ik}^+ n_i dA - T_{0ik}^- n_i dA)_P = 0. \quad (3.448)$$

In order to project the boundary condition on  $S'$  to  $S$ , we progressively reduce the boundary condition at  $P'$  on the deformed surface to  $P$  on the undeformed surface.

Similar to (3.437), we have

$$(\sigma_{ik}^+)_{P'} = (T_{0ik}^+)_P + (T_{ik}^+)_{P'} = (T_{0ik}^+)_P + (T_{ik}^+)_{P'} + \left( \frac{\partial T_{ik}^+}{\partial x_l} \right)_{P'} u_l^+, \quad (3.449)$$

$$(\sigma_{ik}^-)_{P'} = (T_{0ik}^-)_Q + (T_{ik}^-)_{P'} = (T_{0ik}^-)_Q + (T_{ik}^-)_Q + \left( \frac{\partial T_{ik}^-}{\partial x_l} \right)_{P'} u_l^-. \quad (3.450)$$

We also have, according to (3.434),

$$(n'_i dA')_{P'} = \left[ \left( 1 + \frac{\partial u_m^+}{\partial x_m} \right) \delta_{ij} - \frac{\partial u_j^+}{\partial x_i} \right]_P (n_j dA)_P, \quad (3.451)$$

$$(n'_i dA')_{P'} = \left[ \left( 1 + \frac{\partial u_m^-}{\partial x_m} \right) \delta_{ij} - \frac{\partial u_j^-}{\partial x_i} \right]_Q (n_j dA)_Q. \quad (3.452)$$

We obtain, from the above four equations,

$$\begin{aligned} (\sigma_{ik}^+ n'_i dA')_{P'} &= (T_{0ik}^+ n_i dA)_P + \left[ \left( T_{jk}^+ + \frac{\partial u_m^+}{\partial x_m} T_{0jk}^+ - \frac{\partial u_j^+}{\partial x_i} T_{0ik}^+ \right) n_j dA \right]_P \\ &= (T_{0ik}^+ n_i dA)_P + (\tilde{T}_{jk}^+ n_j dA)_P, \end{aligned} \quad (3.453)$$

$$\begin{aligned} (\sigma_{ik}^- n'_i dA')_{P'} &= (T_{0ik}^- n_i dA)_Q + \left[ \left( T_{jk}^- + \frac{\partial u_m^-}{\partial x_m} T_{0jk}^- - \frac{\partial u_j^-}{\partial x_i} T_{0ik}^- \right) n_j dA \right]_Q \\ &= (T_{0ik}^- n_i dA)_Q + (\tilde{T}_{jk}^- n_j dA)_Q, \end{aligned} \quad (3.454)$$

where the definition of  $\tilde{T}_{jk}$ , (3.364), has been used.

In order to use the boundary condition in the undeformed state, i.e., (3.448), we must express the right-hand side of (3.454), which is a value at the point  $Q$ , using the value at the point  $P$ . We have only to formulate the first term, as the second term is itself of the order of displacement, and the difference between it and the value at  $P$  is of second order of displacement, and hence can be ignored. We first derive the relation between  $dA_P$  and  $dA_Q$ . According to (3.451),

$$\begin{aligned} (dA'_{P'})^2 &= (n'_i dA')_{P'} (n'_i dA')_{P'} \\ &= \left\{ \left[ \left( 1 + \frac{\partial u_m^+}{\partial x_m} \right) \delta_{ij} - \frac{\partial u_j^+}{\partial x_i} \right] n_j \left[ \left( 1 + \frac{\partial u_n^+}{\partial x_n} \right) \delta_{ik} - \frac{\partial u_k^+}{\partial x_i} \right] n_k \right\}_P dA_P^2 \\ &= \left[ \left( n_i + \frac{\partial u_m^+}{\partial x_m} n_i - \frac{\partial u_j^+}{\partial x_i} n_j \right) \left( n_i + \frac{\partial u_n^+}{\partial x_n} n_i - \frac{\partial u_k^+}{\partial x_i} n_k \right) \right]_P dA_P^2 \\ &= \left( n_i n_i + \frac{\partial u_m^+}{\partial x_m} n_i n_i - \frac{\partial u_j^+}{\partial x_i} n_i n_j + \frac{\partial u_n^+}{\partial x_n} n_i n_i - \frac{\partial u_k^+}{\partial x_i} n_i n_k \right)_P dA_P^2 \\ &= \left( 1 + 2 \frac{\partial u_k^+}{\partial x_k} - 2 \frac{\partial u_j^+}{\partial x_i} n_i n_j \right)_P dA_P^2. \end{aligned} \quad (3.455)$$

By taking the square root of both sides, we obtain

$$\begin{aligned} dA'_{P'} &= \left( 1 + \frac{\partial u_k^+}{\partial x_k} - \frac{\partial u_j^+}{\partial x_i} n_i n_j \right)_P dA_P \\ &= [1 + \nabla \cdot \vec{u}^+ - \hat{n}(\hat{n} \cdot \nabla) \cdot \vec{u}^+]_P dA_P \\ &= (1 + \nabla_S \cdot \vec{u}^+)_P dA_P, \end{aligned} \quad (3.456)$$

where  $\nabla_S$  is the nabla operator on  $S$  defined by

$$\nabla_S = \nabla - \hat{n}(\hat{n} \cdot \nabla). \quad (3.457)$$

Similarly, we also have

$$\begin{aligned}
 dA'_{P'} &= \left( 1 + \frac{\partial u_k^-}{\partial x_k} - \frac{\partial u_j^-}{\partial x_i} n_i n_j \right)_Q dA_Q \\
 &= [1 + \nabla \cdot \vec{u}^- - \hat{n}(\hat{n} \cdot \nabla) \cdot \vec{u}^-]_Q dA_Q \\
 &= (1 + \nabla_S \cdot \vec{u}^-)_Q dA_Q.
 \end{aligned} \tag{3.458}$$

The relation between  $dA_P$  and  $dA_Q$  can be obtained from (3.456) and (3.458) to be

$$\begin{aligned}
 dA_Q &= (1 + \nabla_S \cdot \vec{u}^+)_P (1 - \nabla_S \cdot \vec{u}^-)_Q dA_P \\
 &= [1 + (\nabla_S \cdot \vec{u}^+)_P - (\nabla_S \cdot \vec{u}^-)_Q] dA_P.
 \end{aligned} \tag{3.459}$$

As the difference between  $(\nabla_S \cdot \vec{u}^-)_Q$  and  $(\nabla_S \cdot \vec{u}^-)_P$  is of second order of displacement, we can write the above formula as

$$dA_Q = dA_P + [\nabla_S \cdot \vec{u}^+ - \nabla_S \cdot \vec{u}^-]_P dA_P. \tag{3.460}$$

The component form of this formula is, according to (3.457),

$$dA_Q = dA_P + \left[ \left( \frac{\partial u_k^+}{\partial x_k} - \frac{\partial u_j^+}{\partial x_i} n_i n_j \right) - \left( \frac{\partial u_k^-}{\partial x_k} - \frac{\partial u_j^-}{\partial x_i} n_i n_j \right) \right]_P dA_P. \tag{3.461}$$

With (3.461), we can express the first term at the right-hand side of (3.454), which is a value at the point  $Q$ , using the value at the point  $P$ . We have, with  $\vec{PQ} = \vec{u}^+ - \vec{u}^-$  as shown in Fig. 3.16,

$$\begin{aligned}
 (T_{0ik}^- n_i dA)_Q &= \left\{ (T_{0ik}^- n_i)_P + \frac{\partial}{\partial x_l} (n_i T_{0ik}^-)_P (u_l^+ - u_l^-) \right\} dA_Q \\
 &= (T_{0ik}^- n_i dA)_P + (n_i T_{0ik}^-)_P \left[ \left( \frac{\partial u_l^+}{\partial x_l} - \frac{\partial u_j^+}{\partial x_l} n_l n_j \right) - \left( \frac{\partial u_l^-}{\partial x_l} - \frac{\partial u_j^-}{\partial x_l} n_l n_j \right) \right]_P dA_P \\
 &\quad + \left\{ \frac{\partial}{\partial x_l} (n_i T_{0ik}^-) (u_l^+ - u_l^-) \right\}_P dA_P.
 \end{aligned} \tag{3.462}$$

where we used the fact that the differences of quantities of the first order of displacement between the points  $P$  and  $Q$  are of second order, and can be neglected. According to (3.444), i.e.,  $\hat{n} \cdot \vec{u}$  is continuous across the boundary, we have

$$(-u_j^+ n_l n_j + u_j^- n_l n_j)_P = 0. \tag{3.463}$$

By adding the left-hand side of this formula to  $u_l^+ - u_l^-$  in the last term at the right-hand side of (3.462), we obtain

$$\begin{aligned}
 (T_{0ik}^- n_i dA)_Q &= (T_{0ik}^- n_i dA)_P \\
 &+ (n_i T_{0ik}^-)_P \left[ \left( \frac{\partial u_l^+}{\partial x_l} - \frac{\partial u_j^+}{\partial x_l} n_l n_j \right) - \left( \frac{\partial u_l^-}{\partial x_l} - \frac{\partial u_j^-}{\partial x_l} n_l n_j \right) \right] dA_P \\
 &+ \left\{ \frac{\partial}{\partial x_l} (n_i T_{0ik}^-) [(u_l^+ - u_j^+ n_l n_j) - (u_l^- - u_j^- n_l n_j)] \right\}_P dA_P \\
 &= (T_{0ik}^- n_i dA)_P + \left[ \frac{\partial}{\partial x_l} (u_l^+ n_i T_{0ik}^-) - \frac{\partial}{\partial x_l} (u_j^+ n_i T_{0ik}^-) n_l n_j \right]_P dA_P \\
 &- \left[ \frac{\partial}{\partial x_l} (u_l^- n_i T_{0ik}^-) - \frac{\partial}{\partial x_l} (u_j^- n_i T_{0ik}^-) n_l n_j \right]_P dA_P. \tag{3.464}
 \end{aligned}$$

This is the expression of the first term at the right-hand side of (3.454) projected to  $P$ .

By substituting (3.453), (3.454), and (3.464) into (3.447), and then using (3.448), we obtain a primitive form of the boundary condition,

$$\begin{aligned}
 &\left[ n_j \tilde{T}_{jk}^+ - \frac{\partial}{\partial x_l} (u_l^+ n_i T_{0ik}^-) + \frac{\partial}{\partial x_l} (u_j^+ n_i T_{0ik}^-) n_l n_j \right]_P \\
 &- \left[ n_j \tilde{T}_{jk}^- - \frac{\partial}{\partial x_l} (u_l^- n_i T_{0ik}^-) + \frac{\partial}{\partial x_l} (u_j^- n_i T_{0ik}^-) n_l n_j \right]_P = 0, \tag{3.465}
 \end{aligned}$$

where we have also neglected the differences of quantities of the first order of displacement between the points  $P$  and  $Q$ . The vector form of the above formula is

$$\begin{aligned}
 &\left\{ \hat{n} \cdot \bar{\tilde{T}}^+ - \nabla \cdot [\vec{u}^+(\hat{n} \cdot \bar{T}_0^-)] + \hat{n}(\hat{n} \cdot \nabla) \cdot [\vec{u}^+(\hat{n} \cdot \bar{T}_0^-)] \right\}_P \\
 &- \left\{ \hat{n} \cdot \bar{\tilde{T}}^- - \nabla \cdot [\vec{u}^-(\hat{n} \cdot \bar{T}_0^-)] + \hat{n}(\hat{n} \cdot \nabla) \cdot [\vec{u}^-(\hat{n} \cdot \bar{T}_0^-)] \right\}_P = 0. \tag{3.466}
 \end{aligned}$$

It can be written using the operator  $\nabla_S$  defined in (3.457) as

$$\left[ \left\{ \hat{n} \cdot \bar{\tilde{T}}^+ - \nabla_S \cdot [\vec{u}^+(\hat{n} \cdot \bar{T}_0^-)] \right\} - \left\{ \hat{n} \cdot \bar{\tilde{T}}^- - \nabla_S \cdot [\vec{u}^-(\hat{n} \cdot \bar{T}_0^-)] \right\} \right]_P = 0. \tag{3.467}$$

We see that the prestress appears in the form of  $\hat{n} \cdot \bar{T}_0^-$  in both pairs of braces, but no  $\hat{n} \cdot \bar{T}_0^+$  is present correspondingly. Actually, in this formula,  $\hat{n} \cdot \bar{T}_0^-$  and  $\hat{n} \cdot \bar{T}_0^+$  can be used interchangeably. We know that  $\hat{n} \cdot \bar{T}_0^+ = \hat{n} \cdot \bar{T}_0^-$ , i.e., the traction is continuous in

the undeformed state. In (3.467), what appear are their derivatives along tangential directions of the boundary, which must also be equal, as the identity  $\hat{n} \cdot \bar{T}_0^+ = \hat{n} \cdot \bar{T}_0^-$  can not be ensured otherwise. Hence, we replace  $\hat{n} \cdot \bar{T}_0^-$  by  $\hat{n} \cdot \bar{T}_0^+$  in the first pair of braces in (3.467) to obtain

$$\left[ \left\{ \hat{n} \cdot \bar{T}^+ - \nabla_S \cdot \left[ \vec{u}^+ (\hat{n} \cdot \bar{T}_0^+) \right] \right\} - \left\{ \hat{n} \cdot \bar{T}^- - \nabla_S \cdot \left[ \vec{u}^- (\hat{n} \cdot \bar{T}_0^-) \right] \right\} \right]_S = 0. \quad (3.468)$$

This is the final form of the stress boundary condition, where we changed the subscript from the point  $P$  to the surface  $S$ , which was straightforward.

### Frictionless Slip Boundaries

On frictionless slip boundaries, the normal component of the traction is still continuous, and the tangential components vanish. In this circumstance, the boundary condition (3.468) still holds. Furthermore, the frictionless property should be also taken into account.

We first take into account of the vanishing of the shear components of the traction in the undeformed state in the boundary condition (3.468). In the undeformed state, the normal component of the traction is

$$\pi_0 = \hat{n} \cdot (\hat{n} \cdot \bar{T}_0) = n_i n_j T_{0ij}, \quad (3.469)$$

and the vanishing of the shear components of the traction implies

$$\left\{ \hat{n} \cdot \bar{T}_0^\pm - \left[ \hat{n} \cdot (\hat{n} \cdot \bar{T}_0^\pm) \right] \hat{n} \right\}_S = 0. \quad (3.470)$$

It can then be obtained from the above two formulae that

$$\left( \hat{n} \cdot \bar{T}_0^\pm \right)_S = (\pi_0^\pm \hat{n})_S. \quad (3.471)$$

By substituting this relation into (3.468), we obtain

$$\left\{ \left[ \hat{n} \cdot \bar{T}^+ - \nabla_S \cdot (\pi_0^+ \vec{u}^+ \hat{n}) \right] - \left[ \hat{n} \cdot \bar{T}^- - \nabla_S \cdot (\pi_0^- \vec{u}^- \hat{n}) \right] \right\}_S = 0. \quad (3.472)$$

This formula will be further developed in the subsequent paragraphs.

We first attempt to take out the unite normal  $\hat{n}$  from under the operator  $\nabla_S$  in (3.472):

$$\begin{aligned} \nabla_S \cdot (\pi_0^\pm \vec{u}^\pm \hat{n}) &= \nabla \cdot (\pi_0^\pm \vec{u}^\pm \hat{n}) - \hat{n} (\hat{n} \cdot \nabla) \cdot (\pi_0^\pm \vec{u}^\pm \hat{n}) \\ &= \left[ \frac{\partial}{\partial x_i} (\pi_0^\pm u_i^\pm n_j) - n_i n_k \frac{\partial}{\partial x_k} (\pi_0^\pm u_i^\pm n_j) \right] \hat{e}_j \end{aligned}$$

$$\begin{aligned}
&= \left[ \frac{\partial}{\partial x_i} (\pi_0^\pm u_i^\pm) n_j - n_i n_k \frac{\partial}{\partial x_k} (\pi_0^\pm u_i^\pm) n_j \right] \hat{e}_j \\
&\quad + \left[ \pi_0^\pm u_i^\pm \frac{\partial n_j}{\partial x_i} - n_i n_k \pi_0^\pm u_i^\pm \frac{\partial n_j}{\partial x_k} \right] \hat{e}_j \\
&= \hat{n} [\nabla \cdot (\pi_0^\pm \vec{u}^\pm)] - \hat{n} [\hat{n}(\hat{n} \cdot \nabla) \cdot (\pi_0^\pm \vec{u}^\pm)] \\
&\quad + \pi_0^\pm \vec{u}^\pm \cdot \nabla \hat{n} - \pi_0^\pm \vec{u}^\pm \cdot [\hat{n}(\hat{n} \cdot \nabla) \hat{n}] \\
&= \hat{n} [\nabla_S \cdot (\pi_0^\pm \vec{u}^\pm)] + \pi_0^\pm \vec{u}^\pm \cdot \nabla_S \hat{n}. \tag{3.473}
\end{aligned}$$

By substituting this formula into (3.472), we obtain

$$\begin{aligned}
&\left\{ \left[ \hat{n} \cdot \vec{T}^+ - \hat{n} \nabla_S \cdot (\pi_0^+ u^+) - \pi_0^+ \vec{u}^+ \cdot \nabla_S \hat{n} \right] \right. \\
&\quad \left. - \left[ \hat{n} \cdot \vec{T}^- - \hat{n} \nabla_S \cdot (\pi_0^- u^-) - \pi_0^- \vec{u}^- \cdot \nabla_S \hat{n} \right] \right\}_S = 0. \tag{3.474}
\end{aligned}$$

In (3.474), the terms  $\vec{u}^\pm \cdot \nabla_S \hat{n}$  can be replaced by  $(\nabla_S \vec{u}^\pm) \cdot \hat{n}$ , which we prove by comparing the formulae of  $\hat{n}_Q - \hat{n}_P$  derived in two ways. A relation between  $\hat{n}'$  and  $\hat{n}_P$  can be derived from (3.456) and (3.451) as

$$\begin{aligned}
(n'_i)_{P'} &= \left[ \left( n_i + \frac{\partial u_m^+}{\partial x_m} n_i - \frac{\partial u_j^+}{\partial x_i} n_j \right) \left( 1 - \frac{\partial u_k^+}{\partial x_k} + \frac{\partial u_l^+}{\partial x_m} n_m n_l \right) \right]_P \\
&= \left( n_i - \frac{\partial u_j^+}{\partial x_i} n_j + \frac{\partial u_l^+}{\partial x_m} n_m n_l n_i \right)_P \\
&= \left[ n_i - \left( \frac{\partial u_j^+}{\partial x_i} - n_i n_m \frac{\partial u_j^+}{\partial x_m} \right) n_j \right]_P. \tag{3.475}
\end{aligned}$$

Its vector form is

$$\hat{n}'_{P'} = \hat{n}_P - \{ [\nabla \vec{u}^+ - \hat{n}(\hat{n} \cdot \nabla) \vec{u}^+] \cdot \hat{n} \}_P = \hat{n}_P - [(\nabla_S \vec{u}^+) \cdot \hat{n}]_P. \tag{3.476}$$

Similarly, a relation between  $\hat{n}'$  and  $\hat{n}_Q$  is

$$\hat{n}'_{P'} = \hat{n}_Q - \{ [\nabla \vec{u}^- - \hat{n}(\hat{n} \cdot \nabla) \vec{u}^-] \cdot \hat{n} \}_Q = \hat{n}_Q - [(\nabla_S \vec{u}^-) \cdot \hat{n}]_Q. \tag{3.477}$$

We obtain, according to (3.476) and (3.477), the first expression of  $\hat{n}_Q - \hat{n}_P$  as,

$$\hat{n}_Q - \hat{n}_P = \{ [(\nabla_S \vec{u}^+) \cdot \hat{n}]_P - [(\nabla_S \vec{u}^-) \cdot \hat{n}]_Q \} = -[(\nabla_S \vec{u}^+) \cdot \hat{n} - (\nabla_S \vec{u}^-) \cdot \hat{n}]_P, \tag{3.478}$$

where the difference between  $[(\nabla_S \vec{u}^-) \cdot \hat{n}]_Q$  and  $[(\nabla_S \vec{u}^-) \cdot \hat{n}]_P$  is of the second order of deformation, and is neglected. A second expression of  $\hat{n}_Q - \hat{n}_P$  can be derived more straightforwardly. By definition, we have, according to Fig. 3.16,

$$\hat{n}_Q - \hat{n}_P = \left( \frac{\partial n_i}{\partial x_l} \right)_P (u_l^+ - u_l^-) \hat{e}_i = \left[ \frac{\partial n_i}{\partial x_l} (u_l^+ - u_l^-) \right]_P \hat{e}_i. \quad (3.479)$$

By adding (3.463) to the term  $u_l^+ - u_l^-$  in the above formula, we obtain the second expression of  $\hat{n}_Q - \hat{n}_P$  as

$$\begin{aligned} \hat{n}_Q - \hat{n}_P &= \left\{ \frac{\partial n_i}{\partial x_l} [u_l^+ - u_j^+ n_l n_j - (u_l^- - u_j^- n_l n_j)] \right\}_P \hat{e}_i \\ &= \left[ u_l^+ \frac{\partial n_i}{\partial x_l} - u_j^+ n_j n_l \frac{\partial n_i}{\partial x_l} - \left( u_l^- \frac{\partial n_i}{\partial x_l} - u_j^- n_j n_l \frac{\partial n_i}{\partial x_l} \right) \right]_P \hat{e}_i \\ &= \left\{ \vec{u}^+ \cdot \nabla \hat{n} - \vec{u}^+ \cdot [\hat{n}(\hat{n} \cdot \nabla) \hat{n}] \right\} - \left\{ \vec{u}^- \cdot \nabla \hat{n} - \vec{u}^- \cdot [\hat{n}(\hat{n} \cdot \nabla) \hat{n}] \right\}_P \\ &= (\vec{u}^+ \cdot \nabla_S \hat{n} - \vec{u}^- \cdot \nabla_S \hat{n})_P. \end{aligned} \quad (3.480)$$

According to the two formulae of  $\hat{n}_Q - \hat{n}_P$ , (3.478) and (3.480), we can obtain

$$[\vec{u}^+ \cdot \nabla_S \hat{n} - \vec{u}^- \cdot \nabla_S \hat{n} + (\nabla_S \vec{u}^+) \cdot \hat{n} - (\nabla_S \vec{u}^-) \cdot \hat{n}]_S = 0, \quad (3.481)$$

where the sunscript ‘P’ is replaced by ‘S’ for generality. Finally, by substituting (3.481) into (3.474), as  $\pi_0^+ = \pi_0^-$ , we obtain

$$\begin{aligned} &\left\{ \left[ \hat{n} \cdot \vec{T}^+ - \hat{n} \nabla_S \cdot (\pi_0 \vec{u}^+) + \pi_0^+ (\nabla_S \vec{u}^+) \cdot \hat{n} \right] \right. \\ &\quad \left. - \left[ \hat{n} \cdot \vec{T}^- - \hat{n} \nabla_S \cdot (\pi_0 \vec{u}^-) + \pi_0^- (\nabla_S \vec{u}^-) \cdot \hat{n} \right] \right\}_S = 0, \end{aligned} \quad (3.482)$$

where the terms  $\vec{u}^\pm \cdot \nabla_S \hat{n}$  become  $(\nabla_S \vec{u}^\pm) \cdot \hat{n}$  as expected. This is the stress boundary condition from the continuity of traction across frictionless slip boundaries.

Another boundary condition on frictionless boundaries is the vanishing of the tangential components of the traction. At the point  $P'$  on the deformed boundary  $S'$ , we have

$$\{\hat{n}' \cdot \vec{\sigma}^\pm - [\hat{n}' \cdot (\hat{n}' \cdot \vec{\sigma}^\pm)] \hat{n}'\}_{P'} = 0. \quad (3.483)$$

On the undeformed surface  $S$ , we have

$$\{\hat{n} \cdot \vec{T}_0^\pm - [\hat{n} \cdot (\hat{n} \cdot \vec{T}_0^\pm)] \hat{n}\}_{P,Q} = 0, \quad (3.484)$$



where the subscript ‘ $P, Q$ ’ is assumed to be in correspondence to the superscript ‘ $\pm$ ’, indicating that the formula holds at both points  $P$  and  $Q$ , corresponding the signs ‘ $+$ ’ and ‘ $-$ ’, respectively. The above two formulae can be written in the following brief forms:

$$[(\hat{n}' \cdot \bar{\sigma}^\pm) \cdot (\bar{I} - \hat{n}'\hat{n}')]_{P'} = 0, \quad (3.485)$$

$$\left[ (\hat{n} \cdot \bar{T}_0^\pm) \cdot (\bar{I} - \hat{n}\hat{n}) \right]_{P,Q} = 0. \quad (3.486)$$

We can write (3.453) and (3.454) together as

$$(\hat{n}' \cdot \bar{\sigma}^\pm dA')_{P'} = \left[ \hat{n} \cdot (\bar{T}_0^\pm + \bar{T}^\pm) dA \right]_{P,Q}. \quad (3.487)$$

We can also obtain, according to (3.476) and (3.477),

$$\begin{aligned} \bar{I} - \hat{n}'\hat{n}' &= \{ \bar{I} - [\hat{n} - (\nabla_S \vec{u}^\pm) \cdot \hat{n}] [\hat{n} - (\nabla_S \vec{u}^\pm) \cdot \hat{n}] \}_{P,Q} \\ &= \{ \bar{I} - \hat{n}\hat{n} + \hat{n}[(\nabla_S \vec{u}^\pm) \cdot \hat{n}] + [(\nabla_S \vec{u}^\pm) \cdot \hat{n}]\hat{n} \}_{P,Q}. \end{aligned} \quad (3.488)$$

By substituting these two formulae into (3.485), and then using (3.486), we obtain

$$\left[ (\hat{n} \cdot \bar{T}^\pm) \cdot (\bar{I} - \hat{n}\hat{n}) + (\hat{n} \cdot \bar{T}_0^\pm) \cdot \{ \hat{n}[(\nabla_S \vec{u}^\pm) \cdot \hat{n}] + [(\nabla_S \vec{u}^\pm) \cdot \hat{n}]\hat{n} \} \right]_{P,Q} = 0. \quad (3.489)$$

This is the vanishing of the tangential components of the traction on the deformed boundary projected to the undeformed boundary, which can be further simplified.

Similar to (3.471), we have

$$(\hat{n} \cdot \bar{T}_0^\pm)_{P,Q} = (\pi_0^\pm \hat{n})_{P,Q}. \quad (3.490)$$

By substitute it into (3.489), we obtain

$$\left[ (\hat{n} \cdot \bar{T}^\pm) \cdot (\bar{I} - \hat{n}\hat{n}) + \pi_0^\pm \hat{n} \cdot \{ \hat{n}[(\nabla_S \vec{u}^\pm) \cdot \hat{n}] \} + \pi_0^\pm \hat{n} \cdot \{ [(\nabla_S \vec{u}^\pm) \cdot \hat{n}]\hat{n} \} \right]_{P,Q} = 0. \quad (3.491)$$

As

$$(\nabla_S \vec{u}^\pm) \cdot \hat{n} = [\nabla \vec{u}^\pm - \hat{n}(\hat{n} \cdot \nabla) \vec{u}^\pm] \cdot \hat{n} = \left( n_j \frac{\partial u_j^\pm}{\partial x_k} - n_k n_l n_j \frac{\partial u_j^\pm}{\partial x_l} \right) \hat{e}_k, \quad (3.492)$$

we have

$$\begin{aligned}\hat{n} \cdot \{[(\nabla_S \vec{u}^\pm) \cdot \hat{n}] \hat{n}\} &= \hat{n} \cdot \left[ \left( n_j n_m \frac{\partial u_j^\pm}{\partial x_k} - n_k n_l n_j n_m \frac{\partial u_j^\pm}{\partial x_l} \right) \hat{e}_k \hat{e}_m \right] \\ &= \left( n_k n_j n_m \frac{\partial u_j^\pm}{\partial x_k} - n_l n_j n_m \frac{\partial u_j^\pm}{\partial x_l} \right) \hat{e}_m = 0.\end{aligned}\quad (3.493)$$

As  $\hat{n} \cdot \hat{n} = n_k n_k = 1$ , we also have

$$\hat{n} \cdot \{\hat{n}[(\nabla_S \vec{u}^\pm) \cdot \hat{n}]\} = (\nabla_S \vec{u}^\pm) \cdot \hat{n}.\quad (3.494)$$

By substituting these two formulae into (3.491), we obtain

$$\{(\hat{n} \cdot \vec{T}^\pm) \cdot (\bar{I} - \hat{n}\hat{n}) + \pi_0^\pm (\nabla_S \vec{u}^\pm) \cdot \hat{n}\}_{P,Q} = 0.\quad (3.495)$$

According to (3.492), we have

$$[(\nabla_S \vec{u}^\pm) \cdot \hat{n}] \cdot (\hat{n}\hat{n}) = \left( n_k n_j \frac{\partial u_j^\pm}{\partial x_k} - n_l n_j \frac{\partial u_j^\pm}{\partial x_l} \right) \hat{n} = 0.\quad (3.496)$$

Finally, we use this formula to write (3.495) as

$$\left\{ [\hat{n} \cdot \vec{T}^\pm + \pi_0^\pm (\nabla_S \vec{u}^\pm) \cdot \hat{n}] \cdot (\bar{I} - \hat{n}\hat{n}) \right\}_S = 0,\quad (3.497)$$

where the subscript  $P, Q$  is replaced by  $S$  for generality. This is the stress boundary condition from the vanishing of the tangential components of the traction.

Finally, we provide a summary of the stress boundary conditions across and on frictionless boundaries. Define

$$\vec{t}^\pm = \hat{n} \cdot \vec{T}^\pm - \hat{n} \nabla_S \cdot (\pi_0 \vec{u}_\pm) + \pi_0^\pm (\nabla_S \vec{u}^\pm) \cdot \hat{n}.\quad (3.498)$$

The continuity of traction, (3.482), can be written as

$$(\vec{t}^+ - \vec{t}^-)_S = 0,\quad (3.499)$$

implying that  $\vec{t}$  is continuous. We now express the formula for the vanishing of the tangential components of the traction, (3.497), in terms of  $\vec{t}^\pm$ . We use (3.496) to express (3.497) as

$$\{\hat{n} \cdot \vec{T}^\pm - [\hat{n} \cdot (\hat{n} \cdot \vec{T}^\pm)] \hat{n} + \pi_0^\pm (\nabla_S \vec{u}^\pm) \cdot \hat{n}\}_S = 0,\quad (3.500)$$

where the term  $[\hat{n} \cdot (\hat{n} \cdot \vec{T}^\pm)]\hat{n}$  is equivalent to the term  $(\hat{n} \cdot \vec{T}^\pm) \cdot (\hat{n}\hat{n})$  in (3.497). This formula is in fact equivalent to (3.495). We also have, taking into account of (3.496),

$$(\hat{n} \cdot \vec{t}^\pm)\hat{n} = [\hat{n} \cdot (\hat{n} \cdot \vec{T}^\pm)]\hat{n} - \hat{n} \nabla_S \cdot (\pi_0 \vec{u}^\pm). \quad (3.501)$$

By subtracting (3.501) from (3.498), and then using (3.500), we obtain finally

$$\vec{t}_S^\pm = [(\hat{n} \cdot \vec{t}^\pm)\hat{n}]_S. \quad (3.502)$$

This formula indicates that  $\vec{t}$  is in the same direction as  $\hat{n}$ , i.e.,  $\vec{t}$  is perpendicular to the boundary. In summary, the boundary conditions are (3.499) and (3.502).

### 3.7.3 Gravitational Potential and Its First Order Derivatives Across All Boundaries

The boundary conditions of the gravitational potential are identical for both welded and slip boundaries. Here we formulate the problem for the simpler case of welded boundary, which can be generalized to the case of slip boundary in a similar fashion as for the stress.

For solid bodies, the gravitational potential as well as its first order derivatives is continuous. According to Fig. 3.15, the boundary conditions in the deformed state are

$$(\phi^+ - \phi^-)_{S'} = 0, \quad (3.503)$$

$$(\nabla\phi^+ - \nabla\phi^-)_{S'} = 0, \quad (3.504)$$

and those in the undeformed state are

$$(\phi_0^+ - \phi_0^-)_S = 0, \quad (3.505)$$

$$(\nabla\phi_0^+ - \nabla\phi_0^-)_S = 0. \quad (3.506)$$

Our objective is to derive the boundary conditions for  $\phi_1 = \phi - \phi_0$  defined in (3.341).

All the derivations will still be accurate to the first order of displacement.

According to the definition of  $\phi_1$ , we have

$$\begin{aligned} \phi_{S'} &= (\phi_0)_{S'} + (\phi_1)_{S'} \\ &= (\phi_0)_S + \left( \frac{\partial\phi_0}{\partial x_k} \right)_S u_k + (\phi_1)_S + \left( \frac{\partial\phi_1}{\partial x_k} \right)_S u_k \\ &= (\phi_0)_S + (\vec{u} \cdot \nabla\phi_0)_S + (\phi_1)_S, \end{aligned} \quad (3.507)$$

where, as  $\vec{u}^+$  and  $\vec{u}^-$  are equal, we did not distinguish them, and used  $\vec{u}$  to represent both of them. According to (3.507), we can use (3.503), (3.505), and (3.506) to obtain

$$(\phi_1^+ - \phi_1^-)_S = 0, \quad (3.508)$$

i.e.,  $\phi_1$  is continuous, which is the first boundary condition of the gravitational potential.

According to (3.507), we can use the definition of  $\phi_1$  to obtain

$$(\nabla\phi^\pm)_{S'} = (\nabla\phi_0^\pm)_{S'} + (\nabla\phi_1^\pm)_{S'} = (\nabla\phi_0^\pm)_S + (\vec{u} \cdot \nabla\nabla\phi_0^\pm)_S + (\nabla\phi_1^\pm)_S. \quad (3.509)$$

By substituting this formula into (3.504), and then using (3.506), we obtain a primitive form of the boundary condition for the derivatives of  $\phi_1$ ,

$$[(\nabla\phi_1^+ + \vec{u} \cdot \nabla\nabla\phi_0^+) - (\nabla\phi_1^- + \vec{u} \cdot \nabla\nabla\phi_0^-)]_S = 0. \quad (3.510)$$

Another condition for  $\phi_1$  can be obtained according to (3.508),

$$(\nabla_S\phi_1^+ - \nabla_S\phi_1^-)_S = 0. \quad (3.511)$$

This is based on the fact that, in order to ensure  $\phi_1$  to be equal at both sides of the boundary, the rate of change of  $\phi_1$  on the boundary, i.e., the directional derivatives along tangential directions, at both sides must also be equal. By subtracting (3.511) from (3.510), we obtain, with  $\nabla\phi_1^\pm = [\nabla_S + \hat{n}(\hat{n} \cdot \nabla)]\phi_1^\pm$ ,

$$\{[\hat{n}(\hat{n} \cdot \nabla)\phi_1^+ + \vec{u} \cdot \nabla\nabla\phi_0^+] - [\hat{n}(\hat{n} \cdot \nabla)\phi_1^- + \vec{u} \cdot \nabla\nabla\phi_0^-]\}_S = 0. \quad (3.512)$$

The term  $\hat{n}(\hat{n} \cdot \nabla)\phi_1^\pm$  can also be written as  $(\partial\phi_1^\pm/\partial n)\hat{n}$ . Hence,

$$\left[ \left( \frac{\partial\phi_1^+}{\partial n} \hat{n} + \vec{u} \cdot \nabla\nabla\phi_0^+ \right) - \left( \frac{\partial\phi_1^-}{\partial n} \hat{n} + \vec{u} \cdot \nabla\nabla\phi_0^- \right) \right]_S = 0. \quad (3.513)$$

According to the same logic to obtain (3.511) from (3.508), we can obtain from (3.506) that  $\phi_0$  satisfy the condition

$$[\nabla_S(\nabla\phi_0^+) - \nabla_S(\nabla\phi_0^-)]_S = 0. \quad (3.514)$$

As  $\nabla\nabla\phi_0^\pm = [\nabla_S + \hat{n}(\hat{n} \cdot \nabla)]\nabla\phi_0^\pm$ , we can use the above formula to reduce (3.513) to

$$\left\{ \left[ \frac{\partial\phi_1^+}{\partial n} \hat{n} + (\hat{n} \cdot \vec{u})(\hat{n} \cdot \nabla)\nabla\phi_0^+ \right] - \left[ \frac{\partial\phi_1^-}{\partial n} \hat{n} + (\hat{n} \cdot \vec{u})(\hat{n} \cdot \nabla)\nabla\phi_0^- \right] \right\}_S = 0. \quad (3.515)$$

By performing a scalar product with  $\hat{n}$  at both sides, we obtain, with  $\hat{n} \cdot (\hat{n} \cdot \nabla) \nabla \phi_0^\pm = \hat{n} \cdot (\hat{n} \cdot \nabla \nabla \phi_0^\pm)$ ,

$$\left\{ \left[ \frac{\partial \phi_1^+}{\partial n} + \hat{n} \cdot (\hat{n} \cdot \nabla \nabla \phi_0^+) (\hat{n} \cdot \vec{u}) \right] - \left[ \frac{\partial \phi_1^-}{\partial n} + \hat{n} \cdot (\hat{n} \cdot \nabla \nabla \phi_0^-) (\hat{n} \cdot \vec{u}) \right] \right\}_S = 0. \quad (3.516)$$

As  $\hat{n} \cdot (\hat{n} \cdot \nabla \nabla \phi_0^\pm) = \partial^2 \phi_0^\pm / \partial n^2$ , we have, according to (3.106),

$$\begin{aligned} & [\hat{n} \cdot (\hat{n} \cdot \nabla \nabla \phi_0^+) - \hat{n} \cdot (\hat{n} \cdot \nabla \nabla \phi_0^-)]_S \\ &= \left( \frac{\partial^2 \phi_0^+}{\partial n^2} - \frac{\partial^2 \phi_0^-}{\partial n^2} \right)_S = (-4\pi G \rho_0^+ + 4\pi G \rho_0^-)_S. \end{aligned} \quad (3.517)$$

Finally, we can use this formula to reduce (3.515) to

$$\left\{ \left[ \frac{\partial \phi_1^+}{\partial n} - 4\pi G \rho_0^+ (\hat{n} \cdot \vec{u}^+) \right] - \left[ \frac{\partial \phi_1^-}{\partial n} - 4\pi G \rho_0^- (\hat{n} \cdot \vec{u}^-) \right] \right\}_S = 0, \quad (3.518)$$

which is the boundary condition for the first order derivative of  $\phi_1$ . This is the second boundary condition of the gravitational potential.

### 3.7.4 Summary • Earth's Surface • Hydrostatic Equilibrium Undeformed State

We have up to present formulated all the boundary conditions. The following is a summary. The boundary conditions for displacement and stress on welded boundaries are (3.424) and (3.440). Here we write them as

$$\{\vec{u}\}_\pm^\pm = 0, \quad (3.519)$$

$$\{\hat{n} \cdot \vec{T}\}_\pm^\pm = 0, \quad (3.520)$$

where the symbol  $\{\cdot\}_\pm^\pm$  indicates the difference of the quantity in the braces at the two sides of the boundary. A vanishing difference implies continuity across the boundary. The boundary condition for stress, (3.520), can also be written as

$$\{\vec{t}\}_\pm^\pm = 0, \quad (3.521)$$

where  $\vec{t}$  is defined in (3.498), i.e.,

$$\vec{t} = \hat{n} \cdot \vec{T} - \hat{n} \nabla_S \cdot (\pi_0 \vec{u}) + \pi_0 (\nabla_S \vec{u}) \cdot \hat{n}. \quad (3.522)$$

This is due to the facts that both  $\pi_0$  and  $\vec{u}$  are continuous across the boundary, implying that both  $\nabla_S \cdot (\pi_0 \vec{u})$  and  $\pi_0 (\nabla_S \vec{u})$  are also continuous across the boundary. The boundary condition for displacement and stress on slip boundaries are (3.444) and (3.468). Here we write them as

$$\{\hat{n} \cdot \vec{u}\}_-^+ = 0, \quad (3.523)$$

$$\{\hat{n} \cdot \vec{T} - \nabla_S \cdot [\vec{u}(\hat{n} \cdot \vec{T}_0)]\}_-^+ = 0. \quad (3.524)$$

On frictionless slip boundaries, the boundary condition for displacement is the same as that of the general slip boundary. The boundary conditions for stress are (3.499) and (3.502). Here we write them as

$$\{\vec{t}\}_-^+ = 0, \quad (3.525)$$

$$\vec{t} = (\hat{n} \cdot \vec{t}) \hat{n}. \quad (3.526)$$

On all boundaries, the boundary conditions for the gravitational potential are (3.508) and (3.518). Here we write them as

$$\{\phi_1\}_-^+ = 0, \quad (3.527)$$

$$\left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G \rho_0 (\hat{n} \cdot \vec{u}) \right\}_-^+ = 0. \quad (3.528)$$

The surface of the Earth  $S_0$  is a special boundary. In general, there is no known boundary condition for  $\vec{u}$ . For stress, the traction is assumed to vanish in the undeformed state, i.e.,

$$(\hat{n} \cdot \vec{T}_0)_{S_0} = 0, \quad (3.529)$$

and in the deformed state, the traction is assumed known to be  $\vec{t}_0^E$ , i.e.,

$$(\hat{n}' \cdot \vec{\sigma})_{S'_0} = \vec{t}_0^E. \quad (3.530)$$

We have particularly used the superscript ‘E’ to indicate that the applied traction is in Eulerian description, i.e., the force per unit areas in the deformed state. According to the above two formulae and (3.438) and (3.456), we obtain

$$(\hat{n} \cdot \vec{T})_{S_0} = \frac{dA'}{dA} \vec{t}_0^E = (1 + \nabla_S \cdot \vec{u}) \vec{t}_0^E, \quad (3.531)$$

where  $dA'$  and  $dA$  are the areas of the same material surface element in the deformed and undeformed states, respectively. Therefore,

$$\vec{t}_0 = \frac{dA'}{dA} \vec{t}_0^E = (1 + \nabla_S \cdot \vec{u}) \vec{t}_0^E \quad (3.532)$$

can be understood as  $dA' \vec{t}_0^E$  divided by  $dA$ , the former being the total traction on a material surface of area  $dA'$  in the deformed state, and the latter the area of this material surface element in the undeformed state. As the traction in the undeformed state vanishes,  $\vec{t}_0^E dA'$  can be understood as the Lagrangian increment of the traction on the material surface element, and consequently,  $\vec{t}_0$  can be interpreted as the Lagrangian increment of traction measured based on the configuration of the undeformed state. As  $\pi_0$  vanishes on the surface of the Earth, the boundary condition (3.531) can also be written as, according to (3.522),

$$\vec{t}_{S_0} = \vec{t}_0. \quad (3.533)$$

Note that  $\vec{t}_0^E$  is the cause of  $\nabla_S \cdot \vec{u}$ . Hence, the term  $\nabla_S \cdot \vec{u}$  in the expression of  $\vec{t}_0$  represents a second order quantity, and can be neglected. For the same reason, it is not necessary to distinguish the Lagrangian and Eulerian descriptions for the traction of the surface of the Earth, and  $\vec{t}_0^E$  could be replaced by  $\vec{t}_0$ . We thus have

$$\vec{t}_{S_0} = \vec{t}_0. \quad (3.534)$$

The boundary conditions of the gravitational potential are the same as (3.527) and (3.528), noting that, if  $\hat{n}$  is defined to point outward, we have  $\rho^+ = 0$ .

If the undeformed state is in hydrostatic equilibrium, the boundary condition for the stress can be further simplified. In this circumstance,  $\bar{T}$  is customarily used instead of  $\tilde{T}$ . Due to the coincidence of the equi-value surfaces of gravity, pressure and density,  $\pi_0 = -p_0$  is constant on all boundaries, and it can be taken out of the operator  $\nabla_S$  in (3.522). We have, taking also reference to (3.419) for the relation between  $\bar{T}$  and  $\tilde{T}$ ,

$$\begin{aligned} \vec{t} &= \hat{n} \cdot \bar{\tilde{T}} + \hat{n} p_0 \nabla_S \cdot \vec{u} - p_0 \nabla_S \vec{u} \cdot \hat{n} \\ &= \hat{n} \cdot \left[ \bar{T} + p_0 (\nabla \vec{u})^T - p_0 (\nabla \cdot \vec{u}) \bar{I} \right] + \hat{n} p_0 \left[ \nabla \cdot \vec{u} - \hat{n} (\hat{n} \cdot \nabla) \cdot \vec{u} \right] \\ &\quad - p_0 \left[ \nabla \vec{u} - \hat{n} (\hat{n} \cdot \nabla) \vec{u} \right] \cdot \hat{n} \\ &= \hat{n} \cdot \bar{T} + p_0 \left( n_i \frac{\partial u_i}{\partial x_j} - n_j \frac{\partial u_k}{\partial x_k} \right) \hat{e}_j + p_0 \left( n_j \frac{\partial u_k}{\partial x_k} - n_j n_i n_k \frac{\partial u_i}{\partial x_k} \right) \hat{e}_j \\ &\quad - p_0 \left( n_i \frac{\partial u_i}{\partial x_j} - n_j n_i n_k \frac{\partial u_i}{\partial x_k} \right) \hat{e}_j \\ &= \hat{n} \cdot \bar{T}. \end{aligned} \quad (3.535)$$

Hence, the stress boundary condition on welded boundaries, (3.521), can be written as

$$\{\hat{n} \cdot \bar{T}\}_\pm^\pm = 0, \quad (3.536)$$

the stress boundary condition on frictionless slip boundaries, (3.525) and (3.526), can be written as

$$\{\hat{n} \cdot \bar{T}\}_-^+ = 0, \quad (3.537)$$

$$\hat{n} \cdot \bar{T} = [\hat{n} \cdot (\hat{n} \cdot \bar{T})]\hat{n}, \quad (3.538)$$

and the stress boundary condition on the Earth's surface, (3.531) or (3.533), can be written as

$$(\hat{n} \cdot \bar{T})_{S_0} = \vec{t}_0. \quad (3.539)$$

The formulation of the boundary conditions is now complete. Their application will be made clear in the formulation of the solution of the equations of the Earth's deformation.

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## Bibliographic Notes

The subjects of Sect. 3.1 is usually taught in courses of *Classical Mechanics*. There are numerous books in this topic, such as Gregory (2012). The subjects of Sect. 3.2 is usually included more comprehensively in books of *Physical Geodesy* dedicated to the static gravity field of the Earth, such as Guo (2023), where several of the topics are presented based on more fundamental mathematics. The subjects of Sect. 3.3 and 3.4 are usually a focus of books in *Continuum Mechanics*, such as Irgens (2008) and Reddy (2013). The author benefited the most from Du and Zheng (1986). The author's main reference includes Moritz and Mueller (1987) for Sect. 3.5; Biot (1965), Dahlen (1972), Dahlen and Tromp (1998), Eringen (1980), Wang (1997) and Woodhouse and Dahlen (1978) for Sects. 3.6 and 3.7. The references listed above are also suggestions as the readers' sources for alternative presentations.

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## Abstract

This chapter systematically presents the Legendre and associated Legendre functions and spherical harmonics. These functions are defined by deriving a finite solution of Laplace's equation in the spherical coordinate system based on Fourier's method of variable separation. Several subjects usually not found in introductory texts are covered. These include some extra orthogonality formulae, an asymptotic expression of the Legendre function, the closed-form formulae of some Legendre sums, the isotropic smoothing of functions defined on a sphere, the expression of a three-dimensional scalar field in spherical harmonics, and the decomposition of a three-dimensional vector field into scalar fields followed with their expressions in spherical harmonics. As an application, the partial differential equations governing the elastic deformation of the Earth are reduced to a system of ordinary differential equations with the distance to the Earth center as the variable. Finally, the discrete spherical transform, together with the discrete Fourier transform and discrete Chebyshev polynomial transform, is presented. As a supplement, a numerical method for solving differential equations, called Chebyshev collocation method, is presented, which is particularly suitable for handling singularities of the equations at a boundary, which is the case of the system of ordinary differential equations mentioned.

## 4.1 Legendre and Associated Legendre Equations

### 4.1.1 Separation of Variables of Laplace's Equation in Spherical Coordinates

In this book, we introduce the Legendre function and spherical harmonics by solving Laplace's equation in the spherical coordinate system. With the Laplacian given

by (2.494), Laplace's equation in spherical coordinates is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi}{\partial \lambda^2} \right] = 0. \quad (4.1)$$

We express the solution of this equation as a product of functions depending on different variables to reduce the partial differential equation to a number of ordinary differential equations depending on different variables. This approach of separating variable is referred to as Fourier method.

First, we separate the dependence on  $r$  from those on  $\theta$  and  $\lambda$  by assuming

$$\phi(r, \theta, \lambda) = f(r)Y(\theta, \lambda). \quad (4.2)$$

Its substitution into (4.1) yields

$$\frac{Y}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) + \frac{f}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \lambda^2} \right] = 0. \quad (4.3)$$

Multiply both sides by  $r^2/(fY)$ , and then move the second term to the right-hand side. We obtain

$$\frac{1}{f} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) = -\frac{1}{Y} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \lambda^2} \right]. \quad (4.4)$$

In this equation, the left-hand side is a function of  $r$ , while the right-hand side is a function of  $\theta$  and  $\lambda$ . The only possibility for the equality to hold is that the left and right-hand sides equal to the same constant, say  $\alpha$ . We thus obtain

$$\frac{d}{dr} \left( r^2 \frac{df}{dr} \right) - \alpha f = 0, \quad (4.5)$$

$$\frac{1}{Y} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \lambda^2} \right] + \alpha = 0. \quad (4.6)$$

In order to separate the dependence on  $\theta$  and  $\lambda$  of (4.6), we assume

$$Y(\theta, \lambda) = B(\theta)L(\lambda). \quad (4.7)$$

Its substitution into (4.6) leads to

$$\frac{1}{B} \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dB}{d\theta} \right) + \frac{1}{L} \frac{1}{\sin^2 \theta} \frac{d^2 L}{d\lambda^2} + \alpha = 0. \quad (4.8)$$

Multiply both sides by  $\sin^2 \theta$ , and then move the second term to the right-hand side. We obtain

$$\frac{1}{B} \sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{dB}{d\theta} \right) + \alpha \sin^2 \theta = -\frac{1}{L} \frac{d^2 L}{d\lambda^2}. \quad (4.9)$$

This is another equation with left and right-hand sides depending on different variables. In order for the equality to hold, the left and right-hand sides must equal to the same constant, say  $\beta$ . We obtain

$$\sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{dB}{d\theta} \right) + (\alpha \sin^2 \theta - \beta) B = 0, \quad (4.10)$$

$$\frac{d^2 L}{d\lambda^2} + \beta L = 0. \quad (4.11)$$

We have thus separated Laplace's partial differential equation into three ordinary differential equations (4.5), (4.10), and (4.11), which we are going to solve.

### 4.1.2 The Eigenvalue-Eigenfunction Solution of the Equation in Longitude

While separating variables, we have introduced two constant parameters  $\alpha$  and  $\beta$ . Conceptually, no constraint has been imposed on these parameters, thus they are considered arbitrary. However, when the background physics of the problem is considered, these parameters can no longer be taken arbitrarily.

We first solve (4.11), and meanwhile determine  $\beta$ . We certainly assume that the function  $\phi$  is a single-valued function. For any fixed values of  $r$  and  $\theta$ ,  $2m\pi + \lambda$  correspond to the same location in space when  $m = 0, 1, 2, \dots$ , and hence,  $\phi$  must have the same value. This requires the  $\lambda$ -dependent function  $L$  to have the following property:

$$L(\lambda) = L(2m\pi + \lambda). \quad (4.12)$$

This property implies that  $L$  must be a periodical function with a period of  $2\pi$ , thus being referred to as periodical condition. While solving (4.11), the value of  $\beta$  must be chosen so that the solution satisfies the periodical condition. It can be easily verified that (4.11) has solutions satisfying the periodical condition only when

$$\beta = k^2, \quad (4.13)$$

and there are two linearly independent solutions:

$$L_1 = \cos k\lambda, \quad L_2 = \sin k\lambda. \quad (4.14)$$

When the sign of  $k$  in these solutions is reversed, the results are still solutions, but these are linearly dependent on the original ones. Hence, we assume  $k \geq 0$  hereafter.

A problem of solving a differential equation with a parameter for non-trivial solutions when the parameter takes certain values is referred to as an eigenvalue problem. The solutions are called eigenfunctions, and the values of the parameter are called eigenvalues. For the above problem,  $\cos k\lambda$  and  $\sin k\lambda$  are eigenfunctions, and  $k^2$  are eigenvalues. In this problem, one eigenvalue corresponds to two eigenfunctions when  $k \neq 0$ .

### 4.1.3 Legendre and Associated Legendre Equations

We now consider the equation depending on the polar angle  $\theta$ , (4.10). For convenience, we substitute  $\theta$  by  $x = \cos \theta$ . We also have  $\sin \theta = (1 - x^2)^{1/2}$  and  $d/d\theta = (dx/d\theta)(d/dx) = -\sin \theta(d/dx) = -(1 - x^2)^{1/2}(d/dx)$ . Hence, (4.10) becomes

$$(1 - x^2) \frac{d}{dx} \left[ (1 - x^2) \frac{dB}{dx} \right] + [\alpha(1 - x^2) - \beta]B = 0. \quad (4.15)$$

Divide both its sides by  $1 - x^2$ , and then substitute (4.13) into it. We obtain

$$(1 - x^2) \frac{d^2 B}{dx^2} - 2x \frac{dB}{dx} + \left( \alpha - \frac{k^2}{1 - x^2} \right) B = 0, \quad (4.16)$$

which is referred to as associated Legendre equation. When  $k = 0$ , the associated Legendre equation degenerates to the Legendre equation

$$(1 - x^2) \frac{d^2 B^0}{dx^2} - 2x \frac{dB^0}{dx} + \alpha B^0 = 0. \quad (4.17)$$

By definition, the domain is  $x \in [-1, 1]$ . We assume that the function  $\phi$  is finite. This requires the function  $B$  to be finite in  $x \in [-1, 1]$ , which is a kind of restriction called finite condition. In subsequent sections, we will solve the Legendre and associated Legendre equations under the finite condition, and meanwhile determine the values that the parameter  $\alpha$  may take. These are also eigenvalue problems.

The solutions of the Legendre and associated Legendre equations are related. We first derive a relation between these equations. Define a function  $\zeta(x)$  according to

$$B(x) = (1 - x^2)^{k/2} \zeta(x) \quad (4.18)$$

to substitute  $B(x)$  in the associated Legendre equation (4.16). We have

$$\frac{dB}{dx} = (1 - x^2)^{k/2} \frac{d\zeta}{dx} - kx(1 - x^2)^{k/2-1} \zeta, \quad (4.19)$$

$$\frac{d^2 B}{dx^2} = (1 - x^2)^{k/2} \frac{d^2 \zeta}{dx^2} - 2kx(1 - x^2)^{k/2-1} \frac{d\zeta}{dx}$$

$$-k \left[ (1-x^2)^{k/2-1} - 2 \left( \frac{k}{2} - 1 \right) x^2 (1-x^2)^{k/2-2} \right] \zeta. \quad (4.20)$$

Substitute the above three formulae into (4.16). After some rearrangement, we divide both sides by  $(1-x^2)^{k/2}$  to obtain

$$(1-x^2) \frac{d^2 \zeta}{dx^2} - 2(k+1)x \frac{d\zeta}{dx} + [\alpha - k(k+1)]\zeta = 0. \quad (4.21)$$

Take a derivative of both sides with respect to  $\zeta$ . We obtain

$$(1-x^2) \frac{d^3 \zeta}{dx^3} - 2(k+2)x \frac{d^2 \zeta}{dx^2} + [\alpha - (k+1)(k+2)] \frac{d\zeta}{dx} = 0. \quad (4.22)$$

While comparing these last two equations, we see that the later can be obtained by replacing  $k$  and  $\zeta$  in the former by  $k+1$  and  $d\zeta/dx$ , respectively. As the former becomes identical to the Legendre equation if we set  $k=0$ , we can then conclude that it can be obtained by taking the derivative of the Legendre equation  $k$  times. Hence, we have  $\zeta = d^k B_0 / dx^k$ , and according to (4.18), we further have

$$B = (1-x^2)^{k/2} \frac{d^k B^0}{dx^k}. \quad (4.23)$$

This relation tells that the two linearly independent solutions of the associated Legendre equation can be determined based on the two linearly independent solutions of the Legendre equation.

## 4.2 Legendre and Associated Legendre Functions

### 4.2.1 A Power Series Solution of the Legendre Equation

We seek the power series solution of the Legendre equation (4.17) in the form of

$$B^0 = \sum_{i=0}^{\infty} C_i x^i. \quad (4.24)$$

we have

$$\frac{dB^0}{dx} = \sum_{i=0}^{\infty} i C_i x^{i-1}, \quad (4.25)$$

$$\frac{d^2 B^0}{dx^2} = \sum_{i=0}^{\infty} i(i-1) C_i x^{i-2}. \quad (4.26)$$

Substitute the above three formulae into (4.17). We obtain

$$\sum_{i=0}^{\infty} i(i-1)C_i x^{i-2} + \sum_{i=0}^{\infty} [-i(i-1)C_i - 2iC_i + \alpha C_i]x^i = 0. \quad (4.27)$$

In the first term, we use a “new”  $i$  to represent the “old”  $i-2$  to obtain

$$\sum_{i=0}^{\infty} i(i-1)C_i x^{i-2} = \sum_{i=0}^{\infty} (i+2)(i+1)C_{i+2} x^i = 0. \quad (4.28)$$

Substitute it back to (4.27). We obtain

$$\sum_{i=0}^{\infty} \{(i+2)(i+1)C_{i+2} - [i(i+1) - \alpha]C_i\}x^i = 0. \quad (4.29)$$

Evidently, in order for the power series to vanish for any  $x$ , all the coefficients must vanish,

$$(i+2)(i+1)C_{i+2} - [i(i+1) - \alpha]C_i = 0. \quad (4.30)$$

A recurrence formula for the coefficients can then be derived,

$$C_{i+2} = \frac{i(i+1) - \alpha}{(i+2)(i+1)} C_i. \quad (4.31)$$

The Legendre equation is a second order homogeneous linear ordinary differential equation, which has two linearly independent solutions. We see that, when  $C_0 \neq 0$  and  $C_1 = 0$ ,  $C_{2i+1} = 0$ , and when  $C_0 = 0$  and  $C_1 \neq 0$ ,  $C_{2i} = 0$ . Hence, the two linearly independent solutions can be written as

$$B_1^0 = \sum_{i=0}^{\infty} C_{2i} x^{2i}, \quad B_2^0 = \sum_{i=0}^{\infty} C_{2i+1} x^{2i+1}. \quad (4.32)$$

The coefficients  $C_{2i}$  and  $C_{2i+1}$  are computed from  $C_0$  and  $C_1$ , respectively, using (4.31). It is evident that  $B_1^0$  includes only even power terms, and  $B_2^0$  includes only odd power terms.

### 4.2.2 Convergence of the Power Series Solution

We now study the convergence properties of the two power series in (4.32). If a series converges to a finite value, it is said to be finite.

It is shown in most calculus textbooks that the convergence radius of a power series like (4.24) is

$$\delta = \lim_{i \rightarrow \infty} \left| \frac{C_i}{C_{i+1}} \right|, \quad (4.33)$$

if the limit exists.

In (4.32), we can take  $B_1^0$  as a power series of  $x^2$ , and  $B_2^0$  the product between  $x$  and a power series of  $x^2$ . Hence, for these two power series, the ratios of coefficients corresponding to that in (4.33) are the ratios between successive coefficients. As the coefficients of these two series have the same recurrence formula (4.31), they have the same convergence radius,

$$\delta = \lim_{i \rightarrow \infty} \left| \frac{C_i}{C_{i+2}} \right| = \lim_{i \rightarrow \infty} \left| \frac{(i+2)(i+1)}{i(i+1)+\alpha} \right| = 1. \quad (4.34)$$

This means that, in  $(-1, 1)$ , both power series in (4.32) converges to a finite value; hence, both solutions are finite.

The problem is that, when  $x = \pm 1$ , both series in (4.32) are infinite, meaning that the solutions are infinite. In order to prove this, we first estimate the magnitudes of the coefficients  $C_i$ . According to (4.31),

$$C_i = \frac{(i-2)(i-1)-\alpha}{i(i-1)} C_{i-2} = \frac{1}{i} \left[ 1 - \frac{\alpha}{(i-1)(i-2)} \right] (i-2) C_{i-2}. \quad (4.35)$$

Using this formula repeatedly, we get

$$C_{i-2} = \frac{1}{i-2} \left[ 1 - \frac{\alpha}{(i-3)(i-4)} \right] (i-4) C_{i-4}, \quad (4.36)$$

.....

$$C_{N+2} = \frac{1}{N+2} \left[ 1 - \frac{\alpha}{(N+1)N} \right] N C_N. \quad (4.37)$$

Substitute (4.36)–(4.37) into (4.35). We obtain

$$C_i = \frac{1}{i} \left[ 1 - \frac{\alpha}{(i-1)(i-2)} \right] \left[ 1 - \frac{\alpha}{(i-3)(i-4)} \right] \cdots \left[ 1 - \frac{\alpha}{(N+1)N} \right] N C_N, \quad (4.38)$$

where  $i$  must be large enough so that  $i-2 \geq N$ . Evidently, the values within all the brackets are positive and smaller than 1 when  $\alpha > 0$ , and larger than 1 when  $\alpha < 0$ . Hence, if we can show that both series  $B_1^0$  and  $B_2^0$  are infinite when  $\alpha > 0$ , we can also conclude that they are infinite when  $\alpha < 0$ . Hence we assume  $\alpha > 0$  below.



We first analyze the order of magnitude of the coefficients. Define

$$\Pi = \left[1 - \frac{\alpha}{(i-1)(i-2)}\right] \left[1 - \frac{\alpha}{(i-3)(i-4)}\right] \cdots \left[1 - \frac{\alpha}{(N+1)N}\right]. \quad (4.39)$$

We chose a value of  $N$  sufficiently large, so that  $1 - \alpha/[(N+1)N] > 0$ . Then, for any  $i+2 > N$ , we have

$$\Pi > \left[1 - \frac{\alpha}{(N+1)N}\right] \cdots \left[1 - \frac{\alpha}{(i-3)(i-4)}\right] \left[1 - \frac{\alpha}{(i-1)(i-2)}\right] \cdots. \quad (4.40)$$

Take the logarithm at both sides,

$$\begin{aligned} \ln \Pi &> \ln \left[1 - \frac{\alpha}{(N+1)N}\right] + \cdots + \ln \left[1 - \frac{\alpha}{(i-3)(i-4)}\right] \\ &\quad + \ln \left[1 - \frac{\alpha}{(i-1)(i-2)}\right] + \cdots. \end{aligned} \quad (4.41)$$

As  $\lim_{i \rightarrow \infty} \{\alpha/[(i-1)(i-2)]\} = 0$ , we have

$$\lim_{i \rightarrow \infty} \left\{ \ln \left[1 - \frac{\alpha}{(i-1)(i-2)}\right] \middle/ \left[-\frac{\alpha}{(i-1)(i-2)}\right] \right\} = 1, \quad (4.42)$$

which can be considered as an alternative form of the formula  $\lim_{x \rightarrow 0} [\ln(1+x)]/x = 1$ . Hence, the infinite series at the right-hand side of (4.41) converges or diverges in the same way as the series

$$\frac{\alpha}{(N+1)N} + \cdots + \frac{\alpha}{(i-3)(i-4)} + \frac{\alpha}{(i-1)(i-2)} + \cdots, \quad (4.43)$$

which does converge, and the proof can be found in most text books of calculus. Therefore, the infinite series at the right-hand side of (4.41) does converge. We denote its sum as  $S$ . We then have  $\ln \Pi > S$ , and thus  $\Pi > e^S$ . Denote  $M = e^S N C_N$ . As  $N$  is fixed,  $M$  is also fixed. It can then be concluded from (4.38) and (4.39) that  $C_i$  and  $M$  have the same sign as  $C_N$ , and when  $i$  is sufficiently large,

$$|C_i| > \frac{|M|}{i}. \quad (4.44)$$

For the two infinite series in (4.32), we consider the case  $M > 0$ . For the case  $M < 0$ , we can change the sign of the series by replacing  $C_i$  and  $M$  by  $-C_i$  and  $-M$ , respectively. According to (4.44), we have

$$C_{2i} > \frac{M}{2i}, \quad C_{2i+1} > \frac{M}{2i+1}. \quad (4.45)$$

The convergence of an infinite series is independent of its first terms of finite numbers. Furthermore, the two series in (4.32) can be regarded as power series of  $x^2$ , which can be considered as positive series since the coefficients have the same sign. Hence, if the series

$$y_1^0(x) = \sum_{i=1}^{\infty} \frac{1}{2i} x^{2i} \quad \text{and} \quad y_2^0(x) = \sum_{i=0}^{\infty} \frac{1}{2i+1} x^{2i+1} \quad (4.46)$$

are infinite, the two series in (4.32) are infinite. Since the constant factor  $M$  in (4.45) does not affect the convergence of the series, we have dropped it.

Explicit formulae in closed form of the sums of the two series in (4.46) can be derived using the following two well-known Taylor series:

$$\begin{cases} \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \\ \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots. \end{cases} \quad (4.47)$$

By adding them together, or subtracting one from another, we obtain

$$\begin{cases} \frac{1}{2}[\ln(1+x) + \ln(1-x)] = -\sum_{i=1}^{\infty} \frac{x^{2i}}{2i}, \\ \frac{1}{2}[\ln(1+x) - \ln(1-x)] = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{2i+1}. \end{cases} \quad (4.48)$$

By comparing these with (4.46), we finally obtain

$$\begin{cases} y_1^0(x) = -\frac{1}{2}[\ln(1+x) + \ln(1-x)], \\ y_2^0(x) = \frac{1}{2}[\ln(1+x) - \ln(1-x)]. \end{cases} \quad (4.49)$$

Evidently, as  $x \rightarrow \pm 1$ ,  $y_1^0(x) \rightarrow +\infty$  and  $y_2^0(x) \rightarrow \pm\infty$ , meaning that both  $y_1^0(x)$  and  $y_2^0(x)$  are infinite when  $x = \pm 1$ . Based on the discussions throughout this subsection, we conclude that  $B_1^0(x)$  and  $B_2^0(x)$  are infinite when  $x = \pm 1$ .

### 4.2.3 Legendre Function

What we are solving for are finite solutions of the Legendre equation in the domain  $x \in [-1, 1]$ . However, we have concluded that, although the two linearly independent power series solutions of the equation,  $B_1^0(x)$  and  $B_2^0(x)$ , are finite in  $(-1, 1)$ , they are infinite when  $x = \pm 1$ . Nevertheless, there exists an occasion when

this conclusion breaks. This happens when  $\alpha = n(n + 1)$ , where  $n$  is an integer not less than 0. In this instance, the recurrence relation (4.31) for the coefficients of the power series solutions becomes

$$C_{i+2} = \frac{i(i + 1) - n(n + 1)}{(i + 2)(i + 1)} C_i. \quad (4.50)$$

Evidently, it can be seen from this relation that  $C_{n+2} = C_{n+4} = \dots = 0$ ; thus one of the two powers series in (4.32) becomes a polynomial. When  $n$  is even,  $B_1^0$  becomes a polynomial, but  $B_2^0$  remains a power series. When  $n$  is odd,  $B_2^0$  becomes a polynomial, but  $B_1^0$  remains a power series. Both polynomials are finite within  $x \in [-1, 1]$ . Therefore, the eigenvalues of the Legendre equation under the finite condition are  $n(n + 1)$ ,  $n = 0, 1, 2, \dots$ . The eigenfunctions are the corresponding polynomials denoted as  $P_n(x)$ , which is referred to as Legendre polynomial or Legendre function of degree  $n$ , where the coefficient of its highest power term is defined as

$$C_n = \frac{(2n)!}{2^n (n!)^2}. \quad (4.51)$$

We now derive an explicit formula for  $P_n(x)$ . Set  $i = n - 2$  in (4.50), and then substitute (4.51) into it. We obtain

$$C_{n-2} = \frac{n(n - 1)}{(n - 2)(n - 1) - n(n + 1)} C_n = -\frac{(2n - 2)!}{2^n (n - 1)!(n - 2)!}. \quad (4.52)$$

We can also obtain by repeatedly applying the above formula,

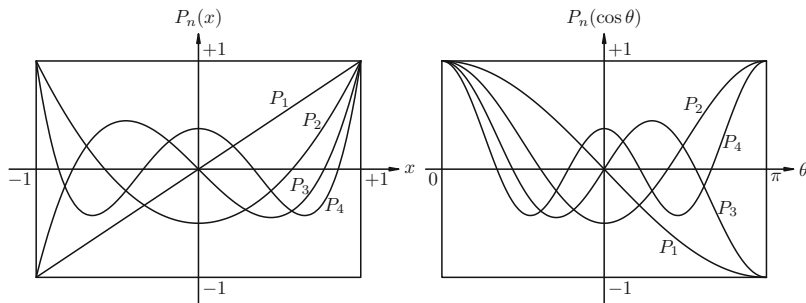
$$C_{n-4} = \frac{(n - 2)(n - 3)}{(n - 4)(n - 3) - n(n + 1)} C_{n-2} = \frac{(2n - 4)!}{2^n \times 2 \times (n - 2)!(n - 4)!}. \quad (4.53)$$

$$C_{n-6} = \frac{(n - 4)(n - 5)}{(n - 6)(n - 5) - n(n + 1)} C_{n-4} = -\frac{(2n - 6)!}{2^n \times 3! \times (n - 3)!(n - 6)!}. \quad (4.54)$$

.....

A general expression can be obtained by induction,

$$C_{n-2s} = (-1)^s \frac{(2n - 2s)!}{2^n s! (n - s)!(n - 2s)!}, \quad s = 0, 1, 2, \dots, \left[ \frac{n}{2} \right], \quad (4.55)$$



**Fig. 4.1** Graphs of some Legendre function

where  $[n/2]$  represents the maximum integer not larger than  $n/2$ . Finally, we obtain an expression of the Legendre function,

$$P_n(x) = \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n s!(n-s)!(n-2s)!} x^{n-2s}. \quad (4.56)$$

The graph of some lower degree Legendre function is shown in Fig. 4.1. Their expressions are included as a part of Table 4.1 in the next subsection.  $P_n(x)$  is a polynomial of degree  $n$ . As will be shown, it has  $n$  zeros of multiplicity 1 in  $(-1, 1)$ . Hence  $x = -1$  and  $x = 1$  are not its zeros. Actually, we will show that  $P_n(\pm 1) = (\pm 1)^n$ . It can also be seen from the figure that  $(d/d\theta)P_n(\cos \theta) = 0$  when  $\theta = 0$  or  $\pi$ , whose proof will be provided later.

Some properties of the Legendre function can be readily inferred from (4.56), such as

$$P_n(-x) = (-1)^n P_n(x) \quad (4.57)$$

and

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}, \quad P_{2n+1}(0) = 0, \quad (4.58)$$

where the value of  $P_n(0)$  is equal to the constant term in (4.56).

Another expression of the Legendre function is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad (4.59)$$

called Rodrigues' formula. It can be proven using the well-known binomial theorem

$$(x^2 - 1)^n = \sum_{s=0}^n \frac{(-1)^s n!}{s!(n-s)!} x^{2n-2s}, \quad (4.60)$$

**Table 4.1** Some lower degree associated Legendre function

|                                                                             |                                                                                                       |
|-----------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------|
| $P_0(x) = 1$                                                                | $P_0(\cos \theta) = 1$                                                                                |
| $P_1(x) = x$                                                                | $P_1(\cos \theta) = \cos \theta$                                                                      |
| $P_1^1(x) = (1 - x^2)^{1/2}$                                                | $P_1^1(\cos \theta) = \sin \theta$                                                                    |
| $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$                                     | $P_2(\cos \theta) = \frac{3}{2}\cos^2 \theta - \frac{1}{2}$                                           |
| $P_2^1(x) = 3x(1 - x^2)^{1/2}$                                              | $P_2^1(\cos \theta) = 3\sin \theta \cos \theta$                                                       |
| $P_2^2(x) = 3(1 - x^2)$                                                     | $P_2^2(\cos \theta) = 3\sin^2 \theta$                                                                 |
| $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$                                    | $P_3(\cos \theta) = \frac{5}{2}\cos^3 \theta - \frac{3}{2}\cos \theta$                                |
| $P_3^1(x) = (1 - x^2)^{1/2} \left( \frac{15}{2}x^2 - \frac{3}{2} \right)$   | $P_3^1(\cos \theta) = \sin \theta \left( \frac{15}{2}\cos^2 \theta - \frac{3}{2} \right)$             |
| $P_3^2(x) = 15x(1 - x^2)$                                                   | $P_3^2(\cos \theta) = 15\sin^2 \theta \cos \theta$                                                    |
| $P_3^3(x) = 15(1 - x^2)^{3/2}$                                              | $P_3^3(\cos \theta) = 15\sin^3 \theta$                                                                |
| $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$                  | $P_4(\cos \theta) = \frac{35}{8}\cos^4 \theta - \frac{15}{4}\cos^2 \theta + \frac{3}{8}$              |
| $P_4^1(x) = (1 - x^2)^{1/2} \left( \frac{35}{2}x^3 - \frac{15}{2}x \right)$ | $P_4^1(\cos \theta) = \sin \theta \left( \frac{35}{2}\cos^3 \theta - \frac{15}{2}\cos \theta \right)$ |
| $P_4^2(x) = (1 - x^2) \left( \frac{105}{2}x^2 - \frac{15}{2} \right)$       | $P_4^2(\cos \theta) = \sin^2 \theta \left( \frac{105}{2}\cos^2 \theta - \frac{15}{2} \right)$         |
| $P_4^3(x) = 105(1 - x^2)^{3/2}x$                                            | $P_4^3(\cos \theta) = 105\sin^3 \theta \cos \theta$                                                   |
| $P_4^4(x) = 105(1 - x^2)^2$                                                 | $P_4^4(\cos \theta) = 105\sin^4 \theta$                                                               |

whence,

$$\frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n = \sum_{s=0}^n \frac{(-1)^s}{2^n s! (n-s)!} \frac{d^n}{dx^n} x^{2n-2s}. \quad (4.61)$$

Evidently,  $(d^n/dx^n)x^{2n-2s}$  vanishes when  $2n - 2s < n$ . Hence, in the above formula, only the terms  $s < [n/2]$  do not vanish. We have

$$\frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n = \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n s! (n-s)! (n-2s)!} x^{n-2s}. \quad (4.62)$$

We see that (4.60) follows by comparing the above formula with (4.59).

#### 4.2.4 Associated Legendre Function

Substitute the two linearly independent power series solutions of the Legendre equation given by (4.32) into (4.23). We obtain the two linearly independent power

series solutions of the associated Legendre equation,

$$\begin{cases} B_1 = (1-x^2)^{k/2} \sum_{2i \geq k} \frac{(2i)!}{(2i-k)!} C_{2i} x^{2i-k}, \\ B_2 = (1-x^2)^{k/2} \sum_{2i+1 \geq k} \frac{(2i+1)!}{(2i-k+1)!} C_{2i+1} x^{2i-k+1}, \end{cases} \quad (4.63)$$

where  $C_{2i}$  and  $C_{2i+1}$  are still determined using (4.31). The limitations  $2i \geq k$  and  $2i+1 \geq k$  are due to the fact that, after taking the derivative of (4.32)  $k$  times, the terms  $2i < k$  in  $B_1^0$  and the terms  $2i+1 < k$  in  $B_2^0$  vanish.

Similar to the cases of  $B_1^0$  and  $B_2^0$ , it can be easily verified that the convergence radius of both  $B_1$  and  $B_2$  is 1, meaning that both  $B_1$  and  $B_2$  are finite in  $(-1, 1)$ . However, as will be shown below, both of them are infinite when  $x = \pm 1$ .

Here we take only the case when  $k$  is even. The case when  $k$  is odd is similar. Assume  $k \geq 2$ . Substitute the summation index  $i$  in (4.63) by  $i + k/2$ . We obtain

$$\begin{cases} B_1 = (1-x^2)^{k/2} \sum_{i=0}^{\infty} \frac{(2i+k)!}{(2i)!} C_{2i+k} x^{2i}, \\ B_2 = (1-x^2)^{k/2} \sum_{i=0}^{\infty} \frac{(2i+k+1)!}{(2i+1)!} C_{2i+k+1} x^{2i+1}. \end{cases} \quad (4.64)$$

When  $i$  is large enough, we have, according to (4.45),

$$\begin{cases} \frac{(2i+k)!}{(2i)!} C_{2i+k} > \frac{(2i+k-1)!}{(2i)!} M > \frac{M}{2i}, \\ \frac{(2i+k+1)!}{(2i+1)!} C_{2i+k+1} > \frac{(2i+k)!}{(2i+1)!} M > \frac{M}{2i+1}. \end{cases} \quad (4.65)$$

Taking reference to the formulations below (4.45), we can readily conclude that the two series in (4.64) are infinite when  $x = \pm 1$ .

The same as the case of Legendre equation, we are solving for finite solutions of the associated Legendre equation in the domain  $x \in [-1, 1]$ , but its two linearly independent power series solutions are infinite when  $x = \pm 1$ , although they are finite in  $(-1, 1)$ . The same instance happens for this conclusion to break, namely, when  $\alpha = n(n+1)$ . In this case,  $B^0(x) = P_n(x)$ , and we denote  $B(x)$  as  $P_n^k(x)$ . According to (4.23), we obtain<sup>1</sup>

$$P_n^k(x) = (1-x^2)^{k/2} \frac{d^k}{dx^k} P_n(x), \quad (4.66)$$

<sup>1</sup> An alternative definition in the literature includes a factor  $(-1)^k$  at the right-hand side. Hence, there is a need to clarify or verify the definition.

which is referred to as associated Legendre function of degree  $n$  and order  $k$ . As  $P_n(x)$  is a polynomial of degree  $n$ ,  $P_n^k(x)$  vanishes when  $n < k$ . Hence, we have  $n \geq k$ . Actually, this can also be inferred from the limitations of summations,  $2i \geq k$  and  $2i + 1 \geq k$ , in the power series in (4.63), where the lower limits of  $2i$  and  $2i + 1$  are actually  $n$ .

Substitute the expressions of the Legendre function (4.56) and (4.59) into (4.66), we obtain the corresponding expressions for the associated Legendre function:

$$P_n^k(x) = (1 - x^2)^{k/2} \sum_{s=0}^{[(n-k)/2]} \frac{(2n - 2s)!}{2^n s! (n - s)! (n - 2s - k)!} x^{n-2s-k}, \quad (4.67)$$

$$P_n^k(x) = \frac{1}{2^n n!} (1 - x^2)^{k/2} \frac{d^{n+k}}{dx^{n+k}} (x^2 - 1)^n. \quad (4.68)$$

When  $k = 0$ , the associated Legendre function  $P_n^k(x)$  degenerates to the Legendre function  $P_n(x)$ . By substituting  $x = \cos \theta$  into (4.67), we obtain

$$P_n^k(\cos \theta) = \sin^k \theta \sum_{s=0}^{[(n-k)/2]} \frac{(2n - 2s)!}{2^n s! (n - s)! (n - 2s - k)!} \cos^{n-k-2s} \theta. \quad (4.69)$$

In the literature, the term *Legendre function* is also used to represent both the Legendre and associated Legendre functions. We will frequently do so in the rest of this book when the distinction between the Legendre and associated Legendre functions is not necessary based on the context.

Some lower degree and order associated Legendre functions are shown in Table 4.1. If  $k \neq 0$ , the factor  $\sin^k \theta$  equals 1 at the equator, and decreases toward the pole, where it equals 0; the larger  $k$  is, the quicker the decrease is. In Fig. 4.2, we show the graphs of  $P_{20}^k(\cos \theta)$  for  $k = 0, 1, \dots, 20$ , which are scaled to fit the height of the boxes for clarity. The increasing effect of the factor  $\sin^k \theta$  as  $k$  increases can be seen in this figure. There is a region around the poles where  $P_{20}^k(\cos \theta)$  remains extremely small, and the size of the region increases as  $k$  increases. Therefore, when  $k$  is extremely large,  $P_n^k(\cos \theta)$  is significantly different from 0 only in a narrow region around the equator.

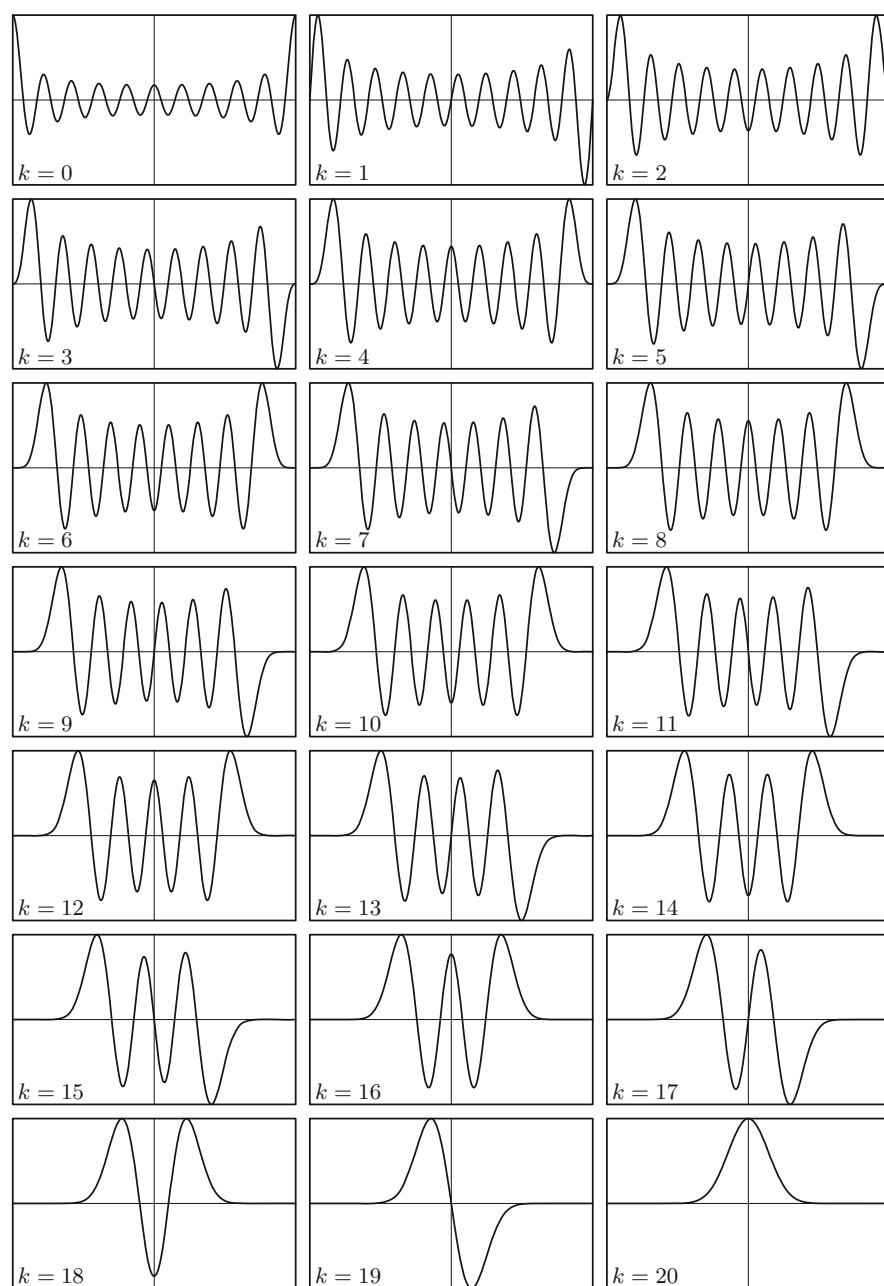
The following properties can be readily inferred from (4.67):

$$P_n^k(-x) = (-1)^{n-k} P_n^k(x); \quad (4.70)$$

$$P_n^k(0) \begin{cases} (-1)^{(n-k)/2} \frac{(n+k)!}{2^n [(n-k)/2]! [(n+k)/2]!}, & \text{even } n-k; \\ 0, & \text{odd } n-k; \end{cases} \quad (4.71)$$

$$P_n^k(\pm 1) = 0, \quad k \neq 0; \quad (4.72)$$

where the value of  $P_n^k(0)$  is the constant term of (4.67).



**Fig. 4.2** The associated Legendre function  $P_{20}^k(\cos \theta)$  scaled to fit the height of the boxes for clarity



The associated Legendre function  $P_n^k(\cos \theta)$  can be understood as a sum in the form of  $\sum_{s=0}^n C_s \cos s\theta$  or  $\sin \theta \sum_{s=0}^{n-1} C_s \sin s\theta$  when  $k$  is even, and  $\sum_{s=0}^n C_s \sin s\theta$  or  $\sin \theta \sum_{s=0}^{n-1} C_s \cos s\theta$  when  $k$  is odd. This can be realized based on the relations

$$\cos^m \theta = \begin{cases} \frac{2}{2^m} \sum_{s=0}^{(m-1)/2} \frac{m!}{s!(m-s)!} \cos[(m-2s)\theta], & \text{odd } m; \\ \frac{1}{2^m} \frac{m!}{[(m/2)!]^2} + \frac{2}{2^m} \sum_{s=0}^{m/2-1} \frac{n!}{s!(m-s)!} \cos[(m-2s)\theta], & \text{even } m; \end{cases} \quad (4.73)$$

and

$$\sin^m \theta = \begin{cases} \frac{2}{2^m} \sum_{s=0}^{(m-1)/2} \frac{m!}{s!(m-s)!} \sin[(m-2s)\theta], & \text{odd } m; \\ \frac{1}{2^m} \frac{m!}{[(m/2)!]^2} + \frac{2}{2^m} \sum_{s=0}^{m/2-1} \frac{(-1)^{m/2-s} n!}{s!(m-s)!} \cos[(m-2s)\theta], & \text{even } m; \end{cases} \quad (4.74)$$

together with the product-to-sum formulae of sine and cosine functions.

The associated Legendre equation remains unchanged when  $k$  is replaced by  $-k$ . Hence, it can be conjectured that what we obtain by replacing  $k$  by  $-k$  in (4.67) or (4.68),  $P_n^{-k}(x)$ , remains a solution of the associated Legendre equation. Actually,  $P_n^{-k}(x)$  is different from  $P_n^k(x)$  only by a constant factor. A proof is provided below.

Assume  $k > 0$ . We obtain from (4.68),

$$P_n^{-k}(x) = \frac{1}{2^n n!} (1-x^2)^{-k/2} \frac{d^{n-k}}{dx^{n-k}} (x^2-1)^n. \quad (4.75)$$

According to the derivative formula

$$\frac{d^m}{dx^m}(uv) = \sum_{i=0}^m \frac{m!}{i!(m-i)!} \frac{d^i u}{dx^i} \frac{d^{m-i} v}{dx^{m-i}}, \quad (4.76)$$

we obtain

$$\begin{aligned} \frac{d^{n+k}}{dx^{n+k}} (x^2-1)^n &= \frac{d^{n+k}}{dx^{n+k}} [(x-1)^n (x+1)^n] \\ &= \sum_{i=0}^{n+k} \frac{(n+k)!}{i!(n+k-i)!} \frac{d^i}{dx^i} (x-1)^n \frac{d^{n+k-i}}{dx^{n+k-i}} (x+1)^n. \end{aligned} \quad (4.77)$$

When  $i > n$ , we have  $d^i (x-1)^n / dx^i = 0$ . When  $i < k$ , we have  $n+k-i > n$ , thus  $d^{n+k-i} (x+1)^n / dx^{n+k-i} = 0$ . We obtain

$$\begin{aligned}
\frac{d^{n+k}}{dx^{n+k}}(x^2 - 1)^n &= \sum_{i=k}^n \frac{(n+k)!}{i!(n+k-i)!} \frac{n!}{(n-i)!} (x-1)^{n-i} \frac{n!}{(i-k)!} (x+1)^{i-k} \\
&= \sum_{j=0}^{n-k} \frac{(n+k)!}{(j+k)!(n-j)!} \frac{n!}{(n-k-j)!} (x-1)^{n-k-j} \frac{n!}{j!} (x+1)^j \\
&= \frac{(n+k)!}{(n-k)!} (x^2 - 1)^{-k} \sum_{j=0}^{n-k} \frac{(n-k)!}{j!(n-k-j)!} \frac{n!}{(n-j)!} (x-1)^{n-j} \\
&\quad \times \frac{n!}{(j+k)!} (x+1)^{j+k} \\
&= \frac{(n+k)!}{(n-k)!} (x^2 - 1)^{-k} \sum_{j=0}^{n-k} \frac{(n-k)!}{j!(n-k-j)!} \frac{d^j}{dx^j} (x-1)^n \frac{d^{n-k-j}}{dx^{n-k-j}} (x+1)^n \\
&= \frac{(n+k)!}{(n-k)!} (x^2 - 1)^{-k} \frac{d^{n-k}}{dx^{n-k}} [(x-1)^n (x+1)^n] \\
&= (-1)^k \frac{(n+k)!}{(n-k)!} (1-x^2)^{-k} \frac{d^{n-k}}{dx^{n-k}} (x^2 - 1)^n. \tag{4.78}
\end{aligned}$$

By comparing (4.68) and (4.75) taking into account of the above formula, we obtain

$$P_n^{-k}(x) = (-1)^k \frac{(n-k)!}{(n+k)!} P_n^k(x). \tag{4.79}$$

Thus, we proved the conclusion that  $P_n^{-k}(x)$  is different from  $P_n^k(x)$  only by a constant factor.

According to (4.79) and the expression of  $P_n^{-k}(x)$  given by (4.75), we obtain an alternative expression of  $P_n^k(x)$ ,

$$P_n^k(x) = (-1)^k \frac{(n+k)!}{2^n n! (n-k)!} (1-x^2)^{-k/2} \frac{d^{n-k}}{dx^{n-k}} (x^2 - 1)^n. \tag{4.80}$$

## 4.3 Some Properties of the Legendre and Associated Legendre Functions

### 4.3.1 A Generating Function for the Legendre Function

Let  $(1+t^2-2xt)^{-1/2}$  be a function of  $t$ , and consider  $x$  as a parameter. The Legendre function appears in its power series expansion

$$(1+t^2-2xt)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n, \quad |t| < 1. \tag{4.81}$$

Hence, the function  $(1 + t^2 - 2xt)^{-1/2}$  is referred to as the generating function of  $P_n(x)$ .

To prove (4.81), we use the power series expansion

$$(1 - z)^{-1/2} = \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i}(i!)^2} z^i. \quad (4.82)$$

Let  $z = t(2x - t)$ . We obtain

$$(1 + t^2 - 2xt)^{-1/2} = \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i}(i!)^2} t^i (2x - t)^i. \quad (4.83)$$

In the binomial theorem

$$(a + b)^i = \sum_{s=0}^i \frac{i!}{s!(i-s)!} a^{i-s} b^s, \quad (4.84)$$

let  $a = 2x$  and  $b = -t$ . We obtain

$$(2x - t)^i = \sum_{s=0}^i (-1)^s \frac{i!}{s!(i-s)!} (2x)^{i-s} t^s. \quad (4.85)$$

Substitute it into (4.83). We obtain

$$\begin{aligned} (1 + t^2 - 2xt)^{-1/2} &= \sum_{i=0}^{\infty} \frac{(2i)!}{2^{2i}(i!)^2} t^i \sum_{s=0}^i (-1)^s \frac{i!}{s!(i-s)!} (2x)^{i-s} t^s \\ &= \sum_{i=0}^{\infty} \sum_{s=0}^i (-1)^s \frac{(2i)!}{2^{i+s}s!i!(i-s)!} x^{i-s} t^{i+s}. \end{aligned} \quad (4.86)$$

Substitute the summation index  $i$  by  $n = i + s$ . We then have  $0 \leq s \leq [n/2]$ , and

$$(1 + t^2 - 2xt)^{-1/2} = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n s!(n-s)!(n-2s)!} x^{n-2s} t^n. \quad (4.87)$$

We see that (4.81) can be obtained by comparing this formula with the expression of  $P_n(x)$  given by (4.56).

Another expression of  $P_n(\cos \theta)$  can be derived based on (4.81) after replacing  $x$  by  $\cos \theta$ ,

$$(1 + t^2 - 2t \cos \theta)^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos \theta) t^n. \quad (4.88)$$

We express the left-hand side as a power series using (4.82),

$$\begin{aligned}
 (1 + t^2 - 2t \cos \theta)^{-1/2} &= (1 - te^{i\theta})^{-1/2} (1 - te^{-i\theta})^{-1/2} \\
 &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} e^{ik\theta} t^k \sum_{s=0}^{\infty} \frac{(2s)!}{2^{2s}(s!)^2} e^{-is\theta} t^s \\
 &= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(2k)!(2s)!}{2^{2(k+s)}(k!s!)^2} e^{i(k-s)\theta} t^{k+s}. \quad (4.89)
 \end{aligned}$$

Substitute the summation index  $k$  by  $n = k + s$ . It follows from  $k \geq 0$  that  $s \leq n$ . The above equation becomes

$$(1 + t^2 - 2t \cos \theta)^{-1/2} = \sum_{n=0}^{\infty} \sum_{s=0}^n \frac{(2n-2s)!(2s)!}{2^{2n}[(n-s)!s!]^2} e^{i(n-2s)\theta} t^n. \quad (4.90)$$

We obtain, by comparing this formula with (4.88),

$$P_n(\cos \theta) = \sum_{s=0}^n \frac{(2n-2s)!(2s)!}{2^{2n}[(n-s)!s!]^2} e^{i(n-2s)\theta}. \quad (4.91)$$

Since  $P_n(\cos \theta)$  is a real function, the sum of all imaginary terms at the right-hand side of this equation must vanish. So we finally have

$$P_n(\cos \theta) = \sum_{s=0}^n \frac{(2n-2s)!(2s)!}{2^{2n}[(n-s)!s!]^2} \cos[(n-2s)\theta]. \quad (4.92)$$

We now study the range of  $P_n(\cos \theta)$ . Setting  $\cos \theta = \pm 1$  in (4.88), we have

$$(1 + t^2 \mp 2t)^{-1/2} = (1 \mp t)^{-1} = \sum_{n=0}^{\infty} (\pm 1)^n t^n. \quad (4.93)$$

This leads to

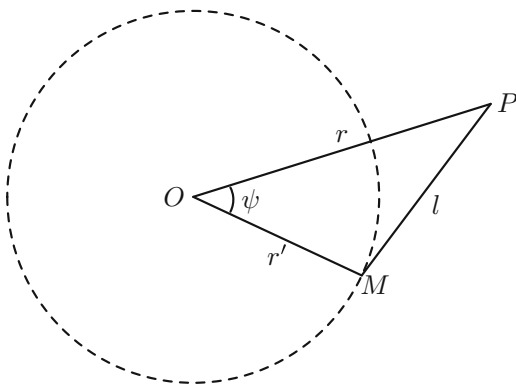
$$P_n(\pm 1) = (-1)^n. \quad (4.94)$$

According to this formula, when  $\theta = 0$ , (4.92) becomes

$$P_n(1) = \sum_{s=0}^n \frac{(2n-2s)!(2s)!}{2^{2n}[(n-s)!s!]^2} = 1. \quad (4.95)$$

Note that all terms under the summation sign in this formula are positive. As  $|\cos[(n-2s)\theta]| \leq 1$ , we conclude from this formula and (4.92) that  $P_n(\cos \theta) \leq 1$ .

**Fig. 4.3** Expression of the reciprocal of a distance into a series of Legendre function



Furthermore, according to (4.94) and (4.57), we obtain

$$-1 \leq P_n(x) = P_n(\cos \theta) \leq 1. \quad (4.96)$$

Finally, as the domain of convergence of the series (4.93) is  $|t| < 1$ , it follows from (4.96) that the domain of convergence of the series (4.81) is  $|t| < 1$  for any  $-1 \leq x \leq 1$ .

A very important application of (4.81) is the development of the reciprocal of a distance into a series of the Legendre function. As shown in Fig. 4.3, the reciprocal of the distance between  $M$  and  $P$  is

$$\frac{1}{l} = (r^2 + r'^2 - 2rR \cos \psi)^{-1/2}. \quad (4.97)$$

According to (4.81), we obtain

$$\frac{1}{l} = \begin{cases} \frac{1}{r} \left[ 1 + \left( \frac{r'}{r} \right)^2 - 2 \frac{r'}{r} \cos \psi \right]^{-1/2} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos \psi), & r > r'; \\ \frac{1}{r'} \left[ 1 + \left( \frac{r}{r'} \right)^2 - 2 \frac{r}{r'} \cos \psi \right]^{-1/2} = \sum_{n=0}^{\infty} \frac{r^n}{r'^{n+1}} P_n(\cos \psi), & r < r'. \end{cases} \quad (4.98)$$

### 4.3.2 Some Recurrence Formulae

#### Recurrence Formulae of the Legendre Function

In practice, very high degree Legendre function is computed. This is most efficiently done using recurrence formulae. Some most useful recurrence formulae of  $P_n(x)$  are

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \quad (4.99)$$

$$\frac{d}{dx}P_{n+1}(x) = x\frac{d}{dx}P_n(x) + (n+1)P_n(x), \quad (4.100)$$

$$\frac{d}{dx}P_{n-1}(x) = x\frac{d}{dx}P_n(x) - nP_n(x), \quad (4.101)$$

$$\frac{d}{dx}P_{n+1}(x) - \frac{d}{dx}P_{n-1}(x) = (2n+1)P_n(x), \quad (4.102)$$

$$(1-x^2)\frac{d}{dx}P_n(x) = nP_{n-1}(x) - nxP_n(x). \quad (4.103)$$

These can be proven based on (4.81).

Take the derivative of (4.81) with respect to  $t$ . We obtain

$$(x-t)(1+t^2-2tx)^{-3/2} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}. \quad (4.104)$$

Multiply both sides by  $(1+t^2-2tx)$ , and then apply (4.81) to the left-hand side. We obtain

$$(x-t) \sum_{n=0}^{\infty} P_n(x)t^n = (1+t^2-2tx) \sum_{n=0}^{\infty} nP_n(x)t^{n-1}. \quad (4.105)$$

Gather terms of the same degree of  $t$ . We obtain

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} (2n+1)xP_n(x)t^n + \sum_{n=0}^{\infty} (n+1)P_n(x)t^{n+1} = 0. \quad (4.106)$$

Some rearrangement of summation index leads it to

$$\sum_{n=0}^{\infty} (n+1)P_{n+1}(x)t^n - \sum_{n=0}^{\infty} (2n+1)xP_n(x)t^n + \sum_{n=1}^{\infty} nP_{n-1}(x)t^n = 0. \quad (4.107)$$

As  $t$  is arbitrary, the sum of the coefficients of  $t^n$  must vanish in order for the identity to hold, which leads to (4.99).

Take the derivative of (4.81) with respect to  $x$ . We obtain

$$t(1+t^2-2tx)^{-3/2} = \sum_{n=0}^{\infty} \frac{d}{dx}P_n(x)t^n. \quad (4.108)$$

Divide both sides by  $(1 + t^2 - 2tx)$ , and then apply (4.81) at the left-hand side. We obtain

$$t \sum_{n=0}^{\infty} P_n(x) t^n = (1 + t^2 - 2tx) \sum_{n=0}^{\infty} \frac{d}{dx} P_n(x) t^n. \quad (4.109)$$

Gather all terms for the same degree of  $t$ . We obtain

$$\sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} \frac{d}{dx} P_n(x) t^{n+1} + \sum_{n=0}^{\infty} \frac{d}{dx} P_n(x) t^{n+2} - 2x \sum_{n=0}^{\infty} \frac{d}{dx} P_n(x) t^{n+1}. \quad (4.110)$$

This formula can be further written as

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(x) t^{n+1} &= \sum_{n=0}^{\infty} \frac{d}{dx} P_{n+1}(x) t^{n+1} + \sum_{n=0}^{\infty} \frac{d}{dx} P_{n-1}(x) t^{n+1} \\ &\quad - 2x \sum_{n=0}^{\infty} \frac{d}{dx} P_n(x) t^{n+1}. \end{aligned} \quad (4.111)$$

As  $t$  is arbitrary, in order for the identity to hold, the sum of the coefficients of  $t^{n+1}$  must vanish. We obtain

$$P_n(x) = \frac{d}{dx} P_{n+1}(x) + \frac{d}{dx} P_{n-1}(x) - 2x \frac{d}{dx} P_n(x). \quad (4.112)$$

Take the derivative of (4.99) with respect to  $x$ , and then eliminate  $dP_{n-1}(x)/dx$  with the above formula. We obtain (4.100).

The rest can be proven as follows: (4.101) can be obtained by eliminating  $dP_{n+1}(x)/dx$  from (4.100) and (4.112); (4.102) can be obtained by eliminating  $dP_n(x)/dx$  from (4.100) and (4.101); (4.103) can be obtained by replacing  $n$  by  $n - 1$  in (4.100), and then eliminating  $dP_{n-1}(x)/dx$  with (4.101).

For easy reference, we rewrite the recurrence formulae (4.99)–(4.103) in terms of  $\theta$ . With  $x = \cos \theta$ ,  $(1 - x^2)^{1/2} = \sin \theta$  and  $d/dx = -(1/\sin \theta)d/d\theta$ , we have

$$(n + 1)P_{n+1}(\cos \theta) - (2n + 1)\cos \theta P_n(\cos \theta) + nP_{n-1}(\cos \theta) = 0, \quad (4.113)$$

$$\frac{d}{d\theta} P_{n+1}(\cos \theta) = \cos \theta \frac{d}{d\theta} P_n(\cos \theta) - (n + 1)\sin \theta P_n(\cos \theta), \quad (4.114)$$

$$\frac{d}{d\theta} P_{n-1}(\cos \theta) = \cos \theta \frac{d}{d\theta} P_n(\cos \theta) + n\sin \theta P_n(\cos \theta), \quad (4.115)$$

$$\frac{d}{d\theta} P_{n+1}(\cos \theta) - \frac{d}{d\theta} P_{n-1}(\cos \theta) = -(2n + 1)\sin \theta P_n(\cos \theta), \quad (4.116)$$

$$-\sin \theta \frac{d}{d\theta} P_n(\cos \theta) = n P_{n-1}(\cos \theta) - n \cos \theta P_n(\cos \theta). \quad (4.117)$$

As a supplement, we reproduce the ordinary differential equation satisfied by  $P_n(\cos \theta)$ ,

$$\frac{d^2}{d\theta^2} P_n(\cos \theta) + \cot \theta \frac{d}{d\theta} P_n(\cos \theta) + n(n+1) P_n(\cos \theta) = 0, \quad (4.118)$$

which looks similar to a recurrence formula.

### Recurrence Formulae of the Associated Legendre Function

Similar to  $P_n(x)$ , it is much more practical to use recurrence formulae to compute  $P_n^k(x)$  when  $n$  and  $k$  are large. Some fundamental recurrence formulae for  $P_n^k(x)$  are

$$(2n+1)x P_n^k(x) = (n+k) P_{n-1}^k(x) + (n-k+1) P_{n+1}^k(x), \quad (4.119)$$

$$(2n+1)(1-x^2)^{1/2} P_n^k(x) = P_{n+1}^{k+1}(x) - P_{n-1}^{k+1}(x), \quad (4.120)$$

$$(2n+1)(1-x^2)^{1/2} P_n^k(x) = (n+k)(n+k-1) P_{n-1}^{k-1}(x) - (n-k+2)(n-k+1) P_{n+1}^{k-1}(x), \quad (4.121)$$

$$(1-x^2) \frac{d}{dx} P_n^k(x) = (1-x^2)^{1/2} P_n^{k+1}(x) - kx P_n^k(x), \quad (4.122)$$

$$(2n+1)(1-x^2) \frac{d}{dx} P_n^k(x) = (n+1)(n+k) P_{n-1}^k(x) - n(n-k+1) P_{n+1}^k(x), \quad (4.123)$$

$$(1-x^2) \frac{d}{dx} P_n^k(x) = (n+k) P_{n-1}^k(x) - nx P_n^k(x). \quad (4.124)$$

These can be proven using the recurrence formulae of  $P_n(x)$ .

Take  $k$  times the derivative of (4.99) with respect to  $x$ . We obtain

$$\begin{aligned} (n+1) \frac{d^k}{dx^k} P_{n-1}(x) - (2n+1)x \frac{d^k}{dx^k} P_n(x) - (2n+1)k \frac{d^{k-1}}{dx^{k-1}} P_n(x) \\ + n \frac{d^k}{dx^k} P_{n+1}(x) = 0. \end{aligned} \quad (4.125)$$

Take  $k-1$  times of the derivative of (4.102) with respect to  $x$ . We obtain

$$\frac{d^k}{dx^k} P_{n+1}(x) - \frac{d^k}{dx^k} P_{n-1}(x) = (2n+1) \frac{d^{k-1}}{dx^{k-1}} P_n(x). \quad (4.126)$$



Eliminate  $d^{k-1}P_n(x)/dx^{k-1}$  from the above two formulae. We obtain

$$(2n+1)x \frac{d^k}{dx^k} P_n(x) = (n+k) \frac{d^k}{dx^k} P_{n-1}(x) + (n-k+1) \frac{d^k}{dx^k} P_{n+1}(x). \quad (4.127)$$

By multiplying both sides by  $(1-x^2)^{k/2}$ , we obtain (4.119) according to the definition of  $P_n^k(x)$  in (4.68).

Take the derivative of (4.126) with respect to  $x$ , and then multiply both sides by  $(1-x^2)^{(k+1)/2}$ . We obtain (4.120) according to the definition of  $P_n^k(x)$  in (4.68).

Eliminate  $x P_n(x)$  from (4.99) and (4.103). We obtain

$$(2n+1)(1-x^2) \frac{d}{dx} P_n(x) = n(n+1)P_{n-1}(x) - n(n+1)P_{n+1}(x). \quad (4.128)$$

Take  $k-1$  times the derivative with respect to  $x$ . We obtain

$$\begin{aligned} (2n+1)(1-x^2) \frac{d^k}{dx^k} P_n(x) - 2(k-1)(2n+1)x \frac{d^{k-1}}{dx^{k-1}} P_n(x) \\ - (k-2)(k-1)(2n+1) \frac{d^{k-2}}{dx^{k-2}} P_n(x) \\ = n(n+1) \frac{d^{k-1}}{dx^{k-1}} P_{n-1}(x) - n(n+1) \frac{d^{k-1}}{dx^{k-1}} P_{n+1}(x). \end{aligned} \quad (4.129)$$

Replace  $k$  by  $k-1$  in (4.126) and (4.127), and then substitute them into the above formula. We obtain

$$\begin{aligned} (2n+1)(1-x^2) \frac{d^k}{dx^k} P_n(x) = (n+k)(n+k-1) \frac{d^{k-1}}{dx^{k-1}} P_{n-1}(x) \\ - (n-k+2)(n-k+1) \frac{d^{k-1}}{dx^{k-1}} P_{n+1}(x). \end{aligned} \quad (4.130)$$

Multiply both sides by  $(1-x^2)^{(k-1)/2}$ . We obtain (4.121) according to the definition of  $P_n^k(x)$  in (4.68).

Take the derivative of (4.68) with respect to  $x$ . We obtain

$$\frac{d}{dx} P_n^k(x) = (1-x^2)^{k/2} \frac{d^{k+1}}{dx^{k+1}} P_n(x) - k(1-x^2)^{k/2-1} \frac{d^k}{dx^k} P_n(x). \quad (4.131)$$

Multiply both sides by  $1-x^2$ . We obtain (4.122) according to (4.68).

The last two formulae can be derived from the first two. Here we leave the derivation to the readers.

A thing to notice in using the recurrence relations is that  $P_n^k(x) = 0$  when  $k > n$ .

For easy reference, we rewrite the recurrence formulae (4.99)–(4.103) in terms of  $\theta$ . With  $x = \cos \theta$ ,  $(1 - x^2)^{1/2} = \sin \theta$  and  $d/dx = -(1/\sin \theta)d/d\theta$ , we have

$$(2n + 1) \cos \theta P_n^k(\cos \theta) = (n + k) P_{n-1}^k(\cos \theta) + (n - k + 1) P_{n+1}^k(\cos \theta), \quad (4.132)$$

$$(2n + 1) \sin \theta P_n^k(\cos \theta) = P_{n+1}^{k+1}(\cos \theta) - P_{n-1}^{k+1}(\cos \theta), \quad (4.133)$$

$$(2n + 1) \sin \theta P_n^k(\cos \theta) = (n + k)(n + k - 1) P_{n-1}^{k-1}(\cos \theta) - (n - k + 2)(n - k + 1) P_{n+1}^{k-1}(\cos \theta), \quad (4.134)$$

$$-\sin \theta \frac{d}{d\theta} P_n^k(\cos \theta) = \sin \theta P_n^{k+1}(\cos \theta) - k \cos \theta P_n^k(\cos \theta), \quad (4.135)$$

$$-(2n + 1) \sin \theta \frac{d}{d\theta} P_n^k(\cos \theta) = (n + 1)(n + k) P_{n-1}^k(\cos \theta) - n(n - k + 1) P_{n+1}^k(\cos \theta), \quad (4.136)$$

$$-\sin \theta \frac{d}{d\theta} P_n^k(\cos \theta) = (n + k) P_{n-1}^k(\cos \theta) - n \cos \theta P_n^k(\cos \theta). \quad (4.137)$$

As a supplement, we reproduce the ordinary differential equation satisfied by  $P_n^k(\cos \theta)$ :

$$\frac{d^2}{d\theta^2} P_n^k(\cos \theta) + \cot \theta \frac{d}{d\theta} P_n^k(\cos \theta) - \frac{k^2}{\sin^2 \theta} P_n^k(\cos \theta) + n(n + 1) P_n^k(\cos \theta) = 0, \quad (4.138)$$

which looks similar to a recurrence formula.

While computing  $P_n^k(\cos \theta)$ , the best choice is to use (4.132), where  $P_k^k(\cos \theta)$  and  $P_{k+1}^k(\cos \theta)$  are obtained from (4.67) as

$$P_k^k(\cos \theta) = \frac{(2k)!}{2^k k!} \sin^k \theta, \quad P_{k+1}^k(\cos \theta) = (2k + 1) \cos \theta P_k^k(\cos \theta). \quad (4.139)$$

The second one can actually be obtained by setting  $n = k$  in (4.132) and using the first one and  $P_{k-1}^k(x) = 0$ .

Note that (4.132) and (4.136) can be used to respectively express  $\cos \theta P_n^k(\cos \theta)$  and  $\sin \theta (d/d\theta) P_n^k(\cos \theta)$  as linear combinations of  $P_{n-1}^k(\cos \theta)$  and  $P_{n+1}^k(\cos \theta)$ . These can be used to express  $\cos^2 \theta P_n^k(\cos \theta)$ ,  $\sin \theta \cos \theta (d/d\theta) P_n^k(\cos \theta)$  and  $\sin^2 \theta P_n^k(\cos \theta)$  as linear combinations of  $P_{n-2}^k(\cos \theta)$ ,  $P_{n-1}^k(\cos \theta)$ ,  $P_{n+1}^k(\cos \theta)$ , and  $P_{n+2}^k(\cos \theta)$  and further as a chain. The readers are referred to Cruz (1986) and Moritz (1980). Note the possible similar applications of (4.133), and (4.134).

None of the formulae provided so far is suitable for computing  $(d/d\theta) P_n^k(\cos \theta)$ . We now derive a formula for it that does not involve any division by  $\sin \theta$  or  $\cos \theta$ .

By adding (4.135) and (4.137), we obtain

$$2 \sin \theta \frac{d}{d\theta} P_n^k(\cos \theta) = (n+k) \left[ \cos \theta P_n^k(\cos \theta) - P_n^{k-1}(\cos \theta) \right] - \sin \theta P_n^{k+1}(\cos \theta). \quad (4.140)$$

Solve for  $\cos \theta P_n^k(\cos \theta)$  from (4.132), and then substitute the result into this equation. We obtain

$$2 \sin \theta \frac{d}{d\theta} P_n^k(\cos \theta) = \frac{(n+k)(n-k+1)}{2n+1} \left[ P_{n+1}^k(\cos \theta) - P_n^{k-1}(\cos \theta) \right] - \sin \theta P_n^{k+1}(\cos \theta), \quad (4.141)$$

Substitute  $k$  by  $k-1$  in (4.133), and then substitute the result into this equation. We obtain

$$2 \frac{d}{d\theta} P_n^k(\cos \theta) = (n+k)(n-k+1) P_n^{k-1}(\cos \theta) - P_n^{k+1}(\cos \theta), \quad (4.142)$$

which is the formula we are seeking. For the special case  $k=0$ , a formula can be inferred from (4.135),

$$\frac{d}{d\theta} P_n(\cos \theta) = -P_n^1(\cos \theta). \quad (4.143)$$

When  $k=n$ , the second term in (4.142) vanishes, and we have

$$\frac{d}{d\theta} P_n^n(\cos \theta) = n P_n^{n-1}(\cos \theta). \quad (4.144)$$

The forms of these formulae do not change after taking derivatives of any order with respect to  $\theta$ . We have

$$2 \frac{d^{i+1}}{d\theta^{i+1}} P_n^k(\cos \theta) = (n+k)(n-k+1) \frac{d^i}{d\theta^i} P_n^{k-1}(\cos \theta) - \frac{d^i}{d\theta^i} P_n^{k+1}(\cos \theta), \quad (4.145)$$

$$\frac{d^{i+1}}{d\theta^{i+1}} P_n(\cos \theta) = -\frac{d^i}{d\theta^i} P_n^1(\cos \theta), \quad (4.146)$$

$$\frac{d^{i+1}}{d\theta^{i+1}} P_n^n(\cos \theta) = n \frac{d^i}{d\theta^i} P_n^{n-1}(\cos \theta). \quad (4.147)$$

These formulae can be used to compute  $(d^{i+1}/d\theta^{i+1}) P_n^k(\cos \theta)$  from the values of  $(d^i/d\theta^i) P_n^k(\cos \theta)$ .

We can infer from (4.143) that  $(d/d\theta)P_n(\cos\theta) = 0$  at  $\theta = 0$  or  $\pi$ , which has already been mentioned.

### 4.3.3 Closed-Form Formulae of Some Legendre Series

Series in the form of  $\sum_{n=0}^{\infty} a_n P_n(x)$  are called Legendre series. The generating relation (4.81) or (4.88) is such a Legendre series. For brevity, we write it in the form of

$$h = (1 + t^2 - 2t \cos \theta)^{1/2}, \quad (4.148)$$

$$\sum_{n=0}^{\infty} t^n P_n(\cos \theta) = \frac{1}{h}. \quad (4.149)$$

The sum of some extra Legendre series can be derived from (4.149). By taking the derivative of both sides of (4.149) with respect to  $t$ , and then multiplying by  $t$ , we obtain

$$\sum_{n=0}^{\infty} n t^n P_n(\cos \theta) = \frac{-t^2 + t \cos \theta}{h^3}. \quad (4.150)$$

In (4.149), move the term  $n = 0$  to the right-hand side, multiply both sides by  $t$ , and then integrate over  $t$  from 0 to  $t$ . We obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} t^n P_n(\cos \theta) = \ln \frac{2}{h + 1 - t \cos \theta}. \quad (4.151)$$

By taking the derivative of both sides of (4.149)–(4.151) with respect to  $\theta$ , we obtain

$$\sum_{n=1}^{\infty} t^n \frac{d}{d\theta} P_n(\cos \theta) = -\frac{t \sin \theta}{h^3}, \quad (4.152)$$

$$\sum_{n=1}^{\infty} n t^n \frac{d}{d\theta} P_n(\cos \theta) = \frac{t \sin \theta (2t^2 - t \cos \theta - 1)}{h^5}, \quad (4.153)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} t^n \frac{d}{d\theta} P_n(\cos \theta) = -\frac{(h+1)t \sin \theta}{h(h+1-t \cos \theta)}. \quad (4.154)$$

By dividing both sides of (4.154) by  $t$ , and then integrating over  $t$  from 0 to  $t$ , we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^2} t^n \frac{d}{d\theta} P_n(\cos \theta) \\ &= -\sin \theta \int_0^t \frac{h+1}{h(h+1-t \cos \theta)} dt = \frac{1}{\sin \theta} \ln \frac{(t-h+1)^2(1-\cos \theta)}{2(h-1+t \cos \theta)(1+\cos \theta)} \\ & \quad - \frac{\cos \theta}{\sin \theta} \ln \frac{t^2 \sin^2 \theta}{2(h-1+t \cos \theta)}. \end{aligned} \quad (4.155)$$

The derivation of this formula is quite tedious, where there is a need to substitute the variable  $t$  by  $x$  according to  $t = [2 \cos \theta x^2 + 2x]/(x^2 - 1)$  for evaluating the integral, and the detail is omitted here. By taking the derivative of both sides of (4.152)–(4.155) with respect to  $\theta$ , we obtain successively

$$\sum_{n=1}^{\infty} t^n \frac{d^2}{d\theta^2} P_n(\cos \theta) = \frac{3t^2 \sin^2 \theta - th^2 \cos \theta}{h^5}, \quad (4.156)$$

$$\begin{aligned} \sum_{n=1}^{\infty} nt^n \frac{d^2}{d\theta^2} P_n(\cos \theta) &= \frac{t \cos \theta (2t^2 - 2t \cos \theta - 1) + t^2}{h^5} \\ & \quad - \frac{5t^2 \sin^2 \theta (2t^2 - t \cos \theta - 1)}{h^7}, \end{aligned} \quad (4.157)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} t^n \frac{d^2}{d\theta^2} P_n(\cos \theta) &= \frac{-th(h+1) \cos \theta (h+1-t \cos \theta) + t^2 \sin^2 \theta (h+1)^2}{h^2(h+1-t \cos \theta)^2} \\ & \quad + \frac{t^2 \sin^2 \theta}{h^3(h+1-t \cos \theta)}, \end{aligned} \quad (4.158)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} t^n \frac{d^2}{d\theta^2} P_n(\cos \theta) &= -\frac{\cos \theta}{\sin^2 \theta} \ln \frac{(t-h+1)^2(1-\cos \theta)}{2(h-1+t \cos \theta)(1+\cos \theta)} \\ & \quad + \frac{1}{\sin^2 \theta} \ln \frac{t^2 \sin^2 \theta}{2(h-1+t \cos \theta)} \\ & \quad + \frac{t(\cos \theta - 1)(1-h)}{(h-1+t \cos \theta)h} - \frac{2t}{(t-h+1)h} + 2. \end{aligned} \quad (4.159)$$

More summations can be obtained from these by taking further derivatives or integration. For example, taking the derivative of both sides of (4.150) with respect to  $t$ , and then multiplying both sides by  $t$  result in a formula for  $\sum_{n=0}^{\infty} n^2 t^n P_n(\cos \theta)$ .

The above formulae (4.149)–(4.159) are the Legendre series we derived. By setting  $t \rightarrow 1$ , as  $h \rightarrow 2 \sin(\theta/2)$ , we obtain successively

$$\sum_{n=0}^{\infty} P_n(\cos \theta) = \frac{1}{2 \sin(\theta/2)}, \quad (4.160)$$

$$\sum_{n=0}^{\infty} n P_n(\cos \theta) = -\frac{1}{4 \sin(\theta/2)}, \quad (4.161)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) = \ln \frac{1}{\sin(\theta/2)[1 + \sin(\theta/2)]}, \quad (4.162)$$

$$\sum_{n=1}^{\infty} \frac{d}{d\theta} P_n(\cos \theta) = \frac{\cos(\theta/2)}{4 \sin^2(\theta/2)}, \quad (4.163)$$

$$\sum_{n=1}^{\infty} n \frac{d}{d\theta} P_n(\cos \theta) = \frac{\cos(\theta/2)}{8 \sin^2(\theta/2)}, \quad (4.164)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{d}{d\theta} P_n(\cos \theta) = -\frac{\cos(\theta/2)[1 + 2 \sin(\theta/2)]}{2 \sin(\theta/2)[1 + \sin(\theta/2)]}, \quad (4.165)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{d}{d\theta} P_n(\cos \theta) &= \frac{1}{\sin \theta} \ln \frac{\sin(\theta/2)[1 - \sin(\theta/2)]}{\cos^2(\theta/2)} \\ &\quad - \frac{\cos \theta}{\sin \theta} \ln \frac{\sin(\theta/2) \cos^2(\theta/2)}{1 - \sin(\theta/2)}, \end{aligned} \quad (4.166)$$

$$\sum_{n=1}^{\infty} \frac{d^2}{d\theta^2} P_n(\cos \theta) = \frac{1 + \cos^2(\theta/2)}{8 \sin^3(\theta/2)}, \quad (4.167)$$

$$\sum_{n=1}^{\infty} n \frac{d^2}{d\theta^2} P_n(\cos \theta) = -\frac{1 + \cos^2(\theta/2)}{16 \sin^3(\theta/2)}, \quad (4.168)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{d^2}{d\theta^2} P_n(\cos \theta) = 1 + \frac{1}{2} \sin \frac{\theta}{2} + \frac{\cos \theta \cos^2(\theta/2)}{4 \sin^2(\theta/2)[1 + \sin(\theta/2)]}, \quad (4.169)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{d^2}{d\theta^2} P_n(\cos \theta) &= -\frac{\cos \theta}{\sin^2 \theta} \ln \frac{\sin(\theta/2)[1 - \sin(\theta/2)]}{\cos^2(\theta/2)} \\ &\quad + \frac{1}{\sin^2 \theta} \ln \frac{\sin(\theta/2) \cos^2(\theta/2)}{1 - \sin(\theta/2)} - \frac{1 - 2 \sin(\theta/2)}{2 \sin(\theta/2)}. \end{aligned} \quad (4.170)$$

### 4.3.4 An Asymptotic Expression of the Legendre Function

In this subsection, we derive an asymptotic expression for the Legendre function  $P_n(\cos \theta)$  when the degree  $n$  is very large.

By setting  $\beta = 0$  and  $\alpha = n(n+1)$  in (4.10), we obtain the differential equation satisfied by the Legendre function  $P_n(\cos \theta)$ ,

$$\frac{d}{d\theta} \left[ \sin \theta \frac{d}{d\theta} P_n(\cos \theta) \right] + n(n+1) \sin \theta P_n(\cos \theta) = 0, \quad (4.171)$$

which is indeed (4.118). By replacing the unknown function  $P_n(\cos \theta)$  by  $\phi$  according to

$$P_n(\cos \theta) = \sin^{-1/2} \theta \phi, \quad (4.172)$$

the differential equation of  $P_n(\cos \theta)$ , (4.171), becomes a differential equation of  $\phi$ ,

$$\frac{d^2 \phi}{d\theta^2} + \left[ n(n+1) + \frac{1}{4} + \frac{1}{4 \sin^2 \theta} \right] \phi = 0. \quad (4.173)$$

In the following, we consider only the domain of  $\theta$  given by

$$\frac{1}{4 \sin^2 \theta} \ll n(n+1) + \frac{1}{4}, \quad (4.174)$$

where (4.173) can be approximated by

$$\frac{d^2 \phi}{d\theta^2} + \left[ n(n+1) + \frac{1}{4} \right] \phi = 0. \quad (4.175)$$

The general solution of this differential equation can be verified to be

$$\phi = A e^{i(n+1/2)\theta} + B e^{-i(n+1/2)\theta}. \quad (4.176)$$

The substitution of this solution into (4.172) yields an expression of  $P_n(\cos \theta)$ . For convenience for further development, we write it as

$$P_n(\cos \theta) = \frac{C}{\sin^{1/2} \theta} \left[ e^{i(n+1/2)\theta + iD} + e^{-i(n+1/2)\theta - iD} \right], \quad (4.177)$$

where the constants to be determined,  $A$  and  $B$ , in (4.176) are substituted by  $C$  and  $D$  to take into account of the fact that  $P_n(\cos \theta)$  is a real function. Evidently, (4.177) is an approximate general solution of (4.171), and in order for it to be an approximate expression of  $P_n(\cos \theta)$ , the constants  $C$  and  $D$  must be determined.

When  $n$  is even, the constants  $C$  and  $D$  in (4.177) can be determined by the values of  $P_n(\cos \theta)$  and  $dP_n(\cos \theta)/d\theta$  at  $\theta = \pi/2$ . According to (4.58) and (4.117),

$$P_{2n}\left(\cos \frac{\pi}{2}\right) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}, \quad \left[\frac{d}{d\theta} P_{2n}(\cos \theta)\right]_{\theta=\pi/2} = 0. \quad (4.178)$$

According to (4.177), we have

$$\begin{cases} P_{2n}\left(\cos \frac{\pi}{2}\right) = \frac{2C}{\sin^{1/2} \theta} \cos\left(n\pi + \frac{\pi}{4} + D\right), \\ \left[\frac{d}{d\theta} P_{2n}(\cos \theta)\right]_{\theta=\pi/2} = \frac{(n+1/2)\pi C}{\sin^{1/2} \theta} \sin\left(n\pi + \frac{\pi}{4} + D\right). \end{cases} \quad (4.179)$$

By comparing (4.178) and (4.179), we obtain  $D = -\pi/4$  and  $2C = (2n)!/[2^{2n}(n!)^2]$ . Hence,

$$P_{2n}(\cos \theta) = \frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{\sin^{1/2} \theta} \cos\left[\left(2n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right]. \quad (4.180)$$

When  $n$  is very large, the following Wallis formula holds:

$$\pi \approx \frac{1}{n} \left[ \frac{2^{2n}(n!)^2}{(2n)!} \right]^2. \quad (4.181)$$

We can thus write (4.180) as

$$P_{2n}(\cos \theta) = \left( \frac{2}{2n\pi \sin \theta} \right)^{1/2} \cos\left[\left(2n + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right]. \quad (4.182)$$

This is the asymptotic expression of even degree Legendre function.

The asymptotic expression of odd degree Legendre function can be derived using the recurrence formula (4.117). We have

$$P_{2n-1}(\cos \theta) = -\frac{\sin \theta}{2n} \frac{d}{d\theta} P_{2n}(\cos \theta) + \cos \theta P_{2n}(\cos \theta). \quad (4.183)$$

We then obtain, by substituting (4.182) into it,

$$P_{2n-1}(\cos \theta) = \left[ \frac{2}{(2n-1)\pi \sin \theta} \right]^{1/2} \cos\left[\left(2n - 1 + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right]. \quad (4.184)$$

This is the asymptotic expression of odd degree Legendre function.

Finally, a general form of the asymptotic expression of the Legendre function can be obtained by combining (4.182) and (4.184),



$$P_n(\cos \theta) = \left( \frac{2}{\pi n \sin \theta} \right)^{1/2} \cos \left[ \left( n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right], \quad (4.185)$$

which can also be written in a complex form as

$$P_n(\cos \theta) = \left( \frac{1}{2\pi n \sin \theta} \right)^{1/2} \left[ e^{i[(n+1/2)\theta - \pi/4]} + e^{-i[(n+1/2)\theta - \pi/4]} \right]. \quad (4.186)$$

It is to be noted that this formula is accurate to the order of  $1/n$ , and is applicable only within the domain of  $\theta$  given by (4.174), here written in the form

$$\sin \theta \gg \frac{1}{[4n(n+1) + 1]^{1/2}} \approx \frac{1}{2n}. \quad (4.187)$$

This domain implies that (4.185) and (4.186) does not apply to the polar regions.

### 4.3.5 Some Orthogonality Formulae

Let  $\phi_1(x), \phi_2(x), \dots$  be a sequence of functions defined in  $a \leq x \leq b$ . If

$$\int_a^b \phi_n(x) \phi_m(x) W(x) dx = \delta_{mn} \lambda_n, \quad (4.188)$$

we say that they are orthogonal in  $a \leq x \leq b$  with the weight  $W(x)$ . In this formulae, no limit is set for the value of the indices  $m$  and  $n$  of the Kronecker Delta  $\delta_{mn}$ . Throughout this subsection, the convention of summation index is not assumed.

#### The First Fundamental Orthogonality Formula

The most used orthogonality formula of the (associated) Legendre function is

$$\int_{-1}^1 P_n^k(x) P_m^k(x) dx = \frac{2\delta_{mn}}{2n+1} \frac{(n+k)!}{(n-k)!}. \quad (4.189)$$

An alternative form is, according to (4.79),

$$\int_{-1}^1 P_n^k(x) P_m^{-k}(x) dx = \delta_{mn} (-1)^k \frac{2}{2n+1}. \quad (4.190)$$

The degenerate form when  $k = 0$  is

$$\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2\delta_{mn}}{2n+1}. \quad (4.191)$$

The proof is provided in the following paragraphs.

When  $m \neq n$ , the differential equations satisfied by  $P_n^k(x)$  and  $P_m^k(x)$  can be obtained by setting  $\alpha = n(n+1)$  and  $\alpha = m(m+1)$  in (4.15). Here we write them as

$$\frac{d}{dx} \left[ (1-x^2) \frac{dP_n^k(x)}{dx} \right] + \left[ n(n+1) - \frac{k^2}{1-x^2} \right] P_n^k(x) = 0, \quad (4.192)$$

$$\frac{d}{dx} \left[ (1-x^2) \frac{dP_m^k(x)}{dx} \right] + \left[ m(m+1) - \frac{k^2}{1-x^2} \right] P_m^k(x) = 0. \quad (4.193)$$

Multiply the first equation by  $P_m^k(x)$ , the second by  $P_n^k(x)$ , and then subtract the second from the first. We obtain

$$\begin{aligned} & [n(n+1) - m(m+1)] P_n^k(x) P_m^k(x) \\ &= P_n^k(x) \frac{d}{dx} \left[ (1-x^2) \frac{dP_m^k(x)}{dx} \right] - P_m^k(x) \frac{d}{dx} \left[ (1-x^2) \frac{dP_n^k(x)}{dx} \right] \\ &= \frac{d}{dx} \left\{ (1-x^2) \left[ P_n^k(x) \frac{dP_m^k(x)}{dx} - P_m^k(x) \frac{dP_n^k(x)}{dx} \right] \right\}. \end{aligned} \quad (4.194)$$

Integrate both sides over  $x$  in the domain  $-1 \leq x \leq 1$ . We obtain

$$[n(n+1) - m(m+1)] \int_{-1}^1 P_n^k(x) P_m^k(x) dx = 0. \quad (4.195)$$

When  $m \neq n$ ,  $n(n+1) - m(m+1) \neq 0$ . Hence,

$$\int_{-1}^1 P_n^k(x) P_m^k(x) dx = 0, \quad m \neq n. \quad (4.196)$$

This is the portion  $m \neq n$  of (4.189).

When  $m = n$ , we obtain, according to (4.68) and (4.80),

$$\int_{-1}^1 [P_n^k(x)]^2 dx = \frac{(-1)^k}{2^{2n}(n!)^2} \frac{(n+k)!}{(n-k)!} J, \quad (4.197)$$

where  $J$  is an integral defined as

$$J = \int_{-1}^1 \frac{d^{n+k}}{dx^{n+k}} (x^2 - 1)^n \frac{d^{n-k}}{dx^{n-k}} (x^2 - 1)^n dx. \quad (4.198)$$

By integrating by parts,  $J$  can be reduced to

$$J = \left[ \frac{d^{n+k}}{dx^{n+k}} (x^2 - 1)^n \frac{d^{n-k-1}}{dx^{n-k-1}} (x^2 - 1)^n \right]_{-1}^1 - \int_{-1}^1 \frac{d^{n+k+1}}{dx^{n+k+1}} (x^2 - 1)^n \frac{d^{n-k-1}}{dx^{n-k-1}} (x^2 - 1)^n dx. \quad (4.199)$$

The first term vanishes, Hence,

$$J = - \int_{-1}^1 \frac{d^{n+k+1}}{dx^{n+k+1}} (x^2 - 1)^n \frac{d^{n-k-1}}{dx^{n-k-1}} (x^2 - 1)^n dx. \quad (4.200)$$

Integrate by parts again and again. We obtain

$$J = \int_{-1}^1 \frac{d^{n+k+2}}{dx^{n+k+2}} (x^2 - 1)^n \frac{d^{n-k-2}}{dx^{n-k-2}} (x^2 - 1)^n dx, \quad (4.201)$$

...

$$J = (-1)^{n-k} \int_{-1}^1 \frac{d^{2n}}{dx^{2n}} (x^2 - 1)^n \times (x^2 - 1)^n dx. \quad (4.202)$$

As  $(x^2 - 1)^n$  is a polynomial of degree  $2n$ ,  $d^{2n}(x^2 - 1)^n/dx^{2n} = (2n)!$ . Substitute it into the above formula. We obtain

$$J = (-1)^{n-k} (2n)! \int_{-1}^1 (x^2 - 1)^n dx = (-1)^k (2n)! \int_{-1}^1 (1 - x^2)^n dx. \quad (4.203)$$

The substitution of this formula into (4.197) yields

$$\int_{-1}^1 [P_n^k(x)]^2 dx = \frac{(2n)!}{2^{2n}(n!)^2} \frac{(n+k)!}{(n-k)!} \int_{-1}^1 (1 - x^2)^n dx. \quad (4.204)$$

Now we only have to compute the integral in this formula. We have

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= \int_{\pi}^0 \sin^{2n} \theta d \cos \theta \\ &= (\sin^{2n} \theta \cos \theta)_{\pi}^0 - \int_{\pi}^0 \cos \theta d \sin^{2n} \theta \\ &= -2n \int_{\pi}^0 \cos^2 \theta \sin^{2n-1} \theta d \theta \\ &= 2n \int_{\pi}^0 (1 - \sin^2 \theta) \sin^{2n-2} \theta d \cos \theta \end{aligned}$$

$$\begin{aligned}
&= 2n \left( \int_{\pi}^0 \sin^{2n-2} \theta d \cos \theta - \int_{\pi}^0 \sin^{2n} \theta d \cos \theta \right) \\
&= 2n \left[ \int_{-1}^1 (1-x^2)^{n-1} dx - \int_{-1}^1 (1-x^2)^n dx \right]. \quad (4.205)
\end{aligned}$$

We obtain from this formula,

$$\int_{-1}^1 (1-x^2)^n dx = \frac{2n}{2n+1} \int_{-1}^1 (1-x^2)^{n-1} dx. \quad (4.206)$$

We can further obtain by repeatedly using this formula,

$$\int_{-1}^1 (1-x^2)^{n-1} dx = \frac{2(n-1)}{2(n-1)+1} \int_{-1}^1 (1-x^2)^{n-2} dx, \quad (4.207)$$

.....

$$\int_{-1}^1 (1-x^2) dx = \frac{2}{3} \int_{-1}^1 dx = \frac{2}{3} \times 2. \quad (4.208)$$

By substituting (4.207)–(4.208) into (4.206), we obtain

$$\int_{-1}^1 (1-x^2)^n dx = \frac{2n[2(n-1)][2(n-2)] \cdots \cdots 2}{(2n+1)(2n-1)(2n-3) \cdots \cdots 3} \times 2 = \frac{2^{2n}(n!)^2}{(2n+1)!} \times 2. \quad (4.209)$$

Finally, by substituting this formula into (4.204), we obtain the desired result,

$$\int_{-1}^1 [P_n^k(x)]^2 dx = \frac{2}{2n+1} \frac{(n+k)!}{(n-k)!}. \quad (4.210)$$

This is the  $m = n$  portion of (4.189).

### The Second Fundamental Orthogonality Formula

Another fundamental orthogonality formula of the Legendre function is

$$\int_{-1}^1 P_n^k(x) P_n^l(x) \frac{dx}{1-x^2} = \frac{\delta_{kl}}{k} \frac{(n+k)!}{(n-k)!}, \quad (4.211)$$

which can be written equivalently as

$$\int_{-1}^1 P_n^k(x) P_n^{-l}(x) \frac{dx}{1-x^2} = (-1)^k \frac{\delta_{kl}}{k}. \quad (4.212)$$

We mention that  $k = l = 0$  is not permitted. In the following paragraphs, we provide a proof for (4.211).

When  $l \neq k$ ,  $P_n^k(x)$  and  $P_n^l(x)$  satisfy

$$\frac{d}{dx} \left[ (1-x^2) \frac{dP_n^k(x)}{dx} \right] + \left[ n(n+1) - \frac{k^2}{1-x^2} \right] P_n^k(x) = 0, \quad (4.213)$$

$$\frac{d}{dx} \left[ (1-x^2) \frac{dP_n^l(x)}{dx} \right] + \left[ n(n+1) - \frac{l^2}{1-x^2} \right] P_n^l(x) = 0, \quad (4.214)$$

respectively. Multiply the first one by  $P_n^l(x)$ , the second one by  $P_n^k(x)$ , and then subtract one from another. We obtain

$$\begin{aligned} & (k^2 - l^2) P_n^k(x) P_n^l(x) \frac{1}{1-x^2} \\ &= P_n^l(x) \frac{d}{dx} \left[ (1-x^2) \frac{dP_n^k(x)}{dx} \right] - P_n^k(x) \frac{d}{dx} \left[ (1-x^2) \frac{dP_n^l(x)}{dx} \right] \\ &= \frac{d}{dx} \left\{ (1-x^2) \left[ P_n^l(x) \frac{dP_n^k(x)}{dx} - P_n^k(x) \frac{dP_n^l(x)}{dx} \right] \right\}. \end{aligned} \quad (4.215)$$

Integrate over  $x$  in the domain  $-1 \leq x \leq 1$ . We obtain

$$(k^2 - l^2) \int_{-1}^1 P_n^k(x) P_n^l(x) \frac{dx}{1-x^2} = 0. \quad (4.216)$$

When  $l \neq k$ , we have  $(k^2 - l^2) \neq 0$ . Hence, this formula is the portion  $l \neq k$  of (4.211).

According to (4.68), we have

$$\int_{-1}^1 [P_n^k(x)]^2 \frac{dx}{1-x^2} = \frac{1}{2^{2n}(n!)^2} J, \quad (4.217)$$

where

$$J = \int_{-1}^1 (1-x^2)^{k-1} \left[ \frac{d^{n+k}}{dx^{n+k}} (x^2-1)^n \right]^2 dx. \quad (4.218)$$

Let

$$G = (1-x^2)^{k-1} \frac{d^{n+k}}{dx^{n+k}} (x^2-1)^n. \quad (4.219)$$

We can then write (4.218) as

$$J = \int_{-1}^1 G \frac{d^{n+k}}{dx^{n+k}} (x^2 - 1)^n dx. \quad (4.220)$$

As  $G(\pm 1) = 0$ , by integrating by part, we obtain

$$\begin{aligned} J &= \left[ G \frac{d^{n+k-1}}{dx^{n+k-1}} (x^2 - 1)^n \right]_{-1}^1 - \int_{-1}^1 \frac{dG}{dx} \frac{d^{n+k-1}}{dx^{n+k-1}} (x^2 - 1)^n dx \\ &= - \int_{-1}^1 \frac{dG}{dx} \frac{d^{n+k-1}}{dx^{n+k-1}} (x^2 - 1)^n dx. \end{aligned} \quad (4.221)$$

As  $G$  includes  $(1 - x^2)^{k-1}$ , we have

$$\left( \frac{d^m G}{dx^m} \right)_{x=\pm 1} = 0, \quad m = 0, 1, 2, \dots, k-2. \quad (4.222)$$

Hence, we can repeatedly apply integration by parts to (4.221) to obtain

$$J = (-1)^{k-1} \int_{-1}^1 \frac{d^{k-1} G}{dx^{k-1}} \frac{d^{n+1}}{dx^{n+1}} (x^2 - 1)^n dx. \quad (4.223)$$

By integrating by parts once again, we obtain

$$J = (-1)^{k-1} \left\{ \left[ \frac{d^{k-1} G}{dx^{k-1}} \frac{d^n}{dx^n} (x^2 - 1)^n \right]_{-1}^1 - \int_{-1}^1 \frac{d^k G}{dx^k} \frac{d^n}{dx^n} (x^2 - 1)^n dx \right\}. \quad (4.224)$$

Substitute this formula into (4.217), and then use (4.59). We obtain

$$\int_{-1}^1 [P_n^k(x)]^2 \frac{dx}{1-x^2} = \frac{(-1)^{k-1}}{2^n n!} \left[ \frac{d^{k-1} G}{dx^{k-1}} P_n(x) \right]_{-1}^1 - \frac{(-1)^{k-1}}{2^n n!} \int_{-1}^1 \frac{d^k G}{dx^k} P_n(x) dx. \quad (4.225)$$

As  $G$  is a polynomial of degree  $2(k-1) + 2n - (n+k) = n+k-2$ ,  $d^k G/dx^k$  should be a polynomial of degree  $n-2$ . As the Legendre function  $P_m(x)$  is a polynomial of degree  $m$ , we can express  $d^k G/dx^k$  as a linear combination of Legendre polynomial up to degree  $n-2$ , i.e.,

$$\frac{d^k G}{dx^k} = \sum_{m=0}^{n-2} a_m P_m(x), \quad (4.226)$$

where the coefficients  $a_m$  are constants. By substituting (4.226) into the second term at the right-hand side of (4.225), and then using (4.191), we conclude that this term

vanishes. Hence, (4.225) becomes

$$\int_{-1}^1 [P_n^k(x)]^2 \frac{dx}{1-x^2} = \frac{(-1)^{k-1}}{2^n n!} \left[ \frac{d^{k-1} G}{dx^{k-1}} P_n(x) \right]_{-1}^1. \quad (4.227)$$

The substitution of  $P_n(\pm 1) = (\pm 1)^n$  into this formula yields

$$\int_{-1}^1 [P_n^k(x)]^2 \frac{dx}{1-x^2} = \frac{(-1)^{k-1}}{2^n n!} \left[ \left( \frac{d^{k-1} G}{dx^{k-1}} \right)_{x=1} - (-1)^n \left( \frac{d^{k-1} G}{dx^{k-1}} \right)_{x=-1} \right]. \quad (4.228)$$

The rest is to formulate  $(d^{k-1} G/dx^{k-1})_{x=\pm 1}$ . First, we have

$$\begin{aligned} & \left( \frac{d^{k-1} G}{dx^{k-1}} \right)_{x=\pm 1} \\ &= \frac{d^{k-1}}{dx^{k-1}} \left[ (1-x^2)^{k-1} \frac{d^{n+k}}{dx^{n+k}} (x^2-1)^n \right]_{x=\pm 1} \\ &= \sum_{l=0}^{k-1} \frac{(k-1)!}{l!(k-1-l)!} \left[ \frac{d^l}{dx^l} (1-x^2)^{k-1} \frac{d^{n+k+k-1-l}}{dx^{n+k+k-1-l}} (x^2-1)^n \right]_{x=\pm 1}. \end{aligned} \quad (4.229)$$

It can be seen that only when  $l = k-1$  the term  $d^l(1-x^2)^{k-1}/dx^l$  does not vanish when  $x \pm 1$ . Hence, the above formula can be written as

$$\left( \frac{d^{k-1} G}{dx^{k-1}} \right)_{x=\pm 1} = \left[ \frac{d^{k-1}}{dx^{k-1}} (1-x^2)^{k-1} \frac{d^{n+k}}{dx^{n+k}} (x^2-1)^n \right]_{x=\pm 1}. \quad (4.230)$$

According to (4.59) and (4.94), we have

$$\begin{aligned} \left[ \frac{d^{k-1}}{dx^{k-1}} (1-x^2)^{k-1} \right]_{x=\pm 1} &= (-1)^{k-1} 2^{k-1} (k-1)! P_{k-1}(\pm 1) \\ &= (\mp 1)^{k-1} 2^{k-1} (k-1)!. \end{aligned} \quad (4.231)$$

We also have

$$\begin{aligned} & \left[ \frac{d^{n+k}}{dx^{n+k}} (x^2-1)^n \right]_{x=\pm 1} \\ &= \sum_{l=0}^{n+k} \frac{(n+k)!}{l!(n+k-l)!} \left[ \frac{d^l}{dx^l} (x-1)^n \frac{d^{n+k-l}}{dx^{n+k-l}} (x+1)^n \right]_{x=\pm 1}. \end{aligned} \quad (4.232)$$

When  $x = 1$ ,  $d^l(x-1)^n/dx^l$  is not vanishing only when  $l = n$ . Hence,

$$\begin{aligned} \left[ \frac{d^{n+k}}{dx^{n+k}} (x^2 - 1)^n \right]_{x=1} &= \frac{(n+k)!}{n!k!} \left[ \frac{d^n}{dx^n} (x-1)^n \frac{d^k}{dx^k} (x+1)^n \right]_{x=1} \\ &= \frac{(n+k)!}{n!k!} n! \frac{n!}{(n-k)!} 2^{n-k} \\ &= \frac{n!}{k!} 2^{n-k} \frac{(n+k)!}{(n-k)!}. \end{aligned} \quad (4.233)$$

Furthermore, when  $x = -1$ ,  $d^{n+k-1}(x+1)^n/dx^{n+k-1}$  is not vanishing only when  $n+k-l = n$ . Hence,

$$\begin{aligned} \left[ \frac{d^{n+k}}{dx^{n+k}} (x^2 - 1)^n \right]_{x=-1} &= \frac{(n+k)!}{k!n!} \left[ \frac{d^k}{dx^k} (x-1)^n \frac{d^n}{dx^n} (x+1)^n \right]_{x=-1} \\ &= \frac{(n+k)!}{k!n!} \frac{n!}{(n-k)!} (-2)^{n-k} n! \\ &= (-1)^{n-k} \frac{n!}{k!} 2^{n-k} \frac{(n+k)!}{(n-k)!}. \end{aligned} \quad (4.234)$$

By substituting (4.231), (4.233), and (4.234) into (4.230), we obtain

$$\begin{aligned} \left( \frac{d^{k-1}G}{dx^{k-1}} \right)_{x=\pm 1} &= (\mp 1)^{k-1} 2^{k-1} (k-1)! (\pm 1)^{n-k} \frac{n!}{k!} 2^{n-k} \frac{(n+k)!}{(n-k)!} \\ &= (\mp 1)^{k-1} (\pm 1)^{n-k} 2^{n-1} \frac{n!}{k} \frac{(n+k)!}{(n-k)!}. \end{aligned} \quad (4.235)$$

Finally, the substitution of this formula into (4.228) yields

$$\begin{aligned} \int_{-1}^1 [P_n^k(x)]^2 \frac{dx}{1-x^2} &= \frac{(-1)^{k-1}}{2^n n!} \left[ (-1)^{k-1} 2^{n-1} \frac{n!}{k} \frac{(n+k)!}{(n-k)!} \right. \\ &\quad \left. - (-1)^n (-1)^{n-k} 2^{n-1} \frac{n!}{k} \frac{(n+k)!}{(n-k)!} \right] \\ &= \frac{1}{k} \frac{(n+k)!}{(n-k)!}. \end{aligned} \quad (4.236)$$

This is the portion  $l = k$  of (4.211).

### Expression of the Orthogonal Formulae Using $\theta$ , and Some Extra Formulae

With  $x = \cos \theta$ , the first orthogonality formula expressed in the forms of (4.189)



and (4.190), and its degenerate form (4.191), become

$$\int_0^\pi P_n^k(\cos \theta) P_m^k(\cos \theta) \sin \theta d\theta = \frac{2\delta_{mn}}{2n+1} \frac{(n+k)!}{(n-k)!}, \quad (4.237)$$

$$\int_0^\pi P_n^k(\cos \theta) P_m^{-k}(\cos \theta) \sin \theta d\theta = \delta_{mn} (-1)^k \frac{2}{2n+1}, \quad (4.238)$$

$$\int_0^\pi P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = \frac{2\delta_{mn}}{2n+1}. \quad (4.239)$$

The second orthogonality formula expressed in the forms of (4.211) and (4.212) are

$$\int_0^\pi P_n^k(\cos \theta) P_n^l(\cos \theta) \frac{1}{\sin \theta} d\theta = \frac{\delta_{kl}}{k} \frac{(n+k)!}{(n-k)!}, \quad (4.240)$$

$$\int_0^\pi P_n^k(\cos \theta) P_n^{-l}(\cos \theta) \frac{1}{\sin \theta} d\theta = (-1)^k \frac{\delta_{kl}}{k}. \quad (4.241)$$

where the special case  $k = l = 0$  is not permitted.

Two extra orthogonality formulae can be derived from (4.237). The differential equation satisfied by  $P_n^k(\cos \theta)$ , (4.138), can be written as

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d}{d\theta} P_n^k(\cos \theta) \right] + \left[ n(n+1) - \frac{k^2}{\sin^2 \theta} \right] P_n^k(\cos \theta) = 0, \quad (4.242)$$

from which we obtain

$$\begin{aligned} & \frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta P_m^k(\cos \theta) \frac{d}{d\theta} P_n^k(\cos \theta) \right] + n(n+1) P_n^k(\cos \theta) P_m^k(\cos \theta) \\ &= \frac{k^2}{\sin^2 \theta} P_n^k(\cos \theta) P_m^k(\cos \theta) + \frac{d}{d\theta} P_n^k(\cos \theta) \frac{d}{d\theta} P_m^k(\cos \theta). \end{aligned} \quad (4.243)$$

Multiply both sides by  $\sin \theta$ , and then integrate over  $\theta$  in the domain  $0 \leq \theta \leq \pi$ . We obtain

$$\begin{aligned} & \int_0^\pi \left[ \frac{k^2}{\sin^2 \theta} P_n^k(\cos \theta) P_m^k(\cos \theta) + \frac{d}{d\theta} P_n^k(\cos \theta) \frac{d}{d\theta} P_m^k(\cos \theta) \right] \sin \theta d\theta \\ &= n(n+1) \int_0^\pi P_n^k(\cos \theta) P_m^k(\cos \theta) \sin \theta d\theta \\ &= \frac{2n(n+1)\delta_{mn}}{2n+1} \frac{(n+k)!}{(n-k)!}. \end{aligned} \quad (4.244)$$

This is the first formula we are proving. The second orthogonality formula is obtained by integrating the equation

$$\begin{aligned} \frac{d}{d\theta} [\cos \theta P_n^k(\cos \theta) P_m^k(\cos \theta)] \\ = -P_n^k(\cos \theta) P_m^k(\cos \theta) \sin \theta + \cos \theta \frac{d}{d\theta} [P_n^k(\cos \theta) P_m^k(\cos \theta)] \end{aligned} \quad (4.245)$$

over  $\theta$  in the domain  $0 \leq \theta \leq \pi$ , yielding

$$\begin{aligned} \int_0^\pi \cos \theta \frac{d}{d\theta} [P_n^k(\cos \theta) P_m^k(\cos \theta)] d\theta \\ = \int_0^\pi P_n^k(\cos \theta) P_m^k(\cos \theta) \sin \theta d\theta + [\cos \theta P_n^k(\cos \theta) P_m^k(\cos \theta)]_0^\pi \\ = \frac{2\delta_{mn}}{2n+1} \frac{(n+k)!}{(n-k)!} - \delta_k [(-1)^{m+n} + 1], \end{aligned} \quad (4.246)$$

where  $\delta_k$  is a Kronecker Delta defined in (2.32).

Another integral formula can be obtained from (4.244) and (4.240),

$$\begin{aligned} \int_0^\pi \left[ \frac{d}{d\theta} P_n^k(\cos \theta) \right]^2 \sin \theta d\theta &= \frac{2n(n+1)}{2n+1} \frac{(n+k)!}{(n-k)!} - \int_0^\pi \frac{k^2}{\sin \theta} [P_n^k(\cos \theta)]^2 d\theta \\ &= \frac{2n(n+1)}{2n+1} \frac{(n+k)!}{(n-k)!} - \frac{k(n+k)!}{(n-k)!}. \end{aligned} \quad (4.247)$$

## 4.4 Spherical Harmonics

### 4.4.1 Solution of Laplace's Equation and Spherical Harmonics

While we were solving Laplace's equation, we have separated the harmonic function  $\phi$  as the product of three of functions  $f(r)$ ,  $B(\theta)$  and  $L(\lambda)$ . We have already obtained  $B(\theta)$  and  $L(\lambda)$ . Now we solve for  $f(r)$ .

Substitute the eigenvalue  $\alpha = n(n+1)$  into the equation of  $f(r)$ , (4.5), we obtain

$$\frac{d}{dr} \left( r^2 \frac{df}{dr} \right) - n(n+1)f = 0. \quad (4.248)$$

It can be easily verified that its two linearly independent solutions are

$$f_1 = r^n, \quad f_2 = \frac{1}{r^{n+1}}. \quad (4.249)$$

As  $f_1 \rightarrow 0$  and  $f_2 \rightarrow \infty$  when  $r \rightarrow 0$ , and  $f_1 \rightarrow \infty$  and  $f_2 \rightarrow 0$  when  $r \rightarrow \infty$ ,  $f_1$  is suitable in studying the solution of Laplace's equation in the interior of a sphere

with center on the origin of the coordinate system  $Or\theta\lambda$ , and  $f_2$  in the exterior of a sphere.

Evidently, all possible combinations  $f(r)B(\theta)L(\lambda)$  is a solution of Laplace's equation. Within a sphere with center on the origin of the coordinate system  $Or\theta\lambda$ , the most general finite solution based on separation of variables is

$$\phi = \sum_{n=0}^{\infty} \sum_{k=0}^n r^n (A_n^k \cos k\lambda + B_n^k \sin k\lambda) P_n^k(\cos \theta), \quad (4.250)$$

where  $A_n^k$  and  $B_n^k$  are arbitrary constants. In the exterior of a sphere, the most general finite solution based on separation of variables is

$$\phi = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{r^{n+1}} (A_n^k \cos k\lambda + B_n^k \sin k\lambda) P_n^k(\cos \theta). \quad (4.251)$$

Within a spherical shell with center on the origin of the coordinate system  $Or\theta\lambda$ , the most general finite solution based on separation of variables is

$$\begin{aligned} \phi = & \sum_{n=0}^{\infty} \sum_{k=0}^n r^n (A_n^k \cos k\lambda + B_n^k \sin k\lambda) P_n^k(\cos \theta) \\ & + \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{r^{n+1}} (C_n^k \cos k\lambda + D_n^k \sin k\lambda) P_n^k(\cos \theta). \end{aligned} \quad (4.252)$$

In these formulae,  $P_n^k(\cos \theta) \cos k\lambda$  and  $P_n^k(\cos \theta) \sin k\lambda$  are referred to as surface spherical harmonics of degree  $n$  and order  $k$ , and their products with  $r$  or  $1/r$  are referred to as solid spherical harmonics. When  $k = 0$ , the coefficients  $A_n^k$  and  $C_n^k$  are also denoted as  $A_n$  and  $C_n$ , respectively. The coefficients  $B_n^0$  and  $D_n^0$  are meaningless, as they multiply with 0.

#### 4.4.2 Geometrical Properties of Surface Spherical Harmonics

We first show that  $P_n^k(x)$  has  $n - k$  zeros of multiplicity 1 in  $(-1, 1)$ . We use the expression of  $P_n^k(x)$  given by (4.80), which is rewritten here for convenience of reference,

$$P_n^k(x) = (-1)^k \frac{(n+k)!}{2^n n! (n-k)!} (1-x^2)^{-k/2} \frac{d^{n-k}}{dx^{n-k}} (x^2-1)^n. \quad (4.253)$$

Evidently,  $P_n^k(x)$  and the polynomial  $(d^{n-k}/dx^{n-k})(x^2-1)^n$  have the same zeros in  $(-1, 1)$ .

As  $(x^2-1)^n$  is a polynomial of degree  $2n$ , it has at most  $2n$  zeros. However, both  $-1$  and  $1$  are its zeros of multiplicity  $n$ , it can not have other zeros. It can be easily

verified that both  $-1$  and  $1$  are zeros of multiplicity  $n - 1$  of  $(d/dx)(x^2 - 1)^n$ , zeros of multiplicity  $n - 2$  of  $(d^2/dx^2)(x^2 - 1)^n$ , ..., and zeros of multiplicity  $k$  of  $(d^{n-k}/dx^{n-k})(x^2 - 1)^n$ . This process of inference requires that the order of derivative to be smaller than  $n$ , which is the reason to chose the expression (4.253) over the standard definition (4.68).

According to Rolle's theorem, as  $(x^2 - 1)^n$  is zero at the boundaries of the closed domain  $x \in [-1, 1]$ , there must be a zero point of  $(d/dx)(x^2 - 1)^n$  in  $(-1, 1)$ , which we denote as  $x'$ . As  $(d/dx)(x^2 - 1)^n$  is a polynomial of degree  $2n - 1$ , it has  $2n - 1$  zeros at most. However, as  $-1$  and  $1$  are its zeros of multiplicity  $n - 1$ , it can have at most  $(2n - 1) - 2(n - 1) = 1$  zero in  $(-1, 1)$ . This means that  $x'$  is its only zero different from  $-1$  and  $1$ , and it is of multiplicity 1. Again, according to Rolle's theorem, as  $-1$ ,  $x'$  and  $1$  are zeros of  $(d/dx)(x^2 - 1)^n$ ,  $(d^2/dx^2)(x^2 - 1)^n$  must have a zero  $x''$  in  $(-1, x')$  and a zero  $x'''$  in  $(x', 1)$ . As  $(d^2/dx^2)(x^2 - 1)^n$  is a polynomial of degree  $2n - 2$ , it has  $2n - 2$  zeros at most. However, as  $-1$  and  $1$  are its zeros of multiplicity  $n - 2$ , it can have at most  $(2n - 2) - 2(n - 2) = 2$  zeros in  $(-1, 1)$ . This means that  $x''$  and  $x'''$  are its only zeros different from  $-1$  and  $1$ , and they are of multiplicity 1. Continuation of this chain of inference leads to the conclusion that  $(d^{n-k}/dx^{n-k})(x^2 - 1)^n$  has, and has only  $n - k$  zeros in  $(-1, 1)$ , which are all of multiplicity 1, which is equivalent to the conclusion that  $P_n^k(x)$  has  $n - k$  zeros of multiplicity 1 in  $(-1, 1)$ . This is illustrated in Fig. 4.2 for the case of  $n = 20$ .

A degenerate case is  $P_n(x)$ , which has  $n$  zeros of multiplicity 1 in  $(-1, 1)$ , and  $P_n(\pm 1) = (-1)^n$ .

We now look back at (4.68), which states that  $P_n^k(x)$  is the product of  $(1 - x^2)^{k/2}$  and a polynomial of degree  $n - k$ . Therefore, the  $n - k$  zeros of  $P_n^k(x)$  in  $(-1, 1)$  are, and are the only zeros of the polynomial, and all of them are of multiplicity 1. Hence, if we denote the zeros as  $x_i$ ,  $i = 1, 2, \dots, n - k$ , we can write  $P_n^k(x)$  as

$$P_n^k(x) = \frac{(2n)!}{2^n n! (n - k)!} (1 - x^2)^{k/2} \prod_{i=1}^{n-k} (x - x_i), \quad (4.254)$$

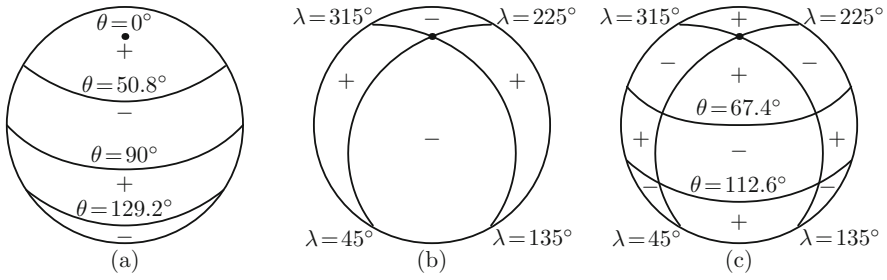
where  $(2n)!/[2^n n! (n - k)!]$  is the coefficient of  $x^{n-k}$  in the polynomial.

According to (4.70), if  $x_i$  is a zero of  $P_n^k(x)$ ,  $-x_i$  is also a zero. When  $n - k$  is even,  $x = 0$  is not a zero of  $P_n^k(x)$ . When  $n - k$  is odd,  $x = 0$  is a zero of  $P_n^k(x)$ . Hence, (4.254) can be written, using only the positive zeros, as

$$P_n^k(x) = \frac{(2n)!}{2^n n! (n - k)!} (1 - x^2)^{k/2} x^v \prod_{i=1}^{(n-k-v)/2} (x^2 - x_i^2), \quad (4.255)$$

where  $v$  is defined as

$$v = \begin{cases} 0, & n - k \text{ even;} \\ 1, & n - k \text{ odd.} \end{cases} \quad (4.256)$$



**Fig. 4.4** Zonal (a), sectorial (b), and tesseral (c) spherical harmonics

Some geometrical properties of the surface spherical harmonics can be inferred from the zeros of the associated Legendre function.

When  $k = 0$ , a surface spherical harmonics degenerates to  $P_n(\cos \theta)$ , which has  $n$  zeros in  $0^\circ < \theta < 180^\circ$ , and changes sign at all zeros. Hence,  $P_n(\cos \theta)$  divides the sphere into  $n+1$  zones of alternate signs, thus being referred to as zonal spherical harmonics. In Fig. 4.4a, we show  $P_3(\cos \theta)$  with zeros  $\theta = 50.8^\circ, 90^\circ$  and  $129.2^\circ$ .

When  $k = n$ ,  $P_n^n(\cos \theta)$  has no zero in  $0^\circ < \theta < 180^\circ$ . It equals zero only at  $\theta = 0^\circ$  and  $180^\circ$ . However,  $\cos n\lambda$  or  $\sin n\lambda$  has  $2n$  zeros in  $0^\circ \leq \lambda < 360^\circ$ , and changes sign at all zeros. Hence,  $P_n^n(\cos \theta) \cos n\lambda$  or  $P_n^n(\cos \theta) \sin n\lambda$  divides the sphere into  $2n$  sectors of alternate signs, thus being referred to as sectorial spherical harmonics. In Fig. 4.4b, we show  $P_2^2(\cos \theta) \cos 2\lambda$  with zeros  $\lambda = 45^\circ, 135^\circ, 225^\circ$  and  $315^\circ$ .

When  $0 < k < n$ ,  $P_n^k(\cos \theta)$  has  $n - k$  zeros in  $0^\circ < \theta < 180^\circ$ , and  $\cos k\lambda$  or  $\sin k\lambda$  has  $2k$  zeros in  $0^\circ \leq \lambda < 360^\circ$ . Both  $P_n^k(\cos \theta)$  and  $\cos k\lambda$  or  $\sin k\lambda$  change sign at their zeros. Hence,  $P_n^k(\cos \theta) \cos k\lambda$  or  $P_n^k(\cos \theta) \sin k\lambda$  divides the sphere into  $2k(n - k + 1)$  quadrangles (triangles at the poles), and thus being referred to as tesseral spherical harmonics. In Fig. 4.4c, we shown  $P_4^2(\cos \theta) \cos 2\lambda$  with zeros  $\theta = 0^\circ, 67.4^\circ, 112.6^\circ$  and  $180^\circ$ , and  $\lambda = 45^\circ, 135^\circ, 225^\circ$  and  $315^\circ$ .

### 4.4.3 Various Forms of Surface Spherical Harmonics

The main component in the spherical harmonic series (4.250)–(4.252) is the surface spherical harmonics of degree  $n$ ,

$$Y_n(\theta, \lambda) = \sum_{k=0}^n (A_n^k \cos k\lambda + B_n^k \sin k\lambda) P_n^k(\cos \theta). \quad (4.257)$$

It can be expressed in various forms.

Define

$$\mathcal{H}_n^k(\theta, \lambda) = \begin{cases} P_n^{|k|}(\cos \theta) \cos |k|\lambda, & k \geq 0; \\ P_n^{|k|}(\cos \theta) \sin |k|\lambda, & k < 0. \end{cases} \quad (4.258)$$

The surface spherical harmonics (4.257) can be written in the following brief form,

$$Y_n(\theta, \lambda) = \sum_{k=-n}^n A_n^k \mathcal{H}_n^k(\theta, \lambda). \quad (4.259)$$

When  $k \geq 0$ , the coefficients  $A_n^k$  in (4.259) is the same as that in (4.257); When  $k < 0$ , the coefficients in  $A_n^k$  in (4.259) is identical to  $B_n^{|k|}$  in (4.257).

The surface spherical harmonics (4.259) can also be expressed in complex form. With Euler's formula

$$e^{ik\lambda} = \cos k\lambda + i \sin k\lambda, \quad (4.260)$$

and taking into account of (4.79), we can express (4.259) in the complex form

$$Y_n(\theta, \lambda) = \sum_{k=-n}^n C_n^k P_n^k(\cos \theta) e^{ik\lambda}. \quad (4.261)$$

For deriving out the relation between the complex coefficient  $C_n^k$  and the real coefficients  $A_n^k$  and  $B_n^k$ , we reduce (4.261) to the form of (4.259):

$$\begin{aligned} Y_n(\theta, \lambda) &= \sum_{k=-n}^n C_n^k (\cos k\lambda + i \sin k\lambda) P_n^k(\cos \theta) \\ &= \sum_{k=0}^n (C_n^k \cos k\lambda + i C_n^k \sin k\lambda) P_n^k(\cos \theta) \\ &\quad + \sum_{k=-n}^{-1} (C_n^k \cos k\lambda + i C_n^k \sin k\lambda) P_n^k(\cos \theta) \\ &= \sum_{k=0}^n (C_n^k \cos k\lambda + i C_n^k \sin k\lambda) P_n^k(\cos \theta) \\ &\quad + \sum_{k=1}^n (C_n^{-k} \cos k\lambda - i C_n^{-k} \sin k\lambda) P_n^{-k}(\cos \theta) \\ &= \sum_{k=0}^n (C_n^k \cos k\lambda + i C_n^k \sin k\lambda) P_n^k(\cos \theta) \\ &\quad + \sum_{k=1}^n (C_n^{-k} \cos k\lambda - i C_n^{-k} \sin k\lambda) (-1)^k \frac{(n-k)!}{(n+k)!} P_n^k(\cos \theta) \end{aligned}$$

$$\begin{aligned}
&= C_n P_n(\cos \theta) + \sum_{k=1}^n \left\{ \left[ C_n^k + (-1)^k \frac{(n-k)!}{(n+k)!} C_n^{-k} \right] \cos k\lambda \right. \\
&\quad \left. + i \left[ C_n^k - (-1)^k \frac{(n-k)!}{(n+k)!} C_n^{-k} \right] \sin k\lambda \right\} P_n^k(\cos \theta). \quad (4.262)
\end{aligned}$$

We obtain, by comparing (4.262) with (4.259), and taking (4.258) into account,

$$\begin{cases} A_n = C_n; \\ A_n^k = C_n^k + (-1)^k \frac{(n-k)!}{(n+k)!} C_n^{-k}, & k \geq 1; \\ A_n^{-k} = i \left[ C_n^k - (-1)^k \frac{(n-k)!}{(n+k)!} C_n^{-k} \right], & k \geq 1. \end{cases} \quad (4.263)$$

For a real function,  $A_n^k$  and  $A_n^{-k}$  are real. Hence,

$$\begin{cases} \text{Im}(C_n) = 0; \\ \text{Im}(C_n^k) + (-1)^k \frac{(n-k)!}{(n+k)!} \text{Im}(C_n^{-k}) = 0, & k \geq 1; \\ \text{Re}(C_n^k) - (-1)^k \frac{(n-k)!}{(n+k)!} \text{Re}(C_n^{-k}) = 0, & k \geq 1. \end{cases} \quad (4.264)$$

where Re and Im are used to indicate the real and imaginary components, respectively. In this case, (4.263) can be simplified to

$$\begin{cases} A_n = \text{Re}(C_n); \\ A_n^k = 2\text{Re}(C_n^k), & k \geq 1; \\ A_n^{-k} = -2\text{Im}(C_n^k), & k \geq 1. \end{cases} \quad (4.265)$$

#### 4.4.4 Orthogonality of Surface Spherical Harmonics

Orthogonality of spherical harmonics can be derived from the orthogonality of associated Legendre function and that of cosine and sine functions. The orthogonality of associated Legendre function is given in (4.237). The cosine and sine have the following well-known orthogonality formulae:

$$\int_0^{2\pi} \cos k\lambda \cos l\lambda d\lambda = \delta_{lk}(1 + \delta_k)\pi, \quad (4.266)$$

$$\int_0^{2\pi} \cos k\lambda \sin l\lambda d\lambda = 0, \quad (4.267)$$

$$\int_0^{2\pi} \sin k\lambda \sin l\lambda d\lambda = \delta_{lk}(1 - \delta_k)\pi. \quad (4.268)$$

The orthogonality formulae of surface spherical harmonics can be obtained using (4.237) and (4.266)–(4.268) as follows:

$$\int_{\omega} P_n^k(\cos \theta) \cos k\lambda P_m^l(\cos \theta) \cos l\lambda d\omega = \delta_{mn} \delta_{lk} (1 + \delta_k) \frac{2\pi}{2n+1} \frac{(n+k)!}{(n-k)!}, \quad (4.269)$$

$$\int_{\omega} P_n^k(\cos \theta) \cos k\lambda P_m^l(\cos \theta) \sin l\lambda d\omega = 0, \quad (4.270)$$

$$\int_{\omega} P_n^k(\cos \theta) \sin k\lambda P_m^l(\cos \theta) \sin l\lambda d\omega = \delta_{mn} \delta_{lk} (1 - \delta_k) \frac{2\pi}{2n+1} \frac{(n+k)!}{(n-k)!}. \quad (4.271)$$

where  $\omega$  is the unit sphere, and  $\delta_k$  is defined in (2.32). With the notation  $\mathcal{H}_n^k(\theta, \lambda)$  defined in (4.258), the above three formulae can be written together as

$$\int_{\omega} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_m^l(\theta, \lambda) d\omega = \delta_{mn} \delta_{kl} \frac{2\pi(1 + \delta_k)}{2n+1} \frac{(n+|k|)!}{(n-|k|)!}. \quad (4.272)$$

The right-hand side is non-vanishing only when  $m = n$  and  $l = k$ .

The orthogonality formulae in complex form can be obtained using (4.237) or (4.238), together with the formula

$$\int_0^{2\pi} e^{ik\lambda} e^{-il\lambda} d\lambda = 2\pi \delta_{lk}. \quad (4.273)$$

It can be written in two alternative forms:

$$\int_{\omega} \left[ P_n^k(\cos \theta) e^{ik\lambda} \right] \left[ P_m^l(\cos \theta) e^{-il\lambda} \right] d\omega = \delta_{mn} \delta_{lk} \frac{4\pi}{2n+1} \frac{(n+k)!}{(n-k)!}, \quad (4.274)$$

$$\int_{\omega} \left[ P_n^k(\cos \theta) e^{ik\lambda} \right] \left[ P_m^{-l}(\cos \theta) e^{-il\lambda} \right] d\omega = \delta_{mn} \delta_{lk} (-1)^k \frac{4\pi}{2n+1}. \quad (4.275)$$

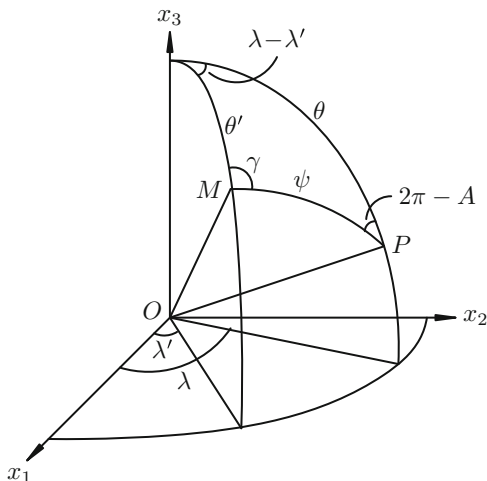
In summary, the orthogonality formulae are (4.269)–(4.271), (4.272), and (4.274) or (4.275). The readers are suggested to expand the right-hand sides of these formulae for each case of  $m = n$ ,  $m \neq n$  and  $k = l$ ,  $k \neq l$  for better understanding.

#### 4.4.5 Addition Theorem and Azimuthal Average of Spherical Harmonics

We will come up with the Legendre function  $P_n(\cos \psi)$ , where  $\psi$  is the angle made by two points on the sphere,  $P$  and  $M$ , with respect to the center of the sphere. We



**Fig. 4.5** An angle made by two point on the sphere with respect to the center of the sphere



denote the coordinates of  $P$  and  $M$  as  $(\theta, \lambda)$  and  $(\theta', \lambda')$ , respectively, as shown in Fig. 4.5. The formula of  $\cos \psi$  is derived in Sect. 1.6. According to (1.11), we have

$$\cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\lambda - \lambda'). \quad (4.276)$$

Here we attempt to express  $P_n(\cos \psi)$  in terms of spherical harmonics of  $\theta, \lambda$  and  $\theta', \lambda'$ . It would be quite complicated to do the derivation by substituting (4.276) into the expression of  $P_n(\cos \psi)$ . Here we start from the variable separation approach of Laplace's equation.

Assume  $M$  is a fixed point. Let  $\phi$  be a harmonic function within the sphere as function of  $r, \theta$  and  $\lambda$ . It can be expressed as a spherical harmonic series

$$\phi = \sum_{n=0}^{\infty} r^n Y_n(\theta, \lambda), \quad (4.277)$$

where  $Y_n(\theta, \lambda)$  is the surface spherical harmonics defined as

$$Y_n(\theta, \lambda) = \sum_{k=-n}^n A_n^k \mathcal{H}_n^k(\theta, \lambda). \quad (4.278)$$

Actually,  $Y_n(\theta, \lambda)$  is a function of the position of  $P$  on the spherical surface. We also define a spherical coordinate system with  $M$  as the pole,  $Or\psi\gamma$ ,<sup>2</sup> where  $(\psi, \gamma)$  are the coordinates of  $P$ . In this coordinate system,  $\phi$  can also be expressed as series of

<sup>2</sup> As compared to  $Or\theta\lambda$ , it is more appropriate to use  $2\pi - \gamma$  instead of  $\gamma$ . However, using  $\gamma$  also serves our purpose; using  $2\pi - \gamma$  is equivalent to change the sign of the coefficient of  $\sin k\gamma$ .

spherical harmonics,

$$\phi = \sum_{n=0}^{\infty} r^n Y_n(\psi, \gamma), \quad (4.279)$$

where  $Y_n(\psi, \gamma)$  is the surface spherical harmonics defined as

$$Y_n(\psi, \gamma) = \sum_{k=-n}^n B_n^k \mathcal{H}_n^k(\psi, \gamma). \quad (4.280)$$

In order that (4.277) and (4.279) express the same harmonic function,  $Y_n(\theta, \lambda)$  and  $Y_n(\psi, \gamma)$  must be equal. We have

$$\sum_{k=-n}^n A_n^k \mathcal{H}_n^k(\theta, \lambda) = \sum_{k=-n}^n B_n^k \mathcal{H}_n^k(\psi, \gamma). \quad (4.281)$$

Therefore, every term at the left-hand side can be expressed as a linear combination of the terms at the right-hand side, and every term at the right hand side can be expressed as a linear combination of the terms at the left-hand side. Particularly, we have

$$\mathcal{H}_n^0(\psi, \gamma) = \sum_{k=-n}^n C_n^k \mathcal{H}_n^k(\theta, \lambda), \quad (4.282)$$

and

$$\mathcal{H}_n^l(\theta, \lambda) = \sum_{k=-n}^n D_n^k \mathcal{H}_n^k(\psi, \gamma). \quad (4.283)$$

These last two formulae are the starting point of our derivation.

Multiply both sides of (4.282) by  $\mathcal{H}_n^l(\theta, \lambda)$ , and then integrate over a unit sphere. According to the orthogonality formula (4.272), we obtain

$$C_n^l = \frac{1}{1 + \delta_l} \frac{2n + 1}{2\pi} \frac{(n - |l|)!}{(n + |l|)!} \int_{\omega} \mathcal{H}_n^0(\psi, \gamma) \mathcal{H}_n^l(\theta, \lambda) d\omega. \quad (4.284)$$

By substituting (4.283) into it, we obtain, using (4.272) again,

$$C_n^l = \frac{2}{1 + \delta_l} \frac{(n - |l|)!}{(n + |l|)!} D_n^l. \quad (4.285)$$

The problem of computing  $C_n^l$  now becomes a problem of computing  $D_n^l$ , which can actually be inferred directly from (4.283).

In (4.283), keeping in mind that  $M$  is the pole of the coordinate system  $Or\psi\gamma$ , we consider the particular case  $\psi = 0$ , and consequently  $\theta = \theta'$  and  $\lambda = \lambda'$ . In this case,  $\mathcal{H}_n^k(\psi, \gamma)$  is independent of  $\gamma$ , and hence, it does not vanish only when  $k = 0$ . We have

$$[\mathcal{H}_n^k(\psi, \gamma)]_{\psi=0} = [P_n^k(\cos \psi)]_{\psi=0} = P_n^k(1) = \delta_k, \quad (4.286)$$

which leads (4.283) to

$$D_n = \mathcal{H}_n^l(\theta', \lambda'). \quad (4.287)$$

Then,  $C_n^l$  is obtained by substituting this formula into (4.285),

$$C_n^l = \frac{2}{1 + \delta_l} \frac{(n - |l|)!}{(n + |l|)!} \mathcal{H}_n^l(\theta', \lambda'). \quad (4.288)$$

Finally, change the index  $l$  to  $k$  in (4.288), and then substitute it into (4.282). We obtain the addition theorem

$$P_n(\cos \psi) = \sum_{k=-n}^n \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_n^k(\theta', \lambda'). \quad (4.289)$$

It can also be written as

$$\begin{aligned} P_n(\cos \psi) &= \sum_{k=0}^n \frac{2}{1 + \delta_k} \frac{(n - k)!}{(n + k)!} P_n^k(\cos \theta) P_n^k(\cos \theta') \cos k(\lambda - \lambda') \\ &= P_n(\cos \theta) P_n(\cos \theta') + 2 \sum_{k=1}^n \frac{(n - k)!}{(n + k)!} P_n^k(\cos \theta) P_n^k(\cos \theta') \cos k(\lambda - \lambda'). \end{aligned} \quad (4.290)$$

A complex form of the addition theorem can be obtained using (4.79):

$$\begin{aligned} P_n(\cos \psi) &= P_n(\cos \theta) P_n(\cos \theta') \\ &\quad + \sum_{k=1}^n (-1)^k P_n^k(\cos \theta) P_n^{-k}(\cos \theta') \left[ e^{ik(\lambda - \lambda')} + e^{-ik(\lambda - \lambda')} \right] \\ &= \sum_{k=-n}^n (-1)^k P_n^k(\cos \theta) P_n^{-k}(\cos \theta') e^{ik(\lambda - \lambda')} \\ &= \sum_{k=-n}^n (-1)^k \left[ P_n^k(\cos \theta) e^{ik\lambda} \right] \left[ P_n^{-k}(\cos \theta') e^{-ik\lambda'} \right]. \end{aligned} \quad (4.291)$$

By setting  $\theta' = \theta$  and  $\lambda' = \lambda$  in (4.290), we obtain an identity of the associated Legendre function

$$\sum_{k=0}^n \frac{2}{1 + \delta_k} \frac{(n-k)!}{(n+k)!} [P_n^k(\cos \theta)]^2 = 1. \quad (4.292)$$

This identity can be used to check the accuracy of the associated Legendre function computed numerically.

The formulae of the azimuthal averages of the surface spherical harmonics can be derived incidentally based on the derivation of the addition theorem. By integrating (4.283) over  $\gamma$  from 0 to  $2\pi$ , we see that only the  $k = 0$  term is non-vanishing. By further taking reference to (4.287), we obtain

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_n^l(\theta, \lambda) d\gamma &= D_n \mathcal{H}_n(\psi, \gamma) \\ &= \mathcal{H}_n(\psi, \gamma) \mathcal{H}_n^l(\theta', \lambda') = P_n(\cos \psi) \mathcal{H}_n^l(\theta', \lambda'). \end{aligned} \quad (4.293)$$

This is the average of  $\mathcal{H}_n^l(\theta, \lambda)$  over the circle that  $P$  draws around  $M$  while keeping  $\psi$  fixed. As shown in Fig. 4.5,  $\gamma$  is the azimuth of  $P$  with respect to  $M$ , i.e., the angle from the north to  $P$  as seen at  $M$ .

In the derivation, we can swap  $M$  and  $P$ ,  $\theta'$  and  $\theta$ ,  $\lambda'$  and  $\lambda$ , as well as  $\gamma$  and the azimuth of  $M$  with respect to  $P$  denoted as  $A$ . This yields

$$\frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_n^l(\theta', \lambda') dA = \mathcal{H}_n(\psi, A) \mathcal{H}_n^l(\theta, \lambda) = P_n(\cos \psi) \mathcal{H}_n^l(\theta, \lambda). \quad (4.294)$$

This is the average of  $\mathcal{H}_n^l(\theta', \lambda')$  over the circle that  $M$  draws around  $P$  while keeping  $\psi$  fixed. We have shown in Fig. 4.5 the angle  $2\pi - A$ , which is more convenient than showing  $A$ .

The results (4.293) and (4.294) are the azimuthal averages of the surface spherical harmonics.

## 4.5 Expression of Functions on a Sphere as Spherical Harmonic Series

### 4.5.1 Definition of the Laplace Series

We understand the spherical harmonic series as an extension of the Fourier series onto a sphere. Let  $f(x)$  be a function with a period of  $2\pi$ , and is piecewise smooth in  $[0, 2\pi]$ . Its Fourier series is defined as

$$S(x) = \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx), \quad (4.295)$$

where the coefficients are

$$\begin{cases} a_k = \frac{1}{1 + \delta_k} \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx, \\ b_k = \frac{1}{1 + \delta_k} \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx. \end{cases} \quad (4.296)$$

The case  $k = 0$  in the second formula is meaningless. Here we include it for similarity to the first one. The Fourier series (4.295) is convergent everywhere, and

$$S(x) = \frac{1}{2}[f_+(x) + f_-(x)], \quad (4.297)$$

where  $f_+(x)$  and  $f_-(x)$  are the right- and left-side limits, respectively.

If  $f(x)$  is a continuous and piecewise smooth function with a period of  $2\pi$ , we have

$$f(x) = \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx), \quad (4.298)$$

where the coefficients are defined in (4.296). In fact, (4.296) can be formally obtained from (4.298) by multiplying both sides by  $\cos kx$  or  $\sin kx$ , integrating over  $x$  in  $[0, 2\pi]$ , and then making use of the orthogonality formulae of  $\cos kx$  and  $\sin kx$ .

Although we have assumed that the readers know Fourier series, it will also be derived as a degenerate case of the spherical harmonic series.

The surface spherical harmonics  $\mathcal{H}_n^k(\theta, \lambda)$  has orthogonality formulae on a sphere in similar forms to those of  $\cos kx$  and  $\sin kx$  in the interval  $[0, 2\pi]$ . Hence, the spherical harmonic series of a function  $f(\theta, \lambda)$  on a sphere is defined as,

$$S(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^{\infty} A_n^k \mathcal{H}_n^k(\theta, \lambda), \quad (4.299)$$

where the coefficient is

$$A_n^k = \frac{1}{1 + \delta_k} \frac{2n + 1}{2\pi} \frac{(n - |k|)!}{(n + |k|)!} \int_{\omega} f(\theta', \lambda') \mathcal{H}_n^k(\theta', \lambda') d\omega. \quad (4.300)$$

We do not know yet when  $S(\theta, \lambda)$  converges, and what function it converges to.

Substitute (4.300) into (4.299). We obtain

$$S(\theta, \lambda) = \sum_{n=0}^{\infty} \frac{2n + 1}{4\pi} \int_{\omega} f(\theta', \lambda') \sum_{k=-n}^n \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_n^k(\theta', \lambda') d\omega. \quad (4.301)$$

According to the addition theorem, this formula can be simplified to

$$S(\theta, \lambda) = \sum_{n=0}^{\infty} \frac{2n+1}{2\pi} \int_{\omega} f(\theta', \lambda') P_n(\cos \psi) d\omega, \quad (4.302)$$

where  $\psi$  is a function of  $\theta, \lambda$  and  $\theta', \lambda'$ , and the integration is performed for the variables  $\theta'$  and  $\lambda'$ .

The spherical harmonic series of a function on a sphere is referred to as Laplace series, which can be written in the forms of (4.299), (4.301), or (4.302).

### 4.5.2 Sum of the Truncated Laplace Series

We establish a new spherical coordinate system on the sphere,  $O\psi\gamma$ , where the point with coordinates  $(\theta, \lambda)$  is the pole,  $\psi$  the polar angle, and  $\gamma$  the longitude. We can then transform the integration variables  $\theta'$  and  $\lambda'$  in (4.302) to  $\psi$  and  $\gamma$ . We obtain

$$S(\theta, \lambda) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_0^{2\pi} d\gamma \int_0^{\pi} f(\psi, \gamma) P_n(\cos \psi) \sin \psi d\psi. \quad (4.303)$$

We use  $x$  to substitute  $\psi$  according to

$$x = \cos \psi, \quad (4.304)$$

and define  $\phi(x)$  as

$$\phi(x) = \phi(\cos \psi) = \frac{1}{2\pi} \int_0^{2\pi} f(\psi, \gamma) d\gamma. \quad (4.305)$$

We can then write (4.303) as

$$S(\theta, \lambda) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \int_{-1}^1 \phi(x) P_n(x) dx. \quad (4.306)$$

We truncate (4.306) up to degree  $N$ . As the summation and integration signs can be switched, we obtain

$$S_N(\theta, \lambda) = \frac{1}{2} \int_{-1}^1 \phi(x) \sum_{n=0}^N (2n+1) P_n(x) dx. \quad (4.307)$$

Add  $(2n + 1)P_n(x)$  at both sides of the recurrence formula (4.99). We can write it in the form of

$$(2n + 1)(1 - x)P_n(x) = -(n + 1)[P_{n+1}(x) - P_n(x)] + n[P_n(x) - P_{n-1}(x)]. \quad (4.308)$$

Set  $n = 0, 1, 2, \dots, N$  to obtain  $N + 1$  formulae, and then add them together. We obtain

$$\sum_{n=0}^N (2n + 1)P_n(x) = -(N + 1) \frac{P_{N+1}(x) - P_N(x)}{1 - x}. \quad (4.309)$$

Substitute this formula into (4.307), we obtain another expression for the partial sum,

$$S_N(\theta, \lambda) = -\frac{1}{2} \int_{-1}^1 \phi(x)(N + 1) \frac{P_{N+1}(x) - P_N(x)}{1 - x} dx. \quad (4.310)$$

Another form of the integrand can be found using the recurrence formulae (4.100) and (4.101). Set  $n = N$  in (4.100),  $n = N + 1$  in (4.101), and then add together to obtain an equation, which can be solved to obtain

$$(N + 1) \frac{P_{N+1}(x) - P_N(x)}{1 - x} = - \left[ \frac{d}{dx} P_{N+1}(x) + \frac{d}{dx} P_N(x) \right]. \quad (4.311)$$

### 4.5.3 An Auxiliary Formula

In order to study the limit of  $S_N(\theta, \lambda)$  as  $N \rightarrow \infty$ , we first prove a limit formula.

Let  $g(x)$  be an arbitrary function defined in  $x \in [-1, 1]$ . We define a number  $G$  as follows:

$$G = \int_{-1}^1 \left[ g(x') - \sum_{n=0}^N \frac{2n + 1}{2} P_n(x') \int_{-1}^1 g(x) P_n(x) dx \right]^2 dx'. \quad (4.312)$$

It can be further derived using the orthogonality formula (4.191) as

$$\begin{aligned} G &= \int_{-1}^1 [g(x')]^2 dx' - 2 \sum_{n=0}^N \frac{2n + 1}{2} \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2 \\ &\quad + \int_{-1}^1 \left[ \sum_{n=0}^N \frac{2n + 1}{2} P_n(x') \int_{-1}^1 g(x) P_n(x) dx \right]^2 dx' \end{aligned}$$

$$\begin{aligned}
&= \int_{-1}^1 [g(x')]^2 dx' - 2 \sum_{n=0}^N \frac{2n+1}{2} \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2 \\
&\quad + \sum_{n=0}^N \left( \frac{2n+1}{2} \right)^2 \int_{-1}^1 [P_n(x')]^2 dx' \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2 \\
&= \int_{-1}^1 [g(x)]^2 dx - \sum_{n=0}^N \frac{2n+1}{2} \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2. \tag{4.313}
\end{aligned}$$

From now on, we assume that  $\int_{-1}^1 [g(x)]^2 dx$  is integrable. As  $G \geq 0$ , we obtain

$$\sum_{n=0}^N \frac{2n+1}{2} \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2 \leq \int_{-1}^1 [g(x)]^2 dx. \tag{4.314}$$

Therefore, the positive term series

$$\sum_{n=0}^N \frac{2n+1}{2} \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2 \tag{4.315}$$

converges.

Number series have the following property: Let  $\sum_{n=0}^{\infty} u_n$  and  $\sum_{n=0}^{\infty} v_n$  be positive term series, and there exists a number  $k > 0$ , such that  $\lim_{n \rightarrow \infty} u_n/v_n = k$ . The two series are then both convergent or divergent. Now let  $\sum_{n=0}^{\infty} u_n$  be convergent, and  $\sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} 1/n$ , which is known to be divergent. We then have  $\lim_{n \rightarrow \infty} u_n/v_n = \lim_{n \rightarrow \infty} nu_n = 0$ . If this identity does not hold,  $\sum_{n=0}^{\infty} u_n$  would be divergent. Apply this property to (4.315). We obtain

$$\lim_{n \rightarrow \infty} \frac{n(2n+1)}{2} \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2 = 0. \tag{4.316}$$

Write  $[n(2n+1)]/2$  as  $n^2[1 + 1/(2n)]$ . We have

$$\lim_{n \rightarrow \infty} n^2 \left[ \int_{-1}^1 g(x) P_n(x) dx \right]^2 = 0. \tag{4.317}$$



Finally, we obtain the required formula

$$\lim_{n \rightarrow \infty} n \int_{-1}^1 g(x) P_n(x) dx = 0. \quad (4.318)$$

We mention again that this formula holds under the condition that  $\int_{-1}^1 [g(x)]^2 dx$  is integrable.

#### 4.5.4 Convergence of the Laplace Series

We now study the limit  $\lim_{N \rightarrow \infty} S_N(\theta, \lambda)$ . A first thought might be to apply (4.318)–(4.310). However, as the function  $g(x) = \phi(x)/(1-x) \rightarrow \infty$  when  $x \rightarrow 1$ ,  $\int_{-1}^1 [g(x)]^2 dx$  is not necessarily integrable (noting that  $\int_{-1}^1 [1/(1-x)]^2 dx$  is not integrable), and some modification of (4.310) is required.

Define a new function

$$g(x) = \begin{cases} \frac{\phi(x)}{1-x}, & -1 \leq x \leq 1-\epsilon; \\ 0, & 1-\epsilon < x \leq 1; \end{cases} \quad (4.319)$$

where  $\epsilon$  is a small positive number. We assume that  $\int_{-1}^1 [\phi(x)]^2 dx$  is integrable from now on. Then  $\int_{-1}^1 [g(x)]^2 dx$  is also integrable for the above defined  $g(x)$ . In this case, we can use (4.319) to rewrite (4.310) as

$$\begin{aligned} S_N(\theta, \lambda) &= -\frac{N+1}{2} \int_{-1}^1 g(x) [P_{N+1}(x) - P_N(x)] dx \\ &\quad - \frac{1}{2} \int_{1-\epsilon}^1 \phi(x) (N+1) \frac{P_{N+1}(x) - P_N(x)}{1-x} dx. \end{aligned} \quad (4.320)$$

Take the limit  $N \rightarrow \infty$  at both sides. As the first term at the right-hand side tends to 0 according to (4.318), we obtain

$$S(\theta, \lambda) = \lim_{N \rightarrow \infty} S_N(\theta, \lambda) = - \lim_{N \rightarrow \infty} \frac{1}{2} \int_{1-\epsilon}^1 \phi(x) (N+1) \frac{P_{N+1}(x) - P_N(x)}{1-x} dx. \quad (4.321)$$

We further assume that  $\phi(x)$  is piecewise smooth, such that it can be treated as constant within  $[1-\epsilon, 1]$ . According to (4.305), we define

$$\phi(1) = \lim_{x \rightarrow 1} \phi(x) = \lim_{\psi \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} f(\psi, \gamma) d\gamma. \quad (4.322)$$

We can then write (4.321) as

$$S(\theta, \lambda) = -\phi(1) \lim_{N \rightarrow \infty} \frac{1}{2} \int_{1-\epsilon}^1 (N+1) \frac{P_{N+1}(x) - P_N(x)}{1-x} dx. \quad (4.323)$$

In such a way, we have taken  $\phi(x)$  out of the integral sign.

In order to evaluate the integral in (4.323), we extend it into the domain  $x \in [-1, 1]$ . Define

$$h(x) = \begin{cases} \frac{1}{1-x}, & -1 \leq x \leq 1-\epsilon; \\ 0, & 1-\epsilon < x \leq 1. \end{cases} \quad (4.324)$$

As  $\int_{-1}^1 [h(x)]^2 dx$  is integrable, we have, according to (4.318),

$$\lim_{N \rightarrow \infty} \int_{-1}^1 h(x) (N+1) [P_{N+1}(x) - P_N(x)] dx = 0. \quad (4.325)$$

By substituting (4.324) into this last formula, we obtain

$$\lim_{N \rightarrow \infty} \int_{-1}^{1-\epsilon} (N+1) \frac{P_{N+1}(x) - P_N(x)}{1-x} dx = 0. \quad (4.326)$$

This formula can be used to rewrite (4.323) as

$$S(\theta, \lambda) = -\phi(1) \lim_{N \rightarrow \infty} \frac{1}{2} \int_{-1}^1 (N+1) \frac{P_{N+1}(x) - P_N(x)}{1-x} dx. \quad (4.327)$$

Finally, by using (4.311) to replace its counter part, we obtain

$$\begin{aligned} S(\theta, \lambda) &= \phi(1) \lim_{N \rightarrow \infty} \frac{1}{2} \int_{-1}^1 \left[ \frac{d}{dx} P_{N+1}(x) + \frac{d}{dx} P_N(x) \right] dx \\ &= \frac{1}{2} \phi(1) \lim_{N \rightarrow \infty} [P_{N+1}(x) + P_N(x)]_{-1}^1 \\ &= \phi(1), \end{aligned} \quad (4.328)$$

where  $P_N(\pm 1) = (\pm 1)^N$  is used. This formula provides the sum of the Laplace series, which holds under the conditions that  $\phi(x)$  is piece wise smooth and  $\int_{-1}^1 [\phi(x)]^2 dx$  is integrable.

Our last task is to express  $\phi(1)$  using  $f(\theta, \lambda)$ . Assume that  $f(\theta, \lambda)$  is finite and piece wise smooth. Then,  $\phi(x)$  is piecewise smooth and  $\int_{-1}^1 [\phi(x)]^2 dx$  is integrable. According to (4.305),  $\phi(1)$  can be understood as the average of  $f(\theta', \lambda')$  on an infinitesimal circle centered on the point  $(\theta, \lambda)$ . Hence, at places where  $f(\theta, \lambda)$  is

continuous, we have

$$S(\theta, \lambda) = \phi(1) = f(\theta, \lambda). \quad (4.329)$$

At places where  $f(\theta, \lambda)$  has a jump across a smooth line, but continuous at both sides of the line, we have

$$S(\theta, \lambda) = \phi(1) = \frac{1}{2}[f_1(\theta, \lambda) + f_2(\theta, \lambda)] \quad (4.330)$$

on the line, where  $f_1(\theta, \lambda)$  and  $f_2(\theta, \lambda)$  are the limits of  $f(\theta, \lambda)$  at the two sides of the line. This formula would be different if the discontinuity is more complicated. For example, the readers may consider an vertex of a broken line across which  $f(\theta, \lambda)$  is discontinuous.

#### 4.5.5 Expression of a Function as a Spherical Harmonic Series

Based on the result of the previous subsection, a finite, continuous, and piecewise smooth function on a sphere can be expressed as the following spherical harmonic series:

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n A_n^k \mathcal{H}_n^k(\theta, \lambda), \quad (4.331)$$

$$A_n^k = \frac{1}{1 + \delta_k} \frac{2n+1}{2\pi} \frac{(n-|k|)!}{(n+|k|)!} \int_{\omega} f(\theta, \lambda) \mathcal{H}_n^k(\theta, \lambda) d\omega. \quad (4.332)$$

Its complex form is

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n C_n^k P_n^k(\cos \theta) e^{ik\lambda}, \quad (4.333)$$

$$\begin{aligned} C_n^k &= (-1)^k \frac{2n+1}{4\pi} \int_{\omega} f(\theta, \lambda) P_n^{-k}(\cos \theta) e^{-ik\lambda} d\omega \\ &= \frac{2n+1}{4\pi} \frac{(n-k)!}{(n+k)!} \int_{\omega} f(\theta, \lambda) P_n^k(\cos \theta) e^{-ik\lambda} d\omega. \end{aligned} \quad (4.334)$$

In the degenerate case when  $f$  is a function of  $\theta$  alone, and independent of  $\lambda$ , the series expression is simplified to, after setting  $x = \cos \theta$ ,

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x), \quad (4.335)$$

$$A_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx, \quad (4.336)$$

called Legendre series of  $f(x)$ . Some special Legendre series have been studied in Sect. 4.3.3.

For computing the spherical harmonic series (4.331) using the fast Fourier transform (FFT), we write it as

$$f(\theta, \lambda) = \sum_{k=0}^{\infty} \left[ \left( \sum_{n=k}^{\infty} A_n^k P_n^k(\cos \theta) \right) \cos k\lambda + \left( \sum_{n=k}^{\infty} A_n^{-k} P_n^k(\cos \theta) \right) \sin k\lambda \right], \quad (4.337)$$

On the equator,  $f$  is a function of  $\lambda$  with  $\theta = \pi/2$ . We can write it in the form of

$$f(\lambda) = \sum_{k=0}^{\infty} (a_k \cos k\lambda + b_k \sin k\lambda). \quad (4.338)$$

As this identity is guaranteed to hold, we can use the orthogonality formulae of the cosine and sine functions given in (4.266)–(4.268) to directly derive the formulae for  $a_k$  and  $b_k$ ,

$$\begin{cases} a_k = \frac{1}{1+\delta_k} \frac{1}{\pi} \int_0^{2\pi} f(\lambda) \cos k\lambda d\lambda, \\ b_k = \frac{1}{1+\delta_k} \frac{1}{\pi} \int_0^{2\pi} f(\lambda) \sin k\lambda d\lambda. \end{cases} \quad (4.339)$$

In this way, we proved the Fourier series of a function as a special case of the spherical harmonic series.

The readers are suggested to use (4.330) to study the cases when the function  $f(\theta, \lambda)$  has a jump across a line on the sphere, including the Legendre series (4.335) and the Fourier series (4.338) as special cases.

### 4.5.6 Resolution of Spherical Harmonics

An immediate observation is that, in a spherical harmonic series, the larger higher degree coefficients are, the richer finer features are, as higher degree spherical harmonics represent variations in shorter space scale. The resolution of the degree  $n$  spherical harmonics can be inferred based on the following facts:

1. At the equator, the spherical harmonics of degree  $n$  degenerates to a Fourier series of order  $n$  with  $\lambda$  as the variable, meaning that the highest resolution is  $n$  cycles of the sine or cosine function from  $0^\circ$  to  $360^\circ$ . The wavelength of each cycle is  $360^\circ/n$ , and the half wavelength is  $180^\circ/n$ .

2. According to the discussion in Sect. 4.4.5, where (4.278) and (4.280) represent the same function, we know that, if a function is a degree  $n$  spherical harmonics in one coordinate system, it remains so in all coordinates systems with the same origin. In the Cartesian coordinate system  $Oxyz$  associated with the spherical coordinate system  $Or\theta\lambda$ , the equator is just a great arc in the  $Oxy$  plane. By rotating the coordinate system, any great arc can be arranged to lie in the  $Oxy$  plane. Hence, the conclusion (1) holds for any great arc.

We thus conclude that the resolution of the degree  $n$  spherical harmonics on the sphere is homogeneous and isotropic. The former means that it is the same everywhere, or position independent. The latter means that it is the same along all directions, or azimuth independent. The spatial resolution is  $360^\circ/n$  in wavelength, or  $180^\circ/n$  in half wavelength. A spherical harmonic series truncated up to degree and order  $N$  has a spatial resolution of wavelength  $360^\circ/N$ , or a half wavelength  $180^\circ/N$ . On the Earth's surface,  $1^\circ$  corresponds to  $2\pi R/360 = 111$  km. In geodesy, the half wavelength is often used to represent the spatial resolution, e.g., a spherical harmonic series truncated up to degree and order 60 corresponds to a spatial resolution of  $3^\circ$ , or 333 km.

#### 4.5.7 Normalization of Spherical Harmonics

In the representation the Earth's gravity field, a fully normalized Legendre function

$$\bar{P}_n^k(\cos \theta) = \left[ \frac{2(2n+1)}{1+\delta_k} \frac{(n-k)!}{(n+k)!} \right]^{1/2} P_n^k(\cos \theta) \quad (4.340)$$

is used. The surface spherical harmonics defined by it,

$$\bar{\mathcal{H}}_n^k(\theta, \lambda) = \begin{cases} \bar{P}_n^k(\cos \theta) \cos k\lambda, & k \geq 0; \\ \bar{P}_n^{|k|}(\cos \theta) \sin |k|\lambda, & k < 0; \end{cases} \quad (4.341)$$

satisfies the normalization condition

$$\frac{1}{4\pi} \int_w \left[ \bar{\mathcal{H}}_n^k(\theta, \lambda) \right]^2 dw = 1. \quad (4.342)$$

The logic behind the normalization is that the left-hand side of (4.342) is considered as the overall magnitudes of  $\bar{\mathcal{H}}_n^k(\theta, \lambda)$ , such that the normalized coefficient  $\bar{A}_n^k$  in the spherical harmonic series

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n A_n^k \bar{\mathcal{H}}_n^k(\theta, \lambda) \quad (4.343)$$

represents the overall magnitude of the signal of the respective term.

In the representation of geomagnetic potential, the normalization is defined by

$$\frac{1}{4\pi} \int_w \left[ \tilde{\mathcal{H}}_n^k(\theta, \lambda) \right]^2 dw = \frac{1}{2n+1}. \quad (4.344)$$

As the right-hand sides still depend on  $n$ ,  $\bar{P}_n^k(\cos \theta)$  is called Semi-normalized Legendre function. It is

$$\bar{P}_n^k(\cos \theta) = \left[ \frac{2}{1 + \delta_k} \frac{(n-k)!}{(n+k)!} \right]^{1/2} P_n^k(\cos \theta). \quad (4.345)$$

In practical computations,  $\bar{P}_n^k(\cos \theta)$  is normally used, as the order of magnitude of  $P_n^k(\cos \theta)$  varies much more strongly with  $n$  and  $k$ , and its values exceed the range of floating point variables of a programming language much more quickly. Therefore, the recurrence formulae used, (4.139) and (4.132), should be converted to those for normalized Legendre function. Even  $\bar{P}_n^k(\cos \theta)$  cannot be computed using the recurrence formulae as is when  $n$  and  $k$  is large, and some more tricks are require. Similarly,  $(d/d\theta) \bar{P}_n^k(\cos \theta)$  should be computed instead of  $(d/d\theta) P_n^k(\cos \theta)$ .

## 4.6 Isotropic Smoothing of a Function Defined on a Sphere

### 4.6.1 General Formulation of Isotropic Smoothing

For a function defined on a sphere, the term smoothing indicate a weighted averaging of the values in the vicinity, leading to a smoother result. For convenience, we use  $\theta'$  and  $\lambda'$  to denote the coordinates for the un-smoothed function,  $f(\theta', \lambda')$ , and use  $\theta$  and  $\lambda$  to denote the coordinates for the smoothed function,  $\bar{f}(\theta, \lambda)$ . We define the weighted average as

$$\bar{f}(\theta, \lambda) = \int_{\omega_{\psi \leq \Theta}} \frac{W(\psi)}{W_s} f(\theta', \lambda') d\omega, \quad (4.346)$$

where  $\psi$  is the geocentric angle between the two points  $(\theta, \lambda)$  and  $(\theta', \lambda')$ ,  $W(\psi)$  the weight supposed to be prescribed, and  $W_s$  the sum of the weight defined by

$$W_s = \int_{\omega_{\psi \leq \Theta}} W(\psi) d\omega. \quad (4.347)$$

We defined the weight to be a function of  $\psi$  alone instead of a more general form as a function of  $\theta, \lambda$  and  $\theta', \lambda'$ . This is the simplest form that conforms with the most direct inference that the weight should decrease with the distance to the point  $(\theta, \lambda)$ . As this kind of weight behave the same in all directions as seen from the point  $(\theta, \lambda)$ , the smoothing is called isotropic. Defining the averaging area to be  $\psi \leq \Theta$  is

a consequence of the simple form of the weight. What we are going to formulate is the relation between the spherical harmonic series of  $f(\theta', \lambda')$  and that of  $\bar{f}(\theta, \lambda)$ .

We use the un-normalized Legendre function to express  $f(\theta', \lambda')$  as

$$f(\theta', \lambda') = \sum_{n=0}^{\infty} \sum_{k=-n}^n a_n^k \mathcal{H}_n^k(\theta', \lambda'). \quad (4.348)$$

The spherical harmonic series of  $\bar{f}(\theta, \lambda)$  can be obtained by substituting (4.348) into (4.346),

$$\bar{f}(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n a_n^k \int_{\omega_{\psi \leq \Theta}} \frac{W(\psi)}{W_s} \mathcal{H}_n^k(\theta', \lambda') d\omega. \quad (4.349)$$

By expressing  $d\omega = \sin \psi d\psi dA$ , where  $A$  is the azimuth of the point  $(\theta', \lambda')$  as observed at the point  $(\theta, \lambda)$ , we can express (4.349) as

$$\bar{f}(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n a_n^k \int_0^{\Theta} \left[ \int_0^{2\pi} \mathcal{H}_n^k(\theta', \lambda') dA \right] \frac{W(\psi)}{W_s} \sin \psi d\psi. \quad (4.350)$$

The integral in the brackets can be evaluated according to the formulae of the azimuthal average of the surface spherical harmonics, (4.294). We obtain

$$\bar{f}(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \left\{ a_n^k \int_0^{\Theta} \left[ 2\pi P_n(\cos \psi) \mathcal{H}_n^k(\theta, \lambda) \right] \frac{W(\psi)}{W_s} \sin \psi d\psi \right\}, \quad (4.351)$$

which can be written in the simple form

$$\bar{f}(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \beta_n a_n^k \mathcal{H}_n^k(\theta, \lambda), \quad (4.352)$$

with  $\beta_n$  defined as

$$\beta_n = 2\pi \int_0^{\Theta} \frac{W(\psi)}{W_s} P_n(\cos \psi) \sin \psi d\psi. \quad (4.353)$$

Hence, any smoothing in the form defined in (4.346) requires only to multiply the spherical harmonic coefficients with a degree only dependent parameter. We realize that this parameter is independent of the normalization of the spherical harmonics used in (4.348) and (4.352).

For any weight  $W(\psi)$  designed to make the variation of the smoothed function smoother than the original one,  $\beta_n$  decreases with  $n$  (even though there may be oscillation while decreasing). Hence, the smoothing is equivalent to a filtering that reduces the contribution of higher degree spherical harmonics. In a more general language, a smoothing in the space domain (the function is considered as depending

on spatial position) is a low pass filtering in the spectral domain (the function is considered as depending on the coefficients of a series expansion, such as the Fourier series, with a higher order term varying with shorter wavelength than a lower order term). Therefore, smoothing is also referred to as filtering in the literature.

The weight  $W(\psi)$  can also be computed from prescribed values of  $\beta_n$ . Even if  $W(\psi)$  is defined within  $\psi \leq \Theta$ , we can always extend its domain to  $0 \leq \psi \leq \pi$  with  $W(\psi) = 0$  when  $\psi > \Theta$ , and write (4.353) as, with the variable substitution  $x = \cos \psi$ ,

$$\beta_n = 2\pi \int_{-1}^1 \frac{W(x)}{W_s} P_n(x) dx. \quad (4.354)$$

Taking reference to the Legendre series expansion (4.335) and (4.336), we obtain a formula to compute  $W(x)/W_s$  from  $\beta_n$ ,

$$\frac{W(x)}{W_s} = \sum_{n=0}^{\infty} \left[ \frac{2n+1}{2} \int_{-1}^1 \frac{W(x)}{W_s} P_n(x) dx \right] P_n(x) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \beta_n P_n(x). \quad (4.355)$$

This formula permits to prescribe  $\beta_n$ , and then compute  $W(x)$ , or  $W(\psi)$  after the variable substitution  $x = \cos \psi$ .

Many smoothing methods can be designed by prescribing  $W$  or  $\beta_n$ . In the space domain, computation is supposed to be done according to (4.346), and in the spectral domain (4.352).

## 4.6.2 Pellinen Smoothing

In Pellinen smoothing,  $W(\psi)$  is prescribed as  $W(\psi) = 1$  and  $\Theta = \psi_0$ , i.e., the smoothing is an averaging within a spherical cap. We have

$$W_s = \int_0^{2\pi} \int_0^{\psi_0} \sin \psi d\psi dA = 2\pi(1 - \cos \psi_0), \quad (4.356)$$

and furthermore,

$$\beta_n = \frac{\int_0^{\psi_0} P_n(\cos \psi) \sin \psi d\psi}{1 - \cos \psi_0} = \frac{\int_{\cos \psi_0}^1 P_n(x) dx}{1 - \cos \psi_0}, \quad (4.357)$$

where a variable substitution  $x = \cos \psi$  is made. Substitute  $P_n(x)$  according to the recurrence formula (4.102). We obtain

$$\beta_n = \frac{\int_{\cos \psi_0}^1 \left[ \frac{d}{dx} P_{n+1}(x) - \frac{d}{dx} P_{n-1}(x) \right] dx}{(2n+1)(1 - \cos \psi_0)} = \frac{P_{n-1}(\cos \psi_0) - P_{n+1}(\cos \psi_0)}{(2n+1)(1 - \cos \psi_0)}, \quad (4.358)$$



where the fact  $P(1) = 1$  is used. This formula permits to compute  $\beta_n$  for  $n > 0$  using values of  $P_n(\cos \psi_0)$  computed using a recurrence formula. The special case  $\beta_0 = 1$  can be obtained from (4.357) with  $P_0(x) = 1$ . As  $\psi_0$  is usually small,  $P_{n-1}(\cos \psi_0)$  and  $P_{n+1}(\cos \psi_0)$  are very close, and several significant digits may be lost in the difference  $P_{n-1}(\cos \psi_0) - P_{n+1}(\cos \psi_0)$ , although the denominator could be computed without losing significant digits with the substitution  $1 - \cos \psi_0 = 2 \sin^2(\psi_0/2)$ . However, using quadruple precision to compute  $\beta_n$  without modifying the formula, which is not a burden for any contemporary computer, would provide enough significant digits for the subsequent computation of  $\hat{f}(\theta, \lambda)$  using (4.352) in double precision.

Graphs of  $\beta_n$  as a function of  $n$  for various values of  $\psi_0$  are shown in Fig. 4.6.

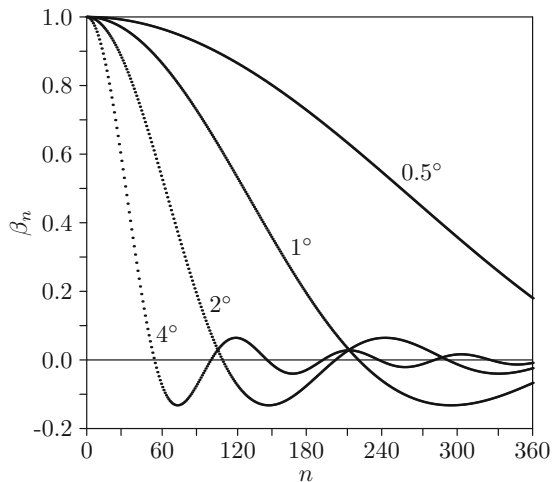
### 4.6.3 Gaussian Smoothing

In Gaussian smoothing, the weight is a generalization of the Gaussian function, or simply Gaussian, which is sometimes referred to as bell curve according to its shape. The Gaussian as a function of  $x$  with the bell centered at  $x = 0$  is defined as

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}, \quad (4.359)$$

which satisfies  $\int_{-\infty}^{\infty} g(x)dx = 1$ , and is the probability density function of the Gaussian or normal distribution. In the language of statistics,  $\sigma$  is the standard deviation defining the width of the bell, such as  $|x| \leq \sigma$  or  $|x| \leq 2\sigma$  or  $|x| \leq 3\sigma$ , with  $\int_{-\sigma}^{\sigma} g(x)dx = 68.27\%$ ,  $\int_{-2\sigma}^{2\sigma} g(x)dx = 95.45\%$  and  $\int_{-3\sigma}^{3\sigma} g(x)dx = 99.73\%$ . A width particularly used in geodesy is the half width at half maximum (HWHM),  $|x_{1/2}|$ , which is defined such that  $g(x_{1/2}) = g(0)/2$ , i.e., the value of

**Fig. 4.6** The factor  $\beta_n$  of Pellinen smoothing for  $\psi_0 = 0.5^\circ, 1^\circ, 2^\circ$  and  $4^\circ$



$g(x)$  at  $x_{1/2}$  is half of its maximum  $g(0)$ . It can be readily obtained that

$$\text{HWHM} = \sqrt{2 \ln 2} \sigma . \quad (4.360)$$

On a sphere, if the probability density function of the normal distribution depending on  $\psi$  is written in the form of (4.359), i.e.,  $g(\psi)$ , we can realize that it can be valid only when  $\sigma$  is small, in which case  $g(\psi)$  practically drops to zero when  $\psi$  is small enough. This is equivalent to approximate the sphere by a plane. Here we adopt a probability density function specifically designed for data on a sphere, called Fisher or Langevin distribution, whose degenerate case depending on  $\psi$  alone is  $Ce^{\kappa \cos \psi}$ , where  $C$  and  $\kappa$  are constants (Fisher et al., 1987). For our purpose here, we write the weight function as

$$W(\psi) = e^{-(1-\cos \psi)/\sigma^2} = e^{-2 \sin^2(\psi/2)/\sigma^2} , \quad (4.361)$$

which can be easily converted to the form  $Ce^{\kappa \cos \psi}$ . When  $\psi$  is very small,  $\sin(\psi/2) = \psi/2$ , (4.361) is simplified to

$$W(\psi) = e^{-\psi^2/(2\sigma^2)} , \quad (4.362)$$

which conforms with the probability density function of the normal distribution (4.359). Hence,  $Ce^{\kappa \cos \psi}$  can indeed be understood as a simple generalization of the probability density function of the normal distribution onto a sphere. In this case, we set  $\Theta = \pi$ , and we have

$$W_s = \int_0^{2\pi} \int_0^\pi e^{-(1-\cos \psi)/\sigma^2} \sin \psi d\psi dA = 2\pi \sigma^2 \left(1 - e^{-2/\sigma^2}\right) . \quad (4.363)$$

Hence,

$$\begin{aligned} \beta_n &= \frac{1}{\sigma^2 (1 - e^{-2/\sigma^2})} \int_0^\pi e^{-(1-\cos \psi)/\sigma^2} P_n(\cos \psi) \sin \psi d\psi \\ &= \frac{1}{\sigma^2 (1 - e^{-2/\sigma^2})} \int_{-1}^1 e^{-(1-x)/\sigma^2} P_n(x) dx , \end{aligned} \quad (4.364)$$

where the variable substitution  $x = \cos \psi$  has been made to obtain the last expression. Analytically evaluating the integral in (4.364) for any  $n$  is not really necessary, as a recurrence formula of  $\beta_n$  can be derived using the recurrence formula of  $P_n(x)$ , (4.102), here written as

$$\frac{d}{dx} P_n(x) = (2n-1)P_{n-1}(x) + \frac{d}{dx} P_{n-2}(x) . \quad (4.365)$$

By integrating (4.364) by part, and then using (4.365), we obtain

$$\beta_n = \beta_{n-2} - (2n-1)\sigma^2 \beta_{n-1} . \quad (4.366)$$

The starting values for the recurrence,  $\beta_0$  and  $\beta_1$ , can be obtained by analytically evaluating (4.364),

$$\beta_0 = 1, \quad \beta_1 = \frac{1 + e^{-2/\sigma^2}}{1 - e^{-2/\sigma^2}} - \sigma^2. \quad (4.367)$$

Integration by part has been required to obtain  $\beta_1$ . We mention that (4.366) and (4.367) are derived by Jekeli (1981).

When Gaussian smoothing is applied in physical geodesy, the HWHM is referred to as smoothing radius, which is, according to (4.360),

$$r_s = \sqrt{2 \ln 2} \sigma. \quad (4.368)$$

This is the measure of the smoothing radius in angle. The radius of the Earth,  $R$ , should be multiplied in order to obtain the smoothing radius in metric length.

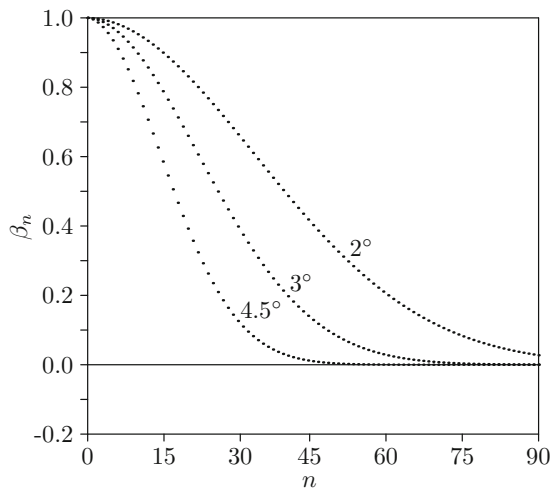
Graphs of  $\beta_n$  as a function of  $n$  for various values of  $r_s$  are shown in Fig. 4.7. We have cut the spherical harmonic degree to 90 for the applications in Chap. 9.

#### 4.6.4 Ideal Filter

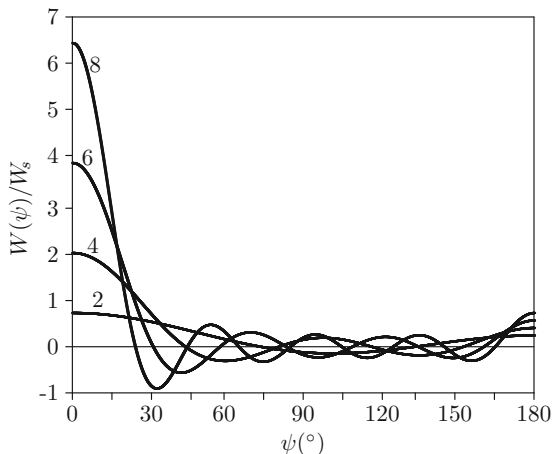
An ideal filter is designed by setting  $\beta_n = 1$  when  $n \leq n_{\max}$ , and  $\beta_n = 0$  when  $n > n_{\max}$ . We have, according to (4.355), with  $x = \cos \psi$ ,

$$\frac{W(\psi)}{W_s} = \sum_{n=0}^{n_{\max}} \frac{2n+1}{4\pi} P_n(\cos \psi). \quad (4.369)$$

**Fig. 4.7** The factor  $\beta_n$  of Gaussian smoothing for  $r_s = 2^\circ, 3^\circ$ , and  $4.5^\circ$



**Fig. 4.8** The weight  $W(\psi)/W_s$  of the ideal filter for  $n_{\max} = 2, 4, 6$  and 8



Graphs of  $W(\psi)/W_s$  for various values of  $n_{\max}$  are shown in Fig. 4.8.

By definition, the weight  $W(\psi)/W_s$  has the property

$$\int_{\omega} \frac{W(\psi)}{W_s} d\omega = 2\pi \int_0^{\pi} \frac{W(\psi)}{W_s} \sin \psi d\psi = 1. \quad (4.370)$$

The limit  $n_{\max} \rightarrow \infty$  corresponds to the unfiltered state. Hence,

$$\lim_{n_{\max} \rightarrow \infty} \frac{W(\psi)}{W_s} = \begin{cases} \infty, & \psi = 0; \\ 0, & \psi \neq 0. \end{cases} \quad (4.371)$$

When  $n_{\max}$  is not very small,  $W(\psi)/W_s$  at  $\psi = 0$  would be too large for graphing. Hence, very small values of  $n_{\max}$  are chosen in Fig. 4.8.

## 4.7 Three-Dimensional Scalar and Vector Fields

### 4.7.1 The nabla Operator and Laplacian in the Spherical Coordinate System

The nabla operator in the spherical coordinate system was formulated in Sect. 2.9, which is copied to here for convenience of reference,

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_{\lambda} \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda}. \quad (4.372)$$

The partial derivatives of the unit base vectors are given in (2.450), which is also copied to here,

$$\frac{\partial(\hat{e}_r, \hat{e}_\theta, \hat{e}_\lambda)}{\partial(r, \theta, \lambda)} = \begin{bmatrix} \frac{\partial \hat{e}_r}{\partial r} & \frac{\partial \hat{e}_\theta}{\partial \theta} & \frac{\partial \hat{e}_\lambda}{\partial \lambda} \\ \frac{\partial \hat{e}_r}{\partial \theta} & \frac{\partial \hat{e}_\theta}{\partial r} & \frac{\partial \hat{e}_\lambda}{\partial r} \\ \frac{\partial \hat{e}_r}{\partial \lambda} & \frac{\partial \hat{e}_\theta}{\partial \lambda} & \frac{\partial \hat{e}_\lambda}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \hat{e}_\theta & -\hat{e}_r & 0 \\ \sin \theta \hat{e}_\lambda & \cos \theta \hat{e}_\lambda & -\sin \theta \hat{e}_r - \cos \theta \hat{e}_\theta \end{bmatrix}. \quad (4.373)$$

In Sect. 3.7.2, we have introduced the nabla operator on a surface  $S$  denoted as  $\nabla_S$ , which is defined by

$$\nabla_S = \nabla - \hat{n}(\hat{n} \cdot \nabla), \quad (4.374)$$

where  $\hat{n}$  is the unit normal vector of the surface. In this section, we consider the surface  $S$  to be a sphere of radius  $r$ , and denote  $\nabla_S$  as  $\nabla_r$ . With  $\hat{n} = \hat{e}_r$  on the sphere, we have

$$\nabla_r = \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\lambda \frac{1}{r \sin \theta} \frac{\partial}{\partial \lambda} = \frac{1}{r} \nabla_1, \quad (4.375)$$

where  $\nabla_1$  is the degenerate case of  $\nabla_r$  with  $r = 1$  defined by

$$\nabla_1 = \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_\lambda \frac{1}{\sin \theta} \frac{\partial}{\partial \lambda}. \quad (4.376)$$

We can use  $\nabla_r$  and  $\nabla_1$  to express  $\nabla$  as

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \nabla_r = \hat{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \nabla_1. \quad (4.377)$$

We defined the Laplacian on a unit sphere as  $\nabla_1^2 = \nabla_1 \cdot \nabla_1$ . According to (4.373), we have

$$\begin{aligned} \nabla_1^2 &= \left( \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_\lambda \frac{1}{\sin \theta} \frac{\partial}{\partial \lambda} \right) \cdot \left( \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_\lambda \frac{1}{\sin \theta} \frac{\partial}{\partial \lambda} \right) \\ &= \hat{e}_\theta \cdot \hat{e}_\theta \frac{\partial^2}{\partial \theta^2} + \hat{e}_\theta \cdot \frac{\partial \hat{e}_\theta}{\partial \theta} \frac{\partial}{\partial \theta} + \hat{e}_\theta \cdot \hat{e}_\lambda \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \lambda} \right) \\ &\quad + \hat{e}_\theta \cdot \frac{\partial \hat{e}_\lambda}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \lambda} + \hat{e}_\lambda \cdot \hat{e}_\theta \frac{1}{\sin \theta} \frac{\partial^2}{\partial \lambda \partial \theta} + \hat{e}_\lambda \cdot \frac{\partial \hat{e}_\theta}{\partial \lambda} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \\ &\quad + \hat{e}_\lambda \cdot \hat{e}_\lambda \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \lambda^2} + \hat{e}_\lambda \cdot \frac{\partial \hat{e}_\lambda}{\partial \lambda} \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \lambda} \\ &= \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \lambda^2} \end{aligned}$$

$$= \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \lambda^2}. \quad (4.378)$$

The expression of the Laplacian  $\nabla^2 = \nabla \cdot \nabla$  in the spherical coordinate system can be formulated using (4.372) and (4.373). Here we infer it from the expression of  $\nabla^2 u$  given by (2.494),

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{\partial}{\partial \lambda} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \lambda} \right) \right]. \quad (4.379)$$

It is related to  $\nabla_1^2$  in the form of

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \nabla_1^2. \quad (4.380)$$

The associated Legendre function  $P_n^k(\cos \theta)$  satisfies the differential equation (4.138), which we write here in the form of

$$\frac{d^2}{d\theta^2} P_n^k(\cos \theta) + \cot \theta \frac{d}{d\theta} P_n^k(\cos \theta) - \frac{k^2}{\sin^2 \theta} P_n^k(\cos \theta) = -n(n+1) P_n^k(\cos \theta). \quad (4.381)$$

We obtain

$$\begin{aligned} \nabla_1^2 \left[ P_n^k(\cos \theta) e^{ik\lambda} \right] &= \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \lambda^2} \right) \left[ P_n^k(\cos \theta) e^{ik\lambda} \right] \\ &= -n(n+1) P_n^k(\cos \theta) e^{ik\lambda}. \end{aligned} \quad (4.382)$$

### 4.7.2 Spherical Harmonic Series of Scalar Fields

According to the spherical harmonic series of functions on a sphere, (4.333), we can express the scalar function  $\phi(r, \theta, \lambda)$  as a spherical harmonic series

$$\phi(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k(r) P_n^k(\cos \theta) e^{ik\lambda}. \quad (4.383)$$

For brevity, we will write  $\phi(r, \theta, \lambda)$ ,  $\phi_n^k(r)$  and  $P_n^k(\cos \theta)$  shortly as  $\phi$ ,  $\phi_n^k$  and  $P_n^k$ , respectively. In this way, the above formula can be written in the shorter form

$$\phi = \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k P_n^k e^{ik\lambda}. \quad (4.384)$$

In the following, we derive the formulae of  $\nabla\phi$  and  $\nabla^2\phi$  based on the spherical harmonic series of  $\phi$ . According to the expression of  $\nabla\phi$  in the spherical coordinate system, (2.463) we have

$$\nabla\phi = \sum_{n=0}^{\infty} \sum_{k=-n}^n \left( \frac{d\phi_n^k}{dr} P_n^k \hat{e}_r + \frac{\phi_n^k}{r} \frac{dP_n^k}{d\theta} \hat{e}_\theta + \frac{ik\phi_n^k}{r \sin\theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda}. \quad (4.385)$$

For  $\nabla^2\phi$ , we have, according to (2.494),

$$\begin{aligned} \nabla^2\phi &= \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{1}{r^2 \sin\theta} \left[ \frac{d}{dr} \left( r^2 \frac{d\phi_n^k}{dr} \right) \sin\theta P_n^k \right. \\ &\quad \left. + \phi_n^k \frac{\partial}{\partial\theta} \left( \sin\theta \frac{dP_n^k}{d\theta} \right) - \phi_n^k \frac{k^2}{\sin\theta} P_n^k \right] e^{ik\lambda} \\ &= \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \left( \frac{d^2\phi_n^k}{dr^2} + \frac{2}{r} \frac{d\phi_n^k}{dr} \right) P_n^k \right. \\ &\quad \left. + \frac{\phi_n^k}{r^2} \left( \frac{d^2P_n^k}{d\theta^2} + \cot\theta \frac{dP_n^k}{d\theta} - \frac{k^2}{\sin^2\theta} P_n^k \right) \right] e^{ik\lambda}. \end{aligned} \quad (4.386)$$

According to (4.381), it can be simplified to

$$\nabla^2\phi = \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \frac{d^2\phi_n^k}{dr^2} + \frac{2}{r} \frac{d\phi_n^k}{dr} - n(n+1) \frac{\phi_n^k}{r^2} \right] P_n^k e^{ik\lambda}. \quad (4.387)$$

### 4.7.3 Decomposition of a Vector Field into a Spheroidal Field and a Toroidal Field

In order to express a vector field in terms spherical harmonics, we first express it using scalars. We decompose a vector field  $\vec{A}$  as (e.g., Jones, 1985),

$$\vec{A} = \vec{S} + \vec{T}, \quad (4.388)$$

where  $\vec{S}$  satisfies

$$(\nabla \times \vec{S}) \cdot \hat{e}_r = 0, \quad (4.389)$$

called spheroidal field, and  $\vec{T}$  satisfies

$$\vec{T} \cdot \hat{e}_r = 0 \quad \text{and} \quad \nabla \cdot \vec{T} = 0, \quad (4.390)$$

called toroidal field. Our approach is to express  $\vec{S}$  and  $\vec{T}$  in terms of scalars, and then prove that the decomposition (4.388) is feasible.

According to the expression of  $\nabla \times \vec{S}$  in the spherical coordinate system, (2.481), we can write (4.389) as

$$(\nabla \times \vec{S}) \cdot \hat{e}_r = \frac{1}{r \sin \theta} \frac{\partial(\sin \theta S_\lambda)}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial S_\theta}{\partial \lambda} = 0. \quad (4.391)$$

This implies that  $\partial(\sin \theta S_\lambda)/\partial \theta = \partial S_\theta/\partial \lambda$ , which holds if  $S_\theta$  and  $S_\lambda$  can be expressed using a scalar  $V$  in the form of

$$S_\theta = \frac{\partial V}{\partial \theta}, \quad S_\lambda = \frac{1}{\sin \theta} \frac{\partial V}{\partial \lambda}. \quad (4.392)$$

We therefore express  $\vec{S}$  using two scalars as

$$\vec{S} = U \hat{e}_r + \frac{\partial V}{\partial \theta} \hat{e}_\theta + \frac{1}{\sin \theta} \frac{\partial V}{\partial \lambda} \hat{e}_\lambda = U \hat{e}_r + \nabla_1 V, \quad (4.393)$$

which does satisfy (4.389). As no constraint is imposed on  $S_r$ , it is included in (4.393) in the general form  $U \hat{e}_r$ .

For  $\vec{T}$ ,  $\vec{T} \cdot \hat{e}_r = 0$  implies

$$\vec{T} = T_\theta \hat{e}_\theta + T_\lambda \hat{e}_\lambda, \quad (4.394)$$

and  $\nabla \cdot \vec{T} = 0$  further implies, according to (2.471),

$$\frac{1}{r \sin \theta} \frac{\partial(\sin \theta T_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_\lambda}{\partial \lambda} = 0. \quad (4.395)$$

We thus have  $\partial(\sin \theta T_\theta)/\partial \theta = -\partial T_\lambda/\partial \lambda$ , which holds if  $T_\theta$  and  $T_\lambda$  can be expressed using a scalar  $W$  in the form of

$$T_\theta = \frac{1}{\sin \theta} \frac{\partial W}{\partial \lambda}, \quad T_\lambda = -\frac{\partial W}{\partial \theta}. \quad (4.396)$$

We therefore express  $\vec{T}$  using a scalar as

$$\vec{T} = \frac{1}{\sin \theta} \frac{\partial W}{\partial \lambda} \hat{e}_\theta - \frac{\partial W}{\partial \theta} \hat{e}_\lambda = -\hat{e}_r \times \nabla_1 W, \quad (4.397)$$

which does satisfy (4.390).

We have proven that the vectors  $\vec{S}$  and  $\vec{T}$  defined in (4.393) and (4.397) satisfy the requirement (4.389) and (4.390), respectively. By substituting (4.393) and (4.397) into (4.388), we obtain the expression for  $\vec{A}$  in terms of the scalar



functions  $U$ ,  $V$  and  $W$ ,

$$\begin{aligned}\vec{A} &= U\hat{e}_r + \nabla_1 V - \hat{e}_r \times \nabla_1 W \\ &= U\hat{e}_r + \left( \frac{\partial V}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial W}{\partial \lambda} \right) \hat{e}_\theta + \left( \frac{1}{\sin \theta} \frac{\partial V}{\partial \lambda} - \frac{\partial W}{\partial \theta} \right) \hat{e}_\lambda, \quad (4.398)\end{aligned}$$

which is often cited in the literature. In the next two paragraphs, we further prove that the decomposition of a vector  $\vec{A}$  as the sum of  $\vec{S}$  and  $\vec{T}$  is unique, or in other words, the representation of a vector  $\vec{A}$  using 3 scalars  $U$ ,  $V$ , and  $W$  in the form of (4.398) is unique.

According to (4.398), we have

$$U = A_r, \quad \frac{\partial V}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial W}{\partial \lambda} = A_\theta, \quad \frac{1}{\sin \theta} \frac{\partial V}{\partial \lambda} - \frac{\partial W}{\partial \theta} = A_\lambda. \quad (4.399)$$

These form a system of equations for the functions  $U$ ,  $V$  and  $W$ . If solutions exist, the decomposition (4.398) is feasible. The first equation is already the solution, i.e.,  $U = A_r$ , thus being decoupled from the rest two. Hence, we only have to solve the last two equations. We can readily obtain the equations satisfied by  $V$  and  $W$ , respectively, which are

$$\begin{cases} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 V}{\partial \lambda^2} = \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{\partial A_\lambda}{\partial \lambda}, \\ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial W}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 W}{\partial \lambda^2} = \frac{\partial A_\theta}{\partial \lambda} - \frac{\partial(\sin \theta A_\lambda)}{\partial \theta}. \end{cases} \quad (4.400)$$

With the expression of  $\nabla_1^2$  given in (4.378), we write these as

$$\begin{cases} \nabla_1^2 V = \frac{1}{\sin \theta} \left[ \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{\partial A_\lambda}{\partial \lambda} \right], \\ \nabla_1^2 W = \frac{1}{\sin \theta} \left[ \frac{\partial A_\theta}{\partial \lambda} - \frac{\partial(\sin \theta A_\lambda)}{\partial \theta} \right]. \end{cases} \quad (4.401)$$

The right-hand sides can be expressed in terms of the divergence and curl of  $\vec{A}$  according to (2.471) and (2.481). We have

$$\begin{Bmatrix} \nabla_1^2 V \\ \nabla_1^2 W \end{Bmatrix} = \begin{Bmatrix} r \nabla \cdot \vec{A} - \frac{1}{r} \frac{\partial(r^2 A_r)}{\partial r} \\ -r(\nabla \times \vec{A})_r \end{Bmatrix}. \quad (4.402)$$

We see that the right-hand sides are finite if  $\vec{A}$ , as well as its divergence and curl, is finite.

We now assume that  $\vec{A}$  has finite and continuous first order derivatives. The right-hand sides of both equations in (4.402) are finite and continuous. As these two

equations have the same form, we write them uniformly as

$$\nabla_1^2 f(r, \theta, \lambda) = \phi(r, \theta, \lambda), \quad (4.403)$$

where  $f$  represents the function  $V$  or  $W$ , and  $\phi$  the right-hand sides of the two equations. The solution of this equation can be found by expressing  $f$  and  $\phi$  as spherical harmonic series according to (4.384),

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^n f_n^k P_n^k e^{ik\lambda}, \quad \phi = \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k P_n^k e^{ik\lambda}. \quad (4.404)$$

By substituting these into (4.403), we obtain

$$\sum_{n=0}^{\infty} \sum_{k=-n}^n f_n^k \nabla_1^2 (P_n^k e^{ik\lambda}) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k P_n^k e^{ik\lambda}, \quad (4.405)$$

which, according to (4.382), can be simplified to

$$-\sum_{n=0}^{\infty} \sum_{k=-n}^n n(n+1) f_n^k P_n^k e^{ik\lambda} = \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k P_n^k e^{ik\lambda}. \quad (4.406)$$

The solution is thus obtained to be

$$f_n^k = -\frac{\phi_n^k}{n(n+1)}. \quad (4.407)$$

The appearance of the factor  $n(n+1)$  in the denominator at the right-hand side of (4.407) implies that no degree zero term should be included in (4.404). In fact, the degree zero terms in  $V$  and  $W$  do not contribute to  $\vec{A}$ , as they are independent of  $\theta$  and  $\lambda$ . Hence, no degree zero term is required in the expression of  $f$  in (4.404). The degree zero term in the expression of  $\phi$  in (4.404) is

$$\phi_0^0 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \phi \sin \theta d\theta d\lambda. \quad (4.408)$$

By substituting the concrete expression of  $\phi$ , i.e., the right-hand sides of (4.401), into this formula, we obtain  $\phi_0^0 = 0$ , i.e., the degree zero term vanishes. We thus conclude that the degree zero terms in (4.404) and (4.407) are vanishing, and consequently,  $f$  is uniquely determined. The uniqueness of  $f$  implies the uniqueness of  $V$  and  $W$ .

In summary, (4.398) is the formula to compute  $\vec{A}$  when the scalar functions  $U$ ,  $V$  and  $W$  are known. When the vector  $\vec{A}$  is known, the scalar functions  $U$ ,  $V$  and  $W$  are to be determined using the first formula of (4.399) and Eqs. (4.401),

or equivalently (4.402), which are differential equations of type (4.403), whose solution is provided in (4.404)–(4.407) using spherical harmonic series.

The spheroidal field  $\vec{S}$  and toroidal field  $\vec{T}$  can also be expressed as

$$\vec{S} = U\hat{e}_r - \hat{e}_r \times [\hat{e}_r \times (r\nabla V)] , \quad (4.409)$$

$$\vec{T} = -\hat{e}_r \times (r\nabla W) ; \quad (4.410)$$

or

$$\vec{S} = U\hat{e}_r + \hat{e}_r \times [\nabla \times (V\vec{r})] , \quad (4.411)$$

$$\vec{T} = \nabla \times (W\vec{r}) . \quad (4.412)$$

The forms (4.409) and (4.410) can be verified straightforwardly, and (4.411) and (4.412) follow from them. Based on (4.409) and (4.410), we can obtain, according to (2.45).

$$\hat{e}_r \times \vec{S} = \hat{e}_r \times (r\nabla V) , \quad (4.413)$$

$$\hat{e}_r \times \vec{T} = -\hat{e}_r \times [\hat{e}_r \times (r\nabla W)] . \quad (4.414)$$

Hence,  $\hat{e}_r \times \vec{S}$  is a toroidal field with  $W = -V$ , and  $\hat{e}_r \times \vec{T}$  is a spheroidal field with  $U = 0$  and  $V = W$ .

#### 4.7.4 Spherical Harmonic Series of Spheroidal and Toroidal Fields

The spherical harmonic series of a vector  $\vec{A}$  is the sum of those of its spheroidal and a toroidal components  $\vec{S}$  and  $\vec{T}$  obtained by expressing the scalars  $U$ ,  $V$ , and  $W$  as spherical harmonic series according to (4.384). We have

$$\begin{aligned} \vec{S} &= \sum_{n=0}^{\infty} \sum_{k=-n}^n \left( U_n^k P_n^k \hat{e}_r + V_n^k \frac{dP_n^k}{d\theta} \hat{e}_\theta + V_n^k \frac{ik}{\sin\theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} \\ &= \sum_{n=0}^{\infty} \sum_{k=-n}^n \left( U_n^k \hat{e}_r + V_n^k \nabla_1 \right) \left( P_n^k e^{ik\lambda} \right) , \end{aligned} \quad (4.415)$$

$$\begin{aligned} \vec{T} &= \sum_{n=0}^{\infty} \sum_{k=-n}^n \left( W_n^k \frac{ik}{\sin\theta} P_n^k \hat{e}_\theta - W_n^k \frac{dP_n^k}{d\theta} \hat{e}_\lambda \right) e^{ik\lambda} \\ &= - \sum_{n=0}^{\infty} \sum_{k=-n}^n W_n^k \hat{e}_r \times \nabla_1 \left( P_n^k e^{ik\lambda} \right) . \end{aligned} \quad (4.416)$$

where  $U_n^k$ ,  $V_n^k$ , and  $W_n^k$  are the spherical harmonic coefficients of  $U$ ,  $V$ , and  $W$ , respectively, which are all assumed to be functions of  $r$  written in a brief form according to the convention in (4.384). The expressions using  $\nabla_1$  are shorter forms sometimes used in the literature.

We also write (4.388), (4.415), and (4.416) in the following form for brevity in subsequent derivations:

$$\vec{A} = \sum_{n=0}^{\infty} \sum_{k=-n}^n (\vec{S}_n^k + \vec{T}_n^k), \quad (4.417)$$

$$\vec{S}_n^k = \left( U_n^k P_n^k \hat{e}_r + V_n^k \frac{dP_n^k}{d\theta} \hat{e}_\theta + V_n^k \frac{ik}{\sin \theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} = \left( U_n^k \hat{e}_r + V_n^k \nabla_1 \right) \left( P_n^k e^{ik\lambda} \right), \quad (4.418)$$

$$\vec{T}_n^k = \left( W_n^k \frac{ik}{\sin \theta} P_n^k \hat{e}_\theta - W_n^k \frac{dP_n^k}{d\theta} \hat{e}_\lambda \right) e^{ik\lambda} = -W_n^k \hat{e}_r \times \nabla_1 \left( P_n^k e^{ik\lambda} \right), \quad (4.419)$$

where  $\vec{S}_n^k$  and  $\vec{T}_n^k$  are the degree  $n$  and order  $k$  spheroidal and toroidal fields, respectively.

We now derive out the formulae of gradient and divergence of  $\vec{S}_n^k$  and  $\vec{T}_n^k$  for future reference. According to (2.467), we have

$$\begin{aligned} \nabla \vec{S}_n^k = & \left\{ \frac{dU_n^k}{dr} P_n^k \hat{e}_r \hat{e}_r + \frac{dV_n^k}{dr} \frac{dP_n^k}{d\theta} \hat{e}_r \hat{e}_\theta + \frac{dV_n^k}{dr} \frac{ik}{\sin \theta} P_n^k \hat{e}_r \hat{e}_\lambda + \frac{U_n^k - V_n^k}{r} \frac{dP_n^k}{d\theta} \hat{e}_\theta \hat{e}_r \right. \\ & + \left( \frac{U_n^k}{r} P_n^k + \frac{V_n^k}{r} \frac{d^2 P_n^k}{d\theta^2} \right) \hat{e}_\theta \hat{e}_\theta + \frac{V_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} - ik \frac{\cos \theta}{\sin^2 \theta} P_n^k \right) \hat{e}_\theta \hat{e}_\lambda \\ & + \frac{U_n^k - V_n^k}{r} \frac{ik}{\sin \theta} P_n^k \hat{e}_\lambda \hat{e}_r + \frac{V_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} - ik \frac{\cos \theta}{\sin^2 \theta} P_n^k \right) \hat{e}_\lambda \hat{e}_\theta \\ & \left. + \left[ \frac{U_n^k}{r} P_n^k + \frac{V_n^k}{r} \left( \cot \theta \frac{dP_n^k}{d\theta} - \frac{k^2}{\sin^2 \theta} P_n^k \right) \right] \hat{e}_\lambda \hat{e}_\lambda \right\} e^{ik\lambda}, \quad (4.420) \end{aligned}$$

$$\begin{aligned} \nabla \vec{T}_n^k = & \left[ \frac{dW_n^k}{dr} \frac{ik}{\sin \theta} P_n^k \hat{e}_r \hat{e}_\theta - \frac{dW_n^k}{dr} \frac{dP_n^k}{d\theta} \hat{e}_r \hat{e}_\lambda - \frac{W_n^k}{r} \frac{ik}{\sin \theta} P_n^k \hat{e}_\theta \hat{e}_r \right. \\ & + \frac{W_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} - ik \frac{\cos \theta}{\sin^2 \theta} P_n^k \right) \hat{e}_\theta \hat{e}_\theta - \frac{W_n^k}{r} \frac{d^2 P_n^k}{d\theta^2} \hat{e}_\theta \hat{e}_\lambda \\ & + \frac{W_n^k}{r} \frac{dP_n^k}{d\theta} \hat{e}_\lambda \hat{e}_r + \frac{W_n^k}{r} \left( \cot \theta \frac{dP_n^k}{d\theta} - \frac{k^2}{\sin^2 \theta} P_n^k \right) \hat{e}_\lambda \hat{e}_\theta \\ & \left. - \frac{W_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} - ik \frac{\cos \theta}{\sin^2 \theta} P_n^k \right) \hat{e}_\lambda \hat{e}_\lambda \right] e^{ik\lambda}. \quad (4.421) \end{aligned}$$

According to (2.471) and (4.381), we have

$$\begin{aligned}
 \nabla \cdot \vec{S}_n^k &= \left[ \frac{1}{r^2} \frac{d(r^2 U_n^k)}{dr} P_n^k + \frac{V_n^k}{r} \frac{1}{\sin \theta} \left( \sin \theta \frac{d^2 P_n^k}{d\theta^2} + \cos \theta \frac{d P_n^k}{d\theta} \right) - \frac{V_n^k}{r} \frac{k^2}{\sin^2 \theta} P_n^k \right] e^{ik\lambda} \\
 &= \left[ \left( \frac{dU_n^k}{dr} + \frac{2U_n^k}{r} \right) P_n^k + \frac{V_n^k}{r} \left( \frac{d^2 P_n^k}{d\theta^2} + \cot^2 \theta \frac{d P_n^k}{d\theta} - \frac{k^2}{\sin^2 \theta} P_n^k \right) \right] e^{ik\lambda} \\
 &= \left[ \frac{dU_n^k}{dr} + \frac{2U_n^k}{r} - n(n+1) \frac{V_n^k}{r} \right] P_n^k e^{ik\lambda}. \tag{4.422}
 \end{aligned}$$

It follows from the definition of the toroidal field that

$$\nabla \cdot \vec{T}_n^k = 0. \tag{4.423}$$

As an exercise, the readers are suggested to express  $\phi$  in (4.384) and  $U$ ,  $V$ , and  $W$  in (4.415) and (4.416) as real spherical harmonic series according to (4.331), and then derive out the other formulae in this subsection.

### 4.7.5 Solenoidal Vector Field as a Special Case

If  $\vec{A}$  is a solenoidal field, i.e., if  $\nabla \cdot \vec{A} = 0$ ,  $\vec{S}$  is also so. In this circumstance, the spheroidal field  $\vec{S}$  is referred to as poloidal field, and its expression can be further simplified. As  $\nabla \cdot \vec{S}_n^k = 0$ , we obtain from (4.422),

$$V_n^k = \frac{r}{n(n+1)} \frac{dU_n^k}{dr} + \frac{2}{n(n+1)} U_n^k = \frac{1}{n(n+1)} \frac{1}{r} \frac{d(r^2 U_n^k)}{dr}. \tag{4.424}$$

By substituting this formula into (4.418), and then redefining the notation  $V_n^k$  to be  $(r^2 U_n^k)/[n(n+1)]$ , we obtain

$$\vec{S}_n^k = \left[ \frac{n(n+1)}{r^2} V_n^k P_n^k \hat{e}_r + \frac{1}{r} \frac{dV_n^k}{dr} \frac{dP_n^k}{d\theta} \hat{e}_\theta + \frac{ik}{r \sin \theta} \frac{dV_n^k}{dr} P_n^k \hat{e}_\lambda \right] e^{ik\lambda}. \tag{4.425}$$

Define  $V = \sum_{n=0}^{\infty} \sum_{k=-n}^n V_n^k Y_n^k$ . We can further use the above formula and (4.382) to write  $\vec{S}$  as

$$\vec{S} = -\frac{1}{r^2} \nabla_1^2 V \hat{e}_r + \frac{1}{r} \frac{\partial^2 V}{\partial r \partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial^2 V}{\partial r \partial \lambda} \hat{e}_\lambda. \tag{4.426}$$

where  $\nabla_1^2$  is the Laplacian on the unit sphere given in (4.378). This formula can also be written as

$$\vec{S} = \nabla \times [\nabla \times (V\hat{e}_r)] = \nabla \times \left[ \nabla \left( \frac{V}{r} \right) \times \vec{r} \right], \quad (4.427)$$

which can be readily verified.

The forms of  $\vec{T}$  corresponding to (4.426) and (4.427) are,

$$\vec{T} = \frac{1}{r \sin \theta} \frac{\partial W}{\partial \lambda} \hat{e}_\theta - \frac{1}{r} \frac{\partial W}{\partial \theta} \hat{e}_\lambda, \quad (4.428)$$

and

$$\vec{T} = \nabla \times (W\hat{e}_r) = \nabla \left( \frac{W}{r} \right) \times \vec{r}, \quad (4.429)$$

respectively, where the notation  $W$  has also been redefined. Its degree  $n$  and order  $k$  spherical harmonic expression corresponding to (4.425) is

$$\vec{T}_n^k = \left( \frac{W_n^k}{r} \frac{ik}{\sin \theta} P_n^k \hat{e}_\theta - \frac{W_n^k}{r} \frac{dP_n^k}{d\theta} \hat{e}_\lambda \right) e^{ik\lambda}. \quad (4.430)$$

In summary, for a solenoidal field, the expressions of the poloidal and toroidal fields are (4.427) and (4.429), respectively. Based on these, we can express a solenoidal field  $\vec{A}$  using two scalars  $V$  and  $W$  as

$$\vec{A} = \nabla \times \left[ \nabla \left( \frac{V}{r} \right) \times \vec{r} \right] + \nabla \left( \frac{W}{r} \right) \times \vec{r} = \sum_{n=0}^{\infty} \sum_{k=-n}^n (\vec{S}_n^k + \vec{T}_n^k), \quad (4.431)$$

where the degree  $n$  and order  $k$  spherical harmonic expressions  $\vec{S}_n^k$  and  $\vec{T}_n^k$  are given in (4.425) and (4.430), respectively.

Some relations between the poloidal and toroidal fields, (4.427) and (4.429), can be obtained. We see immediately that

$$\nabla \times \vec{T} = \nabla \times [\nabla \times (W\hat{e}_r)] \quad (4.432)$$

is a poloidal field with  $V = W$ . We further shown that  $\nabla \times \vec{S}$  is a toroidal field. According to (2.233) and (2.229), we first have

$$\begin{aligned} \nabla \times \vec{S} &= \nabla \times \{ \nabla \times [\nabla \times (V\hat{e}_r)] \} \\ &= \nabla \times \left\{ \nabla [\nabla \cdot (V\hat{e}_r)] - \nabla^2 (V\hat{e}_r) \right\} = -\nabla \times [\nabla^2 (V\hat{e}_r)]. \end{aligned} \quad (4.433)$$

According to (2.498) and (2.481), we further have

$$\begin{aligned}
 \nabla \times \vec{S} &= -\nabla \times [(\nabla^2 V)\hat{e}_r] - \nabla \times \left( -\frac{2}{r^2}V\hat{e}_r + \frac{2}{r^2}\frac{\partial V}{\partial \theta}\hat{e}_\theta + \frac{2}{r^2\sin\theta}\frac{\partial V}{\partial \lambda}\hat{e}_\lambda \right) \\
 &= -\nabla \times [(\nabla^2 V)\hat{e}_r] - \frac{1}{r} \left[ -\frac{1}{\sin\theta}\frac{\partial}{\partial \lambda} \left( \frac{2}{r}\frac{\partial V}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{2}{r}\frac{\partial V}{\partial r} \right) \right] \\
 &= -\nabla \times [(\nabla^2 V)\hat{e}_r] - \nabla \times \left[ -\left( \frac{2}{r}\frac{\partial V}{\partial r} \right) \hat{e}_r \right] \\
 &= \nabla \times \left[ \left( -\nabla^2 V + \frac{2}{r}\frac{\partial V}{\partial r} \right) \hat{e}_r \right] \\
 &= \nabla \times \left[ \left( -\frac{\partial^2 V}{\partial r^2} - \frac{1}{r^2}\nabla_1^2 V \right) \hat{e}_r \right], \tag{4.434}
 \end{aligned}$$

where (4.380) is used to obtain the last expression. This is a toroidal field with  $W = -\partial^2 V / \partial r^2 - (1/r^2)\nabla_1^2 V$ .

In the literature, the definitions of  $V$  and  $W$  may be different. We followed Chandrasekhar (1961). Some authors, e.g., Smylie (1965), defined the scalars to be our  $V/r$  and  $W/r$ , and therefore, the spherical harmonic series obtained are also different.

## 4.8 Reduction of the Elastic Equation of the Spherical Earth Model

### 4.8.1 Equations and Assumptions

The spherically symmetrical elastic model is a primitive approximation of the Earth in the study of various geophysical phenomena, including free oscillations excited by Earthquakes and deformations of the Earth in response to the lunisolar gravitational attraction as well as the surface loading of mass changes in the atmosphere-hydrosphere-cryosphere system. For such an Earth model, the governing equation can be reduced to a system of ordinary differential equations, and hence, the solution is very much simplified.

Here we assume the Earth to be rotating, but neglect the centrifugal force in the equation of motion while keeping the Coriolis force. The equations to be solved are the elastic equation (3.413), the elastic constitutive equation (3.418), and Poisson's equation (3.373), which are written here as

$$\begin{aligned}
 \rho \left( \frac{\partial^2 \vec{u}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{u}}{\partial t} \right) &= \nabla \cdot \bar{T} - \rho(\nabla \cdot \vec{u})\nabla\phi_0 \\
 &\quad + \rho\nabla\vec{u} \cdot \nabla\phi_0 + \rho\vec{u} \cdot \nabla\nabla\phi_0 + \rho\nabla\phi_1 + \vec{f}, \tag{4.435}
 \end{aligned}$$

$$\bar{T} = \lambda \nabla \cdot \vec{u} \bar{I} + \mu [\nabla \vec{u} + (\nabla \vec{u})^T], \quad (4.436)$$

$$\nabla^2 \phi_1 = 4\pi G \nabla \cdot (\rho \vec{u}), \quad (4.437)$$

where we have written  $\rho_0$  as  $\rho$  for brevity.

Due to the assumption of spherical symmetry, the parameters describing the undeformed state are functions of the radial distance alone. First, as  $\phi_0$  is a function of  $r$  alone, we have

$$\nabla \phi_0 = -g \hat{e}_r, \quad (4.438)$$

where  $g$  is the gravity defined positive downward,  $g = -d\phi_0/dr$ , which is also a function of  $r$  alone. With the dependence of  $\phi_0$  and  $g$  on  $r$  alone taken into account, Poisson's equation  $\nabla^2 \phi_0 = -4\pi G \rho_0$  can be readily reduced to

$$\frac{dg}{dr} = -\frac{2g}{r} + 4\pi G \rho. \quad (4.439)$$

An immediate observance by multiplying both sides by  $r$  is that  $g = 0$  when  $r = 0$ . With

$$\frac{1}{r^2} \frac{d(r^2 g)}{dr} = \frac{dg}{dr} + \frac{2g}{r}, \quad (4.440)$$

we can solve (4.439) to obtain

$$g = 4\pi G \frac{1}{r^2} \int_0^r \rho q^2 dq. \quad (4.441)$$

It can also be obtained that, according to (2.467),

$$\nabla \nabla \phi_0 = -\frac{dg}{dr} \hat{e}_r \hat{e}_r - \frac{g}{r} \hat{e}_\theta \hat{e}_\theta - \frac{g}{r} \hat{e}_\lambda \hat{e}_\lambda. \quad (4.442)$$

We have thus expressed  $\nabla \phi_0$  and  $\nabla \nabla \phi_0$  in terms of  $g$ .

As compared to the more accurate ellipsoidal model of the Earth, assuming  $g$  to be function of  $r$  alone implies the neglect of terms of relative magnitude on the order of the flattening, including the centrifugal term.

In the reminder of this subsection, we introduce two assumptions which are necessary for reducing the partial differential equations (4.435)–(4.437) to a system of ordinary differential equations. We then express these equations in the spherical coordinate system. The further development of these equations to a system of ordinary differential equations is quite tedious, which will be the subject of the subsequent two subsections.



First, we assume the phenomena studied is periodical, which is indeed the case for free oscillations and for the tidal deformation arisen from the lunisolar gravitational attraction. The surface loading due to mass transport in the atmosphere-hydrosphere-cryosphere system can also be mostly treated as periodical. According to this assumption, the dependence of all functions in time is in the form of  $e^{i\sigma t}$ , where  $\sigma$  is the angular frequency. Hence, we have

$$\frac{\partial(\cdot)}{\partial t} = i\sigma(\cdot), \quad \frac{\partial^2(\cdot)}{\partial t^2} = -\sigma^2(\cdot), \quad (4.443)$$

where  $(\cdot)$  is used to represent any function in the differential equations (4.435)–(4.437).

Second, we assume the force  $\vec{f}$  to have the form

$$\vec{f} = \rho \nabla \phi_f, \quad \nabla^2 \phi_f = 0. \quad (4.444)$$

Both the lunisolar gravitational attraction and that of the surface load have this form. We can thus substitute  $\rho \nabla \phi_1 + \vec{f}$  by  $\rho \nabla(\phi_f + \phi_1)$  in (4.435), and write (4.437) as

$$\nabla^2(\phi_f + \phi_1) = 4\pi G \nabla \cdot (\rho \vec{u}). \quad (4.445)$$

We now reduce the equations of motion into the spherical coordinate system. It is to be mentioned that  $\lambda$  is used to denote an elastic parameter and the longitude. There should not be any confusion in the context.

According to (2.467), (4.438), and (4.442), we have

$$\nabla \vec{u} \cdot \nabla \phi_0 = -g \frac{\partial u_r}{\partial r} \hat{e}_r + \left( -\frac{g}{r} \frac{\partial u_r}{\partial \theta} + \frac{g}{r} u_\theta \right) \hat{e}_\theta + \left( -\frac{g}{r \sin \theta} \frac{\partial u_r}{\partial \lambda} + \frac{g}{r} u_\lambda \right) \hat{e}_\lambda, \quad (4.446)$$

$$\vec{u} \cdot \nabla \nabla \phi_0 = -\frac{dg}{dr} u_r \hat{e}_r - \frac{g}{r} u_\theta \hat{e}_\theta - \frac{g}{r} u_\lambda \hat{e}_\lambda. \quad (4.447)$$

The angular velocity can be expressed in the spherical coordinate system as

$$\vec{\omega} = \omega \hat{e}_3 = \omega \cos \theta \hat{e}_r - \omega \sin \theta \hat{e}_\theta. \quad (4.448)$$

Hence,

$$\vec{\omega} \times \frac{\partial \vec{u}}{\partial t} = -\omega \sin \theta \frac{\partial u_\lambda}{\partial t} \hat{e}_r - \omega \cos \theta \frac{\partial u_\lambda}{\partial t} \hat{e}_\theta + \left( \omega \sin \theta \frac{\partial u_r}{\partial t} + \omega \cos \theta \frac{\partial u_\theta}{\partial t} \right) \hat{e}_\lambda. \quad (4.449)$$

By substituting (4.438), (4.443)–(4.447), and (4.449) into the vector equation (4.435), we obtain its component form along  $\hat{e}_r$ ,  $\hat{e}_\theta$ , and  $\hat{e}_\lambda$ ,

$$\begin{aligned} & \rho \left( -\sigma^2 u_r - 2i\omega\sigma \sin\theta u_\lambda \right) \\ &= (\nabla \cdot \bar{T})_r + \rho g \nabla \cdot \vec{u} - \rho g \frac{\partial u_r}{\partial r} - \rho \frac{dg}{dr} u_r + \rho [\nabla(\phi_f + \phi_1)]_r, \end{aligned} \quad (4.450)$$

$$\rho \left( -\sigma^2 u_\theta - 2i\omega\sigma \cos\theta u_\lambda \right) = (\nabla \cdot \bar{T})_\theta - \frac{\rho g}{r} \frac{\partial u_r}{\partial \theta} + \rho [\nabla(\phi_f + \phi_1)]_\theta, \quad (4.451)$$

$$\begin{aligned} & \rho \left( -\sigma^2 u_\lambda + 2i\omega\sigma \sin\theta u_r + 2i\omega\sigma \cos\theta u_\theta \right) \\ &= (\nabla \cdot \bar{T})_\lambda - \frac{\rho g}{r \sin\theta} \frac{\partial u_r}{\partial \lambda} + \rho [\nabla(\phi_f + \phi_1)]_\lambda. \end{aligned} \quad (4.452)$$

With  $\rho$  as a function of  $r$  alone, (4.445) is simplified to

$$\nabla^2(\phi_f + \phi_1) = 4\pi G(\rho \nabla \cdot \vec{u} + \nabla \rho \cdot \vec{u}) = 4\pi G \left( \rho \nabla \cdot \vec{u} + \frac{d\rho}{dr} u_r \right). \quad (4.453)$$

These last four equations, together with (4.436), are the equations to be solved.

### 4.8.2 Development into Spherical Harmonic Series

According to our assumption on the dependence on time, we express  $\vec{u}$  in the form of

$$\vec{u} = \sum_{n=0}^{\infty} \sum_{k=-n}^n (\vec{S}_n^k + \vec{T}_n^k) e^{i\sigma t}, \quad (4.454)$$

where  $\vec{S}_n^k$  and  $\vec{T}_n^k$  are the degree  $n$  and order  $k$  spheroidal and toroidal (vector) fields, respectively,

$$\begin{aligned} \vec{S}_n^k = & \left[ U_n^k(r) P_n^k(\cos\theta) \hat{e}_r + V_n^k(r) \frac{d}{d\theta} P_n^k(\cos\theta) \hat{e}_\theta \right. \\ & \left. + V_n^k(r) \frac{ik}{\sin\theta} P_n^k(\cos\theta) \hat{e}_\lambda \right] e^{ik\lambda}, \end{aligned} \quad (4.455)$$

$$\vec{T}_n^k = \left[ W_n^k(r) \frac{ik}{\sin\theta} P_n^k(\cos\theta) \hat{e}_\theta - W_n^k(r) \frac{d}{d\theta} P_n^k(\cos\theta) \hat{e}_\lambda \right] e^{ik\lambda}. \quad (4.456)$$

We also express  $\phi_f + \phi_1$  as

$$\phi_f + \phi_1 = \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k(r) P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}. \quad (4.457)$$

For brevity, we omit the summation sign, assuming that summation over  $n$  and  $k$  is implicit, and as in (4.7.4), we write  $U_n^k(r)$ ,  $V_n^k(r)$ ,  $W_n^k(r)$ ,  $\phi_n^k(r)$ , and  $P_n^k(\cos \theta)$  shortly as  $U_n^k$ ,  $V_n^k$ ,  $W_n^k$ ,  $\phi_n^k$ , and  $P_n^k$ , respectively. The short forms of the above formulae are

$$\vec{u} = (\vec{S}_n^k + \vec{T}_n^k) e^{i\sigma t}, \quad (4.458)$$

$$\vec{S}_n^k = \left( U_n^k P_n^k \hat{e}_r + V_n^k \frac{dP_n^k}{d\theta} \hat{e}_\theta + V_n^k \frac{ik}{\sin \theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda}, \quad (4.459)$$

$$\vec{T}_n^k = \left( W_n^k \frac{ik}{\sin \theta} P_n^k \hat{e}_\theta - W_n^k \frac{dP_n^k}{d\theta} \hat{e}_\lambda \right) e^{ik\lambda}. \quad (4.460)$$

$$\phi_f + \phi_1 = \phi_n^k P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (4.461)$$

Denote

$$\Delta_n^k = \frac{dU_n^k}{dr} + \frac{2U_n^k}{r} - n(n+1) \frac{V_n^k}{r}, \quad (4.462)$$

which can be used to write the divergence of  $\vec{S}_n^k$ , (4.422), as  $\nabla \cdot \vec{S}_n^k = \Delta_n^k P_n^k e^{ik\lambda}$ . According to the formulae of the divergences and gradients of  $\vec{S}_n^k$  and  $\vec{T}_n^k$ , (4.422), (4.423), (4.420), and (4.421), we obtain the components of  $\vec{T}$  in (4.436) as

$$T_{rr} = \left( \lambda \Delta_n^k + 2\mu \frac{dU_n^k}{dr} \right) P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (4.463)$$

$$\begin{aligned} T_{\theta\theta} = & \left[ \left( \lambda \Delta_n^k + 2\mu \frac{U_n^k}{r} \right) P_n^k \right. \\ & \left. + 2\mu \frac{V_n^k}{r} \frac{d^2 P_n^k}{d\theta^2} + 2\mu \frac{W_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} - ik \frac{\cos \theta}{\sin^2 \theta} P_n^k \right) \right] e^{ik\lambda} e^{i\sigma t}, \end{aligned} \quad (4.464)$$

$$\begin{aligned} T_{\lambda\lambda} = & \left[ \left( \lambda \Delta_n^k + 2\mu \frac{U_n^k}{r} - 2\mu \frac{V_n^k}{r} \frac{k^2}{\sin^2 \theta} \right) P_n^k + 2\mu \frac{V_n^k}{r} \cot \theta \frac{dP_n^k}{d\theta} \right. \\ & \left. - 2\mu \frac{W_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} - ik \frac{\cos \theta}{\sin^2 \theta} P_n^k \right) \right] e^{ik\lambda} e^{i\sigma t}, \end{aligned} \quad (4.465)$$

$$T_{r\theta} = T_{\theta r} = \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{dP_n^k}{d\theta} + \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{ik}{\sin \theta} P_n^k \right] e^{ik\lambda} e^{i\sigma t}, \quad (4.466)$$

$$T_{r\lambda} = T_{\lambda r} = \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{ik}{\sin \theta} P_n^k - \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{dP_n^k}{d\theta} \right] e^{ik\lambda} e^{i\sigma t}, \quad (4.467)$$

$$T_{\theta\lambda} = T_{\lambda\theta} = \left[ 2\mu \frac{V_n^k}{r} \left( -ik \frac{\cos \theta}{\sin^2 \theta} P_n^k + \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} \right) + \mu \frac{W_n^k}{r} \left( -\frac{d^2 P_n^k}{d\theta^2} + \cot \theta \frac{dP_n^k}{d\theta} - \frac{k^2}{\sin^2 \theta} P_n^k \right) \right] e^{ik\lambda} e^{i\sigma t}. \quad (4.468)$$

The components of  $\nabla \cdot \bar{T}$  can be obtained according the formula of  $\nabla \cdot \bar{T}$ , (2.476), using the above formulae and (4.381),

$$\begin{aligned} (\nabla \cdot \bar{T})_r &= \left\{ \frac{d}{dr} (\lambda \Delta_n^k + 2\mu \frac{dU_n^k}{dr}) P_n^k + \frac{\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{d^2 P_n^k}{d\theta^2} \right. \\ &\quad + \frac{\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \left( -ik \frac{\cos \theta}{\sin^2 \theta} P_n^k + \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} \right) \\ &\quad - \frac{\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{k^2}{\sin^2 \theta} P_n^k - \frac{\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{ik}{\sin \theta} \frac{dP_n^k}{d\theta} \\ &\quad + \left[ 4\frac{\mu}{r} \left( \frac{dU_n^k}{dr} - \frac{U_n^k}{r} \right) P_n^k - \frac{2\mu}{r} \frac{V_n^k}{r} \left( \frac{d^2 P_n^k}{d\theta^2} + \cot \theta \frac{dP_n^k}{d\theta} - \frac{k^2}{\sin^2 \theta} P_n^k \right) \right] \\ &\quad \left. + \frac{\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \cot \theta \frac{dP_n^k}{d\theta} + \frac{\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{ik \cos \theta}{\sin^2 \theta} P_n^k \right\} e^{ik\lambda} e^{i\sigma t} \\ &= \left\{ \frac{d}{dr} \left( \lambda \Delta_n^k + 2\mu \frac{dU_n^k}{dr} \right) P_n^k - n(n+1) \frac{\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) P_n^k \right. \\ &\quad \left. + \left[ 4\frac{\mu}{r} \left( \frac{dU_n^k}{dr} - \frac{U_n^k}{r} \right) + \frac{2\mu}{r} \frac{V_n^k}{r} n(n+1) \right] P_n^k \right\} e^{ik\lambda} e^{i\sigma t} \\ &= \left\{ \frac{d}{dr} \left( \lambda \Delta_n^k + 2\mu \frac{dU_n^k}{dr} \right) + \frac{\mu}{r} \left[ 4\frac{dU_n^k}{dr} - 4\frac{U_n^k}{r} \right. \right. \\ &\quad \left. \left. - n(n+1) \left( \frac{dV_n^k}{dr} + \frac{U_n^k - 3V_n^k}{r} \right) \right] \right\} P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (4.469) \end{aligned}$$

$$(\nabla \cdot \bar{T})_\theta = \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] \frac{dP_n^k}{d\theta} + \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \frac{ik}{\sin \theta} P_n^k \right.$$

$$\begin{aligned}
& + \frac{\lambda}{r} \Delta_n^k \frac{dP_n^k}{d\theta} + \frac{2\mu}{r} \frac{U_n^k}{r} \frac{dP_n^k}{d\theta} + \frac{2\mu}{r} \frac{V_n^k}{r} \frac{d^3 P_n^k}{d\theta^3} + \frac{2\mu}{r} \frac{W_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{d^2 P_n^k}{d\theta^2} \right. \\
& - 2ik \frac{\cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} + \frac{ik}{\sin \theta} P_n^k + 2ik \frac{\cos^2 \theta}{\sin^3 \theta} P_n^k \left. \right) + \frac{2\mu}{r} \frac{V_n^k}{r} \left( k^2 \frac{\cos \theta}{\sin^3 \theta} P_n^k \right. \\
& - \frac{k^2}{\sin^2 \theta} \frac{dP_n^k}{d\theta} \left. \right) + \frac{\mu}{r} \frac{W_n^k}{r} \left( - \frac{ik}{\sin \theta} \frac{d^2 P_n^k}{d\theta^2} + ik \frac{\cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} - \frac{ik^3}{\sin^3 \theta} P_n^k \right) \\
& + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{dP_n^k}{d\theta} + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{ik}{\sin \theta} P_n^k \\
& + \frac{2\mu}{r} \frac{V_n^k}{r} \cot \theta \frac{d^2 P_n^k}{d\theta^2} + \frac{2\mu}{r} \frac{V_n^k}{r} \frac{k^2 \cos \theta}{\sin^3 \theta} P_n^k - \frac{2\mu}{r} \frac{V_n^k}{r} \frac{\cos^2 \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} \\
& + \frac{4\mu}{r} \frac{W_n^k}{r} \left( ik \frac{\cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} - ik \frac{\cos^2 \theta}{\sin^3 \theta} P_n^k \right) \left. \right\} e^{ik\lambda} e^{i\sigma t} \\
& = \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] \frac{dP_n^k}{d\theta} + \frac{\lambda}{r} \Delta_n^k \frac{dP_n^k}{d\theta} + \frac{2\mu}{r} \frac{U_n^k}{r} \frac{dP_n^k}{d\theta} \right. \\
& + \frac{2\mu}{r} \frac{V_n^k}{r} \left[ \frac{d^3 P_n^k}{d\theta^3} + \frac{d}{d\theta} \left( \cot \theta \frac{dP_n^k}{d\theta} \right) - \frac{d}{d\theta} \left( \frac{k^2}{\sin^2 \theta} P_n^k \right) + \frac{dP_n^k}{d\theta} \right] \\
& + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{dP_n^k}{d\theta} + \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \frac{ik}{\sin \theta} P_n^k \\
& + \frac{\mu}{r} \frac{W_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{d^2 P_n^k}{d\theta^2} + ik \frac{\cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} - \frac{ik^3}{\sin^3 \theta} P_n^k + 2 \frac{ik}{\sin \theta} P_n^k \right) \\
& + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{ik}{\sin \theta} P_n^k \left. \right\} e^{ik\lambda} e^{i\sigma t} \\
& = \left[ \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r} \frac{U_n^k}{r} + \frac{2\mu}{r} \frac{V_n^k}{r} [-n(n+1) + 1] \right. \right. \\
& + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \left. \right\} \frac{dP_n^k}{d\theta} + \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \right. \\
& + \frac{\mu}{r} \frac{W_n^k}{r} [-n(n+1) + 2] + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \left. \right\} \frac{ik}{\sin \theta} P_n^k \left. \right] e^{ik\lambda} e^{i\sigma t} \\
& = \left[ \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r^2} [U_n^k - (n^2 + n - 1)V_n^k] \right. \right. \\
& + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \left. \right\} \frac{dP_n^k}{d\theta} + \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \right.
\end{aligned}$$

$$- (n-1)(n+2) \frac{\mu}{r^2} W_n^k + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \left\{ \frac{ik}{\sin \theta} P_n^k \right\} e^{ik\lambda} e^{i\sigma t}. \quad (4.470)$$

$$\begin{aligned} (\nabla \cdot \bar{T})_\lambda = & \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] \frac{ik}{\sin \theta} P_n^k - \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \frac{dP_n^k}{d\theta} \right. \\ & + \frac{2\mu}{r} \frac{V_n^k}{r} \left( \frac{ik}{\sin \theta} P_n^k + 2ik \frac{\cos^2 \theta}{\sin^3 \theta} P_n^k - 2ik \frac{\cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} + \frac{ik}{\sin \theta} \frac{d^2 P_n^k}{d\theta^2} \right) \\ & + \frac{\mu}{r} \frac{W_n^k}{r} \left( -\frac{d^3 P_n^k}{d\theta^3} + \cot \theta \frac{d^2 P_n^k}{d\theta^2} - \frac{1}{\sin^2 \theta} \frac{dP_n^k}{d\theta} - \frac{k^2}{\sin^2 \theta} \frac{dP_n^k}{d\theta} + 2 \frac{k^2 \cos \theta}{\sin^3 \theta} P_n^k \right) \\ & + \frac{\lambda}{r} \Delta_n^k \frac{ik}{\sin \theta} P_n^k + \frac{2\mu}{r} \frac{U_n^k}{r} \frac{ik}{\sin \theta} P_n^k - \frac{2\mu}{r} \frac{V_n^k}{r} \frac{ik^3}{\sin^3 \theta} P_n^k + \frac{2\mu}{r} \frac{V_n^k}{r} \frac{ik \cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} \\ & - \frac{2\mu}{r} \frac{W_n^k}{r} \left( -\frac{k^2}{\sin^2 \theta} \frac{dP_n^k}{d\theta} + k^2 \frac{\cos \theta}{\sin^3 \theta} P_n^k \right) \\ & + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{ik}{\sin \theta} P_n^k e^{ik\lambda} \\ & - \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{dP_n^k}{d\theta} + \frac{4\mu}{r} \frac{V_n^k}{r} \left( -ik \frac{\cos^2 \theta}{\sin^3 \theta} P_n^k + ik \frac{\cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} \right) \\ & + \frac{2\mu}{r} \frac{W_n^k}{r} \left( -\cot \theta \frac{d^2 P_n^k}{d\theta^2} + \frac{\cos^2 \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} - k^2 \frac{\cos \theta}{\sin^3 \theta} P_n^k \right) \left. \right\} e^{ik\lambda} e^{i\sigma t} \\ = & \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] \frac{ik}{\sin \theta} P_n^k + \frac{\lambda}{r} \Delta_n^k \frac{ik}{\sin \theta} P_n^k + \frac{2\mu}{r} \frac{U_n^k}{r} \frac{ik}{\sin \theta} P_n^k \right. \\ & + \frac{2\mu}{r} \frac{V_n^k}{r} \left( \frac{ik}{\sin \theta} \frac{d^2 P_n^k}{d\theta^2} + ik \frac{\cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} - \frac{ik^3}{\sin^3 \theta} P_n^k + \frac{ik}{\sin \theta} P_n^k \right) \\ & + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \frac{ik}{\sin \theta} P_n^k - \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \frac{dP_n^k}{d\theta} \\ & - \frac{\mu}{r} \frac{W_n^k}{r} \left[ \frac{d^3 P_n^k}{d\theta^3} + \frac{d}{d\theta} \left( \cot \theta \frac{dP_n^k}{d\theta} \right) - \frac{d}{d\theta} \left( \frac{k^2}{\sin^2 \theta} P_n^k \right) + 2 \frac{dP_n^k}{d\theta} \right] \\ & - \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \frac{dP_n^k}{d\theta} \left. \right\} e^{ik\lambda} e^{i\sigma t} \\ = & \left[ \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r} \frac{U_n^k}{r} + \frac{2\mu}{r} \frac{V_n^k}{r} [-n(n+1) + 1] \right. \right. \\ & \left. \left. + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right\} \frac{ik}{\sin \theta} P_n^k - \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\mu}{r} \frac{W_n^h}{r} [-n(n+1) + 2] + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \left\{ \frac{dP_n^k}{d\theta} \right\} e^{ik\lambda} e^{i\sigma t} \\
= & \left[ \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r^2} [U_n^k - (n^2 + n - 1)V_n^k] \right. \right. \\
& + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \left\{ \frac{ik}{\sin \theta} P_n^k - \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \right. \right. \\
& \left. \left. - (n-1)(n+2) \frac{\mu}{r^2} W_n^k + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right\} \frac{dP_n^k}{d\theta} \right\} e^{ik\lambda} e^{i\sigma t}. \quad (4.471)
\end{aligned}$$

These are the formulae related to stress, which are the most tedious to derive.

By substituting (4.458)–(4.461) and (4.469)–(4.471) into (4.450)–(4.453), taking reference to (4.385) and (4.387), and with  $\nabla \cdot \vec{u} = \Delta_n^k P_n^k e^{ik\lambda} e^{i\sigma t}$ , we obtain, after eliminating  $e^{i\sigma t}$  from both sides of the equations, and adding back the summation sign over  $n$  and  $k$ ,  $\sum_{n=0}^{\infty} \sum_{k=-n}^n$ , which is written briefly as  $\sum_{n,k}$ ,

$$\begin{aligned}
& \sum_{n,k} \left[ \left( -\rho \sigma^2 U_n^k + 2k\rho \omega \sigma V_n^k \right) P_n^k + 2i\rho \omega \sigma W_n^k \sin \theta \frac{dP_n^k}{d\theta} \right] e^{ik\lambda} \\
= & \sum_{n,k} \left\{ \frac{d}{dr} \left( \lambda \Delta_n^k + 2\mu \frac{dU_n^k}{dr} \right) + \frac{\mu}{r} \left[ 4 \frac{dU_n^k}{dr} - 4 \frac{U_n^k}{r} - n(n+1) \left( \frac{dV_n^k}{dr} \right. \right. \right. \\
& \left. \left. + \frac{U_n^k - 3V_n^k}{r} \right) \right] + \rho g \Delta_n^k - \rho \frac{d}{dr} (g U_n^k) + \rho \frac{d\phi_n^k}{dr} \left\{ P_n^k e^{ik\lambda}, \quad (4.472)
\end{aligned}$$

$$\begin{aligned}
& \sum_{n,k} \left[ -\rho \sigma^2 V_n^k \frac{dP_n^k}{d\theta} - \rho \sigma^2 W_n^k \frac{ik}{\sin \theta} P_n^k + 2k\rho \omega \sigma V_n^k \cot \theta P_n^k \right. \\
& \left. + 2i\rho \omega \sigma W_n^k \cos \theta \frac{dP_n^k}{d\theta} \right] e^{ik\lambda} \\
= & \sum_{n,k} \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r^2} [U_n^k - (n^2 + n - 1)V_n^k] \right. \\
& + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) - \frac{\rho g}{r} U_n^k + \frac{\rho}{r} \phi_n^k \left\{ \frac{dP_n^k}{d\theta} e^{ik\lambda} \right. \\
& + \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] - (n-1)(n+2) \frac{\mu}{r^2} W_n^k \right. \\
& \left. \left. + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right\} \frac{ik}{\sin \theta} P_n^k e^{ik\lambda}, \quad (4.473)
\end{aligned}$$

$$\begin{aligned}
& \sum_{n,k} \left[ -\rho\sigma^2 V_n^k \frac{ik}{\sin\theta} P_n^k + \rho\sigma^2 W_n^k \frac{dP_n^k}{d\theta} + 2i\rho\omega\sigma U_n^k \sin\theta P_n^k \right. \\
& \quad \left. + 2i\rho\omega\sigma V_n^k \cos\theta \frac{dP_n^k}{d\theta} - 2k\rho\omega\sigma W_n^k \cot\theta P_n^k \right] e^{ik\lambda} \\
& = \sum_{n,k} \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r^2} \left[ U_n^k - (n^2 + n - 1)V_n^k \right] \right. \\
& \quad \left. + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) - \frac{\rho g}{r} U_n^k + \frac{\rho}{r} \phi_n^k \right\} \frac{ik}{\sin\theta} P_n^k e^{ik\lambda} \\
& \quad - \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] - (n-1)(n+2) \frac{\mu}{r^2} W_n^k \right. \\
& \quad \left. + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right\} \frac{dP_n^k}{d\theta} e^{ik\lambda}, \tag{4.474}
\end{aligned}$$

$$\sum_{n,k} \left[ \frac{d^2 \phi_n^k}{dr^2} + \frac{2}{r} \frac{d\phi_n^k}{dr} - n(n+1) \frac{\phi_n^k}{r^2} \right] P_n^k e^{ik\lambda} = 4\pi G \sum_{n,k} \left( \rho \Delta_n^k + \frac{d\rho}{dr} U_n^k \right) P_n^k e^{ik\lambda}. \tag{4.475}$$

For each of the above equations, multiply both sides by  $e^{-il\lambda}$ , and then integrate over  $\lambda$  in the domain  $[0, 2\pi]$ . According to (4.273), only the terms with  $k = l$  is non-vanishing. This removes the summation over  $k$ . After changing  $l$  back to  $k$ , these equations become

$$\begin{aligned}
& \sum_n \left[ \left( -\rho\sigma^2 U_n^k + 2k\rho\omega\sigma V_n^k \right) P_n^k + 2i\rho\omega\sigma W_n^k \sin\theta \frac{dP_n^k}{d\theta} \right] \\
& = \sum_n \left\{ \frac{d}{dr} \left( \lambda \Delta_n^k + 2\mu \frac{dU_n^k}{dr} \right) + \frac{\mu}{r} \left[ 4 \frac{dU_n^k}{dr} - 4 \frac{U_n^k}{r} - n(n+1) \left( \frac{dV_n^k}{dr} \right. \right. \right. \\
& \quad \left. \left. + \frac{U_n^k - 3V_n^k}{r} \right) \right] + \rho g \Delta_n^k - \rho \frac{d}{dr} \left( g U_n^k \right) + \rho \frac{d\phi_n^k}{dr} \right\} P_n^k, \tag{4.476}
\end{aligned}$$

$$\begin{aligned}
& \sum_n \left[ -\rho\sigma^2 V_n^k \frac{dP_n^k}{d\theta} - \rho\sigma^2 W_n^k \frac{ik}{\sin\theta} P_n^k + 2k\rho\omega\sigma V_n^k \cot\theta P_n^k \right. \\
& \quad \left. + 2i\rho\omega\sigma W_n^k \cos\theta \frac{dP_n^k}{d\theta} \right] \\
& = \sum_n \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r^2} \left[ U_n^k - (n^2 + n - 1)V_n^k \right] \right.
\end{aligned}$$



$$\begin{aligned}
& + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) - \frac{\rho g}{r} U_n^k + \frac{\rho}{r} \phi_n^k \left\} \frac{dP_n^k}{d\theta} e^{ik\lambda} + \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} \right. \right. \right. \\
& \left. \left. \left. - \frac{W_n^k}{r} \right) \right] - (n-1)(n+2) \frac{\mu}{r^2} W_n^k + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right\} \frac{ik}{\sin \theta} P_n^k, \\
\end{aligned} \tag{4.477}$$

$$\begin{aligned}
& \sum_n \left[ -\rho \sigma^2 V_n^k \frac{ik}{\sin \theta} P_n^k + \rho \sigma^2 W_n^k \frac{dP_n^k}{d\theta} + 2i\rho \omega \sigma U_n^k \sin \theta P_n^k \right. \\
& \left. + 2i\rho \omega \sigma V_n^k \cos \theta \frac{dP_n^k}{d\theta} - 2k\rho \omega \sigma W_n^k \cot \theta P_n^k \right] \\
& = \sum_n \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) \right] + \frac{\lambda}{r} \Delta_n^k + \frac{2\mu}{r^2} \left[ U_n^k - (n^2 + n - 1) V_n^k \right] \right. \\
& \left. + \frac{3\mu}{r} \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right) - \frac{\rho g}{r} U_n^k + \frac{\rho}{r} \phi_n^k \right\} \frac{ik}{\sin \theta} P_n^k - \left\{ \frac{d}{dr} \left[ \mu \left( \frac{dW_n^k}{dr} \right. \right. \right. \\
& \left. \left. \left. - \frac{W_n^k}{r} \right) \right] - (n-1)(n+2) \frac{\mu}{r^2} W_n^k + \frac{3\mu}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right\} \frac{dP_n^k}{d\theta}, \\
\end{aligned} \tag{4.478}$$

$$\sum_n \left[ \frac{d^2 \phi_n^k}{dr^2} + \frac{2}{r} \frac{d\phi_n^k}{dr} - n(n+1) \frac{\phi_n^k}{r^2} \right] P_n^k = 4\pi G \sum_n \left( \rho \Delta_n^k + \frac{d\rho}{dr} U_n^k \right) P_n^k. \tag{4.479}$$

We remark that the summation over  $n$  is for  $n = |k|, |k| + 1, \dots$ .

### 4.8.3 Reduction to Ordinary Differential Equations

Equations (4.476)–(4.479) derived in the last subsection can be further developed to remove the dependence on  $\theta$  as well as the summation over  $n$ . Customarily, the following notations are used in such formulation,<sup>3</sup>

$$y_1^n = U_n^k, \quad (u_r \text{ of the spheroidal field}); \tag{4.480}$$

$$y_2^n = \lambda \Delta_n^k + 2\mu \frac{dU_n^k}{dr}, \quad (T_{rr} \text{ of the spheroidal field}); \tag{4.481}$$

$$y_3^n = V_n^k, \quad (u_\theta \text{ and } u_\lambda \text{ of the spheroidal field}); \tag{4.482}$$

<sup>3</sup> Some authors use a different definition of  $y_6^n$ . Hence, there is always a need to verify the meaning of this symbol.

$$y_4^n = \mu \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right), \quad (T_{r\theta} \text{ and } T_{r\lambda} \text{ of the spheroidal field}); \quad (4.483)$$

$$y_5^n = \phi_n^k, \quad (\phi_f + \phi_1 \text{ of the spheroidal field}); \quad (4.484)$$

$$y_6^n = \frac{d\phi_n^k}{dr} - 4\pi G\rho U_n^k, \quad (\text{Auxiliary}); \quad (4.485)$$

$$y_7^n = iW_n^k, \quad (u_\theta \text{ and } u_\lambda \text{ of the toroidal field}); \quad (4.486)$$

$$y_8^n = i\mu \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right), \quad (T_{r\theta} \text{ and } T_{r\lambda} \text{ of the toroidal field}). \quad (4.487)$$

It is noted that the  $y_s$  are defined to represent the functions  $U_n^k$ ,  $V_n^k$ ,  $W_n^k$ , and  $\phi_n^k$ , or their combinations, thus depending on  $k$ . We have made the dependence on  $k$  implicit for brevity.

To see more clearly the meanings of the  $y_s$ , we use them to express the spherical harmonic expressions of  $\vec{u}$ ,  $\hat{e}_r \cdot \vec{T}$ , and  $\phi_f + \phi_1$ ,

$$\begin{aligned} \vec{u} = \sum_{n,k} \left[ \left( y_1^n P_n^k \hat{e}_r + y_3^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_3^n \frac{ik}{\sin\theta} P_n^k \hat{e}_\lambda \right) \right. \\ \left. + \left( y_7^n \frac{k}{\sin\theta} P_n^k \hat{e}_\theta + iy_7^n \frac{d}{d\theta} P_n^k \hat{e}_\lambda \right) \right] e^{ik\lambda} e^{i\sigma t}, \end{aligned} \quad (4.488)$$

$$\begin{aligned} \hat{e}_r \cdot \vec{T} = T_{rr} \hat{e}_r + T_{r\theta} \hat{e}_\theta + T_{r\lambda} \hat{e}_\lambda \\ = \sum_{n,k} \left[ \left( y_2^n P_n^k \hat{e}_r + y_4^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_4^n \frac{ik}{\sin\theta} P_n^k \hat{e}_\lambda \right) \right. \\ \left. + \left( y_8^n \frac{k}{\sin\theta} P_n^k \hat{e}_\theta + iy_8^n \frac{d}{d\theta} P_n^k \hat{e}_\lambda \right) \right] e^{ik\lambda} e^{i\sigma t}, \end{aligned} \quad (4.489)$$

$$\phi_f + \phi_1 = \sum_{n,k} y_5^n P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (4.490)$$

$$\frac{\partial(\phi_f + \phi_1)}{\partial r} - 4\pi G\rho(\hat{e}_r \cdot \vec{u}) = \sum_{n,k} y_6^n P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (4.491)$$

Finally, the symbol  $\Delta_n^k$  expressed in terms of the  $y_s$  is

$$\Delta_n^k = \frac{dy_1^n}{dr} + 2\frac{y_1^n}{r} - n(n+1)\frac{y_3^n}{r}. \quad (4.492)$$

Again, we emphasize that the  $y$ s depend on  $k$  implicitly, meaning that they can be different for different  $k$ . Such examples will be given in the discussion of the free oscillations in Sect. 6.4.3.

We now derive the ordinary differential equations for the functions  $y_1^n, \dots, y_8^n$ .

The first ordinary differential equation can be obtained straightforwardly by substituting (4.480) and (4.492) into the definition of  $y_2^n$ , (4.481),

$$\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r} y_1^n + \frac{1}{\lambda + 2\mu} y_2^n + \frac{n(n+1)\lambda}{(\lambda + 2\mu)r} y_3^n. \quad (4.493)$$

An new expression of  $\Delta_n^k$  for later reference can be obtained by substituting this equation into (4.492),

$$\Delta_n^k = \frac{4\mu}{\lambda + 2\mu} \frac{y_1^n}{r} + \frac{1}{\lambda + 2\mu} y_2^n - \frac{2n(n+1)\mu}{\lambda + 2\mu} \frac{y_3^n}{r}. \quad (4.494)$$

By substituting (4.480)–(4.487) into (4.476), we obtain

$$\begin{aligned} \sum_n \left[ \frac{dy_2^n}{dr} + \frac{4\mu}{r} \frac{dy_1^n}{dr} - 4\mu \frac{y_1^n}{r^2} - n(n+1) \frac{y_4^n}{r} + 2n(n+1)\mu \frac{y_3^n}{r^2} + \rho g \Delta_n^k \right. \\ \left. - \rho \frac{d}{dr} (gy_1^n) + \rho y_6^n + 4\pi G \rho^2 y_1^n + \rho \sigma 2y_1^n - 2k\rho\omega\sigma y_3^n \right] P_n^k \\ - 2\rho\omega\sigma \sum_n y_7^n \sin\theta \frac{dP_n^k}{d\theta} = 0. \end{aligned} \quad (4.495)$$

As mentioned in Sect. 4.3.2, we have, according to the recurrence formulae of (4.136),

$$\sin\theta \frac{dP_n^k}{d\theta} = \frac{n(n-k+1)}{2n+1} P_{n+1}^k - \frac{(n+1)(n+k)}{2n+1} P_{n-1}^k. \quad (4.496)$$

Hence,

$$\begin{aligned} \sum_n y_7^n \sin\theta \frac{dP_n^k}{d\theta} &= \sum_n \left[ \frac{n(n-k+1)}{2n+1} y_7^n P_{n+1}^k - \frac{(n+1)(n+k)}{2n+1} y_7^n P_{n-1}^k \right] \\ &= \sum_n \left[ \frac{(n-1)(n-k)}{2n-1} y_7^{n-1} - \frac{(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right] P_n^k. \end{aligned} \quad (4.497)$$

After substituting this formula into (4.495), we can eliminate  $P_n^k$  using its orthogonality to obtain an equation that is no longer a summation over  $n$ ,

$$\begin{aligned}
& \frac{dy_2^n}{dr} + \frac{4\mu}{r} \frac{dy_1^n}{dr} - 4\mu \frac{y_1^n}{r^2} - n(n+1) \frac{y_4^n}{r} + 2n(n+1)\mu \frac{y_3^n}{r^2} + \rho g \Delta_n^k \\
& - \rho \frac{d}{dr} (gy_1^n) + \rho y_6^n + 4\pi G \rho^2 y_1^n + \rho \sigma^2 y_1^n - 2k\rho\omega\sigma y_3^n \\
& - 2\rho\omega\sigma \left[ \frac{(n-1)(n-k)}{2n-1} y_7^{n-1} - \frac{(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right] = 0.
\end{aligned} \tag{4.498}$$

The term  $d(gy_1^n)/dr$  can be further developed using (4.439) as

$$\frac{d}{dr} (gy_1^n) = g \frac{dy_1^n}{dr} + \frac{dg}{dr} y_1^n = g \frac{dy_1^n}{dr} + \left( 4\pi G \rho - \frac{2g}{r} \right) y_1^n. \tag{4.499}$$

By substituting (4.494), (4.499), and (4.493) into (4.498), we obtain the second ordinary differential equation,

$$\begin{aligned}
\frac{dy_2^n}{dr} = & \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda+2\mu)r} y_2^n \\
& + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} + 2k\rho\omega\sigma \right] y_3^n + \frac{n(n+1)}{r} y_4^n \\
& - \rho y_6^n + 2\rho\omega\sigma \left[ \frac{(n-1)(n-k)}{2n-1} y_7^{n-1} - \frac{(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right].
\end{aligned} \tag{4.500}$$

The third ordinary differential equation can be obtained by substituting (4.480) and (4.482) into (4.483),

$$\frac{dy_3^n}{dr} = -\frac{1}{r} y_1^n + \frac{1}{r} y_3^n + \frac{1}{\mu} y_4^n. \tag{4.501}$$

By substituting (4.480)–(4.487) into (4.473) and (4.474), we obtain

$$\begin{aligned}
& \sum_n \left[ \frac{dy_4^n}{dr} + \frac{\lambda}{r} \Delta_n^k + 2\mu \frac{y_1^n}{r^2} - 2(n^2+n-1)\mu \frac{y_3^n}{r^2} + 3 \frac{y_4^n}{r} - \frac{\rho g}{r} y_1^n + \frac{\rho}{r} y_5^n \right. \\
& \quad \left. + \rho\sigma^2 y_3^n \right] \frac{dP_n^k}{d\theta} + \sum_n \left[ \frac{dy_8^n}{dr} - (n-1)(n+2)\mu \frac{y_7^n}{r^2} + 3 \frac{y_8^n}{r} + \rho\sigma^2 y_7^n \right] \frac{k}{\sin\theta} P_n^k \\
& - \sum_n \left( 2k\rho\omega\sigma y_3^n \cot\theta P_n^k + 2\rho\omega\sigma y_7^n \cos\theta \frac{dP_n^k}{d\theta} \right) = 0,
\end{aligned} \tag{4.502}$$

$$\sum_n \left[ \frac{dy_4^n}{dr} + \frac{\lambda}{r} \Delta_n^k + 2\mu \frac{y_1^n}{r^2} - 2(n^2+n-1)\mu \frac{y_3^n}{r^2} + 3 \frac{y_4^n}{r} - \frac{\rho g}{r} y_1^n + \frac{\rho}{r} y_5^n \right]$$

$$\begin{aligned}
& + \rho \sigma^2 y_3^n \left[ \frac{k}{\sin \theta} P_n^k + \sum_n \left[ \frac{dy_8^n}{dr} - (n-1)(n+2) \mu \frac{y_7^n}{r^2} + 3 \frac{y_8^n}{r} + \rho \sigma^2 y_7^n \right] \frac{dP_n^k}{d\theta} \right. \\
& \left. - \sum_n (2\rho \omega \sigma y_1^n \sin \theta P_n^k + 2\rho \omega \sigma y_3^n \cos \theta \frac{dP_n^k}{d\theta} + 2k\rho \omega \sigma y_7^n \cot \theta P_n^k) = 0. \right. \\
& \qquad \qquad \qquad (4.503)
\end{aligned}$$

Two ordinary differential equations can be derived from these two equations. We first derive one of them, and postpone the derivation of the other as the last ordinary differential equation.

Multiply both sides of (4.502) by  $\sin \theta$ , take the derivative with respect to  $\theta$ , and then subtract the product of (4.503) with  $k$ . We obtain

$$\begin{aligned}
& \sum_n \left[ \frac{dy_4^n}{dr} + \frac{\lambda}{r} \Delta_n^k + 2\mu \frac{y_1^n}{r^2} - 2(n^2 + n - 1) \mu \frac{y_3^n}{r^2} + 3 \frac{y_4^n}{r} - \frac{\rho g}{r} y_1^n \right. \\
& \left. + \frac{\rho}{r} y_5^n + \rho \sigma^2 y_3^n \right] \left( \sin \theta \frac{d^2 P_n^k}{d\theta^2} + \cos \theta \frac{dP_n^k}{d\theta} - \frac{k^2}{\sin^2 \theta} P_n^k \right) \\
& + \sum_n \left\{ 2k\rho \omega \sigma y_1^n \sin \theta P_n^k + 2k\rho \omega \sigma y_3^n \sin \theta P_n^k - 2\rho \omega \sigma y_7^n \left[ \sin \theta \cos \theta \frac{d^2 P_n^k}{d\theta^2} \right. \right. \\
& \left. \left. + (\cos^2 \theta - \sin^2 \theta) \frac{dP_n^k}{d\theta} - k^2 \cot \theta P_n^k \right] \right\} = 0. \qquad (4.504)
\end{aligned}$$

By dividing both sides by  $\sin \theta$ , and then using (4.381), we obtain

$$\begin{aligned}
& \sum_n \left[ \frac{dy_4^n}{dr} + \frac{\lambda}{r} \Delta_n^k + 2\mu \frac{y_1^n}{r^2} - 2(n^2 + n - 1) \mu \frac{y_3^n}{r^2} + 3 \frac{y_4^n}{r} - \frac{\rho g}{r} y_1^n \right. \\
& \left. + \frac{\rho}{r} y_5^n + \rho \sigma^2 y_3^n \right] [-n(n+1) P_n^k + \sum_n (2k\rho \omega \sigma y_1^n + 2k\rho \omega \sigma y_3^n) P_n^k \\
& + 2\rho \omega \sigma \sum_n \left[ n(n+1) y_7^n \cos \theta P_n^k + y_7^n \sin \theta \frac{dP_n^k}{d\theta} \right] = 0. \qquad (4.505)
\end{aligned}$$

The term  $y_7^n \sin \theta (dP_n^k/d\theta)$  has already been expressed using  $P_n^k$  in (4.497). Here we further express  $n(n+1) y_7^n \cos \theta P_n^k$  using  $P_n^k$ . Again, as mentioned in Sect. 4.3.2, we have, according to the recurrence formula (4.132),

$$\cos \theta P_n^k = \frac{n-k+1}{2n+1} P_{n+1}^k + \frac{n+k}{2n+1} P_{n-1}^k. \qquad (4.506)$$

Hence,

$$\begin{aligned}
 & \sum_n n(n+1)y_7^n \cos \theta P_n^k \\
 &= \sum_n \left[ \frac{n(n+1)(n-k+1)}{2n+1} y_7^n P_{n+1}^k + \frac{n(n+1)(n+k)}{2n+1} y_7^n P_{n-1}^k \right] \\
 &= \sum_n \left[ \frac{n(n-1)(n-k)}{2n-1} y_7^{n-1} + \frac{(n+1)(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right] P_n^k.
 \end{aligned} \tag{4.507}$$

After substituting (4.497) and (4.507) into (4.505), we can eliminate  $P_n^k$  using its orthogonality to obtain an equation that is no longer a summation over  $n$ ,

$$\begin{aligned}
 \frac{dy_4^n}{dr} &= \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} + \frac{2k\rho\omega\sigma}{n(n+1)} \right] y_1^n - \frac{\lambda}{(\lambda+2\mu)r} y_2^n + \left\{ -\rho\sigma^2 \right. \\
 &\quad \left. + \frac{2\mu}{(\lambda+2\mu)r^2} \left[ (2n^2+2n-1)\lambda + 2(n^2+n-1)\mu \right] + \frac{2k\rho\omega\sigma}{n(n+1)} \right\} y_3^n - \frac{3}{r} y_4^n \\
 &\quad - \frac{\rho}{r} y_5^n + \frac{2\rho\omega\sigma}{n(n+1)} \left[ \frac{(n^2-1)(n-k)}{2n-1} y_7^{n-1} + \frac{n(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right].
 \end{aligned} \tag{4.508}$$

This is our fourth ordinary differential equation.

The fifth ordinary differential equation can be readily obtained by substituting (4.480) and (4.484) into (4.485),

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n. \tag{4.509}$$

The sixth ordinary differential equation can be obtained from (4.479) using (4.484), (4.485), (4.494), and (4.509),

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r} y_3^n + \frac{n(n+1)}{r^2} y_5^n - \frac{2}{r} y_6^n. \tag{4.510}$$

The seventh ordinary differential equation can be obtained by substituting (4.486) into (4.487),

$$\frac{dy_7^n}{dr} = \frac{1}{r} y_7^n + \frac{1}{\mu} y_8^n. \tag{4.511}$$

Finally, we derive the eighth ordinary differential equation from (4.502) and (4.503). Multiply both sides of (4.503) by  $\sin \theta$ , take the derivative with

respect to  $\theta$ , and then subtract the product of (4.502) with  $k$ . We obtain

$$\begin{aligned} \sum_n \left[ \frac{dy_8^n}{dr} - (n-1)(n+2)\mu \frac{y_7^n}{r^2} + 3\frac{y_8^n}{r} + \rho\sigma^2 y_7^n \right] & \left( \sin\theta \frac{d^2 P_n^k}{d\theta^2} + \cos\theta \frac{dP_n^k}{d\theta} \right. \\ & \left. - \frac{k^2}{\sin\theta} P_n^k \right) + \sum_n \left\{ -2\rho\omega\sigma y_1^n \left( \sin^2\theta \frac{dP_n^k}{d\theta} + 2\sin\theta \cos\theta P_n^k \right) \right. \\ & \left. - 2\rho\omega\sigma y_3^n \left[ \sin\theta \cos\theta \frac{d^2 P_n^k}{d\theta^2} + (\cos^2\theta - \sin^2\theta) \frac{dP_n^k}{d\theta} - k^2 \cot\theta P_n^k \right] \right. \\ & \left. + 2k\rho\omega\sigma y_7^n \sin\theta P_n^k \right\} = 0. \end{aligned} \quad (4.512)$$

By multiplying both sides by  $\sin\theta$  and then using (4.381), we obtain

$$\begin{aligned} \sum_n \left\{ \left[ \frac{dy_8^n}{dr} - (n-1)(n+2)\mu \frac{y_7^n}{r^2} + 3\frac{y_8^n}{r} + \rho\sigma^2 y_7^n \right] [-n(n+1)] + 2k\rho\omega\sigma y_7^n \right\} & P_n^k \\ + 2\rho\omega\sigma \sum_n \left\{ - \left( y_1^n \sin\theta \frac{dP_n^k}{d\theta} + 2y_1^n \cos\theta P_n^k \right) \right. \\ \left. + \left[ n(n+1)y_3^n \cos\theta P_n^k + y_3^n \sin\theta \frac{dP_n^k}{d\theta} \right] \right\} & = 0. \end{aligned} \quad (4.513)$$

The terms  $y_1^n \sin\theta (dP_n^k/d\theta)$  and  $y_3^n \sin\theta (dP_n^k/d\theta)$  can be expressed in similar forms as (4.497), the term  $n(n+1)y_3^n \cos\theta P_n^k$  can be expressed in a similar form as (4.507), and we can further obtain, according to (4.506),

$$\begin{aligned} \sum_n y_1^n \cos\theta P_n^k &= \sum_n \left( \frac{n-k+1}{2n+1} y_1^n P_{n+1}^k + \frac{n+k}{2n+1} y_1^n P_{n-1}^k \right) \\ &= \sum_n \left( \frac{n-k}{2n-1} y_1^{n-1} + \frac{n+k+1}{2n+3} y_1^{n+1} \right) P_n^k. \end{aligned} \quad (4.514)$$

With these terms substituted by their expressions in terms of  $P_n^k$ , we can develop (4.513) to the eighth ordinary differential equation

$$\begin{aligned} \frac{dy_8^n}{dr} &= \left[ -\rho\sigma^2 + \frac{(n-1)(n+2)\mu}{r^2} + \frac{2k\rho\omega\sigma}{n(n+1)} \right] y_7^n - \frac{3}{r^2} y_8^n \\ &+ \frac{2\rho\omega\sigma}{n(n+1)} \left[ -\frac{(n+1)(n-k)}{2n-1} y_1^{n-1} + \frac{(n^2-1)(n-k)}{2n-1} y_3^{n-1} \right. \end{aligned}$$

$$+ \frac{n(n+k+1)}{2n+3} y_1^{n+1} + \frac{n(n+2)(n+k+1)}{2n+3} y_3^{n+1} \Big], \quad (4.515)$$

where  $P_n^k$  is eliminated using its orthogonality.

We have up to present, derived all the ordinary differential equations for the  $y_s$ , which are (4.493), (4.500), (4.501), (4.508), (4.509), (4.510), (4.511), and (4.515). These eight ordinary differential equations form a system, which are gathered together here,

$$\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda+2\mu)r} y_1^n + \frac{1}{\lambda+2\mu} y_2^n + \frac{n(n+1)\lambda}{(\lambda+2\mu)r} y_3^n, \quad (4.516a)$$

$$\begin{aligned} \frac{dy_2^n}{dr} = & \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda+2\mu)r} y_2^n \\ & + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} + 2k\rho\omega\sigma \right] y_3^n + \frac{n(n+1)}{r} y_4^n \\ & - \rho y_6^n + 2\rho\omega\sigma \left[ \frac{(n-1)(n-k)}{2n-1} y_7^{n-1} - \frac{(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right], \end{aligned} \quad (4.516b)$$

$$\frac{dy_3^n}{dr} = -\frac{1}{r} y_1^n + \frac{1}{r} y_3^n + \frac{1}{\mu} y_4^n, \quad (4.516c)$$

$$\begin{aligned} \frac{dy_4^n}{dr} = & \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} + \frac{2k\rho\omega\sigma}{n(n+1)} \right] y_1^n - \frac{\lambda}{(\lambda+2\mu)r} y_2^n + \left\{ -\rho\sigma^2 \right. \\ & + \frac{2\mu}{(\lambda+2\mu)r^2} [(2n^2+2n-1)\lambda + 2(n^2+n-1)\mu] + \frac{2k\rho\omega\sigma}{n(n+1)} \Big\} y_3^n - \frac{3}{r} y_4^n \\ & - \frac{\rho}{r} y_5^n + \frac{2\rho\omega\sigma}{n(n+1)} \left[ \frac{(n^2-1)(n-k)}{2n-1} y_7^{n-1} + \frac{n(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right], \end{aligned} \quad (4.516d)$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \quad (4.516e)$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r} y_3^n + \frac{n(n+1)}{r^2} y_5^n - \frac{2}{r} y_6^n, \quad (4.516f)$$

$$\frac{dy_7^n}{dr} = \frac{1}{r} y_7^n + \frac{1}{\mu} y_8^n, \quad (4.516g)$$

$$\begin{aligned} \frac{dy_8^n}{dr} = & \left[ -\rho\sigma^2 + \frac{(n-1)(n+2)\mu}{r^2} + \frac{2k\rho\omega\sigma}{n(n+1)} \right] y_7^n - \frac{3}{r} y_8^n \\ & + \frac{2\rho\omega\sigma}{n(n+1)} \left\{ -\frac{(n+1)(n-k)}{2n+1} \left[ y_1^{n-1} - (n-1)y_3^{n-1} \right] \right. \end{aligned}$$



$$+ \frac{n(n+k+1)}{2n+3} \left[ y_1^{n+1} + (n+2)y_3^{n+1} \right] \Big\}. \quad (4.516h)$$

These equations are for the degree  $n$  deformation described by  $y_1^n, \dots, y_8^n$ . However, the degrees  $n-1$  and  $n+1$  are coupled to them. This is actually a chain of mutually coupled equations, and it must be truncated for a solution. More concretely speaking, as  $y_1^n, \dots, y_6^n$  represent the spheroidal deformation, and  $y_7^n$  and  $y_8^n$  represent the toroidal deformation, the coupling is that the spheroidal deformation of a certain degree produces toroidal deformations of one degree higher and one degree lower through coupling, and that the toroidal deformation of a certain degree produces spheroidal deformations of one degree higher and one degree lower through coupling. The order  $k$  is the same in all the equations.

In practical applications, the Coriolis term is often neglected, or corrected for, and thus we set  $\omega = 0$  in the equations. In this case, the equations for  $y_1^n, \dots, y_6^n$  and those for  $y_7^n$  and  $y_8^n$  are decoupled. The equations for the spheroidal deformation are

$$\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda+2\mu)r}y_1^n + \frac{1}{\lambda+2\mu}y_2^n + \frac{n(n+1)\lambda}{(\lambda+2\mu)r}y_3^n, \quad (4.517a)$$

$$\begin{aligned} \frac{dy_2^n}{dr} = & \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda+2\mu)r}y_2^n \\ & + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} \right] y_3^n + \frac{n(n+1)}{r}y_4^n \\ & - \rho y_6^n, \end{aligned} \quad (4.517b)$$

$$\frac{dy_3^n}{dr} = -\frac{1}{r}y_1^n + \frac{1}{r}y_3^n + \frac{1}{\mu}y_4^n, \quad (4.517c)$$

$$\begin{aligned} \frac{dy_4^n}{dr} = & \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda+2\mu)}{(\lambda+2\mu)r^2} \right] y_1^n - \frac{\lambda}{(\lambda+2\mu)r}y_2^n + \left\{ -\rho\sigma^2 \right. \\ & + \frac{2\mu}{(\lambda+2\mu)r^2} \left[ (2n^2+2n-1)\lambda + 2(n^2+n-1)\mu \right] \Big\} y_3^n \\ & - \frac{3}{r}y_4^n - \frac{\rho}{r}y_5^n, \end{aligned} \quad (4.517d)$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \quad (4.517e)$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r}y_3^n + \frac{n(n+1)}{r^2}y_5^n - \frac{2}{r}y_6^n. \quad (4.517f)$$

The equations for the toroidal deformation are

$$\begin{cases} \frac{dy_7^n}{dr} = \frac{1}{r}y_7^n + \frac{1}{\mu}y_8^n, \\ \frac{dy_8^n}{dr} = \left[ -\rho\sigma^2 + \frac{(n-1)(n+2)\mu}{r^2} \right] y_7^n - \frac{3}{r}y_8^n. \end{cases} \quad (4.518)$$

We see that a change of the gravitational field is associated with the spheroidal deformation, but not with the toroidal deformation. These two systems of equations no longer depend on the order  $k$ .

Equations (4.517) and (4.518) are quite standard in the literature, which were originally credited to Alterman et al. (1959). To the author's knowledge, Eq. (4.516) was provided only in Crossley and Rochester (1980).

#### 4.8.4 Boundary Conditions

First, we provide a summary of the boundary conditions derived in Sect. 3.7.

The condition of displacement across welded boundaries is

$$\{\vec{u}\}_{\pm}^+ = 0. \quad (4.519)$$

The condition of displacement across slip boundaries is

$$\{\hat{n} \cdot \vec{u}\}_{\pm}^+ = 0. \quad (4.520)$$

A condition of the stress holds across all boundaries is

$$\{\hat{n} \cdot \vec{T}\}_{\pm}^+ = 0. \quad (4.521)$$

When applied to compute the global deformation of the Earth, the only slip boundaries in the Earth's interior are the inner core boundary (ICB) and core-mantle boundary (CMB). These are both frictionless slip boundaries, where a second condition of stress,  $\hat{n} \cdot \vec{T} = [\hat{n} \cdot (\hat{n} \cdot \vec{T})]\hat{n}$ , should hold. However, this identity is automatically satisfied at the side of the fluid outer core, and its satisfaction at the solid side is ensured by the continuity (4.521). Hence, it is not discussed further. On the Earth's surface, we assumed that there is no constraint for the displacement  $\vec{u}$ , and the condition for the stress  $\vec{T}$  is

$$(\hat{n} \cdot \vec{T})_{S_0} = \vec{t}_0, \quad (4.522)$$

where  $\vec{t}_0$  is a known traction exerted on the Earth's surface. The conditions for the gravitational potential across all kinds of boundaries are

$$\{\phi_1\}_-^+ = 0, \quad (4.523)$$

$$\left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right\}_-^+ = 0. \quad (4.524)$$

We assume that  $\phi_f$  as well as its first order derivative is continuous within whole the Earth's interior. In this way,  $\phi_1$  can be replaced by  $\phi_f + \phi_1$  in (4.523) and (4.524). The subject of this subsection is to express the boundary conditions summarized above in terms of the functions  $y_1^n, \dots, y_8^n$ .

For welded boundaries, we can use (4.488) to write the condition of displacement  $\vec{u}$ , (4.519), in terms of the components as

$$\{u_r\}_-^+ = \left\{ \sum_{n=0}^{\infty} \sum_{k=-n}^n y_1^n P_n^k e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0, \quad (4.525)$$

$$\{u_\theta\}_-^+ = \left\{ \sum_{n=0}^{\infty} \sum_{k=-n}^n \left( y_3^n \frac{dP_n^k}{d\theta} + y_7^n \frac{k}{\sin\theta} P_n^k \right) e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0, \quad (4.526)$$

$$\{u_\lambda\}_-^+ = \left\{ \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ i \left( y_3^n \frac{k}{\sin\theta} P_n^k + y_7^n \frac{dP_n^k}{d\theta} \right) \right] e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0. \quad (4.527)$$

It can be readily seen that  $e^{i\sigma t}$  can be eliminated. Furthermore,  $e^{ik\lambda}$  as well as the summation sign over  $k$  can be eliminated using the orthogonality of  $e^{ik\lambda}$ . We have, by eliminating also the factor  $i$  in the last equation,

$$\left\{ \sum_n y_1^n P_n^k \right\}_-^+ = 0, \quad (4.528)$$

$$\left\{ \sum_n \left( y_3^n \frac{dP_n^k}{d\theta} + y_7^n \frac{k}{\sin\theta} P_n^k \right) \right\}_-^+ = 0, \quad (4.529)$$

$$\left\{ \sum_n \left( y_3^n \frac{k}{\sin\theta} P_n^k + y_7^n \frac{dP_n^k}{d\theta} \right) \right\}_-^+ = 0. \quad (4.530)$$

Similar to the derivation of (4.505) and (4.513) from (4.502) and (4.503), we can obtain from (4.529) and (4.530),

$$\left\{ \sum_n y_3^n [-n(n+1)] P_n^k \right\}_-^+ = 0, \quad (4.531)$$

$$\left\{ \sum_n y_7^n [-n(n+1)] P_n^k \right\}_-^+ = 0. \quad (4.532)$$

Equations (4.528), (4.531), and (4.532) imply that (by eliminating  $P_n^k$  and the summation sign over  $n$  using the orthogonality of  $P_n^k$ )

$$\{y_1^n\}_-^+ = 0, \quad (4.533)$$

$$\{y_3^n\}_-^+ = 0, \quad (4.534)$$

$$\{y_7^n\}_-^+ = 0. \quad (4.535)$$

These are the results for the displacement across welded boundaries.

For the stress, as our Earth model is spherically symmetrical, we have  $\hat{n} = \hat{e}_r$ . Hence, we can use (4.489) to write the condition across a welded boundary, (4.521), as

$$\{T_{rr}\}_-^+ = \left\{ \sum_{n,k} y_2^n P_n^k e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0, \quad (4.536)$$

$$\{T_{r\theta}\}_-^+ = \left\{ \sum_{n,k} \left( y_4^n \frac{dP_n^k}{d\theta} + y_8^n \frac{k}{\sin\theta} P_n^k \right) e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0, \quad (4.537)$$

$$\{T_{r\lambda}\}_-^+ = \left\{ \sum_{n,k} \left[ i \left( y_4^n \frac{k}{\sin\theta} P_n^k + y_8^n \frac{dP_n^k}{d\theta} \right) \right] e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0. \quad (4.538)$$

These equations are similar to (4.525)–(4.527). Hence, we can similarly obtain

$$\{y_2^n\}_-^+ = 0, \quad (4.539)$$

$$\{y_4^n\}_-^+ = 0, \quad (4.540)$$

$$\{y_8^n\}_-^+ = 0. \quad (4.541)$$

These are results for the stress across welded boundaries.

For the gravitational potential, as already mentioned, we assume that  $\phi_f$  as well as its first order derivative is continuous within whole the Earth's interior, and thus,  $\phi_1$  can be replaced by  $\phi_f + \phi_1$  in (4.523) and (4.524). As we have  $\hat{n} = \hat{e}_r$ , we can use (4.490) and (4.491) to write (4.523) and (4.524) as

$$\left\{ \sum_{n,k} y_5^n P_n^k e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0, \quad (4.542)$$

$$\left\{ \sum_{n,k} y_6^n P_n^k e^{ik\lambda} e^{i\sigma t} \right\}_-^+ = 0. \quad (4.543)$$

These yield

$$\{y_5^n\}_-^+ = 0, \quad (4.544)$$

$$\{y_6^n\}_-^+ = 0, \quad (4.545)$$

which are the results for the gravitational potential across all boundaries.

As a summary, the functions  $y_1^n, \dots, y_8^n$  are all continuous across welded boundaries.

For internal frictionless slip boundaries, the condition of the displacement (4.520) can be readily reduced to

$$\{y_1^n\}_-^+ = 0, \quad (4.546)$$

and the conditions for the stress and the gravitational potential are the same as those for the welded boundaries, i.e.,

$$\{y_2^n\}_-^+ = 0, \quad (4.547)$$

$$\{y_4^n\}_-^+ = 0, \quad (4.548)$$

$$\{y_8^n\}_-^+ = 0, \quad (4.549)$$

$$\{y_5^n\}_-^+ = 0, \quad (4.550)$$

$$\{y_6^n\}_-^+ = 0. \quad (4.551)$$

These mean that no condition is applied to  $y_3^n$  and  $y_7^n$ , and the rest of the functions  $y_1^n, \dots, y_8^n$  are continuous.

On the Earth's surface, we assume that no condition for displacement is applied. As  $\hat{n} = \hat{e}_r$ , we use (4.489) to write the condition for the stress, (4.522), as

$$\left\{ \sum_{n,k} y_2^n P_n^k e^{ik\lambda} e^{i\sigma t} \right\}_{r=R} = t_{0r}, \quad (4.552)$$

$$\left\{ \sum_{n,k} \left( y_4^n \frac{dP_n^k}{d\theta} + y_8^n \frac{k}{\sin\theta} P_n^k \right) e^{ik\lambda} e^{i\sigma t} \right\}_{r=R} = t_{0\theta}, \quad (4.553)$$

$$\left\{ \sum_{n,k} \left[ i \left( y_4^n \frac{k}{\sin\theta} P_n^k + y_8^n \frac{dP_n^k}{d\theta} \right) \right] e^{ik\lambda} e^{i\sigma t} \right\}_{r=R} = t_{0\lambda}, \quad (4.554)$$

where  $t_{0r}$ ,  $t_{0\theta}$ , and  $t_{0\lambda}$  are the components of  $\vec{t}_0$ . By expressing the vector  $\vec{t}_0$  in spherical harmonic series in the form of

$$t_{0r} = \sum_{n,k} (t_{0r}^S)_n^k P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (4.555)$$

$$t_{0\theta} = \sum_{n,k} \left[ (t_{0r}^S)_n^k \frac{dP_n^k}{d\theta} + (t_{0t}^T)_n^k \frac{k}{\sin\theta} P_n^k \right] e^{ik\lambda} e^{i\sigma t}, \quad (4.556)$$

$$t_{0\lambda} = \sum_{n,k} \left\{ i \left[ (t_{0r}^S)_n^k \frac{k}{\sin\theta} P_n^k + (t_{0t}^T)_n^k \frac{dP_n^k}{d\theta} \right] \right\} e^{ik\lambda} e^{i\sigma t}, \quad (4.557)$$

where  $(t_{0r}^S)_n^k$  and  $(t_{0t}^T)_n^k$  represent the spheroidal field, and  $(t_{0t}^T)_n^k$  the toroidal field. We can obtain, by substituting (4.555) to (4.557) into (4.552)–(4.554), respectively,

$$y_2^n(R) = (t_{0r}^S)_n^k, \quad (4.558)$$

$$y_4^n(R) = (t_{0t}^S)_n^k, \quad (4.559)$$

$$y_8^n(R) = (t_{0t}^T)_n^k, \quad (4.560)$$

where  $y_2^n(R)$ ,  $y_4^n(R)$ , and  $y_8^n(R)$  are the values of  $y_2^n$ ,  $y_4^n$ , and  $y_8^n$ , respectively, on the Earth's surface considered as a sphere of radius  $R$ .

The formulation of the boundary condition for the gravitational potential on the Earth's surface requires some more attention. The first thing we should notice is that the gravitational potential outside the Earth is meaningful, while the other quantities are meaningless. We use '+' to denote the outer side of the Earth's surface, and '−' the inner side. According to (4.542) and (4.543), we have, at the inner side,

$$(\phi_1)_{r=R}^- = (\phi_f + \phi_1)_{r=R}^- - (\phi_f)_{r=R}^- = \sum_{n,k} y_5^n(R) P_n^k e^{ik\lambda} e^{i\sigma t} - (\phi_f)_{r=R}^-, \quad (4.561)$$

$$\begin{aligned} \left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right\}_{r=R}^- &= \left\{ \left[ \frac{\partial(\phi_f + \phi_1)}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right] - \frac{\partial \phi_f}{\partial n} \right\}_{r=R}^- \\ &= \sum_{n,k} y_6^n(R) P_n^k e^{ik\lambda} e^{i\sigma t} - \left( \frac{\partial \phi_f}{\partial r} \right)_{r=R}^-. \end{aligned} \quad (4.562)$$

where  $y_5^n(R)$  and  $y_6^n(R)$  are the values of  $y_5^n$  and  $y_6^n$  on the Earth's surface. We have assumed  $\phi_f$  to be a harmonic function in the interior of the Earth. Hence, it can be expressed as a spherical harmonic series in the form of

$$\phi_f = \sum_{n,k} \phi_{fn}^k \left( \frac{r}{R} \right)^n P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (4.563)$$

By substituting this formula into (4.561) and (4.562), we obtain

$$(\phi_1)_{r=R}^- = \sum_{n,k} \left[ y_5^n(R) - \phi_{fn}^k \right] P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (4.564)$$

$$\left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right\}_{r=R}^- = \sum_{n,k} \left[ y_6^n(R) - \frac{n}{R} \phi_{fn}^k \right] P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (4.565)$$

As  $\phi_1$  is harmonic outside the Earth, it can be obtained based on its continuity (4.523) and its value at the inner side of the Earth's surface given by (4.564),

$$(\phi_1)^+ = \sum_{n,k} \left[ y_5^n(R) - \phi_{fn}^k \right] \left( \frac{R}{r} \right)^{n+1} P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (4.566)$$

Since the density  $\rho$  vanishes outside the Earth, we can further obtain,

$$\begin{aligned} \left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right\}_{r=R}^+ \\ = \left( \frac{\partial \phi_1}{\partial r} \right)_{r=R}^+ = - \sum_{n,k} \frac{n+1}{R} \left[ y_5^n(R) - \phi_{fn}^k \right] P_n^k e^{ik\lambda} e^{i\sigma t}. \end{aligned} \quad (4.567)$$

By substituting (4.565) and (4.567) into the boundary condition (4.524), we finally obtain

$$y_6^n(R) - \frac{n}{R} \phi_{fn}^k = - \frac{n+1}{R} \left[ y_5^n(R) - \phi_{fn}^k \right], \quad (4.568)$$

which can be written in the more brief form

$$\frac{n+1}{R} y_5^n(R) + y_6^n(R) = \frac{2n+1}{R} \phi_{fn}^k. \quad (4.569)$$

This is the boundary condition of the gravitational potential on the Earth's surface.

In summary, the boundary conditions on the Earth's surface are (4.558), (4.559), (4.560), and (4.569), where terms depending on the order  $k$  are present. We remark that the system of ordinary differential equations (4.516) does depend on  $k$ , but not its degenerate forms (4.517) and (4.518). These could be simplified in the applications to real problems.

We now assume that the Earth is completely solid. We have 8 ordinary differential equations for 8 functions to be solved, but we have only 4 boundary conditions on the Earth's surface. Therefore, we need 4 more boundary conditions for the solution to be uniquely determined. These conditions are actually implicitly included in the ordinary differential equations themselves at the Earth center. For the solutions to be finite, thus being physically acceptable, the functions must satisfy 4 conditions at the Earth center  $r = 0$ , called regularity boundary conditions.

We now show that 4 regularity boundary conditions are present at the Earth center provided that the Earth's inner-most layer is solid, which is indeed the case. By multiplying (4.516a) and (4.516c) by  $r$ , and then setting  $r = 0$ , we obtain  $-[2\lambda/(\lambda +$

$2\mu)y_1^n(0) + [n(n+1)\lambda/(\lambda+2\mu)]y_3^n(0) = 0$  and  $-y_1^n(0) + y_3^n(0) = 0$ , where ‘(0)’ implies when  $r = 0$ . These two identities lead to  $y_1^n(0) = y_3^n(0) = 0$  when  $n \neq 1$ , but only  $-y_1^n(0) + y_3^n(0) = 0$  when  $n = 1$ . Multiplying (4.516b) and (4.516d) by  $r^2$  and then setting  $r = 0$  do not lead to new conditions. By multiplying (4.516f) by  $r^2$ , and then setting  $r = 0$ , we obtain  $y_5^n(0) = 0$  when  $n \neq 0$ . By multiplying (4.516g) by  $r$ , and then setting  $r = 0$ , we obtain  $y_7^n(0) = 0$ . Thus, we have all the conditions when  $n \geq 2$ , and the cases  $n = 0$  and  $n = 1$  are in need of special treatment. When  $n = 0$ ,  $y_1^0(0) = y_3^0(0) = 0$ , and  $y_7^0(0) = 0$  still hold, and the multiplication of (4.516f) by  $r$ , and then setting  $r = 0$  yields  $y_6^0(0) = 0$ . When  $n = 1$ , we have  $-y_1^1(0) + y_3^1(0) = 0$ ,  $y_5^1(0) = 0$ , and  $y_7^1(0) = 0$ , and another condition is required. Multiplying (4.516b) and (4.516d) by  $r^2$  and then setting  $r = 0$  lead to  $-y_1^1(0) + y_3^1(0) = 0$  again. Hence, we attempt to eliminate the terms on the order of  $1/r^2$  by adding the product of (4.516d) and 2 to (4.516b), leading to, with  $n = 1$  and keeping terms on the order of  $1/r$  only at the right-hand side,

$$\begin{aligned} \frac{d}{dr} (y_2^1 + 2y_4^1) = & - \left( \rho\sigma^2 + \frac{2\rho g}{r} \right) y_1^1 + \left( -2\rho\sigma^2 + \frac{2\rho g}{r} \right) y_3^1 \\ & - \frac{2}{r} (y_2^1 + 2y_4^1) - \frac{2\rho}{r} y_5^1 - \rho y_6^1 + \dots, \end{aligned} \quad (4.570)$$

where ‘ $\dots$ ’ represents the terms arisen from the coupling between the spheroidal and toroidal fields through the angular velocity  $\omega$  of Earth rotation, which evidently vanish when  $r \rightarrow 0$ . Multiply both sides of (4.570) by  $r$ , and then set  $r = 0$ . As  $g(0) = 0$  and  $y_5^1(0) = 0$ , we obtain  $y_2^1(0) + 2y_4^1(0) = 0$ , which is the last boundary condition we are seeking. In summary, the regularity boundary conditions at  $r = 0$  are

$$\begin{cases} n \geq 2: & y_1^n(0) = 0, & y_3^n(0) = 0, & y_5^n(0) = 0, & y_7^n(0) = 0; \\ n = 0: & y_1^0(0) = 0, & y_3^0(0) = 0, & y_6^0(0) = 0, & y_7^0(0) = 0; \\ n = 1: & -y_1^1(0) + y_3^1(0) = 0, & y_2^1(0) + 2y_4^1(0) = 0, & y_5^1(0) = 0, & y_7^1(0) = 0. \end{cases} \quad (4.571)$$

The readers are suggested to compare (4.570) with the consistency relation in the formulation of the degree 1 static load Love numbers in Sect. 7.6.3. For the case of degree 1, it is more convenient to solve the system of ordinary differential equation if  $y_2^1$  or  $y_4^1$  is substituted by the new function  $y_2^1 + 2y_4^1$ .

In practical computations, as the ordinary differential equations are singular when  $r = 0$ , i.e., some terms at the right hand are infinitely large, some more caution is required. One approach is to assume a small central sphere to be homogeneous, for which the analytic solution exists, and start to numerically integrate the ordinary differential equations from the surface of the small sphere. Another approach is to express the functions as power series of  $r$  around the Earth center,  $r \leq \epsilon$ , to find the power series solution of the ordinary differential equations. In this book, we will introduce in Sect. 4.9.5 a method—the Chebyshev collocation method—that can circumvent the particular treatment at the Earth center. More explanation will



be given when we deal with specific problems in subsequent chapters including free oscillations and Love numbers.

## 4.9 Discrete Fourier, Chebyshev and Spherical Harmonic Transforms

### 4.9.1 Discrete Fourier Transform

#### Discrete Fourier Transform

Consider the function  $u(x)$  defined in  $x \in [0, 2\pi]$ . We approximate it using the truncated Fourier series

$$u(x) = \sum_{k=[-(N-1)/2]}^{[(N-1)/2]} \hat{u}_k e^{ikx}, \quad (4.572)$$

where the brackets represent the integer not larger than the value of the quantity inside them. The lower and upper summation limits are  $-(N-1)/2$  and  $(N-1)/2$ , respectively, when  $N$  is odd, and  $-N/2$  and  $N/2-1$ , respectively, when  $N$  is even. All terms at the right-hand side of (4.572) are periodical with a period of  $2\pi$ . Hence, the function  $u(x)$  must have the property  $u(2\pi) = u(0)$  in order to be approximate this way. The coefficient  $\hat{u}_k$  is related to the function  $u(x)$  in the form of

$$\hat{u}_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx. \quad (4.573)$$

We can understand (4.572) and (4.573) as the complex form of the Fourier series given by (4.338) and (4.339) truncated to include only a finite number of terms.

For both cases of even and odd  $N$ , the number of the coefficients  $\hat{u}_k$  is  $N$ . Therefore, in the discrete Fourier transform (DFT), the function  $u(x)$  is chosen to be represented by its values at a set of  $N$  equidistant discrete points in the domain  $x \in [0, 2\pi]$ ,

$$x_j = \frac{2\pi j}{N}, \quad j = 0, \dots, N-1. \quad (4.574)$$

The approximation of  $u(x)$  expressed in the form of (4.572) is assumed equal to  $u(x_j)$  when  $x = x_j$ , i.e.,

$$u(x_j) = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{u}_k e^{ikx_j}. \quad (4.575)$$

In this way, there is a one-to-one correspondence between the  $N$  values  $u(x_j)$  and the  $N$  coefficients  $\hat{u}_k$ . Therefore, a single-valued inverse formula, i.e., the formula of  $\hat{u}_k$  in terms of  $u(x_j)$ , can be derived. To do so, we make use of the following

orthogonality formula

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{ipx_j} = \frac{1}{N} \sum_{j=0}^{N-1} \left( e^{i\frac{2\pi p}{N}} \right)^j = \begin{cases} 1, & p = 0, \pm N, \pm 2N, \dots, \\ 0, & \text{Otherwise.} \end{cases} \quad (4.576)$$

The proof for the case  $p = 0, \pm N, \pm 2N, \dots$  is straightforward. The case *Otherwise* can be proven using the sum of the truncated geometric series: for any complex constant  $z \neq 1$ ,  $\sum_{j=0}^{N-1} z^j = (1 - z^N)/(1 - z)$ , whose proof is left to the readers.

We readily obtain  $\sum_{j=0}^{N-1} \left( e^{i\frac{2\pi p}{N}} \right)^j = [1 - e^{i(2\pi p)}]/(1 - e^{i\frac{2\pi p}{N}}) = 0$ , which is the result we are seeking. Multiply both sides of (4.575) by  $(1/N)e^{-ilx_j}$ , and then sum up for  $j = 0, \dots, N-1$ . We obtain the inverse formula

$$\frac{1}{N} \sum_{j=0}^{N-1} u(x_j) e^{-ilx_j} = \sum_{k=-N/2}^{N/2-1} \hat{u}_k \frac{1}{N} \sum_{j=0}^{N-1} e^{i(k-l)x_j} = \sum_{k=N/2}^{N/2-1} \hat{u}_k \delta_{kl} = \hat{u}_l. \quad (4.577)$$

We write it in the form

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j) e^{-ikx_j}. \quad (4.578)$$

This last formula is called discrete Fourier transform (DFT), and (4.575), inverse discrete Fourier transform (IDFT). They could be computed using the fast Fourier transform (FFT) technique. The set of the Fourier coefficients  $\hat{u}_k$  is called spectral domain, and that of the values of the function  $u(x_j)$  is called space domain.

When  $u(x)$  is a real function,  $\hat{u}_k$  has the property

$$\hat{u}_k = \hat{u}_{-k}^*, \quad (4.579)$$

where the superscript  $*$  denotes the conjugate. Hence, when  $N$  is odd, (4.572) is simplified to, with the lower and higher summation limits being  $-(N-1)/2$  and  $(N-1)/2$ , respectively,

$$u(x) = \hat{u}_0 + \sum_{k=1}^{(N-1)/2} (2\hat{u}_k^R \cos kx + 2\hat{u}_{-k}^I \sin kx). \quad (4.580)$$

When  $N$  is even, (4.572) is simplified to, with the lower and higher summation limits being  $-N/2$  and  $N/2-1$ , respectively,

$$u(x) = \hat{u}_0 + \sum_{k=1}^{N/2-1} (2\hat{u}_k^R \cos kx + 2\hat{u}_{-k}^I \sin kx) + \hat{u}_{-N/2}^R \cos \frac{N}{2}x + i\hat{u}_{-N/2}^R \sin \frac{N}{2}x. \quad (4.581)$$

In this case, the real function  $u(x)$  is actually approximated by a complex function. Nevertheless, it can be seen that the imaginary term equals zero at the nodes given by (4.574), and it is neglected in order for the approximation to be real.

### Integration

We derive a formula for the integral of the product of two functions  $u(x)v(x)$  from 0 to  $2\pi$ . Denote the Fourier coefficients of  $v(x)$  as  $v_m$ . According to (4.572), we have

$$\begin{aligned} \int_0^{2\pi} u(x)v(x)dx &= \sum_{k=[-(N-1)/2]}^{[(N-1)/2]} \sum_{m=[-(N-1)/2]}^{[(N-1)/2]} \int_0^{2\pi} \hat{u}_k \hat{v}_m e^{i(k+m)x} dx \\ &= 2\pi \sum_{m=[-(N-1)/2]}^{[(N-1)/2]} \hat{u}_{-m} \hat{v}_m, \end{aligned} \quad (4.582)$$

where the second identity is due to the fact that  $\int_0^{2\pi} e^{i(k+m)x} dx$  equals zero when  $k \neq -m$ , and  $2\pi$  when  $k = -m$ .

The formula (4.582) is the expression of the integral in terms of the Fourier coefficients. A formula using the values of the functions  $u(x_j)$  and  $v(x_l)$  can be derived by substituting (4.578) into it,

$$\int_0^{2\pi} u(x)v(x)dx = \frac{2\pi}{N^2} \sum_{m=[-(N-1)/2]}^{[(N-1)/2]} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} u(x_j)v(x_l)e^{im(x_j-x_l)}. \quad (4.583)$$

A special case with  $v(x) = e^{-ikx}$  can be shown to be, by using (4.576),

$$\int_0^{2\pi} u(x)e^{-ikx}dx = \frac{2\pi}{N} \sum_{j=0}^{N-1} u(x_j)e^{-ikx_j}, \quad (4.584)$$

which is indeed the equality between (4.573) and (4.578).

### Differentiation

The derivative of  $u(x)$  can be computed based on the expression (4.572). We have

$$\frac{du(x)}{dx} = \sum_{k=[-(N-1)/2]}^{[(N-1)/2]} ik \hat{u}_k e^{ikx}. \quad (4.585)$$

This is a formula using the Fourier coefficients. By substituting (4.578) into it, we obtain a formula using the values of the function,  $u(x_j)$ ,

$$\frac{du(x)}{dx} = \sum_{k=[-(N-1)/2]}^{[(N-1)/2]} \left[ \frac{ik}{N} \sum_{j=0}^{N-1} u(x_j) e^{-ikx_j} \right] e^{ikx}. \quad (4.586)$$

Its value at  $x_l$  is

$$\left[ \frac{du(x)}{dx} \right]_{x=x_l} = \sum_{k=[-(N-1)/2]}^{[(N-1)/2]} \frac{ik}{N} \sum_{j=0}^{N-1} u(x_j) e^{ik(x_l - x_j)} = \sum_{j=0}^{N-1} D_{lj}^F u(x_j), \quad (4.587)$$

where the differential operator  $D_{lj}^F$  is defined as

$$D_{lj}^F = \frac{1}{N} \sum_{k=[-(N-1)/2]}^{[(N-1)/2]} ike^{2ik(l-j)\pi/N}. \quad (4.588)$$

The super script  $F$  is for Fourier. It can be readily verified that the differential operator  $D_{lj}^F$  is a real function,

$$D_{lj}^F = -\frac{2}{N} \sum_{k=1}^{[(N-1)/2]} k \sin \frac{2\pi(l-j)k}{N}, \quad (4.589)$$

where the term with  $k = -N/2$  is neglected when  $N$  is even. Its closed-form expression can be found in Canuto et al. (2006).

### 4.9.2 Gauss-Type Quadrature and Discrete Polynomial Transform

The orthogonality of a system of function has been defined in Sect. 4.3.5. Here we instantiate the definition to a system of polynomial defined in  $x \in [-1, 1]$ . Let  $L_k(x)$  be a polynomial of degree  $k$ , and  $L_1(x), L_2(x), \dots$  a system of polynomial. If

$$\int_{-1}^1 L_k(x) L_j(x) W(x) dx = \delta_{kj} \lambda_k, \quad (4.590)$$

the polynomial system is said to be orthogonal in  $x \in [-1, 1]$  with respect to the weight function  $W(x)$ . A smooth function  $u(x)$  defined in  $x \in [-1, 1]$  can be approximated by a truncated polynomial series

$$u(x) = \sum_{k=0}^N \hat{u}_k L_k(x), \quad (4.591)$$

where the coefficient is

$$\hat{u}_k = \frac{1}{\lambda_k} \int_{-1}^1 u(x) L_k(x) W(x) dx. \quad (4.592)$$

The approximate expressibility of the function  $u(x)$  in the form of (4.591) can be proven using the Legendre series:  $u(x)$  can be approximated as a truncated series of  $P_n(x)$ , and  $P_n(x)$  can be expressed as a linear combination of  $L_k(x)$ ,  $k = 0, 1, n$ . The formula for the coefficient can be proven using (4.590).

Orthogonal polynomials can be used to compute integrals numerically, called Gauss-type quadrature. We follow Canuto et al. (2006) in the following text.

*Gauss.* Let  $x_0, \dots, x_N$  be the  $N + 1$  roots of  $L_{N+1}(x) = 0$ , and  $w_0, \dots, w_N$  the solution of the linear equation system

$$\sum_{j=0}^N (x_j)^k w_j = \int_{-1}^1 x^k W(x) dx, \quad k = 0, \dots, N. \quad (4.593)$$

The formula

$$\sum_{j=0}^N L(x_j) w_j = \int_{-1}^1 L(x) W(x) dx \quad (4.594)$$

holds for any polynomial  $L(x)$  of degree not higher than  $2N + 1$ .

*Gauss-Radau.* Define a polynomial

$$q(x) = L_{N+1}(x) + a L_N(x), \quad (4.595)$$

where  $a$  is chosen so that  $q(1) = 0$  [note that Canuto et al. (2006) choose  $q(-1) = 0$ ]. Let  $x_0, \dots, x_N = 1$  be the  $N + 1$  roots of  $q(x) = 0$ , and  $w_0, \dots, w_N$  the solution of the linear equation system

$$\sum_{j=0}^N (x_j)^k w_j = \int_{-1}^1 x^k W(x) dx, \quad k = 0, \dots, N. \quad (4.596)$$

The formula

$$\sum_{j=0}^N L(x_j) w_j = \int_{-1}^1 L(x) W(x) dx \quad (4.597)$$

holds for any polynomial  $L(x)$  of degree not higher than  $2N$ .

*Gauss-Lobatto*. Define a polynomial

$$q(x) = L_{N+1}(x) + aL_N(x) + bL_{N-1}(x), \quad (4.598)$$

where  $a$  and  $b$  are chosen so that  $q(-1) = q(1) = 0$ . Let  $-1 = x_0, \dots, x_N = 1$  be the  $N + 1$  roots of  $q(x) = 0$ , and  $w_0, \dots, w_N$  the solution of the linear equation system

$$\sum_{j=0}^N (x_j)^k w_j = \int_{-1}^1 x^k W(x) dx, \quad k = 0, \dots, N. \quad (4.599)$$

The formula

$$\sum_{j=0}^N L(x_j) w_j = \int_{-1}^1 L(x) W(x) dx \quad (4.600)$$

holds for any polynomial  $L(x)$  of degree not higher than  $2N - 1$ .

The proof of the first theorem could be found in many textbooks of numerical analysis (e.g., Devis, 1975), and those of the last two in Canuto et al. (2006).

The Legendre polynomial is most used in the numerical evaluation of integrals. As  $W(x) = 1$ , what (4.594), (4.597), and (4.600) give are just  $\int_{-1}^1 L(x) dx$ . For any function  $f(x)$ , the integral  $\int_{-1}^1 f(x) dx$  is computed by replacing  $L(x)$  with  $f(x)$  in (4.594), (4.597), and (4.600). This is to implicitly approximate  $f(x)$  with the polynomial  $L(x)$  up to the degree that the theorems claim for.

The basis for discrete polynomial transform is to approximate a smooth function  $u(x)$  in  $x \in [-1, 1]$  using its values at the Gauss-type quadrature points  $x_0, \dots, x_N$  by polynomial interpolation. Evidently the interpolation polynomial is of degree  $N$ , which is equivalent to approximate the function as a truncated polynomial series,

$$u(x) = \sum_{k=0}^N \hat{u}_k L_k(x), \quad (4.601)$$

and the formula for computing  $u(x_j)$  from  $\hat{u}_k$  is

$$u(x_j) = \sum_{k=0}^N \hat{u}_k L_k(x_j), \quad j = 0, \dots, N. \quad (4.602)$$

We now formulate its inverse formula. Denote

$$\gamma_k = \sum_{j=0}^N L_k^2(x_j) w_j. \quad (4.603)$$

According to the orthogonal formula (4.590) and the relevant Gauss-type quadrature formulas, we have

$$\sum_{j=0}^N L_l(x_j) L_k(x_j) w_j = \gamma_k \delta_{kl} . \quad (4.604)$$

This is the discrete form of the orthogonal formula. The inverse formula of (4.602) can be easily obtained from it,

$$\hat{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^N u(x_j) L_k(x_j) w_j . \quad (4.605)$$

This last formula is called discrete polynomial transform, and (4.602) inverse discrete polynomial transform. Similar to the case of Fourier transform, the set of the polynomial coefficients  $\hat{u}_k$  is called spectral domain, and that of the values  $u(x_j)$  is called space domain. The Gauss-type quadrature points are also referred to as spectral points.

The discrete Chebyshev transform (DCT) to be formulated in the next subsection will be an instantiation of the general formulation of this subsection.

### 4.9.3 Discrete Chebyshev Transform

#### Discrete Chebyshev Transform

The degree  $k$  Chebyshev polynomial denoted as  $T_k(x)$ ,  $x \in [-1, 1]$ , is defined by

$$T_k(x) = \cos(k \arccos x) . \quad (4.606)$$

A few lower degree cases can be easily obtained based on the definition,

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1 . \quad (4.607)$$

Several properties can be verified based on the definition. A recurrence formula,

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) , \quad (4.608)$$

can be obtained from  $\cos(k+1)\alpha + \cos(k-1)\alpha = 2\cos\alpha \cos k\alpha$  with  $x = \cos\alpha$  and  $T_k(x) = \cos k\alpha$ . A recurrence formula involving derivatives,

$$2T_k(x) = \frac{1}{k+1} \frac{d}{dx} T_{k+1}(x) - \frac{1}{k-1} \frac{d}{dx} T_{k-1}(x) , \quad (4.609)$$

can be obtained from  $\sin(k+1)\alpha - \sin(k-1)\alpha = 2\sin\alpha \cos k\alpha$ . Its degenerate form when  $k = 0$  or  $1$  is, as can be verified based on (4.607),

$$2T_k(x) = \frac{c_k}{k+1} \frac{d}{dx} T_{k+1}(x), \quad (4.610)$$

where  $c_k$  is defined such that  $c_0 = 2$  and  $c_k = 1$  otherwise. Some other properties are

$$|T_k(x)| \leq 1 \quad (4.611)$$

$$T_k(\pm 1) = (\pm 1)^k \quad (4.612)$$

$$\left| \frac{d}{dx} T_k(x) \right| \leq k^2 \quad (4.613)$$

$$\left[ \frac{d}{dx} T_k(x) \right]_{x=\pm 1} = (\pm 1)^k k^2 \quad (4.614)$$

It can be readily shown that

$$\int_{-1}^1 \frac{T_k(x) T_l(x)}{\sqrt{1-x^2}} dx = \frac{\pi c_k}{2} \delta_{kl} \quad (4.615)$$

with the variable substitution  $x = \cos\alpha$ . This implies that the Chebyshev polynomial constitutes an orthogonal polynomial system with the weight function  $1/\sqrt{1-x^2}$ . Hence, a smooth function defined in  $x \in [-1, 1]$  can be approximated by a truncated Chebyshev series

$$u(x) = \sum_{k=0}^N \hat{u}_k T_k(x), \quad (4.616)$$

where the coefficient is

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^1 \frac{u(x) T_k(x)}{\sqrt{1-x^2}} dx. \quad (4.617)$$

To introduce the discrete Chebyshev transform (DCT), we note that  $N+1$  coefficients  $\hat{u}_0, \hat{u}_1, \dots, \hat{u}_N$  are used to represent a continuous function in (4.616). This is to assume that the function  $u(x)$  is smooth enough so that the terms of order higher than  $N$  can be neglected. The discrete Chebyshev transform is to further use the values of  $u(x)$  at  $N+1$  nodes  $x_0, x_1, \dots, x_N$  to determine the  $N+1$  coefficients  $\hat{u}_0, \hat{u}_1, \dots, \hat{u}_N$ . This is to solve the equations

$$u(x_j) = \sum_{k=0}^N \hat{u}_k T_k(x_j), \quad j = 0, 1, \dots, N \quad (4.618)$$



for  $\hat{u}_k$ . The most favorable choice of  $x_0, x_1, \dots, x_N$  is those of a Gauss-type quadrature, and the solution has been shown to be

$$\hat{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^N u(x_j) w_j T_k(x_j), \quad k = 0, 1, \dots, N, \quad (4.619)$$

where  $w_j$  is the weight as defined in the Gauss-type quadrature, and  $\gamma_k$  is the normalization factor as defined in (4.603). The last formula is called discrete Chebyshev transform (DCT), and (4.618) the inverse discrete Chebyshev transform (IDCT). Here again, the set of the Chebyshev coefficients  $\hat{u}_k$  is referred to as spectral domain, and that of the values of the function  $u(x_j)$  space domain.

There are three kinds of Gauss-type quadrature. Here, contrary to Canuto et al. (2006), we order the quadrature points  $x_j$  from  $-1$  to  $1$ , which is how we have implemented. In the following formulae, we will use the symbol  $c_k$  defined below (4.610) and another symbol  $\bar{c}_k$  defined such that  $\bar{c}_0 = \bar{c}_N = 2$  and  $\bar{c}_k = 1$  otherwise.

- *Gauss*: None of  $x = \pm 1$  is a quadrature point, i.e.,  $x_0 \neq -1, x_N \neq 1$ . We have

$$x_j = \cos \frac{[(2(N-j)+1)]\pi}{2N+2}, \quad w_j = \frac{\pi}{N+1}, \quad \gamma_k = \frac{c_k}{2} \pi. \quad (4.620)$$

- *Gauss-Radau*: One of  $x = \pm 1$  is a quadrature point, i.e.,  $x_0 \neq -1, x_N = 1$ . We have

$$x_j = \cos \frac{2\pi(N-j)}{2N+1}, \quad w_j = \frac{2\pi}{c_{N-j}(2N+1)}, \quad \gamma_k = \frac{c_k}{2} \pi. \quad (4.621)$$

- *Gauss-Lobatto*: Both  $x = \pm 1$  are quadrature points, i.e.,  $x_0 = -1, x_N = 1$ . We have

$$x_j = \cos \frac{\pi(N-j)}{N}, \quad w_j = \frac{\pi}{\bar{c}_j N}, \quad \gamma_k = \frac{\bar{c}_k}{2} \pi. \quad (4.622)$$

The differences in the characteristics of the three cases are whether one or both of  $x = \pm 1$  are defined as quadrature points. Each of them have specific applications.

For a function  $u(r)$  defined in  $r \in [a, b]$ , we use

$$r = \frac{b+a}{2} + \frac{b-a}{2}x, \quad x = \frac{2r-(b+a)}{b-a}, \quad \frac{d}{dr} = \frac{2}{b-a} \frac{d}{dx}, \quad \frac{d}{dx} = \frac{b-a}{2} \frac{d}{dr} \quad (4.623)$$

to transform between the two variables  $r$  and  $x$ . The DCT (4.619) and the IDCT (4.618) can be written as

$$\hat{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^N u(r_j) w_j T_k(x_j), \quad u(r_j) = \sum_{k=0}^N \hat{u}_k T_k(x_j), \quad (4.624)$$

where the relation between  $r_j$  and  $x_j$  is defined by (4.623). We will also refer  $r_j$  as quadrature points.

### Integration

The formula of the integral  $\int_{-1}^1 u(x) dx$  for practical computation can be derived out using (4.616), (4.609), and (4.610). We have (Guo, 2001)

$$\begin{aligned} \int_{-1}^1 u(x) dx &= \sum_{k=0}^N \left[ \hat{u}_k \int_{-1}^1 T_k(x) dx \right] \\ &= \hat{u}_0 \int_{-1}^1 dx + \hat{u}_1 \int_{-1}^1 x dx \\ &\quad + \frac{1}{2} \sum_{k=2}^N \hat{u}_k \int_{-1}^1 \left[ \frac{1}{k+1} \frac{d}{dx} T_{k+1}(x) - \frac{1}{k-1} \frac{d}{dx} T_{k-1}(x) \right] dx \\ &= 2\hat{u}_0 + \frac{1}{2} \sum_{k=2}^N \hat{u}_k \left[ \frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1} \right]_{-1}^1 \\ &= 2\hat{u}_0 + \sum_{k=2, k \text{ even}}^N \hat{u}_k \frac{-2}{(k-1)(k+1)} \\ &= -2 \sum_{k=0, k \text{ even}}^N \frac{\hat{u}_k}{(k-1)(k+1)}. \end{aligned} \quad (4.625)$$

This is the formula using the coefficients of the Chebyshev series. With (4.619), we can convert it to a form using the values  $u(x_j)$ ,

$$\int_{-1}^1 u(x) dx = \sum_{j=0}^N W_j u(x_j), \quad (4.626)$$

where

$$W_j = -2w_j \sum_{k=0, k \text{ even}}^N \frac{T_k(x_j)}{\gamma_k (k-1)(k+1)}. \quad (4.627)$$

For the function  $u(t)$  defined in  $t \in [a, b]$ , we have

$$\int_a^b u(r) dt = \frac{b-a}{2} \int_{-1}^1 u(r) dx = \frac{b-a}{2} \sum_{j=0}^N W_j u(r_j). \quad (4.628)$$

Another integral to be considered is  $\int_{-1}^1 [u(x)/\sqrt{1-x^2}] dx$ . Since  $W(x) = 1/\sqrt{1-x^2}$  is the weight of orthogonality of the Chebyshev polynomial, we can apply (4.594), (4.597), or (4.600) directly by approximating  $u(x)$  as a polynomial. Hence,

$$\int_{-1}^1 \frac{u(x)}{\sqrt{1-x^2}} dx = \sum_{j=0}^N w_j u(x_j), \quad (4.629)$$

where  $w_j$  is the weight given by (4.620), (4.621), or (4.622). In the case of Gauss, Gauss-Radau, and Gauss-Lobatto quadrature points,  $u(x)$  is approximated by a polynomial of degree  $2N+1$ ,  $2N$ , and  $2N-1$ , respectively.

### Differentiation

The derivative of a function can be conveniently computed using its values at the quadrature points. The derivative of  $u(x)$  is, according to (4.616),

$$\frac{du(x)}{dx} = \sum_{k=1}^N \hat{u}_k \frac{d}{dx} T_k(x), \quad (4.630)$$

where the degree 0 is vanishing as  $T_0(x) = 1$ . We seek a formula in the form of

$$\frac{du(x)}{dx} = \sum_{k=0}^N \hat{u}_k^{(1)} T_k(x). \quad (4.631)$$

In order to obtain a relation between  $\hat{u}_k^{(1)}$  and  $\hat{u}_k$ , we reduce (4.631) to the form of (4.630). By substituting (4.609) and (4.610) into (4.631), we obtain (remembering that  $c_k$  is defined such that  $c_0 = 2$  and  $c_k = 1$  otherwise)

$$\begin{aligned} \frac{du(x)}{dx} &= \sum_{k=0}^N \hat{u}_k^{(1)} \frac{c_k}{2(k+1)} \frac{d}{dx} T_{k+1}(x) - \sum_{k=2}^N \hat{u}_k^{(1)} \frac{1}{2(k-1)} \frac{d}{dx} T_{k-1}(x) \\ &= \sum_{k=1}^N \hat{u}_{k-1}^{(1)} \frac{c_{k-1}}{2k} \frac{d}{dx} T_k(x) - \sum_{k=1}^N \hat{u}_{k+1}^{(1)} \frac{1}{2k} \frac{d}{dx} T_k(x) \end{aligned}$$

$$= \sum_{k=1}^N \left[ \frac{c_{k-1}}{2k} \hat{u}_{k-1}^{(1)} - \frac{1}{2k} \hat{u}_{k+1}^{(1)} \right] \frac{d}{dx} T_k(x). \quad (4.632)$$

By comparing it with (4.630) we obtain  $\hat{u}_k = (c_{k-1}/2k) \hat{u}_{k-1}^{(1)} - (1/2k) \hat{u}_{k+1}^{(1)}$ . We write it as

$$c_{k-1} \hat{u}_{k-1}^{(1)} = 2k \hat{u}_k + \hat{u}_{k+1}^{(1)}. \quad (4.633)$$

By using this formula in a nested way, we obtain a formula of  $\hat{u}_k^{(1)}$ ,

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{p=k+1, p+k \text{ odd}}^{N-1} p \hat{u}_p = \sum_{p=k+1}^{N-1} \hat{D}_{kp} \hat{u}_p, \quad (4.634)$$

where

$$\hat{D}_{kp} = \begin{cases} \frac{2}{c_k} p, & p+k \text{ odd}; \\ 0, & \text{otherwise.} \end{cases} \quad (4.635)$$

A formula of the derivative using the values  $u(x_j)$  can be derived. By substituting (4.634) into (4.631), we obtain

$$\frac{du(x)}{dx} = \sum_{k=0}^N \sum_{p=k+1}^{N-1} \hat{D}_{kp} \hat{u}_p T_k(x). \quad (4.636)$$

By substituting  $\hat{u}_p$  by  $u(x_j)$  according to (4.619), we obtain the value of  $du(x)/dx$  at the quadrature points to be

$$\left[ \frac{du(x)}{dx} \right]_{x=x_l} = \sum_{k=0}^N \sum_{p=k+1}^{N-1} \hat{D}_{kp} \frac{1}{\gamma_p} \sum_{j=0}^N u(x_j) T_p(x_j) w_j T_k(x_l) = \sum_{j=0}^N D_{lj}^C u(x_j), \quad (4.637)$$

where the coefficient  $D_{lj}^C$  is defined as

$$D_{lj}^C = \sum_{k=0}^N \sum_{p=k+1}^{N-1} \hat{D}_{kp} \frac{1}{\gamma_p} T_p(x_j) w_j T_k(x_l). \quad (4.638)$$

The super script  $C$  is for Chebyshev. Note that  $p$  cannot be 0 or  $N$ , and hence, the factor  $c_p$  or  $\bar{c}_p$  in the expression of  $\gamma_p$  can be ignored. This formula is applicable to any kind quadrature points. For the Gauss-Lobatto points, a closed form of  $D_{lj}^C$  can be found in Canuto et al. (2006).

For the function  $u(r)$  define in  $r \in [a, b]$ , the formula of derivative is

$$\left[ \frac{du(r)}{dt} \right]_{r=r_l} = \frac{2}{b-a} \sum_{j=0}^N D_{lj}^C u(r_j), \quad (4.639)$$

where the relation between  $r_j$  and  $x_j$  is given in (4.623).

#### 4.9.4 Discrete Spherical Harmonic Transform

We adopt the complex form. Based on (4.333), we write the truncated spherical harmonic series as

$$f(\theta, \lambda) = \sum_{n=0}^{b-1} \sum_{k=-n}^n \bar{C}_n^k \bar{P}_n^k(\cos \theta) e^{ik\lambda}, \quad (4.640)$$

where  $\bar{P}_n^k(\cos \theta)$  is a type of normalized Legendre function. Instead of deriving the formula of the coefficients  $\bar{C}_n^k$  according to (4.334) for the specific type of normalized Legendre function, here we infer a formula directly from (4.640) without referring how the Legendre function is normalized. This can be done by changing the summation indices  $n$  and  $k$  to  $m$  and  $l$ , respectively, in (4.640), multiply both sides by  $\bar{P}_n^k(\cos \theta) e^{-ik\lambda}$ , and then integrate over the unit sphere  $\omega$ . We obtain

$$\bar{C}_n^k = \gamma_{nk}^2 \int_{\omega} f(\theta, \lambda) \bar{P}_n^k(\cos \theta) e^{-ik\lambda} d\omega, \quad (4.641)$$

where the factor  $\gamma_{nk}$  is defined according to the orthogonality formula

$$\int_{\omega} [\bar{P}_n^k(\cos \theta) e^{-ik\lambda}] [\bar{P}_m^l(\cos \theta) e^{il\lambda}] d\omega = \delta_{mn} \delta_{lk} \gamma_{nk}^{-2}. \quad (4.642)$$

According to (4.274), this orthogonality formula does hold, but the factor  $\gamma_{nk}$  is different for different type of normalization. Therefore, for a specific normalized Legendre function, the factor  $\gamma_{nk}$  can be obtained by substituting  $\bar{P}_n^k(\cos \theta)$  by the definition of  $\bar{P}_n^k(\cos \theta)$  into (4.274). Finally, we convert the integral over  $\omega$  in (4.640) to an integral over  $\theta$  and  $\lambda$ ,

$$\bar{C}_n^k = \gamma_{nk}^2 \int_0^{2\pi} \int_0^{\pi} f(\theta, \lambda) \bar{P}_n^k(\cos \theta) \sin \theta e^{-ik\lambda} d\theta d\lambda. \quad (4.643)$$

The discrete spherical harmonic transform (DSHT) consists of evaluating  $\bar{C}_n^k$  using the values of  $f(\theta, \lambda)$  over a grid on the sphere.

We follow Driscoll and Healy (1994) and Kostelec et al. (2000) to define an equi-angular grid of up to  $2b$  points in both  $\theta$  and  $\lambda$ , and truncate the spherical harmonic series at degree and order  $b - 1$ . The formulation requires considerable efforts, and is beyond the scope of this book. Below is a summary of the result. Driscoll and Healy (1994) adopted the grid

$$\theta_j = \frac{j\pi}{2b}, \quad \lambda_l = \frac{l\pi}{b}; \quad j, l = 0, 1, \dots, 2b - 1. \quad (4.644)$$

Kostelec et al. (2000) adopted the grid

$$\theta_j = \frac{(2j + 1)\pi}{4b}, \quad \lambda_l = \frac{l\pi}{b}; \quad j, l = 0, 1, \dots, 2b - 1. \quad (4.645)$$

For both types of grid, the relation between the values of the function at the grid points and its spherical harmonic coefficients can be written as

$$f(\theta_j, \lambda_l) = \sum_{n=0}^{b-1} \sum_{k=-n}^n \bar{C}_n^k \bar{P}_n^k(\cos \theta_j) e^{ik\lambda_l}, \quad (4.646)$$

$$\bar{C}_n^k = \gamma_{nk}^2 \frac{\pi}{b} \sum_{j=0}^{2b-1} \sum_{l=0}^{2b-1} a_j f(\theta_j, \lambda_l) \bar{P}_n^k(\cos \theta_j) e^{-ik\lambda_l}, \quad (4.647)$$

where the factor  $a_j$  is

$$a_j = \frac{2}{b} \sin \theta_j \sum_{q=0}^{b-1} \frac{\sin(2q + 1)\theta_j}{2q + 1}, \quad (4.648)$$

called sampling weight. We understand (4.647) as a discrete form of (4.643) based on the grid (4.644) or (4.645). The need of  $2b$  points over  $\theta$  is for evaluating  $\bar{C}_n^k$  precisely, but not a requirement for sampling  $f(\theta, \lambda)$ . In practical computations,  $f(\theta, \lambda)$  could be sampled over a  $\theta$ -grid of  $b$  points, and then interpolate to the  $2b$  points for using (4.647). Evidently, (4.646) and (4.647) involve an IDFT a DFT in  $\lambda$ , respectively.

The sampling weight can also be obtained by evaluating the integral (4.641) using the discrete point values of  $f(\theta, \lambda)$  according to (4.626), (4.627), and (4.584). With  $x = \cos \theta$ , we write (4.641) as

$$\bar{C}_n^k = \gamma_{nk}^2 \int_0^{2\pi} \left\{ \left[ \int_{-1}^1 f(x, \lambda) \bar{P}_n^k(x) dx \right] e^{-ik\lambda} \right\} d\lambda. \quad (4.649)$$

Assume that, when  $\lambda = \text{Constant}$ ,  $f(x, \lambda)$  is (or can be approximated by) a polynomial of order  $b - 1$ , i.e., the maximum degree of (4.646). In this case, the

integrand  $f(x, \lambda) \bar{P}_n^k(x)$  is (or can be approximated by) a polynomial with an order that may attain  $2(b-1)$ . We follow (4.647) to set  $N = 2b-1$  in (4.626) and (4.627), although the minimum required value is  $2(b-1)$ . To evaluate the integral over  $\lambda$ , we also follow (4.647) to set  $N = 2b$  in (4.584), although the minimum required value is  $2b-1$ . The numbers of grid points in both  $\theta$  and  $\lambda$  thus chosen are  $2b$ . We can readily obtain

$$a_j = -2w_j \sum_{k=0, k \text{ even}}^{2b-1} \frac{T_k(x_j)}{\gamma_k(k-1)(k+1)}. \quad (4.650)$$

By definition,  $T_k(x_j) = \cos(k\theta_j)$ , and the parameters  $x_j = \cos\theta_j$ ,  $w_j$ , and  $\gamma_k$  are defined in (4.620)–(4.622) for the three kinds of quadrature points. Note that (4.620) is identical to (4.645), and (4.622) to (4.644). The values of  $a_j$  computed using (4.650) and (4.648) are identical for the grids (4.620) or (4.645) that we adopted in our computations. It is beyond the scope of this book to prove analytically that (4.650) and (4.648) are the same.

The formula (4.647) is called discrete spherical harmonic transform (DSHT) or spherical harmonic analysis (SHA), and (4.646) is called inverse discrete spherical harmonic transform (IDSHT) or spherical harmonic synthesis (SHS).

In practical computations, fast Fourier transform (FFT) can be used. For this purpose, we write (4.646) and (4.647) in the forms of

$$f(\theta_j, \lambda_l) = \sum_{k=-(b-1)}^{b-1} \alpha_{jk} e^{ik\lambda_l}, \quad \alpha_{jk} = \sum_{n=|k|}^{b-1} \bar{C}_n^k \bar{P}_n^k(\cos\theta_j); \quad (4.651)$$

$$\bar{C}_n^k = \gamma_{nk}^2 \frac{\pi}{b} \sum_{j=0}^{2b-1} (a_j \beta_{kj}) \bar{P}_n^k(\cos\theta_j), \quad \beta_{kj} = \sum_{l=0}^{2b-1} f(\theta_j, \lambda_l) e^{-ik\lambda_l}. \quad (4.652)$$

The FFT technique can be used to evaluate  $f(\theta_j, \lambda_l)$  in (4.651) and  $\beta_{kj}$  in (4.652). We mention that a fast algorithms exist for evaluating the sums of Legendre functions such as the expression of  $\alpha_{jk}$  in (4.651) and that of  $\bar{C}_n^k$  in (4.652), which are again beyond the scope of this book. The readers are referred to Driscoll and Healy (1994).

When  $f(\theta, \lambda)$  is a real function, the IDSHT and DSHT are

$$f(\theta_j, \lambda_l) = \sum_{n=0}^{b-1} \sum_{k=0}^n \left( \bar{A}_n^k \cos k\lambda_l + \bar{B}_n^k \sin k\lambda_l \right) \bar{P}_n^k(\cos\theta_j), \quad (4.653)$$

$$\begin{pmatrix} \bar{A}_n^k \\ \bar{B}_n^k \end{pmatrix} = \gamma_{nk}^2 \frac{\pi}{b} \sum_{j=0}^{2b-1} \sum_{l=0}^{2b-1} a_j f(\theta_j, \lambda_l) \bar{P}_n^k(\cos\theta_j) \begin{pmatrix} \cos k\lambda_l \\ \sin k\lambda_l \end{pmatrix}, \quad (4.654)$$

respectively, where  $\gamma_{nk}$  is defined by (which is distinct from that of the complex form, although the same symbol is used)

$$\int_{\omega} [\bar{P}_n^k(\cos \theta) \cos k\lambda] [\bar{P}_m^l(\cos \theta) \cos l\lambda] d\omega = \delta_{mn} \delta_{lk} \gamma_{nk}^{-2}. \quad (4.655)$$

The formulae in favor of the FFT technique are

$$f(\theta_j, \lambda_l) = \sum_{k=0}^{b-1} \left\{ \left[ \sum_{n=k}^{b-1} \bar{A}_n^k \bar{P}_n^k(\cos \theta_j) \right] \cos k\lambda_l + \left[ \sum_{n=k}^{b-1} \bar{B}_n^k \bar{P}_n^k(\cos \theta_j) \right] \sin k\lambda_l \right\}, \quad (4.656)$$

$$\begin{pmatrix} \bar{A}_n^k \\ \bar{B}_n^k \end{pmatrix} = \gamma_{nk}^2 \frac{\pi}{b} \sum_{j=0}^{2b-1} \left\{ a_j \left[ \sum_{l=0}^{2b-1} f(\theta_j, \lambda_l) \begin{pmatrix} \cos k\lambda_l \\ \sin k\lambda_l \end{pmatrix} \right] \right\} \bar{P}_n^k(\cos \theta_j). \quad (4.657)$$

Even though there are  $2b$  grid points over  $\theta$  in (4.645) or (4.644), the required resolution of the DSHT and IDSHT is equivalent to that of  $b$  grid points. Hence, the data over the grid of only  $b$  points are actually required for the computation. In practice, the data over the  $2b$  grid points can be obtained by interpolating those over the  $b$  grid points, which consists of increasing the number grid points without increasing the resolution of the data. One such approach is to perform a discrete Chebyshev transform (DCT) with  $x = \cos \theta$ , increase the number of the Chebyshev coefficients from  $b$  to  $2b$  by appending  $b$  zeros, and then perform an inverse discrete Chebyshev transform (IDCT). Note that a Chebyshev series over  $x$  is equivalent to a cosine series over  $\theta$ , and that the DCT and IDCT over the variable  $x$  are equivalent to the discrete cosine transform and its inverse over  $\theta$ . This is equivalent to compute the values over the  $2b$  grid points using the Chebyshev series, or the cosine series, up to order  $b - 1$  obtained from the values over the  $b$  grid points.

It is noted that, as the number of spherical harmonic coefficients,  $b^2$ , is half of that of the grid values,  $2b^2$  ( $b$  grid points over  $\theta$  and  $2b$  over  $\lambda$ ), the grid and spherical harmonic series representations are not necessarily consistent. On the surface of the sphere, the distance between two meridians of longitude interval  $\delta\lambda$  is proportional to  $\sin \theta$ , thus decreasing toward the poles. This implies that the resolution of the grid data on the grid (4.645) or (4.644) in terms of the distance on the sphere can be higher in the longitudinal direction toward the poles. The resolution of the spherical harmonic series is isotropic and homogeneous in terms of the distance on the sphere. If the consistency is required, e.g., both the grid values and the spherical harmonic coefficients are used in an algorithm, the grid values must be generated using the spherical harmonic series. The simplest way is to transform the original grid values to a spherical harmonic series with a DSHT, and then transform back to grid values with an IDSHT.

Applications of the DSHT and IDSHT will be presented in Sects. 5.6.3 and 7.9.3.



As a supplement to this subsection, we mention that a DCT and then an IDCT procedure can also be used to convert the data from one type of grid over  $\theta$  to another, e.g., from (4.645)–(4.644) by using the Gauss-Lobatto grid in the DCT and the Gauss grid in the IDCT. The number of grid points, or equally the number of coefficients, in the DCT is determined by the input grid data, while that in the subsequent IDCT could be reassigned by appending some zeros beyond the highest order coefficient or by truncating some coefficients away. In this procedure, the resolution of the data remains the same if the number of grid points in the IDCT is equal or larger than that in the DCT. Otherwise, the resolution decreases to that supported by the number of grid in the IDCT. This procedure can be combined with a discrete Fourier transform (DFT) and its inverse (IDFT) for transforming from one type of  $\theta - \lambda$  grid to another (Guo & Shum, 2009).

#### 4.9.5 Numerical Solution of Differential Equations: The Chebyshev Collocation Method

The formulae (4.587) and (4.639) convert a derivative to an algebraic expression. They can therefore be used to convert differential equations to algebraic equations. Here we outline the application of (4.639) in the numerical solution of a system of linear differential-algebraic equations of the variable  $r \in [a, b]$ , which we write in the form of

$$a_{m0}(r) \frac{du_m(r)}{dr} = \sum_{n=0}^M a_{mn}(r) u_n(r) + f_m(r), \quad m = 1, \dots, M. \quad (4.658)$$

The inclusion of the coefficient  $a_{m0}(r)$  is for the implicit inclusion of algebraic equation: The  $m$ th equation is an algebraic equation if  $a_{m0}(r) \equiv 0$ .

The most rudimentary principle to convert this system of differential-algebraic equations to a system of purely algebraic equations is to assume the solution of each function to be a truncated Chebyshev series in the form of (4.616), and then force the differential-algebraic equations to be satisfied exactly at the Gauss-type quadrature points  $r_l$ ,  $l = 0, \dots, N$ . At a first glance, this appears to be solving for the coefficients of the Chebyshev series of the unknown functions. However, for any function, the coefficients of its Chebyshev series are related to its values at the quadrature points according to the DCT and IDCT. Hence, it is equivalent to solve for the values of the unknown functions at the quadrature points by forcing the differential-algebraic equations to be satisfied exactly at these points. By setting (4.658) to hold at the quadrature points, and then substituting (4.639) into it, we obtain, by moving all the unknowns to the left-hand side,

$$\left( \frac{2}{b-a} \right) a_{m0}(r_l) \sum_{j=0}^N D_{lj}^C u_m(r_j) - \sum_{n=0}^M a_{mn}(r_l) u_n(r_l)$$

$$= f_m(r_l), \quad m = 0, \dots, M \text{ and } l = 0, \dots, N. \quad (4.659)$$

This method is referred to as Chebyshev collocation method, and hence, the quadrature points are also called collocation points. Generally speaking, the collocation method is to force the differential equations to hold exactly at a set of discrete points.

When a function is expressed in the form of a series, the coefficients constitute the spectral domain, and the values of the function the space domain. A method is said to be spectral if the unknown functions are represented by series, and the coefficients are determined. In the case of the Chebyshev collocation method, the functions are indeed expressed as series, but those are the values of the functions which are actually computed. Hence, this method is also referred to as pseudo-spectral method or spectral-element method (as compared to the finite element method).

Differential equations are accompanied by boundary conditions. In the best circumstance, we assume that no singularity is present in the differential equations in (4.658). In this case, we adopt the Gauss-Lobatto collocation points in (4.659) including both the boundaries,  $r_0 = a$  and  $r_N = b$ , which correspond to  $x_0 = -1$  and  $x_N = 1$ , respectively. Assuming that all the equations in (4.658) are differential equations, there is then a need of  $M$  boundary conditions to uniquely determine the solution. Each boundary condition can be expressed in the form of an algebraic equation, no matter if it is in the simple form of  $u_m(r_0) = u_m(a)$  or  $u_m(r_N) = u_m(b)$ , in a complicated form of some kind of combination of more than one functions, or even if it includes derivatives, which can be transforms to algebraic forms according to (4.639). The approach to apply boundary conditions is to replace the algebraic equation at a boundary, either the one at  $r_0 = a$  or the one at  $r_N = b$ , originated from each differential equation by a boundary condition. The system of linear algebraic equations can then be solved to obtain the solutions of the functions at the collocation points, i.e., the values of  $u_m(r_j)$ .

As an example, we consider the differential equation of the gravity  $g$  within a spherically symmetrical Earth model provided in (4.439). Here we write it as

$$\frac{dg}{dr} + \frac{2}{r}g = 4\pi G\rho, \quad (4.660)$$

where  $\rho$  is a function of  $r$ . The Earth model is considered to be composed of several continuous layers with a jump of density across the boundaries. We number the continuous layers of the Earth model with  $i$ , starting with  $i = 0$  for the inner most layer, i.e., the inner core, to the outer most layer, i.e., the ocean. The boundaries of the  $i$ th layer are the inner and outer radii, and we denote them as  $a_i$  and  $b_i$ , with  $a_{i+1} = b_i$ . For the  $i$ th layer the algebraic equations are

$$\left(\frac{2}{b_i - a_i}\right) \sum_{j=0}^{N_i} D_{lj}^C g(r_{i,j}) + \frac{2}{r_{i,l}} g(r_{i,l}) = 4\pi G\rho(r_{i,l}), \quad l = 0, \dots, N_i, \quad (4.661)$$

where  $r_{i,j}$  is the  $j$ th collocation point in the  $i$ th layer, and the matrix  $D_{ij}^C$  is computed using  $N_i$  as the total number of collocation points. Assuming the density at the collocation points to be known, the solution is to be computed layer by layer starting from the innermost layer, i.e., the inner core, which is numbered as  $i = 0$ . An evident fact is that  $r = 0$  at the Earth center, which causes a singularity. Hence, the treatment of the inner core needs some caution, and is to be discussed in the next two paragraphs. For a layer other than the inner core, i.e., when  $i \neq 0$ , no singularity is present. In this case, we use the Gauss-Lobatto collocation points as discussed in the previous paragraph, which include both ends of the layer,  $r_{i,0} = a_i$  and  $r_{i,N} = b_i$ . As  $g$  is continuous across the boundaries between the layers, its value at the inner boundary of the  $i$ th layer is the same as its value at the outer boundary of the  $(i - 1)$ th layer. Therefore, we just have to replace the algebraic equation at  $r_{i,0}$  by  $g(r_{i,0}) = g(r_{i-1,N_{i-1}})$  and then solve the system of algebraic equations to obtain the solution of  $g(r_{i,j})$ .

We now assume that all the  $M$  equations in (4.658) are differential equations, but singularities are present at  $t = a$ , which is due to the fact that some of the coefficients  $a_{mn}(r)$ ,  $n > 0$ , tend to infinitely large when  $t \rightarrow a$ . In the simplest case, a singular differential equation can be reduced to a valuable algebraic equation at  $r = a$  by multiplying both sides of it with a common multiplier so that the coefficients of the derivative term vanishes and certain other coefficients do not vanish, as in Sect. 4.8.4. The algebraic equation obtained is then a boundary condition for the system of differential equations, called regularity boundary condition. If all the regularity boundary conditions are known, the differential equations can still be solved using the Chebyshev collocation method, using Gauss-Lobatto collocation points by replacing the algebraic equations at  $r_0 = a$  by boundary conditions, including the regularity boundary conditions. This is in fact the case of the differential equation (4.660) and the corresponding algebraic equations (4.661) in the inner core, i.e., when  $i = 0$ . By multiplying both sides of the differential equation (4.660) by  $r$  and then setting  $r = a_0 = 0$ , we obtain  $g(0) = 0$ , which is a regularity boundary condition in agreement with the physics of the problem. Therefore, for the case of inner core where  $i = 0$ , we just have to replace the algebraic equation at  $r_{0,0} = 0$  by the boundary condition  $g(r_{0,0}) = 0$  to solve for the solution of  $g(r_{0,j})$ .

The more complicated situation is that no information can be easily drawn at the singular point. If the total number of regularity boundary conditions is known to be  $M_r$ , then our problem is to solve the differential equations (4.658) with  $M - M_r$  boundary conditions at  $r = b$ . The trick to solve this kind of problem is to use the Gauss-Radau collocation points excluding the boundary  $r = a$ . The algebraic equations are still in the same form as (4.661), but with  $r_0 \neq a$ . They are solved by replacing  $M - M_r$  of them at  $r_N = b$  by the boundary conditions. This approach does not need the explicit expression of the regularity boundary conditions which can only be obtained by further developing the differential equations, which is sometimes quite tedious. Normally the total number of regularity boundary conditions  $M_r$  can be inferred from the physical background of the problem to be

solved. If the number of regularity boundary conditions is not known, the author's experience is to determine it by numerical test. Impose  $d$  boundary conditions at  $r_N = b$ , and solve the problem with various values of  $N$ . If the solution is the same for all values of  $N$ , we have  $M_r = d$ . Otherwise, different results would be obtained for different values of  $N$ .

We now assume that there are  $M_d$  differential equations and  $M_a$  algebraic equations in (4.658), with  $M = M_d + M_a$ . In this case, the algebraic equations are also included in (4.659) with  $a_{m0}(r) \equiv 0$ . The only difference in the solution is that the number of boundary conditions is  $M_d$ .

In this book, we deal with a lot of differential equations like (4.660), mostly a system of several equations, which are singular only at the Earth center. Hence, the Gauss-Radau collocation points are chosen for the inner core, and the Gauss-Lobatto collocation points for the other layers of the Earth model. The detail is discussed in each application.

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## Bibliographic Notes

The author's main reference includes Guo (2023) and Hobson (1931) for Sects. 4.1 and 4.2; Bosch (2000), Fukushima (2012a), Fukushima (2012b), Guo (2023), Guo et al. (2004), Hobson (1931) and Lapwood and Usami (1981) for Sect. 4.3; Guo (2023) for Sect. 4.4; Guo (2023), Jekeli (1981) and Rapp (1986) for Sect. 4.6; Chandrasekhar (1961), Lapwood and Usami (1981) and Smylie (1965) for Sect. 4.7; Alterman et al. (1959), Crossley and Rochester (1980) and Lapwood and Usami (1981) for Sect. 4.8; Canuto et al. (2006), Driscoll and Healy (1994), Kostelec et al. (2000), and Guo and Shum (2009) for Sect. 4.9. These are also suggested as the readers' sources for alternative presentations.

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## Abstract

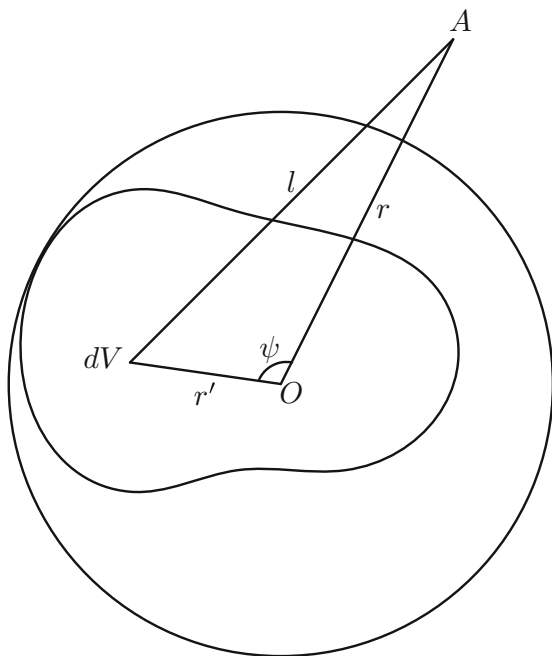
This chapter presents the fundamentals of the gravity and shape of the Earth. The spherical harmonic series of the Earth's external gravitational potential, along with the gravitational attraction and its gradient, is derived. The determination of the Earth's external gravity field using data of ground gravity is briefly formulated with approximations. The concepts introduced include the geoid, the normal gravity field, the reference Earth ellipsoid, height systems, the geoidal height, the deflection of the vertical, the disturbing gravity field, gravity reductions, and isostatic assumptions. A global model of the gravitational potential is presented as a reference. The internal shape and gravity of the Earth is formulated assuming the Earth to be in hydrostatic equilibrium state, i.e., the Earth behaves like a fluid body rotating as a whole without any relative motion inside. Finally, the gravitational potential of a layer of mass on the Earth's surface considered as either spherical or aspherical is formulated.

## 5.1 Spherical Harmonic Representation of the Gravitational Field

### 5.1.1 Gravitational Potential: The Stokes Series

As the gravitational potential of the Earth satisfies Laplace's equation outside the Earth, it can be expressed as a spherical harmonic series. Let  $M$  be the mass of the Earth. We use  $GM/r$  as the scale of gravitational potential, and  $a$  the scale of length. We can then express the gravitational potential of the Earth according to (4.251) and

**Fig. 5.1** Gravitational potential of a solid body



(4.259) in the form of

$$\phi = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{k=-n}^n \left(\frac{a}{r}\right)^n A_n^k \mathcal{H}_n^k(\theta, \lambda). \quad (5.1)$$

This formula is referred to as Stokes series, where the coefficients  $A_n^k$  are dimensionless, and are referred to as Stokes coefficients or potential coefficients.

To interpret the Stokes coefficients physically in terms of mass distribution within the Earth, we derive the Stokes series based on the definition of the gravitational potential. As shown in Fig. 5.1, the gravitational potential  $\phi$  of a solid body  $V$  at a point  $A$  is, by definition,

$$\phi = G \int_V \frac{\rho}{l} dV = G \int_V \rho \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos \psi) dV, \quad (5.2)$$

where the Legendre series expansion of  $1/l$ , (4.98), has been used. In this formula,  $r$ ,  $\theta$  and  $\lambda$  are the coordinates of the point  $A$ , and the variables of integration are  $r'$ ,  $\theta'$  and  $\lambda'$ , i.e.,  $r'$ ,  $\theta'$  and  $\lambda'$  are the coordinates of the volume element  $dV$ , and  $dV = r'^2 \sin \theta' d\theta' d\lambda'$ . The domain of integration  $V$  is whole the inside of the Earth. The series of  $P_n(\cos \psi)$  converges when  $A$  is outside the sphere with the smallest possible radius that enclose the entire body  $V$  within it, called Brillouin sphere. By

substituting the addition theorem (4.289) into (5.2), we obtain

$$\phi = G \int_V \rho \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{r'^n}{r^{n+1}} \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \mathcal{H}_n^k(\theta', \lambda') \mathcal{H}_n^k(\theta, \lambda) dV, \quad (5.3)$$

where the variables of integration are still  $r'$ ,  $\theta'$  and  $\lambda'$ . It can be written in the form of (5.1) as

$$\begin{aligned} \phi &= \frac{GM}{r} \sum_{n=0}^n \sum_{k=-n}^n \left(\frac{a}{r}\right)^n \left[ \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \right. \\ &\quad \left. \times \int_V \frac{\rho}{M} \left(\frac{r'}{a}\right)^n \mathcal{H}_n^k(\theta', \lambda') dV' \right] \mathcal{H}_n^k(\theta, \lambda). \end{aligned} \quad (5.4)$$

By comparing with (5.1), the Stokes coefficients expressed in terms of  $\rho$  is the expression within the brackets, i.e.,

$$A_n^k = \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \int_V \frac{\rho}{M} \left(\frac{r}{a}\right)^n \mathcal{H}_n^k(\theta, \lambda) dV, \quad (5.5)$$

where we have replaced the variables of integration  $r'$ ,  $\theta'$  and  $\lambda'$  by  $r$ ,  $\theta$  and  $\lambda$ , which does not alter the value of the integral.

### 5.1.2 The Stokes Coefficients of Degree 0, 1, and 2

For relating the degree 0, 1, and 2 Stokes coefficients to the physical parameters of the Earth, we develop the integrals of  $r$ ,  $\theta$ , and  $\lambda$  in (5.5) to integrals of  $x_1$ ,  $x_2$ , and  $x_3$  with  $dV = dx_1 dx_2 dx_3$ . This can be readily done by substituting the expression of  $\mathcal{H}_n^k(\theta, \lambda)$  defined in (4.258) into (5.5), and then using the formulae of the Legendre function listed in Table 4.1. The result is listed in Table 5.1.

We see readily that

$$A_0 = \frac{1}{M} \int_V \rho dv = 1. \quad (5.6)$$

The coordinates of the center of mass of the Earth are

$$\frac{1}{M} \int_V \rho x_i dV = X_i. \quad (5.7)$$

These can be used to write the degree 1 Stokes coefficients as

$$A_1 = \frac{X_3}{a}, \quad A_1^1 = \frac{X_1}{a}, \quad A_1^{-1} = \frac{X_2}{a}. \quad (5.8)$$



**Table 5.1** Some lower degree and order Stokes coefficients

|                                                                                                                       |                                                             |
|-----------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------|
| $A_0 = \int_v \frac{\rho}{M} dv$                                                                                      | $= \frac{1}{M} \int_v \rho dv$                              |
| $A_1 = \int_v \frac{\rho}{M} \frac{r}{a} \cos \theta dv$                                                              | $= \frac{1}{Ma} \int_v \rho x_3 dv$                         |
| $A_1^1 = \int_v \frac{\rho}{M} \frac{r}{a} \sin \theta \cos \lambda dV$                                               | $= \frac{1}{Ma} \int_v \rho x_1 dv$                         |
| $A_1^{-1} = \int_v \frac{\rho}{M} \frac{r}{a} \sin \theta \sin \lambda dv$                                            | $= \frac{1}{Ma} \int_v \rho x_2 dv$                         |
| $A_2 = \int_v \frac{\rho}{M} \left(\frac{r}{a}\right)^2 \left(\frac{3}{2} \cos^2 \theta - 1\right) dv$                | $= \frac{1}{2Ma^2} \int_v \rho (2x_3^2 - x_1^2 - x_2^2) dv$ |
| $A_2^1 = \frac{1}{3} \int_v \frac{\rho}{M} \left(\frac{r}{a}\right)^2 (3 \sin \theta \cos \theta) \cos \lambda dv$    | $= \frac{1}{Ma^2} \int_v x_1 x_3 dV$                        |
| $A_2^{-1} = \frac{1}{3} \int_v \frac{\rho}{M} \left(\frac{r}{a}\right)^2 (3 \sin \theta \cos \theta) \sin \lambda dv$ | $= \frac{1}{Ma^2} \int_v \rho x_2 x_3 dv$                   |
| $A_2^2 = \frac{1}{12} \int_v \frac{\rho}{M} \left(\frac{r}{a}\right)^2 (3 \sin^2 \theta) \cos 2\lambda dv$            | $= \frac{1}{4Ma^2} \int_v \rho (x_1^2 - x_2^2) dv$          |
| $A_2^{-2} = \frac{1}{12} \int_v \frac{\rho}{M} \left(\frac{r}{a}\right)^2 (3 \sin^2 \theta) \sin 2\lambda dv$         | $= \frac{1}{2Ma^2} \int_v \rho x_1 x_2 dv$                  |

The moments of inertia around the  $x_1$ -,  $x_2$ - and  $x_3$ -axes are

$$A = \int_V \rho (x_2^2 + x_3^2) dV, \quad B = \int_V \rho (x_3^2 + x_1^2) dV, \quad C = \int_V \rho (x_1^2 + x_2^2) dV. \quad (5.9)$$

The products of inertia are

$$D = \int_V \rho x_1 x_2 dV, \quad E = \int_V \rho x_2 x_3 dV, \quad F = \int_V \rho x_1 x_3 dV. \quad (5.10)$$

The above two sets of parameters can be used to write the degree 2 Stokes coefficients as

$$\begin{cases} A_2 = -\frac{C - (A + B)/2}{Ma^2}, \\ A_2^1 = \frac{F}{Ma^2}, \\ A_2^2 = \frac{B - A}{4Ma^2}, \end{cases} \quad \begin{cases} A_2^{-1} = \frac{E}{Ma^2}, \\ A_2^{-2} = \frac{D}{2Ma^2}. \end{cases} \quad (5.11)$$

The physical meanings of the degree 1 and 2 Stokes coefficients are then clearly seen based on their expressions (5.8) and (5.11).

The geocentric terrestrial reference system, i.e., the coordinate system with the origin on the center of mass of the Earth, is most used in expressing the gravitational

potential. In this coordinate system,  $X_i = 0$ . Hence,

$$A_1 = A_1^1 = A_1^{-1} = 0, \quad (5.12)$$

and the expression of the Stokes series becomes

$$\phi = \frac{GM}{r} \left[ 1 + \sum_{n=2}^{\infty} \sum_{k=-n}^n \left( \frac{a}{r} \right)^n A_n^k \mathcal{H}_n^k(\theta, \lambda) \right]. \quad (5.13)$$

The coefficient

$$J_2 = -A_2 = \frac{C - (A + B)/2}{Ma^2} \quad (5.14)$$

is called dynamic form factor of the Earth.

At present,  $GM$  and  $J_2$  can be determined very accurately by measuring satellite orbit, the former by measuring the period a satellite orbiting the Earth, and the latter by measuring the precession rate of the intersection point between the satellite orbit and the equator.

### 5.1.3 Gravitational Attraction and Its Gradient

We denote the Stokes coefficients as  $C_n^k$  and  $S_n^k$  with  $k \geq 0$ , the former being  $A_n^k$ , and the later  $A_n^{-k}$ . The Stokes series expressed using  $C_n^k$  and  $S_n^k$  is

$$\phi = \frac{GM}{r} \sum_{n=0}^{\infty} \left( \frac{a}{r} \right)^n \sum_{k=0}^n (C_n^k \cos k\lambda + S_n^k \sin k\lambda) P_n^k(\cos \theta). \quad (5.15)$$

As compared to (5.13),  $C_0^0 = 1$ , and the degree 1 terms are included in this formula. This formula is to be used to formulate the gravitational attraction and its gradient.

According to (2.463), we have

$$F_r = \frac{\partial \phi}{\partial r}, \quad F_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad F_\lambda = \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \lambda}. \quad (5.16)$$

Evidently, (5.15) is more convenient than (5.13) when partial derivatives with respect to  $\theta$  and  $\lambda$  are involved. By substituting (5.15) into (5.16), it follows that

$$F_r = -\frac{GM}{r^2} \sum_{n=0}^{\infty} (n+1) \left( \frac{a}{r} \right)^n \sum_{k=0}^n (C_n^k \cos k\lambda + S_n^k \sin k\lambda) P_n^k(\cos \theta), \quad (5.17)$$

$$F_\theta = \frac{GM}{r^2} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \sum_{k=0}^n (C_n^k \cos k\lambda + S_n^k \sin k\lambda) \frac{d}{d\theta} P_n^k(\cos \theta), \quad (5.18)$$

$$F_\lambda = \frac{GM}{r^2} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \sum_{k=0}^n (S_n^k \cos k\lambda - C_n^k \sin k\lambda) \frac{1}{\sin \theta} P_n^k(\cos \theta), \quad (5.19)$$

where the independence of the degree 0 term from  $\theta$  and  $\lambda$  is taken into account.

By comparing (5.16)–(4.393), we see that the gravitational attraction is a spheroidal field with  $U = \partial\phi/\partial r$  and  $V = \phi/r$ . Hence, the gradient of the gravitational attraction can be inferred from the spherical harmonic series of the gradient of a spheroidal field, (4.420). Here we write it in real form corresponding to (5.15) component by component (a superscript  $G$  is used to indicate gradient):

$$F_{rr}^G = \frac{GM}{r^3} \sum_{n=0}^{\infty} (n+1)(n+2) \left(\frac{a}{r}\right)^n \sum_{k=0}^n (C_n^k \cos k\lambda + S_n^k \sin k\lambda) P_n^k(\cos \theta), \quad (5.20)$$

$$F_{r\theta}^G = F_{\theta r}^G = -\frac{GM}{r^3} \sum_{n=0}^{\infty} (n+2) \left(\frac{a}{r}\right)^n \sum_{k=0}^n (C_n^k \cos k\lambda + S_n^k \sin k\lambda) \frac{d}{d\theta} P_n^k(\cos \theta), \quad (5.21)$$

$$F_{r\lambda}^G = F_{\lambda r}^G = -\frac{GM}{r^3} \sum_{n=0}^{\infty} (n+2) \left(\frac{a}{r}\right)^n \sum_{k=0}^n (S_n^k \cos k\lambda - C_n^k \sin k\lambda) \frac{k}{\sin \theta} P_n^k(\cos \theta), \quad (5.22)$$

$$\begin{aligned} F_{\theta\theta}^G &= \frac{GM}{r^3} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n \sum_{k=0}^n (C_n^k \cos k\lambda + S_n^k \sin k\lambda) \\ &\quad \times \left[ \frac{d^2}{d\theta^2} P_n^k(\cos \theta) - (n+1) P_n^k(\cos \theta) \right], \end{aligned} \quad (5.23)$$

$$\begin{aligned} F_{\theta\lambda}^G &= F_{\lambda\theta}^G = \frac{GM}{r^3} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n \sum_{k=0}^n (S_n^k \cos k\lambda - C_n^k \sin k\lambda) \\ &\quad \times \frac{k}{\sin \theta} \left[ \frac{d}{d\theta} P_n^k(\cos \theta) - \cot \theta P_n^k(\cos \theta) \right], \end{aligned} \quad (5.24)$$

$$F_{\lambda\lambda}^G = \frac{GM}{r^3} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n \sum_{k=0}^n (C_n^k \cos k\lambda + S_n^k \sin k\lambda) \times \left\{ \cot \theta \frac{d}{d\theta} P_n^k(\cos \theta) - \left[ (n+1) + \frac{k^2}{\sin^2 \theta} \right] P_n^k(\cos \theta) \right\}. \quad (5.25)$$

The detail of the derivation is left to the readers as an exercise.

The global model of the Earth's gravitational field, often referred to as Earth gravitational model (EGM), is provided as a Stokes series, where the fully normalized Legendre function defined by (4.340), or equivalently, the surface spherical harmonics defined by (4.341), is used. The expression of the gravitational potential is a truncated Stokes series,

$$\phi = \frac{GM}{r} \left[ 1 + \sum_{n=2}^N \sum_{k=-n}^n \left(\frac{a}{r}\right)^n \bar{A}_n^k \bar{\mathcal{H}}_n^k(\theta, \lambda) \right], \quad (5.26)$$

where  $N$  is the maximum degree of the model. It is left to the readers to express the gravitational attraction and gradient in terms of  $\bar{A}_n^k$ .

## 5.2 The External Gravity Field of the Earth

### 5.2.1 Basic Concepts

#### 5.2.1.1 Gravity and Geoid

In the context of this subsection, we are concerned with the static gravity field of the Earth, based on which the normal gravity field and reference Earth ellipsoid are defined.

The gravity considered here is restricted to be the time-invariable part of the Earth's gravitational attraction and centrifugal acceleration. Therefore, the mass distribution within the Earth is assumed to be time-invariable, and the angular velocity of the Earth is chosen to be a kind of Earth's mean angular velocity assumed to represent Earth's long-term time-invariable rotation. The meaning of the Earth's mean angular velocity is twofold: it includes the definitions of a mean angular rate of the Earth rotation with respect to the inertial space, and a mean pole defining the mean axis of the Earth rotation as seen on the Earth.

The gravity measured by a gravimeter is the free fall acceleration of a particle not subjecting to electromagnetic force, which is the instantaneous true value of gravity; it is not the time-invariable gravity as defined in the previous paragraph, but it varies. Hence, there is a need to correct the measured value by removing the effect of extraterrestrial celestial bodies and the time-variable part to obtain the time-invariable gravity. The corrections commonly include: the gravitational effect of the Sun, the Moon, and some other planets (i.e., their tidal-generating force); the gravitational effect of mass transport within the Earth system, particularly in the

hydrosphere-cryosphere-atmosphere system; the effect of Earth rotation variation including rotation rate variation and polar motion on centrifugal acceleration; and the gravitational effect of Earth's deformation in response to all the above.

The equipotential surface of gravity, i.e., the equi-value surface of the gravity potential, plays an important role in the theory of the Earth's gravity field. By definition, the equation of an equipotential surface of gravity is  $W = \alpha$ , where  $W$  is the gravity potential, and  $\alpha$  a constant. When  $\alpha$  takes a series values, we obtain a series of equipotential surfaces of gravity. As the gravity  $\vec{g}$  is the gradient of  $W$ ,  $\vec{g}$  is perpendicular to the equipotential surface of gravity. Adjacent equipotential surfaces of gravity are not necessarily parallel, and they cannot intersect or be tangent each other. The distance between two infinitesimally close equipotential surfaces of gravity is inversely proportional to the gravity. To demonstrate these, we choose a direction  $\hat{H}$  that is perpendicular to the equipotential surface of gravity, and points opposite to the direction of gravity. This means that  $\hat{H}$  is chosen to point upward, while the gravity points downward. The gravity can then be expressed as  $\vec{g} = -g\hat{H}$ , where the minus sign ensures the magnitude  $g$  to be positive, and we have  $g = -\partial W / \partial H$ . Hence, the distance between two adjacent equipotential surfaces of gravity with a difference of gravity potential  $dW$  is

$$dH = -\frac{dW}{g}. \quad (5.27)$$

On an equipotential surface of gravity,  $g$  is finite, but it is not necessarily constant. Therefore,  $dH$  could be different from place to place.

The geoid is the most important equipotential surface of gravity. First of all, we imagine an ideal state that all the oceans are at standstill. In this case, the gravity on the surface of the oceans must be perpendicular to the surface, as the water would flow if this is not true. Hence, the surface of the oceans at standstill coincides with an equipotential surface of gravity. This particular equipotential surface of gravity is referred to as the geoid. In reality, the oceans cannot be at standstill, and the mean surface of the oceans after removing the time-variable part is neither an equipotential surface of gravity. What we can do is to choose an equipotential surface of gravity as close as possible to the mean surface of the oceans as the geoid. Without relying on the gravity potential that is not something we can feel, the geoid could be depicted as a closed surface satisfying the following conditions: (1) the gravity is perpendicular to it everywhere; (2) it is chosen to be as close as possible to the mean ocean surface. The geoid is considered as a physical surface representing the shape of the Earth.

### 5.2.1.2 Normal Gravity Field and Reference Earth Ellipsoid

By definition, the Earth's gravitational potential and attraction are irregular functions of position, and the shape of the geoid is also irregular. The Earth's normal gravity field is used as an approximation of the Earth's gravity field, and is described analytically using simple functions. Currently, in physical geodesy, the Earth's normal gravity field is defined as the gravity field of a fictitious Earth of regular shape, called reference Earth ellipsoid. The reference Earth ellipsoid is defined as a

reference surface in Sect. 1.2, which best fits the geoid, for representing positions. Here we further consider it as a solid body that best fits the Earth.

Nomenclatures of the normal gravity field include normal gravitational potential, normal gravitational attraction, normal gravity potential, and normal gravity. The difference between the real and normal gravity fields is referred to as disturbing gravity field.

The reference Earth ellipsoid is a rotating solid body as an approximation of the real Earth. Here we provide a full definition of the reference Earth ellipsoid with some repetition with Sect. 1.2. Based on observations, the shape of the geoid is very close to an ellipsoid of revolution with its minor axis along the Earth rotation axis. Hence the reference Earth ellipsoid is defined to have the following properties: (1) its surface is an ellipsoid of revolution with its center on the center of mass of the Earth system and its minor axis along the rotational axis; (2) its rotation rate is equal to that of the Earth; (3) its mass is equal to that of the Earth; (4) its surface is an equipotential surface of normal gravity, i.e., the gravity of the reference Earth ellipsoid itself; (5) the normal gravity potential on the surface of the reference Earth ellipsoid is equal to the gravity potential of the Earth on the geoid. The reference Earth ellipsoid defined in this way best fits the geoid globally, whose proof can be found in books of *Physical Geodesy* dedicated to the static gravity field of the Earth (e.g., Guo, 2023).

Traditionally, the reference Earth ellipsoid thus defined is referred to as the mean Earth ellipsoid, as it is practically a global mean, and the term *reference Earth ellipsoid* is used in a less restricted sense, which could be an ellipsoid that best fits the geoid of one or several countries. At present, as global measurements from satellites are always used, any reference Earth ellipsoid could be considered as a realization of the mean Earth ellipsoid. Hence, we choose to use the term *reference Earth ellipsoid* alone for all realizations of ellipsoidal bodies used as an approximation of the Earth.

### 5.2.1.3 Elementary Ellipsoidal Geometry

Ellipsoidal geometry is a sub-discipline of geodesy, and is often taught in a course like *Geometric Geodesy*. Here we formulate some fundamental properties for the need of this book.

The equation of the reference Earth ellipsoid in the rectangular coordinate system  $Ox_1x_2x_3$  is

$$\frac{x_1^2 + x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1, \quad (5.28)$$

where  $a$  and  $b$  are the semi-major and semi-minor axes. In the spherical coordinate system, the equation is

$$r^2 \left( \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} \right) = 1. \quad (5.29)$$

The flattening of the ellipsoid is defined as

$$f = \frac{a - b}{a}, \quad (5.30)$$

which is known to be on the order of  $\sim 1/300$  based on observations. We can express the equation of the ellipsoid using  $f$  as

$$\begin{aligned} r &= \left( \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} \right)^{-1/2} \\ &= a \left[ \sin^2 \theta + \left( \frac{a}{b} \right)^2 \cos^2 \theta \right]^{-1/2} \\ &= a [\sin^2 \theta + (1 - f)^{-2} \cos^2 \theta]^{-1/2}. \end{aligned} \quad (5.31)$$

Accurate to the first order of  $f$ , we obtain

$$r = a [\sin^2 \theta + (1 + 2f) \cos^2 \theta]^{-1/2} = a(1 + 2f \cos^2 \theta)^{-1/2} = a(1 - f \cos^2 \theta). \quad (5.32)$$

Define an equi-volumetric average radius  $R$ , such that the volume of the sphere of radius  $R$  is equal to that of the ellipsoid. As the volume of the ellipsoid is  $(3/4)\pi a^2 b$ , we have

$$R = (a^2 b)^{1/3} = a(1 - f)^{1/3}. \quad (5.33)$$

Accurate to the first order of  $f$ , we have

$$R = a \left( 1 - \frac{1}{3} f \right). \quad (5.34)$$

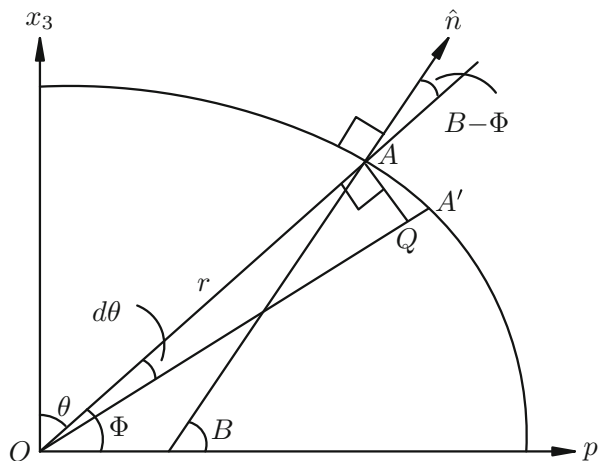
The equation of the ellipsoid accurate to the first order of  $f$ , (5.32), can thus be written as

$$r = R \left[ 1 - f \left( \cos^2 \theta - \frac{1}{3} \right) \right] = R \left[ 1 - \frac{2}{3} f P_2(\cos \theta) \right], \quad (5.35)$$

where  $P_2(\cos \theta) = (3/2) \cos^2 \theta - (1/2)$  is the degree 2 Legendre function.

We now derive the formula for the difference between the geodetic and geocentric latitudes of a point  $A$  on the reference Earth ellipsoid, as shown in Fig. 5.2. The geodetic latitude  $B$  is the angle made by the normal of the reference Earth ellipsoid and its equator, and the geocentric latitude  $\Phi$  is the complementary of the polar angle  $\theta$ . We take an adjacent point  $A'$  on the same meridian with a polar angle  $\theta + d\theta$ . Draw a line  $AQ$  perpendicular to  $OA'$ . We have  $B - \Phi = \angle QAA'$ . As  $AQ = rd\theta$ ,

**Fig. 5.2** Difference between the geodetic latitude  $B$  and the geocentric latitude  $\Phi$ . The unit vector  $\hat{n}$  is the outward normal of the reference Earth ellipsoid



$A'Q = (dr/d\theta)d\theta$ , we obtain

$$\tan(B - \Phi) = \frac{A'Q}{AQ} = \frac{1}{r} \frac{dr}{d\theta}. \quad (5.36)$$

By substituting (5.32) into this formula, we obtain, accurate to the first order of  $f$ ,

$$B - \Phi = \tan(B - \Phi) = \frac{a(2f \sin \theta \cos \theta)}{a(1 - f \cos^2 \theta)} = 2f \sin \theta \cos \theta. \quad (5.37)$$

As seen in Fig. 5.2, the unit outward normal vector of the reference Earth ellipsoid is

$$\hat{n} = \cos(B - \Phi)\hat{e}_r - \sin(B - \Phi)\hat{e}_\theta. \quad (5.38)$$

Accurate to the first order of  $f$ , we have  $\cos(B - \Phi) = 1$  and  $\sin(B - \Phi) = B - \Phi$ . Hence, we obtain, according to (5.37),

$$\hat{n} = \hat{e}_r - 2f \sin \theta \cos \theta \hat{e}_\theta. \quad (5.39)$$

## 5.2.2 The Normal Gravity Field

### 5.2.2.1 The Normal Gravitational Potential

In order to formulate the Earth's normal gravity field, we have to solve the following mathematical problem: Given an reference Earth ellipsoid with semi-major axis  $a$ , semi-minor axis  $b$ , rotation rate  $\Omega$ , and normal gravity potential  $W_0$  on its surface, derive the formula for the external normal gravity potential  $U$ . As the centrifugal potential is known, and can be subtracted from the normal gravity potential to obtain



the normal gravitational potential, the problem is equivalent to derive the formula of the normal gravitational potential outside the reference Earth ellipsoid when it is known on the surface of the reference Earth ellipsoid.

We use  $\phi$  to denote the normal gravitational potential outside the reference Earth ellipsoid. We know that it can be expressed as a spherical harmonic series in the form of (5.13). Based on the symmetry of the problem,  $\phi$  is independent of  $\lambda$ , and symmetrical with respect to the equator. Hence, only the even degree and order 0 terms in the harmonic series are non-vanishing, which we write in the form of

$$\phi = \frac{GM}{r} \left[ 1 - \sum_{n=1}^{\infty} J_{2n} \left( \frac{a}{r} \right)^{2n} P_{2n}(\cos \theta) \right]. \quad (5.40)$$

We use  $U_0$  to denote the value of the normal gravity potential on the surface of the reference Earth ellipsoid. As the normal gravity potential outside the reference Earth ellipsoid is  $U = \phi + (1/2)\omega^2 r^2 \sin^2 \theta$ , the value of  $\phi$  on the surface of the reference Earth ellipsoid is

$$\phi_0 = U_0 - \frac{1}{2}\omega^2 r^2 \sin^2 \theta. \quad (5.41)$$

It is to be remembered that  $U_0$  is equal to the gravity potential of the Earth on the geoid,  $W_0$ . The problem to be solved is then to determine the coefficients  $J_{2n}$  in (5.40) using (5.41) and the equation of the reference Earth ellipsoid (5.29). Define a parameter

$$q = \frac{\omega^2 a}{GM/a^2} = \frac{\omega^2 a^3}{GM}, \quad (5.42)$$

which is approximately the ratio between the centrifugal acceleration and gravitational attraction at the equator of the Earth, and is on the same order of magnitude as the flattening  $f$ . We can write (5.41) as

$$\phi_0 = U_0 - \frac{1}{2} \frac{GM}{r} q \left( \frac{r}{a} \right)^3 \sin^2 \theta. \quad (5.43)$$

We seek the solution only accurate to the first order of the flattening  $f$ . Hence, the equation of the surface of the reference Earth ellipsoid accurate to the first order of  $f$ , (5.32), can be used, and the boundary condition (5.43) can be written as, accurate to the first order of  $f$ ,

$$\begin{aligned} \phi_0 &= U_0 - \frac{1}{2} \frac{GM}{a(1-f\cos^2\theta)} q \left[ \frac{a(1-f\cos^2\theta)}{a} \right]^3 \sin^2 \theta \\ &= \left( U_0 - \frac{1}{2} \frac{GM}{a} q \right) + \frac{1}{2} \frac{GM}{a} q \cos^2 \theta. \end{aligned} \quad (5.44)$$

In the expression of the normal gravitational potential (5.40), as  $P_{2n}(\cos \theta)$  is a degree  $2n$  polynomial of  $\cos \theta$ , and only the term  $\cos^2 \theta$  is present in (5.44), we can conclude that, in (5.40), only the degree 2 term is on the first order of  $f$ . Hence, accurate to the first order of  $f$ , we have

$$\phi = \frac{GM}{r} \left[ 1 - J_2 \left( \frac{a}{r} \right)^2 P_2(\cos \theta) \right]. \quad (5.45)$$

We can conclude furthermore that  $J_2$  is on the same order of magnitude as  $f$ . By substituting  $P_2(\cos \theta) = (3/2) \cos^2 \theta - (1/2)$  into this formula, we obtain

$$\begin{aligned} \phi_0 &= \frac{GM}{a(1 - f \cos^2 \theta)} \left[ 1 - J_2 \left( \frac{1}{1 - f \cos^2 \theta} \right)^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \right] \\ &= \frac{GM}{a} (1 + f \cos^2 \theta) \left[ 1 - J_2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \right] \\ &= \frac{GM}{a} \left[ 1 + \frac{1}{2} J_2 + \left( f - \frac{3}{2} J_2 \right) \cos^2 \theta \right]. \end{aligned} \quad (5.46)$$

The substitution of this formula into (5.44) yields the equation

$$U_0 - \frac{GM}{a} \left[ 1 + \frac{1}{2} J_2 + \frac{1}{2} q + \left( f - \frac{3}{2} J_2 - \frac{1}{2} q \right) \cos^2 \theta \right] = 0. \quad (5.47)$$

In order for this equation to hold, the coefficient of  $\cos^2 \theta$  must vanish. We obtain

$$f = \frac{3}{2} J_2 + \frac{1}{2} q, \quad (5.48)$$

which is referred to as Clairaut's theorem. The normal gravity potential on the surface of the reference Earth ellipsoid is

$$U_0 = \frac{GM}{a} \left( 1 + \frac{1}{2} J_2 + \frac{1}{2} q \right). \quad (5.49)$$

Outside the reference Earth ellipsoid, the normal gravity potential accurate to the first order of  $f$  is

$$\begin{aligned} U &= \frac{GM}{r} \left[ 1 - J_2 \left( \frac{a}{r} \right)^2 P_2(\cos \theta) \right] + \frac{1}{2} \omega^2 r^2 \sin^2 \theta \\ &= \frac{GM}{r} \left[ 1 - J_2 \left( \frac{a}{r} \right)^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) + \frac{1}{2} q \left( \frac{r}{a} \right)^3 \sin^2 \theta \right]. \end{aligned} \quad (5.50)$$

### 5.2.2.2 The Normal Gravity

The magnitude of the normal gravity is

$$\gamma = |\nabla U| = \left[ \left( \frac{\partial U}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial U}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial U}{\partial \lambda} \right)^2 \right]^{1/2} \quad (5.51)$$

According to (5.50),  $(1/r^2)(\partial U/\partial \theta)^2$  is on the order of  $f^2$  and  $\partial U/\partial \lambda = 0$ , Hence, accurate to the first order on  $f$ , we simply have

$$\gamma = \left| \frac{\partial U}{\partial r} \right| = -\frac{\partial U}{\partial r}, \quad (5.52)$$

where the negative sign is added considering that the normal gravity is actually pointing toward below. Finally, by substituting (5.50) into (5.52), we obtain

$$\begin{aligned} \gamma &= \frac{GM}{r^2} \left[ 1 - J_2 \left( \frac{a}{r} \right)^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) + \frac{1}{2} q \left( \frac{r}{a} \right)^3 \sin^2 \theta \right] \\ &\quad - \frac{GM}{r} \left[ J_2 \left( 2 \frac{a^2}{r^3} \right) \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) + \frac{1}{2} q \left( 3 \frac{r^2}{a^3} \right) \sin^2 \theta \right] \\ &= \frac{GM}{r^2} \left[ 1 - 3J_2 \left( \frac{a}{r} \right)^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) - q \left( \frac{r}{a} \right)^3 \sin^2 \theta \right]. \end{aligned} \quad (5.53)$$

This is the general expression of the normal gravity outside the reference Earth ellipsoid.

The substituting of (5.32) into (5.53) yields the normal gravity on the surface of the reference Earth ellipsoid,

$$\begin{aligned} \gamma_0 &= \frac{GM}{a^2(1-f\cos^2\theta)^2} \left[ 1 - 3J_2 \frac{1}{(1-f\cos^2\theta)^2} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \right. \\ &\quad \left. - q(1-f\cos^2\theta)^3 \sin^2 \theta \right] \\ &= \frac{GM}{a^2} (1+2f\cos^2\theta) \left[ 1 - 3J_2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) - q \sin^2 \theta \right] \\ &= \frac{GM}{a^2} \left[ 1 + \frac{3}{2} J_2 - q + \left( 2f - \frac{q}{2} J_2 + q \right) \cos^2 \theta \right], \end{aligned} \quad (5.54)$$

which can be written as, by further using (5.48),

$$\gamma_0 = \frac{GM}{a^2} \left[ 1 + f - \frac{3}{2} q + \left( -f + \frac{5}{2} q \right) \cos^2 \theta \right]. \quad (5.55)$$

The normal gravity at the equator and at the poles can be obtained by setting  $\theta = \pi/2$  and  $\theta = 0$ ,

$$\gamma_e = \frac{GM}{a^2} \left( 1 + f - \frac{3}{2}q \right), \quad (5.56)$$

$$\gamma_p = \frac{GM}{a^2} (1 + q). \quad (5.57)$$

Define a gravimetric flattening

$$\beta = \frac{\gamma_p - \gamma_e}{\gamma_e}. \quad (5.58)$$

We obtain, from (5.56) and (5.57),

$$\beta = -f + \frac{5}{2}q, \quad (5.59)$$

and we can write (5.55) as

$$\gamma_0 = \gamma_e (1 + \beta \cos^2 \theta). \quad (5.60)$$

This is the first order practical formula of normal gravity at the surface of the reference Earth ellipsoid.

Finally, we derive the formula of the normal gravity at a point of geodetic height  $h$ , i.e., the height above the reference Earth ellipsoid. Accurate to the first order of  $h$ , we have

$$\gamma_h = \gamma_0 + \frac{\partial \gamma}{\partial n} h. \quad (5.61)$$

where  $\hat{n}$  is the outward normal of the ellipsoid. Accurate to the first order of  $f$ , we have shown in (5.39) that

$$\hat{n} = \hat{e}_r - 2f \sin \theta \cos \theta \hat{e}_\theta. \quad (5.62)$$

We thus obtain, accurate to the first order of  $f$ ,

$$\frac{\partial \gamma}{\partial n} = \nabla \gamma \cdot \hat{n} = \frac{\partial \gamma}{\partial r} - 2f \sin \theta \cos \theta \left( \frac{1}{r} \frac{\partial \gamma}{\partial \theta} \right) = \frac{\partial \gamma}{\partial r}. \quad (5.63)$$

where the fact that  $\partial\gamma/\partial\theta$  is on the order of  $f$  has been taken into account. According to (5.53), we have

$$\begin{aligned}\frac{\partial\gamma}{\partial r} &= -2\frac{GM}{r^3}\left[1 - 3J_2\left(\frac{a}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + q\left(\frac{r}{a}\right)^3\sin^2\theta\right] \\ &\quad + \frac{GM}{r^2}\left[-3J_2\left(-2\frac{a^2}{r^3}\right)\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + q\left(3\frac{r^2}{a^3}\right)\sin^2\theta\right] \\ &= -2\frac{GM}{r^3}\left[1 - 6J_2\left(\frac{a}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \frac{1}{2}q\left(\frac{r}{a}\right)^3\sin^2\theta\right].\end{aligned}\quad (5.64)$$

By substituting this formula as well as (5.32) into (5.63), we obtain

$$\begin{aligned}\frac{\partial\gamma}{\partial n} &= -2\frac{GM}{a^3(1-f\cos^2\theta)^3}\left[1 - 6J_2\frac{1}{(1-f\cos^2\theta)^2}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \frac{1}{2}q(1-f\cos^2\theta)^3\sin^2\theta\right] \\ &= -2\frac{GM}{a^3}(1+3f\cos^2\theta)\left[1 - 6J_2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \frac{1}{2}q\sin^2\theta\right] \\ &= -2\frac{GM}{a^3}\left[1 + 3J_2 + \frac{1}{2}q + \left(3f - 9J_2 - \frac{1}{2}q\right)\cos^2\theta\right].\end{aligned}\quad (5.65)$$

The substitution of (5.48) and (5.56) into this formula yields

$$\begin{aligned}\frac{\partial\gamma}{\partial n} &= -2\frac{GM}{a^3}\left[1 + 2f - \frac{1}{2}q + \left(-3f + \frac{5}{2}q\right)\cos^2\theta\right] \\ &= -2\frac{\gamma_e}{a}\left[1 + f + q + \left(-3f + \frac{5}{2}q\right)\cos^2\theta\right].\end{aligned}\quad (5.66)$$

We finally obtain

$$\gamma_h = \gamma_0 - 2\gamma_e\left[1 + f + q + \left(-3f + \frac{5}{2}q\right)\cos^2\theta\right]\frac{h}{a}.\quad (5.67)$$

We mention that  $h$  should not be too large in this formula. An approximate formula, where  $\partial\gamma/\partial n$  is approximated by its global average,  $(1/4\pi)\int_\omega(\partial\gamma/\partial n)d\omega$ , is often used,

$$\gamma_h = \gamma_0 + \left(\frac{1}{4\pi}\int_\omega\frac{\partial\gamma}{\partial n}d\omega\right)h = \gamma_0 - 2\frac{\gamma_e}{a}\left(1 + \frac{11}{6}q\right)h = \gamma_0 - 0.3086h,\quad (5.68)$$

where in the term  $-0.3086h$ , the unit of  $h$  is m, and that of  $\gamma$  is mGal. In practice,  $-0.3086$  is often used as an approximation of the vertical gradient of the gravity just above the Earth's surface.

### 5.2.2.3 Parameters of the Reference Earth Ellipsoid

As can be seen from the derivation, four parameters,  $GM$ ,  $a$ ,  $f$  and  $\omega$ , are required to fully determine a reference Earth ellipsoid. In current practice,  $f$  is replaced by  $J_2$ , and thus the four fundamental parameters are  $GM$ ,  $a$ ,  $J_2$  and  $\omega$ . The parameters  $GM$  and  $J_2$  can be determined very accurately by observing artificial satellites, and  $\omega$  can be determined very accurately by observing distant celestial bodies. By definition,  $a$  is chosen such that the normal gravity potential on the surface of the reference ellipsoid is equal to the gravity potential of the Earth on the geoid. The most used standard values are those of the Geodetic Reference System 1980 (GRS80), with

$$a = 6,378,137 \text{ m}, \quad (5.69)$$

$$GM = 3.986005 \times 10^{14} \text{ m}^3 \text{ s}^{-2}, \quad (5.70)$$

$$J_2 = 1.08263 \times 10^{-3}, \quad (5.71)$$

$$\Omega = 7.292115 \times 10^{-5} \text{ rad s}^{-1}. \quad (5.72)$$

Other parameters can be computed from these. Some parameters computed using the formulae accurate to  $f^2$  are (Guo, 2023)

$$b = 6,356,752.4 \text{ m}, \quad (5.73)$$

$$f = 1/298.25792, \quad (5.74)$$

$$R = 6,371,000.8 \text{ m}. \quad (5.75)$$

A formula for the normal gravity accurate to  $f^2$  is

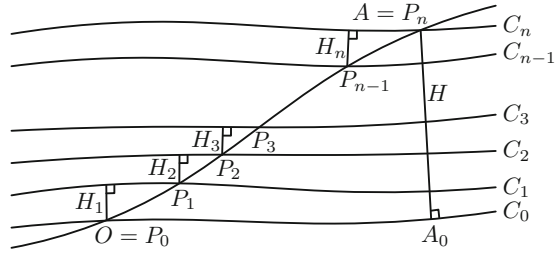
$$\gamma_0 = 978.03272(1 + 0.0053022 \sin^2 B - 0.0000058 \sin^2 2B) \text{ Gal}. \quad (5.76)$$

It is noted that for internal consistency to commensurate with current accuracy of observations, computation using the closed-form formulae, which are usually formulated in books of *Physical Geodesy* dedicated to the static gravity field of the Earth, is required.

## 5.2.3 Height Systems

In surveying, height is traditionally determined using spirit leveling. The main component of a level is a telescope turning on an axis strictly perpendicular to its line of sight. While doing observations, a level is leveled by pointing its turning axis to the direction of gravity, so that its line of sight turns in the horizontal

**Fig. 5.3** Determination of height by spirit leveling



plane. Hence, when two identically graduated poles adjusted to stand along the direction of gravity are read at the line of sight, the difference between the readings is then the height difference between the bottoms of the two poles. Based on the principle, we can realize that what a level measures is the local distance between two equipotential surfaces of gravity. As shown in Fig. 5.3, the height difference  $H_i$  could be understood as that measured by a level by reading two poles standing at  $P_{i-1}$  and  $P_i$ , which is the distance between the equipotential surface of gravity passing through  $P_{i-1}$  and that passing through  $P_i$ . Now we assume that  $O$  is the point where the height is defined to be 0. The height of  $A$  obtained by leveling through the specific path is then

$$H_L = \sum_{i=1}^n H_i, \quad (5.77)$$

where the subscript  $L$  indicates leveling.

As the distance between two equipotential surfaces of gravity is different at different locations, the height of a point determined by leveling through different paths may be different; therefore, it cannot be used as a standard measure of height, and must be converted to something else which is unique. A simple inference is that the height of an equipotential surface of gravity be identical everywhere, which obeys strictly the law that water flows from higher places to lower places. The quantity that is unique for an equipotential surface of gravity is the gravity potential on it. Hence a quantity named potential number  $C$  is defined for any point  $A$ , which is the difference between the gravity potential at the zero height point  $O$ ,  $W_O$ , and that at  $A$ ,  $W_A$ ,

$$C = W_O - W_A = \sum_{i=1}^n (W_{i-1} - W_i) = \sum_{i=1}^n g_{i-1} H_i = \int_O^A g dH_L, \quad (5.78)$$

where  $W_i$  is the gravity potential at the equipotential surface of gravity  $C_i$  as shown in Fig. 5.3, (5.27) is used to infer the third identity, and the last identity states the fact that every step of leveling is infinitesimally small as compared to the distance between  $O$  and  $A$ . We see that measurements of gravity is necessary in order to uniquely determine height.

Although  $C$  is unique, and it is the most important parameter to define height, it is not a length, while a height should be. Hence, it must be divided by a gravity to be converted to a length. Different types of height are defined following the choices of different kinds of gravity. A straightforward choice is to use the average gravity in the region where the height system is used, or even the normal gravity at around the center of the region. This type of height is referred to as dynamic height,

$$H_D = \frac{C}{\bar{g}}. \quad (5.79)$$

The height thus defined obeys the inference that water flows from higher places to lower places. However, this height is not the length of a line explicitly defined.

By definition, the geoid is a level surface; the surface of the oceans would coincide with it if the oceans were at standstill. Therefore, it represents an idealized sea level, and thus serves as an ideal reference surface to measure height; the height of any point on it is zero. Hence, the point  $O$  is defined to be on the geoid. For any point around the Earth's surface, the distance from the point to the geoid is defined as the orthometric height, which is also referred to as height above sea level. When the point is above the geoid, the height is positive; when the point is below the geoid, the height is negative. Therefore, the orthometric height  $H$  is the length of the line  $A_0A$  as shown in Fig. 5.3, which is measured along the line perpendicular to the geoid. We can also obtain  $C$  by integrating  $g$  along  $A_0A$ ,

$$C = \int_{A_0}^A g(H') dH' = H \left( \frac{1}{H} \int_{A_0}^A g(H') dH' \right). \quad (5.80)$$

Define

$$g_m = \frac{1}{H} \int_{A_0}^A g(H') dH', \quad (5.81)$$

which is the average of  $g$  over  $A_0A$ . We have

$$H = \frac{C}{g_m}. \quad (5.82)$$

The key to compute the orthometric height using leveling data is to compute  $g_m$ , which requires to estimate the gravity below the ground, and is beyond the scope of this book. The readers are referred to books of *Physical Geodesy* dedicated to the static gravity field of the Earth.

The computation of  $g_m$  requires to know  $g$  between the ground and geoid, which cannot be measured. To overcome this weakness, Molodensky defined a new height, the normal height denoted as  $H_\gamma$ , by modifying the definition of the orthometric height (5.80)–(5.82). Again, define  $O$  to be on the geoid where  $H_\gamma$  is defined to be zero, and spirit leveling has been performed from  $O$  to  $A$ , as shown in Fig. 5.3. The



normal height at  $A$  is defined as

$$H_\gamma = \frac{C}{\gamma_m}, \quad (5.83)$$

where  $C$  is the same as that in the orthometric height defined in (5.78), and

$$\gamma_m = \frac{1}{H_\gamma} \int_0^{H_\gamma} \gamma(H') dH'. \quad (5.84)$$

In this definition,  $C$  can be determined using measurements of spirit leveling and gravity as in the case of orthometric height, and  $\gamma_m$  can be computed exactly using parameters of the reference Earth ellipsoid. In (5.84),  $H'$  is actually the height above the reference Earth ellipsoid, which is equivalent to  $h$  in the formula of normal gravity (5.68). We thus obtain

$$\gamma_m = \frac{1}{H_\gamma} \int_0^{H_\gamma} (\gamma_0 - 0.3086H') dH' = \gamma_0 - 0.1543H_\gamma. \quad (5.85)$$

A more accurate formula can be obtained based on (5.67). In the numerical result, the dimension of  $H_\gamma$  is m, and that of  $\gamma_m$  is mGal.

A relation between orthometric height and normal height can be readily obtained from (5.83) and (5.84) to be  $C = \gamma_m H_\gamma = g_m H$ , yielding

$$H = H_\gamma - \frac{g_m - \gamma_m}{g_m} H_\gamma, \quad H_\gamma = H + \frac{g_m - \gamma_m}{\gamma_m} H, \quad (5.86)$$

which can be used to convert one height to another. Actually, the difference between these two heights is very small, 0 over the oceans, as the ocean surface is assumed to be extremely close to the geoid, a few centimeters over most continental regions, and may attain a meter over extremely mountainous areas.

We have already defined a height in Sect. 5.2.2, called geodetic height, which is the height above the surface of the reference Earth ellipsoid, and can now be routinely measured using Global Navigation Satellite System (GNSS) receivers. While the geodetic height and orthometric height are both defined as the height above a predefined reference surface, it is to be noted that the normal height is not defined with respect to a predefined reference surface; instead, it is defined by modifying the definition of orthometric height in favor of computability without making approximations for computing the gravity in between the Earth's surface and the geoid. For mapping purpose, either orthometric height or normal height is adopted by different country or group of countries, as these heights are very close to the inference that water flows from higher places to lower places.

Although the normal height  $H_\gamma$  is not defined with respect to a predefined reference surface, such a surface, called quasigeoid, is defined based on normal height, which is the surface  $H_\gamma$  below the Earth's surface. Accurate to the difference

between the orthometric and normal heights, the quasigeoid and normal height could be taken as approximations of the geoid and orthometric height.

The readers are referred to books of *Physical Geodesy* dedicated to the static gravity field of the Earth for more comprehensive and in-depth descriptions of the normal height.

## 5.2.4 The Disturbing Gravity Field

### 5.2.4.1 Disturbing Potential and Disturbing Gravity

In the practical determination, the gravity field of the Earth is separated into a normal gravity field and a disturbing gravity field. The normal gravity field, which has been formulated in the previous subsection, is assumed to be known. Therefore, the determination of the disturbing gravity field remains to be formulated.

By definition, the disturbing gravity field is the difference between the real and normal gravity fields. Therefore, we define

$$T = W - U, \quad \delta \vec{g} = \vec{g} - \vec{\gamma}, \quad (5.87)$$

where  $T$  is referred to as disturbing potential, and  $\delta \vec{g}$  gravity disturbance. Furthermore,  $\delta \vec{g}$  is related to  $T$  in the form of

$$\delta \vec{g} = \nabla W - \nabla U = \nabla T, \quad (5.88)$$

whose components are

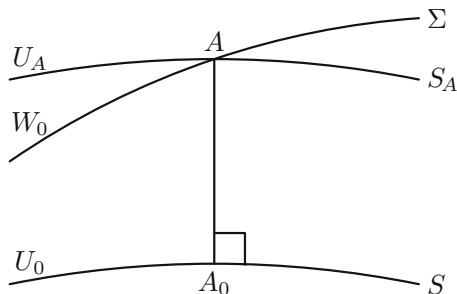
$$\delta g_r = \frac{\partial T}{\partial r}, \quad \delta g_\theta = \frac{1}{r} \frac{\partial T}{\partial \theta}, \quad \delta g_\lambda = \frac{1}{r \sin \theta} \frac{\partial T}{\partial \lambda}. \quad (5.89)$$

As the same centrifugal potential is included in both the real and normal gravity potentials, the disturbing potential is actually the difference between the real and normal gravitational potentials. Hence, the disturbing potential has all the properties of the gravitational potential. Consequently, the gravity disturbance has all the properties of the gravitational attraction. Furthermore, The disturbing potential and gravity disturbance are small quantities, since the normal gravity field is very close to the real gravity field.

### 5.2.4.2 Geoidal Height and Deflection of the Vertical

The geoid, which is defined as an equipotential surface of gravity chosen as close as possible to the mean ocean surface, is a geometrical representation of the gravity field. Its deviation from the reference Earth ellipsoid is then a representation of the disturbing gravity field. Two quantities are used to describe the deviation of the geoid from the reference Earth ellipsoid: geoidal height and deflection of the vertical.

The geoidal height  $N$ , as shown in Fig. 5.4, is a measure of the distance between the geoid and the reference Earth ellipsoid. We use  $\Sigma$  to denote the geoid, and  $S$

**Fig. 5.4** Geoidal height

the reference Earth ellipsoid. Let  $A$  be a point on the geoid, and its projection on the reference Earth ellipsoid is  $A_0$ . The geoidal height  $N$  at the point  $A$  is then the distance from  $A_0$  to  $A$ , positive upward.

The relation between the geoidal height  $N$  and the disturbing potential  $T$  can be derived quite straightforwardly. We use  $U_0$  to denote the normal gravity potential on the reference Earth ellipsoid,  $W_0$  the gravity potential on the geoid, and  $U_A$  the normal gravity potential at  $A$ . Also shown in Fig. 5.4 is the normal gravitational equipotential surface  $S_A$ . The geoidal height can then be expressed as

$$N = -\frac{U_A - U_{A_0}}{\gamma_{A_0}}. \quad (5.90)$$

Taking into account of (5.87), and with the facts  $W = W_0$  on  $\Sigma$ , and  $U = U_0$  on  $S$ , we obtain

$$N = -\frac{(W_A - T_A) - U_{A_0}}{\gamma_{A_0}} = -\frac{W_0 - U_0}{\gamma_0} + \frac{T_A}{\gamma_{A_0}}. \quad (5.91)$$

As the reference Earth ellipsoid is defined such that  $U_0 = W_0$ , the first term at the right-hand side vanishes. After replacing  $T_A$  and  $\gamma_{A_0}$  by  $T_0$  and  $\gamma_0$ , respectively, we obtain a general relation between the geoidal height and the disturbing potential:

$$N = \frac{T_0}{\gamma_0}. \quad (5.92)$$

As the disturbing potential is introduced as a small quantity, replacing  $\gamma_0$  by its global average  $\bar{\gamma}_0$  in this formula would introduce an error of even smaller order of magnitude. Therefore, we have

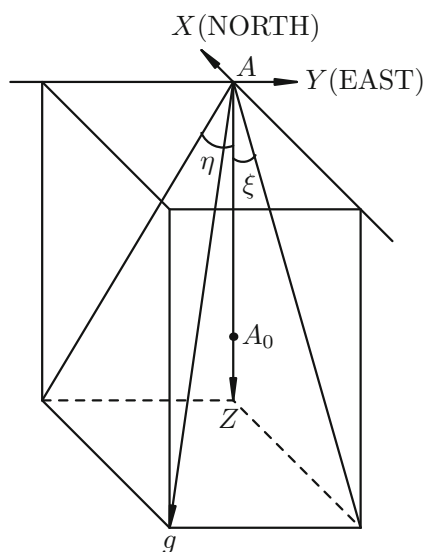
$$N = \frac{T_0}{\bar{\gamma}_0 + \gamma_0 - \bar{\gamma}_0} = \frac{T_0}{\bar{\gamma}_0} \left[ 1 - \frac{\gamma_0 - \bar{\gamma}_0}{\bar{\gamma}_0} \right] = \frac{T_0}{\bar{\gamma}_0}. \quad (5.93)$$

The relation between the geoidal height and the disturbing potential, either (5.91), (5.92), or (5.93), is called **Bruns' formula**.

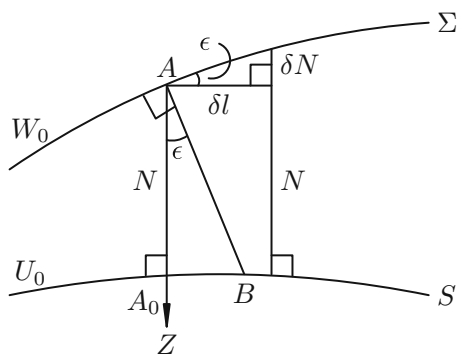
The deflection of the vertical, or more specifically, gravimetric deflection of the vertical here, is used to measure the angle between the geoid and the reference Earth ellipsoid. For a point  $A$  on the geoid, the deflection of the vertical is defined to be the angle between the normal to the geoid and that to the reference Earth ellipsoid, or equivalently, the angle between the gravity at  $A$  and the normal gravity at  $A_0$ , the projection of  $A$  on the reference Earth ellipsoid as defined in Fig. 5.4. Evidently, besides magnitude, the direction should also be specified. Hence, two components are necessary to completely describe the deflection of the vertical. Customarily, the component to the south,  $\xi$ , and that to the west,  $\eta$ , are used, as shown in Fig. 5.5. The  $Z$ -axis is perpendicular to the reference Earth ellipsoid, thus  $A_0$  is on it.

As shown in Fig. 5.6, the projection of the deflection of the vertical in any plane passing through the  $Z$ -axis is  $\angle A_0AB$ , where  $AB$  is the projection of the direction of gravity. It actually is equal to the angle between the intersection lines of this plane

**Fig. 5.5** Components of the deflection of the vertical, briefly referred to as deflection components



**Fig. 5.6** Projection of the deflection of the vertical in a plane passing through the  $Z$ -axis



with the geoid and reference Earth ellipsoid. Hence, we have

$$\epsilon = \lim_{\delta l \rightarrow 0} \frac{\delta N}{\delta l} = \frac{\partial N}{\partial l}. \quad (5.94)$$

The south and west components defined in Fig. 5.5 can then be expressed as

$$\xi = -\frac{\partial N}{\partial X}, \quad \eta = -\frac{\partial N}{\partial Y}. \quad (5.95)$$

As the deflection of the vertical is a small quantity, approximating the reference Earth ellipsoid by a sphere would introduce an error of even smaller order of magnitude. Hence, the above formulae can be approximated as

$$\xi = \frac{1}{R} \frac{\partial N}{\partial \theta}, \quad \eta = -\frac{1}{R \sin \theta} \frac{\partial N}{\partial \lambda}, \quad (5.96)$$

where the radius  $R$  is chosen to be the equi-volumetric average radius of the reference Earth ellipsoid. By substituting (5.93) into this formula, we obtain an expression of the deflection of the vertical expressed using the disturbing potential,

$$\xi = \frac{1}{R \bar{\gamma}_0} \frac{\partial T_0}{\partial \theta}, \quad \eta = -\frac{1}{R \sin \theta \bar{\gamma}_0} \frac{\partial T_0}{\partial \lambda}. \quad (5.97)$$

According to (5.89), these formulae can also be written as

$$\xi = \frac{\delta g_{0\theta}}{\bar{\gamma}_0}, \quad \eta = -\frac{\delta g_{0\lambda}}{\bar{\gamma}_0}, \quad (5.98)$$

where  $\delta g_{0\theta}$  and  $\delta g_{0\lambda}$  are the values of  $\delta g_\theta$  and  $\delta g_\lambda$ , respectively, on the geoid.

Here, the geoidal height and deflection of the vertical are defined for the static gravity field. However, once the reference Earth ellipsoid is defined, the geoidal height and deflection of the vertical may be defined for the variable gravity field as well. Therefore, these nomenclature should be understood in the specific context in the literature.

#### 5.2.4.3 Gravity Anomaly

We assume that the magnitude of gravity at all points on the geoid is known, and is denoted as  $g_0$ . We also denote the magnitude of normal gravity on the reference Earth ellipsoid as  $\gamma_0$ . For brevity, we will use gravity and normal gravity to indicate their magnitudes. Let  $A$  be a point on the geoid, and  $A_0$  its projection on the reference Earth ellipsoid as shown in Fig. 5.4. The difference between the gravity  $g_0$  at  $A$  and the normal gravity  $\gamma_0$  at  $A_0$  is referred to as the gravity anomaly  $\Delta g$ ,

$$\Delta g = g_0 - \gamma_0. \quad (5.99)$$

We now assume that the gravity anomaly is known, and the disturbing potential is to be determined.

We first formulate a relation between  $\Delta g$  and  $T$  on the geoid. We use  $\hat{e}_{n_\Sigma}$  and  $\hat{e}_{n_S}$  to denote the upward normals of the geoid  $\Sigma$  and the reference Earth ellipsoid  $S$ , respectively. We then have

$$g_0 = -\left(\frac{\partial W}{\partial n_\Sigma}\right)_\Sigma = -\left(\frac{\partial U}{\partial n_\Sigma}\right)_\Sigma - \left(\frac{\partial T}{\partial n_\Sigma}\right)_\Sigma, \quad \gamma_0 = -\left(\frac{\partial U}{\partial n_S}\right)_S. \quad (5.100)$$

Hence,

$$\Delta g = -\left(\frac{\partial U}{\partial n_\Sigma}\right)_\Sigma + \left(\frac{\partial U}{\partial n_S}\right)_S - \left(\frac{\partial T}{\partial n_\Sigma}\right)_\Sigma. \quad (5.101)$$

The first term can be further developed as

$$\left(\frac{\partial U}{\partial n_\Sigma}\right)_\Sigma = \left(\frac{\partial U}{\partial n_S}\right)_S + N \left(\frac{\partial^2 U}{\partial n_\Sigma^2}\right)_S = \left(\frac{\partial U}{\partial n_S}\right)_S \cos(\hat{e}_{n_\Sigma}, \hat{e}_{n_S}) + N \left(\frac{\partial^2 U}{\partial n_\Sigma^2}\right)_S. \quad (5.102)$$

where  $(\hat{e}_{n_\Sigma}, \hat{e}_{n_S})$  is the magnitude of the deflection of the vertical, which seldom exceeds one arc minute. Hence, we make the approximation

$$\left(\frac{\partial U}{\partial n_\Sigma}\right)_\Sigma = \left(\frac{\partial U}{\partial n_S}\right)_S + N \left(\frac{\partial^2 U}{\partial n_S^2}\right)_S = \left(\frac{\partial U}{\partial n_S}\right)_S - N \left(\frac{\partial \gamma}{\partial n_S}\right)_S, \quad (5.103)$$

where we also neglected the variation of direction of the normal gravity over height. Finally, we obtain

$$\Delta g = N \left(\frac{\partial \gamma}{\partial n_S}\right)_S - \left(\frac{\partial T}{\partial n_\Sigma}\right)_\Sigma, \quad (5.104)$$

which can be written as, by substituting Bruns' formula into it,

$$\Delta g = \frac{T_0}{\gamma_0} \left(\frac{\partial \gamma}{\partial n_S}\right)_S - \left(\frac{\partial T}{\partial n_\Sigma}\right)_\Sigma. \quad (5.105)$$

This formula can also be expressed using the geodetic height  $h$  and the orthometric height  $H$  as

$$\Delta g = \frac{T_0}{\gamma_0} \left(\frac{\partial \gamma}{\partial h}\right)_{h=0} - \left(\frac{\partial T}{\partial H}\right)_{H=0}, \quad (5.106)$$

which is referred to as fundamental gravimetric equation.

As both  $\Delta g$  and  $T$  are small quantities, spherical approximation can be applied to the fundamental gravimetric equation with reasonable precision, which includes two aspects. First, the reference Earth ellipsoid is to be considered as in spherical symmetry when computing  $\gamma_0$  and  $(\partial\gamma/\partial h)_{h=0}$ , meaning that,

$$\gamma_0 = \frac{GM}{r^2}, \quad \left( \frac{\partial\gamma}{\partial h} \right)_{h=0} = -2 \frac{GM}{R^3}. \quad (5.107)$$

where  $R$  is the radius of the sphere used to approximate the reference Earth ellipsoid. Second, when the coordinates of a point on the geoid are used as arguments of  $T$ , we approximate the geoid by the sphere of radius  $R$ , meaning that

$$T_0 = T_\Sigma \approx T_R, \quad \left( \frac{\partial T}{\partial H} \right)_{H=0} \approx \left( \frac{\partial T}{\partial r} \right)_R. \quad (5.108)$$

where the subscript  $R$  stands for  $r = R$ . We thus obtain an approximate fundamental gravimetric equation on a spherical surface of radius  $R$ ,

$$\Delta g = -2 \frac{T_0}{R} - \left( \frac{\partial T}{\partial r} \right)_R. \quad (5.109)$$

Our purpose is to determine the Earth's external gravity potential, which is separated into a normal gravity potential and a disturbing potential. As the normal gravity potential is known, we have only to determine the disturbing potential, which satisfies Laplace's equation outside the Earth, and satisfies the fundamental gravimetric equation on the geoid.

If there exists no mass outside the geoid, the disturbing potential then satisfies Laplace's equation outside the geoid, and satisfies the fundamental gravimetric equation on the geoid. This is actually a third-type boundary value problem of Laplace's equation with the geoid as the boundary, and the fundamental gravimetric equation as the boundary condition. This problem is referred to as Stokes' boundary value problem. As the boundary—the geoid—depends on the disturbing potential which is to be solved for, thus being unknown, Stokes' boundary value problem is a free boundary value problem.

Nevertheless, although the geoid as a boundary is not known, it deviates little from the reference Earth ellipsoid; the geoidal height seldom exceeds one hundred meters, and the deflection of the vertical seldom exceeds one arc minute. Hence, in solving Stokes' boundary value problem, replacing the free boundary, i.e., the geoid, by the fixed boundary, i.e., the reference Earth ellipsoid, leads to very accurate result. Furthermore, the flattening of the reference Earth ellipsoid is on the order of  $\sim 1/300$ , meaning that approximating the reference Earth ellipsoid with a sphere would introduce an error of the same level. This leads to the conclusion that spherical approximation is a reasonable first order approximation.

Finally, we mention that the spherical approximation is to be applied to small quantities like the disturbing potential, or to parameters multiplied to a small

quantity. In the computation of gravity anomaly according to (5.99), the normal gravity  $\gamma_0$  must be computed using its formula on the reference Earth ellipsoid, rather than its spherical approximation understood as its global average.

#### 5.2.4.4 Solution of the Disturbing Potential in Spherical Harmonic Series

We now assume that no mass is existent outside the geoid, so that the disturbing potential  $T$  satisfies Laplace's equation. As the mass of the reference Earth ellipsoid is chosen to be identical to that of the Earth, the degree 0 term is missing from the spherical harmonic series of the disturbing potential. Furthermore, we assume the center of the reference Earth ellipsoid is on the center of mass of the Earth. In this way, the degree 1 terms are also missing. Hence, the spherical harmonic series of the disturbing potential is

$$T = \frac{GM}{r} \sum_{n=2}^{\infty} \sum_{k=-n}^n \left(\frac{R}{r}\right)^n A_n^k \mathcal{H}_n^k(\theta, \lambda). \quad (5.110)$$

This is a general form of the solution of Laplace's equation for our case, where the scale length  $a$  is replaced by  $R$  under spherical approximation.

By substituting (5.110) into the fundamental gravimetric equation (5.109), we obtain an expression of  $\Delta g$  in terms of the spherical harmonic coefficients of  $T$ ,

$$\Delta g = \frac{GM}{R^2} \sum_{n=2}^{\infty} \sum_{k=-n}^n (n-1) A_n^k \mathcal{H}_n^k(\theta, \lambda). \quad (5.111)$$

As  $\Delta g$  is assumed to be known all over the geoid, it can be expressed using spherical harmonic series as

$$\Delta g = \sum_{n=2}^{\infty} \sum_{k=-n}^n a_n^k \mathcal{H}_n^k(\theta, \lambda), \quad (5.112)$$

where the degree 0 and 1 terms are not included, as they are known to be missing according to (5.111). The coefficients are, according to (4.332),

$$a_n^k = \frac{1}{1 + \delta_k} \frac{2n+1}{2\pi} \frac{(n-|k|)!}{(n+|k|)!} \int_{\omega} \Delta g(\theta, \lambda) \mathcal{H}_n^k(\theta, \lambda) d\omega. \quad (5.113)$$

By comparing (5.111) and (5.112), we obtain the spherical harmonic coefficients of  $T$  as functions of those of  $\Delta g$ ,

$$A_n^k = \frac{R^2}{GM} \frac{a_n^k}{n-1}. \quad (5.114)$$



Note that the computation of  $a_n^k$  using data of  $\Delta g(\theta, \lambda)$  over a  $\theta$ - $\lambda$  grid is in fact a discrete spherical harmonic transform (DSHT) presented in Sect. 4.9.4.

The formulae (5.110), (5.113), and (5.114) constitute the solution of the disturbing potential in spherical harmonic series. When the disturbing potential is added to the normal gravitational potential, we obtain the gravitational potential of the Earth. Formulation of the geoidal height and the deflection of the vertical based on the spherical harmonic series of the disturbing potential is left to the readers.

We see that the degree 1 terms are missing in  $\Delta g$ , even if it is existent in  $T$ . Hence, strictly speaking, the solution of the boundary value problem is not unique; arbitrary degree 1 terms can be added to  $T$  without altering the fact that  $T$  is a solution. The solution is made unique by assuming degree 1 terms of  $T$  to vanish. Physically, this is assured by setting the origin of the coordinate system on the center of the reference Earth ellipsoid, which is on the center of mass of the Earth.

### 5.3 Gravity Reduction and Isostasy

The solution of  $T$  is obtained based on two assumptions: (1) the gravity on the geoid is known; (2) no mass is existent above the geoid. For the real Earth, neither of these can be true. Therefore, assuming that the gravity is measured on the Earth's surface, the gravity on the geoid should be computed from the measurements, which is referred to as gravity reduction. The purpose of gravity reduction is to correct the gravity by removing the mass above the geoid or relocating it to under the geoid, and then projecting the gravity onto the geoid, assuming the orthometric height and the density of the crust to be known. The gravity anomaly thus obtained can then be used to compute  $T$ . However, removing or relocating the mass above the geoid is in fact an adjustment of the real Earth to a fictitious Earth, and consequently, what we obtain is the solution of  $T$  for the fictitious Earth. Therefore, the result of  $T$  obtained should be corrected by recovering the fictitious Earth to the real Earth, and the correction is referred to as indirect effect. In this section, we provide the elementary corrections to the gravity observed on the ground for computing gravity anomaly.

#### 5.3.1 Free Air Correction and Free air Anomaly

The most primitive gravity reduction is to ignore the mass above the geoid, and compute the gravity on the geoid,  $g_0$ , from that on the ground,  $g$ , using the gradient of the gravity,  $g_0 = g - (\partial g / \partial H)_\Sigma H$ , where  $H$  is the orthometric height, and  $\Sigma$  the geoid. The gradient of gravity  $(\partial g / \partial H)_\Sigma$  is not known, and is approximated by the gradient of the normal gravity on the reference Earth ellipsoid,  $\partial \gamma / \partial n$ , given by (5.66). For most applications, using the global average of  $\partial \gamma / \partial n$  is accurate enough. Hence, we define, with (5.68) taken into account,

$$g_0 - \gamma_0 = g + \Delta_1 g - \gamma_0, \quad (5.115)$$

with

$$\Delta_1 g = 0.3086 H , \quad (5.116)$$

where the unit of  $g$  is mGal, and that of  $H$  is m. The gravity anomaly thus defined is called free air anomaly, and the correction  $\Delta_1 g$  is called free air correction.

### 5.3.2 Bouguer Plate Correction and Incomplete Bouguer Anomaly

In the free air correction, we assumed that no mass exists above the geoid, which is not the case in reality. In this subsection, we remove the effect of the mass above the geoid under a simple assumption: the geoid and the Earth's surface are planes parallel each other. The correction is called Bouguer plate correction, or simply plate correction, and is denoted as  $\Delta_2 g$ .

The plate can be considered as a cylinder with an extremely large radius as compared to its height, e.g., a radius of 100 km with a height of 2 km, and the gravity is measured at the center of the top. The formula of gravitational attraction of a cylinder is provided in (3.97). As removing the effect of the plate always brings in a negative correction, we obtain, by setting  $z = 0$  and  $h = H$  in (3.97),

$$\begin{aligned} \Delta_2 g &= -2\pi G\rho \left[ (a + H) - (a^2 + H^2)^{1/2} \right] \\ &= -2\pi G\rho a \left[ 1 + \frac{H}{a} - \left( 1 + \frac{H^2}{a^2} \right)^{1/2} \right] \\ &\approx -2\pi G\rho a \left[ 1 + \frac{H}{a} - \left( 1 + \frac{H^2}{2a^2} \right) \right] \\ &= -2\pi G\rho H \left( 1 - \frac{H}{2a} \right) \\ &\approx -2\pi G\rho H \\ &= -0.1119 H . \end{aligned} \quad (5.117)$$

This formula is approximated two times, and the numerical value is obtained with the average crustal density  $\rho = 2.67 \text{ g cm}^{-3}$ . As in the case of free air correction, the unit of  $\Delta_2 g$  is mGal, and that of  $H$  is m. Based on the derivation, we understand  $\Delta_2 g$  as an approximation of the gravitational attraction of a thin cylinder.

The gravity anomaly obtained by applying both the corrections  $\Delta_1 g$  and  $\Delta_2 g$  is

$$g_0 - \gamma_0 = g + \Delta_1 g + \Delta_2 g - \gamma_0 , \quad (5.118)$$

which is referred to as incomplete Bouguer anomaly. It is computed by removing the effect of the mass above the geoid assuming the ground to be a plane parallel to the geoid, and then project the gravity to the geoid using the free air correction.

### 5.3.3 Terrain Correction and Complete Bouguer Anomaly

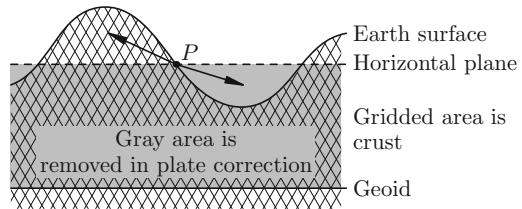
The plate correction removes the effect of a layer of mass between the local horizontal plane (that passes through the ground point where gravity is measured) and the geoid. Evidently, it is more accurate to consider the actual topography. Therefore, we add a correction, called terrain correction denoted as  $\Delta_3g$ , on top of the plate correction  $\Delta_2g$ , such that their sum  $\Delta_2g + \Delta_3g$ , which is referred to as topographic correction, removes the effect of all masses above the geoid and below the Earth's surface.

As shown in Fig. 5.7, the terrain correction at a point  $P$  includes two parts. One part is from places where the ground is above the horizontal plane through  $P$ , and the mass between the plane and the ground is to be removed. The other part is from places where the ground is below the horizontal plane through  $P$ , and the hollow between the plane and the ground should be filled with mass; this is due to the fact that mass is assumed to exist there and its effect is removed in the plate correction.

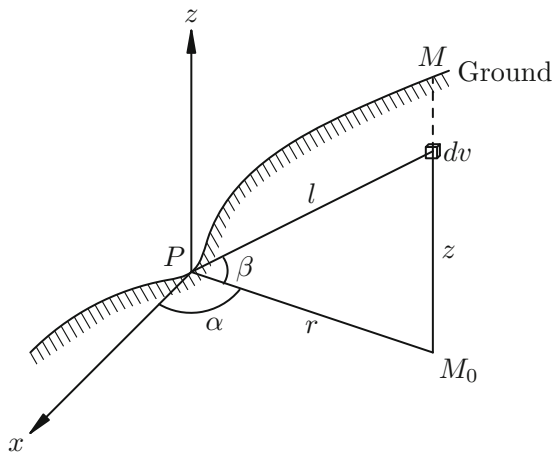
The contribution to  $\Delta_3g$  from any place is positive, also as shown in Fig. 5.7. For a place above the horizontal plane through  $P$ , the contribution of the mass above the horizontal plane to the gravitational attraction at  $P$  points upward. Therefore, the gravity at  $P$  would become larger after removing this contribution. So, the correction is positive. For a place below the horizontal plain through  $P$ , adding mass to fill the hollow makes the gravitational attraction at  $P$  larger. So, the correction is also positive.

In order to formulate  $\Delta_3g$ , we use a local cylindrical coordinate system  $Pr\alpha z$ , as shown in Fig. 5.8, where the gravity correction at  $P$  is to be computed. Consider a volume element  $dv$ . Its vertical projection on the ground is  $M$ , and that on the horizontal plane is  $M_0$ . The contribution of the volume element to the gravitational attraction at  $P$  along the upward direction is  $(\rho dv/l^2) \sin \beta$ , where  $\rho$  is the density assumed to be constant, and other symbols are defined in Fig. 5.8. Evidently,  $\sin \beta = z/l$ ,  $l^2 = r^2 + z^2$  and  $dv = r d\alpha dr dz$ . We consider only the total

**Fig. 5.7** Terrain correction



**Fig. 5.8** Local coordinate system  $Ox\alpha z$



contribution of the region within the distance  $a$  measured in the horizontal plane. Hence,

$$\Delta_3 g = G\rho \int_0^{2\pi} d\alpha \int_0^a dr \int_0^{\Delta H} \frac{zr}{(r^2 + z^2)^{3/2}} dz, \quad (5.119)$$

where  $\Delta H = M_0M = H_M - H_P$  is the difference between the heights of the ground points  $M$  and  $P$ . The integral over  $z$  can be easily obtained analytically. We obtain

$$\Delta_3 g = G\rho \int_0^{2\pi} d\alpha \int_0^a \left[ 1 - \frac{r}{(r^2 + \Delta H^2)^{1/2}} \right] dr, \quad (5.120)$$

which can be evaluated numerically using a digital ground elevation model.

The formula (5.120) is the projection of the gravitational attraction of the terrain mass in upward direction at  $P$  assuming  $M$  to be above  $P$ ; hence it is positive. It can be readily shown that the formula of terrain correction is the same when  $M$  is below  $P$ , where the correction is the projection of the gravitational attraction of the terrain mass in downward direction.

The gravity anomaly computed according to

$$g_0 - \gamma_0 = g + \Delta_1 g + \Delta_2 g + \Delta_3 g - \gamma_0 \quad (5.121)$$

is referred to as complete Bouguer anomaly. It is computed by removing the effect of all masses above the geoid, and then projecting the gravity to the geoid using the free air correction.

The gravity anomaly obtained by applying the free air and terrain corrections,

$$g_0 - \gamma_0 = g + \Delta_1 g + \Delta_3 g - \gamma_0, \quad (5.122)$$

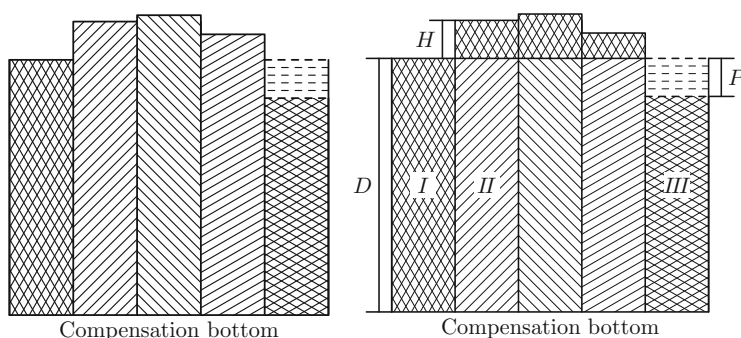
is called Faye anomaly. It is computed by adjusting the ground to a horizontal plane through the point where gravity is measured, and then projecting the gravity to the geoid using the free air correction. In the literature, Faye correction and anomaly are sometimes used as synonyms of free-air correction and anomaly. Hence, there is a need to verify or clarify how it is defined.

### 5.3.4 Isostasy

If the mass above the geoid is the main cause of the gravity anomaly, the complete Bouguer anomaly should be very small, as the effect of the mass above the geoid on gravity is completely removed. However, the reality is the reverse. The complete Bouguer anomaly obtained is very large in magnitude in mountainous regions, and the sign is negative. This implies that there may be some kind of mass deficit under mountains. Observations of the deflection of the vertical demonstrate the same phenomenon; the magnitude of the deflection of the vertical near a mountain is far smaller than what is caused by the visible topographic masses above the geoid, which may also imply a mass deficit under mountains. These observed phenomenon have led to the hypothesis of isostasy: the mass above the geoid is compensated by a mass deficit under the geoid, called isostatic compensation. Several models of isostasy have been proposed, which may be considered as models of crustal density based on gravity measurements.

#### 5.3.4.1 The Pratt and Pratt-Hayford Models

Pratt devise an isostatic model as shown in Fig. 5.9 (left). The idea is that the crustal density is independent of height above a compensation bottom, but it varies horizontally such that all columns of equal cross section area from the compensation bottom to the Earth's surface have the same mass. A modified version of this model, the Pratt-Hayford isostatic model, which is usually used for gravity reduction, is shown in Fig. 5.9 (right). This model is based on the assumption that the crustal density above the geoid is constant (to agree with that in the complete Bouguer



**Fig. 5.9** The Pratt (left) and Pratt-Hayford (right) isostatic models

anomaly),  $\rho = 2.67 \text{ g cm}^{-3}$ , but the crustal density below the geoid and above a compensation bottom of depth  $D$  varies horizontally, such that all columns of equal cross section area from the compensation bottom to the Earth's surface have the same mass. With this assumption, the pressure at the compensation bottom is identical under all columns, meaning that it is position independent. We still use  $H$  to denote the orthometric height over land, and use  $\rho'$  to denote the density between the geoid and the compensation bottom. When  $H = 0$ , such as the column *I* in Fig. 5.9 (right), there is no compensation under the geoid, and the density of the column under the geoid is assumed to be  $\rho' = \rho$ . When  $H > 0$ , such as the column *II* in Fig. 5.9(right), the density above the geoid is  $\rho$ , and that below is  $\rho'$ . As the cross section areas of the columns *I* and *II* are equal, and their masses from the compensation bottom to the Earth's surface are also equal, we have

$$\rho D = \rho H + \rho' D. \quad (5.123)$$

This leads to a density deficit of

$$\Delta\rho = \rho - \rho' = \frac{H}{D} \rho. \quad (5.124)$$

Over oceans, such as the column *III* in Fig. 5.9 (right), we have

$$\rho D = \rho_w P + \rho' (D - P), \quad (5.125)$$

where  $\rho_w$  is the density of ocean water, and  $P$  the ocean depth. The density deficit is

$$\Delta\rho = \rho - \rho' = -\frac{(\rho - \rho_w)P}{D - P}. \quad (5.126)$$

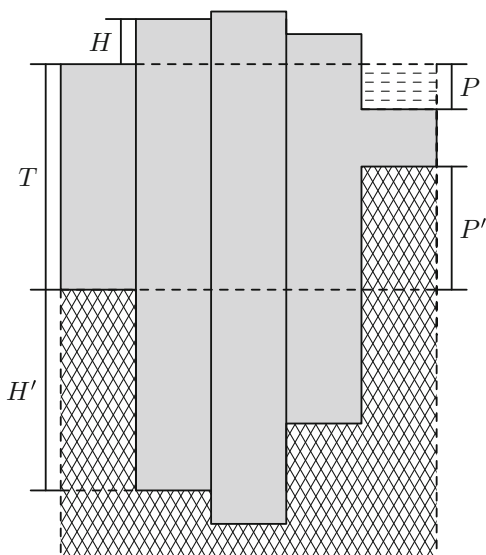
This deficit has a negative sign, meaning that it is indeed a surplus of mass in compensation to the deficit of mass in the ocean.

It is to be noted that the compensation region under the ground is from the geoid to the compensation bottom, but that under oceans is from the ocean bottom to the compensation bottom. The compensation depth is mostly  $D = 100 \text{ km}$ .

### 5.3.4.2 The Airy-Heiskanen Models

Heiskanen used a model devised by Airy as shown in Fig. 5.10 for gravity reduction. A homogeneous crust of density  $\rho$  floats on a homogeneous mantle of larger density  $\rho_1$ , as if all crustal columns in the figure are detached from each other, and the weight of each column is balanced by the buoyancy. Normally, the densities  $\rho = 2.67 \text{ g cm}^{-3}$  and  $\rho_1 = 3.27 \text{ g cm}^{-3}$  are adopted. We denote the crustal thickness when  $H = 0$  as  $T$ , which is referred to as the normal thickness of the crust. In most regions,  $T \approx 30 \text{ km}$ . In this model, the topographic mass above the geoid is compensated by the deficit of mass within the crust below the depth  $T$  in contrast to

**Fig. 5.10** The Airy-Heiskanen isostatic model



the larger density of the mantle. We use  $H'$  to denote the depth of the crustal bottom below the surface of depth  $T$ , as shown in Fig. 5.10. We then have, over land,

$$\rho H = (\rho_1 - \rho) H'. \quad (5.127)$$

Hence,

$$H' = \frac{\rho}{\rho_1 - \rho} H. \quad (5.128)$$

As  $\rho/(\rho_1 - \rho) = 4.45$ , this formula implies that a 1 km high mountain should have a 4.45 km deep root. Over the oceans, the balance between the mass and buoyancy of a column leads to

$$(\rho - \rho_w)P = (\rho_1 - \rho)P', \quad (5.129)$$

where  $P$  is the ocean depth, and  $P'$  the height of anti-root of the ocean, as shown in Fig. 5.10. We obtain

$$P' = \frac{\rho - \rho_w}{\rho_1 - \rho} P. \quad (5.130)$$

In practical computations, oceans and lands can be handled altogether with the formulation for lands. The idea is to “compress water” of depth  $P$  to a “rock-equivalent” layer of thickness  $t$  with the density of crust, i.e.,  $\rho t = \rho_w P$ ,  $t = (\rho_w/\rho)P$ . In this way, oceans behave the same as lands with a height of  $-(P - t)$ .

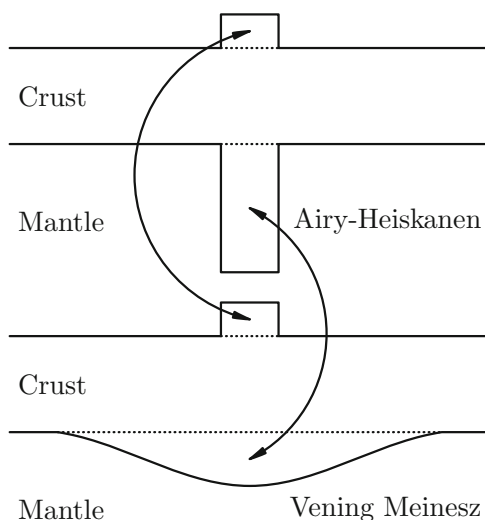
The Airy-Heiskanen model is also based on an idealized assumption: the crustal mass above the geoid at any place is fully compensated by a deficit of mass strictly below. This is in fact based on an assumption that all the columns as shown in Fig. 5.10 can move independently in vertical direction, meaning that no friction exists between the vertical walls of adjacent columns. This assumption appears totally unrealistic at a first glance, as both the crust and mantle are solid. However, the topographic mass is a long term load of millions of years. Under the action of such a long time scale forcing, both the crust and mantle may flow slowly to reach an ultimate balanced state. The Airy-Heiskanen isostatic model is in fact the simplest model of such a balance; the balance is fully achieved vertically, but not horizontally.

### 5.3.4.3 The Vening Meinesz Model

Vening Meinesz generalized the idea of the Airy-Heiskanen model by taking into account of more geophysical considerations. In Fig. 5.10, imagine a crustal column that is very high above the geoid, but its bottom is not as deep as that of the root required to achieve isostatic compensation in the sense of the Airy-Heiskanen model. An evident fact would be that this column tends to sink under the action of its weight. The reality would not be as ideal as the Airy-Heiskanen model where adjacent columns simply let this column slide down by the walls. Instead, adjacent columns are attached to this column, and would yield and also sink under the downward dragging of this column. As a result, adjacent columns would also contribute to the final balance. A conceptual comparison between the Airy-Heiskanen and Vening Meinesz isostatic models is demonstrated in Fig. 5.11.

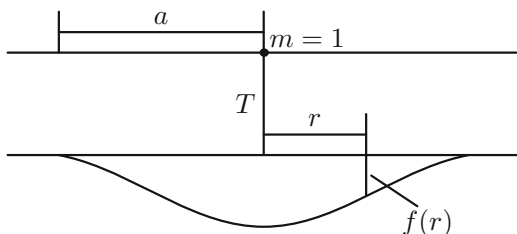
In order to develop a mathematical theory for the Vening Meinesz model, we consider first the load of a unit point mass  $m = 1$  as shown in Fig. 5.12. Under

**Fig. 5.11** A conceptual comparison between the Airy-Heiskanen and Vening Meinesz isostatic models





**Fig. 5.12** Definition of compensation function as the compensation to a unit point mass



the action of this load, both the top and bottom of the crust would yield and curve, leading to a concave on the top and a convex at the bottom. We assume that the concave on the top is refilled with crustal material, such that the top remains a plane. As a result, the convex at bottom would be larger, which is to be also included in the mathematical model. We use  $T$  to denote the normal thickness of the crust, i.e., the thickness of the crust when  $H = 0$ , and express the depth of the convex below  $T$  as  $f(r)$ , called compensation function, where  $r$  is the horizontal distance to the unit point mass. This implies that the compensation is assumed to be symmetrical about the load, which is rational. We further assume that the compensation is limited to the region within the distance  $a$ , and denote the region as  $S$ . The requirement of mass conservation (deficit of mass below the geoid is equal to the mass above the geoid) is then

$$(\rho_1 - \rho) \int_S f(r) dS = 1. \quad (5.131)$$

We assume that the lower convex is the result of bending (without breaking) of the crust. An additional constraint to  $f(r)$  would then be

$$\left[ \frac{d}{dr} f(r) \right]_{r=a} = 0, \quad \left[ \frac{d}{dr} f(r) \right]_{r=0} = 0. \quad (5.132)$$

The first identity implies that the lower convex is tangential to the horizontal plane at its boundary where  $r = a$ , and the second one is a consequence of the fact that the lower convex must be smooth and symmetrical around the point mass.

The compensation function  $f(r)$  can be empirically assigned based on a theoretical model of the bending of an elastic plate, or determined empirically. When  $f(r)$  is known, the compensation depth below  $T$  can be obtained by multiplying  $f(r)$  with the mass per unit area of the horizontal plane computed using topography,  $\rho H$ , and then integrating over the whole region of interest. This is in fact a convolution, and is not to be further discussed in this book.

### 5.3.5 Isostatic Correction and Isostatic Anomaly

The formulation of the isostatic correction is almost identical to that of the terrain correction (5.119). We still consider only the total contribution of a region within the distance  $a$  (to be distinguished from the parameter  $a$  in the Vening Meinesz model, which is specific to the model only) measured in the horizontal plane. We take the Airy-Heiskanen model as example. What we need is the effect of gravity in the downward direction by filling the mass deficit region with the density  $\Delta\rho = \rho_1 - \rho$ . Hence, we need to replace  $\rho$  by  $\Delta\rho$  in (5.119), and the limits of integration for  $z$  should be from  $-(H_P + T + H')$  to  $-(H_P + T)$ ,  $H_P$  being the orthometric height of the point  $P$  where the correction is computed. The detail of the formulation is not to be further discussed. We denote this correction as  $\Delta_4g$ , called isostatic correction. The gravity anomaly defined by

$$g_0 - \gamma_0 = g + \Delta_1g + \Delta_2g + \Delta_3g + \Delta_4g - \gamma_0 \quad (5.133)$$

is referred to as isostatic anomaly. It is computed by removing all the masses above the geoid, refill the masses removed to a region below the geoid where the mass deficit is supposed to be present, and then project the gravity to the geoid using the free air correction.

Imagine an extreme Pratt-Hayford isostatic model whose compensation depth  $D$  is infinitesimally small. In this case, the mass deficiency is restricted in a thin layer on the geoid. Removing the topographic mass and filling the mass deficiency in this thin layer is equivalent to compress the topographic mass to within this layer. This case is referred to as Helmert condensation. Hence, in this case, the isostatic correction is referred to as Helmert correction, and the isostatic anomaly is referred to as Helmert anomaly.

### 5.3.6 Determination of Isostatic Models: The Concept

As the isostatic anomaly is computed by adjusting the crustal mass to the most possible homogeneous distribution, and is projected to the geoid, it should be extremely small if the adjustment is perfect, and other heterogeneity in mass distribution is minimal. Hence, a reasonable approach to determine an isostatic model using gravity measurements and topography data is

$$\sum_i (g_i + \Delta_1g_i + \Delta_2g_i + \Delta_3g_i + \Delta_4g_i - \gamma_{0i})^2 = \min, \quad (5.134)$$

where the subscript  $i$  indicates the  $i$ th gravity measurement station. The logic is using this relation to estimate the parameters of the isostatic model, e.g.,  $D$  for the Pratt-Hayford isostatic model and  $T$  for the Airy-Heiskanen model.

In all the isostatic models discussed so far, the geoid is taken as the upper surface of compensation. This is mostly due to the fact that our objective here is gravity reduction to remove or relocate the mass above the geoid while maintaining the Earth's total mass unchanged. However, in the context of geophysics where the reality is sought other than making models in favor of gravity reduction, the constraints in establishing isostatic models could be loosened. For example, instead of fixing on the geoid, the top surface of compensation could be above or below the geoid depending on the region studied. Furthermore, the full excess-deficit mass compensation could also be compromised, e.g., in the case of Airy-Heiskanen model, a parameter like percentage of compensation could be included at the right-hand side of (5.128).

The Mohorovičić discontinuity, or briefly Moho, which is the bottom of the crust, is not a surface purely determined using gravity data; the crust is also mapped using seismic tomography with finer and finer resolution, from averages over  $5^\circ \times 5^\circ$  grids to  $2^\circ \times 2^\circ$ , and then to  $1^\circ \times 1^\circ$ , which are provided as crust models named CRUST5.1, CRUST2.0 and CRUST1.0 (Laske et al., 2013), respectively. The last one is extended to include the lithospheric mantle in a model referred to as LITHO1.0 (Pasyanos et al., 2014). Data of all the models mentioned above are available over the internet, including the density that we need. We have compared the LITHO1.0 crust with the Airy-Heiskanen style isostatic model, which can be regarded as a verification using independent data whether isostasy is realistic. The result confirms that the LITHO1.0 is indeed in reasonably good agreement with the Airy-Heiskanen style isostasy, except some particular regions like tectonic subduction zones. However, we must keep in mind that the full isostatic compensation as discussed in this book is an idealized assumption, but not the exact reality. Considering that the LITHO1.0 is provided as averages over  $1^\circ \times 1^\circ$  grids, it confirms isostasy only down to this spatial scale, i.e., the order of  $\sim 100$  km or above (a  $1^\circ$  great arc on the Earth's surface corresponds to 111 km). This is about the lower limit of spatial scale that the crust is supposed to be in isostatic compensation; the topographic mass in shorter spatial scale is supposed to be supported by the rigidity of the lithosphere.

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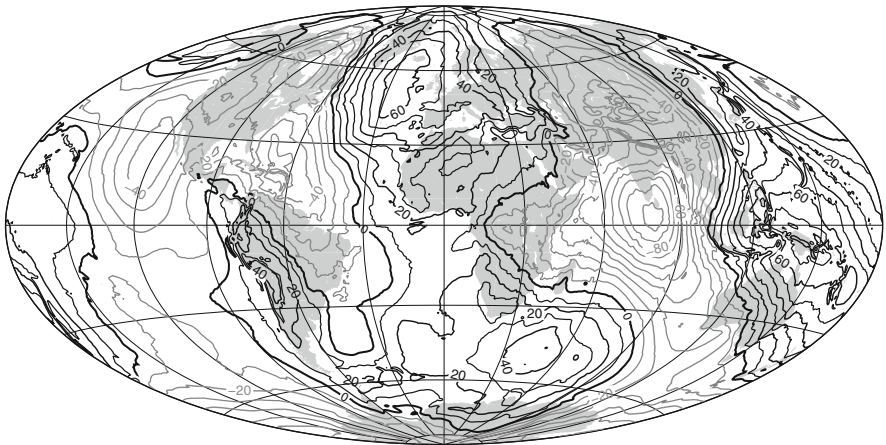
## 5.4 A Global Model of the Earth's Gravitational Potential

The global model of the Earth's gravitational potential is referred to as Earth gravitational model (EGM). For illustrative purpose, we chose to show the EGM GGM05C (Ries et al., 2016), which is computed by combining data from the Gravity Recovery and Climate Experiment (GRACE) satellite mission, the Gravity Field and Steady-State Ocean Circulation Explorer (GOCE) satellite mission, satellite altimetry and ground gravimetry. This model provides the Stokes coefficients up to degree and order 360. Here we list them up to degree and order 10 in Table 5.2. In Fig. 5.13, we show the geoidal height computed from this model by subtracting a model of normal gravitational potential.

**Table 5.2** Coefficients ( $\times 10^7$ ) of the EGM GGM05C up to degree and order 10

| $n$ | $k$ | $\overline{C}_n^k$ | $\overline{S}_n^k$ | $n$ | $k$ | $\overline{C}_n^k$ | $\overline{S}_n^k$ | $n$ | $k$ | $\overline{C}_n^k$ | $\overline{S}_n^k$ |
|-----|-----|--------------------|--------------------|-----|-----|--------------------|--------------------|-----|-----|--------------------|--------------------|
| 2   | 0   | $\overline{C}_2$   | 0                  | 3   | 0   | 9.57165            | 0                  | 4   | 0   | 5.39982            | 0                  |
| 5   | 0   | 0.68650            | 0                  | 6   | 0   | -1.49976           | 0                  | 7   | 0   | 0.90501            | 0                  |
| 8   | 0   | 0.49478            | 0                  | 9   | 0   | 0.28023            | 0                  | 10  | 0   | 0.53345            | 0                  |
| 2   | 1   | -0.00310           | 0.01411            | 3   | 1   | 20.30447           | 2.48241            | 4   | 1   | -5.36181           | -4.73577           |
| 5   | 1   | -0.62915           | -0.94343           | 6   | 1   | -0.75943           | 0.26526            | 7   | 1   | 2.80886            | 0.95160            |
| 8   | 1   | 0.23141            | 0.58905            | 9   | 1   | 1.42149            | 0.21416            | 10  | 1   | 0.83754            | -1.31094           |
| 2   | 2   | 24.39373           | -14.00294          | 3   | 2   | 9.04765            | -6.19007           | 4   | 2   | 3.50492            | 6.62505            |
| 5   | 2   | 6.52059            | -3.23343           | 6   | 2   | 0.48636            | -3.73769           | 7   | 2   | 3.30396            | 0.93020            |
| 8   | 2   | 0.80020            | 0.65300            | 9   | 2   | 0.21412            | -0.31680           | 10  | 2   | -0.93978           | -0.51265           |
| 3   | 3   | 7.21285            | 14.14400           | 4   | 3   | 9.90861            | -2.00951           | 5   | 3   | -4.51831           | -2.14942           |
| 6   | 3   | 0.57251            | 0.08973            | 7   | 3   | 2.50463            | -2.17097           | 8   | 3   | -0.19359           | -0.85942           |
| 9   | 3   | -1.60602           | -0.74250           | 10  | 3   | -0.06991           | -1.54128           | 4   | 4   | -1.88492           | 3.08819            |
| 5   | 4   | -2.95323           | 0.49811            | 6   | 4   | -0.85997           | -4.71427           | 7   | 4   | -2.74986           | -1.24051           |
| 8   | 4   | -2.44345           | 0.69812            | 9   | 4   | -0.09349           | 0.19899            | 10  | 4   | -0.84459           | -0.79024           |
| 5   | 5   | 1.74814            | -6.69355           | 6   | 5   | -2.67163           | -5.36496           | 7   | 5   | 0.01647            | 0.17931            |
| 8   | 5   | -0.25702           | 0.89206            | 9   | 5   | -0.16315           | -0.54044           | 10  | 5   | -0.49286           | -0.50617           |
| 6   | 6   | 0.09480            | -2.37384           | 7   | 6   | -3.58799           | 1.51793            | 8   | 6   | -0.65971           | 3.08945            |
| 9   | 6   | 0.62784            | 2.22964            | 10  | 6   | -0.37587           | -0.79771           | 7   | 7   | 0.01523            | 0.24103            |
| 8   | 7   | 0.67257            | 0.74864            | 9   | 7   | -1.17981           | -0.96921           | 10  | 7   | 0.08257            | -0.03049           |
| 8   | 8   | -1.24033           | 1.20540            | 9   | 8   | 1.88132            | -0.03002           | 10  | 8   | 0.40590            | -0.91711           |
| 9   | 9   | -0.47558           | 0.96877            | 10  | 9   | 1.25385            | -0.37942           | 10  | 10  | 1.00423            | -0.23864           |

$\overline{C}_2 = -4841.69457$  (after multiplication by  $10^7$ )



**Fig. 5.13** The geoidal height based on GGM05C and GRS80. Thick black line: 0 geoidal height; Black line: positive geoidal height; Gray line: negative geoidal height

## 5.5 Earth's Internal Shape and Gravity Assuming Hydrostatic Equilibrium

### 5.5.1 Wavre's Formula of the Gravity Field

We know that, in hydrostatic equilibrium, the equi-value surfaces of density, gravity potential, and pressure coincide. The coincidence of the equi-value surfaces of density and gravity potential is a sufficient condition to determine the shapes of these surfaces. Guo (2023) followed Jeffreys (1976) to adopt a simpler formulation that relies on the spherical harmonic series of the gravitational potential that are, strictly speaking, divergent. In this section, we present a formulation that avoids the divergence problem.

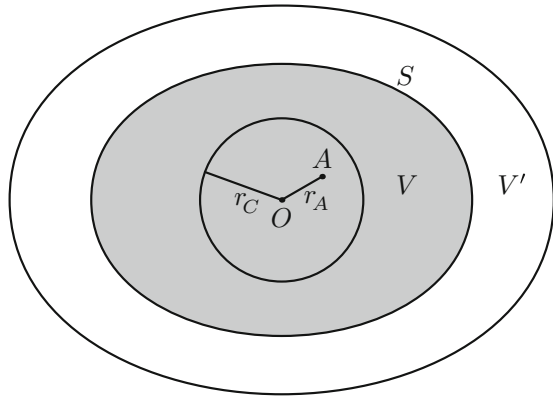
First of all, we derive a formula that relates density to the equipotential surface of gravity, called Wavre's formula. This formula does not require the assumption that the Earth's interior is in hydrostatic equilibrium, which is the reason why the combination of this formula with the coincidence of the equi-value surfaces of density and gravity potential determines the shapes of these surfaces.

As shown in Fig. 5.14, the equipotential surface of gravity  $W = W_S$  denoted as  $S$  divides the Earth into an inner region  $V$ , which is shaded, and an outer region  $V'$ . By applying Green's third theorem (2.256) to  $V$  and its surface  $S$ , and setting  $v = W$ , the gravity potential, we obtain

$$\int_V \frac{\nabla^2 W}{l} dV = \oint_S \frac{1}{l} \frac{\partial W}{\partial n} dS - W_S \oint_S \frac{\partial}{\partial n} \left( \frac{1}{l} \right) dS - pW(A), \quad (5.135)$$

where  $l$  is the distance to a fixed point  $A$ , and  $p$  is defined in (2.257), which takes different values when  $A$  is inside, on or outside  $S$ . The gravitational potential  $\phi$  satisfies Poisson's equation (3.82). It can be readily shown that the centrifugal potential  $\psi$  satisfies Poisson's equation  $\nabla^2 \psi = 2\omega^2$ . Hence, the gravity potential

**Fig. 5.14** Separation of the Earth's interior into two parts by an equipotential surface of gravity



satisfies Poisson's equation

$$\nabla^2 W = \nabla^2 \phi + \nabla^2 \psi = -4\pi G\rho + 2\omega^2. \quad (5.136)$$

By substituting this equation into (5.135), and then using (2.258), we obtain

$$-4\pi G \int_V \frac{\rho}{l} dV + 2\omega^2 \int_V \frac{1}{l} dV = \oint_S \frac{1}{l} \frac{\partial W}{\partial n} dS + p[W_S - W(A)]. \quad (5.137)$$

By definition, the gravity potential at  $A$  is

$$W(A) = G \int_V \frac{\rho}{l} dV + G \int_{V'} \frac{\rho}{l} dV + \psi(A), \quad (5.138)$$

where  $\psi(A)$  is the centrifugal potential at  $A$ . By eliminating the term  $\int_V (\rho/l) dV$  in the above two equations, we obtain

$$\begin{aligned} & \frac{\omega^2}{2\pi} \int_V \frac{1}{l} dV + G \int_{V'} \frac{\rho}{l} dV + \frac{1}{4\pi} \oint_S \frac{1}{l} \frac{\partial W}{\partial n} dS + \psi(A) \\ &= W(A) + \frac{p}{4\pi} [W_S - W(A)]. \end{aligned} \quad (5.139)$$

Finally, the substitution of the concrete values of  $p$  given by (2.257) yields, with  $\partial W/\partial n = -g$ ,

$$\frac{\omega^2}{2\pi} \int_V \frac{1}{l} dV + G \int_{V'} \frac{\rho}{l} dV - \frac{1}{4\pi} \oint_S \frac{g}{l} dS + \psi(A) = \begin{cases} W(A), & A \text{ outside } S; \\ W_S, & A \text{ inside or on } S; \end{cases} \quad (5.140)$$

which is referred to as Wavre's formula. It is noted that  $W(A) = W_S$  when  $A$  is on  $S$ .

In the study of the Earth's internal figure, it is the case of (5.140) when  $A$  is inside  $S$  that is used,

$$\frac{\omega^2}{2\pi} \int_V \frac{1}{l} dV + G \int_{V'} \frac{\rho}{l} dV - \frac{1}{4\pi} \oint_S \frac{g}{l} dS + \psi(A) = W_S. \quad (5.141)$$

In order to develop  $1/l$  into a convergent spherical harmonic series, we draw a spherical body  $C$  of radius  $r_C$  completely inside  $S$  with center on the origin  $O$ , and let  $A$  be inside the sphere. As  $G \int_C (1/l) dV$  is the gravitational potential of the spherical body  $C$  with density  $\rho = 1$  at the point  $A$ , we have, according to (3.88),

$$G \int_C \frac{1}{l} dV = \frac{2}{3} \pi G (3r_C^2 - r_A^2), \quad (5.142)$$

where  $r_A$  is the radial distance of  $A$ . This formula can be used to develop (5.141) to

$$\frac{\omega^2}{2\pi} \int_{V-C} \frac{1}{l} dV + G \int_{V'} \frac{\rho}{l} dV + \frac{1}{4\pi} \oint_S \frac{g}{l} dS + \psi(A) = W_S - \omega^2 \left( r_C^2 - \frac{1}{3} r_A^2 \right). \quad (5.143)$$

We further choose the polar axis to be the rotation axis of the Earth. The centrifugal potential  $\psi(A)$  can then be expressed as a spherical harmonic series

$$\psi(A) = \frac{1}{2} \omega^2 r_A^2 \sin^2 \theta_A = \frac{1}{2} \omega^2 r_A^2 (1 - \cos^2 \theta_A) = \frac{1}{3} \omega^2 r_A^2 - \frac{1}{3} \omega^2 r_A^2 P_2(\cos \theta_A). \quad (5.144)$$

The substitution of this formula into (5.143) yields

$$\frac{\omega^2}{2\pi} \int_{V-C} \frac{1}{l} dV + G \int_{V'} \frac{\rho}{l} dV + \frac{1}{4\pi} \oint_S \frac{g}{l} dS - \frac{1}{3} \omega^2 r_A^2 P_2(\cos \theta_A) = W_S - \omega^2 r_C^2, \quad (5.145)$$

which is the equation to be used to study the internal figure of the Earth. We use  $r$  to denote the radial distance of the volume element  $dV$  or the area element  $dS$ . As  $r > r_A$  always holds, we can express  $1/l$  as the convergent Legendre series

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r_A^n}{r^{n+1}} P_n(\cos \psi), \quad (5.146)$$

where  $\psi$  is the angle between  $dV$  or  $dS$  and  $A$  with respect to the origin  $O$  (it is not the centrifugal potential). By substituting (5.146) into (5.145), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} r_A^n \left\{ \frac{\omega^2}{2\pi} \int_{V-C} \frac{P_n(\cos \psi)}{r^{n+1}} dV + G \int_{V'} \frac{\rho P_n(\cos \psi)}{r^{n+1}} dV \right. \\ \left. + \frac{1}{4\pi} \oint_S \frac{g P_n(\cos \psi)}{r^{n+1}} dS \right\} - \frac{1}{3} \omega^2 r_A^2 P_2(\cos \theta_A) = W_S - \omega^2 r_C^2. \end{aligned} \quad (5.147)$$

This is an identity depending on the position of  $A$ . As the position of  $A$  can be arbitrarily chosen within the sphere  $C$ , the coefficients of all powers of  $r_A$  must vanish, resulting in

$$\begin{aligned} \frac{\omega^2}{2\pi} \int_{V-C} \frac{P_n(\cos \psi)}{r^{n+1}} dV + G \int_{V'} \frac{\rho P_n(\cos \psi)}{r^{n+1}} dV \\ + \frac{1}{4\pi} \oint_S \frac{g P_n(\cos \psi)}{r^{n+1}} dS = \begin{cases} W_S - \omega^2 r_C^2, & n = 0; \\ \frac{1}{3} \omega^2 P_2(\cos \theta_A), & n = 2; \\ 0, & \text{Otherwise.} \end{cases} \end{aligned} \quad (5.148)$$

This is the equation we will use to derive the equation for the internal shape.

### 5.5.2 Expression of Equi-value Surfaces of Density in Spherical Harmonic Series

The radial distance of any point on an equi-value surface of density is a function of the coordinates  $\theta$  and  $\lambda$ , which we write as

$$r = q \left[ 1 + \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right]. \quad (5.149)$$

where each value of  $q$  corresponds to an equi-value surface of density, and the coefficients  $\epsilon_m^l$  are assumed to be functions of  $q$  alone. We assume that the coefficients  $\epsilon_m^l \ll 1$ , i.e., the deviation of the equi-value surfaces of density from a sphere is infinitesimally small. In this way, our subsequent derivations will be accurate to the first order of  $\epsilon_m^l$ .

In hydrostatic equilibrium, the gravity potential  $W$  is constant on any equi-value surface of density given by (5.149), and hence, it is a function of  $q$  alone, and its equation can be written as

$$W(r, \theta, \lambda) = W(q), \quad (5.150)$$

where we emphasize that  $r, \theta$  and  $\lambda$  are related to  $q$  according to (5.149). We remark that the density and pressure depend on position in the same form. The gravity is then

$$\vec{g} = \nabla W = \frac{dW}{dq} \frac{\partial q}{\partial r} \hat{e}_r + \frac{dW}{dq} \frac{1}{r} \frac{\partial q}{\partial \theta} \hat{e}_\theta + \frac{dW}{dq} \frac{1}{r \sin \theta} \frac{\partial q}{\partial \lambda} \hat{e}_\lambda, \quad (5.151)$$

Accurate to first order of  $\epsilon_m^l$ , we obtain from (5.149).

$$q = r \left[ 1 - \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right]. \quad (5.152)$$

By substituting this formula into (5.151), we obtain, accurate to the first order of  $\epsilon_m^l$ ,

$$\begin{aligned} \vec{g} = \frac{dW}{dq} \left\{ \left[ 1 - \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( \epsilon_m^l + q \frac{d\epsilon_m^l}{dq} \right) \mathcal{H}_m^l(\theta, \lambda) \right] \hat{e}_r \right. \\ \left. - \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \frac{\partial}{\partial \theta} \mathcal{H}_m^l(\theta, \lambda) \hat{e}_\theta - \frac{1}{\sin \theta} \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \frac{\partial}{\partial \lambda} \mathcal{H}_m^l(\theta, \lambda) \hat{e}_\lambda \right\}. \end{aligned} \quad (5.153)$$



Two formulae can be inferred from (5.153). First, the formula of the norm of the gravity,  $g = |\vec{g}|$ , is

$$\begin{aligned} g &= -\frac{dW}{dq} \left\{ \left[ 1 - \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( \epsilon_m^l + q \frac{d\epsilon_m^l}{dq} \right) \mathcal{H}_m^l(\theta, \lambda) \right]^2 \right. \\ &\quad \left. + \left[ \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \frac{\partial}{\partial \theta} \mathcal{H}_m^l(\theta, \lambda) \right]^2 + \left[ \frac{1}{\sin \theta} \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \frac{\partial}{\partial \lambda} \mathcal{H}_m^l(\theta, \lambda) \right]^2 \right\}^{1/2} \\ &= -\frac{dW}{dq} \left[ 1 - \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( \epsilon_m^l + q \frac{d\epsilon_m^l}{dq} \right) \mathcal{H}_m^l(\theta, \lambda) \right], \end{aligned} \quad (5.154)$$

where the negative sign is due to the fact that  $dW/dq$  is negative. Second, as  $\vec{g}$  is normal to the equipotential surface, it is in the same direction as the unit outward normal of the equi-value surface of density (5.149),  $\hat{n} = -\vec{g}/|\vec{g}|$  (as  $\hat{n}$  points upward,  $\vec{g}$  downward, a negative sign appears), which leads to, accurate to the first order of  $\epsilon_m^l$ ,

$$\hat{n} = \hat{e}_r - \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \frac{\partial}{\partial \theta} \mathcal{H}_m^l(\theta, \lambda) \hat{e}_\theta - \frac{1}{\sin \theta} \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \frac{\partial}{\partial \lambda} \mathcal{H}_m^l(\theta, \lambda) \hat{e}_\lambda. \quad (5.155)$$

From  $\hat{e}_r \times \hat{n}$ , we realize that the angle between  $\hat{e}_r$  and  $\hat{n}$  is on the order of the magnitude of  $\epsilon_m^l$ .

In hydrostatic equilibrium, the equipotential surface  $S$  in Fig. 5.14 is also an equi-value surface of density. Hence, its equation is in the same form as (5.149), and we written it as

$$r_S = r_0 \left[ 1 + \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right]. \quad (5.156)$$

The equation of  $\epsilon_m^l$  will be obtained by substituting this formula and (5.149) into (5.148). Here we first derive some preparatory formulae.

The mass enclosed within  $S$  is

$$\begin{aligned} M_S &= \int_V \rho dV = \int_w \left( \int_0^{r_S} \rho r^2 dr \right) dw = \int_w \left( \frac{1}{3} \int_0^{r_S} \rho dr^3 \right) dw \\ &= \int_w \left\{ \frac{1}{3} \int_0^{r_0} \rho d \left[ q^3 \left( 1 + 3 \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right) \right] \right\} dw \\ &= \int_w \left\{ \int_0^{r_0} \rho \left[ q^2 \left( 1 + 3 \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right) \right] \right. \end{aligned}$$

$$\begin{aligned}
& + q^3 \sum_{m=1}^{\infty} \sum_{l=-m}^m \frac{d\epsilon_m^k}{dq} \mathcal{H}_m^l(\theta, \lambda) \Big] dq \Big\} dw \\
& = \int_w \left\{ \int_0^{r_0} \rho q^2 \left[ 1 + \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( 3\epsilon_m^l + q \frac{d\epsilon_m^k}{dq} \right) \mathcal{H}_m^l(\theta, \lambda) \right] dq \right\} dw \\
& = 4\pi \int_0^{r_0} \rho q^2 dq, \tag{5.157}
\end{aligned}$$

where the orthogonality of the spherical harmonics has been used. Setting  $\rho = 1$  leads to the volume within the surface  $S$ ,

$$V_S = 4\pi \int_0^{r_0} q^2 dq = \frac{4}{3}\pi r_0^3. \tag{5.158}$$

This implies that  $r_0$  is the equi-volumetric average radius of  $S$ , i.e., the volume within  $S$  is equal to that of the sphere with a radius of  $r_0$ . The average density of the mass within  $S$  can be obtained from (5.157) and (5.158) to be

$$\bar{\rho} = \frac{M_S}{V_S} = \frac{3}{r_0^3} \int_0^{r_0} \rho q^2 dq. \tag{5.159}$$

According to (5.154), we express the magnitude of gravity on  $S$  as

$$g_S = g_0 \left[ 1 - \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( \epsilon_m^l + r_0 \frac{d\epsilon_m^l}{dr_0} \right) \mathcal{H}_m^l(\theta, \lambda) \right], \tag{5.160}$$

where  $g_0 = -(dW/dq)_{q=r_0}$  remains to be further formulated. According to Green's first identity (2.254) and Poisson's equation of  $W$ , (5.136), we have

$$\oint_S \frac{\partial W}{\partial n} dS = \int_V \nabla^2 W dV = -4\pi G \int_V \rho dV + 2\omega^2 \int_V dV. \tag{5.161}$$

With  $g_S = -\partial W / \partial n$ , this formula can be written as

$$\oint_S g_S dS = 4\pi G M_S - 2\omega^2 V_S, \tag{5.162}$$

where  $M_S$  is the mass within  $S$ , and  $V_S$  the volume within  $S$ . In order to reduce the integral over  $S$  to an integral over the unit sphere  $\omega$ , a relation between  $dS$  and  $d\omega$  is required. Evidently, we have  $dS = (r_S^2 / \cos \alpha) d\omega$ , where  $\alpha$  is the angle between  $\hat{n}$  and  $\hat{e}_r$ , i.e.,  $\alpha = (\hat{n}, \hat{e}_r)$ , which is on the order of magnitude of  $\epsilon_m^l$  according to (5.155). Therefore, accurate to the first order of  $\epsilon_m^l$ , we have

$$dS = r_S^2 d\omega. \tag{5.163}$$

Hence, we can write (5.162) as

$$\int_{\omega} g_S r_S^2 d\omega = 4\pi G M_S - 2\omega^2 V_S. \quad (5.164)$$

According to (5.156) and (5.160), and accurate to the first order of  $\epsilon_m^l$ , we have

$$\begin{aligned} \int_{\omega} g_S r_S^2 d\omega &= \int_{\omega} g_0 \left[ 1 + \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( \epsilon_m^l + q \frac{d\epsilon_m^l}{dq} \right) \mathcal{H}_m^l(\theta, \lambda) \right] \\ &\quad \times r_0^2 \left[ 1 + 2 \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right] d\omega \\ &= \int_{\omega} g_0 r_0^2 \left[ 1 + \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( 3\epsilon_m^l + q \frac{d\epsilon_m^l}{dq} \right) \mathcal{H}_m^l(\theta, \lambda) \right] d\omega \\ &= 4\pi g_0 r_0^2, \end{aligned} \quad (5.165)$$

where the orthogonality of spherical harmonics was used. By substituting this formula into (5.164), and then using (5.157) and (5.158), we obtain

$$g_0 = G \frac{M_S}{r_0^2} - \frac{\omega^2}{2\pi} \frac{V_S}{r_0^2} = \frac{4\pi G}{r_0^2} \int_0^{r_0} \rho q^2 dq - \frac{2}{3} \omega^2 r_0. \quad (5.166)$$

The formula of  $W_S$  will be derived from (5.148) in the next subsection.

### 5.5.3 Equation of the Spherical Harmonic Coefficients

We first derive the analytic formulae for the integrals in (5.148) according to (5.149), (5.156), (5.159), (5.160), and (5.163), and then substitute them back into (5.148) to derive the equation for  $\epsilon_m^l$ . For convenience, we write (5.148) in the form of

$$\frac{\omega^2}{2\pi} I_1 + G I_2 + \frac{1}{4\pi} I_3 = \begin{cases} W_S - \omega^2 r_C^2, & n = 0; \\ \frac{1}{3} \omega^2 P_2(\cos \theta_A), & n = 2; \\ 0, & \text{Otherwise;} \end{cases} \quad (5.167)$$

where the symbols  $I_1$ ,  $I_2$  and  $I_3$  are defined as

$$I_1 = \int_{V-C} \frac{P_n(\cos \psi)}{r^{n+1}} dV, \quad I_2 = \int_{V'} \frac{\rho P_n(\cos \psi)}{r^{n+1}} dV, \quad I_3 = \oint_S \frac{g P_n(\cos \psi)}{r^{n+1}} dS, \quad (5.168)$$

which are integrals to be evaluated analytically.

The integral  $I_1$  can be reduced to

$$I_1 = \int_{\omega} d\omega \int_{r_C}^{r_S} \frac{P_n(\cos \psi)}{r^{n+1}} r^2 dr, \quad (5.169)$$

where  $\omega$  is the unit sphere, and  $d\omega = \sin \theta d\theta d\lambda$ . As  $P_n(\cos \psi)$  is independent of  $r$ , the integral of  $r$  can be evaluated first,

$$I_1 = \int_{\omega} \left( \int_{r_C}^{r_S} \frac{dr}{r^{n-1}} \right) P_n(\cos \psi) d\omega = -\frac{1}{n-2} \int_{\omega} \left( \frac{1}{r_S^{n-2}} - \frac{1}{r_C^{n-2}} \right) P_n(\cos \psi) d\omega. \quad (5.170)$$

According to (5.156), we have

$$\frac{1}{r_S^{n-2}} = \frac{1}{r_0^{n-2}} \left[ 1 - (n-2) \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right]. \quad (5.171)$$

According to this formula and the addition theorem of the spherical harmonics, (4.289), we can write (5.170) as

$$\begin{aligned} I_1 = & -\frac{1}{n-2} \int_{\omega} \left( \frac{1}{r_0^{n-2}} - \frac{1}{r_C^{n-2}} \right) \sum_{k=-n}^n \frac{2}{1+\delta_k} \frac{(n-|k|)!}{(n+|k|)!} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_n^k(\theta_A, \lambda_A) d\omega \\ & + \int_{\omega} \frac{1}{r_0^{n-2}} \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \sum_{k=-n}^n \frac{2}{1+\delta_k} \frac{(n-|k|)!}{(n+|k|)!} \mathcal{H}_n^k(\theta, \lambda) \\ & \times \mathcal{H}_n^k(\theta_A, \lambda_A) d\omega. \end{aligned} \quad (5.172)$$

As both  $r_0$  and  $r_C$  are independent of  $\theta$  and  $\lambda$ , we see that, based on the orthogonality of the spherical harmonics, the first term at the right-hand side is non-vanishing only when  $n = k = 0$ , and the second term is non-vanishing only when  $m = n$  and  $l = k$ . We obtain, using (4.272),

$$I_1 = \begin{cases} 2\pi(r_0^2 - r_C^2), & n = 0; \\ \frac{4\pi}{2n+1} \frac{1}{r_0^{n-2}} \sum_{k=-n}^n \epsilon_n^k \mathcal{H}_n^k(\theta_A, \lambda_A), & n \geq 1. \end{cases} \quad (5.173)$$

The appearance of  $\mathcal{H}_n^k(\theta_A, \lambda_A)$  is due to the fact that  $\psi$  depends on  $\theta_A$  and  $\lambda_A$ .

The integral  $I_2$  can be written as

$$I_2 = \int_{\omega} d\omega \int_{r_S}^R \frac{\rho P_n(\cos \psi)}{r^{n+1}} r^2 dr, \quad (5.174)$$

where  $R$  is the radial distance of the Earth's surface. The Earth's surface is also an equi-value surface of density and gravity potential, and the value of  $q$  on it is  $R$ . We substitute  $r$  by  $q$  according to (5.149),

$$\begin{aligned}
 \frac{1}{r^{n+1}} r^2 dr &= \frac{1}{r^{n-1}} dr = -\frac{1}{n-2} d\left(\frac{1}{r^{n-2}}\right) \\
 &= -\frac{1}{n-2} d\left\{\frac{1}{q^{n-2}} \left[1 - (n-2) \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda)\right]\right\} \\
 &= \frac{1}{q^{n-1}} \left[1 - (n-2) \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda)\right] dq \\
 &\quad + \frac{1}{q^{n-2}} \sum_{m=1}^{\infty} \sum_{l=-m}^m \frac{d\epsilon_m^l}{dq} \mathcal{H}_m^l(\theta, \lambda) dq \\
 &= \frac{1}{q^{n-1}} \left\{1 + \sum_{m=1}^{\infty} \sum_{l=-m}^m \left[-(n-2)\epsilon_m^l + q \frac{d\epsilon_m^l}{dq}\right] \mathcal{H}_m^l(\theta, \lambda)\right\} dq.
 \end{aligned} \tag{5.175}$$

This also transforms the equipotential surface  $S$  and the Earth's surface to spheres of radii  $r_0$  and  $R$ , respectively. According to this formula and the addition theorem of the spherical harmonics, (4.289), we can develop (5.174) to

$$\begin{aligned}
 I_2 &= \int_{\omega} \left( \int_{r_0}^R \frac{\rho}{q^{n-1}} dq \right) \sum_{k=-n}^n \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_n^k(\theta_A, \lambda_A) d\omega \\
 &\quad + \int_{\omega} \sum_{m=1}^{\infty} \sum_{l=-m}^m \left\{ \int_{r_0}^R \frac{\rho}{q^{n-1}} \left[ -(n-2)\epsilon_m^l + q \frac{d\epsilon_m^l}{dq} \right] dq \right\} \mathcal{H}_m^l(\theta, \lambda) \\
 &\quad \times \sum_{k=-n}^n \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_n^k(\theta_A, \lambda_A) d\omega.
 \end{aligned} \tag{5.176}$$

As  $\rho$  depends on  $q$  alone, both of them can be treated as constants while evaluating the integral of over  $\omega$ . Similar to the case of  $I_1$ , the first term is non-vanishing only when  $n = 0 = k$ , and the second term is non-vanishing only when  $m = n$  and  $l = k$ . We thus obtain

$$I_2 = \begin{cases} 4\pi \int_{r_0}^R \rho q dq, & n = 0; \\ \frac{4\pi}{2n+1} \sum_{k=-n}^n \left\{ \int_{r_0}^R \frac{\rho}{q^{n-1}} \left[ -(n-2)\epsilon_n^k + q \frac{d\epsilon_n^k}{dq} \right] dq \right\} \mathcal{H}_n^k(\theta_A, \lambda_A), & n \geq 1. \end{cases} \tag{5.177}$$

The term  $I_3$  is a surface integral. We note that the symbol  $g$  under the integral sign is the value of  $g$  on  $S$ ,  $g_S$ , which is provided in (5.160). As (5.163), i.e.,  $dS = r_S^2 dw$ , holds accurate to the first order of  $\epsilon_m^l$ , (5.163) can be simplified to

$$I_3 = \int_{\omega} \frac{g_S P_n(\cos \psi)}{r_S^{n+1}} r_S^2 d\omega = \int_{\omega} \frac{g_S P_n(\cos \psi)}{r_S^{n-1}} d\omega. \quad (5.178)$$

According to (5.156) and (5.160), we have

$$\begin{aligned} \frac{g_S}{r_S^{n-1}} &= \left\{ g_0 \left[ 1 - \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( \epsilon_m^l + r_0 \frac{d\epsilon_m^l}{dr_0} \right) \mathcal{H}_m^l(\theta, \lambda) \right] \right\} \\ &\quad \times \left\{ r_0^{n-1} \left[ 1 + (n-1) \sum_{m=1}^{\infty} \sum_{l=-m}^m \epsilon_m^l \mathcal{H}_m^l(\theta, \lambda) \right] \right\}^{-1} \\ &= \frac{g_0}{r_0^{n-1}} \left\{ 1 + \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( -n\epsilon_n^k - r_0 \frac{d\epsilon_n^k}{dr_0} \right) \mathcal{H}_m^l(\theta, \lambda) \right\}. \end{aligned} \quad (5.179)$$

According to this formula and the addition theorem of the spherical harmonics, (4.289), we can write (5.178) as

$$\begin{aligned} I_3 &= \int_{\omega} \frac{g_0}{r_0^{n-1}} \sum_{k=-n}^n \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_n^k(\theta_A, \lambda_A) d\omega \\ &\quad + \int_{\omega} \frac{g_0}{r_0^{n-1}} \sum_{m=1}^{\infty} \sum_{l=-m}^m \left( -n\epsilon_n^k - r_0 \frac{d\epsilon_n^k}{dr_0} \right) \mathcal{H}_m^l(\theta, \lambda) \\ &\quad \times \sum_{k=-n}^n \frac{2}{1 + \delta_k} \frac{(n - |k|)!}{(n + |k|)!} \mathcal{H}_n^k(\theta, \lambda) \mathcal{H}_n^k(\theta_A, \lambda_A) d\omega. \end{aligned} \quad (5.180)$$

Similar to the cases of  $I_1$  and  $I_2$ , according to the orthogonality of the spherical harmonics, the first term is non-vanishing only when  $n = k = 0$ , and the second term is non-vanishing only when  $m = n$  and  $l = k$ . We finally obtain the expression of  $I_3$  to be

$$I_3 = \begin{cases} 4\pi r_0 g_0, & n = 0; \\ \frac{4\pi}{2n+1} \frac{g_0}{r_0^{n-1}} \sum_{k=-n}^n \left( -n\epsilon_n^k - r_0 \frac{d\epsilon_n^k}{dr_0} \right) \mathcal{H}_n^k(\theta_A, \lambda_A), & n > 0; \end{cases} \quad (5.181)$$

where  $g_0$  is given in (5.166).

By substituting the expressions of  $I_1$ ,  $I_2$ , and  $I_3$  into (5.167), we obtain, when  $n = 0$ ,

$$\omega^2(r_0^2 - r_C^2) + 4\pi G \int_{r_0}^R \rho q dq + r_0 g_0 = W_S - \omega^2 r_C^2, \quad (5.182)$$

whence,

$$W_S = r_0 g_0 + 4\pi G \int_{r_0}^R \rho q dq + \omega^2 r_0^2, \quad (5.183)$$

and when  $n \neq 0$ ,

$$\begin{aligned} \sum_{k=-n}^n \mathcal{H}_n^k(\theta_A, \lambda_A) \left\{ \frac{2}{2n+1} \frac{\omega^2}{r_0^{n-2}} \epsilon_n^k + \frac{4\pi G}{2n+1} \int_{r_0}^R \frac{\rho}{q^{n-1}} \left[ -(n-2)\epsilon_n^k + q \frac{d\epsilon_n^k}{dq} \right] dq \right. \\ \left. + \frac{1}{2n+1} \frac{g_0}{r_0^{n-1}} \left( -n\epsilon_n^k - r_0 \frac{d\epsilon_n^k}{dr_0} \right) \right\} = \begin{cases} \frac{1}{3} \omega^2 P_2(\cos \theta_A), & n = 2; \\ 0, & n = 1 \text{ or } n > 2. \end{cases} \end{aligned} \quad (5.184)$$

The result for  $n = 0$ , (5.183), is the expression for  $W_S$ , which can be written in a more concrete form using (5.166),

$$W_S = \frac{4\pi G}{r_0} \int_0^{r_0} \rho q^2 dq + 4\pi G \int_{r_0}^R \rho q dq + \frac{1}{3} \omega^2 r_0^2, \quad (5.185)$$

and that for  $n \neq 0$ , (5.184), will be used to derive the equation for  $\epsilon_n^k$ .

As  $A$  can be arbitrarily chosen within the sphere  $C$  of radius  $r_C$ ,  $\theta_A$  and  $\lambda_A$  in (5.184) are arbitrary. Hence, in order for the identity to hold, the coefficients of  $\mathcal{H}_n^k(\theta_A, \lambda_A)$  must vanish. With  $P_2(\cos \theta_A) = \mathcal{H}_2^0(\theta_A, \lambda_A)$ , we obtain

$$\begin{aligned} \frac{2}{2n+1} \frac{\omega^2}{r_0^{n-2}} \epsilon_n^k + \frac{4\pi G}{2n+1} \int_{r_0}^R \frac{\rho}{q^{n-1}} \left[ -(n-2)\epsilon_n^k + q \frac{d\epsilon_n^k}{dq} \right] dq \\ + \frac{1}{2n+1} \frac{1}{r_0^{n-1}} \left( 4\pi G \frac{1}{r_0^2} \int_0^{r_0} \rho q^2 dq - \frac{2}{3} \omega^2 r_0 \right) \left( -n\epsilon_n^k - r_0 \frac{d\epsilon_n^k}{dr_0} \right) \\ - \frac{1}{3} \omega^2 = 0, \end{aligned} \quad (5.186)$$

where we have substituted  $g_0$  according to (5.166), and the term underlined with ' $n = 2, k = 0$ ' is existent only when  $n = 2$  and  $k = 0$ . This is an equation for  $\epsilon_n^k$  with  $\omega$  and  $\rho$  assumed to be known. The square of the angular rate of rotation,  $\omega^2$ ,

is assumed to be on the same order of magnitude as  $\epsilon_n^k$ , as the former is in fact the cause of the later, and will be seen more clearly at a later stage, e.g., (5.189) below. Hence, the products between  $\omega^2$  and terms on the order of  $\epsilon_n^k$  can be neglected in the above equation. With

$$\frac{1}{q^{n-1}} \left[ -(n-2)\epsilon_n^k + q \frac{d\epsilon_n^k}{dq} \right] = -\frac{(n-2)}{q^{n-1}}\epsilon_n^k + \frac{1}{q^{n-2}} \frac{d\epsilon_n^k}{dq} = \frac{d}{dq} \left( \frac{\epsilon_n^k}{q^{n-2}} \right), \quad (5.187)$$

we can write (5.186) as

$$\begin{aligned} & \frac{4\pi G}{2n+1} \int_{r_0}^R \rho \frac{d}{dq} \left( \frac{\epsilon_n^k}{q^{n-2}} \right) dq \\ & - \frac{4\pi G}{2n+1} \frac{1}{r_0^{n+1}} \int_0^{r_0} \rho q^2 dq \left( n\epsilon_n^k + r_0 \frac{d\epsilon_n^k}{dr_0} \right) - \frac{1}{\frac{3}{n=2, k=0}} \omega^2 = 0, \end{aligned} \quad (5.188)$$

which can be further written as

$$\left( \frac{1}{r_0^n} \frac{d\epsilon_n^k}{dr_0} + \frac{n}{r_0^{n+1}} \epsilon_n^k \right) \int_0^{r_0} \rho q^2 dq - \int_{r_0}^R \rho \frac{d}{dq} \left( \frac{\epsilon_n^k}{q^{n-2}} \right) dq + \frac{5\omega^2}{\frac{12\pi G}{n=2, k=0}} = 0. \quad (5.189)$$

Take the derivative with respect to  $r_0$ , we obtain, by using the Newton-Leibniz formula,

$$\begin{aligned} & \left( \frac{1}{r_0^n} \frac{d^2\epsilon_n^k}{dr_0^2} - \frac{n}{r_0^{n+1}} \frac{d\epsilon_n^k}{dr_0} + \frac{n}{r_0^{n+1}} \frac{d\epsilon_n^k}{dr_0} - \frac{n(n+1)}{r_0^{n+2}} \epsilon_n^k \right) \int_0^{r_0} \rho q^2 dq \\ & + \left( \frac{1}{r_0^n} \frac{d\epsilon_n^k}{dr_0} + \frac{n}{r_0^{n+1}} \epsilon_n^k \right) \rho r_0^2 + \rho \frac{d}{dr_0} \left( \frac{\epsilon_n^k}{r_0^{n-2}} \right) = 0. \end{aligned} \quad (5.190)$$

After multiplying both sides by  $r_0^n$ , some rearrangements lead to

$$\left( \frac{d^2\epsilon_n^k}{dr_0^2} - \frac{n(n+1)}{r_0^2} \epsilon_n^k \right) \int_0^{r_0} \rho q^2 dq + 2\rho r_0^2 \left( \frac{d\epsilon_n^k}{dr_0} + \frac{\epsilon_n^k}{r_0} \right) = 0. \quad (5.191)$$

This equation can be written as, using the average density  $\bar{\rho}$  of the mass within  $S$  provided in (5.159),

$$\frac{d^2\epsilon_n^k}{dr_0^2} + \frac{6}{r_0} \frac{\rho}{\bar{\rho}} \frac{d\epsilon_n^k}{dr_0} - \frac{1}{r_0^2} \left[ n(n+1) - 6 \frac{\rho}{\bar{\rho}} \right] \epsilon_n^k = 0. \quad (5.192)$$

Both (5.189) and (5.192) are the ordinary differential equations of  $\epsilon_n^k$ .



### 5.5.4 Clairaut's Equation of Internal Flattening

We now proceed to show that  $\epsilon_2^0 \neq 0$ , but  $\epsilon_n^k = 0$  for all other  $n$  and  $k$ .

First, we shown that  $|\epsilon_n^k|$  increases with  $r_0$  when  $r_0$  is extremely small. We know that  $\rho$  decreases with  $r_0$ —under the long term action of gravity, Earth materials flow, and heavier materials sink down with time. Hence, we have  $\bar{\rho} = \rho$  when  $r_0 = 0$ , and  $\bar{\rho} > \rho$  when  $r_0 > 0$ . When  $r_0$  is extremely small, we can approximately take  $\bar{\rho} = \rho$ , and (5.192) is simplified to

$$\frac{d^2 \epsilon_n^k}{dr_0^2} - \frac{n(n+1)}{r_0^2} \epsilon_n^k + \frac{6}{r_0} \left( \frac{d\epsilon_n^k}{dr_0} + \frac{\epsilon_n^k}{r_0} \right) = 0. \quad (5.193)$$

The general solution of this second order differential equation is

$$\epsilon_n^k = A_n^k r_0^{n-2} + B_n^k r_0^{-n-3}, \quad (5.194)$$

where  $A_n^k$  and  $B_n^k$  are arbitrary coefficients. Since  $\epsilon_n^k$  must be finite, we have  $B_n^k = 0$ . When  $n > 2$ , (5.193) implies that  $|\epsilon_n^k|$  increases with  $r_0$ . The cases when  $n = 1$  and  $n = 2$  have to be studied particularly. When  $n = 1$ , it can be easily verified that  $\epsilon_1^k = A_1^k/r_0$  is in fact an exact solution of (5.192). We now substitute this solution into the equation of equi-value surface of density (5.156). After further substituting  $\mathcal{H}_n^k(\theta, \lambda)$  by its definition (4.258), and then the Legendre function by their expressions in Table 4.1, we obtain

$$r_s(\theta, \lambda) = r_0 + A_1^0 \cos \theta + A_1^1 \sin \theta \cos \lambda + A_1^{-1} \sin \theta \sin \lambda. \quad (5.195)$$

This is a sphere with the center's rectangular coordinate to be  $(-A_1^1, -A_1^{-1}, -A_1^0)$ , implying that this solution is a rigid shift of the whole sphere. In order for  $r_s = 0$  when  $r_0 = 0$ , we must have  $A_1^k = 0$ . Hence, the case  $n = 1$  is irrelevant, and there is no need to be further considered. When  $n = 2$ , (5.194) implies that  $\epsilon_2^k = A$  is a constant. Some refinement is required. We further approximate  $\rho/\bar{\rho}$  and  $\epsilon_2^k$  as

$$\frac{\rho}{\bar{\rho}} = 1 - H r_0^\alpha, \quad \epsilon_2^k = A + B r_0^s, \quad (5.196)$$

where  $H > 0$  and  $\alpha > 0$  must hold, as  $\rho/\bar{\rho} < 0$ . The substitution of the above formulae into (5.192) yields

$$s(s+5)B r_0^{s-2} - 6H A r_0^{\alpha-2} = 0. \quad (5.197)$$

For this relation to hold, we must have  $s = \alpha$ , and  $A$  and  $B$  must have the same sign. These implies that  $|\epsilon_2^k|$  increases with  $r_0$ . This completes our proof that, for any  $n$ ,  $|\epsilon_n^k|$  increases with  $r_0$  when  $r_0$  is extremely small.

It can be further shown that  $|\epsilon_n^k|$  increases with  $r_0$  throughout  $0 \leq r_0 \leq R$ . It is known that  $|\epsilon_n^k|$  increases with  $r_0$  when  $r_0$  is extremely small. In order for  $|\epsilon_n^k|$  to start decreasing with  $r_0$ ,  $d\epsilon_n^k/dr_0 = 0$  must hold at a certain point, where (5.192) becomes

$$\frac{d^2\epsilon_n^k}{dr_0^2} = \left[ n(n+1) - 6\frac{\rho}{\bar{\rho}} \right] \frac{\epsilon_n^k}{r_0^2}. \quad (5.198)$$

When  $n \geq 2$ , the value within the brackets at the right-hand side is always positive, meaning that  $d^2\epsilon_n^k/dr_0^2$  and  $\epsilon_n^k$  have the same sign. This implies that, according to Taylor's series

$$\epsilon_n^k(r_0 + \delta r_0) = \epsilon_n^k(r_0) + \frac{d\epsilon_n^k}{dr_0}\delta r_0 + \frac{1}{2} \frac{d^2\epsilon_n^k}{dr_0^2}\delta r_0^2, \quad (5.199)$$

$|\epsilon_n^k|$  keeps increasing with  $r_0$  after this point. In summary, we proved that  $|\epsilon_n^k|$  increases with  $r_0$  throughout  $0 \leq r_0 \leq R$  without changing sign.

Finally, we examine the value of  $\epsilon_n^k$  on the Earth's surface,  $\epsilon_n^k(R)$ . By setting  $r_0 = R$  in (5.189), and then dividing both sides by  $R^{n-1} \int_0^R \rho q^2 dq$ , we obtain

$$n\epsilon_n^k(R) + R \left( \frac{d\epsilon_n^k}{dr_0} \right)_{r_0=R} = - \frac{5\Omega^2}{4\pi G \bar{\rho}(R)}, \quad (5.200)$$

$\bar{\rho} = \bar{\rho}(R)$

where

$$\bar{\rho}(R) = \frac{3}{R^3} \int_0^R \rho q^2 dq \quad (5.201)$$

is the average density of whole the Earth. As  $|\epsilon_n^k|$  increases with  $r_0$  throughout  $0 \leq r_0 \leq R$ , the two terms at the left-hand side of (5.200) have the same sign, and consequently, they must vanish when the right-hand side of the equation vanishes. This implies that  $\epsilon_n^k = 0$  throughout the Earth's interior except the case of  $n = 2$  and  $k = 0$ .

We have shown that only  $\epsilon_2^0$  is non-vanishing, which is the result of rotation, as can be seen from (5.200). In this circumstance, the equipotential surface of gravity (5.156) is simplified to

$$r_s = r_0 \left[ 1 + \epsilon_2^0 \mathcal{H}_2^0(\theta, \lambda) \right] = r_0 \left[ 1 + \epsilon_2^0 P_2(\cos \theta) \right]. \quad (5.202)$$

This is in fact the equation of an ellipsoid of revolution accurate to the first order of the flattening as given by (5.35), where the flattening  $f$  is related to  $\epsilon_2^0$  in the form of

$$\epsilon_2^0 = -\frac{2}{3}f. \quad (5.203)$$

The equation of  $S$  can then be written as

$$r_S = r_0 \left[ 1 - \frac{2}{3}f P_2(\cos \theta) \right]. \quad (5.204)$$

The substitution (5.203) into (5.189) and (5.192) yields the ordinary differential equations of  $f$ ,

$$\left( \frac{1}{r_0^2} \frac{df}{dr_0} + \frac{2}{r_0^3} f \right) \int_0^{r_0} \rho q^2 dq - \int_{r_0}^R \rho \frac{df}{dq} dq - \frac{5\omega^2}{8\pi G} = 0, \quad (5.205)$$

$$\frac{d^2 f}{dr_0^2} + \frac{6}{r_0} \frac{\rho}{\bar{\rho}} \frac{df}{dr_0} - \frac{6}{r_0^2} \left( 1 - \frac{\rho}{\bar{\rho}} \right) f = 0. \quad (5.206)$$

Both of them are referred to as Clairaut's equation, which are to be solved to compute the internal flattening  $f$  as a function of  $r_0$  using data of internal density, i.e., the density of a spherically symmetrical Earth model. Every sphere of the spherically symmetrical Earth model corresponds to an ellipsoid in hydrostatic equilibrium with the action of the centrifugal force.

Here we solve (5.206), which is a second order ordinary differential equation, and is in need of two boundary conditions for the solution to be unique. A boundary condition on the Earth's surface  $r_0 = R$  can be readily obtained by substituting (5.203) into (5.200),

$$2f(R) + R \left( \frac{df}{dr_0} \right)_{r_0=R} = \frac{15\omega^2}{8\pi G \bar{\rho}(R)} = \frac{5}{2}m, \quad (5.207)$$

where the parameter  $m$  is defined as

$$m = \frac{3\omega^2}{4\pi G \bar{\rho}(R)} = \frac{\omega^2 R^3}{G[(4/3)\pi R^3 \bar{\rho}(R)]} = \frac{\omega^2 R^3}{GM}, \quad (5.208)$$

which is equivalent to the parameter  $q$  defined in (5.42) accurate to the first order of themselves. The boundary condition (5.207) implies that  $f > 0$  holds throughout the Earth's interior, since the two terms at its left-hand side must have the same sign considering that  $|f|$  always increases with  $r_0$ , and its right-hand side is positive. A boundary condition at the Earth center, i.e., at  $r_0 = 0$ , is implicitly included

in (5.206) itself. We take  $\bar{\rho} = \rho$  as an approximation for an infinitesimally small sphere  $r \leq \delta r$ . Clairaut's equation (5.206) then degenerates to

$$\frac{d^2 f}{dr_0^2} + \frac{6}{r_0} \frac{df}{dr_0} = 0. \quad (5.209)$$

Its general solution is

$$f = A + Br_0^{-5}. \quad (5.210)$$

where  $A$  and  $B$  are arbitrary constants. Evidently, if  $B \neq 0$ ,  $f$  would be infinitely large at  $r_0 = 0$ , which contradicts the physics. Hence,  $B = 0$  must hold. The other constant  $A$  can be determined using the boundary condition (5.207). The boundary condition at the Earth center implied by  $B = 0$  is referred to as regularity boundary condition. We can write it as

$$r_0 \frac{df}{dr_0} = 0, \quad (5.211)$$

which indeed ensure  $B = 0$  in (5.210).

Spherically symmetrical models of the Earth's interior have discontinuities, i.e., there are jumps of the value of  $\rho$  at certain values of  $r_0$ . In order that no cavity or overlap happen,  $f$  must be continuous across boundaries of discontinuity. It can be inferred from (5.205) that  $df/dr_0$  is also continuous, since both integrals in it are continuous.

Here we proceed to solve Clairaut's equation (5.206) numerically. Define

$$f_1 = r_0 \frac{df}{dr_0}. \quad (5.212)$$

We can transform the second order ordinary differential equation (5.206) to a system of two first order ordinary differential equations,

$$\begin{cases} \frac{df}{dr_0} = \frac{1}{r_0} f_1, \\ \frac{df_1}{dr_0} = \frac{6}{r_0} \left(1 - \frac{\rho}{\bar{\rho}}\right) f + \frac{1}{r_0} \left(1 - 6\frac{\rho}{\bar{\rho}}\right) f_1. \end{cases} \quad (5.213)$$

The boundary conditions (5.207) and (5.211) are reduced to

$$2f(R) + f_1(R) = \frac{5}{2}m, \quad f_1(0) = 0. \quad (5.214)$$

Two complexities arise while solving (5.213) numerically. First, the right-hand sides of both equations in (5.213) tend to infinitely large when  $r_0 = 0$ , and special treatment is necessary there. Second, as can be inferred from the special case of

a homogeneous sphere with solution given in (5.210), the solution is unstable if (5.213) is integrated from the Earth's surface to the center. In this approach, two boundary conditions have to be prescribed on the Earth's surface, which do not necessarily conform with the regularity boundary condition at the Earth center, rendering  $B \neq 0$  in the solution (5.210). This introduces an extremely large error while approaching the Earth center.

Hence, a practical procedure of computation is as follows: (1) We start the integration from the Earth center. For  $r_0 \leq \delta r$ , we approximately take  $\bar{\rho} = \rho$ . The solution is known to be  $f = A$ , and  $f_1 = 0$ . Hence, for  $r_0 \leq \delta r$ , we set  $f = A$ , and  $f_1 = 0$ , where  $A$  is a constant arbitrarily chosen. (2) Numerically integrate the system of equations (5.213) from  $r_0 = \delta r$  to the Earth's surface  $r_0 = R$  using a standard procedure such as the Runge-Kutta method. At places where  $\rho$  has a jump, the integration is continued by keeping both  $f$  and  $f_1$  continuous. (3) At the Earth's surface, multiply both  $f(R)$  and  $f_1(R)$  by a constant, so that the first boundary condition in (5.214) is satisfied. (4) Multiply  $f$  and  $f_1$  throughout  $0 \leq r_0 \leq R$  by the constant. The result is what we are seeking. In Sect. 5.5.6, we will solve this problem using the Chebyshev collocation method.

### 5.5.5 Gravity, Pressure and Moments of Inertia

In this subsection, we derive the formula of some other relevant physical parameters related to the internal flattening of the Earth. All quantities are related to the internal equi-value surface of density and gravity potential denoted as  $S$  given by (5.204), which we copy to here for convenience of reference,

$$r_S = r_0 \left[ 1 - \frac{2}{3} f P_2(\cos \theta) \right]. \quad (5.215)$$

This formula projects the equi-value surface of the ellipsoidal Earth model to a sphere of radius  $r_0$  of the spherical Earth model.

The formula of the gravity potential on  $S$  is given by (5.185), which we copy here also for convenience of reference,

$$W_S = \frac{4\pi G}{r_0} \int_0^{r_0} \rho q^2 dq + 4\pi G \int_{r_0}^R \rho q dq + \frac{1}{3} \omega^2 r_0^2. \quad (5.216)$$

It depends on  $r_0$  alone. The term  $(1/3)\omega^2 r_0^2$  is the average of the centrifugal potential on the sphere of radius  $r_0$ . Therefore,  $W_S$  includes not only the gravitational potential of the spherically symmetrical Earth model, but also the spherical average of the centrifugal potential.

The gravity on  $S$  can be inferred from (5.153) by setting  $r = r_S$  and  $q = r_0$ , with the only non-vanishing spherical harmonic coefficient  $\epsilon_2^0$  substituted by the

flattening  $f$  according to (5.203),

$$\vec{g}_S = \frac{dW}{dr_0} \left\{ \left[ 1 + \frac{2}{3} \left( f + r_0 \frac{df}{dr_0} \right) P_2(\cos \theta) \right] \hat{e}_r + \frac{2}{3} f \frac{d}{d\theta} P_2(\cos \theta) \hat{e}_\theta \right\}, \quad (5.217)$$

where  $g_0 = -dW/dr_0$  is given by (5.166), and is copied to here for convenience of reference,

$$g_0 = -\frac{dW}{dr_0} = \frac{4\pi G}{r_0^2} \int_0^{r_0} \rho q^2 dq - \frac{2}{3} \omega^2 r_0. \quad (5.218)$$

This formula can also be obtained by taking the derivative of (5.216) with respect to  $r_0$ . The magnitude of  $\vec{g}_S$  accurate to the first order of  $f$  is

$$g_S = g_0 \left[ 1 + \frac{2}{3} \left( f + r_0 \frac{df}{dr_0} \right) P_2(\cos \theta) \right]. \quad (5.219)$$

The equation for the pressure  $p$  can be derived based on the relation  $\nabla p = \rho \nabla W$ . We just have to realize that  $\vec{g}_S$  given by (5.217) is the value of  $\nabla W$  on  $S$ , and that the value of  $\nabla p$  on  $S$  is in the same form as (5.217) with  $dW/dr_0$  replaced by  $dp/dr_0$ . Hence,  $\nabla p = \rho \nabla W$  leads to

$$\frac{dp}{dr_0} = \rho \frac{dW}{dr_0} = -\rho g_0. \quad (5.220)$$

This ordinary differential equation can be solved to compute  $p$ . We mention again that, like  $W$ ,  $p$  is constant on the equi-value surfaces of density.

We also derive the formulae for the moments of inertia. Let  $x_1$  and  $x_2$  be in the equatorial plane, and  $x_3$  along the rotation axis. We use  $A$ ,  $B$ , and  $C$  to denote the moments of inertial of the mass within  $S$  around  $x_1$ ,  $x_2$ , and  $x_3$  respectively. By definition,

$$A = \int_V \rho(x_2^2 + x_3^2) dV, \quad B = \int_V \rho(x_3^2 + x_1^2) dV, \quad C = \int_V \rho(x_1^2 + x_2^2) dV, \quad (5.221)$$

where the domain of integration  $V$  is the inside of  $S$ . Due to symmetry, we have  $A = B$ . For convenience we derive the formulae for  $I = (A + B + C)/3$  and  $C - A$ .

The average of the three components of moments of inertia is

$$I = \frac{2}{3} \int_V \rho(x_1^2 + x_2^2 + x_3^2) dV = \frac{2}{3} \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_S} \rho r^4 \sin \theta dr. \quad (5.222)$$

In this formula,  $r_S$  is defined by (5.204). Upon substituting  $r$  by  $q$  according to

$$r = q \left[ 1 - \frac{2}{3} f P_2(\cos \theta) \right], \quad (5.223)$$

$r_S$  is reduced to  $r_0$ , and we obtain

$$\begin{aligned} r^4 dr &= \frac{1}{5} dr^5 = \frac{1}{5} d \left\{ q^5 \left[ 1 - \frac{10}{3} f P_2(\cos \theta) \right] \right\} \\ &= q^4 \left[ 1 - \frac{10}{3} f P_2(\cos \theta) \right] dq - \frac{2}{3} q^5 \frac{df}{dq} P_2(\cos \theta) dq \\ &= q^4 \left[ 1 - \left( \frac{10}{3} f + \frac{2}{3} q \frac{df}{dq} \right) P_2(\cos \theta) \right] dq. \end{aligned} \quad (5.224)$$

By substituting these into (5.222), and then using the orthogonality of the Legendre function, the integral of  $P_2(\cos \theta)$  vanishes, and we obtain

$$I = \frac{8\pi}{3} \int_0^{r_0} \rho q^4 dq, \quad (5.225)$$

where  $\rho$  is the density within  $S$  as function of  $q$  defining an equi-value surface of density.

The difference  $C - A$  can be obtained from (5.221) to be

$$\begin{aligned} C - A &= \int_V \rho (x_1^2 - x_3^2) dV \\ &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_S} \rho r^4 (\sin^2 \theta \cos^2 \lambda - \cos^2 \theta) \sin \theta dr \\ &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_S} \rho r^4 \left[ \cos^2 \lambda - (1 + \cos^2 \lambda) \cos^2 \theta \right] \sin \theta dr \\ &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_S} \rho r^4 \left[ \cos^2 \lambda - \frac{2}{3} (1 + \cos^2 \lambda) \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} + \frac{1}{2} \right) \right] \\ &\quad \times \sin \theta dr \\ &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_S} \rho r^4 \left[ -\frac{1}{3} + \frac{2}{3} \cos^2 \lambda - \frac{2}{3} (1 + \cos^2 \lambda) P_2(\cos \theta) \right] \\ &\quad \times \sin \theta dr. \end{aligned} \quad (5.226)$$

By substitute (5.224) into this formula, and then using the orthogonality of the Legendre function, we obtain

$$\begin{aligned}
 C - A &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_0} \rho q^4 \left( -\frac{1}{3} + \frac{2}{3} \cos^2 \lambda \right) \sin \theta dq \\
 &\quad + \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_0} \rho q^4 \left( \frac{2}{3} + \frac{2}{3} \cos^2 \lambda \right) \left( \frac{10}{3} f + \frac{2}{3} q \frac{df}{dq} \right) \\
 &\quad \times [P_2(\cos \theta)]^2 \sin \theta dq \\
 &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_0} \frac{1}{3} \rho q^4 \cos 2\lambda \sin \theta dq \\
 &\quad + \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^{r_0} \frac{2}{3} \rho \left( 5q^4 f + q^5 \frac{df}{dq} \right) \left( 1 - \frac{1}{3} \cos 2\lambda \right) \\
 &\quad \times [P_2(\cos \theta)]^2 \sin \theta dq \\
 &= (2\pi) \left( \frac{2}{5} \right) \int_0^{r_0} \frac{2}{3} \rho \left( 5q^4 f + q^5 \frac{df}{dq} \right) dq \\
 &= \frac{8\pi}{15} \int_0^{r_0} \rho \left( 5q^4 f + q^5 \frac{df}{dq} \right) dq \\
 &= \frac{8\pi}{15} \int_0^{r_0} \rho \frac{d}{dq} (q^5 f) dq, \tag{5.227}
 \end{aligned}$$

where the meaning of  $\rho$  is the same as that in the expression of  $I$ .

### 5.5.6 Practical Computations and Numerical Results

Even though the parameters  $\bar{\rho}$ ,  $g_0$ ,  $p$ ,  $I$ , and  $C - A$  can be computed by directly evaluating integrals numerically, here we convert the integrals to ordinary differential equations, and then solve them using the Chebyshev collocation method described in Sect. 4.9.5. The philosophy of the method is to convert the ordinary differential equation of the parameter to a system of algebraic equations, where the unknowns to be solved are the values of the parameter at a set of collocation points. The procedure of computation was discussed in Sect. 4.9.5 for the case of  $g$  as an example, where the ordinary differential equation is given by (4.660), and the algebraic equation to be solved is given by (4.661).

The average density  $\bar{\rho}$  is given by (5.159) as an integral. Here we replace  $r_0$  by  $r$  to obtain  $\bar{\rho}(r) = (3/r^3) \int_0^r \rho(q) q^2 dq$ . By multiplying both sides with  $r^3$ , taking the derivative with respect to  $r$ , and then dividing both sides by  $r^3$ , we obtain the



differential equation of  $\bar{\rho}$  as

$$\frac{d\bar{\rho}}{dr} + \frac{3}{r}\bar{\rho} = \frac{3\rho}{r}. \quad (5.228)$$

The gravity is given in (5.218). Here we consider a spherically symmetrical non-rotating Earth model by setting  $\omega = 0$ , and write (5.218) as  $g(r) = (4\pi G/r^2) \int_0^r \rho q^2 dq$ . By multiplying both sides of this formula by  $r^2$ , taking the derivative with respect to  $r$ , and then dividing both sides by  $r^2$ , we obtain the ordinary differential equation for  $g$ ,

$$\frac{dg}{dr} + \frac{2}{r}g = 4\pi G\rho. \quad (5.229)$$

This equation has also been derived out from Poisson's equation in Sect. 4.8.1, and was used in Sect. 4.9.5 as an example of application of the Chebyshev collocation method. The ordinary differential equation for  $p$  is (5.220), which we write here as

$$\frac{dp}{dr} = -\rho g. \quad (5.230)$$

The ordinary differential equation for  $I$  can be readily obtained to be  $dI/dr = (8\pi/3)\rho r^4$ . However, the magnitude of  $I$  varies too much across the range of  $r$ ; the solution is  $I = (8\pi/15)\rho r^5$  if  $\rho$  is a constant. Therefore, we choose to solve the ordinary differential equation of the auxiliary parameter  $I' = I/r^4$ , which is

$$\frac{dI'}{dr} + \frac{4}{r}I' = \frac{8\pi}{3}\rho. \quad (5.231)$$

We have chosen  $I'$  to be  $I/r^4$  instead of  $I/r^5$ , as the former still vanishes at  $r = 0$ , but the latter does not. Nevertheless, the latter can also serve the purpose if we choose it. The solution of  $I$  is obtained by multiplying the solution of  $I'$  by  $r^4$ . Similarly, for  $C - A$ , we solve the ordinary differential equation of  $D' = (C - A)/r^4$ , which can be readily obtained to be

$$\frac{dD'}{dr} + \frac{4}{r}D' = \frac{8\pi}{15}\rho(5f + f_1). \quad (5.232)$$

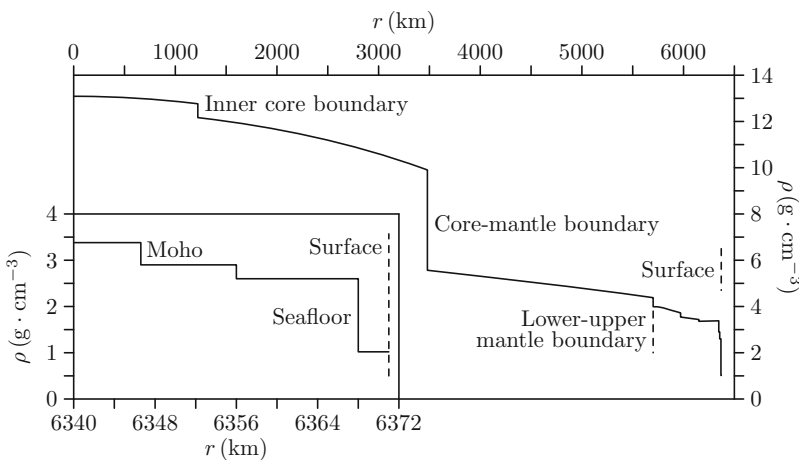
where  $f_1 = r(df/dr)$  is defined in (5.212). This equation is to be solved after solving (5.213) with the boundary condition (5.214). The solution of  $C - A$  is obtained by multiplying the solution of  $D'$  by  $r^4$ .

The ordinary differential equations of  $\bar{\rho}$ ,  $g$ ,  $I'$ , and  $D'$  are all similar. Therefore, the procedure of the numerical solution is the same as that of  $g$  explained in Sect. 4.9.5, and is not to be repeated here. The only exception is that of  $p$ , which does not have any singularity. The known boundary condition is  $p = 0$  on the Earth's surface  $r = R$  (neglecting the atmospheric pressure). Therefore, this ordinary

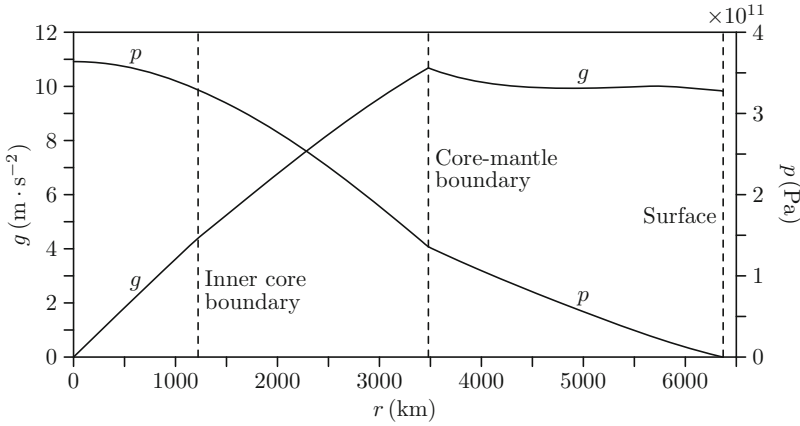
differential equation is to be solved layer by layer starting from the outermost layer, where the value of  $p$  at the outer boundary is imposed as boundary condition. The Gauss-Lobatto collocation points should be used for specifying the boundary condition at the outer boundary of a layer to be the solution of the layer above at the inner boundary, which reflects the continuity of  $p$ . In the innermost layer, i.e., the inner core, either Gauss-Lobatto or Gauss-Radau collocation points could be used.

The computation of the flattening  $f$  and  $f_1 = r(df/dr)$  by solving (5.213) with the boundary condition (5.214) has already been discussed. The computation should be done layer by layer starting from the innermost layer. As this is a system of two ordinary differential equations, and only one regularity boundary condition at  $r = 0$ ,  $f_1 = 0$ , is present, there is a need to specify another boundary condition for the solution in the inner core. If the Gauss-Lobatto collocation points are used, we can use  $f_1 = 0$  and  $f = A$ , where  $A$  is any constant, to replace the algebraic equations at the Earth center. If the Gauss-Radau collocation points are used, only one boundary condition,  $f = B$ , where  $B$  is a constant, should be specified at the inner core boundary (ICB). This can be done by using this boundary condition to replace the algebraic equation resulting from the first ordinary differential equation in (5.213) at the ICB. In other layers, the Gauss-Lobatto collocation points are used, and the boundary condition at the inner boundary is applied based on the solution in the layer below. Finally, as already mentioned, a constant should be multiplied to the solution throughout  $0 \leq r \leq R$  to match the first boundary condition in (5.214).

In current spherically symmetrical Earth models,  $\rho$  is provided either in the form of a polynomial in each continuous layer, or in the form of discrete values at a series of nodes. The profile of the density  $\rho$  of the Preliminary Reference Earth Model (PREM) (Dziewonski & Anderson, 1981) is shown in Fig. 5.15, which is to be discussed in more detail in Sect. 6.8.6 with numerical values of the polynomial model listed in Table 6.2. We mention that  $G = 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$  was

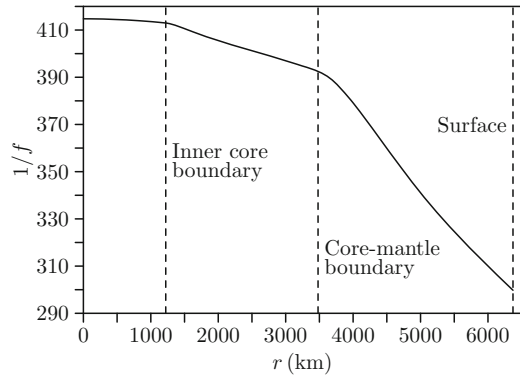


**Fig. 5.15** Profile of density of the Preliminary Reference Earth Model (PREM)



**Fig. 5.16** Profile of gravity and pressure of the PREM

**Fig. 5.17** Profile of the reciprocal of flattening of the PREM



adopted in this classical model. The values needed in the computations are those at the collocation points. The computation is straightforward for a polynomial model. For a discrete model, we use the cubic spline interpolation in each layer with the second order derivative assumed to vanish at all endpoints, except the Earth center, where the first order derivative is assumed to vanish.

The profiles of the gravity  $g$  and pressure  $p$  computed are shown in Fig. 5.16, which agree with those in the original document if we set  $G = 6.67 \times 10^{-11} \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$ . The Earth's flattening computed by solving (5.213) with the boundary condition (5.214) is a theoretical value under the assumption of hydrostatic equilibrium. The result of computation shows that the value of  $f$  is  $1/414.8$  at the Earth center,  $1/413.0$  at the inner core boundary (ICB),  $1/392.5$  at the core-mantle boundary (CMB), and  $1/299.8$  on the Earth's surface. By comparing the value of  $1/299.8$  in hydrostatic equilibrium to the value of  $1/298.3$  determined by observations, we see that hydrostatic equilibrium is indeed very close to reality. The profile of the reciprocal of the flattening is shown in Fig. 5.17.

Finally, we mention briefly the results of other parameters. The value of the mean density is 5.514, 10.987, and 12.894 for whole the Earth, whole the core, and the inner core, respectively. The value of  $I$  is  $8.020 \times 10^{37}$ ,  $9.125 \times 10^{36}$ , and  $5.858 \times 10^{34}$  for whole the Earth, whole the core, and the inner core, respectively. The value of  $C - A$  is  $2.603 \times 10^{35}$ ,  $2.323 \times 10^{34}$ , and  $1.418 \times 10^{32}$  for whole the Earth, whole the core, and the inner core, respectively.

## 5.6 Gravitational Potential of a Thin Layer of Mass on the Earth's Surface

In this section, we first formulate the gravitational potential of a thin layer of mass on the Earth's surface, and then apply the result inversely, i.e., to determine the density of the thin layer of mass using the external gravitational potential of this layer. The forward modeling, i.e., the computation of the gravitational potential of the thin layer of mass, is straightforward. The inverse can be understood as an instantiation of the following theorem: whatever the shape of the Earth's surface, the external gravitational field of the Earth can always be understood as that of a unique thin layer of mass on the Earth's surface (Guan & Ning, 1981; Guo, 2023). Guo's (2023) (2.55) demonstrates that the external gravitational potential of the Earth can be expressed as that of a thin layer of mass on the Earth's surface. The density of the layer must depend on its gravitational attraction in its vicinities inside and outside according to (3.111). This aligns with Guo's (2023) (2.56), which can be readily realized based on the facts that (1) the gravitational potential of the layer,  $V$ , is continuous across the layer, (2) the function  $U$  inside the layer is uniquely determined by its value  $V + C$  on the Earth's surface through Dirichlet's boundary value problem of the Laplace's equation, and (3) the constant  $C$  does not affect the value of  $(\partial U / \partial n)_i$ , as  $U = V + C$  holds throughout the inside of the layer. This implies that the density of the layer given by Guo's (2023) (2.56) is unique. The formulation of the inverse problem in this section also serves as a proof of the uniqueness of the density of the layer.

### 5.6.1 Spherical Earth

Mass transport in the atmosphere-hydrosphere-cryosphere system is ceaselessly occurring. At satellite altitude, this system can be considered as a thin layer, where the mass transport can be represented by a material surface with time-dependent density. We first approximate the Earth as a sphere of radius  $R$ , and use  $s(\theta, \lambda, t)$  to denote a time-dependent surface mass density within the thin layer, which is the mass per unit area. We express  $s(\theta, \lambda, t)$  in the form of a spherical harmonic series,

$$s(\theta, \lambda, t) = \sum_{n=0}^{\infty} \sum_{k=-n}^n s_n^k(t) P_n^k(\cos \theta) e^{ik\lambda}, \quad (5.233)$$

and derive the formula of the gravitational potential at satellite altitude. As implied by this expression, we formulate the problem based on complex spherical harmonics. This is in favor of future application in the formulation of elastic loading in Chap. 7.

By definition, the gravitational potential is

$$\phi(r, \theta, \lambda, t) = GR^2 \int_0^\pi \int_0^{2\pi} \frac{s(\theta', \lambda', t)}{l(R, r, \psi)} \sin \theta' d\theta' d\lambda', \quad (5.234)$$

where  $l(R, r, \psi)$  is the distance between the points  $(R, \theta', \lambda')$  and  $(r, \theta, \lambda)$ , and  $\psi$  the angle between these points with respect to the Earth center. Taking reference to Fig. 4.3 with  $r'$  replaced by  $R$ , we understand the points  $(R, \theta', \lambda')$  and  $(r, \theta, \lambda)$  as  $M$  and  $P$ , respectively. In this way, the symbol  $l(R, r, \psi)$  here corresponds to  $l$  in the figure, and  $\psi$  has the same meaning. We can thus express  $1/l(R, r, \psi)$  as a series of  $P_n(\cos \psi)$  according to (4.98), and then apply the addition theorem (4.291) to  $P_n(\cos \psi)$ . Replace  $\theta$  and  $\lambda$  by  $\theta'$  and  $\lambda'$ , respectively, in (5.233), and then substitute it into (5.234). We obtain, using and the orthogonality (4.275),

$$\phi(r, \theta, \lambda, t) = \begin{cases} \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi GR^2}{2n+1} s_n^k(t) \frac{R^n}{r^{n+1}} P_n^k(\cos \theta) e^{ik\lambda}, & r \geq R; \\ \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi GR^2}{2n+1} s_n^k(t) \frac{r^n}{R^{n+1}} P_n^k(\cos \theta) e^{ik\lambda}, & r \leq R. \end{cases} \quad (5.235)$$

In the result, we have changed the  $>$  and  $<$  as obtained from (4.98) to  $\geq$  and  $\leq$ , respectively. This is to assume that the spherical harmonic series converges when  $r = R$ , which is guaranteed by the convergence of the spherical harmonic series in (5.233). In the derivation, we have only provided an outline from (5.234) to (5.235). The readers can easily fill in the detail following the procedure as explained. In fact, the detail will be a part of the formulation in the next subsection for a more complicated problem. In practical applications, it is convenient to write the gravitational potential (5.235) in the form of

$$\phi(r, \theta, \lambda, t) = \begin{cases} Rg_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k(t) \left(\frac{R}{r}\right)^{n+1} P_n^k(\cos \theta) e^{ik\lambda}, & r \geq R; \\ Rg_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k(t) \left(\frac{r}{R}\right)^n P_n^k(\cos \theta) e^{ik\lambda}, & r \leq R; \end{cases} \quad (5.236)$$

where  $g_R = GM/R^2$  is the gravity on the Earth's surface considered to be a sphere of radius  $R$ , whence,  $Rg_R = GM/R$ . We see that, in the above formula, the dependence on  $GM$ ,  $r$ , and  $R$  agrees with that in (5.1) with  $a$  replaced by  $R$ . The spherical harmonic coefficients of the potential are related to those of the surface mass density

according to

$$\phi_n^k(t) = \frac{4\pi G}{(2n+1)g_R} s_n^k(t), \quad s_n^k(t) = \frac{(2n+1)g_R}{4\pi G} \phi_n^k(t). \quad (5.237)$$

This relation between the spherical harmonic coefficients can be used to convert between  $s(r, \theta, \lambda, t)$  and  $\phi(r, \theta, \lambda, t)$ . Note that  $g_R/(4\pi G) = (1/3)R\bar{\rho}_R$ , where  $\bar{\rho}_R$  is the mean density of the Earth, and  $s_n^k(t)$  and  $\phi_n^k(t)$  are the complex spherical harmonic coefficients..

In real spherical harmonics, (5.233) and (5.236) become, by redefining  $s_n^k(t)$  and  $\phi_n^k(t)$  as the real spherical harmonic coefficients,

$$s(\theta, \lambda, t) = \sum_{n=0}^{\infty} \sum_{k=-n}^n s_n^k(t) \mathcal{H}_n^k(\theta, \lambda), \quad (5.238)$$

$$\phi(r, \theta, \lambda, t) = \begin{cases} Rg_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k(t) \left(\frac{R}{r}\right)^{n+1} \mathcal{H}_n^k(\theta, \lambda), & r \geq R; \\ Rg_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k(t) \left(\frac{r}{R}\right)^n \mathcal{H}_n^k(\theta, \lambda), & r \leq R; \end{cases} \quad (5.239)$$

and the relation between the spherical harmonic coefficients is the same as (5.237). Both  $s$  and  $\phi$  are real functions, and their real and complex coefficients have the property (4.264) or (4.265), which can be used to infer the formulae in real spherical harmonics from those in complex spherical harmonics.

The result can also applied to time-independent problems as well, in which cases, we write the surface mass density and gravitational potential as  $s(\theta, \lambda)$  and  $\phi(r, \theta, \lambda)$ , and their spherical harmonic coefficients as  $s_n^k$  and  $\phi_n^k$ . All the formulae remain the same without the dependence on  $t$ .

We show a simple inverse problem, which is to study the Earth's internal structure using data of the gravitational potential. The few lowest degree terms of the Earth's gravitational potential beyond degree 2, e.g., degrees 3 to 6, were suggested to be related to the undulation of the core-mantle boundary (CMB) (e.g., Melchior, 1986), where the jump of density is as large as  $\rho_j = \rho_{\text{core}} - \rho_{\text{mantle}} = 9.9 - 5.6 = 4.3 \text{ g}\cdot\text{cm}^{-3}$  based on PREM. We understand the CMB as a sphere of radius  $R_C$  in the average sense, and use  $h(\theta, \lambda)$  to denote the height of the undulation with respect to the sphere. The deviation of mass distribution from spherical symmetry related to the undulation can be approximated by a material surface of mass density  $s(\theta, \lambda) = \rho_j h(\theta, \lambda)$  on the sphere. For the problem here, the radius  $R$  should be replaced by  $R_C$  throughout (5.237)–(5.239), and the spherical harmonic series is limited to

degrees 3 to 6. Therefore, we have, outside the Earth,

$$\phi(r, \theta, \lambda) = \sum_{n=3}^6 \sum_{k=-n}^n \frac{4\pi G R_C^2}{2n+1} \rho_j h_n^k \frac{R_C^n}{r^{n+1}} \mathcal{H}_n^k(\theta, \lambda), \quad (5.240)$$

where  $h_n^k$  is the spherical harmonic coefficient of  $h$ . We use (5.1) to represent the Earth's gravitational potential, and write it in the form of

$$\phi(r, \theta, \lambda) = \sum_{n=3}^6 \sum_{k=-n}^n \left( GM A_n^k \right) \frac{R^n}{r^{n+1}} \mathcal{H}_n^k(\theta, \lambda), \quad (5.241)$$

where  $M$  is the mass of the Earth, and we have replaced the scale length  $a$  by the Earth's equi-volumetric average radius  $R$ . By comparing the above two formulae, we obtain

$$h_n^k = \frac{(2n+1) M A_n^k}{4\pi R_C^2 \rho_j} \left( \frac{R}{R_C} \right)^n. \quad (5.242)$$

The computation of  $h(\theta, \lambda)$  using its spherical harmonic coefficient  $h_n^k$  is straightforward, and is not to be further elaborated here. With the values of  $A_n^k$  determined by satellites, the magnitude of  $h(\theta, \lambda)$  thus obtained attains several hundred meters. We remark that this result is based on an assumption. Our purpose here is to demonstrate the use of the formulation from (5.237) to (5.239), which can also be applied to other boundaries of discontinuity, such as the Moho and the lower-upper mantle boundary.

### 5.6.2 Aspherical Earth

In this case, the radial distance of a point on the Earth's surface is position dependent. We express it as

$$r_S(\theta, \lambda) = R + h(\theta, \lambda) = R [1 + \epsilon(\theta, \lambda)]. \quad (5.243)$$

This implies that the actual surface of the Earth is  $h$  above the sphere of radius  $R$ , with

$$\epsilon(\theta, \lambda) = \frac{h(\theta, \lambda)}{R} \quad (5.244)$$

as a relative measure of the height. We define the surface mass density at a point as the mass per unit area on the equipotential surface of gravity through the point, which conforms with the mass of a vertical water column of unit bottom area. We

still express the time-variable surface mass density as a spherical harmonic series

$$s(\theta, \lambda, t) = \sum_{n=0}^{\infty} \sum_{k=-n}^n s_n^k(t) P_n^k(\cos \theta) e^{ik\lambda}. \quad (5.245)$$

Based on the definition of  $s(\theta, \lambda, t)$ , the mass of an infinitesimal area at  $(\theta', \lambda')$  is

$$dm(\theta', \lambda', t) = \frac{r_S^2(\theta', \lambda') s(\theta', \lambda', t) \sin \theta' d\theta' d\lambda'}{\cos(-\hat{e}_r, \hat{e}_g)}, \quad (5.246)$$

where  $(-\hat{e}_r, \hat{e}_g)$  is the angle between the directions of the origin of the coordinate system and the gravity. The difference between  $\cos(-\hat{e}_r, \hat{e}_g)$  and 1 is of second order of magnitude of the angle. Even the largest effect of the Earth's flattening, where  $(-\hat{e}_r, \hat{e}_g) = B - \Phi$ , is on the order of  $5.6 \times 10^{-6}$ . Hence, accurate to the first order of magnitude of the asphericity, we can make the approximation  $\cos(-\hat{e}_r, \hat{e}_g) = 1$ , which leads to the approximation

$$dm(\theta', \lambda', t) = r_S^2(\theta', \lambda') s(\theta', \lambda', t) \sin \theta' d\theta' d\lambda'. \quad (5.247)$$

The gravitational potential of this infinitesimal mass at  $(r, \theta, \lambda)$  is

$$d\phi(r, \theta, \lambda, t) = G \frac{r_S^2(\theta', \lambda')}{l[(r, \theta, \lambda), (r_S, \theta', \lambda')]} s(\theta', \lambda', t) \sin \theta' d\theta' d\lambda', \quad (5.248)$$

where  $l$  is the distance between the two points  $(r, \theta, \lambda)$  and  $(r_S, \theta', \lambda')$ . The total gravitational potential is thus

$$\phi(r, \theta, \lambda, t) = G \int_{\omega} \frac{r_S^2(\theta', \lambda')}{l[(r, \theta, \lambda), (r_S, \theta', \lambda')]} s(\theta', \lambda', t) d\omega, \quad (5.249)$$

where  $\omega$  is the unit sphere with  $d\omega = \sin \theta' d\theta' d\lambda'$ .

In the derivation of (5.249), we have made the approximation  $\cos(-\hat{e}_r, \hat{e}_g) = 1$ , which brings an error on a relative order of magnitude of  $5.6 \times 10^{-6}$ . In fact, this approximation can be avoided by using  $s'(\theta', \lambda', t) = s(\theta', \lambda', t) / \cos(-\hat{e}_r, \hat{e}_g)$  throughout the formulation instead of  $s(\theta', \lambda', t)$ .

From now on, we keep implicit the variables that the functions depend on. Denote the geocentric angle between the two points  $(r, \theta, \lambda)$  and  $(r_S, \theta', \lambda')$  as  $\psi$ . We have

$$l = \left( r^2 + r_S^2 - 2rr_S \cos \psi \right)^{1/2}, \quad (5.250)$$

$$\cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\lambda - \lambda'). \quad (5.251)$$



According to (4.98), we express  $1/l$  as

$$\frac{1}{l} = \begin{cases} \sum_{p=0}^{\infty} \frac{r_S^p}{r^{p+1}} P_p(\cos \psi), & r > r_S; \\ \sum_{p=0}^{\infty} \frac{r^p}{r_S^{p+1}} P_p(\cos \psi), & r < r_S. \end{cases} \quad (5.252)$$

Replace  $\theta$  and  $\lambda$  by  $\theta'$  and  $\lambda'$ , respectively, in (5.245), and then substitute it into (5.249). By further using (5.252), we obtain

$$\phi = \begin{cases} G \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_{p=0}^{\infty} s_n^k \int_{\omega} \frac{r_S^{p+2}}{r^{p+1}} P_p(\cos \psi) P_n^k(\cos \theta') e^{ik\lambda'} d\omega, & r > \max(r_S); \\ G \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_{p=0}^{\infty} s_n^k \int_{\omega} \frac{r^p}{r_S^{p+1}} P_p(\cos \psi) P_n^k(\cos \theta') e^{ik\lambda'} d\omega, & r < \min(r_S). \end{cases} \quad (5.253)$$

Note the difference in the domain of convergence between (5.252) and (5.253);  $r_S$  is a fixed value in the former, while  $\min(r_S) \leq r_S \leq \max(r_S)$  in the latter. We write the addition theorem (4.291) as

$$P_p(\cos \psi) = \sum_{q=-p}^p \left[ (-1)^q P_p^{-q}(\cos \theta') e^{-iq\lambda'} \right] P_p^q(\cos \theta) e^{iq\lambda}. \quad (5.254)$$

It can be used to write (5.253) as

$$\phi = \begin{cases} GR^2 \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_{p=0}^{\infty} \sum_{q=-p}^p s_n^k I_{nkpq}^{(e)} \frac{R^p}{r^{p+1}} P_p^q(\cos \theta) e^{iq\lambda}, & r > \max(r_S); \\ GR^2 \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_{p=0}^{\infty} \sum_{q=-p}^p s_n^k I_{nkpq}^{(i)} \frac{r^p}{R^{p+1}} P_p^q(\cos \theta) e^{iq\lambda}, & r < \min(r_S); \end{cases} \quad (5.255)$$

where  $I_{nkpq}^{(e)}$  and  $I_{nkpq}^{(i)}$  are defined as

$$\begin{cases} I_{nkpq}^{(e)} = \int_{\omega} \left( \frac{r_S}{R} \right)^{p+2} (-1)^q P_p^{-q}(\cos \theta') e^{-iq\lambda'} P_n^k(\cos \theta') e^{ik\lambda'} d\omega, \\ I_{nkpq}^{(i)} = \int_{\omega} \left( \frac{R}{r_S} \right)^{p-1} (-1)^q P_p^{-q}(\cos \theta') e^{-iq\lambda'} P_n^k(\cos \theta') e^{ik\lambda'} d\omega. \end{cases} \quad (5.256)$$

According to (5.243), we make the arrangements

$$\left(\frac{r_s}{R}\right)^{p+2} = 1 + \left[(1 + \epsilon)^{p+2} - 1\right], \quad \left(\frac{R}{r_s}\right)^{p-1} = 1 + \left[\frac{1}{(1 + \epsilon)^{p-1}} - 1\right]. \quad (5.257)$$

Define

$$\begin{cases} \delta k_{pq}^{h_e} = \left[(1 + \epsilon)^{p+2} - 1\right] (-1)^q P_p^{-q}(\cos \theta') e^{-iq\lambda'}, \\ \delta k_{pq}^{h_i} = \left[\frac{1}{(1 + \epsilon)^{p-1}} - 1\right] (-1)^q P_p^{-q}(\cos \theta') e^{-iq\lambda'}, \end{cases} \quad (5.258)$$

and further define  $\delta k_{mlpq}^{h_e}$  and  $\delta k_{mlpq}^{h_i}$  through

$$\begin{cases} \delta k_{pq}^{h_e} = \sum_{m=0}^{\infty} \sum_{l=-m}^m \delta k_{mlpq}^{h_e} (-1)^l P_m^{-l}(\cos \theta') e^{-il\lambda'}, \\ \delta k_{pq}^{h_i} = \sum_{m=0}^{\infty} \sum_{l=-m}^m \delta k_{mlpq}^{h_i} (-1)^l P_m^{-l}(\cos \theta') e^{-il\lambda'}. \end{cases} \quad (5.259)$$

As  $\delta k_{pq}^{h_e}$  and  $\delta k_{pq}^{h_i}$  are known,  $\delta k_{mlpq}^{h_e}$  and  $\delta k_{mlpq}^{h_i}$  can be computed by expressing  $\delta k_{pq}^{h_e}$  and  $\delta k_{pq}^{h_i}$  in spherical harmonic series. The substitution of (5.257), (5.258), and (5.259) into (5.256) leads to

$$\begin{aligned} I_{nkpq}^{(e)} &= (-1)^q \int_{\omega} P_p^{-q}(\cos \theta') e^{-iq\lambda'} P_n^k(\cos \theta') e^{ik\lambda'} d\omega \\ &\quad + \sum_{m=0}^{\infty} \sum_{l=-m}^m \delta k_{mlpq}^{h_e} (-1)^l \int_{\omega} P_m^{-l}(\cos \theta') e^{-il\lambda'} P_n^k(\cos \theta') e^{ik\lambda'} d\omega, \end{aligned} \quad (5.260a)$$

$$\begin{aligned} I_{nkpq}^{(i)} &= (-1)^q \int_{\omega} P_p^{-q}(\cos \theta') e^{-iq\lambda'} P_n^k(\cos \theta') e^{ik\lambda'} d\omega \\ &\quad + \sum_{m=0}^{\infty} \sum_{l=-m}^m \delta k_{mlpq}^{h_i} (-1)^l \int_{\omega} P_m^{-l}(\cos \theta') e^{-il\lambda'} P_n^k(\cos \theta') e^{ik\lambda'} d\omega. \end{aligned} \quad (5.260b)$$

These can be simplified to, according to the orthogonality (4.275),

$$\begin{cases} I_{nkpq}^{(e)} = \frac{4\pi}{2n+1} \delta_{pn} \delta_{qk} + \frac{4\pi}{2n+1} \delta k_{nkpq}^{h_e}, \\ I_{nkpq}^{(i)} = \frac{4\pi}{2n+1} \delta_{pn} \delta_{qk} + \frac{4\pi}{2n+1} \delta k_{nkpq}^{h_i}. \end{cases} \quad (5.261)$$

Finally the result of  $\phi$  is obtained by substituting (5.261) into (5.255). We express it as

$$\phi = \phi_0 + \delta\phi \Rightarrow \begin{cases} \phi^{(e)} = \phi_0^{(e)} + \delta\phi^{(e)}, & r \geq R; \\ \phi^{(i)} = \phi_0^{(i)} + \delta\phi^{(i)}, & r \leq R; \end{cases} \quad (5.262)$$

where

$$\begin{cases} \phi_0^{(e)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G}{(2n+1)g_R} s_n^k \left(\frac{R}{r}\right)^{n+1} P_n^k(\cos\theta) e^{ik\lambda}, \\ \phi_0^{(i)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G}{(2n+1)g_R} s_n^k \left(\frac{r}{R}\right)^n P_n^k(\cos\theta) e^{ik\lambda}; \end{cases} \quad (5.263)$$

$$\begin{cases} \delta\phi^{(e)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \frac{4\pi G}{(2n+1)g_R} s_n^k \sum_{p=0}^{\infty} \sum_{q=-p}^p \delta k_{nkpq}^{h_e} \left(\frac{R}{r}\right)^{p+1} P_p^q(\cos\theta) e^{iq\lambda} \right], \\ \delta\phi^{(i)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \frac{4\pi G}{(2n+1)g_R} s_n^k \sum_{p=0}^{\infty} \sum_{q=-p}^p \delta k_{nkpq}^{h_i} \left(\frac{r}{R}\right)^p P_p^q(\cos\theta) e^{iq\lambda} \right]. \end{cases} \quad (5.264)$$

We see that  $\phi_0^{(e)}$  and  $\phi_0^{(i)}$  are already given in (5.235) for the spherical Earth. In this regards,  $\delta\phi^{(e)}$  and  $\delta\phi^{(i)}$  are corrections due to the asphericity, i.e., the deviation from sphericity.

In the result (5.262)–(5.264), we have changed the domain of convergence  $r > \max(r_S)$  and  $r < \min(r_S)$  to  $r \geq R$  and  $r \leq R$ , respectively. As already indicated, the superscripts  $(e)$  and  $(i)$  are used to denote the domains  $r \geq R$  and  $r \leq R$ , respectively. Physically,  $\phi^{(e)}$  is meaningless for  $r < \max(r_S)$ , and  $\phi^{(i)}$  for  $r > \min(r_S)$ . However, to facilitate mathematical treatment, the variation of  $\phi$  in the layer  $\min(r_S) < r < \max(r_S)$  can be treated as a jump through the sphere of radius  $R$ . This is to extend  $\phi^{(e)}$  to the sphere from the exterior,  $\phi^{(i)}$  from the interior, and treat the difference  $\phi^{(e)} - \phi^{(i)}$  on the sphere as a jump. Such an approach has already been used in Sect. 3.7 in the formulation of boundary conditions for the elastic deformation of the Earth to project a deformed boundary to its undeformed position. We see that  $\phi_0$  is continuous across the sphere. It is not evident to formulate the jump  $\delta\phi^{(e)} - \delta\phi^{(i)}$  across the sphere based on (5.264). Instead, we use (5.253). The expression of  $\phi^{(e)}$  and  $\phi^{(i)}$  are obtained by setting  $r_S = R$ ,

$$\begin{cases} \phi^{(e)} = G \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_{p=0}^{\infty} s_n^k \int_{\omega} \frac{R^{p+2}}{r^{p+1}} P_p(\cos\psi) P_n^k(\cos\theta') e^{ik\lambda'} d\omega, \\ \phi^{(i)} = G \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_{p=0}^{\infty} s_n^k \int_{\omega} \frac{r^p}{R^{p-1}} P_p(\cos\psi) P_n^k(\cos\theta') e^{ik\lambda'} d\omega. \end{cases} \quad (5.265)$$

On the sphere, we have  $r = R$  and  $\phi_0^{(e)} = \phi_0^{(i)}$ . Hence, we can readily obtain

$$\begin{aligned} \left[ \delta\phi^{(e)} - \delta\phi^{(i)} \right]_{r=R} &= G \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_{p=0}^{\infty} s_n^k \int_{\omega} \left( \frac{r_S^{p+2}}{R^{p+1}} - \frac{R^p}{r_S^{p-1}} \right) \\ &\quad \times P_p(\cos \psi) P_n^k(\cos \theta') e^{ik\lambda'} d\omega. \end{aligned} \quad (5.266)$$

Accurate to only the first order of  $h$ , we obtain, according to (5.243),

$$\frac{r_S^{p+2}}{R^{p+1}} - \frac{R^p}{r_S^{p-1}} = R \left[ 1 + (p+2) \frac{h}{R} \right] - R \left[ 1 - (p-1) \frac{h}{R} \right] = (2p+1)h. \quad (5.267)$$

By substituting this formula into (5.266), and then using (5.265), we obtain

$$\left[ \delta\phi^{(e)} - \delta\phi^{(i)} \right]_{r=R} = -h \left[ \frac{\partial \phi_0^{(e)}}{\partial r} - \frac{\partial \phi_0^{(i)}}{\partial r} \right]_{r=R}, \quad (5.268)$$

which can be written briefly as

$$(\delta\phi)_{-}^{+} = -h \left( \frac{\partial \phi_0}{\partial r} \right)_{-}^{+}. \quad (5.269)$$

This condition is the consequence of the continuity of  $\phi_0$  across the sphere of radius  $R$ , and the continuity of  $\phi$  across the Earth's aspherical surface defined by (5.243).

It is noted that (5.269) is accurate to the first order of  $h$ , or equivalently, the first order of  $\epsilon$ , while (5.264) is an exact expression that holds only in  $r > \max(r_S)$ .

We have formulated the problem using complex spherical harmonics. By redefining all the coefficients for real spherical harmonics, we have (keeping implicit the variables that the functions depend on)

$$s = \sum_{n=0}^{\infty} \sum_{k=-n}^n s_n^k \mathcal{H}_n^k(\theta, \lambda); \quad (5.270)$$

$$\begin{cases} \delta k_{pq}^{h_e} = \left[ (1 + \epsilon)^{p+2} - 1 \right] \frac{2}{1 + \delta_q} \frac{(p - |q|)!}{(p + |q|)!} \mathcal{H}_p^q(\theta', \lambda'), \\ \delta k_{pq}^{h_i} = \left[ \frac{1}{(1 + \epsilon)^{p-1}} - 1 \right] \frac{2}{1 + \delta_q} \frac{(p - |q|)!}{(p + |q|)!} \mathcal{H}_p^q(\theta', \lambda'); \end{cases} \quad (5.271)$$

$$\begin{cases} \delta k_{pq}^{h_e} = \sum_{m=0}^{\infty} \sum_{l=-m}^m \delta k_{mlpq}^{h_e} \frac{2}{1+\delta_l} \frac{(m-|l|)!}{(m+|l|)!} \mathcal{H}_m^l(\theta', \lambda'), \\ \delta k_{pq}^{h_i} = \sum_{m=0}^{\infty} \sum_{l=-m}^m \delta k_{mlpq}^{h_i} \frac{2}{1+\delta_l} \frac{(m-|l|)!}{(m+|l|)!} \mathcal{H}_m^l(\theta', \lambda'); \end{cases} \quad (5.272)$$

$$\phi = \phi_0 + \delta\phi \Rightarrow \begin{cases} \phi^{(e)} = \phi_0^{(e)} + \delta\phi^{(e)}, & r \geq R; \\ \phi^{(i)} = \phi_0^{(i)} + \delta\phi^{(i)}, & r \leq R; \end{cases} \quad (5.273)$$

$$\begin{cases} \phi_0^{(e)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G}{(2n+1)g_R} s_n^k \left(\frac{R}{r}\right)^{n+1} \mathcal{H}_n^k(\theta, \lambda), \\ \phi_0^{(i)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G}{(2n+1)g_R} s_n^k \left(\frac{r}{R}\right)^n \mathcal{H}_n^k(\theta, \lambda); \end{cases} \quad (5.274)$$

$$\begin{cases} \delta\phi^{(e)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \frac{4\pi G}{(2n+1)g_R} s_n^k \sum_{p=0}^{\infty} \sum_{q=-p}^p \delta k_{nkpq}^{h_e} \left(\frac{R}{r}\right)^{p+1} \mathcal{H}_p^q(\theta, \lambda) \right], \\ \delta\phi^{(i)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \frac{4\pi G}{(2n+1)g_R} s_n^k \sum_{p=0}^{\infty} \sum_{q=-p}^p \delta k_{nkpq}^{h_i} \left(\frac{r}{R}\right)^p \mathcal{H}_p^q(\theta, \lambda) \right]. \end{cases} \quad (5.275)$$

Since  $s$  and  $\phi$  are real functions, their real and complex spherical harmonic coefficients have the properties (4.264) and (4.265). These properties may be used to infer the formulae in real spherical harmonics from those in complex spherical harmonics. However, the treatment of the coefficients  $\delta k_{nkpq}^{h_e}$  and  $\delta k_{nkpq}^{h_i}$ , which are defined in (5.259) for the formulation in complex form, and in (5.271) and (5.272) for that in real form, require some care. In fact, directly formulating the problem using real spherical harmonics is more straightforward, which is what we followed. The detail is omitted here, and is left to the readers as an exercise.

In (5.273)–(5.275), we have expressed the gravitational potential  $\phi$  of the surface layer of density  $s$  as the sum of two terms,  $\phi_0$  and  $\delta\phi$ , the former being the gravitational potential assuming the Earth to be spherical, and the latter a correction related to the deviation from spherical symmetry. Here we further write the gravitational potential in the exterior of the Earth,  $\phi^{(e)} = \phi_0^{(e)} + \delta\phi^{(e)}$ , in an alternative form in favor of practical application. We omit the superscript  $(e)$  for brevity, and express  $\phi$  using its spherical harmonic coefficients as

$$\phi = R g_R \sum_{p=0}^{\infty} \sum_{q=-p}^p \phi_p^q \left(\frac{R}{r}\right)^{p+1} \mathcal{H}_p^q(\theta, \lambda). \quad (5.276)$$

According to (5.273)–(5.275), we have

$$\begin{aligned}\phi_p^q &= \frac{4\pi G}{(2p+1)g_R} s_p^q + \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G}{(2n+1)g_R} s_n^k \delta k_{nkpq}^{h_e} \\ &= \frac{4\pi G}{g_R} \left[ \frac{1}{2p+1} s_p^q + \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{1}{2n+1} s_n^k \delta k_{nkpq}^{h_e} \right].\end{aligned}\quad (5.277)$$

This is a formula for computing the spherical harmonic coefficients of the gravitational potential from those of the surface mass density. The inverse, i.e., the computation of the spherical harmonic coefficients of the surface mass density from those of the gravitational potential, should be done by solving the system of equations of (5.277) for all values of  $p$  and  $q$  truncated at a degree of, say,  $N$ . An iterative form for the solution is

$$s_p^q = (2p+1) \left[ \frac{g_R}{4\pi G} \phi_p^q - \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{1}{2n+1} s_n^k \delta k_{nkpq}^{h_e} \right]. \quad (5.278)$$

In (5.278), the second term in the brackets is supposed to be of smaller magnitude than the first term. We first solve for  $s_p^q$  without the second term, leading to the same result as that of the spherical Earth model given by (5.237). By substituting the result back into the second term in the brackets of (5.278), we obtain an approximate formula for  $s_p^q$  accurate to the first order of  $\delta k_{nkpq}^{h_e}$ , i.e.,

$$s_p^q = \frac{(2p+1)g_R}{4\pi G} \left[ \phi_p^q - \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k \delta k_{nkpq}^{h_e} \right]. \quad (5.279)$$

This approximate formula should be suitable for most practical applications as far as the truncation degree is not too high such as that of the Gravity Recovery and Climate Experiment satellite mission (GRACE) data<sup>1</sup> of  $\phi_n^k$  to be discussed in more detail in Chap. 9 (noting that here  $\phi_n^k$  is a difference of the GRACE monthly solutions). In this case, the topographic height  $h$  can be smoothed to be commensurate with the resolution of the data of  $\phi_n^k$ , so that  $\delta k_{nkpq}^{h_e} \ll 1$  can be assumed.

### 5.6.3 Practical Computation

There are two problems of computation. One is the foreword problem, i.e., computing the spherical harmonic coefficients of the gravitational potential when

<sup>1</sup> We use GRACE to implicitly indicate both the GRACE and GRACE Follow-On missions.

the density of the mass is known. The other is the inverse problem, i.e., computing the density of the mass when the spherical harmonic coefficients of the gravitational potential are known. In both cases, we need the data of  $h$ , which is the height of the Earth's surface above the sphere of radius  $R$ .

In view of practical applications, we discuss only the external case in terms of real spherical harmonics. In geodesy, the fully normalized Legendre function and spherical harmonics defined by (4.340) and (4.341) are used. The surface mass density (5.270) is rewritten

$$s = \sum_{n=0}^{\infty} \sum_{k=-n}^n \bar{s}_n^k \bar{\mathcal{H}}_n^k(\theta, \lambda), \quad (5.280)$$

where the normalized spherical harmonics  $\bar{\mathcal{H}}_n^k(\theta, \lambda)$  and coefficient  $\bar{s}_n^k$  are defined through

$$\mathcal{H}_n^k(\theta, \lambda) = \left[ \frac{1 + \delta_k}{2(2n+1)} \frac{(n+|k|)!}{(n-|k|)!} \right]^{1/2} \bar{\mathcal{H}}_n^k(\theta, \lambda) \quad (5.281)$$

and

$$s_n^k = \left[ \frac{2(2n+1)}{1 + \delta_k} \frac{(n-|k|)!}{(n+|k|)!} \right]^{1/2} \bar{s}_n^k, \quad (5.282)$$

respectively. Define  $\bar{\delta}k_{pq}^{he}$  in the same way as  $\bar{s}_n^k$ , and

$$\delta k_{mlpq}^{he} = \left[ \frac{1 + \delta_k}{2(2m+1)} \frac{(m+|l|)!}{(m-|l|)!} \right]^{1/2} \left[ \frac{2(2p+1)}{1 + \delta_k} \frac{(p-|q|)!}{(p+|q|)!} \right]^{1/2} \bar{\delta}k_{mlpq}^{he}. \quad (5.283)$$

It can be obtained from (5.271) to (5.275) that

$$\delta \bar{k}_{pq}^{he} = \frac{1}{2p+1} \left[ (1 + \epsilon)^{p+2} - 1 \right] \bar{\mathcal{H}}_p^q(\theta', \lambda'), \quad (5.284)$$

$$\delta \bar{k}_{pq}^{he} = \sum_{m=0}^{\infty} \sum_{l=-m}^m \frac{1}{2m+1} \delta \bar{k}_{mlpq}^{he} \bar{\mathcal{H}}_m^l(\theta', \lambda'), \quad (5.285)$$

$$\phi_0^{(e)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G}{(2n+1)g_R} \bar{s}_n^k \left( \frac{R}{r} \right)^{n+1} \bar{\mathcal{H}}_n^k(\theta, \lambda), \quad (5.286)$$

$$\delta \phi^{(e)} = R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \frac{4\pi G}{(2n+1)g_R} \bar{s}_n^k \sum_{p=0}^{\infty} \sum_{q=-p}^p \delta \bar{k}_{nkpq}^{he} \left( \frac{R}{r} \right)^{p+1} \bar{\mathcal{H}}_p^q(\theta, \lambda) \right]; \quad (5.287)$$

and (5.276)–(5.279) can be written as

$$\phi = R g_R \sum_{p=0}^{\infty} \sum_{q=-p}^p \bar{\phi}_p^q \left( \frac{R}{r} \right)^{p+1} \bar{\mathcal{H}}_p^q(\theta, \lambda), \quad (5.288)$$

$$\bar{\phi}_p^q = \frac{4\pi G}{g_R} \left[ \frac{1}{2p+1} \bar{s}_p^q + \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{1}{2n+1} \bar{s}_n^k \delta \bar{k}_{nkpq}^{he} \right], \quad (5.289)$$

$$\bar{s}_p^q = (2p+1) \left[ \frac{g_R}{4\pi G} \bar{\phi}_p^q - \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{1}{2n+1} \bar{s}_n^k \delta \bar{k}_{nkpq}^{he} \right], \quad (5.290)$$

$$\bar{s}_p^q = \frac{(2p+1)g_R}{4\pi G} \left[ \bar{\phi}_p^q - \sum_{n=0}^{\infty} \sum_{k=-n}^n \bar{\phi}_n^k \delta \bar{k}_{nkpq}^{he} \right]. \quad (5.291)$$

The derivation is straightforward, and is left to the readers.

The key is to compute  $\delta \bar{k}_{mnpq}^{he}$  according to (5.285). As  $\delta \bar{k}_{mnpq}^{he}/(2m+1)$  is the spherical harmonic coefficient of  $\bar{k}_{pq}^{he}$ , it can be computed using the discrete spherical harmonic transform (DSHT) presented in Sect. 4.9.4. This requires to compute  $\bar{k}_{pq}^{he}$  over a  $\theta$ - $\lambda$  grid, and then perform the DSHT. The sum of a spherical harmonic series like (5.280) over a  $\theta$ - $\lambda$  grid can be computed using the inverse discrete spherical harmonic transform (IDSHT).

We use extensively the DSHT and IDSHT presented in Sect. 4.9.4. Therefore, the values of  $s$  and  $h$  over a  $\theta$ - $\lambda$  grid are used. We adopt the grid (4.645) and write it here as

$$\theta_i = \frac{(2i+1)\pi}{4b}, \quad \lambda_j = \frac{j\pi}{b}; \quad i, j = 0, 1, \dots, 2b-1. \quad (5.292)$$

This grid does not include the poles; so we prefer it over (4.644). As the DSHT and IDSHT assume, we truncate all the spherical harmonic series to degree and order  $b-1$ . The grid values do not necessarily agree with the spherical harmonic series, as the number of spherical harmonic coefficients is about half of that of the grid values (Guo & Shum, 2009). To maintain the consistency between them, the grid values must be generated using the spherical harmonic series. This implies that

$$\epsilon(\theta_i, \lambda_j) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \bar{\epsilon}_n^k \bar{\mathcal{H}}_n^k(\theta_i, \lambda_j) \quad (5.293)$$

must hold for the data of  $\epsilon = h/R$ . Similarly, for the data of the density of the mass to be consistent,

$$s(\theta_i, \lambda_j) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \bar{s}_n^k \bar{\mathcal{H}}_n^k(\theta_i, \lambda_j) \quad (5.294)$$



must hold. A simple way to ensure these is to first transform the grid data to spherical harmonic coefficients using the DSHT, and then transform back to grid data using the IDSHT.

For the spherical Earth surface model, the computation is straightforward, where all we need is (5.237) with the DSHT or IDSHT based on (5.294).

For the aspherical Earth surface model, we should first compute  $\delta\bar{k}_{mlpq}^{he}$  based on (5.284) and (5.285). In these formulae,  $\delta\bar{k}_{pq}^{he}$  and  $\epsilon$  are implicitly understood as functions of  $\theta'$  and  $\lambda'$ . We first evaluate

$$\delta\bar{k}_{pq}^{he}(\theta_i, \lambda_j) = \frac{1}{2p+1} \left\{ [1 + \epsilon(\theta_i, \lambda_j)]^{p+2} - 1 \right\} \bar{\mathcal{H}}_p^q(\theta_i, \lambda_j). \quad (5.295)$$

According to (5.285), performing a DSHT to

$$\delta\bar{k}_{pq}^{he}(\theta_i, \lambda_j) = \sum_{m=0}^{\infty} \sum_{l=-m}^m \frac{1}{2m+1} \delta\bar{k}_{mlpq}^{he} \bar{\mathcal{H}}_m^l(\theta_i, \lambda_j) \quad (5.296)$$

yields  $\delta\bar{k}_{mlpq}^{he}/(2m+1)$ , thus obtaining  $\delta\bar{k}_{mlpq}^{he}$ . For the forward problem, the spherical harmonic coefficient  $\bar{\phi}_p^q$  in (5.288) can be computed according to (5.289) from  $\bar{s}_n^k$  and  $\delta\bar{k}_{mlpq}^{he}$ . For the inverse problem, the spherical harmonic coefficient  $\bar{s}_p^q$  can be computed iteratively according to (5.290) from  $\bar{\phi}_p^q$  and  $\delta\bar{k}_{mlpq}^{he}$ , or approximately according to (5.291). The grid values of the density of the mass can be computed using the IDSHT according to (5.294).

It is noted that the topographic height in the formulation is defined as the height above the sphere of radius  $R$ . The largest component is the height of the ellipsoidal surface above the sphere, which can be computed according to (5.35),  $r - R = -Rf(\cos^2 \theta - 1/3)$ . The height of the topographic surface above the ellipsoidal surface can be obtained from a digital elevation model (DEM).

The accuracy of the formulation is verified using a topographic model with  $R = 6371$  km,  $f = 1.0/298.257$ , and the top surface of the CRUST1.0 with a resolution of  $1^\circ \times 1^\circ$  (Laske et al., 2013). The model is first expanded into a spherical harmonics according to (5.293), and is truncated to degree and order 63, which corresponds to a grid of  $b = 64$  in (5.292). As we are just quantifying the accuracy of the formulation, the mass density could be arbitrarily chosen. We made the computation using two sets of surface mass data. One is to mimic the surface mass density by the top surface height of the CRUST1.0 without taking into account of the flattening, i.e., choosing  $\bar{s}_n^k$  to be  $\bar{\epsilon}_n^k$  computed without adding the effect of the flattening. This choice allows the readers to easily duplicate the result with limited effort. We remind that, in this case, the topographic height of CRUST1.0 used to mimic the mass changes vanishes over the oceans. The other one is a linear trend of mass change computed from GRACE Level 2 data. The potential coefficients  $\bar{\phi}_p^q$  are computed according to (5.289). The inversion of mass density assuming the potential coefficients  $\bar{\phi}_p^q$  to be known is performed using three formulae, (5.237),

(5.291), and (5.290), which are referred to as spherical (equivalent to assuming the Earth to be spherical in the inversion), approximate, and exact (in the sense that the exact formula is used), respectively. The mass density over the grid (5.292) can be computed using the IDSHT, and are denoted as  $s_s(\theta_i, \lambda_j)$ ,  $s_a(\theta_i, \lambda_j)$ , and  $s_e(\theta_i, \lambda_j)$ , respectively, for the three cases.

We use the root mean square (RMS) to define the error. By definition, the RMS of the density over the sphere is defined according to

$$[\text{RMS}(s)]^2 = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi [s(\theta, \lambda)]^2 \sin \theta d\theta d\lambda. \quad (5.297)$$

For the grid values, we define the RMS according to

$$[\text{RMS}(s)]^2 = \frac{1}{2b \times 2b} \sum_{i=0}^{2b-1} \sum_{j=0}^{2b-1} [s(\theta_i, \lambda_j)]^2 \sin \theta_i. \quad (5.298)$$

This is an average of the square weighted with  $\sin \theta_i$ , which is proportional to the area of the grid centered by  $(\theta_i, \lambda_j)$ . The RMS of the difference  $s_s(\theta_i, \lambda_j) - s(\theta_i, \lambda_j)$  is defined according to

$$[\text{RMS}(s_s - s)]^2 = \frac{1}{2b \times 2b} \sum_{i=0}^{2b-1} \sum_{j=0}^{2b-1} [s_s(\theta_i, \lambda_j) - s(\theta_i, \lambda_j)]^2 \sin \theta_i. \quad (5.299)$$

The relative error of  $s_s(\theta_i, \lambda_j)$  is defined by

$$e_s = \frac{\text{RMS}(s_s - s)}{\text{RMS}(s)}. \quad (5.300)$$

The relative errors of  $s_a(\theta_i, \lambda_j)$  and  $s_e(\theta_i, \lambda_j)$ ,  $e_a$  and  $e_e$ , are defined similarly.

Our results for the two data sets are  $e_s = 2.0\%, 5.0\%$ ;  $e_a = 0.098\%, 0.33\%$ ; and  $e_e = 0.0068\%, 0.023\%$ . In the computation, the exact formula (5.290) is used to compute  $e_e$  by substituting into it the result of  $\bar{s}_p^q$  computed according to (5.291) without further iteration. The results imply that the errors brought by approximating the Earth as a sphere are on order of  $e_s = 2.0\%, 5.0\%$ ; the approximate formula (5.291) is accurate to  $e_a = 0.098\%, 0.33\%$ ; and one iteration of the exact formula (5.290) brings the errors down to  $e_e = 0.0068\%, 0.023\%$ . Remember that enough iterations can bring the error of the exact formula down to the level of roundoff error.

We have also quantified the accuracy of inversion with only the ellipsoidal correction. The potential coefficients  $\bar{\phi}_p^q$  are computed in exactly the same way as in the previous paragraph. In the inversion of mass density, we use a different set of values of  $\delta \bar{k}_{mlpq}^{he}$ , which are computed for an ellipsoidal Earth surface, i.e., without adding the elevation of the CRUST1.0. The results are  $e_s = 2.0\%, 5.0\%$ ,  $e_a = 0.93\%, 0.43\%$ , and  $e_e = 0.97\%, 0.40\%$ . Our experience demonstrates that

more iterations do not bring any significant improvement. This means that  $e_e = 0.97\%$ ,  $0.40\%$  are the levels of error caused by ignoring the topography above the ellipsoid in the inversion. The value of  $e_e$  is larger for the data set mimicked by the topographic height of the CRUST1.0, since the topographic height vanishes over the oceans considered to be on the ellipsoid.

The results in the above two paragraphs are suggestive of the accuracy of the inversion of mass transport on the Earth's surface using the variation of the potential coefficients provided by the GRACE. For our examples, the errors brought by completely neglecting the Earth's nonsphericity are on the orders of  $e_s = 2.0\%$ ,  $5.0\%$ ; and those by neglecting the height of the Earth's surface above the ellipsoid are on the order of  $e_e = 0.97\%$ ,  $0.40\%$ . If there is no mass change over the land, it suffices to consider the Earth to be ellipsoidal in the inversion. We have redone the computation assuming the Earth's surface to be an ellipsoid throughout, i.e., the layer of mass is on the ellipsoid for the computation of the gravitational potential, and the inversion of the gravitational potential is done for a layer of mass on the same ellipsoid. The results are  $e_s = 1.9\%$ ,  $5.0\%$ ;  $e_a = 0.11\%$ ,  $0.32\%$ ; and  $e_e = 0.0085\%$ ,  $0.024\%$ ; noting that  $e_e$  could be brought down to the level of roundoff error with enough iterations. This implies that, for signals purely originated from a layer of mass over the oceans, the errors by approximating the Earth as a sphere are on the order of  $e_s = 1.9\%$ ,  $5.0\%$ .

We have also attempted to express the disturbing potential of the Earth Gravitational Model 2008 (EGM2008) (Pavlis et al., 2012) with respect to the reference Earth ellipsoid of the world geodetic system 1984 (WGS84) as the gravitational potential of a thin layer of mass on the topographic surface of the Earth. To compute the disturbing potential, we have converted the values of  $GM$  and  $a$  to those of the WGS84 reference Earth ellipsoid using the exact formulae (Guo, 2023; Pavlis et al., 2012), and then subtracted the normal potential of the WGS84 reference Earth ellipsoid. For illustrative purpose based on our immediate computational capacity, we did the computation by truncating the disturbing potential up to degree and order 255, i.e., with  $b = 256$ . The topography of the corresponding resolution above the WGS84 ellipsoid is obtained by truncating the spherical harmonic series of the global relief model ETOPO5 through a DSHT-truncate-IDSHT procedure, and then setting the negative heights to zeros. Since this topography serves as a synthetic model for numerical experiments, the latitude of the ETOPO5 model is taken as the geocentric latitude. The distance from the average sphere to the ellipsoid is then added based on the exact formula (Guo, 2023). As this problem requires extreme accuracy, we first evaluate the density of the layer by iterating twenty times, and consider the result as the reference value (the relative difference between the results of the twentieth and twenty-first iterations is  $5.0 \times 10^{-11}\%$ ) for verifying how the accuracy depends on the number of iterations. The relative accuracy of the approximate solution is on the order of  $2.1\%$ , and those corresponding to one to fifteen iterations are  $5.6 \times 10^{-1}\%$ ,  $1.7 \times 10^{-1}\%$ ,  $5.3 \times 10^{-2}\%$ ,  $1.8 \times 10^{-2}\%$ ,  $6.4 \times 10^{-3}\%$ ,  $2.4 \times 10^{-3}\%$ ,  $9.0 \times 10^{-4}\%$ ,  $3.5 \times 10^{-4}\%$ ,  $1.4 \times 10^{-4}\%$ ,  $5.3 \times 10^{-5}\%$ ,  $2.1 \times 10^{-5}\%$ ,  $8.5 \times 10^{-6}\%$ ,  $3.4 \times 10^{-6}\%$ ,  $1.4 \times 10^{-6}\%$ , and  $5.5 \times 10^{-7}\%$ . Our numerical experiments demonstrated that, the larger the truncation degree and

order of the spherical harmonic series, the slower the convergence of the iterative procedure.

What we achieved is to express the known disturbing potential outside the Brillouin sphere as the gravitational potential of a unique thin layer of mass on the Earth's surface based on spherical harmonic series. However, it is known that, between the Earth's surface and the Brillouin sphere, the spherical harmonic series of the disturbing potential is not guaranteed to converge, and even the gradient of the series is not guaranteed to represent the gravity disturbance (e.g. Bevis et al., 2024; Guo, 2023). Nevertheless, we know that the disturbing potential throughout the outside of the Earth's surface can be uniquely expressed by the gravitational potential of a thin layer of mass on the Earth's surface. The density of the layer defined as a function on the Earth's surface does not suffer from the problems of the spherical harmonic series mentioned above, and hence, it can serve as an alternative representation of the disturbing gravity field valid throughout the outside of the Earth's surface. Besides the adoption of the spherical harmonics series for the disturbing potential on and above the Brillouin sphere as formulated in this section, two other problems are in need of formulation for practical applications: (1) the determination the density of the layer making use of the airborne and surface gravity data, and (2) the representation and computation of the disturbing potential and gravity disturbance between the Earth's surface and the Brillouin sphere using the density of the layer.

It is straightforward, although tedious, to generalize the formulation from that based on a spherical reference surface and the spherical harmonic expansion to that based on an ellipsoidal reference surface and the ellipsoidal harmonic expansion, the latter being adopted in the development of the EGM2008 (Pavlis et al., 2012).

We note that, according to (5.284), (5.285), and (5.288)–(5.291), the topography of resolution above the truncation degree and order, e.g., 63 or 255 in our examples, also contributes to the result in both the forward and inverse problems. The truncated topographies in our examples serve as models of the topography in our demonstrative computations. Certainly, the choice of the truncation degree and order depends on the nature of the concrete physical problem that the formulation is applied to.

Finally, we mention that an approximate analytic formula of  $\delta\bar{k}_{mlpq}^{he}$  can be derived assuming the Earth's surface to be ellipsoidal based on (5.284) and (5.285). This is to keep only the first order terms of the flattening  $f$ . The procedure is as follows: (a) In (5.284), make the approximation  $(1 + \epsilon)^{p+2} - 1 \approx (p + 2)\epsilon$  and then substitute  $\epsilon$  by  $\epsilon = (r - R)/R = -f(\cos^2 \theta - 1/3)$ ; (b) Develop  $\cos^2 \theta \bar{\mathcal{H}}_p^q(\theta', \lambda')$  into the form of  $\alpha_p^q \bar{\mathcal{H}}_{p-2}^q(\theta', \lambda') + \beta_p^q \bar{\mathcal{H}}_p^q(\theta', \lambda') + \gamma_p^q \bar{\mathcal{H}}_{p+2}^q(\theta', \lambda')$ , which, as mentioned in Sect. 4.3.2, can be achieved by successively applying two times the normalized version of (4.132) (the un-normalized case can be found in Moritz (1980)); (c) Compare with (5.285) to infer the formulae of  $\delta\bar{k}_{mlpq}^{he}$ . One can also derive the result based on (5.255) and (5.256) for the complex case. The similar can be done for the real case. This permits to jump the subsequent formulation through the definition of  $\delta\bar{k}_{mlpq}^{he}$ . The details of the derivations are left to the readers as an exercise.

## Bibliographic Notes

The author's main reference include Guo (2023) and Moritz (1990) for Sect. 5.1 through 5.4; Guo (2001), Jardtetzky (1958), Jeffreys (1976), Lanzano (1982), Moritz (1990) and Smylie et al. (1984) for Sect. 5.5; Melchior (1986) for Sect. 5.6. These are also suggested as the readers' sources for alternative presentations.

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# Seismology and the Spherical Elastic Earth Model

## 6

### Abstract

This chapter serves as an introduction to theoretical seismology. The seismic body waves, seismic surface waves, seismic ray theory, and the free oscillations of the Earth are formulated systematically based on respective adequate models of the Earth. Particular emphasis is dedicated to the free oscillations. First, the ordinary differential equations of the free oscillation of the spherical Earth model taking into account of the Coriolis force are derived in detail. Second, a perturbation theory of the free oscillation is provided in detail. These are usually not found in introductory books. The relations among the free oscillation theory, surface wave theory, and body wave theory are also formulated. Finally, the determination of the Earth's internal structure and physical parameters using astro-geodetic data, seismic body wave data, and free oscillation data are summarized, including a discussion on the stratification within the Earth.

## 6.1 Seismic Body Waves

### 6.1.1 Earth's Elastic Equation for Studying Seismic Waves

The displacement associated with waves is periodical in both time and space. Based on these periodical properties, the elastic equation of the Earth can be simplified as follows:

1. The angular frequency of the wave,  $\sigma$ , is much larger than the angular rate of Earth rotation,  $\omega$ , and the Coriolis term  $2\vec{\omega} \times (\partial \vec{u} / \partial t) \sim 2\omega\sigma$  can be neglected as compared to the second order derivative of the displacement with respect to time,  $\partial^2 \vec{u} / \partial t^2 \sim \sigma^2$ .
2. The amplitude of the displacement is small, but its rate of changes in space is large (as the wavelength is short), and the effect of gravity, which depends on the

displacement or its first order derivatives with respect to spatial coordinates, can be neglected as compared to the elastic stress  $\nabla \cdot \bar{T}$ , which depends on the second order derivatives of the displacement with respect to spatial coordinates.

3. The external force can be neglected.

With these approximations, the elastic equation of the Earth is simplified to

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \bar{T}, \quad (6.1)$$

and the Eulerian and Lagrangian coordinates are no longer in need to be distinguished. The isotropic elastic constitutive equation is

$$\bar{T} = \lambda \nabla \cdot \vec{u} \bar{I} + \mu [\nabla \vec{u} + (\nabla \vec{u})^T]. \quad (6.2)$$

The component forms of these equations are

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_i}, \quad (6.3)$$

$$T_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right). \quad (6.4)$$

These equations are identical to those of the non-rotating and unstressed case. The only difference is that the stress in the non-rotating and unstressed case,  $\bar{\sigma}$ , is replaced by the Lagrangian increment of the stress,  $\bar{T}$ , here.

We also assume that the variations of the elastic parameters  $\lambda$  and  $\mu$  in space are much slower than that of the displacement, such that, as compared to the derivatives of  $\vec{u}$  with respect to coordinates, those of  $\lambda$  and  $\mu$  are negligible. We can obtain, by substituting (6.4) into (6.3),

$$\begin{aligned} \rho \frac{\partial^2 u_j}{\partial t^2} &= \lambda \delta_{ij} \frac{\partial}{\partial x_i} \left( \frac{\partial u_k}{\partial x_k} \right) + \mu \left[ \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_i} \right) + \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_j} \right) \right] \\ &= \lambda \frac{\partial}{\partial x_j} \left( \frac{\partial u_k}{\partial x_k} \right) + \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_i} \right) + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_i} \\ &= (\lambda + \mu) \frac{\partial}{\partial x_j} \left( \frac{\partial u_k}{\partial x_k} \right) + \mu \frac{\partial^2 u_j}{\partial x_i \partial x_i}, \end{aligned} \quad (6.5)$$

which can be written in vector form as

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) + \mu \nabla^2 \vec{u}. \quad (6.6)$$

For convenience, we use (2.233) to write this equation in the form of

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu) \nabla(\nabla \cdot \vec{u}) - \mu \nabla \times (\nabla \times \vec{u}). \quad (6.7)$$

The substitution of  $\lambda$  by  $k$  yields

$$\frac{\partial^2 \vec{u}}{\partial t^2} = \frac{k + (4/3)\mu}{\rho} \nabla(\nabla \cdot \vec{u}) - \frac{\mu}{\rho} \nabla \times (\nabla \times \vec{u}). \quad (6.8)$$

In subsequent derivations, we will also assume that the derivatives of  $\rho$  with respect to the coordinates are negligible as compared to those of  $\vec{u}$ , and hence, so are those of  $\lambda/\rho$ ,  $\mu/\rho$ ,  $(\lambda + 2\mu)/\rho$ , and  $[k + (4/3)\mu]/\rho$ .

### 6.1.2 The Concept of Seismic Body Wave

Now we discuss the solution of (6.8) in a very simple case. We assume that  $\vec{u}$  is a function of  $x_1$  and  $t$  only,

$$\vec{u} = \vec{u}(x_1, t) = u_i(x_1, t) \hat{e}_i. \quad (6.9)$$

We can then obtain

$$\nabla(\nabla \cdot \vec{u}) = \frac{\partial^2 u_1}{\partial x_1^2} \hat{e}_1, \quad (6.10)$$

$$\nabla \times (\nabla \times \vec{u}) = -\frac{\partial^2 u_2}{\partial x_1^2} \hat{e}_2 - \frac{\partial^2 u_3}{\partial x_1^2} \hat{e}_3. \quad (6.11)$$

Upon substituting of these formulae into (6.8), we obtain a vector equation, which can be written in components as

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{k + (4/3)\mu}{\rho} \frac{\partial^2 u_1}{\partial x_1^2}, \quad (6.12)$$

$$\frac{\partial^2 u_2}{\partial t^2} = \frac{\mu}{\rho} \frac{\partial^2 u_2}{\partial x_1^2}, \quad (6.13)$$

$$\frac{\partial^2 u_3}{\partial t^2} = \frac{\mu}{\rho} \frac{\partial^2 u_3}{\partial x_1^2}. \quad (6.14)$$

These equations are in the same form, which we write as

$$\frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial x^2}. \quad (6.15)$$



This equation is referred to as wave equation, as will be explained in the next two paragraphs.

In order to find the solution of (6.15), we introduce the variable substitution

$$\xi = x - Vt, \quad \eta = x + Vt. \quad (6.16)$$

We then further have

$$\frac{\partial u}{\partial t} = -V \frac{\partial u}{\partial \xi} + V \frac{\partial u}{\partial \eta}, \quad (6.17)$$

$$\frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial \xi^2} - 2V^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + V^2 \frac{\partial^2 u}{\partial \eta^2}, \quad (6.18)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \quad (6.19)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}. \quad (6.20)$$

By substituting (6.18) and (6.20) into (6.15), we obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \quad (6.21)$$

The integration over  $\eta$  yields

$$\frac{\partial u}{\partial \xi} = f(\xi), \quad (6.22)$$

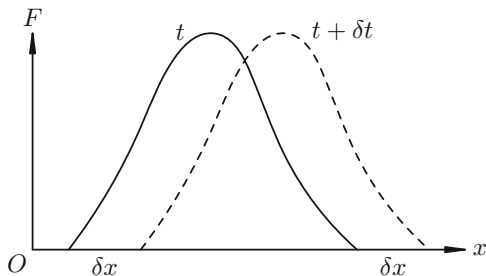
where  $f(\xi)$  is a constant of integration. The constant of integration is an arbitrary function of  $\xi$ , since  $\xi$  is considered as a constant in the integration over  $\eta$ . A further integration over  $\xi$  leads to

$$u = \int f(\xi) d\xi = F(\xi) + G(\eta). \quad (6.23)$$

where  $G(\eta)$  is a constant of integration. The constant of integration is an arbitrary function of  $\eta$ , since  $\eta$  is considered as a constant in the integration over  $\xi$ . Finally, by substituting (6.16) into this formula, we obtain

$$u = F(x - Vt) + G(x + Vt), \quad (6.24)$$

which is referred to as d'Alembert's solution of the wave equation (6.15), where  $F$  and  $G$  are arbitrary functions differentiable up to second order.

**Fig. 6.1** Propagation of a wave

In (6.24), if we assume the increments of  $x$  and  $t$  to have the relation

$$\delta x = V \delta t, \quad (6.25)$$

we then have

$$F[(x + \delta x) - V(t + \delta t)] = F(x - Vt). \quad (6.26)$$

As shown in Fig. 6.1, where the solid line is the  $F$ - $x$  curve at the time  $t$ , and the dashed line that at  $t + \delta t$ , this formula implies that the value of  $F$  at  $t + \delta t$  and  $x + \delta x$  is equal to the values of  $F$  at  $t$  and  $x$ , i.e., the dashed line is obtained by shifting the solid line along the  $x$ -axis by  $\delta x$  without altering the shape. This property can be interpreted as that the  $F$ - $x$  curve is moving in the direction of the  $x$ -axis with the velocity  $V$ . Physically, this is a wave propagating in the direction of the  $x$ -axis with the velocity  $V$ . Similarly, the function  $G(x - Vt)$  represents a wave propagating along the opposite direction of the  $x$ -axis with the velocity  $V$ . For these reasons, (6.15) is referred to as wave equation.

By applying the solution of the wave equation to the elastic equations of the Earth's deformation, (6.12), (6.13), and (6.14), we readily find that (6.12) is the equation of a wave with displacement in the same direction of propagation, thus being a longitudinal or compression wave, and (6.13) and (6.14) represent a wave with displacement perpendicular to the direction of propagation, thus being a transverse or shear wave. In seismology, as the longitudinal wave arrives earlier than the transverse wave, they are referred to as primary and secondary waves, respectively, and are denoted as  $P$  and  $S$  waves. The velocities of the  $P$  and  $S$  waves are

$$V_P = \sqrt{\frac{k + (4/3)\mu}{\rho}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad (6.27)$$

$$V_S = \sqrt{\frac{\mu}{\rho}}, \quad (6.28)$$

respectively. In fluids, as  $\mu = 0$ , only the  $P$  wave is propagated, and the sound wave in the air is such an example. The  $P$  and  $S$  waves are referred to as body waves,

as their velocities at a location depend only on the density and elastic parameters at that location; they travel within the body.

The displacement of the  $S$  wave can be further decomposed into two components: one is in the horizontal direction, and the wave associated with it is referred to as  $SH$  wave; the rest is referred to as  $SV$  wave. If the  $x_1x_2$  coordinate plane is assumed to be the horizontal plane, (6.13) represents an  $SH$  wave, and (6.14) an  $SV$  wave.

In the above, we have discussed a special case where the waves depend on  $x_1$  and  $t$  only, and always propagate in the direction of the  $x_1$ -axis, which is perpendicular to the  $x_2x_3$  plane. In other words, on any plane parallel to the  $x_2x_3$  plane, the dependence of the wave on  $x_1$  and  $t$  is the same. Hence, the wavefronts, i.e., the surfaces perpendicular to the direction of the wave everywhere on it, are planes parallel to the  $x_2x_3$  plane. This kind of wave is referred to as plane wave.

The elastic waves produced by Earthquakes are referred to as seismic waves. The plane waves discussed in this subsection is the simplest special case.

### 6.1.3 The Equation of Elastic Waves

According to (6.27) and (6.28), the governing equation of elastic deformation, (6.8), can be written as

$$\frac{\partial^2 \vec{u}}{\partial t^2} = V_P^2 \nabla(\nabla \cdot \vec{u}) - V_S^2 \nabla \times (\nabla \times \vec{u}). \quad (6.29)$$

By decomposing  $\vec{u}$  according to Helmholtz's theorem,

$$\vec{u} = \nabla \phi + \nabla \times \vec{\psi}, \quad \nabla \cdot \vec{\psi} = 0, \quad (6.30)$$

we obtain

$$\frac{\partial^2}{\partial t^2} (\nabla \phi + \nabla \times \vec{\psi}) = V_P^2 \nabla(\nabla^2 \phi) + V_S^2 \nabla \times (\nabla^2 \vec{\psi}), \quad (6.31)$$

where the three formulae  $\nabla \cdot (\nabla \times \vec{\psi}) = 0$ ,  $\nabla \times (\nabla \phi) = 0$ , and  $\nabla \times (\nabla \times \vec{\psi}) = \nabla(\nabla \cdot \vec{\psi}) - \nabla^2 \vec{\psi} = -\nabla^2 \vec{\psi}$  have been used in the derivation. The two functions  $\phi$  and  $\vec{\psi}$  are independent. Hence, this equation can be decomposed into two equations, one for  $\phi$ , and one for  $\vec{\psi}$ ,

$$\frac{\partial^2 \phi}{\partial t^2} = V_P^2 \nabla^2 \phi, \quad (6.32)$$

$$\frac{\partial^2 \vec{\psi}}{\partial t^2} = V_S^2 \nabla^2 \vec{\psi}, \quad (6.33)$$

which are the wave equations for the  $P$  and  $S$  waves, respectively. At this point, we just understand that any linear combination of the solutions of (6.32) and (6.33) is a solution of (6.31).

With the basic knowledge of wave and wave equation, we derive a special form of the elastic wave equation general enough for subsequent uses in this book. We assume that all functions are independent of  $x_2$ , i.e.,

$$\phi = \phi(x_1, x_3, t), \quad \vec{\psi} = \vec{\psi}(x_1, x_3, t). \quad (6.34)$$

We can obtain, by definition,

$$\vec{u} = \frac{\partial \phi}{\partial x_1} \hat{e}_1 + \frac{\partial \phi}{\partial x_3} \hat{e}_3 - \frac{\partial \psi_2}{\partial x_3} \hat{e}_1 + \left( \frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \right) \hat{e}_2 + \frac{\partial \psi_2}{\partial x_1} \hat{e}_3. \quad (6.35)$$

For brevity, denote  $\psi_2$  as  $\psi$  and use  $u_2$  instead of  $\partial \psi_1 / \partial x_3 - \partial \psi_3 / \partial x_1$ . We can write the above formula as

$$\vec{u} = \left( \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3} \right) \hat{e}_1 + u_2 \hat{e}_2 + \left( \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1} \right) \hat{e}_3. \quad (6.36)$$

The equations of  $\phi$ ,  $u_2$  and  $\psi$  can be obtained according to (6.32) and (6.33) to be

$$\frac{\partial^2 \phi}{\partial t^2} = V_P^2 \left( \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_3^2} \right), \quad (6.37)$$

$$\frac{\partial^2 \psi}{\partial t^2} = V_S^2 \left( \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_3^2} \right), \quad (6.38)$$

$$\frac{\partial^2 u_2}{\partial t^2} = V_S^2 \left( \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_3^2} \right). \quad (6.39)$$

When these are understood to represent plane waves, (6.37) represents a  $P$  wave, and if the  $x_1x_2$ -plane is understood as the horizontal plane, (6.38) and (6.39) represent an  $SV$  wave and an  $SH$  wave, respectively. Here we realize that the displacement associated with the  $SV$  wave is not necessarily in the vertical direction.

## 6.1.4 Reflection and Refection of Seismic Body Waves

### 6.1.4.1 A General Expression of the Plane Harmonic Wave

We have assumed the wave velocity to be constant while introducing the concept of body waves. In reality, the wave velocity inside the Earth varies from place to place. The variation may be continuous or involving jumps. If a jump of the wave velocity happens across a surface, this surface is referred to as discontinuous surface. In this subsection, we discuss the reflection and refraction of seismic body waves at discontinuous surfaces.

We study the simple case of plane waves independent of the  $x_2$ -coordinate traveling in the  $x_1x_3$  plane, and the discontinuous surface is the plane  $x_3 = 0$ , which is assumed to be welded if it is not a free surface. We still decompose the  $S$  wave into an  $SH$  wave and an  $SV$  wave taking the  $x_1x_2$ -plane as the horizontal plane. We realize immediately that the displacement of the  $SH$  wave, i.e.,  $u_2$  in (6.38), is parallel to the  $x_2$ -axis, and those of both the  $SV$  and  $P$  waves, i.e., the  $\phi$  and  $\psi$  terms in (6.37) and (6.39), are parallel to the  $x_1x_3$ -plane and perpendicular to the  $x_2$ -axis. Hence, an incident  $SH$  wave can only produce reflected and refracted  $SH$  waves, and an incident  $SV$  or  $P$  wave may produce both reflected and refracted  $SV$  and  $P$  waves.

We will be limited to study the reflection and refraction of plane waves, for which we first express the plane waves in an appropriate form for subsequent formulations. Taking reference to Fig. 6.1, we know that the expression

$$u = F(x - Vt) \quad (6.40)$$

represents a plane wave traveling along the direction of the  $x$ -axis. For a plane wave traveling along the direction of  $\vec{l}$  in Fig. 6.2, the expression should be the same as (6.40) if we set the  $x$ -axis to be along  $\vec{l}$ , where the value of  $x$  in (6.40) is in fact the distance to the origin  $O$ . Therefore, the expression of the wave traveling in the direction of  $\vec{l}$  is

$$u = F(x_1 \sin i + x_3 \cos i - Vt), \quad (6.41)$$

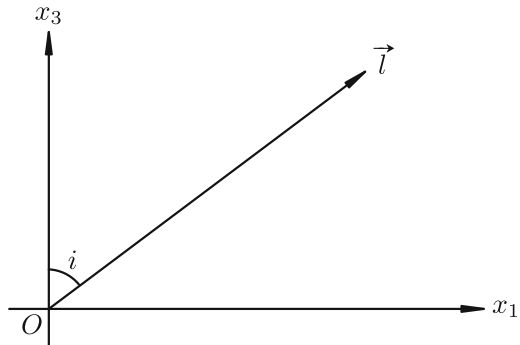
which can also be written as, by deviding the variables by  $\sin i$ .

$$u = F\left(x_1 + x_3 \cot i - \frac{V}{\sin i}t\right). \quad (6.42)$$

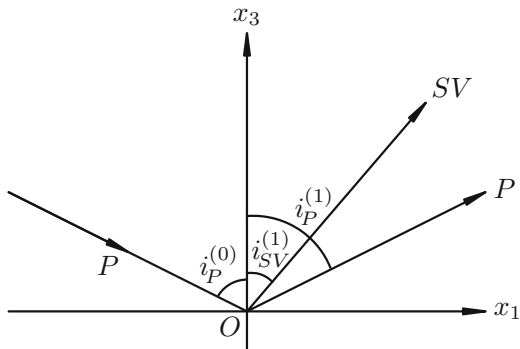
In subsequent formulations, we will use the simplified form

$$u = A \exp \left\{ i \left( x_1 + x_3 \cot i - \frac{V}{\sin i}t \right) \right\}, \quad (6.43)$$

**Fig. 6.2** A plane wave with wavefront in the  $x_1x_3$ -plane



**Fig. 6.3** Reflection of a free plane surface to an incident  $P$  wave



where  $\exp\{\}$  is the natural exponential function. This is in fact the harmonic wave, or more perceptibly, sine wave. It can be readily verified that, if  $u$  is replaced by  $\phi$ ,  $\psi$  or  $u_2$ , they all satisfy their respective Eqs. (6.37)–(6.39)

As examples, we will formulate the simple cases of the reflection of an incident  $P$  wave at a free plane surface and the reflection and refraction of an incident  $SH$  wave at a plane surface of discontinuity.

#### 6.1.4.2 Reflection of the $P$ Wave at a Free Plane Surface

As shown in Fig. 6.3, let  $i_P^{(0)}$  be the incident angle of the incident  $P$  wave, and  $i_P^{(1)}$  and  $i_{SV}^{(1)}$  the reflection angles of the reflected  $P$  and  $SV$  waves, respectively. According to (6.43), the incident  $P$  wave represented by  $\phi^{(0)}$ , and the reflected  $P$  and  $SV$  waves represented by  $\phi^{(1)}$  and  $\psi^{(1)}$  are, respectively,

$$\phi^{(0)} = A^{(0)} \exp \left\{ i \left[ x_1 - x_3 \cot i_P^{(0)} - \frac{V_P}{\sin i_P^{(0)}} t \right] \right\}, \quad (6.44)$$

$$\phi^{(1)} = A^{(1)} \exp \left\{ i \left[ x_1 + x_3 \cot i_P^{(1)} - \frac{V_P}{\sin i_P^{(1)}} t \right] \right\}, \quad (6.45)$$

$$\psi^{(1)} = B^{(1)} \exp \left\{ i \left[ x_1 + x_3 \cot i_{SV}^{(1)} - \frac{V_S}{\sin i_{SV}^{(1)}} t \right] \right\}. \quad (6.46)$$

All of them satisfy their respective wave equations. Hence, the only formulation left is to determine  $A^{(1)}$ ,  $i_P^{(1)}$  and  $B^{(1)}$ ,  $i_{SV}^{(1)}$  assuming  $A^{(0)}$  and  $i_P^{(0)}$  to be known. The wave velocities  $V_P$  and  $V_S$  are material parameters which are assumed to be known.

As the plane  $x_3 = 0$  is assumed to be a free surface, the stress on it vanishes. For the present problem, as  $u_2 = 0$ , the components concerned are  $T_{31} = 0$  and  $T_{33} = 0$ . We have, according to the stress-displacement relation,

$$\left[ T_{31} = \mu \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \right]_{x_3=0} = 0, \quad (6.47)$$

$$\left[ T_{33} = \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} \right) + 2\mu \frac{\partial u_3}{\partial x_3} \right]_{x_3=0} = 0. \quad (6.48)$$

Evidently, the first condition can be simplified to

$$\left[ \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right]_{x_3=0} = 0. \quad (6.49)$$

As for the second condition, the substitution of  $\lambda$  and  $\mu$  by  $V_P$  and  $V_S$  leads to

$$\left[ V_P^2 \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} \right) - 2V_S^2 \frac{\partial u_1}{\partial x_1} \right]_{x_3=0} = 0. \quad (6.50)$$

In order to use the boundary conditions (6.49) and (6.50), we derive the formula of the displacement, which is straightforward by setting  $\phi = \phi^{(0)} + \phi^{(1)}$  and  $\psi = \psi^{(1)}$  in (6.36), and we have

$$\begin{aligned} u_1 = & iA^{(0)} \exp \left\{ i \left[ x_1 - x_3 \cot i_P^{(0)} - \frac{V_P}{\sin i_P^{(0)}} t \right] \right\} \\ & + iA^{(1)} \exp \left\{ i \left[ x_1 + x_3 \cot i_P^{(1)} - \frac{V_P}{\sin i_P^{(1)}} t \right] \right\} \\ & - iB^{(1)} \cot i_{SV}^{(1)} \exp \left\{ i \left[ x_1 + x_3 \cot i_{SV}^{(1)} - \frac{V_S}{\sin i_{SV}^{(1)}} t \right] \right\} \end{aligned} \quad (6.51)$$

$$\begin{aligned} u_3 = & -iA^{(0)} \cot i_P^{(0)} \exp \left\{ i \left[ x_1 - x_3 \cot i_P^{(0)} - \frac{V_P}{\sin i_P^{(0)}} t \right] \right\} \\ & + iA^{(1)} \cot i_P^{(1)} \exp \left\{ i \left[ x_1 + x_3 \cot i_P^{(1)} - \frac{V_P}{\sin i_P^{(1)}} t \right] \right\} \\ & + iB^{(1)} \exp \left\{ i \left[ x_1 + x_3 \cot i_{SV}^{(1)} - \frac{V_S}{\sin i_{SV}^{(1)}} t \right] \right\} \end{aligned} \quad (6.52)$$

The substitution of the above two formulae into (6.49) and (6.50) yields

$$\begin{aligned} & 2A^{(0)} \cot i_P^{(0)} \exp \left\{ i \left[ x_1 - \frac{V_P}{\sin i_P^{(0)}} t \right] \right\} \\ & - 2A^{(1)} \cot i_P^{(1)} \exp \left\{ i \left[ x_1 - \frac{V_P}{\sin i_P^{(1)}} t \right] \right\} \\ & - B^{(1)} \left( 1 - \cot^2 i_{SV}^{(1)} \right) \exp \left\{ i \left[ x_1 - \frac{V_S}{\sin i_{SV}^{(1)}} t \right] \right\} = 0, \end{aligned} \quad (6.53)$$

$$\begin{aligned}
& -A^{(0)}(1 + \cot^2 i_P^{(0)})V_P^2 - 2V_S^2 \exp \left\{ i \left[ x_1 - \frac{V_P}{\sin i_P^{(0)}} t \right] \right\} \\
& -A^{(1)}(1 + \cot^2 i_P^{(1)})V_P^2 - 2V_S^2 \exp \left\{ i \left[ x_1 - \frac{V_P}{\sin i_P^{(1)}} t \right] \right\} \\
& -2B^{(1)}V_S^2 \cot i_{SV}^{(1)} \exp \left\{ i \left[ x_1 - \frac{V_S}{\sin i_{SV}^{(1)}} t \right] \right\} = 0.
\end{aligned} \tag{6.54}$$

All the quantities we are seeking can be obtained by solving (6.53) and (6.54). As the coefficients of  $\exp\{\}$  are all independent of  $x_1$  and  $t$ , the expressions in all the braces of  $\exp\{\}$  must be the same at any time for any fixed value of  $x_1$ . This requires the coefficients of  $t$  to be the same, i.e.,

$$\frac{V_P}{\sin i_P^{(0)}} = \frac{V_P}{\sin i_P^{(1)}} = \frac{V_S}{\sin i_{SV}^{(1)}}. \tag{6.55}$$

This is the relation among the incident and reflection angles, particularly,  $i_P^{(1)} = i_P^{(0)}$  holds.

To obtain the amplitudes of the reflected waves, we substitute (6.55) into (6.53) and (6.54), yielding

$$2(A^{(0)} - A^{(1)}) \cot i_P^{(0)} - B^{(1)}(1 - \cot^2 i_{SV}^{(1)}) = 0. \tag{6.56}$$

$$(A^{(0)} + A^{(1)})[(1 + \cot^2 i_P^{(0)}) \sin^2 i_P^{(0)} - 2 \sin^2 i_{SV}^{(1)}] + 2B^{(1)} \sin i_{SV}^{(1)} \cos i_{SV}^{(1)} = 0. \tag{6.57}$$

The amplitudes  $A^{(1)}$  and  $B^{(1)}$  can then be obtained by solving these last two equations, which is left to the readers as an exercise.

In the mantle,  $\mu = \lambda$  is a good approximation. We have  $V_P/V_S = \sqrt{3}$ , and (6.57) is simplified to

$$(A^{(0)} + A^{(1)})(1 + 3 \cot^2 i_P^{(0)}) + 2B^{(1)} \cot i_{SV}^{(1)} = 0. \tag{6.58}$$

This kind of elastic body is referred to as an ideal Poisson body.

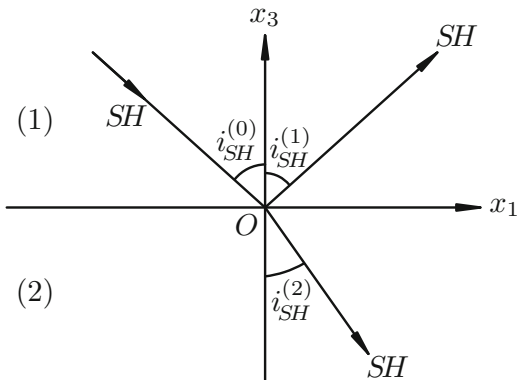
#### 6.1.4.3 Reflection and Refraction of the $SH$ Wave at a Discontinuous Plane Surface

As shown in Fig. 6.4, we use the superscript (1) to represent the medium and quantities at the side  $x_3 > 0$ , the superscript (2) those at the side  $x_3 < 0$ , and the superscript (0) the quantities of the incident wave. According to (6.43), we can write the incident, reflected, and refracted  $SH$  waves as

$$u_2^{(0)} = B^{(0)} \exp \left\{ i \left[ x_1 - x_3 \cot i_{SH}^{(0)} - \frac{V_S^{(1)}}{\sin i_{SH}^{(0)}} t \right] \right\}, \tag{6.59}$$



**Fig. 6.4** Reflection and refraction of the  $SH$  wave at a discontinuous plane surface



$$u_2^{(1)} = B^{(1)} \exp \left\{ i \left[ x_1 + x_3 \cot i_{SH}^{(1)} - \frac{V_S^{(1)}}{\sin i_{SH}^{(1)}} t \right] \right\}, \quad (6.60)$$

$$u_2^{(2)} = B^{(2)} \exp \left\{ i \left[ x_1 - x_3 \cot i_{SH}^{(2)} - \frac{V_S^{(2)}}{\sin i_{SH}^{(2)}} t \right] \right\}. \quad (6.61)$$

Evidently,  $u_2 = u_2^{(0)} + u_2^{(1)}$  in the medium (1), and  $u_2 = u_2^{(2)}$  in the medium (2). Our purpose is to determine  $B^{(1)}$ ,  $i_{SH}^{(1)}$  and  $B^{(2)}$ ,  $i_{SH}^{(2)}$  assuming  $B^{(0)}$  and  $i_{SH}^{(0)}$  to be known.

We assume that the discontinuous surface is welded, i.e., no slip may happen, meaning that both the displacement and stress are continuous. The continuity of displacement implies

$$\left[ u_2^{(0)} + u_2^{(1)} \right]_{x_3=0} = \left[ u_2^{(2)} \right]_{x_3=0}. \quad (6.62)$$

The continuity of stress degenerates to the continuity of the single component  $T_{32}$ . With the stress-displacement relation (6.4), we have

$$\mu^{(1)} \left[ \frac{\partial u_2^{(0)}}{\partial x_3} + \frac{\partial u_2^{(1)}}{\partial x_3} \right]_{x_3=0} = \mu^{(2)} \left[ \frac{\partial u_2^{(2)}}{\partial x_3} \right]_{x_3=0}. \quad (6.63)$$

The substitution of (6.59)–(6.61) into these two boundary conditions yields

$$\begin{aligned} & B^{(0)} \exp \left\{ i \left[ x_1 - \frac{V_S^{(1)}}{\sin i_{SH}^{(0)}} t \right] \right\} + B^{(1)} \exp \left\{ i \left[ x_1 - \frac{V_S^{(1)}}{\sin i_{SH}^{(1)}} t \right] \right\} \\ &= B^{(2)} \exp \left\{ i \left[ x_1 - \frac{V_S^{(2)}}{\sin i_{SH}^{(2)}} t \right] \right\}, \end{aligned} \quad (6.64)$$

$$\begin{aligned}
& \mu^{(1)} \left[ -iB^{(0)} \cot i_{SH}^{(0)} \exp \left\{ i \left[ x_1 - \frac{V_S^{(1)}}{\sin i_{SH}^{(0)}} t \right] \right\} \right. \\
& \quad \left. + iB^{(1)} \cot i_{SH}^{(1)} \exp \left\{ i \left[ x_1 - \frac{V_S^{(1)}}{\sin i_{SH}^{(1)}} t \right] \right\} \right] \\
& = -iB^{(2)} \mu^{(2)} \cot i_{SH}^{(2)} \exp \left\{ i \left[ x_1 - \frac{V_S^{(2)}}{\sin i_{SH}^{(2)}} t \right] \right\}. \tag{6.65}
\end{aligned}$$

Similar to the case of incident  $P$  wave to a free surface, all quantities we are seeking can be obtained by solving the Eqs. (6.64) and (6.65). The relation among the incident, reflection and refraction angles is

$$\frac{V_S^{(1)}}{\sin i_{SH}^{(0)}} = \frac{V_S^{(1)}}{\sin i_{SH}^{(1)}} = \frac{V_S^{(2)}}{\sin i_{SH}^{(2)}}. \tag{6.66}$$

The substitution of this formula into (6.64) and (6.65) yields

$$B^{(0)} + B^{(1)} = B^{(2)}, \tag{6.67}$$

$$\mu^{(1)} (B^{(0)} - B^{(1)}) \cot i_{SH}^{(0)} = \mu^{(2)} B^{(2)} \cot i_{SH}^{(2)}, \tag{6.68}$$

which can be solved to obtain  $B^{(1)}$  and  $B^{(2)}$ .

#### 6.1.4.4 General Relation Among the Incident, Reflection, and Refraction Angles

Although we have studied only two simple cases, the results of the relations among the incident, reflection and refraction angles obtained hold in general, and we write the general relation as

$$\frac{V_P^{(1)}}{\sin i_P^{(0)}} \left[ \text{or } \frac{V_S^{(1)}}{\sin i_S^{(0)}} \right] = \frac{V_P^{(1)}}{\sin i_P^{(1)}} = \frac{V_S^{(1)}}{\sin i_S^{(1)}} = \frac{V_P^{(2)}}{\sin i_P^{(2)}} = \frac{V_S^{(2)}}{\sin i_S^{(2)}}, \tag{6.69}$$

where the  $S$  wave includes both the  $SH$  and  $SV$  waves. In the case of incident  $SH$  wave, the  $P$  wave part in the formula disappears.

## 6.2 Seismic Surface Waves

### 6.2.1 The Concept of Surface Wave

The velocities of the seismic body waves at a particular location depend on the material parameters at the specific location only, thus can be understood as material

parameters. This implies that how body waves propagate is strictly a local property. However, there exists another kind of seismic waves traveling along surfaces—their amplitudes are largest at the surfaces, and decreases to zero as the distance to the surfaces tends to infinitely large, which are referred to as seismic surface waves. Here we take the Earth's surface as a plane, and study some simple plane waves as examples. Let the  $x_1x_2$ -plane be the Earth's surface, and the  $x_3$ -axis points downward. The decrease of the amplitudes with the distance to the surface implies that the amplitudes are functions of the depth, which is the  $x_3$ -coordinate. Hence, the quantities  $\phi$ ,  $\psi$ , and  $u_2$  of the plane waves traveling along the  $x_1$ -axis can be expressed as

$$\phi = f(x_3) \exp\{i\kappa(x_1 - vt)\}, \quad (6.70)$$

$$\psi = g(x_3) \exp\{i\kappa(x_1 - vt)\}, \quad (6.71)$$

$$u_2 = h(x_3) \exp\{i\kappa(x_1 - vt)\}. \quad (6.72)$$

By definition, we have

$$\phi, \psi, u_2 \xrightarrow{x_3 \rightarrow \infty} 0, \quad \text{or} \quad f, g, h \xrightarrow{x_3 \rightarrow \infty} 0, \quad (6.73)$$

which is the physical characteristic of surface waves. This mathematical setting simplifies the formulation to be independent of the  $x_2$ -coordinate.

The waves expressed by (6.70)–(6.72) are harmonic waves, with  $\kappa$  as wave number,  $v$  as wave velocity,  $2\pi/\kappa$  as wave length,  $\kappa v$  as angular frequency, and  $(2\pi)/(\kappa v)$  as wave period.

The amplitudes  $f(x_3)$ ,  $g(x_3)$ , and  $h(x_3)$  cannot be arbitrarily chosen, since  $\phi$ ,  $\psi$ , and  $u_2$  must satisfy their respective wave equations. Taking  $\phi$  as example, the substitution of (6.70) into (6.37) yields

$$\frac{d^2 f(x_3)}{dx_3^2} - \kappa^2 \left( 1 - \frac{v^2}{V_P^2} \right) f(x_3) = 0. \quad (6.74)$$

Define

$$P^2 = 1 - \frac{v^2}{V_P^2}. \quad (6.75)$$

We assume  $P$  to be positive when it is real, and  $P = ai$  with  $a > 0$  when it is imaginary. The general solution of (6.74) is then

$$f(x_3) = A_\phi \exp\{-\kappa P x_3\} + B_\phi \exp\{\kappa P x_3\}, \quad (6.76)$$

where  $A_\phi$  and  $B_\phi$  are constant parameters independent of  $x_3$ . Similarly, we also have

$$g(x_3) = A_\psi \exp\{-\kappa S x_3\} + B_\psi \exp\{\kappa S x_3\}, \quad (6.77)$$

$$h(x_3) = A \exp\{-\kappa S x_3\} + B \exp\{\kappa S x_3\}, \quad (6.78)$$

where  $A_\psi$ ,  $B_\psi$  and  $A$ ,  $B$  are constant parameters independent of  $x_3$  (as  $A_\phi$  and  $B_\phi$ ), and  $S$  is defined as

$$S^2 = 1 - \frac{v^2}{V_S^2}. \quad (6.79)$$

The same as for the case of  $P$ , we assume  $S$  to be positive when it is real, and  $S = ai$  with  $a > 0$  when it is imaginary

By substituting (6.76)–(6.78) into (6.70)–(6.72), we obtain the expressions of  $\phi$ ,  $\psi$  and  $u_2$  satisfying their respective wave equations,

$$\phi = [A_\phi \exp\{-\kappa P x_3\} + B_\phi \exp\{\kappa P x_3\}] \exp\{i\kappa(x_1 - vt)\}, \quad (6.80)$$

$$\psi = [A_\psi \exp\{-\kappa S x_3\} + B_\psi \exp\{\kappa S x_3\}] \exp\{i\kappa(x_1 - vt)\}, \quad (6.81)$$

$$u_2 = [A \exp\{-\kappa S x_3\} + B \exp\{\kappa S x_3\}] \exp\{i\kappa(x_1 - vt)\}. \quad (6.82)$$

The physical property of surface waves given by (6.73) has not yet been taken into account in these solutions. In order for  $\phi \rightarrow 0$  when  $x_3 \rightarrow \infty$ ,  $P$  must be real and positive, and  $B_\phi$  must vanish. Hence the solution of  $\phi$  is finally

$$\phi = A_\phi \exp\{-\kappa P x_3\} \exp\{i\kappa(x_1 - vt)\}. \quad (6.83)$$

Similarly, for  $\psi$  and  $u_2$ ,  $S$  must be real and positive, and  $B_\psi$  and  $B$  must vanish, and their final solutions are

$$\psi = A_\psi \exp\{-\kappa S x_3\} \exp\{i\kappa(x_1 - vt)\}, \quad (6.84)$$

$$u_2 = A \exp\{-\kappa S x_3\} \exp\{i\kappa(x_1 - vt)\}. \quad (6.85)$$

The expressions (6.80)–(6.82) and (6.83)–(6.85) are the basic equations for our subsequent formulation of seismic surface waves. We mention that (6.80)–(6.82) satisfy only their respective wave equations, and (6.83)–(6.85) not only satisfy their respective wave equations, but also decrease to zero as the distance to the surface increases to infinitely large.

As already recognized, the displacement of the  $P$  wave represented by  $\phi$  and that of the  $SV$  wave represented by  $\psi$  are both parallel to the  $x_1 x_3$ -plane, while the displacement of the  $SH$  wave represented by  $u_2$  is perpendicular to this plane. Hence, the  $P$  and  $SV$  waves may affect each other, but the  $SH$  wave is independent

**Fig. 6.5** A homogeneous half space



of them. Hence, in the formulation,  $P$  and  $SV$  have to be handled together, called Rayleigh wave, and the  $SH$  wave can be handled individually, called Love wave.

Surface waves may exist not only around free surfaces like the Earth's surface, but also around internal surfaces of discontinuity, called Stoneley wave or interface wave.

### 6.2.2 Rayleigh Wave of a Homogeneous Half Space

As shown in Fig. 6.5, we consider a simple model of a free plane surface filled with homogeneous elastic medium below. For this model,  $\phi$  and  $\psi$  should satisfy (6.73), and hence, the expressions (6.83) and (6.84) should be adopted. We have

$$\phi = A_\phi \exp\{-\kappa P x_3\} \exp\{i\kappa(x_1 - vt)\}, \quad (6.86)$$

$$\psi = A_\psi \exp\{-\kappa S x_3\} \exp\{i\kappa(x_1 - vt)\}. \quad (6.87)$$

The substitution of these expressions into (6.36) yields the expression of the displacement,

$$u_1 = [i\kappa A_\phi \exp\{-\kappa P x_3\} + \kappa A_\psi S \exp\{-\kappa S x_3\}] \exp\{i\kappa(x_1 - vt)\}, \quad (6.88)$$

$$u_3 = [-\kappa A_\phi P \exp\{-\kappa P x_3\} + i\kappa A_\psi \exp\{-\kappa S x_3\}] \exp\{i\kappa(x_1 - vt)\}. \quad (6.89)$$

The velocity of the surface wave,  $v$ , is to be determined based on the boundary conditions.

The boundary conditions are that the stress components vanish on the surface  $x_3 = 0$ . For our simplified mathematical setting, the components concerned are  $T_{31} = 0$  and  $T_{33} = 0$ . This is the same as in the case of reflection and refraction of an incident  $P$  wave studied in the previous section, and therefore, the boundary conditions expressed in terms of displacement remain the same, i.e., (6.49) and (6.50). The substitution of (6.88) and (6.89) into these boundary conditions leads

to

$$-2i\kappa^2 A_\phi P - (1 + S^2)\kappa^2 A_\psi = 0, \quad (6.90)$$

$$\kappa^2 A_\phi \left[ -(1 - P^2)V_P^2 + 2V_S^2 \right] - 2i\kappa^2 A_\psi S V_S^2 = 0. \quad (6.91)$$

Multiply both sides of the second equation by  $2iP$ , and then substitute  $2i\kappa^2 A_\phi P$  by its expression solved from the first equation. We obtain

$$\left[ -(1 - P^2)V_P^2 + 2V_S^2 \right] (1 + S^2) - 4P S V_S^2 = 0. \quad (6.92)$$

As both  $P$  and  $S$  are positive real numbers, the substitution of (6.75) and (6.79) into this equation yields

$$\left( 2 - \frac{v^2}{V_S^2} \right)^2 = 4 \left[ \left( 1 - \frac{v^2}{V_P^2} \right) \left( 1 - \frac{v^2}{V_S^2} \right) \right]^{1/2}, \quad (6.93)$$

where the square root is assumed to be positive. This equation is referred to as Rayleigh's equation, whose solution is the velocity of the Rayleigh wave.

Although Rayleigh's equation is an algebraic equation, it is still very complicated that it can only be solved numerically using computers. Evidently, if a nontrivial solution of  $v$  exists, it must satisfy  $0 < v < V_S < V_P$ . As a supplement, we show that such a solution does exist. Take the square of both sides of (6.93), we can obtain, after some rearrangement of terms,

$$\frac{v^6}{V_S^6} - 8\frac{v^4}{V_S^4} + \frac{v^2}{V_S^2} \left( 24 - 16\frac{V_S^2}{V_P^2} \right) - 16 \left( 1 - \frac{V_S^2}{V_P^2} \right) = 0. \quad (6.94)$$

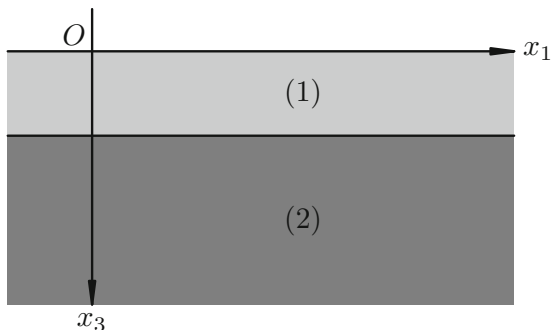
The left-hand side of this equation is negative when  $v = 0$ , and positive when  $v = V_S$ , meaning that there must be a solution of  $v$  in  $(0, V_S)$ , and the Rayleigh wave does exist.

### 6.2.3 Love Wave of a Homogeneous Layer Overlying a Homogeneous Half Space

It can be shown that no Love wave exists in the simple model of homogeneous half space. Therefore, we consider a model composed of two layers, as shown in Fig. 6.6. Under the free surface  $x_3 = 0$  is a layer of homogeneous elastic medium of constant depth  $H$ , which is denoted as later (1). Under this layer is a homogeneous half space of elastic medium denoted as layer (2). The discontinuous surface between the two layers,  $x_3 = H$ , is assumed welded, i.e., no slip can happen.

For this model,  $u_2$  is required to satisfy (6.73) only in the lower layer, i.e., layer (2). We still use the superscripts (1) and (2) to represent the quantities in

**Fig. 6.6** A homogeneous layer overlying a homogeneous half space



the respective layers. We can assert that the expressions of  $u_2^{(1)}$  and  $u_2^{(2)}$  should be in the forms of (6.82) and (6.85), respectively, i.e.,

$$u_2^{(1)} = [A^{(1)} \exp\{-\kappa S^{(1)} x_3\} + B^{(1)} \exp\{\kappa S^{(1)} x_3\}] \exp\{i\kappa(x_1 - vt)\}, \quad (6.95)$$

$$u_2^{(2)} = A^{(2)} \exp\{-\kappa S^{(2)} x_3\} \exp\{i\kappa(x_1 - vt)\}. \quad (6.96)$$

The velocity of the wave,  $v$ , is to be determined by the boundary conditions.

Boundary conditions at both the free surface and the intersection between the two layers, i.e.,  $x_3 = 0$  and  $x_3 = H$ , respectively, must be imposed. Due to the simplicity of the model, the vanishing of the stress on the surface  $x_3 = 0$  is simplified to  $T_{32} = 0$ , which can be expressed in terms of displacement as

$$\left[ \mu^{(1)} \frac{\partial u_2^{(1)}}{\partial x_3} \right]_{x_3=0} = 0. \quad (6.97)$$

On the surface  $x_3 = H$ , both the displacement and the stress are continuous, which can be written as, with the stress component concerned,  $T_{32}$ , expressed in terms of the displacement,

$$\left[ u_2^{(1)} \right]_{x_3=H} = \left[ u_2^{(2)} \right]_{x_3=H}, \quad (6.98)$$

$$\left[ \mu^{(1)} \frac{\partial u_2^{(1)}}{\partial x_3} \right]_{x_3=H} = \left[ \mu^{(2)} \frac{\partial u_2^{(2)}}{\partial x_3} \right]_{x_3=H}. \quad (6.99)$$

The substitution of the displacement (6.95) and (6.96) into the boundary conditions (6.97)–(6.99) yields

$$-A^{(1)} + B^{(1)} = 0, \quad (6.100)$$

$$A^{(1)} \exp\{-\kappa S^{(1)} H\} + B^{(1)} \exp\{\kappa S^{(1)} H\} = A^{(2)} \exp\{-\kappa S^{(2)} H\}, \quad (6.101)$$

$$\begin{aligned}
 \mu^{(1)} S^{(1)} [-A^{(1)} \exp\{-\kappa S^{(1)} H\} + B^{(1)} \exp\{\kappa S^{(1)} H\}] \\
 = -\mu^{(2)} S^{(2)} A^{(2)} \exp\{-\kappa S^{(2)} H\}.
 \end{aligned} \tag{6.102}$$

Substitute the first equation into the last two, and then eliminate  $A^{(2)} \exp\{-\kappa S^{(2)} H\}$ . we obtain

$$\frac{\exp\{\kappa S^{(1)} H\} - \exp\{-\kappa S^{(1)} H\}}{\exp\{\kappa S^{(1)} H\} + \exp\{-\kappa S^{(1)} H\}} = \frac{\mu^{(2)} S^{(2)}}{\mu^{(1)} S^{(1)}}, \tag{6.103}$$

which is to be solved for  $v$ .

The Eq. (6.103) can be reduced to two different forms, as  $S^{(1)}$  may be either real or imaginary, while  $S^{(2)}$  must be real and positive. When  $S^{(1)}$  is real, (6.103) is reduced to, with the hyperbolic tangent function  $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$ ,

$$\tanh \left\{ \kappa \left[ 1 - \left( \frac{v}{V_S^{(1)}} \right)^2 \right]^{1/2} H \right\} = \frac{\mu^{(2)} \left[ 1 - \left( \frac{v}{V_S^{(2)}} \right)^2 \right]^{1/2}}{\mu^{(1)} \left[ 1 - \left( \frac{v}{V_S^{(1)}} \right)^2 \right]^{1/2}}. \tag{6.104}$$

The solution of this equation must satisfy  $v \leq V_S^{(1)}$  and  $v \leq V_S^{(2)}$ . When  $S^{(1)}$  is imaginary, (6.103) is reduced to

$$\tan \left\{ \kappa \left[ \left( \frac{v}{V_S^{(1)}} \right)^2 - 1 \right]^{1/2} H \right\} = \frac{\mu^{(2)} \left[ 1 - \left( \frac{v}{V_S^{(2)}} \right)^2 \right]^{1/2}}{\mu^{(1)} \left[ \left( \frac{v}{V_S^{(1)}} \right)^2 - 1 \right]^{1/2}}. \tag{6.105}$$

The solution of this equation must satisfy  $V_S^{(1)} \leq v \leq V_S^{(2)}$ . We note that the parameter  $\kappa$  is also an unknown parameter, and both  $\kappa$  and  $v$  can be adjusted for the Eqs. (6.104) or (6.105) to hold. Note that at the right-hand sides of both (6.104) and (6.105),  $\mu^{(2)}/\mu^{(1)} = (\rho^{(2)}/\rho^{(1)})[(V_S^{(2)})^2/(V_S^{(1)})^2]$ .

For the Earth's crust and the mantle immediately below the Moho, we have  $\rho^{(1)} < \rho^{(2)}$  and  $V_S^{(1)} < V_S^{(2)}$ . In this case, the left hand side of (6.104) is smaller than 1, but its right-hand side is larger than 1. Hence, this equation does not have a solution. Therefore, it is (6.105) that is relevant, where for whatever value of  $v$  satisfying  $V_S^{(1)} \leq v \leq V_S^{(2)}$ , there must be a value of  $\kappa$  for the equation to hold.

Finally, we introduce the concepts of nodal plane and mode.



By substituting (6.100) into (6.95), and then replacing  $S^{(1)}$  by  $i[(v/V_S^{(1)})^2 - 1]^{1/2}$ , we obtain the formula of the displacement of the Love wave,

$$u_2^{(1)} = 2A^{(1)} \cos \left\{ \kappa \left[ \left( \frac{v}{V_S^{(1)}} \right)^2 - 1 \right]^{1/2} x_3 \right\} \exp\{i\kappa(x_1 - vt)\}. \quad (6.106)$$

Due to the presence of the cosine function,  $u_2$  vanishes when  $\kappa[(v/V_S^{(1)})^2 - 1]^{1/2}x_3$  takes the special values  $(1/2)\pi, (3/2)\pi, (5/2)\pi, \dots$ . These are planes parallel to the free surface, and are referred to as nodal planes. When  $\kappa[(v/V_S^{(1)})^2 - 1]^{1/2}H$  is smaller than  $(1/2)\pi$ , there is no nodal plane within the top layer; when it is larger than  $(1/2)\pi$  and smaller than  $(3/2)\pi$ , there is one nodal plane within the top layer; and so fourth.

The equation of the velocity of the Love wave, (6.105), can also be written as

$$\kappa \left[ \left( \frac{v}{V_S^{(1)}} \right)^2 - 1 \right]^{1/2} H = \arctan \left\{ \frac{\mu^{(2)} \left[ 1 - \left( \frac{v}{V_S^{(2)}} \right)^2 \right]^{1/2}}{\mu^{(1)} \left[ \left( \frac{v}{V_S^{(1)}} \right)^2 - 1 \right]^{1/2}} \right\} + n\pi, \quad (6.107)$$

where  $n = 0, 1, \dots$ . The appearance of  $n\pi$  at the right-hand side implies that one value of  $v$  within  $(V^{(1)}, V^{(2)})$  may lead to multiple solutions of  $\kappa$ . Each solution is referred to as a mode. The solution with  $n = 0$  is called fundamental mode, and those with  $n > 0$  overtones. The value of the left-hand side is within  $(n\pi, (n + 1/2)\pi)$ . By comparing (6.106) and (6.107), we see that the number of nodal planes in the layer equals  $n$ .

The Rayleigh wave is existent for the physical model of a homogeneous layer of constant depth overlying on a half space. However, the mathematical formulation is lengthy, and is beyond the scope of this book.

### 6.2.4 Phase Velocity, Group Velocity and Dispersion Curve

We find that, in the Love wave of a homogeneous layer overlying a half space studied in the previous subsection, the velocity and wave number are related. Actually, the Rayleigh wave for the same model also has this property. In this subsection, we provide some fundamental properties of this kind of waves, and introduce the concept of phase velocity and group velocity.

First, we discuss the simpler case of the sum of two harmonic waves,

$$u = A \exp\{i(\kappa_1 x - \omega_1 t)\} + A \exp\{i(\kappa_2 x - \omega_2 t)\}, \quad (6.108)$$

where the wave velocity  $v$  is substituted by the angular frequency  $\omega = \kappa v$ . This formula can be further developed as

$$\begin{aligned}
 u &= \left[ A \exp \left\{ -i \left[ \frac{1}{2}(\kappa_2 - \kappa_1)x - \frac{1}{2}(\omega_2 - \omega_1)t \right] \right\} \right. \\
 &\quad \left. + A \exp \left\{ i \left[ \frac{1}{2}(\kappa_2 - \kappa_1)x - \frac{1}{2}(\omega_2 - \omega_1)t \right] \right\} \right] \\
 &\quad \times \exp \left\{ i \left[ \frac{1}{2}(\kappa_1 + \kappa_2)x - \frac{1}{2}(\omega_1 + \omega_2)t \right] \right\} \\
 &= 2A \cos \left[ \frac{1}{2}(\kappa_2 - \kappa_1)x - \frac{1}{2}(\omega_2 - \omega_1)t \right] \\
 &\quad \times \exp \left\{ i \left[ \frac{1}{2}(\kappa_1 + \kappa_2)x - \frac{1}{2}(\omega_1 + \omega_2)t \right] \right\}. \tag{6.109}
 \end{aligned}$$

Assume  $\kappa_1$  and  $\omega_1$  are very close to  $\kappa_2$  and  $\omega_2$ , respectively, and define

$$\kappa = \frac{1}{2}(\kappa_1 + \kappa_2), \quad \omega = \frac{1}{2}(\omega_1 + \omega_2); \tag{6.110}$$

$$\delta\kappa = \kappa_2 - \kappa = \kappa - \kappa_1, \quad \delta\omega = \omega_2 - \omega = \omega - \omega_1. \tag{6.111}$$

We can write (6.109) briefly as

$$u = 2A \cos(\delta\kappa x - \delta\omega t) \exp\{i(\kappa x - \omega t)\}. \tag{6.112}$$

This is a harmonic wave with the amplitude modulated. Its equiphas surfaces, i.e., the surfaces on which  $\kappa x - \omega t$  is a constant, travel with the velocity  $v = \omega/\kappa$  along the direction of the  $x$ -axis. Hence,  $v$  is referred to as phase velocity. Its equi-amplitude surfaces, i.e., the surfaces on which  $2A \cos(\delta\kappa x - \delta\omega t)$  is a constant, travel with the velocity

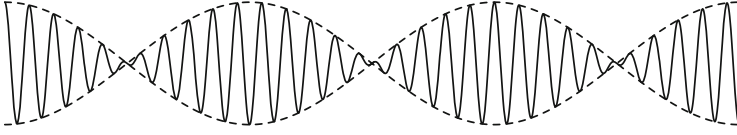
$$C = \frac{\delta\omega}{\delta\kappa} \tag{6.113}$$

along the direction of the  $x$ -axis. This velocity is referred to as group velocity; it is the velocity of a group of waves.

The physical meanings of the phase and group velocities can be demonstrated using a graph of (6.112). Here we use its real part

$$u = 2A \cos(\delta\kappa x - \delta\omega t) \cos(\kappa x - \omega t), \tag{6.114}$$

which is shown in Fig. 6.7. The phase velocity  $v$  is the propagation velocity of the smaller waves in solid line, and the group velocity  $C$  is the propagation velocity of the larger waves in dashed line.



**Fig. 6.7** Superimposition of two harmonic waves

The example discussed above is the superimposition of two harmonic waves with the same amplitude. In reality,  $\omega$  may be continuous, and the amplitude may be a function of  $\omega$ . Hence, in the general instance, (6.108) should be replaced by

$$u = \int_{-\infty}^{\infty} A(\omega) \exp\{i[\kappa(\omega)x - \omega t]\} d\omega. \quad (6.115)$$

For demonstrative purpose, we assume that  $A(\omega)$  is non-vanishing only within a narrow interval with  $\omega_0$  as the center, which can be expressed as

$$\omega_0 - \delta\omega \leq \omega \leq \omega_0 + \delta\omega, \quad \delta\omega/\omega_0 \ll 1. \quad (6.116)$$

Within this interval,  $\kappa(\omega)$  can be developed into a Taylor series as

$$\kappa(\omega) = \kappa(\omega_0) + \left(\frac{d\kappa}{d\omega}\right)_{\omega_0} (\omega - \omega_0) + \frac{1}{2} \left(\frac{d^2\kappa}{d\omega^2}\right)_{\omega_0} (\omega - \omega_0)^2. \quad (6.117)$$

We further assume that the amplitude  $A(\omega)$  is

$$A(\omega) = \exp\{-\tau^2(\omega - \omega_0)^2\}, \quad (6.118)$$

where  $\tau^2$  is a large positive value. The reasons for choosing (6.118) are that (1) it decreases quickly with the increase of  $|\omega - \omega_0|$ ; (2) the exclusion of the first order term of  $\omega - \omega_0$  is for ensuring the  $A-\omega$  curve to be smooth; (3) the maximum of  $A(\omega)$  is set to be 1 for brevity. The value of  $A(\omega)$  given by (6.118) is very small outside the small interval given by (6.116). Hence, the restriction (6.116) is no longer needed. By substituting (6.117) and (6.118) into (6.115), we obtain

$$u = U(x, t) \exp\{i[\kappa(\omega_0)x - \omega_0 t]\}, \quad (6.119)$$

with  $U(x, t)$  defined by

$$U(x, t) = \int_{-\infty}^{\infty} \exp \left\{ i \left[ \left( \frac{d\kappa}{d\omega} \right)_{\omega_0} x - t \right] (\omega - \omega_0) \right\} \\ \times \exp \left\{ - \left[ \tau^2 - \frac{i}{2} \left( \frac{d^2\kappa}{d\omega^2} \right)_{\omega_0} x \right] (\omega - \omega_0)^2 \right\} d(\omega - \omega_0)$$

$$= \int_0^\infty 2 \cos \left\{ \left[ \left( \frac{d\kappa}{d\omega} \right)_{\omega_0} x - t \right] \xi \right\} \exp \left\{ - \left[ \tau^2 - \frac{i}{2} \left( \frac{d^2\kappa}{d\omega^2} \right)_{\omega_0} x \right] \xi^2 \right\} d\xi. \quad (6.120)$$

In the last expression,  $\omega - \omega_0$  is replaced by  $\xi$ . This is an integral of the type  $\int_0^\infty \cos bx e^{-ax^2} dx$ , which can be found in a table of integral to be

$$\int_0^\infty \cos bx e^{-ax^2} dx = \frac{1}{2} \left( \frac{\pi}{a} \right)^{1/2} \exp \left\{ - \frac{b^2}{4a} \right\}. \quad (6.121)$$

We thus obtain

$$U(x, t) = \frac{\sqrt{\pi} \left[ \tau^2 + \frac{i}{2} \left( \frac{d^2\kappa}{d\omega^2} \right)_{\omega_0} x \right]^{1/2}}{\left[ \tau^4 + \frac{1}{4} \left( \frac{d^2\kappa}{d\omega^2} \right)_{\omega_0}^2 x^2 \right]^{1/2}} \exp \left\{ - \frac{\left[ \left( \frac{d\kappa}{d\omega} \right)_{\omega_0} x - t \right]^2 \left[ \tau^2 + \frac{i}{2} \left( \frac{d^2\kappa}{d\omega^2} \right)_{\omega_0} x \right]}{4 \left[ \tau^4 + \frac{1}{4} \left( \frac{d^2\kappa}{d\omega^2} \right)_{\omega_0}^2 x^2 \right]} \right\}. \quad (6.122)$$

If the Taylor series of  $\kappa(\omega)$  is accurate to the first order of  $\omega - \omega_0$  only, we have

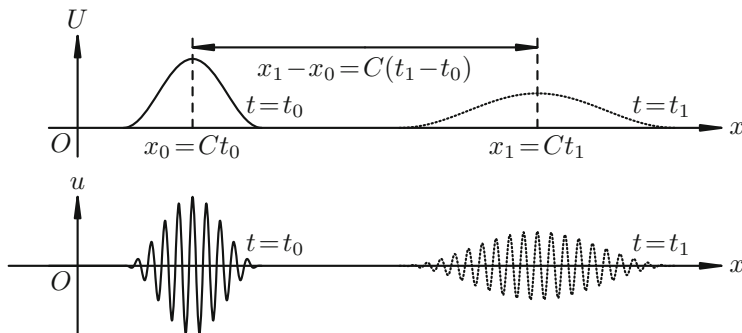
$$U(x, t) = \frac{\sqrt{\pi}}{\tau} \exp \left\{ - \left[ \left( \frac{d\kappa}{d\omega} \right)_{\omega_0} x - t \right]^2 / (4\tau^2) \right\}. \quad (6.123)$$

It can be seen from (6.119) that the equi-phase surfaces propagate with the velocity  $v = \omega_0/\kappa(\omega_0)$ , which is the phase velocity. By comparing (6.123) with  $F(x - Vt)$  in (6.24), we see that the equi-amplitude surfaces propagate with the velocity

$$C = \left( \frac{d\kappa}{d\omega} \right)^{-1} = \frac{d\omega}{d\kappa} = \frac{d(\kappa v)}{d\kappa} = v + \kappa \frac{dv}{d\kappa}, \quad (6.124)$$

which is the group velocity. It can be readily inferred that this is the limiting case of (6.113) with  $\delta\kappa \rightarrow 0$ . However, the amplitude  $U(x, t)$  given in (6.123) is significantly non-vanishing only within a narrow interval of  $x$ . At any time  $t$ , let  $x = x_0 + (x - x_0)$  with  $x_0 = t/[(d\kappa/d\omega)_{\omega_0}]$ . We see that  $U(x, t)$  is largest at  $x_0$ , and decreases quickly to a negligibly small magnitude as  $(x - x_0)^2$  increases. Therefore, the wave expressed by (6.119) is a wave packet.

We must keep in mind that (6.123) is an approximation of (6.122). However, the former is a real function, but the later is a complex function. Therefore, they must be understood after being substituted into (6.119). With (6.122), the maximum amplitude (i.e., the quantity in front of the function  $\exp\{\}$ ) decreases as  $x$  increases, but this does not happen with (6.123). As illustrated in Fig. 6.8, which can be



**Fig. 6.8** Dispersion of a wave according to (6.119) and (6.122). On top is the amplitude, and at the bottom the displacement. Solid lines are the curves at  $t = t_0$ , and dotted lines at  $t_1$

understood as either the real or imaginary part of  $u$  obtained by substituting (6.122) into (6.119), the decrease of the amplitude is accompanied by a widening, which is the consequence that different harmonic waves travel with different phase velocity. This phenomenon is called dispersion.

Finally we discuss the Love wave of a homogeneous layer overlying a half space formulated in the previous subsection. Let  $C$  be the group velocity corresponding to a particular value of  $\kappa$ . We have

$$\frac{C}{V_S^{(1)}} = \frac{1}{V_S^{(1)}} \frac{d(\kappa v)}{d\kappa} = \frac{v}{V_S^{(1)}} + \kappa H \frac{d(v/V_S^{(1)})}{d(\kappa H)}. \quad (6.125)$$

Assume  $\mu^{(2)}/\mu^{(1)}$  and  $V_S^{(2)}/V_S^{(1)}$  to be known, we can draw the curve of  $v/V_S^{(1)}$  as a function of  $\kappa H$  according to (6.107) for each Love wave mode, i.e., each value of  $n$ . These curves, together with (6.125), can be used to obtain the curves of  $C/V_S^{(1)}$  as a functions of  $v/V_S^{(1)}$  or  $\kappa H$ , which are referred to as dispersion curves. These curves can be used to estimate  $H$ . The detail is referred to books of seismology.

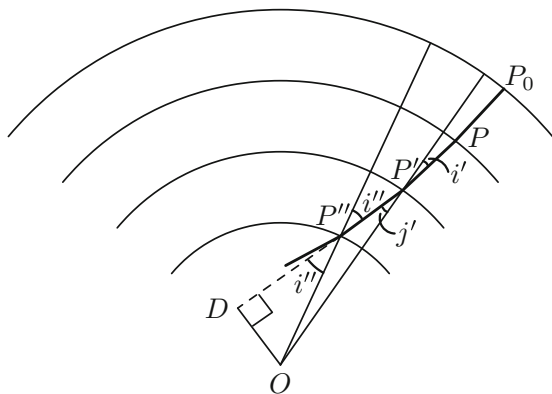
## 6.3 Seismic Ray Theory

### 6.3.1 The Ray Parameter and Bendorf's Law

The path of a seismic wave within the Earth is called a seismic ray, which is a curve always perpendicular to the wavefront, or whose tangent at any point is in the propagation direction of the wave. For a spherically symmetrical Earth model, any seismic ray is within a plane. Hence, we limit ourselves to discuss the problem within a plane.

We divide the Earth into a series of concentric spherical layers, and approximately take the wave velocity in each layer as a constant. The seismic ray is then

**Fig. 6.9** Refraction of seismic wave



the path of the seismic wave refracted at each of the boundaries between adjacent layers. As shown in Fig. 6.9, we examine the portion  $PP'P''$  of a seismic ray, with  $P$ ,  $P'$ , and  $P''$  on boundaries. We denote the incident and refraction angles at  $P'$  as  $i'$  and  $j'$ , respectively, and  $i''$  the incident angle at  $P''$ . We have, according to the relation between incident and refraction angles,

$$\frac{OP' \sin i'}{v'} = \frac{OP' \sin j'}{v''}, \quad (6.126)$$

where  $v'$  and  $v''$  are the velocities of the seismic wave through the line segments  $PP'$  and  $P'P''$ , respectively. We can also obtain from the rectangular triangles  $OP'D$  and  $OP''D$ , where  $D$  is on the extension line of  $P'P''$ ,

$$OD = OP' \sin j' = OP'' \sin i''. \quad (6.127)$$

The substitution of this formula into (6.126) leads to

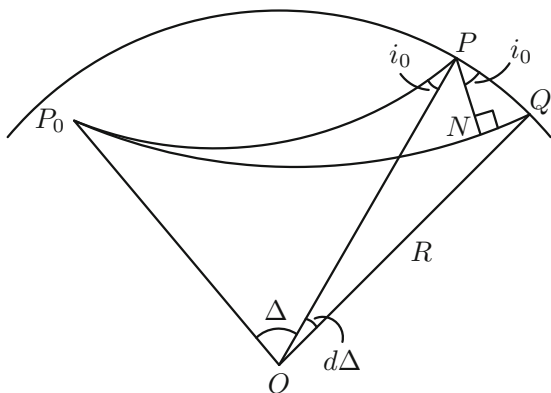
$$\frac{OP' \sin i'}{v'} = \frac{OP'' \sin i''}{v''}. \quad (6.128)$$

As the general case, we use  $i$  to denote the angle between the seismic ray and the radial direction,  $r$  the radial distance, and  $v$  the wave velocity understood as a function of  $r$ . Based on the above formula, we obtain that

$$p = \frac{r \sin i}{v} \quad (6.129)$$

is a constant for each seismic ray, called ray parameter. Define another parameter

$$\eta = \frac{r}{v}. \quad (6.130)$$

**Fig. 6.10** Bendorf's law

We can express the ray parameter as

$$p = \eta \sin i. \quad (6.131)$$

As the parameter  $p$  is a ray constant, the property of the ray is fully determined by it.

We now derive a principle to determine the ray parameter using data of travel time. As shown in Fig. 6.10, let  $P_0P$  be a seismic ray with  $p$  as the ray parameter. We denote the geocentric angle  $\angle P_0OP$  as  $\Delta$ , and the travel time from  $P_0$  to  $P$  as  $T$ . Let  $P_0Q$  be an adjacent seismic ray with  $p + dp$ ,  $\Delta + d\Delta$  and  $T + dT$ . Draw a line perpendicular to  $P_0Q$  from  $P$ , which intersects  $P_0Q$  at  $N$ . With  $PQ = R d\Delta$  and  $NQ = v_0 dT$ , where  $R$  is the Earth's radius and  $v_0$  is the wave velocity just below the Earth's surface, we obtain from the triangle  $PNQ$ ,

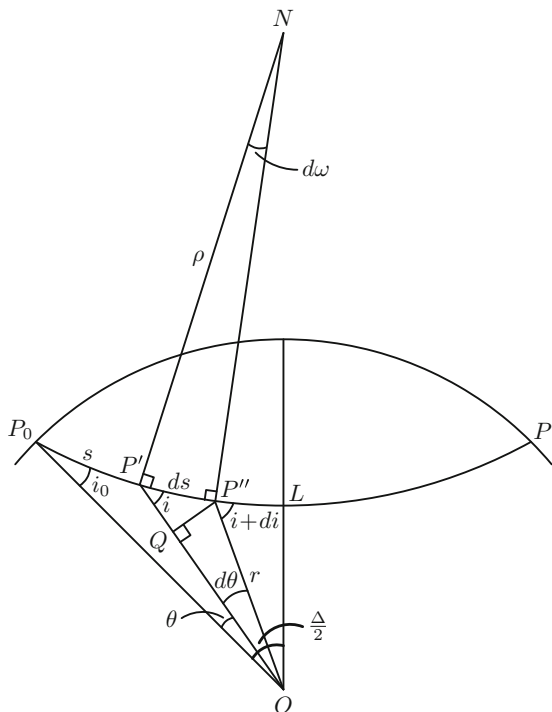
$$\sin i_0 = \frac{NQ}{PQ} = \frac{v_0}{R} \frac{dT}{d\Delta}. \quad (6.132)$$

Finally,

$$\frac{dT}{d\Delta} = \frac{R \sin i_0}{v_0} = p, \quad (6.133)$$

meaning that the rate of change of travel time with respect to the geocentric angle equals the ray parameter. This formula is referred to as Bendorf's law. The quantity  $\bar{v}_0 = PQ/dT = R(d\Delta/dT)$  is referred to as visual velocity, whose physical meaning can be seen from Fig. 6.10: it is the propagation velocity as seen on the Earth's surface. As both  $T$  and  $\Delta$  can be measured, Bendorf's law is the formula to compute the ray parameter  $p$  using the  $T$ - $\Delta$  curve, or travel-time curve, determined using observations.

**Fig. 6.11** Curvature of a seismic ray



### 6.3.2 Curvature of a Seismic Ray

The curvature represents a fundamental property of the ray. As shown in Fig. 6.11, let  $P_0P$  be a seismic ray with  $L$  as its lowest point. What we look for is the curvature at  $P'$ . Let  $P''$  be a neighboring point on the seismic ray. Draw straight lines perpendicular to the seismic ray through  $P'$  and  $P''$ , which intersect at the point  $N$  with an angle  $d\omega$ . Denote the length of the seismic ray from  $P_0$  to  $P'$  as  $s$ , and the incident angle at  $P'$  as  $i$ . These parameters for the point  $P''$  are different by an infinitesimal increment, and are  $s + ds$  and  $i + di$ . The curvature  $\kappa$  at  $P'$  is, by definition,

$$\kappa = \frac{1}{\rho} = \frac{d\omega}{ds}, \quad (6.134)$$

where  $\rho$  is the curvature radius as shown in Fig. 6.11. We mention that the seismic ray is assumed to concave upward or bulge downward. We thus have  $d\omega = di - d\theta$  based on the quadrilateral  $NP'OP''$ , and we can write (6.134) as

$$\kappa = \frac{1}{\rho} = \frac{di}{ds} - \frac{d\theta}{ds}. \quad (6.135)$$



Draw a line perpendicular to  $OP'$  through  $P''$ , intersecting  $OP'$  at  $Q$ . We obtain

$$\sin i = \frac{P''Q}{P'P''} = \frac{rd\theta}{ds}, \quad \cos i = \frac{P'Q}{P'P''} = -\frac{dr}{ds}. \quad (6.136)$$

Hence,  $d\theta/ds = \sin i/r$  and  $dr/ds = -\cos i$ , and (6.135) can be written as

$$\kappa = \frac{1}{\rho} = -\cos i \frac{di}{dr} - \frac{\sin i}{r}. \quad (6.137)$$

In order to compute  $\cos[i(di/dr)]$ , we take the derivative of both sides of (6.129) with respect to  $r$ . As  $p$  is a constant, we obtain

$$\frac{\sin i}{v} + \frac{r \cos i}{v} \frac{di}{dr} - \frac{r \sin i}{v^2} \frac{dv}{dr} = 0, \quad (6.138)$$

whence

$$\cos i \frac{di}{dr} = \frac{\sin i}{v} \frac{dv}{dr} - \frac{\sin i}{r}. \quad (6.139)$$

By substituting this formula into (6.137), we finally obtain the formula of the curvature,

$$\kappa = \frac{1}{\rho} = -\frac{\sin i}{v} \frac{dv}{dr}. \quad (6.140)$$

The curvature formula (6.140) can be used to study the geometrical properties of the seismic ray:

1. If  $v$  is a constant, we have  $dv/dr = 0$ , and hence  $\kappa = 0$ , implying that the seismic ray is a straight line.
2. If  $dv/dr < 0$ , i.e.,  $v$  increases with depth, we have  $\kappa > 0$ , implying that the seismic ray bulges toward below, and a lowest point exists.
3. If  $0 < dv/dr < v/r$ , i.e.,  $v$  decreases with depth, but the rate of decrease is small without exceeding  $v/r$ , we have  $0 < -\kappa < \sin i/r < 1/r$ , implying that the seismic ray bulges upward, but its curvature is smaller than the curvature of the concentric sphere, and the seismic ray still arrive on the Earth's surface.
4. If  $dv/dr = v/r$ , we have  $-\kappa = \sin i/r$ , implying that the seismic ray bulges upward, and  $-\kappa = 1/r$  after the highest point where  $i = \pi/2$ . This means that the seismic ray becomes a concentric circle after the point  $i = \pi/2$ , and cannot arrive on the Earth's surface.
5. If  $dv/dr > v/r$ , we have  $-\kappa > \sin i/r$ , implying that the seismic ray bulges upward with a curvature larger than that of the concentric sphere, and twists toward the Earth center.

### 6.3.3 Determination of Wave Velocity Using Travel Time

We assume that the starting and end points of the seismic ray are both on the Earth's surface, and that the angle with respect to the Earth center is  $\Delta$ . Before deriving the formula for determining wave velocity, we first derive a formula for determining  $\Delta$  using the ray parameter  $p$ .

Taking reference to Fig. 6.11, we have

$$ds^2 = P'Q^2 + P''Q^2 = dr^2 + r^2 d\theta^2. \quad (6.141)$$

We can also obtain, according to the first formula of (6.136) and (6.131),

$$ds = \frac{r}{\sin i} d\theta = r \frac{\eta}{p} d\theta. \quad (6.142)$$

The elimination of  $ds$  from the above two formulae yields

$$r^2 \frac{\eta^2}{p^2} = \left( \frac{dr}{d\theta} \right)^2 + r^2. \quad (6.143)$$

This equation can be solved to obtain

$$d\theta = \pm pr^{-1}(\eta^2 - p^2)^{-1/2} dr, \quad (6.144)$$

where  $\pm$  takes the sign of  $dr$ , which is negative from the starting point to the lowest point, and positive from the lowest point to the end point. Denote the lowest point of the ray as  $L$ , and its radial distance as  $r_L$ . We can integrate (6.144) from  $P_0$  to the lowest point  $L$ , yielding

$$\frac{1}{2}\Delta = p \int_{r_L}^R r^{-1}(\eta^2 - p^2)^{-1/2} dr, \quad (6.145)$$

which is the formula to compute  $\Delta$  once  $p$  is known.

We now derive the formula of wave velocity starting from (6.145). We rewrite it as

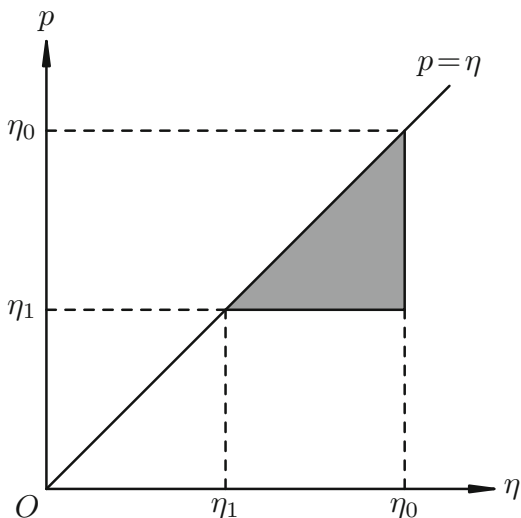
$$\Delta = \int_{\eta_L}^{\eta_0} 2pr^{-1}(\eta^2 - p^2)^{-1/2} \frac{dr}{d\eta} d\eta, \quad (6.146)$$

where  $\eta_0$  and  $\eta_L$  are values of  $\eta$  on the Earth's surface and the lowest point. As  $i = \pi/2$  at the lowest point, we have, according to (6.131) and (6.133),

$$\eta_L = \frac{r_L}{v_L} = p = \frac{dT}{d\Delta}. \quad (6.147)$$

Our logic is to take (6.146) as a function  $\Delta = \Delta(L)$  to solve for  $r_L$ , and then to obtain  $v_L$  from (6.147).

**Fig. 6.12** The integration domain



Define  $\eta_1$  as a function of  $r_1$  according to (6.130). Multiply both sides of (6.146) by  $(p^2 - \eta_1^2)^{-1/2}$ , and then integrate with respect to  $p$  from  $\eta_1$  to  $\eta_0$ . We obtain

$$\int_{\eta_1}^{\eta_0} \Delta(p^2 - \eta_1^2)^{-1/2} dp = \int_{\eta_1}^{\eta_0} \left\{ (p^2 - \eta_1^2)^{-\frac{1}{2}} \int_{\eta_L}^{\eta_0} 2pr^{-1}(\eta^2 - p^2)^{-1/2} \frac{dr}{d\eta} d\eta \right\} dp. \quad (6.148)$$

The right-hand side is a multivariable integral. As  $\eta_L = p$ , its domain of integration is as shown in Fig. 6.12. By interchanging the order of integration variables, we obtain

$$\begin{aligned} & \int_{\eta_1}^{\eta_0} \Delta(p^2 - \eta_1^2)^{-1/2} dp \\ &= \int_{\eta_1}^{\eta_0} \left\{ 2r^{-1} \frac{dr}{d\eta} \left[ \int_{\eta_1}^{\eta} p(\eta^2 - p^2)^{-1/2} (p^2 - \eta_1^2)^{-1/2} dp \right] \right\} d\eta. \end{aligned} \quad (6.149)$$

Assuming  $\eta_1 < \eta$ , the integral within the brackets is  $\pi/2$ . We thus obtain

$$\int_{\eta_1}^{\eta_0} \Delta(p^2 - \eta_1^2)^{-1/2} dp = \int_{\eta_1}^{\eta_0} \pi \frac{d \ln r}{d\eta} d\eta = \pi \ln \frac{R}{r_1}, \quad (6.150)$$

and hence (note that the integration variable is always  $p$ ),

$$\begin{aligned}
 \ln \frac{R}{r_1} &= \frac{1}{\pi} \int_{\eta_1}^{\eta_0} \Delta (p^2 - \eta_1^2)^{-1/2} dp \\
 &= \frac{1}{\pi} \int_{\eta_1}^{\eta_0} \Delta \left[ \left( \frac{p}{\eta_1} \right)^2 - 1 \right]^{-1/2} d \left( \frac{p}{\eta_1} \right) \\
 &= \frac{1}{\pi} \int_{\eta_1}^{\eta_0} \Delta d \ln \left\{ \frac{p}{\eta_1} + \left[ \left( \frac{p}{\eta_1} \right)^2 - 1 \right]^{1/2} \right\} \\
 &= \frac{1}{\pi} \left[ \Delta \ln \left\{ \frac{p}{\eta_1} + \left[ \left( \frac{p}{\eta_1} \right)^2 - 1 \right]^{1/2} \right\} \right]_{\eta_1}^{\eta_0} \\
 &\quad - \frac{1}{\pi} \int_{\eta_1}^{\eta_0} \ln \left\{ \frac{p}{\eta_1} + \left[ \left( \frac{p}{\eta_1} \right)^2 - 1 \right]^{1/2} \right\} \frac{d\Delta}{dp} dp. \tag{6.151}
 \end{aligned}$$

According to (6.147),  $p = \eta_0$  defines a seismic ray with the lowest point on the Earth's surface, for which  $\Delta = 0$ . Hence, the first term of (6.151) vanishes. Again, according to (6.147),  $p = \eta_1$  defines a seismic ray with  $r_1$  as the radial distance of the lowest point. Denoting the angle of this seismic ray with respect to the Earth center as  $\Delta_1$ , we can write (6.151) as an integral over  $\Delta$ ,

$$\ln \frac{R}{r_1} = \frac{1}{\pi} \int_0^{\Delta_1} \ln \left\{ \frac{p}{\eta_1} + \left[ \left( \frac{p}{\eta_1} \right)^2 - 1 \right]^{1/2} \right\} d\Delta. \tag{6.152}$$

where  $p = dT/d\Delta$  is understood as a single valued function of  $\Delta$  as given by (6.133) and  $\eta_1 = (dT/d\Delta)_{\Delta=\Delta_1}$ .

Denote the integral in (6.152) as  $I(\Delta_1)$ , which can be numerically evaluated based on the  $T$ - $\Delta$  curve observed. We obtain from (6.152),

$$r_1 = R \exp\{-I(\Delta_1)/\pi\}. \tag{6.153}$$

This is the formula to compute the radial distance of the lowest point of the seismic ray with  $\Delta_1$  as the angle with respect to the Earth center. The formula of the wave velocity at this point can be obtained from (6.147),

$$v(r_1) = \frac{r_1}{\eta_1}. \tag{6.154}$$

In summary, the procedure for computing the wave velocity inside the Earth is as follows: Compute  $r_1$  using  $\Delta_1$  according to (6.153) at first, and then compute the wave velocity at the radial distance  $r_1$  according to (6.154).

### 6.3.4 Correction of Travel Time Due to the Earth's Ellipticity

The Earth is not a spherically symmetrical body. The largest deviation from spherical symmetry is the ellipticity. As it is much more convenient to study the Earth's interior based on spherical symmetry, the effect of ellipticity is customarily corrected using an internal profile of flattening as formulated in Sect. 5.5.

Accurate to the first order of flattening, the equation of the equiwave surfaces of wave velocity is

$$r' = r \left\{ 1 + f \left( \frac{1}{3} - \cos^2 \theta' \right) \right\}, \quad (6.155)$$

where the flattening  $f$  is a function of  $r$ . We note that the symbol  $\theta'$  is the spherical coordinate polar angle. For a spherical Earth model, we have

$$T = \int_0^\Delta \frac{1}{v} ds, \quad (6.156)$$

$$ds^2 = dr^2 + r^2 d\theta^2, \quad (6.157)$$

where  $\theta$  is the angle with respect to the Earth center from the starting point of the seismic ray,  $P_0$ , as defined in Fig. 6.11. For an ellipsoidal Earth model, we have

$$T' = \int_0^\Delta \frac{1}{v} ds', \quad (6.158)$$

$$ds'^2 = dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2 \theta d\lambda^2, \quad (6.159)$$

where  $\lambda$  is the longitude with the starting point of the seismic ray,  $P_0$ , as the pole, and is not to be confused with the geodetic longitude. Define

$$\delta r = r' - r = rf \left( \frac{1}{3} - \cos^2 \theta' \right), \quad (6.160)$$

which is an infinitesimal quantity, and only its first order term will be kept in subsequent formulation. We also note that, since  $r'd\lambda$  is an effect of the ellipticity, it is on the order of magnitude of  $d(\delta r)$ . Hence, the last term at the right-hand side of (6.159) is on the order of magnitude of  $[d(\delta r)]^2$ , and can be neglected. We obtain, considering that the difference between  $dr'$  and  $dr$  is infinitesimal as compared to themselves,

$$ds'^2 - ds^2 = 2drd(\delta r) + 2r\delta r d\theta^2. \quad (6.161)$$

As the difference between  $ds'$  and  $ds$  is also infinitesimal as compared to themselves, we have

$$ds' - ds = \frac{ds'^2 - ds^2}{ds' + ds} = \frac{ds'^2 - ds^2}{2ds}. \quad (6.162)$$

The use of (6.157) and (6.161) leads to

$$ds' - ds = \left[ \left( \frac{dr}{d\theta} \right)^2 + r^2 \right]^{-1/2} \left[ \frac{dr}{d\theta} \frac{d(\delta r)}{d\theta} + r\delta r \right] d\theta, \quad (6.163)$$

which can be further developed to, according to (6.143),

$$ds' - ds = \frac{pv}{r^2} \left[ \frac{dr}{d\theta} \frac{d(\delta r)}{d\theta} + r\delta r \right] d\theta. \quad (6.164)$$

The ellipticity correction of travel time for this infinitesimal fraction of the ray is  $(ds' - ds)/v$ . Therefore, the ellipticity correction of travel time for the entire ray is

$$\delta T = T' - T = p \int_0^\Delta \left[ \frac{1}{r^2} \frac{dr}{d\theta} \frac{d(\delta r)}{d\theta} + \frac{\delta r}{r} \right] d\theta, \quad (6.165)$$

which can be further simplified.

We can write (6.165) as, by integrating its first term of the integrand by part,

$$\delta T = p \left[ \frac{1}{r^2} \frac{dr}{d\theta} \delta r \right]_0^\Delta + p \int_0^\Delta \left[ \frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} \right] \delta r dr, \quad (6.166)$$

where  $(1/r^2)(dr/d\theta) = -(d/d\theta)(1/r)$  is used in the second term. We can use (6.144) to write the first term at the right-hand side as

$$\begin{aligned} p \left[ \frac{1}{r^2} \frac{dr}{d\theta} \delta r \right]_0^\Delta &= \left[ r^{-1}(\eta^2 - p^2)^{1/2} \delta r \right]_\Delta + \left[ r^{-1}(\eta^2 - p^2)^{1/2} \delta r \right]_0 \\ &= \left[ r^{-1}(\eta^2 - p^2)^{1/2} \delta r \right]_{0,\Delta}. \end{aligned} \quad (6.167)$$

The second term at the right-hand side of (6.166) can also be simplified. According to the two formulae of (6.136), we have

$$\frac{dr}{d\theta} = -r \frac{\cos i}{\sin i}. \quad (6.168)$$

Hence,

$$\frac{d}{d\theta} \left( \frac{1}{r} \right) = -\frac{1}{r^2} \frac{dr}{d\theta} = \frac{1}{r} \frac{\cos i}{\sin i}, \quad (6.169)$$

$$\frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) = -\frac{1}{r^2} \frac{\cos i}{\sin i} \frac{dr}{d\theta} - \frac{1}{r} \frac{1}{\sin^2 i} \frac{di}{dr} \frac{dr}{d\theta} = \frac{1}{r} \frac{\cos^2 i}{\sin^2 i} + \frac{\cos i}{\sin^3 i} \frac{di}{dr}. \quad (6.170)$$

By substituting (6.139), (6.131), and (6.130) into this formula, we obtain

$$\frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) = \frac{\eta^3}{p^2 r} \frac{dv}{dr} - \frac{1}{r}. \quad (6.171)$$

With this formula, we can write the second term at the right-hand side of (6.166) as

$$p \int_0^\Delta \left[ \frac{d^2}{d\theta^2} \left( \frac{1}{r} \right) + \frac{1}{r} \right] \delta r dr = p^{-1} \int_0^\Delta r^{-1} \delta r \eta^3 \frac{dv}{dr} d\theta. \quad (6.172)$$

Finally, by substituting (6.167) and (6.172) into (6.166), and then using (6.160), we obtain

$$\delta T = \left[ f \left( \frac{1}{3} - \cos^2 \theta' \right) (\eta^2 - p^2)^{1/2} \right]_{0, \Delta} + p^{-1} \int_0^\Delta f \left( \frac{1}{3} - \cos^2 \theta' \right) \eta^3 \frac{dv}{dr} d\theta. \quad (6.173)$$

This formula can be used to compute the ellipticity correction, where the subscripts 0,  $\Delta$  is defined in (6.167), and the difference between the symbols  $\theta$  and  $\theta'$  is to be kept in mind. Evidently, there is a need of a velocity model in order to compute this correction.

## 6.4 Free Oscillations of the Spherical Elastic Earth Model

### 6.4.1 General Formulation

After a bell is struck, it vibrates free of external forcing to produce a sound. The Earth also does so after being struck by, e.g., Earthquakes. The sound of a bell is composed of multiple frequencies depending on the material, size and shape of the bell. The vibration of the Earth is also composed of multiple frequencies, thus being referred to as “free oscillations”, which are below the audible frequency band of the human being, and can only be “heard” by seismometers. As the free oscillations depend on the deep interior of the Earth, their records are used to study the deep interior of the Earth.

The frequencies of Earth’s free oscillations can be computed using the density and elastic parameters of the Earth. The theory is quite complicated when the deviation from spherical symmetry is considered. However, the influence of the largest deviation from sphericity, the Earth’s flattening, is only at a level of  $\sim 1/300$ , and that of lateral heterogeneity should be even smaller. Therefore, reasonably accurate results can be obtained by approximating the Earth as a spherically symmetrical sphere. The formulation of the free oscillations of such an Earth model was included in Sect. 4.8. where a slightly more general case including external forcing has been considered. Here we provide a complete description of the physical formulae of the free oscillation problem while taking the result of mathematical formulation from Sect. 4.8. The governing equations are the Earth’s elastic equations

without external forcing, which are indeed (4.435)–(4.437) with  $\phi_f = 0$ . We have

$$\begin{aligned} \rho \left( \frac{\partial^2 \vec{u}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{u}}{\partial t} \right) &= \nabla \cdot \vec{T} - \rho(\nabla \cdot \vec{u})\nabla\phi_0 \\ &\quad + \rho\nabla\vec{u} \cdot \nabla\phi_0 + \rho\vec{u} \cdot \nabla\nabla\phi_0 + \rho\nabla\phi_1, \end{aligned} \quad (6.174)$$

$$\vec{T} = \lambda\nabla \cdot \vec{u}\vec{I} + \mu[\nabla\vec{u} + (\nabla\vec{u})^T], \quad (6.175)$$

$$\nabla^2\phi_1 = 4\pi G\nabla \cdot (\rho\vec{u}). \quad (6.176)$$

The boundary conditions are (4.519)–(4.524) with  $\vec{t}_0 = 0$ . The condition of displacement across welded boundaries is

$$\{\vec{u}\}_{\pm}^{\pm} = 0. \quad (6.177)$$

The condition of displacement across frictionless slip boundaries is

$$\{\hat{n} \cdot \vec{u}\}_{\pm}^{\pm} = 0. \quad (6.178)$$

A condition of stress across all boundaries is

$$\{\hat{n} \cdot \vec{T}\}_{\pm}^{\pm} = 0. \quad (6.179)$$

The only frictionless slip boundaries in the Earth's interior are the inner core boundary (ICB) and core-mantle boundary (CMB). In the fluid core, as  $\hat{n} \cdot \vec{T} = [\hat{n} \cdot (\hat{n} \cdot \vec{T})]\hat{n}$ , the condition that the shear stress vanishes at the ICB and CMB is automatically satisfied, and is not to be discussed further. We assumed that the Earth's surface is stress free, i.e.,  $\vec{t}_0 = 0$ , which leads to

$$(\hat{n} \cdot \vec{T})_{S_0} = 0. \quad (6.180)$$

The conditions for the gravitational potential across all kinds of boundaries are

$$\{\phi_1\}_{\pm}^{\pm} = 0, \quad (6.181)$$

$$\left\{ \frac{\partial\phi_1}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right\}_{\pm}^{\pm} = 0. \quad (6.182)$$

Following the formulation in Sect. 4.8, the partial differential equations as well as the boundary conditions of the free oscillation problem described in the previous paragraph can be reduced to a system of ordinary differential equations by expressing the unknown functions as spherical harmonic series. Here we directly use



the functions  $y_1^n, \dots, y_8^n$  defined in (4.480)–(4.487) to write the series expansions as

$$\begin{aligned} \vec{u} = \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \left( y_1^n P_n^k \hat{e}_r + y_3^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_3^n \frac{ik}{\sin \theta} P_n^k \hat{e}_\lambda \right) \right. \\ \left. + \left( y_7^n \frac{k}{\sin \theta} P_n^k \hat{e}_\theta + iy_7^n \frac{d}{d\theta} P_n^k \hat{e}_\lambda \right) \right] e^{ik\lambda} e^{i\sigma t}, \end{aligned} \quad (6.183)$$

$$\begin{aligned} \hat{e}_r \cdot \vec{T} = \sum_{n=0}^{\infty} \sum_{k=-n}^n \left[ \left( y_2^n P_n^k \hat{e}_r + y_4^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_4^n \frac{ik}{\sin \theta} P_n^k \hat{e}_\lambda \right) \right. \\ \left. + \left( y_8^n \frac{k}{\sin \theta} P_n^k \hat{e}_\theta + iy_8^n \frac{d}{d\theta} P_n^k \hat{e}_\lambda \right) \right] e^{ik\lambda} e^{i\sigma t}, \end{aligned} \quad (6.184)$$

$$\phi_1 = \sum_{n=0}^{\infty} \sum_{k=-n}^n y_5^n P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (6.185)$$

$$\frac{\partial \phi_1}{\partial r} - 4\pi G\rho(\hat{e}_r \cdot \vec{u}) = \sum_{n=0}^{\infty} \sum_{k=-n}^n y_6^n P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (6.186)$$

where  $\sigma$  is the angular frequency of the free oscillation. The formulation of the partial differential equations is the same as that in Sect. 4.8, and the result is

$$\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r} y_1^n + \frac{1}{\lambda + 2\mu} y_2^n + \frac{n(n+1)\lambda}{(\lambda + 2\mu)r} y_3^n, \quad (6.187a)$$

$$\begin{aligned} \frac{dy_2^n}{dr} = \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda + 2\mu)r} y_2^n \\ + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} + 2k\rho\omega\sigma \right] y_3^n + \frac{n(n+1)}{r} y_4^n \\ - \rho y_6^n + 2\rho\omega\sigma \left[ \frac{(n-1)(n-k)}{2n-1} y_7^{n-1} - \frac{(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right], \end{aligned} \quad (6.187b)$$

$$\frac{dy_3^n}{dr} = -\frac{1}{r} y_1^n + \frac{1}{r} y_3^n + \frac{1}{\mu} y_4^n, \quad (6.187c)$$

$$\frac{dy_4^n}{dr} = \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} + \frac{2k\rho\omega\sigma}{n(n+1)} \right] y_1^n - \frac{\lambda}{(\lambda + 2\mu)r} y_2^n + \left\{ -\rho\sigma^2 \right.$$

$$\begin{aligned}
& + \frac{2\mu}{(\lambda + 2\mu)r^2} [(2n^2 + 2n - 1)\lambda + 2(n^2 + n - 1)\mu] + \frac{2k\rho\omega\sigma}{n(n+1)} \left\{ y_3^n - \frac{3}{r} y_4^n \right. \\
& \left. - \frac{\rho}{r} y_5^n + \frac{2\rho\omega\sigma}{n(n+1)} \left[ \frac{(n^2 - 1)(n - k)}{2n - 1} y_7^{n-1} + \frac{n(n+2)(n+k+1)}{2n+3} y_7^{n+1} \right] \right\},
\end{aligned} \tag{6.187d}$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \tag{6.187e}$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r} y_3^n + \frac{n(n+1)}{r^2} y_5^n - \frac{2}{r} y_6^n, \tag{6.187f}$$

$$\frac{dy_7^n}{dr} = \frac{1}{r} y_7^n + \frac{1}{\mu} y_8^n, \tag{6.187g}$$

$$\begin{aligned}
\frac{dy_8^n}{dr} = & \left[ -\rho\sigma^2 + \frac{(n-1)(n+2)\mu}{r^2} + \frac{2k\rho\omega\sigma}{n(n+1)} \right] y_7^n - \frac{3}{r} y_8^n \\
& + \frac{2\rho\omega\sigma}{n(n+1)} \left\{ -\frac{(n+1)(n-k)}{2n+1} \left[ y_1^{n-1} - (n-1)y_3^{n-1} \right] \right. \\
& \left. + \frac{n(n+k+1)}{2n+3} \left[ y_1^{n+1} + (n+2)y_3^{n+1} \right] \right\}.
\end{aligned} \tag{6.187h}$$

All the boundary conditions are the same as those derived in Sect. 4.8 after setting  $(t_{0r}^S)_n^k = (t_{0r}^S)_n^k = (t_{0r}^T)_n^k = 0$  as implied by  $\vec{t}_0 = 0$ , and  $\phi_{fn}^k = 0$  as implied by  $\phi_f = 0$ . We have:

- On internal welded boundaries,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_3^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = \{y_7^n\}_-^+ = \{y_8^n\}_-^+ = 0. \tag{6.188}$$

- On internal frictionless slip boundaries, i.e., the ICB and CMB,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = \{y_8^n\}_-^+ = 0, \tag{6.189}$$

and no condition applies to  $y_3^n$  and  $y_7^n$ , meaning that they can be discontinuous.

- On the Earth's surface,

$$y_2^n(R) = y_4^n(R) = y_8^n(R) = 0, \tag{6.190}$$

$$\frac{n+1}{R} y_5^n(R) + y_6^n(R) = 0. \tag{6.191}$$

We see that we have a system of eight ordinary differential equations for eight unknown functions. But we have only four boundary conditions on the Earth's surface. The other four boundary conditions are either regularity conditions at the

Earth center, or conditions at the internal boundaries, such as  $y_8^n = 0$  at the ICB and CMB, which is evident.

For the free oscillations governed by elastic deformation, the periods are mostly a few minutes; even the longest (except the Slichter modes as will be detailed in Sect. 6.4.3) is below an hour. Hence, the Coriolis acceleration,  $2\vec{\omega} \times (\partial \vec{u}/\partial t)$ , is much smaller than the acceleration of the free oscillation,  $\partial^2 \vec{u}/\partial t^2$ , and can be neglected as a first order approximation. However, for the free oscillations governed by fluid flow in the outer core, the periods (including those of the Slichter modes) may attain hours. Therefore, the magnitude of the Coriolis acceleration,  $2\vec{\omega} \times (\partial \vec{u}/\partial t)$ , is comparable to that of the acceleration of the oscillation,  $\partial^2 \vec{u}/\partial t^2$ , and can no longer be neglected.

Here we focus on the free oscillations dominated by the elastic deformation, for which we can neglect the Coriolis term as an approximation. In this case, the terms involving  $\omega$  disappears, and the system of differential equations (6.187) splits to two independent systems of ordinary differential equations:

- One system includes the functions  $y_1^n, \dots, y_6^n$ , as already given in Eq. (4.517),

$$\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r}y_1^n + \frac{1}{\lambda + 2\mu}y_2^n + \frac{n(n+1)\lambda}{(\lambda + 2\mu)r}y_3^n, \quad (6.192a)$$

$$\begin{aligned} \frac{dy_2^n}{dr} = & \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda + 2\mu)r}y_2^n \\ & + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_3^n + \frac{n(n+1)}{r}y_4^n \\ & - \rho y_6^n, \end{aligned} \quad (6.192b)$$

$$\frac{dy_3^n}{dr} = -\frac{1}{r}y_1^n + \frac{1}{r}y_3^n + \frac{1}{\mu}y_4^n, \quad (6.192c)$$

$$\begin{aligned} \frac{dy_4^n}{dr} = & \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{\lambda}{(\lambda + 2\mu)r}y_2^n + \left\{ -\rho\sigma^2 \right. \\ & + \frac{2\mu}{(\lambda + 2\mu)r^2} \left[ (2n^2 + 2n - 1)\lambda + 2(n^2 + n - 1)\mu \right] \left. \right\} y_3^n \\ & - \frac{3}{r}y_4^n - \frac{\rho}{r}y_5^n, \end{aligned} \quad (6.192d)$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \quad (6.192e)$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r}y_3^n + \frac{n(n+1)}{r^2}y_5^n - \frac{2}{r}y_6^n. \quad (6.192f)$$

The associated boundary conditions are

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_3^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0 \quad (6.193)$$

across internal welded boundaries,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0 \quad (6.194)$$

across internal frictionless slip boundaries, i.e., the ICB and CMB, and

$$y_2^n(R) = y_4^n(R) = 0, \quad \frac{n+1}{R} y_5^n(R) + y_6^n(R) = 0 \quad (6.195)$$

on the Earth's surface.

- The other system includes the functions  $y_7^n$  and  $y_8^n$ , as already given in (4.518),

$$\begin{cases} \frac{dy_7^n}{dr} = \frac{1}{r} y_7^n + \frac{1}{\mu} y_8^n, \\ \frac{dy_8^n}{dr} = \left[ -\rho \sigma^2 + \frac{(n-1)(n+2)\mu}{r^2} \right] y_7^n - \frac{3}{r} y_8^n. \end{cases} \quad (6.196)$$

The associated boundary conditions are

$$\{y_7^n\}_-^+ = \{y_8^n\}_-^+ = 0 \quad (6.197)$$

across internal welded boundaries,

$$\{y_8^n\}_-^+ = 0 \quad (6.198)$$

across internal frictionless slip boundaries, i.e., the ICB and CMB, and

$$y_8^n(R) = 0 \quad (6.199)$$

on the Earth's surface.

## 6.4.2 The Concepts of Spheroidal and Toroidal Free Oscillations

The system of ordinary differential equations and boundary conditions (6.187)–(6.191) is an eigenvalue problem, whose nontrivial solutions exist only when  $\sigma$  takes some specific values. These values of  $\sigma^2$  and the associated solutions of the functions  $y_1^n, \dots, y_8^n$  are referred to as eigenvalues and eigenfunctions. The same is for the two degenerate systems (6.192)–(6.195) and (6.196)–(6.199). Every eigenvalue as well as the corresponding eigenfunction is referred to as a normal mode. For this reason, the Earth's free oscillations are also called normal modes.

Since the ordinary differential equations as well as the boundary conditions are all linear and homogeneous, only the relative amplitudes of the functions of a solution can be obtained, as the product of all functions of a solution with a constant is still a solution. The degenerate system (6.192)–(6.195) concerns only the spheroidal component of the displacement, and (6.196)–(6.199) the toroidal component; consequently, their solutions are referred to as spheroidal and toroidal free oscillations, respectively. We see that a change of the gravitational field is associated with the spheroidal free oscillations, but not with the toroidal free oscillations.

In the discussions of this subsection, we refer the system of ordinary differential equations and boundary conditions (6.187)–(6.191) as the coupled system, (6.192)–(6.195) as the spheroidal system, and (6.196)–(6.199) as the toroidal system.

In the rest of this subsection, we first outline conceptually how to solve the free oscillation equations of the non-rotating Earth model, i.e., the spheroidal and toroidal systems, then we indicate how their solutions are related to those of the rotating Earth model, i.e., the coupled system. The procedure of solution of the coupled system will be outlined afterwards. More detail on the computation of the free oscillations governed by elastic deformation will be discussed in the next subsection.

We first consider the spheroidal system (6.192)–(6.195), where the six functions have three regularity conditions at the Earth center, and three boundary conditions on the Earth's surface. We solve this system of ordinary differential equations numerically starting from the Earth center assumed to be solid. Certainly we have to prescribe a value for  $\sigma$ . As there are six unknown functions and three regularity conditions at the Earth center, there exist three independent solutions that satisfy all the three regularity conditions. Therefore, we prescribe the initial conditions of three independent solutions at the Earth center, and then integrate the ordinary differential equations from the center to the surface of the Earth. At solid-to-solid boundaries, which are always assumed to be welded, the solution can be transferred to the next layer according to the continuity conditions (6.193). In a fluid layer, as  $\mu = 0$ , we have  $y_4^n = 0$  from (6.192c), and consequently, we can obtain an algebraic expression of  $y_3^n$  in terms of  $y_1^n$ ,  $y_2^n$  and  $y_5^n$  from (4.517d), and the six ordinary differential equations are reduced to four by dropping (6.192c) and (6.192d). At solid-to-fluid boundaries, which is of frictionless slip type by its nature, two independent solutions satisfying  $y_4^n = 0$  can be obtained from the three independent solutions from the solid side, and the other functions can be transferred to the fluid side according to the continuity conditions (6.193). This means that there are two independent solutions in the fluid layers. At fluid-to-solid boundaries, as no constraint exists for  $y_3^n$ , the two independent solutions at the fluid side can be used to construct three sets of independent boundary conditions at the solid side: (a) for the two independent solutions at the fluid side, setting  $y_3^n = 0$  at the solid side, and transfer other functions to the solid side based on the continuity conditions (6.193); and (b) start a new solution at the solid side with  $y_1^n = y_2^n = y_4^n = y_5^n = y_6^n = 0$  and  $y_3^n = 1$  at the boundary. Hence, we have again three independent solutions in the solid layer. We continue this way up to the Earth's surface. If the top layer is solid, there are

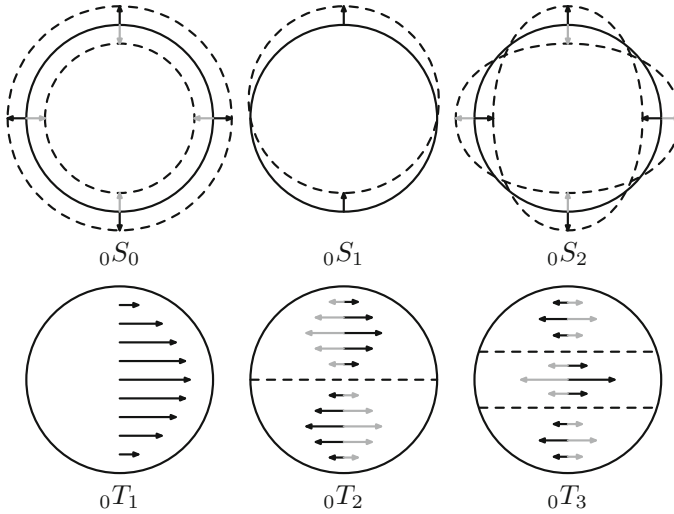
three independent solutions, and three boundary conditions are to be imposed. If the top layer is fluid, there are two independent solutions, both satisfying  $y_4^n = 0$ , and hence, only the two other boundary conditions are to be imposed. The last step is to check if a nontrivial solution satisfying all the boundary conditions to within a prescribed threshold on the Earth's surface can be obtained by linearly combining the independent solutions. If it is,  $\sigma$  is a free oscillation angular frequency. If not, prescribe a neighboring value to  $\sigma$ , and redo the computation. In this way, all the free oscillation modes can be found.

We now consider the toroidal system (6.196)–(6.199). From the first equation, we have  $y_8^n = 0$  in fluid layers. This implies that, at the CMB,  $y_8^n = 0$  and no constraint exists for  $y_7^n$ , meaning that the solutions in the mantle are not influenced by the core. Hence, we need to seek solutions only within the mantle with the boundary condition

$$(y_8^n)_{r=r_{\text{CMB}}} = 0 \quad (6.200)$$

at the CMB, together with  $y_8^n = 0$  on the Earth's outermost solid surface as can be easily inferred from (6.197) to (6.199). The computation in this case is simpler. Only one solution exists that satisfy (6.200). We chose a value of  $\sigma$ , and integrate the ordinary differential equations from the CMB to the Earth's top solid surface. If the condition  $y_8^n = 0$  is satisfied there, the value of  $\sigma$  chosen is a free oscillation angular frequency.

For any degree  $n$ , both the spheroidal and toroidal systems have multiple solutions of  $\sigma$ , which are customarily ordered from smaller to larger magnitudes, and each of them represents a free oscillation mode. The spheroidal free oscillation modes ordered this way are denoted as  ${}_0S_n$ ,  ${}_1S_n$ ,  ${}_2S_n$ ,  $\dots$ , and the toroidal free oscillation modes,  ${}_0T_n$ ,  ${}_1T_n$ ,  ${}_2T_n$ ,  $\dots$ . The mode with the lowest frequency or longest period,  ${}_0S_n$  for spheroidal free oscillations, and  ${}_0T_n$  for toroidal free oscillations, is referred to as the fundamental mode, and the rest overtones. In the discussions of the properties of the solutions, we use the functions  $y_1^n$ ,  $y_3^n$ , and  $y_7^n$ , which are used to represent the displacement, and  $y_5^n$ , which is used to represent the gravitational potential. These functions are considered more fundamental than the rest. It has been observed that the toroidal free oscillation mode  ${}_mT_n$  has the property that the solution of  $y_7^n$  in  $0 < r < R$  has  $m$  nodes, which are the values of  $r$  where  $y_7^n$  changes sign. For the spheroidal free oscillation mode  ${}_mS_n$ , the solutions of  $y_1^n$ ,  $y_3^n$ , and  $y_5^n$  are more complicated. Customarily, the mode  ${}_0S_n$  is always assigned to the solution where no node of  $y_1^n$  and  $y_3^n$  is present, such as the mode  ${}_0S_1$  as will be explained in the next subsection. The displacement of the Earth's surface of a few simple spheroidal and toroidal free oscillation modes are shown in Fig. 6.13, where the direction of the displacement is alternative following the directions of the dark and gray arrows. The modes  ${}_0S_1$  and  ${}_0T_1$  have a zero frequency, thus being time independent. The mode  ${}_0S_1$  is a translational displacement of the Earth as a whole, and  ${}_0T_1$  a rotation of the Earth as a whole, which are to be discussed in the next subsection.



**Fig. 6.13** The displacement of the Earth's surface of a few free oscillation modes. The direction of the displacement is alternative following the directions of the dark and gray arrows. The modes  ${}_0S_1$  and  ${}_0T_1$  have a zero frequency, thus being time independent, and are to be discussed in more detail in the next subsection

Before preceding to discuss the solutions of the coupled system (6.187)–(6.191), we first take a look at the relation between its solutions and those of the spheroidal and toroidal systems. We use  $\sigma_n$  to represent the eigenvalues of either the spheroidal or the toroidal systems,  $S_n$  the solutions of the spheroidal system, and  $T_n$  the solutions of the toroidal system, which are independent of  $k$ . We use  $\sigma_n^k$  to represent the eigenvalues of the coupled system,  $S_n^k$  the solutions of the spheroidal component, i.e.,  $y_1^n$  to  $y_6^n$ , and  $T_n^k$  the solutions of the toroidal component, i.e.,  $y_7^n$  and  $y_8^n$ . The inclusion of the order  $k$  is due to the dependence of (6.187) on  $k$ . As the effect of rotation is small, the solutions of the spheroidal and toroidal systems are approximate solutions of the coupled system, and a solution of the coupled system can be understood as a solution of the spheroidal or toroidal system with an Earth rotation correction. We now assume that  $\sigma_n \Leftrightarrow S_n$  is a set of solution of the spheroidal system. While looking to find the corresponding solution of the coupled system, we realize that  $S_n^k$  is coupled to  $T_{n-1}^k$  and  $T_{n+1}^k$  based on (6.187b) and (6.187d), and based on (6.187h),  $T_{n-1}^k$  is further coupled to  $S_{n-2}^k$ , and  $T_{n+1}^k$  is further coupled to  $S_{n+2}^k$ . Hence, solving the spheroidal system for  $\sigma_n \Leftrightarrow S_n$  becomes a problem of solving the coupled system for

$$\underline{\sigma}_n^k \Leftrightarrow \cdots T_{n-3}^k, S_{n-2}^k, T_{n-1}^k, \underline{S}_n^k, T_{n+1}^k, S_{n+2}^k, T_{n+3}^k, \cdots \quad (6.201)$$

Since  $k$  could be any of the  $2n + 1$  values  $0, \pm 1, \dots, \pm n$ , one solution of the spheroidal system,  $\sigma_n \Leftrightarrow S_n$ , corresponds to  $2n + 1$  solutions of the coupled system

as expressed in (6.201). Similarly, solving the toroidal system for  $\sigma_n \Leftrightarrow T_n$  becomes a problem of solving the coupled system for

$$\underline{\sigma}_n^k \Leftrightarrow \cdots S_{n-3}^k, T_{n-2}^k, S_{n-1}^k, \underline{T}_n^k, S_{n+1}^k, T_{n+2}^k, S_{n+3}^k, \cdots \quad (6.202)$$

We see that the effect of the rotation is to split one spheroidal or toroidal free oscillation mode of the non-rotating Earth model to  $2n + 1$ . The Slichter modes  ${}_1S_1^k$ ,  $k = -1, 0, 1$ , will be taken as an example of rotational splitting in the next subsection.

The coupled system includes an infinite number of unknown functions and equations coupled according to (6.201) or (6.202). In order to solve it, the coupling chain in (6.201) or (6.202) must be truncated, e.g., by including a finite number of terms at each sides of  $\underline{S}_n^k$  or  $\underline{T}_n^k$ , to reduce the number of equations to a finite number. The solution of this system of finite number of ordinary differential equations is similar to the spheroidal and toroidal systems, where the only complication is that more unknown functions, equations and boundary conditions are involved.

We mention that, the effect of all the complexities of the real Earth, including rotation, flattening, as well as lateral heterogeneity, is to split each degree  $n$  free oscillation mode of the spherically symmetrical non-rotating Earth model to  $2n + 1$ . The splitting effects of Earth rotation and flattening have been formulated and are provided as corrections (Dahlen & Sailor, 1979).

### 6.4.3 Practical Computations

In this subsection, we first provide some results of the spheroidal and toroidal free oscillation modes based on their uncoupled systems of equations (6.192) and (6.196), respectively. We then present the result of computation for the Slichter modes, i.e.,  ${}_1S_1^{-1}$ ,  ${}_1S_1^0$  and  ${}_1S_1^{+1}$ , as an example of the solution of the coupled system of equations (6.187).

#### 6.4.3.1 The Mode ${}_0S_1$

For the degree 1 spheroidal system (6.192)–(6.195), it can be verified that

$$\sigma = 0, \quad (6.203)$$

$$\left\{ y_1^1, y_2^1, y_3^1, y_4^1, y_5^1, y_6^1 \right\} = \left\{ C_1^k, 0, C_1^k, 0, gC_1^k, -\frac{2g}{r}C_1^k \right\}, \quad (6.204)$$

where  $C_1^k$  is an arbitrary constant, is a solution. Remembering that the  $y$ s depend on  $k$  implicitly, we used the subscript 1 to indicate the degree, and the superscript  $k$  to indicate that the constant can be different for different  $k$ . This mode is the degree 1 fundamental spheroidal free oscillation denoted as  ${}_0S_1$ . The vanishing of the frequency, i.e.,  $\sigma = 0$ , implies that the period is infinitely long, meaning that the displacement of this mode is time-independent, as given by  $y_1^1 = y_3^1 = C_1^k$ . The



stress of this mode vanishes as given by  $y_2^1 = y_4^1 = 0$ , meaning that no deformation is present.

As shown in Fig. 6.13, the mode  ${}_0S_1$  is a translational displacement of the Earth as a whole. Here we provide the proof. The displacement of this mode is, according to (6.183),

$$\vec{u} = \sum_{k=-1}^1 C_1^k \left( P_1^k \hat{e}_r + \frac{d}{d\theta} P_1^k \hat{e}_\theta + \frac{ik}{\sin \theta} P_1^k \hat{e}_\lambda \right) e^{ik\lambda}. \quad (6.205)$$

By definition, the constant  $C_1^k$  is a complex number, while  $\vec{u}$  is a real function. With  $P_1^0(\cos \theta) = \cos \theta$ ,  $P_1^1(\cos \theta) = \sin \theta$ ,  $P_1^{-1}(\cos \theta) = -(1/2) \sin \theta$  and  $e^{ik\lambda} = \cos k\lambda + i \sin k\lambda$ , we can further write the above formula as

$$\begin{aligned} \vec{u} = & \left\{ C_1^0 \cos \theta + \left[ \left( C_1^1 - \frac{C_1^{-1}}{2} \right) \cos \lambda + i \left( C_1^1 + \frac{C_1^{-1}}{2} \right) \sin \lambda \right] \sin \theta \right\} \hat{e}_r \\ & + \left\{ -C_1^0 \sin \theta + \left[ \left( C_1^1 - \frac{C_1^{-1}}{2} \right) \cos \lambda + i \left( C_1^1 + \frac{C_1^{-1}}{2} \right) \sin \lambda \right] \cos \theta \right\} \hat{e}_\theta \\ & + \left[ i \left( C_1^1 + \frac{C_1^{-1}}{2} \right) \cos \lambda - \left( C_1^1 - \frac{C_1^{-1}}{2} \right) \sin \lambda \right] \hat{e}_\lambda. \end{aligned} \quad (6.206)$$

In the rectangular coordinate system  $Ox_1x_2x_3$ , the expression of this vector is, according to (2.448),

$$\vec{u} = \left( C_1^1 - \frac{C_1^{-1}}{2} \right) \hat{e}_1 + i \left( C_1^1 + \frac{C_1^{-1}}{2} \right) \hat{e}_2 + C_1^0 \hat{e}_3. \quad (6.207)$$

This is a displacement of components  $(C_1^1 - C_1^{-1}/2)$ ,  $i(C_1^1 + C_1^{-1}/2)$  and  $C_1^0$  along the  $x_1$ -,  $x_2$ - and  $x_3$ -axes, respectively. For the displacement to be a real value, these components must be real values. The readers are suggested to compare this requirement with (4.264) while keeping in mind the definition of  $y_1^1$  and  $y_3^1$  in (4.480) and (4.482), respectively.

#### 6.4.3.2 The Mode ${}_0T_1$

For the degree 1 toroidal free oscillation modes (6.196)–(6.199), it can be verified that

$$\sigma = 0, \quad (6.208)$$

$$\{y_7^1, y_8^1\} = \{C_1^k r, 0\}, \quad (6.209)$$

where  $C_1^k$  is an arbitrary constant, is a solution. The meanings of the subscript 1 and superscript  $k$  are the same as for the case of  ${}_0S_1$ . This mode is the degree 1

fundamental toroidal free oscillation denoted as  ${}_0T_1$ . Again, the vanishing of the frequency, i.e.,  $\sigma = 0$ , implies that the period is infinitely long, meaning that the displacement of this mode is time-independent, as given by  $y_7^1 = C_1^k r$ . The stress of this mode vanishes as given by  $y_8^1 = 0$ , meaning that no deformation is present.

As shown in Fig. 6.13, the mode  ${}_0T_1$  is a rotation of the Earth as a whole. Here we provide the proof. The displacement of this mode is, according to (6.183),

$$\vec{u} = \sum_{k=-1}^1 C_1^k r \left( \frac{k}{\sin \theta} P_1^k \hat{e}_\theta + i \frac{d}{d\theta} P_1^k \hat{e}_\lambda \right) e^{ik\lambda}. \quad (6.210)$$

Similar to the case of  ${}_0S_1$ , this formula can be developed to

$$\begin{aligned} \vec{u} = & r \left[ \left( C_1^1 + \frac{C_1^{-1}}{2} \right) \cos \lambda + i \left( C_1^1 - \frac{C_1^{-1}}{2} \right) \sin \lambda \right] \hat{e}_\theta \\ & + r \left\{ -i C_1^0 \sin \theta + \left[ i \left( C_1^1 - \frac{C_1^{-1}}{2} \right) \cos \lambda - \left( C_1^1 + \frac{C_1^{-1}}{2} \right) \sin \lambda \right] \cos \theta \right\} \hat{e}_\lambda. \end{aligned} \quad (6.211)$$

For a vector

$$\vec{\alpha} = \alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2 + \alpha_3 \hat{e}_3, \quad (6.212)$$

we have, according to (2.448),

$$\begin{aligned} \vec{r} \times \vec{\alpha} &= r \hat{e}_r \times \vec{\alpha} \\ &= r \alpha_1 (\cos \theta \cos \lambda \hat{e}_\lambda + \sin \lambda \hat{e}_\theta) \\ &\quad + r \alpha_2 (\cos \theta \sin \lambda \hat{e}_\lambda - \cos \lambda \hat{e}_\theta) + r \alpha_3 (-\sin \theta \hat{e}_\lambda) \\ &= r (-\alpha_2 \cos \lambda + \alpha_1 \sin \lambda) \hat{e}_\theta \\ &\quad + r [-\alpha_3 \sin \theta + (\alpha_1 \cos \lambda + \alpha_2 \sin \lambda) \cos \theta] \hat{e}_\lambda. \end{aligned} \quad (6.213)$$

By comparing this formula with (6.211), we express  $\vec{u}$  as

$$\vec{u} = \vec{r} \times \vec{\alpha}, \quad (6.214)$$

with

$$\alpha_1 = i \left( C_1^1 - \frac{C_1^{-1}}{2} \right), \quad \alpha_2 = - \left( C_1^1 + \frac{C_1^{-1}}{2} \right), \quad \alpha_3 = i C_1^0. \quad (6.215)$$

Similar to the relation between the velocity  $\vec{v}$  and the angular velocity  $\vec{\beta}$  of a rigid rotation, i.e.,  $\vec{v} = \vec{r} \times \vec{\beta}$ , we can interpret  $\vec{\alpha}$  as an infinitesimal angle of rigid

rotation of the Earth, i.e., the displacement  $\vec{u}$  is the result of the Earth's rotating an angle  $\vec{\alpha}$  as a whole. Again, the components of  $\vec{\alpha}$  defined in (6.215) must have a real value, so that  $\vec{u}$  has a real values. The readers are suggested to compare this requirement with (4.264) while keeping in mind the definition of  $y_7^1$  in (4.486).

If we take the derivatives of all the differential equations and boundary conditions (6.174)–(6.182) with respect to  $t$ , and solve for the velocity  $\vec{v}$  of this mode, the solution of the velocity would be  $\vec{v} = \vec{r} \times \vec{\beta}$  with  $\vec{\beta}$  in the same form as that of  $\vec{\alpha}$  given by (6.214) and (6.215). This is a rigid rotation with the constant angular velocity  $\vec{\beta}$ .

### 6.4.3.3 Computation of Spheroidal and Toroidal Free Oscillation Modes

This is to solve two eigenvalue problems: one is (6.192)–(6.195) for the spheroidal free oscillations, and the other is (6.196)–(6.199) for the toroidal free oscillations. The general procedure of computation has already been presented in the previous subsection.

Some extra care should be taken for the degree 0 spheroidal free oscillation. By the definition given by (6.183)–(6.186), the solution of the degree zero mode  $_mS_0$  is independent of  $\theta$  and  $\lambda$ . We therefore can drop  $y_3^0$  and  $y_4^0$ . The system of ordinary differential equation (6.192) is simplified to, with  $n = 0$ ,

$$\frac{dy_1^0}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r}y_1^0 + \frac{1}{\lambda + 2\mu}y_2^0, \quad (6.216a)$$

$$\frac{dy_2^0}{dr} = \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^0 - \frac{4\mu}{(\lambda + 2\mu)r}y_2^0 - \rho y_6^0, \quad (6.216b)$$

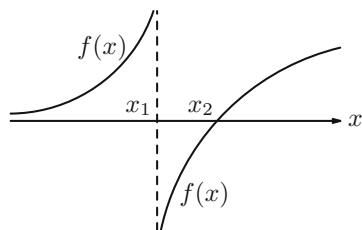
$$\frac{dy_5^0}{dr} = 4\pi G\rho y_1^0 + y_6^0, \quad (6.216c)$$

$$\frac{dy_6^0}{dr} = -\frac{2}{r}y_6^0. \quad (6.216d)$$

The equation of  $y_6^0$  is independent of the others, whose general solution is  $y_6^0 = A/r^2$ , where  $A$  is an arbitrary constant. Noting that (6.216d) has the regularity boundary condition  $y_6^0(0) = 0$  (the value at  $r = 0$ ), we must set  $A = 0$  in the inner core. The solution is then  $y_6^0 = 0$  throughout the Earth's interior with the boundary conditions (6.193) and (6.194) taken into account. We can thus set  $y_6^0 = 0$  in the above system of equation, reducing the number of equations to three. With the solution of the above system of ordinary differential equations, we have  $\vec{u} = y_1^0 e^{i\sigma t} \hat{e}_r$ ,  $\hat{e}_r \cdot \vec{T} = y_2^0 e^{i\sigma t} \hat{e}_r$ , and  $\phi_1 = y_5^0 e^{i\sigma t}$ . We also obtain  $y_5^0(R) = 0$  (the value at  $r = R$ ) from the boundary condition (6.195). The readers are suggested to infer its physical significance.

The eigenvalues are sought using the bisection method. Suppose we are solving the equation  $f(x) = 0$  in the domain  $x \in [a, b]$ . We cut the domain into small

**Fig. 6.14** A particular case of change of sign of the function  $f(x)$  to be handled in the bisection method



subdomains  $[a, a_1], [a_1, a_2], \dots, [a_i, a_{i+1}], \dots$ . For the subdomain  $[a_i, a_{i+1}]$ , if  $f(a_i)$  and  $f(a_{i+1})$  have different signs, we assume that a solution is present in it, and find the solution using the bisection method; the solution is the value of  $x$ -say,  $x_i$ -where  $f(x)$  changes sign. If  $f(a_i)$  and  $f(a_{i+1})$  have the same sign, we assume that no solution is present in the subdomain, and proceed to the next subdomain. Evidently, if the subdomain is too wide that two solutions are present in it, we will miss them. Hence, there is no guaranty that all solutions are found in the procedure. For free oscillations, Woodhouse (1988) developed an algorithm to ensure that all solutions in a subdomain are found. In our computations, we have found situations like that shown in Fig. 6.14, where  $x_1$  and  $x_2$  are very close,  $f(x) \rightarrow \pm\infty$  as  $x \rightarrow x_1$  and  $f(x) \rightarrow 0$  as  $x \rightarrow x_2$ . Therefore, before concluding  $x_i$  is indeed a solution, we need to check whether  $f(x) \rightarrow 0$  as  $x \rightarrow x_i$  even if  $f(x)$  changes sign there. Further study for the physics behind this phenomenon is beyond the scope of this book.

The ordinary differential equations are solved using the Chebyshev collocation method presented in Sect. 4.9.5. We circumvent the singularity of the equations at the Earth center by adopting the Gauss-Radau collocation points in the inner core with boundary conditions prescribed at the inner core boundary (ICB).<sup>1</sup> We then use the Gauss-Lobatto collocation points in the other layers for the solutions to be transferred from layer to layer using the boundary conditions. Note that the degree 0 should be handled particularly as given by (6.216).

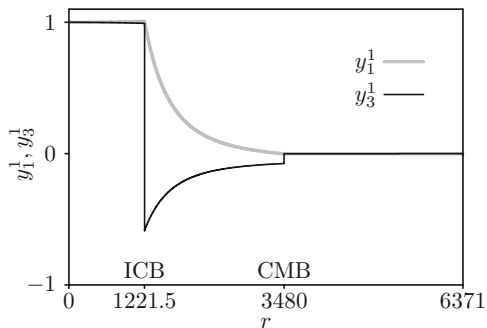
For the spheroidal free oscillations, we first solve for  $y_2^n$  on the Earth's surface, i.e., at  $r = R$ , by linearly combining the solutions propagated from the inner core to the surface by satisfying the other boundary conditions on the surface. If the top layer is solid, three solutions are to be combined to satisfy two boundary conditions. If the top layer is fluid, two boundary conditions are to be combined to satisfy one boundary condition. As  $y_2^n = 0$  on the surface, we then solve the equation  $y_2^n(\sigma) = 0$  for  $\sigma$  using the bisection method. For the toroidal free oscillations,  $y_8^n = 0$  on the surface of the outermost solid layer. Therefore, we solve the equation  $y_8^n(\sigma) = 0$  for  $\sigma$  using the bisection method. We list in Table 6.1 some results of the free oscillation periods computed with  $G = 6.67 \times 10^{-11} \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$  for the PREM

<sup>1</sup> The Gauss-Lobatto collocation points can also be used in the inner core. In this case, the algebraic equations of the singular equations at the Earth center are replaced by the regularity boundary conditions (Guo & Zhang, 1996; Bos & Scherneck, 2012).

**Table 6.1** Periods of some free oscillation modes

| $n$ | ${}_0S_n$ | ${}_1S_n$ | ${}_2S_n$ | ${}_3S_n$ | ${}_0T_n$ | ${}_1T_n$ | ${}_2T_n$ | ${}_3T_n$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0   | 1228.03   | 612.99    | 398.30    | 305.65    |           |           |           |           |
| 1   | $\infty$  | 19513.77  | 2475.53   | 1059.40   | $\infty$  |           |           |           |
| 2   | 3233.58   | 1471.21   | 1065.94   | 904.04    | 2639.92   | 757.81    | 448.47    | 309.26    |
| 3   | 2134.47   | 1064.31   | 805.18    | 705.41    | 1708.01   | 695.19    | 435.94    | 304.95    |
| 4   | 1545.72   | 852.80    | 725.32    | 545.31    | 1307.91   | 631.04    | 420.40    | 299.44    |
| 5   | 1190.17   | 729.82    | 660.51    | 447.40    | 1079.12   | 571.59    | 402.60    | 292.87    |
| 6   | 963.49    | 657.00    | 595.39    | 392.29    | 928.77    | 519.63    | 383.32    | 285.42    |
| 7   | 812.14    | 604.05    | 536.59    | 372.35    | 821.07    | 475.64    | 363.31    | 277.27    |
| 8   | 707.81    | 555.78    | 488.32    | 354.77    | 739.31    | 438.86    | 343.32    | 268.59    |
| 9   | 633.98    | 509.23    | 448.96    | 338.94    | 674.62    | 408.06    | 324.02    | 259.54    |
| 10  | 579.62    | 465.45    | 416.38    | 324.34    | 621.84    | 381.98    | 305.91    | 250.26    |
| 11  | 537.34    | 425.96    | 388.96    | 310.63    | 577.76    | 359.59    | 289.32    | 240.90    |
| 12  | 502.82    | 391.36    | 365.46    | 297.65    | 540.26    | 340.06    | 274.36    | 231.61    |
| 13  | 473.67    | 361.48    | 344.94    | 285.34    | 507.87    | 322.83    | 260.99    | 222.54    |
| 14  | 448.52    | 336.02    | 326.48    | 273.69    | 479.53    | 307.45    | 249.07    | 213.81    |
| 15  | 426.55    | 315.38    | 308.59    | 262.70    | 454.49    | 293.63    | 238.40    | 205.57    |
| 16  | 407.14    | 299.49    | 290.42    | 252.37    | 432.15    | 281.12    | 228.80    | 197.88    |
| 17  | 389.86    | 286.16    | 273.42    | 242.68    | 412.07    | 269.74    | 220.11    | 190.79    |
| 18  | 374.36    | 274.28    | 258.11    | 233.61    | 393.89    | 259.35    | 212.18    | 184.29    |
| 19  | 360.37    | 263.50    | 225.14    | 159.34    | 377.35    | 249.82    | 204.90    | 178.34    |
| 20  | 347.66    | 253.60    | 217.22    | 153.70    | 362.21    | 241.05    | 198.19    | 172.88    |
| 21  | 336.05    | 244.48    | 209.82    | 148.56    | 348.30    | 232.96    | 191.98    | 167.85    |
| 22  | 325.37    | 236.03    | 202.90    | 143.90    | 335.45    | 225.47    | 186.21    | 163.17    |
| 23  | 315.50    | 228.18    | 196.42    | 138.03    | 323.56    | 218.52    | 180.82    | 158.81    |
| 24  | 306.34    | 220.86    | 192.97    | 134.69    | 312.50    | 212.06    | 175.78    | 154.72    |
| 25  | 297.80    | 214.02    | 185.29    | 131.61    | 302.19    | 206.03    | 171.05    | 150.87    |
| 26  | 289.79    | 207.62    | 179.38    | 128.72    | 292.56    | 200.40    | 166.60    | 147.22    |
| 27  | 282.27    | 201.62    | 174.32    | 125.99    | 283.53    | 195.13    | 162.40    | 143.76    |
| 28  | 275.17    | 195.98    | 169.56    | 123.39    | 275.06    | 190.18    | 158.44    | 140.48    |
| 29  | 268.46    | 190.68    | 165.08    | 121.99    | 267.08    | 185.52    | 154.69    | 137.35    |
| 30  | 262.10    | 185.68    | 160.84    | 116.32    | 259.56    | 181.14    | 151.13    | 134.37    |
| 31  | 256.04    | 180.96    | 156.84    | 113.72    | 252.45    | 176.99    | 147.76    | 131.52    |
| 32  | 250.28    | 176.51    | 153.05    | 111.24    | 245.73    | 173.08    | 144.54    | 128.80    |
| 33  | 244.77    | 172.30    | 149.45    | 108.87    | 239.35    | 169.36    | 141.48    | 126.20    |
| 34  | 239.51    | 168.31    | 135.97    | 103.51    | 233.30    | 165.84    | 138.56    | 123.71    |
| 35  | 234.47    | 164.53    | 132.08    | 101.61    | 227.56    | 162.48    | 135.78    | 121.33    |
| 36  | 229.64    | 160.95    | 128.41    | 99.79     | 222.08    | 159.29    | 133.11    | 119.05    |
| 37  | 225.00    | 157.54    | 124.93    | 98.04     | 216.87    | 156.24    | 130.56    | 116.86    |
| 38  | 220.55    | 154.31    | 121.65    | 96.37     | 211.90    | 153.32    | 128.13    | 114.75    |
| 39  | 216.26    | 151.22    | 118.53    | 94.77     | 207.15    | 150.53    | 125.79    | 112.74    |
| 40  | 212.13    | 148.29    | 115.57    | 93.23     | 202.61    | 147.86    | 123.55    | 110.80    |
| 41  | 208.15    | 145.49    | 113.50    | 93.17     | 198.26    | 145.30    | 121.41    | 108.93    |
| 42  | 204.31    | 142.81    | 111.53    | 90.32     | 194.10    | 142.83    | 119.35    | 107.14    |
| 43  | 200.61    | 140.26    | 109.63    | 88.94     | 190.11    | 140.46    | 117.37    | 105.41    |
| 44  | 197.03    | 137.81    | 107.81    | 87.61     | 186.28    | 138.18    | 115.47    | 103.74    |
| 45  | 193.58    | 135.47    | 106.05    | 86.33     | 182.60    | 135.98    | 113.64    | 102.14    |

**Fig. 6.15** The eigenfunctions  $y_1^1$  and  $y_3^1$  of the spheroidal free oscillation mode  ${}_1S_1$



(see Sect. 6.8.6) as Dziewonski and Anderson (1981) did. More comprehensive list can be found in, e.g., Dziewonski and Anderson (1981) and Kennett et al. (1995). Graphical representations of the free oscillation frequencies can be found in Dahlen and Tromp (1998), where figures of the eigenfunctions of many modes can be found.

The splitting by rotation will be the topic of Sect. 6.6.1. Dahlen and Sailor (1979) provided a comprehensive list of splitting due to rotation and ellipticity of the Earth.

#### 6.4.3.4 The Slichter Modes

The mode  ${}_1S_1$  with a period of about 5.42 hours is called Slichter mode. According to (4.571), we have the regularity boundary condition  $y_1^1(0) = y_3^1(0)$ . This condition must be satisfied even if it is not explicitly used. It is automatically satisfied in our approach of using the Chebyshev collocation method based on the Gauss-Radau points in the inner core that exclude the Earth center. We draw in Fig. 6.15 its eigenfunctions  $y_1^1$  and  $y_3^1$  scaled to  $y_1^1 = y_3^1 = 1$  at  $r = 0$ , which are computed with the period precomputed with an accuracy of up to  $1.0 \times 10^{-8}$  s. We see that  $y_1^1$  and  $y_3^1$  are almost equal and constant within the inner core, and they are of much smaller order of magnitude in the mantle and crust. Taking reference to the discussion on the mode  ${}_0S_1$ , which has  $y_1^1 = y_3^1$  and represents a translational displacement of the Earth as a whole, we see that the mode  ${}_1S_1$  represents a translational free oscillation of the inner core almost as a whole with respect to the mantle and crust. Since its period is several hours long, the effect of the Earth rotation on it is significant. This means that the periods for  $k = -1$ ,  $k = 0$  and  $k = 1$  are significantly different. In the literature, the plural *Slichter modes* is used to indicate that there are three distinct modes of different  $k$ ,  ${}_1S_1^k$ ,  $k = -1, 0, 1$ .

Here we take the Slichter modes as an example to show how to solve the coupled system of differential equations (6.187). For a spheroidal free oscillation mode  $S_n^k$ , the fundamental equations are (6.187a)–(6.187f) for  $y_1^n, \dots, y_6^n$ . We see that  $y_7^{n-1}$  and  $y_7^{n+1}$ , which belong to the modes  $T_{n-1}^k$  and  $T_{n+1}^k$ , respectively, are present in these equations. When we inspect (6.187g) and (6.187h) after setting  $n$  to  $n-1$ , we see that  $T_{n-1}^k$  is coupled to  $S_{n-2}^k$  through  $y_1^{n-2}$  and  $y_3^{n-2}$ . When we inspect (6.187g) and (6.187h) again after setting  $n$  to  $n+1$ , we see that  $T_{n+1}^k$  is coupled to  $S_{n+2}^k$

through  $y_1^{n+2}$  and  $y_3^{n+2}$ . This infinite chain of coupling has already been mentioned in (6.201). As no  $T_0^k$  mode is present ( $y_7^0$  and  $y_8^0$  are meaningless), the Slichter mode  ${}_1S_1^k$  couples only to  $T_2^k$  (the coefficient of  $y_7^0$  vanishes in (6.187b) and (6.187d) when  $n = 1$ ), which in turn couples to  $S_2^k$  and so forth. Here we truncate the chain beyond  $T_2^k$  for demonstrative purpose. Base on (6.187), we first write the equations for  $y_1^1, \dots, y_6^1$ , and then for  $y_7^2, \dots, y_8^2$ . The system of equations we obtain is

$$\frac{dy_1^1}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r}y_1^1 + \frac{1}{\lambda + 2\mu}y_2^1 + \frac{2\lambda}{(\lambda + 2\mu)r}y_3^1, \quad (6.217a)$$

$$\begin{aligned} \frac{dy_2^1}{dr} = & \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^1 - \frac{4\mu}{(\lambda + 2\mu)r}y_2^1 \\ & + \left[ \frac{2\rho g}{r} - \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} + 2k\rho\omega\sigma \right] y_3^1 + \frac{2}{r}y_4^1 \\ & - \rho y_6^1 - \frac{6(2+k)\rho\omega\sigma}{5}y_7^2, \end{aligned} \quad (6.217b)$$

$$\frac{dy_3^1}{dr} = -\frac{1}{r}y_1^1 + \frac{1}{r}y_3^1 + \frac{1}{\mu}y_4^1, \quad (6.217c)$$

$$\begin{aligned} \frac{dy_4^1}{dr} = & \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} + k\rho\omega\sigma \right] y_1^1 - \frac{\lambda}{(\lambda + 2\mu)r}y_2^1 \\ & + \left[ -\rho\sigma^2 + \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} + k\rho\omega\sigma \right] y_3^1 - \frac{3}{r}y_4^1 \\ & - \frac{\rho}{r}y_5^1 + \frac{3(2+k)\rho\omega\sigma}{5}y_7^2, \end{aligned} \quad (6.217d)$$

$$\frac{dy_5^1}{dr} = 4\pi G\rho y_1^1 + y_6^1, \quad (6.217e)$$

$$\frac{dy_6^1}{dr} = -\frac{8\pi G\rho}{r}y_3^1 + \frac{2}{r^2}y_5^1 - \frac{2}{r}y_6^1, \quad (6.217f)$$

$$\frac{dy_7^2}{dr} = \frac{1}{r}y_7^2 + \frac{1}{\mu}y_8^2, \quad (6.217g)$$

$$\begin{aligned} \frac{dy_8^2}{dr} = & -\frac{(2-k)\rho\omega\sigma}{5}y_1^1 + \frac{(2-k)\rho\omega\sigma}{5}y_3^1 \\ & + \left( -\rho\sigma^2 + \frac{4\mu}{r^2} + \frac{k\rho\omega\sigma}{3} \right) y_7^2 - \frac{3}{r}y_8^2. \end{aligned} \quad (6.217h)$$

Note that the  $y_1^3$  and  $y_3^3$  terms in (6.217h) are dropped due to the truncation. In the fluid outer core, we have  $\mu = 0$  and  $y_8^2 = 0$ . The boundary conditions are given by (6.188)–(6.191).

The procedure of numerical solution of (6.217) is similar to that of the spheroidal free oscillations. We just summarize the main differences here. In the solid inner core we have eight functions and four independent solutions satisfying the regulatory boundary conditions at the Earth center. In fluid layers, we have two independent solutions with  $y_4^1 = y_8^2 = 0$  and (6.217d) and (6.217h) becoming algebraic equations. Therefore,  $y_8^2$  is treated like  $y_4^1$ , and  $y_7^2$  like  $y_3^1$ , in the transition of boundary conditions between solid and fluid layers. The complication is that, at solid-to-fluid boundaries, the boundary conditions of  $y_3^1$  and  $y_7^2$  at the fluid side should be computed by solving (6.217d) and (6.217h) together from the two independent solutions with  $y_4^1 = y_8^2 = 0$  at the solid side obtained by linearly combining the four independent solutions. Since both  $y_3^1$  and  $y_7^2$  are discontinuous across the boundaries between solid and fluid layers, two more independent boundary conditions can be assigned at the solid side of fluid-to-solid boundaries, which increase the number of independent solutions from two in the fluid layer to four in the solid layer. The result of  $\sigma$  is different for different values of  $k$ . We thus replace  $\sigma$  by  $\sigma_k$ , and use  $T_k = 2\pi/\sigma_k$  to denote the period. The numerical result of the periods for the PREM (see Sect. 6.8.6) is (in hours)

$$T_{-1} = 6.02 \quad T_0 = 5.36, \quad T_{+1} = 4.79. \quad (6.218)$$

We now study the how the inner core oscillates corresponding to  $k = -1, 0$ , and 1. Since the angular frequency  $\sigma$  is different for different  $k$ , we can no longer follow the above discussions on the modes  ${}_0S_1$  and  ${}_0T_1$  where the effect of rotation is ignored. The cases for different values of  $k$  must be studied separately. We make the approximation  $y_1^1 = y_3^1$  for each value of  $k$ , and denote them as  $\tilde{y}_k$ . Based on the definition of the displacement given by (6.183), the displacement corresponding to  $k = -1, 0$ , and 1 are (neglecting the minor contribution from  $y_7^2$  and  $y_8^2$ )

$$\begin{cases} \vec{u}_{-1} = \frac{1}{2} \tilde{y}_{-1} (-\sin \theta \hat{e}_r - \cos \theta \hat{e}_\theta + i \hat{e}_\lambda) e^{i(\sigma_{-1}t - \lambda)}, \\ \vec{u}_0 = \tilde{y}_0 (\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta) e^{i\sigma_0 t}, \\ \vec{u}_{+1} = \tilde{y}_{+1} (\sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta + i \hat{e}_\lambda) e^{i(\sigma_{+1}t + \lambda)}, \end{cases} \quad (6.219)$$

respectively, where  $P_1^0 = \cos \theta$ ,  $P_1^1 = \sin \theta$  and  $P_1^{-1}(\cos \theta) = -(1/2) \sin \theta$  are substituted. The case  $k = 0$  is simplest. As  $\hat{e}_3 = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta$  according to (2.448), we have

$$\vec{u}_0 = \tilde{y}_0 e^{i\sigma_0 t} \hat{e}_3. \quad (6.220)$$

This is a translational oscillation along the rotation axis, called axial mode. The cases  $k = -1$  and  $k = 1$  are similar. We choose to discuss the case  $k = 1$ . According to (2.335), we can express  $\vec{u}_{+1}$  as

$$\vec{u}_{+1} = \tilde{y}_{+1} [(\cos \lambda \hat{e}_1 + \sin \lambda \hat{e}_2) + i(-\sin \lambda \hat{e}_1 + \cos \lambda \hat{e}_2)] e^{i(\sigma_{+1}t + \lambda)}. \quad (6.221)$$



Both the real and imaginary part is a solution. According to Euler's formula  $e^{ikx} = \cos kx + i \sin kx$ , the real part is

$$\begin{aligned}
 \vec{u}_{+1}^r &= \tilde{y}_{+1} [(\cos \lambda \hat{e}_1 + \sin \lambda \hat{e}_2) \cos(\sigma_{+1}t + \lambda) \\
 &\quad - (-\sin \lambda \hat{e}_1 + \cos \lambda \hat{e}_2) \sin(\sigma_{+1}t + \lambda)] \\
 &= \tilde{y}_{+1} \{ [\cos \lambda \cos(\sigma_{+1}t + \lambda) + \sin \lambda \sin(\sigma_{+1}t + \lambda)] \hat{e}_1 \\
 &\quad + [\sin \lambda \cos(\sigma_{+1}t + \lambda) - \cos \lambda \sin(\sigma_{+1}t + \lambda)] \hat{e}_2 \} \\
 &= \tilde{y}_{+1} [\cos(-\sigma_{+1}t) \hat{e}_1 + \sin(-\sigma_{+1}t) \hat{e}_2]. \tag{6.222}
 \end{aligned}$$

This is a circular motion in (or parallel to) the equatorial plane in the retrograde direction, i.e., the motion is toward the west, thus being referred to as the retrograde mode. For the imaginary part, we can similarly obtain

$$\vec{u}_{+1}^i = \tilde{y}_{+1} \left\{ \cos \left[ - \left( \sigma_{+1}t - \frac{\pi}{2} \right) \right] \hat{e}_1 + \sin \left[ - \left( \sigma_{+1}t - \frac{\pi}{2} \right) \right] \hat{e}_2 \right\}. \tag{6.223}$$

It behaves in the same way as the real part except for the phase difference of  $\pi/2$ . The discussion of the mode  $k = -1$  is left to the readers. It is a circular motion in (or parallel to) the equatorial plane in the prograde direction, i.e., the motion is toward the east, thus being referred to as the prograde mode. Any linear combination of all the above solutions is a solution, since the differential equations and the boundary conditions are all linear and homogeneous. Therefore, the three modes can coexist.

As already mentioned,  $y_1^1$  and  $y_3^1$  are almost equal and constant within the inner core, and they are of much smaller order of magnitude in the mantle and crust. Therefore, the largest force restoring the inner core to its equilibrium position is the gravitational attraction of the rest of the Earth to it. An immediate inference is then that the periods of the Slichter modes depend strongly on the difference between the mean density of the inner core and that of the fluid outer core surrounding the inner core, which is critical to the density jump across the inner core boundary (ICB) in Earth models like PREM. Hence, the detection of the Slichter modes would greatly improve our knowledge on the density of the inner core. During the last forty years or so, the community of the worldwide superconducting gravimeter network has dedicated a lot of efforts to their detection. However, they remain elusive up to present; although there were declarations of their detection, but none was solidly confirmed by the community.

Our result in (6.218) is primitive, as we have coupled only up to  $T_2^k$ . The splitting of the Slichter modes is included in Dahlen and Sailor (1979) with much higher accuracy. The Slichter modes are also closely related to the dynamics of the fluid outer core. The readers are referred to Rochester and Peng (1993) and Smith (1976) for a comprehensive theoretical discussion, including the splitting.

## 6.5 A Perturbation Theory of Free Oscillations: General Formulation

### 6.5.1 A Weak Form of the Elastic Equation

A very important application of free oscillation data is the determination of the physical parameters within the Earth, where the main approach is to use the data to improve an existing model. Hence, from now on, we formulate the influences of small changes of the physical parameters to the free oscillation periods. For this purpose, we first express the equations of elastic deformation of the Earth in an integral form.

The starting equations are still (4.435)–(4.437). In the study of free oscillations, the functions  $\vec{u}$  and  $\phi_1$  depend on time in the form of  $e^{i\sigma t}$ , and (4.443) holds. Hence,

$$\frac{\partial \vec{u}}{\partial t} = i\sigma \vec{u}, \quad \frac{\partial^2 \vec{u}}{\partial t^2} = -\sigma^2 \vec{u}. \quad (6.224)$$

Furthermore, no external force is present in free oscillations. The Eqs. (4.435)–(4.437) are then reduced to

$$\begin{aligned} -\rho\sigma^2 \vec{u} + 2i\rho\sigma \vec{\omega} \times \vec{u} &= \nabla \cdot \bar{T} - \rho(\nabla \cdot \vec{u})\nabla W \\ &+ \rho\nabla \vec{u} \cdot \nabla W + \rho\vec{u} \cdot \nabla \nabla W + \rho\nabla \phi_1, \end{aligned} \quad (6.225)$$

$$\bar{T} = \bar{C} : \nabla \vec{u}, \quad (6.226)$$

$$\nabla^2 \phi_1 = 4\pi G \nabla \cdot (\rho \vec{u}). \quad (6.227)$$

We have expressed the elastic constitutive equation (4.436) in the general form (6.226), which can be verified using the components,

$$T_{ij} = C_{ijkl}\epsilon_{kl} = \frac{1}{2}C_{ijkl}\left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l}\right) = C_{ijkl}\frac{\partial u_l}{\partial x_k}, \quad (6.228)$$

where the symmetry of  $C_{ijkl}$  with respect to  $k$  and  $l$  has been used to derive the last identity. This formula is more convenient in the derivation, and can be applied to more general cases like non-isotropic Earth models. Besides, we assume that the elastic parameters  $\bar{C}$  are complex numbers, so that  $\sigma$  is also a complex number, and the formulation can be applied to non-purely elastic Earth models. The boundary conditions are also rewritten here for convenience of reference. The continuity of displacement across welded (solid-solid) boundaries is

$$\{\vec{u}\}_{\pm}^+ = 0. \quad (6.229)$$

The continuity of displacement across frictionless slip (solid-fluid) boundaries is

$$\{\hat{n} \cdot \vec{u}\}_{-}^{+} = 0. \quad (6.230)$$

The continuity of traction across both kinds of boundaries is

$$\{\hat{n} \cdot \vec{T}\}_{-}^{+} = 0. \quad (6.231)$$

The traction on frictionless slip (solid-fluid) boundaries also satisfies the condition

$$\hat{n} \cdot \vec{T} = [\hat{n} \cdot (\hat{n} \cdot \vec{T})]\hat{n}. \quad (6.232)$$

On the Earth's surface, no condition for displacement is present, and the condition of traction is

$$\{\hat{n} \cdot \vec{T}\}_{S_0} = 0, \quad (6.233)$$

where the symbol  $S_0$  is understood as the Earth's surface. The conditions of gravitational potential across all boundaries are

$$\{\phi_1\}_{-}^{+} = 0, \quad (6.234)$$

$$\left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right\}_{-}^{+} = 0. \quad (6.235)$$

The free oscillation problem is to find the particular values of  $\sigma$  for the differential equations (6.225)–(6.227) to have nontrivial solutions satisfying the boundary conditions (6.229)–(6.235), which has already been discussed in the previous section.

The problem now is to express the equations of elastic deformation in the form of an integral equation. Take the complex conjugate at both sides of 6.225, perform a scalar product with  $\vec{u}$ , and then integrate over whole the Earth's interior. We obtain

$$\begin{aligned} & -(\sigma^2)^* \int_V \rho \vec{u} \cdot \vec{u}^* dV + 2\sigma^* \int_V \rho \vec{u} \cdot (i\vec{\omega} \times \vec{u})^* dV \\ & = \int_V \vec{u} \cdot (\nabla \vec{T})^* dV - \int_V \rho \vec{u} \cdot [(\nabla \cdot \vec{u})\nabla W]^* dV + \int_V \rho \vec{u} \cdot (\nabla \vec{u} \cdot \nabla W)^* dV \\ & \quad + \int_V \rho \vec{u} \cdot (\vec{u} \cdot \nabla \nabla W)^* dV + \int_V \rho \vec{u} \cdot (\nabla \phi_1)^* dV, \end{aligned} \quad (6.236)$$

where '\*' is used to denote the complex conjugate. The next is to further develop this equation, where Gauss' theorem will be repeatedly used.

We use  $S_0$  to denote the Earth's surface,  $S_i$ , all internal boundaries of discontinuity,  $S_w$ , all welded, i.e., solid-solid, boundaries, and  $S_s$ , all frictionless slip, i.e., solid-fluid, boundaries.

We use  $I_i$  to denote the  $i$ th integral at the right-hand side of (6.236), which are further developed as follows.

The first integral at the right-hand side of (6.236) can be developed as

$$I_1 = \int_V u_j \frac{\partial T_{ij}^*}{\partial x_i} dV = \int_V \frac{\partial}{\partial x_i} (u_j T_{ij}^*) dV - \int_V \frac{\partial u_j}{\partial x_i} T_{ij}^* dV. \quad (6.237)$$

According to Gauss' theorem, the first term can be developed to

$$\begin{aligned} \int_V \frac{\partial}{\partial x_i} (u_j T_{ij}^*) dV &= \int_{S_0} n_i u_j T_{ij}^* dS + \int_{S_w} \{n_i u_j T_{ij}^*\}_-^+ dS + \int_{S_s} \{n_i u_j T_{ij}^*\}_-^+ dS \\ &= \int_{S_0} (\hat{n} \cdot \bar{T}^*) \cdot \vec{u} dS + \int_{S_w} \{(\hat{n} \cdot \bar{T}^*) \cdot \vec{u}\}_-^+ dS + \int_{S_s} \{(\hat{n} \cdot \bar{T}^*) \cdot \vec{u}\}_-^+ dS. \end{aligned} \quad (6.238)$$

It can be inferred that the first term vanishes according to the boundary condition (6.233), the second term vanishes according to the boundary conditions (6.229) and (6.231), and the last term vanishes according to the boundary conditions (6.232), (6.230), and (6.231). This implies that the first term at the right-hand side of (6.237) vanishes, and with (6.226), we have

$$I_1 = - \int_V C_{ijkl}^* \frac{\partial u_j}{\partial x_i} \frac{\partial u_l^*}{\partial x_k} dV = - \int_V \nabla \vec{u} : \bar{C}^* : \nabla \vec{u}^* dV. \quad (6.239)$$

This formula can also be written in the alternative form

$$I_1 = - \int_V \bar{\epsilon} : \bar{C}^* : \bar{\epsilon}^* dV = - \int_V \bar{T}^* : \bar{\epsilon} = - \int_V \bar{T}^* : \nabla \vec{u} dV, \quad (6.240)$$

whose proof is left to the readers as an exercise.

The sum of the second and third integrals at the right-hand side of (6.236) can be developed as, by swapping their orders,

$$I_2 + I_3 = \int_V \rho \nabla W \cdot (\vec{u} \cdot \nabla \vec{u}^* - \vec{u} \nabla \cdot \vec{u}^*) dV. \quad (6.241)$$

Application of the relation between pressure and gravity potential in hydrostatic equilibrium,  $\nabla p = \rho \nabla W$ , leads to

$$\begin{aligned} I_2 + I_3 &= \int_V \nabla p \cdot (\vec{u} \cdot \nabla \vec{u}^* - \vec{u} \nabla \cdot \vec{u}^*) dV \\ &= \int_V \left( \frac{\partial p}{\partial x_j} u_i \frac{\partial u_j^*}{\partial x_i} - \frac{\partial p}{\partial x_j} u_j \frac{\partial u_i^*}{\partial x_i} \right) dV \end{aligned}$$

$$\begin{aligned}
&= \int_V \left( -p \frac{\partial u_i}{\partial x_j} \frac{\partial u_j^*}{\partial x_i} + p \frac{\partial u_j}{\partial x_j} \frac{\partial u_i^*}{\partial x_i} \right) dV \\
&\quad + \int_V \left[ \frac{\partial}{\partial x_j} (p u_i) \frac{\partial u_j^*}{\partial x_i} - \frac{\partial}{\partial x_j} (p u_j) \frac{\partial u_i^*}{\partial x_i} \right] dV \\
&= - \int_V p \left[ (\nabla \vec{u})^T : \nabla \vec{u}^* - |\nabla \cdot \vec{u}|^2 \right] dV \\
&\quad + \int_V \left[ \frac{\partial}{\partial x_j} \left( p u_i \frac{\partial u_j^*}{\partial x_i} \right) - \frac{\partial}{\partial x_j} \left( p u_j \frac{\partial u_i^*}{\partial x_i} \right) \right] dV. \quad (6.242)
\end{aligned}$$

The second term at the right-hand side can be reduced to the surface integral

$$\int_{S_0} \left( n_j p u_i \frac{\partial u_j^*}{\partial x_i} - n_j p u_j \frac{\partial u_i^*}{\partial x_i} \right) dS + \int_{S_i} \left\{ n_j p u_i \frac{\partial u_j^*}{\partial x_i} - n_j p u_j \frac{\partial u_i^*}{\partial x_i} \right\}_+^- dS. \quad (6.243)$$

Its first term vanishes as  $p = 0$  on  $S_0$ , and its second term can be written in the vector form

$$\int_{S_i} \{ p \vec{u} \cdot \nabla \vec{u}^* \cdot \hat{n} - p (\hat{n} \cdot \vec{u}) (\nabla \cdot \vec{u}^*) \}_+^- dS. \quad (6.244)$$

By expressing  $\nabla$  as the sum of a tangential and a normal components on the boundaries,

$$\nabla = \nabla_S + \hat{n}(\hat{n} \cdot \nabla), \quad (6.245)$$

we can write (6.244) as

$$\begin{aligned}
&\int_{S_i} \{ p \vec{u} \cdot \nabla_S \vec{u}^* \cdot \hat{n} - p (\hat{n} \cdot \vec{u}) (\nabla_S \cdot \vec{u}^*) \}_+^- dS \\
&\quad + \int_{S_i} \{ p \vec{u} \cdot [\hat{n}(\hat{n} \cdot \nabla) \vec{u}^*] \cdot \hat{n} - p (\hat{n} \cdot \vec{u}) [\hat{n}(\hat{n} \cdot \nabla)] \cdot \vec{u}^* \}_+^- dS, \quad (6.246)
\end{aligned}$$

where the integrand of the second term can be seen to vanish if written in the component form as follows,

$$p u_i n_i n_j \frac{\partial u_k^*}{\partial x_j} n_k - p n_i u_i n_k n_j \frac{\partial u_k}{\partial x_j} = 0. \quad (6.247)$$

Hence, in the expression of  $I_2 + I_3$ , only the first term of (6.242) and the first term of (6.246) are non-vanishing. In order for  $\vec{u}$  to be continuous across welded, i.e., solid-solid, boundaries,  $\nabla_S \vec{u}$  and  $\nabla_S \cdot \vec{u}$  must also be continuous. Hence, the integrand of the first term of (6.246) vanishes on welded, i.e., solid-solid, boundaries. In

summary, the final expression of  $I_2 + I_3$  is

$$I_2 + I_3 = - \int_V p[(\nabla \vec{u})^T : \nabla \vec{u}^* - |\nabla \cdot \vec{u}|^2] dV \\ + \int_{S_s} \{p \vec{u} \cdot \nabla_S \vec{u}^* \cdot \hat{n} - p(\hat{n} \cdot \vec{u})(\nabla_S \cdot \vec{u}^*)\}_-^+ dS. \quad (6.248)$$

The fourth and fifth integrals at the right-hand side of (6.236) are

$$I_4 = \int_V \rho u_j u_i^* \frac{\partial^2 W}{\partial x_i \partial x_j} dV = \int_V \rho \vec{u} \cdot \nabla \nabla W \cdot \vec{u}^* dV, \quad (6.249)$$

$$I_5 = \int_V \rho \vec{u} \cdot \nabla \phi_1^* dV. \quad (6.250)$$

where the fact that  $W$  is a real number has been taken into account in the first formula.

In summary, we can write (6.236) as

$$(\sigma^2)^* K - 2\sigma^* D - E - F = 0, \quad (6.251)$$

where

$$K = \int_V \rho \vec{u} \cdot \vec{u}^* dV, \quad (6.252)$$

$$D = \int_V \rho \vec{u} \cdot (i\vec{\omega} \times \vec{u})^* dV, \quad (6.253)$$

$$E = -I_1 = \int_V \nabla \vec{u} : \vec{C}^* : \nabla \vec{u}^* dV, \quad (6.254)$$

$$F = -(I_2 + I_3 + I_4 + I_5) \\ = \int_V \left\{ p[(\nabla \vec{u})^T : \nabla \vec{u}^* - |\nabla \cdot \vec{u}|^2] - \rho \vec{u} \cdot \nabla \nabla W \cdot \vec{u}^* - \rho \vec{u} \cdot \nabla \phi_1^* \right\} dV \\ - \int_{S_s} \left\{ p \vec{u} \cdot \nabla_S \vec{u}^* \cdot \hat{n} - p(\hat{n} \cdot \vec{u})(\nabla_S \cdot \vec{u}^*) \right\}_-^+ dS. \quad (6.255)$$

By adopting the original expression of  $I_2 + I_3$ , (6.241),  $F$  can also be written as

$$F = \int_V [-\rho \nabla W \cdot (\vec{u} \cdot \nabla \vec{u}^* - \vec{u} \nabla \cdot \vec{u}^*) - \rho \vec{u} \cdot \nabla \nabla W \cdot \vec{u}^* - \rho \vec{u} \cdot \nabla \phi_1^*] dV. \quad (6.256)$$

If no frictionless slip boundary is present in the interior of the Earth, the surface integral in the expression of  $F$ , (6.255), disappears.

The Eq. (6.251) is the integral equation we are seeking, called weak form of the equation of elastic deformation.<sup>2</sup>

### 6.5.2 A Perturbation Theory

We now formulate the perturbation of rotation and small changes of the density and elastic parameters to the angular frequency of the free oscillation,  $\sigma$ , based on (6.251). As we have adopted hydrostatic equilibrium as the undeformed state, the non-rotating initial model of the Earth must be spherically symmetrical, and the model including rotation and small changes of density and elastic parameters must be ellipsoidal. However, the longest free oscillation period observed using seismometers is only on the order of 54 minutes (except the Slichter modes), which is the mode  ${}_0S_2$ . This period is less than 4% of a day, i.e.,  $\omega/\sigma < 4\%$ . Hence,  $\omega$  can be considered as infinitesimally small as compared to  $\sigma$ , which is the basis why the effect of rotation can be treated as a small perturbation to free oscillations. With this consideration, we will include only first order terms of  $\omega$  in this book, and neglect the effect of centrifugal force and Earth's ellipticity, which are both on the relative order of magnitude of  $\sim 1/300$ .

All functions and parameters for the initial model are denoted with a subscript '0', including  $\rho_0, \bar{C}_0, W_0, \vec{u}_0, \phi_{10}$ , and  $\sigma_0$ . We assume that the initial model is purely elastic, and hence,  $\bar{C}_0$  is real. The increments with respect to the initial model is denoted using ' $\delta$ ', including  $\delta\rho, \delta\bar{C}, \delta W, \delta\vec{u}, \delta\phi_1$ , and  $\delta\sigma$ . In this way, the relations among the quantities of the initial and perturbed models are

$$\begin{cases} \rho = \rho_0 + \delta\rho, & \bar{C} = \bar{C}_0 + \delta\bar{C}, & W = W_0 + \delta W, \\ \vec{u} = \vec{u}_0 + \delta\vec{u}, & \phi_1 = \phi_{10} + \delta\phi_1, & \sigma = \sigma_0 + \delta\sigma. \end{cases} \quad (6.257)$$

We understand  $\bar{T}_0$  and  $\delta\bar{T}$  as derived quantities from the constitutive equation.

What we are going to formulate are the influences of  $\omega, \delta\rho, \delta\bar{C}$ , and  $\delta W$  to  $\delta\sigma$ . We consider only a simplified model of the Earth: the parameters  $\delta\rho$  and  $\delta\bar{C}$  are continuous everywhere within the Earth. In this way, jump of  $\delta\rho$  or  $\delta\bar{C}$  would not appear.

By comparing  $\rho_0, \bar{C}_0, W_0, \vec{u}_0, \bar{T}_0, \phi_{10}$ , and  $\sigma_0$  with  $\rho, \bar{C}, W, \vec{u}, \bar{T}, \phi_1$ , and  $\sigma$ , the equations for the latter set of quantities would become those of the former if the Coriolis term were ignored, and the boundary conditions of both sets are the same. This implies that the boundary conditions of  $\delta\vec{u}, \delta\bar{T}$  and  $\delta\phi_1$  are also the same, but with one exception—as the product between  $\rho$  and  $\vec{u}$  is present in the boundary

<sup>2</sup> Some authors multiply the equation of  $\phi_1$ , (6.227), by  $\phi_1^*/4\pi G$ , integrate it over whole the Earth's interior, and then add it into the Eq. (6.251). Some authors also treat the boundary conditions similarly so that they are added into (6.251) as surface integrals. Hence, different weak forms are present in the literature.

condition of  $\partial\phi_1/\partial n$ , the boundary condition of  $\partial(\delta\phi_1)/\partial n$  has to be particularly treated. These are the basis for later formulations.

By substituting (6.257) into (6.252)–(6.256), we formulate how  $K$ ,  $D$ ,  $E$  and  $F$  depend on  $\delta\rho$ ,  $\delta\bar{C}$ ,  $\delta W$ ,  $\delta\vec{u}$ ,  $\delta\phi_1$  and  $\delta\sigma$ . We denote  $K$ ,  $D$ ,  $E$  and  $F$  for the initial non-rotating Earth model as  $K_0$ ,  $D_0$ ,  $E_0$  and  $F_0$ , where  $D_0 = 0$  is evident. As the effect of ellipticity of the Earth is neglected, the domains of the integrals for both the non-rotating initial and the rotating perturbed models are identical. Furthermore, we will keep only the first order terms of the small perturbations.

The formula of  $K$  is

$$\begin{aligned} K &= \int_V (\rho_0 + \delta\rho)(\vec{u}_0 + \delta\vec{u}) \cdot (\vec{u}_0^* + \delta\vec{u}^*) dV \\ &= K_0 + \int_V \delta\rho \vec{u}_0 \cdot \vec{u}_0^* dV + \int_V \rho_0 \vec{u}_0 \cdot \delta\vec{u}^* dV + \int_V \rho_0 \delta\vec{u} \cdot \vec{u}_0^* dV. \end{aligned} \quad (6.258)$$

The formula of  $D$  is

$$\begin{aligned} D &= \int_V (\rho_0 + \delta\rho)(\vec{u}_0 + \delta\vec{u}) \cdot [i\vec{\omega} \times (\vec{u}_0 + \delta\vec{u})]^* dV \\ &= \int_V \rho_0 \vec{u}_0 \cdot (i\vec{\omega} \times \vec{u}_0)^* dV. \end{aligned} \quad (6.259)$$

It is mentioned that  $D$  is multiplied with  $\sigma$ , and that  $\vec{\omega}$  is considered as an infinitesimal quantity as compared to  $\sigma$ . Hence, products of  $\vec{\omega}$  with other infinitesimal quantities can be neglected, as what we did in (6.259).

The formula of  $E$  is, taking into account of the fact that  $\bar{C}_0$  is real,

$$\begin{aligned} E &= \int_V (\nabla \vec{u}_0 + \nabla \delta\vec{u}) : (\bar{C}_0 + \delta\bar{C}^*) : (\nabla \vec{u}_0^* + \nabla \delta\vec{u}^*) dV \\ &= E_0 + \int_V \nabla \vec{u}_0 : \delta\bar{C}^* : \nabla \vec{u}_0^* dV + \int_V \nabla \vec{u}_0 : \bar{C}_0 : \nabla \delta\vec{u}^* dV \\ &\quad + \int_V \nabla \delta\vec{u} : \bar{C}_0 : \nabla \vec{u}^* dV. \end{aligned} \quad (6.260)$$

Similar to the formulation of  $I_1$ , with the boundary conditions of  $T_0$  and  $\delta\vec{u}$ , the last two terms can be shown to be  $-\int_V (\nabla \cdot \bar{T}_0) \cdot \delta\vec{u}^* dV$  and  $-\int_V \delta\vec{u} \cdot (\nabla \cdot \bar{T}_0^*) dV$ , respectively. We obtain

$$\begin{aligned} E &= E_0 + \int_V \nabla \vec{u}_0 : \delta\bar{C}^* : \nabla \vec{u}_0^* dV \\ &\quad - \int_V (\nabla \cdot \bar{T}_0) \cdot \delta\vec{u}^* dV - \int_V \delta\vec{u} \cdot (\nabla \cdot \bar{T}_0^*) dV. \end{aligned} \quad (6.261)$$



The formula of  $F$  is

$$\begin{aligned}
 F = \int_V \bigg\{ & -(\rho_0 + \delta\rho) \nabla(W_0 + \delta W) \cdot [(\vec{u}_0 + \delta\vec{u}) \cdot \nabla(\vec{u}_0^* + \delta\vec{u}^*) \\
 & - (\vec{u}_0 + \delta\vec{u}_0) \nabla \cdot (\vec{u}_0^* + \delta\vec{u}^*)] \\
 & - (\rho_0 + \delta\rho)(\vec{u}_0 + \delta\vec{u}) \cdot \nabla \nabla(W_0 + \delta W) \cdot (\vec{u}_0^* + \delta\vec{u}^*) \\
 & - (\rho_0 + \delta\rho)(\vec{u}_0 + \delta\vec{u}) \cdot \nabla(\phi_{10}^* + \delta\phi_1^*) \bigg\} dV. \quad (6.262)
 \end{aligned}$$

We write it as

$$F = F_0 + \delta F_1 + \delta F_2 + \delta F_3, \quad (6.263)$$

where

$$\begin{aligned}
 \delta F_1 = \int_V & [-\delta\rho \nabla W_0 \cdot (\vec{u}_0 \cdot \nabla \vec{u}_0^* - \vec{u}_0 \nabla \cdot \vec{u}_0^*) \\
 & - \delta\rho \vec{u}_0 \cdot \nabla \nabla W_0 \cdot \vec{u}_0^* - \delta\rho \vec{u}_0 \cdot \nabla \phi_{10}^*] dV \\
 & + \int_V [-\rho_0 \nabla \delta W \cdot (\vec{u}_0 \cdot \nabla \vec{u}_0^* - \vec{u}_0 \nabla \cdot \vec{u}_0^*) - \rho_0 \vec{u}_0 \cdot \nabla \nabla \delta W \cdot \vec{u}_0^*] dV \\
 & + \int_V [-\rho_0 \nabla W_0 \cdot (\delta\vec{u} \cdot \nabla \vec{u}_0^* - \delta\vec{u} \nabla \cdot \vec{u}_0^*) \\
 & - \rho_0 \delta\vec{u} \cdot \nabla \nabla W_0 \cdot \vec{u}_0^* - \rho_0 \delta\vec{u} \cdot \nabla \phi_{10}^*] dV \\
 & + \int_V [-\rho_0 \vec{u}_0 \cdot \nabla \nabla W_0 \cdot \delta\vec{u}^*] dV, \quad (6.264)
 \end{aligned}$$

$$\delta F_2 = \int_V [-\rho_0 \nabla W_0 \cdot (\vec{u}_0 \cdot \nabla \delta\vec{u}^* - \vec{u}_0 \nabla \cdot \delta\vec{u}^*)] dV, \quad (6.265)$$

$$\delta F_3 = \int_V [-\rho_0 \vec{u}_0 \cdot \nabla \delta\phi_1^*] dV. \quad (6.266)$$

The last two formulae have to be further developed so that no derivative of  $\delta\vec{u}^*$  or  $\delta\phi_1^*$  is present.

According to the relation between the pressure and the gravity potential,  $\nabla p = \rho \nabla W$ , (6.265) can be further developed as

$$\begin{aligned}
 \delta F_2 &= \int_V \left( -\frac{\partial p}{\partial x_j} u_{0i} \frac{\partial \delta u_j^*}{\partial x_i} + \frac{\partial p}{\partial x_j} u_{0j} \frac{\partial \delta u_i^*}{\partial x_i} \right) dV \\
 &= \int_V \left( \frac{\partial p}{\partial x_j} \frac{\partial u_{0i}}{\partial x_i} \delta u_j^* - \frac{\partial p}{\partial x_j} \frac{\partial u_{0j}}{\partial x_i} \delta u_i^* \right) dV
 \end{aligned}$$

$$\begin{aligned}
& + \int_V \left[ -\frac{\partial p}{\partial x_j} \frac{\partial}{\partial x_i} (u_{0i} \delta u_j^*) + \frac{\partial p}{\partial x_j} \frac{\partial}{\partial x_i} (u_{0j} \delta u_i^*) \right] dV \\
& = \int_V \nabla p_0 \cdot [(\nabla \cdot \vec{u}_0) \delta \vec{u}^* - \delta \vec{u}^* \cdot \nabla \vec{u}_0] dV + \delta \tilde{F}_2, \tag{6.267}
\end{aligned}$$

where  $p_0$  is the pressure of the initial model, and

$$\delta \tilde{F}_2 = \int_V \left\{ -\frac{\partial}{\partial x_j} \left[ p \frac{\partial}{\partial x_i} (u_{0i} \delta u_j^*) \right] + \frac{\partial}{\partial x_j} \left[ p \frac{\partial}{\partial x_i} (u_{0j} \delta u_i^*) \right] \right\} dV. \tag{6.268}$$

It can be shown that  $\delta \tilde{F}_2 = 0$ . To do so, we convert the volume integral in its expression to surface integrals, which we write as

$$\delta \tilde{F}_2 = \delta \tilde{F}_2' + \delta \tilde{F}_2'', \tag{6.269}$$

$$\delta \tilde{F}_2' = \int_{S_0} \left[ -pn_j \frac{\partial}{\partial x_i} (u_{0i} \delta u_j^*) + pn_j \frac{\partial}{\partial x_i} (u_{0j} \delta u_i^*) \right] dS, \tag{6.270}$$

$$\delta \tilde{F}_2'' = \int_{S_i} \left\{ -pn_j \frac{\partial}{\partial x_i} (u_{0i} \delta u_j^*) + pn_j \frac{\partial}{\partial x_i} (u_{0j} \delta u_i^*) \right\}_-^+ dS. \tag{6.271}$$

Similar to the formulation of (6.243)–(6.247), we can develop the expression of  $\delta \tilde{F}_2''$  to

$$\delta \tilde{F}_2'' = \int_{S_i} \{ -p \hat{n} \cdot [\nabla_S \cdot (\vec{u}_0 \delta \vec{u}^*)] + p \hat{n} \cdot [\nabla_S \cdot (\delta \vec{u}^* \vec{u}_0)] \}_-^+ dS. \tag{6.272}$$

As  $p$  is constant on  $S_i$ , it can be moved to within the operator  $\nabla_S$ ,

$$\delta \tilde{F}_2'' = \int_{S_i} \{ -\hat{n} \cdot [\nabla_S \cdot (p \vec{u}_0 \delta \vec{u}^*)] + \hat{n} \cdot [\nabla_S \cdot (p \delta \vec{u}^* \vec{u}_0)] \}_-^+ dS. \tag{6.273}$$

By substituting  $\nabla_S$  back to  $\nabla$ , we obtain in a similar way,

$$\begin{aligned}
\delta \tilde{F}_2'' & = \int_{S_i} \{ -\hat{n} \cdot [\nabla \cdot (p \vec{u} \cdot \delta \vec{u}^*)] + \hat{n} \cdot [\nabla \cdot (p \delta \vec{u}^* \vec{u}_0)] \}_-^+ dS \\
& = \int_{S_i} \left\{ -n_j \frac{\partial}{\partial x_i} (p u_{0i} \delta u_j^*) + n_j \frac{\partial}{\partial x_i} (p \delta u_i^* u_{0j}) \right\}_-^+ ds. \tag{6.274}
\end{aligned}$$

It can be readily inferred that  $\delta \tilde{F}_2'$  vanishes, since  $p$  vanishes on the Earth's surface. Anyway, we still formally combine the expressions of both  $\delta \tilde{F}_2'$  and  $\delta \tilde{F}_2''$ , and then transform the result to a volume integral that vanishes,

$$\delta \tilde{F}_2 = \int_V \left[ -\frac{\partial^2}{\partial x_i \partial x_j} (p u_{0i} \delta u_j^*) + \frac{\partial^2}{\partial x_i \partial x_j} (p \delta u_i^* u_{0j}) \right] dV = 0. \tag{6.275}$$

Hence, with the relation between the pressure and the gravity potential,  $\nabla p = \rho \nabla W$ , we obtain from (6.267),

$$\delta F_2 = \int_V \rho_0 \nabla W_0 \cdot [(\nabla \cdot \vec{u}_0) \delta \vec{u}^* - \delta \vec{u}^* \cdot \nabla \vec{u}_0] dV. \quad (6.276)$$

In this formula, the derivative of  $\delta \vec{u}^*$  is no longer present.

In the further development of the expression of  $\delta F_3$ , (6.266), we will need Poisson's equation and the boundary conditions of  $\delta \phi_1$ . As  $\phi_1$  satisfies Poisson's equation (6.227) and the boundary conditions (6.234) and (6.235), and Poisson's equation and the boundary conditions of  $\phi_{10}$  are in the same form, we can obtained Poisson's equation and the boundary conditions of  $\delta \phi_1$  to be

$$\nabla^2 \delta \phi_1 = 4\pi G \nabla \cdot (\rho_0 \delta \vec{u} + \delta \rho \vec{u}_0), \quad (6.277)$$

$$\{\delta \phi_1\}_-^+ = 0, \quad (6.278)$$

$$\left\{ \frac{\partial \delta \phi_1}{\partial n} - 4\pi G \hat{n} \cdot (\rho_0 \delta \vec{u} + \delta \rho \vec{u}_0) \right\}_-^+ = 0. \quad (6.279)$$

We then have

$$\begin{aligned} \delta F_3 &= \int_V (-\rho_0 \vec{u}_0 \cdot \nabla \delta \phi_1^*) dV \\ &= \int_V [-\nabla \cdot (\rho_0 \vec{u}_0 \delta \phi_1^*) + \nabla \cdot (\rho_0 \vec{u}_0) \delta \phi_1^*] dV \\ &= \int_V \left[ -\nabla \cdot (\rho_0 \vec{u}_0 \delta \phi_1^*) + \frac{1}{4\pi G} \nabla^2 \phi_{10} \delta \phi_1^* \right] dV \\ &= \int_V \left[ -\nabla \cdot (\rho_0 \vec{u}_0 \delta \phi_1^*) + \frac{1}{4\pi G} \nabla \cdot (\nabla \phi_{10} \delta \phi_1^*) - \frac{1}{4\pi G} \nabla \phi_{10} \cdot \nabla \delta \phi_1^* \right] dV \\ &= \int_{S_0+S_i} \left\{ -\hat{n} \cdot (\rho_0 \vec{u}_0 \delta \phi_1^*) + \frac{1}{4\pi G} \hat{n} \cdot \nabla \phi_{10} \delta \phi_1^* \right\}_-^+ dS \\ &\quad + \int_V \left[ -\frac{1}{4\pi G} \nabla \cdot (\phi_{10} \nabla \delta \phi_1^*) + \frac{1}{4\pi G} \phi_{10} \nabla^2 \delta \phi_1^* \right] dV \\ &= \int_{S_0+S_i} \left\{ \frac{\delta \phi_1^*}{4\pi G} [-4\pi G \hat{n} \cdot (\rho_0 \vec{u}_0) + \frac{\partial \phi_{10}}{\partial n}] \right\}_-^+ dS \\ &\quad + \int_V \left[ -\frac{1}{4\pi G} \nabla \cdot (\phi_{10} \nabla \delta \phi_1^*) + \phi_{10} \nabla \cdot (\rho_0 \delta \vec{u}^* + \delta \rho \vec{u}_0^*) \right] dV \\ &= \int_V \left\{ -\frac{1}{4\pi G} \nabla \cdot (\phi_{10} \nabla \delta \phi_1^*) + \nabla \cdot [\phi_{10} (\rho_0 \delta \vec{u}^* + \delta \rho \vec{u}_0^*)] \right\} dV \end{aligned}$$

$$\begin{aligned}
& -\nabla\phi_{10} \cdot (\rho_0\delta\vec{u}^* + \delta\rho\vec{u}_0^*) \Big\} dV \\
& = \int_{S_0+S_i} \left\{ -\frac{1}{4\pi G} \hat{n} \cdot (\phi_{10}\nabla\delta\phi_1^*) + \hat{n} \cdot [\phi_{10}(\rho_0\delta\vec{u}^* + \delta\rho\vec{u}_0^*)] \right\}_-^+ dS \\
& \quad + \int_V (-\rho_0\nabla\phi_{10} \cdot \delta\vec{u}^* - \delta\rho\nabla\phi_{10} \cdot \vec{u}_0^*) dV \\
& = \int_{S_0+S_i} \left\{ \frac{\phi_{10}}{4\pi G} \left[ -\frac{\partial\phi_1^*}{\partial n} + 4\pi G\hat{n} \cdot (\rho_0\delta\vec{u}^* + \delta\rho\vec{u}_0^*) \right] \right\}_-^+ dS \\
& \quad + \int_V (-\rho\nabla\phi_{10} \cdot \delta\vec{u}^* - \delta\rho\nabla\phi_{10} \cdot \vec{u}_0^*) dV \\
& = \int_V (-\rho_0\nabla\phi_{10} \cdot \delta\vec{u}^* - \delta\rho\nabla\phi_{10} \cdot \vec{u}_0^*) dV. \tag{6.280}
\end{aligned}$$

In the derivation, the boundary conditions of  $\phi_{10}$  have also been used. In the last formula, the derivative of  $\delta\phi_1^*$  is no longer present.

Finally, by substituting (6.264), (6.276), and (6.280) into the expression of  $F$ , (6.263), we obtain, after some rearrangement of the terms,

$$\begin{aligned}
F & = F_0 + \int_V \delta\rho[-\nabla W_0 \cdot (\vec{u}_0 \cdot \nabla\vec{u}_0^* - \vec{u}_0\nabla \cdot \vec{u}_0^*) \\
& \quad - \vec{u}_0 \cdot \nabla\nabla W_0 \cdot \vec{u}_0^* - \vec{u}_0 \cdot \nabla\phi_{10}^* - \nabla\phi_{10} \cdot \vec{u}_0^*] dV \\
& \quad + \int_V [-\rho_0\nabla\delta W \cdot (\vec{u}_0 \cdot \nabla\vec{u}_0^* - \vec{u}_0\nabla \cdot \vec{u}_0^*) - \rho_0\vec{u}_0 \cdot \nabla\nabla\delta W \cdot \vec{u}_0^*] dV \\
& \quad + \int_V \{[\rho_0(\nabla\vec{u}_0)\nabla W_0 - \rho_0\nabla\vec{u}_0 \cdot \nabla W_0 - \rho_0\vec{u}_0 \cdot \nabla\nabla W_0 - \rho_0\nabla\phi_{10}] \cdot \delta\vec{u}^*\} dV \\
& \quad + \int_V \{\delta\vec{u} \cdot [\rho_0(\nabla \cdot \vec{u}_0^*)\nabla W_0 - \rho_0\nabla\vec{u}_0^* \cdot \nabla W_0 - \rho_0\vec{u}_0^* \cdot \nabla\nabla W_0 - \rho_0\nabla\phi_{10}^*]\} dV. \tag{6.281}
\end{aligned}$$

Now, all terms in the weak form of the elastic equation (6.251), have been derived, which are (6.258), (6.259), (6.261), and (6.281). For the non-rotating initial Earth model, this weak form degenerates to

$$\sigma_0^2 K_0 - E_0 - F_0 = 0, \tag{6.282}$$

and the moment equation of deformation is

$$-\rho_0\sigma_0^2\vec{u}_0 = \nabla \cdot \vec{T}_0 - \rho_0(\nabla \cdot \vec{u}_0)\nabla W_0 + \rho_0\nabla\vec{u}_0 \cdot \nabla W_0 + \rho_0\vec{u}_0 \cdot \nabla\nabla W_0 + \rho_0\nabla\phi_{10}, \tag{6.283}$$

where  $\sigma_0$  is a real number. By substituting (6.258), (6.259), (6.261), and (6.281) into (6.251), and then using (6.282) and (6.283), we obtain

$$\begin{aligned}
 & 2\sigma_0\delta\sigma^*K_0 + \sigma_0^2 \int_V \delta\rho \vec{u}_0 \cdot \vec{u}_0^* dV - 2\sigma_0 \int_V \rho_0 \vec{u}_0 \cdot (i\vec{\omega} \times \vec{u}_0)^* dV \\
 & - \int_V \nabla \vec{u}_0 : \delta\bar{C}^* : \nabla \vec{u}_0^* dV - \int_V \delta\rho \{ -\nabla W_0 \cdot (\vec{u}_0 \cdot \nabla \vec{u}_0^* - \vec{u}_0 \nabla \cdot \vec{u}_0^*) \\
 & - \vec{u}_0 \cdot \nabla \nabla W_0 \cdot \vec{u}_0^* - \vec{u}_0 \cdot \nabla \phi_{10}^* - \nabla \phi_{10} \cdot \vec{u}_0^* \} dV \\
 & + \int_V [-\rho_0 \nabla \delta W \cdot (\vec{u}_0 \cdot \nabla \vec{u}_0^* - \vec{u}_0 \nabla \cdot \vec{u}_0^*) - \rho_0 \vec{u}_0 \cdot \nabla \nabla \delta W \cdot \vec{u}_0^*] dV = 0,
 \end{aligned} \tag{6.284}$$

where  $K_0$  is defined as

$$K_0 = \int_V \rho_0 \vec{u}_0 \cdot \vec{u}_0^* dV. \tag{6.285}$$

A formula for the perturbation  $\delta\sigma$  can be obtained from (6.284) to be

$$\begin{aligned}
 \delta\sigma^* &= \frac{1}{K_0} \int_V \rho_0 \vec{u}_0 \cdot (i\vec{\omega} \times \vec{u}_0)^* dV \\
 &+ \frac{1}{2\sigma_0 K_0} \int_V \left[ -\delta\rho\sigma_0^2 \vec{u}_0 \cdot \vec{u}_0^* - (\delta\rho \nabla W_0 + \rho_0 \nabla \delta W) \cdot (\vec{u}_0 \cdot \nabla \vec{u}_0^* - \vec{u}_0 \nabla \cdot \vec{u}_0^*) \right. \\
 &- \vec{u}_0 \cdot (\delta\rho \nabla \nabla W_0 + \rho_0 \nabla \nabla \delta W) \cdot \vec{u}_0^* - \delta\rho (\vec{u}_0 \cdot \nabla \phi_{10}^* + \nabla \phi_{10} \cdot \vec{u}_0^*) \left. \right] dV \\
 &+ \frac{1}{2\sigma_0 K_0} \int_V \nabla \vec{u}_0 : \delta\bar{C}^* : \nabla \vec{u}_0^* dV.
 \end{aligned} \tag{6.286}$$

This is the formula to compute, from the result of a non-rotating Earth model, the free oscillation frequency of a rotating Earth model with slightly different density and elastic parameters.

In (6.286), no particular assumption is made on the rheology of the Earth. It can be used to handle non-isotropy and even viscoelasticity (the free oscillation equation of a viscoelasticity Earth model is in the same form as that of an elastic Earth model, the differences are that the elastic parameters and frequencies are complex numbers for the viscoelasticity Earth model). We now reduce (6.286) to the case of an isotropic elastic Earth model. According to (3.269), the elastic parameters are

$$\delta C_{ijkl} = \delta\lambda \delta_{ij} \delta_{kl} + \delta\mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \tag{6.287}$$

where  $\delta\lambda$  and  $\delta\mu$  are real numbers. We thus have

$$\begin{aligned}
 \nabla \vec{u}_0 : \delta \vec{C}^* : \nabla \vec{u}_0^* &= \frac{\partial u_{0j}}{\partial x_i} \delta C_{ijkl} \frac{\partial u_{0l}^*}{\partial x_k} \\
 &= \delta\lambda \frac{\partial u_{0i}}{\partial x_i} \frac{\partial u_{0k}^*}{\partial x_k} + \delta\mu \left( \frac{\partial u_{0j}}{\partial x_i} \frac{\partial u_{0j}^*}{\partial x_i} + \frac{\partial u_{0j}}{\partial x_i} \frac{\partial u_{0i}^*}{\partial x_j} \right) \\
 &= \delta\lambda (\nabla \cdot \vec{u}_0) (\nabla \cdot \vec{u}_0^*) + \delta\mu [\nabla \vec{u}_0 : \nabla \vec{u}_0^* + (\nabla \vec{u}_0)^T : \nabla \vec{u}_0^*]. \quad (6.288)
 \end{aligned}$$

The substituting of this formula into the third term at the right-hand side of (6.286) leads to the result of the perturbation of the angular frequency for the isotropic elastic Earth model. Here we write it out for convenience of later reference ( $\delta\sigma$  is real now),

$$\begin{aligned}
 \delta\sigma &= \frac{1}{K_0} \int_V \rho_0 \cdot \vec{u}_0 \cdot (i\vec{\omega} \times \vec{u}_0)^* dV \\
 &+ \frac{1}{2\sigma_0 K_0} \int_V \left[ -\delta\rho \sigma_0^2 \vec{u}_0 \cdot \vec{u}_0^* - (\delta\rho \nabla W_0 + \rho_0 \nabla \delta W) \cdot (\vec{u}_0 \cdot \nabla \vec{u}_0^* - \vec{u}_0 \nabla \cdot \vec{u}_0^*) \right. \\
 &\quad \left. - \vec{u}_0 \cdot (\delta\rho \nabla \nabla W_0 + \rho_0 \nabla \nabla \delta W) \cdot \vec{u}_0^* - \delta\rho (\vec{u}_0 \cdot \nabla \phi_{10}^* + \nabla \phi_{10} \cdot \vec{u}_0^*) \right] dV \\
 &+ \frac{1}{2\sigma_0 K_0} \int_V \delta\lambda (\nabla \cdot \vec{u}_0) (\nabla \cdot \vec{u}_0^*) dV \\
 &+ \frac{1}{2\sigma K_0} \int_V \delta\mu [\nabla \vec{u}_0 : \nabla \vec{u}_0^* + (\nabla \vec{u}_0)^T : \nabla \vec{u}_0^*] dV. \quad (6.289)
 \end{aligned}$$

Finally, we make some comments on the applicability of (6.289) or (6.286). In deriving them, we have made the assumption that both the non-rotating initial and the rotating and slightly disturbed Earth models are in hydrostatic equilibrium as the undeformed state, and that the effects of centrifugal acceleration and ellipticity of the Earth have been neglected. Based on these assumptions, the formulae can only be used to compute the difference between the free oscillation frequencies of two Earth models in hydrostatic equilibrium, one Earth model being obtained by perturbing the other. However, for a three dimensional Earth model, hydrostatic equilibrium is non-existent. As the deviation of a three dimensional Earth model from a spherically symmetrical Earth model is small by its nature, it is still a reasonable approximation to use a spherically symmetrical Earth model in hydrostatic equilibrium as the undeformed state. Based on this perspective, the formulae derived can be approximately used to compute the effect of three dimensional perturbation of density and elastic parameters on free oscillation frequencies. We mention that, in the derivation, we assumed that  $\delta\rho$  and  $\delta\vec{C}$  (or  $\delta\lambda$  and  $\delta\mu$ ) are continuous throughout the Earth's interior, and that there is no perturbation in the positions of boundaries of discontinuity. More complete perturbation theories, including part of or all the effects of Earth rotation, ellipticity, three dimensional perturbations of density and

elastic parameters, and shift of boundaries of discontinuity, have been developed (e.g., Dahlen & Tromp, 1998, as a comprehensive reference). The relatively simple model presented in this book is a compromise of the detail that can be provided in the permissible pages and the limitation of the knowledge of mathematics assumed for the readers. However, this simple model does provide reasonable insights into the major concerns of the problem in physics and mathematics.

## 6.6 Reduction of the Perturbation Theory to Formulae for Practical Computation

### 6.6.1 Perturbation of Rotation to Free Oscillations

We know that, under the spherically symmetrical assumption, each free oscillation mode of the non rotating Earth model,  ${}_m S_n$  or  ${}_m T_n$ , is split into  $2n + 1$  modes for the corresponding rotating Earth model,  ${}_m S_n^k$  or  ${}_m T_n^k$ , with  $k = 0, \pm 1, \pm 2, \dots, \pm n$ . In fact, this remains true for the general case, i.e., each free oscillation mode of the non-rotating spherically symmetrical Earth model is split into  $2n + 1$  modes for an Earth model with rotation, ellipticity and laterally heterogeneous structure inside. The topic here is to formulate the splitting by rotation, which is in fact the effect of the Coriolis force. The starting formula has already been derived in the previous section, i.e., the first term at the right-hand side of (6.289). We provide a derivation only for the simpler case of toroidal free oscillations, and provide the result for the spheroidal free oscillations directly. The readers interested in the derivation are suggested to do it by themselves as an exercise.

In subsequent formulations, we will omit the subscript 0 of density, elastic parameters, and the solution of the non-rotating initial Earth model introduced in the previous section, which would not lead to any confusion.

For the spherically symmetrical non-rotating Earth model as formulated in Sects. 4.8 and 6.4, the symbols  $y_1^n = U_n^k$ ,  $y_3^n = V_n^k$  and  $y_7^n = iW_n^k$  are independent of  $k$ , and the terms of different  $n$  are decoupled each other. In this subsection, the initial model is assumed to be spherically symmetrical and non-rotating. Hence, terms of different  $n$  and  $k$  are independent each other, and can be treated separately. In the formulation, we will always keep the super script  $k$  even for the initial non-rotating Earth model.

The effect of rotation on the free oscillation angular frequency is, according to (6.289),

$$\delta\sigma = \frac{1}{K_0} \int_V \rho \cdot \vec{u} \cdot (i\vec{\omega} \times \vec{u})^* dV. \quad (6.290)$$

Based on the formulations in Sects. 4.8 and 6.4, the toroidal displacement of degree  $n$  and order  $k$  is (noting that  $\lambda$  is used as both the elastic parameter  $\lambda$  and the longitude  $\lambda$ , which would not bring any confusion based on it appears in the

formulae)

$$\vec{u}_n^k = \vec{T}_n^k e^{i\sigma t} = W_n^k \left( \frac{ik}{\sin \theta} P_n^k \hat{e}_\theta - \frac{dP_n^k}{d\theta} \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (6.291)$$

where  $\vec{T}_n^k e^{i\sigma t}$  is the degree  $n$  order  $k$  toroidal field defined in (4.456) with the dependence on  $t$  added. The angular velocity of the Earth rotation is, in the spherical coordinate system,

$$\vec{\omega} = \omega(\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta). \quad (6.292)$$

To simplify the formulation, we take all the functions  $y_1^n, \dots, y_8^n$  defined in (4.480)–(4.487) to be real, since they are the solution of the system of ordinary equations (4.516), whose coefficients are all real. It can be inferred based on the physics of the problem that this definition simplifies the computation and formulation, but does not affect the result. Hence, in the formulations hereafter, we take  $U_n^k$ ,  $V_n^k$ , and  $\phi_n^k$  as real functions, and  $W_n^k$  as an imaginary function. In this way, the conjugate of  $\vec{u}_n^k$  is (noting that the conjugate of  $ae^{ix}$  is  $ae^{-ix}$  when  $a$  is real, and  $-ae^{-ix}$  when  $a$  is imaginary)

$$(\vec{u}_n^k)^* = (\vec{T}_n^k e^{i\sigma t})^* = W_n^k \left( \frac{ik}{\sin \theta} P_n^k \hat{e}_\theta + \frac{dP_n^k}{d\theta} \hat{e}_\lambda \right) e^{-ik\lambda} e^{-i\sigma t}. \quad (6.293)$$

According to (6.285), we have

$$\begin{aligned} K_0 &= \int_V \rho \vec{u} \cdot \vec{u}^* dV \\ &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^R \rho r^2 \left\{ -(W_n^k)^2 \left[ \frac{k^2}{\sin^2 \theta} (P_n^k)^2 + \left( \frac{dP_n^k}{d\theta} \right)^2 \right] \right\} \sin \theta dr \\ &= -2\pi \int_0^\pi \left[ \frac{k^2}{\sin^2 \theta} (P_n^k)^2 + \left( \frac{dP_n^k}{d\theta} \right)^2 \right] \sin \theta d\theta \int_0^R \rho r^2 (W_n^k)^2 dr \\ &= -\frac{4\pi n(n+1)}{2n+1} \frac{(n+k)!}{(n-k)!} \int_0^R \rho r^2 (W_n^k)^2 dr, \end{aligned} \quad (6.294)$$

where the orthogonality formula (4.244) with  $m = n$  was used to derive the last identity. Since

$$\vec{\omega} \times \vec{u}_n^k = \omega W_n^k \left( \sin \theta \frac{dP_n^k}{d\theta} \hat{e}_r + \cos \theta \frac{dP_n^k}{d\theta} \hat{e}_\theta + \frac{ik \cos \theta}{\sin \theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (6.295)$$



we have, taking  $W_n^k$  as an imaginary function as above,

$$\begin{aligned}
 & \int_V \rho \vec{u}_n^k \cdot (i\vec{\omega} \times \vec{u}_n^k)^* dV \\
 &= \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^R \rho r^2 \left\{ -2\omega (W_n^k)^2 \frac{k \cos \theta}{\sin \theta} P_n^k \frac{dP_n^k}{d\theta} \right\} \sin \theta d\theta \\
 &= -4\pi \omega k \int_0^\pi \cos \theta P_n^k \frac{dP_n^k}{d\theta} d\theta \int_0^R \rho r^2 (W_n^k)^2 dr \\
 &= -\frac{4\pi \omega k}{2n+1} \frac{(n+k)!}{(n-k)!} \int_0^R \rho r^2 (W_n^k)^2 dr, \tag{6.296}
 \end{aligned}$$

where the orthogonality formula (4.246) with  $m = n$  was used to derive the last identity. As the integral over  $\theta$  is multiplied by  $k$ , the second term in (4.246), which is non-vanishing only when  $k = 0$ , do not contribute to the result. Finally, by substituting (6.294) and (6.296) into (6.290), we obtain the perturbation of rotation to the angular frequencies of toroidal free oscillations,

$$\delta\sigma = \frac{\omega k}{n(n+1)}, \tag{6.297}$$

which is surprisingly independent of  $W_n^k$ !

For spheroidal free oscillations, we write out the result directly:

$$K_0 = \frac{4\pi}{2n+1} \frac{(n+k)!}{(n-k)!} \int_0^R \rho r^2 [(U_n^k)^2 + n(n+1)(V_n^k)^2] dr, \tag{6.298}$$

$$\int_V \rho \vec{u} \cdot (i\vec{\omega} \times \vec{u})^* dV = \frac{4\pi}{2n+1} \frac{(n+k)!}{(n-k)!} \omega k \int_0^R \rho r^2 [2U_n^k V_n^k + (V_n^k)^2] dr, \tag{6.299}$$

$$\delta\sigma = \frac{\omega k \int_0^R \rho r^2 [2U_n^k V_n^k + (V_n^k)^2] dr}{\int_0^R \rho r^2 [(U_n^k)^2 + n(n+1)(V_n^k)^2] dr}. \tag{6.300}$$

In the two formulae (6.297) and (6.300), the order  $k$  can take any of the  $2n+1$  values  $0, \pm 1, \dots, \pm n$ . This demonstrates again that the rotation splits each free oscillation mode of the spherically symmetrical non-rotating Earth model to  $2n+1$ . Finally, we mention that both (6.297) and (6.300) are accurate to the first order of  $\omega$ , and that the result accurate to  $\omega^2$  can be found in the literature (e.g., Dahlen & Sailor, 1979).

### 6.6.2 Perturbations of Density and Elastic Parameters to Free Oscillations

In this subsection, we derive the formulae of the effects of  $\delta\rho$ ,  $\delta\lambda$ , and  $\delta\mu$  to  $\delta\sigma$ , assuming the perturbed Earth model to remain spherically symmetrical. As in the previous subsection, we also omit the subscript 0 of density, elastic parameters, and the solution of the non-rotating Earth model, which would not lead to any confusion. For brevity, we rewrite (6.289) as

$$\delta\sigma = \frac{1}{2\sigma K_0} (K_\rho + K_\lambda + K_\mu), \quad (6.301)$$

where  $K_0$  is defined in (6.285), the term related to rotation has already been formulated in the previous subsection, and is omitted here. The parameters  $K_\rho$ ,  $K_\lambda$ , and  $K_\mu$  are defined as

$$\begin{aligned} K_\rho &= \int_V \left[ -\delta\rho\sigma^2 \vec{u} \cdot \vec{u}^* - (\delta\rho\nabla W + \rho\nabla\delta W) \cdot (\vec{u} \cdot \nabla\vec{u}^* - \vec{u}\nabla \cdot \vec{u}^*) \right. \\ &\quad \left. - \vec{u} \cdot (\delta\rho\nabla\nabla W + \rho\nabla\nabla\delta W) \cdot \vec{u}^* - \delta\rho(\vec{u} \cdot \nabla\phi_1^* + \nabla\phi_1 \cdot \vec{u}^*) \right] dV \\ &= \int_V \left\{ \delta\rho \left[ -\sigma^2 \vec{u} \cdot \vec{u}^* - \nabla W \cdot (\vec{u} \cdot \nabla\vec{u}^* - \vec{u}\nabla \cdot \vec{u}^*) - \vec{u} \cdot \nabla\nabla W \cdot \vec{u}^* \right. \right. \\ &\quad \left. \left. - (\vec{u} \cdot \nabla\phi_1^* + \nabla\phi_1 \cdot \vec{u}^*) \right] + \rho \left[ -\nabla\delta W \cdot (\vec{u} \cdot \nabla\vec{u}^* - \vec{u}\nabla \cdot \vec{u}^*) \right. \right. \\ &\quad \left. \left. - \vec{u} \cdot (\nabla\nabla\delta W) \cdot \vec{u}^* \right] \right\} dV, \end{aligned} \quad (6.302)$$

$$K_\lambda = \int_V \delta\lambda (\nabla \cdot \vec{u}) (\nabla \cdot \vec{u}^*) dV, \quad (6.303)$$

$$K_\mu = \int_V \delta\mu [\nabla\vec{u} + (\nabla\vec{u})^T] : \nabla\vec{u}^* dV = \int_V \delta\mu (2\bar{\epsilon} : \nabla\vec{u}^*) dV, \quad (6.304)$$

where  $\bar{\epsilon} = (1/2)[\nabla\vec{u} + (\nabla\vec{u})^T]$  is the strain tensor. The perturbation of the parameters,  $\delta\rho$ ,  $\delta\lambda$ , and  $\delta\mu$  are all assumed to be functions of  $r$  alone, and all other parameters and functions at the right-hand sides are of those of the non-rotating initial Earth model. The formulae of  $K_0$  have already been derived in the previous subsection, i.e., (6.294) for the toroidal free oscillation modes and (6.298) for the spheroidal free oscillation modes, and what we need is to further formulate  $K_\rho$ ,  $K_\lambda$ , and  $K_\mu$ .

The formulae of  $K_\rho$  and  $K_\mu$ , i.e., (6.302) and (6.304), need some further development.

A first observance is that, the expression of  $K_\rho$  should be further developed to substitute  $\delta W$  by  $\delta\rho$ , as the former is completely determined by the later. The relation between  $\delta W$  and  $\delta\rho$  is the same as that between  $W$  and  $\rho$  (noting that  $\lambda$  is used as both the elastic parameter  $\lambda$  and the longitude  $\lambda$ , which would not bring any

confusion based on it appears in the formulae),

$$\nabla \delta W = -\delta g \hat{e}_r, \quad (6.305)$$

$$\frac{d\delta g}{dr} = 4\pi G \delta \rho - \frac{2\delta g}{r}, \quad (6.306)$$

$$\delta g = 4\pi G \frac{1}{r^2} \int_0^r \delta \rho q^2 dq, \quad (6.307)$$

$$\begin{aligned} \nabla \nabla \delta W &= -\frac{d\delta g}{dr} \hat{e}_r \hat{e}_r - \frac{\delta g}{r} \hat{e}_\theta \hat{e}_\theta - \frac{\delta g}{r} \hat{e}_\lambda \hat{e}_\lambda \\ &= -\left( \frac{d\delta g}{dr} + \frac{2\delta g}{r} \right) \hat{e}_r \hat{e}_r + \frac{3\delta g}{r} \hat{e}_r \hat{e}_r - \frac{\delta g}{r} (\hat{e}_r \hat{e}_r + \hat{e}_\theta \hat{e}_\theta + \hat{e}_\lambda \hat{e}_\lambda) \\ &= -4\pi G \delta \rho \hat{e}_r \hat{e}_r + \frac{3\delta g}{r} \hat{e}_r \hat{e}_r - \frac{\delta g}{r} (\hat{e}_r \hat{e}_r + \hat{e}_\theta \hat{e}_\theta + \hat{e}_\lambda \hat{e}_\lambda). \end{aligned} \quad (6.308)$$

We then have

$$\begin{aligned} \nabla \delta W \cdot (\vec{u} \cdot \nabla \vec{u}^*) &= \vec{u} \cdot (\nabla \vec{u}^* \cdot \nabla \delta W) \\ &= -\delta g \vec{u} \cdot (\nabla \vec{u}^* \cdot \hat{e}_r) \\ &= -\delta g \vec{u} \cdot \left[ \frac{\partial u_r^*}{\partial r} \hat{e}_r + \left( \frac{1}{r} \frac{\partial u_r^*}{\partial \theta} - \frac{u_\theta^*}{r} \right) \hat{e}_\theta + \left( \frac{1}{r \sin \theta} \frac{\partial u_r^*}{\partial \lambda} - \frac{u_\lambda^*}{r} \right) \hat{e}_\lambda \right] \\ &= -\delta g \vec{u} \cdot \left( \frac{\partial u_r^*}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial u_r^*}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial u_r^*}{\partial \lambda} \hat{e}_\lambda \right) + \frac{\delta g}{r} \vec{u} \cdot (u_\theta^* \hat{e}_\theta + u_\lambda^* \hat{e}_\lambda) \\ &= -\delta g \vec{u} \cdot \nabla u_r^* + \frac{\delta g}{r} \vec{u} \cdot (u_r^* \hat{e}_r + u_\theta^* \hat{e}_\theta + u_\lambda^* \hat{e}_\lambda) - \frac{\delta g}{r} \vec{u} \cdot (u_r^* \hat{e}_r) \\ &= -\delta g (\vec{u} \cdot \nabla) u_r^* + \frac{\delta g}{r} \vec{u} \cdot \vec{u}^* - \frac{\delta g}{r} u_r u_r^*, \end{aligned} \quad (6.309)$$

$$\nabla \delta W \cdot (\vec{u} \nabla \cdot \vec{u}^*) = -\delta g u_r (\nabla \cdot \vec{u}^*), \quad (6.310)$$

$$\vec{u} \cdot \nabla \nabla \delta W \cdot \vec{u}^* = -4\pi G \delta \rho u_r u_r^* + \frac{3\delta g}{r} u_r u_r^* - \frac{\delta g}{r} \vec{u} \cdot \vec{u}^*. \quad (6.311)$$

With the above three formulae, we obtain

$$\begin{aligned} \nabla \delta W \cdot (\vec{u} \cdot \nabla \vec{u}^* - \vec{u} \nabla \cdot \vec{u}^*) + \vec{u} \cdot \nabla \nabla \delta W \cdot \vec{u}^* \\ = -\delta g (\vec{u} \cdot \nabla) u_r^* + \delta g u_r (\nabla \cdot \vec{u}^*) + \frac{2\delta g}{r} u_r u_r^* - 4\pi G \delta \rho u_r u_r^* \end{aligned}$$

$$\begin{aligned}
&= -\delta g \left[ (\vec{u} \cdot \nabla) u_r^* - u_r (\nabla \cdot \vec{u}^*) - \frac{2}{r} u_r u_r^* \right] - 4\pi G \delta \rho u_r u_r^* \\
&= -\delta g \Lambda - 4\pi G \delta \rho u_r u_r^*,
\end{aligned} \tag{6.312}$$

where  $\Lambda$  is defined as

$$\Lambda = (\vec{u} \cdot \nabla) u_r^* - u_r (\nabla \cdot \vec{u}^*) - \frac{2}{r} u_r u_r^*. \tag{6.313}$$

We realize immediately that a similar formula holds for  $W$ ,

$$\nabla W \cdot (\vec{u} \cdot \nabla \vec{u}^* + \vec{u} \nabla \cdot \vec{u}^*) + \vec{u} \cdot \nabla \nabla W \cdot \vec{u}^* = -g \Lambda - 4\pi G \rho u_r u_r^*. \tag{6.314}$$

Upon substituting (6.312) and (6.314) into (6.302), we obtain

$$\begin{aligned}
K_\rho &= \int_V \left\{ \delta \rho \left[ -\sigma^2 \vec{u} \cdot \vec{u}^* - (\vec{u} \cdot \nabla \phi_1^* + \nabla \phi_1 \cdot \vec{u}^*) \right] \right. \\
&\quad \left. + 8\pi G \delta \rho u_r u_r^* + (\delta \rho g + \rho \delta g) \Lambda \right\} dV.
\end{aligned} \tag{6.315}$$

The term  $\delta g$  is in need of further attention. Define

$$S(r) = \int_r^R \rho \Lambda dq, \tag{6.316}$$

where  $\rho$  and  $\Lambda$  within the integrand are understood as functions of  $q$ . We have  $\rho \Lambda = -dS/dr$ , and

$$\begin{aligned}
\int_V \rho \delta g \Lambda dV &= \int_\omega \left( r^2 \delta g \rho \Lambda dr \right) d\omega \\
&= -4\pi G \int_\omega \left[ \int_0^R \left( \int_0^r \delta \rho q^2 dq \right) \frac{dS}{dr} dr \right] d\omega \\
&= -4\pi G \int_\omega \left\{ \left[ \left( \int_0^r \delta \rho q^2 dq \right) \left( \int_r^R \rho \Lambda dq \right) \right]_0^R - \int_0^R r^2 \delta \rho S dr \right\} d\omega \\
&= 4\pi G \int_\omega \left( \int_0^R r^2 \delta \rho S dr \right) d\omega \\
&= 4\pi G \int_V \left( \delta \rho \int_r^R \rho \Lambda dq \right) dV.
\end{aligned} \tag{6.317}$$

Finally, we can write the expression of  $K_\rho$  as

$$K_\rho = \int_V \left\{ \delta\rho \left[ 8\pi G\rho u_r u_r^* - \sigma^2 \vec{u} \cdot \vec{u}^* + g\Lambda \right. \right. \\ \left. \left. + 4\pi G \int_r^R \rho \Lambda dq - (\vec{u} \cdot \nabla \phi_1^* + \nabla \phi_1 \cdot \vec{u}^*) \right] \right\} dV, \quad (6.318)$$

where the dependence on  $\delta\rho$  is completely explicit.

The expression of  $K_\mu$ , (6.304), can also be further developed to a more manageable form. The kernel of the integral can be developed as follows:

$$\begin{aligned} (2\bar{\epsilon} : \nabla \vec{u}^*) &= [\nabla \vec{u} + (\nabla \vec{u})^T] : \nabla \vec{u}^* = \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \frac{\partial u_j^*}{\partial x_i} \\ &= \frac{\partial}{\partial x_i} \left[ \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) u_j^* \right] - \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) u_j^* \\ &= \nabla \cdot \left\{ [\nabla \vec{u} + (\nabla \vec{u})^T] \cdot \vec{u}^* \right\} - \left\{ \nabla \cdot [\nabla \vec{u} + (\nabla \vec{u})^T] \right\} \cdot \vec{u}^* \\ &= \nabla \cdot (2\bar{\epsilon} \cdot \vec{u}^*) - [\nabla \cdot (2\bar{\epsilon})] \cdot \vec{u}^*. \end{aligned} \quad (6.319)$$

Hence,

$$K_\mu = \int_V \delta\mu \left\{ \nabla \cdot (2\bar{\epsilon} \cdot \vec{u}^*) - [\nabla \cdot (2\bar{\epsilon})] \cdot \vec{u}^* \right\} dV. \quad (6.320)$$

This is the formula of  $K_\mu$  we will use for subsequent derivations.

Subsequent formulation will be based on (6.301), (6.303), (6.318), and (6.320), which are applicable to both the spheroidal and toroidal free oscillation modes.

We provide the derivation for the toroidal free oscillation modes only, and then directly write out the results for the spheroidal free oscillation modes. We use again the notations defined in Sects. 4.8 and 6.4,  $y_1^n = U_n^k$ ,  $y_3^n = V_n^k$ ,  $y_5^n = \phi_n^k$ , and  $y_7^n = iW_n^k$ . As  $u_r = 0$  and  $\phi_1 = 0$  for the toroidal free oscillation modes, following the derivation of  $K_0$  in (6.294), we readily obtain

$$K_\rho = -\sigma^2 \int_V \delta\rho \vec{u} \cdot \vec{u}^* dV = \sigma^2 \frac{4\pi n(n+1)}{2n+1} \frac{(n+k)!}{(n-k)!} \int_0^R \delta\rho r^2 (W_n^k)^2 dr. \quad (6.321)$$

With  $\nabla \cdot \vec{u} = 0$ , we readily have

$$K_\lambda = 0. \quad (6.322)$$

The derivation for the formula of  $K_\mu$  is more tedious. If we follow (6.304), we would come up with an integral of the Legendre function that we have no clue to handle, which is the reason we developed it to the form of (6.320) for use in

subsequent derivations. We first formulate the first term in the kernel of (6.320), whose expression in terms of components is

$$\begin{aligned}
 2\bar{\epsilon} \cdot \vec{u}^* &= [\nabla \vec{u} + (\nabla \vec{u})^T] \cdot \vec{u}^* \\
 &= \{2(\nabla \vec{u})_{rr}u_r^* + [(\nabla \vec{u})_{r\theta} + (\nabla \vec{u})_{\theta r}]u_\theta^* + [(\nabla \vec{u})_{r\lambda} + (\nabla \vec{u})_{\lambda r}]u_\lambda^*\}\hat{e}_r \\
 &\quad + \{[(\nabla \vec{u})_{\theta r} + (\nabla \vec{u})_{r\theta}]u_r^* + 2(\nabla \vec{u})_{\theta\theta}u_\theta^* + [(\nabla \vec{u})_{\theta\lambda} + (\nabla \vec{u})_{\lambda\theta}]u_\lambda^*\}\hat{e}_\theta \\
 &\quad + \{[(\nabla \vec{u})_{\lambda r} + (\nabla \vec{u})_{r\lambda}]u_r^* + [(\nabla \vec{u})_{\lambda\theta} + (\nabla \vec{u})_{\theta\lambda}]u_\theta^* + 2(\nabla \vec{u})_{\lambda\lambda}u_\lambda^*\}\hat{e}_\lambda.
 \end{aligned} \tag{6.323}$$

Here,  $\vec{u}$  and  $\vec{u}^*$  are assumed to be the degree  $n$  and order  $k$  toroidal fields given by (6.291) and (6.293), respectively. By comparing (4.436) with (4.463)–(4.468), we immediately realize that  $\nabla \vec{u} + (\nabla \vec{u})^T$  is the  $W_n^k$  terms without  $\mu$ . Hence, according to (6.293) and (4.463)–(4.468), we obtain the degree  $n$  and order  $k$  term of  $2\bar{\epsilon} \cdot \vec{u}^*$ ,

$$\begin{aligned}
 \{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k &= \left\{ [\nabla \vec{u} + (\nabla \vec{u})^T] \cdot \vec{u}^* \right\}_n^k \\
 &= -W_n^k \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \left[ \left( \frac{dP_n^k}{d\theta} \right)^2 + \frac{k^2}{\sin^2 \theta} (P_n^k)^2 \right] \hat{e}_r \\
 &\quad + (\{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k)_\theta \hat{e}_\theta + (\{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k)_\lambda \hat{e}_\lambda,
 \end{aligned} \tag{6.324}$$

where

$$\begin{aligned}
 (\{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k)_\theta &= \frac{(W_n^k)^2}{r} \left[ -\frac{3k^2}{\sin^2 \theta} \frac{dP_n^k}{d\theta} P_n^k + \frac{2k^2 \cos \theta}{\sin^3 \theta} (P_n^k)^2 \right. \\
 &\quad \left. - \frac{d^2 P_n^k}{d\theta^2} \frac{dP_n^k}{d\theta} + \cot \theta \left( \frac{dP_n^k}{d\theta} \right)^2 \right],
 \end{aligned} \tag{6.325}$$

$$\begin{aligned}
 (\{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k)_\lambda &= \frac{(W_n^k)^2}{r} \left[ -\frac{ik}{\sin \theta} \frac{d^2 P_n^k}{d\theta^2} P_n^k + \frac{3ik \cos \theta}{\sin^2 \theta} \frac{dP_n^k}{d\theta} P_n^k \right. \\
 &\quad \left. - \frac{ik^3}{\sin^3 \theta} (P_n^k)^2 - \frac{2ik}{\sin \theta} \left( \frac{dP_n^k}{d\theta} \right)^2 \right].
 \end{aligned} \tag{6.326}$$

The detail of these last two expressions are not to be required in subsequent formulations. With the divergence formula of vectors in spherical coordinates, (2.471), we obtain

$$\begin{aligned}
 \nabla \cdot \{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k &= -\frac{1}{r^2} \frac{d}{dr} \left[ r^2 W_n^k \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \left[ \left( \frac{dP_n^k}{d\theta} \right)^2 + \frac{k^2}{\sin^2 \theta} (P_n^k)^2 \right] \\
 &\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta (\{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k)_\theta \right],
 \end{aligned} \tag{6.327}$$

where we have taken into account the fact that the component  $(\{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k)_\lambda$  is independent of  $\lambda$ . For the second term in the kernel of (6.320), the expression of  $\nabla \cdot (2\bar{\epsilon}) = \nabla \cdot [\nabla \vec{u} + (\nabla \vec{u})^T]$  is actually included in the expressions of  $\nabla \cdot \vec{T}$ , i.e., the  $W_n^k$  terms in (4.469)–(4.471) without  $\mu$ ,

$$\begin{aligned} \{\nabla \cdot (2\bar{\epsilon})\}_n^k &= \left\{ \nabla \cdot [\nabla \vec{u} + (\nabla \vec{u})^T] \right\}_n^k \\ &= \left[ \frac{d}{dr} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) - (n-1)(n+2) \frac{W_n^k}{r^2} + \frac{3}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \\ &\quad \times \frac{ik}{\sin \theta} P_n^k e^{ik\lambda} e^{i\sigma t} \hat{e}_\theta \\ &\quad - \left[ \frac{d}{dr} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) - (n-1)(n+2) \frac{W_n^k}{r^2} + \frac{3}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \\ &\quad \times \frac{dP_n^k}{d\theta} e^{ik\lambda} e^{i\sigma t} \hat{e}_\lambda. \end{aligned} \quad (6.328)$$

Hence, according to (6.293), we have

$$\begin{aligned} \{[\nabla \cdot (2\bar{\epsilon})] \cdot \vec{u}^*\}_n^k &= \left\{ \nabla \cdot [\nabla \vec{u} + (\nabla \vec{u})^T] \cdot \vec{u}^* \right\}_n^k \\ &= -W_n^k \left[ \frac{d}{dr} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) - (n-1)(n+2) \frac{W_n^k}{r^2} + \frac{3}{r} \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right) \right] \\ &\quad \times \left[ \left( \frac{dP_n^k}{d\theta} \right)^2 + \frac{k^2}{\sin^2 \theta} (P_n^k)^2 \right]. \end{aligned} \quad (6.329)$$

Finally, by substituting (6.327) and (6.329) into (6.320), we obtain

$$\begin{aligned} K_\mu &= -2\pi \int_0^\pi \left[ \left( \frac{dP_n^k}{d\theta} \right)^2 + \frac{k^2}{\sin^2 \theta} (P_n^k)^2 \right] \sin \theta d\theta \\ &\quad \times \int_0^R \delta \mu r^2 \left[ \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right)^2 + (n-1)(n+2) \left( \frac{W_n^k}{r} \right)^2 \right] dr \\ &\quad + \int_0^{2\pi} d\lambda \int_0^\pi d\theta \int_0^R \frac{\partial}{\partial \theta} \left[ \sin \theta (\{2\bar{\epsilon} \cdot \vec{u}^*\}_n^k)_\theta \right] r dr. \end{aligned} \quad (6.330)$$

The integration over  $\theta$  in the first term is referred to (6.294). The second term vanishes, which is evident by integrating over  $\theta$  first with  $(\sin \theta)_{\theta=0,\pi} = 0$ . We

finally obtain

$$K_\mu = -\frac{4\pi n(n+1)}{2n+1} \frac{(n+k)!}{(n-k)!} \\ \times \int_0^R \delta\mu r^2 \left[ \left( \frac{dW_n^k}{dr} - \frac{W_n^k}{r} \right)^2 + (n-1)(n+2) \left( \frac{W_n^k}{r} \right)^2 \right] dr, \quad (6.331)$$

which is the result we are seeking.

We have now derived all the formulae for the perturbation to the toroidal free oscillation modes, which are (6.301), (6.294), (6.321), (6.322), and (6.331). As the solution of  $W_n^k$  is free from a constant multiplier, we normalize it according to

$$\int_0^R \rho r^2 (iW_n^k)^2 dr = 1, \quad (6.332)$$

remembering that we defined  $W_n^k$  to be imaginary following the definition  $y_7^n = iW_n^k$  assuming  $y_7^n$  to be real. We can then substitute (6.294), (6.321), (6.322), and (6.331) into (6.301) to obtain

$$\delta\sigma = \frac{1}{2\sigma} \int_0^R [\delta\rho K_\rho^r + \delta\lambda K_\lambda^r + \delta\mu K_\mu^r] dr, \quad (6.333)$$

where the functions of  $r$ ,  $K_\rho^r$ ,  $K_\lambda^r$ , and  $K_\mu^r$ , are

$$K_\rho^r = -\sigma^2 r^2 (iW_n^k)^2, \quad (6.334)$$

$$K_\lambda^r = 0, \quad (6.335)$$

$$K_\mu^r = r^2 \left\{ \left[ \frac{d(iW_n^k)}{dr} - \frac{iW_n^k}{r} \right]^2 + (n-1)(n+2) \left( \frac{iW_n^k}{r} \right)^2 \right\}. \quad (6.336)$$

The results for the spheroidal free oscillation modes are (6.298) and

$$K_\rho = \frac{4\pi}{2n+1} \frac{(n+k)!}{(n-k)!} \int_0^R \delta\rho r^2 \left\{ 8\pi G\rho (U_n^k)^2 - \sigma^2 \left[ (U_n^k)^2 + n(n+1)(V_n^k)^2 \right] \right. \\ - 2g \left[ \frac{2(U_n^k)^2}{r} - n(n+1) \frac{U_n^k V_n^k}{r} \right] - 8\pi G \int_r^R \rho \left[ \frac{2(U_n^k)^2}{q} \right. \\ \left. \left. - n(n+1) \frac{U_n^k V_n^k}{q} \right] dq - 2 \left[ U_n^k \frac{d\phi_n^k}{dr} + n(n+1) \frac{V_n^k \phi_n^k}{r} \right] \right\} dr, \quad (6.337)$$



$$K_\lambda = \frac{4\pi}{2n+1} \frac{(n+k)!}{(n-k)!} \int_0^R \delta\lambda r^2 \left[ \frac{dU_n^k}{dr} + \frac{2U_n^k}{r} - n(n+1) \frac{V_n^k}{r} \right]^2 dr, \quad (6.338)$$

$$K_\mu = \frac{4\pi}{2n+1} \frac{(n+k)!}{(n-k)!} \int_0^R \delta\mu r^2 \left[ 2 \left( \frac{dU_n^k}{dr} \right)^2 + 4 \left( \frac{U_n^k}{r} \right)^2 + n(n+1) \left\{ \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right)^2 - \frac{2V_n^k}{r^2} [2U_n^k - (n^2 + n - 1)V_n^k] \right\} \right] dr. \quad (6.339)$$

By normalizing the solution  $U_n^k$  and  $V_n^k$  according to

$$\int_0^R \rho r^2 \left[ (U_n^k)^2 + n(n+1)(V_n^k)^2 \right] dr = 1, \quad (6.340)$$

we can write the result (6.301) as

$$\delta\sigma = \frac{1}{2\sigma} \int_0^R [\delta\rho K_\rho^r + \delta\lambda K_\lambda^r + \delta\mu K_\mu^r] dr, \quad (6.341)$$

where the functions of  $r$ ,  $K_\rho^r$ ,  $K_\lambda^r$ , and  $K_\mu^r$ , are

$$K_\rho^r = r^2 \left\{ 8\pi G\rho (U_n^k)^2 - \sigma^2 \left[ (U_n^k)^2 + n(n+1)(V_n^k)^2 \right] - 2g \left[ \frac{2(U_n^k)^2}{r} - n(n+1) \frac{U_n^k V_n^k}{r} \right] - 8\pi G \int_r^R \rho \left[ \frac{2(U_n^k)^2}{q} - n(n+1) \frac{U_n^k V_n^k}{q} \right] dq - 2 \left[ U_n^k \frac{d\phi_n^k}{dr} + n(n+1) \frac{V_n^k \phi_n^k}{r} \right] \right\}, \quad (6.342)$$

$$K_\lambda^r = r^2 \left[ \frac{dU_n^k}{dr} + \frac{2U_n^k}{r} - n(n+1) \frac{V_n^k}{r} \right]^2, \quad (6.343)$$

$$K_\mu^r = r^2 \left\{ 2 \left( \frac{dU_n^k}{dr} \right)^2 + 4 \left( \frac{U_n^k}{r} \right)^2 + n(n+1) \left[ \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right)^2 - \frac{2V_n^k}{r} [2U_n^k - (n^2 + n - 1)V_n^k] \right] \right\}. \quad (6.344)$$

The density and elastic parameters we used so far are  $\rho$ ,  $\lambda$  and  $\mu$ . For both the toroidal and spheroidal free oscillation modes, we can convert the formulae to a

form using another set of density and elastic parameters, i.e.,  $\rho$ ,  $k$  and  $\mu$ ,

$$\delta\sigma = \frac{1}{2\sigma} \int_0^R \left[ \delta\rho \tilde{K}_\rho^r + \delta k \tilde{K}_k^r + \delta\mu \tilde{K}_\mu^r \right] dr, \quad (6.345)$$

where  $\tilde{K}_\rho^r$ ,  $\tilde{K}_k^r$ , and  $\tilde{K}_\mu^r$  can be obtained by substituting  $\delta\lambda = \delta k - (2/3)\delta\mu$  into (6.333) and (6.341) for the toroidal and spheroidal free oscillation modes, respectively. Evidently, we have  $\tilde{K}_\rho^r = K_\rho^r$ ,  $\tilde{K}_k^r = K_\lambda^r$ , and  $\tilde{K}_\mu^r = K_\mu^r - (2/3)K_\lambda^r$  for both cases. Here we provide the formula of  $\tilde{K}_\mu^r$  for the spheroidal case,

$$\begin{aligned} \tilde{K}_\mu^r = r^2 \left\{ \frac{1}{3} \left[ 2 \frac{dU_n^k}{dr} - \left( \frac{2U_n^k}{r} - n(n+1) \frac{V_n^k}{r} \right) \right]^2 + n(n+1) \left( \frac{dV_n^k}{dr} + \frac{U_n^k - V_n^k}{r} \right)^2 \right. \\ \left. + (n-1)n(n+1)(n+2) \left( \frac{V_n^k}{r} \right)^2 \right\}. \end{aligned} \quad (6.346)$$

Other formulae are evident.

Finally, we mention that possible perturbations of the boundaries of discontinuity are not included in our formulations. The readers are referred to Woodhouse (1976).

## 6.7 Relations Among the Free Oscillation, Surface Wave, and Body Wave Theories

The body waves, surface waves, and free oscillations are all solutions of the equations governing the Earth's elastic deformation. Hence, they are expected to be the same solution expressed differently, and it is indeed so, as will be demonstrated in this section.

### 6.7.1 Correspondence Between Free Oscillations and Surface Waves

We discuss only the case of the spherically symmetrical non-rotating Earth model, in which case, spheroidal and toroidal free oscillation modes are decoupled, and the components of different degrees and orders are also independent. Particularly, the solutions depend only on the degree, but independent of the order.

As can be inferred from the free oscillation theory, all motions of the Earth without the effect of external force can be represented by a combination of spheroidal and toroidal free oscillations of different degrees and orders. The reason is that, except free oscillation modes, there exists no other solutions of the Earth's elastic equation satisfying all the boundary conditions. In other words, free oscillations, surface waves and body waves are different expressions of the same solutions.

A correspondence between higher degree free oscillations and surface waves can be easily found using the asymptotic expression of the Legendre function, (4.186). Define the location of the wave source as the pole. The displacement is then (6.183) with  $k = 0$ . With  $x = R\theta$ , the displacement associated with  ${}_mS_n$  is

$$u_r = y_1^n \left( \frac{1}{2\pi n \sin \theta} \right)^{1/2} \left\{ \exp \left[ i \frac{n+1/2}{R} \left( x + \frac{R \cdot {}_m\sigma_n}{n+1/2} t \right) - i \frac{\pi}{4} \right] + \exp \left[ -i \frac{n+1/2}{R} \left( x - \frac{R \cdot {}_m\sigma_n}{n+1/2} t \right) + i \frac{\pi}{4} \right] \right\}, \quad (6.347)$$

$$u_\theta = i n y_3^n \left( \frac{1}{2\pi n \sin \theta} \right)^{1/2} \left\{ \exp \left[ i \frac{n+1/2}{R} \left( x + \frac{R \cdot {}_m\sigma_n}{n+1/2} t \right) - i \frac{\pi}{4} \right] - \exp \left[ -i \frac{n+1/2}{R} \left( x - \frac{R \cdot {}_m\sigma_n}{n+1/2} t \right) + i \frac{\pi}{4} \right] \right\}, \quad (6.348)$$

and the that associated with  ${}_mT_n$  is

$$u_\lambda = -n y_7^n \left( \frac{1}{2\pi n \sin \theta} \right)^{1/2} \left\{ \exp \left[ i \frac{n+1/2}{R} \left( x + \frac{R \cdot {}_m\sigma_n}{n+1/2} t \right) - i \frac{\pi}{4} \right] - \exp \left[ -i \frac{n+1/2}{R} \left( x - \frac{R \cdot {}_m\sigma_n}{n+1/2} t \right) + i \frac{\pi}{4} \right] \right\}, \quad (6.349)$$

where terms of relative order of magnitude  $1/n$  or smaller have been neglected.

By comparing (6.347) and (6.348) with (6.88) and (6.89), we see that the first terms at the right-hand sides of (6.347) and (6.348) are a Rayleigh wave propagating from the pole  $\theta = \pi$  to the pole  $\theta = 0$ , and the second terms a Rayleigh wave propagating from the pole  $\theta = 0$  to the pole  $\theta = \pi$ . Note that  $u_r$  corresponds to  $u_3$ , and  $u_\theta$  to  $u_1$ . By comparing (6.349) with (6.95) or (6.96), the first term at the right-hand side of (6.349) is a Love wave propagating from the pole  $\theta = \pi$  to the pole  $\theta = 0$ , and the second term a Love wave propagating from the pole  $\theta = 0$  to the pole  $\theta = \pi$ . The phase velocities of all of these surface waves are

$$({}_m v_n)_{SW} = \frac{R \cdot {}_m\sigma_n}{n+1/2}, \quad (6.350)$$

where the subscript  $SW$  is used to indicate surface wave. The wave numbers are all  $[n + (1/2)]/R$ , and the wavelengths and angular frequencies are all

$$({}_m \lambda_n)_{SW} = \frac{2\pi R}{n+1/2}, \quad ({}_m \sigma_n)_{SW} = {}_m \sigma_n, \quad (6.351)$$

i.e., the angular frequencies of the free oscillation and the corresponding surface wave are identical. The group velocities of the surface waves are, according to the

definition (6.124),

$$({}_m C_n)_{SW} = \frac{d_m \sigma_n}{d[(n + 1/2)/R]} = R \frac{d_m \sigma_n}{dn}, \quad (6.352)$$

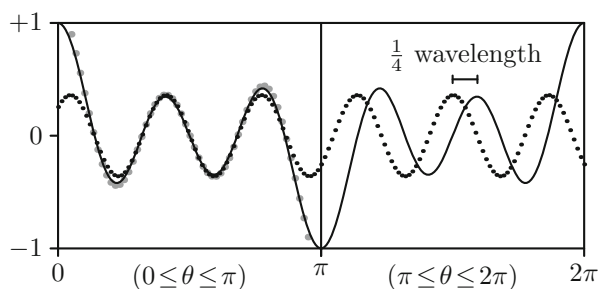
where the derivative should be computed using differences. The appearance of the factor  $1/\sin^{1/2} \theta$  in (6.347)–(6.349) is due to the fact that the total energy on all circles with a constant  $\theta$  are equal. In practical applications, the angular frequencies of higher frequency free oscillation modes are often computed according to (6.350) using observation data of the surface wave velocity.

The poles, i.e., the location of the source of seismic waves and its antipode, where  $\sin \theta = 0$ , are singular points in (6.347)–(6.349). An extra problem arises around the antipode. As the wave number of the surface wave is  $(n + 1/2)/R$ , there should be  $n + 1/2$  full cycles within the domain  $0 \leq \theta \leq 2\pi$ , where  $\pi < \theta \leq 2\pi$  is understood as an extension beyond the antipode. However,  $P_n(\cos \theta)$  has only  $n$  zero points within  $0 \leq \theta \leq \pi$ , implying that there are actually  $n$  full cycles within  $0 \leq \theta \leq 2\pi$ . This is due to the fact that the cycle of  $P_n(\cos \theta)$  passing through the antipode is  $1/4$  wavelength longer. A comparison between  $P_5(\cos \theta)$  and its asymptotic expression  $0.36 \cos[5.5\theta - (\pi/4)]$  is shown in Fig. 6.16, where the coefficient 0.36 is the value of the factor  $[2/(\pi n \sin \theta)]^{-1/2}$  in the asymptotic expression (4.185) at  $\theta = \pi/2$ . Evidently, in the domain  $\pi < \theta < 2\pi$ ,  $P_5(\cos \theta)$  is  $1/4$  wavelength behind  $0.36 \cos[5.5\theta - (\pi/4)]$ . Hence, the asymptotic expressions of free oscillation, (6.347)–(6.349), are not to be used beyond the antipode. We have also shown in Fig. 6.16 the asymptotic values of  $P_5(\cos \theta)$  according to (4.185), which, as expected, approximates  $P_5(\cos \theta)$  better than  $0.36 \cos[5.5\theta - (\pi/4)]$ , particularly, close to the poles.

We mention that, as can be seen from (6.347) and (6.348), for a spheroidal free oscillation mode, the order of magnitude of  $u_r$  is the same as  $y_1^n$  or  $U_n^k$ , and that of  $u_\theta$  the same as that of  $ny_3^n$  or  $nV_n^k$ .

In fact, the theories of surface waves and free oscillations are also referred to as mode theories of the half space and the spherical Earth model, respectively.

**Fig. 6.16** The lag of  $1/4$  wavelength across the antipode. Solid line is the function  $P_5(\cos \theta)$ , black dots the expression  $0.36 \cos(5.5\theta - \pi/4)$ , and larger gray dots the asymptotic expression (4.185)



### 6.7.2 Correspondence Between Body Waves and Free Oscillations

The solutions of free oscillations also includes body waves. The equation of the seismic ray of parameter  $p$  is

$$p = \frac{r \sin i}{v} = \frac{R \sin i_0}{v_0} = \frac{dT}{d\Delta}. \quad (6.353)$$

An immediate observation is that the phase velocity of the free oscillations interpreted as surface waves given by (6.350) can be understood as the visual velocity of a body wave, i.e.,

$$\bar{v}_0 = R \frac{d\Delta}{dT} = \frac{R \cdot {}_m\sigma_n}{n + 1/2}. \quad (6.354)$$

The relation between a free oscillation mode and the corresponding body wave can be obtained to be

$${}_m p_n = \frac{n + 1/2}{{}_m \sigma_n}, \quad (6.355)$$

where  ${}_m p_n$  is the ray parameter of the body wave corresponding to the free oscillation mode  ${}_m S_n$  or  ${}_m T_n$ .

The wavelengths of seismic rays are short. Hence, they correspond to high frequency free oscillation modes. Naturally,  ${}_m S_n$  includes both  $P$  and  $SV$  waves, whereas  ${}_m T_n$  includes only  $SH$  waves.

As free oscillations include all solutions of body waves, and free oscillation frequencies are discrete, the frequencies of body waves must also be discrete, which is a property not found based on the wave theory.

### 6.7.3 Correspondence Between Surface Waves and Body Waves

While free oscillations can be used to explain both body and surface waves, there must be a direct relation between body and surface waves. Here we discuss only the Love wave of a homogeneous layer overlying a half space.

For this simple model, the Love wave can be understood as the interference between the  $SH$  wave and its total internal reflection from the boundary between the layer and the half space. Hence, we first discuss the total internal reflection of the  $SH$  wave, which should satisfy the solution of the model studied in Sect. 6.1.4, i.e., (6.66), (6.67), and (6.68).

Total internal reflection happens when the value of the incident angle  $i_{SH}^{(0)}$  is in the range that no solution of the refraction angle  $i_{SH}^{(2)}$  as a real value exists according to (6.66). This requires

$$\sin i_{SH}^{(2)} = \frac{V_S^{(2)}}{V_S^{(1)}} \sin i_{SH}^{(0)} > 1, \quad (6.356)$$

which can be solved to obtain

$$\cot i_{SH}^{(2)} = \pm i \left\{ 1 - \left[ \frac{V_S^{(1)}}{V_S^{(2)}} \right]^2 \frac{1}{\sin^2 i_{SH}^{(0)}} \right\}^{1/2}. \quad (6.357)$$

Substitute this solution into (6.61), and take reference to Fig. 6.4. We see that, when the distance to the boundary,  $-x_3$ , tends to infinitely large, the solution of  $u_2$  with ‘+’ tends to infinitely large, and that with ‘-’ tends to zero. Hence, the solution with ‘-’ is the physically acceptable solution,

$$\cot i_{SH}^{(2)} = -i \left\{ 1 - \left[ \frac{V_S^{(1)}}{V_S^{(2)}} \right]^2 \frac{1}{\sin^2 i_{SH}^{(0)}} \right\}^{1/2}. \quad (6.358)$$

Substitute this solution into (6.67) and (6.68), and take reference to Fig. 6.4. We can solve them to obtain

$$B^{(1)} = \frac{\mu^{(1)} \cot i_{SH}^{(0)} + i\mu^{(2)} \left\{ 1 - \left[ \frac{V_S^{(1)}}{V_S^{(2)}} \right]^2 \frac{1}{\sin^2 i_{SH}^{(0)}} \right\}^{1/2}}{\mu^{(1)} \cot i_{SH}^{(0)} - i\mu^{(2)} \left\{ 1 - \left[ \frac{V_S^{(1)}}{V_S^{(2)}} \right]^2 \frac{1}{\sin^2 i_{SH}^{(0)}} \right\}^{1/2}} B^{(0)}. \quad (6.359)$$

This formula can be further developed to

$$B^{(1)} = e^{2i\zeta} B^{(0)}, \quad (6.360)$$

where  $\zeta$  satisfies

$$\tan \zeta = \frac{\mu^{(2)} \left\{ 1 - \left[ \frac{V_S^{(1)}}{V_S^{(2)}} \right]^2 \frac{1}{\sin^2 i_{SH}^{(0)}} \right\}^{1/2}}{\mu^{(1)} \cot i_{SH}^{(0)}} = \frac{\mu^{(2)} \left\{ \sin^2 i_{SH}^{(0)} - \left[ \frac{V_S^{(1)}}{V_S^{(2)}} \right]^2 \right\}^{1/2}}{\mu^{(1)} \cos i_{SH}^{(0)}}. \quad (6.361)$$

Hence, the reflected wave is, with  $B^{(1)}$  given by (6.360),

$$u_2^{(1)} = B^{(0)} \exp \left\{ i \left[ \left( x_1 + x_3 \cot i_{SH}^{(0)} - \frac{V_S^{(1)}}{\sin i_{SH}^{(0)}} t \right) + 2\zeta \right] \right\}, \quad (6.362)$$

where the fact  $i_{SH}^{(1)} = i_{SH}^{(0)}$  is used. It can be seen that, as compared to the incident wave, the reflected wave is  $2\zeta$  advance in phase. When the half space is absent, i.e., the bottom of the layer is a free surface, we have  $\mu^{(2)} = 0$ , and we can obtain



where  $v = V_S^{(1)}/\sin i_{SH}^{(0)}$  and  $\kappa = (2\pi/l) \sin i_{SH}^{(0)}$ . This is the equation of the Love wave of velocity  $v$  and wave number  $\kappa$ .

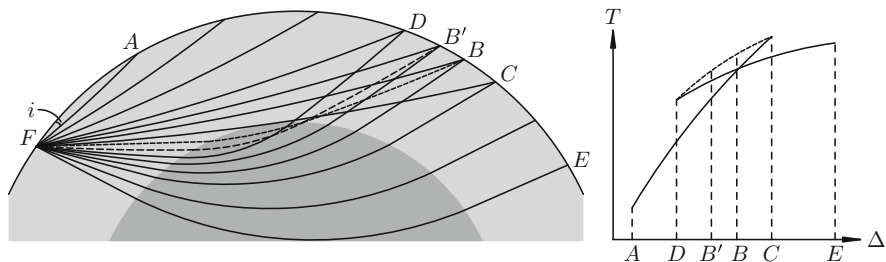
## 6.8 Determination of Density and Elastic Parameters Within the Earth

### 6.8.1 Determination of the Earth's Internal Structure Using Data of Seismic Rays

The travel time  $T$  of a seismic ray can be observed using seismometers, and the angles between the source and seismometer with respect to the Earth center,  $\Delta$ , can be computed using their locations. A relation between the travel time  $T$  and the geocentric angle  $\Delta$  can then be derived from the data of observations at different locations, called travel-time curve or  $T$ - $\Delta$  curve.

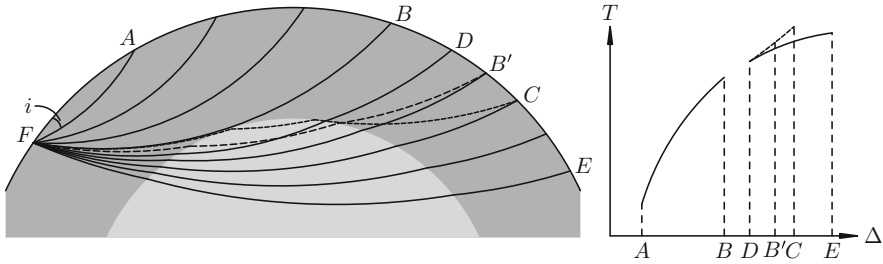
The internal structure of the Earth can be studied using  $T$ - $\Delta$  curves. In general, the velocity  $v$  of a seismic wave increases regularly with depth. However, irregularities and jumps also happen. All the factors influence the geometrical shapes and travel times of seismic rays, thus impacting  $T$ - $\Delta$  curves. Thinking reversely, we can use the shapes of  $T$ - $\Delta$  curves to study the Earth's internal structure.

If the rate of increase of the velocity with depth,  $dv/dh = -dv/dr$ , suddenly becomes much larger, as shown in Fig. 6.18, the bending of a seismic ray would become quicker as well. Consequently, a seismic ray that entered the higher velocity gradient zone would arrive at the same places where a seismic ray without entering the zone also arrives, and a crossover appears in the  $T$ - $\Delta$  curve. If the rate of increase of the velocity with depth suddenly becomes much smaller (the rate could even becomes negative, as far as not causing the ray to twist toward the Earth center), as shown in Fig. 6.19, the bending of a seismic ray would become slower as well. Consequently, a seismic ray that entered the lower velocity gradient zone would



**Fig. 6.18** When the rate of increase of wave velocity,  $dv/dh = -dv/dr$ , suddenly becomes much quicker at a certain depth, a crossover appears in the  $T$ - $\Delta$  curve (not to scale). The dashed  $T$ - $\Delta$  curve between  $D$  and  $C$  in the right side figure correspond to the dashed rays in the left side figure. With the increase of the incident angle  $i$ , a seismic ray reaches the Earth's surface in the order  $A$ - $D$ - $B'$ - $B$ - $C$ - $E$ . Three rays of different incident angle  $i$  would arrive at a location between  $D$  and  $C$ , which is referred to as triplication





**Fig. 6.19** When the rate of increase of wave velocity,  $dv/dh = -dv/dr$ , suddenly becomes much slower, or even becomes negative, at a certain depth, a shadow appears in the  $T$ - $\Delta$  curve (not to scale). Again, the dashed  $T$ - $\Delta$  curve between  $D$  and  $C$  in the right side figure correspond to the dashed rays in the left side figure. With the increase of the incident angle  $i$ , a seismic ray reaches the Earth's surface in the order  $A$ - $B$ -JUMP- $C$ - $B'$ - $D$ - $B'$ - $C$ - $E$

arrive at a farther place, and a shadow would appear in the  $T$ - $\Delta$  curve. If the velocity itself,  $v$ , has a sudden increase or decrease across a boundary instead of its rate of increase with depth,  $dv/dh$ , reflection and refraction happen, and a crossover or a shadow would also appear in the  $T$ - $\Delta$  curve. In this case, the dashed area of the refracted ray in Figs. 6.18 or 6.19 would not be present. However, a reflected ray would be present throughout the left side of  $C$  in Fig. 6.18 and that of  $B$  in Fig. 6.19. More complicated changes in velocity would make the  $T$ - $\Delta$  curve more complicated (e.g., Bormann, 2012, Chapter 2).

Based on the principles established so far, we can study the Earth's interior layer by layer. A conceptual procedure is as follows:

1. We take the outermost layer as a regular layer, where the wave velocity changes smoothly, use (6.153) and (6.154) to compute the wave velocity till a crossover or shadow appears, and then use the data of the crossover or shadow to compute the depth where the second regular layer begins.
2. We take away the first layer, and repeat the step (1) to determine the velocity in the second layer and find the depth where the third layer begins.
3. We repeat step (2) down to the Earth center.

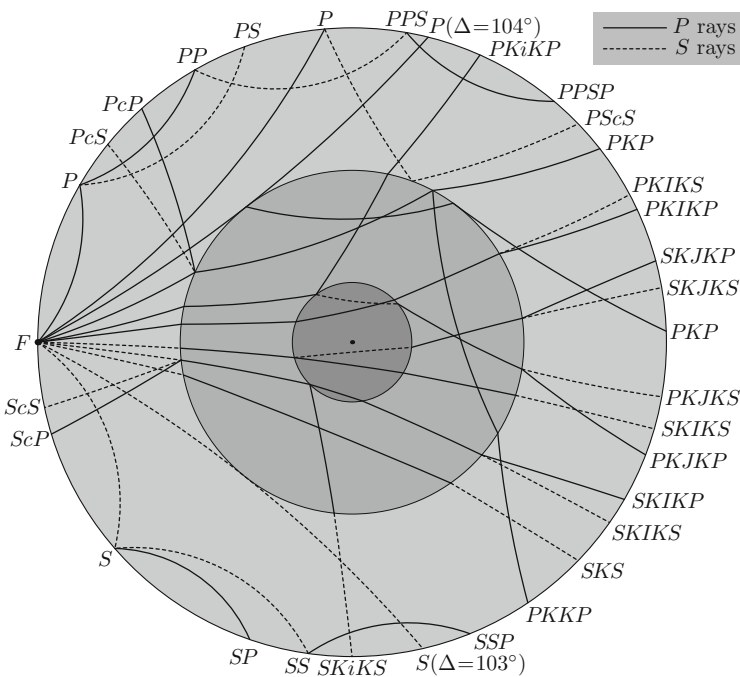
Another approach is to add a fictitious layer after removing each layer, which can be considered as a whole with the layer below, where wave velocity changes smoothly throughout, so that (6.153) and (6.154) can be used to compute the wave velocity.

The Earth's interior, as a big picture, is composed of mainly four layers which are called, from the surface to the center, crust, mantle, outer core and inner core. Except the outer core that is fluid, all other layers are solid. All the boundaries in between these adjacent layers have been discovered based on crossovers and shadows of  $T$ - $\Delta$  curves during the first half of the previous century. For convenience of distinguishing different seismic rays in the Earth, a naming convention was introduced:

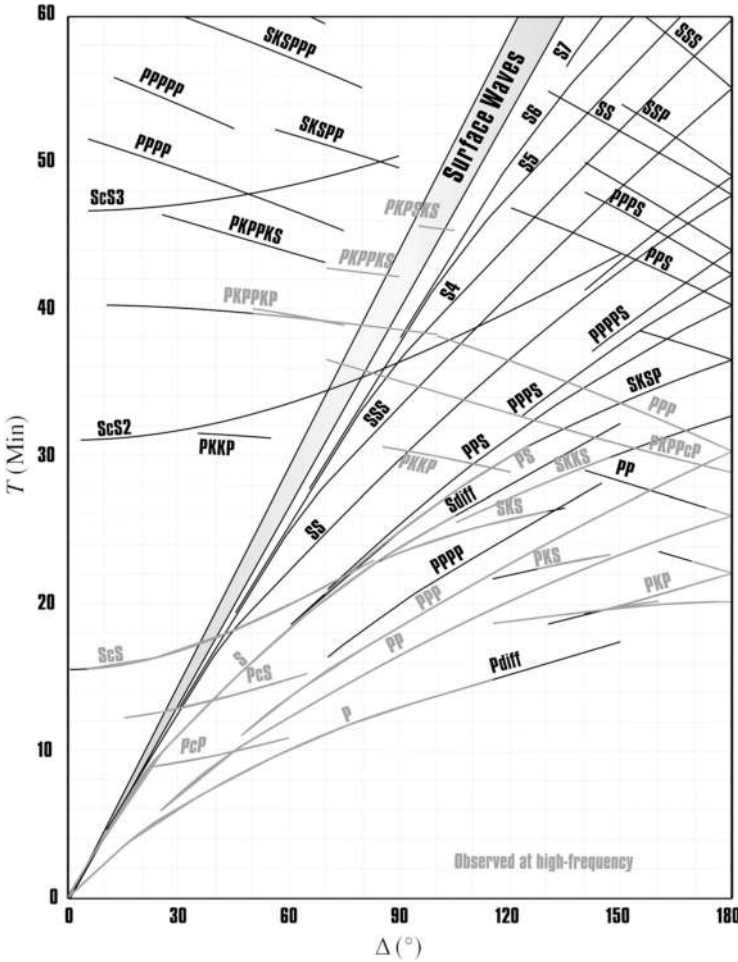
- $P$  and  $S$ :  $P$  and  $S$  waves in the crust and mantle, respectively.
- $K$ :  $P$  wave in the outer core.
- $I$  and  $J$ :  $P$  and  $S$  wave in the inner core, respectively.
- $c$  and  $i$ : A reflection on the core-mantle boundary (CMB) and inner core boundary (ICB), respectively.

A  $P$  wave from the source arriving directly to the station is denoted as  $P$ . Let a  $P$  wave from the source arrive at a point on the Earth's surface where it is reflected to produce a  $P$  wave and an  $S$  wave, and both arrive at stations; the former is denoted as  $PP$ , and the later  $PS$ . A  $P$  wave that goes through every layer of the Earth arriving at the opposite side of the source is denoted as  $PKIKP$ . Some more examples are shown in Fig. 6.20. Each of these rays is referred to as a seismic phase. These are only the most common ones. The readers are referred to Storchak et al. (2011) for a comprehensive list. Different seismic phases are used to study different layers of the Earth.

Observational  $T$ - $\Delta$  curves for various seismic phases can be established statistically using large amounts of data. Theoretical  $T$ - $\Delta$  curves can be computed



**Fig. 6.20** Illustration of the naming convention of seismic phases (not to relate to reality). These are only the most fundamental ones. The readers are referred to Storchak et al. (2011) for a comprehensive list, and to the AK135 travel time tables (Kennett et al., 1995) for illustrations showing the reality with more details



**Fig. 6.21** The IASP91 travel time table (Credit: U.S. Geological Survey). The seismic phases in gray are observed only at frequencies higher than  $1/6$  Hz, or at periods shorter than 6 s (Astiz et al., 1996). The number  $n$  following a string indicates that the string repeats  $n$  times, e.g.,  $S4$  is a short notation of  $SSSS$ , and  $ScS2$  of  $ScSScS$ . The readers are referred to Storchak et al. (2011) for a comprehensive list of seismic phases and their symbols, among which only some most common ones are explained in this book

using a model of seismic wave velocity in the Earth's interior. A collection of these  $T$ - $\Delta$  curves is referred to as a travel time table. In Fig. 6.21, we show the IASP91 travel time table derived from a spherically symmetrical velocity model (Kennett & Engdahl, 1991), which was the result of an effort of the Sub-Commission on Earthquake Algorithms of the International Association of Seismology and the Physics of the Earth's Interior (IASPEI). The starting and end points of the seismic phases are assumed to be on the Earth's surface. The effects of the Earth's ellipticity

and other deviations from the spherically symmetrical velocity model should be corrected in the observed data in order to compare with this model.

### 6.8.2 Constraint of Astro-Geodetic Data to Density Distribution

The following geometrical and physical parameters of the Earth can be determined using astro-geodetic observations: (1) the radius of the equator  $a = 6,378,137$  m; (2) the flattening  $f = 1/298.257$ ; (3) the mass  $M = 5.974 \times 10^{24}$  kg; (4) the dynamic form factor  $J_2 = 10,826.3 \times 10^{-7}$ ; (5) the dynamic flattening  $H = 1/305.4$ . The dynamic form factor and dynamic flattening are defined as

$$J_2 = \frac{C - A}{Ma^2}, \quad H = \frac{C - A}{C}, \quad (6.366)$$

where the Earth is assumed to be an ellipsoid of revolution, and  $A$  and  $C$  are the moments of inertia around an axis in the equator and the polar axis, respectively. The determinations of (1)–(4) have been mentioned in Sect. 5.2.2, and (5) is determined from the precession rate of the Earth rotation axis with respect to the inertial space, which will be discussed in Chap. 8.

Some overall constraint to the density distribution inside the Earth is derived from the above parameters. As a first order approximation, the Earth is assumed to be spherically symmetrical with (1) a radius equal to the equivolumetric radius of the ellipsoid,  $R = a(1 - f)^{1/3} = 6371$  km; (2) a total mass of  $M = 5.974 \times 10^{24}$  kg; (3) a moment of inertia factor of  $y = I/(MR^2) = 0.3307$ , where  $I = (2A + C)/3$  is the average of the moments of inertia. It can be verified that  $y$  is related to the parameters determined from observations in the form of

$$y = \frac{J_2}{H} \left[ 1 + \frac{2}{3}(f - H) \right], \quad (6.367)$$

which is the formula used to compute the numerical value of  $y$ . The relation between  $y$  and the internal density is, according to the expression of  $I$  in terms of the density  $\rho(r)$  given by (5.225),

$$y = \frac{2}{R^5 \bar{\rho}(R)} \int_0^R \rho(r) r^4 dr, \quad (6.368)$$

where  $\bar{\rho}(R)$  is the average density of the Earth that can be determined using  $M$  and  $R$ . It is straightforward to verify that  $y = 0.4$  when  $\rho$  is a constant. Hence, the actual value  $y = 0.0.3307 < 0.4$  implies that the density is larger toward the center.

### 6.8.3 Determination of Density and Elastic Parameters Using Seismic Velocity Data

The physical parameters  $\rho$ ,  $k$ , and  $\mu$  depend on the chemical compositions, material phases, temperature, pressure, etc. By inspecting the dependence of the  $P$  and  $S$  waves on  $\rho$ ,  $k$ , and  $\mu$ , we define a parameter

$$\phi = V_P^2 - \frac{4}{3}V_S^2 = \frac{k}{\rho}. \quad (6.369)$$

It combines the two seismic wave velocities into one parameter by eliminating the parameter  $\mu$ . This implies that  $k/\rho$  can be determined using  $V_P$  and  $V_S$ . In order to determine  $\rho$  and  $k$  individually, another relation between  $\rho$  and  $k$  is required. Once  $\rho$  is determined,  $\mu$  can be determined using  $\rho$  and  $V_S$ .

No other measurements are available to establish a relation between  $\rho$  and  $k$  independent of (6.369). In order to determine  $\rho$  and  $k$  individually, we must make some kind of assumption on the physical state of the Earth's interior to establish a new relation between  $\rho$  and  $k$ .

First, in the spherical approximation,  $\rho$ ,  $k$ , and  $\mu$  are all functions of  $r$  alone. As a static state, we assume the Earth to be in hydrostatic equilibrium described by (3.410). Here it degenerates to

$$\frac{dp}{dr} = -\rho g, \quad (6.370)$$

where we have dropped the subscript '0'. Although this relation includes  $\rho$ , it also includes another parameter  $p$ . We now make another assumption to derive a new relation among  $p$ ,  $\rho$ , and  $k$ . We assume that the region inside the Earth we are studying is homogeneous in both chemical compositions and material phases, and that the variation of density is purely caused by adiabatic compression and expansion, i.e., when the pressure of a lower pressure point is increased adiabatically (i.e., no heat transfer) to that at a higher pressure point, the densities at the two points become identical. The definition of the bulk modulus is  $k = -dp/(dV/V)$ , where  $dV$  is the volume increment caused by the pressure increment  $dp$ . We use  $d\rho$  to denote the corresponding density increment. The conservation of mass is  $\rho V = (\rho + d\rho)(V + dV)$ , whence,  $dV = -(V/\rho)d\rho$ . We obtain

$$\frac{dp}{d\rho} = \frac{\rho}{k}, \quad (6.371)$$

which is the relation among  $p$ ,  $\rho$ , and  $k$  that we need. The adiabatic assumption is in agreement with the nature of seismic wave propagation, where the movement of the Earth material is so quick that the associated heat transfer is negligible, i.e., the parameter  $k$  in the wave velocity is in fact the adiabatic impressibility. Finally, we obtain from (6.369) to (6.371),

$$\frac{d\rho}{dr} + \frac{\rho g}{\phi} = 0, \quad (6.372)$$

which is an equation for determining the density using wave velocities, called Adams-Williamson equation.

A more general equation of density can be derived without making the assumptions of homogeneity and adiabaticity. We can obtain from (6.369),

$$\frac{dk}{dp} = \rho \frac{d\phi}{dp} + \phi \frac{d\rho}{dp}. \quad (6.373)$$

With the equation of hydrostatic equilibrium (6.370), we obtain

$$\frac{d\rho}{dr} + \left( \frac{dk}{dp} + \frac{1}{g} \frac{d\phi}{dr} \right) \frac{\rho g}{\phi} = 0. \quad (6.374)$$

Define

$$\eta = \frac{dk}{dp} + \frac{1}{g} \frac{d\phi}{dr}, \quad (6.375)$$

called Bullen's parameter. We can write (6.374) as

$$\frac{d\rho}{dr} + \eta \frac{\rho g}{\phi} = 0. \quad (6.376)$$

We note that only the hydrostatic equilibrium is assumed to derive this equation, but not the homogeneity and adiabaticity. It can be seen that (6.376) degenerates to (6.372) when  $\eta = 1$ . Hence, Bullen's parameter  $\eta$  is a measure of the deviation from homogeneity and adiabaticity. As  $dk/dp$  can only be determined in laboratory, but not using geophysical observations, Bullen's parameter  $\eta$  can not be determined using geophysical observations either.

Both (6.372) and (6.376) are ordinary differential equations. In order to solve them assuming the Earth can be approximated by homogeneous (in both chemical compositions and material phases) and adiabatic layers, the density on the Earth's surface as well as the jumps across internal surfaces of discontinuity must be known. These values can either be determined using geophysical observations, laboratory experiments, or constraint of the astro-geodetic data.

We mention that (6.372) has been usually used in the lower mantle and the core, and a linear empirical  $\rho$ - $V_P$  relation

$$\rho = a + bV_P \quad (6.377)$$

has been used in the upper mantle and the crust, where  $a$  and  $b$  are constant parameters. An early model is  $\rho = 0.252 + 0.3788V_P$ , where the unit of  $\rho$  is  $\text{g}\cdot\text{cm}^{-3}$ , and that of  $V_P$  is  $\text{km}\cdot\text{s}^{-1}$  (Birch, 1961). Earth materials are different at different

locations and depths, and the relation (6.377) is different for different materials. Therefore, the values of  $a$  and  $b$  at different locations and depths are different in more sophisticated models (e.g., Christensen & Mooney, 1995).

#### 6.8.4 Stratification in the Earth's Interior

Bullen's parameter  $\eta$  is closely related to another parameter  $N$ , called Brunt-väisälä frequency, or buoyancy frequency, which describes the stability inside the Earth, particularly in the fluid outer core. We consider again a layer of the spherically symmetrical Earth model, and denote the radial distance using  $r$ . We assume that an infinitesimal element of material has displaced vertically from  $r$  to  $r + \xi$ , where  $\xi$  is an infinitesimal quantity. We further assume that the displacement is adiabatic, i.e., the infinitesimal element of material has no heat exchange with the surroundings. We now compute the total force exerted on the infinitesimal element of material. After the displacement, the pressure on the element increases by

$$dp = \frac{dp}{dr} \xi = -\rho g \xi ; \quad (6.378)$$

and hence, the density of the element increases by

$$d\rho_i = \frac{\rho}{k} dp = -\frac{\rho^2 g}{k} \xi , \quad (6.379)$$

where  $k$  is the adiabatic bulk modulus. Outside the element, the density of the surrounding material increases by

$$d\rho_e = \frac{d\rho}{dr} \xi . \quad (6.380)$$

We denote the instantaneous volume of the material element after displacement as  $dV$ . The total force exerted on the fluid element, i.e., the sum of the weight and buoyant force, along the direction  $\hat{e}_r$  is

$$g(d\rho_e - d\rho_i)dV = \xi g \left( \frac{d\rho}{dr} + \frac{\rho^2 g}{k} \right) dV . \quad (6.381)$$

With the parameter  $N$  defined by

$$N^2 = -\frac{g}{\rho} \frac{d\rho}{dr} - \frac{\rho g^2}{k} , \quad (6.382)$$

we can write the total force exerted on the material element as

$$g(d\rho_e - d\rho_i)dV = -\rho \xi N^2 dV . \quad (6.383)$$

We neglect the friction between this material element and the surrounding material. Since  $d^2\xi/dt^2$  is the acceleration of the material element, we have, according to Newton's second law of motion,

$$\rho \frac{d^2\xi}{dt^2} dV = g(d\rho_e - d\rho_i) dV = -\rho \xi N^2 dV, \quad (6.384)$$

which we write in the form of

$$\frac{d^2\xi}{dt^2} + N^2 \xi = 0. \quad (6.385)$$

This is a homogeneous second order ordinary differential equation. Its general solution is

$$\xi = A \exp\{\sqrt{-N^2}t\} + B \exp\{-\sqrt{-N^2}t\}, \quad (6.386)$$

where  $A$  and  $B$  are arbitrary constants. This solution has the following properties:

1. When  $N^2 > 0$ , the solution is  $\xi = A \exp\{i|N|t\} + B \exp\{-i|N|t\}$ , meaning that the material element oscillates around the initial position. We say that the layer considered is in a stable stratification.
2. When  $N^2 = 0$ , the solution is  $\xi = A + B$ , meaning that the material element stays at the new position after the displacement. We say that the layer considered is in a neutral stratification.
3. When  $N^2 < 0$ , the solution is  $\xi = A \exp\{|N|t\} + B \exp\{-|N|t\}$ , meaning that the material element moves away further after the displacement. We say that the fluid is in an unstable stratification. A convection may happen in this state.

Evidently,  $N$  is a parameter describing the stability. As it has the dimension of a frequency, it is referred to as Brunt-väisälä frequency, or buoyancy frequency.

Convection is a property of fluids. How the mantle considered to be solid may convect? In fact, treating a substance as solid or liquid is relative depending on the property of the physical process involved. For short period phenomena such as seismic waves, the mantle does resist the shear stress, and behaves as a solid. However, for extremely long term actions, the mantle does yield to the shear stress, and flows. For example, the long term action of the centrifugal force caused the permanent bulging at the equator, rendering the Earth to be extremely close to an ellipsoid of revolution. In a thermal convection (assuming the substance to be the same everywhere), due to thermal expansion, a higher temperature region has lower density, and the buoyant force acting upward exceeds the weight acting downward, causing the material in the region to rise. Contrarily, the material in a lower temperature region sinks. For the Earth's interior, heating from below is a long term phenomenon, leading to an extremely slow convection, where higher temperature material rises, and lower temperature material sinks. In this perspective, The value of  $N^2$  in a convecting system should be negative and small in magnitude.



An alternative parameter describing the stratification is the Pekeris-Accad parameter defined as

$$\beta = -\frac{k}{\rho g^2} N^2 = 1 + \frac{k}{\rho^2 g} \frac{d\rho}{dr}. \quad (6.387)$$

According to (6.376), this parameter is related to Bullen's parameter in the form of

$$\eta = 1 - \beta = -\frac{k}{\rho^2 g} \frac{d\rho}{dr}. \quad (6.388)$$

For the spherical Earth model, we can write (6.387) as

$$\frac{d\rho}{dr} = (1 - \beta) \frac{\rho^2}{k} \frac{dW}{dr}, \quad (6.389)$$

where  $W$  is the gravity potential. This formula can be generalized to the rotating Earth model in hydrostatic equilibrium, where the gradients of  $\rho$  and  $W$  are in the same direction. We have (Rochester & Peng, 1993)

$$\nabla \rho = (1 - \beta) \frac{\rho^2}{k} \nabla W. \quad (6.390)$$

The stability parameter is much harder to determine as compared to  $\rho$ ,  $k$ , or  $\mu$ . In the Earth's interior, the deviation from homogeneity and adiabaticity is very small, i.e., the deviation  $\delta\eta = \eta - 1$  is very small. In order to determine this small deviation, very accurate determinations of  $k$  and  $d\rho/dr$  independent of any assumption are required, which can hardly be achieved. A neutral stratification is assumed in the Adams-Williamson equation (6.372).

### 6.8.5 Determination of Density and Elastic Parameters Using Free Oscillation Data

For using free oscillation data, we assumed that the Earth's internal layering structure is known, and an initial spherically symmetrical non-rotating Earth model of density and elastic parameters is available. The use of free oscillation data is for improving the initial model.

The procedure includes two steps. The first step is to convert the free oscillation data to those of a spherically symmetrical non-rotating Earth model, and the second step is to use the data to improve the initial model.

We know that, when the Earth is assumed to be spherically symmetrical, each free oscillation mode of the non-rotating model,  ${}_mS_n$  or  ${}_mT_n$ , corresponds to  $2n + 1$  modes of the rotating model,  ${}_mS_n^k$  or  ${}_mT_n^k$ , with  $k = 0, \pm 1, \pm 2, \dots, \pm n$ . In reality, this property remains the same when the effects of rotation, ellipticity, and lateral heterogeneities of the Earth's interior are all taken into account, meaning that

each free oscillation mode of the spherically symmetrical non-rotating Earth model corresponds to  $2n + 1$  modes of the real Earth. The effect of rotation has been formulated in this book accurate to only one term. More accurate formulation of the effect of rotation and that of the Earth's ellipticity are available in the literature (e.g., Dahlen & Sailor, 1979), which can be used to convert the free oscillation data. In summary, each free oscillation angular frequency of the spherically symmetrical non-rotating Earth model is deduced from  $2n + 1$  values.

The formulae for using free oscillation angular frequencies are (6.333)–(6.336) and (6.341)–(6.344), or their alternates in the form of (6.345) complemented with the corresponding expressions of  $\tilde{K}_\rho^r$ ,  $\tilde{K}_k^r$ , and  $\tilde{K}_\mu^r$ . However, the parameters provided in Earth models are customarily  $\rho$ ,  $V_P$ , and  $V_S$ . Hence, we use (6.27) and (6.28) to express  $\delta\lambda$  and  $\delta\mu$  in terms of  $\delta\rho$ ,  $\delta V_P$ , and  $\delta V_S$ , and then develop (6.333) and (6.341) to the form (e.g., Dahlen & Tromp, 1998)

$$\delta\sigma = \frac{1}{2\sigma} \int_0^R \left( \delta\rho \tilde{K}_\rho^r + \delta V_P \tilde{K}_{V_P}^r + \delta V_S \tilde{K}_{V_S}^r \right) dr, \quad (6.391)$$

where we re-emphasize that  $\sigma$  is the theoretical value of the initial Earth model, and  $\delta\sigma$  is the difference between the observed value and  $\sigma$ . Improving the Earth model is to determine  $\delta\rho$ ,  $\delta V_P$ , and  $\delta V_S$ . Since the kernels  $\tilde{K}_\rho^r$ ,  $\tilde{K}_{V_P}^r$ , and  $\tilde{K}_{V_S}^r$  are different for each mode, the observation equations based on (6.391) are

$$\delta\sigma_i = \frac{1}{2\sigma_i} \int_0^R \left( \delta\rho \tilde{K}_{\rho i}^r + \delta V_P \tilde{K}_{V_P i}^r + \delta V_S \tilde{K}_{V_S i}^r \right) dr, \quad (6.392)$$

where  $i = 1, 2, \dots, N$  indicate the numbering of observations. The problem is that  $\delta\rho$ ,  $\delta V_P$ , and  $\delta V_S$  are functions of  $r$ , while the total number of observations is finite. Hence, however large the amount of observation is, they are not enough to uniquely determine the parameters  $\delta\rho$ ,  $\delta V_P$ , and  $\delta V_S$ . A particular method named after the authors, Backus-Gilbert method, or shortly BG method, has been developed to solve the integral equation (6.392), not to obtain the values of  $\delta\rho$ ,  $\delta V_P$ , and  $\delta V_S$  as is, but their smoothed values (Backus & Gilbert, 1967, 1968, 1970). The improved Earth model is obtained by adding  $\delta\rho$ ,  $\delta V_P$ , and  $\delta V_S$  to the respective values of the initial model.

### 6.8.6 The Preliminary Reference Earth Model (PREM)

The Preliminary Reference Earth Model, briefly referred to as PREM (Dziewonski & Anderson, 1981), is a spherically symmetrical model of the Earth's interior divided into 13 continuous layers with jumps across adjacent layers. The parameters provided are the density, the  $P$  and  $S$  wave velocities, and the quality factors describing the deviation from pure elasticity to be briefly introduced below. A polynomial representation of the model is given in Table 6.2. It is derived using

**Table 6.2** The polynomial model of PREM ( $r$  is the radial distance,  $R$  is the radius of the Earth,  $d = R - r$  is the depth, and  $x = r/R$ )

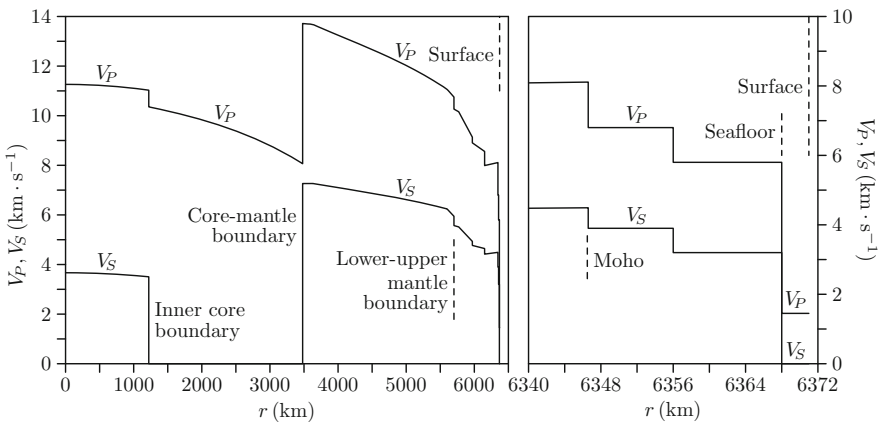
| Region       | $r$ (km) | $d$ (km) | $\rho$ (g·cm <sup>-3</sup> )                | $V_P$ (km·s <sup>-1</sup> )                    | $V_S$ (km·s <sup>-1</sup> )                   | $Q_k$    | $Q_\mu$  |
|--------------|----------|----------|---------------------------------------------|------------------------------------------------|-----------------------------------------------|----------|----------|
| Inner core   | 0        | 6370     | $13.0885 - 8.8381x^2$                       | $11.2622 - 6.3640x^2$                          | $3.6678 - 4.4475x^2$                          | $1327.7$ | $84.6$   |
|              | 1221.5   | 5149.5   |                                             |                                                |                                               |          |          |
| Outer core   | 1221.5   | 5149.5   | $12.5815 - 1.2638x - 3.6426x^2 - 5.5281x^3$ | $11.0487 - 4.0362x + 4.8023x^2 - 13.5732x^3$   | 0                                             | 57823    | $\infty$ |
|              | 3480     | 2891     |                                             |                                                |                                               |          |          |
| Lower mantle | 3480     | 2891     | $7.9565 - 6.4761x + 5.5283x^2 - 3.0807x^3$  | $15.3891 - 5.3181x + 5.5242x^2 - 2.5514x^3$    | $6.9254 + 1.4672x - 2.0834x^2 + 0.9783x^3$    | 57823    | 312      |
|              | 3630     | 2741     |                                             |                                                |                                               |          |          |
|              | 3630     | 2741     | $7.9565 - 6.4761x + 5.5283x^2 - 3.0807x^3$  | $24.9520 - 40.4673x + 51.4832x^2 - 26.6419x^3$ | $11.1671 - 13.7818x + 17.4575x^2 - 9.2777x^3$ | 57823    | 312      |
|              | 5600     | 771      |                                             |                                                |                                               |          |          |
|              | 5600     | 771      | $7.9565 - 6.4761x + 5.5283x^2 - 3.0807x^3$  | $29.2766 - 23.6027x + 5.5242x^2 - 2.5514x^3$   | $22.3459 - 17.2473x - 2.0834x^2 + 0.9783x^3$  | 57,823   | 312      |
|              | 5701     | 670      |                                             |                                                |                                               |          |          |
| Upper mantle | 5701     | 670      | $5.3197 - 1.4836x$                          | $19.0957 - 9.8672x$                            | $9.9839 - 4.9324x$                            | 57,823   | 143      |
|              | 5771     | 600      |                                             |                                                |                                               |          |          |
|              | 5771     | 600      | $11.2494 - 8.0298x$                         | $39.7027 - 32.6166x$                           | $22.3512 - 18.5856x$                          | 57,823   | 143      |
|              | 5971     | 400      |                                             |                                                |                                               |          |          |
|              | 5971     | 400      | $7.1089 - 3.8045x$                          | $20.3926 - 12.2569x$                           | $8.9496 - 4.4597x$                            | 57,823   | 143      |
|              | 6151     | 220      |                                             |                                                |                                               |          |          |
| Crust        | 6151     | 220      | $2.6910 + 0.6924x$                          | $4.1875 + 3.9382x$                             | $2.1519 + 2.3481x$                            | 57,823   | 80       |
|              | 6291     | 80       |                                             |                                                |                                               |          |          |
|              | 6291     | 80       | $2.6910 + 0.6924x$                          | $4.1875 + 3.9382x$                             | $2.1519 + 2.3481x$                            | 57,823   | 600      |
|              | 6346.6   | 24.4     |                                             |                                                |                                               |          |          |
|              | 6346.6   | 24.4     | 2.900                                       | 6.800                                          | 3.900                                         | 57,823   | 600      |
|              | 6356     | 15       |                                             |                                                |                                               |          |          |
| Ocean        | 6356     | 15       | 2.600                                       | 5.800                                          | 3.200                                         | 57,823   | 600      |
|              | 6368     | 3        |                                             |                                                |                                               |          |          |
|              | 6368     | 3        | 1.020                                       | 1.450                                          | 0                                             | 57,823   | $\infty$ |
|              | 6371     | 0        |                                             |                                                |                                               |          |          |

astro-geodetic, seismic body wave, long period surface wave, and free oscillation data. The model includes mainly the following layers from the surface to the center:

1. The ocean;
2. The upper and lower crusts;
3. The LID above the low velocity zone (LVZ);
4. The LVZ;
5. The region between the LVZ and the 400 km depth discontinuity;
6. The transition zone between the 400 km and 670 km discontinuities;
7. The lower mantle;
8. The outer core;
9. The inner core.

The upper part of the mantle, i.e., from the bottom of the crust to the 670 km depth discontinuity is referred to as upper mantle. The profile of density is provided in Fig. 5.15. The profiles of  $P$  and  $S$  wave velocities are shown in Fig. 6.22.

We are familiar with the concepts of density and  $P$  and  $S$  wave velocities. The quality factors  $Q_k$  and  $Q_\mu$  describes the deviation from pure elasticity. In a pure elastic medium, the seismic waves and free oscillations are not damped. However, they are damped in reality, implying that the Earth is not a purely elastic body. The damping of the waves and free oscillations causes the elastic parameters, and consequently the wave velocities, to be frequency dependent, i.e., the Earth material can still be treated as pure elastic for periodical phenomena, but the elastic parameters and wave velocities are different for different frequencies. The detail of the study of such rheology is beyond the scope of this book. We just take the relation



**Fig. 6.22** Profiles of body wave velocities of the Preliminary Reference Earth Model (PREM)

between the wave velocity and the quality factors as reference (Liu et al., 1976),

$$\begin{cases} V_S(T) = V_S(1) \left( 1 - \frac{\ln T}{\pi} \frac{1}{Q_\mu} \right), \\ V_P(T) = V_P(1) \left[ 1 - \frac{\ln T}{\pi} \left( \frac{1-E}{Q_k} + \frac{E}{Q_\mu} \right) \right], \end{cases} \quad (6.393)$$

where  $E = (4/3)[V_S(1)/V_P(1)]^2$ . The data of  $V_P$  and  $V_S$  in Table 6.2 are their values when the period of seismic wave or free oscillation is 1 s. The above formula provides their values when the period of seismic wave or free oscillation is  $T$  seconds.

In the PREM, the difference between the elastic behaviors in the vertical and horizontal directions is also considered, which is a kind of anisotropy referred to as transverse isotropy, and the corresponding wave velocities in the vertical and horizontal directions are provided. The readers are referred to the original literature for the detail. Based on our computations of gravity, pressure, and free oscillation periods, the value of the universal gravitational constant adopted in PREM is  $G = 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$ , which must be kept in mind when using the model. Several other spherical Earth models have been built afterward, such as the IASP91 and AK135 models (e.g., Bormann, 2012).

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## Bibliographic Notes

Numerous books are dedicated to seismology. The purpose of this chapter is to provide the background knowledge in physics on how models of the Earth's interior are built and with an emphasis on the free oscillation theory, which pertains to the global deformation theory. The author's main reference include Bullen and Bolt (1985), Dziewonski and Gilbert (1976), Fu et al. (1985), Jeffreys (1976), Stein and Wysession (1981) and Zeng (1984) for Sects. 6.1 through 6.3 and 6.7; Alterman et al. (1959), Backus and Gilbert (1967), Crossley and Rochester (1980), Dahlen and Sailor (1979), Guo et al. (2001), Lapwood and Usami (1981), Smith and Dahlen (1981), Takeuchi and Saito (1972) and Zhu (1991) for Sects. 6.4 and 6.5; Backus and Gilbert (1967), Birch (1961), Bullen and Bolt (1985), Dahlen and Sailor (1979), Dziewonski and Anderson (1981), Lapwood and Usami (1981), and Melchior (1986) for Sect. 6.8. These are also suggested as the readers' sources for alternative presentations.

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# Earth Tide and Surface Loading of the Spherical Elastic Earth Model

# 7

## Abstract

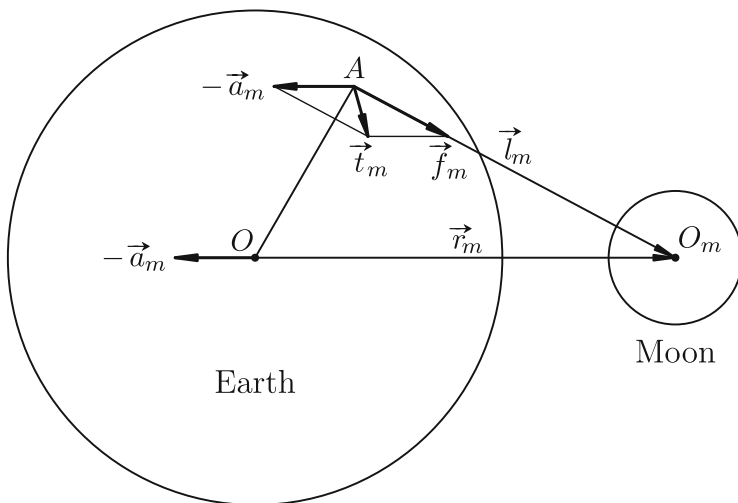
This chapter formulates systematically the Earth tide and surface loading of the spherical elastic Earth model. The Earth tide is the deformation of the Earth in response to the variable gravitational attraction of extraterrestrial celestial bodies. The surface loading is the deformation of the Earth in response to the variable gravitational attraction and surface pressure of the mass transport in the atmosphere-hydrosphere-cryosphere system. The subjects concerned include tidal-generating force and potential, development of the tidal-generating potential into harmonic waves, the tidal Love numbers, the observables affected by Earth tide, the dynamic and static load Love numbers, the asymptotic expressions of the load Love numbers, the load Green's function, and the practical computation of loading effects. Some closely related geophysical problems including the definition of the geocenter, the gravitational potential of ocean tide and its loading effect, and the pattern of sea level change as a result of present-day continent-ocean water exchange, are also formulated.

## 7.1 Tidal-Generating Force and Potential

### 7.1.1 Tidal-Generating Force and Potential

Besides the gravitational force of the mass of the Earth itself and the centrifugal force due to the rotation of the Earth, the sum of two other forces related to the Moon, the Sun and other extraterrestrial celestial bodies are observed on the Earth: (1) the gravitational force of the extraterrestrial celestial bodies; (2) the inertial force related to the acceleration of the center of mass of the Earth in the inertial reference system, which is a response of the translational motion (i.e., the displacement without changing orientation) of the Earth to the gravitational force of extraterrestrial celestial bodies. The sum of these two forces is referred to as tidal-





**Fig. 7.1** The lunar tidal-generating force at a point within the Earth (not to scale)

generating force, which is in fact the force  $\vec{f}$  in the equation of elastic deformation of the Earth, (3.333), and is defined in (3.334). In this subsection, we first derive the formula of the tidal-generating force, and then define the tidal-generating potential. In this book, we limit to discuss the Moon and the Sun, as the effects of other planets are much smaller.

We first consider the effect of the Moon as shown in Fig. 7.1. By definition, the lunar tidal-generating force exerted on a particle of unit point mass located at a point A is, with the subscript  $m$  indicating the Moon,

$$\vec{t}_m = \vec{f}_m - \vec{a}_m, \quad (7.1)$$

where  $\vec{f}_m$  is the gravitational attraction of the Moon, and  $\vec{a}_m$  the acceleration of the center of mass of the Earth in response to the gravitational attraction of the Moon. We use  $M$  and  $M_m$  to denote the masses of the Earth and the Moon, respectively, and  $O$  and  $O_m$  their centers of mass. We also define  $\vec{l}_m = \vec{AO}_m$ ,  $l_m = |\vec{l}_m|$ ,  $\vec{r}_m = \vec{OO}_m$ , and  $r_m = |\vec{r}_m|$ . Taking the Moon as a point mass, we can express  $\vec{f}_m$  and  $\vec{a}_m$  as

$$\vec{f}_m = G \frac{M_m}{l_m^3} \vec{l}_m, \quad (7.2)$$

$$\vec{a}_m = \frac{1}{M} \left( GM_m \int_V \frac{\rho}{l_m^3} \vec{l}_m dV \right). \quad (7.3)$$

The term within the parentheses is the total gravitational force of the Moon on the Earth, where  $V$  is the space domain occupied by the Earth, and  $\rho$  the density of the Earth. Noted that  $\vec{f}_m$  is a function of the position of  $A$ , while  $\vec{a}_m$  is position independent. As the expression (3.334) is the force exerted on the mass within a unit volume of density  $\rho$ , we have  $\vec{f} = \rho \vec{t}_m$ ,  $\vec{f}_e = \rho \vec{f}_m$  and  $\vec{a}_0 = \vec{a}_m$  for the lunar tidal-generating force.

If the Earth is approximately considered as spherically symmetrical, we can simplify the term in the parentheses in (7.3) to  $(GM M_m / r_m^3) \vec{r}_m$ . Hence, (7.3) is simplified to

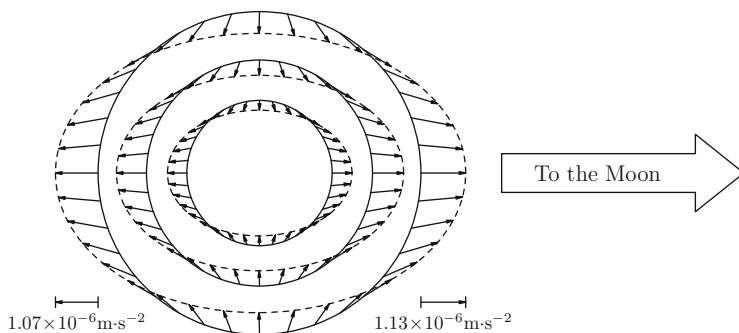
$$\vec{a}_m = G \frac{M_m}{r_m^3} \vec{r}_m. \quad (7.4)$$

The lunar tidal-generating force on a unit point mass located at  $A$ , (7.1), can then be written in the simple form

$$\vec{t}_m = G \frac{M_m}{l_m^3} \vec{l}_m - G \frac{M_m}{r_m^3} \vec{r}_m, \quad (7.5)$$

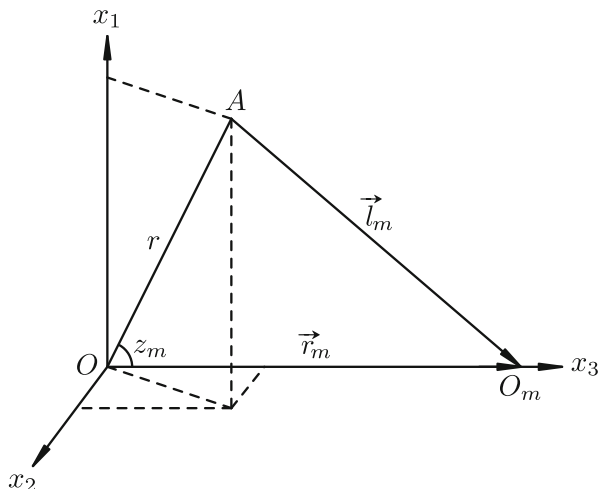
which is evidently the difference between the gravitational attractions of the Moon at  $A$  and  $O$ . We will simply refer  $\vec{t}_m$  defined in (7.5) to as the lunar tidal-generating force hereafter, which is in reality the intensity of the force, and has the dimension of acceleration. In descriptive physics, the tidal-generating force is the difference of gravitational attraction. The global distribution of the tidal-generating force is demonstrated in Fig. 7.2. It is left to the readers as an exercise to explain why the lunar tidal-generating force points upward at both sides facing and opposite to the Moon, and why the magnitude of the tidal-generating force at the side facing the Moon is a little larger than that at the opposite side.

In order to introduce the lunar tidal-generating potential, we use the coordinate system  $Ox_1x_2x_3$  defined in Fig. 7.3. We denote the coordinates of the point  $A$  as



**Fig. 7.2** Global distribution of the lunar tidal-generating force

**Fig. 7.3** Definition of a coordinate system  $Ox_1x_2x_3$  (not to scale)



$(x_1, x_2, x_3)$ . We then have

$$\vec{l}_m = -x_1\hat{e}_1 - x_2\hat{e}_2 - (x_3 - r_m)\hat{e}_3, \quad (7.6)$$

$$l_m = [x_1^2 + x_2^2 + (x_3 - r_m)^2]^{1/2}, \quad (7.7)$$

$$\vec{r}_m = r_m\hat{e}_3. \quad (7.8)$$

Upon substitution of these three formulae into (7.5), we obtain

$$\vec{l}_m = GM_m \left[ -\frac{x_1}{l_m^3}\hat{e}_1 - \frac{x_2}{l_m^3}\hat{e}_2 - \left( \frac{x_3 - r_m}{l_m^3} + \frac{1}{r_m^2} \right) \hat{e}_3 \right], \quad (7.9)$$

where  $r_m$  is independent of the coordinates of the point A. Define a scalar function

$$\phi_m = GM_m \left( \frac{1}{l_m} - \frac{x_3}{r_m^2} - \frac{1}{r_m} \right). \quad (7.10)$$

It can be readily verified that

$$\begin{cases} \frac{\partial \phi_m}{\partial x_1} = GM_m \left( -\frac{x_1}{l_m^3} \right), \\ \frac{\partial \phi_m}{\partial x_2} = GM_m \left( -\frac{x_2}{l_m^3} \right), \\ \frac{\partial \phi_m}{\partial x_3} = GM_m \left[ -\left( \frac{x_3 - r_m}{l_m^3} + \frac{1}{r_m^2} \right) \right]. \end{cases} \quad (7.11)$$

By comparing this formula with (7.9), we see that

$$\vec{t}_m = \nabla \phi_m. \quad (7.12)$$

Hence,  $\phi_m$  is the potential of  $\vec{t}_m$ , and is referred to as lunar tidal-generating potential. Note that the relations in (7.11) also hold without the constant term  $-1/r_m$  in (7.10). This constant term is included so that the lunar tidal-generating potential vanishes at the Earth center  $O$ .

The readers are referred to Dahlen (1993) for a formulation of the difference between the exact expression (7.3) and the approximate expression (7.4).

As both the gravitational attraction and the acceleration obey the addition rule of vectors, the total tidal-generating force and potential of all celestial bodies is the sum of each of them, which can be inferred based on (7.9) and (7.10).

### 7.1.2 Expansion of the Tidal-Generating Potential as a Spherical Harmonic Series

Like the gravitational potential, the lunar tidal-generating potential can also be expressed as a spherical harmonic series. As shown in Fig. 7.3, we use  $z_m$  to denote the angle  $\angle AOO_m$ , i.e., the angle between  $A$  and  $O_m$  with respect to  $O$ . We then have

$$\frac{1}{l_m} = \sum_{n=0}^{\infty} \frac{r^n}{r_m^{n+1}} P_n(\cos z_m). \quad (7.13)$$

By substituting this formula into (7.10), we obtain, with  $P_0(\cos z_m) = 1$  and  $P_1(\cos z_m) = \cos z_m = x_3/r$ ,

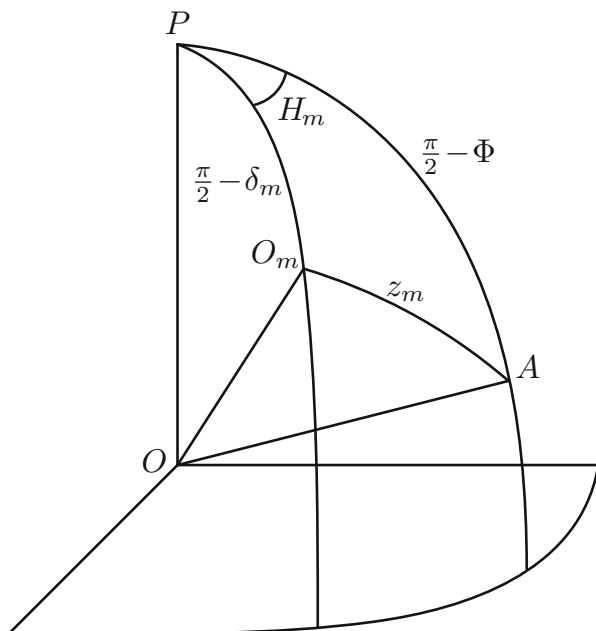
$$\phi_m = GM_m \sum_{n=2}^{\infty} \frac{r^n}{r_m^{n+1}} P_n(\cos z_m). \quad (7.14)$$

The angle  $z_m$  is the zenith angle of the Moon at  $A$ . We use  $H_m$  to denote the hour angle of the Moon at  $A$ ,  $\Phi$  the geocentric latitude of  $A$ , and  $\delta_m$  the declination of the Moon, as shown in Fig. 7.4, where  $O_m$  and  $A$  are the projection of these two points on the celestial sphere. Considering the spherical triangle  $O_mAP$ , we have

$$\cos z_m = \sin \Phi \sin \delta_m + \cos \Phi \cos \delta_m \cos H_m. \quad (7.15)$$

According to the addition theorem of spherical harmonics, we further have

$$P_n(\cos z_m) = \sum_{k=0}^n \frac{2}{1 + \delta_k} \frac{(n-k)!}{(n+k)!} P_n^k(\sin \Phi) P_n^k(\sin \delta_m) \cos k H_m. \quad (7.16)$$



**Fig. 7.4** The zenith angle of the Moon at A

Finally, the spherical harmonic series of the lunar tidal-generating potential is obtained by substituting (7.16) into (7.14),

$$\phi_m = GM_m \sum_{n=2}^{\infty} \sum_{k=0}^n \frac{r^n}{r_m^{n+1}} \frac{2}{1 + \delta_k} \frac{(n-k)!}{(n+k)!} P_n^k(\sin \Phi) P_n^k(\sin \delta_m) \cos k H_m. \quad (7.17)$$

The spherical harmonic series of the solar tidal-generating potential has the same form,

$$\phi_s = GM_s \sum_{n=2}^{\infty} \sum_{k=0}^n \frac{r^n}{r_s^{n+1}} \frac{2}{1 + \delta_k} \frac{(n-k)!}{(n+k)!} P_n^k(\sin \Phi) P_n^k(\sin \delta_s) \cos k H_s, \quad (7.18)$$

where the meaning of the symbols  $M_s$ ,  $\delta_s$ , and  $H_s$  are evident by comparing with the case of the Moon.

As already mentioned, we consider only the Moon and the Sun. The total tidal-generating potential is

$$\phi = \phi_m + \phi_s. \quad (7.19)$$

We continue to denote the equatorial radius of the Earth as  $a$ . As illustration, we follow Xi and Hou (1987) to adopt  $a = 6,378,140$  m,  $GM = 3.986005 \times$

$10^{14} \text{ m}^3 \text{ s}^{-2}$ , and

$$M_m/M = 1/81.30068, \quad a/r_m = 1/60.26816; \quad (7.20)$$

$$M_s/M = 332,946, \quad a/r_s = 1/23,454.8. \quad (7.21)$$

We limit to consider the lunar tidal-generating potential up to degree 3, and the solar one up to degree 2. According to Table 4.1, we have

$$P_2(\sin \alpha) = \frac{3}{2} \sin^2 \alpha - \frac{1}{2}, \quad (7.22)$$

$$P_2^1(\sin \alpha) = 3 \sin \alpha \cos \alpha, \quad (7.23)$$

$$P_2^2(\sin \alpha) = 3 \cos^2 \alpha, \quad (7.24)$$

$$P_3(\sin \alpha) = \frac{5}{2} \sin^3 \alpha - \frac{3}{2} \sin \alpha, \quad (7.25)$$

$$P_3^1(\sin \alpha) = \frac{3}{2} \cos \alpha (5 \sin^2 \alpha - 1), \quad (7.26)$$

$$P_3^2(\sin \alpha) = 15 \cos^2 \alpha \sin \alpha, \quad (7.27)$$

$$P_3^3(\sin \alpha) = 15 \cos^3 \alpha. \quad (7.28)$$

By substituting these formulae into (7.17) and (7.18), we can write  $\phi$  as

$$\phi = \phi_{m_2} + \phi_{m_3} + \phi_{s_2}, \quad (7.29)$$

where  $\phi_{m_2}$  and  $\phi_{m_3}$  represent the degree 2 and 3 lunar tide, respectively, and  $\phi_{s_2}$  the degree 2 solar tide. We have

$$\begin{aligned} \phi_{m_2} = GM_m \frac{r^2}{r_m^3} & \left[ \left( \frac{3}{2} \sin^2 \Phi - \frac{1}{2} \right) \left( \frac{3}{2} \sin^2 \delta_m - \frac{1}{2} \right) \right. \\ & + 3 \sin \Phi \cos \Phi \sin \delta_m \cos \delta_m \cos H_m \\ & \left. + \frac{3}{4} \cos^2 \Phi \cos^2 \delta_m \cos 2H_m \right], \end{aligned} \quad (7.30)$$

$$\begin{aligned} \phi_{m_3} = GM_m \frac{r^3}{r_m^4} & \left[ \left( \frac{5}{2} \sin^3 \Phi - \frac{3}{2} \sin \Phi \right) \left( \frac{5}{2} \sin^3 \delta_m - \frac{3}{2} \sin \delta_m \right) \right. \\ & + \frac{3}{8} \cos \Phi (5 \sin^2 \Phi - 1) \cos \delta_m (5 \sin^2 \delta_m - 1) \cos H_m \\ & \left. + \frac{15}{4} \cos^2 \Phi \sin \Phi \cos^2 \delta_m \sin \delta_m \cos 2H_m \right] \end{aligned}$$

$$+ \frac{15}{24} \cos^3 \Phi \cos^3 \delta_m \cos 3H_m \Big], \quad (7.31)$$

$$\begin{aligned} \phi_{s_2} = & GM_s \frac{r^2}{r_s^3} \left[ \left( \frac{3}{2} \sin^2 \Phi - \frac{1}{2} \right) \left( \frac{3}{2} \sin^2 \delta_s - \frac{1}{2} \right) \right. \\ & + 3 \sin \Phi \cos \Phi \sin \delta_s \cos \delta_s \cos H_s \\ & \left. + \frac{3}{4} \cos^2 \Phi \cos^2 \delta_s \cos 2H_s \right]. \end{aligned} \quad (7.32)$$

These are the major components of the tide observed on the Earth.

Define a constant

$$D = \frac{3}{4} GM_m \frac{a^2}{C_m^3}, \quad (7.33)$$

called Doodson's constant, where  $C_m$  is the mean of  $r_m$ , i.e., the mean distance between the centers of the Earth and the Moon. Its numerical value given by Xi and Hou (1987) is

$$D = 2.6335838 \text{ m}^2 \cdot \text{s}^{-2}. \quad (7.34)$$

This constant can be used to write (7.30)–(7.32) as

$$\begin{aligned} \phi_{m_2} = & D \left( \frac{r}{a} \right)^2 \left( \frac{C_m}{r_m} \right)^3 \left[ 3 \left( \frac{1}{3} - \sin^2 \Phi \right) \left( \frac{1}{3} - \sin^2 \delta_m \right) \right. \\ & \left. + \sin 2\Phi \sin 2\delta_m \cos H_m + \cos^2 \Phi \cos^2 \delta_m \cos 2H_m \right], \end{aligned} \quad (7.35)$$

$$\begin{aligned} \phi_{m_3} = & D \frac{a}{C_m} \left( \frac{r}{a} \right)^3 \left( \frac{C_m}{r_m} \right)^4 \left[ \frac{1}{3} \sin \Phi (3 - 5 \sin^2 \Phi) \sin \delta_m (3 - 5 \sin^2 \delta_m) \right. \\ & + \frac{1}{2} \cos \Phi (1 - 5 \sin^2 \Phi) \cos \delta_m (1 - 5 \sin^2 \delta_m) \cos H_m \\ & + 5 \cos^2 \Phi \sin \Phi \cos^2 \delta_m \sin \delta_m \cos 2H_m \\ & \left. + \frac{5}{6} \cos^3 \Phi \cos^3 \delta_m \cos 3H_m \right], \end{aligned} \quad (7.36)$$

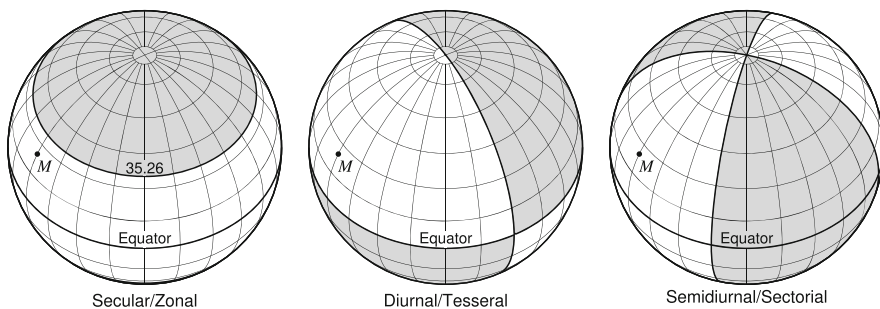
$$\begin{aligned} \phi_{s_2} = & D \left( \frac{r}{a} \right)^2 \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \left[ 3 \left( \frac{1}{3} - \sin^2 \Phi \right) \left( \frac{1}{3} - \sin^2 \delta_s \right) \right. \\ & \left. + \sin 2\Phi \sin 2\delta_s \cos H_s + \cos^2 \Phi \cos^2 \delta_s \cos 2H_s \right], \end{aligned} \quad (7.37)$$

where  $C_s$  is the mean of  $r_s$ , i.e., the mean distance between the centers of the Earth and the Sun.

### 7.1.3 Classification Based on Periods

In the expressions of tidal-generating potential (7.35)–(7.37), different terms in the brackets depend on time in different ways. Laplace classified the degree 2 tide into three species. Here we take the case of the lunar tide (7.35) as example, whose characteristics of geographical distributions of the three terms are shown in Fig. 7.5.

- In the first term, the time dependent part is  $1/3 - \sin^2 \delta_m$ , which varies with a period of half a month, as the period of variation of  $\delta_m$  is a month. This term is called secular tide (as compared to the diurnal and semidiurnal tides to be discussed below), or tide of the first species according to Laplace. The geographical distribution of this tide is determined by the expression  $1/3 - \sin^2 \Phi$ . The solutions of  $1/3 - \sin^2 \Phi = 0$  are  $\Phi = \pm 35.26^\circ$ , which represent two parallels dividing the Earth's surface into three zones of alternative signs of the expression. For this reason, this term is also referred to as zonal tide. As  $\delta_m$  varies in between  $-28.8^\circ$  and  $+28.8^\circ$ , the expression  $1/3 - \sin^2 \delta_m$  is always positive. Hence, the zonal tidal-generating potential is positive in the equatorial region, and negative in the polar regions, as the expression  $1/3 - \sin^2 \Phi$ .
- In the second term, the temporal variation is dominated by  $\cos H_m$ , whose period is about a day. This term is called diurnal tide, or tide of the second species according to Laplace. The period of  $\sin 2\delta_m$  is half a month, which varies little within a day, and its variation can be understood as a variation of the amplitude of the diurnal tide. The geographical distribution of this tide is mainly determined by the expression  $\sin 2\Phi \cos H_m$ . The solutions of  $\sin 2\Phi \cos H_m = 0$  are  $\Phi = 0, \pm\pi/2$  and  $H_m = \pm\pi/2$ , which represent the equator, the two poles, and two



**Fig. 7.5** Three species of tides according to Laplace. The Moon tides are shown, where the direction of the Moon is denoted as  $M$



meridians. Hence, the Earth's surface is divided into four regions of alternative signs of this term by these lines. This term is also referred to as tesseral tide.

- In the last term, the temporal variation is dominated by  $\cos 2H_m$ , whose period is about half a day. This term is called semidiurnal tide, or tide of the third species according to Laplace. The period of  $\cos^2 \delta_m$  is half a month, which varies little within half a day, and its variation can be understood as a variation of the amplitude of the semidiurnal tide. The geographical distribution of this tide is mainly determined by the expression  $\cos^2 \Phi \cos 2H_m$ . The solutions of  $\cos^2 \Phi \cos 2H_m = 0$  are  $\Phi = \pm\pi/2$  and  $H_m = \pm\pi/4, \pm3\pi/4$ , which represent the two poles and four meridians, dividing the Earth's surface into four sectors of alternative signs of this term. Hence, this term is also referred to as sectorial tide.

The case of the degree 2 solar tide (7.37) is similar, for which we just have to replace the term *month* by the term *year* and recognize the fact that  $\delta_s$  varies in between  $-23.4^\circ$  and  $+23.4^\circ$ . There are four terms in the brackets of the expression of the degree 3 lunar tide (7.36). The first three terms are secular, diurnal, and semidiurnal tides. However, their geographical distributions are not the same as the case of degree 2 tide as shown in Fig. 7.5. The fourth term has a period of a third of a day, thus being referred to as terdiurnal tide.

Base on the classification according to periods, we can write the tidal-generating potential as the sum of the secular, diurnal, semidiurnal, and terdiurnal tides:

$$\phi = \phi_{L_0} + \phi_{L_1} + \phi_{L_2} + \phi_{L_3}. \quad (7.38)$$

The expressions of the four types of tide are

$$\begin{aligned} \phi_{L_0} = G_0 & \left[ \left( \frac{C_m}{r_m} \right)^3 \left( \frac{2}{3} - 2 \sin^2 \delta_m \right) + \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \left( \frac{2}{3} - 2 \sin^2 \delta_s \right) \right] \\ & + G'_0 \left[ \frac{2}{3\sqrt{5}} \frac{a}{C_m} \left( \frac{C_m}{r_m} \right)^4 \sin \delta_m (3 - 5 \sin^2 \delta_m) \right], \end{aligned} \quad (7.39)$$

$$\begin{aligned} \phi_{L_1} = G_1 & \left[ \left( \frac{C_m}{r_m} \right)^3 \sin 2\delta_m \cos H_m + \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \sin 2\delta_s \cos H_s \right] \\ & + G'_1 \left[ \frac{8}{3\sqrt{15}} \frac{a}{C_m} \left( \frac{C_m}{r_m} \right)^4 \cos \delta_m (1 - 5 \sin^2 \delta_m) \cos H_m \right], \end{aligned} \quad (7.40)$$

$$\begin{aligned} \phi_{L_2} = G_2 & \left[ \left( \frac{C_m}{r_m} \right)^3 \cos^2 \delta_m \cos 2H_m + \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \cos^2 \delta_s \cos 2H_s \right] \\ & + G'_2 \left[ \frac{10}{3\sqrt{3}} \frac{a}{C_m} \left( \frac{C_m}{r_m} \right)^4 \cos^2 \delta_m \sin \delta_m \cos 2H_m \right], \end{aligned} \quad (7.41)$$

$$\phi_{L_3} = G'_3 \left[ \frac{5}{6} \frac{a}{C_m} \left( \frac{C_m}{r_m} \right)^4 \cos^3 \delta_m \cos 3H_m \right], \quad (7.42)$$

where

$$G_0 = \frac{1}{2} D \left( \frac{r}{a} \right)^2 (1 - 3 \sin^2 \Phi), \quad G'_0 = \frac{\sqrt{5}}{2} D \left( \frac{r}{a} \right)^3 \sin \Phi (3 - 5 \sin^2 \Phi), \quad (7.43)$$

$$G_1 = D \left( \frac{r}{a} \right)^2 \sin 2\Phi, \quad G'_1 = \frac{3\sqrt{15}}{16} D \left( \frac{r}{a} \right)^3 \cos \Phi (1 - 5 \sin^2 \Phi), \quad (7.44)$$

$$G_2 = D \left( \frac{r}{a} \right)^2 \cos^2 \Phi, \quad G'_2 = \frac{3\sqrt{3}}{2} D \left( \frac{r}{a} \right)^3 \cos^2 \Phi \sin \Phi, \quad (7.45)$$

$$G'_3 = D \left( \frac{r}{a} \right)^3 \cos^3 \Phi, \quad (7.46)$$

are referred to as geodetic coefficients.

In (7.39)–(7.42), within the brackets are the time dependent part, and outside the brackets are the geodetic coefficients depending on the position alone, which are given in (7.43)–(7.46). The geodetic coefficients are defined such that the absolute maximums of all of them when  $r = a$  are  $D$ . An exception is  $G'_0$ , where the coefficient  $\sqrt{5}/2$  should be replaced by  $1/2$  for the absolute maximum to be  $D$ . This should be due to a mistake made by Doodson (1921), who may have taken the extreme value of  $G'_0$  at  $\sin \Phi = 1/\sqrt{5}$  as the absolute maximum, while the absolute maximum is actually attained at  $\sin \Phi = \pm 1$ , which can be readily verified (e.g., Xi, 2007). However, Doodson's choice is considered as a convention, and is followed in later development for consistency.

## 7.2 Development of the Tidal-Generating Potential into Waves

### 7.2.1 Orbits of the Sun and the Moon

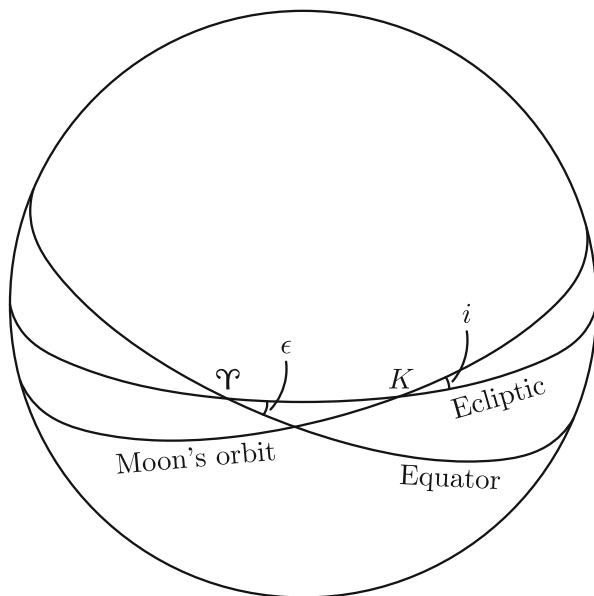
The tidal-generating potential depends on the positions of the Moon and the Sun. In order to express the tidal-generating potential as a simple function of geographical position and time, we first analyze the orbit of the Moon and the Sun with respect to the Earth, and express the positions of the Moon and the Sun in terms of some appropriate parameters.

We know that, if the solar system were composed of only the Sun and the Earth without any other planet or satellite, the orbit of the Earth with respect to the Sun would be an ellipse with the Sun at one of the foci. In reality, due to the influence of other planets, the orbit of the Earth with respect to the Sun is more complicated.

However, for convenience, we still approximate the instantaneous orbit of the Earth with respect to the Sun using an ellipse, on which the point closest to the Sun is called the perihelion of the Earth, and its position changes as a function of time. While observing the Sun on the Earth, the Sun is closest when the Earth is at the perihelion, and the corresponding position of the Sun is called perigee of the Sun. Similarly, we also use an ellipse to approximate the instantaneous orbit of the Moon around the Earth, on which the point closest to the Earth is called perigee of the Moon. The orbits of the Sun and the Moon with respect to the Earth on the celestial sphere are shown in Fig. 7.6. The orbit of the sun with respect to the Earth is called ecliptic, and may be approximately considered as time-independent. However, the orbit of the Moon should be considered as time-dependent. We have also shown in Fig. 7.6 two particular points. The intersection between the equator and the ecliptic,  $\Upsilon$ , is called vernal equinox, where the Sun crosses the equator from the southern hemisphere to the northern hemisphere around March 21. The intersection between the orbit of the Moon and the ecliptic,  $K$ , is called ascending node of the Moon, where the Moon moves from the south side of the ecliptic to the north side. The angle between the celestial equatorial and ecliptic planes,  $\epsilon$ , is called obliquity of the ecliptic, which is about  $23^\circ 27'$ , and varies very little (besides a very slow long term variation to be detained below, the effect of nutation is on the order of  $\pm 10''$ ). The inclination of the orbit of the Moon to the ecliptic plane, i.e., the angle  $i$ , is about  $5^\circ 8'$ , and also varies very little (about  $\pm 10'$ ).

The positions of celestial bodies can be represented using either the equatorial coordinates, i.e., the right ascension  $\alpha$  and the declination (equatorial latitude)  $\delta$ , or the ecliptic coordinates, i.e., the ecliptic longitude  $\Lambda$  and ecliptic latitude  $\beta$ . Due to

**Fig. 7.6** Relative positions of the ecliptic, Moon's orbit and equator



the effect of precession-nutation, the position of the equator varies considerably with time. Hence, the ecliptic longitude  $\Lambda$  and ecliptic latitude  $\beta$  are used to represent the positions of the Moon and the Sun in the development of the tidal-generating potential into waves. In order to further develop the expression of the tidal-generating potential, we first provide the expressions of the instantaneous ecliptic longitudes and ecliptic latitudes of the Sun and Moon, and their distance to the Earth.

We first introduce the concept of mean longitude of celestial bodies (the Moon, the Sun or a point like the perigee of the Moon or the Sun). We first consider the ecliptic longitude. The variation of the ecliptic longitude of a celestial body with time includes two components: one component is a secular trend depending on time in the form of a polynomial, and the other component is periodical depending on time in the form of a sum of sine and/or cosine functions. The mean ecliptic longitude is the remaining part of the ecliptic longitude after removing the periodical component, and is a function of time in the form of a polynomial. The mean right ascension can be defined similarly. An immediate observation is that the mean ecliptic longitude and the mean right ascension are equal, and they could be simply referred to as mean longitude. Since the pioneer work of Doodson (1921), the fundamental parameters of the orbits of the Sun and the Moon for expressing tidal-generating potential are chosen to be

- $s$ , the mean longitude of the Moon;
- $h$ , the mean longitude of the Sun;
- $p$ , the mean longitude of the perigee of the Moon;
- $N' = -N$ ,  $N$  being the mean longitude of the ascending node of the Moon;
- $p_s$ , the mean longitude of the perigee of the Sun.

The variations of these parameters with time are determined from observations. Their expressions adopted by Xi (2007) are

$$\begin{cases} s = 218.31643^\circ + 481,267.88128^\circ T - 0.00161^\circ T^2 + 0.000005^\circ T^3, \\ h = 280.46607^\circ + 36,000.76980^\circ T + 0.00030^\circ T^2, \\ p = 83.35345^\circ + 4069.01388^\circ T - 0.01031^\circ T^2 - 0.00001^\circ T^3, \\ N = 125.04452^\circ - 1934.13626^\circ T + 0.00207^\circ T^2 + 0.000002^\circ T^3, \\ p_s = 282.93835^\circ + 1.71946^\circ T + 0.00046^\circ T^2 + 0.00000^\circ T^3, \end{cases} \quad (7.47)$$

where  $T$  is the number of Julian century from the epoch  $J2000.0$ . One Julian century is defined to be 36,525 mean solar days, and the epoch  $J2000.0$  is defined as January 1, 2000, at noon. See Sect. 1.5 for more details. Xi (2007) has also taken into account of the long term variation of  $\epsilon$  with time,

$$\epsilon = 23.43929^\circ - 0.01300^\circ T - 0.00000016^\circ T^2 + 0.0000005^\circ T^3. \quad (7.48)$$

The development of the expressions of the instantaneous positions of the Sun and the Moon in terms of the parameters in (7.47) is a topic of celestial mechanics. Here we follow Doodson (1921) to demonstrate the logic. The ecliptic longitude of the Sun is

$$\Lambda_s = h + \sum_i Y_i \sin(c_i h + f_i p_s). \quad (7.49)$$

By neglecting the variation of the ecliptic with time, the ecliptic latitude of the Sun vanishes, i.e.,

$$\beta_s = 0. \quad (7.50)$$

The reciprocal of the distance between the sun and the Earth is

$$\frac{C_s}{r_s} = 1 + \sum_i Y_i \cos(c_i h + f_i p_s). \quad (7.51)$$

The ecliptic longitude of the Moon is

$$\Lambda_m = s + \sum_i Y_i \sin(b_i s + c_i h + d_i p + e_i N' + f_i p_s). \quad (7.52)$$

The ecliptic latitude of the Moon is

$$\beta_m = \sum_i Y_i \sin(b_i s + c_i h + d_i p + e_i N' + f_i p_s). \quad (7.53)$$

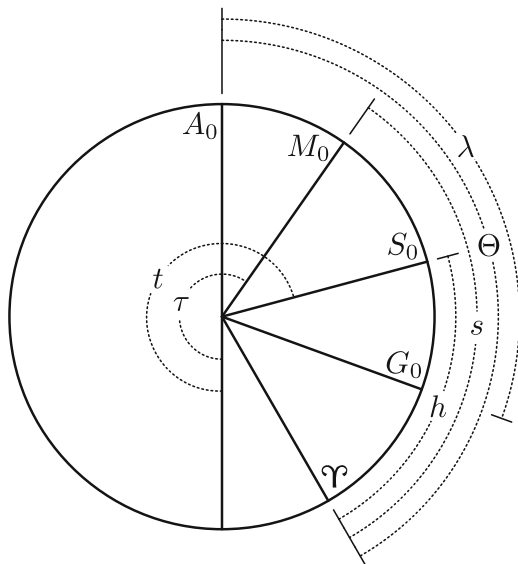
The reciprocal of the distance between the Moon and the Earth is

$$\frac{C_m}{r_m} = 1 + \sum_i Y_i \cos(b_i s + c_i h + d_i p + e_i N' + f_i p_s). \quad (7.54)$$

In these expressions, the coefficients  $b_i$ ,  $c_i$ ,  $d_i$ ,  $e_i$  and  $f_i$  are all integers. They are provided as tables together with the amplitude  $Y_i$ , which are not to be listed here. We see that the parameters of the orbit of the Moon is not present in (7.49)–(7.51). This is due to the fact that the Earth center in these expressions is actually approximated by the center of mass of the Earth-Moon system, and the Earth-Moon interaction does not alter the center of mass of the system. The sine and cosine terms with amplitudes denoted as  $Y_i$  are all a deviation from a mean, and are small quantities.

A geographical location dependent parameter, the mean lunar time  $\tau$ , is also used in the development. The mean ecliptic longitude of the Sun is defined as the ecliptic longitude of the ecliptic mean Sun with respect to the mean vernal equinox, which also equals the right ascension of the equatorial mean Sun with respect to the mean vernal equinox. The mean solar time is defined as the hour angle of the equatorial

**Fig. 7.7** Relations among  $\tau$ ,  $s$ ,  $h$ ,  $t$ , and  $\Theta$



mean Sun with respect to the lower meridian, i.e., the meridian of the nadir. The mean lunar time is defined similarly. We define an equatorial mean Moon, whose right ascension with respect to the mean vernal equinox is equal to its mean ecliptic longitude. The mean lunar time is defined as the hour angle of this equatorial mean Moon with respect to the lower meridian.

As the mean lunar time  $\tau$  is not usually used, we derive a formula for computing it using the universal time. The relations among  $\tau$ ,  $s$ ,  $h$ , the universal time  $t$ , and the local sidereal time  $\Theta$  are shown in Fig. 7.7, where  $\mathcal{P}$ ,  $S_0$ , and  $M_0$  are the mean vernal equinox, the equatorial mean Sun, and the equatorial mean Moon, respectively, and  $G_0$  and  $A_0$  are the intersections of the meridians through Greenwich and the observer, respectively. By definition,  $\Theta$  is the hour angle of the mean vernal equinox. We have

$$\Theta = \tau + s - \pi, \quad \Theta = t + h - \pi. \quad (7.55)$$

Consequently,

$$\tau = t + h - s. \quad (7.56)$$

With the universal time  $t_{UT}$ , we have

$$\tau = t_{UT} + h - s + \lambda, \quad (7.57)$$

where  $\lambda$  is the geodetic longitude of the observer.

**Table 7.1** Physical properties of the astronomical parameters

| Parameter | Rate ( $^{\circ}/h$ ) | Period                                                  |
|-----------|-----------------------|---------------------------------------------------------|
| $t$       | 15.0000000            | 1 mean solar day                                        |
| $\Theta$  | 15.0410686            | 1 sidereal day = 0.99727 mean solar day                 |
| $\tau$    | 14.4920521            | 1 mean lunar day = 1.035050 mean solar days             |
| $s$       | 0.5490165             | 1 tropical month = 27.321582 mean solar days            |
| $h$       | 0.0410686             | 1 tropical year = 365.242189 mean solar days            |
| $p$       | 0.0046418             | Period of the perigee of the Moon = 8.847 years         |
| $N'$      | 0.0022064             | Period of the ascending node of the Moon = 18.613 years |
| $p_s$     | 0.0000020             | Period of the perigee of the Sun = 20,940 years         |
| $s - h$   | 0.5079479             | 1 synodic month = 29.53059 mean solar day               |

Finally, we show in Table 7.1 some physical properties of the astronomical parameters, where the rate and period are with respect to observers on the Earth.

## 7.2.2 Development of the Tidal-Generating Potential into Waves

With the preparation so far, we can now consider developing the tidal-generating potential into a sum of harmonic waves, i.e., sine waves. Our purpose is to express the time-dependent part in the expression of the tidal-generating potential, i.e., the expressions in the brackets at the right-hand sides of (7.39)–(7.42), in the form of

$$\sum_i Y_i \sin(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s) \quad (7.58)$$

or

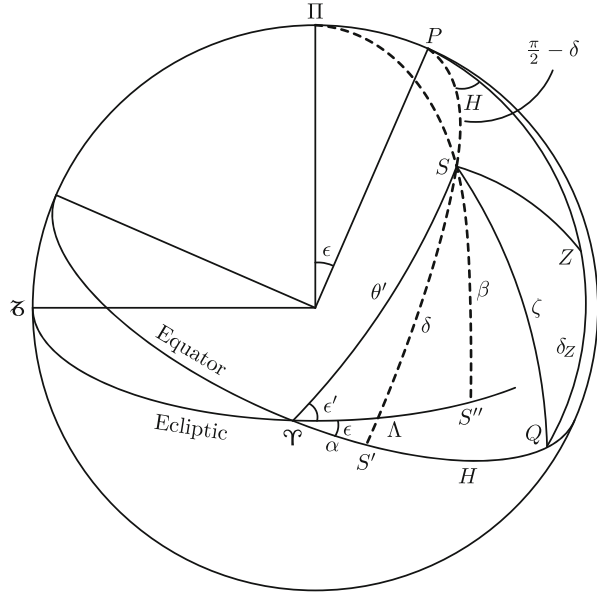
$$\sum_i Y_i \cos(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s). \quad (7.59)$$

As the period of variation of  $\tau$  is about 1 day, while those of  $s$ ,  $h$ ,  $p$ ,  $N'$  and  $p_s$  are much longer, the value of  $a_i$  is approximately the number of cycles of the wave in 1 day. Hence,  $a_i = 0, 1, 2$ , and  $3$  correspond to secular tide, diurnal tide, semidiurnal tide, and terdiurnal tide, respectively.

The mean lunar time  $\tau$  and the five lunar and solar orbital parameters  $s$ ,  $h$ ,  $p$ ,  $N'$ , and  $p_s$  are referred to as Doodson's arguments.

We see that (7.39)–(7.42) include the declinations and hour angles of the Moon and the Sun, while what we have in (7.49)–(7.54) are their ecliptic longitudes and latitudes. Hence, we need the expressions of the declination and hour angle in terms of ecliptic longitude and latitude. We take reference to Fig. 7.8, where  $S$  is a celestial body,  $Z$  the zenith of the observation station, and  $Q$  the upper culmination of the equator.

**Fig. 7.8** Relations of the declination and hour angle to the ecliptic latitude and longitude. The arcs defining the ecliptic latitude and the declination are drawn in thicker dashed lines for avoiding confusion with the other arcs through  $S$



We first derive the formula for the declination  $\delta$ . Based on the spherical triangle  $S\Upsilon S'$ , we have

$$\sin \delta = \sin \theta' \sin(\epsilon + \epsilon') = \sin \epsilon \sin \theta' \cos \epsilon' + \cos \epsilon \sin \theta' \sin \epsilon'. \quad (7.60)$$

Based on the spherical triangle  $S\Upsilon S''$ , we also have

$$\sin \theta' \cos \epsilon' = \sin \Lambda \cos \beta, \quad (7.61)$$

and

$$\sin \theta' \sin \epsilon' = \sin \beta. \quad (7.62)$$

From these three formulae we can find an expression of  $\delta$ ,

$$\sin \delta = \sin \epsilon \sin \Lambda \cos \beta + \cos \epsilon \sin \beta. \quad (7.63)$$

We now derive the formula for the hour angle  $H$ . First, we note that  $\alpha + H$  is the local sidereal time. Hence, as given in (1.1), we have

$$\Theta = \alpha + H. \quad (7.64)$$

Based on the spherical triangle  $PSQ$ , we have

$$\cos \zeta = \cos \delta \cos H. \quad (7.65)$$



Based on the spherical triangle  $S\Upsilon Q$ , we have

$$\cos \zeta = \cos \theta' \cos \Theta + \sin \theta' \sin \Theta \cos(\epsilon + \epsilon'). \quad (7.66)$$

Based on the spherical triangle  $S\Upsilon S''$ , we have

$$\cos \theta' = \cos \Lambda \cos \beta. \quad (7.67)$$

From (7.62) and (7.65)–(7.67) we can obtain a formula for the hour angle  $H$ ,

$$\cos \zeta = \cos \delta \cos H = \cos \Lambda \cos \beta \cos \Theta + \sin \Theta (\cos \epsilon \sin \Lambda \cos \beta - \sin \epsilon \sin \beta). \quad (7.68)$$

Logically, further development of (7.39)–(7.42) can be performed directly by substituting (7.63) and (7.68) into them. However, developing  $\sin \delta$  according to (7.63) and  $\cos \zeta$  according to (7.68) into waves beforehand can substantially reduce the amount of formulation. Hence, we substitute (7.48)–(7.54) into (7.63) and (7.68) to obtain, for both the Moon and the Sun

$$\sin \delta = \sum_i Y_i \sin(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s) \quad (7.69)$$

and

$$\cos \zeta = \sum_i Y_i \cos(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s). \quad (7.70)$$

A subscript  $m$  should be added to  $\delta$  and  $\zeta$  for the Moon, and  $s$  for the Sun. In the derivation, the product-to-sum trigonometric formulae

$$\begin{cases} 2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta), \\ 2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta), \\ 2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta), \end{cases} \quad (7.71)$$

are repeatedly used. Furthermore, the formulae

$$\begin{aligned} \sin(a \sin \alpha) &= a \sin \alpha - \frac{a^3}{3!} \sin^3 \alpha + \dots \\ &= a \sin \alpha - \frac{a^3}{3!} \left( \frac{3}{4} \sin \alpha - \frac{1}{4} \sin 3\alpha \right) + \dots, \end{aligned} \quad (7.72a)$$

$$\begin{aligned} \cos(a \sin \alpha) &= 1 - \frac{a^2}{2!} \sin^2 \alpha + \dots \\ &= 1 - \frac{a^2}{2!} \left( \frac{1}{2} - \frac{1}{2} \cos 2\alpha \right) + \dots, \end{aligned} \quad (7.72b)$$

have also been used, where the quantity  $a \sin \alpha$  involved are from the summation terms in (7.49), (7.52), and (7.53), and are all small.

With (7.69) and (7.70), together with (7.51) and (7.54), we can straightforwardly develop the expressions in the brackets at the right-hand sides of (7.39)–(7.42) into waves by repeatedly using (7.71). Certainly, only the terms larger than a prescribed lower limit of the amplitude are kept. For the secular tide (7.39), as  $(M_s/M_m)(C_m/C_s)^3$  and  $R/C_m$  are constant, we obtain

$$\begin{aligned} & \left(\frac{C_m}{r_m}\right)^3 \left(\frac{2}{3} - 2 \sin^2 \delta_m\right) + \left(\frac{M_s}{M_m}\right) \left(\frac{C_m}{C_s}\right)^3 \left(\frac{C_s}{r_s}\right)^3 \left(\frac{2}{3} - 2 \sin^2 \delta_s\right) \\ &= \sum_i Y_i \cos(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s), \end{aligned} \quad (7.73)$$

$$\begin{aligned} & \frac{2}{3\sqrt{5}} \frac{a}{C_m} \left(\frac{C_m}{r_m}\right)^4 \sin \delta_m (3 - 5 \sin^2 \delta_m) \\ &= \sum_i Y_i \sin(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s). \end{aligned} \quad (7.74)$$

For the diurnal tide (7.40), with  $\sin 2\delta \cos H = 2 \sin \delta \cos \delta \cos H$  and  $\cos \delta \cos H = \cos \zeta$ , we obtain

$$\begin{aligned} & \left(\frac{C_m}{r_m}\right)^3 \sin 2\delta_m \cos H_m + \left(\frac{M_s}{M_m}\right) \left(\frac{C_m}{C_s}\right)^3 \left(\frac{C_s}{r_s}\right)^3 \sin 2\delta_s \cos H_s \\ &= \sum_i Y_i \sin(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s), \end{aligned} \quad (7.75)$$

$$\begin{aligned} & \frac{8}{3\sqrt{15}} \frac{a}{C_m} \left(\frac{C_m}{r_m}\right)^4 \cos \delta_m (1 - 5 \sin^2 \delta_m) \cos H_m \\ &= \sum_i Y_i \cos(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s). \end{aligned} \quad (7.76)$$

For the semidiurnal tide (7.41), with  $\cos^2 \delta \cos 2H = 2 \cos^2 \zeta - (1 - \sin^2 \delta)$ , we obtain

$$\begin{aligned} & \left(\frac{C_m}{r_m}\right)^3 \cos^2 \delta_m \cos 2H_m + \left(\frac{M_s}{M_m}\right) \left(\frac{C_m}{C_s}\right)^3 \left(\frac{C_s}{r_s}\right)^3 \cos^2 \delta_s \cos 2H_s \\ &= \sum_i Y_i \cos(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s), \end{aligned} \quad (7.77)$$

$$\begin{aligned} & \frac{10}{3\sqrt{3}} \frac{a}{C_m} \left( \frac{C_m}{r_m} \right)^4 \cos^2 \delta_m \sin \delta_m \cos 2H_m \\ &= \sum_i Y_i \sin(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s). \end{aligned} \quad (7.78)$$

For the terdiurnal tide (7.42), with  $\cos^3 \delta \cos 3H = 4 \cos^3 \zeta - 3(1 - \sin^2 \delta) \cos \zeta$ , we obtain

$$\frac{5}{6} \frac{a}{C_m} \left( \frac{C_m}{r_m} \right)^4 \cos^3 \delta_m \cos 3H_m = \sum_i Y_i \cos(a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s). \quad (7.79)$$

In all the expressions in the form of a sum of harmonic waves, i.e., (7.73)–(7.79), the only location-dependent argument is  $\tau$ , which depends on the longitude, as can be seen from (7.57). Evidently, (7.73) includes a constant term, which is related to the means of  $2/3 - 2 \sin^2 \delta_m$  and  $2/3 - 2 \sin^2 \delta_s$ , and will be further discussed in Sect. 7.4.6.

It is noted that what we have presented is only a primitive theory where the effects of nutation and polar motion are neglected.

### 7.2.3 The Argument-Number and Table of Tidal Waves

We have developed the tidal-generating potential into a sum of harmonic waves. The waves corresponding to the secular, diurnal, semidiurnal, and terdiurnal tides are referred to as secular, diurnal, semidiurnal and terdiurnal tidal waves, respectively. The argument of each wave is a combination of the six independent parameters of the orbits of the Moon and the Sun:

$$a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s. \quad (7.80)$$

All the parameters increase with time, the linear term being dominating. As the mean longitude of the ascending node of the Moon  $N$  decreases with time, its negative  $N'$  is chosen. The six parameters are ordered based on their rate of change in descending order, as listed in Table 7.1.

Among the six coefficients  $a_i, b_i, c_i, d_i, e_i$ , and  $f_i$ ,  $a_i$  is equal to the order  $k$  as compared to the original expression of the tidal-generating potential (7.17) and (7.18), which is approximately the cycle per day of the wave, and it is always set to be positive. Even if the tidal-generating potential is to be accurate to degree and order 5,  $a_i$  attains a maximum of 5 only. The values of the other five coefficients  $b_i, c_i, d_i, e_i$ , and  $f_i$  are mostly within  $-4$  and  $4$ . Based on these facts, Doodson (1921) introduced the concept of argument number

$$a_i(b_i + 5)(c_i + 5)(d_i + 5)(e_i + 5)(f_i + 5) \quad (7.81)$$

to denote tidal waves. It is a six digits number with the first digit called species-number, the first two digits called group-number, and the first three digits called constituent-number. In some rare instances, a digit in Doodson's argument number may exceed 9. If this happens,  $X$  is used to denote 10,  $E$  is used to denote 11, etc.

The largest lunar and solar tidal waves are listed in Table 7.2 based on Melchior (1966, 1983), where the angular rate is with respect to observers on the Earth. Darwin's symbols, as implied by the name, was given by Darwin in an earlier work, and are conventionally used afterwards. In contemporary developments, as the total number of waves is huge, not each wave is given a name like Darwin's symbol. In the table, only the terdiurnal lunar tidal wave  $M_3$  is from degree 3, and all other waves are from degree 2, where it is straightforward for the readers to infer which term of (7.39)–(7.42) each wave is originated from. The symbol  $t' = \tau + s = t + h$  is used according to (7.55).

Some tidal waves listed with particular characteristics are as follows:  $M_0$  and  $S_0$  do not vary with time, which correspond to the constant term in (7.73);  $O_1$ ,  $K_1$ , and  $P_1$  are the largest diurnal waves;  $M_2$ ,  $S_2$ , and  $N_2$  are the largest semidiurnal waves; the period of  $S_1$  is different from a mean solar day by only one over 7.5 million; the period of  $S_2$  is exactly half a mean solar day; the periods of  $M_2$  and  $M_3$  are exactly half and one third of a mean lunar day, respectively; and the period of  $K_1$  is exactly one sidereal day.

The origin of each wave is also provided in Table 7.2 with the notations explained in the note at the bottom, where the meanings of L, S, C, and P are evident, and are not to be further explained. The term *elliptic* denoted by E refers to the variation of the Earth-Moon or Earth-Sun distance due to orbital ellipticity, which brings time dependent terms in  $C_m/r_m$  or  $C_s/r_s$ . The term *declinational* refers to the non-vanishing of the declination  $\delta_m$  or  $\delta_s$ , which is the reason for the non-vanishing of the degree 2 diurnal tidal-generating potential, as can be seen in (7.75). In this formula, we can roughly infer how the product-to-sum development of  $\sin 2\delta_m \cos H_m$  results in a pair of waves,  $O_1$  with the argument  $\tau - s$ , and  ${}^mK_1$  with the argument  $\tau + s$ , where  $\tau$  is from  $H_m$ , and  $s$  is from  $2\delta_m$ . The same is for  $\sin 2\delta_s \cos H_s$  and the pair of waves  $P_1$  and  ${}^sK_1$ . The terms *evection* and *variation* denoted by Ev and V, respectively, refer to the deviations of the Moon's orbit with respect to the Earth from an elliptic Kepler orbit as in a two-body system due to the Sun's action, the former being larger than the later. We mention that the precessions of the perigee and the ascension node of the Moon is also due to the action of the Sun. A tidal wave placed in a pair of parentheses after E or D means that the result is originated from an elliptic or declinational perturbation to the amplitude and argument of this wave.

As the variation of all the six parameters with time is dominated by the linear term, we have, with very high accuracy,

$$a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s = \omega_i t + \phi_i, \quad (7.82)$$

where  $\omega_i$  is the angular frequency or angular rate of the wave, which depends mostly on the value of  $a_i \tau$ , with smaller contributions from other terms.

**Table 7.2** Major lunar and solar tidal waves

| Darwin's<br>symbol       | Argument<br>-number | Argument               | Angular rate<br>( $^{\circ}$ )/ $h$ | Amplitude<br>$\times 10^5$ | Origin            |
|--------------------------|---------------------|------------------------|-------------------------------------|----------------------------|-------------------|
| Secular tidal waves      |                     |                        |                                     |                            |                   |
| $M_0$                    | 055.555             | 0                      | 0.000000                            | 50,458                     | L, C              |
| $s_0$                    | 055.555             | 0                      | 0.000000                            | 23,411                     | S, C              |
| $s_a$                    | 056.554             | $h - p_s$              | 0.041067                            | 1176                       | S, E              |
| $s_{sa}$                 | 057.555             | $2h$                   | 0.082137                            | 7287                       | S, D              |
| $M_m$                    | 065.455             | $s - p$                | 0.544375                            | 8254                       | L, E              |
| $M_f$                    | 075.555             | $2s$                   | 1.098033                            | 15,642                     | L, D              |
| Diurnal tidal waves      |                     |                        |                                     |                            |                   |
| $Q_1$                    | 135.655             | $(\tau - s) - (s - p)$ | 13.398661                           | 7216                       | L, E ( $O_1$ )    |
| $O_1$                    | 145.555             | $\tau - s$             | 13.943036                           | 37,689                     | L, P              |
| $M_1$                    | 155.655             | $(\tau + s) - (s - p)$ | 14.496694                           | -2964                      | L, E ( $^m K_1$ ) |
| $\pi_1$                  | 162.556             | $(t - h) - (h - p_s)$  | 14.917865                           | 1029                       | S, E ( $P_1$ )    |
| $P_1$                    | 163.555             | $t - h$                | 14.958931                           | 17,554                     | S, P              |
| $S_1$                    | 164.556             | $(t + h) - (h - p_s)$  | 15.000002                           | -423                       | S, E ( $^s K_1$ ) |
| $^m K_1$                 | 165.555             | $\tau + s = t'$        | 15.041069                           | -36,233                    | L, D              |
| $^s K_1$                 | 165.555             | $t + h = t'$           | 15.041069                           | -16,817                    | S, D              |
| $\psi_1$                 | 166.554             | $(t + h) + (h - p_s)$  | 15.082135                           | -423                       | S, E ( $^s K_1$ ) |
| $\phi_1$                 | 167.555             | $t + 3h$               | 15.123206                           | -756                       | S, D              |
| $J_1$                    | 175.455             | $(\tau + s) + (s - p)$ | 15.585443                           | -2964                      | L, E ( $^m K_1$ ) |
| $OO_1$                   | 185.555             | $\tau + 3s$            | 16.139102                           | -1623                      | L, D              |
| Semi-diurnal tidal waves |                     |                        |                                     |                            |                   |
| $2N_2$                   | 235.755             | $2\tau - 2(s - p)$     | 27.895355                           | 2301                       | L, E ( $M_2$ )    |
| $\mu_2$                  | 237.555             | $2\tau - 2(s - h)$     | 27.968208                           | 2777                       | L, V              |
| $N_2$                    | 245.655             | $2\tau - (s - p)$      | 28.439730                           | 17,387                     | L, E ( $M_2$ )    |
| $\nu_2$                  | 247.455             | $2\tau - (s - 2h + p)$ | 28.512583                           | 3303                       | L, Ev             |
| $M_2$                    | 255.555             | $2\tau$                | 28.984104                           | 90,812                     | L, P              |
| $\lambda_2$              | 263.655             | $2\tau + (s - 2h + p)$ | 29.455625                           | -670                       | L, Ev             |
| $L_2$                    | 265.455             | $2\tau + (s - p)$      | 29.528479                           | -2567                      | L, E ( $M_2$ )    |
| $T_2$                    | 272.556             | $2t - (h - p_s)$       | 29.958933                           | 2479                       | S, E ( $S_2$ )    |
| $S_2$                    | 273.555             | $2t$                   | 30.000000                           | 42,286                     | S, P              |
| $R_2$                    | 274.554             | $2t + (h - p_s)$       | 30.041067                           | -354                       | S, E ( $S_2$ )    |
| $^m K_2$                 | 275.555             | $2(\tau + s) = 2t'$    | 30.082137                           | 7858                       | L, D              |
| $^s K_2$                 | 275.555             | $2(t + h) = 2t'$       | 30.082137                           | 3648                       | S, D              |
| Terdiurnal tidal waves   |                     |                        |                                     |                            |                   |
| $M_3$                    | 355.555             | $3\tau$                | 43.476156                           | -1188                      | L, P              |

*L* Lunar, *S* Solar, *C* Constant, *P* Principal, *E* Elliptic, *D* Declinational, *Ev* Evection, *V* Variation

With contemporary accuracy of observations, especially the gravity observed using superconducting gravimeters, much more care has to be taken in the development of tidal-generating potential into waves. Besides the requirement of more accurate ephemerides of the Moon and the Sun and the inclusion of higher degree terms, the effects of some other planets must also be considered. These consideration leads to the more convenient catalog of tidal waves defined by Hartmann and Wenzel (1995),

$$\phi = \sum_{n=2}^{\infty} \sum_{k=0}^n \left(\frac{r}{a}\right)^n \bar{P}_n^k(\cos \theta) \sum_i \left\{ C_i^{nk}(t) \cos[\alpha_i(t)] + S_i^{nk}(t) \sin[\alpha_i(t)] \right\}, \quad (7.83)$$

where  $r$ ,  $\theta$ , and  $\lambda$  are the geocentric spherical coordinates of the station. For the lunar and solar degree 2 and 3 tides, the derivation of the relation between this catalog and Doodson's one is left to the readers. For these terms, the coefficients  $C_i^{nk}(t)$  and  $S_i^{nk}(t)$  are constant, and the argument  $\alpha_i(t)$  is the same as (7.80), but written in the form of

$$\alpha_i(t) = a_i \lambda + (a_i \tau_0 + b_i s + c_i h + d_i p + e_i N' + f_i p_s), \quad (7.84)$$

where  $a_i = k$ , and  $\tau_0$  is the Greenwich mean lunar time. In more accurate catalogs, the expressions of the coefficients  $C_i^{nk}(t)$  and  $S_i^{nk}(t)$  include a linear term besides the constant term, and the ecliptic longitudes of some other planets considered are included in the expression of  $\alpha_i(t)$ .

Currently, the development is performed using computers. One way is to use a computer to derive trigonometric formulae analytically (e.g., Xi & Hou, 1987). The other way is to compute the values of  $\phi$  as a time series using numerical ephemerides according to (7.17) and (7.18), and then fit the time series with the expression (7.83) (e.g., Kudryavtsev, 2004). In the analytical development, the nutation components provided analytically, which may reach an order of magnitude of 10 seconds, can be taken into account, but the effect of the nutation component provided numerically and that of the polar motion have to be treated particularly. In fact, when separation of the waves is not necessary, it is more convenient to compute the tidal-generating potential using numerical ephemerides (Cartwright & Tayler, 1971; Petit & Luzum, 2010).

## 7.3 Earth Tide

### 7.3.1 Governing Equation of Tidal Deformation

The solid Earth deforms under the action of the tidal-generating force, called Earth tide, or body tide of the Earth. Here we formulate the deformation of a spherically symmetrical non-rotating elastic Earth model assuming the reference state, i.e., the undeformed state, to be in hydrostatic equilibrium. The elastic equations of

such an Earth model were summarized in Sect. 4.8.1, and the associated boundary conditions in Sect. 4.8.4. In Sect. 4.8.1, the external force  $\vec{f}$  was originally defined in (3.334) while formulating the elastic deformation of the more general Earth model. The tidal-generating force in Sect. 7.1.1 was defined based on this definition. According to the discussion below (7.3), the tidal-generating force of the Moon is  $\vec{f} = \rho \vec{t}_m = \rho \nabla \phi_m$ . It agrees with the assumptions made for  $\vec{f}$  and  $\phi_f$  in Sect. 4.8.1, or more specifically, in (4.444), considering that the tidal-generating potential is harmonic. The formula for the sun is in the same form. Hence, the development made in Sect. 4.8 is applicable as envisaged.

In the expression of the tidal-generating potential given by (7.83), the dependence of  $C_i^{nk}(t) \cos[\alpha_i(t)]$  and  $S_i^{nk}(t) \sin[\alpha_i(t)]$  on time is dominated by that of  $\alpha_i(t)$ , which is in turn dominated by the term  $\omega_i t + \phi_i$ , where  $\omega_i$  is the angular frequency of the tidal wave, and  $\phi_i$  its phase. The local mean solar time  $t$  depends on the longitude in the form of  $t = t_{UT} + \lambda$ , where  $t_{UT}$  is the universal time. Based on these discussions, in the formulation of Earth tide, we write the tidal-generating potential in the complex form as,

$$\phi = \sum_{n=2}^{\infty} \sum_{k=0}^n \left(\frac{r}{a}\right)^n \sum_i A_n^k(i) \left[ P_n^k(\cos \theta) e^{ik\lambda} e^{i\omega_i t} \right]. \quad (7.85)$$

For brevity of notations, we first formulate the effect of a single tidal wave of degree  $n$  and order  $k$ . We replace  $a$  by  $R$  as the scale of the length, since the Earth is assumed to be spherical here, and also use  $\phi$  to denote the tidal-generating potential of this tidal wave,

$$\begin{aligned} \phi &= \frac{GM}{R} \phi_n^k \left(\frac{r}{R}\right)^n P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t} \\ &= R g_R \phi_n^k \left(\frac{r}{R}\right)^n P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t} \\ &= \phi(R) \left(\frac{r}{R}\right)^n, \end{aligned} \quad (7.86)$$

where  $M$  is the mass of the Earth,  $R$  the radius of the Earth,  $g_R = GM/R^2$  the gravity at the surface of the Earth,  $\sigma$  the angular frequency of the tidal wave, and  $\phi(R)$  the value of  $\phi$  on the Earth's surface, i.e., when  $r = R$ .

In the rest of this subsection,  $\phi$  is understood as defined by (7.86). The summing up of the effects of all tidal waves will be discussed in the Sect. 7.3.3.

The governing equations of the Earth's elastic deformation are (4.435)–(4.437). Here we drop the Coriolis term in (4.435), and write these equation as, with  $\phi_f$  replaced by  $\phi$ ,

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \vec{T} - \rho(\nabla \cdot \vec{u}) \nabla \phi_0 + \rho \nabla \vec{u} \cdot \nabla \phi_0 + \rho \vec{u} \cdot \nabla \nabla \phi_0 + \rho \nabla(\phi + \phi_1), \quad (7.87)$$

$$\bar{T} = \lambda \nabla \cdot \vec{u} \bar{I} + \mu [\nabla \vec{u} + (\nabla \vec{u})^T], \quad (7.88)$$

$$\nabla^2(\phi + \phi_1) = 4\pi G \nabla \cdot (\rho \vec{u}), \quad (7.89)$$

where we used the fact  $\nabla^2 \phi = 0$  in the last equation. The term  $\phi_1$ , which is the gravitational potential due to the deformation of the Earth, is referred to as additional potential. The inclusion of the inertial term  $\rho \partial^2 \vec{u} / \partial t^2$  in (7.87), but not the Coriolis term, is a trick for practical computation at a later stage in solving the ordinary differential equations. The rigorous treatment of neglecting the inertial term will be discussed later in the formulation of static load Love numbers in Sect. 7.6.2.

The boundary conditions are summarized in Sect. 4.8.4. Here we list them again for convenience of reference, with modifications for the particular case of Earth tide. The condition of displacement across welded boundaries is

$$\{\vec{u}\}_{\pm}^+ = 0. \quad (7.90)$$

The condition of displacement across frictionless slip boundaries is

$$\{\hat{n} \cdot \vec{u}\}_{\pm}^+ = 0. \quad (7.91)$$

The condition of stress across all boundaries is

$$\{\hat{n} \cdot \bar{T}\}_{\pm}^+ = 0. \quad (7.92)$$

The only frictionless slip boundaries in the Earth's interior are the inner core boundary (ICB) and core-mantle boundary (CMB). In the fluid core, as  $\hat{n} \cdot \bar{T} = [\hat{n} \cdot (\hat{n} \cdot \bar{T})]\hat{n}$ , the condition that the shear stress vanishes at the ICB and CMB is automatically satisfied, and is not to be discussed further. On the Earth's surface, there is no constraint for the displacement  $\vec{u}$  in this case, and the condition for the stress  $\bar{T}$  is

$$(\hat{n} \cdot \bar{T})_{S_0} = 0, \quad (7.93)$$

i.e., no force is acting on the Earth's surface. The conditions for the gravitational potential across all kinds of boundaries are

$$\{\phi_1\}_{\pm}^+ = 0, \quad (7.94)$$

$$\left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G \rho (\hat{n} \cdot \vec{u}) \right\}_{\pm}^+ = 0, \quad (7.95)$$

noting also that  $\phi$  as well as its first order derivative is continuous throughout the Earth's interior.



### 7.3.2 Tidal Love Numbers

We continue to understand that  $\phi$  is defined by (7.86) in this subsection. It is straightforward to relate (7.86)–(7.85) and then to other forms of the tidal-generating potential given by (7.83), (7.38), and (7.29), particularly, through the value on the surface of the Earth,  $\phi(R)$ . Note that, since the Earth is assumed to be spherical here, the radius  $R$  will be used as the scale of length.

The tidal-generating potential  $\phi$  is the forcing term in the equations of the Earth's tidal deformation. It can be inferred from the formulation in Sect. 4.8 that the solution of the displacement should be a spheroidal field, as the toroidal field of the displacement is not associated with any change in the gravitational potential. Although the solution in Sect. 4.8 can be directly applied, here we first scale the unknown functions  $\vec{u}$ ,  $\bar{T}$ , and  $\phi + \phi_1$  by the factor  $\phi_n^k$ , i.e., the coefficient of the tidal-generating potential as given by (7.86), and then proceed to solve the problem according to the procedure of the formulation in Sect. 4.8. In this way,  $\phi_n^k$  is a common multiplier of all the functions and can be cancelled, and it is then straightforward to define the tidal Love numbers and provide the expressions of the solution of the displacement and gravitational potential. Based on the formulation in Sect. 4.8, more specifically, (4.488)–(4.491) and (4.480)–(4.487), we have, for the degree  $n$  and order  $k$  term alone,

$$\vec{u} = \phi_n^k \left( y_1^n P_n^k \hat{e}_r + y_3^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_3^n \frac{ik}{\sin\theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (7.96)$$

$$\hat{e}_r \cdot \bar{T} = \phi_n^k \left( y_2^n P_n^k \hat{e}_r + y_4^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_4^n \frac{ik}{\sin\theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (7.97)$$

$$\phi + \phi_1 = \phi_n^k y_5^n P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (7.98)$$

$$\frac{\partial}{\partial r}(\phi + \phi_1) - 4\pi G\rho(\hat{e}_r \cdot \vec{u}) = \phi_n^k y_6^n P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (7.99)$$

The only difference from the formulation in Sect. 4.8 is the presence of the extra factor  $\phi_n^k$  here, which is common to all functions. Therefore, all the differential equations and boundary conditions derived in Sect. 4.8 remain valid here, as only linear terms of the functions  $y_1^n, \dots, y_6^n$  are present. The result of the ordinary differential equations is the same as (4.517), which we copy to here for convenience of reference,

$$\begin{aligned} \frac{dy_1^n}{dr} &= -\frac{2\lambda}{(\lambda + 2\mu)r} y_1^n + \frac{1}{\lambda + 2\mu} y_2^n + \frac{n(n+1)\lambda}{(\lambda + 2\mu)r} y_3^n, \\ \frac{dy_2^n}{dr} &= \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda + 2\mu)r} y_2^n \\ &\quad + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_3^n + \frac{n(n+1)}{r} y_4^n \end{aligned} \quad (7.100a)$$

$$-\rho y_6^n, \quad (7.100b)$$

$$\frac{dy_3^n}{dr} = -\frac{1}{r}y_1^n + \frac{1}{r}y_3^n + \frac{1}{\mu}y_4^n, \quad (7.100c)$$

$$\begin{aligned} \frac{dy_4^n}{dr} = & \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{\lambda}{(\lambda + 2\mu)r} y_2^n + \left\{ -\rho\sigma^2 \right. \\ & \left. + \frac{2\mu}{(\lambda + 2\mu)r^2} \left[ (2n^2 + 2n - 1)\lambda + 2(n^2 + n - 1)\mu \right] \right\} y_3^n \\ & - \frac{3}{r}y_4^n - \frac{\rho}{r}y_5^n, \end{aligned} \quad (7.100d)$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \quad (7.100e)$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r}y_3^n + \frac{n(n+1)}{r^2}y_5^n - \frac{2}{r}y_6^n. \quad (7.100f)$$

The following boundary conditions also remain the same as those derived in Sect. 4.8:

- On internal welded boundaries,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_3^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0. \quad (7.101)$$

- On internal frictionless slip boundaries, i.e., the ICB and CMB,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0, \quad (7.102)$$

and no condition applies to  $y_3^n$ , meaning that it can be discontinuous.

- On the Earth's surface,

$$y_2^n(R) = y_4^n(R) = 0. \quad (7.103)$$

The only boundary condition derived in Sect. 4.8 in need of modification is (4.569), where the spherical harmonic coefficient of the potential of the external force appears. By definition of the functions  $y_1^n, \dots, y_6^n$ , the symbols  $y_5^n(R)$  and  $y_6^n(R)$  should be replaced by  $\phi_n^k y_5^n(R)$  and  $\phi_n^k y_6^n(R)$ , respectively. By comparing the definition of  $\phi_{fn}^k$  in (4.563) and that of  $\phi_n^k$  in (7.86), we realize that  $\phi_{fn}^k$  should be replaced by  $Rg_R\phi_n^k$ . These replacements lead to the boundary condition

$$\frac{n+1}{R}y_5^n(R) + y_6^n(R) = (2n+1)g_R. \quad (7.104)$$

This completes the formulation of the ordinary differential equations and boundary conditions to be solved numerically. An immediate observation is that the functions

$y_1^n, \dots, y_6^n$  depend on the degree  $n$ , but not the order  $k$ , which is a fact we already realized in Sect. 4.8. Hence, we have indicated its dependence on  $n$  only with the superscript  $n$ .

The ordinary differential equations (7.100) with the boundary conditions (7.101)–(7.104) can be solved using a numerical method to be addressed after this paragraph. The solution is customarily provided in terms of some parameters called tidal Love numbers. We first define the  $r$ -dependent tidal Love numbers as

$$H_n(r) = \frac{y_1^n(r)}{R}, \quad L_n(r) = \frac{y_3^n(r)}{R}, \quad K_n(r) = \frac{y_5^n(r)}{Rg_R} - 1. \quad (7.105)$$

On the surface of the Earth, we define

$$h_n = H_n(R) = \frac{y_1^n(R)}{R}, \quad l_n = L_n(R) = \frac{y_3^n(R)}{R}, \quad k_n = K_n(R) = \frac{y_5^n(R)}{Rg_R} - 1, \quad (7.106)$$

which are the parameters that the tidal Love numbers usually refer to. In fact, the numbers  $h_n$  and  $k_n$  were defined by Love, and the number  $l_n$  was defined by Shida. Therefore, the whole set of numbers  $h_n$ ,  $l_n$ , and  $k_n$  should be referred to as Love-Shida numbers, although they are mostly referred to as Love numbers.

The procedure of solution of the system of differential equations-boundary conditions (7.100)–(7.104) is almost identical to that of the system (6.192)–(6.195) of spheroidal free oscillation. We have six ordinary differential equations for six unknown functions with three regularity boundary conditions at the Earth center, and three boundary conditions on the Earth's surface. The characteristic of the problem here is that, as the boundary condition (7.104) is not homogeneous, i.e., it is not the vanishing of a linear combination of the unknown functions, a unique solution exists for any value of  $\sigma$ . We prescribe the value for  $\sigma$  as the tidal angular frequency, and start to integrate the ordinary differential equations from the Earth center assumed to be solid. The computation of solution from the center to the surface of the Earth is the same as in the case of spheroidal free oscillations presented in Sect. 6.4.2. For the present problem, we always assume the top layer to be solid. The ocean layer is either ignored or averaged with the top crust layer maintaining the mass. On the Earth's surface, there are three independent solutions and three boundary conditions. The last step is to find a linear combination of the three independent solutions that satisfies all the boundary conditions, which is the solution we are seeking.

The neglect of the Coriolis term can be verified a posteriori by computing it from the result of the solution, which is indeed of much smaller order of magnitude than the terms at the right-hand side of (7.87).

It can also be verified from the result of the solution that the inertial term  $\rho \partial^2 \vec{u} / \partial t^2$  in (7.87) is much smaller than the terms at the right-hand side, or equivalently, the term  $\rho \sigma^2$  in the system of equations (7.100) is negligibly small. This means that the effects of the tidal-generating and additional potentials are practically fully balanced by the stress of elastic deformation, which is actually

expected. The homogeneous part of the tidal elastic deformation equation is the same as the elastic free oscillation equations. We use  ${}_0\sigma_n$  to denote the angular frequency of the fundamental mode of the degree  $n$  spheroidal free oscillation, i.e., the mode with the lowest frequency. It is known that the tidal angular frequency  $\sigma$  is much smaller than  ${}_0\sigma_n$ . Hence, the resonance of the tidal deformation with free oscillations is extremely small, as it is proportional to  $1/|{}_0\sigma_n - \sigma|$ , i.e., the reciprocal of the frequency difference. We did the computation by assigning  $\sigma$  to be much smaller than  ${}_0\sigma_n$ . The results show that, as far as  $\sigma < (1/10){}_0\sigma_n$ , the differences among the tidal Love numbers obtained assuming different values of  $\sigma$  lie within 0.1%. Hence, in the practical numerical solution, we can assign any small enough value to  $\sigma$  without referring what the angular frequency of the tide actually is. We suggest  $\sigma$  to be chosen on the order of  $(1/10){}_0\sigma_n$ , since the non-rotating spherical model is at best accurate to the order of  $\sim 1/300$ . We mention that a too large value would bring non-negligible resonance effect into the solution, and a too small value would bring numerical instability in the numerical computation. With this property, we can consider the tidal Love numbers obtained as frequency independent.

As the Earth tide is strictly a periodical phenomenon, the inertial term  $\rho \partial^2 \vec{u} / \partial t^2$  in the original partial differential equation (7.87) exactly corresponds to the term  $\rho \sigma^2$  in the ordinary differential equations (7.100) to be solved numerically. Therefore, keeping them is the most natural way, although it turns out that the result of the tidal Love numbers is practically independent of the value of  $\sigma$ . However, knowing that the influence of the inertial term is negligibly small, we may think of formulating the problem without keeping this term, leading to a system of differential equations in the same form as (7.100) without the term  $\rho \sigma^2$ , and the same boundary conditions as (7.101)–(7.104). The problem is that, the application of (7.100) without the term  $\rho \sigma^2$  in the fluid outer core and the boundary conditions (7.101)–(7.104) at the inner core boundary (ICB) and core-mantle boundary (CMB) leads to a conflict: there is no solution that satisfies all the differential equations as well as all the boundary conditions. Therefore, completely neglecting the inertial term in the formulation requires the consideration of the Earth's elastic deformation equation based on a different perspective. We postpone the formulation without the inertial term in Sect. 7.6.2, where the static deformation of the Earth, i.e., the solution of the Earth's elastic equation without the inertial term is to be studied.

The tidal Love numbers computed based on the formulation of this subsection with a specific value of the angular frequency  $\sigma$  are called dynamic tidal Love numbers. The tidal Love numbers computed based on the formulation without the inertial term, which will be the subject of Sect. 7.6.2, is referred to as static tidal Love numbers. The term dynamic tidal Love numbers is really required only when the small differences among different tidal frequencies are to be distinguished. For the spherically symmetrical non-rotating elastic Earth model, the dependence on  $\sigma$  is extremely weak, and it suffices to classify the dynamic tidal Love numbers based on the value of Doodson's species-number  $a_i$ , i.e., to classify them as terdiurnal, semidiurnal, diurnal, secular, etc. The dependence of the dynamic tidal Love numbers on the tidal angular frequency  $\sigma$  is more complicated in higher

accuracy models when the ellipticity and rotation of the Earth are taken into account, where a resonance in the diurnal frequency band happens in the core (e.g., Mathews et al., 1995; Petit & Luzum, 2010), which is beyond the scope of this book.

### 7.3.3 Representation of Earth Tide Using Love Numbers

For the particular tidal wave (7.86), the displacement and gravitational potential inside the Earth can be expressed as, according to (7.96), (7.98), and (7.105),

$$\vec{u} = R\phi_n^k \left( H_n P_n^k \hat{e}_r + L_n \frac{d}{d\theta} P_n^k \hat{e}_\theta + L_n \frac{ik}{\sin \theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (7.107)$$

$$\phi + \phi_1 = Rg_R(1 + K_n)\phi_n^k P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (7.108)$$

The appearance of the complex number in these formulae is inconvenient. With the expression (7.86), we further write these formulae in the more practical forms

$$u_r = \frac{H_n}{g_R} \phi(R), \quad u_\theta = \frac{L_n}{g_R} \frac{\partial \phi(R)}{\partial \theta}, \quad u_\lambda = \frac{L_n}{g_R \sin \theta} \frac{\partial \phi(R)}{\partial \lambda}, \quad (7.109)$$

$$\phi + \phi_1 = (1 + K_n)\phi(R). \quad (7.110)$$

These expressions use the tidal-generating potential itself, not its spherical harmonic coefficient. Hence, they are no longer relying on the expression of the tidal-generating potential in terms of a complex spherical harmonic series.

The formulae (7.109) and (7.110) are for the particular tidal wave (7.86). However, with the judgment that the dependence of the tidal Love numbers on the tidal frequency can be neglected, these formulae can be applied as is to the degree  $n$  tide as a whole. We can then write the total tidal effect as the sum of all degrees,

$$u_r = \sum_n \frac{H_n}{g_R} \phi_n(R), \quad u_\theta = \sum_n \frac{L_n}{g_R} \frac{\partial \phi_n(R)}{\partial \theta}, \quad u_\lambda = \sum_n \frac{L_n}{g_R \sin \theta} \frac{\partial \phi_n(R)}{\partial \lambda}, \quad (7.111)$$

$$\phi + \phi_1 = \sum_n (1 + K_n) \phi_n(R), \quad (7.112)$$

where  $\phi$  is understood as defined by (7.85), and  $\phi_n$  its degree  $n$  term. It is straightforward to infer  $\phi_n$  according to the tidal-generating potential expressed in (7.17) and (7.18) or (7.83). With the tidal Love numbers defined in (7.106), the

displacement and potential on the Earth's surface are

$$\begin{cases} u_r(R) = \sum_n \frac{h_n}{g_R} \phi_n(R), \\ u_\theta(R) = \sum_n \frac{l_n}{g_R} \frac{\partial \phi_n(R)}{\partial \theta}, \\ u_\lambda(R) = \sum_n \frac{l_n}{g_R \sin \theta} \frac{\partial \phi_n(R)}{\partial \lambda}, \end{cases} \quad (7.113)$$

$$\phi(R) + \phi_1(R) = \sum_n (1 + k_n) \phi_n(R). \quad (7.114)$$

The gravitational potential outside the Earth, which is really useful, can be derived from (7.114). First, we obtain the value of  $\phi_1(R)$  to be

$$\phi_1(R) = \sum_n k_n \phi_n(R). \quad (7.115)$$

As  $\phi_1$  is due to the change of mass distribution inside the Earth caused by the tidal deformation, it is harmonic outside the Earth. Hence, outside the Earth, we have

$$\phi_1 = \sum_n k_n \left( \frac{R}{r} \right)^{n+1} \phi_n(R), \quad r \geq R. \quad (7.116)$$

We finally obtain, together with (7.17) and (7.18), or (7.83),

$$\begin{aligned} \phi + \phi_1 &= \sum_n \left[ \left( \frac{r}{R} \right)^n + k_n \left( \frac{R}{r} \right)^{n+1} \right] \phi_n(R) \\ &= \sum_n \left[ 1 + k_n \left( \frac{R}{r} \right)^{2n+1} \right] \phi_n, \quad r \geq R. \end{aligned} \quad (7.117)$$

It is to be noted that, at the right-hand side, the first term in the brackets represents the tidal-generating potential  $\phi$ , and the second term the gravitational potential of the Earth tide,  $\phi_1$ , which is referred to as additional potential.

The formulation of the problem for computing the tidal deformation is now complete.

### 7.3.4 Secular and Fluid Love Numbers

A secular Love number  $k_2^s$  can be defined assuming the Earth's oblateness to be the Earth's response to the Earth rotation, which can be considered as time-independent

(the meaning of *secular* here). The formula of the centrifugal potential is

$$\psi = \frac{1}{2}\omega^2 r^2 \sin^2 \theta = \frac{1}{2}\omega^2 r^2 (1 - \cos^2 \theta) = \frac{1}{3}\omega^2 r^2 - \frac{1}{3}\omega^2 r^2 P_2(\cos \theta), \quad (7.118)$$

where the first term is spherically symmetrical, and the second term is in the same form as a degree 2 and order 0 tidal-generating potential. The Earth's response to the centrifugal potential as given by the second term in (7.118) is the degree 2 and order 0 term of the Earth's gravitational potential, which represents the Earth's oblateness. Base on the spherical harmonic series of the gravitational potential given in (5.13) and the definition of  $J_2$  given in (5.14), the degree 2 and order 0 term of the Earth's gravitational potential on the Earth's surface is  $-(GM/R)J_2 P_2(\cos(\theta))$ . According to (7.114), the secular Love number  $k_2^s$  is defined through

$$-\frac{GM}{R}J_2 P_2(\cos(\theta)) = -k_2^s \frac{1}{3}\omega^2 R^2 P_2(\cos \theta). \quad (7.119)$$

Finally we obtain

$$k_2^s = \frac{3GMJ_2}{\omega^2 R^3} = \frac{3G(C-A)}{\omega^2 R^5} = 0.94, \quad (7.120)$$

where we made the approximations  $A = B$  and  $a = R$  in the expression of  $J_2$  defined in (5.14). The value 0.94 is calculated using  $J_2 = 1.083 \times 10^{-3}$  and  $q = \omega^2 R^3/GM = 3.461 \times 10^{-3}$ .

We also consider the height of a point on the reference Earth ellipsoid above the surface of the averaged spherical Earth model as the vertical displacement in response to the centrifugal potential as given by the second term in (7.118). This height can be inferred from (5.35) to be  $-(2/3)Rf P_2(\cos \theta)$ . Therefore, according to (7.113), we define the secular Love number  $h_2^s$  through

$$-(2/3)Rf P_2(\cos \theta) = -\frac{h_2^s}{g_R} \frac{1}{3}\omega^2 R^2 P_2(\cos \theta). \quad (7.121)$$

We finally obtain, with  $g_R = GM/R^2$ ,

$$h_2^s = \frac{2fg_R}{\omega^2 R} = \frac{2fGM}{\omega^2 R^3} = 1.94, \quad (7.122)$$

The numerical value 1.94 is obtained from the values  $f = 3.353 \times 10^{-3}$  and  $q = \omega^2 R^3/GM = 3.461 \times 10^{-3}$  of the reference Earth ellipsoid.

If the Earth were in strict hydrostatic equilibrium under the action of purely its own gravitational and centrifugal forces, the secular Love numbers  $k_2^s$  and  $h_2^s$  would be the Love numbers as if the Earth were fluid and respond to external forcing in hydrostatic equilibrium. The Love numbers of a fluid Earth model are called

fluid Love numbers, and we denote them as  $k_2^f$  and  $h_2^f$ . With the values of  $C - A = 2.603 \times 10^{35}$  and  $f = 3.336 \times 10^{-3}$  in hydrostatic equilibrium calculated in Sect. 5.5.6, together with the values of  $GM$ ,  $\omega$  and  $R$ , we obtain

$$k_2^f = 0.93, \quad h_2^f = 1.93. \quad (7.123)$$

The secular Love numbers are affected by the deviation of the Earth from hydrostatic equilibrium. As expected, this effect is very small, which is confirmed by the small difference between the secular and fluid Love numbers. An application of these Love numbers will be discussed in Sect. 7.4.6.

## 7.4 Observable Earth Tide Effects

In this section, we formulate the observable effects of Earth tide assuming the Love numbers are only degree dependent, and are independent of tidal frequency.

### 7.4.1 Absolute and Relative Measures of the Equilibrium Ocean Tide

We assume the ocean surface to be an equipotential surface of gravity, and the equilibrium ocean tide is the change of this surface due to the addition of the tidal-generating and additional potentials.

Without the tidal-generating and additional potentials, the ocean surface can be expressed as

$$W = W_0, \quad (7.124)$$

where  $W$  is the gravity potential of the undeformed Earth, and  $W_0$  its value on the ocean surface, which is a constant.

We first measure the ocean tide as the rise of the ocean surface with respect to the unperturbed position,  $\zeta^a$ . This is in fact a measure of the ocean tide with respect to the Earth center, and we refer it to as the absolute measure of ocean tide. The equation of the ocean surface including the tidal-generating and additional potentials is

$$W_0 + \zeta^a \left( \frac{\partial W}{\partial H} \right)_{W=W_0} + \sum_n (1 + k_n) \phi_n(R) = W_0, \quad (7.125)$$

where  $H$  is the height above the geoid, and  $n$  the degree of the tide. This equation assumes that the ocean surface reaches equilibrium instantaneously. In other words, the effect of ocean flow to reach the equilibrium is neglected. The sum of the first two terms at the left-hand side is the value of  $W$  on the deformed ocean surface



accurate to the first order of  $\zeta^a$ . The tidal-generating and additional potentials, i.e., the third term at the left-hand side, are small quantities, and their values on the ocean surface are approximated by their value on the sphere of radius  $R$ . As the gravity on the ocean surface,  $(\partial W/\partial H)_{W=W_0}$ , is multiplied by the small quantity  $\zeta^a$ , it can be approximated as  $g_R = GM/R^2$ . By definition, the average of  $\phi_n$  over the sphere vanishes. Hence, the third term at the left-hand side does not bring an offset to the gravity potential of the equipotential surface of gravity, which is the reason the right-hand side of (7.125) remains to be  $W_0$ . In the formulae, the effect of the redistribution of ocean water mass is neglected. Furthermore, the limited coverage of the Earth's surface by the oceans and the dynamic effect of ocean water flow are also neglected. Therefore, (7.125) is only a primitive approximation.

We obtain from (7.125),

$$\zeta^a = \sum_n (1 + k_n) \frac{\phi_n(R)}{g_R} = \sum_n \gamma_n^a \bar{\zeta}_n, \quad (7.126)$$

where the parameter  $\gamma_n^a$  is defined as

$$\gamma_n^a = 1 + k_n, \quad (7.127)$$

and  $\bar{\zeta}_n = \phi_n(R)/g_R$  is the degree  $n$  term of the spherical harmonic series of the ocean surface rise from its unperturbed position due to the tidal-generating potential alone. The quantity  $\bar{\zeta} = \sum_n \bar{\zeta}_n$  is referred to as equilibrium tide, which represents the perturbation of the tidal-generating potential to the ocean surface assumed to reach equilibrium instantaneously. It can also be understood as the perturbation of the tidal-generating potential to the geoid assuming the Earth to be completely rigid, thus also being referred to geoid tide. With the contemporary satellite observation technologies, the absolute measure of sea level  $\zeta^a$  is supposed to be observed by satellite altimetry. We call the parameter  $\gamma_n^a$  tidal factor of ocean tide with respect to the Earth center.

Customarily, the amplitude of ocean tide is understood as the variation of ocean depth in response to the tidal-generating and additional potentials. This is a measure of ocean tide with respect to the ocean bottom, and we refer it to as relative sea level change of ocean tide. It includes two parts. One is the rise of the ocean surface in response to the tidal-generating and additional potentials, which is given in (7.126). The other one is the rise of ocean bottom in response to the tidal-generating and additional potentials, which is given in (7.113). Here we write it in the form of

$$u_r(R) = \sum_n h_n \frac{\phi_n(R)}{g_R}. \quad (7.128)$$

We see a physical interpretation of the tidal Love number  $h_n$ : it is the ratio between the degree  $n$  ground uplift tide and geoid tide. The amplitude of ocean tide is obtained by taking away the rise of the ocean bottom from the rise of the ocean

surface,

$$\zeta = \sum_n (1 + k_n - h_n) \frac{\phi_n(R)}{g_R} = \sum_n \gamma_n \bar{\zeta}_n, \quad (7.129)$$

where the parameter  $\gamma_n$  is defined as

$$\gamma_n = 1 + k_n - h_n, \quad (7.130)$$

and  $\bar{\zeta}_n = \phi_n(R)/g_R$  is the degree  $n$  the geoid tide. The parameter  $\gamma_n$  is called tidal factor of ocean tide, which can be estimated using data of the ocean tide amplitude  $\zeta$  as measured by tide gauges.

We emphasize again that the use of (7.126) implies that the ocean surface is assumed to be in equilibrium, and the effect of the redistribution of ocean water mass is neglected. Therefore, the accuracy of (7.126) and (7.129) is limited.

### 7.4.2 Variation of Gravity at a Ground Station

The variation of gravity as measured by a gravimeter installed on the Earth's surface is composed of three parts. The first part is the tidal-generating force in the downward direction,

$$\Delta g' = - \left( \frac{\partial \phi}{\partial r} \right)_{r=R}. \quad (7.131)$$

The second part is the gravitational attraction in the downward direction related to the Earth tide, i.e., the tidal deformation of the Earth,

$$\Delta g'' = - \left( \frac{\partial \phi_1}{\partial r} \right)_{r=R}. \quad (7.132)$$

The third part is the gravity variation due to the vertical motion of the ground where the gravimeter is installed,

$$\Delta g''' = u_r(R) \left( \frac{\partial g}{\partial r} \right)_{r=R}. \quad (7.133)$$

The detailed expressions of the first and second parts can be obtained from the two terms at the right-hand side of (7.117),

$$\Delta g' = - \sum_n \left[ \frac{d}{dr} \left( \frac{r}{R} \right)^n \right]_{r=R} \frac{\phi_n(R)}{R} = - \sum_n \frac{n \phi_n(R)}{R}, \quad (7.134)$$

$$\Delta g'' = - \sum_n k_n \left[ \frac{d}{dr} \left( \frac{R}{r} \right)^{n+1} \right]_{r=R} \phi_n(R) = \sum_n \frac{(n+1) k_n \phi_n(R)}{R}. \quad (7.135)$$

As for the third part, since  $u_r(R)$  is very small, we make the spherical approximation to  $(\partial g / \partial r)_{r=R}$ ,

$$g(r) = -\frac{\partial}{\partial r} \left( \frac{GM}{r} \right) = \frac{GM}{r^2}, \quad (7.136)$$

$$\left( \frac{\partial g}{\partial r} \right)_{r=R} = -2 \frac{GM}{R^3} = -\frac{2g_R}{R}. \quad (7.137)$$

Hence,

$$\Delta g''' = - \left( \frac{\partial g}{\partial r} \right)_{r=R} \sum_n \frac{h_n \phi_n(R)}{g_R} = - \sum_n \frac{2h_n \phi_n(R)}{R}. \quad (7.138)$$

Finally, the variation of gravity  $\Delta g$  is the sum of  $\Delta g'$ ,  $\Delta g''$ , and  $\Delta g$ ,

$$\Delta g = - \sum_n \left( 1 - \frac{n+1}{n} k_n + \frac{2}{n} h_n \right) \frac{n \phi_n(R)}{R} = \sum_n \delta_n \overline{\Delta g_n}, \quad (7.139)$$

where the parameter  $\delta_n$  is defined as

$$\delta_n = 1 - \frac{n+1}{n} k_n + \frac{2}{n} h_n, \quad (7.140)$$

and  $\overline{\Delta g_n} = (n+1)k_n \phi_n(R)/R$  is the contribution of the tidal-generating potential alone. The parameter  $\delta_n$  is called tidal factor of gravity, which can be determined by measuring gravity.

If the gravity is measured on an airborne platform, the third part should not be included. Therefore, the last term at the right-hand side of (7.140) is not to be included.

### 7.4.3 Deviation of the Vertical with Respect to the Crust at a Station

This is the deviation of the vertical, i.e., the change of the direction of the gravity, that a tiltmeter fixed on the ground measures. It includes two parts. One is the effect of the tidal-generating and additional potentials on the direction of the vertical. The other is the tilt of the crust with respect to the undeformed state due the tidal deformation of the Earth. The variation of the direction of the vertical with respect to the crust is the difference between the two. According to the definition of the deflection of the vertical in Sect. 5.2.4, the effect of the tidal-generating and additional potentials is

$$\begin{cases} \xi' = \sum_n (1 + k_n) \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \partial \theta}, \\ \eta' = - \sum_n (1 + k_n) \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \sin \theta \partial \lambda}. \end{cases} \quad (7.141)$$

The tilt of the crust with respect to the undeformed state due to tidal deformation is

$$\begin{cases} \xi'' = \frac{\partial u_r(R)}{R \partial \theta} = \sum_n h_n \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \partial \theta}, \\ \eta'' = - \frac{1}{R \sin \theta} \frac{\partial u_r(R)}{\partial \lambda} = - \sum_n h_n \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \sin \theta \partial \lambda}. \end{cases} \quad (7.142)$$

The deviation of the vertical with respect to the crust is their difference,

$$\begin{cases} \xi = \sum_n (1 + k_n - h_n) \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \partial \theta} = \sum_n \gamma_n \bar{\xi}_n, \\ \eta = \sum_n (1 + k_n - h_n) \left[ - \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \sin \theta \partial \lambda} \right] = \sum_n \gamma_n \bar{\eta}_n, \end{cases} \quad (7.143)$$

where the parameter  $\gamma_n$  is defined as

$$\gamma_n = 1 + k_n - h_n, \quad (7.144)$$

and  $\bar{\xi}_n$  and  $\bar{\eta}_n$  are the contribution of the tidal-generating potential alone. The parameter  $\gamma_n$  is referred to as tidal factor of crustal tilt. Note that the components of the tilt of the crust with respect to the vertical is  $-\xi$  and  $-\eta$ .

Of particular interest is the deviation of the vertical as measured by an instrument installed along a horizontal direction of azimuth  $A$ ,

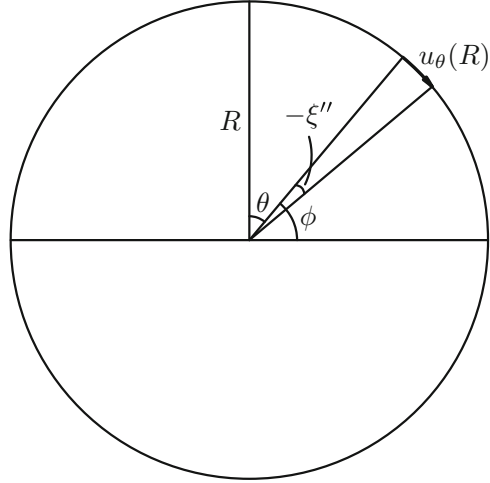
$$\epsilon_A = -\xi \cos A - \eta \sin A = - \sum_n \gamma_n (\bar{\xi}_n \cos A + \bar{\eta}_n \sin A). \quad (7.145)$$

This formula could be used in the reverse sense: if the orientation of the instrument is not known, its azimuth  $A$  could be determined by analyzing the Earth tide data of the tilt it measures,  $\epsilon_A$ , particularly, for stations deep in a cave or a mine.

#### 7.4.4 Deviation of the Vertical at a Station with Respect to the Celestial Reference System

This is the deviation of the vertical as determined by observing celestial objects at a station. It can be understood as the effect on the astronomical latitude and longitude of the station. It includes two parts. One is the change of the direction

**Fig. 7.9** Deviation of the vertical with respect to the space related to the displacement



of the gravity related to the tidal-generating and additional potentials, which is the same as in the case of the deviation of the direction of the gravity with respect to the crust as given by (7.141). The other is the contribution of the displacement of the astronomical instrument following the displacement of the crust. The case for the latitudinal component by approximating the Earth as a sphere is shown in Fig. 7.9, noting that the displacement toward the south causes the gravity to deviate toward the north for the instrument. This is in fact the angle between the verticals at the displaced and original positions. The contribution to the two components are

$$\begin{cases} \xi'' = -\frac{u_\theta(R)}{R} = -\sum_n l_n \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \partial \theta}, \\ \eta'' = \frac{u_\lambda(R)}{R} = \sum_n l_n \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \sin \theta \partial \lambda}. \end{cases} \quad (7.146)$$

The total deviation of the vertical is the sum of the two. We have

$$\begin{cases} \xi = \sum_n (1 + k_n - l_n) \frac{1}{g_R} \frac{\partial \phi_n(R)}{R \partial \theta} = \sum_n \Lambda_n \bar{\xi}_n, \\ \eta = \sum_n (1 + k_n - l_n) \left[ -\frac{1}{g_R} \frac{\partial \phi_n(R)}{R \sin \theta \partial \lambda} \right] = \sum_n \Lambda_n \bar{\eta}_n, \end{cases} \quad (7.147)$$

where the parameter  $\Lambda_n$  is defined as

$$\Lambda_n = 1 + k_n - l_n, \quad (7.148)$$

and  $\bar{\xi}_n$  and  $\bar{\eta}_n$  are the contribution of the tidal-generating potential alone. The parameter  $\Lambda_n$  is called tidal factor of astronomical coordinates.

### 7.4.5 The Strain Tensor

The strain-displacement relation in the spherical coordinate system can be derived direction by expressing the relation (3.211) in the spherical coordinate system, which is straightforward using (2.467). Here we first directly derive the formula of the normal components of the strain tensor,  $\epsilon_{rr}$ ,  $\epsilon_{\theta\theta}$ , and  $\epsilon_{\lambda\lambda}$ , as functions of the displacement components,  $u_r$ ,  $u_\theta$ , and  $u_\lambda$ . This illustrates more clearly the physical meaning of the normal strain components in the spherical coordinate system.

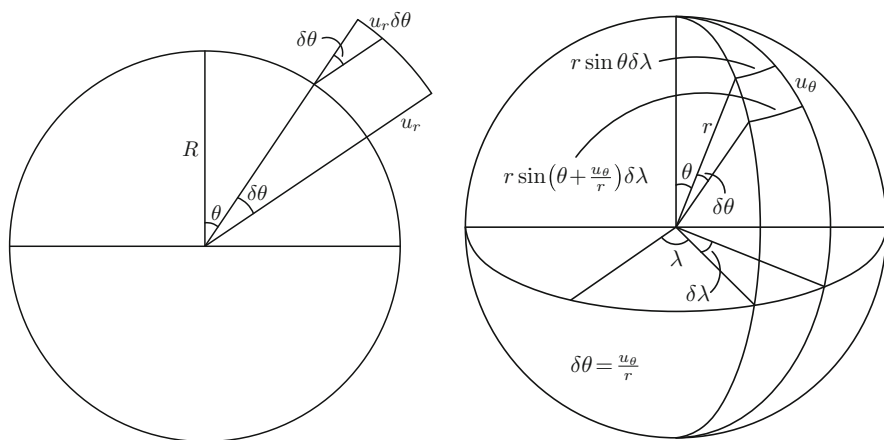
The component  $\epsilon_{rr}$  is the relative elongation of a line segment along the direction of  $\hat{e}_r$ . We take a line segment of  $\delta r$  long in the undeformed state. The prolongation of the line segment due to tidal deformation is  $(\partial u_r / \partial r) \delta r$ . Hence, the normal strain component  $\epsilon_{rr}$ , i.e., the relative elongation of the line segment, is

$$\epsilon_{rr} = \left( \frac{\partial u_r}{\partial r} \delta r \right) \delta r^{-1} = \frac{\partial u_r}{\partial r}. \quad (7.149)$$

The component  $\epsilon_{\theta\theta}$  is the relative elongation of a line segment along the direction of  $\hat{e}_\theta$ . In the undeformed state, the length of the line segment corresponding to the coordinate increment  $\delta\theta$  is  $r\delta\theta$ . The displacement component  $u_\theta$  causes the line segment to prolong  $(\partial u_\theta / \partial \theta) \delta\theta$ . As shown in Fig. 7.10 (left), the displacement component  $u_r$  causes the line segment to prolong  $u_r \delta\theta$ . Hence, the normal strain component  $\epsilon_{\theta\theta}$ , i.e., the relative elongation of the line segment, is

$$\epsilon_{\theta\theta} = \left( \frac{\partial u_\theta}{\partial \theta} \delta\theta + u_r \delta\theta \right) (r \delta\theta)^{-1} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}. \quad (7.150)$$

The component  $\epsilon_{\lambda\lambda}$  is the relative elongation of a line segment along the direction of  $\hat{e}_\lambda$ . In the undeformed state, the length of the line segment corresponding to the



**Fig. 7.10** Relation between displacement and strain. Left: The contribution of  $u_r$  to  $\epsilon_{\theta\theta}$ ; Right: The contribution of  $u_\theta$  to  $\epsilon_{\lambda\lambda}$ .

coordinate increment  $\delta\lambda$  is  $r \sin\theta \delta\lambda$ . The displacement component  $u_\lambda$  causes the line segment to prolong  $(\partial u_\lambda / \partial \lambda) \delta\lambda$ . The displacement component  $u_r$  causes the line segment to prolong  $u_r \sin\theta \delta\lambda$ . As shown in Fig. 7.10 (right), the displacement component  $u_\theta$  causes the line segment to prolong  $r \sin[\theta + u_\theta/r] \delta\lambda - r \sin\theta \delta\lambda$ , which can also be written as  $r \sin\theta \cos(u_\theta/r) \delta\lambda + r \cos\theta \sin(u_\theta/r) \delta\lambda - r \sin\theta \delta\lambda$ . Noting that the difference between  $\cos(u_\theta/r)$  and 1 is on the order of  $(u_\theta/r)^2$ , and the first order approximation of  $\sin(u_\theta/r)$  is  $u_\theta/r$ , we can write this part of prolongation as  $u_\theta \cos\theta \delta\lambda$  under first order approximation. Hence, the normal strain component  $\epsilon_{\lambda\lambda}$ , i.e., the relative elongation of the line segment, is

$$\begin{aligned} \epsilon_{\lambda\lambda} &= \left( \frac{\partial u_\lambda}{\partial \lambda} \delta\lambda + u_r \sin\theta \delta\lambda + u_\theta \cos\theta \delta\lambda \right) (r \sin\theta \delta\lambda)^{-1} \\ &= \frac{1}{r \sin\theta} \frac{\partial u_\lambda}{\partial \lambda} + \frac{u_r}{r} + \frac{\cot\theta}{r} u_\theta. \end{aligned} \quad (7.151)$$

It can be seen that  $\epsilon_{rr}$  is not affected by  $u_\theta$  and  $u_\lambda$ , and  $\epsilon_{\theta\theta}$  is not affected by  $u_\lambda$ .

By substituting (7.109) and (7.113) into (7.149)–(7.151), we obtain the three normal strain components on the Earth's surface,

$$\begin{aligned} \epsilon_{rr}(R) &= \sum_n \left\{ \left[ \frac{dH_n(r)}{dr} \right]_{r=R} \frac{\phi_n(R)}{g_R} + \frac{H_n(R)}{g_R} \left[ \frac{d\phi_n(r)}{dr} \right]_{r=R} \right\} \\ &= \sum_n (Rh'_n + nh_n) \frac{\phi_n(R)}{Rg_R}, \end{aligned} \quad (7.152a)$$

$$\epsilon_{\theta\theta}(R) = \sum_n \left[ \frac{l_n}{Rg_R} \frac{\partial^2 \phi_n(R)}{\partial \theta^2} + \frac{h_n}{Rg_R} \phi_n(R) \right], \quad (7.152b)$$

$$\epsilon_{\lambda\lambda}(R) = \sum_n \left[ \frac{l_n}{Rg_R \sin^2\theta} \frac{\partial^2 \phi_n(R)}{\partial \lambda^2} + \frac{l_n}{Rg_R} \cot\theta \frac{\partial \phi_n(R)}{\partial \theta} + \frac{h_n}{Rg_R} \phi_n(R) \right], \quad (7.152c)$$

where the parameter  $h'_n$  is defined as

$$h'_n = \left[ \frac{dH_n(r)}{dr} \right]_{r=R} = \frac{1}{R} \left[ \frac{d}{dr} y_1^n(r) \right]_{r=R}. \quad (7.153)$$

By definition,  $h'_n$  can be obtained according to the first equation of (7.100), together with the definitions of the tidal Love numbers and boundary condition of  $y_2^n$ ,

$$h'_n = -\frac{2\lambda_R}{R(\lambda_R + 2\mu_R)} h_n + \frac{n(n+1)\lambda_R}{R(\lambda_R + 2\mu_R)} l_n = \frac{\lambda_R}{R(\lambda_R + 2\mu_R)} [-2h_n + n(n+1)l_n]. \quad (7.154)$$

Different parameters can be determined using different combinations of  $\epsilon_{\theta\theta}$  and  $\epsilon_{\lambda\lambda}$ . Their sum is

$$\epsilon_{\theta\theta}(R) + \epsilon_{\lambda\lambda}(R) = \sum_n \left\{ \frac{2h_n}{Rg_R} \phi_n(R) + \frac{l_n}{Rg_R} \left[ \frac{\partial^2 \phi_n(R)}{\partial \theta^2} + \cot \theta \frac{\partial \phi_n(R)}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi_n(R)}{\partial \lambda^2} \right] \right\}. \quad (7.155)$$

As  $\phi_n(R)$  is in fact a surface spherical harmonics of degree  $n$ ,  $r^n \phi_n(R)$  must satisfy Laplace's equation, and we have

$$\frac{\partial^2 \phi_n(R)}{\partial \theta^2} + \cot \theta \frac{\partial \phi_n(R)}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi_n(R)}{\partial \lambda^2} = -n(n+1)\phi_n(R). \quad (7.156)$$

By substituting (7.156) into (7.155), we obtain

$$\epsilon_{\theta\theta}(R) + \epsilon_{\lambda\lambda}(R) = \sum_n [2h_n - n(n+1)l_n] \frac{\phi_n(R)}{Rg_R}. \quad (7.157)$$

This implies that the combination  $\epsilon_{\theta\theta} + \epsilon_{\lambda\lambda}$  can be used to determine the Love numbers combination  $2h_n - n(n+1)l_n$ . Furthermore, the difference between  $\epsilon_{\theta\theta}$  and  $\epsilon_{\lambda\lambda}$  can be used to determine  $l_n$ ,

$$\epsilon_{\theta\theta}(R) - \epsilon_{\lambda\lambda}(R) = \sum_n \frac{l_n}{Rg_R} \left[ \frac{\partial^2 \phi_n(R)}{\partial \theta^2} - \cot \theta \frac{\partial \phi_n(R)}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi_n(R)}{\partial \lambda^2} \right]. \quad (7.158)$$

As already mentioned, the strain-displacement relation in the spherical coordinate system can be derived directly by expressing the relation (3.211) in the spherical coordinate system, which is straightforward using (2.467). The normal strain components have already been given above. Here we list the shear strain components:

$$\begin{cases} \epsilon_{r\theta} = \epsilon_{\theta r} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right), \\ \epsilon_{\theta\lambda} = \epsilon_{\lambda\theta} = \frac{1}{2} \left[ \frac{1}{r} \left( \frac{\partial u_\lambda}{\partial \theta} - u_\lambda \cot \theta \right) + \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \lambda} \right], \\ \epsilon_{\lambda r} = \epsilon_{r\lambda} = \frac{1}{2} \left( \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \lambda} + \frac{\partial u_\lambda}{\partial r} - \frac{u_\lambda}{r} \right). \end{cases} \quad (7.159)$$

By substituting (7.111) into these equations and then using (7.106), we obtain, after some derivations similar to those of (7.152),



$$\begin{cases} \epsilon_{r\theta}(R) = \epsilon_{\theta r}(R) = \frac{1}{2} \sum_n [Rl'_n + (n-1)l_n + h_n] \frac{1}{Rg_R} \frac{\partial \phi_n(R)}{\partial \theta}, \\ \epsilon_{\theta\lambda}(R) = \epsilon_{\lambda\theta}(R) = \frac{1}{2} \sum_n \left[ \frac{2l_n}{Rg_R \sin \theta} \frac{\partial^2 \phi_n(R)}{\partial \theta \partial \lambda} - \frac{2l_n \cos \theta}{Rg_R \sin^2 \theta} \frac{\partial \phi_n(R)}{\partial \lambda} \right], \\ \epsilon_{\lambda r}(R) = \epsilon_{r\lambda}(R) = \frac{1}{2} \sum_n [h + Rl'_n + (n-1)l_n] \frac{1}{Rg_R \sin \theta} \frac{\partial \phi_n(R)}{\partial \lambda}. \end{cases} \quad (7.160)$$

The parameter  $l'_n$  is defined as

$$l'_n = \left[ \frac{dL_n(r)}{dr} \right]_{r=R} = \frac{1}{R} \left[ \frac{d}{dr} y_3^n(r) \right]_{r=R}, \quad (7.161)$$

which can be computed according to the third equation of (7.100),

$$l'_n = \frac{1}{R} (-h_n + l_n). \quad (7.162)$$

Note that the notations  $h'_n$  and  $l'_n$  here are different from the load Love numbers  $h'_n$  and  $l'_n$  to be defined later.

Of particular interest is the normal component of the strain tensor along a horizontal direction of azimuth  $A$ ,

$$\epsilon_{AA} = \hat{e}_A \cdot (\hat{e}_A \cdot \bar{\epsilon}) = (\hat{e}_A \cdot \bar{\epsilon}) \cdot \hat{e}_A, \quad (7.163)$$

where  $\hat{e}_A = -\hat{e}_\theta \cos A + \hat{e}_\lambda \sin A$  is the unit vector along this direction, and  $\bar{\epsilon}$  the strain tensor. It is most convenient to write this formula in matrix form to derive the expression of  $\epsilon_{AA}$  in terms of the components of the strain tensor  $\bar{\epsilon}$ . We obtain

$$\begin{aligned} \epsilon_{AA} &= \left\{ [0 - \cos A \sin A] \begin{bmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{r\lambda} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta\lambda} \\ \epsilon_{\lambda r} & \epsilon_{\lambda\theta} & \epsilon_{\lambda\lambda} \end{bmatrix} \right\} \begin{Bmatrix} 0 \\ -\cos A \\ \sin A \end{Bmatrix} \\ &= \epsilon_{\theta\theta} \cos^2 A - \epsilon_{\theta\lambda} \sin 2A + \epsilon_{\lambda\lambda} \sin^2 A. \end{aligned} \quad (7.164)$$

The expression in terms of the tidal Love numbers and tidal-generating potential can be obtained by substituting (7.152) and (7.160) into this formula. The derivation is left to the readers as an exercise. We mention that this formula could be used in the reverse sense: if the orientation of the instrument is not known, its azimuth  $A$  could be determined by analyzing the Earth tide data of the strain it measures,  $\epsilon_{AA}$ , particularly, for stations in a cave or a mine.

#### 7.4.6 Computation and Some Characteristics of the Earth Tide

The computation of Earth tide is straightforward using a model of tidal waves. Here we follow Doodson's convention using the data listed in Table 7.2. We consider

here only the degree 2 tide. According to (7.46)–(7.38) and the tidal waves defined in Sect. 7.2.2, we can write the degree 2 tidal-generating potential as

$$\phi_2 = \sum_{k=0}^2 \left[ G_k \sum_i Y_i \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s) \right]. \quad (7.165)$$

This formula is understood as the sum of (7.73), (7.75), and (7.77) multiplied with the respective geodetic coefficients  $G_k$  defined in (7.43)–(7.46). The meaning of the symbols are evident. The subscript ‘2’ of  $\phi_2$  means degree 2,  $k$  is the order of the spherical harmonics,  $i$  indicates the tidal waves of order  $k$ , and cos or sin is what appears in (7.73), (7.75), or (7.77). We see that  $a_i = k$ .

The degree 2 tidal-generating force is

$$\vec{t}_2 = \nabla \phi_2 = \frac{\partial \phi_2}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi_2}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \phi_2}{\partial \lambda} \hat{e}_\lambda. \quad (7.166)$$

By definition,  $G_k$  is a function of  $r$  and  $\Phi$ , with  $\Phi = \pi/2 - \theta$ . The longitude of the station enters through the mean lunar time,  $\tau = \tau_0 + \lambda$ , where  $\tau_0$  is the Greenwich mean lunar time. We thus have

$$\begin{cases} \frac{\partial \phi_2}{\partial r} = \sum_{k=0}^2 \left[ \frac{\partial G_k}{\partial r} \sum_i Y_i \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s) \right], \\ \frac{\partial \phi_2}{\partial \theta} = \sum_{k=0}^2 \left[ - \frac{\partial G_k}{\partial \Phi} \sum_i Y_i \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s) \right], \\ \frac{\partial \phi_2}{\partial \lambda} = \sum_{k=0}^2 \left[ G_k \sum_i Y_i a_i \left\{ \begin{matrix} -\sin \\ \cos \end{matrix} \right\} (a_i \tau + b_i s + c_i h + d_i p + e_i N' + f_i p_s) \right]. \end{cases} \quad (7.167)$$

It is left to the reader to derive the formulae for  $\partial G_k / \partial r$  and  $-\partial G_k / \partial \Phi$ . Note that  $\partial G_k / \partial r = 2G_k / r$ , leading to  $\partial \phi_2 / \partial r = 2\phi_2 / r$ , which is a special case of  $\partial \phi_n / \partial r = n\phi_n / r$ . Second order derivatives are required for computing the stress.

The computation using a model of tidal waves in the form of (7.83) is similar; all the need is to realize how  $\phi$  depends on  $r$ ,  $\theta$ , and  $\lambda$ . The computation can also be done using numerical ephemerides according to Cartwright and Tayler (1971).

The tidal Love numbers are required to compute the observable quantities. We take the values obtained by solving the ordinary differential equation (7.100) with the associated boundary conditions (7.101)–(7.104),

$$h_2 = 0.61, \quad l_2 = 0.085, \quad k_2 = 0.30. \quad (7.168)$$

Their practical computation is the topic of Sects. 7.3.2 and 7.8.1. Here they are accurate to two significant digits assumed to be independent of the tidal frequency for demonstrative purpose.

We are now ready to compute all the Earth tide observables we have formulated. We can compute all, a subset, or just one of the tidal waves for any location. Here we are not going to actually get into the detail of the numerical result. Instead, we infer some general characteristics of the Earth tide.

First, we look at the largest lunar tidal wave  $M_2$  with the magnitude 0.91 in Table 7.2. The maximum of  $G_2$  on the ground (with  $r \approx a$ ) is  $D$  at  $\Phi = 0$  (i.e., on the equator). The amplitude of the geoid tide there is  $\bar{\zeta} = \phi_{M_2}/g_R = 0.24$  m (i.e., 0.48 m peak to peak), that of the vertical displacement there,  $h_2\bar{\zeta} = 0.15$  m (i.e., 0.30 m peak to peak), and that of the gravity on the ground there,  $[1 - (3/2)k_2 + h_2](2\phi_{M_2}/R) = 8.7 \times 10^{-7} \text{ m} \cdot \text{s}^{-2} = 0.087 \text{ mGal}$  (i.e.,  $1.7 \times 10^{-6} \text{ m} \cdot \text{s}^{-2} = 0.17 \text{ mGal}$  peak to peak). These values represent roughly the magnitude of the lunar tide.

Second, we look at the largest solar tidal wave  $S_2$  with the magnitude 0.42 in Table 7.2. Its amplitude is also maximum on the equator. In summary, on the equator, the amplitude of the geoid tide is 0.11 m (i.e., 0.22 m peak to peak), that of the vertical displacement, 0.067 m (i.e., 0.13 m peak to peak), and that of the gravity on the ground,  $4.0 \times 10^{-7} \text{ m} \cdot \text{s}^{-2} = 0.040 \text{ mGal}$  (i.e.,  $8.0 \times 10^{-6} \text{ m} \cdot \text{s}^{-2} = 0.080 \text{ mGal}$  peak to peak). These values represent roughly the magnitude of the solar tide.

The waves discussed,  $M_2$  and  $S_2$ , are the largest semidiurnal lunar and solar tidal waves, respectively. The readers are suggested to discuss the largest lunar and solar diurnal and secular waves in a similar manner.

Lastly, we look at the sum of the waves  $M_0$  and  $s_0$ , the total amplitude being 0.74. These are time-independent, thus being referred to as permanent tide. It is straightforward to infer that the fluid Love numbers should be used for the permanent tide.<sup>1</sup> The geodetic coefficient  $G_0$  has a maximum of  $(1/2)D$  on the equator and a minimum of  $-D$  at the pole. The amplitude of the geoid tide is 0.099 m on the equator and  $-0.20$  m at the poles. Hence, the geoid tide at the equator is 0.29 m higher than that at the poles. The vertical deformation corresponds to a bulge that the equator is 0.57 m higher than the poles. In the literature, the tidal Love numbers are often used for the permanent tide, in which case, the results should be interpreted as nominal, but not real.

Our discussion is based on a spherical, non-rotating, and elastic Earth model for demonstrative purpose. The more accurate model based on an ellipsoidal, rotating, and anelastic (which means non-pure elastic in the Earth tide community) Earth model is required for the current data accuracy (Mathews et al., 1995; Petit & Luzum, 2010).

<sup>1</sup> The fluid Love numbers are used instead of the secular Love numbers, since the former represents strictly the response of the Earth to the centrifugal force, while the latter are affected by the actual structure of the Earth, such as the distribution of continents, which is not a result of the response to the centrifugal force.

## 7.5 Surface Elastic Loading and Dynamic Load Love Numbers

### 7.5.1 The Dynamic Surface Elastic Loading Problem

The present-day variations of masses in the hydrosphere-cryosphere-atmosphere system—such as the motion of air in the atmosphere and of water in the oceans, the freezing and melting of ice sheets and glaciers, and the impoundment and pumping of groundwater—behave as loads to the solid Earth. The Earth does deform in response to these variations, which is predominantly elastic. Therefore, the theory of the Earth's deformation in response to such variations is referred to as elastic loading theory. In the study of the Earth's deformation, we approximately take the whole hydrosphere-cryosphere-atmosphere system as a thin layer on the surface of the Earth.

We approximate the Earth as a sphere of radius  $R$ , and use  $s(\theta, \lambda, t)$  to denote the time-variable part of the surface mass density within the thin layer, which is the mass change per unit area. In the formulation, we first assume that  $s(\theta, \lambda, t)$  varies periodically with the angular frequency  $\sigma$ , and express it in the form of a spherical harmonic series,

$$s = \sum_{n=0}^{\infty} \sum_{k=-n}^n s_n^k P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}. \quad (7.169)$$

Note that  $s_n^k e^{i\sigma t}$  in this equation, where  $s_n^k$  is constant, corresponds to  $s_n^k(t)$  in (5.233). It will be proven that, as far as the temporal variation is slow, i.e., the value of  $\sigma$  is much smaller than the angular frequency of the fundamental spheroidal free oscillation mode of the same degree, the dependence of the Earth's elastic deformation on  $\sigma$  is weak and can be neglected. The inclusion of the periodical dependence on time is mainly for convenience of formulation. In subsequent formulations, we simply refer  $s(\theta, \lambda, t)$  or  $s_n^k$  as surface mass.

The surface mass  $s(\theta, \lambda, t)$  deforms the Earth by two actions. One is its gravitational attraction to the mass inside the Earth, and the other is the pressure exerted on the Earth's surface. The gravitational potential has been formulated in Sect. 5.6.1. We have

$$\phi = \begin{cases} \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G R^2}{2n+1} s_n^k \frac{R^n}{r^{n+1}} P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}, & r \geq R; \\ \sum_{n=0}^{\infty} \sum_{k=-n}^n \frac{4\pi G R^2}{2n+1} s_n^k \frac{r^n}{R^{n+1}} P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}, & r \leq R. \end{cases} \quad (7.170)$$

In forthcoming formulations we write it as

$$\phi = \begin{cases} R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k \left( \frac{R}{r} \right)^{n+1} P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}, & r \geq R; \\ R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k \left( \frac{r}{R} \right)^n P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}, & r \leq R; \end{cases} \quad (7.171)$$

where we used the relation  $R g_R = GM/R$ . The relation between the spherical harmonic coefficients of the surface mass density and those of the gravitational potential is

$$\phi_n^k = \frac{4\pi G}{(2n+1)g_R} s_n^k, \quad s_n^k = \frac{(2n+1)g_R}{4\pi G} \phi_n^k. \quad (7.172)$$

The normal stress exerted by the mass layer on the Earth's surface is the negative of the pressure, i.e.,  $t_{0r}(\theta, \lambda, t) = -g_R s(\theta, \lambda, t)$ . Its spherical harmonic expression is

$$\begin{aligned} t_{0r}^s &= - \sum_{n=0}^{\infty} \sum_{k=-n}^n g_R s_n^k P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t} \\ &= - \sum_{n=0}^{\infty} \sum_{k=-n}^n \left( \frac{2n+1}{4\pi G} g_R^2 \right) \phi_n^k P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}. \end{aligned} \quad (7.173)$$

No shear stress is present. Hence,

$$t_{0\theta}^s = 0, \quad t_{0\lambda}^s = 0. \quad (7.174)$$

The surface elastic loading problem is the study of the Earth's elastic deformation in response to the gravitational potential given by (7.171) and the surface stress given by (7.173) and (7.174).

For brevity of notations, we consider first the degree  $n$  and order  $k$  term of (7.169) in the formulation, and denote it using the same symbol  $s$ ,

$$s = s_n^k P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}. \quad (7.175)$$

We also denote the gravitational potential of this term in the interior of the Earth using  $\phi$ ,

$$\phi = R g_R \phi_n^k \left( \frac{r}{R} \right)^n P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}. \quad (7.176)$$

The stress on the Earth's surface is

$$t_{0r} = - \left( \frac{2n+1}{4\pi G} g_R^2 \right) \phi_n^k P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma t}, \quad t_{0\theta} = t_{0\lambda} = 0. \quad (7.177)$$

In the following, we solve the Earth's elastic equation in response to the forcing of  $\phi$  and  $t_{0r}$  as defined in (7.176) and (7.177).

The governing equations of the Earth's elastic deformation are (4.435)–(4.437). Here we neglect the Coriolis term in (4.435), and write these equation as, with  $\phi_f$  replaced by  $\phi$ ,

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \bar{T} - \rho(\nabla \cdot \vec{u})\nabla\phi_0 + \rho\nabla\vec{u} \cdot \nabla\phi_0 + \rho\vec{u} \cdot \nabla\nabla\phi_0 + \rho\nabla(\phi + \phi_1), \quad (7.178)$$

$$\bar{T} = \lambda \nabla \cdot \vec{u} \bar{I} + \mu [\nabla \vec{u} + (\nabla \vec{u})^T], \quad (7.179)$$

$$\nabla^2(\phi + \phi_1) = 4\pi G \nabla \cdot (\rho \vec{u}), \quad (7.180)$$

where we used the fact  $\nabla^2\phi = 0$  in the last equation. The inclusion of the inertial term  $\rho\partial^2\vec{u}/\partial t^2$  in (7.87), but not the Coriolis term, is a trick for practical computation, as in the case of tidal deformation. This will be mentioned later after having developed these equations into ordinary differential equations. The treatment of neglecting the inertial term will be discussed later in the formulation of static load Love numbers in Sect. 7.6.2.

The boundary conditions are summarized in Sect. 4.8.4. Here we list them again for convenience of reference, with modifications for the particular case of surface loading. The condition of displacement across welded boundaries is

$$\{\vec{u}\}_{\pm}^+ = 0. \quad (7.181)$$

The condition of displacement across frictionless slip boundaries is

$$\{\hat{n} \cdot \vec{u}\}_{\pm}^+ = 0. \quad (7.182)$$

The condition of stress across all boundaries is

$$\{\hat{n} \cdot \bar{T}\}_{\pm}^+ = 0. \quad (7.183)$$

The only frictionless slip boundaries in the Earth's interior are the inner core boundary (ICB) and core-mantle boundary (CMB). In the fluid core, as  $\hat{n} \cdot \bar{T} = [\hat{n} \cdot (\hat{n} \cdot \bar{T})]\hat{n}$ , the condition that the shear stress vanishes at the ICB and CMB is automatically satisfied, and is not to be discussed further. On the Earth's surface, there is no constraint for the displacement  $\vec{u}$  as in the tidal deformation. The condition for the stress  $\bar{T}$  is

$$(\hat{n} \cdot \bar{T})_{S_0} = t_{0r} \hat{e}_r, \quad (7.184)$$

i.e., only the normal stress component is present, and the shear stress components vanish, as given by (7.173) and (7.174). The conditions for the gravitational potential across all kinds of boundaries are

$$\{\phi_1\}_-^+ = 0, \quad (7.185)$$

$$\left\{ \frac{\partial \phi_1}{\partial n} - 4\pi G\rho(\hat{n} \cdot \vec{u}) \right\}_-^+ = 0, \quad (7.186)$$

noting also that  $\phi$  as well as its first order derivative is continuous throughout the Earth's interior. Only one boundary condition, i.e., (7.184), is different from the case of Earth tide: In the case of Earth tide, there is no stress exerted on the Earth's surface; in the case of surface loading, the Earth's surface is subject to the pressure from the mass of load.

### 7.5.2 Dynamic Load Love Numbers

The dynamic load Love numbers are used to represent the solution of the surface elastic loading problem described in the previous subsection.

In this subsection, we continue to understand the gravitational potential given in (7.176) and the surface normal stress given in (7.177) as the forcing terms. It can be inferred from the formulation in Sect. 4.8 that the solution of the displacement should be a spheroidal field, as the toroidal field of the displacement is not associated with any change in the gravitational potential or normal stress. Although the solution in Sect. 4.8 can be directly applied, here we first scale the unknown functions  $\vec{u}$ ,  $\vec{T}$ , and  $\phi + \phi_1$  by the factor  $\phi_n^k$  as defined in (7.176), and then proceed to solve the problem according to the procedure of the development in Sect. 4.8. As will be seen, it is more convenient to define the load Love numbers and provide the expressions of the solution of the displacement and gravitational potential in this way. Based on the formulation in Sect. 4.8, more specifically, (4.488)–(4.491) and (4.480)–(4.487), we have, for the degree  $n$  and order  $k$  term alone,

$$\vec{u} = \phi_n^k \left( y_1^n P_n^k \hat{e}_r + y_3^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_3^n \frac{ik}{\sin\theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (7.187)$$

$$\hat{e}_r \cdot \vec{T} = \phi_n^k \left( y_2^n P_n^k \hat{e}_r + y_4^n \frac{d}{d\theta} P_n^k \hat{e}_\theta + y_4^n \frac{ik}{\sin\theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (7.188)$$

$$\phi + \phi_1 = \phi_n^k y_5^n P_n^k e^{ik\lambda} e^{i\sigma t}, \quad (7.189)$$

$$\frac{\partial}{\partial r}(\phi + \phi_1) - 4\pi G\rho(\hat{e}_r \cdot \vec{u}) = \phi_n^k y_6^n P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (7.190)$$

The only difference from the formulation in Sect. 4.8 is the presence of the extra factor  $\phi_n^k$  here, which is common to all functions. Therefore, all the differential

equations and boundary conditions derived in Sect. 4.8 remain correct here as far as only linear terms of the functions  $y_1^n, \dots, y_6^n$  are present. The result of the ordinary differential equations is the same as (4.517), which we copy to here for convenience of reference,

$$\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r}y_1^n + \frac{1}{\lambda + 2\mu}y_2^n + \frac{n(n+1)\lambda}{(\lambda + 2\mu)r}y_3^n, \quad (7.191a)$$

$$\begin{aligned} \frac{dy_2^n}{dr} = & \left[ -\rho\sigma^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda + 2\mu)r}y_2^n \\ & + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_3^n + \frac{n(n+1)}{r}y_4^n \\ & - \rho y_6^n, \end{aligned} \quad (7.191b)$$

$$\frac{dy_3^n}{dr} = -\frac{1}{r}y_1^n + \frac{1}{r}y_3^n + \frac{1}{\mu}y_4^n, \quad (7.191c)$$

$$\begin{aligned} \frac{dy_4^n}{dr} = & \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{\lambda}{(\lambda + 2\mu)r}y_2^n + \left\{ -\rho\sigma^2 \right. \\ & + \frac{2\mu}{(\lambda + 2\mu)r^2} \left[ (2n^2 + 2n - 1)\lambda + 2(n^2 + n - 1)\mu \right] \left. \right\} y_3^n \\ & - \frac{3}{r}y_4^n - \frac{\rho}{r}y_5^n, \end{aligned} \quad (7.191d)$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \quad (7.191e)$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r}y_3^n + \frac{n(n+1)}{r^2}y_5^n - \frac{2}{r}y_6^n. \quad (7.191f)$$

The boundary conditions derived in Sect. 4.8 that remain valid are

- On internal welded boundaries,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_3^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0. \quad (7.192)$$

- On internal frictionless slip boundaries, i.e., the ICB and CMB,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0, \quad (7.193)$$

and no condition applies to  $y_3^n$ , meaning that it can be discontinuous.

- On the Earth's surface,

$$y_4^n(R) = 0. \quad (7.194)$$



Two boundary conditions are in need of modification. One is (4.558), where  $y_2^n(R)$  should be replaced by  $\phi_n^k y_2^n(R)$ . Based on the definition of  $(t_{0r}^S)_n^k$  by (4.555),  $(t_{0r}^S)_n^k$  should be replaced by  $t_{0r}$  given in (7.177). We obtain

$$y_2^n(R) = -\frac{2n+1}{4\pi G} g_R^2. \quad (7.195)$$

The other one is (4.569). As in the case of Earth tide,  $y_5^n(R)$  and  $y_6^n(R)$  should be replaced by  $\phi_n^k y_5^n(R)$  and  $\phi_n^k y_6^n(R)$ , respectively. According to the definition of  $\phi_{fn}^k$  in (4.563) and that of  $\phi_n^k$  in (7.176),  $\phi_{fn}^k$  should be replaced by  $R g_R \phi_n^k$ . We obtain

$$\frac{n+1}{R} y_5^n(R) + y_6^n(R) = (2n+1) g_R. \quad (7.196)$$

This completes the formulation of the solution. An immediate observation is that the functions  $y_1^n, \dots, y_6^n$  depend on the degree  $n$ , but not the order  $k$ , which is a fact we already realized in Sect. 4.8. Hence, we have indicated its dependence on  $n$  only with the superscript  $n$ .

Like Earth tide, the loading effect is also expressed in terms of three parameters, i.e., the load Love numbers. We first define the  $r$ -dependent load Love numbers as

$$H'_n(r) = \frac{y_1^n(r)}{R}, \quad L'_n(r) = \frac{y_3^n(r)}{R}, \quad K'_n(r) = \frac{y_5^n(r)}{R g_R} - 1. \quad (7.197)$$

On the surface of the Earth, we define

$$h'_n = H'_n(R) = \frac{y_1^n(R)}{R}, \quad l'_n = L'_n(R) = \frac{y_3^n(R)}{R}, \quad k'_n = K'_n(R) = \frac{y_5^n(R)}{R g_R} - 1, \quad (7.198)$$

which are the parameters that load Love numbers are mostly referred to.

The procedure of the solution of the system of differential equations-boundary conditions (7.191)–(7.196) is practically identical to that of the system (7.100)–(7.104) for computing the tidal Love numbers, as explained in the paragraph below the Eq. (7.106). The only difference between these two systems is in the boundary condition of  $y_2^n$  on the Earth's surface, which influences only the coefficients in the linear combination of the three independent solutions in the last step. Hence, this procedure is not to be repeated here.

The neglection of the Coriolis term can be quantified a posteriori by computing it from the result of the solution, which is indeed of much smaller order of magnitude than the terms at the right-hand side of (7.178).

The same as in the case of Earth tide, the load Love numbers computed based on the formulation of this subsection with a specific value of the angular frequency  $\sigma$  are called dynamic load Love numbers, and those computed based on the formulation without the inertial term to be presented Sect. 7.6.2 are called static load

Love numbers. When the term dynamic load Love number is explicitly mentioned, the angular frequency  $\sigma$  should be explicitly indicated as well.

The formulation of the dynamic load Love numbers is now complete. We mention again that the load Love numbers should be more strictly referred to as load Love-Shida numbers, as the definition of  $l'_n$  is originated from Shida.

The most fundamental observable quantities are the displacement of the crust and the gravitational potential outside the Earth. For the specific degree  $n$  and order  $k$  load with the angular frequency  $\sigma$  given in (7.175)–(7.177), the displacement and gravitational potential inside the Earth can be expressed as, according to (7.187), (7.189), and (7.197),

$$\vec{u} = R\phi_n^k \left( H'_n P_n^k \hat{e}_r + L'_n \frac{d}{d\theta} P_n^k \hat{e}_\theta + L'_n \frac{ik}{\sin \theta} P_n^k \hat{e}_\lambda \right) e^{ik\lambda} e^{i\sigma t}, \quad (7.199)$$

$$\phi + \phi_1 = R g_R \phi_n^k (1 + K'_n) P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (7.200)$$

The appearance of the complex number in these formulae is inconvenient. With the expression of  $\phi$  given by (7.176), we further write these formulae in the more practical form

$$u_r = \frac{H'_n}{g_R} \phi(R), \quad u_\theta = \frac{L'_n}{g_R} \frac{\partial \phi(R)}{\partial \theta}, \quad u_\lambda = \frac{L'_n}{g_R \sin \theta} \frac{\partial \phi(R)}{\partial \lambda}; \quad (7.201)$$

$$\phi + \phi_1 = (1 + K'_n) \phi(R). \quad (7.202)$$

These expressions use the gravitational potential of the mass of load itself, not its spherical harmonic coefficients, thus being much more convenient to be generalized to the forms for practical application.

### 7.5.3 Static Loading Expressed in Terms of Dynamic Load Love Numbers

In practical applications, the expressions (7.201) and (7.202) can be very much simplified.

The same as in the case of tidal Love numbers, as far as the angular frequency of the loading phenomenon,  $\sigma$ , is much smaller than that of the fundamental mode of the degree  $n$  spheroidal free oscillation,  ${}_0\sigma_n$ , the effect of the term  $\rho\sigma^2$  in the system of equations (7.191) is negligibly small. Geophysical processes with observable loading effects are generally slow enough so that  $\sigma$  could be assumed to be much smaller than  ${}_0\sigma_n$ . The results of numerical experiments are similar to the case of tidal Love numbers. As far as  $\sigma < (1/10){}_0\sigma_n$ , the differences among the load Love numbers obtained assuming different values of  $\sigma$  lie within 0.1%. See Sect. 7.8.1 for detail. Hence, in practical applications, we still can choose  $\sigma$  to be on the order of  $(1/10){}_0\sigma_n$ , and treat the result as  $\sigma$ -independent.

Although most geophysical phenomena in the hydrosphere-cryosphere-atmosphere system do have some kinds of periodicity, the angular frequency  $\sigma$  cannot be assumed known. Therefore, in a later section, we will formulate the surface elastic loading problem without the inertial term  $\rho \partial^2 \vec{u} / \partial t^2$  in (7.178), i.e., without the term  $\rho \sigma^2$  in the ordinary differential equations (7.191) to be solved numerically. The same problem as in the case of Earth tide is present: when simply setting  $\sigma = 0$  in (7.191), and apply it to the fluid outer core, together with the boundary conditions (7.192)–(7.196) applied to the inner core boundary (ICB) and core-mantle boundary (CMB), there is no longer a solution that satisfies the differential equations (7.191) as well as all the boundary conditions. Therefore, the static elastic loading problem is to be formulated based on a different perspective in the next section.

Now we take into account of the fact that the load Love numbers depend on the degree, but not the frequency of the variation of the load. We assume that the actual density of the mass of load varies with time with multiple frequencies, and we write it in the general form

$$s = \sum_{n=0}^{\infty} \sum_{k=-n}^n \sum_i s_n^k(\sigma_i) P_n^k(\cos \theta) e^{ik\lambda} e^{i\sigma_i t}, \quad (7.203)$$

where  $s_n^k(\sigma_i)$  is the degree  $n$ , order  $k$ , and angular frequency  $\sigma_i$  spherical harmonic coefficient of  $s$ . As compared to (5.233),  $\sum_i s_n^k(\sigma_i) e^{i\sigma_i t}$  can be understood as the Fourier series of  $s_n^k(t)$ . When the load Love numbers are independent of the angular frequency of the load, the theory developed for the time-dependent mass of load in the form of (7.169) equally applies to that in the form of (7.203). This implies that we can practically drop the dependence on the frequency in the applications of the dynamical loading theory. We integrate over a time interval, say from  $t_0$  to  $t_0 + \delta t$ , to obtain the change. The degree  $n$  and order  $k$  change of the mass of load  $\delta s$  and the change of its gravitational potential  $\delta \phi$  can be obtained by integrating (7.169) and (7.171) over time from  $t_0$  to  $t_0 + \delta t$ . The sums of all degree and order terms are

$$\delta s(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{k=-n}^n \delta s_n^k P_n^k(\cos \theta) e^{ik\lambda}, \quad (7.204)$$

$$\delta \phi(r, \theta, \lambda) = \begin{cases} R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \delta \phi_n^k \left( \frac{R}{r} \right)^{n+1} P_n^k(\cos \theta) e^{ik\lambda}, & r \geq R; \\ R g_R \sum_{n=0}^{\infty} \sum_{k=-n}^n \delta \phi_n^k \left( \frac{r}{R} \right)^n P_n^k(\cos \theta) e^{ik\lambda}, & r \leq R. \end{cases} \quad (7.205)$$

The relation between  $\delta s_n^k$  and  $\delta \phi_n^k$  can be obtained from (7.172) to be

$$\delta \phi_n^k = \frac{4\pi G}{(2n+1)g_R} \delta s_n^k, \quad \delta s_n^k = \frac{(2n+1)g_R}{4\pi G} \delta \phi_n^k. \quad (7.206)$$

The degree  $n$  and order  $k$  loading effect in the interior of the Earth, i.e., the displacement and the gravitational potential, can be obtained by integrating (7.201) and (7.202) over time from  $t_0$  to  $t_0 + \delta t$ . The sums of all degree and order terms are

$$\delta u_r = \sum_n \frac{H'_n}{g_R} \delta \phi_n(R), \quad \delta u_\theta = \sum_n \frac{L'_n}{g_R} \frac{\partial \delta \phi_n(R)}{\partial \theta}, \quad \delta u_\lambda = \sum_n \frac{L'_n}{g_R \sin \theta} \frac{\partial \delta \phi_n(R)}{\partial \lambda}; \quad (7.207)$$

$$\delta \phi + \delta \phi_1 = \sum_n (1 + K'_n) \delta \phi_n(R); \quad (7.208)$$

where  $\delta \phi_n$  is the degree  $n$  term of  $\delta \phi$ , i.e.,  $\delta \phi = \sum_n \delta \phi_n$ . In the formulation of this paragraph, (7.204) gives the density of the mass of load, (7.205) and (7.206) the gravitational potential of the mass of load, (7.207) the loading displacement, and  $\delta \phi_1$  in (7.208) the gravitational potential induced by the deformation. The results (7.204)–(7.208) can be applied to any slow change of the mass of load expressed by the surface mass density  $\delta s(\theta, \lambda)$ , either a net change or a rate of change.

The displacement and gravitational potential on the Earth's surface are, by taking into account of (7.198),

$$\begin{cases} \delta u_r(R) = \sum_n \frac{h'_n}{g_R} \delta \phi_n(R), \\ \delta u_\theta(R) = \sum_n \frac{l'_n}{g_R} \frac{\partial \delta \phi_n(R)}{\partial \theta}, \\ \delta u_\lambda(R) = \sum_n \frac{l'_n}{g_R \sin \theta} \frac{\partial \delta \phi_n(R)}{\partial \lambda}; \end{cases} \quad (7.209)$$

$$\delta \phi(R) + \delta \phi_1(R) = \sum_n (1 + k'_n) \delta \phi_n(R). \quad (7.210)$$

The gravitational potential outside the Earth, which is really useful, can be derived from (7.210). First, we obtain the value of  $\delta \phi_1(R)$  to be

$$\delta \phi_1(R) = \sum_n k'_n \delta \phi_n(R). \quad (7.211)$$

As  $\delta \phi_1$  is due to the change of mass distribution inside the Earth caused by the tidal deformation, it is harmonic outside the Earth. Hence, its expression outside the Earth is

$$\delta \phi_1 = \sum_n k'_n \left( \frac{R}{r} \right)^{n+1} \delta \phi_n(R), \quad r \geq R. \quad (7.212)$$

We finally obtain, with  $\delta\phi$  given by (7.205),

$$\delta\phi + \delta\phi_1 = \sum_n (1 + k'_n) \left(\frac{R}{r}\right)^{n+1} \delta\phi_n(R) = \sum_n (1 + k'_n) \delta\phi_n, \quad r \geq R. \quad (7.213)$$

Note that this formula is different from that of the Earth tide given by (7.117). Here the mass changes as the source of both  $\delta\phi$  and  $\delta\phi_1$  are on or within the Earth's surface, while in the case of the Earth tide, the masses as the source of the tidal-generating potential are beyond the scope of  $r$  where (7.117) is to be applied.

We used  $\delta\phi_n(R)$  as the parameter of the mass of load in the previous paragraph. According to (7.204)–(7.206), we have

$$\delta\phi_n(R) = \frac{4\pi GR}{(2n+1)} \delta s_n, \quad \delta s_n = \frac{(2n+1)}{4\pi GR} \delta\phi_n(R), \quad (7.214)$$

where  $\delta s_n$  is the degree  $n$  term of  $\delta s$ , i.e.,  $\delta s = \sum_n \delta s_n$ . With  $g_R = GM/R^2 = (4/3)\pi GR \bar{\rho}$ , where  $\bar{\rho}$  is the mean density of the Earth, we can substitute (7.214) into (7.209) and (7.210) to obtain

$$\begin{cases} \delta u_r(R) = \frac{4\pi R^3}{M} \sum_n \frac{h'_n}{2n+1} \delta s_n = \frac{3}{\bar{\rho}} \sum_n \frac{h'_n}{2n+1} \delta s_n, \\ \delta u_\theta(R) = \frac{4\pi R^3}{M} \sum_n \frac{l'_n}{2n+1} \frac{\partial \delta s_n}{\partial \theta} = \frac{3}{\bar{\rho}} \sum_n \frac{l'_n}{2n+1} \frac{\partial \delta s_n}{\partial \theta}, \\ \delta u_\lambda(R) = \frac{4\pi R^3}{M} \sum_n \frac{l'_n}{2n+1} \frac{1}{\sin \theta} \frac{\partial \delta s_n}{\partial \lambda} = \frac{3}{\bar{\rho}} \sum_n \frac{l'_n}{2n+1} \frac{1}{\sin \theta} \frac{\partial \delta s_n}{\partial \lambda}; \end{cases} \quad (7.215)$$

$$\delta\phi(R) + \delta\phi_1(R) = \frac{GM}{R} \frac{4\pi R^2}{M} \sum_n \frac{1 + k'_n}{2n+1} \delta s_n = \frac{GM}{R} \frac{3}{R \bar{\rho}} \sum_n \frac{1 + k'_n}{2n+1} \delta s_n. \quad (7.216)$$

We can write (7.213) as

$$\begin{aligned} \delta\phi + \delta\phi_1 &= \frac{GM}{R} \sum_n \delta\Phi_n(R) \left(\frac{R}{r}\right)^{n+1} \\ &= \frac{GM}{R} \frac{4\pi R^2}{M} \sum_n \frac{1 + k'_n}{2n+1} \left(\frac{R}{r}\right)^{n+1} \delta s_n \\ &= \frac{GM}{R} \frac{3}{R \bar{\rho}} \sum_n \frac{1 + k'_n}{2n+1} \left(\frac{R}{r}\right)^{n+1} \delta s_n, \quad r \geq R. \end{aligned} \quad (7.217)$$

Note that  $[\delta\phi(R) + \delta\phi_1(R)]/g_R$  is the change of the geoidal height. With  $g_R = GM/R^2$ , we write (7.216) in the form of

$$\delta\phi(R) + \delta\phi_1(R) = \frac{GM}{R} \sum_n \delta\Phi_n(R) = \frac{GM}{R} \frac{3}{R\bar{\rho}} \sum_n \frac{1+k'_n}{2n+1} \delta s_n; \quad (7.218)$$

$$\delta N = R \sum_n \delta\Phi_n(R) = \frac{3}{\bar{\rho}} \sum_n \frac{1+k'_n}{2n+1} \delta s_n, \quad r \geq R; \quad (7.219)$$

where  $\Phi_n(R)$  is the degree  $n$  term of the geoidal height  $\delta N$  scaled by the length  $R$ , which is defined implicitly in (7.219).

As in the case of Earth tide, the displacement of the Earth's surface and the gravitational potential outside the Earth are the most fundamental observable quantities of the loading effect. The formulation of other observable quantities of surface loading is almost identical to that of the Earth tide in Sect. 7.4, with the notations replaced by those in this section. Only one major difference needs attention. In the case of Earth tide, the degree  $n$  term of the tidal-generating potential depends on the radial distance in the form of  $(r/R)^n$  in both the interior and exterior of the Earth. In the case of elastic loading, the degree  $n$  term of the gravitational potential of the mass of load depends on the radial distance in the form of  $(r/R)^n$  in the Earth's interior, but in the form of  $(R/r)^{n+1}$  in the Earth's exterior. Remember that the mass of load is approximated as a surface mass density in the formulation.

Practical applications may be more complicated than that in the previous paragraph; approximating the mass of load by a surface mass density may not be accurate enough. In the observations of loading effect, the contribution of the gravitational attraction of the mass of load, i.e., the terms without a load Love number in their formula, is called direct effect, and the contribution of the Earth's deformation, i.e., the terms with a load Love number in their formula, is called indirect effect. The quantities including a direct effect term are the sea level, gravity, deviation of the vertical with respect to the crust, and the deviation of the vertical with respect to the celestial reference system (CRS). We take for example the loading due to mass changes in the atmosphere. If the station is below the ground, e.g., in a cave, the mass of load is mostly above the station; if the station is on top of a high mountain, the mass of load is mostly below the station. Even for oceanic loading near the shore, the gravitational effect of the mass of load could strongly depend on the height of the station, e.g., on a cliff or at sea level. Therefore, the direct effect is in need of more accurate formulation. The most straightforward way is to divide the mass of load into small volume elements, and then add the contribution of each volume element together.

Before concluding this section, we mention some other instances of loading. Imaging that a wind is blowing on the surface of a spherical Earth. The friction of the air with the ground produces a shear traction to the ground, which renders  $t_{0\theta}(\theta, \lambda, t)$  and  $t_{0\lambda}(\theta, \lambda, t)$  in (7.174) non-vanishing. This would produce a toroidal deformation of the Earth which can be obtained by solving the toroidal deformation equation (4.518) with appropriate boundary conditions, and a set of shear load Love

numbers can be defined (Saito, 1978). For the real Earth, a wind blowing to a slope surface, such as a hillside, would produce both a shear and a normal traction to the ground. The normal traction to the ground would induce a spheroidal deformation, where no mass of load is involved. Such a problem has been formulated by Guo et al. (2004). A set of dislocation Love numbers has been defined to study the deformation of the Earth in response to Earthquakes (e.g., Sun & Okubo, 1993). These models are beyond the scope of this book.

## 7.6 Static Load Love Numbers

### 7.6.1 The Static Case Treated as a Zero Frequency Dynamic Case: Longman's Core Paradox

We consider the problem of neglecting the inertial term in the dynamic elastic loading problem, i.e., the term  $\rho \partial^2 \vec{u} / \partial t^2$  in the governing equation (7.178), which leads to the missing of the term  $\rho \sigma^2$  in the system of ordinary equations (7.191). This is equivalent to setting  $\sigma = 0$  in the dynamic loading problem. Hence, the problem can be understood as a zero frequency dynamic loading problem. It is referred to as static elastic loading problem, and the load Love numbers defined for it are referred to as static load Love numbers. In the literature, the load Love numbers are implicitly understood as the static load Love numbers in most instances.

In setting  $\sigma = 0$ , the term  $e^{i\sigma t}$  throughout (7.169)–(7.202) disappears. We can re-understand these derivations by setting the density of the mass of load in (7.169) to equal the density increment implied in (7.204)–(7.219), i.e., setting  $s = \delta s$ . In doing so, the system of ordinary differential equations to be solved is

$$\frac{dy_1^n}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r} y_1^n + \frac{1}{\lambda + 2\mu} y_2^n + \frac{n(n+1)\lambda}{(\lambda + 2\mu)r} y_3^n, \quad (7.220a)$$

$$\begin{aligned} \frac{dy_2^n}{dr} = & \left[ -\frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{4\mu}{(\lambda + 2\mu)r} y_2^n \\ & + \left[ \frac{n(n+1)\rho g}{r} - \frac{2n(n+1)\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_3^n + \frac{n(n+1)}{r} y_4^n \\ & - \rho y_6^n, \end{aligned} \quad (7.220b)$$

$$\frac{dy_3^n}{dr} = -\frac{1}{r} y_1^n + \frac{1}{r} y_3^n + \frac{1}{\mu} y_4^n, \quad (7.220c)$$

$$\begin{aligned} \frac{dy_4^n}{dr} = & \left[ \frac{\rho g}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1^n - \frac{\lambda}{(\lambda + 2\mu)r} y_2^n + \\ & + \frac{2\mu}{(\lambda + 2\mu)r^2} \left[ (2n^2 + 2n - 1)\lambda + 2(n^2 + n - 1)\mu \right] y_3^n \\ & - \frac{3}{r} y_4^n - \frac{\rho}{r} y_5^n, \end{aligned} \quad (7.220d)$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \quad (7.220e)$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r}y_3^n + \frac{n(n+1)}{r^2}y_5^n - \frac{2}{r}y_6^n. \quad (7.220f)$$

The associated boundary conditions are the same, which are

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_3^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0 \quad (7.221)$$

on internal welded boundaries,

$$\{y_1^n\}_-^+ = \{y_2^n\}_-^+ = \{y_4^n\}_-^+ = \{y_5^n\}_-^+ = \{y_6^n\}_-^+ = 0 \quad (7.222)$$

on internal frictionless slip boundaries, i.e., the inner core boundary (ICB) and core-mantle boundary (CMB), and

$$y_2^n(R) = -\frac{2n+1}{4\pi G}g_R^2, \quad y_4^n(R) = 0, \quad \frac{n+1}{R}y_5^n(R) + y_6^n(R) = (2n+1)g_R \quad (7.223)$$

on the Earth's surface.

Apart from the missing of the term  $\rho\sigma^2$ , the system of differential equations-boundary conditions (7.220)–(7.223) is identical to the system (7.191)–(7.196). We may then imagine that the procedure of solution is also identical. However, a problem of the existence of solution arises. When  $n = 0$ , we have  $y_3^n = y_4^n = 0$  by definition, and the equations can be solved by setting  $\mu = 0$  in the core. The problem occurs when  $n \neq 0$ .

The problem is that, when the system of ordinary differential equations-boundary conditions (7.220)–(7.223) is applied to the fluid outer core as well as its boundaries by setting  $\mu = 0$ , no solution exists, unless the Earth's fluid core has a particular property. We first derive the formula that demonstrates the problem. With  $\mu = 0$ , we have  $y_4^n = 0$ , and the system of ordinary equations (7.220) becomes a system differential-algebraic equations,

$$\frac{dy_1^n}{dr} = -\frac{2}{r}y_1^n + \frac{1}{\lambda}y_2^n + \frac{n(n+1)}{r}y_3^n, \quad (7.224a)$$

$$\frac{dy_2^n}{dr} = -\frac{4\rho g}{r}y_1^n + \frac{n(n+1)\rho g}{r}y_3^n - \rho y_6^n, \quad (7.224b)$$

$$\rho g y_1^n - y_2^n - \rho y_5^n = 0, \quad (7.224c)$$

$$\frac{dy_5^n}{dr} = 4\pi G\rho y_1^n + y_6^n, \quad (7.224d)$$

$$\frac{dy_6^n}{dr} = -\frac{4\pi G\rho n(n+1)}{r}y_3^n + \frac{n(n+1)}{r^2}y_5^n - \frac{2}{r}y_6^n. \quad (7.224e)$$



By multiplying (7.224a) with  $\rho g$ , and then subtracting (7.224b) from it, we obtain

$$\rho g \frac{dy_1^n}{dr} - \frac{dy_2^n}{dr} = \frac{2\rho g}{r} y_1^n + \frac{\rho g}{\lambda} y_2^n + \rho y_6^n. \quad (7.225)$$

By taking the derivative of (7.224c) with respect to  $r$ , we obtain

$$\rho g \frac{dy_1^n}{dr} - \frac{dy_2^n}{dr} = -\frac{d(\rho g)}{dr} y_1^n + \frac{d\rho}{dr} y_5^n + \rho \frac{dy_5^n}{dr}. \quad (7.226)$$

The left-hand sides of the above two equations are identical. Hence, their right-hand sides are equal,

$$\frac{2\rho g}{r} y_1^n + \frac{\rho g}{\lambda} y_2^n + \rho y_6^n = -\frac{d(\rho g)}{dr} y_1^n + \frac{d\rho}{dr} y_5^n + \rho \frac{dy_5^n}{dr}. \quad (7.227)$$

The gravity  $g$  obeys the Eq.(4.439), which we copy to here for convenience of reference,

$$\frac{dg}{dr} = 4\pi G\rho - \frac{2g}{r}. \quad (7.228)$$

It can then be obtained, by eliminating  $y_6^n$  and  $dg/dr$  from (7.224d) and the above two equations,

$$\frac{\rho g}{\lambda} y_2^n = \frac{d\rho}{dr} (-gy_1^n + y_5^n) = -\frac{1}{\rho} \frac{d\rho}{dr} y_2^n, \quad (7.229)$$

where (7.224c) is used to derive the second identity. Finally, we write this equation as

$$\left( \frac{1}{\rho} \frac{d\rho}{dr} + \frac{\rho g}{\lambda} \right) y_2^n = 0. \quad (7.230)$$

This equation implies that, if  $(1/\rho)(d\rho/dr) + \rho g/\lambda \neq 0$ , we have

$$y_2^n = 0, \quad y_5^n = gy_1^n. \quad (7.231)$$

The same problem arises in the computation of the static tidal Love numbers, i.e., the tidal Love numbers with the inertial term neglected in the equation of motion (Pekeris & Luzum, 1972; Takeuchi, 1950).

With  $y_2^n = 0$ , the differential equation (7.224b) degenerates to an algebraic equation. Consequently, as we have one differential equation less, we have one free parameter less in the solution to be adjusted for the solution to obey the boundary conditions. This can be understood based on the procedure of the solution of the system (7.220)–(7.223). At the ICB, the three independent solutions in the inner

core can be combined to obtain only one set of independent boundary conditions with  $y_4'' = y_2'' = 0$ . Hence, only one independent solution can be obtained in the outer core, which splits into two independent solutions across the CMB by taking into account of the permissible discontinuity of  $y_3''$ . Therefore we can only obtain two independent solutions on the Earth's surface assumed to be solid, but there are three boundary conditions to be satisfied. As such, the situation we face is that we have one free parameters less than the number of boundary conditions, and no solution satisfying all the boundary conditions is possible. This conflict is referred to as Longman's core paradox (Longman, 1963). An immediate remediation to this paradox is to assume

$$\frac{1}{\rho} \frac{d\rho}{dr} + \frac{\rho g}{\lambda} = 0, \quad (7.232)$$

in the fluid outer core, which is equivalent to the Adams-Williamson equation (6.372) taking into account of (6.369) where  $k = \lambda$  in a fluid. Under this assumption, the Eq. (7.230) is automatically satisfied, so that (7.231) can be disregarded. However, the Eqs. (7.224a) and (7.224b) can no longer be considered as independent, and one of them should be dropped. As a result, although we still have one differential equation less, we also have one more unknown function than the number of equations, which provide the free parameter to be adjusted for the solution to obey the boundary conditions. Longman (1963) computed the static load Love numbers in this way.

The assumption (7.232), i.e., the Adams-Williamson equation, implies that the fluid outer core is assumed to be homogeneous (in both chemical compositions and material phases) and adiabatic (or neutrally stratified according to the nomenclature of stratification). However, it is physically illogical that the Earth's core must satisfy (7.232) for the static deformation problem as formulated above to be solvable. Therefore, the correctness of the formulation must be examined, which will be subject of the next subsection.

## 7.6.2 Formulation of the Static Load Love Numbers

The formulation in this subsection concerns only the equation of motion of the fluid outer core and the boundary conditions across the ICB and CMB, which is also applicable to the computation of static tidal Love numbers. The only difference between the computations of the load Love numbers and the tidal Love numbers is in the boundary conditions on the Earth's surface

In the formulation of oscillatory deformation leading to Longman's core paradox, the ordinary differential equations-boundary conditions (7.220)–(7.223) are derived for solids, but they are applied to the fluid core as is by simply setting  $\mu = 0$ , which is questionable. There are two assumptions in the formulation: (1) a material element is assumed to be oscillating around the original position; (2) the component of the displacement normal to frictionless slip boundary is assumed to be

continuous. However, in the static deformation, the displacement at the two sides of a solid-fluid boundary may behave differently: the solid side may bulge into the fluid side, propelling the fluid to flow away tangentially along the boundary, meaning that a fluid element at the vicinity of the boundary can flow far away from its original position and then stay at where it is. This causes the component of the displacement normal to the boundary discontinuous.

The discontinuity of the displacement normal to the boundary was first pointed out by Smylie and Mansinha (1971). A comprehensive treatment can be found in Dahlen (1974) and Dahlen and Fels (1978). A review of the literature can be found in Sabadini et al. (2004) including both correct and incorrect formulations. Here we follow Crossley and Gubbins (1975) and Chinnery (1975), who provided simplified, yet physically rigorous, formulations emphasizing different aspects of the problem starting from the first principles of physics, i.e., the fluid outer core is in hydrostatic equilibrium both before and after the deformation. As discussed in Sect. 6.8.4, for the Earth's core to be in stable hydrostatic equilibrium, its stratification must be neutral or positive. However, the Earth's core is undoubtedly in ceaseless motion, such as the flow sustaining the geodynamo. Therefore, any deformation is superimposed on the existing flow, including the case of dynamic tidal and load Love numbers, and the assumption of hydrostatic equilibrium here is just another simplification for the complicated physical problem to become treatable. Although most formulae can be inferred from former formulations of the book, here we provide a complete derivation for convenience of reference.

We first write out the fundamental formulae of the problem. We use  $\rho$ ,  $\phi_0$ ,  $g$  and  $p$  to denote the density, gravitational potential, gravity and pressure in the undeformed state. For a spherically symmetrical non-rotating Earth model, all of them are functions of  $r$  alone. We have, according to  $\vec{g} = \nabla\phi_0$  and  $\nabla p = \rho\nabla\phi_0$ ,

$$g = |\vec{g}| = -\frac{d\phi_0}{dr}, \quad \frac{dp}{dr} = -\rho g. \quad (7.233)$$

An equation of  $g$  can be derived from Poisson's Equation  $\nabla^2\phi_0 = -4\pi G\rho_0$ , as given by (4.439) and is copied to here for convenience of reference,

$$\frac{dg}{dr} = 4\pi G\rho - \frac{2g}{r}. \quad (7.234)$$

In the deformed state, we use  $\rho_1$ ,  $\phi_1$  and  $p_1$  to denote the Eulerian increment of the respective quantity, and use  $\phi$  to denote the gravitational potential of the mass of load which satisfies  $\nabla^2\phi = 0$ . The assumption of hydrostatic equilibrium yields

$$\nabla(p + p_1) = (\rho + \rho_1)\nabla[\phi_0 + (\phi + \phi_1)], \quad (7.235)$$

$$\nabla^2[\phi_0 + (\phi + \phi_1)] = -4\pi G(\rho + \rho_1). \quad (7.236)$$

The equations of  $p_1$  and  $\phi_1$  are then

$$\nabla p_1 = \rho \nabla(\phi + \phi_1) + \rho_1 \nabla \phi_0, \quad (7.237)$$

$$\nabla^2(\phi + \phi_1) = -4\pi G \rho_1. \quad (7.238)$$

The expressions of  $p_1$  and  $\rho_1$  in terms of  $(\phi + \phi_1)$  can be obtained from (7.237). In spherical coordinates, (7.237) is

$$\begin{aligned} & \frac{\partial p_1}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial p_1}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial p_1}{\partial \lambda} \hat{e}_\lambda \\ &= \rho \left[ \frac{\partial(\phi + \phi_1)}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial(\phi + \phi_1)}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial(\phi + \phi_1)}{\partial \lambda} \hat{e}_\lambda \right] - g \rho_1 \hat{e}_r. \end{aligned} \quad (7.239)$$

The component of this formula along  $\hat{e}_r$  is

$$\frac{\partial p_1}{\partial r} = \rho \frac{\partial(\phi + \phi_1)}{\partial r} - g \rho_1, \quad (7.240)$$

and its components along  $\hat{e}_\theta$  and  $\hat{e}_\lambda$  yield

$$p_1 = \rho(\phi + \phi_1) + f(r), \quad (7.241)$$

where  $f(r)$  is an arbitrary function of  $r$ , which is evidently a degree 0 term. By taking the derivative of (7.241) with respect to  $r$ , and then substituting the result into (7.240), we obtain the expression of  $\rho_1$  in terms of  $(\phi + \phi_1)$ ,

$$\rho_1 = -\frac{1}{g} \frac{d\rho}{dr} (\phi + \phi_1), \quad (7.242)$$

where we have assumed that no degree 0 term is present in  $\rho_1$  and  $(\phi + \phi_1)$ , leading to  $df(r)/dr = 0$ . In this case,  $f(r)$  degenerates to a constant, and (7.241) is simplified to

$$p_1 = \rho(\phi + \phi_1) + C, \quad (7.243)$$

where  $C$  is a constant. This implies that even if no degree 0 term is present in  $\rho_1$  and  $(\phi + \phi_1)$ , a constant term may appear in  $p_1$ .

In the fluid core, the displacement of fluid elements is no longer suitable for describing the static deformation, as will be explained in the next paragraph. Instead, we use the radial displacement of equipotential surfaces. In the following, we use  $u_r$  to denote the radial displacement of equipotential surfaces in the fluid, but maintain its original definition in the solid, i.e., the radial displacement of material elements.

As an equipotential surface is assumed to deform to an equipotential surface, we have

$$\phi_0 = \phi_0 + u_r \frac{d\phi_0}{dr} + (\phi + \phi_1), \quad (7.244)$$

where the left-hand side is the potential in the undeformed state, and the right-hand side the potential in the deformed state. We obtain

$$u_r = \frac{1}{g}(\phi + \phi_1). \quad (7.245)$$

It can be readily inferred from (7.242) and (7.245) that the Lagrangian increment of the density on an equipotential surface from the undeformed state to its deformed state vanishes,

$$\delta\rho = \rho_1 + u_r \frac{d\rho}{dr} = 0, \quad (7.246)$$

and the Lagrangian increment of the pressure on an equipotential surface from the undeformed state to its deformed state is

$$\delta p = p_1 + u_r \frac{dp}{dr} = p_1 - u_r \rho g = \rho \phi_1 + C - \frac{1}{g} \phi_1 \rho g = C, \quad (7.247)$$

meaning that  $\delta p$  can only have a degree 0 term. By substituting (7.242) into (7.238), we obtain an equation for  $(\phi + \phi_1)$ ,

$$\nabla^2(\phi + \phi_1) = \frac{4\pi G}{g} \frac{d\rho}{dr}(\phi + \phi_1). \quad (7.248)$$

Noting that  $u_r$  is the radial displacement of equipotential surfaces, the formula (7.246) implies that an equipotential surface in the deformed state is also an equi-value surface of density, just as in the undeformed state. It can then be readily inferred that the exchange of positions of two equi-volume fluid elements on the same equipotential surface does not alter the hydrostatic equilibrium of the fluid core. Therefore, the displacement of fluid elements is no longer suitable for describing the static deformation.

We now proceed to derive the ordinary differential equations for the fluid outer core. We express  $u_r$  and  $(\phi + \phi_1)$  in spherical harmonic series excluding the degree 0 term as

$$u_r = \sum_{n=1}^{\infty} \sum_{k=-n}^n U_n^k(r) P_n^k(\cos \theta) e^{ik\lambda}, \quad (7.249)$$

$$(\phi + \phi_1) = \sum_{n=1}^{\infty} \sum_{k=-n}^n \phi_n^k(r) P_n^k(\cos \theta) e^{ik\lambda}. \quad (7.250)$$

The substitution of (7.250) into (7.248) yields

$$\frac{d^2 \phi_n^k}{dr^2} + \frac{2}{r} \frac{d\phi_n^k}{dr} - \left[ \frac{n(n+1)}{r^2} + \frac{4\pi G}{g} \frac{d\rho}{dr} \right] \phi_n^k = 0. \quad (7.251)$$

We define

$$z_1^n = U_n^k(r), \quad z_5^n = \phi_n^k(r), \quad z_6^n = \frac{d\phi_n^k}{dr} - 4\pi G\rho U_n^k(r). \quad (7.252)$$

We then obtain, according to (7.245),

$$z_1^n = \frac{1}{g} z_5^n. \quad (7.253)$$

The definition of  $z_6^n$  leads to

$$\frac{dz_5^n}{dr} = z_6^n + 4\pi G\rho z_1^n = \frac{4\pi G\rho}{g} z_5^n + z_6^n. \quad (7.254)$$

The reduction of (7.251) is similar to that of (4.479), and the result is

$$\frac{dz_6^n}{dr} = \left[ -\frac{16\pi G\rho}{rg} + \frac{n(n+1)}{r^2} \right] z_5^n - \left[ \frac{2}{r} + \frac{4\pi G\rho}{g} \right] z_6^n. \quad (7.255)$$

These last two ordinary differential equations are to be solved in the fluid core to obtain  $z_5^n$  and  $z_6^n$ . Other quantities can be computed from the solution of  $z_5^n$  and  $z_6^n$ . We already know that  $z_1^n$  is related to  $z_5^n$  according to (7.253). By definition, we have  $T_{rr} = -\delta p$ , where  $T_{rr}$  is the Lagrangian increment of the normal stress component on the equipotential surface. Therefore, the spherical harmonic coefficients of  $T_{rr}$ ,  $z_2^n$ , vanishes according to (7.247), i.e.,

$$z_2^n = 0. \quad (7.256)$$

The shear stress vanishes in both the undeformed and deformed states. Hence its Lagrangian increment vanishes, leading to the vanishing of its spherical harmonic coefficients,

$$z_4^n = 0. \quad (7.257)$$

The horizontal components of the displacement is undefined. Hence,  $z_3^n$  is not defined in the core.

With (7.256) and (7.257), we mention that (7.253)–(7.255) can be obtained from (7.224) by setting  $y_2^n = 0$ , eliminating  $y_3^n$ , and then replace the  $y$ s by  $z$ s. However, the physics behind is not made clear in this way.

Lastly, we derive the boundary conditions across the inner core boundary (ICB) and core-mantle boundary (CMB). The boundary conditions on these two boundaries are in the same form. We use superscripts  $S_0$  and  $S$  to denote a boundary in the undeformed and deformed states, respectively. The use  $y$ s and  $z$ s at the solid and fluid sides, respectively, of a boundary is evident. We also use the subscripts  $s$  and  $f$  to indicate the solid and fluid sides, respectively, of a boundary.

An immediate inference is that  $z_1^n$  is not continuous with  $y_1^n$ , i.e., the value of  $z_1^n$  at the fluid side does not equal the value of  $y_1^n$  at the solid side. This could be understood based on the fact that  $y_1^n$  and  $z_1^n$  are defined differently at the two sides of a boundary:  $z_1^n$  represents the radial displacement of the equipotential surfaces which may be penetrated by the boundary represented by  $y_1^n$  when  $n \geq 1$ . We thus have (Smylie & Mansinha, 1971)

$$y_1^n = z_1^n + K_1, \quad (7.258)$$

where  $K_1$  is a constant to be determined by the boundary conditions on the Earth's surface.

The boundary condition for  $y_2^n$  is to be inferred based on the continuity of the pressure (to be understood as the negative of the normal stress at the solid side) in both the undeformed and deformed state. We have

$$p_s^{S_0} = p_f^{S_0}, \quad (7.259)$$

$$(p + p_1)_s^S = (p + p_1)_f^S. \quad (7.260)$$

The displacement of the boundary is described by the displacement of its solid side. The radial displacement of the boundary with respect to the undeformed state is  $\{u_r\}_s$ . Hence, we have

$$p_s^S = p_s^{S_0} + \left\{ \frac{dp}{dr} \right\}_s \{u_r\}_s, \quad p_f^S = p_f^{S_0} + \left\{ \frac{dp}{dr} \right\}_f \{u_r\}_s. \quad (7.261)$$

By subtracting (7.259) from (7.260), we obtain,

$$\{p_1\}_s + \left\{ \frac{dp}{dr} \right\}_s \{u_r\}_s = \{p_1\}_f + \left\{ \frac{dp}{dr} \right\}_f \{u_r\}_s. \quad (7.262)$$

We see that the left-hand side is the Lagrangian increment  $\{\delta p\}_s$ . In the fluid, we have  $\{\delta p\}_f = \{p_1\}_f + \{dp/dr\}_f \{u_r\}_f$ . Hence, with  $\{dp/dr\}_f = -\rho_f g$ , we obtain

$$\{\delta p\}_s = \{\delta p\}_f + \left\{ \frac{dp}{dr} \right\}_f (\{u_r\}_s - \{u_r\}_f) = \{\delta p\}_f - \rho_f g (\{u_r\}_s - \{u_r\}_f). \quad (7.263)$$

As pressure has the opposite sign as the normal stress, the boundary condition for the normal stress is

$$y_2^n = z_2^n + K_1 g \rho_f = K_1 \rho_f g, \quad (7.264)$$

where the fact  $z_2^n = 0$  in the fluid core was used to derive the second identity. This is a jump of the Lagrangian increment of the normal stress associated with the jump of  $K_1$  from  $y_1^n$  to  $z_1^n$  as given by (7.258).

The tangential component of displacement is discontinuous. We write it as

$$y_3^n = K_2, \quad (7.265)$$

where  $K_2$  is a constant to be determined by the boundary conditions on the Earth's surface. It is noted that no corresponding variable is defined in the fluid core.

The shear stress is continuous. We readily have

$$y_4^n = z_4^n = 0, \quad (7.266)$$

where the fact  $z_4^n = 0$  in the fluid core was used to derive the second identity.

The gravitational potential as well as its first order derivatives is continuous in both the undeformed and deformed states. We have

$$\{\phi_0\}_s^{S_0} = \{\phi_0\}_f^{S_0}, \quad \left\{ \frac{d\phi_0}{dr} \right\}_s^{S_0} = \left\{ \frac{d\phi_0}{dr} \right\}_f^{S_0}; \quad (7.267)$$

$$\left\{ \begin{aligned} & \{[\phi_0 + (\phi + \phi_1)]\}_s^S = \{[\phi_0 + (\phi + \phi_1)]\}_f^S, \\ & \left\{ \frac{\partial[\phi_0 + (\phi + \phi_1)]}{\partial r} \right\}_s^S = \left\{ \frac{\partial[\phi_0 + (\phi + \phi_1)]}{\partial r} \right\}_f^S. \end{aligned} \right. \quad (7.268)$$

Again, the radial displacement of the boundary with respect to the undeformed state is  $\{u_r\}_s$ . We obtain, in a similar way as deriving (7.262),

$$(\phi + \phi_1)_s + \left\{ \frac{d\phi_0}{dr} \right\}_s \{u_r\}_s = (\phi + \phi_1)_f + \left\{ \frac{d\phi_0}{dr} \right\}_f \{u_r\}_s, \quad (7.269)$$

$$\left\{ \frac{\partial(\phi + \phi_1)}{\partial r} \right\}_s + \left\{ \frac{d^2\phi_0}{dr^2} \right\}_s \{u_r\}_s = \left\{ \frac{\partial(\phi + \phi_1)}{\partial r} \right\}_f + \left\{ \frac{d^2\phi_0}{dr^2} \right\}_f \{u_r\}_s. \quad (7.270)$$

Across a boundary,  $d\phi_0/dr$  is continuous, but  $d^2\phi_0/dr^2$  is not. we have

$$\left\{ \frac{d\phi_0}{dr} \right\}_s = \left\{ \frac{d\phi_0}{dr} \right\}_f, \quad \left\{ \frac{d^2\phi_0}{dr^2} \right\}_s - \left\{ \frac{d^2\phi_0}{dr^2} \right\}_f = -4\pi G(\rho_s - \rho_f). \quad (7.271)$$



We obtain from (7.269),

$$(\phi + \phi_1)_s = (\phi + \phi_1)_f, \quad (7.272)$$

which yields

$$y_5^n = z_5^n. \quad (7.273)$$

We obtain from (7.270),

$$\left\{ \frac{\partial \phi_1}{\partial r} \right\}_s - 4\pi G \rho_s \{u_r\}_s = \left\{ \frac{\partial \phi_1}{\partial r} \right\}_f - 4\pi G \rho_f \{u_r\}_f - 4\pi G \rho_f (\{u_r\}_s - \{u_r\}_f), \quad (7.274)$$

which yields

$$y_6^n = z_6^n - 4\pi G K_1 \rho_f. \quad (7.275)$$

In summary, the boundary conditions are (7.258), (7.264), (7.265), (7.266), (7.273), and (7.275), which are grouped together as

$$\{y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n\} = \{z_1^n + K_1, K_1 g \rho_f, K_2, 0, z_5^n, z_6^n - 4\pi G K_1 \rho_f\}. \quad (7.276)$$

With the algebraic-ordinary differential equations (7.253)–(7.257) in the fluid core and the boundary conditions (7.276) across the ICB and CMB, we can now design a procedure to compute the static load Love numbers.

At the ICB, the three independent solutions of (7.220) in the inner core can be combined linearly to obtain only one solution satisfying the boundary condition (7.276). Denote the  $i$ th solution of  $y_j^n$  as  $(y_j^n)_i$ . We are to find three parameters  $\alpha_i$  such that

$$\frac{1}{g} z_5^n + K_1^{\text{ICB}} = \sum_{i=1}^3 \alpha_i (y_1^n)_i, \quad (7.277)$$

$$K_1^{\text{ICB}} g \rho_f = \sum_{i=1}^3 \alpha_i (y_2^n)_i, \quad (7.278)$$

$$0 = \sum_{i=1}^3 \alpha_i (y_4^n)_i, \quad (7.279)$$

$$z_5^n = \sum_{i=1}^3 \alpha_i (y_5^n)_i, \quad (7.280)$$

$$z_6^n - 4\pi G K_1^{\text{ICB}} \rho_f = \sum_{i=1}^3 \alpha_i (y_6^n)_i, \quad (7.281)$$

where (7.253) is used to substitute  $z_1^n$  by  $z_5^n$  in the first condition. There is no constraint on  $y_3^n$  and  $z_3^n$ . We now define  $z_j^n = K_1^{\text{ICB}} \tilde{z}_j^n$  and develop the boundary conditions (7.277)–(7.281) into the boundary conditions of  $\tilde{z}_j^n$  in two groups (noting that this is equivalent to define  $\alpha'_i = \alpha_i / K_1^{\text{ICB}}$ ):

$$\begin{cases} 1 = \sum_{i=1}^3 \alpha'_i \left[ (y_1^n)_i - \frac{1}{g} (y_5^n)_i \right], \\ g\rho_f = \sum_{i=1}^3 \alpha'_i (y_2^n)_i, \\ 0 = \sum_{i=1}^3 \alpha'_i (y_4^n)_i; \end{cases} \quad (7.282)$$

$$\begin{cases} \tilde{z}_5^n = \sum_{i=1}^3 \alpha'_i (y_5^n)_i, \\ \tilde{z}_6^n - 4\pi G\rho_f = \sum_{i=1}^3 \alpha'_i (y_6^n)_i. \end{cases} \quad (7.283)$$

The objective is clear: (7.282) is to be used to solve for  $\alpha'_i$ , and (7.283) is to be used to compute the boundary values for  $\tilde{z}_5^n$  and  $\tilde{z}_6^n$ . The ordinary differential equations (7.254) and (7.255) also hold for  $\tilde{z}_5^n$  and  $\tilde{z}_6^n$ , which are to be solved throughout the fluid outer core.

At the CMB, three independent sets of boundary conditions can be assigned for the functions  $y_j^n$ :

$$\{y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n\}_1 = \left\{ \frac{1}{g} \tilde{z}_5^n, 0, 0, 0, \tilde{z}_5^n, \tilde{z}_6^n \right\}, \quad (7.284)$$

$$\{y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n\}_2 = \{1, g\rho_f, 0, 0, 0, -4\pi G\rho_f\}, \quad (7.285)$$

$$\{y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n\}_3 = \{0, 0, 1, 0, 0, 0\}. \quad (7.286)$$

The system of ordinary differential equations (7.220) is to be solved from the CMB to the Earth's surface. The general solution is  $K_1^{\text{ICB}}\{y_j^n\}_1 + K_1^{\text{CMB}}\{y_j^n\}_2 + K_2^{\text{CMB}}\{y_j^n\}_3$ . The solution we seek can be finally obtained by determining the three parameters  $K_1^{\text{ICB}}$ ,  $K_1^{\text{CMB}}$ , and  $K_2^{\text{CMB}}$  using the boundary conditions on the Earth's surface.

The formulation of the spherical harmonic expressions of the observable quantities based on static load Love numbers is the same as that based on dynamic load Love numbers in Sect. 7.5, which only needs to replace the dynamic load Love numbers by the static load Love numbers.

### 7.6.3 The Degree One Static Deformation

The degree 1 static deformation has some particular properties. We first consider the case for Earth models without a fluid outer core. We then formulate the solution in the fluid core as well as in the inner core.

#### The Case Without a Core

Extra constraint is required for uniquely defining the degree 1 static deformation. We first consider the degenerate case assuming no core were present.

It can be verified with (7.234) that, the same as for the free oscillation mode  ${}_0S_1$ ,

$$\left\{ y_1^1, y_2^1, y_3^1, y_4^1, y_5^1, y_6^1 \right\} = \left\{ C, 0, C, 0, gC, -\frac{2g}{r}C \right\} \quad (7.287)$$

is a solution of the differential equations (7.220), where  $C$  is an arbitrary constant. This implies that, besides this one, two other linearly independent solutions of (7.220) satisfying the regularity boundary conditions at  $r = 0$  exist. On the Earth's surface, this solution is, with  $1 + 1$  denoting  $n + 1$  for  $n = 1$ ,

$$y_2^1 = 0, \quad y_4^1 = 0, \quad \frac{1+1}{R}y_5^1(R) + y_6^1(R) = 0. \quad (7.288)$$

An immediate inference is that the sum of (7.287) with an arbitrary value of  $C$  and a solution of the differential equation-boundary condition system (7.220)–(7.223) is still a solution of this system. Hence, the solution of the differential equation-boundary condition system (7.220)–(7.223) is not unique when  $n = 1$ . Therefore, some other kind of constraint is required to make the solution unique.

The solution (7.287) is in fact a rigid shift of whole the Earth. We consider here as an example the degree 1 and order 0 term. The expression of the displacement is, according to (7.187),

$$\vec{u} = C \cos \theta \hat{e}_r - C \sin \theta \hat{e}_\theta. \quad (7.289)$$

Its expression in the rectangular coordinate system is, according to (2.335),

$$\vec{u} = C \hat{e}_3. \quad (7.290)$$

This is a rigid shift of whole the Earth along the  $x_3$ -axis. It can be readily verified that the degree 1 and order  $-1$  or  $1$  term represents a rigid shift of whole the Earth along a direction in the  $x_1x_2$ -plane. These are the same as the  ${}_0S_1$  free oscillation mode formulated in Sect. 6.4.3.

The rigid shift of whole the Earth can be interpreted as a shift of the origin of the coordinate system with respect to the Earth. Consequently, the solution of the differential equation-boundary condition system (7.220)–(7.223) can be made

unique by specifically defining the origin of the coordinate system. Customarily, the origin is defined to be the center of mass of the solid Earth denoted as CE in the computation of load Love numbers. In the coordinate system thus defined, the degree 1 spherical harmonic coefficients of  $\phi_1$  on and outside the Earth vanish. The spherical harmonic series of  $\phi_1$  on and outside the Earth is given by (4.566). By taking into account of the difference between the definitions of  $\phi_{fn}^k$  in (4.563) and  $\phi_n^k$  in (7.176), and the difference between the definitions of  $y_5^n$  in (4.490) and (7.189),  $y_5^n(R)$  should be replaced by  $\phi_n^k y_5^n(R)$ , and  $\phi_{fn}^k$  by  $R g_R \phi_n^k$ . We have

$$(\phi_1)^+ = \sum_{n=0}^{\infty} \sum_{k=-n}^n \phi_n^k [y_5^n(R) - R g_R] \left(\frac{R}{r}\right)^{n+1} P_n^k e^{ik\lambda} e^{i\sigma t}. \quad (7.291)$$

The vanishing of the degree 1 coefficients yield

$$y_5^1(R) = R g_R. \quad (7.292)$$

After replacing the last condition in (7.223) by this one, the solution of the differential equation-boundary condition system (7.220)–(7.223) becomes unique. Note that the degree 1 load Love numbers thus computed is with respect to the CE, and alternative definitions of the degree 1 load Love numbers will be discussed in Sect. 7.9.1.

However, the last condition in (7.223) must be satisfied. We would therefore ask the following question: does the last condition in (7.223) still hold even though it is replaced by (7.292) in the computation? The answer is yes, which is ensured by the consistency relation to be derived from the differential equations (7.220). With  $n = 1$ , we can readily obtain

$$\frac{d}{dr} \left( y_2^1 + 2y_4^1 \right) = -\frac{2}{r} \left( y_2^1 + 2y_4^1 \right) + \frac{2\rho g}{r} \left( y_3^1 - y_1^1 \right) - \rho \left( y_6^1 + \frac{2}{r} y_5^1 \right). \quad (7.293)$$

$$\frac{d}{dr} \left( \frac{2}{r} y_5^1 + y_6^1 \right) = -8\pi G \rho - \frac{1}{r} \left( y_3^1 - y_1^1 \right). \quad (7.294)$$

With (7.234), these two equations can be combined to obtain

$$\frac{d}{dr} \left[ y_2^1 + 2y_4^1 + \frac{g}{4\pi G} \left( \frac{2}{r} y_5^1 + y_6^1 \right) \right] = -\frac{2}{r} \left[ y_2^1 + 2y_4^1 + \frac{g}{4\pi G} \left( \frac{2}{r} y_5^1 + y_6^1 \right) \right], \quad (7.295)$$

which can be written as the following consistency relation

$$\frac{d}{dr} \left\{ r^2 \left[ y_2^1 + 2y_4^1 + \frac{g}{4\pi G} \left( \frac{2}{r} y_5^1 + y_6^1 \right) \right] \right\} = 0. \quad (7.296)$$

The solution is

$$y_2^1 + 2y_4^1 + \frac{g}{4\pi G} \left( \frac{2}{r} y_5^1 + y_6^1 \right) = \frac{C_i}{r^2}, \quad (7.297)$$

where  $C_i$  is a constant with the subscript  $i$  referring to the  $i$ th continuous layer of the Earth counted from the innermost layer, i.e., the inner core, with  $i = 1$  to the outermost layer. After multiplying both sides of (7.297) by  $r^2$  and then setting  $r = 0$ , we obtain  $C_1 = 0$ , which holds throughout the innermost layer. With the boundary conditions (7.221), i.e., all the functions  $y_s$  are continuous, we obtain  $C_i = 0$  for all values of  $i$ , i.e., in all layers. Hence, we obtain

$$y_2^1 + 2y_4^1 + \frac{g}{4\pi G} \left( \frac{2}{r} y_5^1 + y_6^1 \right) = 0 \quad (7.298)$$

throughout the Earth. On the surface, have

$$y_2^1(R) + 2y_4^1(R) + \frac{g_R}{4\pi G} \left[ \frac{2}{r} y_5^1(R) + y_6^1(R) \right] = 0. \quad (7.299)$$

We immediately realize that the substitution of the first two conditions in (7.223) into this equation leads to the last condition. This means that the last condition in (7.223) on the Earth's surface is satisfied by the solution which satisfies the regularity boundary conditions at  $r = 0$ , the continuity conditions (7.221) across the internal boundaries, and the first two conditions in (7.223) on the Earth's surface.

### The Case with a Fluid Core

An analytic solution can be obtained for the degree 1 static deformation of the core when a fluid core is present. We first derive the analytic solution in the fluid core, and then infer the solution in the solid inner core.

Following the derivation of the consistency relation (7.296) in solid layers, with  $n = 1$ , we obtain from (7.254) and (7.255),

$$\frac{d}{dr} \left( \frac{2}{r} z_5^1 + z_6^1 \right) = -\frac{4\pi G \rho}{g} \left( \frac{2}{r} z_5^1 + z_6^1 \right). \quad (7.300)$$

This equation leads to, with  $d(r^2 g)/dr = r^2 (dg/dr) + 2rg = 4\pi Gr^2 \rho$  according to (7.234),

$$\frac{d}{dr} \left[ r^2 g \left( \frac{2}{r} z_5^1 + z_6^1 \right) \right] = r^2 g \frac{d}{dr} \left( \frac{2}{r} z_5^1 + z_6^1 \right) + 4\pi Gr^2 \rho \left( \frac{2}{r} z_5^1 + z_6^1 \right) = 0. \quad (7.301)$$

The solution is

$$\frac{2}{r}z_5^1 + z_6^1 = \frac{D}{r^2g}, \quad (7.302)$$

where  $D$  is a constant. According to (7.254) and (7.234), we have

$$\frac{d}{dr} \left( \frac{z_5^1}{g} \right) = \frac{4\pi G\rho}{g^2} z_5^1 + \frac{1}{g} z_6^1 - \frac{1}{g^2} \frac{dg}{dr} z_5^1 = \frac{1}{g} \left( \frac{2}{r} z_5^1 + z_6^1 \right). \quad (7.303)$$

We can then write (7.302) as

$$\frac{d}{dr} \left( \frac{z_5^1}{g} \right) = \frac{D}{r^2g^2}. \quad (7.304)$$

Another way to derive this formula is to use (7.251) with  $n = 1$  and  $z_5^1 = \phi_1^k$ .

If no solid inner core is present, we obtain  $D = 0$  by multiplying both sides of (7.302) by  $r^2g^2$  and then setting  $r = 0$ . Hence, (7.304) degenerates to

$$\frac{d}{dr} \left( \frac{z_5^1}{g} \right) = 0. \quad (7.305)$$

The solution is

$$z_5^1 = gA, \quad (7.306)$$

where  $A$  is a constant. The solution of  $z_1^1$  can be obtained according to (7.253),

$$z_1^1 = A. \quad (7.307)$$

The solution of  $z_6^1$  can be obtained according to (7.254) and (7.234),

$$z_6^1 = \frac{dz_5^1}{dr} - 4\pi G\rho z_1^1 = \left( \frac{dg}{dr} - 4\pi G\rho \right) A = -\frac{2g}{r} A. \quad (7.308)$$

The result (7.307) implies that, if no solid inner core were present, the bulk of the fluid core can only move as a whole, meaning that the equipotential surfaces are maintained undeformed.

### The Case with a Solid Inner Core

When a solid inner core is present, we first derive the consistency relation. In this case, we only know that the consistency relation (7.298) holds in the inner core.

At the inner core boundary (ICB), by substituting the boundary conditions (7.276) into (7.298), we obtain

$$\left[ \frac{g}{4\pi G} \left( \frac{2}{r} z_5^1 + z_6^1 \right) \right]_{\text{ICB}} = 0. \quad (7.309)$$

In the fluid outer core, (7.302) holds. By comparing with the above formula, we obtain  $D = 0$ . As  $z_2^1 = z_4^1 = 0$  in the fluid outer core, we can write (7.302) in the form of

$$z_2^1 + 2z_4^1 + \frac{g}{4\pi G} \left( \frac{2}{r} z_5^1 + z_6^1 \right) = 0, \quad (7.310)$$

which is in the same form as the consistency relation in the inner core given by (7.298). At the CMB, by substituting the boundary conditions (7.276) into (7.310), we obtain

$$\left[ y_2^1 + 2y_4^1 + \frac{g}{4\pi G} \left( \frac{2}{r} y_5^1 + y_6^1 \right) \right]_{\text{CMB}} = 0. \quad (7.311)$$

This implies that (7.298) holds in the lowest layer of the mantle, and therefore, it holds throughout the mantle as a result of the continuity of all the functions across all internal boundaries. Hence, the consistency relation (7.298) holds throughout the Earth model with a fluid outer core and a solid inner core; in the fluid outer core, the functions  $y_j^1$  are replaced by the functions  $z_j^1$ .

The degree 1 deformation of the inner core can be inferred from that of the outer core and the boundary conditions at the inner core boundary (ICB). The values of the  $z$ s at the ICB are, according to the solution in the outer core (remembering that  $z_3^n$  is not defined in the fluid core),

$$\left\{ z_1^1, z_2^1, z_4^1, z_5^1, z_6^1 \right\}_{\text{ICB}} = \left\{ A, 0, 0, gA, -\frac{2g}{r}A \right\}. \quad (7.312)$$

The boundary values of the solution in the inner core can be obtained from the boundary conditions (7.276),

$$\begin{aligned} & \left\{ y_1^1, y_2^1, y_3^1, y_4^1, y_5^1, y_6^1 \right\}_{\text{ICB}} \\ &= \left\{ A + K_1, K_1 g \rho_f, K_2, 0, gA, -\frac{2g}{r}A - 4\pi G K_1 \rho_f \right\}_{\text{ICB}}. \end{aligned} \quad (7.313)$$

where  $K_1$  and  $K_2$  are constants which can be arbitrarily chosen. As two arbitrary constants are present in the six boundary conditions to be satisfied, there is a need of four independent solutions of the system of differential equations (7.220) in

order for all the boundary conditions to be satisfied. However, we know that this system of differential equations has only three independent solutions in the inner core satisfying the regularity condition at the origin where  $r = 0$ . This appears that a solution is not possible at a first glance. Nevertheless, this system of differential equations does have a solution in the inner core in the form of (7.287), which can be interpreted as a rigid shift of the inner core, and is copied to here for convenience of reference,

$$\{y_1^1, y_2^1, y_3^1, y_4^1, y_5^1, y_6^1\} = \left\{ C, 0, C, 0, gC, -\frac{2g}{r}C \right\}. \quad (7.314)$$

We realize immediately that this solution satisfies the boundary conditions given by (7.313) with  $C = A$ ,  $K_1 = 0$  and  $K_2 = A$ . The solution in the inner core is thus

$$\{y_1^1, y_2^1, y_3^1, y_4^1, y_5^1, y_6^1\} = \left\{ A, 0, A, 0, gA, -\frac{2g}{r}A \right\}. \quad (7.315)$$

This is a rigid shift following the flow of the outer core without any deformation, which has been previously suggested based on results of numerical computations (Okubo & Endo, 1986; Smith, 1976).

With the analytic solution in the core to be known, we can compute the degree 1 load Love numbers by integrating the system of differential equations (7.220) in the mantle only. The boundary conditions of three independent solutions can be assigned on the CMB through the three parameters to be determined: the parameter  $A$  in the solution (7.306)–(7.308), and the parameters  $K_1$  and  $K_2$  in the boundary conditions (7.276). As the ICB is not concerned here, these parameters are implicitly understood at assigned on the CMB. The three boundary conditions are

$$\{y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n\}_1 = \left\{ 1, 0, 0, 0, g, -\frac{2g}{r} \right\}, \quad (7.316)$$

$$\{y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n\}_2 = \{1, g\rho_f, 0, 0, 0, -4\pi G\rho_f\}, \quad (7.317)$$

$$\{y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n\}_3 = \{0, 0, 1, 0, 0, 0\}, \quad (7.318)$$

where the parameters  $r$ ,  $g$ , and  $\rho_f$  take values on the CMB. The system of ordinary differential equations (7.220) is to be solved from the CMB to the Earth's surface. The general solution is  $A\{y_j^n\}_1 + K_1\{y_j^n\}_2 + K_2\{y_j^n\}_3$ . The solution we are seeking can be finally obtained by determining the three parameters  $A$ ,  $K_1$ , and  $K_2$  using the boundary conditions on the Earth's surface.

It can be readily inferred that, just as in the case without a core, the solution suffers from non-uniqueness, which can be made unique by replacing the last boundary condition in (7.223) by (7.292). We mention again that the degree 1 load Love numbers thus computed is with respect to the CE, and alternative definitions of the degree 1 load Love numbers will be discussed in Sect. 7.9.1.



## 7.7 The Unit Point Load and Load Green's Function

### 7.7.1 The Unit Point Load and Load Green's Function

In this section, we formulate the loading of a unit point mass on the Earth's surface. The loading effect of a unit point mass is called load Green's function. The loading effect of any distribution of mass on the Earth's surface is the convolution of the surface mass density of the mass of load with the load Green's function. This approach allows very accurate computation of the loading effect of the mass changes near the observation station. When this approach is used, the direct effect is computed separately by taking into account of the relative positions of the station and the mass of load. If a gravimeter is installed in a tunnel on the seafloor, the mass of load of sea level change is above the gravimeter; if a gravimeter is installed on a cliff by the shore, the mass of load of sea level change is below the gravimeter. The direct effect in these two cases are of opposite signs. The case of atmospheric loading is even more complicated. The direct effects of the masses above and below the gravimeter have opposite signs. A primitive way to compute the direct effect is to divide the mass of load into small cells and then add together the contribution of all the cells. In this section, we formulate only the indirect effect.

We formulate the problem by approximating the unit point mass as a homogeneous material spherical cap of surface mass density  $\mu$  and angular radius  $\alpha$ . We further assume for convenience that the center of the cap is situated at the pole  $\theta = 0$ . In this way, the spherical cap is the region  $\theta \leq \alpha$  on the sphere. By definition, the mass of the material spherical cap is 1, i.e.,

$$2\pi R^2 \mu \int_0^\alpha \sin \theta d\theta = 1. \quad (7.319)$$

We obtain  $\mu$  to be

$$\mu = \frac{1}{2\pi R^2(1 - \cos \alpha)}. \quad (7.320)$$

The surface mass density of the mass of load over whole the surface of the Earth is then

$$\delta s = \begin{cases} \mu, & 0 \leq \theta \leq \alpha; \\ 0, & \alpha < \theta \leq \pi. \end{cases} \quad (7.321)$$

Its spherical harmonic series is

$$\delta s = \sum_{n=0}^{\infty} \delta s_n^0 P_n(\cos \theta), \quad (7.322)$$

where the coefficient is given by

$$\begin{aligned}\delta s_n^0 &= \frac{2n+1}{2} \int_0^\pi \delta s P_n(\cos \theta) \sin \theta d\theta \\ &= \frac{2n+1}{2} \mu \int_0^\alpha P_n(\cos \theta) \sin \theta d\theta .\end{aligned}\quad (7.323)$$

According to the recurrence formula (4.116), we have

$$(2n+1) \sin \theta P_n(\cos \theta) = \frac{d}{d\theta} P_{n-1}(\cos \theta) - \frac{d}{d\theta} P_{n+1}(\cos \theta) . \quad (7.324)$$

We thus obtain

$$\delta s_n^0 = \begin{cases} \frac{\mu}{2} [P_{n-1}(\cos \alpha) - P_{n+1}(\cos \alpha)] , & n \geq 1 ; \\ \frac{\mu}{2} (1 - \cos \alpha) , & n = 0 . \end{cases} \quad (7.325)$$

The substitution of (7.320) into this formula yields

$$\delta s_n^0 = \begin{cases} \frac{P_{n-1}(\cos \alpha) - P_{n+1}(\cos \alpha)}{4\pi R^2 (1 - \cos \alpha)} , & n \geq 1 ; \\ \frac{1}{4\pi R^2} , & n = 0 . \end{cases} \quad (7.326)$$

The formulae for the loading effect of this spherical cap of unit mass can be obtained by substituting the above formula into the result derived in Sect. 7.5.3 (e.g., Bevis et al., 2016, who considered a spherical cap of unit density), noting that  $\delta s_n^k \neq 0$  only when  $k = 0$  here.

The expression of  $\delta s_n^0$  for the unit point mass is obtained by setting  $\alpha \rightarrow 0$  in (7.326). By applying l'Hopital's rule and then using the recurrence formula (7.324), we obtain

$$\delta s_n^0 = \frac{2n+1}{4\pi R^2} . \quad (7.327)$$

Consequently, we have

$$\delta s = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi R^2} P_n(\cos \theta) . \quad (7.328)$$

With this formula, we can also express the gravitational potential of the unit point mass in the form of spherical harmonic series according to (7.171). As the problem

here is independent of  $\lambda$ , we have

$$\delta\phi = \begin{cases} Rg_R \sum_{n=0}^{\infty} \delta\phi_n^0 \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta), & r \geq R; \\ Rg_R \sum_{n=0}^{\infty} \delta\phi_n^0 \left(\frac{r}{R}\right)^n P_n(\cos\theta), & r \leq R. \end{cases} \quad (7.329)$$

The coefficients can be obtained according (7.206),

$$\delta\phi_n^0 = \frac{4\pi G}{(2n+1)g_R} \delta s_n^0 = \frac{G}{R^2 g_R} = \frac{1}{M}. \quad (7.330)$$

The surface displacement and gravitational potential in response to the loading of the unit point mass can be obtained by substituting (7.328) into (7.215) and (7.217),

$$\delta u_r(R) = \frac{R}{M} \sum_{n=0}^{\infty} h'_n P_n(\cos\theta), \quad \delta u_\theta(R) = \frac{R}{M} \sum_{n=1}^{\infty} l'_n \frac{d}{d\theta} P_n(\cos\theta), \quad \delta u_\lambda(R) = 0, \quad (7.331)$$

$$\delta\phi_1 = \frac{GM}{R} \frac{1}{M} \sum_{n=0}^{\infty} k'_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta) = \frac{Rg_R}{M} \sum_{n=0}^{\infty} k'_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta), \quad (7.332)$$

where we have indicated explicitly that the summation is from  $n = 0$  or  $n = 1$  to  $\infty$ . We have kept only the indirect effect in the gravitational potential, as the indirect effect alone is concerned in this section. These formulae are fundamental. We mention that  $\delta\phi_1(R)/g_R$  is the indirect effect of the geoidal height. The loading effect of other observable quantities can be derived from the above two formulae.

The following formulations are similar to those in Sect. 7.4 for the Earth tide. Hence we only state the logic of physics when the formulation is straightforward based on that in Sect. 7.4.

The variation of gravity as measured by a gravimeter installed on the Earth's surface includes two components. One is related to the surface displacement that displaces the gravimeter, and the other to the mass redistribution associated with the Earth's deformation that alters the gravitational potential of the Earth by  $\delta\phi_1$ . We have, following the formulation in Sect. 7.4,

$$\delta g^E = \frac{g_R}{M} \sum_{n=0}^{\infty} [(n+1)k'_n - 2h'_n] P_n(\cos\theta). \quad (7.333)$$

The deviation of the vertical with respect to the crust as observed by a tiltmeter installed on the Earth's surface also includes two components. One is the change of the direction of the gravity associated with  $\delta\phi_1$  caused by the mass redistribution,

and the other the tilt of the crust with respect to the undeformed state related to the deformation of the Earth. We have, following the formulation in Sect. 7.4,

$$\delta\epsilon^E = \frac{1}{M} \sum_{n=1}^{\infty} (k'_n - h'_n) \frac{d}{d\theta} P_n(\cos \theta), \quad (7.334)$$

which is defined positive along the direction away from the point mass of load. In the literature, the tilt of the crust with respect to the direction of the gravity,  $t^E = -\delta\epsilon^E$ , is also often used. Due to the symmetry of the problem, only the three normal strain components,  $\epsilon_{rr}$ ,  $\epsilon_{\theta\theta}$ , and  $\epsilon_{\lambda\lambda}$ , and the shear strain component  $\epsilon_{r\theta}$  may be non-vanishing. Taking reference to Sect. 7.4, we have, knowing that  $\delta u_\lambda = 0$  due to the symmetry of the problem,

$$\delta\epsilon_{rr} = \left( \frac{\partial \delta u_r}{\partial r} \right)_{r=R}, \quad (7.335a)$$

$$\delta\epsilon_{\theta\theta} = \left( \frac{1}{r} \frac{\partial \delta u_\theta}{\partial \theta} + \frac{\delta u_r}{r} \right)_{r=R}, \quad (7.335b)$$

$$\delta\epsilon_{\lambda\lambda} = \left( \frac{\delta u_r}{r} + \frac{\cot \theta}{r} \delta u_\theta \right)_{r=R}, \quad (7.335c)$$

$$\delta\epsilon_{r\theta} = \frac{1}{2} \left( \frac{\partial \delta u_\theta}{\partial r} - \frac{\delta u_\theta}{r} + \frac{1}{r} \frac{\partial \delta u_r}{\partial \theta} \right)_{r=R}. \quad (7.335d)$$

By definition, we have  $\delta u_r = (1/M) \sum_{n=0}^{\infty} y_1^n P_n(\cos \theta)$ , and hence,

$$\delta\epsilon_{rr} = \frac{1}{M} \sum_{n=0}^{\infty} \left( \frac{dy_1^n}{dr} \right)_{r=R} P_n(\cos \theta). \quad (7.336)$$

We also have, according to (7.220a),

$$\left( \frac{dy_1^n}{dr} \right)_{r=R} = -\frac{2\lambda_R}{\lambda_R + 2\mu_R} h'_n + \frac{1}{\lambda_R + 2\mu_R} y_2^n(R) + \frac{n(n+1)\lambda_R}{\lambda_R + 2\mu_R} l'_n. \quad (7.337)$$

Substitute this formula into the former. As  $(1/g_R) \sum_{n=0}^{\infty} y_2^n(R) P_n(\cos \theta)$  is the total radial normal stress on the Earth's surface which is known to vanish when  $\theta \neq 0$ , we obtain

$$\delta\epsilon_{rr} = -\frac{2\lambda_R}{R\sigma_R} \delta u_r(R) + \frac{\lambda_R}{M\sigma_R} \sum_{n=0}^{\infty} n(n+1) l'_n P_n(\cos \theta), \quad (7.338)$$

where  $\sigma_R = \lambda_R + 2\mu_R$ . The component  $\delta\epsilon_{\theta\theta}$  can be readily obtained according to the expression of  $\delta u_\theta$ ,

$$\delta\epsilon_{\theta\theta} = \frac{1}{R} \delta u_r(R) + \frac{1}{M} \sum_{n=1}^{\infty} l'_n \frac{d^2}{d\theta^2} P_n(\cos \theta). \quad (7.339)$$

The component  $\delta\epsilon_{\lambda\lambda}$  can be computed using

$$\delta\epsilon_{\lambda\lambda} = \frac{1}{R} \delta u_r(R) + \frac{\cot \theta}{R} \delta u_\theta(R). \quad (7.340)$$

With  $\delta u_r = (1/M) \sum_{n=0}^{\infty} y_1^n P_n(\cos \theta)$  and  $\delta u_\theta = (1/M) \sum_{n=0}^{\infty} y_3^n (dP_n(\cos \theta)/dr)$ , we can readily show using (7.220c) that the shear component  $\delta\epsilon_{r\theta}$  vanishes,

$$\delta\epsilon_{r\theta} = 0. \quad (7.341)$$

Further formulation is required for evaluating the Legendre series in (7.331), (7.333), (7.334), (7.338), and (7.339). In order to make these Legendre series numerically convergent for practical computation, the asymptotic formulae of the load Love numbers when  $n$  is very large are required.

### 7.7.2 Asymptotic Formulae of the Load Love Numbers

Farrell (1972) inferred the asymptotic expressions of the load Love numbers from the solution of the Boussinesq problem, i.e., the point load on a homogeneous half-space. Here we follow Guo et al. (2004) to derive them by seeking the asymptotic solutions of the governing ordinary differential equations and boundary conditions.

#### Scaling of the Governing Equations

We first infer the order of magnitude of the functions  $y_1^n, \dots, y_6^n$  as power of  $n$  when  $n$  is very large, i.e., when  $n$  is on the order of thousands. In the formulation, we have defined  $y_1^n$  and  $y_3^n$  as the spherical harmonic coefficients of  $u_r$  and  $u_\theta$ , respectively. The degree  $n$  terms are

$$u_{rn} = y_1^n P_n(\cos \theta), \quad u_{\theta n} = y_3^n \frac{d}{d\theta} P_n(\cos \theta), \quad (7.342)$$

where we have dropped the common multiplier  $\phi_n^k$  and the dependence on  $\lambda$  for brevity. By inspecting the recurrence formulae (4.114)–(4.116), we realize that  $dP_n(\cos \theta)/d\theta$  is on the order of  $n$  times  $P_n(\cos \theta)$ ,  $dP_n(\cos \theta)/d\theta \sim n P_n(\cos \theta)$ . We assume that  $u_{rn}$  and  $u_{\theta n}$  are of the same order of magnitude, and that  $y_1^n$  is of the order  $O(n^0)$ . We can then see that  $y_3^n$  should be of the order  $O(n^{-1})$ . Similarly, the degree  $n$  terms of the spherical harmonic series of the radial and tangential

components of the stress on a sphere of radius  $r$ ,  $T_{rr}$  and  $T_{r\theta}$ , are

$$T_{rrn} = y_2^n P_n(\cos \theta), \quad T_{r\theta n} = y_4^n \frac{d}{d\theta} P_n(\cos \theta). \quad (7.343)$$

We assume that  $T_{rrn}$  and  $T_{r\theta n}$  are of the same order of magnitude and that, according to the first boundary condition in (7.223),  $y_2^n$  is of the order  $O(n^1)$ . We can then see that  $y_4^n$  is of the order  $O(n^0)$ . Finally, in the last boundary condition in (7.223), we assume that the two terms at the left-hand side are of the same order of magnitude, which leads to the assumption that  $y_5^n$  and  $y_6^n$  are on the orders of  $O(n^0)$  and  $O(n^1)$ , respectively. In summary, we assumed that the orders of magnitudes of the functions  $y_1^n, \dots, y_6^n$  are

$$y_1^n \sim (n^0), \quad y_2^n \sim (n^1), \quad y_3^n \sim (n^{-1}), \quad y_4^n \sim (n^0), \quad y_5^n \sim (n^0), \quad y_6^n \sim (n^1). \quad (7.344)$$

This assumption is based on an informal analysis. Its validity will be later seen from the asymptotic solution. With this assumption, we define, for convenience of further formulation, a new set of functions  $z_1, \dots, z_6$  according to

$$y_1^n = z_1, \quad y_2^n = \frac{n}{r} z_2, \quad y_3^n = \frac{1}{n} z_3, \quad y_4^n = \frac{1}{r} z_4, \quad y_5^n = r z_5, \quad y_6^n = n z_6, \quad (7.345)$$

which are all on the order  $O(n^0)$ . Although the superscript  $n$  is dropped, these new functions are still  $n$  dependent.

The differential equations and boundary conditions in terms of the new set of functions can be readily obtained by substituting (7.345) into (7.220)–(7.223). The differential equations are

$$\frac{r}{n} \frac{dz_1}{dr} = -\frac{2\lambda}{n(\lambda + 2\mu)} z_1 + \frac{1}{\lambda + 2\mu} z_2 + \frac{(n+1)\lambda}{n(\lambda + 2\mu)} z_3, \quad (7.346a)$$

$$\begin{aligned} \frac{r}{n} \frac{dz_2}{dr} = & \left[ -\frac{4r\rho g}{n^2} + \frac{4\mu(3\lambda + 2\mu)}{n^2(\lambda + 2\mu)} \right] z_1 + \frac{\lambda - 2\mu}{n(\lambda + 2\mu)} z_2 \\ & + \left[ \frac{(n+1)r\rho g}{n^2} - \frac{2(n+1)\mu(3\lambda + 2\mu)}{n^2(\lambda + 2\mu)} \right] z_3 \\ & + \frac{n+1}{n} z_4 - \frac{r^2\rho}{n} z_6, \end{aligned} \quad (7.346b)$$

$$\frac{r}{n} \frac{dz_3}{dr} = -z_1 + \frac{1}{n} z_3 + \frac{1}{\mu} z_4, \quad (7.346c)$$

$$\begin{aligned} \frac{r}{n} \frac{dz_4}{dr} = & \left[ \frac{r\rho g}{n} - \frac{2\mu(3\lambda + 2\mu)}{n(\lambda + 2\mu)} \right] z_1 - \frac{\lambda}{\lambda + 2\mu} z_2 \\ & + \frac{2\mu}{\lambda + 2\mu} \left[ \lambda \frac{2n^2 + 2n - 1}{n^2} + 2\mu \frac{n^2 + n - 1}{n^2} \right] z_3 \end{aligned}$$

$$-\frac{2}{n}z_4 - \frac{r^2\rho}{n}z_5, \quad (7.346d)$$

$$\frac{r}{n} \frac{dz_5}{dr} = \frac{4\pi G\rho}{n}z_1 - \frac{1}{n}z_5 + z_6, \quad (7.346e)$$

$$\frac{r}{n} \frac{dz_6}{dr} = -\frac{4\pi G(n+1)\rho}{n^2}z_3 + \frac{n+1}{n}z_5 - \frac{2}{n}z_6. \quad (7.346f)$$

The boundary conditions are: at solid-solid boundaries,  $z_1, \dots, z_6$  are continuous; at solid-fluid boundaries, no condition exists for  $z_3$  and the others are continuous; and on the Earth's surface (when  $n \geq 2$ ),

$$z_2(R) = -\frac{(2n+1)R}{4\pi Gn}g_R^2, \quad z_4(R) = 0, \quad z_6(R) + \frac{n+1}{n}z_5(R) = \frac{2n+1}{n}g_R. \quad (7.347)$$

As we are studying the case when  $n$  is very large, the cases for  $n = 0$  and  $n = 1$  are no longer kept. It must be noticed that, as the dependence of  $z_1, \dots, z_6$  on  $r$  are not known, the orders of magnitude of  $dz_1/dr, \dots, dz_6/dr$  cannot be determined yet at this stage. Therefore, the terms on the left-hand side in equations (7.346) must be kept. The load Love numbers are

$$h'_n = \frac{z_1(R)}{R}, \quad l'_n = \frac{z_3(R)}{nR}, \quad k'_n = \frac{z_5(R)}{g_R} - 1. \quad (7.348)$$

Normally, in Earth models of internal physical parameters used to compute the load Love numbers, the upper-most layer is homogeneous, i.e. the parameters  $\rho$ ,  $\lambda$  and  $\mu$  are constant. We take as example the PREM, which has a homogeneous ocean layer and a homogeneous upper-crustal layer. For computing the load Love numbers, the ocean layer and the upper-crustal layer are normally averaged to form a single homogeneous layer. It will be shown that, when  $n$  is very large, the solutions of  $z_1, \dots, z_6$  depend on  $r$  in the form of  $(r/R)^n$  and  $\ln(r/R)^n(r/R)^n$ , which decrease very fast from the surface to the interior of the Earth. For the PREM, these two terms are  $5.8 \times 10^{-11}$  and  $-1.4 \times 10^{-9}$  at the bottom of the upper-most layer when  $n = 10,000$ . Thus, accurate to at least to eight significant digits, we need only to consider the upper-most homogeneous layer. For this reason, we shall treat the parameters  $\rho$ ,  $\lambda$ , and  $\mu$  as constants when searching for asymptotic solutions. This is the same logic as in Farrell (1972), where the Earth is approximated by a homogeneous half-space when  $n \rightarrow \infty$ .

### The Classic Asymptotic Solution

First of all, we search for an asymptotic solution at the same level of accuracy as Farrell (1972). For this reason, we call this asymptotic solution *classic*.

For this purpose, we simplify the differential equations (7.346) and the associated boundary conditions by retaining only the terms of order  $O(n^0)$  and neglecting those

of order  $O(n^{-1})$  and smaller. We denote this solution as  $z_1^0, \dots, z_6^0$ . The differential equations then become

$$\frac{r}{n} \frac{dz_1^0}{dr} = \frac{1}{\lambda + 2\mu} z_2^0 + \frac{\lambda}{\lambda + 2\mu} z_3^0, \quad (7.349a)$$

$$\frac{r}{n} \frac{dz_2^0}{dr} = z_4^0, \quad (7.349b)$$

$$\frac{r}{n} \frac{dz_3^0}{dr} = -z_1^0 + \frac{1}{\mu} z_4^0, \quad (7.349c)$$

$$\frac{r}{n} \frac{dz_4^0}{dr} = -\frac{\lambda}{\lambda + 2\mu} z_2^0 + \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} z_3^0, \quad (7.349d)$$

$$\frac{r}{n} \frac{dz_5^0}{dr} = z_6^0, \quad (7.349e)$$

$$\frac{r}{n} \frac{dz_6^0}{dr} = z_5^0. \quad (7.349f)$$

The boundary conditions are: at solid-solid boundaries,  $z_1^0, \dots, z_6^0$  are continuous; at solid-fluid boundaries, no condition exists for  $z_3^0$ , and the rest are continuous; and on the Earth's surface (when  $n \geq 2$ ),

$$z_2^0(R) = -\frac{R}{2\pi G} g_R^2, \quad z_4^0(R) = 0, \quad z_5^0(R) + z_6^0(R) = 2g_R. \quad (7.350)$$

We see that the use of the factors  $1/r$  and  $r$  in the definitions of the functions  $z_1^0, \dots, z_6^0$  in (7.345) rendered that  $r$  does not appear at the right-hand sides of all the equations in (7.349).

The first four and the last two equations in (7.349) are decoupled. Hence, we treat (7.349a)–(7.349d) as one group, and (7.349e) and (7.349f) as another group. The Eqs. (7.349a)–(7.349d) have four linearly independent solutions. It can be verified that two of them are

$$z_1^0 = \left(\frac{r}{R}\right)^{\pm n}, \quad z_2^0 = \pm 2\mu \left(\frac{r}{R}\right)^{\pm n}, \quad z_3^0 = \pm \left(\frac{r}{R}\right)^{\pm n}, \quad z_4^0 = 2\mu \left(\frac{r}{R}\right)^{\pm n}; \quad (7.351)$$

and the other two are

$$z_1^0 = \left[ \ln \left(\frac{r}{R}\right)^n \mp \frac{\lambda + 3\mu}{\lambda + \mu} \right] \left(\frac{r}{R}\right)^{\pm n}, \quad (7.352a)$$

$$z_2^0 = \left[ \pm 2\mu \ln \left(\frac{r}{R}\right)^n - \frac{2\mu(\lambda + 2\mu)}{\lambda + \mu} \right] \left(\frac{r}{R}\right)^{\pm n}, \quad (7.352b)$$

$$z_3^0 = \left[ \pm \ln \left(\frac{r}{R}\right)^n \right] \left(\frac{r}{R}\right)^{\pm n}, \quad (7.352c)$$



$$z_4^0 = \left[ 2\mu \ln \left( \frac{r}{R} \right)^n \mp \frac{2\mu^2}{\lambda + \mu} \right] \left( \frac{r}{R} \right)^{\pm n}. \quad (7.352d)$$

As the solutions depending on  $(r/R)^{-n}$  may be unreasonably large when  $r < R$ , they are not in agreement with the physical problem we are studying, and are no longer considered. The solution satisfying the necessary boundary conditions can be readily obtained to be

$$z_1^0 = \frac{Rg_R^2}{4\pi G} \left[ \frac{1}{\mu} \ln \left( \frac{r}{R} \right)^n - \frac{\lambda + 2\mu}{\mu(\lambda + \mu)} \right] \left( \frac{r}{R} \right)^n, \quad (7.353a)$$

$$z_2^0 = \frac{Rg_R^2}{4\pi G} \left[ 2 \ln \left( \frac{r}{R} \right)^n - 2 \right] \left( \frac{r}{R} \right)^n, \quad (7.353b)$$

$$z_3^0 = \frac{Rg_R^2}{4\pi G} \left[ \frac{1}{\mu} \ln \left( \frac{r}{R} \right)^n + \frac{1}{\lambda + \mu} \right] \left( \frac{r}{R} \right)^n, \quad (7.353c)$$

$$z_4^0 = \frac{Rg_R^2}{4\pi G} \left[ 2 \ln \left( \frac{r}{R} \right)^n \right] \left( \frac{r}{R} \right)^n. \quad (7.353d)$$

The Eqs. (7.349e) and (7.349f) have two independent solutions, which are

$$z_5^0 = \left( \frac{r}{R} \right)^{\pm n}, \quad z_6^0 = \pm \left( \frac{r}{R} \right)^{\pm n}. \quad (7.354)$$

The solution satisfying the necessary boundary conditions is:

$$z_5^0 = g_R \left( \frac{r}{R} \right)^n, \quad z_6^0 = g_R \left( \frac{r}{R} \right)^n. \quad (7.355)$$

The solutions of  $z_5^0$  and  $z_6^0$  given by (7.355) are, however, not accurate enough as approximations of  $z_5$  and  $z_6$  for our purpose. To obtain a more accurate solutions of  $z_5$  and  $z_6$ , we reanalyze (7.346e) and (7.346f) more accurately. We use the perturbation method to define

$$z_i = z_i^0 + \frac{1}{n} z_i', \quad (i = 1, \dots, 6), \quad (7.356)$$

where  $z_i'$  are assumed to be on the order of  $O(n^0)$ . We multiply both sides of (7.346e) and (7.346f) with  $n$ , and then substitute (7.356) into them. We obtain, after neglecting the terms of order  $O(n^{-1})$  or smaller,

$$\begin{cases} \frac{r}{n} \frac{dz_5'}{dr} = z_6' + 4\pi G \rho z_1^0 - z_5^0, \\ \frac{r}{n} \frac{dz_6'}{dr} = z_5' - 4\pi G \rho z_3^0 + z_5^0 - 2z_6^0. \end{cases} \quad (7.357)$$

The boundary conditions for  $z'_5$  and  $z'_6$  are obtained from those of  $z_5, z_6$  and  $z_5^0, z_6^0$  as: both  $z'_5$  and  $z'_6$  are continuous across solid-solid as well as solid-liquid boundaries, and on the Earth's surface (when  $n \geq 2$ ),

$$z'_5(R) + z'_6(R) = -z_5^0(R) + g_R. \quad (7.358)$$

By substituting (7.353) and (7.355) into (7.357) and (7.358), we obtain, for  $z'_5$  and  $z'_6$ , the differential equations

$$\begin{cases} \frac{r}{n} \frac{dz'_5}{dr} = z'_6 + Rg_R^2 \rho \left[ \frac{1}{\mu} \ln \left( \frac{r}{R} \right)^n - \frac{\lambda + 2\mu}{\mu(\lambda + \mu)} - \frac{1}{Rg_R \rho} \right] \left( \frac{r}{R} \right)^n, \\ \frac{r}{n} \frac{dz'_6}{dr} = z'_5 - Rg_R^2 \rho \left[ \frac{1}{\mu} \ln \left( \frac{r}{R} \right)^n + \frac{1}{\lambda + \mu} + \frac{1}{Rg_R \rho} \right] \left( \frac{r}{R} \right)^n; \end{cases} \quad (7.359)$$

and the boundary condition

$$z'_5(R) + z'_6(R) = 0. \quad (7.360)$$

The homogeneous parts of (7.359) are the same as those of (7.349e) and (7.349f). Hence, their two sets of linearly-independent solutions are the same as (7.354). Base on the forms of (7.359) and (7.354), we assume that the differential equations (7.359) have a special solution in the form of

$$z'_i = \left[ A_i \ln \left( \frac{r}{R} \right)^n + B_i \right] \left( \frac{r}{R} \right)^n, \quad (i = 5, 6), \quad (7.361)$$

where  $A_i$  and  $B_i$  are constants to be determined. We have

$$\frac{r}{n} \frac{dz'_i}{dr} = \left[ A_i \ln \left( \frac{r}{R} \right)^n + A_i + B_i \right] \left( \frac{r}{R} \right)^n, \quad (i = 5, 6). \quad (7.362)$$

The substitution of (7.361) and (7.362) into (7.359) leads to a special solution. Here we omit the somewhat lengthy algebraic operations, and directly write out the result:

$$\begin{cases} A_5 = -Rg_R^2 \rho \left[ \frac{1}{\lambda + \mu} + \frac{1}{Rg_R \rho} \right], & B_5 = 0; \\ A_6 = -Rg_R^2 \rho \left[ \frac{\lambda + 2\mu}{\mu(\lambda + \mu)} + \frac{1}{Rg_R \rho} \right], & B_6 = \frac{Rg_R^2 \rho}{\mu}. \end{cases} \quad (7.363)$$

The solution of (7.359) satisfying the boundary condition (7.360) can then be determined using this special solution and (7.354). The result is

$$\begin{cases} z'_5 = Rg_R^2 \rho \left\{ - \left[ \frac{1}{\lambda + \mu} + \frac{1}{Rg_R \rho} \right] \ln \left( \frac{r}{R} \right)^n - \frac{1}{2\mu} \right\} \left( \frac{r}{R} \right)^n, \\ z'_6 = Rg_R^2 \rho \left\{ - \left[ \frac{\lambda + 2\mu}{\mu(\lambda + \mu)} + \frac{1}{Rg_R \rho} \right] \ln \left( \frac{r}{R} \right)^n + \frac{1}{2\mu} \right\} \left( \frac{r}{R} \right)^n. \end{cases} \quad (7.364)$$

From the solutions given above, we can verify that  $y_1^n, \dots, y_6^n$  are actually on the order of magnitudes we have assumed in (7.344). This confirms that the assumptions made in (7.344) are valid, and that the above analysis is correct.

Finally, we derive Farrell's (1972) well-known asymptotic expressions of the load Love numbers according to our asymptotic solutions  $z_1^0, \dots, z_4^0$  and  $z_5^0 + (1/n)z'_5, z_6^0 + (1/n)z'_6$ . We have

$$\begin{cases} \lim_{n \rightarrow \infty} h'_n = \lim_{n \rightarrow \infty} \frac{z_1^0(R)}{R} = h'_\infty, \\ \lim_{n \rightarrow \infty} nl'_n = \lim_{n \rightarrow \infty} \frac{z_3^0(R)}{R} = l'_\infty, \\ \lim_{n \rightarrow \infty} nk'_n = \lim_{n \rightarrow \infty} \frac{z_5^0(R)}{g_R} = k'_\infty, \end{cases} \quad (7.365)$$

where

$$h'_\infty = - \frac{g_R^2(\lambda_R + 2\mu_R)}{4\pi G \mu_R(\lambda_R + \mu_R)}, \quad l'_\infty = \frac{g_R^2}{4\pi G(\lambda_R + \mu_R)}, \quad k'_\infty = - \frac{R\rho_R g_R}{2\mu_R}. \quad (7.366)$$

In these asymptotic expressions,  $z_1, \dots, z_4$  are accurate to  $O(n^0)$ , and  $z_5$  and  $z_6$  to  $O(n^{-1})$ .

### A More Accurate Asymptotic Solution

Now we search for an asymptotic solution that is an order of magnitude more accurate than the one given above, i.e.,  $z_1, \dots, z_4$  are to be accurate to the order of  $O(n^{-1})$ , and  $z_5$  and  $z_6$  to the order of  $O(n^{-2})$ .

Following the perturbation method, let

$$\begin{cases} z_i = z_i^0 + \frac{1}{n} z'_i, & (i = 1, \dots, 4); \\ z_i = z_i^0 + \frac{1}{n} z'_i + \frac{1}{n^2} z''_i, & (i = 5, 6); \end{cases} \quad (7.367)$$

where  $z'_i$  and  $z''_i$  are assumed to be of the order  $O(n^0)$ .

We multiply both sides of (7.346a)–(7.346d) by  $n$ , (7.346e) and (7.346f) by  $n^2$ , and then substitute (7.367) into them. We then make use of (7.349) and (7.357) to

obtain, keeping only the terms of order  $O(n^0)$  or larger,

$$\frac{r}{n} \frac{dz'_1}{dr} = \frac{1}{\lambda + 2\mu} z'_2 + \frac{\lambda}{\lambda + 2\mu} z'_3 - \frac{2\lambda}{\lambda + 2\mu} z_1^0 + \frac{\lambda}{\lambda + 2\mu} z_3^0, \quad (7.368a)$$

$$\frac{r}{n} \frac{dz'_2}{dr} = z'_4 + \frac{\lambda - 2\mu}{\lambda + 2\mu} z_2^0 + \left[ r\rho g - \frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \right] z_3^0 + z_4^0 - r^2 \rho z_6^0, \quad (7.368b)$$

$$\frac{r}{n} \frac{dz'_3}{dr} = -z'_1 + \frac{1}{\mu} z'_4 + z_3^0, \quad (7.368c)$$

$$\begin{aligned} \frac{r}{n} \frac{dz'_4}{dr} = & -\frac{\lambda}{\lambda + 2\mu} z'_2 + \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} z'_3 + \left[ r\rho g - \frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu} \right] z_1^0 \\ & + \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} z_3^0 - 2z_4^0 - r^2 \rho z_5^0, \end{aligned} \quad (7.368d)$$

$$\frac{r}{n} \frac{dz''_5}{dr} = z''_6 + 4\pi G \rho z'_1 - z'_5, \quad (7.368e)$$

$$\frac{r}{n} \frac{dz''_6}{dr} = z''_5 - 4\pi G \rho z_3^0 - 4\pi G \rho z'_3 + z'_5 - 2z'_6. \quad (7.368f)$$

This is a system of linear ordinary differential equations of the functions  $z'_1, \dots, z'_4$  and  $z''_5, z''_6$ . The boundary conditions for them are, at the same level of accuracy: at solid-solid boundaries, all the functions are continuous; at solid-fluid boundaries, no condition exists for  $z'_3$ , and the rest of the functions are continuous; and on the Earth's surface (when  $n \geq 2$ ),

$$z'_2(R) = -\frac{R g_R^2}{4\pi G}, \quad z'_4(R) = 0, \quad z''_6(R) + z''_5(R) = -z'_5(R). \quad (7.369)$$

We see that, in (7.368), the last two differential equations depend upon the solution of the first four that can be solved alone. Therefore, we first solve the first four, and then the last two.

By substituting the solutions of  $z_1^0, \dots, z_6^0$  into (7.368a)–(7.368d), we obtain

$$\begin{aligned} \frac{r}{n} \frac{dz'_1}{dr} = & \frac{1}{\lambda + 2\mu} z'_2 + \frac{\lambda}{\lambda + 2\mu} z'_3 + \frac{R g_R^2}{4\pi G} \left\{ -\frac{\lambda}{\mu(\lambda + 2\mu)} \ln \left( \frac{r}{R} \right)^n \right. \\ & \left. + \frac{\lambda(2\lambda + 5\mu)}{\mu(\lambda + \mu)(\lambda + 2\mu)} \right\} \left( \frac{r}{R} \right)^n, \end{aligned} \quad (7.370a)$$

$$\begin{aligned} \frac{r}{n} \frac{dz'_2}{dr} = & z'_4 + \frac{R g_R^2}{4\pi G} \left\{ \left[ -2 + \frac{r\rho g}{\mu} \right] \ln \left( \frac{r}{R} \right)^n - \frac{2\lambda}{\lambda + \mu} \right. \\ & \left. + \frac{r\rho g}{\lambda + \mu} - \frac{4\pi G r^2 \rho}{R g_R} \right\} \left( \frac{r}{R} \right)^n, \end{aligned} \quad (7.370b)$$

$$\frac{r}{n} \frac{dz'_3}{dr} = -z'_1 + \frac{1}{\mu} z'_4 + \frac{Rg_R^2}{4\pi G} \left\{ \frac{1}{\mu} \ln \left( \frac{r}{R} \right)^n + \frac{1}{\lambda + \mu} \right\} \left( \frac{r}{R} \right)^n, \quad (7.370c)$$

$$\begin{aligned} \frac{r}{n} \frac{dz'_4}{dr} = & -\frac{\lambda}{\lambda + 2\mu} z'_2 + \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} z'_3 + \frac{Rg_R^2}{4\pi G} \left\{ \left[ -\frac{2(3\lambda + 4\mu)}{\lambda + 2\mu} + \frac{r\rho g}{\mu} \right] \ln \left( \frac{r}{R} \right)^n \right. \\ & \left. + \frac{6\lambda^2 + 20\lambda\mu + 12\mu^2}{(\lambda + \mu)(\lambda + 2\mu)} - \frac{r\rho g(\lambda + 2\mu)}{\mu(\lambda + \mu)} - \frac{4\pi Gr^2\rho}{Rg_R} \right\} \left( \frac{r}{R} \right)^n. \end{aligned} \quad (7.370d)$$

The boundary conditions do not need to be reduced further.

The homogeneous part of (7.370) is the same as (7.349a)–(7.349d). Its four sets of linearly independent solutions are the same as (7.351) and (7.352). We need to find a special solution of (7.370). As (7.370) is similar to (7.359), we naturally assume that this special solution is similar to that of (7.359), i.e., (7.361),

$$z'_i = \left[ A_i \ln \left( \frac{r}{R} \right)^n + B_i \right] \left( \frac{r}{R} \right)^n, \quad (i = 1, \dots, 4). \quad (7.371)$$

Unlike (7.359), (7.370) contains, in the coefficients of  $(r/R)^n$  and  $\ln(r/R)^n(r/R)^n$ , the quantities  $r$  and  $g$  that are not constant. Hence,  $A_i$  and  $B_i$  cannot be assumed constant at a first glance. However, we have, according to (7.371),

$$\frac{r}{n} \frac{dz'_i}{dr} = \left[ \left( A_i + \frac{r}{n} \frac{dA_i}{dr} \right) \ln \left( \frac{r}{R} \right)^n + \left( A_i + B_i + \frac{r}{n} \frac{dB_i}{dr} \right) \right] \left( \frac{r}{R} \right)^n. \quad (7.372)$$

We see that, accurate to the order  $O(n^0)$ , the terms  $(r/n)(dA_i/dr)$  and  $(r/n)(dB_i/dr)$  do not need to be considered assuming  $dA_i/dr$  and  $dB_i/dr$  to be on the order of  $O(n^0)$ . After substituting (7.371) and (7.372) into (7.370), we can obtain, after some tedious algebraic operations, a solution with

$$\begin{cases} A_1 = \frac{Rg_R^2}{4\pi G} \left[ -\frac{1}{\lambda + \mu} + \frac{r\rho g}{2\mu(\lambda + \mu)} \right], \\ B_1 = \frac{Rg_R^2}{4\pi G} \left[ \frac{\lambda + 4\mu}{2\mu(\lambda + \mu)} - \frac{r\rho g}{2\mu(\lambda + \mu)} - \frac{2\pi Gr^2\rho}{Rg_R(\lambda + \mu)} \right]; \end{cases} \quad (7.373a)$$

$$\begin{cases} A_2 = \frac{Rg_R^2}{4\pi G} \left[ -2 + \frac{r\rho g}{\mu} \right], \\ B_2 = \frac{Rg_R^2}{4\pi G} \left[ \frac{2\mu}{\lambda + \mu} - \frac{r\rho g\lambda}{\mu(\lambda + \mu)} - \frac{4\pi Gr^2\rho}{Rg_R} \right]; \end{cases} \quad (7.373b)$$

$$\begin{cases} A_3 = \frac{Rg_R^2}{4\pi G} \left[ \frac{\lambda + 2\mu}{\mu(\lambda + \mu)} - \frac{r\rho g}{2\mu(\lambda + \mu)} \right] = 0, \\ B_3 = \frac{Rg_R^2}{4\pi G} \left[ -\frac{3(\lambda + 2\mu)}{2\mu(\lambda + \mu)} + \frac{r\rho g}{\mu(\lambda + \mu)} + \frac{2\pi Gr^2\rho}{Rg_R(\lambda + \mu)} \right]; \end{cases} \quad (7.373c)$$

$$\begin{cases} A_4 = 0, \\ B_4 = 0. \end{cases} \quad (7.373d)$$

This is a special solution of (7.370). The linearly independent solutions of its homogeneous part are (7.351) and (7.352). We can thus obtain the solutions of (7.370) satisfying the boundary conditions (7.369) to be

$$\begin{aligned} z'_1 = \frac{Rg_R^2}{4\pi G} \left\{ \left[ \frac{1}{2\mu} - \frac{r\rho g(\lambda - \mu)}{2\mu^2(\lambda + \mu)} - \frac{2\pi Gr^2\rho}{Rg_R\mu} \right] \ln\left(\frac{r}{R}\right)^n \right. \\ \left. - \frac{\mu}{(\lambda + \mu)^2} + \frac{r\rho g(\lambda^2 + \lambda\mu - \mu^2)}{2\mu^2(\lambda + \mu)^2} + \frac{2\pi Gr^2\rho}{Rg_R\mu} \right\} \left(\frac{r}{R}\right)^n, \end{aligned} \quad (7.374a)$$

$$z'_2 = \frac{Rg_R^2}{4\pi G} \left\{ \left[ -\frac{\lambda - \mu}{\lambda + \mu} + \frac{r\rho g}{\lambda + \mu} - \frac{4\pi Gr^2\rho}{Rg_R} \right] \ln\left(\frac{r}{R}\right)^n - 1 \right\} \left(\frac{r}{R}\right)^n, \quad (7.374b)$$

$$\begin{aligned} z'_3 = \frac{Rg_R^2}{4\pi G} \left\{ \left[ \frac{3\lambda + 7\mu}{2\mu(\lambda + \mu)} - \frac{r\rho g}{2\mu^2} - \frac{2\pi Gr^2\rho}{Rg_R\mu} \right] \ln\left(\frac{r}{R}\right)^n \right. \\ \left. - \frac{3\lambda^2 + 8\lambda\mu + 3\mu^2}{2\mu(\lambda + \mu)^2} + \frac{r\rho g(\lambda + 2\mu)}{2\mu(\lambda + \mu)^2} \right\} \left(\frac{r}{R}\right)^n, \end{aligned} \quad (7.374c)$$

$$z'_4 = \frac{Rg_R^2}{4\pi G} \left\{ \left[ \frac{\lambda + 3\mu}{\lambda + \mu} - \frac{r\rho g\lambda}{\mu(\lambda + \mu)} - \frac{4\pi Gr^2\rho}{Rg_R} \right] \ln\left(\frac{r}{R}\right)^n \right\} \left(\frac{r}{R}\right)^n. \quad (7.374d)$$

We should remember that this solution is accurate to the order  $O(n^0)$ , i.e. the terms of order  $O(n^{-1})$  and smaller are neglected. This would introduce errors of order  $O(n^{-2})$  into the final results of  $z_1, \dots, z_4$ , which is in agreement with the accuracy requirement of the order  $O(n^{-1})$ .

By substituting (7.353), (7.364), and (7.374) into (7.368e) and (7.368f), we obtain the differential equations for  $z''_5$  and  $z''_6$ ,

$$\begin{aligned} \frac{r}{n} \frac{dz''_5}{dr} = z''_6 + Rg_R^2\rho \left\{ \left[ \frac{\lambda + 3\mu}{2\mu(\lambda + \mu)} - \frac{r\rho g(\lambda - \mu)}{2\mu^2(\lambda + \mu)} + \frac{1}{Rg_R\rho} - \frac{2\pi Gr^2\rho}{Rg_R\mu} \right] \ln\left(\frac{r}{R}\right)^n \right. \\ \left. + \frac{\lambda^2 + 2\lambda\mu - \mu^2}{2\mu(\lambda + \mu)^2} + \frac{r\rho g(\lambda^2 + \lambda\mu - \mu^2)}{2\mu^2(\lambda + \mu)^2} + \frac{2\pi Gr^2\rho}{Rg_R\mu} \right\} \left(\frac{r}{R}\right)^n, \end{aligned} \quad (7.375a)$$

$$\begin{aligned} \frac{r}{n} \frac{dz_6''}{dr} = z_5'' + Rg_R^2 \rho \left\{ \left[ -\frac{\lambda + 3\mu}{2\mu(\lambda + \mu)} + \frac{r\rho g}{2\mu^2} + \frac{1}{Rg_R\rho} + \frac{2\pi Gr^2\rho}{Rg_R\mu} \right] \ln\left(\frac{r}{R}\right)^n \right. \\ \left. - \frac{\mu}{(\lambda + \mu)^2} - \frac{r\rho g(\lambda + 2\mu)}{2\mu(\lambda + \mu)^2} \right\} \left(\frac{r}{R}\right)^n. \end{aligned} \quad (7.375b)$$

The boundary condition on the Earth's surface is

$$z_6''(R) + z_5''(R) = \frac{Rg_R^2\rho_R}{2\mu_R}. \quad (7.376)$$

As the homogeneous parts of (7.375) is the same as that of (7.349e) and (7.349f), its two sets of linearly independent solutions are the same as (7.354). The equations (7.375) do not have a special solution in the same form as (7.361). We therefor assume a more complicated form,

$$z_i'' = \left[ A_i \ln^2\left(\frac{r}{R}\right)^n + B_i \ln\left(\frac{r}{R}\right)^n + C_i \right] \left(\frac{r}{R}\right)^n. \quad (7.377)$$

Similar to the derivation of (7.373), we can obtain a special solution accurate to the order of  $O(n^{-1})$  with

$$\begin{cases} A_5 = Rg_R^2\rho \left[ \frac{r\rho g}{4\mu(\lambda + \mu)} + \frac{1}{2Rg_R\rho} \right], \\ B_5 = Rg_R^2\rho \left[ \frac{\lambda^2 + 3\lambda\mu}{2\mu(\lambda + \mu)^2} - \frac{r\rho g(\lambda + 3\mu)}{4\mu(\lambda + \mu)^2} \right], \\ C_5 = Rg_R^2\rho \left[ \frac{\lambda}{4\mu(\lambda + \mu)} + \frac{r\rho g(2\lambda + \mu)}{8\mu^2(\lambda + \mu)} + \frac{\pi Gr^2\rho}{Rg_R\mu} \right]; \end{cases} \quad (7.378)$$

$$\begin{cases} A_6 = Rg_R^2\rho \left[ \frac{r\rho g}{4\mu(\lambda + \mu)} + \frac{1}{2Rg_R\rho} \right], \\ B_6 = Rg_R^2\rho \left[ -\frac{\lambda + 3\mu}{2(\lambda + \mu)^2} + \frac{r\rho g(2\lambda^2 + \lambda\mu - 3\mu^2)}{4\mu^2(\lambda + \mu)^2} + \frac{2\pi Gr^2\rho}{Rg_R\mu} \right], \\ C_6 = Rg_R^2\rho \left[ \frac{\lambda + 2\mu}{4\mu(\lambda + \mu)} - \frac{r\rho g(2\lambda + \mu)}{8\mu^2(\lambda + \mu)} - \frac{\pi Gr^2\rho}{Rg_R\mu} \right]. \end{cases} \quad (7.379)$$

The solution of (7.375) satisfying the boundary condition (7.376) is

$$\begin{aligned} z_5'' = Rg_R^2\rho \left\{ \left[ \frac{r\rho g}{4\mu(\lambda + \mu)} + \frac{1}{2Rg_R\rho} \right] \ln^2\left(\frac{r}{R}\right)^n + \left[ \frac{\lambda^2 + 3\lambda\mu}{2\mu(\lambda + \mu)^2} \right. \right. \\ \left. \left. - \frac{r\rho g(\lambda + 3\mu)}{4\mu(\lambda + \mu)^2} \right] \ln\left(\frac{r}{R}\right)^n + \frac{\lambda}{4\mu(\lambda + \mu)} + \frac{r\rho g(2\lambda + \mu)}{8\mu^2(\lambda + \mu)} \right\} \end{aligned}$$

$$+ \frac{\pi Gr^2 \rho}{R g_R \mu} \left\} \left( \frac{r}{R} \right)^n, \quad (7.380a)$$

$$\begin{aligned} z_6'' = R g_R^2 \rho \left\{ \left[ \frac{r \rho g}{4 \mu (\lambda + \mu)} + \frac{1}{2 R g_R \rho} \right] \ln^2 \left( \frac{r}{R} \right)^n + \left[ - \frac{\lambda + 3 \mu}{2 (\lambda + \mu)^2} \right. \right. \\ \left. \left. + \frac{r \rho g (2 \lambda^2 + \lambda \mu - 3 \mu^2)}{4 \mu^2 (\lambda + \mu)^2} + \frac{2 \pi G r^2 \rho}{R g_R \mu} \right] \ln \left( \frac{r}{R} \right)^n + \frac{\lambda + 2 \mu}{4 \mu (\lambda + \mu)} \right. \\ \left. - \frac{r \rho g (2 \lambda + \mu)}{8 \mu^2 (\lambda + \mu)} - \frac{\pi G r^2 \rho}{R g_R \mu} \right\} \left( \frac{r}{R} \right)^n. \end{aligned} \quad (7.380b)$$

According to our asymptotic solutions  $z_1^0 + (1/n)z_1', \dots, z_4^0 + (1/n)z_4'$  and  $z_5^0 + (1/n)z_5' + (1/n^2)z_5'', z_6^0 + (1/n)z_6' + (1/n^2)z_6''$ , we obtain the more accurate asymptotic expressions of the load Love numbers:

$$\begin{cases} \lim_{n \rightarrow \infty} n(h_n' - h_\infty') = \lim_{n \rightarrow \infty} \frac{z_1'(R)}{R} = h_\infty'', \\ \lim_{n \rightarrow \infty} n(l_n' - l_\infty') = \lim_{n \rightarrow \infty} \frac{z_3'(R)}{R} = l_\infty'', \\ \lim_{n \rightarrow \infty} n(k_n' - k_\infty') = \lim_{n \rightarrow \infty} \frac{z_5''(R)}{g_R} = k_\infty'', \end{cases} \quad (7.381)$$

where

$$\begin{aligned} h_\infty'' = \frac{g_R^2}{4 \pi G (\lambda_R + \mu_R)} \left[ - \frac{\mu_R}{\lambda_R + \mu_R} + \frac{R \rho_R g_R (\lambda_R^2 + \lambda_R \mu_R - \mu_R^2)}{2 \mu_R^2 (\lambda_R + \mu_R)} \right. \\ \left. + \frac{2 \pi G R \rho_R (\lambda_R + \mu_R)}{g_R \mu_R} \right], \end{aligned} \quad (7.382a)$$

$$k_\infty'' = \frac{g_R^2}{4 \pi G (\lambda_R + \mu_R)} \left[ - \frac{3 \lambda_R^2 + 8 \lambda_R \mu_R + 3 \mu_R^2}{2 \mu_R (\lambda_R + \mu_R)} + \frac{R \rho_R g_R (\lambda_R + 2 \mu_R)}{2 \mu_R (\lambda_R + \mu_R)} \right], \quad (7.382b)$$

$$k_\infty'' = \frac{R g_R \rho_R}{\mu_R} \left[ \frac{\lambda_R}{4 (\lambda_R + \mu_R)} + \frac{R \rho_R g_R (2 \lambda_R + \mu_R)}{8 \mu_R (\lambda_R + \mu_R)} + \frac{\pi G R \rho_R}{g_R} \right]. \quad (7.382c)$$

Finally, we write the more accurate asymptotic expressions of the load Love numbers in the form

$$h_n' = h_\infty' + \frac{1}{n} h_\infty'', \quad n l_n' = l_\infty' + \frac{1}{n} l_\infty'', \quad n k_n' = k_\infty' + \frac{1}{n} k_\infty'', \quad (7.383)$$

where  $h_\infty', l_\infty'$  and  $k_\infty'$  are given in (7.366), and  $h_\infty'', l_\infty''$  and  $k_\infty''$  in (7.382).



### Application to the Computation of the Load Green's Function

The asymptotic expressions of the load Love numbers can be used to further formulate the Legendre series in the load Green's function to improve their convergence in numerical computation. The closed-form formulae of Legendre series provided in Sect. 4.3.3 will be used to convert Legendre series to closed-form formulae. This approach is referred to as Kummer's transformation. The formulation is quite straightforward.

The surface vertical displacement given in (7.331) can be further formulated as

$$\begin{aligned}\delta u_r(R) &= \frac{Rh'_\infty}{M} \sum_{n=0}^{\infty} P_n(\cos \theta) + \frac{R}{M} \sum_{n=0}^{\infty} (h'_n - h'_\infty) P_n(\cos \theta) \\ &= \frac{Rh'_\infty}{2M \sin(\theta/2)} + \frac{R}{M} \sum_{n=0}^{\infty} (h'_n - h'_\infty) P_n(\cos \theta),\end{aligned}\quad (7.384)$$

where the Legendre series is to be evaluated numerically. It can be further developed as

$$\begin{aligned}\delta u_r(R) &= \frac{Rh'_\infty}{2M \sin(\theta/2)} + \frac{R}{M} (h'_0 - h'_\infty) + \frac{Rh''_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) \\ &\quad + \frac{R}{M} \sum_{n=1}^{\infty} \left( h'_n - h'_\infty - \frac{h''_\infty}{n} \right) P_n(\cos \theta) \\ &= \frac{Rh'_\infty}{2M \sin(\theta/2)} + \frac{R}{M} (h'_0 - h'_\infty) + \frac{Rh''_\infty}{M} \ln \frac{1}{\sin(\theta/2)[1 + \sin(\theta/2)]} \\ &\quad + \frac{R}{M} \sum_{n=1}^{\infty} \left( h'_n - h'_\infty - \frac{h''_\infty}{n} \right) P_n(\cos \theta),\end{aligned}\quad (7.385)$$

where the Legendre series is to be evaluated numerically. The coefficients of the series in (7.385) decreases faster than that in (7.384) by an order of  $O(n^{-1})$ .

The treatment of other quantities are similar. We just provide the formulae as Legendre series. For the Legendre series with a closed-form formulae provided in Sect. 4.3.3, it is left to the readers to replace them by the closed-form formulae.

The horizontal displacement given in (7.331) is

$$\delta u_\theta(R) = \frac{Rl'_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n} \frac{d}{d\theta} P_n(\cos \theta) + \frac{R}{M} \sum_{n=1}^{\infty} \left( l'_n - \frac{l'_\infty}{n} \right) \frac{d}{d\theta} P_n(\cos \theta),\quad (7.386)$$

where the first Legendre series has a closed-form formula, and the second one is to be computed numerically. It can be further developed to

$$\begin{aligned} \delta u_\theta(R) = & \frac{Rl'_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n} \frac{d}{d\theta} P_n(\cos \theta) + \frac{Rl''_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{d}{d\theta} P_n(\cos \theta) \\ & + \frac{R}{M} \sum_{n=1}^{\infty} \left( l'_n - \frac{l'_\infty}{n} - \frac{l''_\infty}{n^2} \right) \frac{d}{d\theta} P_n(\cos \theta), \end{aligned} \quad (7.387)$$

where the first two Legendre series have a closed-form formula, and the third one is to be computed numerically.

The gravitational potential on the Earth's surface,  $\delta\phi_1(R)$ , is, according to (7.332),

$$\delta\phi_1(R) = \frac{Rg_R k'_0}{M} + \frac{Rg_R k'_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) + \frac{Rg_R}{M} \sum_{n=1}^{\infty} \left( k'_n - \frac{k'_\infty}{n} \right) P_n(\cos \theta), \quad (7.388)$$

where the first Legendre series has a closed-form formula, and the second one is to be evaluated numerically. In this formula, the coefficients of the Legendre series to be evaluated numerically is already on the same order of magnitude as in the case of vertical displacement, and this formula is not to be further developed.

The gravity given in (7.333) is

$$\begin{aligned} \delta g^E = & \frac{g_R(k'_\infty - 2h'_\infty)}{M} \sum_{n=0}^{\infty} P_n(\cos \theta) \\ & + \frac{g_R}{M} \sum_{n=0}^{\infty} [(nk'_n - k'_\infty) + k'_n - 2(h'_n - h'_\infty)] P_n(\cos \theta), \end{aligned} \quad (7.389)$$

where the first Legendre series has a closed-form formula, and the second one is to be evaluated numerically. It can be further developed to

$$\begin{aligned} \delta g^E = & \frac{g_R(k'_\infty - 2h'_\infty)}{M} \sum_{n=0}^{\infty} P_n(\cos \theta) + \frac{g_R}{M} [-k'_\infty + k'_0 - 2(h'_0 - h'_\infty)] \\ & + \frac{g_R}{M} (k''_\infty + k'_\infty - 2h''_\infty) \sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) + \frac{g_R}{M} \sum_{n=1}^{\infty} \left[ \left( nk'_n - k'_\infty - \frac{k''_\infty}{n} \right) \right. \\ & \left. + \left( k'_n - \frac{k'_\infty}{n} \right) - 2 \left( h'_n - h'_\infty - \frac{h''_\infty}{n} \right) \right] P_n(\cos \theta). \end{aligned} \quad (7.390)$$

where the first two Legendre series have a closed-form formula, and the third one is to be evaluated numerically.

The deviation of the vertical with respect to the crust defined positive along the direction away from the point mass of load given in (7.334) is

$$\delta\epsilon^E = -\frac{h'_\infty}{M} \sum_{n=1}^{\infty} \frac{d}{d\theta} P_n(\cos\theta) + \frac{1}{M} \sum_{n=1}^{\infty} [k'_n - (h'_n - h'_\infty)] \frac{d}{d\theta} P_n(\cos\theta). \quad (7.391)$$

where the first Legendre series has a closed-form formula, and the second one is to be evaluated numerically. It can be further developed to

$$\begin{aligned} \delta\epsilon^E = & -\frac{h'_\infty}{M} \sum_{n=1}^{\infty} \frac{d}{d\theta} P_n(\cos\theta) + \frac{k'_\infty - h''_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n} \frac{d}{d\theta} P_n(\cos\theta) \\ & + \frac{1}{M} \sum_{n=1}^{\infty} \left[ \left( k'_n - \frac{k'_\infty}{n} \right) - \left( h'_n - h'_\infty - \frac{h''_\infty}{n} \right) \right] \frac{d}{d\theta} P_n(\cos\theta), \end{aligned} \quad (7.392)$$

where the first two Legendre series have a closed-form formula, and the third one is to be evaluated numerically.

The strain component  $\delta\epsilon_{rr}$  given in (7.338) is, with  $\sigma_R = \lambda_R + 2\mu_R$ ,

$$\begin{aligned} \delta\epsilon_{rr} = & -\frac{2\lambda_R}{R\sigma_R} \delta u_r(R) + \frac{\lambda_R l'_\infty}{M\sigma_R} \sum_{n=0}^{\infty} n P_n(\cos\theta) \\ & + \frac{\lambda_R}{M\sigma_R} \sum_{n=0}^{\infty} [n(nl'_n - l'_\infty) + nl'_n] P_n(\cos\theta). \end{aligned} \quad (7.393)$$

where the first Legendre series has a closed-form formula, and the second one is to be evaluated numerically. It can be further developed to

$$\begin{aligned} \delta\epsilon_{rr} = & -\frac{2\lambda_R}{R\sigma_R} \delta u_r(R) + \frac{\lambda_R l'_\infty}{M\sigma_R} \sum_{n=0}^{\infty} n P_n(\cos\theta) + \frac{\lambda_R}{M\sigma_R} (l''_\infty + l'_\infty) \sum_{n=0}^{\infty} P_n(\cos\theta) \\ & + \frac{\lambda_R}{\sigma_R M} \sum_{n=0}^{\infty} \{ [n(nl'_n - l'_\infty) - l''_\infty] + (nl'_n - l'_\infty) \} P_n(\cos\theta), \end{aligned} \quad (7.394)$$

where the first two Legendre series have a closed-form formula, and the third one is to be evaluated numerically.

The strain component  $\delta\epsilon_{\theta\theta}$  given in (7.339) is

$$\delta\epsilon_{\theta\theta} = \frac{1}{R} \delta u_r(R) + \frac{l'_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n} \frac{d^2}{d\theta^2} P_n(\cos\theta) + \frac{1}{M} \sum_{n=1}^{\infty} \left( l'_n - \frac{l'_\infty}{n} \right) \frac{d^2}{d\theta^2} P_n(\cos\theta). \quad (7.395)$$

where the first Legendre series has a closed-form formula, and the second one is to be evaluated numerically. It can be further developed to

$$\begin{aligned} \delta\epsilon_{\theta\theta} = & \frac{1}{R} \delta u_r(R) + \frac{l'_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n} \frac{d^2}{d\theta^2} P_n(\cos \theta) + \frac{l''_\infty}{M} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{d^2}{d\theta^2} P_n(\cos \theta) \\ & + \frac{1}{M} \sum_{n=1}^{\infty} \left( l'_n - \frac{l'_\infty}{n} - \frac{l''_\infty}{n^2} \right) \frac{d^2}{d\theta^2} P_n(\cos \theta), \end{aligned} \quad (7.396)$$

where the first two Legendre series has a closed-form formula, and the third one is to be evaluated numerically.

The strain component  $\delta\epsilon_{\lambda\lambda}$  given in (7.340) can be computed using the results of  $\delta u_r(R)$  and  $\delta u_\theta(R)$ . We copy the formula to here for convenience of reference,

$$\delta\epsilon_{\lambda\lambda} = \frac{1}{R} \delta u_r(R) + \frac{\cot \theta}{R} \delta u_\theta(R). \quad (7.397)$$

All other strain components vanish.

The details of Kummer's transformation using the more accurate asymptotic expressions of the load Love numbers were not explicitly provided by Guo et al. (2004), but they are provided by Martens et al. (2016).

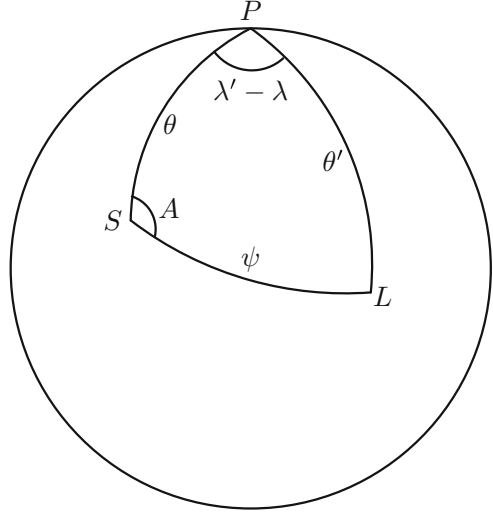
### 7.7.3 The Load Green's Function for Practical Computation of the Loading Effect

We have assumed, in the formulation of the load Green's function, that the unit point mass is located at the north pole. In this subsection, we express the load Green's function in terms of the coordinates of the station and that of the unit point mass of load, which is the form suitable for practical computation of the loading effect. This is done based on the fact that  $\theta$  in the formulation is the geocentric angle between the station and the unit point mass of load. As shown in Fig. 7.11, let  $S$  be the station,  $L$  the unit point mass of load, and  $\psi$  the geocentric angle between  $S$  and  $L$ . The loading effect of  $L$  at  $S$  can then be obtained by replacing  $\theta$  by  $\psi$  in the formulae of the Green's function. We thus obtain, in the spherical coordinate system with  $L$  as the pole, and  $\psi$  the polar angle, the loading effect of displacement and gravitational potential at  $S$ ,

$$\delta u_r(R) = \frac{R}{M} \sum_{n=0}^{\infty} h'_n P_n(\cos \psi), \quad \delta u_\psi(R) = \frac{R}{M} \sum_{n=0}^{\infty} l'_n \frac{d}{d\psi} P_n(\cos \psi), \quad (7.398)$$

$$\delta\phi_1(R) = \frac{R g_R}{M} \sum_{n=0}^{\infty} k'_n P_n(\cos \psi). \quad (7.399)$$

**Fig. 7.11** The position of the unit point mass of load  $L$  with respect to the station  $S$ , where  $A$  is the azimuth of the former with respect to the latter



We use  $(\theta, \lambda)$  to denote the coordinates of the station  $S$ , and  $(\theta', \lambda')$  that of the unit point mass of load  $L$ . We have

$$\cos \psi = \cos \theta' \cos \theta + \sin \theta' \sin \theta \cos(\lambda' - \lambda). \quad (7.400)$$

It can be readily inferred that the  $\theta$ - and  $\lambda$ -components of the horizontal displacement are

$$\delta u_{\theta}(R) = \delta u_{\psi}(R) \cos A, \quad \delta u_{\lambda}(R) = -\delta u_{\psi}(R) \sin A, \quad (7.401)$$

where  $A$  is the azimuth of the unit point mass of load  $L$  with respect to the station  $S$ , as shown in Fig. 7.11, which can be computed according to following fundamental spherical trigonometrical formulae,

$$\sin \psi \sin A = \sin \theta' \sin(\lambda' - \lambda), \quad (7.402)$$

$$\sin \psi \cos A = \cos \theta' \sin \theta - \sin \theta' \cos \theta \cos(\lambda' - \lambda). \quad (7.403)$$

Similarly, the components of the deviation of the vertical with respect to the crust toward south and west are

$$\delta \xi = \delta \epsilon_{\psi}^E \cos A, \quad \delta \eta = \delta \epsilon_{\psi}^E \sin A, \quad (7.404)$$

where  $\delta \epsilon_{\psi}^E$  is the value of  $\delta \epsilon^E$  computed with  $\theta$  replaced by  $\psi$ . It is left to the readers to formula the indirect effect of the deviation of the vertical with respect to the celestial reference system (CRS) at a ground station.

The formulation of the strain components requires to consider the problem in a more general way. Let  $Or\psi\gamma$  be a spherical coordinate system with  $L$  as the pole. The formulae of the load Green's function derived in the preceding subsections can then be used in  $Or\psi\gamma$  with  $\theta$  and  $\lambda$  replaced by  $\psi$  and  $\gamma$ , respectively. Our formulation is hence a coordinate transformation of the quantities from  $Or\psi\gamma$  to  $Or\theta\lambda$ . The relation between the base vectors of  $Or\psi\gamma$  and  $Or\theta\lambda$  at  $S$  can be inferred from Fig. 7.11. The directions of  $\hat{e}_r$ ,  $\hat{e}_\theta$ ,  $\hat{e}_\lambda$ , and  $\hat{e}_\psi$  are evident. The direction of  $\hat{e}_\gamma$  can be inferred from  $\hat{e}_r$  and  $\hat{e}_\psi$  based on the right-hand rule. We have

$$\hat{e}_r = \hat{e}_r, \quad \hat{e}_\psi = \hat{e}_\theta \cos A - \hat{e}_\lambda \sin A, \quad \hat{e}_\gamma = \hat{e}_\theta \sin A + \hat{e}_\lambda \cos A; \quad (7.405)$$

$$\hat{e}_r = \hat{e}_r, \quad \hat{e}_\theta = \hat{e}_\psi \cos A + \hat{e}_\gamma \sin A, \quad \hat{e}_\lambda = -\hat{e}_\psi \sin A + \hat{e}_\gamma \cos A. \quad (7.406)$$

Note that the results (7.401) and (7.404) can be obtained by using (7.405). The strain components in  $Or\theta\lambda$  are computed from those in  $Or\psi\gamma$  according to the coordinate transformation of tensor components. For convenience, we denote the coordinates  $r\theta\lambda$  as  $q'_1q'_2q'_3$ , and  $r\psi\gamma$  as  $q_1q_2q_3$ . We denote the components of the strain in  $q'_1q'_2q'_3$  as  $\delta\epsilon'_{mn}$ , and in  $q_1q_2q_3$  as  $\delta\epsilon_{ij}$ . We then have

$$\delta\epsilon'_{mn} = n_{mi}n_{nj}\delta\epsilon_{ij}, \quad (7.407)$$

where summation over repeated indices is assumed,  $n_{mi}$  is the cosine of the angle between  $q'_m$  and  $q_i$ , and  $n_{nj}$  is defined similarly. According to (7.405) or (7.406), we have

$$\begin{cases} n_{11} = 1, n_{12} = 0, & n_{13} = 0; \\ n_{21} = 0, n_{22} = \cos A, & n_{23} = \sin A; \\ n_{31} = 0, n_{32} = -\sin A, & n_{33} = \cos A. \end{cases} \quad (7.408)$$

In  $Or\psi\gamma$ , only the strain components  $\delta\epsilon_{rr}$ ,  $\delta\epsilon_{\psi\psi}$  and  $\delta\epsilon_{\gamma\gamma}$  are non-vanishing, which are in fact respectively the values of  $\delta\epsilon_{rr}$ ,  $\delta\epsilon_{\theta\theta}$  and  $\delta\epsilon_{\lambda\lambda}$  formulated in the preceding subsections and computed with  $\theta$  and  $\lambda$  replaced by  $\psi$  and  $\gamma$ , respectively. Hence,  $\delta\epsilon_{ij} = 0$  when  $i \neq j$ , and we can simplify (7.407) to

$$\delta\epsilon'_{mn} = n_{m1}n_{n1}\delta\epsilon_{11} + n_{m2}n_{n2}\delta\epsilon_{22} + n_{m3}n_{n3}\delta\epsilon_{33}. \quad (7.409)$$

Finally, we obtain, written in terms of the coordinates  $r\theta\lambda$  and  $r\psi\gamma$ ,

$$\begin{cases} \delta\epsilon_{r\theta} = \delta\epsilon_{\theta r} = 0, & \delta\epsilon_{r\lambda} = \delta\epsilon_{\lambda r} = 0; \\ \delta\epsilon_{\theta\theta} = \delta\epsilon_{\psi\psi} \cos^2 A + \delta\epsilon_{\gamma\gamma} \sin^2 A, & \delta\epsilon_{\theta\lambda} = (-\delta\epsilon_{\psi\psi} + \delta\epsilon_{\gamma\gamma}) \sin A \cos A; \\ \delta\epsilon_{\lambda\theta} = (-\delta\epsilon_{\psi\psi} + \delta\epsilon_{\gamma\gamma}) \sin A \cos A, & \delta\epsilon_{\lambda\lambda} = \delta\epsilon_{\psi\psi} \sin^2 A + \delta\epsilon_{\gamma\gamma} \cos^2 A, \end{cases} \quad (7.410)$$

which are symmetrical as they should be.

The loading effect of the mass of load distributed on the Earth's surface is the convolution of the load Green's function with the density of the mass of load. For the gravity, we have

$$\delta g_t^E(\theta, \lambda) = R^2 \int_0^\pi \int_0^{2\pi} \delta s(\theta', \lambda') \delta g^E(\psi) \sin \theta' d\theta' d\lambda', \quad (7.411)$$

where we used the subscript  $t$  to indicate the total contribution from the mass on the Earth's surface,  $\psi$  is a function of  $\theta, \lambda$  and  $\theta', \lambda'$  given in (7.400), and  $\delta g^E(\psi)$  is the value of  $\delta g^E$  formulated in the preceding subsections and computed with  $\theta$  replaced by  $\psi$ . For the horizontal displacement, we have

$$\begin{Bmatrix} \delta u_{\theta t}(R) \\ \delta u_{\lambda t}(R) \end{Bmatrix} = R^2 \int_0^\pi \int_0^{2\pi} \delta s(\theta', \lambda') \delta u_\psi(R) \begin{Bmatrix} \cos A \\ -\sin A \end{Bmatrix} \sin \theta' d\theta' d\lambda', \quad (7.412)$$

where  $A$  is a function of  $\theta, \lambda$  and  $\theta', \lambda'$  given in (7.402) and (7.403). The formulae for other quantities are similar to one of the above two formulae, and their derivation is left to the readers.

## 7.8 Computation of Love Numbers and Load Green's Function

### 7.8.1 Static and Dynamic Love Numbers

The Love numbers are computed by solving a system of linear ordinary differential equations with relevant boundary conditions. The general procedure of computation is described in Sect. 7.3.2 for the dynamic tidal Love numbers. The computations in other cases, the static tidal Love number, the dynamic and static load Love numbers, are similar. Note that the degree 0 and degree 1 should be handled particularly. For the degree 0, the ordinary differential equations degenerates to 6.216, where  $\sigma = 0$  for the static load Love numbers. For the degree 1, particular care should be taken to the definition of the Earth center as to be discussed in the next subsection.

The same as for the case of spheroidal free oscillation in Sect. 6.4.3, we use the Chebyshev collocation method presented in Sect. 4.9.5 to solve the ordinary differential equations (Guo et al., 2001; Bos et al., 2014). To circumvent the singularity of the equations at the Earth center, we adopt the Gauss-Radau collocation points in the inner core with boundary conditions prescribed at the inner core boundary (ICB). We then use the Gauss-Lobatto collocation points in the other layers for the solutions to be transferred from layer to layer using the boundary conditions. We mention again the difference of the equations in the core between the dynamic and static Love numbers and the need of care for degree 0 and 1, e.g., for the degree 1 static load Love numbers, the equation should be solved only from the core-mantle boundary (CMB) to the surface of the Earth as formulated in Sect. 7.6.3.

In the literature, the static load Love numbers are also computed following an approach described by Longman (1963), which imposes homogeneity and

adiabatic stratification in the outer core. The author has programmed both cases, i.e., the Smylie and Mansinha approach as formulated in Sects. 7.6.2 and 7.6.3 and the Longman approach. We list in Table 7.3 some low degree values computed according to both approaches using the same physical parameters for PREM ( $G = 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$  and 200s seismic wave velocity). The differences between the results are small. They are the same up to 10 significant digits at degree 20 and beyond. The largest differences is in the values of  $k'_n$ , whose relative differences from degree 2 to 8 are 1.1%, 2.4%, 1.1%, 0.12%, 0.16%, 0.052%, and 0.016%, respectively. Therefore, even though the Longman approach is physically inconsistent with static deformation, the results computed according to it can still be used for practical applications up to an accuracy within the differences.

We also make a comparison between the static and dynamic load Love numbers. The relative differences in percentage between the dynamic and static load Love numbers are listed in Table 7.4. The dynamic load Love numbers are computed with periods of 10, 20, 50, and 100 times the periods of the fundamental modes of spheroidal free oscillations. As the physics of the static deformation is different from that of the dynamics deformations, it is expected that the dynamic load Love numbers do not converge to the static load Love numbers exactly. In fact, the asymptotic values of the dynamic load Love numbers with very long periods are closer to the static load Love numbers computed according to the Longman approach. The reason is that the Longman approach is a compromise of the dynamic deformation assuming the fluid outer core to be neutrally stratified. Even though the stratification of the fluid outer core is expected to be close to neutral, assuming it to be strictly neutral has no firm physical evidence. The relative differences between the dynamic load Love numbers computed with periods of 50 and 100 times the periods of the fundamental modes of spheroidal free oscillations are all within 0.04%.

Finally, we provide in Table 7.5 some static load Love numbers beyond those of Table 7.4. The asymptotic formulae are

$$\begin{cases} h'_n = (-6.232755633 \times 10^{+0}) + (6.261527398 \times 10^{+0})/n, \\ l'_n = (1.890974650 \times 10^{+0})/n + (-2.594842435 \times 10^{-1})/n^2, \\ k'_n = (-2.696795193 \times 10^{+0})/n + (8.125146193 \times 10^{+0})/n^2. \end{cases} \quad (7.413)$$

## 7.8.2 Load Green's Function

For computing the Legendre series in the load Green's function, we need to compute  $P_n(\cos \theta)$ ,  $(d/d\theta)P_n(\cos \theta)$ , and  $(d^2/d\theta^2)P_n(\cos \theta)$ . These can be readily inferred from the recurrence formulae (4.113) and (4.116),

$$P_{n+1}(\cos \theta) = \frac{2n+1}{n+1} \cos \theta P_n(\cos \theta) + \frac{n}{n+1} P_{n-1}(\cos \theta), \quad (7.414)$$



Table 7.3 Some low degree static load Love numbers: a comparison between the approaches

| Smylie and Mansinha |                                 | Longman                        |                                 | $k'_n$                          | $h'_n$                         | $l'_n$                         | $k''_n$                         | $h''_n$                         | $l''_n$                        | $k'''_n$                        |
|---------------------|---------------------------------|--------------------------------|---------------------------------|---------------------------------|--------------------------------|--------------------------------|---------------------------------|---------------------------------|--------------------------------|---------------------------------|
| $n$                 | $h'_n$                          | $l'_n$                         | $k'_n$                          | $h'_n$                          | $l'_n$                         | $l'_n$                         | $k'_n$                          | $h'_n$                          | $l'_n$                         | $k'_n$                          |
| 0                   | -1.321710713 × 10 <sup>-1</sup> | 0.000000000 × 10 <sup>+0</sup> | 0.000000000 × 10 <sup>+0</sup>  | -1.321710713 × 10 <sup>-1</sup> | 0.000000000 × 10 <sup>+0</sup> | 0.000000000 × 10 <sup>+0</sup> | 0.000000000 × 10 <sup>+0</sup>  | -1.321710713 × 10 <sup>-1</sup> | 0.000000000 × 10 <sup>+0</sup> | 0.000000000 × 10 <sup>+0</sup>  |
| 1                   | -2.861682826 × 10 <sup>-1</sup> | 1.024158768 × 10 <sup>-1</sup> | 0.000000000 × 10 <sup>+0</sup>  | -2.861445611 × 10 <sup>-1</sup> | 1.024396800 × 10 <sup>-1</sup> | 1.024396800 × 10 <sup>-1</sup> | 0.000000000 × 10 <sup>+0</sup>  | -2.861445611 × 10 <sup>-1</sup> | 1.024396800 × 10 <sup>-1</sup> | 0.000000000 × 10 <sup>+0</sup>  |
| 2                   | -9.970648449 × 10 <sup>-1</sup> | 2.170906920 × 10 <sup>-2</sup> | -3.101007855 × 10 <sup>-1</sup> | -9.940409150 × 10 <sup>-1</sup> | 2.198444132 × 10 <sup>-2</sup> | 2.198444132 × 10 <sup>-2</sup> | -3.066652912 × 10 <sup>-1</sup> | -9.940409150 × 10 <sup>-1</sup> | 2.198444132 × 10 <sup>-2</sup> | -3.066652912 × 10 <sup>-1</sup> |
| 3                   | -1.058460879 × 10 <sup>+0</sup> | 6.914071177 × 10 <sup>-2</sup> | -2.018688182 × 10 <sup>-1</sup> | -1.054513972 × 10 <sup>+0</sup> | 6.892743542 × 10 <sup>-2</sup> | 6.892743542 × 10 <sup>-2</sup> | -1.971726116 × 10 <sup>-1</sup> | -1.054513972 × 10 <sup>+0</sup> | 6.892743542 × 10 <sup>-2</sup> | -1.971726116 × 10 <sup>-1</sup> |
| 4                   | -1.059599771 × 10 <sup>+0</sup> | 5.779346823 × 10 <sup>-2</sup> | -1.362282219 × 10 <sup>-1</sup> | -1.058395016 × 10 <sup>+0</sup> | 5.768775860 × 10 <sup>-2</sup> | 5.768775860 × 10 <sup>-2</sup> | -1.346843384 × 10 <sup>-1</sup> | -1.058395016 × 10 <sup>+0</sup> | 5.768775860 × 10 <sup>-2</sup> | -1.346843384 × 10 <sup>-1</sup> |
| 5                   | -1.092508626 × 10 <sup>+0</sup> | 4.513459661 × 10 <sup>-2</sup> | -1.055392639 × 10 <sup>-1</sup> | -1.092600757 × 10 <sup>+0</sup> | 4.514337019 × 10 <sup>-2</sup> | 4.514337019 × 10 <sup>-2</sup> | -1.056639731 × 10 <sup>-1</sup> | -1.092600757 × 10 <sup>+0</sup> | 4.514337019 × 10 <sup>-2</sup> | -1.056639731 × 10 <sup>-1</sup> |
| 6                   | -1.151590935 × 10 <sup>+0</sup> | 3.757182519 × 10 <sup>-2</sup> | -9.141629191 × 10 <sup>-2</sup> | -1.151485902 × 10 <sup>+0</sup> | 3.756187100 × 10 <sup>-2</sup> | 3.756187100 × 10 <sup>-2</sup> | -9.126976884 × 10 <sup>-2</sup> | -1.151485902 × 10 <sup>+0</sup> | 3.756187100 × 10 <sup>-2</sup> | -9.126976884 × 10 <sup>-2</sup> |
| 7                   | -1.221462789 × 10 <sup>+0</sup> | 3.319982777 × 10 <sup>-2</sup> | -8.304136404 × 10 <sup>-2</sup> | -1.221432282 × 10 <sup>+0</sup> | 3.319703852 × 10 <sup>-2</sup> | 3.319703852 × 10 <sup>-2</sup> | -8.299823741 × 10 <sup>-2</sup> | -1.221432282 × 10 <sup>+0</sup> | 3.319703852 × 10 <sup>-2</sup> | -8.299823741 × 10 <sup>-2</sup> |
| 8                   | -1.294421216 × 10 <sup>+0</sup> | 3.051299633 × 10 <sup>-2</sup> | -7.749765222 × 10 <sup>-2</sup> | -1.294412366 × 10 <sup>+0</sup> | 3.051222631 × 10 <sup>-2</sup> | 3.051222631 × 10 <sup>-2</sup> | -7.748509737 × 10 <sup>-2</sup> | -1.294412366 × 10 <sup>+0</sup> | 3.051222631 × 10 <sup>-2</sup> | -7.748509737 × 10 <sup>-2</sup> |
| 9                   | -1.366689516 × 10 <sup>+0</sup> | 2.871013614 × 10 <sup>-2</sup> | -7.337699534 × 10 <sup>-2</sup> | -1.366686947 × 10 <sup>+0</sup> | 2.870992477 × 10 <sup>-2</sup> | 2.870992477 × 10 <sup>-2</sup> | -7.337335821 × 10 <sup>-2</sup> | -1.366686947 × 10 <sup>+0</sup> | 2.870992477 × 10 <sup>-2</sup> | -7.337335821 × 10 <sup>-2</sup> |
| 10                  | -1.436749404 × 10 <sup>+0</sup> | 2.741319010 × 10 <sup>-2</sup> | -7.007575351 × 10 <sup>-2</sup> | -1.436748658 × 10 <sup>+0</sup> | 2.741313217 × 10 <sup>-2</sup> | 2.741313217 × 10 <sup>-2</sup> | -7.007470204 × 10 <sup>-2</sup> | -1.436748658 × 10 <sup>+0</sup> | 2.741313217 × 10 <sup>-2</sup> | -7.007470204 × 10 <sup>-2</sup> |
| 11                  | -1.504092660 × 10 <sup>+0</sup> | 2.643788451 × 10 <sup>-2</sup> | -6.730503316 × 10 <sup>-2</sup> | -1.504092443 × 10 <sup>+0</sup> | 2.643786863 × 10 <sup>-2</sup> | 2.643786863 × 10 <sup>-2</sup> | -6.730472946 × 10 <sup>-2</sup> | -1.504092443 × 10 <sup>+0</sup> | 2.643786863 × 10 <sup>-2</sup> | -6.730472946 × 10 <sup>-2</sup> |
| 12                  | -1.568613521 × 10 <sup>+0</sup> | 2.568406846 × 10 <sup>-2</sup> | -6.490731869 × 10 <sup>-2</sup> | -1.568613458 × 10 <sup>+0</sup> | 2.568406410 × 10 <sup>-2</sup> | 2.568406410 × 10 <sup>-2</sup> | -6.490723101 × 10 <sup>-2</sup> | -1.568613458 × 10 <sup>+0</sup> | 2.568406410 × 10 <sup>-2</sup> | -6.490723101 × 10 <sup>-2</sup> |
| 13                  | -1.630345520 × 10 <sup>+0</sup> | 2.508940783 × 10 <sup>-2</sup> | -6.278579853 × 10 <sup>-2</sup> | -1.630345501 × 10 <sup>+0</sup> | 2.508940663 × 10 <sup>-2</sup> | 2.508940663 × 10 <sup>-2</sup> | -6.278577322 × 10 <sup>-2</sup> | -1.630345501 × 10 <sup>+0</sup> | 2.508940663 × 10 <sup>-2</sup> | -6.278577322 × 10 <sup>-2</sup> |
| 14                  | -1.689359980 × 10 <sup>+0</sup> | 2.461067843 × 10 <sup>-2</sup> | -6.087553361 × 10 <sup>-2</sup> | -1.689359974 × 10 <sup>+0</sup> | 2.461067809 × 10 <sup>-2</sup> | 2.461067809 × 10 <sup>-2</sup> | -6.087552630 × 10 <sup>-2</sup> | -1.689359974 × 10 <sup>+0</sup> | 2.461067809 × 10 <sup>-2</sup> | -6.087552630 × 10 <sup>-2</sup> |
| 15                  | -1.745732873 × 10 <sup>+0</sup> | 2.421603141 × 10 <sup>-2</sup> | -5.913054586 × 10 <sup>-2</sup> | -1.745732871 × 10 <sup>+0</sup> | 2.421603132 × 10 <sup>-2</sup> | 2.421603132 × 10 <sup>-2</sup> | -5.913054375 × 10 <sup>-2</sup> | -1.745732871 × 10 <sup>+0</sup> | 2.421603132 × 10 <sup>-2</sup> | -5.913054375 × 10 <sup>-2</sup> |
| 16                  | -1.799537522 × 10 <sup>+0</sup> | 2.388142802 × 10 <sup>-2</sup> | -5.751727408 × 10 <sup>-2</sup> | -1.799537522 × 10 <sup>+0</sup> | 2.388142800 × 10 <sup>-2</sup> | 2.388142800 × 10 <sup>-2</sup> | -5.751727347 × 10 <sup>-2</sup> | -1.799537522 × 10 <sup>+0</sup> | 2.388142800 × 10 <sup>-2</sup> | -5.751727347 × 10 <sup>-2</sup> |
| 17                  | -1.850845521 × 10 <sup>+0</sup> | 2.358866479 × 10 <sup>-2</sup> | -5.601080413 × 10 <sup>-2</sup> | -1.850845521 × 10 <sup>+0</sup> | 2.358866478 × 10 <sup>-2</sup> | 2.358866478 × 10 <sup>-2</sup> | -5.601080395 × 10 <sup>-2</sup> | -1.850845521 × 10 <sup>+0</sup> | 2.358866478 × 10 <sup>-2</sup> | -5.601080395 × 10 <sup>-2</sup> |
| 18                  | -1.899729257 × 10 <sup>+0</sup> | 2.332405844 × 10 <sup>-2</sup> | -5.459245949 × 10 <sup>-2</sup> | -1.899729257 × 10 <sup>+0</sup> | 2.332405844 × 10 <sup>-2</sup> | 2.332405844 × 10 <sup>-2</sup> | -5.459245944 × 10 <sup>-2</sup> | -1.899729257 × 10 <sup>+0</sup> | 2.332405844 × 10 <sup>-2</sup> | -5.459245944 × 10 <sup>-2</sup> |
| 19                  | -1.946263929 × 10 <sup>+0</sup> | 2.307746099 × 10 <sup>-2</sup> | -5.324814997 × 10 <sup>-2</sup> | -1.946263929 × 10 <sup>+0</sup> | 2.307746099 × 10 <sup>-2</sup> | 2.307746099 × 10 <sup>-2</sup> | -5.324814995 × 10 <sup>-2</sup> | -1.946263929 × 10 <sup>+0</sup> | 2.307746099 × 10 <sup>-2</sup> | -5.324814995 × 10 <sup>-2</sup> |
| 20                  | -1.990528611 × 10 <sup>+0</sup> | 2.284148266 × 10 <sup>-2</sup> | -5.196719151 × 10 <sup>-2</sup> | -1.990528611 × 10 <sup>+0</sup> | 2.284148266 × 10 <sup>-2</sup> | 2.284148266 × 10 <sup>-2</sup> | -5.196719151 × 10 <sup>-2</sup> | -1.990528611 × 10 <sup>+0</sup> | 2.284148266 × 10 <sup>-2</sup> | -5.196719151 × 10 <sup>-2</sup> |
| 21                  | -2.032606479 × 10 <sup>+0</sup> | 2.261086946 × 10 <sup>-2</sup> | -5.074144238 × 10 <sup>-2</sup> | -2.032606479 × 10 <sup>+0</sup> | 2.261086946 × 10 <sup>-2</sup> | 2.261086946 × 10 <sup>-2</sup> | -5.074144238 × 10 <sup>-2</sup> | -2.032606479 × 10 <sup>+0</sup> | 2.261086946 × 10 <sup>-2</sup> | -5.074144238 × 10 <sup>-2</sup> |
| 22                  | -2.072584435 × 10 <sup>+0</sup> | 2.238200467 × 10 <sup>-2</sup> | -4.956466214 × 10 <sup>-2</sup> | -2.072584435 × 10 <sup>+0</sup> | 2.238200467 × 10 <sup>-2</sup> | 2.238200467 × 10 <sup>-2</sup> | -4.956466214 × 10 <sup>-2</sup> | -2.072584435 × 10 <sup>+0</sup> | 2.238200467 × 10 <sup>-2</sup> | -4.956466214 × 10 <sup>-2</sup> |
| 23                  | -2.110552356 × 10 <sup>+0</sup> | 2.215251197 × 10 <sup>-2</sup> | -4.843203225 × 10 <sup>-2</sup> | -2.110552356 × 10 <sup>+0</sup> | 2.215251197 × 10 <sup>-2</sup> | 2.215251197 × 10 <sup>-2</sup> | -4.843203225 × 10 <sup>-2</sup> | -2.110552356 × 10 <sup>+0</sup> | 2.215251197 × 10 <sup>-2</sup> | -4.843203225 × 10 <sup>-2</sup> |
| 24                  | -2.146602143 × 10 <sup>+0</sup> | 2.192094182 × 10 <sup>-2</sup> | -4.733979613 × 10 <sup>-2</sup> | -2.146602143 × 10 <sup>+0</sup> | 2.192094182 × 10 <sup>-2</sup> | 2.192094182 × 10 <sup>-2</sup> | -4.733979613 × 10 <sup>-2</sup> | -2.146602143 × 10 <sup>+0</sup> | 2.192094182 × 10 <sup>-2</sup> | -4.733979613 × 10 <sup>-2</sup> |
| 25                  | -2.180826720 × 10 <sup>+0</sup> | 2.168652572 × 10 <sup>-2</sup> | -4.628498838 × 10 <sup>-2</sup> | -2.180826720 × 10 <sup>+0</sup> | 2.168652572 × 10 <sup>-2</sup> | 2.168652572 × 10 <sup>-2</sup> | -4.628498838 × 10 <sup>-2</sup> | -2.180826720 × 10 <sup>+0</sup> | 2.168652572 × 10 <sup>-2</sup> | -4.628498838 × 10 <sup>-2</sup> |

**Table 7.4** Relative difference in percentage between the static load Love numbers and dynamic load Love numbers computed with periods of 10, 20, 50, and 100 times the periods of the fundamental modes of spheroidal free oscillation

| $n$ | 10×                  |                      | 20×                  |                      | 50×                  |                      | 100×                 |                      |
|-----|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
|     | $h'_n$               | $l'_n$               | $h'_n$               | $l'_n$               | $h'_n$               | $l'_n$               | $h'_n$               | $l'_n$               |
| 0   | $6.4 \times 10^{-1}$ |                      | $1.6 \times 10^{-1}$ |                      | $2.5 \times 10^{-2}$ |                      | $6.4 \times 10^{-3}$ |                      |
| 1   | $1.8 \times 10^{-2}$ | $1.1 \times 10^{-2}$ | $1.3 \times 10^{-3}$ | $3.9 \times 10^{-3}$ | $1.7 \times 10^{-2}$ | $1.9 \times 10^{-2}$ | $9.6 \times 10^{-3}$ | $1.1 \times 10^{-2}$ |
| 2   | $3.5 \times 10^{-1}$ | $6.9 \times 10^{-1}$ | $1.4 \times 10^{-1}$ | $1.1 \times 10^{+0}$ | $2.7 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.1 \times 10^{+0}$ | $1.3 \times 10^{+0}$ |
| 3   | $2.0 \times 10^{-1}$ | $6.6 \times 10^{-1}$ | $2.3 \times 10^{-1}$ | $6.2 \times 10^{-2}$ | $3.5 \times 10^{+0}$ | $2.7 \times 10^{-1}$ | $2.3 \times 10^{+0}$ | $3.0 \times 10^{-1}$ |
| 4   | $3.9 \times 10^{-1}$ | $8.8 \times 10^{-1}$ | $1.1 \times 10^{-2}$ | $8.5 \times 10^{-2}$ | $9.5 \times 10^{-1}$ | $9.3 \times 10^{-2}$ | $1.1 \times 10^{+0}$ | $1.7 \times 10^{-1}$ |
| 5   | $4.7 \times 10^{-1}$ | $1.1 \times 10^{+0}$ | $1.2 \times 10^{-1}$ | $2.9 \times 10^{-1}$ | $2.9 \times 10^{-1}$ | $2.7 \times 10^{-2}$ | $1.5 \times 10^{-1}$ | $1.3 \times 10^{-2}$ |
| 6   | $4.6 \times 10^{-1}$ | $1.1 \times 10^{+0}$ | $1.1 \times 10^{-1}$ | $2.5 \times 10^{-1}$ | $9.3 \times 10^{-3}$ | $9.6 \times 10^{-3}$ | $1.3 \times 10^{-1}$ | $4.4 \times 10^{-3}$ |
| 7   | $4.9 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.2 \times 10^{-1}$ | $2.8 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $1.7 \times 10^{-2}$ | $2.4 \times 10^{-2}$ | $2.4 \times 10^{-3}$ |
| 8   | $5.2 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.3 \times 10^{-1}$ | $3.0 \times 10^{-1}$ | $1.6 \times 10^{-1}$ | $2.0 \times 10^{-2}$ | $1.3 \times 10^{-2}$ | $4.5 \times 10^{-3}$ |
| 9   | $5.4 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.4 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $1.8 \times 10^{-1}$ | $2.1 \times 10^{-2}$ | $2.5 \times 10^{-2}$ | $5.2 \times 10^{-3}$ |
| 10  | $5.6 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.4 \times 10^{-1}$ | $3.2 \times 10^{-1}$ | $1.9 \times 10^{-1}$ | $2.2 \times 10^{-2}$ | $2.9 \times 10^{-2}$ | $5.5 \times 10^{-3}$ |
| 11  | $5.7 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.4 \times 10^{-1}$ | $3.2 \times 10^{-1}$ | $1.9 \times 10^{-1}$ | $2.3 \times 10^{-2}$ | $3.0 \times 10^{-2}$ | $5.7 \times 10^{-3}$ |
| 12  | $5.8 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.4 \times 10^{-1}$ | $3.2 \times 10^{-1}$ | $1.9 \times 10^{-1}$ | $2.3 \times 10^{-2}$ | $3.1 \times 10^{-2}$ | $5.8 \times 10^{-3}$ |
| 13  | $5.9 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.5 \times 10^{-1}$ | $3.2 \times 10^{-1}$ | $1.9 \times 10^{-1}$ | $2.3 \times 10^{-2}$ | $3.1 \times 10^{-2}$ | $5.9 \times 10^{-3}$ |
| 14  | $5.9 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.5 \times 10^{-1}$ | $3.2 \times 10^{-1}$ | $1.9 \times 10^{-1}$ | $2.4 \times 10^{-2}$ | $3.1 \times 10^{-2}$ | $5.9 \times 10^{-3}$ |
| 15  | $6.0 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.5 \times 10^{-1}$ | $3.2 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.4 \times 10^{-2}$ | $3.1 \times 10^{-2}$ | $6.0 \times 10^{-3}$ |
| 16  | $6.1 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.5 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.4 \times 10^{-2}$ | $3.1 \times 10^{-2}$ | $6.1 \times 10^{-3}$ |
| 17  | $6.1 \times 10^{-1}$ | $1.3 \times 10^{+0}$ | $1.5 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.1 \times 10^{-3}$ |
| 18  | $6.2 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.5 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.2 \times 10^{-3}$ |
| 19  | $6.2 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.6 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.2 \times 10^{-3}$ |
| 20  | $6.3 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.6 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.3 \times 10^{-3}$ |
| 21  | $6.3 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.6 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.3 \times 10^{-3}$ |
| 22  | $6.3 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.6 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.3 \times 10^{-3}$ |
| 23  | $6.3 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.6 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.3 \times 10^{-3}$ |
| 24  | $6.3 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.6 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.3 \times 10^{-3}$ |
| 25  | $6.4 \times 10^{-1}$ | $1.2 \times 10^{+0}$ | $1.6 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.5 \times 10^{-2}$ | $3.2 \times 10^{-2}$ | $6.4 \times 10^{-3}$ |

**Table 7.5** Some static load Love numbers

| $n$    | $h'_n$                        | $l'_n$                       | $k'_n$                        |
|--------|-------------------------------|------------------------------|-------------------------------|
| 30     | $-2.327778490 \times 10^{+0}$ | $2.047186613 \times 10^{-2}$ | $-4.150216829 \times 10^{-2}$ |
| 40     | $-2.532897280 \times 10^{+0}$ | $1.801455599 \times 10^{-2}$ | $-3.392147925 \times 10^{-2}$ |
| 60     | $-2.764334677 \times 10^{+0}$ | $1.396531545 \times 10^{-2}$ | $-2.411625099 \times 10^{-2}$ |
| 100    | $-3.010125134 \times 10^{+0}$ | $9.126858299 \times 10^{-3}$ | $-1.467184848 \times 10^{-2}$ |
| 160    | $-3.293268467 \times 10^{+0}$ | $5.715777059 \times 10^{-3}$ | $-9.156383325 \times 10^{-3}$ |
| 240    | $-3.673429189 \times 10^{+0}$ | $3.792006470 \times 10^{-3}$ | $-6.362939148 \times 10^{-3}$ |
| 340    | $-4.142520210 \times 10^{+0}$ | $2.827816880 \times 10^{-3}$ | $-4.924196299 \times 10^{-3}$ |
| 500    | $-4.802452167 \times 10^{+0}$ | $2.282081823 \times 10^{-3}$ | $-3.901900151 \times 10^{-3}$ |
| 700    | $-5.404978328 \times 10^{+0}$ | $1.996251175 \times 10^{-3}$ | $-3.206348153 \times 10^{-3}$ |
| 1000   | $-5.903411404 \times 10^{+0}$ | $1.671117517 \times 10^{-3}$ | $-2.508032379 \times 10^{-3}$ |
| 1500   | $-6.171131360 \times 10^{+0}$ | $1.231356117 \times 10^{-3}$ | $-1.772101521 \times 10^{-3}$ |
| 2500   | $-6.229005219 \times 10^{+0}$ | $7.559253238 \times 10^{-4}$ | $-1.077121609 \times 10^{-3}$ |
| 4000   | $-6.231191150 \times 10^{+0}$ | $4.727272176 \times 10^{-4}$ | $-6.736912102 \times 10^{-4}$ |
| 6000   | $-6.231713578 \times 10^{+0}$ | $3.151553364 \times 10^{-4}$ | $-4.492403507 \times 10^{-4}$ |
| 8000   | $-6.231973805 \times 10^{+0}$ | $2.363678202 \times 10^{-4}$ | $-3.369725212 \times 10^{-4}$ |
| 10,000 | $-6.232130032 \times 10^{+0}$ | $1.890948924 \times 10^{-4}$ | $-2.695983074 \times 10^{-4}$ |

$$\frac{d}{d\theta} P_{n+1}(\cos \theta) = \frac{d}{d\theta} P_{n-1}(\cos \theta) - (2n+1) \sin \theta P_n(\cos \theta), \quad (7.415)$$

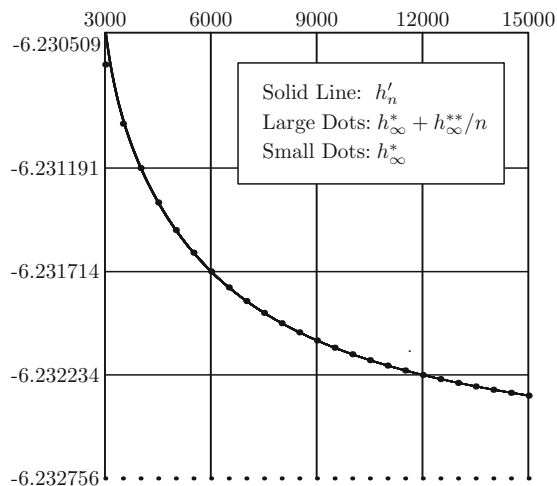
$$\begin{aligned} \frac{d^2}{d\theta^2} P_{n+1}(\cos \theta) &= \frac{d^2}{d\theta^2} P_{n-1}(\cos \theta) \\ &\quad - (2n+1) \left[ \cos \theta P_n(\cos \theta) + \sin \theta \frac{d}{d\theta} P_n(\cos \theta) \right], \end{aligned} \quad (7.416)$$

The initial values for degrees 0 and 1 can be assigned with  $P_0(\cos \theta) = 1$  and  $P_1(\cos \theta) = \cos \theta$ . It is noted that the division by  $\sin \theta$  or  $\cos \theta$  would amplify the error and thus should be avoided. Based on the above formulae, we can infer that  $P_n(\cos \theta) \sim O(1)$ ,  $(d/d\theta)P_n(\cos \theta) \sim O(n)$ , and  $(d^2/d\theta^2)P_n(\cos \theta) \sim O(n^2)$ .

As formulated in Sect. 7.7.2, asymptotic expressions of the load Love numbers are required in the computation of the load Green's function according to Kummer's transformation. We have extended Farrell's (1972) asymptotic theory to included one more term, resulting in an improvement of accuracy by an order of magnitude  $1/n$ . Beyond  $n > 4000$ , our asymptotic values agree with the true values up to seven significant digits. As a demonstration of the improvement of our asymptotic theory, a comparison between Farrell's and our improved asymptotic values of  $h'_n$  is given in Fig. 7.12. However, as to be detailed in the next paragraph, some of the Legendre series of the load Green's function are still divergent, and further care must be taken.

We focus on the evaluation of the Legendre series in the last term of the expressions of the functions concerned; the other terms can be computed using

**Fig. 7.12** Comparison between the classical and our improved asymptotic values of  $h'_n$



closed-form formulae. For  $\delta u_r(R)$  given by (7.385), as  $h'_n - h'_\infty - h''_\infty/n \sim O(1/n^2)$  and  $P_n(\cos \theta) \sim O(1)$ , the series behave like  $\sum_{n=1}^{\infty} O(1/n^2) \times O(1) = \sum_{n=1}^{\infty} O(1/n^2)$ . After analyzing the other quantities similarly, we put the result together as

- for  $\delta u_r(R)$ ,  $\sum_{n=1}^{\infty} O(1/n^2) \times O(1) = \sum_{n=1}^{\infty} O(1/n^2)$ ;
- for  $\delta u_\theta(R)$ ,  $\sum_{n=1}^{\infty} O(1/n^3) \times O(n) = \sum_{n=1}^{\infty} O(1/n^2)$ ;
- for  $\delta \phi_1(R)$ ,  $\sum_{n=1}^{\infty} O(1/n^2) \times O(1) = \sum_{n=1}^{\infty} O(1/n^2)$ ;
- for  $\delta g^E$ ,  $\sum_{n=1}^{\infty} O(1/n^2) \times O(1) = \sum_{n=1}^{\infty} O(1/n^2)$ ;
- for  $\delta \epsilon^E$ ,  $\sum_{n=1}^{\infty} O(1/n^2) \times O(n) = \sum_{n=1}^{\infty} O(1/n)$ ;
- for  $\delta \epsilon_{rr}$ ,  $\sum_{n=1}^{\infty} O(1/n) \times O(1) = \sum_{n=1}^{\infty} O(1/n)$ ;
- for  $\delta \epsilon_{\theta\theta}$ ,  $\sum_{n=1}^{\infty} O(1/n^3) \times O(n^2) = \sum_{n=1}^{\infty} O(1/n)$ .

The series on the order of  $\sum_{n=1}^{\infty} O(1/n^2)$  converge, but those on the order of  $\sum_{n=1}^{\infty} O(1/n)$  do not. Even though we reduced the magnitude of the terms of

the Legendre series by a factor of  $1/n$ , some further techniques like the Euler transformation that Farrell (1972) used is still desired.

We apply the Euler transformation in the following way. We separate a series into two sums:  $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^n a_i + \sum_{i=n+1}^{\infty} a_i$ . We compute the first sum as is. The second sum is

$$\sum_{i=n+1}^{\infty} a_i. \quad (7.417)$$

We write it in the form

$$\frac{1}{2}a_{n+1} + \sum_{i=n+1}^{\infty} b_i, \quad b_i = \frac{1}{2}(a_i + a_{i+1}). \quad (7.418)$$

We continue this procedure to write it as

$$\frac{1}{2}a_{n+1} + \frac{1}{2}b_{n+1} + \sum_{i=n+1}^{\infty} c_i, \quad c_i = \frac{1}{2}(b_i + b_{i+1}), \quad (7.419)$$

$$\begin{aligned} \frac{1}{2}a_{n+1} + \frac{1}{2}b_{n+1} + \frac{1}{2}c_{n+1} + \sum_{i=n+1}^{\infty} d_i, \quad d_i = \frac{1}{2}(c_i + c_{i+1}), \\ \dots \end{aligned} \quad (7.420)$$

Now we truncate to  $i = N$ . The above formulae become

$$\left\{ \begin{array}{l} a_{n+1} + \dots + a_{N-4} + a_{N-3} + a_{N-2} + a_{N-1} + a_N \\ a_{n+1} + \dots + a_{N-4} + a_{N-3} + a_{N-2} + a_{N-1} + \frac{1}{2}a_N \\ a_{n+1} + \dots + a_{N-4} + a_{N-3} + a_{N-2} + \frac{3}{4}a_{N-1} + \frac{1}{4}a_N \\ a_{n+1} + \dots + a_{N-4} + a_{N-3} + \frac{7}{8}a_{N-2} + \frac{1}{2}a_{N-1} + \frac{1}{8}a_N \\ \dots \end{array} \right. \quad (7.421)$$

This procedure is equivalent to write the sum as

$$\sum_{i=n+1}^N \alpha_i^j a_i, \quad (7.422)$$

where  $j = 1, 2, 3, 4, \dots$  correspond to the first, second, third, fourth,  $\dots$  equation of (7.421), respectively. The parameter  $\alpha_i^j$  can be computed up to  $j = N - n + 1$

using the recurrence formula

$$\alpha_i^{j+1} = \frac{1}{2}(\alpha_i^j + \alpha_{i+1}^j), \quad i \leq N-1; \quad \alpha_N^{j+1} = \frac{1}{2}\alpha_N^j. \quad (7.423)$$

It is evident that the parameter  $\alpha_i^j$  decreases quickly with  $i$  when  $j$  is large.

The adoption of (7.422) and (7.423) for computing the Legendre series in the load Green's function is proven to be effective. We did the computation with various values of  $n$  and  $N-n$ . The result is stable with  $n$  up to around 4000. Considering that the asymptotic values of the load Love numbers beyond degree 4000 agree with the exact values up to seven significant digits, the contribution of the terms beyond is known to be negligible. The result is less sensitive to  $N-n$ , which we fixed at 50. Some result of the load Green's function is tabulated in Table 7.6, where  $\theta$  is in radian in the normalization factors.

## 7.9 Some Geophysical Problems of Loading

### 7.9.1 The Geocenter and the Degree One Load Love Numbers

We have been using the term *Earth center* as the origin of the coordinate system, which can be defined differently, such as the center of mass of whole the Earth system or the center of mass of the solid Earth, which we have already come across. In this subsection, we elaborate the different definitions of the Earth center. When the definition of the Earth center is a concern in the context, we use the term *geocenter*.

In the definition of the degree 1 static load Love numbers in Sect. 7.6.3, we have to specify the origin of the coordinate system  $Or\theta\lambda$ ,  $O$ , as the center of mass of the solid Earth (CE) for the solution to be unique. This coordinate system is referred to as the CE frame. In the exact language, the CE is the center of mass of the Earth system without the mass of load. Hence, the degree 1 spherical harmonics of the gravitational potential due to the deformation of the Earth,  $\phi_1$ , vanishes on and outside the Earth. This leads to the boundary condition (7.292), which is copied to here for comparison with other kinds of Earth center to be defined,

$$y_5^1(R) = Rg_R \quad (\text{for the CE}). \quad (7.424)$$

Other kinds of geocenter are defined (e.g., Blewitt, 2003). The most referred are the center of mass of the Earth system including the mass of load, which is referred to as center of mass denoted as CM, and the geometric average point of the deforming surface of the Earth, which is referred to as center of figure denoted as CF. The CM frame and CF frame are defined accordingly.

The transformations of the solution among the different frames are just parallel shifts of the coordinate system  $Ox_1x_2x_3$  for the origin  $O$  to move from one kind of geocenters to another. Following Sect. 7.6.3, we define the transformation from the

**Table 7.6** Load Green’s function per kg mass of load

| $\theta$<br>( $^{\circ}$ ) | $\delta u_r(R)$<br>$10^{12}(R\theta)$ | $\delta u_{\theta}(R)$<br>$10^{12}(R\theta)$ | $\delta g^E$<br>$10^{18}(R\theta)$ | $\delta \epsilon^E$<br>$10^{12}(R\theta)^2$ | $\delta \epsilon_{rr}$<br>$10^{18}(R\theta)$ | $\delta \epsilon_{\theta\theta}$<br>$10^{12}(R\theta)^2$ | $\delta \epsilon_{\lambda\lambda}$<br>$10^{12}(R\theta)^2$ |
|----------------------------|---------------------------------------|----------------------------------------------|------------------------------------|---------------------------------------------|----------------------------------------------|----------------------------------------------------------|------------------------------------------------------------|
| 0.00001                    | -42.351                               | -12.850                                      | -102.267                           | 42.354                                      | 5.489                                        | 12.850                                                   | -12.850                                                    |
| 0.00005                    | -42.343                               | -12.850                                      | -102.246                           | 42.354                                      | 5.383                                        | 12.850                                                   | -12.850                                                    |
| 0.0001                     | -42.332                               | -12.850                                      | -102.221                           | 42.353                                      | 5.250                                        | 12.850                                                   | -12.850                                                    |
| 0.0005                     | -42.246                               | -12.850                                      | -102.019                           | 42.353                                      | 4.186                                        | 12.849                                                   | -12.850                                                    |
| 0.001                      | -42.140                               | -12.849                                      | -101.767                           | 42.352                                      | 2.856                                        | 12.849                                                   | -12.850                                                    |
| 0.005                      | -41.290                               | -12.841                                      | -99.758                            | 42.347                                      | -7.767                                       | 12.855                                                   | -12.844                                                    |
| 0.01                       | -40.234                               | -12.813                                      | -97.259                            | 42.335                                      | -20.936                                      | 12.880                                                   | -12.820                                                    |
| 0.03                       | -36.111                               | -12.525                                      | -87.481                            | 42.120                                      | -70.340                                      | 13.141                                                   | -12.544                                                    |
| 0.06                       | -30.473                               | -11.641                                      | -74.038                            | 40.837                                      | 125.149                                      | 13.796                                                   | -11.673                                                    |
| 0.1                        | -24.510                               | -9.996                                       | -59.613                            | 36.915                                      | 149.748                                      | 14.274                                                   | -10.039                                                    |
| 0.15                       | -19.742                               | -7.991                                       | -47.699                            | 30.076                                      | 121.132                                      | 13.181                                                   | -8.043                                                     |
| 0.2                        | -17.259                               | -6.591                                       | -41.080                            | 23.930                                      | -72.406                                      | 10.746                                                   | -6.651                                                     |
| 0.3                        | -15.579                               | -5.524                                       | -35.796                            | 17.579                                      | -9.243                                       | 6.389                                                    | -5.605                                                     |
| 0.4                        | -15.183                               | -5.520                                       | -34.000                            | 16.153                                      | 8.010                                        | 4.720                                                    | -5.626                                                     |
| 0.6                        | -14.637                               | -5.882                                       | -31.940                            | 16.448                                      | 6.000                                        | 5.017                                                    | -6.035                                                     |
| 0.8                        | -13.936                               | -5.969                                       | -30.113                            | 16.775                                      | 1.054                                        | 5.925                                                    | -6.163                                                     |
| 1.0                        | -13.181                               | -5.854                                       | -28.393                            | 16.802                                      | -1.453                                       | 6.495                                                    | -6.084                                                     |
| 1.2                        | -12.433                               | -5.647                                       | -26.781                            | 16.636                                      | -2.578                                       | 6.781                                                    | -5.906                                                     |
| 1.6                        | -11.049                               | -5.145                                       | -23.889                            | 15.954                                      | -3.178                                       | 6.891                                                    | -5.453                                                     |
| 2                          | -9.856                                | -4.645                                       | -21.417                            | 15.028                                      | -3.043                                       | 6.708                                                    | -4.987                                                     |
| 2.5                        | -8.624                                | -4.083                                       | -18.849                            | 13.755                                      | -2.623                                       | 6.310                                                    | -4.456                                                     |
| 3                          | -7.640                                | -3.603                                       | -16.763                            | 12.497                                      | -2.169                                       | 5.840                                                    | -4.000                                                     |
| 4                          | -6.222                                | -2.864                                       | -13.672                            | 10.266                                      | -1.409                                       | 4.887                                                    | -3.294                                                     |
| 6                          | -4.696                                | -1.989                                       | -10.114                            | 7.090                                       | -0.472                                       | 3.274                                                    | -2.474                                                     |
| 8                          | -4.005                                | -1.579                                       | -8.306                             | 5.271                                       | 0.023                                        | 2.077                                                    | -2.128                                                     |
| 10                         | -3.646                                | -1.405                                       | -7.249                             | 4.329                                       | 0.260                                        | 1.293                                                    | -2.027                                                     |
| 15                         | -3.122                                | -1.311                                       | -5.692                             | 3.569                                       | 0.361                                        | 0.567                                                    | -2.099                                                     |
| 20                         | -2.648                                | -1.281                                       | -4.592                             | 3.400                                       | 0.286                                        | 0.537                                                    | -2.153                                                     |
| 25                         | -2.120                                | -1.217                                       | -3.646                             | 3.349                                       | 0.191                                        | 0.711                                                    | -2.063                                                     |
| 30                         | -1.532                                | -1.114                                       | -2.786                             | 3.305                                       | 0.092                                        | 1.035                                                    | -1.813                                                     |
| 40                         | -0.270                                | -0.815                                       | -1.299                             | 2.901                                       | -0.100                                       | 1.995                                                    | -0.866                                                     |
| 50                         | 0.890                                 | -0.471                                       | -0.234                             | 1.832                                       | -0.233                                       | 2.868                                                    | 0.433                                                      |
| 60                         | 1.730                                 | -0.207                                       | 0.325                              | 0.273                                       | -0.284                                       | 3.138                                                    | 1.686                                                      |
| 70                         | 2.140                                 | -0.117                                       | 0.415                              | -1.381                                      | -0.264                                       | 2.664                                                    | 2.563                                                      |
| 80                         | 2.112                                 | -0.223                                       | 0.151                              | -2.785                                      | -0.198                                       | 1.597                                                    | 2.893                                                      |
| 90                         | 1.691                                 | -0.492                                       | -0.329                             | -3.747                                      | -0.112                                       | 0.195                                                    | 2.656                                                      |
| 100                        | 0.954                                 | -0.853                                       | -0.904                             | -4.217                                      | -0.022                                       | -1.294                                                   | 1.928                                                      |
| 110                        | -0.015                                | -1.227                                       | -1.486                             | -4.249                                      | 0.060                                        | -2.695                                                   | 0.829                                                      |
| 120                        | -1.133                                | -1.538                                       | -2.020                             | -3.944                                      | 0.130                                        | -3.912                                                   | -0.513                                                     |
| 130                        | -2.321                                | -1.722                                       | -2.480                             | -3.423                                      | 0.188                                        | -4.923                                                   | -1.988                                                     |
| 140                        | -3.507                                | -1.739                                       | -2.864                             | -2.790                                      | 0.234                                        | -5.758                                                   | -3.505                                                     |
| 160                        | -5.601                                | -1.202                                       | -3.451                             | -1.448                                      | 0.300                                        | -7.176                                                   | -6.420                                                     |
| 180                        | -6.913                                | -0.000                                       | -3.914                             | -0.000                                      | 0.346                                        | -8.815                                                   | -8.815                                                     |

CE frame to other frames by

$$\left\{ \bar{y}_1^1, \bar{y}_2^1, \bar{y}_3^1, \bar{y}_4^1, \bar{y}_5^1, \bar{y}_6^1 \right\} = \left\{ \bar{C}, 0, \bar{C}, 0, g\bar{C}, -\frac{2g}{r}\bar{C} \right\}. \quad (7.425)$$

This is a solution of the degree 1 deformation equation, which will be added to the solution in the CE frame to infer the solution in the CM frame or CF frame. We see that only the degree 1 load Love numbers are different in different frames.

For the CM frame, as the spherical harmonic coefficient  $y_5^n$  includes the contributions of both of the mass of load and the deformation of the Earth, the boundary condition is evidently

$$y_5^1(R) = 0 \quad (\text{for the CM}), \quad (7.426)$$

meaning that  $O$  is on the CM. We see that adding  $-Rg_R$  to (7.424) leads to (7.426). This implies that the solution in the CM frame can be obtained by adding (7.425) with  $\bar{C} = -R$  to the solution in the CE frame. We obtain

$$\left\{ \begin{matrix} y_1^1 \\ y_3^1 \\ y_5^1 \end{matrix} \right\}_{\text{CM}} = \left\{ \begin{matrix} y_1^1 \\ y_3^1 \\ y_5^1 \end{matrix} \right\}_{\text{CE}} + \left\{ \begin{matrix} -R \\ -R \\ -gR \end{matrix} \right\}. \quad (7.427)$$

The load Love numbers are, with their definitions given by (7.198),

$$\left\{ \begin{matrix} h'_1 \\ l'_1 \\ k'_1 \end{matrix} \right\}_{\text{CM}} = \left\{ \begin{matrix} h'_1 \\ l'_1 \\ k'_1 \end{matrix} \right\}_{\text{CE}} + \left\{ \begin{matrix} -1 \\ -1 \\ -1 \end{matrix} \right\}. \quad (7.428)$$

With (7.424) and the definition of  $k'_1$  in (7.198), we have  $k'_1 = 0$  in the CE frame and  $k'_1 = -1$  in the CM frame. It is left to the readers to interpret these result in physics.

In the CF frame, the average of the surface deformation vanishes, which is the meaning that the CF is the geometric average point of the deforming surface. The definition “ $O$  is on the CF” implies that  $\int_{\omega} \vec{u}(R)d\omega = 0$ , where  $\omega$  is the unit sphere and  $\vec{u}(R)$  the surface displacement. We use its alternative form  $\int_{\omega} \hat{e}_l \cdot \vec{u}(R)d\omega = 0$  with  $l = 1, 2, 3$  in subsequent formulations. With  $\vec{u}$  defined in (7.187), we obtain

$$\int_{\omega} \hat{e}_l \cdot \left[ y_1^n(R) P_n^k \hat{e}_r + y_3^n(R) \frac{d}{d\theta} P_n^k \hat{e}_{\theta} + y_3^n(R) \frac{ik}{\sin \theta} P_n^k \hat{e}_{\lambda} \right] e^{ik\lambda} d\omega = 0. \quad (7.429)$$

We take the simplest case:  $n = 1, k = 0, l = 3$ . With  $P_1(\cos \theta) = \cos \theta$ ,  $\hat{e}_l \cdot \hat{e}_r = \cos \theta$  and  $\hat{e}_l \cdot \hat{e}_{\theta} = -\sin \theta$ , we obtain

$$\int_0^{2\pi} d\lambda \int_0^{\pi} \left[ y_1^1(R) \cos^2 \theta + y_3^1(R) \sin^2 \theta \right] \sin \theta d\theta = 0. \quad (7.430)$$



This yields the boundary condition

$$y_1^1(R) + 2y_3^1(R) = 0 \quad (\text{for the CF}). \quad (7.431)$$

The relation between this solution and that in the CE frame is then

$$\{y_1^1(R)\}_{\text{CF}} = \{y_1^1(R)\}_{\text{CE}} + \bar{C}, \quad \{y_3^1(R)\}_{\text{CF}} = \{y_3^1(R)\}_{\text{CE}} + \bar{C}. \quad (7.432)$$

The constant  $\bar{C}$  is determined by (7.431), i.e.,

$$\left[ \{y_1^1(R)\}_{\text{CE}} + \bar{C} \right] + 2 \left[ \{y_3^1(R)\}_{\text{CE}} + \bar{C} \right] = 0, \quad (7.433)$$

whence,

$$\bar{C} = -\frac{1}{3} \left[ \{y_1^1(R)\}_{\text{CE}} + 2\{y_3^1(R)\}_{\text{CE}} \right]. \quad (7.434)$$

We finally obtain

$$\left\{ \begin{matrix} y_1^1 \\ y_3^1 \\ y_5^1 \end{matrix} \right\}_{\text{CF}} = \left\{ \begin{matrix} y_1^1 - (1/3)[y_1^1(R) + 2y_3^1(R)] \\ y_3^1 - (1/3)[y_1^1(R) + 2y_3^1(R)] \\ y_5^1 - (g/3)[y_1^1(R) + 2y_3^1(R)] \end{matrix} \right\}_{\text{CE}}. \quad (7.435)$$

The load Love numbers are, with their definitions given by (7.198),

$$\left\{ \begin{matrix} h_1' \\ l_1' \\ k_1' \end{matrix} \right\}_{\text{CF}} = \left\{ \begin{matrix} (2/3)(h_1' - l_1') \\ -(1/3)(h_1' - l_1') \\ k_1' - (1/3)h_1' - (2/3)l_1' \end{matrix} \right\}_{\text{CE}}, \quad (7.436)$$

remembering that  $k_1' = 0$  for the CE frame. The readers are suggested to discuss all the cases with  $n = 1$ ;  $k = -1, 0, 1$ ; and  $l = 1, 2, 3$ . All these are either trivial or leading to the same result as (7.436). Furthermore, (7.429) is trivial for all cases with  $n = 0$  or  $n > 1$ .

The dynamic load Love numbers defined in Sect. 7.5.2 are computed by solving the Eq. (7.191) with the surface boundary conditions (7.194)–(7.196). For the degree 1, the solution is not unique. We thus replace (7.196) by (7.424), (7.426), or (7.431) to obtain the solution for the CE frame, CM frame, or CF frame, respectively. For the dynamic case, as (7.425) cannot be added to a solution to yield another solution, the transformations (7.428) and (7.436) are approximate; the smaller  $\sigma$  is, the more accurate. As demonstrated numerically in the previous section, the difference between the dynamic and static case can be neglected as far as  $\sigma$  is small enough.

Two more Earth centers are defined in the literature (e.g., Blewitt, 2003). One is the center of surface height figure (CH) defined by  $\int_{\omega} u_r(R) \hat{e}_r d\omega = 0$  and the other

is the center of surface lateral figure (CL) defined by  $\int_{\omega} [u_{\theta}(R)\hat{e}_{\theta} + u_{\lambda}(R)\hat{e}_{\lambda}]d\omega = 0$ . By inspecting (7.429)–(7.431), we can readily infer that, for the former,  $y_1^1(R) = 0$  that leads to  $\{h_1'\}_{\text{CH}} = 0$ , and for the latter,  $y_3^1(R) = 0$  that leads to  $\{l_1'\}_{\text{CL}} = 0$ . It is left to the readers to complete the formulations.

## 7.9.2 Loading Effect of the Ocean Tide

Ocean tide is well determined using tide gauge and satellite altimetry data. We follow Doodson's convention to express ocean tide using the Greenwich mean lunar time  $\tau_0$  and the five fundamental parameters of the orbits of the Sun and the Moon given by (7.47), i.e.,  $s$ ,  $h$ ,  $p$ ,  $N'$ , and  $p_s$ . We consider only one ocean tide constituent here. For this constituent, we follow (7.80) to define

$$\Psi(t) = a\tau_0(t) + bs(t) + ch(t) + dp(t) + eN'(t) + fp_s(t), \quad (7.437)$$

which is the phase of Doodson's tidal wave expression at Greenwich. As Petit and Luzum (2010), we express the height of this ocean tide constituent in the form of

$$\zeta(\theta, \lambda, t) = A(\theta, \lambda) \cos[\Psi(t) - \psi(\theta, \lambda)]. \quad (7.438)$$

Note that the difference between the local and Greenwich mean lunar time is the longitude of the location, which is absorbed into  $\psi$ . The above two formulae can be applied to any ocean tide constituent, such as those listed in Lyard et al. (2006). For example, we get the  $M_2$  constituent by assigning the parameters  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ ,  $f$ ,  $A$ , and  $\phi$  to those of the  $M_2$  ocean tide model.

The determination of ocean tide in the form of (7.438) using tide gauge and satellite altimetry data is beyond the scope of this book. We assume that we do have a model of ocean tide with constituents provided in the form of (7.438), the largest being evidently  $M_2$ ,  $S_2$ ,  $N_2$ ,  $K_1$ ,  $O_1$ ,  $P_1$ , and  $M_f$  based on the magnitude of tidal-generating potential listed in Table 7.2.

In order to use the load Love numbers for displacement and gravitational potential, we express  $\zeta$  as a spherical harmonic series. We rewrite (7.438) as

$$\zeta(\theta, \lambda, t) = A(\theta, \lambda) \cos[\psi(\theta, \lambda)] \cos[\Psi(t)] + A(\theta, \lambda) \sin[\psi(\theta, \lambda)] [\sin \Psi(t)]. \quad (7.439)$$

We start with the truncated spherical harmonic series

$$\begin{cases} A(\theta, \lambda) \cos \psi(\theta, \lambda) = \sum_{n=0}^N \sum_{k=-n}^n a_n^k \mathcal{H}_n^k(\theta, \lambda), \\ A(\theta, \lambda) \sin \psi(\theta, \lambda) = \sum_{n=0}^N \sum_{k=-n}^n b_n^k \mathcal{H}_n^k(\theta, \lambda). \end{cases} \quad (7.440)$$

These can be used to write (7.439) as

$$\zeta(\theta, \lambda, t) = \sum_{n=0}^N \sum_{k=-n}^n \{a_n^k \cos[\Psi(t)] + b_n^k \sin[\Psi(t)]\} \mathcal{H}_n^k(\theta, \lambda). \quad (7.441)$$

With the spherical harmonic series expression (7.441) for the height of ocean tide, we can readily use (7.215) and (7.217) to write out the surface displacement and gravitational potential (in a shorter form to keep the formulae within a line):

$$\begin{cases} u_r = \frac{3\rho_w}{\bar{\rho}} \sum_{n=0}^N \sum_{k=-n}^n \frac{h'_n}{2n+1} (a_n^k \cos \Psi + b_n^k \sin \Psi) \mathcal{H}_n^k(\theta, \lambda), \\ u_\theta = \frac{3\rho_w}{\bar{\rho}} \sum_{n=0}^N \sum_{k=-n}^n \frac{l'_n}{2n+1} (a_n^k \cos \Psi + b_n^k \sin \Psi) \frac{\partial}{\partial \theta} \mathcal{H}_n^k(\theta, \lambda), \\ u_\lambda = \frac{3\rho_w}{\bar{\rho}} \sum_{n=0}^N \sum_{k=-n}^n \frac{l'_n}{2n+1} (a_n^k \cos \Psi + b_n^k \sin \Psi) \frac{1}{\sin \theta} \frac{\partial}{\partial \lambda} \mathcal{H}_n^k(\theta, \lambda); \end{cases} \quad (7.442)$$

$$V = \frac{GM}{R} \frac{3\rho_w}{R\bar{\rho}} \sum_{n=0}^N \sum_{k=-n}^n \frac{1+k'_n}{2n+1} \left(\frac{R}{r}\right)^{n+1} (a_n^k \cos \Psi + b_n^k \sin \Psi) \mathcal{H}_n^k(\theta, \lambda); \quad (7.443)$$

where  $\rho_w$  is the density of the ocean water.

For very precise observations near a shore, such as the gravity recorded by a superconducting gravimeter or the crustal displacement recorded by a GNSS receiver, the resolution of the spherical harmonic series may be insufficient. In this case, the loading effect should be computed by evaluating the convolution of the density of the mass of load,  $\rho_w \zeta$ , with the load Green's function.

Ocean tide is much better known than the non-tidal variation in the ocean, the variations in the atmosphere, and the hydrologic and glaciologic variations. Hence, ocean tide effect is often used as a correction to the data for studying other phenomenon.

### 7.9.3 Sea Level Change as a Result of Present-Day Continent-Ocean Water Exchange

We know that ice sheets are melting, glaciers are melting, groundwater is pumped out for agricultural irrigation. A large portion of the land surface water may evaporate into the atmosphere or flow ultimately into the oceans. Ocean water also evaporates into the atmosphere. The water in the atmosphere goes back to land or oceans in the form of rain. In summary, there is a ceaseless exchange of water

between the land and oceans, the net exchange at present being a discharge of land water into the oceans. The continents are understood as land. In this subsection, we use *land* and *continents* interchangeably.

The water discharged from the land into the oceans causes the sea level to rise. Even in the ideal case that the ocean water achieves an equilibrium, the sea level rise is still different from place to place. The sea level is controlled by the gravitational attraction between the water at different places (both land and oceans) and to a lesser extent by the loading effect of gravitational attraction of the solid Earth to the ocean water. For example, the depletion of ice sheet over Greenland would cause the sea level around Greenland to decrease, since the depletion causes a decrease of the gravitational attraction of the ice sheet to the ocean water around. The water redistribution on the surface also induces a polar motion, which affects sea level as a feedback.

In this subsection, we formulate the ideal case: the water distribution over the continents has changed, the net loss of the water over the continents is all gone into the oceans, and the oceans have attained an equilibrium state, i.e., the surface is an equipotential surface of gravity. We consider only the self-gravitational attraction of the water and the loading effect of the solid Earth. The effect on Earth rotation variation and the consequent feedback are the topic of Sect. 8.5.5.

We describe here an approximate algorithm using extensively the discrete spherical harmonic transform (DSHT) and its inverse (IDSHT) presented in Sect. 4.9.4. Therefore, the computation is based on a regular latitude-longitude grid. Our preference is (4.645), i.e.,

$$\theta_j = \frac{(2j+1)\pi}{4b}, \quad \lambda_l = \frac{l\pi}{b}; \quad j, l = 0, 1, \dots, 2b-1, \quad (7.444)$$

which does not include the poles. The grid intervals are  $\delta\theta = \pi/(2b)$  and  $\delta\lambda = \pi/b$ . We take  $(\theta_j, \lambda_l)$  as the center of the grid cell defined by  $\theta_j - \delta\theta/2 \leq \theta \leq \theta_j + \delta\theta/2$  and  $\lambda_l - \delta\lambda/2 \leq \lambda \leq \lambda_l + \delta\lambda/2$ , which is denoted as  $C_{jl}$ . Its area is

$$S_{jl} = \int_{\lambda_l - \delta\lambda/2}^{\lambda_l + \delta\lambda/2} \int_{\theta_j - \delta\theta/2}^{\theta_j + \delta\theta/2} \sin\theta d\theta d\lambda = 2 \sin\theta_j \sin\frac{\delta\theta}{2} \delta\lambda. \quad (7.445)$$

We use the change of the equivalent water height (EWH),  $\delta H$ , to express the mass change over both the continents and oceans. The change of surface mass density is then  $\delta\mu = \rho_w \delta H$ , where  $\rho_w$  is the water density. For fresh water with  $\rho_w = 1 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$ ,  $\delta H = 1 \text{ mm}$  is equivalent to  $\delta\mu = 1 \text{ kg} \cdot \text{m}^{-2}$ . We suppose that the procedure of water exchange is over, so that  $\delta H$  is independent of time, thus being a function of  $\theta$  and  $\lambda$  only. We presume that the grid size is small enough such that  $\delta H$  can be treated as a constant in each grid cell. This allows us to compute the mass change in a cell using  $\rho_w \delta H(\theta_j, \lambda_l) S_{jl}$ . We also presume that the spherical harmonic coefficients of  $\delta H$  can be computed using  $\delta H(\theta_j, \lambda_l)$  according to the DSHT. This requires again that the grid size be small enough, so that  $\delta H$  at any point can be well approximated by an interpolation using  $\delta H(\theta_j, \lambda_l)$ .

The distinction between land and oceans is described by the ocean function, which is 1 over the oceans and 0 over the land; a grid cell is either completely land or completely ocean. We see that the grid size also represents the resolution of the shorelines. In the following, we will distinguish land and ocean by the superscript  $L$  and  $O$ . For example,  $(\theta_j^L, \lambda_l^L)$  or  $(\theta_j^O, \lambda_l^O)$  implies that the grid cell is land or ocean, respectively.

We provide our algorithm as a procedure of computation ( $\rho_w$  is a prescribed value, e.g.,  $\rho_w = 1 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$ ):

1. Prepare all data of  $\rho_w \delta H(\theta_j^L, \lambda_l^L)$ . Set  $\rho_w \delta H(\theta_j^L, \lambda_l^L) = 0$  if no mass change has happened in the cell or the cell is not concerned. For example, if our concern is the sea level change due to the depletion of ice sheet over Greenland, we set  $\rho_w \delta H(\theta_j^L, \lambda_l^L) = 0$  over all other continents.
2. Compute the total mass loss over land (a positive value of  $\delta H$  means a gain):

$$\delta M = - \sum_{j,l} \rho_w \delta H(\theta_j^L, \lambda_l^L). \quad (7.446)$$

3. Compute the total area of the oceans:

$$S^O = \sum_{j,l} S_{jl}^O. \quad (7.447)$$

4. Homogeneously distribute  $\delta M$  over the oceans (the loss over the continents is the gain over the oceans):

$$\delta H(\theta_j^O, \lambda_l^O) = \frac{\delta M}{S^O \rho_w} \text{ for all } (\theta_j^O, \lambda_l^O). \quad (7.448)$$

5. We now have the value of  $\delta H(\theta_j, \lambda_l)$  all over the globe. Use the DSHT to compute its spherical harmonic coefficient  $\delta H_n^k$  in

$$\delta H(\theta_j, \lambda_l) = \sum_{n=0}^{b-1} \sum_{k=-n}^n \delta H_n^k \mathcal{H}_n^k(\theta_j, \lambda_l). \quad (7.449)$$

6. Use  $\delta H_n^k$  to compute the surface vertical displacement  $\delta u_r(\theta_j, \lambda_l)$  according to (7.215) and the change of geoidal height  $\delta N(\theta_j, \lambda_l)$  according to (7.219):

$$\delta u_r(\theta_j, \lambda_l) = \sum_{n=0}^{b-1} \sum_{k=-n}^n \left( \frac{h'_n}{2n+1} \frac{3\rho_w}{\bar{\rho}} \delta H_n^k \right) \mathcal{H}_n^k(\theta_j, \lambda_l), \quad (7.450)$$

$$\delta N(\theta_j, \lambda_l) = \sum_{n=0}^{b-1} \sum_{k=-n}^n \left( \frac{1+k'_n}{2n+1} \frac{3\rho_w}{\bar{\rho}} \delta H_n^k \right) \mathcal{H}_n^k(\theta_j, \lambda_l). \quad (7.451)$$

This is a IDSHT. This is the change of the geoidal height with respect to the Earth center caused by a small change of the gravitational potential on the Earth's surface, which is the same as in the case of equilibrium tide discussed Sect. 7.4.1. As the ocean surface is assumed to be an equipotential surface of gravity after the equilibrium is realized,  $\delta N$  is the change of height of the ocean surface with respect to the Earth center. Note that water redistribution in the ocean induces a variation in Earth rotation that feedbacks to water redistribution in the ocean. This requires to add a Earth rotation feedback term to the degree 2 spherical harmonic coefficients in (7.450) and (7.450). The formulation will be the topic of Sect. 8.5.5.

7. As  $\delta u_r$  is the change of the ocean bottom height with respect to the Earth center, the change of ocean water thickness now becomes  $\delta N - \delta u_r$ . Over the grid, we first approximate the change of ocean water thickness by

$$\delta H_a(\theta_j^O, \lambda_l^O) = \delta N(\theta_j^O, \lambda_l^O) - \delta u_r(\theta_j^L, \lambda_l^L) \quad (7.452)$$

The corresponding total change of mass over the oceans is

$$\delta M' = \sum_{j,l} \rho_w \delta H_a(\theta_j^O, \lambda_l^O), \quad (7.453)$$

Which is not necessarily equal to  $\delta M$ . The total mass of the ocean should be conserved. We obtain  $\delta H(\theta_j^O, \lambda_l^O)$  by adding a homogeneous layer to  $\delta H_a(\theta_j^O, \lambda_l^O)$ ,

$$\delta H(\theta_j^O, \lambda_l^O) = \delta H_a(\theta_j^O, \lambda_l^O) + \frac{\delta M - \delta M'}{S^O \rho_w} \text{ for all } (\theta_j^O, \lambda_l^O). \quad (7.454)$$

8. With the updated  $\delta H(\theta_j^O, \lambda_l^O)$ , iterate from step 5 to 7 till a prescribed accuracy is achieved. The criteria of the accuracy is usually a value that the result must be smaller than, e.g., the maximum absolute difference between the values of the last and the second last steps of  $\delta H(\theta_j^O, \lambda_l^O)$ ,

$$\epsilon = \max_{j,l} \left| \delta H(\theta_j^O, \lambda_l^O)_{\text{last}} - \delta H(\theta_j^O, \lambda_l^O)_{\text{second last}} \right|. \quad (7.455)$$

It is evident that the change of shoreline associated with the sea level change has been neglected.

Note that the degree 1 load Love numbers depend on the definition of the geocenter, and that the effect of the accompanying Earth rotation variation is not included, which is to be formulated in Sect. 8.5.5.

The property of the result is apparent:

- $\delta H(\theta_j^O, \lambda_l^O)$  is the ocean surface height change with respect to the seafloor, or ocean depth change, which is the quantity observed by tide gauges (should be scaled by  $\rho_w/\rho_{\text{ocean}}$ ).
- $\delta N(\theta_j^O, \lambda_l^O)$  is the ocean surface height change with respect to the Earth center, which is the quantity observed by satellite altimetry.
- $\delta u_r(\theta_j^O, \lambda_l^O)$  is the vertical crustal displacement as observed by GNSS.
- The horizontal crustal displacement as observed by GNSS can be computed according to (7.215),

$$\begin{cases} \delta u_\theta(\theta_j, \lambda_l) = \sum_{n=0}^{b-1} \sum_{k=-n}^n \left( \frac{l'_n}{2n+1} \frac{3\rho_w}{\bar{\rho}} \delta H_n^k \right) \left[ \frac{\partial \mathcal{H}_n^k(\theta, \lambda)}{\partial \theta} \right]_{jl}, \\ \delta u_\lambda(\theta_j, \lambda_l) = \sum_{n=0}^{b-1} \sum_{k=-n}^n \left( \frac{l'_n}{2n+1} \frac{3\rho_w}{\bar{\rho}} \delta H_n^k \right) \frac{1}{\sin \theta} \left[ \frac{\partial \mathcal{H}_n^k(\theta, \lambda)}{\partial \lambda} \right]_{jl}, \end{cases} \quad (7.456)$$

where  $[\partial \mathcal{H}_n^k(\theta, \lambda)/\partial \theta]_{jl}$  and  $[\partial \mathcal{H}_n^k(\theta, \lambda)/\partial \lambda]_{jl}$  are the values of the derivatives at  $(\theta_j, \lambda_l)$ . These can also be computed using the fast Fourier transform (FFT) as in the cases of DSHT and IDSHT.

The algorithm applies to two cases: (1) the total change from a starting epoch to an ending epoch, (2) a rate of change during a long time span (replace the total changes in the formulation by the rate of changes). For case (1), the ocean is assumed to be in hydrostatic equilibrium before the starting epoch and after the ending epoch. For case (2), the rate of change is so slow that the ocean can be constantly approximated as in hydrostatic equilibrium.

The procedure described in this subsection is for computing the sea level change associated with present-day continent-ocean water exchanges (e.g., Mitrovica et al., 2001). It is the degenerate case of the theory of the sea level change associated with the glacial isostatic adjustment (GIA) (e.g., Farrell & Clark, 1976; Mitrovica et al., 2001; Sabadini et al., 2004).

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## Bibliographic Notes

The author's main reference include Agnew (2015), Doodson (1921), Lambeck (1980), Melchior (1966, 1983), Moritz and Mueller (1985), Munk and Macdonald (1960), Wenzel (1997) and Xi and Hou (1987) for Sects. 7.1, 7.2, 7.3 and 7.4; Bos et al. (2014), Chinnery (1975), Crossley and Gubbins (1975), Farrell (1972), Guo et al. (2004), Longman (1962, 1963), Martens et al. (2016) and Melchior (1983) for Sects. 7.5, 7.6 and 7.7, Guo et al. (2001, 2004) for Sect. 7.8; Blewitt (2003), Farrell and Clark (1976), McCarthy and Petit (2003), Mitrovica and Milne (2003)

and Mitrovica et al. (2001) for Sect. 7.9. These are also suggested as the readers' sources for alternative presentations.

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## Abstract

In this chapter, the formulations of precession-nutation and polar motion of the rigid and elastic Earth models are formulated in detail. For the rigid Earth model, the precession-nutation and polar motion caused by the lunisolar gravitational attraction are first presented, and then the normal modes of rotation including the Eulerian wobble, the axial spin mode, and the tilt-over mode are presented. For the elastic Earth model, the deformation caused by the variation of the Earth rotation, notably the pole tide, is formulated, and then the precession-nutation and polar motion caused by the lunisolar gravitational attraction, as well as the normal modes of rotation, are formulated. Lastly, the variation of the Earth rotation as a result of the interaction between the solid Earth and the atmosphere-hydrosphere-cryosphere system is formulated for the elastic Earth model, and then the up-to-date theory is introduced by incorporating the effects of mantle anelasticity (i.e., non-pure elasticity) and the Earth's core. As an application, the Earth rotation variation in response to the present-day continent-ocean water exchange as well as its feedback to the sea level change is formulated.

## 8.1 Rotation of the Rigid Earth Model

### 8.1.1 The Lunisolar Gravitational Torque Exerted on the Earth

The variation of the Earth's orientation in the space is mainly caused by the gravitational torque exerted on the Earth by the Moon and the Sun. The second largest cause is the geophysical processes, such as the motion and mass transport in the atmosphere-hydrosphere-cryosphere system. The influence of other celestial bodies is much smaller, and is beyond the scope of this book.

We first compute the lunisolar gravitational torque exerted on the Earth. We provide a formulation for the Moon. The case for the Sun is similar. As shown in

**Fig. 8.1** Formulation of the torque of the gravitational attraction of the Moon on the Earth (not to scale)

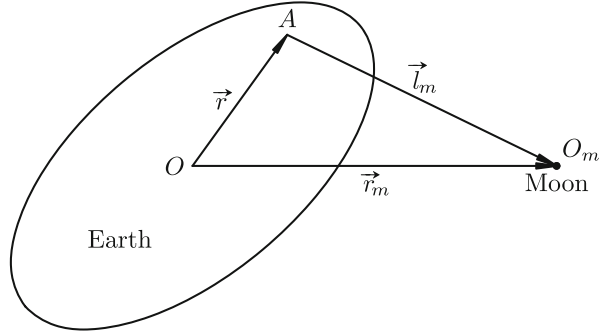


Fig. 8.1, the gravitational torque of the Moon exerted on the Earth around the point  $O$  is

$$\vec{N}_m = \int_V \vec{r} \times \vec{f} dV, \quad (8.1)$$

where  $O$  is the point around which the Earth rotation is referred to, and  $\vec{f}$  is the density of the gravitational force, i.e., the gravitational force per unit volume, from the Moon. By approximately taking the Moon as a point mass, we have  $\vec{f} = (G\rho M_m/l_m^3) \vec{l}_m$ , where  $\rho$  is the density of the Earth,  $M_m$  the mass of the Moon,  $\vec{l}_m$  as shown 8.1, and  $l_m$  its norm. We can thus write (8.1) in the more detailed form

$$\vec{N}_m = \int_V \vec{r} \times \left( G\rho \frac{M_m}{l_m^3} \vec{l}_m \right) dV, \quad (8.2)$$

which can be further developed as

$$\vec{N}_m = \int_V (\vec{r}_m - \vec{l}_m) \times \left( G\rho \frac{M_m}{l_m^3} \vec{l}_m \right) dV = -M_m \vec{r}_m \times \left( -G \int_V \frac{\rho}{l_m^3} \vec{l}_m dV \right). \quad (8.3)$$

The expression in the parentheses of the last term is actually the gravitational attraction of the Earth (to a unit point mass) at the Moon's position  $O_m$ . We can thus write the last formula as, using  $\phi$  to denote the gravitational potential of the Earth,

$$\vec{N}_m = -M_m \vec{r}_m \times \nabla \phi, \quad (8.4)$$

where  $\nabla \phi$  should be evaluated at the position of the Moon, i.e., at  $O_m$ . Actually, (8.4) can be inferred directly by analyzing the Earth-Moon system. As only the interaction between the Earth and the Moon is considered, the Earth-Moon system behaves as a closed system, and the total torque on the system vanishes. As the gravitational force is the only action between the Earth and the Moon, and the

gravitational torque exerted by the Earth on the Moon can be readily inferred to be  $M_m \vec{r}_m \times \nabla \phi$ , the gravitational torque exerted by the Moon on the Earth must be the opposite of it, which is (8.4).

In this book, we consider only the dominant term in the spherical harmonic series of the Earth's gravitational potential, i.e., the  $J_2$  term, which is the effect of the Earth's flattening. We can write  $\phi$  simply as

$$\phi = \frac{GM}{r} \left[ 1 - J_2 \left( \frac{a}{r} \right)^2 P_2(\cos \theta) \right] = \frac{GM}{r} \left[ 1 - \frac{C - A}{Ma^2} \left( \frac{a}{r} \right)^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \right], \quad (8.5)$$

where we have implicitly assumed  $A = B$ , and the vanishing of the products of inertia.

As  $\phi$  is independent of  $\lambda$ , we have

$$\nabla \phi = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta. \quad (8.6)$$

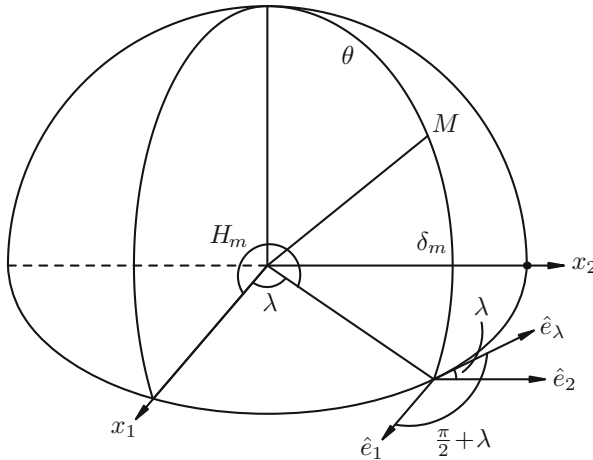
With  $\hat{e}_r \times \hat{e}_r = 0$  and  $\hat{e}_r \times \hat{e}_\theta = \hat{e}_\lambda$ , we can write (8.4) as

$$\vec{N}_m = -M_m \frac{\partial \phi}{\partial \theta} \hat{e}_\lambda = -\frac{3}{2} G \frac{M_m}{r_m^3} (C - A) \sin 2\theta \hat{e}_\lambda. \quad (8.7)$$

where  $\theta$  and  $\lambda$  are the spherical coordinates of the Moon.

The position of the Moon is represented by its hour angle with respect to the Greenwich meridian,  $H_m$ , and the declination  $\delta_m$ . Taking reference to Fig. 8.2, we can convert  $\theta$  and  $\lambda$  to  $H_m$  and  $\delta_m$ ,

$$\theta = \frac{\pi}{2} - \delta_m, \quad \lambda = 2\pi - H_m. \quad (8.8)$$



**Fig. 8.2** Conversion of  $\theta$  and  $\lambda$  to  $H_m$  and  $\delta_m$

Hence, we can write the torque as

$$\vec{N}_m = -\frac{3}{2}G \frac{M_m}{r_m^3} (C - A) \sin 2\delta_m \hat{e}_\lambda. \quad (8.9)$$

Its components are, taking reference to Fig. 8.2,

$$\begin{cases} N_{m1} = -\frac{3}{2}G \frac{M_m}{r_m^3} (C - A) \sin 2\delta_m \sin H_m, \\ N_{m2} = -\frac{3}{2}G \frac{M_m}{r_m^3} (C - A) \sin 2\delta_m \cos H_m, \end{cases} \quad (8.10)$$

where  $\sin H_m$  and  $\cos H_m$  are derived from  $\cos(\pi/2 + \lambda)$  and  $\cos \lambda$  according to (8.8). Similarly, the gravitational torque exerted by the Sun on the Earth is

$$\begin{cases} N_{s1} = -\frac{3}{2}G \frac{M_s}{r_s^3} (C - A) \sin 2\delta_s \sin H_s, \\ N_{s2} = -\frac{3}{2}G \frac{M_s}{r_s^3} (C - A) \sin 2\delta_s \cos H_s. \end{cases} \quad (8.11)$$

The torques (8.10) and (8.11) can be further expressed using the diurnal tidal waves. With the Doodson's constant  $D$  defined in (7.33), we can write the above two formulae as

$$\begin{cases} N_{m1} = -2 \frac{D}{R^2} (C - A) \left( \frac{C_m}{r_m} \right)^3 \sin 2\delta_m \sin H_m, \\ N_{m2} = -2 \frac{D}{R^2} (C - A) \left( \frac{C_m}{r_m} \right)^3 \sin 2\delta_m \cos H_m; \end{cases} \quad (8.12)$$

$$\begin{cases} N_{s1} = -2 \frac{D}{R^2} (C - A) \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \sin 2\delta_s \sin H_s, \\ N_{s2} = -2 \frac{D}{R^2} (C - A) \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \sin 2\delta_s \cos H_s. \end{cases} \quad (8.13)$$

The total lunisolar gravitational torque exerted on the Earth is the sum of them,

$$\begin{cases} N_1 = -2 \frac{D}{R^2} (C - A) \psi'_1, \\ N_2 = -2 \frac{D}{R^2} (C - A) \psi_1, \end{cases} \quad (8.14)$$

with

$$\begin{cases} \psi'_1 = \left(\frac{C_m}{r_m}\right)^3 \sin 2\delta_m \sin H_m + \left(\frac{M_s}{M_m}\right) \left(\frac{C_m}{C_s}\right)^3 \left(\frac{C_s}{r_s}\right)^3 \sin 2\delta_s \sin H_s, \\ \psi_1 = \left(\frac{C_m}{r_m}\right)^3 \sin 2\delta_m \cos H_m + \left(\frac{M_s}{M_m}\right) \left(\frac{C_m}{C_s}\right)^3 \left(\frac{C_s}{r_s}\right)^3 \sin 2\delta_s \cos H_s. \end{cases} \quad (8.15)$$

The function  $\psi_1$  is the time-dependent portion of the degree 2 diurnal lunisolar tidal-generating potential at the Greenwich meridian, as given in (7.40). It can be developed into a series of harmonic waves, as given in (7.75),

$$\psi_1 = \sum_i Y_i^d \sin(\tau_0 + b_i^d s + c_i^d h + d_i^d p + e_i^d N' + f_i^d p_s), \quad (8.16)$$

where the superscript  $d$  is for *diurnal*. Use  $H$  to denote either  $H_m$  or  $H_s$ . As  $\psi_1$  represents the degree 2 diurnal tidal-generating potential on the Greenwich meridian, and  $\sin H = \cos(H - \pi/2)$ , we can infer that  $\psi'_1$  represents the degree 2 diurnal tidal-generating potential on the meridian of hour angle  $H - \pi/2$ , i.e., at west longitude  $\pi/2$ , where the mean lunar time is  $\tau' = \tau_0 - \pi/2$ . Hence,  $\psi'_1$  can be obtained by replacing  $\tau_0$  by  $\tau_0 - \pi/2$  in (8.16),

$$\psi'_1 = - \sum_i Y_i^d \cos(\tau_0 + b_i^d s + c_i^d h + d_i^d p + e_i^d N' + f_i^d p_s). \quad (8.17)$$

With the above two formulae, the formula of the lunisolar gravitational torque exerted on the Earth, (8.14), can be written as

$$\begin{cases} N_1 = 2 \frac{D}{R^2} (C - A) \sum_i Y_i^d \cos(\tau_0 + b_i^d s + c_i^d h + d_i^d p + e_i^d N' + f_i^d p_s), \\ N_2 = -2 \frac{D}{R^2} (C - A) \sum_i Y_i^d \sin(\tau_0 + b_i^d s + c_i^d h + d_i^d p + e_i^d N' + f_i^d p_s), \end{cases} \quad (8.18)$$

where the values of  $Y_i^d$  and  $b_i^d$ ,  $c_i^d$ ,  $d_i^d$ ,  $e_i^d$ , and  $f_i^d$  can be found in a table of tidal waves. As all the parameters  $\tau$ ,  $s$ ,  $h$ ,  $p$ ,  $N'$ , and  $p_s$  vary almost linearly, and (7.82) holds, we can further write the torque (8.18) as

$$\begin{cases} N_1 = 2 \frac{D}{R^2} (C - A) \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ N_2 = -2 \frac{D}{R^2} (C - A) \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.19)$$

This is the final form of the formula of the torque we are formulating. We note that  $\omega_i^d$  is the angular frequency of the tidal wave,  $x_1$  points to the Greenwich meridian, and  $\tau_0$  is the mean lunar time at the Greenwich meridian.

For comparison with other expressions in the literature, we express the torque (8.19) in the complex form

$$\tilde{N} = N_1 + iN_2 = \frac{D}{R^2}(C - A) \sum_i Y_i^d e^{-i(\omega_i^d t + \phi_i^d)}. \quad (8.20)$$

Another form is (e.g., Mathews & Shapiro, 1992)

$$N_1 \hat{e}_1 + N_2 \hat{e}_2 = -\hat{e}_3 \times (-N_2 \hat{e}_1 + N_1 \hat{e}_2) \quad (8.21)$$

with

$$-N_2 + iN_1 = \frac{D}{R^2}(C - A) \sum_i Y_i^d e^{-i(\omega_i^d t + \phi_i^d - \pi/2)}. \quad (8.22)$$

In the Earth rotation community, instead of Doodson's arguments, the Delaunay arguments are used as lunar and solar orbital parameters, which are (e.g., Dehant & Mathews, 2015)

- $\Omega$ , the mean longitude of the ascending node of the Moon;
- $l$ , the mean anomalies of the Moon;
- $l'$ , the mean anomalies of the Sun;
- $D$ , the mean elongation of the Moon from the Sun;
- $F = L - \Omega$ , where  $L$  is the Mean Longitude of the Moon.

The relations between these parameters and Doodson's arguments are

$$l = s - p, \quad l' = h - p_s, \quad D = s - h, \quad F = s + N', \quad \Omega = -N'. \quad (8.23)$$

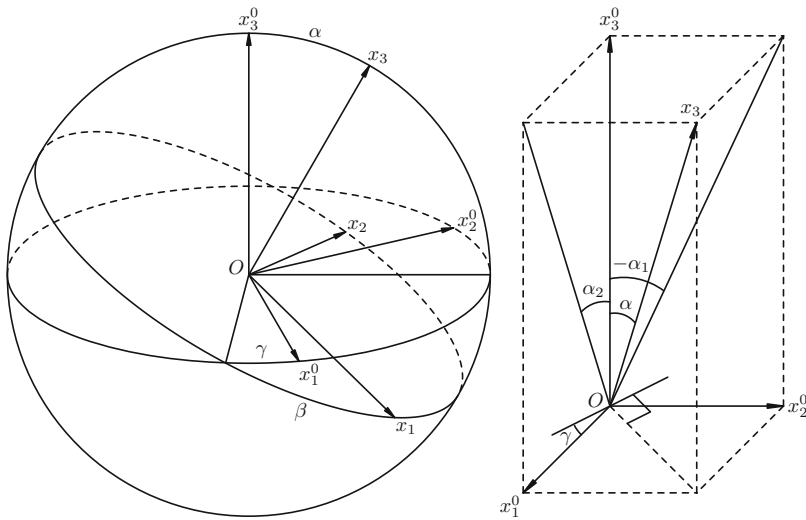
Here we understand these relations as the definition of the Delaunay arguments. In this book, Doodson's arguments are used throughout, as the readers are supposed to be geodesists or geophysicists.

### 8.1.2 Linearization of Euler's Equations

We formulate the Earth rotation with respect to the inertial space with the help of three coordinate systems: A coordinate system  $O\xi_1\xi_2\xi_3$  with coordinate directions fixed with respect to the inertial space, a coordinate system  $Ox_1x_2x_3$  fixed on the Earth, and an auxiliary coordinate system  $Ox_1^0x_2^0x_3^0$  with the  $x_3^0$ -axis pointing to the celestial pole and the  $x_1^0$  and  $x_2^0$ -axes very close to the  $x_1$ - and  $x_2$ -axes, respectively. All the three coordinate systems have the same origin  $O$ , which is defined to be the center of mass of the Earth system. Therefore,  $O\xi_1\xi_2\xi_3$  is in fact the geocentric celestial reference system (GCRS) and is referred to as space-fixed

or inertial,  $Ox_1x_2x_3$  is in fact the geocentric terrestrial reference system (GTRS) or international terrestrial reference system (ITRS) and is referred to as Earth-fixed. The  $x_3^0$ -axis representing the celestial pole can be defined as the Earth rotation axis, the Earth's angular momentum axis, or otherwise. The definitions of the  $x_1^0$ - and  $x_2^0$ -axes will be progressively made clear in the formulation. We realize immediately that the Euler's equations governing the rotation of a solid body around its center of mass can be straightforwardly applied to study the rotation of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$ . Our focus is to study the variation of the direction of the  $x_3^0$ -axis with respect to the coordinate systems  $O\xi_1\xi_2\xi_3$  and  $Ox_1x_2x_3$ , i.e., the precession-nutation and polar motion, respectively. Therefore, it is advantageous to develop the Euler's equations governing the rotation of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$  to a form suitable to study the rotation of  $Ox_1^0x_2^0x_3^0$  with respect to both  $O\xi_1\xi_2\xi_3$  and  $Ox_1x_2x_3$ .

As shown in Fig. 8.3 (Left), we use three angles,  $\alpha$ ,  $\beta$  and  $\gamma$ , to describe the orientation of  $Ox_1x_2x_3$  with respect to  $Ox_1^0x_2^0x_3^0$ . The rotation of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$  is then the combination of the rotation of  $Ox_1^0x_2^0x_3^0$  with respect to  $O\xi_1\xi_2\xi_3$  and that of  $Ox_1x_2x_3$  with respect to  $Ox_1^0x_2^0x_3^0$ ; the later can be represented by the angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . Based on Fig. 8.3 (Left), the expressions of



**Fig. 8.3** Left: Position of the Earth-fixed coordinate system  $Ox_1x_2x_3$  with respect to the auxiliary coordinate system  $Ox_1^0x_2^0x_3^0$  expressed in terms of  $\alpha$ ,  $\beta$ , and  $\gamma$ ; Right: The direction of the  $x_3$ -axis in  $Ox_1^0x_2^0x_3^0$  expressed in terms of  $\alpha_1$  and  $\alpha_2$ , which can be understood as angles rotated around  $x_1^0$  and  $x_2^0$ , respectively, according to the right-hand rule



the base vectors of  $Ox_1x_2x_3$  in terms of those of  $Ox_1^0x_2^0x_3^0$  are

$$\begin{cases} \hat{e}_1 = \hat{e}_1^0(\cos \gamma \cos \beta + \sin \gamma \cos \alpha \sin \beta) \\ \quad + \hat{e}_2^0(-\sin \gamma \cos \beta + \cos \gamma \cos \alpha \sin \beta) - \hat{e}_3^0 \sin \alpha \sin \beta, \\ \hat{e}_2 = \hat{e}_1^0(-\cos \gamma \sin \beta + \sin \gamma \cos \alpha \cos \beta) \\ \quad + \hat{e}_2^0(\sin \gamma \sin \beta + \cos \gamma \cos \alpha \cos \beta) - \hat{e}_3^0 \sin \alpha \cos \beta, \\ \hat{e}_3 = \hat{e}_1^0 \sin \gamma \sin \alpha + \hat{e}_2^0 \cos \gamma \sin \alpha + \hat{e}_3^0 \cos \alpha, \end{cases} \quad (8.24)$$

and the angular velocity of  $Ox_1x_2x_3$  with respect to  $Ox_1^0x_2^0x_3^0$  is

$$\vec{\omega}' = \hat{e}_1^0(-\dot{\alpha} \cos \gamma + \dot{\beta} \sin \alpha \sin \gamma) + \hat{e}_2^0(\dot{\alpha} \sin \gamma + \dot{\beta} \sin \alpha \cos \gamma) + \hat{e}_3^0(\dot{\beta} \cos \alpha - \dot{\gamma}). \quad (8.25)$$

The auxiliary coordinate system  $Ox_1^0x_2^0x_3^0$  is assumed to be prescribed, which defines the celestial pole, and is to be further discussed in more detail later. Here we make the assumption that  $Ox_1^0x_2^0x_3^0$  is extremely close to  $Ox_1x_2x_3$ , so that  $\alpha$  and  $\beta - \gamma$  are infinitesimally small. We further define

$$\alpha_1 = -\alpha \cos \gamma, \quad \alpha_2 = \alpha \sin \gamma, \quad \alpha_3 = \beta - \gamma, \quad (8.26)$$

where the meanings of  $\alpha_1$  and  $\alpha_2$  are shown in Fig. 8.3 (Right). We have

$$\dot{\alpha}_1 = -\dot{\alpha} \cos \gamma + \alpha \dot{\gamma} \sin \gamma, \quad \dot{\alpha}_2 = \dot{\alpha} \sin \gamma + \alpha \dot{\gamma} \cos \gamma, \quad \dot{\alpha}_3 = \dot{\beta} - \dot{\gamma}. \quad (8.27)$$

Accurate to the first order of  $\alpha$  and  $\beta - \gamma$ , we have

$$\sin \alpha = \alpha, \quad \cos \alpha = 1, \quad \sin \alpha_3 = \alpha_3, \quad \cos \alpha_3 = 1. \quad (8.28)$$

These formulae can be used to simplify (8.24) and (8.25) to

$$\hat{e}_1 = \hat{e}_1^0 + \alpha_3 \hat{e}_2^0 - \alpha_2 \hat{e}_3^0, \quad \hat{e}_2 = -\alpha_3 \hat{e}_1^0 + \hat{e}_2^0 + \alpha_1 \hat{e}_3^0, \quad \hat{e}_3 = \alpha_2 \hat{e}_1^0 - \alpha_1 \hat{e}_2^0 + \hat{e}_3^0; \quad (8.29)$$

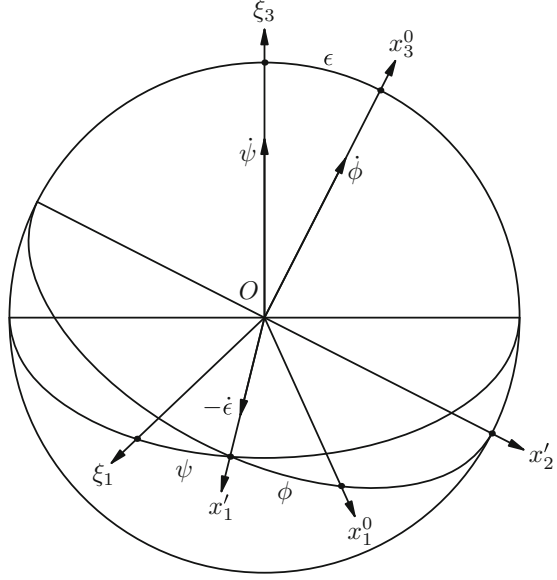
$$\vec{\omega}' = \dot{\alpha}_1 \hat{e}_1^0 + \dot{\alpha}_2 \hat{e}_2^0 + \dot{\alpha}_3 \hat{e}_3^0. \quad (8.30)$$

Remember that  $\vec{\omega}'$  is the angular velocity of  $Ox_1x_2x_3$  with respect to  $Ox_1^0x_2^0x_3^0$ , and that  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  will be explicitly defined later.

We now formulate the angular velocity of  $Ox_1^0x_2^0x_3^0$  with respect to  $O\xi_1\xi_2\xi_3$ . We use  $\psi$ ,  $\epsilon$  and  $\phi$  to denote the Euler angles of  $Ox_1^0x_2^0x_3^0$  with respect to  $O\xi_1\xi_2\xi_3$ , as shown in Fig. 8.4. We write the angular velocity of  $Ox_1^0x_2^0x_3^0$  with respect to  $O\xi_1\xi_2\xi_3$  expressed in  $Ox_1^0x_2^0x_3^0$  as

$$\vec{\omega}^0 = \delta \omega_1^0 \hat{e}_1^0 + \delta \omega_2^0 \hat{e}_2^0 + \omega_0 \hat{e}_3^0. \quad (8.31)$$

**Fig. 8.4** The Euler angles of the auxiliary coordinate system  $Ox_1^0x_2^0x_3^0$  with respect to the space-fixed coordinate system  $O\xi_1\xi_2\xi_3$ , where the coordinate system  $Ox_1'x_2'x_3^0$  is referred to as precession-nutation coordinate system



The components  $\delta\omega_1^0$  and  $\delta\omega_2^0$  represent the precession-nutation (as to be expressed explicitly later), and  $\omega_0$  the rotation rate of  $Ox_1^0x_2^0x_3^0$  around the  $x_3^0$ -axis, which is approximately the rotation rate of the Earth. Hence,  $\delta\omega_1^0$  and  $\delta\omega_2^0$  are infinitesimal as compared to  $\omega_0$ , thus being written as increments. It is straightforward to infer that the components of the angular velocity,  $\delta\omega_1^0$ ,  $\delta\omega_2^0$ , and  $\omega_0$ , are related to the Euler angles according to

$$\begin{cases} \delta\omega_1^0 = -\dot{\psi} \sin \epsilon \sin \phi - \dot{\epsilon} \cos \phi, \\ \delta\omega_2^0 = -\dot{\psi} \sin \epsilon \cos \phi + \dot{\epsilon} \sin \phi, \\ \omega_0 = \dot{\psi} \cos \epsilon + \dot{\phi}, \end{cases} \quad (8.32)$$

which is the Euler's kinematic equation. Accurate to the first order of the small quantities  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ , the inverse of (8.29) is

$$\hat{e}_1^0 = \hat{e}_1 - \alpha_3 \hat{e}_2 + \alpha_2 \hat{e}_3, \quad \hat{e}_2^0 = \alpha_3 \hat{e}_1 + \hat{e}_2 - \alpha_1 \hat{e}_3, \quad \hat{e}_3^0 = -\alpha_2 \hat{e}_1 + \alpha_1 \hat{e}_2 + \hat{e}_3. \quad (8.33)$$

Its substitution into (8.30) and (8.31) leads to, accurate to the first order of the small quantities  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ ,

$$\vec{\omega}' = \dot{\alpha}_1 \hat{e}_1 + \dot{\alpha}_2 \hat{e}_2 + \dot{\alpha}_3 \hat{e}_3, \quad (8.34)$$

$$\vec{\omega}^0 = (-\alpha_2 \omega_0 + \delta\omega_1^0) \hat{e}_1 + (\alpha_1 \omega_0 + \delta\omega_2^0) \hat{e}_2 + \omega_0 \hat{e}_3. \quad (8.35)$$

The former is the angular velocity of  $Ox_1x_2x_3$  with respect to  $Ox_1^0x_2^0x_3^0$ , and the latter that of  $Ox_1^0x_2^0x_3^0$  with respect to  $O\xi_1\xi_2\xi_3$ , both expressed in  $Ox_1x_2x_3$ .

We are now ready to derive the dynamic equation of Earth rotation. According to (8.34) and (8.35), we can express the angular velocity of  $Ox_1x_2x_3$  with respect to  $O\xi_1\xi_2\xi_3$ ,  $\vec{\omega} = \vec{\omega}^0 + \vec{\omega}'$ , as

$$\begin{aligned}\vec{\omega} &= \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3 \\ &= \left( \dot{\alpha}_1 - \alpha_2 \omega_0 + \delta \omega_1^0 \right) \hat{e}_1 + \left( \dot{\alpha}_2 + \alpha_1 \omega_0 + \delta \omega_2^0 \right) \hat{e}_2 + (\dot{\alpha}_3 + \omega_0) \hat{e}_3.\end{aligned}\quad (8.36)$$

The substitution of (8.36) into Euler's dynamic equation (3.313) leads to the linearized dynamic equation of Earth rotation

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_E) \dot{\alpha}_2 + \omega_0 \sigma_E \alpha_1 + \frac{d\delta\omega_1^0}{dt} + \sigma_E \delta\omega_2^0 = \frac{N_1}{A}, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_E) \dot{\alpha}_1 + \omega_0 \sigma_E \alpha_2 + \frac{d\delta\omega_2^0}{dt} - \sigma_E \delta\omega_1^0 = \frac{N_2}{A}, \\ \ddot{\alpha}_3 + \dot{\omega}_0 = \frac{N_3}{C}, \end{cases}\quad (8.37)$$

where the physical meaning of the parameter  $\sigma_E$  defined by

$$\sigma_E = \frac{C - A}{A} \omega_0 \quad (8.38)$$

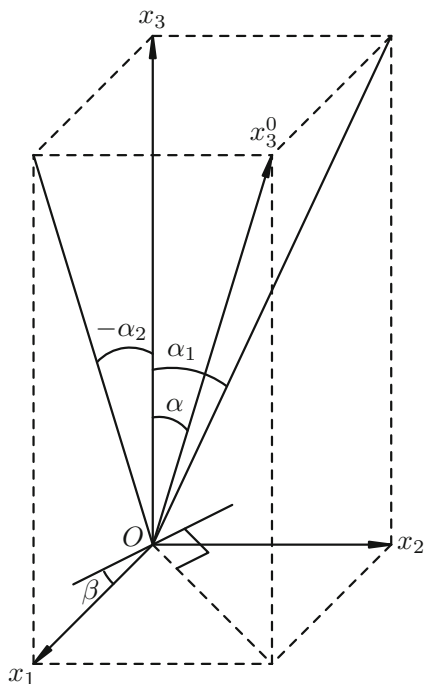
will be explained later. As the Earth rotation rate is almost constant,  $\dot{\omega}_0$  is also infinitesimal.

The Eqs. (8.32) and (8.37) are the linearized Euler's kinematic and dynamic equations for studying the Earth rotation. Two sets of infinitesimal quantities, one set is  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ , and the other set is  $\delta\omega_1^0$  and  $\delta\omega_2^0$ , are defined to express the three components of the Earth's angular velocity with respect to the inertial space as given in (8.36),  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$ . Only first order terms of the infinitesimal quantities are kept. The precession-nutation can be computed from the result of  $\delta\omega_1^0$  and  $\delta\omega_2^0$ , and the polar motion from that of  $\alpha_1$  and  $\alpha_2$ . The precession-nutation is represented by the Euler angles  $\psi$  and  $\epsilon$  as shown in Fig. 8.4, which can be readily obtained from the Euler's kinematic equation (8.32),

$$\begin{cases} \dot{\psi} \sin \epsilon = -\delta\omega_1^0 \sin \phi - \delta\omega_2^0 \cos \phi, \\ \dot{\epsilon} = -\delta\omega_1^0 \cos \phi + \delta\omega_2^0 \sin \phi. \end{cases}\quad (8.39)$$

The angles  $\alpha_1$  and  $\alpha_2$  defined in Fig. 8.3 represent the direction of the Earth-fixed axis  $x_3$  in the auxiliary coordinate system  $Ox_1^0x_2^0x_3^0$ . These angles can also be interpreted to represent the direction of the celestial axis  $x_3^0$  in the Earth-fixed coordinate system  $Ox_1x_2x_3$ , as shown in Fig. 8.5. The two components along the

**Fig. 8.5** Polar motion, i.e., the motion of the  $x_3^0$ -axis with respect to the coordinate system  $Ox_1x_2x_3$ , expressed using  $\alpha_1$  and  $\alpha_2$ . The angle  $\beta$  was defined in Fig. 8.3 and  $\beta - \gamma = \alpha_3$  is assumed to be infinitesimal



prime meridian and west longitude  $90^\circ$  are

$$x_p = -\alpha_2, \quad y_p = -\alpha_1, \quad (8.40)$$

where the fact that  $x_2$  points to east longitude  $90^\circ$  is taken into account. The variation of  $x_p$  and  $y_p$  is the polar motion.

In the systems of equations (8.32) and (8.37), there are nine unknown functions, but there are only six equations. We can actually consider (8.37) alone, as three more functions are included in the three equations in (8.32). We need three more equations to uniquely determine the solution. This is due to the fact that the auxiliary coordinate system  $Ox_1^0x_2^0x_3^0$  is in fact free of choice—the only assumption we made is that it is very close to the Earth-fixed coordinate system  $Ox_1x_2x_3$ , so that  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  are infinitesimal. In the following, we always assume  $\omega_0 = \text{Constant}$  to be the Earth's average rotation rate, thus  $\dot{\omega}_0 = 0$ . Three different choices of the celestial axis  $x_3^0$  are of basic importance physically.

1. The figure axis. We choose the  $x_3^0$ -axis to coincide with the  $x_3$ -axis, which is defined to be the symmetry axis of the Earth, and is also an axis of principal moment of inertia. Evidently, it performs precession-nutation, but not polar

motion. We have

$$\alpha_1 = 0, \quad \alpha_2 = 0. \quad (8.41)$$

2. The rotation axis. We choose the  $x_3^0$ -axis to be the direction of the angular velocity of  $Ox_1x_2x_3$  with respect  $O\xi_1\xi_2\xi_3$ , i.e., the direction of  $\vec{\omega}$  in (8.36). Taking reference to Fig. 8.5, we have

$$-\alpha_2 = \frac{\omega_1}{\omega_3}, \quad \alpha_1 = \frac{\omega_2}{\omega_3}. \quad (8.42)$$

With (8.36), we obtain the relations between  $\alpha_1, \alpha_2$  and  $\delta\omega_1^0, \delta\omega_2^0$ ,

$$\dot{\alpha}_1 + \delta\omega_1^0 = 0, \quad \dot{\alpha}_2 + \delta\omega_2^0 = 0. \quad (8.43)$$

3. The angular momentum axis. The angular momentum vector is

$$\vec{L} = A\omega_1\hat{e}_1 + A\omega_2\hat{e}_2 + C\omega_3\hat{e}_3. \quad (8.44)$$

In Fig. 8.5, by choosing the  $x_3^0$ -axis to be in the same direction as  $\vec{L}$ , we obtain

$$-\alpha_2 = \frac{A\omega_1}{C\omega_3}, \quad \alpha_1 = \frac{A\omega_2}{C\omega_3}. \quad (8.45)$$

With (8.36) and (8.38), we obtain the relations between  $\alpha_1, \alpha_2$  and  $\delta\omega_1^0, \delta\omega_2^0$ ,

$$\dot{\alpha}_1 + \sigma_E\alpha_2 + \delta\omega_1^0 = 0, \quad \dot{\alpha}_2 - \sigma_E\alpha_1 + \delta\omega_2^0 = 0. \quad (8.46)$$

By choosing the celestial axis  $x_3^0$  to be the figure axis, rotation axis, or angular momentum axis, we can use (8.41), (8.43), or (8.46), respectively, to reduce the first two of Euler's dynamic equations in (8.37) to equations of  $\alpha_1$  and  $\alpha_2$  or of  $\delta\omega_1^0$  and  $\delta\omega_2^0$ . We mention that the celestial axis  $x_3^0$  can also be defined otherwise beside these three cases, such as the CIP to be addressed later.

### 8.1.3 Forced Rotation

We first discuss the effect of the lunisolar gravitational torque given by (8.19). The governing equation can be obtained by substituting (8.19) into the linearized Euler's dynamic equation (8.37), together with an equation defining the auxiliary coordinate system, such as (8.41) for the figure axis, (8.43) for the rotation axis, or (8.46) for the angular momentum axis. Precession-nutation and polar motion can be obtained using (8.39) and (8.40) from the solution of  $\delta\omega_1^0, \delta\omega_2^0$  and  $\alpha_1, \alpha_2$ . As  $N_3 = 0$ , the

last equation in (8.37) is not to be further discussed in this subsection, and hence, only precession-nutation and polar motion are to be concerned.

### 8.1.3.1 The Figure Axis

For the figure axis, by substituting (8.19) into (8.37), and then applying (8.41), we obtain

$$\begin{cases} \frac{d\delta\omega_1^0}{dt} + \sigma_E \delta\omega_2^0 = 2 \frac{D}{R^2} \frac{C-A}{A} \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \frac{d\delta\omega_2^0}{dt} - \sigma_E \delta\omega_1^0 = -2 \frac{D}{R^2} \frac{C-A}{A} \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.47)$$

The solution is

$$\begin{cases} \delta\omega_1^0 = 2 \frac{D}{R^2} \frac{C-A}{A} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \delta\omega_2^0 = 2 \frac{D}{R^2} \frac{C-A}{A} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.48)$$

The precession-nutation rate can be obtained from (8.39) to be

$$\begin{cases} \dot{\psi} \sin \epsilon = -2 \frac{D}{R^2} \frac{C-A}{A} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \cos(\omega_i^d t + \phi_i^d - \phi), \\ \dot{\epsilon} = -2 \frac{D}{R^2} \frac{C-A}{A} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \sin(\omega_i^d t + \phi_i^d - \phi). \end{cases} \quad (8.49)$$

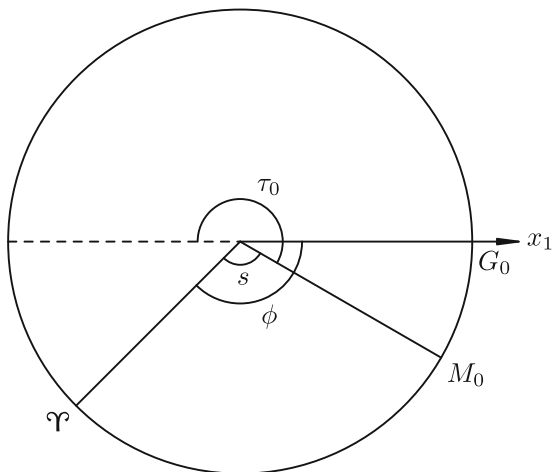
In (8.49),  $\omega_i^d t + \phi_i^d$  is the argument of a tidal wave, which is a linear combination of the six fundamental astronomical parameters, as given in (7.82). We copy it to here for convenience of reference,

$$\omega_i^d t + \phi_i^d = \tau_0 + b_i^d s + c_i^d h + d_i^d p + e_i^d N' + f_i^d p_8. \quad (8.50)$$

This formula has been used to derive (8.19) from (8.18). It is reminded that  $\tau_0$  is the mean lunar time at the prime meridian. The coefficient of  $\tau_0$  is 1 because only the degree 2 diurnal tidal waves are concerned here. The Euler angle  $\phi$  is the angle from the vernal equinox to the  $x_1^0$ -axis measured in the equator of the  $x_3^0$ -axis as shown in Fig. 8.4. By definition, the angle between the  $x_1$ - and  $x_1^0$  axes, i.e., the angle between  $\hat{e}_1$  and  $\hat{e}_1^0$ , is infinitesimal, as can be inferred analytically by computing  $\hat{e}_1 \times \hat{e}_1^0$  according to (8.29) or (8.33). By ignoring this infinitesimal angle, we obtain, according to Fig. 8.6,

$$\phi = \tau_0 + s - \pi. \quad (8.51)$$

**Fig. 8.6** The angle from the vernal equinox  $\Upsilon$  to the prime meridian  $G_0$  in the equator,  $\phi$ , where  $M_0$  is the mean Moon



We thus have

$$\omega_i^d t + \phi_i^d - \phi = (b_i^d - 1)s + c_i^d h + d_i^d p + e_i^d N' + f_i^d p_s + \pi. \quad (8.52)$$

The angular rate of the Earth rotation around the  $x_3^0$ -axis with respect to the vernal equinox is

$$\dot{\phi} = \dot{\tau}_0 + \dot{s}, \quad (8.53)$$

which is referred to as the Earth's sidereal rotation rate. We have

$$(b_i^d - 1)s + c_i^d h + d_i^d p + e_i^d N' + f_i^d p_s = \Delta\omega_i^d t + \zeta_i^d, \quad (8.54)$$

where

$$\Delta\omega_i^d = \omega_i^d - \dot{\phi} \quad (8.55)$$

is the difference between the angular frequency of the tidal wave and the Earth's sidereal rotation rate. In summary, we have

$$\omega_i^d t + \phi_i^d - \phi = \Delta\omega_i^d t + \zeta_i^d + \pi. \quad (8.56)$$

The substitution of (8.56) into (8.49) leads to

$$\begin{cases} \dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{C-A}{A} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i \cos(\Delta\omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = 2 \frac{D}{R^2} \frac{C-A}{A} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i \sin(\Delta\omega_i^d t + \zeta_i^d). \end{cases} \quad (8.57)$$

Between  $\dot{\phi}$  defined in (8.53) and  $\omega_0$  in (8.32), there is an extremely small difference  $\dot{\psi} \cos \epsilon$ , where  $\dot{\psi}$  is the precession rate of the equator, and  $\epsilon$  the obliquity of the ecliptic. By neglecting the effect of this small difference in the amplitude, we have, according to (8.55) and (8.38),

$$\frac{1}{\omega_i^d + \sigma_E} = \frac{A}{C} \frac{1}{\omega_0} \left( 1 - \frac{A}{C} \frac{\Delta \omega_i^d}{\omega_0} \right), \quad (8.58)$$

which is accurate to the first order of  $\Delta \omega_i^d / \omega_0$ . Finally, we can simplify (8.57) to

$$\begin{cases} \dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{C-A}{C} \frac{1}{\omega_0} \sum_i \left( 1 - \frac{A}{C} \frac{\Delta \omega_i^d}{\omega_0} \right) Y_i \cos(\Delta \omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = 2 \frac{D}{R^2} \frac{C-A}{C} \frac{1}{\omega_0} \sum_i \left( 1 - \frac{A}{C} \frac{\Delta \omega_i^d}{\omega_0} \right) Y_i \sin(\Delta \omega_i^d t + \zeta_i^d). \end{cases} \quad (8.59)$$

By observing (8.59), (8.54), and the tidal waves in Table 7.2, we immediately realize that  $\Delta \omega_i^d t + \zeta_i^d = 0$  for the tidal wave  $K_1$  with the Doodson's argument number number 165.555. We define  $i = 0$  for this tidal wave, and write its effect on precession-nutation individually as

$$\dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{1}{\omega_0} \frac{C-A}{C} Y_0^d, \quad \dot{\epsilon} = 0. \quad (8.60)$$

This formula describes the precession. With the tidal amplitude  $Y_0 = -0.5305$  (the sum of the amplitudes of  ${}^m K_1$  and  ${}^s K_1$ ), the observed precession rate of the equator  $\dot{\psi} = -50.38''$  per year, the observed obliquity of the ecliptic  $\epsilon = 23^\circ 27'$ , and the observed Earth rotation rate  $\omega = 7.292 \times 10^{-5} \text{ s}^{-1}$ , we obtain

$$H = \frac{C-A}{C} = \frac{1}{305.4}. \quad (8.61)$$

This is how the dynamic flattening  $H$  is determined from the observation of the precession rate. As the Earth is ceaselessly deforming,  $H$  is not constant. Therefore, the precession rate is neither a constant. We will use another symbol

$$H' = \frac{C-A}{A} \quad (8.62)$$

in subsequent texts for writing formulae shorter.

Define (Melchior, 1983)

$$E = 2 \frac{D}{R^2} \frac{H}{\omega_0^2} = 0.01644''. \quad (8.63)$$



We can write the nutation rate (8.59) as

$$\begin{cases} \dot{\psi} \sin \epsilon = E \omega_0 \sum_i \left( 1 - \frac{A}{C} \frac{\Delta \omega_i^d}{\omega_0} \right) Y_i^d \cos(\Delta \omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = E \omega_0 \sum_i \left( 1 - \frac{A}{C} \frac{\Delta \omega_i^d}{\omega_0} \right) Y_i^d \sin(\Delta \omega_i^d t + \zeta_i^d). \end{cases} \quad (8.64)$$

By integrating from  $t_0$  to  $t$ , we obtain the nutation angle (excluding the term  $i = 0$  in the sum),

$$\begin{cases} \delta \psi \sin \epsilon = E \omega_0 \sum_i \left( \frac{Y_i^d}{\Delta \omega_i^d} - \frac{A}{C} \frac{Y_i^d}{\omega_0} \right) \delta \sin(\Delta \omega_i^d t + \zeta_i^d), \\ \delta \epsilon = -E \omega_0 \sum_i \left( \frac{Y_i^d}{\Delta \omega_i^d} - \frac{A}{C} \frac{Y_i^d}{\omega_0} \right) \delta \cos(\Delta \omega_i^d t + \zeta_i^d), \end{cases} \quad (8.65)$$

where  $\delta$  is used to denote the increment from  $t_0$  to  $t$ . The physical interpretation of the term  $Y_i^d / \Delta \omega_i^d$  in the amplitude is that there exists a resonance at the sidereal diurnal frequency. The precession can be interpreted as the result of an exact resonance of the Earth's nutation to the tidal wave  $K_1$  with  $i = 0$ ,  $\Delta \omega_0^d = 0$ , and  $Y_0^d / \Delta \omega_0^d$  infinitely large.

We arrange the diurnal tidal waves into pairs of  $i$  and  $-i$ , which are symmetrical with respect to the sidereal angular rate of Earth rotation, i.e.,

$$\Delta \omega_{-i}^d t + \zeta_{-i}^d = -(\Delta \omega_i^d t + \zeta_i^d) \quad (8.66)$$

according to (8.54). This can be achieved as follows:

$$\begin{cases} i : a_i^d (b_i^d + 5)(c_i^d + 5) \cdot (d_i^d + 5)(e_i^d + 5)(f_i^d + 5), & \Delta \omega_i^d > 0; \\ -i : a_i^d [(-b_i^d + 2) + 5](-c_i^d + 5) \cdot (-d_i^d + 5)(-e_i^d + 5)(-f_i^d + 5), & \Delta \omega_{-i}^d < 0. \end{cases} \quad (8.67)$$

Consequently, we can write the nutation angle as (keeping  $N$  pairs of tidal waves)

$$\begin{cases} \delta \psi \sin \epsilon = E \omega_0 \sum_{i=1}^N \left( \frac{Y_i^d + Y_{-i}^d}{\Delta \omega_i^d} - \frac{A}{C} \frac{Y_i^d - Y_{-i}^d}{\omega_0} \right) \delta \sin(\Delta \omega_i^d t + \zeta_i^d), \\ \delta \epsilon = -E \omega_0 \sum_{i=1}^N \left( \frac{Y_i^d - Y_{-i}^d}{\Delta \omega_i^d} - \frac{A}{C} \frac{Y_i^d + Y_{-i}^d}{\omega_0} \right) \delta \cos(\Delta \omega_i^d t + \zeta_i^d). \end{cases} \quad (8.68)$$

It is noted that the nutation wave corresponding to each tidal wave expressed in (8.65) is circular, and that the nutation wave corresponding to each pair of tidal waves expressed in (8.68) is elliptical. Each elliptical wave is the combination of one prograde and one retrograde circular waves of the same frequency. The precession

**Table 8.1** Precession and main nutation waves

| Doodson's argument | Argument of tidal wave | Magnitude of tidal wave | Argument of nutation | Period of nutation | Magnitude of $\delta\psi$ and $\delta\epsilon$ | Type cause         |
|--------------------|------------------------|-------------------------|----------------------|--------------------|------------------------------------------------|--------------------|
| 165.555            | $\tau + s$             | -53,002                 | 0                    | $\infty$           |                                                | Precession         |
| 165.565            | $\tau + s + N'$        | -7185                   | $N'$                 | 18.6 years         | -17.23''                                       | Principle nutation |
| 165.545            | $\tau + s - N'$        | 1051                    |                      |                    | 9.21''                                         | Lunar node         |
| 167.555            | $\tau + s + 2h$        | -754                    | $2h$                 | 183 days           | -1.27''                                        | Semi-annual        |
| 163.555            | $\tau + s - 2h$        | 17,531                  |                      |                    | 0.55''                                         | Sun                |
| 185.555            | $\tau + 3s$            | -1623                   | $2s$                 | 13.7 days          | -0.20''                                        | Semi-monthly       |
| 145.555            | $\tau - s$             | 37,690                  |                      |                    | 0.09''                                         | Moon               |

and some largest nutation waves are listed in Table 8.1 (Melchior, 1983; Moritz & Mueller, 1987). The magnitudes of the tidal waves, i.e.,  $Y_i^d$  and  $Y_{-i}^d$ , are taken from Xi and Hou (1987) scaled in the same way as in Table 7.2. As an exercise, the readers are suggested to compute the nutation magnitudes using the data of the tidal waves.

### 8.1.3.2 The Rotation Axis

For the rotation axis, by substituting (8.19) into (8.37), and then using (8.43), we obtain,

$$\begin{cases} \dot{\alpha}_1 + \sigma_E \alpha_2 = -2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \dot{\alpha}_2 - \sigma_E \alpha_1 = -2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d), \end{cases} \quad (8.69)$$

where  $H'$  is defined in (8.62). In subsequent text,  $H$  defined in (8.61) and  $H'$  will be used without further explanation. The solution is

$$\begin{cases} \alpha_1 = 2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.70)$$

With (8.58),  $H'/H = C/A$ , and (8.63), we obtain

$$\begin{cases} \alpha_1 = E \sum_i \left(1 - \frac{A}{C} \frac{\Delta \omega_i^d}{\omega_0}\right) Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -E \sum_i \left(1 - \frac{A}{C} \frac{\Delta \omega_i^d}{\omega_0}\right) Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.71)$$

According to (8.40), the two components of polar motion,  $x_p$  and  $y_p$ , are  $-\alpha_2$  and  $-\alpha_1$ , respectively.

To formulate precession-nutation, we substitute (8.70) into (8.43) to obtain

$$\begin{cases} \delta\omega_1^0 = 2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i \frac{\omega_i^d}{\omega_i^d + \sigma_E} Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \delta\omega_2^0 = 2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i \frac{\omega_i^d}{\omega_i^d + \sigma_E} Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.72)$$

We then obtain from (8.39) that

$$\begin{cases} \dot{\psi} \sin \epsilon = -2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i \frac{\omega_i^d}{\omega_i^d + \sigma_E} Y_i \cos(\omega_i^d t + \phi_i^d - \phi), \\ \dot{\epsilon} = -2 \frac{D}{R^2} \frac{H'}{\omega_0} \sum_i \frac{\omega_i^d}{\omega_i^d + \sigma_E} Y_i \sin(\omega_i^d t + \phi_i^d - \phi). \end{cases} \quad (8.73)$$

We have, like (8.58),

$$\frac{\omega_i^d}{\omega_i^d + \sigma_E} = \frac{A}{C} \left( 1 + H \frac{\Delta\omega_i^d}{\omega_0} \right). \quad (8.74)$$

We can then use this formula and (8.56) to simplify (8.71) to

$$\begin{cases} \dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \left( 1 + H \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i \cos(\Delta\omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \left( 1 + H \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i \sin(\Delta\omega_i^d t + \zeta_i^d). \end{cases} \quad (8.75)$$

The precession can be readily inferred to be

$$\dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{H}{\omega_0} Y_0^d, \quad \dot{\epsilon} = 0, \quad (8.76)$$

which is the same as that of the figure axis. Similar to the case of figure axis, the nutation rate can be written as, with (8.63),

$$\begin{cases} \dot{\psi} \sin \epsilon = E \omega_0 \sum_i \left( 1 + H \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \cos(\Delta\omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = E \omega_0 \sum_i \left( 1 + H \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \sin(\Delta\omega_i^d t + \zeta_i^d), \end{cases} \quad (8.77)$$

where  $E$  is defined in (8.63). The integration from  $t_0$  to  $t$  yields the nutation angle

$$\begin{cases} \delta\psi \sin \epsilon = E\omega_0 \sum_i \left( \frac{Y_i^d}{\Delta\omega_i^d} + H \frac{Y_i^d}{\omega_0} \right) \delta \sin(\Delta\omega_i^d t + \zeta_i^d), \\ \delta\epsilon = -E\omega_0 \sum_i \left( \frac{Y_i^d}{\Delta\omega_i^d} + H \frac{Y_i^d}{\omega_0} \right) \delta \cos(\Delta\omega_i^d t + \zeta_i^d). \end{cases} \quad (8.78)$$

After arranging the tidal waves as pairs according to (8.67), we can use (8.66) to express the nutation as

$$\begin{cases} \delta\psi \sin \epsilon = E\omega_0 \sum_{i=1}^N \left( \frac{Y_i^d + Y_{-i}^d}{\Delta\omega_i^d} + H \frac{Y_i^d - Y_{-i}^d}{\omega_0} \right) \delta \sin(\Delta\omega_i^d t + \zeta_i^d), \\ \delta\epsilon = -E\omega_0 \sum_{i=1}^N \left( \frac{Y_i^d - Y_{-i}^d}{\Delta\omega_i^d} + H \frac{Y_i^d + Y_{-i}^d}{\omega_0} \right) \delta \cos(\Delta\omega_i^d t + \zeta_i^d). \end{cases} \quad (8.79)$$

### 8.1.3.3 The Angular Momentum Axis

For the angular momentum axis, we substitute (8.19) into (8.37) and then use (8.46) to obtain

$$\begin{cases} (\omega_0 + \sigma_E)\delta\omega_1^0 = -\frac{N_2}{A} = 2\frac{D}{R^2}H' \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ (\omega_0 + \sigma_E)\delta\omega_2^0 = \frac{N_1}{A} = 2\frac{D}{R^2}H' \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.80)$$

The solution is, with  $\omega_0 + \sigma_E = (C/A)\omega_0$ ,

$$\begin{cases} \delta\omega_1^0 = 2\frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \delta\omega_2^0 = 2\frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.81)$$

The precession-nutation rate is, according to (8.39) and (8.56),

$$\begin{cases} \dot{\psi} \sin \epsilon = 2\frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \cos(\Delta\omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = 2\frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \sin(\Delta\omega_i^d t + \zeta_i^d). \end{cases} \quad (8.82)$$

The precession is

$$\dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{H}{\omega_0} Y_0^d, \quad \dot{\epsilon} = 0, \quad (8.83)$$

which is the same as the cases of figure axis and rotation axis. The nutation rate is, with (8.63),

$$\begin{cases} \dot{\psi} \sin \epsilon = E \omega_0 \sum_i Y_i^d \cos(\Delta \omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = E \omega_0 \sum_i Y_i^d \sin(\Delta \omega_i^d t + \zeta_i^d). \end{cases} \quad (8.84)$$

The integration from  $t_0$  to  $t$  leads to the nutation angle

$$\begin{cases} \delta \psi \sin \epsilon = E \omega_0 \sum_i \frac{Y_i^d}{\Delta \omega_i^d} \delta \sin(\Delta \omega_i^d t + \zeta_i^d), \\ \delta \epsilon = -E \omega_0 \sum_i \frac{Y_i^d}{\Delta \omega_i^d} \delta \cos(\Delta \omega_i^d t + \zeta_i^d). \end{cases} \quad (8.85)$$

After arranging the tidal waves as pairs according to (8.67), we can use (8.66) to express the nutation as

$$\begin{cases} \delta \psi \sin \epsilon = E \omega_0 \sum_{i=1}^N \frac{Y_i^d + Y_{-i}^d}{\Delta \omega_i^d} \delta \sin(\Delta \omega_i^d t + \zeta_i^d), \\ \delta \epsilon = -E \omega_0 \sum_{i=1}^N \frac{Y_i^d - Y_{-i}^d}{\Delta \omega_i^d} \delta \cos(\Delta \omega_i^d t + \zeta_i^d). \end{cases} \quad (8.86)$$

The difference between it and those of the figure axis or rotation axis is very small.

The equations for  $\alpha_1$  and  $\alpha_2$  can be obtained by substituting (8.81) into (8.46),

$$\begin{cases} \dot{\alpha}_1 + \sigma_E \alpha_2 = -2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \dot{\alpha}_2 - \sigma_E \alpha_1 = -2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.87)$$

The solution is

$$\begin{cases} \alpha_1 = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \frac{1}{\omega_i^d + \sigma_E} Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.88)$$

With (8.58) and (8.63), we finally obtain

$$\begin{cases} \alpha_1 = E \frac{A}{C} \sum_i \left( 1 - \frac{A}{C} \frac{\Delta \omega_i}{\omega_0} \right) Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -E \frac{A}{C} \sum_i \left( 1 - \frac{A}{C} \frac{\Delta \omega_i}{\omega_0} \right) Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.89)$$

According to (8.40), the two components of polar motion,  $x_p$  and  $y_p$ , are  $-\alpha_2$  and  $-\alpha_1$ , respectively.

We see that, for both the rotation and angular momentum axes, the amplitude of the polar motion is on the order of  $\Delta \omega_i^d / \omega_0$  of the amplitude of the nutation. The small differences among the nutations of different axes, the figure axis, the rotation axis, and the angular momentum axis, are called Oppolzer terms.

### 8.1.4 Normal Modes of Earth Rotation

The governing equations of the Earth rotation, (8.37), are a system of nonhomogeneous linear ordinary differential equations. The system of complementary homogeneous ordinary differential equations is

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_E) \dot{\alpha}_2 + \omega_0 \sigma_E \alpha_1 + \frac{d\delta\omega_1^0}{dt} + \sigma_E \delta\omega_2^0 = 0, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_E) \dot{\alpha}_1 + \omega_0 \sigma_E \alpha_2 + \frac{d\delta\omega_2^0}{dt} - \sigma_E \delta\omega_1^0 = 0, \\ \ddot{\alpha}_3 = 0. \end{cases} \quad (8.90)$$

The sum of a solution of this system of equations and a solution of (8.37) is still a solution of (8.37). Hence, the solutions of this system of equations are also components of Earth rotation. These solutions are independent of any external forcing, and are solutions of an eigenvalue problem. Hence, they are referred to as free rotations or normal modes. We will see that the precession is completely caused by the lunisolar gravitational torque, and no free precession is present.

To solve (8.90), relations between  $\alpha_1$ ,  $\alpha_2$  and  $\delta\omega_1^0$ ,  $\delta\omega_2^0$  are required, which are indeed provided by the definition of the auxiliary coordinate system.

#### 8.1.4.1 The Figure Axis

The figure axis does not have a polar motion, and we have  $\alpha_1 = \alpha_2 = 0$ . Hence, (8.90) is simplified to

$$\frac{d\delta\omega_1^0}{dt} + \sigma_E \delta\omega_2^0 = 0, \quad \frac{d\delta\omega_2^0}{dt} - \sigma_E \delta\omega_1^0 = 0, \quad \ddot{\alpha}_3 = 0. \quad (8.91)$$

Define a complex number

$$\delta\omega^0 = \delta\omega_1^0 + i\delta\omega_2^0. \quad (8.92)$$

We can write (8.91) as

$$\frac{d\delta\omega^0}{dt} - i\sigma_E\omega^0 = 0, \quad \ddot{\alpha}_3 = 0. \quad (8.93)$$

We assume that the solution is in the form of

$$\delta\omega^0 = De^{i\sigma t}, \quad \dot{\alpha}_3 = D'e^{i\sigma t}, \quad (8.94)$$

where  $D$  and  $D'$  are constants. The differential equations (8.93) are simplified to the algebraic equations

$$D\sigma - D\sigma_E = 0, \quad D'\sigma = 0. \quad (8.95)$$

We are seeking nontrivial solutions, i.e., solutions with nonzero  $D$  and  $D'$ . We write (8.95) in the matrix form

$$\begin{bmatrix} \sigma - \sigma_E & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} D \\ D' \end{bmatrix} = 0. \quad (8.96)$$

This is a system of homogeneous linear algebraic equation. In order for  $D$  and  $D'$  to have nonzero solutions, the determinant of the coefficient matrix must vanish, yielding two solutions of  $\sigma$ ,

$$\sigma_1 = \sigma_E = \frac{C - A}{A}\omega_0, \quad \sigma_2 = 0. \quad (8.97)$$

Each of these two solutions lead to a non-vanishing solution of (8.91).

By substituting the first solution  $\sigma = \sigma_1 = \sigma_E$  into (8.95), we obtain that  $D$  could be any constant and  $D' = 0$ . We can thus obtain a nontrivial solution of (8.91) according to (8.94),

$$\delta\omega_1^0 = D \cos \sigma_E t, \quad \delta\omega_2^0 = D \sin \sigma_E t; \quad \dot{\alpha}_3 = 0. \quad (8.98)$$

The solution of nutation in  $\dot{\psi} \sin \epsilon$  and  $\dot{\epsilon}$  can be computed according to (8.39). With  $\phi = \dot{\phi}t + \phi_0$ , where  $\phi_0$  is the initial phase, we obtain

$$\begin{cases} \dot{\psi} \sin \epsilon = -D \sin [(\dot{\phi} + \sigma_E)t + \phi_0] \\ \dot{\epsilon} = -D \cos [(\dot{\phi} + \sigma_E)t + \phi_0] \end{cases} \quad (8.99)$$

The free nutation is nearly diurnal with an angular frequency  $\dot{\phi} + \sigma_E \approx \omega_0 + \sigma_E = (C/A)\omega_0$ . It is left to the reader to integrate (8.99) over time to convert the rate of change to amplitude. See also below for the case of the rotation axis.

By substituting the second solution  $\sigma = \sigma_2 = 0$  into (8.95), we obtain that  $D = 0$  and  $D'$  could be any constant. We thus obtain the second nontrivial solution of (8.91) according to (8.94),

$$\delta\omega_1^0 = \delta\omega_2^0 = 0, \quad \dot{\alpha}_3 = D'. \quad (8.100)$$

This is a rotation of the Earth around the figure axis  $x_3$  with a constant incremental rate  $D'$ , i.e., with an angular rate of  $D'$  in addition to  $\omega^0$ , thus being referred to as axial spin mode.

#### 8.1.4.2 The Rotation Axis

For the rotation axis, we use the relations between  $\alpha_1, \alpha_2$  and  $\delta\omega_1^0, \delta\omega_2^0$ , (8.43), to simplify (8.90) to

$$\dot{\alpha}_1 + \sigma_E \alpha_2 = 0, \quad \dot{\alpha}_2 - \sigma_E \alpha_1 = 0, \quad \ddot{\alpha}_3 = 0. \quad (8.101)$$

They are in the same form as (8.91) with the unknown functions  $\delta\omega_1^0$  and  $\delta\omega_2^0$  replaced by  $\alpha_1$  and  $\alpha_2$ .

The first nontrivial solution of (8.101) is in the same form as (8.98). We have

$$\alpha_1 = D \cos \sigma_E t, \quad \alpha_2 = D \sin \sigma_E t; \quad \ddot{\alpha}_3 = 0. \quad (8.102)$$

This is a periodical variation of the rotation axis with respect to the Earth with an angular frequency of  $\sigma_E$ , called Eulerian wobble. Its period is  $[A/(C-A)](2\pi/\omega_0)$ , i.e., about  $A/(C-A) = 304.4$  sidereal days (noting that the difference between the exact length of a sidereal day defined by  $2\pi/\dot{\phi}$  and the time for the Earth to rotate one cycle with respect to the inertial space defined by  $2\pi/\omega_0$  is on the order of only  $1.0^{-7}$  day). According to the definitions of  $x_p$  and  $-y_p$  in (8.40), the  $x_1$ - and  $x_2$ -coordinates of the pole are  $x_p$  and  $-y_p$ , i.e.,  $-\alpha_2$  and  $\alpha_1$ . We thus write (8.102) as

$$-\alpha_2 = D \cos \left( \sigma_E t + \frac{\pi}{2} \right), \quad \alpha_1 = D \sin \left( \sigma_E t + \frac{\pi}{2} \right); \quad \ddot{\alpha}_3 = 0, \quad (8.103)$$

for physical interpretation. The appearance of the phase  $\pi/2$  is due to the definition of  $\alpha_1$  and  $\alpha_2$ . The expression (8.103) means that the  $x_0^3$ -axis draws a circle of radius  $D$  around the  $x_3$ -axis progradely every 304.4 sidereal days. In reality, a wobble with such a period has not been observed. This is due to the fact that the Earth is not rigid. With the complexity of the Earth taken into account, the observed period of this wobble is 434.2 sidereal days, called Chandler wobble in honor to the discoverer.

To solve for nutation, we substitute (8.102) into (8.43) to obtain

$$\delta\omega_1^0 = D\sigma_E \sin \sigma_E t, \quad \delta\omega_2^0 = -D\sigma_E \cos \sigma_E t. \quad (8.104)$$



Similar to the derivation of (8.99) from (8.98) using (8.39), we have

$$\begin{cases} \dot{\psi} \sin \epsilon = D \sigma_E \cos [(\dot{\phi} + \sigma_E) t + \phi_0] , \\ \dot{\epsilon} = -D \sigma_E \sin [(\dot{\phi} + \sigma_E) t + \phi_0] . \end{cases} \quad (8.105)$$

With  $\dot{\phi} + \sigma_E = (C/A)\omega_0$ , the amplitude of this nutation is obtained by integrating over time from  $t_0$  to  $t$ ,

$$\begin{cases} \delta\psi \sin \epsilon = D \frac{A}{C} \frac{\sigma_E}{\omega_0} \delta \sin [(\dot{\phi} + \sigma_E) t + \phi_0] = DH \delta \sin [(\dot{\phi} + \sigma_E) t + \phi_0] , \\ \delta\epsilon = D \frac{A}{C} \frac{\sigma_E}{\omega_0} \delta \cos [(\dot{\phi} + \sigma_E) t + \phi_0] = DH \delta \cos [(\dot{\phi} + \sigma_E) t + \phi_0] , \end{cases} \quad (8.106)$$

By comparing the nutation (8.106) with the polar motion (8.101), we see differences in their amplitudes and frequencies. The polar motion is slow with a frequency of  $1/304.4$  cycle per sidereal day, but the nutation is nearly diurnal with a frequency of  $1 + 1/304.4$  cycle per sidereal day. The amplitude of the nutation is only on the order of  $H = 1/305.4$  of that of the polar motion.

The second nontrivial solution of (8.101) is

$$\alpha_1 = \alpha_2 = 0, \quad \dot{\alpha}_3 = D', \quad (8.107)$$

which is the axial spin mode. It is not associated with a nutation.

### 8.1.4.3 The Angular Momentum Axis

For the angular momentum axis, we can use (8.46) to simplify (8.90) to

$$(\omega_0 + \sigma_E) \delta \omega_2^0 = 0, \quad -(\omega_0 + \sigma_E) \delta \omega_1^0 = 0; \quad \ddot{\alpha}_3 = 0. \quad (8.108)$$

These equations have only one solution

$$\delta \omega_1^0 = 0, \quad \delta \omega_2^0 = 0, \quad \dot{\alpha}_3 = D', \quad (8.109)$$

which is the axial spin mode. No free nutation mode is present.

For the polar motion, we obtain, by substituting (8.109) into (8.46),

$$\dot{\alpha}_1 + \sigma_E \alpha_2 = 0, \quad \dot{\alpha}_2 - \sigma_E \alpha_1 = 0. \quad (8.110)$$

This is a system of homogeneous linear ordinary differential equations. The non-trivial solution is

$$\alpha_1 = D \cos \sigma_E t, \quad \alpha_2 = D \sin \sigma_E t, \quad (8.111)$$

which is the Eulerian wobble.

It is seen that the Eulerian wobble appears only in polar motion, but not in nutation. The physical significance of this property is evident: Euler's dynamic equation is in fact the equation of the angular momentum. The angular momentum changes with respect to the inertial space only under the action of torques. Here, as the torque is vanishing, the angular momentum does not change, and therefore, its direction does not change, meaning that no free nutation exists.

#### 8.1.4.4 The Auxiliary Coordinate System Defined by a Space Fixed $x_3^0$ -Axis

When the direction of the  $x_3^0$ -axis is defined to not vary in space, i.e.,  $\delta\omega_1^0 = \delta\omega_2^0 = 0$ , (8.90) is simplified to

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_E)\dot{\alpha}_2 + \omega_0\sigma_E\alpha_1 = 0, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_E)\dot{\alpha}_1 + \omega_0\sigma_E\alpha_2 = 0, \end{cases} \quad (8.112)$$

where the third equation leading to the axial spin mode is dropped. With

$$\alpha = \alpha_1 + i\alpha_2, \quad (8.113)$$

we can express (8.112) as

$$\ddot{\alpha} + i(\omega_0 - \sigma_E)\dot{\alpha} + \omega_0\sigma_E\alpha = 0. \quad (8.114)$$

Assuming its solution is in the form of

$$\alpha = De^{i\sigma t}, \quad (8.115)$$

we obtain an algebraic equation.

$$[-\sigma^2 - (\omega_0 - \sigma_E)\sigma + \omega_0\sigma_E]D = 0. \quad (8.116)$$

In order for  $D$  to have nontrivial solutions, its coefficient must vanish. We obtain two solutions of  $\sigma$ ,

$$\sigma_1 = \sigma_E, \quad \sigma_2 = -\omega_0. \quad (8.117)$$

In both solutions,  $D$  can be any constant. The first solution is the Eulerian wobble, which has already been discussed for the rotation and angular momentum axes, and is not to be repeated here. The second solution yields

$$\alpha_1 = D \cos \omega_0 t, \quad \alpha_2 = -D \sin \omega_0 t. \quad (8.118)$$

The angular velocity of the Earth can be obtained from (8.36) to be  $\vec{\omega} = \omega_0 \hat{e}_3$ , meaning that the  $x_3$ -axis is the rotation axis as well as the angular momentum axis.

The physical meaning of this solution is that the  $x_0^3$ -axis draws a circle of radius  $D$  around the  $x_3$ -axis retrogradely every sidereal day (note again that this is the day defined by the Earth rotation rate with respect to the inertial space, not the vernal equinox). While observing from the inertial space, the directions of both the  $x_3^0$ - and  $x_3$ -axes do not vary, but they are not in the same direction; the rotation axis  $x_3$  is tilted to the celestial axis  $x_3^0$  by a constant angle  $D$ . Based on the physical property, this solution is called tilt-over mode.

For this auxiliary coordinate system, the forced polar motion is the negative of the forced precession-nutation of the figure axis.

Except the Tilt-over mode, the solutions for all types of the  $x_3^0$ -axes—figure, rotation, or angular momentum—can be derived one from another, which is left to interested readers as an exercise.

### 8.1.5 The Celestial Intermediate Pole

The celestial intermediate pole (CIP) is defined with the complexity of whole the Earth system taken into account. Here we mimic the definition of the CIP for the rigid Earth model within the accuracy of our formulation in this section.

Table 8.2 is a summary of the properties of nutation and polar motion of the axes we studied, i.e., the figure axis, the rotation axis and the angular momentum axis. The nutation and polar motion of all these axes have two periodical components, one diurnal and one secular (note the exception that the figure axis does not have any polar motion). The CIP is an empirically defined axis that moves with respect to the coordinate system  $O\xi_1\xi_2\xi_3$  with the figure axis' precession and its nutations with periods longer than 2 sidereal days. According to Table 8.2, the CIP should include only the precession and forced nutation of the figure axis. This is equivalent to assign the CIP as the  $x_3^0$ -axis defined through (8.47). Here we write it as

$$\frac{d\delta\omega_1^0}{dt} + \sigma_E\delta\omega_2^0 = \frac{N_1}{A}, \quad \frac{d\delta\omega_2^0}{dt} - \sigma_E\delta\omega_1^0 = \frac{N_2}{A}. \quad (8.119)$$

**Table 8.2** Characteristics of nutation and polar motion of different axes

|                       | Forced<br>nutation<br>(~Secular) | Free<br>nutation<br>(~Diurnal) | Forced<br>polar<br>motion<br>(~Diurnal) | Free polar<br>motion<br>(~Secular) |
|-----------------------|----------------------------------|--------------------------------|-----------------------------------------|------------------------------------|
| Figure axis           | Yes                              | Yes                            | No                                      | No                                 |
| Rotation axis         | Yes                              | Yes                            | Yes                                     | Yes                                |
| Angular momentum axis | Yes                              | No                             | Yes                                     | Yes                                |

~Secular: The period of variation is much longer than a day

~Diurnal: The period of variation is about a day

The equation for polar motion can be obtained by substituting these into the governing equations of the Earth rotation, (8.37),

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_E)\dot{\alpha}_2 + \omega_0\sigma_E\alpha_1 = 0, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_E)\dot{\alpha}_1 + \omega_0\sigma_E\alpha_2 = 0. \end{cases} \quad (8.120)$$

The solution of nutation is given in (8.68), which is secular as expected. The equation governing the polar motion, (8.120), is identical to (8.112). Hence, it includes two free rotation modes, the Eulerian wobble and the tilt-over mode, but no forced polar motion is present. The Eulerian wobble is secular, as indicated in Table 8.2. The tilt-over mode is an angle between the  $x_3^0$ - and  $x_3$ -axes which is invariable as seen in the inertial reference system, but the polar motion, i.e., the motion of the  $x_3^0$ -axis with respect to the  $x_3$ -axis as seen in the Earth-fixed reference system, is circular, retrograde, and diurnal. This polar motion mode is eliminated by properly defining the  $x_3^0$ -axes, i.e., the CIP, based on the precession-nutation of the  $x_3$ -axis. As a result, the polar motion includes only one component—the Eulerian wobble.

## 8.2 Earth's Deformation in Response to the Variation of Its Angular Velocity, the Pole Tide

### 8.2.1 Centrifugal Acceleration and Potential

When the Earth's angular velocity varies, the centrifugal acceleration varies, and the Earth deforms as a response.

We take as reference state a constant angular velocity  $\vec{\omega}_0 = \omega_0\hat{e}_3$ , and express the instantaneous angular velocity as  $\vec{\omega} = \vec{\omega}_0 + \delta\vec{\omega} = \omega_0\hat{e}_3 + \delta\omega_i\hat{e}_i$ . The centrifugal acceleration of the reference state is

$$\vec{P}_0 = -\vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r}) = (\vec{\omega}_0 \cdot \vec{\omega}_0)\vec{r} - (\vec{\omega}_0 \cdot \vec{r})\vec{\omega}_0 = \omega_0^2 x_1 \hat{e}_1 + \omega_0^2 x_2 \hat{e}_2. \quad (8.121)$$

The instantaneous centrifugal acceleration is, accurate to the first order of  $\delta\vec{\omega}$ ,

$$\begin{aligned} \vec{P} &= -\vec{\omega} \times (\vec{\omega} \times \vec{r}) = (\vec{\omega} \cdot \vec{\omega})\vec{r} - (\vec{\omega} \cdot \vec{r})\vec{\omega} = \omega_i \omega_j x_j \hat{e}_i - \omega_i x_i \omega_j \hat{e}_j \\ &= \omega_0^2 x_j \hat{e}_j + 2\omega_0 \delta\omega_3 x_j \hat{e}_j - \omega_0^2 x_3 \hat{e}_3 - \omega_0 x_3 \delta\omega_j \hat{e}_j - \omega_0 \delta\omega_i x_i \hat{e}_3 \\ &= \omega_0^2 x_1 \hat{e}_1 + \omega_0^2 x_2 \hat{e}_2 + (2\omega_0 \delta\omega_3 x_1 - \omega_0 \delta\omega_1 x_3) \hat{e}_1 \\ &\quad + (2\omega_0 \delta\omega_3 x_2 - \omega_0 \delta\omega_2 x_3) \hat{e}_2 - (\omega_0 \delta\omega_1 x_1 + \omega_0 \delta\omega_2 x_2) \hat{e}_3. \end{aligned} \quad (8.122)$$

The increment of the centrifugal acceleration with respect to the reference state is

$$\delta\vec{P} = \omega_0(2\delta\omega_3 x_1 - \delta\omega_1 x_3) \hat{e}_1 + \omega_0(2\delta\omega_3 x_2 - \delta\omega_2 x_3) \hat{e}_2 - \omega_0(\delta\omega_1 x_1 + \delta\omega_2 x_2) \hat{e}_3. \quad (8.123)$$

The corresponding increment of the centrifugal potential is

$$\delta\psi = \omega_0\delta\omega_3(x_1^2 + x_2^2) - \omega_0(\delta\omega_1x_1x_3 + \delta\omega_2x_2x_3). \quad (8.124)$$

The identity  $\vec{\delta P} = \nabla\delta\psi$  can be easily verified. In the spherical coordinate system, the expression of  $\delta\psi$  is

$$\begin{aligned} \delta\psi &= \omega_0\delta\omega_3r^2\sin^2\theta - \omega_0r^2(\delta\omega_1\sin\theta\cos\theta\cos\lambda + \delta\omega_2\sin\theta\cos\theta\sin\lambda) \\ &= \frac{2}{3}\omega_0\delta\omega_3r^2 - \omega_0r^2\left[\frac{2}{3}\delta\omega_3\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \right. \\ &\quad \left. + (\delta\omega_1\cos\lambda + \delta\omega_2\sin\lambda)\sin\theta\cos\theta\right] \\ &= \frac{2}{3}r^2\omega_0^2\frac{\delta\omega_3}{\omega_0} - \frac{r^2\omega_0^2}{3}\left[2\frac{\delta\omega_3}{\omega_0}P_2(\cos\theta) + \right. \\ &\quad \left.\left(\frac{\delta\omega_1}{\omega_0}\cos\lambda + \frac{\delta\omega_2}{\omega_0}\sin\lambda\right)P_2^1(\cos\theta)\right], \end{aligned} \quad (8.125)$$

remembering that the centrifugal potential of the reference state is  $(1/2)r^2\omega_0^2\sin^2\theta$ . The first term is spherically symmetrical, and the second term has the same form as the degree 2 spherical harmonic expression of the tidal-generating potential in (7.17) or (7.18), with  $\cos\theta = \sin\Phi$ .

### 8.2.2 The Pole Tide

We first consider the second term in (8.125) which is in the same form as the degree 2 tidal-generating potential. Therefore, the Earth's response to it is the same as that to the degree 2 tidal-generating potential. The value of this term on the Earth's surface is

$$\delta\psi^t(R) = -\frac{R^2\omega_0^2}{3}\left[2\frac{\delta\omega_3}{\omega_0}P_2(\cos\theta) + \left(\frac{\delta\omega_1}{\omega_0}\cos\lambda + \frac{\delta\omega_2}{\omega_0}\sin\lambda\right)P_2^1(\cos\theta)\right]. \quad (8.126)$$

The three components of the surface displacement are

$$\begin{cases} u_r(R) = -\frac{R^2\omega_0^2}{3}\frac{h_2}{g_R}\left[2\frac{\delta\omega_3}{\omega_0}P_2(\cos\theta) + \left(\frac{\delta\omega_1}{\omega_0}\cos\lambda + \frac{\delta\omega_2}{\omega_0}\sin\lambda\right)P_2^1(\cos\theta)\right], \\ u_\theta(R) = -\frac{R^2\omega_0^2}{3}\frac{l_2}{g_R}\left[2\frac{\delta\omega_3}{\omega_0}\frac{dP_2(\cos\theta)}{d\theta} \right. \\ \quad \left. + \left(\frac{\delta\omega_1}{\omega_0}\cos\lambda + \frac{\delta\omega_2}{\omega_0}\sin\lambda\right)\frac{dP_2^1(\cos\theta)}{d\theta}\right], \\ u_\lambda(R) = -\frac{R^2\omega_0^2}{3}\frac{l_2}{g_R}\left(-\frac{\delta\omega_1}{\omega_0}\sin\lambda + \frac{\delta\omega_2}{\omega_0}\cos\lambda\right)P_2^1(\cos\theta). \end{cases} \quad (8.127)$$

The change of the Earth's gravitational potential on the surface,  $k_2 \delta \psi^t(R)$ , is

$$\delta \psi_1^t(R) = -\frac{\omega_0^2 R^2}{3} k_2 \left[ 2 \frac{\delta \omega_3}{\omega_0} P_2(\cos \theta) + \left( \frac{\delta \omega_1}{\omega_0} \cos \lambda + \frac{\delta \omega_2}{\omega_0} \sin \lambda \right) P_2^1(\cos \theta) \right]. \quad (8.128)$$

Consequently, the change of the Earth's gravitational potential outside the Earth is

$$\begin{aligned} \delta \psi_1^t(r) = & -\frac{\omega_0^2 R^2}{3} k_2 \left( \frac{R}{r} \right)^3 \left[ 2 \frac{\delta \omega_3}{\omega_0} P_2(\cos \theta) \right. \\ & \left. + \left( \frac{\delta \omega_1}{\omega_0} \cos \lambda + \frac{\delta \omega_2}{\omega_0} \sin \lambda \right) P_2^1(\cos \theta) \right]. \end{aligned} \quad (8.129)$$

The deformation of the Earth in response to  $m_1 = \delta \omega_1 / \omega_0$  and  $m_2 = \delta \omega_2 / \omega_0$  is referred to as pole tide.

Besides the solid Earth, the oceans also respond to the variation of centrifugal acceleration.

### 8.2.3 Deformation in the Radial Direction

The first term in (8.125) is spherically symmetrical, thus being a degree 0 term. It does not induce any change in the gravitational field outside the Earth, but it does induce a radial displacement. This degree 0 term is not harmonic. Hence, the general formulation of the Earth's elastic deformation in Sect. 4.8, where the potential of the external force was assumed harmonic, cannot be applied. We have to formulate it particularly. Analogous to (7.86) with  $n = k = 0$ , we rewrite this term in the form of

$$\delta \psi^r = \frac{2}{3} r^2 \omega_0^2 \frac{\delta \omega_3}{\omega_0} = \frac{GM}{R} \frac{2}{3} \frac{\omega_0^2 R^3}{GM} \frac{\delta \omega_3}{\omega_0} \left( \frac{r}{R} \right)^2 = \frac{GM}{R} \phi^r \left( \frac{r}{R} \right)^2 = R g_R \phi^r \left( \frac{r}{R} \right)^2, \quad (8.130)$$

with  $\phi^r$  defined as

$$\phi^r = \frac{2}{3} \frac{\omega_0^2 R^3}{GM} \frac{\delta \omega_3}{\omega_0} = \frac{2}{3} q \frac{\delta \omega_3}{\omega_0}. \quad (8.131)$$

The parameter  $q$  was defined in (5.42) with  $a$  replaced by  $R$  here, introducing a relative error on the order of the magnitude of the Earth's flattening. We follow the formulation of Earth tide in Sects. 7.3.1–7.3.3. Although there is no degree 0 Earth tide, the formulation in those sections does apply to a degree 0 forcing with a harmonic potential. Here the potential of the external forcing is not harmonic, and the formulation needs some modification. The governing equations (7.87) and

(7.88) remain valid here with  $\phi$  replaced by  $\delta\psi^r$ , and (7.89) should be replaced by

$$\nabla^2(\delta\psi^r + \phi_1) = 4\pi G \nabla \cdot (\rho \vec{u}) + \nabla^2 \delta\psi^r. \quad (8.132)$$

The replacement of  $\phi$  by  $\delta\psi^r$  is natural, as the latter is the forcing here. The additional term  $\nabla^2 \delta\psi^r$  at the right-hand side is to account for the fact that  $\delta\psi^r$  is not harmonic (in the formulation of Earth tide,  $\phi$  was added to  $\phi_1$  at the left-hand side with  $\nabla^2 \phi = 0$  taken into consideration). In this way, the symbols  $y_i^n$  defined in (7.96)–(7.99) can still be used. Here we write them simply as  $y_i$ . With  $n = k = 0$ , we have

$$\begin{cases} \vec{u} = \phi^r y_1 \hat{e}_r, \\ \hat{e}_r \cdot \vec{T} = \phi^r y_2 \hat{e}_r, \\ \phi_1 + \delta\psi^r = \phi^r y_5, \\ \frac{\partial(\delta\psi^r + \phi_1)}{\partial r} - 4\pi G \rho (\hat{e}_r \cdot \vec{u}) = \phi^r y_6. \end{cases} \quad (8.133)$$

It is evident that  $y_3$  and  $y_4$  vanishes. According to (8.130),

$$\nabla^2 \delta\psi^r = \frac{1}{r^2} \frac{d}{dr} \left\{ r^2 \frac{d}{dr} \left[ R g_R \phi^r \left( \frac{r}{R} \right)^2 \right] \right\} = 6 \frac{g_R}{R} \phi^r. \quad (8.134)$$

The system of ordinary differential equations (7.100) becomes

$$\frac{dy_1}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)r} y_1 + \frac{1}{\lambda + 2\mu} y_2, \quad (8.135a)$$

$$\frac{dy_2}{dr} = \left[ -\frac{4\rho g}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2} \right] y_1 - \frac{4\mu}{(\lambda + 2\mu)r} y_2 - \rho y_6, \quad (8.135b)$$

$$\frac{dy_5}{dr} = 4\pi G \rho y_1 + y_6, \quad (8.135c)$$

$$\frac{dy_6}{dr} = -\frac{2}{r} y_6 + 6 \frac{g_R}{R}. \quad (8.135d)$$

As expected, the extra term  $\nabla^2 \delta\psi^r$  in (8.132) affects only the last equation, (8.135d). In fact, the derivation of (8.135d) from (8.132) is straightforward keeping in mind that all functions depend on  $r$  only. The disappearance of the term  $-\rho\sigma^2$  is equivalent to neglecting the inertial term or assuming the Earth's response to be static. All the boundary conditions across internal boundaries are the same as in the case of Earth tide. We have

$$\{y_1\}_-^+ = \{y_2\}_-^+ = \{y_5\}_-^+ = \{y_6\}_-^+ = 0, \quad (8.136)$$

i.e., all the  $y$ s are continuous across the internal boundaries. As the Earth's surface is free of any action, which is the same as in the case of Earth tide, the boundary

condition

$$y_2(R) = 0 \quad (8.137)$$

remains valid. The boundary condition of the gravitational potential has to be reformulated. Taking reference of the formulation from (4.561)–(4.569). With  $\phi_f$  replaced by  $\delta\psi^r$  defined in (8.130), we have, at the inner side of the Earth's surface,

$$(\phi_1)_{r=R}^- = (\delta\psi^r + \phi_1)_{r=R}^- - (\delta\psi^r)_{r=R}^- = \phi^r y_5(R) - R g_R \phi^r, \quad (8.138)$$

$$\left\{ \frac{\partial \phi_1}{\partial r} - 4\pi G \rho (\hat{e}_r \cdot \vec{u}) \right\}_{r=R}^- = \left\{ \left[ \frac{\partial (\delta\psi^r + \phi_1)}{\partial r} - 4\pi G \rho (\hat{e}_r \cdot \vec{u}) \right] - \frac{\partial \delta\psi^r}{\partial r} \right\}_{r=R}^- \\ = \phi^r y_6(R) - 2g_R \phi^r. \quad (8.139)$$

As  $\phi_1$  is harmonic outside the Earth, it can be obtained based on its continuity (7.94) and its value at the inner side of the Earth's surface given by (8.138), remembering that it is a degree 0 term,

$$(\phi_1)^+ = \phi^r [y_5(R) - R g_R] \left( \frac{R}{r} \right). \quad (8.140)$$

As the density  $\rho$  vanishes outside the Earth, we can further obtain, from this formula,

$$\left\{ \frac{\partial \phi_1}{\partial r} - 4\pi G \rho (\hat{e}_r \cdot \vec{u}) \right\}_{r=R}^+ = \left( \frac{\partial \phi_1}{\partial r} \right)_{r=R}^+ = -\frac{\phi^r}{R} [y_5(R) - R g_R]. \quad (8.141)$$

By substituting (8.139) and (8.141) into the boundary condition (7.95), we finally obtain

$$\phi^r [y_6(R) - 2g_R] = -\frac{\phi^r}{R} [y_5(R) - R g_R], \quad (8.142)$$

which can be written in the more brief form

$$\frac{y_5(R)}{R} + y_6(R) = 3g_R. \quad (8.143)$$

We have four ordinary differential equations (8.135) and two boundary conditions (8.137) and (8.143) on the Earth's surface. Two regularity boundary conditions at the Earth center,  $y_1(0) = 0$  and  $y_6(0) = 0$ , can be readily derived. Together with the continuity boundary conditions (8.136), the solution is uniquely defined.

To express the deformation, we define the Love numbers

$$H_0^r(r) = \frac{y_1(r)}{R}, \quad K_0^r(r) = \frac{y_5(r)}{R g_R} - 1, \quad (8.144)$$



where the superscript “ $r$ ” indicates *rotation rate* and *radial*. According to (8.133) and (8.130), the results of the displacement and the potential can be written as

$$u_r(r) = \frac{H_0^r(r)}{g_R} \delta\psi^r(R), \quad \delta\psi^r(r) + \phi_1(r) = [1 + K_0^r(r)] \delta\psi^r(R). \quad (8.145)$$

The deformation is a spherically symmetrical inflation or contraction, which does not bring any change to the gravitational potential outside the Earth, i.e.,  $(\phi_1)^+ = 0$ , which can be readily verified. The only acceptable solution (finite at  $r = 0$ ) of (8.135d) is  $y_6 = 2(g_R/R)r$ . We then obtain from (8.143) that  $y_5(R) = Rg_R$ , which leads to  $(\phi_1)^+ = 0$  upon its substitution into (8.140).

On the Earth’s surface, we define, with  $y_5(R) = Rg_R$ ,

$$h_0^r = H_0^r(R) = \frac{y_1(R)}{R}, \quad k_0^r = K_0^r(R) = \frac{y_5(R)}{Rg_R} - 1 = 0, \quad (8.146)$$

and write the results of the displacement and the potential as

$$u_r(R) = \frac{h_0^r}{g_R} \delta\psi^r(R), \quad \delta\psi^r(R) + \phi_1(R) = (1 + k_0^r) \delta\psi^r(R) = \delta\psi^r(R). \quad (8.147)$$

The above formulae (8.144)–(8.147) have the same forms as in the case of Earth tide.

The numerical solution of the equations is similar to the case of Earth tide, and we integrate them starting from the Earth center. However, as these equations are not all homogeneous, there is a need to find one solution of these equations and two independent solutions of its homogeneous equations, all satisfying the regularity boundary conditions at the Earth center, i.e.,  $y_1(0) = 0$  and  $y_6(0) = 0$ . The solution obeying the boundary conditions on the Earth’s surface is obtained by adding a linear combination of the two independent solutions of the homogeneous equations to the solution of themselves.

## 8.3 Rotation of the Elastic Earth Model

### 8.3.1 Linearization of the Euler-Liouville Equation

As the deformation of the Earth is involved, the linearization of the Euler-Liouville equation should be complemented by the formulation of the change of the Earth’s moment of inertia tensor due to Earth tide and angular velocity variation.

#### 8.3.1.1 Linearization of the Euler-Liouville Equation

Similar to the case of the rigid Earth model, we use  $Ox_1x_2x_3$  to denote the “Earth-fixed” coordinate system defined by the Tisserand axes, which is referred to as the Tisserand frame, and  $Ox_1^0x_2^0x_3^0$  the auxiliary coordinate system. We still express

the angular velocity of  $Ox_1^0x_2^0x_3^0$  with respect to the inertial space represented by  $O\xi_1\xi_2\xi_3$  as

$$\vec{\omega}^0 = \delta\omega_1^0\hat{e}_1^0 + \delta\omega_2^0\hat{e}_2^0 + \omega_0\hat{e}_3^0. \quad (8.148)$$

As in the case of the rigid Earth model, we use the angles  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  to describe the relation between  $Ox_1x_2x_3$  and  $Ox_1^0x_2^0x_3^0$ , as shown in Fig. 8.3. The angular velocity of  $Ox_1x_2x_3$  with respect to  $Ox_1^0x_2^0x_3^0$  is the same as (8.36). Here we write it in the form of

$$\vec{\omega} = (\omega_{01} + \delta\omega_1)\hat{e}_1 + (\omega_{02} + \delta\omega_2)\hat{e}_2 + (\omega_{03} + \delta\omega_3)\hat{e}_3, \quad (8.149)$$

where

$$\omega_{01} = 0, \quad \omega_{02} = 0, \quad \omega_{03} = \omega_0. \quad (8.150)$$

$$\delta\omega_1 = \dot{\alpha}_1 - \alpha_2\omega_0 + \delta\omega_1^0, \quad \delta\omega_2 = \dot{\alpha}_2 + \alpha_1\omega_0 + \delta\omega_2^0, \quad \delta\omega_3 = \dot{\alpha}_3. \quad (8.151)$$

Due to the deformation, the moment of inertia tensor is a function of time. We define the undeformed reference state to be axially symmetrical with the moment of inertia tensor given by

$$C_{11}^0 = A, \quad C_{22}^0 = A, \quad C_{33}^0 = C; \quad C_{ij}^0 = 0 \text{ when } i \neq j. \quad (8.152)$$

We then express the instantaneous moment of inertia tensor  $C_{ij}$  as the sum of  $C_{ij}^0$  and a small quantity  $c_{ij}$ , i.e.,

$$C_{ij} = C_{ij}^0 + c_{ij}. \quad (8.153)$$

The substitution of (8.149) and (8.153) into the Euler-Liouville equation based on the Tisserand axes, (3.327), leads to, keeping only the linear terms of the infinitesimal quantities,

$$\begin{aligned} C_{ij}^0 \frac{d\delta\omega_j}{dt} + \frac{dc_{ij}}{dt} \omega_{0j} + \epsilon_{ijk} C_{kl}^0 \omega_{0j} \omega_{0l} + \epsilon_{ijk} c_{kl} \omega_{0j} \omega_{0l} \\ + \epsilon_{ijk} C_{kl}^0 \delta\omega_j \omega_{0l} + \epsilon_{ijk} C_{kl}^0 \omega_{0j} \delta\omega_l = N_i. \end{aligned} \quad (8.154)$$

The use of (8.150) and (8.152) yields

$$\begin{cases} A \frac{d\delta\omega_1}{dt} + \frac{dc_{13}}{dt} \omega_0 - c_{23} \omega_0^2 + C \delta\omega_2 \omega_0 - A \omega_0 \delta\omega_2 = N_1, \\ A \frac{d\delta\omega_2}{dt} + \frac{dc_{23}}{dt} \omega_0 + c_{13} \omega_0^2 - C \delta\omega_1 \omega_0 + A \omega_0 \delta\omega_1 = N_2, \\ C \frac{d\delta\omega_3}{dt} + \frac{dc_{33}}{dt} \omega_0 = N_3. \end{cases} \quad (8.155)$$

Finally, the use of (8.151) yields

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_E)\dot{\alpha}_2 + \omega_0\sigma_E\alpha_1 + \frac{d\delta\omega_1^0}{dt} + \sigma_E\delta\omega_2^0 + \frac{1}{A}\frac{dc_{13}}{dt}\omega_0 - \frac{c_{23}}{A}\omega_0^2 = \frac{N_1}{A}, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_E)\dot{\alpha}_1 + \omega_0\sigma_E\alpha_2 + \frac{d\delta\omega_2^0}{dt} - \sigma_E\delta\omega_1^0 + \frac{1}{A}\frac{dc_{23}}{dt}\omega_0 + \frac{c_{13}}{A}\omega_0^2 = \frac{N_2}{A}, \\ \ddot{\alpha}_3 + \frac{1}{C}\frac{dc_{33}}{dt}\omega_0 = \frac{N_3}{C}. \end{cases} \quad (8.156)$$

In order to solve this system of equations, the formulae of  $c_{13}$ ,  $c_{23}$  and  $c_{33}$  must be derived.

### 8.3.1.2 Change of the Moment of Inertia Tensor Due to Earth Tide

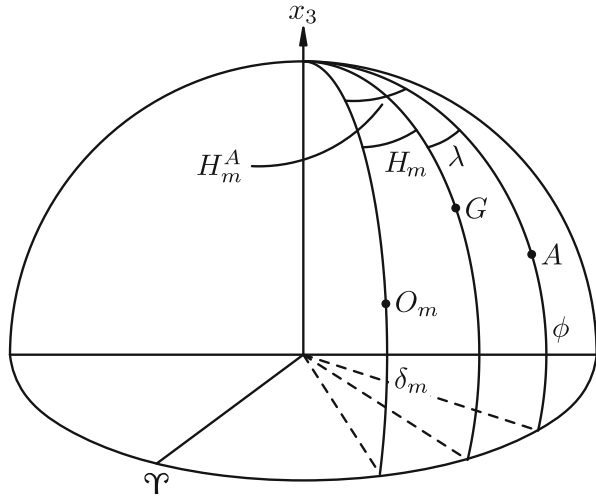
We first formulate the contribution of Earth tide to  $c_{13}$ ,  $c_{23}$  and  $c_{33}$ . They can be inferred from the spherical harmonic coefficients of the additional potential. For this purpose, we express the additional potential in the Tisserand frame considered to be “Earth fixed” here.

As shown in Fig. 8.7, here we denote the hour angle of the Moon with respect to the prime meridian as  $H_m$ , and that with respect to the point  $A$  as  $H_m^A$ . We have

$$H_m^A = H_m + \lambda. \quad (8.157)$$

By substituting this relation into (7.17), noting that the notation  $H_m^A$  here corresponds to the notation  $H_m$  in (7.17), we can write the degree 2 lunar tidal-generating

**Fig. 8.7** The hour angle of the Moon with respect to the prime meridian and any point



potential on the Earth's surface as

$$\begin{aligned} \phi_m^t(R) = & GM_m \frac{R^2}{r_m^3} \sum_{k=0}^2 \frac{2}{1+\delta_k} \frac{(2-k)!}{(2+k)!} P_2^k(\sin \phi) P_n^k(\sin \delta_m) \\ & \times (\cos k H_m \cos k \lambda - \sin k H_m \sin k \lambda). \end{aligned} \quad (8.158)$$

According to the definition of the degree 2 potential Love number  $k_2$ , the additional potential on the Earth's surface is  $k_2 \phi_m^t$ , i.e.,

$$\begin{aligned} \phi_{1m}(R) = & \frac{GM}{R} \sum_{k=0}^2 \frac{2k_2}{1+\delta_k} \frac{(2-k)!}{(2+k)!} \left(\frac{M_m}{M}\right) \left(\frac{R}{r_m}\right)^3 P_2^k(\sin \delta_m) \\ & \times (\cos k H_m \cos k \lambda - \sin k H_m \sin k \lambda) P_2^k(\sin \phi). \end{aligned} \quad (8.159)$$

By comparing with the spherical harmonic series of the gravitational potential (5.1) (with  $a$  replaced by  $R$ ), and taking into account of the definition of  $\mathcal{H}_n^k(\theta, \lambda)$ , we have the following spherical harmonic coefficients of the additional potential,

$$A_2 = k_2 \left(\frac{M_m}{M}\right) \left(\frac{R}{r_m}\right)^3 P_2(\sin \delta_m), \quad (8.160a)$$

$$A_2^1 = \frac{1}{3} k_2 \left(\frac{M_m}{M}\right) \left(\frac{R}{r_m}\right)^3 P_2^1(\sin \delta_m) \cos H_m, \quad (8.160b)$$

$$A_2^{-1} = -\frac{1}{3} k_2 \left(\frac{M_m}{M}\right) \left(\frac{R}{r_m}\right)^3 P_2^1(\sin \delta_m) \sin H_m, \quad (8.160c)$$

$$A_2^2 = \frac{1}{12} k_2 \left(\frac{M_m}{M}\right) \left(\frac{R}{r_m}\right)^3 P_2^2(\sin \delta_m) \cos 2H_m, \quad (8.160d)$$

$$A_2^{-2} = -\frac{1}{12} k_2 \left(\frac{M_m}{M}\right) \left(\frac{R}{r_m}\right)^3 P_2^2(\sin \delta_m) \sin 2H_m. \quad (8.160e)$$

As compared to the physical meanings of the degree 2 spherical harmonic coefficients of the gravitational potential, (5.11), the quantities  $c_{11}$ ,  $c_{22}$ ,  $c_{33}$ ,  $c_{13}$  and  $c_{23}$  are equivalent to the quantities  $A$ ,  $B$ ,  $C$ ,  $-F$ , and  $-E$  in (5.11). We particularly have ( $c_{33}$  cannot be uniquely determined using the coefficients)

$$c_{33}^m - \frac{1}{2}(c_{11}^m + c_{22}^m) = -MR^2 A_2, \quad c_{13}^m = -MR^2 A_2^1, \quad c_{23}^m = -MR^2 A_2^{-1}. \quad (8.161)$$

With (8.160), we obtain

$$\begin{cases} c_{33}^m - \frac{1}{2}(c_{11}^m + c_{22}^m) = -k_2 M_m \frac{R^5}{r_m^3} P_2(\sin \delta_m), \\ c_{13}^m = -\frac{1}{3} k_2 M_m \frac{R^5}{r_m^3} P_2^1(\sin \delta_m) \cos H_m, \\ c_{23}^m = \frac{1}{3} k_2 M_m \frac{R^5}{r_m^3} P_2^1(\sin \delta_m) \sin H_m. \end{cases} \quad (8.162)$$

By substituting  $P_2(\sin \delta_m)$  and  $P_2^1(\sin \delta_m)$  by their expressions given in (7.22) and (7.23), respectively, we obtain

$$\begin{cases} c_{33}^m - \frac{1}{2}(c_{11}^m + c_{22}^m) = -k_2 M_m \frac{R^5}{r_m^3} \left( \frac{3}{2} \sin^2 \delta_m - \frac{1}{2} \right), \\ c_{13}^m = -\frac{1}{2} k_2 M_m \frac{R^5}{r_m^3} \sin 2\delta_m \cos H_m, \\ c_{23}^m = \frac{1}{2} k_2 M_m \frac{R^5}{r_m^3} \sin 2\delta_m \sin H_m. \end{cases} \quad (8.163)$$

For the same reason, the contribution of the Sun is

$$\begin{cases} c_{33}^s - \frac{1}{2}(c_{11}^s + c_{22}^s) = -k_2 M_s \frac{R^5}{r_s^3} \left( \frac{3}{2} \sin^2 \delta_s - \frac{1}{2} \right), \\ c_{13}^s = -\frac{1}{2} k_2 M_s \frac{R^5}{r_s^3} \sin 2\delta_s \cos H_s, \\ c_{23}^s = \frac{1}{2} k_2 M_s \frac{R^5}{r_s^3} \sin 2\delta_s \sin H_s. \end{cases} \quad (8.164)$$

These two sets of equations can be written as, using the Doodson's constant  $D$  defined in (7.33),

$$\begin{cases} c_{33}^m - \frac{1}{2}(c_{11}^m + c_{22}^m) = \frac{k_2}{G} R^3 D \left( \frac{C_m}{r_m} \right)^3 \left( \frac{2}{3} - 2 \sin^2 \delta_m \right), \\ c_{13}^m = -\frac{2}{3} \frac{k_2}{G} R^3 D \left( \frac{C_m}{r_m} \right)^3 \sin 2\delta_m \cos H_m, \\ c_{23}^m = \frac{2}{3} \frac{k_2}{G} R^3 D \left( \frac{C_m}{r_m} \right)^3 \sin 2\delta_m \sin H_m; \end{cases} \quad (8.165)$$

$$\begin{cases} c_{33}^s - \frac{1}{2}(c_{11}^s + c_{22}^s) = \frac{k_2}{G} R^3 D \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \left( \frac{2}{3} - 2 \sin^2 \delta_s \right), \\ c_{13}^s = -\frac{2}{3} \frac{k_2}{G} R^3 D \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \sin 2\delta_s \cos H_s, \\ c_{23}^s = \frac{2}{3} \frac{k_2}{G} R^3 D \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \sin 2\delta_s \sin H_s. \end{cases} \quad (8.166)$$

The total is the sum of the lunar and solar contributions,

$$\begin{cases} c_{33} - \frac{1}{2}(c_{11} + c_{22}) = \frac{k_2}{G} R^3 D \psi_0, \\ c_{13} = -\frac{2}{3} \frac{k_2}{G} R^3 D \psi_1, \\ c_{23} = \frac{2}{3} \frac{k_2}{G} R^3 D \psi_1', \end{cases} \quad (8.167)$$

where

$$\begin{cases} \psi_0 = \left( \frac{C_m}{r_m} \right)^3 \left( \frac{2}{3} - 2 \sin^2 \delta_m \right) + \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \left( \frac{2}{3} - 2 \sin^2 \delta_s \right), \\ \psi_1 = \left( \frac{C_m}{r_m} \right)^3 \sin 2\delta_m \cos H_m + \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \sin 2\delta_s \cos H_s, \\ \psi_1' = \left( \frac{C_m}{r_m} \right)^3 \sin 2\delta_m \sin H_m + \left( \frac{M_s}{M_m} \right) \left( \frac{C_m}{C_s} \right)^3 \left( \frac{C_s}{r_s} \right)^3 \sin 2\delta_s \sin H_s. \end{cases} \quad (8.168)$$

In these formula,  $\psi_0$  and  $\psi_1$  are respectively the time-dependent parts of the order 0 (secular) and order 1 (diurnal) terms of the degree 2 tidal-generating potential. The quantity  $\psi_1'$  has been discussed in (8.16) and (8.17). In this way, we can write (8.167) using tidal waves as

$$\begin{cases} c_{33} - \frac{1}{2}(c_{11} + c_{22}) = \frac{k_2}{G} R^3 D \sum_i Y_i^s \cos(\omega_i^s t + \phi_i^s), \\ c_{13} = -\frac{2}{3} \frac{k_2}{G} R^3 D \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ c_{23} = -\frac{2}{3} \frac{k_2}{G} R^3 D \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.169)$$

The superscripts  $s$  and  $d$  stand for *secular* and *diurnal*, respectively.

In (8.169), we have the result of  $c_{13}$  and  $c_{23}$ , but we are still in need of an additional equation to determine  $c_{33}$ . Here we compute  $c_{11} + c_{22} + c_{33}$ . In the undeformed reference state, we have

$$C_{11}^0 + C_{22}^0 + C_{33}^0 = 2 \int_{V_0} \rho_0 r_0^2 dV_0. \quad (8.170)$$

In the deformed state, we have

$$C_{11} + C_{22} + C_{33} = 2 \int_V \rho r^2 dV. \quad (8.171)$$

We assume that  $dV$  and  $dV_0$  contain the same material element inside, i.e.,  $dV$  is the result of deformation of  $dV_0$ . We thus have  $\rho dV = \rho_0 dV_0$ , and the radial distance of this material element in the deformed state is  $r = r_0 + u_r$ . Hence, we can write (8.171) as

$$C_{11} + C_{22} + C_{33} = 2 \int_{V_0} \rho_0 (r_0 + u_r)^2 dV_0, \quad (8.172)$$

where  $u_r$  is a function of  $r_0, \theta$  and  $\lambda$ . By subtracting (8.170) from (8.172), we obtain, accurate to the first order of  $u_r$ ,

$$c_{11} + c_{22} + c_{33} = 4 \int_{V_0} \rho_0 r_0 u_r dV_0. \quad (8.173)$$

In the first equation of (8.169), the degree 2 and order 0 tide is concerned. According to the first formula of (7.109), we have

$$c_{11} + c_{22} + c_{33} = 4 \int_{V_0} \rho_0 r_0 H_2(r_0) \frac{\phi_2^{t0}(R)}{g_R} dV_0, \quad (8.174)$$

where  $\phi_2^{t0}(R)$  is the degree 2 and order 0 tidal-generating potential on the Earth's surface. As  $\phi_2^{t0}(R)$  is a degree 2 and order 0 spherical harmonics, the integral at the right-hand side vanishes. Hence, we have

$$c_{11} + c_{22} + c_{33} = 0. \quad (8.175)$$

Finally, we obtain  $c_{33}$  from this formula and the first formula of (8.169),

$$c_{33} = \frac{2}{3} \frac{k_2}{G} R^3 D \sum_i Y_i^s \cos(\omega_i^s t + \phi_i^s). \quad (8.176)$$

### 8.3.1.3 Change of the Moment of Inertia Tensor Due to Angular Velocity Variation

The Earth's deformation in response to the variation of its angular velocity was formulated in Sect. 8.2. Here we summarise the logic for our present purpose. As in (8.149)–(8.151), we take as the reference state a constant angular velocity  $\vec{\omega}_0 = \omega_0 \hat{e}_3$ , and express the instantaneous angular velocity as  $\vec{\omega} = \vec{\omega}_0 + \delta\vec{\omega} = \omega_0 \hat{e}_3 + \delta\omega_i \hat{e}_i$ . The increment of the instantaneous centrifugal potential with respect to the reference state was given in (8.125), which is copied to here for convenience of

reference,

$$\begin{aligned} \delta\psi = & \frac{2}{3}r^2\omega_0^2\frac{\delta\omega_3}{\omega_0} - \frac{r^2\omega_0^2}{3}\left[2\frac{\delta\omega_3}{\omega_0}P_2(\cos\theta) \right. \\ & \left. + \left(\frac{\delta\omega_1}{\omega_0}\cos\lambda + \frac{\delta\omega_2}{\omega_0}\sin\lambda\right)P_2^1(\cos\theta)\right]. \end{aligned} \quad (8.177)$$

The second term in (8.177) is in the same form as the degree 2 tidal-generating potential, and hence, the Earth tide theory is applicable. On the Earth's surface, the change of the gravitational potential caused by this term is given in (8.128). Here we write it in the form of

$$\begin{aligned} \delta\psi_1 = & \frac{GM}{R}\left[-\frac{2k_2}{3}\frac{R^3\omega_0\delta\omega_3}{GM}P_2(\cos\theta) + \left(-\frac{k_2}{3}\frac{R^3\omega_0\delta\omega_1}{GM}\cos\lambda \right. \right. \\ & \left. \left. - \frac{k_2}{3}\frac{R^3\omega_0\delta\omega_2}{GM}\sin\lambda\right)P_2^1(\cos\theta)\right]. \end{aligned} \quad (8.178)$$

The spherical harmonic coefficients are

$$A_2 = -\frac{2k_2}{3}\frac{R^3\omega_0\delta\omega_3}{GM}, \quad A_2^1 = -\frac{k_2}{3}\frac{R^3\omega_0\delta\omega_1}{GM}, \quad A_2^{-1} = -\frac{k_2}{3}\frac{R^3\omega_0\delta\omega_2}{GM}. \quad (8.179)$$

The relation between these coefficients and the moment of inertia tensor is in the same form as (8.161). We obtain, with (8.151),

$$\begin{cases} c_{33} - \frac{1}{2}(c_{11} + c_{22}) = \frac{2}{3}\frac{k_2}{G}R^5\omega_0\dot{\alpha}_3, \\ c_{13} = \frac{1}{3}\frac{k_2}{G}R^5\omega_0(\dot{\alpha}_1 - \alpha_2\omega_0 + \delta\omega_1^0), \\ c_{23} = \frac{1}{3}\frac{k_2}{G}R^5\omega_0(\dot{\alpha}_2 + \alpha_1\omega_0 + \delta\omega_2^0). \end{cases} \quad (8.180)$$

As this deformation is in the same form as the Earth tide, the relation (8.175) still holds for this term. We obtain

$$c_{33} = \frac{4}{9}\frac{k_2}{G}R^5\omega_0\dot{\alpha}_3. \quad (8.181)$$

The Earth's deformation due to the first term in (8.177) was formulated in Sect. 8.2.3. We have written this term as

$$\delta\psi^r = \frac{2}{3}r^2\omega_0^2\frac{\delta\omega_3}{\omega_0} = Rg_R\phi^r\left(\frac{r}{R}\right)^2, \quad (8.182)$$

which is copied to here for convenience of reference. This term is spherically symmetrical, and can be considered as a degree 0 term. However, as it is not harmonic, the formulation in Sect. 4.8 is not applicable, and we had to formulate it particularly. Due to the spherical symmetry, it influences the diagonal terms of



the inertial tensor in the same way, i.e.,  $c_{11} = c_{22} = c_{33}$ , and has no effect on the off-diagonal terms. The formula of  $c_{11} + c_{22} + c_{33}$ , (8.173), is still applicable to this case. As a degree 0 term,  $u_r$  is independent of  $\theta$  and  $\lambda$ . To conform with the notations in Sect. 8.2.3 in subsequent formulations, we replace  $\rho_0$  by  $\rho$  and  $r_0$  by  $q$  in (8.173), and write it in the form of

$$I(R) = \frac{1}{3} (c_{11} + c_{22} + c_{33}) = \frac{16\pi}{3} \int_0^R \rho(q) q^3 u_r(q) dq, \quad (8.183)$$

where  $I(R)$  is the value of the function  $I(r)$  when  $r = R$ , which is defined by

$$I(r) = \frac{16\pi}{3} \int_0^r \rho(q) q^3 u_r(q) dq. \quad (8.184)$$

By taking the derivative with respect to  $r$  at both sides, we obtain

$$\frac{dI(r)}{dr} = \frac{16\pi}{3} \rho(r) r^3 u_r(r). \quad (8.185)$$

Analogous to (8.133), we introduce a function  $z$  through

$$I(r) = \phi^r z(r). \quad (8.186)$$

This permits us to write (8.185) as the ordinary differential equation

$$\frac{dz}{dr} = \frac{16\pi}{3} \rho r^3 y_1, \quad (8.187)$$

where  $y_1$  is defined in (8.133). This equation can be solved together with (8.135). The boundary condition associated with it is  $z(0) = 0$ , i.e., it vanishes at the Earth center, which is evident. The advantage of converting the integral to a differential equation is that we can introduce a dimensionless Love number in favor of further application,

$$C_0^r(r) = \frac{z(r)}{MR^2}, \quad c_0^r = \frac{z(R)}{MR^2}. \quad (8.188)$$

In this way, we can express  $I$  as

$$I(r) = \frac{R^3 C_0^r(r)}{G} \delta\psi^r(R), \quad I(R) = \frac{R^3 c_0^r}{G} \delta\psi^r(R). \quad (8.189)$$

Finally, we obtain, with the expression of  $\delta\psi^r(R)$ ,

$$c_{33} = I(R) = \frac{2}{3} \frac{c_0^r}{G} R^5 \omega_0 \delta\omega_3 = \frac{2}{3} \frac{c_0^r}{G} R^5 \omega_0 \dot{\alpha}_3. \quad (8.190)$$

Rochester and Smylie (1974) discussed this term, but did not formulate it.

We mention that the total contribution to  $c_{33}$  from the Earth's deformation caused by the variation of the Earth rotation rate is the sum of two terms, i.e., (8.181) and (8.190).

### 8.3.1.4 The Linearized Euler-Liouville Equation

We are now ready to write out the final form of the linearized Euler-Liouville equation. We will use the secular Love number defined in Sect. 7.3.4,

$$k_s = \frac{3G(C - A)}{\omega_0^2 R^5} = 0.94, \quad (8.191)$$

in the equations. Here it is not used to represent a deformation of any kind; instead, it is used as a geophysical parameter. The total increment of the moment of inertia tensor is written as

$$\begin{aligned} c_{13} = & -2 \frac{k_2}{k_s} \frac{D}{R^2 \omega_0^2} (C - A) \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d) \\ & + \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A) (\dot{\alpha}_1 - \alpha_2 \omega_0 + \delta \omega_1^0), \end{aligned} \quad (8.192a)$$

$$\begin{aligned} c_{23} = & -2 \frac{k_2}{k_s} \frac{D}{R^2 \omega_0^2} (C - A) \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d) \\ & + \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A) (\dot{\alpha}_2 + \alpha_1 \omega_0 + \delta \omega_2^0), \end{aligned} \quad (8.192b)$$

$$\begin{aligned} c_{33} = & 2 \frac{k_2}{k_s} \frac{D}{R^2 \omega_0^2} (C - A) \sum_i Y_i^s \cos(\omega_i^s t + \phi_i^s) \\ & + \frac{4}{3} \frac{\tilde{k}_2}{k_s} \frac{1}{\omega_0} (C - A) \dot{\alpha}_3, \end{aligned} \quad (8.192c)$$

where we used the notation

$$\tilde{k}_2 = k_2 + \frac{3}{2} c_0^r. \quad (8.193)$$

The term  $(3/2)c_0^r$  is rarely mentioned in the literature (Rochester & Smylie, 1974). In the depth of this book, the effect of the  $\tilde{k}_2$  term is negligible, as to be detailed in the relevant context. The substitution of (8.19) and (8.192) into (8.156) leads to the linearized Euler-Liouville equation,

$$\left(1 + H' \frac{k_2}{k_s}\right) \ddot{\alpha}_1 - \omega_0 \left[ \left(1 + H' \frac{k_2}{k_s}\right) - H' \left(1 - \frac{k_2}{k_s}\right) \right] \dot{\alpha}_2 + H' \omega_0^2 \left(1 - \frac{k_2}{k_s}\right) \alpha_1,$$

$$\begin{aligned}
& + \left(1 + H' \frac{k_2}{k_s}\right) \frac{d\delta\omega_1^0}{dt} + H' \omega_0 \left(1 - \frac{k_2}{k_s}\right) \delta\omega_2^0 \\
& = 2 \frac{D}{R^2} H' \sum_i \left(1 + \frac{k_2}{k_s} \frac{\omega_i^d - \omega_0}{\omega_0}\right) Y_i^d \cos(\omega_i^d t + \phi_i^d), \quad (8.194a)
\end{aligned}$$

$$\begin{aligned}
& \left(1 + H' \frac{k_2}{k_s}\right) \ddot{\alpha}_2 + \omega_0 \left[ \left(1 + H' \frac{k_2}{k_s}\right) - H' \left(1 - \frac{k_2}{k_s}\right) \right] \dot{\alpha}_1 + H' \omega_0^2 \left(1 - \frac{k_2}{k_s}\right) \alpha_2 \\
& + \left(1 + H' \frac{k_2}{k_s}\right) \frac{d\delta\omega_2^0}{dt} - H' \omega_0 \left(1 - \frac{k_2}{k_s}\right) \delta\omega_1^0 \\
& = -2 \frac{D}{R^2} H' \sum_i \left(1 + \frac{k_2}{k_s} \frac{\omega_i^d - \omega_0}{\omega_0}\right) Y_i^d \sin(\omega_i^d t + \phi_i^d) \quad (8.194b)
\end{aligned}$$

$$\left(1 + \frac{4}{3} H' \frac{\tilde{k}_2}{k_s}\right) \ddot{\alpha}_3 = 2 \frac{k_2}{k_s} \frac{D}{R^2} H' \sum_i \frac{\omega_i^s}{\omega_0} Y_i^s \sin(\omega_i^s t + \phi_i^s) + \frac{N_3}{C}, \quad (8.194c)$$

where we have kept  $N_3$  for later use. Define a parameter

$$\sigma_C = \sigma_E \left(1 - \frac{k_2}{k_s}\right) \left(1 + \frac{k_2}{k_s} \frac{\sigma_E}{\omega_0}\right)^{-1} = H' \omega_0 \left(1 - \frac{k_2}{k_s}\right) \left(1 + H' \frac{k_2}{k_s}\right)^{-1}. \quad (8.195)$$

We can finally write the linearized Euler-Liouville equation as

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_C) \dot{\alpha}_2 + \omega_0 \sigma_C \alpha_1 + \frac{d\delta\omega_1^0}{dt} + \sigma_C \delta\omega_2^0 = \frac{\tilde{N}_1}{A}, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_C) \dot{\alpha}_1 + \omega_0 \sigma_C \alpha_2 + \frac{d\delta\omega_2^0}{dt} - \sigma_C \delta\omega_1^0 = \frac{\tilde{N}_2}{A}, \\ \ddot{\alpha}_3 = \frac{\tilde{N}_3}{C}, \end{cases} \quad (8.196)$$

where

$$\begin{cases} \tilde{N}_1 = 2 \frac{D}{R^2} (C - A) \left(1 + H' \frac{k_2}{k_s}\right)^{-1} \sum_i \left(1 + \frac{k_2}{k_s} \frac{\omega_i^d - \omega_0}{\omega_0}\right) Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \tilde{N}_2 = -2 \frac{D}{R^2} (C - A) \left(1 + H' \frac{k_2}{k_s}\right)^{-1} \sum_i \left(1 + \frac{k_2}{k_s} \frac{\omega_i^d - \omega_0}{\omega_0}\right) Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \tilde{N}_3 = \left(1 + \frac{4}{3} H' \frac{\tilde{k}_2}{k_s}\right)^{-1} \left[ 2 \frac{k_2}{k_s} \frac{D}{R^2} (C - A) \sum_i \frac{\omega_i^s}{\omega_0} Y_i^s \sin(\omega_i^s t + \phi_i^s) + N_3 \right]. \end{cases} \quad (8.197)$$

Note that  $\tilde{N}_1$ ,  $\tilde{N}_2$  and  $\tilde{N}_3$  are not the components of the lunisolar gravitational torque.

We still consider 3 choices of the celestial axis  $x_3^0$ : the Tisserand axis, the rotation axis, and the angular momentum axis. For the deformable Earth model, the term *figure axis* may indicate the Tisserand axis  $x_3$  or the instantaneous principal moment of inertia axis. Here we choose the Tisserand axis.

For the Tisserand axis, there is, again, no polar motion,

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad (8.198)$$

and only precession-nutation is present. This is due to the fact that the Tisserand axes are defined to constitute the Earth-fixed coordinate system.

The case of rotation axis is also the same as that of a rigid Earth model. The relation between precession-nutation and polar motion is the same as (8.43), and is copied to here for convenience of reference,

$$\dot{\alpha}_1 + \delta\omega_1^0 = 0, \quad \dot{\alpha}_2 + \delta\omega_2^0 = 0. \quad (8.199)$$

The case of angular momentum axis is more complicated, as the moment of inertia tensor is no longer diagonal. The angular momentum is

$$\vec{L} = \vec{C} \cdot \vec{\omega} = (c_{13}\omega_0 + A\delta\omega_1)\hat{e}_1 + (c_{23}\omega_0 + A\delta\omega_2)\hat{e}_2 + (C\omega_0 + c_{33}\omega_0 + C\delta\omega_3)\hat{e}_3. \quad (8.200)$$

We have, similar to (8.45),

$$-\alpha_2 = \frac{c_{13}\omega_0 + A\delta\omega_1}{C\omega_0 + c_{33}\omega_0 + C\delta\omega_3}, \quad \alpha_1 = \frac{c_{23}\omega_0 + A\delta\omega_2}{C\omega_0 + c_{33}\omega_0 + C\delta\omega_3}, \quad (8.201)$$

which reduce to, accurate to the first order of the small quantities,

$$-C\omega_0\alpha_2 = c_{13}\omega_0 + A\delta\omega_1, \quad C\omega_0\alpha_1 = c_{23}\omega_0 + A\delta\omega_2. \quad (8.202)$$

The use of (8.151) and (8.192) leads to, after some tedious algebraic operations,

$$\begin{cases} \dot{\alpha}_1 + \sigma_C\alpha_2 + \delta\omega_1^0 = 2\frac{k_2}{k_s}\frac{D}{R^2}\frac{H'}{\omega_0}\left(1 + H'\frac{k_2}{k_s}\right)^{-1} \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \dot{\alpha}_2 - \sigma_C\alpha_1 + \delta\omega_2^0 = 2\frac{k_2}{k_s}\frac{D}{R^2}\frac{H'}{\omega_0}\left(1 + H'\frac{k_2}{k_s}\right)^{-1} \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.203)$$

This is the relation between precession-nutation and polar motion for the angular momentum axis.

### 8.3.2 Precession-Nutation and Polar Motion

The equations to be solved are (8.196) and (8.197). They have the same form as the equations solved for the rigid Earth model, (8.37) and (8.19). Hence the solutions

can be readily inferred from the solutions of the rigid Earth model. The forced rotation and free rotation modes of the rigid Earth model were studied separately. Here we put them together to save some pages.

### 8.3.2.1 The Tisserand Axes

The substitution of (8.197) and (8.198) into (8.196) leads to the equation to be solved,

$$\begin{cases} \frac{d\delta\omega_1^0}{dt} + \sigma_C \delta\omega_2^0 \\ \quad = 2 \frac{D}{R^2} H' \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \frac{d\delta\omega_2^0}{dt} - \sigma_C \delta\omega_1^0 \\ \quad = -2 \frac{D}{R^2} H' \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \sin(\omega_i^d t + \phi_i^d), \end{cases} \quad (8.204)$$

where, as in the case of the rigid Earth model, the approximation  $(\omega_i^d - \omega_0)/\omega_0 = \Delta\omega_i^d/\omega_0$  is made in the amplitude.

The system of equations (8.204) is composed of nonhomogeneous ordinary differential equations. The free rotation modes are the solution of its complementary homogeneous equations, which are exactly in the same form as those for the rigid Earth model, (8.90), with  $\sigma_E$  replaced by  $\sigma_C$ . Hence replacing  $\sigma_E$  by  $\sigma_C$  in the solution of the rigid Earth model leads to the solution here.

The solution of the forced rotation is

$$\begin{cases} \delta\omega_1^0 = 2 \frac{D}{R^2} H' \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \frac{1}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \delta\omega_2^0 = 2 \frac{D}{R^2} H' \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \frac{1}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.205)$$

The same as in the case of the rigid Earth model, we define  $\Delta\omega_i^d = \omega_i^d - \dot{\phi}$ . Neglecting the effect of the difference between  $\omega_0$  and  $\dot{\phi}$  and accurate to the first order of  $\Delta\omega_i^d/\omega_0$  in the amplitude, we have, with (8.195) and  $\sigma_E = H'\omega_0$ ,

$$\begin{aligned} & \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \frac{1}{\omega_i^d + \sigma_C} \\ &= \left[ \omega_0 \left( 1 + H' \frac{k_2}{k_s} \right) + \sigma_E \left( 1 - \frac{k_2}{k_s} \right) + \left( 1 + H' \frac{k_2}{k_s} \right) \Delta\omega_i^d \right]^{-1} \\ &= \left[ \omega_0 + \sigma_E + \left( 1 + H' \frac{k_2}{k_s} \right) \Delta\omega_i^d \right]^{-1} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\omega_0 + \sigma_E} \left[ 1 + \left( 1 + H' \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0 + \sigma_E} \right]^{-1} \\
&= \frac{1}{\omega_0 + \sigma_E} \left[ 1 - \left( 1 + H' \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0 + \sigma_E} \right] \\
&= \frac{A}{C} \frac{1}{\omega_0} \left[ 1 - \frac{A}{C} \left( 1 - H' \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right]. \tag{8.206}
\end{aligned}$$

$$\begin{aligned}
&\left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \frac{1}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta \omega_i^d}{\omega_0} \right) \\
&= \frac{A}{C} \frac{1}{\omega_0} \left[ 1 - \frac{A}{C} \left( 1 + H' \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} + \frac{k_2}{k_s} \frac{\Delta \omega_i^d}{\omega_0} \right] \\
&= \frac{A}{C} \frac{1}{\omega_0} \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right]. \tag{8.207}
\end{aligned}$$

We obtain, by substituting (8.205) into (8.39) and then using (8.207) and (8.56),

$$\begin{cases} \dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right] Y_i^d \cos(\Delta \omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right] Y_i^d \sin(\Delta \omega_i^d t + \zeta_i^d), \end{cases} \tag{8.208}$$

where the argument  $\Delta \omega_i^d t + \zeta_i^d$  is to be computed according to (8.54).

Taking reference to the case of the rigid Earth model, we obtain the precession

$$\dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{H}{\omega} Y_0, \quad \dot{\epsilon} = 0, \tag{8.209}$$

which is the same as in the case of the rigid Earth model, meaning that elastic deformation does not affect precession. The nutation angle from  $t_0$  to  $t$  is, with (8.63),

$$\begin{cases} \delta \psi \sin \epsilon = E \omega_0 \sum_i \left[ \frac{Y_i}{\Delta \omega_i^d} - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{Y_i}{\omega_0} \right] \delta \sin(\Delta \omega_i t + \zeta_i), \\ \delta \epsilon = -E \omega_0 \sum_i \left[ \frac{Y_i}{\Delta \omega_i^d} - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{Y_i}{\omega_0} \right] \delta \cos(\Delta \omega_i t + \zeta_i); \end{cases} \tag{8.210}$$

which can also be written as, by combining the two terms  $\Delta\omega_i t + \zeta_i$  and  $-(\Delta\omega_i t + \zeta_i)$  together

$$\begin{cases} \delta\psi \sin \epsilon = E\omega_0 \sum_{i=1}^N \left[ \frac{Y_i + Y_{-i}}{\Delta\omega_i^d} - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{Y_i - Y_{-i}}{\omega_0} \right] \delta \sin(\Delta\omega_i t + \zeta_i), \\ \delta\epsilon = -E\omega_0 \sum_{i=1}^N \left[ \frac{Y_i - Y_{-i}}{\Delta\omega_i^d} - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{Y_i + Y_{-i}}{\omega_0} \right] \delta \cos(\Delta\omega_i t + \zeta_i). \end{cases} \quad (8.211)$$

The result of the rigid Earth model is given in (8.68). In the results of both the rigid and elastic Earth models, the second terms in the brackets are only on the order of  $\Delta\omega_i/\omega_0$  of the first term, and the elastic deformation affects only the second terms.

### 8.3.2.2 The Rotation Axis

By substituting (8.197) and (8.199) into (8.196), we obtain the equations for polar motion (after swapping the order of the two equations)

$$\begin{cases} \dot{\alpha}_1 + \sigma_C \alpha_2 = -2 \frac{D}{R^2} \frac{H'}{\omega_0} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \dot{\alpha}_2 - \sigma_C \dot{\alpha}_1 = -\frac{D}{R^2} \frac{H'}{\omega_0} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \cos(\omega_i^d t + \phi_i^d), \end{cases} \quad (8.212)$$

where the approximation  $(\omega_i^d - \omega_0)/\omega_0 = \Delta\omega_i^d/\omega_0$  is also made in the amplitude.

Setting the right-hand side to zero leads to the equations for the free polar motion. As compared to the case of the rigid Earth model, the only difference is that  $\sigma_E$  becomes  $\sigma_C$  throughout. The same is for the free nutation, which can be obtained by substituting the solution of free polar motion into (8.199). This similar process of solution is not to be repeated here. We mention that the wobble with the angular rate  $\sigma_C$  is referred to as Chandler wobble, thus the subscript  $C$ . As already mentioned, the Chandler wobble with a period of 434.2 sidereal days is the free rotation mode of the Earth actually observed, which corresponds to the theoretical Eulerian wobble of the rigid Earth model.

The solution of the forced polar motion is

$$\begin{cases} \alpha_1 = 2 \frac{D}{R^2} \frac{H'}{\omega_0} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \frac{1}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -2 \frac{D}{R^2} \frac{H'}{\omega_0} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \frac{1}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.213)$$

It can be simplified to, using (8.207),

$$\begin{cases} \alpha_1 = E \sum_i \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right] Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -E \sum_i \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right] Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.214)$$

The solution of the forced precession-nutation is obtained by substituting (8.213) into (8.199),

$$\begin{cases} \delta \omega_1^0 = 2 \frac{D}{R^2} \frac{H'}{\omega_0} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \frac{\omega_i^d}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta \omega_i^d}{\omega_0} \right) Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \delta \omega_2^0 = 2 \frac{D}{R^2} \frac{H'}{\omega_0} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \sum_i \frac{\omega_i^d}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta \omega_i^d}{\omega_0} \right) Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.215)$$

With (8.207), we have

$$\begin{aligned} & \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \frac{\omega_i^d}{\omega_i^d + \sigma_C} \left( 1 + \frac{k_2}{k_s} \frac{\Delta \omega_i^d}{\omega_0} \right) \\ &= \frac{A}{C} \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right] \left( 1 + \frac{\Delta \omega_i^d}{\omega_0} \right) \\ &= \frac{A}{C} \left[ 1 + \left( H + \frac{A}{C} \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right]. \end{aligned} \quad (8.216)$$

With this formula, we can obtain, by substituting (8.215) into (8.39) and then using (8.56),

$$\begin{cases} \dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \left[ 1 + \left( H + \frac{A}{C} \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right] Y_i^d \cos(\Delta \omega_i^d t + \zeta_i^d), \\ \dot{\epsilon} = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i \left[ 1 + \left( H + \frac{A}{C} \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right] Y_i^d \sin(\Delta \omega_i^d t + \zeta_i^d). \end{cases} \quad (8.217)$$

The precession is

$$\dot{\psi} \sin \epsilon = 2 \frac{D}{R^2} \frac{H}{\omega_0} Y_0^d, \quad \dot{\epsilon} = 0, \quad (8.218)$$



which is again the same as that of the case of the rigid Earth model. The nutation angle from  $t_0$  to  $t$  is, with (8.63),

$$\begin{cases} \delta\psi \sin \epsilon = E\omega_0 \sum_i \left[ \frac{Y_i^d}{\Delta\omega_i^d} + \left( H + \frac{A}{C} \frac{k_2}{k_s} \right) \frac{Y_i^d}{\omega_0} \right] \delta \sin(\Delta\omega_i^d t + \zeta_i^d), \\ \delta\epsilon = -E\omega_0 \sum_i \left[ \frac{Y_i^d}{\Delta\omega_i^d} + \left( H + \frac{A}{C} \frac{k_2}{k_s} \right) \frac{Y_i^d}{\omega_0} \right] \delta \cos(\Delta\omega_i^d t + \zeta_i^d), \end{cases} \quad (8.219)$$

which can also be written as, by combining the two terms  $\Delta\omega_i t + \zeta_i$  and  $-(\Delta\omega_i t + \zeta_i)$  together

$$\begin{cases} \delta\psi \sin \epsilon = E\omega_0 \sum_{i=1}^N \left[ \frac{Y_i^d + Y_{-i}^d}{\Delta\omega_i^d} + \left( H + \frac{A}{C} \frac{k_2}{k_s} \right) \frac{Y_i^d - Y_{-i}^d}{\omega_0} \right] \delta \sin(\Delta\omega_i^d t + \zeta_i^d), \\ \delta\epsilon = -E\omega_0 \sum_{i=1}^N \left[ \frac{Y_i^d + Y_{-i}^d}{\Delta\omega_i^d} + \left( H + \frac{A}{C} \frac{k_2}{k_s} \right) \frac{Y_i^d - Y_{-i}^d}{\omega_0} \right] \delta \cos(\Delta\omega_i^d t + \zeta_i^d). \end{cases} \quad (8.220)$$

The same as in the case of the rigid Earth model, the second terms in the brackets at the right-hand sides are on the order of magnitude of  $\Delta\omega_i^d/\omega_0$  of the first terms, and the elastic deformation affects only the second terms. Remember that this is also the case for the figure axis.

### 8.3.2.3 The Angular Momentum Axis

With (8.197) and (8.203), we can reduce the system of differential equations (8.196) to a system of algebraic equations. The solution is

$$\begin{cases} \delta\omega_1^0 = 2 \frac{D}{R^2} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \frac{H'}{\omega_0 + \sigma_C} \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \delta\omega_2^0 = 2 \frac{D}{R^2} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \frac{H'}{\omega_0 + \sigma_C} \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.221)$$

With

$$\left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \frac{1}{\omega_0 + \sigma_C} = \frac{1}{\omega_0 + \sigma_E} = \frac{A}{C} \frac{1}{\omega_0}, \quad (8.222)$$

the solution can be written as

$$\begin{cases} \delta\omega_1^0 = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \sin(\omega_i^d t + \phi_i^d), \\ \delta\omega_2^0 = 2 \frac{D}{R^2} \frac{H}{\omega_0} \sum_i Y_i^d \cos(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.223)$$

This is exactly the same as for the case of rigid Earth model, which can be understood straightforwardly. Seen from the inertial space, the Euler's dynamic equation  $d\vec{L}/dt = \vec{N}$  is the same for both the rigid and elastic Earth models. As the torque  $\vec{N}$  for both cases is the same (the effect of the deformation on the torque is neglected in our formulation), the rate of change of the angular momentum for both cases is also the same. Only a torque can alter the angular momentum. Hence, no free nutation is present for the angular momentum axis. Further reduction of (8.223) for the precession-nutation is the same as in the case of the rigid Earth model, and is not to be repeated here.

The equation for the polar motion can be obtained by substituting (8.223) into (8.203),

$$\begin{cases} \dot{\alpha}_1 + \sigma_C \alpha_2 = -2 \frac{D}{R^2} \frac{H}{\omega_0} \left[ 1 - \frac{C}{A} \frac{k_2}{k_s} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \right] \sum_i Y_i \sin(\omega_i t + \phi_i), \\ \dot{\alpha}_2 - \sigma_C \alpha_1 = -2 \frac{D}{R^2} \frac{H}{\omega_0} \left[ 1 - \frac{C}{A} \frac{k_2}{k_s} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \right] \sum_i Y_i \cos(\omega_i t + \phi_i). \end{cases} \quad (8.224)$$

Again, replacing  $\sigma_E$  by  $\sigma_C$  in the formula of free polar motion of the rigid Earth model yields the formula of free polar motion of the elastic Earth model, which is not to be produced here.

The solution of the forced polar motion is

$$\begin{cases} \alpha_1 = 2 \frac{D}{R^2} \frac{H}{\omega_0} \left[ 1 - \frac{C}{A} \frac{k_2}{k_s} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \right] \sum_i \frac{1}{\omega_i^d + \sigma_C} Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -2 \frac{D}{R^2} \frac{H}{\omega_0} \left[ 1 - \frac{C}{A} \frac{k_2}{k_s} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \right] \sum_i \frac{1}{\omega_i^d + \sigma_C} Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.225)$$

With (8.206), we have

$$\begin{aligned} & \left[ 1 - \frac{C}{A} \frac{k_2}{k_s} \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \right] \frac{1}{\omega_i^d + \sigma_C} \\ &= \left( 1 + H' \frac{k_2}{k_s} - \frac{C}{A} \frac{k_2}{k_s} \right) \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \frac{1}{\omega_i^d + \sigma_C} \\ &= \frac{A}{C} \frac{1}{\omega_0} \left( 1 - \frac{k_2}{k_s} \right) \left[ 1 - \frac{A}{C} \left( 1 - H' \frac{k_2}{k_s} \right) \frac{\Delta \omega_i^d}{\omega_0} \right]. \end{aligned} \quad (8.226)$$

We finally obtain, with (8.63),

$$\begin{cases} \alpha_1 = E \frac{A}{C} \left(1 - \frac{k_2}{k_s}\right) \sum_i \left[1 - \frac{A}{C} \left(1 + H' \frac{k_2}{k_s}\right) \frac{\Delta \omega_i^d}{\omega_0}\right] Y_i^d \cos(\omega_i^d t + \phi_i^d), \\ \alpha_2 = -E \frac{A}{C} \left(1 - \frac{k_2}{k_s}\right) \sum_i \left[1 - \frac{A}{C} \left(1 + H' \frac{k_2}{k_s}\right) \frac{\Delta \omega_i^d}{\omega_0}\right] Y_i^d \sin(\omega_i^d t + \phi_i^d). \end{cases} \quad (8.227)$$

### 8.3.3 The Celestial Intermediate Pole

Putting together the results of the forced and free nutations and polar motions, we obtain the same as Table 8.2 with  $\sigma_E$  replaced by  $\sigma_C$  for the free polar motion and free nutation. The angular frequency of the free polar motion of the elastic Earth model is  $\sigma_C$ , while that of the rigid Earth model is  $\sigma_E$ ; the angular frequency of the free nutation of the elastic Earth model is  $\omega_0 - \sigma_C$ , while that of the rigid Earth model is  $\omega_0 - \sigma_E = (C/A)\omega_0$ .

The free polar motion with the angular frequency  $\sigma_C$  is the generalization of the Eulerian wobble of the rigid Earth model to more realistic Earth models, called Chandler wobble, as already mentioned. With  $k_2 = 0.30$  and  $k_s = 0.94$ , the period of the Chandler wobble is 436 sidereal days, which is very close to the observed value of 434.2 sidereal days. However, the effect of the oceans and the core are about 29.8 and  $-50.5$  sidereal days, respectively. Even a much more realistic Earth model leads to a theoretical value of 426.8 sidereal days, which is 7 days shorter than the observed value (e.g. Gross, 2015).

The celestial intermediate pole (CIP) can be defined in a similar way to the case of the rigid Earth model. Besides the precession that is the same for all cases we studied, we defined the nutation of the CIP as the forced part of that of the Tisserand axis  $x_3$ . By substituting (8.204) and (8.197) into (8.196), we obtain

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_C)\dot{\alpha}_2 + \omega_0\sigma_C\alpha_1 = 0, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_C)\dot{\alpha}_1 + \omega_0\sigma_C\alpha_2 = 0, \end{cases} \quad (8.228)$$

from the first two equations. This system of differential equations is the same as (8.120) with  $\sigma_E$  replaced by  $\sigma_C$ . Therefore, the solution is also the same as that of (8.120) with  $\sigma_E$  replaced by  $\sigma_C$ . One solution is the Chandler wobble with the angular frequency  $\sigma_C$ , and the other is the diurnal tilt-over mode. By properly defining the  $x_3^0$ -axes as explained for the case of the rigid Earth model, the tilt-over mode can be excluded in the polar motion. Hence, the polar motion of the CIP includes only one component, i.e., the Chandler wobble, which is a free rotation mode.

### 8.3.4 The Transfer Function

Within the context of the formulation in this section, we define the transfer function for the nutation as the ratio between the amplitudes of the nutation of the elastic and rigid Earth models. The same can be defined for the polar motion. We take as example the rotation axis.

We represent the nutation using a complex value defined as

$$\tilde{\eta} = \delta\epsilon + i\delta\psi \sin \epsilon. \quad (8.229)$$

According to (8.78) and (8.219), for the  $i$ th wave, we have,

$$\tilde{\eta}_i^r = E\omega_0 \left( \frac{Y_i^d}{\Delta\omega_i^d} + H \frac{Y_i^d}{\omega_0} \right) \delta e^{-i(\Delta\omega_i^d t + \zeta_i^d - \pi)}, \quad (8.230)$$

$$\tilde{\eta}_i^e = E\omega_0 \left[ \frac{Y_i^d}{\Delta\omega_i^d} + \left( H + \frac{A k_2}{C k_s} \right) \frac{Y_i^d}{\omega_0} \right] \delta e^{-i(\Delta\omega_i^d t + \zeta_i^d - \pi)}, \quad (8.231)$$

where the superscripts  $r$  and  $e$  are for the *rigid* and *elastic* Earth models, respectively. By definition, the transfer function for the nutation is, accurate to the first order of  $\Delta\omega_i^d/\omega_0$ ,

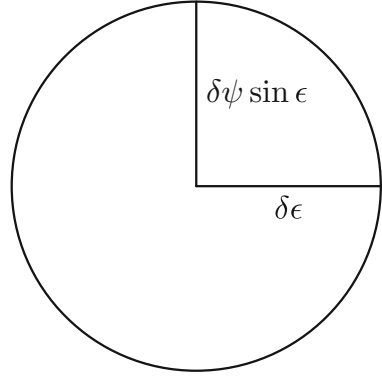
$$\begin{aligned} T_i^\eta &= \left[ \frac{Y_i^d}{\Delta\omega_i^d} + \left( H + \frac{A k_2}{C k_s} \right) \frac{Y_i^d}{\omega_0} \right] \left( \frac{Y_i^d}{\Delta\omega_i^d} + H \frac{Y_i^d}{\omega_0} \right)^{-1} \\ &= 1 + \left( \frac{A k_2}{C k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) \left( 1 + H \frac{\Delta\omega_i^d}{\omega_0} \right)^{-1} \\ &= 1 + \frac{A k_2}{C k_s} \frac{\Delta\omega_i^d}{\omega_0}. \end{aligned} \quad (8.232)$$

The geometrical meaning of the complex expression (8.229) is shown in Fig. 8.8. For any  $i$ , the amplitude of  $\delta\epsilon$  and  $\delta\psi \sin \epsilon$  are equal, and  $\tilde{\eta} = \eta e^{i(\sigma t + \phi)}$  represents a prograde circular motion with the angular rate  $\sigma$ . Compare this figure with Figs. 1.15–1.18 for the relative orientations of  $\delta\epsilon$  and  $\delta\psi \sin \epsilon$ . This definition is adopted by Moritz and Mueller (1987). Another form,  $\delta\psi \sin \epsilon + i\delta\epsilon$ , is also present in the literature (e.g., Mathews & Bretagnon, 2003).

We can also define the transfer function for the polar motion. Recognizing the fact that the  $x_1$ - and  $x_2$ -coordinates of the pole are  $x_p$  and  $-y_p$  or  $-\alpha_2$  and  $\alpha_1$ , respectively, we define a complex pole coordinate

$$\tilde{\alpha} = -\alpha_2 + i\alpha_1. \quad (8.233)$$

**Fig. 8.8** The nutation angles  $\delta\epsilon$  and  $\delta\psi \sin \epsilon$



According to (8.71) and (8.214), for the  $i$ th wave, we have

$$\tilde{\alpha}_i^r = E \left( 1 - \frac{A}{C} \frac{\Delta\omega_i^d}{\omega_0} \right) Y_i^d e^{-i(\omega_i^d t + \phi_i^d - \pi/2)}, \quad (8.234)$$

$$\tilde{\alpha}_i^e = E \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta\omega_i^d}{\omega_0} \right] Y_i^d e^{-i(\omega_i^d t + \phi_i^d - \pi/2)}. \quad (8.235)$$

By definition, the transfer function for the polar motion is, accurate to the first order of  $\Delta\omega_i^d/\omega_0$ ,

$$\begin{aligned} T_i^\alpha &= \left[ 1 - \frac{A}{C} \left( 1 - \frac{k_2}{k_s} \right) \frac{\Delta\omega_i^d}{\omega_0} \right] \left( 1 - \frac{A}{C} \frac{\Delta\omega_i^d}{\omega_0} \right)^{-1} \\ &= 1 + \left( \frac{A}{C} \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0} \right) \left( 1 + \frac{A}{C} \frac{\Delta\omega_i^d}{\omega_0} \right)^{-1} \\ &= 1 + \frac{A}{C} \frac{k_2}{k_s} \frac{\Delta\omega_i^d}{\omega_0}. \end{aligned} \quad (8.236)$$

We see that, for the rotation axis, the transfer function for the nutation and that for the polar motion are identical, i.e.,  $T_i^\eta = T_i^\alpha$ . This identity holds strictly in much more accurate formulations for much more complicated Earth models. If damping happens, like viscous friction in the mantle or the core, the transfer function becomes a complex value. This is the reason it is defined based on complex expressions of nutation and polar motion.

With the expressions in complex values, we see much more clearly the direction of the polar motion and nutation. The polar motion is retrograde diurnal (the frequency is equal to that of the tide). For  $\Delta\omega_i^d > 0$ , the nutation is retrograde secular (the frequency is equal to the difference obtained by subtracting the sidereal frequency from the tidal frequency).

The research on precession-nutation is separated into two problems: one provides the precession-nutation of the rigid Earth model, and the other provides the transfer function of the more realistic Earth model. The numerical determination of precession-nutation is performed by fitting the observed Earth orientation data based on a model of rigid Earth rotation and a model of transfer function. Since the models cannot be perfect, some parameters of the transfer function model, e.g., the Chandler wobble period, are also determined from the data. The periods of the forced nutation waves are very accurate, since they depend on the well determined orbits of the forcing celestial bodies. The principle of numerical procedure established by Zhu et al. (1990) has been followed, although additional complexity of the transfer function has been introduced to accommodate the more complicated Earth models (Mathews et al., 1991, 2002). See Dehant and Mathews (2015) for details.

## 8.4 Change of the Rotation Rate: The Effect of Earth Tide

We now analyze the last equation in (8.196). We first consider the effect of the first term in the expression of  $\tilde{N}_3$ . The equation to be solved is

$$\ddot{\alpha}_3 = 2 \frac{k_2}{k_s} \frac{D}{R^2} H \left( 1 + \frac{4}{3} \frac{\tilde{k}_2}{k_s} H \right)^{-1} \sum_i \frac{\omega_i^s}{\omega_0} Y_i^s \sin(\omega_i^s t + \phi_i^s). \quad (8.237)$$

The integration of both sides over time yields

$$\dot{\alpha}_3 = -2 \frac{k_2}{k_s} \frac{D}{R^2} \frac{H}{\omega_0} \left( 1 + \frac{4}{3} \frac{\tilde{k}_2}{k_s} H \right)^{-1} \sum_i Y_i^s \cos(\omega_i^s t + \phi_i^s), \quad (8.238)$$

where we have omitted the integration constant. Approximating the factor  $[1 + (4/3)(\tilde{k}_2/k_s)H]^{-1}$  by 1 brings in a relative error on the order of  $\sim 1/300$ . We will keep it in the formulation, which does not hurt. The instantaneous angular rate of the Earth rotation is  $\omega_3 = \dot{\alpha}_3 + \omega_0$ . The change of length of day due to  $\dot{\alpha}_3$  is

$$\frac{2\pi}{\dot{\alpha}_3 + \omega_0} - \frac{2\pi}{\omega_0} = -2\pi \frac{\dot{\alpha}_3}{\omega_0^2}. \quad (8.239)$$

This variation is periodical, with semi-monthly, monthly, semi-annual, and annual components. It is in fact due to the change of the axial moment of inertia associated with Earth tide, and the variation of the rotation rate is the result of the conservation of the angular momentum. The change in Earth rotation angle can be obtained by integrating (8.238) over time,

$$\alpha_3 = -2 \frac{k_2}{k_s} \frac{D}{R^2} \frac{H}{\omega_0} \left( 1 + \frac{4}{3} \frac{\tilde{k}_2}{k_s} H \right)^{-1} \sum_i \frac{Y_i^s}{\omega_i^s} \sin(\omega_i^s t + \phi_i^s), \quad (8.240)$$

where an integration constant is dropped again. The periods of the secular tidal waves could be half a month, a month, half a year, or a year. Hence, the value of  $\omega_i^s$  is very small.

The equation for the effect of  $N_3$  is

$$\ddot{\alpha}_3 = \frac{N_3}{C} \left( 1 + \frac{4}{3} \frac{\tilde{k}_2}{k_s} H \right)^{-1}. \quad (8.241)$$

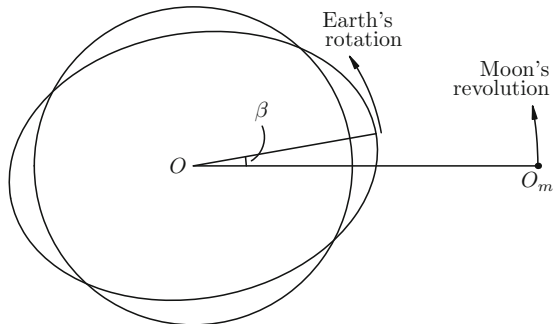
In our formulation, we have  $N_3 = 0$  in the lunisolar gravitational torque, which is a consequence of our approximation to consider only the  $J_2$  term, i.e., the degree 2 and order 0 term, of the Earth's gravitational potential. Other terms, most evidently, the degree 2 and order 2 term that is in elliptical form in the equator, would contribute to  $N_3$ , causing a semidiurnal variation of the Earth rotation rate.

Here we consider another effect, i.e., the delay of the Earth's tidal deformation, which causes a permanent slowing down of the Earth rotation. The Earth is not purely elastic, causing the tidal deformation to be lagged a small angle  $\epsilon$  to the tidal-generating force, as shown in Fig. 8.9. The small deviation from pure elasticity is referred to anelasticity in the Earth tide literature. Loosely speaking, the Earth rotates from west to east one cycle every day, and the Moon revolves around the Earth from west to east one cycle every month. As seen from the Earth, the Moon moves from east to west. Hence, the lag is to the east of the Moon. More specifically speaking, the tidal bulge is  $\epsilon$  east of the Moon. In other words, the lagged tidal deformation appears to be that of the elastic tidal deformation when the hour angle of the Moon or Sun is diminished by  $\epsilon$ , as the hour angle is measured from east to west.

We first consider the lunar tide. Taking reference to (8.159), we write the additional potential of the elastic tidal deformation as

$$\begin{aligned} \phi_{1m} = & \frac{GM_m}{r} \left( \frac{R}{r} \right)^3 \left( \frac{R}{r_m} \right)^3 \\ & \times \sum_{k=0}^2 \frac{2k_2}{1 + \delta_k} \frac{(2-k)!}{(2+k)!} P_2^k(\sin \delta_m) P_2^k(\sin \phi) \cos k(H_m + \lambda), \end{aligned} \quad (8.242)$$

**Fig. 8.9** Lag angle  $\epsilon$  of the Earth's tidal deformation



where  $H_m$  is the hour angle of the Moon at the prime meridian. The function  $\phi_{1m}$  is the value of the additional potential at the point of geocentric coordinate  $(r, \phi, \lambda)$ . Only the degree 2 tide is considered. The additional potential of the lagged tidal deformation is

$$\phi_{1m} = \frac{GM_m}{r} \left(\frac{R}{r}\right)^3 \left(\frac{R}{r_m}\right)^3 \sum_{k=0}^2 \frac{2k_2}{1+\delta_k} \frac{(2-k)!}{(2+k)!} P_2^k(\sin \delta_m) \times P_2^k(\sin \phi) \cos k(H_m + \lambda - \beta). \quad (8.243)$$

The gravitational torque of the Moon on the Earth related to the Earth's tidal deformation is, according to (8.4),

$$\vec{N}_m = -M_m \vec{r}_m \times \nabla \phi_{1m} = M_m r_m \left( \frac{1}{r_m \sin \theta} \frac{\partial \phi_{1m}}{\partial \lambda} \hat{e}_\theta - \frac{1}{r_m} \frac{\partial \phi_{1m}}{\partial \theta} \hat{e}_\lambda \right)_{\theta=\theta_m, \lambda=\lambda_m}, \quad (8.244)$$

where  $r_m, \theta_m = \pi/2 - \delta_m$  and  $\lambda_m = 2\pi - H_m$  are the geocentric coordinates of the Moon. The components of this torque along the  $x_1$ - and  $x_2$ -axes are very small as compared to the effect of the Earth's ellipticity, and are not to be investigated here. The component along the  $x_3$ -axis is

$$\begin{aligned} N_{3m} &= M_m r_m \left[ \frac{1}{r_m \sin \theta} \frac{\partial \phi_{1m}}{\partial \lambda} \cos \left( \frac{\pi}{2} + \theta \right) \right]_{\theta=\theta_m, \lambda=\lambda_m} \\ &= -M_m \left( \frac{\partial \phi_{1m}}{\partial \lambda} \right)_{\theta=\theta_m, \lambda=\lambda_m}. \end{aligned} \quad (8.245)$$

The substitution of (8.243) into this formula yields,

$$N_{m3} = -\frac{GM_m^2}{R} \left(\frac{R}{r_m}\right)^6 \sum_{k=0}^2 \frac{2k_2}{1+\delta_k} \frac{(2-k)!}{(2+k)!} [P_2^k(\sin \delta_m)]^2 k(\sin k\beta). \quad (8.246)$$

The substitution of  $P_2^k(\sin \delta_m)$  by its expression given in (7.22)–(7.24) yields

$$N_{m3} = -\frac{GM_m^2}{R} \left(\frac{R}{r_m}\right)^6 k_2 \left( 3 \sin^2 \delta_m \cos^2 \delta_m \sin \beta + \frac{3}{2} \cos^4 \delta_m \sin 2\beta \right). \quad (8.247)$$

As  $\beta$  is extremely small, we can make the approximations  $\sin \beta = \beta$  and  $\sin 2\beta = 2\beta$ . This leads to

$$N_{m3} = -3 \frac{GM_m^2}{R} \left(\frac{R}{r_m}\right)^6 k_2 \cos^2 \delta_m \beta. \quad (8.248)$$

It is proportional to  $M_m^2$ , as both the deformation and the gravitational attraction depend on  $M_m$ . The solar torque is similar. We assume that the lag angle  $\beta$  is the



same for both the lunar and solar tides. By adding them together, we obtain

$$N_3 = -3 \frac{GM_m^2}{R} \left( \frac{R}{r_m} \right)^6 k_2 \left[ \cos^2 \delta_m + \left( \frac{M_s}{M_m} \right)^2 \left( \frac{r_m}{r_s} \right)^6 \cos^2 \delta_s \right] \beta. \quad (8.249)$$

The obliquity of the ecliptic is  $23.5^\circ$ . The inclination of the Moon's orbit to the ecliptic is  $5.1^\circ$ . Therefore,  $-23.5^\circ \leq \delta_s \leq 23.5^\circ$  and  $-28.6^\circ \leq \delta_m \leq 28.6^\circ$ . The deviations of  $\cos^2 \delta_m$  and  $\cos^2 \delta_s$  from 1 are small. By setting  $\cos^2 \delta_m \sim 1$  and  $\cos^2 \delta_s \sim 1$ , we obtain a rough approximation

$$N_3 \approx -3 \frac{GM_m^2}{R} \left( \frac{R}{r_m} \right)^6 k_2 \left[ 1 + \left( \frac{M_s}{M_m} \right)^2 \left( \frac{r_m}{r_s} \right)^6 \right] \beta = -1.3 \times 10^{18} \beta, \quad (8.250)$$

the unit being  $\text{N} \cdot \text{m}$ . The contribution from the Sun is 21% of that of the Moon.

The value of  $N_3$  given by (8.250) is constant, and it decelerates the Earth rotation constantly. However, this contribution to the long term retardation of the Earth rotation is very small as compared to the contribution of the ocean tide. Even the lag  $\beta$  is difficult to determine. The mean lengthening rate of the day determined using data of the last 2735 years is 1.8 ms per century (Stephenson et al., 2016). As for the effect of Earth tide, a value of  $\beta \sim 0.20^\circ$  (Ray et al., 2001) leads to  $N_3 = -4.5 \times 10^{-15} \text{N} \cdot \text{m}$ ,  $\ddot{\alpha} = -5.6 \times 10^{-23} \text{rad} \cdot \text{s}^{-2}$ , with a lengthening rate of the day of 0.21 ms per century, which is on the order of 10% of the total amount.

The sum of Earth rotation deceleration caused by the lags of the Earth and ocean tides can be inferred from lunar laser ranging (LLR). Consider the Earth-Moon system as an isolated system. For the torques exerted on the Earth due to the lags of both the Earth and ocean tides, an opposite torque is exerted on the Moon. The torque exerted on the Moon is also with respect to the Earth center, which accelerate the Moon's orbital motion, causing the Earth-Moon distance to increase. The total lunar tide contribution to Earth rotation deceleration can then be computed from the rate of the increase of the Earth-Moon distance. The contribution of the solar tide can be obtained by properly scaling the contribution of the lunar tide. The rate of Earth rotation deceleration thus obtained is  $\ddot{\alpha} = -6.1 \times 10^{-23} \text{rad} \cdot \text{s}^{-2}$  (Stephenson et al., 2016), corresponding to a lengthening rate of the day of 2.3 ms per century. Note that this value is obtained using data of only the last few decades, while the value of 1.8 ms per century is obtained using data of over two thousand years.

## 8.5 Excitation by the Atmosphere-Hydrosphere-Cryosphere System

### 8.5.1 Formulation for the Elastic Earth Model

The governing equation is the linearized Euler-Liouville equation (8.156), which is copied to here for convenience of reference,

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_E)\dot{\alpha}_2 + \omega_0\sigma_E\alpha_1 + \frac{d\delta\omega_1^0}{dt} + \sigma_E\delta\omega_2^0 + \frac{1}{A}\frac{dc_{13}}{dt}\omega_0 - \frac{c_{23}}{A}\omega_0^2 = \frac{N_1}{A}, \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_E)\dot{\alpha}_1 + \omega_0\sigma_E\alpha_2 + \frac{d\delta\omega_2^0}{dt} - \sigma_E\delta\omega_1^0 + \frac{1}{A}\frac{dc_{23}}{dt}\omega_0 + \frac{c_{13}}{A}\omega_0^2 = \frac{N_2}{A}, \\ \ddot{\alpha}_3 + \frac{1}{C}\frac{dc_{33}}{dt}\omega_0 = \frac{N_3}{C}. \end{cases} \quad (8.251)$$

Now the torque  $\vec{N} = N_1\hat{e}_1 + N_2\hat{e}_2 + N_3\hat{e}_3$  is exerted by the atmosphere-hydrosphere-cryosphere system. Note that  $\alpha_1$  and  $\alpha_2$  are related to the Greenwich and west longitude  $90^\circ$  pole coordinates according to (8.40).

In this problem, we consider the Earth system as an isolated system, assuming that the response to external forcings, e.g., the lunisolar gravitational attraction, is studied in a separate problem. Therefore, a torque of  $-\vec{N}$  is exerted by the solid Earth on the atmosphere-hydrosphere-cryosphere system. The rate of change of the total angular momentum of the atmosphere-hydrosphere-cryosphere system is then  $-\vec{N}$ . According to (3.317)–(3.320), the angular momentum equation of the atmosphere-hydrosphere-cryosphere system expressed in the Earth-fixed coordinate system  $Ox_1x_2x_3$  (defined by the Tisserand axes of the solid Earth) is

$$\vec{N} = -\frac{d}{dt}(\vec{C}' \cdot \vec{\omega} + \vec{h}') - \vec{\omega} \times (\vec{C}' \cdot \vec{\omega} + \vec{h}'), \quad (8.252)$$

where

$$\vec{C}' \cdot \vec{\omega} = \int_{V'} \rho' \vec{r} \times (\vec{\omega} \times \vec{r}) dV', \quad (8.253)$$

$$\vec{h}' = \int_{V'} \rho' \vec{r} \times \vec{v}' dV'. \quad (8.254)$$

We emphasize that, in these equations, the domain of integration,  $V'$ , refers to the atmosphere-hydrosphere-cryosphere system,  $\vec{C}'$  the moment of inertia tensor of the system,  $\vec{\omega}$  the angular velocity of the Earth with respect to the inertial space,  $\vec{v}'$  the velocity with respect to the Earth-fixed system, and  $\vec{r}$  the position vector in the Earth-fixed coordinate system. We further emphasize that the derivative  $d/dt$  is defined in the Earth-fixed coordinate system, i.e., it is equivalent to  $(d/dt)_{\text{rot}}$  in (3.46). We assume that the average state of the atmosphere-hydrosphere-cryosphere system rotates with the solid Earth as a whole. In this way, the interaction between the average atmosphere-hydrosphere-cryosphere system and the solid Earth is excluded. Strictly speaking, the moment of inertia components  $A$  and  $C$  should include the effect of the average atmosphere-hydrosphere-cryosphere system. We now then have to consider only the difference between the instant and average states of the atmosphere-hydrosphere-cryosphere system, i.e., the mass transport. We can

thus write (8.252) and (8.253) as

$$\vec{N} = -\frac{d}{dt}[\vec{c}' \cdot (\omega_0 \hat{e}_3) + \vec{h}'] - (\omega_0 \hat{e}_3) \times [\vec{c}' \cdot (\omega_0 \hat{e}_3) + \vec{h}'], \quad (8.255)$$

$$\vec{c}' \cdot (\omega_0 \hat{e}_3) = \int_{V'} \delta \rho' \vec{r} \times [(\omega_0 \hat{e}_3) \times \vec{r}] dV', \quad (8.256)$$

where  $\vec{c}'$  and  $\delta \rho$  are the increments of the moment of inertia tensor and density with respect to the average state, respectively. Since  $\vec{c}$ ,  $\delta \rho$ , and  $\vec{h}$  are small quantities, we omitted their products with the change of Earth rotation velocity as the first order approximation. In (8.254), we do not have to distinguish the density in the average state and that in the instantaneous state, since the contribution of  $\delta \rho$  is apparently of smaller order of magnitude.

We need the expressions of the components of  $\vec{N}$  in order to solve (8.251). As in the derivation of (3.307) in Sect. 3.5.2, we can develop (8.256) to

$$\vec{c}' \cdot (\omega_0 \hat{e}_3) = \omega_0 c'_{13} \hat{e}_1 + \omega_0 c'_{23} \hat{e}_2 + \omega_0 c'_{33} \hat{e}_3, \quad (8.257)$$

where

$$c'_{13} = -\int_{V'} \delta \rho x_1 x_3 dV', \quad c'_{23} = -\int_{V'} \delta \rho x_2 x_3 dV', \quad c'_{33} = \int_{V'} (x_1^2 + x_2^2) dV'. \quad (8.258)$$

We thus obtain the components of  $\vec{N}$  as

$$\begin{cases} N_1 = -\frac{d}{dt}(\omega_0 c'_{13} + h'_1) + \omega_0(\omega_0 c'_{23} + h'_2), \\ N_2 = -\frac{d}{dt}(\omega_0 c'_{23} + h'_2) - \omega_0(\omega_0 c'_{13} + h'_1), \\ N_3 = -\frac{d}{dt}(\omega_0 c'_{33} + h'_3). \end{cases} \quad (8.259)$$

The change of the moment of inertia tensor components,  $c'_{13}$ ,  $c'_{23}$ , and  $c'_{33}$ , indicates a change of the degree 2 gravitational potential of the atmosphere-hydrosphere-cryosphere system, which is induced by a mass redistribution. This mass redistribution is actually a mass of load that induces a change of the moment of inertia tensor of the solid Earth. The corresponding components are, according to (7.212),  $c_{13} = k'_2 c'_{13}$ ,  $c_{23} = k'_2 c'_{23}$ , and  $c_{33} = k'_2 c'_{33}$ . The variation of the Earth rotation velocity also deforms the solid Earth, causing a change of the moment of inertia tensor, which is given by the second terms of each component in (8.192). We

thus have

$$\begin{cases} c_{13} = k'_2 c'_{13} + \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A)(\dot{\alpha}_1 - \alpha_2 \omega_0 + \delta \omega_1^0), \\ c_{23} = k'_2 c'_{23} + \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A)(\dot{\alpha}_2 + \alpha_1 \omega_0 + \delta \omega_2^0), \\ c_{33} = k'_2 c'_{33} + \frac{4}{3} \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A) \dot{\alpha}_3. \end{cases} \quad (8.260)$$

The substitution of (8.259) and (8.260) into (8.251) yields

$$\begin{aligned} \ddot{\alpha}_1 - (\omega_0 - \sigma_C) \dot{\alpha}_2 + \omega_0 \sigma_C \alpha_1 + \frac{d\delta \omega_1^0}{dt} + \sigma_C \delta \omega_2^0 \\ = \omega_0 \sigma_C \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \left( \chi_2 - \frac{1}{\omega_0} \frac{d\chi_1}{dt} \right), \end{aligned} \quad (8.261a)$$

$$\begin{aligned} \ddot{\alpha}_2 + (\omega_0 - \sigma_C) \dot{\alpha}_1 + \omega_0 \sigma_C \alpha_2 + \frac{d\delta \omega_2^0}{dt} - \sigma_C \delta \omega_1^0 \\ = \omega_0 \sigma_C \left( 1 + H' \frac{k_2}{k_s} \right)^{-1} \left( -\chi_1 - \frac{1}{\omega_0} \frac{d\chi_2}{dt} \right), \end{aligned} \quad (8.261b)$$

$$\ddot{\alpha}_3 = \omega_0 \left( 1 + \frac{4}{3} H' \frac{k_2}{k_s} \right)^{-1} \frac{d\chi_3}{dt}. \quad (8.261c)$$

The Chandler wobble angular rate  $\sigma_C$  is defined in (8.195). The functions  $\chi_1$ ,  $\chi_2$ , and  $\chi_3$  defined as

$$\begin{cases} \chi_1 = \frac{\omega_0}{\sigma_C} \frac{\omega_0(1 + k'_2)c'_{13} + h'_1}{A\omega_0}, \\ \chi_2 = \frac{\omega_0}{\sigma_C} \frac{\omega_0(1 + k'_2)c'_{23} + h'_2}{A\omega_0}, \\ \chi_3 = \frac{\omega_0(1 + k'_2)c'_{33} + h'_3}{C\omega_0} \end{cases} \quad (8.262)$$

are referred to as excitation function.

From the derivation of forced nutation, we realize that the frequency of the nutation of either the Tisserand, rotation, or angular momentum axis is 1 cycle per sidereal day less than the frequency of the torque expressed in the Earth-fixed coordinate system. Now the variation of the excitation function is much slower than 1 cycle per sidereal day. Therefore, its effect on the nutation of any of the axis mentioned would be retrograde with a frequency of about 1 cycle per sidereal day. Based on the definition of the celestial intermediate pole (CIP), we attribute the variation of the Earth rotation in response to the excitation function to the polar

motion, i.e., the nutation is set to zero, or  $\delta\omega_1^0 = \delta\omega_2^0 = 0$ . The governing equations (8.251) simplify to

$$\begin{cases} \ddot{\alpha}_1 - (\omega_0 - \sigma_C)\dot{\alpha}_2 + \omega_0\sigma_C\alpha_1 = \omega_0\sigma_C\left(\chi_2 - \frac{1}{\omega_0}\frac{d\chi_1}{dt}\right), \\ \ddot{\alpha}_2 + (\omega_0 - \sigma_C)\dot{\alpha}_1 + \omega_0\sigma_C\alpha_2 = \omega_0\sigma_C\left(-\chi_1 - \frac{1}{\omega_0}\frac{d\chi_2}{dt}\right), \\ \ddot{\alpha}_3 = \omega_0\frac{d\chi_3}{dt}, \end{cases} \quad (8.263)$$

where an approximation is made by neglecting the terms on the order of  $H \sim 1/300$ .

The first two equations of (8.263) can be simplified to

$$-\dot{\alpha}_2 + \sigma_C\alpha_1 = \sigma_C\chi_2, \quad \dot{\alpha}_1 + \sigma_C\alpha_2 = -\sigma_C\chi_1. \quad (8.264)$$

Based on (8.40), we express the excitation function and pole coordinates in the complex forms

$$\tilde{\chi} = \chi_1 + i\chi_2, \quad \tilde{p} = -\alpha_2 + i\alpha_1, \quad (8.265)$$

noting that  $-\alpha_2$  and  $\alpha_1$  are the  $x_1$ - and  $x_2$ -coordinates of the pole, respectively. We finally obtain the governing equation for the CIP,

$$\tilde{p} + \frac{i}{\sigma_C} \frac{d\tilde{p}}{dt} = \tilde{\chi}. \quad (8.266)$$

For the historical reason that the CIP was introduced much later than the other axes, the usual procedure of formulation in the literature is to derive the governing equation for the rotation axis, and then transform to that for the CIP. Define

$$\alpha_1^r = \alpha_1 + \frac{1}{\omega_0}\dot{\alpha}_2, \quad \alpha_2^r = \alpha_2 - \frac{1}{\omega_0}\dot{\alpha}_1. \quad (8.267)$$

The first two of the governing equations (8.263) become

$$\begin{cases} \dot{\alpha}_2^r - \sigma_C\alpha_1^r = \sigma_C\left(\chi_2 - \frac{1}{\omega_0}\frac{d\chi_1}{dt}\right), \\ \dot{\alpha}_1^r + \sigma_C\alpha_2^r = \sigma_C\left(-\chi_1 - \frac{1}{\omega_0}\frac{d\chi_2}{dt}\right). \end{cases} \quad (8.268)$$

This is in fact the governing equation of the rotation pole, which can be obtained by substituting (8.199) into (8.251) and then adding a subscript  $r$  to distinguish from the CIP. Define the complex rotation pole coordinate as

$$\tilde{m} = -\alpha_2^r + i\alpha_1^r. \quad (8.269)$$

We can write (8.267) as

$$\tilde{m} = \tilde{p} - \frac{i}{\omega_0} \frac{d\tilde{p}}{dt} \quad (8.270)$$

and (8.268) as

$$\tilde{m} + \frac{i}{\sigma_C} \frac{d\tilde{m}}{dt} = \tilde{\chi} - \frac{i}{\omega_0} \frac{d\tilde{\chi}}{dt}. \quad (8.271)$$

The governing equation for the CIP, (8.266), can be obtained by substituting (8.270) into that of the rotation pole, (8.271).

We mention that, as the average state of the atmosphere-hydrosphere-cryosphere system is assumed to rotate with the solid Earth as a whole, even the response of the atmosphere-hydrosphere-cryosphere system to the polar motion should be included in the excitation function.

### 8.5.2 Formulae for Practical Applications

The theory based on the elastic Earth model is not accurate enough for interpreting current data. An Earth model composed of an anelastic (i.e., non-pure elastic) mantle and a fluid core is required. The detail of the strict formulation for such a complicated Earth model is beyond the scope of this book. Here we infer the result based on the analysis of the observed physical facts.

Since there is a damping, the Chandler wobble frequency becomes complex. We write it as

$$\sigma_0 = \frac{2\pi}{T_0} \left( 1 + \frac{i}{2Q_0} \right). \quad (8.272)$$

The observed values are 1.185 years and  $Q_0 = 179$  (Wilson & Vicente, 1990). Therefore,  $\omega_C$  in (8.266) is replaced by  $\sigma_0$ , yielding

$$\tilde{p} + \frac{i}{\sigma_0} \frac{d\tilde{p}}{dt} = \tilde{\chi}. \quad (8.273)$$

For the rotation with periods as long as those concerned here, the mantle is practically decoupled from the core. Therefore, the moments of inertia in the excitation functions,  $A$  and  $C$ , should be replaced by those of the mantle,  $A_m$  and  $C_m$ . We write them as

$$\begin{cases} \chi_1 = \frac{\omega_0}{\sigma_C} \frac{C - A}{A_m} \frac{\omega_0(1 + k'_2)c'_{13} + h'_1}{(C - A)\omega_0} = \frac{1.608[0.695\omega_0 c'_{13} + h'_1]}{(C - A)\omega_0}, \\ \chi_2 = \frac{\omega_0}{\sigma_C} \frac{C - A}{A_m} \frac{\omega_0(1 + k'_2)c'_{23} + h'_2}{(C - A)\omega_0} = \frac{1.608[0.695\omega_0 c'_{23} + h'_2]}{(C - A)\omega_0}, \\ \chi_3 = \frac{\omega_0(1 + k'_2)c'_{33} + h'_3}{C_m\omega_0} = \frac{0.695\omega_0 c'_{33} + h'_3}{C_m\omega_0}, \end{cases} \quad (8.274)$$

where the numerical values are computed using the observed Chandler wobble angular frequency. The strict formulation for the Earth model composed of an anelastic mantle and a fluid core yields the result (Gross, 2015)

$$\begin{cases} \chi_1 = \frac{1.608[0.684\omega_0 c'_{13} + h'_1]}{(C - A)\omega_0}, \\ \chi_2 = \frac{1.608[0.684\omega_0 c'_{23} + h'_2]}{(C - A)\omega_0}, \\ \chi_3 = \frac{0.997[0.750\omega_0 c'_{33} + h'_3]}{C_m \omega_0}. \end{cases} \quad (8.275)$$

The change of 0.695 to 0.684 in  $\chi_1$  and  $\chi_2$  attributes mainly to the inclusion of mantle viscosity. The change of 0.695 to 0.750 includes not only the effect of mantle viscosity, but also the effect of scaling the incremental moment of inertia component of the Earth,  $c_{33}$ , to that of the mantle (Merriam, 1980; Wahr, 1983). The coefficient 0.997 is from minor influences, e.g., the factor dropped in deriving (8.262) from (8.261) is  $1 - 0.0014$  (without the radial expansion represented by  $c'_0$ ). The factor dropped for the  $\chi_1$  and  $\chi_2$  components is  $1 - 0.0011$ .

The governing equation (8.273) with the excitation function given by (8.275) is usually used in current practice.

The polar motion  $\tilde{p}$  and the excitation function  $\tilde{\chi}$  can be converted from one to another using (8.273). The computation of  $\tilde{\chi}$  from  $\tilde{p}$  is straightforward. In practice, both  $\tilde{p}$  and  $d\tilde{p}/dt$  are computed in the processing of observations for  $\tilde{p}$  (Gross et al., 1998). Otherwise,  $d\tilde{p}/dt$  can also be computed from  $\tilde{p}$  using a simple numerical differentiation algorithm. The solution of the differential equation (8.273) is required for computing  $\tilde{p}$  from  $\tilde{\chi}$ , which is

$$\tilde{p} = \left[ \tilde{p}(t_0)e^{-i\sigma_0 t_0} - i\sigma_0 \int_{t_0}^t \tilde{\chi} e^{-i\sigma_0 t} dt \right] e^{i\sigma_0 t}. \quad (8.276)$$

It can be verified by substituting it into the equation.

The last equation in (8.263) results in, with (8.239),

$$\chi_3 = \frac{\dot{\alpha}_3}{\omega_0} = - \left( 2\pi \frac{\dot{\alpha}_3}{\omega_0^2} \right) \left( \frac{2\pi}{\omega_0} \right)^{-1} = - \frac{\Delta\Lambda}{\Lambda_0}, \quad (8.277)$$

where  $-\Delta\Lambda$  is the change of the length of day as defined in (8.239), and  $\Lambda_0$  is the length-of-day with a nominal value of 86,400 seconds.

Finally, we provide the formulae for computing  $c'_{13}$ ,  $c'_{23}$ , and  $c'_{33}$  defined in (8.258), and  $h'_1$ ,  $h'_2$ , and  $h'_3$  defined in (8.254), which are required for computing the excitation function. In spherical coordinates, we denote the  $r$ -coordinates of the bottom and top of the atmosphere-hydrosphere-cryosphere system as  $r_b$  and  $r_t$ ,

respectively, and the change of pressure at the bottom as  $\delta p$ . We have

$$\begin{aligned} c'_{13} &= - \int_0^{2\pi} \int_0^\pi \int_{r_b}^{r_t} \delta \rho r^4 \sin^2 \theta \cos \theta \cos \lambda dr d\theta d\lambda \\ &\approx - \frac{R^4}{g_R} \int_0^{2\pi} \int_0^\pi \delta p \sin^2 \theta \cos \theta \cos \lambda d\theta d\lambda, \end{aligned} \quad (8.278a)$$

$$\begin{aligned} c'_{23} &= - \int_0^{2\pi} \int_0^\pi \int_{r_b}^{r_t} \delta \rho r^4 \sin^2 \theta \cos \theta \sin \lambda dr d\theta d\lambda \\ &\approx - \frac{R^4}{g_R} \int_0^{2\pi} \int_0^\pi \delta p \sin^2 \theta \cos \theta \sin \lambda d\theta d\lambda, \end{aligned} \quad (8.278b)$$

$$\begin{aligned} c'_{33} &= \int_0^{2\pi} \int_0^\pi \int_{r_b}^{r_t} \delta \rho r^4 \sin^3 \theta dr d\theta d\lambda \\ &\approx \frac{R^4}{g_R} \int_0^{2\pi} \int_0^\pi \delta p \sin^3 \theta d\theta d\lambda. \end{aligned} \quad (8.278c)$$

These are the mass terms that act like surface loading. For the velocity term, we neglect the vertical velocity that is much smaller than the horizontal one. With the relations

$$v'_1 = v'_\theta \cos \theta \cos \lambda - v'_\lambda \sin \lambda, \quad v'_2 = v'_\theta \cos \theta \sin \lambda + v'_\lambda \cos \lambda, \quad v'_3 = -v'_\theta \sin \theta; \quad (8.279)$$

$$\begin{aligned} \vec{r} \times \vec{v}' &= (x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3) \times (v'_1 \hat{e}_1 + v'_2 \hat{e}_2 + v'_3 \hat{e}_3) \\ &= (x_2 v'_3 - x_3 v'_2) \hat{e}_1 + (x_3 v'_1 - x_1 v'_3) \hat{e}_2 + (x_1 v'_2 - x_2 v'_1) \hat{e}_3 \\ &= r(-v'_\theta \sin \lambda - v'_\lambda \cos \theta \cos \lambda) \hat{e}_1 \\ &\quad + r(v'_\theta \cos \lambda - v'_\lambda \cos \theta \sin \lambda) \hat{e}_2 \\ &\quad + r(v'_\lambda \sin \theta) \hat{e}_3; \end{aligned} \quad (8.280)$$

we have

$$\begin{cases} h'_1 = \int_0^{2\pi} \int_0^\pi \int_{r_b}^{r_t} \rho' (-v'_\theta \sin \lambda - v'_\lambda \cos \theta \cos \lambda) r^3 \sin \theta dr d\theta d\lambda, \\ h'_2 = \int_0^{2\pi} \int_0^\pi \int_{r_b}^{r_t} \rho' (v'_\theta \cos \lambda - v'_\lambda \cos \theta \sin \lambda) r^3 \sin \theta dr d\theta d\lambda, \\ h'_3 = \int_0^{2\pi} \int_0^\pi \int_{r_b}^{r_t} \rho' v'_\lambda r^3 \sin^2 \theta dr d\theta d\lambda. \end{cases} \quad (8.281)$$



The equations in this subsection link the polar motion and length of day variation to atmospheric, oceanic, glacial, and hydrologic models. Gross et al. (2003, 2004) are ideal sources to learn the applications of the equations. A comprehensive review of the topic is given by Gross (2015).

### 8.5.3 Some Examples of Excitation Function for the Polar Motion

To understand how the pole coordinates respond to the excitation function, we study a few examples.

First, we consider an excitation function like a step function. We assume that the change of the excitation function happens only between  $t_0$  and  $t_0 + \delta t$ . For convenience of mathematical manipulation, we write the excitation function in the form of

$$\tilde{\chi}(t) = \begin{cases} 0, & t \leq t_0 \\ \frac{\tilde{a}}{2} \left\{ 1 + \sin \left[ \frac{\pi}{\delta t} (t - t_0) - \frac{\pi}{2} \right] \right\}, & t_0 \leq t \leq t_0 + \delta t; \\ \tilde{a}, & t \geq t_0 + \delta t. \end{cases} \quad (8.282)$$

where  $\tilde{a}$  is a constant complex number. This implies that the excitation function smoothly increases from 0 at  $t \leq t_0$  to  $\tilde{a}$  at  $t \geq t_0 + \delta t$ . In the limit  $\delta t \rightarrow 0$ ,  $\tilde{\chi}(t)$  becomes a step function.

We assume that  $\delta t$  is short enough, so that the approximation  $e^{-i\sigma_0 \delta t} = 1$  can be made, and thus  $e^{-i\sigma_0 t}$  can be approximated by  $e^{-i\sigma_0 t_0}$  within the time-interval  $t_0 \leq t \leq t_0 + \delta t$ . Since  $\sigma_0$  is  $2\pi / (1.19 \text{ years})$ , this assumption should hold reasonably well with  $\delta t$  ranging from a fraction of a second up to a month. By substituting (8.282) into (8.276), we obtain, for  $t \geq t_0 + \delta t$  (the integral  $\int_{t_0}^{t_0 + \delta t} \sin[(\pi/\delta t)(t - t_0) - (\pi/2)] e^{-i\sigma_0 t} dt$  vanishes with  $e^{-i\sigma_0 t}$  approximated by  $e^{-i\sigma_0 t_0}$ )

$$\begin{aligned} \tilde{p} &= \left\{ \tilde{p}(t_0) e^{-i\sigma_0 t_0} + \frac{\tilde{a}}{2} \left[ e^{-i\sigma_0(t_0 + \delta t)} - e^{-i\sigma_0 t_0} \right] - i\sigma_0 \tilde{a} \int_{t_0 + \delta t}^t e^{-i\sigma_0 t} dt \right\} e^{i\sigma_0 t} \\ &= \left\{ \tilde{p}(t_0) e^{-i\sigma_0 t_0} + \frac{\tilde{a}}{2} \left[ e^{-i\sigma_0(t_0 + \delta t)} - e^{-i\sigma_0 t_0} \right] + \tilde{a} \left[ e^{-i\sigma_0 t} - e^{-i\sigma_0(t_0 + \delta t)} \right] \right\} e^{i\sigma_0 t} \\ &= \tilde{a} + \left\{ \tilde{p}(t_0) e^{-i\sigma_0 t_0} - \frac{\tilde{a}}{2} \left[ e^{-i\sigma_0(t_0 + \delta t)} - e^{-i\sigma_0 t_0} \right] \right\} e^{i\sigma_0 t} \\ &= \tilde{a} + [\tilde{p}(t_0) - \tilde{a}] e^{i\sigma_0(t - t_0)}, \end{aligned} \quad (8.283)$$

where the approximation  $e^{-i\sigma_0(t_0 + \delta t)} = e^{-i\sigma_0 t_0}$  is made in the last step. We see that the excitation function (8.282) not only shifts the pole by  $\tilde{a}$  but also alters the Chandler wobble. What we can see particularly is that

$$\tilde{p}(t_0 + \delta t) = \tilde{a} + [\tilde{p}(t_0) - \tilde{a}] e^{i\sigma_0 \delta t} = \tilde{p}(t_0). \quad (8.284)$$

This is obtained within the accuracy of our formulation with the approximation  $e^{i\sigma_0\delta t} = 1$ . In fact,  $\tilde{p}(t_0 + \delta t) = \tilde{p}(t_0)$  holds strictly as can be seen by setting  $\delta t \rightarrow 0$ . This means that the excitation, even if it is from a sudden shock like an Earthquake (e.g., Gross, 1986; Gross & Chao, 2006), does not induce a jump of the pole position; it only brings a change to the direction of polar motion.

Second, we consider an excitation like a trend during a long time interval,

$$\tilde{\chi}(t) = \begin{cases} 0, & t \leq t_0 \\ \frac{\tilde{a}}{t_1 - t_0} (t - t_0), & t_0 \leq t \leq t_1; \\ \tilde{a}, & t \geq t_1. \end{cases} \quad (8.285)$$

where  $\tilde{a}$  is a constant complex number. Its substitution into (8.276) yields, for  $t_0 \leq t \leq t_1$ ,

$$\begin{aligned} \tilde{p} &= \left[ \tilde{p}(t_0) e^{-i\sigma_0 t_0} - i \frac{\sigma_0 \tilde{a}}{t_1 - t_0} \int_{t_0}^t (t - t_0) e^{-i\sigma_0 t} dt \right] e^{i\sigma_0 t} \\ &= \left\{ \tilde{p}(t_0) e^{-i\sigma_0 t_0} - i \frac{\sigma_0 \tilde{a}}{t_1 - t_0} \left[ \frac{1}{\sigma_0^2} e^{-i\sigma_0 t} (i\sigma_0 t + 1) - i \frac{t_0}{\sigma_0} e^{-i\sigma_0 t} \right]_{t_0}^t \right\} e^{i\sigma_0 t} \\ &= \left\{ \tilde{p}(t_0) e^{-i\sigma_0 t_0} - i \frac{\sigma_0 \tilde{a}}{t_1 - t_0} \left[ \frac{i\sigma_0(t - t_0) + 1}{\sigma_0^2} e^{-i\sigma_0 t} - \frac{1}{\sigma_0^2} e^{-i\sigma_0 t_0} \right] \right\} e^{i\sigma_0 t} \\ &= \frac{\sigma_0(t - t_0) - i}{\sigma_0(t_1 - t_0)} \tilde{a} + \left\{ \tilde{p}(t_0) + i \frac{\tilde{a}}{\sigma_0(t_1 - t_0)} \right\} e^{i\sigma_0(t - t_0)}. \end{aligned} \quad (8.286)$$

We thus have

$$\tilde{p}(t_1) = \left[ 1 - i \frac{1}{\sigma_0(t_1 - t_0)} \right] \tilde{a} + \left[ \tilde{p}(t_0) + i \frac{\tilde{a}}{\sigma_0(t_1 - t_0)} \right] e^{i\sigma_0(t_1 - t_0)}. \quad (8.287)$$

For  $t \geq t_1$ , we obtain

$$\begin{aligned} \tilde{p} &= \left\{ \tilde{p}(t_1) e^{-i\sigma_0 t_1} - i\sigma_0 \tilde{a} \int_{t_1}^t e^{-i\sigma_0 t} dt \right\} e^{i\sigma_0 t} \\ &= \left\{ \tilde{p}(t_1) e^{-i\sigma_0 t_1} + \tilde{a} (e^{-i\sigma_0 t} - e^{-i\sigma_0 t_1}) \right\} e^{i\sigma_0 t} \\ &= \tilde{a} + \left\{ -i \frac{\tilde{a}}{\sigma_0(t_1 - t_0)} e^{-i\sigma_0 t_1} + \left[ \tilde{p}(t_0) + i \frac{\tilde{a}}{\sigma_0(t_1 - t_0)} \right] e^{-i\sigma_0 t_0} \right\} e^{i\sigma_0 t} \\ &= \tilde{a} + \left[ \tilde{p}(t_0) - i \frac{e^{-i\sigma_0(t_1 - t_0)} - 1}{\sigma_0(t_1 - t_0)} \tilde{a} \right] e^{i\sigma_0(t - t_0)} \end{aligned} \quad (8.288)$$

We can particularly obtain, using L'Hôpital's rule,

$$\lim_{t_1 \rightarrow t_0} \tilde{p} = \tilde{a} + [\tilde{p}(t_0) - \tilde{a}]e^{i\sigma_0(t-t_0)}, \quad (8.289)$$

which is the same as (8.283), and is not to be discussed further.

The result can be applied to two kinds of processes. One is to a process started at  $t_0$  and ended at  $t_1$ . In this case, the excitation function  $\tilde{a}$  should be computed using the difference between the ending epoch  $t_1$  and the starting epoch  $t_0$ . Without excitation, the pole coordinate would follow the path  $\tilde{p}(t_0)e^{i\sigma_0(t-t_0)}$ . The change of the pole coordinate caused the excitation is

$$\tilde{p} - \tilde{p}(t_0)e^{i\sigma_0(t-t_0)} = \tilde{a} - i \frac{e^{-i\sigma_0(t_1-t_0)} - 1}{\sigma_0(t_1 - t_0)} \tilde{a}e^{i\sigma_0(t-t_0)}, \quad (8.290)$$

where the first term at the right-hand side is a shift of the average pole position, and the second an alternation to the periodical Chandler wobble.

The other is a long time ongoing process with constant rate,

$$\dot{\tilde{a}} = \frac{\tilde{a}}{t_1 - t_0}. \quad (8.291)$$

The change of the pole coordinate caused by the excitation is, according to (8.286),

$$\tilde{p} - \tilde{p}(t_0)e^{i\sigma_0(t-t_0)} = \left[ (t - t_0) - \frac{i}{\sigma_0} \right] \dot{\tilde{a}} + i \frac{\dot{\tilde{a}}}{\sigma_0} e^{i\sigma_0(t-t_0)}. \quad (8.292)$$

The rate of change of the pole coordinate caused by the excitation is

$$\frac{d}{dt} \left[ \tilde{p} - \tilde{p}(t_0)e^{i\sigma_0(t-t_0)} \right] = \dot{\tilde{a}} - \dot{\tilde{a}}e^{i\sigma_0(t-t_0)}, \quad (8.293)$$

where the first term at the right-hand side is the rate of shift of the average pole position, and the second term the alternation to the rate of the Chandler wobble.

The rotation axis can be computed from the CIP according to (8.270) or obtained by solving its governing equation

$$\tilde{m} + \frac{i}{\sigma_0} \frac{d\tilde{m}}{dt} = \tilde{\chi} - \frac{i}{\omega_0} \frac{d\tilde{\chi}}{dt}. \quad (8.294)$$

This equation can be obtained by substituting (8.270) into (8.273), or by replacing  $\sigma_C$  by the observed complex Chandler angular frequency,  $\sigma_0$ , in (8.271). For the CIP, the solution of (8.273) is (8.276). Analogously, the solution of (8.294) is

$$\tilde{m} = \left[ \tilde{m}(t_0)e^{-i\sigma_0 t_0} - i\sigma_0 \int_{t_0}^t \left( \tilde{\chi} - \frac{i}{\omega_0} \frac{d\tilde{\chi}}{dt} \right) e^{-i\sigma_0 t} dt \right] e^{i\sigma_0 t}. \quad (8.295)$$

Except sudden processes like Earthquakes, where the excitation function behaves like a jump from  $t_0$  to  $t_1$  with a very small  $\delta t = t_1 - t_0$ , we usually have  $(1/\omega_0)d\tilde{p}/dt \ll \tilde{p}$  and  $(1/\omega_0)d\tilde{\chi}/dt \ll \tilde{\chi}$ . Therefore, for slow processes, we can approximately ignore the difference between  $\tilde{p}$  and  $\tilde{m}$ .

We mention that for the excitation function given in (8.282), both  $\tilde{\chi}$  and  $d\tilde{\chi}/dt$  are continuous. For the one given in (8.285),  $\tilde{\chi}$  is continuous, but  $d\tilde{\chi}/dt$  has jumps at  $t_0$  and  $t_1$ . However, a jump in either  $\tilde{\chi}$  or  $d\tilde{\chi}/dt$  does not bring a jump to  $\tilde{p}$  or  $\tilde{m}$ .

### 8.5.4 Relation with the Change of Gravitational Potential

In geodesy, the gravitational potential of the Earth expressed in terms of a spherical harmonic series has been routinely monitored for decades by measuring satellite orbit. The degree 2 spherical harmonic coefficients are related to the moment of inertia tensor as expressed in (5.11), which are already used in formulating the pole tide and linearizing the Euler-Liouville equation. We note that  $A, B, C, D, E$ , and  $F$  in (5.11) correspond to  $C_{11}, C_{22}, C_{33}, -C_{12}, -C_{23}$ , and  $-C_{13}$ , respectively. We also note that the moment of inertia tensor is symmetrical, i.e.,  $C_{ij} = C_{ji}$ . In geodesy, the fully normalized Legendre function is adopted in the spherical harmonic series. Here we use the un-normalized version in the formulation for convenience.

We continue to assume that the average atmosphere-hydrosphere-cryosphere system rotates with the solid Earth as a whole in the reference state. We also assume that the model of the gravitational potential of the Earth is preprocessed such that the variation of the spherical harmonic coefficients with respect to the average state is obtained. This measured change includes both the contributions from the motion in the atmosphere-hydrosphere-cryosphere system and the deformation of the solid Earth, as satellite orbits are affected by both of them. Specifically, we have (with  $a$  replaced by  $R$ ),

$$\left\{ \begin{array}{l} \delta A_2 = -\frac{(c_{33} + c'_{33}) - (c_{11} + c'_{11} + c_{22} + c'_{22})/2}{MR^2}, \\ \delta A_2^1 = -\frac{c_{13} + c'_{13}}{MR^2}, \\ \delta A_2^2 = \frac{c_{22} + c'_{22} - (c_{11} + c'_{11})}{4MR^2}, \end{array} \right. \quad \delta A_2^{-1} = -\frac{c_{23} + c'_{23}}{MR^2}, \quad \delta A_2^{-2} = -\frac{c_{12} + c'_{12}}{2MR^2}. \quad (8.296)$$

These equations can be solved to obtain

$$\left\{ \begin{array}{l} c_{33}^+ = c_{33} + c'_{33} = \frac{1}{3}[T^+ - 2MR^2\delta A_2], \\ c_{13}^+ = c_{13} + c'_{13} = -MR^2\delta A_2^1, \end{array} \right. \quad c_{13}^+ = c_{23} + c'_{23} = -MR^2\delta A_2^{-1}, \quad (8.297)$$

with

$$T^+ = c_{11} + c'_{11} + c_{22} + c'_{22} + c_{33} + c'_{33}. \quad (8.298)$$

The relation between the un-normalized and normalized coefficients are  $\delta A_2 = \sqrt{5}\delta\bar{A}_2$ ,  $\delta A_2^1 = \sqrt{5/3}\delta\bar{A}_2^1$ , and  $\delta A_2^{-1} = \sqrt{5/3}\delta\bar{A}_2^{-1}$ . Denote the value of  $r$  at the top of the atmosphere as  $r^a$  and the change of density anywhere in the entire Earth system as  $\delta\rho$ . By definition, we have

$$\begin{aligned} T^+ &= 2 \int_0^{r^a} r^2 \left[ \int_0^{2\pi} \int_0^\pi \delta\rho(r, \theta, \lambda) r^2 \sin\theta d\theta d\lambda \right] dr \\ &= 2 \int_0^{r^a} r^2 [4\pi r^2 \delta\bar{\rho}(r)] dr, \end{aligned} \quad (8.299)$$

where  $\delta\bar{\rho}(r) = [1/(4\pi r^2)] \int_0^{2\pi} \int_0^\pi \delta\rho(r, \theta, \lambda) r^2 \sin\theta d\theta d\lambda$  is the average density on the sphere of radius  $r$ . This implies that the change of the trace of the total moment of inertia tensor,  $T^+ = \text{tr}(\bar{c} + \bar{c}')$ , reflexes only the change of mass in the radial direction in a spherically symmetrical manner. The change of mass in a spherically symmetrical manner does not alter the gravity outside the Earth system as far as the total mass is conserved, which is the case of the Earth system. Therefore,  $T$  cannot be inferred from gravity measured by satellites. We will keep it as is in subsequent formulations.

Both the loading and pole tide are included in  $c_{ij}$ , as expressed in (8.260) for the elastic Earth model. For the CIP, as  $\delta\omega_1^0 = \delta\omega_2^0 = 0$ , we write (8.260) as

$$\begin{cases} c_{13} = k'_2 c'_{13} + \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A)(\dot{\alpha}_1 - \alpha_2 \omega_0), \\ c_{23} = k'_2 c'_{23} + \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A)(\dot{\alpha}_2 + \alpha_1 \omega_0), \\ c_{33} = k'_2 c'_{33} + \frac{4}{3} \frac{\tilde{k}_2}{k_s} \frac{1}{\omega_0} (C - A)\dot{\alpha}_3. \end{cases} \quad (8.300)$$

This is obtained assuming that the effect of Earth rotation variation on Earth deformation is not removed from the gravitational potential. We thus can obtain  $c'_{13}$ ,  $c'_{23}$ , and  $c'_{33}$  as those in the excitation functions (8.262):

$$\begin{cases} c'_{13} = \frac{1}{1 + k'_2} \left[ c_{13}^+ - \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A)(\dot{\alpha}_1 - \alpha_2 \omega_0) \right], \\ c'_{23} = \frac{1}{1 + k'_2} \left[ c_{23}^+ - \frac{k_2}{k_s} \frac{1}{\omega_0} (C - A)(\dot{\alpha}_2 + \alpha_1 \omega_0) \right], \\ c'_{33} = \frac{1}{1 + k'_2} \left[ c_{33}^+ - \frac{4}{3} \frac{\tilde{k}_2}{k_s} \frac{1}{\omega_0} (C - A)\dot{\alpha}_3 \right]. \end{cases} \quad (8.301)$$

We first formulate the excitation function components for the polar motion,  $\chi_1$  and  $\chi_2$ . Assuming no motion is present, i.e.,  $h'_1 = h'_2 = 0$ , the substitution of these into (8.262) yields the  $x_1$  and  $x_2$  mass components of the excitation function,

$$\begin{cases} \chi_1 = \frac{\omega_0}{\sigma_C} \frac{1}{A} \left[ c_{13}^+ - \frac{k_2}{k_s} (C - A) \left( \frac{1}{\omega_0} \dot{\alpha}_1 - \alpha_2 \right) \right], \\ \chi_2 = \frac{\omega_0}{\sigma_C} \frac{1}{A} \left[ c_{23}^+ - \frac{k_2}{k_s} (C - A) \left( \frac{1}{\omega_0} \dot{\alpha}_2 + \alpha_1 \right) \right]. \end{cases} \quad (8.302)$$

With the  $\alpha$  angles for the rotation axis provided in (8.267), we obtain, with (8.191),

$$\begin{cases} \chi_1 = \frac{\omega_0}{\sigma_C} \frac{1}{A} \left( c_{13}^+ + \frac{\omega_0^2 R^5}{3G} k_2 \alpha_2^r \right), \\ \chi_2 = \frac{\omega_0}{\sigma_C} \frac{1}{A} \left( c_{23}^+ - \frac{\omega_0^2 R^5}{3G} k_2 \alpha_1^r \right). \end{cases} \quad (8.303)$$

The excitation in complex form for the polar motion is thus

$$\tilde{\chi} = \chi_1 + i\chi_2 = \frac{\omega_0}{\sigma_C} \frac{1}{A} \left[ (c_{13}^+ + ic_{23}^+) - \frac{\omega_0^2 R^5}{3G} k_2 \tilde{m} \right], \quad (8.304)$$

where  $\tilde{m}$  is the direction of the rotation axis in complex form defined in (8.269). Note that we could have adopted the rotation axis as early as in (8.300).

Following Gross (2015), we adopt the polar motion equation (8.273), and write the excitation functions in the form of

$$\begin{aligned} \tilde{\chi} &= \frac{\omega_0}{\sigma_C} \frac{C - A}{A_m} \frac{1}{C - A} \left[ (c_{13}^+ + ic_{23}^+) - \frac{\omega_0^2 R^5}{3G} k_2 \tilde{m} \right] \\ &= \frac{1.608}{C - A} \left[ (c_{13}^+ + ic_{23}^+) - \frac{\omega_0^2 R^5}{3G} k_2 \tilde{m} \right], \end{aligned} \quad (8.305)$$

where  $A$  is replaced by  $A_m$ . The second term in the brackets is for removing the pole tide, which is directly related to the rotation axis. This term should be dropped if the pole tide is removed from the model of the gravitational potential. As expressed in (8.251), the excitation function represents the torque at the right-hand side, and the pole tide effect is included at the left-hand side. Therefore, the pole tide should not be part of the excitation function.

We now consider the excitation function component  $\chi_3$ . We have, from (8.301) and (8.191),

$$c'_{33} = \frac{1}{1 + k'_2} \left( c_{33}^+ - \frac{4}{3} \frac{\omega_0^2 R^5}{3G} \tilde{k}_2 \frac{\dot{\alpha}_3}{\omega_0} \right). \quad (8.306)$$

With (8.275) cited from Gross (2015) and  $h_3 = 0$ , we obtain the  $x_3$  mass component of the excitation function

$$\chi_3 = \frac{0.997}{C_m} (0.750 c'_{33})$$

$$\begin{aligned}
&= \frac{0.997}{C_m} \frac{0.750}{1+k'_2} \left( c_{33}^+ - \frac{4}{3} \frac{\omega_0^2 R^5}{3G} \tilde{k}_2 \frac{\dot{\alpha}_3}{\omega_0} \right) \\
&= \frac{0.997}{C_m} \frac{0.750}{0.684} \left( c_{33}^+ - \frac{4}{3} \frac{\omega_0^2 R^5}{3G} \tilde{k}_2 \frac{\dot{\alpha}_3}{\omega_0} \right), \tag{8.307}
\end{aligned}$$

where  $1 + k'_2 = 0.695$  for the elastic Earth model is replaced by  $1 + k'_2 = 0.684$  for the anelastic Earth model. The second term at the right-hand side is the response to the variation of Earth rotation rate. It should be dropped or kept depending on whether it is removed in model of the gravitational potential, like the pole tide term in (8.305).

### 8.5.5 Rotational Feedback in the Sea Level Change as a Result of Present-Day Continent-Ocean Water Exchange

In Sect. 7.9.3, we presented an algorithm to compute the sea level change as a result of present-day continent-ocean water exchange without considering the accompanying Earth rotation variation and its feedback. In this section, we provide the detail of the formulation of the feedback of the Earth rotation variation to be added in the step 6 of the computation procedure. The algorithm presented in Sect. 7.9.3 applies to both the following cases

- CASE 1: The total change from a starting epoch to an ending epoch assuming the oceans to be in hydrostatic equilibrium before the starting epoch and after the ending epoch. The formulation in Sect. 7.9.3 is based on this case.
- CASE 2: The change with a slow constant rate that the ocean can be constantly approximated as in hydrostatic equilibrium. The formulation is the same as the CASE 1 with the total change replaced by the rate of change.

For both cases, the excitation function is assumed to be in the form of (8.285), i.e., the process is assumed to be a change with constant rate. The formulae of Earth rotation variation and its feedback to the sea level change are also the same, as to be detailed below.

For the present problem, it is physically more apparent to use the rotation axis, which is to be referred as rotation pole. We assume that the processes concerned are very slow, so that the difference between the CIP and the rotation pole can be neglected, as discussed in Sect. 8.5.3.

We used  $\alpha_1$  and  $\alpha_2$ ,  $x_p$  and  $y_p$ , and  $\tilde{m} = m_1 + im_2$  to represent the position of the pole (here for the rotation pole). The  $x_1$ - and  $x_2$ -coordinates are

$$m_1 = x_p = -\alpha_2 \quad \text{and} \quad m_2 = -y_p = \alpha_1, \tag{8.308}$$

respectively. In the following, we use  $m_1$  and  $m_2$ .

We need to compute the excitation function according to (8.274). By neglecting  $\vec{h}'$  under the hydrostatic equilibrium assumption, we have

$$\chi_1 = \frac{1.608 \times 0.695}{(C - A)} c'_{13}, \quad \chi_2 = \frac{1.608 \times 0.695}{(C - A)} c'_{23}, \quad \chi_3 = \frac{0.695}{C_m} c'_{33}, \quad (8.309)$$

where  $c'_{13}$ ,  $c'_{23}$ , and  $c'_{33}$  are components of the change of the moment of inertia tensor due to the water redistribution. According to (5.11) and (7.451), the changes of the relevant degree 2 spherical harmonic coefficients of the gravitational potential due to the water redistribution (without the loading effects) are

$$\begin{cases} \delta A'_2 = -\frac{c'_{33} - (c'_{11} + c'_{22})/2}{MR^2} = \frac{3\rho_w}{5R\bar{\rho}} \delta H_2, \\ \delta A'_2 = -\frac{c'_{13}}{MR^2} = \frac{3\rho_w}{5R\bar{\rho}} \delta H_2^1, \end{cases} \quad \delta A_2'^{-1} = -\frac{c'_{23}}{MR^2} = \frac{3\rho_w}{5R\bar{\rho}} \delta H_2^{-1}. \quad (8.310)$$

Note that the coefficients  $\delta H_2$ ,  $\delta H_2^1$ , and  $H_2^{-1}$  are defined based on the unnormalized spherical harmonics. If the full normalization (4.340) is used in the spherical harmonic series of  $\delta H$ , these coefficients should be computed using the normalized ones according to  $\delta H_2 = \sqrt{5} \delta \bar{H}_2$ ,  $\delta H_2^1 = \sqrt{5/3} \delta \bar{H}_2^1$ , and  $H_2^{-1} = \sqrt{5/3} \delta \bar{H}_2^{-1}$ . We thus obtain

$$\begin{cases} c'_{33} = \frac{1}{3} \left( T' - \frac{6\rho_w}{5\bar{\rho}} MR \delta H_2 \right), \\ c'_{13} = -\frac{3\rho_w}{5\bar{\rho}} MR \delta H_2^1, \end{cases} \quad c'_{23} = -\frac{3\rho_w}{5\bar{\rho}} MR \delta H_2^{-1}, \quad (8.311)$$

with

$$T' = c'_{11} + c'_{22} + c'_{33}. \quad (8.312)$$

The trace of the moment of inertia tensor,  $T' = \text{tr}(\vec{c}')$ , cannot be determined by the gravitational potential. We keep it as a known parameter in the derivation. Some remarks about it will be made afterwards. The substitution of (8.311) into (8.309) leads to the formulae of the excitation function, which is straightforward.

We still formulate the CASE 1, where  $c'_{13}$ ,  $c'_{23}$ , and  $c'_{33}$  represent the total changes. The excitation function can be computed by substituting (8.311) into (8.309). What we obtain is the excitation function in (8.285),  $\tilde{a} = \tilde{\chi}$ . Remember that the difference between the CIP and the rotation pole can be neglected for the slow processes. We are concerned with the shift of the average rotation pole, which is, according to (8.290),  $\tilde{m} = \tilde{\chi}$ . For more apparent further reference, we write it more explicitly as

$$m_1 = \chi_2, \quad m_2 = \chi_2. \quad (8.313)$$



The second term at the right-hand side of (8.290) is periodical. Its integral within each Chandler wobble period vanishes if the very small decay due to anelastic damping is neglected. It is dropped because our interest here is the long term change. The relation between the rotation rate and the excitation function is given by (8.277). Here we write it as

$$m_3 = \frac{\dot{\alpha}_3}{\omega_0} = \chi_3. \quad (8.314)$$

The change of the centrifugal potential is given by (8.125). Here, we write it as, with  $m_1 = \delta\omega_1/\omega_0$ ,  $m_2 = \delta\omega_2/\omega_0$ , and  $m_3 = \delta\omega_3/\omega_0 = \dot{\alpha}_3/\omega_0$  by definition,

$$\delta\psi(R) = \delta\psi^r(R) + \delta\psi^t(R), \quad (8.315)$$

with

$$\delta\psi^r(R) = \frac{2}{3}R^2\omega_0^2m_3, \quad (8.316)$$

and

$$\begin{aligned} \delta\psi^t(R) &= -\frac{R^2\omega_0^2}{3} \left[ 2m_3P_2(\cos\theta) + (m_1\cos\lambda + m_2\sin\lambda)P_2^1(\cos\theta) \right] \\ &= -\frac{2R^2\omega_0^2}{3}m_3\mathcal{H}_2^0(\theta, \lambda) - \frac{R^2\omega_0^2}{3}m_1\mathcal{H}_2^1(\theta, \lambda) - \frac{R^2\omega_0^2}{3}m_2\mathcal{H}_2^{-1}(\theta, \lambda). \end{aligned} \quad (8.317)$$

The spherically symmetrical term  $\delta\psi^r(R)$  is the same everywhere on the Earth's surface approximated as a sphere of radius  $R$ . It does not alter the water distribution, and is dropped. Therefore, only the term  $\delta\psi^t(R)$  is to be considered hereafter. The radial displacement is, according to (8.127),

$$u_r(R) = \frac{h_2}{g_R}\delta\psi^t(R). \quad (8.318)$$

The potential due to the Earth's deformation is, according to and (8.128),

$$\delta\psi_1(R) = k_2\delta\psi^t(R). \quad (8.319)$$

The change of the geoidal height is

$$\delta N = \frac{\delta\psi^t(R) + \delta\psi_1(R)}{g_R} = \frac{1+k_2}{g_R}\delta\psi^t(R). \quad (8.320)$$

The expressions of the spherical harmonic coefficients of  $u_r(R)$  and  $\delta N$  can be readily obtained by substituting (8.317) into (8.318) and (8.320), respectively, which is left to the readers. In the step 6 of the procedure of computation presented in Sect. 7.9.3, it is required to add the spherical harmonic coefficients of  $u_r(R)$  and  $\delta N$  to the corresponding coefficients in (7.450) and (7.450), respectively. Note that if a normalized Legendre function is used in (7.450) and (7.450), (8.317) must be converted to the same normalization. For example, if the full normalization (4.340) is used, the three terms in (8.317) must be multiplied by  $\sqrt{1/5}$ ,  $\sqrt{3/5}$  and  $\sqrt{3/5}$ , respectively.

For CASE 2, the same formulae applies by replacing the total amount of change to the rate of change. This is a consequence of the similarity between the first term at the right-hand side of (8.290) and that of (8.293), and the similarity between (8.277) and the last identity in (8.263).

Finally, we discuss the trace of the moment of inertia tensor,  $T' = \text{tr}(\bar{c}')$ . An approximation is made in the formulation in Sect. 7.9.3: both the continents and the oceans are on the sphere of radius  $R$ . If this approximation is kept, we therefore have

$$T' = 2 \int_0^{2\pi} \int_0^\pi R^2(\rho_w \delta H) R^2 \sin \theta d\theta d\lambda = 0, \quad (8.321)$$

since the total water storage is assumed to be the same. The reality is more complicated. On the surface of the reference Earth ellipsoid, the equator is 21 km higher than the poles with respect to the Earth center. Mountain glaciers could be several km above the ellipsoid. Approximating the ellipsoid by a sphere brings in a relative error on the order of  $\sim 1/300$ . The error brought in by ignoring the height of mountain glaciers above the ellipsoid is smaller if the process is not extreme. In the extreme case such as all the water discharged into the oceans is from the top of the Mount Everest (highest above the geoid), or from the Mount Chimborazo (highest with respect to the Earth center), which are both close to the equator, the effect on the rotation rate,  $\dot{\alpha}_3$ , may need to be particularly treated. This is unlikely, and therefore, we can set  $T' = 0$  in our spherical approximation.

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## Bibliographic Notes

Several books are dedicated to Earth rotation. Dehant and Mathews (2015) and Moritz and Mueller (1987) focus more on lunisolar attraction and nutation, and Lambeck (1980) and Munk and Macdonald (1960) on geophysical processes and polar motion. The author's main reference include Melchior (1983), Moritz and Mueller (1987) and Rochester and Smylie (1974) for Sects. 8.1 through 8.4; and Barnes et al. (1983), Gross (2015), and Wahr (1982, 1983) for Sect. 8.5. Sasao et al. (1980) provided an easy to approach formulation for the effect of the Earth's core on Earth rotation, which is the next to be considered beyond elasticity. It is followed by Mathews et al. (1991, 2002), leading to the standard adopted of the International

Astronomical Union (IAU). Mathews and Bretagnon (2003) provided a formulation for the effect of the lunisolar gravitational torque other than the far more important retrograde diurnal one considered in this book. These are suggested as the readers' sources for alternative presentations and further learning.

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## Abstract

This chapter describes briefly the most relevant geodetic technologies and data, geophysical models, and their usages as applications of the theories presented in this book. The background principles of physics are emphasized. These are foci of current researches, and are advancing fast. For the state-of-the-art knowledge, the best learning sources are the recent papers, particularly review papers if available.

## 9.1 A Summary of the Global Geodetic Observing System

The geodetic technologies as well as the data they provide relevant to this book are coordinated by a component of the International Association of Geodesy (IAG)—the Global Geodetic Observing System (GGOS). Central to the GGOS is a terrestrial reference system (TRS) materialized as a terrestrial reference frame (TRF), which is a network of stations on the Earth. The most important task of the GGOS is certainly to establish the most accurate/precise TRF using all kinds of data available; the accuracy/precision of the TRF controls the accuracy/precision of most other data provided. Below we first provide a brief description of some of the most important technologies, and then a summary of the international terrestrial reference system (ITRS) and frame (ITRF) provided in the framework of the International Earth Rotation and Reference Systems Service (IERS). The data products will be briefly summarized in the next section. Comprehensive information can be found at or through the GGOS website.<sup>1</sup>

<sup>1</sup> <https://ggos.org>

### 9.1.1 Very Long Baseline Interferometry

The very long baseline interferometry (VLBI) is composed of a network of stations (actually antennas) receiving radio signals from a set of quasars. The quasars are very distant celestial objects with extremely stable relative directions, or positions on the celestial sphere. The positions of these quasars are very accurately determined and are used as the celestial reference frame (CRF), which is a materialization of the celestial reference system (CRS). The geocentric celestial reference system (GCRS) materialized as the CRF is considered as the inertial reference system in the studies of the dynamics of the Earth system, including Earth rotation. Since the quasars are extremely far from the Earth, the radio signals arriving at the Earth can be considered as exact plane waves. The differences among the arrival times of a wave crest or trough at the stations are measured. The product of the differences of the arrival time with the speed of light yields the distances between the projection points of the stations in the direction of the quasar (draw a line parallel to the direction of a quasar, draw another line from a station perpendicular to this line, the intersection between the two lines is the intersection). Earth rotation and relative variations of positions among the stations cause these distances to vary. In the opposite way, Earth rotation and relative variations of positions among the stations can be determined by monitoring these distances through measuring the arrival times. Currently, the VLBI is the most accurate technology for monitoring Earth rotation. Certainly, the VLBI stations are considered as a part of the TRF, or attached to the TRF (meaning that the positions of the VLBI stations with respect to the TRF are accurately/precisely known), and the data are used to derive/maintain/provide the IERS ITRF and the Earth orientation parameters (EOPs).

### 9.1.2 Satellite Laser Ranging

The satellite laser ranging (SLR) is the most accurate technology for monitoring the distance between a station (composed of antennas sending and receiving radio signals) on the Earth and an artificial satellite. A laser pulse is sent from the station to the satellite, which is then reflected back to the station by a reflector equipped on the satellite. The round-trip time of the pulse is measured and used to determine the instantaneous distance between the station and the satellite; the product between this time and the speed of light is twice the station-satellite distance. When one or multiple satellites are monitored by a network of SLR stations on the Earth, the data can be used to determine the position of the center of mass of the Earth system (CM) with respect to the network. The physical principle is that the CM can only be “felt” by satellite orbits. Satellite orbits are most affected by the gravitational attraction of the Earth system. Therefore, data of satellite orbits can be used to determine the gravity field of the Earth. Currently the few lowest degree spherical harmonic coefficients of the Earth’s gravitational potential are most accurately determined by SLR data, where the degree 1 coefficients represent the position of the CM in the

coordinate system represented by the coordinates of the SLR stations. As satellite orbits are above the entire Earth system, the CM “seen” by satellite orbits is that of the Earth system, which moves with respect to the SLR stations due to the mass transport in the Earth system, e.g., global atmospheric and oceanic circulations, accumulation and depletion of ground water, glaciers, ice sheets, and so on. The SLR stations are also considered as a part of or attached to the TRF, and the data are used to derive/maintain/provide the IERS ITRF and EOPs.

### 9.1.3 Global Navigation Satellite System

The Global Navigation Satellite System (GNSS) is made widely known by the U.S. Global Positioning System (GPS). At present, three other GNSS are in operation: the Russian Global Navigation Satellite System (GLONASS), the Chinese BeiDou Navigation Satellite System (BDS), and the European Galileo satellite navigation system (Galileo). GNSS is now available to everyone for real-time travel navigation. The root control of a GNSS is a network of stations on the Earth with known coordinates representing a reference system, such as the WGS84 (composed of a network and a reference Earth ellipsoid) for the GPS. The positions of the GNSS satellites are determined in this reference system through observations linking to the control network. The coordinates of the users are then determined through observations linking to the satellites. A continuously operating GNSS receiver fixed on the Earth, which is referred to as permanent GNSS station, monitors the variation of the coordinates of the station in this reference system. The variations of the coordinates can be interpreted as station displacement if the reference system is stable and accurate/precise enough, so that the errors of the coordinates are of smaller magnitude than the station displacement. The GNSS is much more affordable than the VLBI and SLR. Certainly, the majority of a TRF are permanent GNSS stations. The ultimate data product is time series of station coordinates which are widely used for studying the changes in the Earth system related to Earth deformation, including the use in deriving/maintaining/providing the IERS ITRF and EOPs.

### 9.1.4 Satellite Altimetry

As implied by the name, satellite altimetry measures the height of a surface from satellites. A radar or laser pulse is sent from the satellite to the surface which reflects the pulse back to the satellite. Like the SLR, the round-trip time of the pulse is measured and then used to determine the satellite-surface distance, i.e., the height of the satellite above the surface. The satellite position is determined by an onboard GNSS receiver or by receiving radio signals sent from a ground network of stations with very accurately/precisely known coordinates, such as that of the Doppler Orbitography and Radio-positioning Integrated by Satellite instrument (DORIS). The position of the surface is determined using the coordinates of the satellite and

the satellite-surface distance. Usually, the height of the surface above a reference Earth ellipsoid is provided as the product. Repeated measurements provide temporal height variation of the surface. The footprint, i.e., the spot on the surface that reflects the pulse, of the radar is much larger than that of the laser. A radar or laser altimeter is selected based on the purpose of a mission. For oceanic dedicated missions, radar altimeters with footprint diameters on the order of several to tens of kilometers are used, so that the reflected signal represents some kind of average within the footprint to avoid the effects of the shorter scale waves. For missions dedicated to measuring rough or small scale surfaces, e.g., mountain glaciers or inland lakes, laser altimeters with very small footprints are used for focusing on the target. For missions that requires a compromise, like measuring the ice sheet surface, radar altimeters with footprint diameters on the order of a few hundred meters are used, so that the extremely small scale relieves are averaged out while maintaining the result to have a reasonably high spatial resolution. There have been many terminated missions and several missions are in operation, each designed for its specific purpose or that of its predecessor. The data products are the surface height of the footprint at the time of measurement throughout the time-span of the mission. The data products of multiple missions are also processed together to provide combined data products, such as the time series of surface height over a grid.

### 9.1.5 Satellite Gravimetry

Here we take the GCRS as the reference system, which is considered to be a geocentric inertial reference system. The free fall acceleration of a satellite is due to the actions of the gravitational attraction of the Earth and the tidal-generating force of extraterrestrial celestial bodies, which are both conservative forces. It can be used to model the Earth's gravitational field if it is measured and the effect of the tidal-generating force is removed.

In reality, the motion of a satellite cannot be free fall; it is affected by the light pressure from the Sun, the atmospheric drag (even though the atmosphere is extremely sparse at satellite altitude), etc., which are non-conservative forces. The contribution of the non-conservative forces to the acceleration can be measured by a set of accelerometers (providing the reading in three perpendicular directions) equipped on the satellite. Suppose that the total acceleration of the satellite is determined from the measurement of an onboard GNSS receiver. The free fall acceleration can then be obtained by subtracting that determined using the accelerometers from that determined using the GNSS receiver. Therefore, the simplest design is a single satellite equipped with a GNSS receiver and a set of accelerometers, such as the Challenging Mini-satellite Payload (CHAMP) mission, which was followed by more sophisticated missions.

A more accurate design is a pair of satellites flying one after another, such as the Gravity Recovery and Climate Experiment (GRACE) mission as well as the



GRACE Follow-On mission.<sup>2</sup> A GNSS receiver and a set of accelerometers are equipped on each of them, and a microwave or laser ranging instrument is equipped on them to very accurately/precisely monitor the rate of change of the distance between them, called range rate. Conceptually, the rate of change of the range rate is the projection of the difference between the accelerations of the two satellites along their line-of-sight, which can be determined with far higher accuracy/precision than the accelerations of the two satellites determined using the GNSS receivers. Therefore, the gravitational field of the Earth can be much better modeled than the single satellite design. The main final data product of the GRACE mission is the monthly Earth gravitational models in the form of spherical harmonic series. These models are considered accurate up to degree and order 60, although products with higher degree and order are also provided, and their changes over time are suitable to study the mass redistribution in the Earth system. Another type of data product is the monthly average of the mass in the form of masscon, i.e., the cell average over a grid on the Earth's surface.

A third design is the satellite gradiometry, such as the Gravity Field and Steady-State Ocean Circulation Explorer (GOCE) mission. Accelerometers equipped on a satellite cannot sense the free fall acceleration of the satellite. However, two accelerometers aligned in parallel can sense the difference of the gravitational attraction between the two accelerometers. The spatial rate of change obtained by dividing the difference by the distance between the accelerometers is the gradient of the gravitational attraction that the accelerometers measure in the direction from one accelerometer to the other. When expressed in a coordinate system, each of the three components of the gravitational attraction has three components of gradient. Therefore, there are a total of nine components of gradient, such as the gradient of the  $x_i$ -component of the gravitational attraction along the  $x_j$ -axis. The non-conservative forces act on the outer shell of the satellite. They result in an acceleration of the satellite as a whole, and hence, do not contribute to the difference of the readings of the accelerometers. The conservative force, whose gradient is measured, includes the gravitational attraction of the Earth and the tidal-generating force. These forces have a potential, and the potential obeys Laplace's equation. We denote the total conservative specific force (i.e., the force exerted on a unit point mass) as  $\vec{f}$ , and its potential as  $\phi$ . We can readily show that  $\partial f_i / \partial x_j = \partial f_j / \partial x_i$ , both equaling  $\partial^2 \phi / \partial x_i \partial x_j$ , and Laplace's equation implies  $\partial f_1 / \partial x_1 + \partial f_2 / \partial x_2 + \partial f_3 / \partial x_3 = 0$ . Hence, only five of the nine components are independent each other. Certainly, after removing the contribution of the tidal-generating force, data of the gradient components can be used to model the Earth's gravity field. The GOCE mission should have attained the highest resolution achievable by a satellite gravimetry mission. The ultimate data product is an Earth gravitational model up to degree and order of around 200.

<sup>2</sup> In the subsequent text, GRACE is to be understood to indicate both the GRACE and GRACE Follow-On missions.

### 9.1.6 Superconducting Gravimeter

Superconducting gravimeter is the most accurate gravimeter designed to measure gravity variations at permanent stations as time series, reaching a sensitivity on the order of 1 nGal, i.e.,  $1 \times 10^{-11} \text{ nm} \cdot \text{s}^{-2}$ . Therefore, superconducting gravimeter data are the default choices as the ground truth of gravity variations for other kinds of technologies of gravity variation measurement. With a sensitivity much higher than other types of gravimeters, superconducting gravimeter data are the most used to detect new geophysical processes associated with a gravity variation signal, such as the Slichter modes discussed in Sect. 6.4.3.

### 9.1.7 The ITRS/ITRF

We now focus on the realization of the TRS/TRF that requires a combination of all kinds of data that may bring improvements. As mentioned in Sect. 1.2, this is a task within the framework of the IERS, and the realization of the TRF is referred to as ITRF. The root control is a realization of GCRS referred to as the (geocentric) CRF. The VLBI, SLR, and GNSS data are the most important. The VLBI data link the orientation of the ITRF to the CRF, and are the most sensitive to Earth rotation. Therefore, they are the most critical for the data of EOPs. The SLR data provide the most accurate location of the CM with respect to the ITRF stations. Hence, they are the most critical for attaching the ITRS origin on the CM, which is the physical point of the Earth system most accurately determined. The GNSS is far more cost-effective per station. Therefore, as compared to the VLBI and SLR, there is a huge number of GNSS stations all over the world. The most important role that the GNSS data play is to continuously monitor the relative motion of the stations. These data are the most crucial for defining the ITRF to be free from the effect of individual station displacement. Although the ITRF stations are moving one respect to another, the ITRS, i.e., the coordinate axes implied by the station coordinates, is defined as a rigid triad tied on a rigid body representing the Earth as a whole. Hence, the rigid body that the triad is tied on should be defined. This rigid body can be defined as that represented by the network of stations after removing all the relative motions. The separation between the relative motions among the stations and their rotation as a whole is not unique. The ideal is to locate the triad on the Tisserand frame of the mantle, i.e., the  $\vec{h}$  term discussed in Sect. 3.5.3 vanishes. Plate tectonic motion is the largest in magnitude and the most difficult to handle. A kind of plate motion model, called no-net-rotation (NNR) model is conceived for such a purpose. The meaning of NNR is that relative velocity among the plates do not include an overall rotation as a whole. The triad of the ITRS can then be determined by minimizing the residual station displacement after removing all those known. The causes and characteristics of station displacements are quite complicated in nature. Therefore, establishing the ITRF requires careful analysis of all the data and geophysical processes. The readers interested in the detail are referred to the publications dedicated to ITRF, the most

recent one being Altamimi et al. (2023), through which other relevant publications can be traced. There have been several realizations of the ITRF, the latest being the ITRF2020.

Although the ITRS origin is defined to be on the CM, it cannot be true in any of its ITRF realization. In practice, data are always expressed in an ITRF. Therefore, when data are concerned, we use the terms *ITRF* and *ITRF origin* to represent the triad of the coordinate system and its origin, respectively.

The ITRS is the most accurate/precise for global applications. For national mappings, an alternative choice is to attach the reference system, through a network, on the plate that the country is on for the coordinates of an object to not vary following the plate motion, so that the reference system can remain valid for a longer time-span. For monitoring deformation in a small region, a reference system attached on the nearby stable crust through a network can be used.

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## 9.2 A Summary of the Data and Models

Besides the data provided by the technologies as summarized in the previous section, the deformation of the Earth is concerned with many geophysical models. There is no net distinction between data and models. For example, the GRACE monthly gravitational models are models for those who use the (usually preprocessed) observation data to derive the models, but they are data for those who use them to study geophysical processes. Therefore, we put data and models together in the same section.

A model can be understood as the set of parameters describing the state of a body or substance, which is referred to as static, or the temporal evolution of a process, which is referred to as dynamic. The static and dynamic models are not mutually exclusive. For example, the Earth tide is a dynamic process, but the tidal Love numbers are physical parameters of the Earth independent of the tidal process generated by the gravitational attraction of extraterrestrial celestial bodies.

Strictly speaking, all data and models involving mass transport and Earth deformation are relevant to the contents of this book. Geodetic data are distributed through the support of many agencies worldwide, such as the NASA Earthdata archive,<sup>3</sup> the Geodetic Facility for the Advancement of Geoscience (GAGE)<sup>4</sup> funded by the U.S. Natural Science Foundation (NSF), and the Open Science Catalog<sup>5</sup> funded by the Earth Observation program of the European Space Agency (ESA). We list in this section a few of the data sets and models most relevant to the contents of this book with a brief description. The description here is elementary and brief. More detail can be found through the websites or references provided.

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<sup>3</sup> <https://www.earthdata.nasa.gov/>

<sup>4</sup> <https://www.unavco.org>

<sup>5</sup> <https://opensciencedata.esa.int>

### 9.2.1 Earth's Internal Physical Parameters

We have included the polynomial Preliminary Reference Earth Model (PREM) in Sect. 6.8. A discrete model, i.e., the values of the parameters at a series of nodes over the radius, is also provided in the original paper (Dziewonski & Anderson, 1981). Some other models of the same type, i.e., spherically symmetrical or one dimensional (1D), have been derived, particularly the discrete model AK135 (Kennett et al., 1995). A review of the major models is provided by Bormann (2012, Chapter 2 and Datasheet 2.1). The AK135 has a sediment layer between the crust and ocean, which should be treated in the same way as the ocean (either ignored or averaged with the crust) in the computation of the load Love numbers.

The Earth has a three-dimensional (3D) structure. A general review of 3D models of the deep Earth is provided by Bormann (2012, Chapter 2). The most relevant to the contents of this book are the crust model CRUST1.0 (Laske et al., 2013) and the lithospheric model LITHO1.0 (Pasyanos et al., 2014); the latter being an extension of the former by including the lithospheric mantle. These models provide the radial profiles of the physical parameters over a grid of geographical locations, which can be understood as local 1D models. The LITHO1.0 is accompanied with a program to compute the radial profile of the physical parameters at any geological location by interpolation using the model data. The higher degree load Love numbers are sensitive to the physical parameters of the shallow layer. Therefore, using the local profile of the physical parameters around the instrument location instead of the global 1D model to compute them is expected to provide more accurate interpretation of the elastic loading data. Certainly, the more inclusive model LITHO1.0 is preferred for this purpose.

### 9.2.2 Earth Orientation Parameters

The EOPs are available through the IERS website in several formats.<sup>6</sup> The Rapid data and predictions are the predictions of a year into the future with or without the data since 1992. Two long term series, C01 and C04, are provided, the former being a long-term combined data series since 1846, and the latter a more accurate data series since 1962 (at 0hUTC) or 1984 (at 12hUTC) subjecting to frequent updates. The terms IAU1980, IAU2000, and IAU2000A refer to the nutation models adopted by the International Astronomical Union (IAU), and 14 and 20 refer to the ITRF2014 and ITRF2020, respectively. To achieve the maximum accuracy/precision, the realization of the ITRF and its EOPs are derived together by combining all the data that contribute. Besides the metadata of the data files, the readers are also suggested to consult the manuals of the SOFA and NOVAS,<sup>7</sup> where

<sup>6</sup> <https://www.iers.org/>

<sup>7</sup> SOFA: Standard of Fundamental Astronomy software provided by the International Astronomical Union (<http://www.iausofa.org>); NOVAS: Naval Observatory Vector Astronomy Software provided by the United States Naval Observatory ([https://aa.usno.navy.mil/software/novas\\_info](https://aa.usno.navy.mil/software/novas_info)).

the data are used from the point of view of users. The Jet Propulsion Laboratory (JPL) of the National Aeronautics and Space Administration (NASA) also routinely provides the EOPs.<sup>8</sup> The readers are referred to the IERS Convention 2010 (Petit & Luzum, 2010) for an official description of the definitions and data of the EOPs.

The excitation functions of the atmosphere, ocean, and hydrology are provided by the respective special bureaus of the Global Geophysical Fluids Center (GGFC) of the IERS, and are referred to as effective atmospheric (AAM), oceanic (OAM), and hydrologic angular momentum (HAM), which can also be accessed through the IERS website.

### 9.2.3 Earth's Gravitational Potential

A large number of global gravitational potential models of the Earth can be found at the website of the International Centre for Global Earth Models (ICGEM),<sup>9</sup> which can also be accessed through the GGOS website. There are two categories of models: static and temporal. Different types of data, GRACE, GOCE, and terrestrial data are complementary for modelling the static field: GRACE data are best up to degree and order  $\sim 100$ , GOCE data are best from degree  $\sim 100$  to  $\sim 200$ , and terrestrial data are best beyond degree  $\sim 200$  (van der Meijde et al., 2015, Fig. 3). For temporal variations, only the GRACE data can provide solutions with errors below the magnitude of the signal up to degree  $\sim 60$  (Pail, 2020, Fig. 2). Therefore, best static models are obtained by combining the data of all the three types, while only the GRACE data can produce meaningful temporal models. Note that the SLR data provides better results for the few lowest degrees of the gravitational potential (e.g. Loomis et al., 2020). The currently most used static model is the EGM2008 (Pavlis et al., 2012; Petit & Luzum, 2010).

The reason for the appearance of the term *steady-state ocean circulation* in the name of the GOCE is that the geoidal height obtained using GOCE data can be used to significantly improve the modelling of the steady-state ocean circulation. The mean dynamic ocean-surface topography (MDOT), i.e., the height of the mean ocean-surface above the geoid, is a consequence of the steady-state ocean circulation. There were several satellite altimetry missions that have determined the height of the mean ocean-surface above an ellipsoid reasonably well. However, very limited marine gravimetry data are available, and the GRACE data can only reach degree and order  $\sim 100$ . The GOCE mission extended the model of oceanic geoid up to degree and order  $\sim 200$ . Therefore, a resolution of the MDOT up to degree and order  $\sim 200$  spherical harmonics can be reached by combining GRACE, GOCE, and satellite altimetry data, which can be used as constraint in the modeling of the steady-state ocean circulation.

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<sup>8</sup> <https://keof.jpl.nasa.gov>

<sup>9</sup> <https://icgem.gfz-potsdam.de>

GRACE data provide the temporal gravitational potential starting from degree 2. The comprehensive temporal models of the gravitational potential should also include the degree 1 term, which is equivalent to providing the coordinates of the CM. As already mentioned, the origin realized by any ITRF cannot be strictly on the CM. The geocenter motion, i.e., the small offset of the CM with respect to the ITRF origin as a function of time, is provided as a time series by further analyzing the SLR data (e.g. Cheng et al., 2013). Different realizations of the ITRF, ITRF2005, ITRF2008, ITRF2014, and ITRF2020, used data of different time spans (Altamimi et al., 2023). Therefore, the origins of different realizations of the ITRF do not coincide, and hence, the realization of the ITRF must be specified for the geocenter motion data. The GRACE data are insensitive to the realization of the ITRF.

It is noted that any model of the gravitational potential is supposed to be a kind of average over a certain time span during which the data used to derive the model are collected. All the known time-dependent signals as to be detailed in subsequent subsections in the data should be removed before deriving the model.

### 9.2.4 Earth Tide

A complete Earth tide model includes two types of data: the tidal Love numbers and the tidal-generating potential. The observables of Earth tide can be computed from these data. The tidal-generating potential can be computed in two ways: one way is to use the discrete numerical ephemeris of the Sun and the Moon, and the other is to use a table of harmonic tidal waves. The Earth tide is the best known temporal process of Earth deformation. For most applications, the Earth tide is removed as a correction based on a model. For the need as a correction, the ephemeris method is more convenient (e.g. Petit & Luzum, 2010). For the need of harmonic analysis with extremely high accuracy such as in the processing and use of superconducting gravimeter data, the tidal wave method is more convenient (e.g. Kudryavtsev, 2004). The latter is in the framework of the International Geodynamics and Earth Tide Service (IGETS).<sup>10</sup>

To match the accuracy/precision of current observations, the tidal Love numbers of an ellipsoidal, rotating, and anelastic (meaning non-pure elastic) Earth model with a fluid outer core and a solid inner core have to be used. Several complexities arise as compared to the static tidal Love numbers of the spherical Earth model:

- They depend on the frequency represented by the order (secular, diurnal, semidiurnal etc.), which is from the acceleration term in the governing equation of the Earth's deformation. The effect of resonances with the free oscillation modes is included in this dependency.
- The deformation of the Earth caused by the degree  $n$  tidal-generating potential is no longer limited to the same degree, which is an effect of the rotation and

<sup>10</sup> <http://igets.u-strasbg.fr>, which can also be accessed through the GGOS website.

ellipticity. For example, associated with the degree 2 tidal-generating potential are the degree 0, 2, and 4 spheroidal and degree 1 and 3 toroidal deformations (Mathews et al., 1995).

- Due to the anelasticity, the tidal Love numbers become complex with the imaginary part representing a lag of deformation.
- There is a resonance with some normal modes of Earth rotation, namely the Chandler wobble and two modes related to the core.
- Due to the ellipticity of the Earth, latitude-dependent Love numbers are introduced to describe the surface displacement without explicitly referring to the coordinate and tidal-generating potential on the ellipsoidal Earth surface (Mathews et al., 1995).

The detail for computing the degree 2 and 3 Earth tide (gravitational potential and displacement) based on the ephemeris approach is provided in the IERS convention 2010 (Petit & Luzum, 2010). Even in the ephemeris approach, the tidal waves affected by the resonance with Earth rotation normal modes have to be handled individually.

### 9.2.5 Ocean Tide

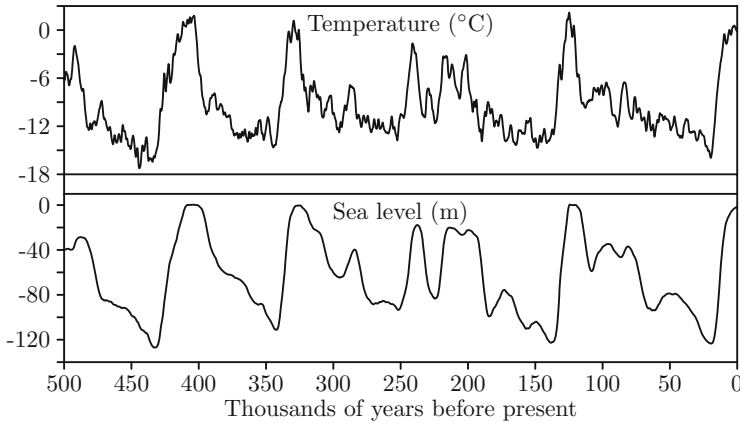
Current ocean tide models are derived from satellite altimetry data. The Earth deforms under its loading. The computation of the effects on the gravitational potential and surface displacement is straightforward and is provided in the IERS convention 2010 (Petit & Luzum, 2010).

The computation of the direct effect of gravity, i.e., the contribution of the gravitational attraction of the ocean water, needs more careful consideration for stations not far from the coast; it depends strongly on the height of the stations. This is well known to the community processing very accurate gravity time series.

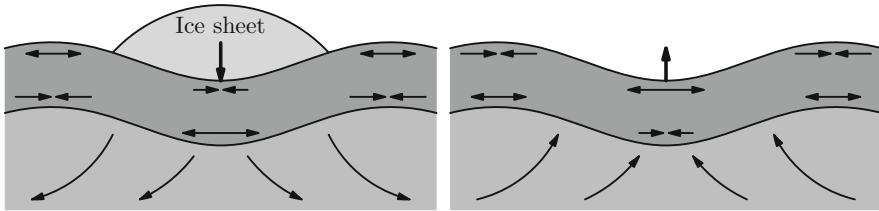
Tide also exists in the atmosphere, which is mainly driven by the heating from the Sun. Therefore, atmospheric tide is dominated by a diurnal and a semidiurnal frequencies. It is much smaller than the ocean tide and is not to be mentioned further. The readers are referred to the IERS convention 2010 (Petit & Luzum, 2010).

### 9.2.6 Glacial Isostatic Adjustment

As already mentioned in Sect. 1.1, the glacial isostatic adjustment (GIA) refers to the Earth's viscoelastic deformation as a result of surface loading associated with the glacial-interglacial cycles on the order of 120 thousand years. Shown in Fig. 9.1 are the glacial-interglacial cycles of the last 500 thousand years drawn using the data provided by Bintanja et al. (2005). In each cycle, a significant part of the ocean water is frozen to become ice sheets, which are then melted to become ocean water again. The main features of the GIA process associated with a glacial-interglacial cycle is illustrated in Fig. 9.2. The GIA signal observed at present is mainly the



**Fig. 9.1** The ice age cycles

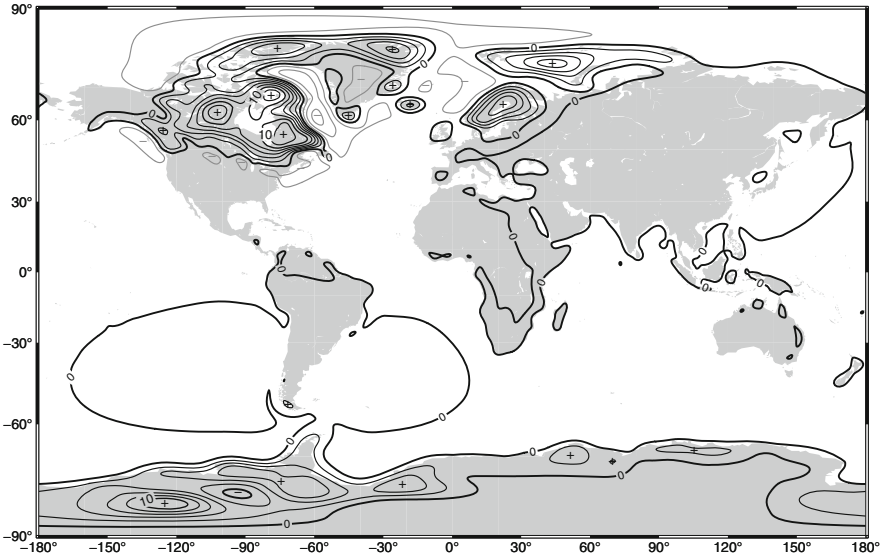


**Fig. 9.2** The glacial viscoelastic loading (left) and post-glacial rebound (right) (Courtesy of Dr. V. Klemann, GFZ)

Earth's ongoing response to the last glacial cycle ended about six thousands years ago, called post-glacial rebound (PGR), which consists of an elastic bouncing-back of the lithosphere accompanied by a viscous flow of the mantle below from the very large deformation caused by the last glacial cycle. At present, the time-dependence of the GIA, or PGR, can be considered as linear in the interpretation of decades-long observations. Figure 9.3 serves as an illustration of the present-day GIA vertical displacement rate of the crust drawn using the data provided by Peltier et al. (2018).

In GIA modeling, the ocean surface is assumed to be in hydrostatic equilibrium, meaning that it coincides with an equipotential surface of gravity that extends to the continents. The changes of the height of this surface with respect to the origin of the reference system and the ocean bottom are referred to as absolute and relative sea level changes, respectively. The independent data sets of a GIA model include the rate of the crustal displacement, the rate of the absolute sea level change, and the rate of the polar motion. The rate of the gravitational potential can be computed from the rate of the absolute sea level change discarding the degree 0 term that represents the global average of the absolute sea level change. The rate of the relative sea level change can be obtained by subtracting the rate of the vertical crustal displacement from the rate of the absolute sea level change. As the total ocean water volume





**Fig. 9.3** A model of GIA crustal uplift rate. Contours are in 2 mm/yr interval. Thick black line: 0 uplift rate; Black line: positive uplift rate; Gray line: negative uplift rate

is assumed to be conserved, it can be inferred that the global average of the rate of absolute sea level change equals the average of the rate of the vertical crustal displacement over the oceans, and that the average of the rate of the relative sea level change over the oceans vanishes.

The GIA signals in the current data of gravity variation, crustal displacement, and absolute (i.e., satellite altimetry measured) and relative (i.e., tide gauge measured) sea level changes are significant. To study current mass changes in the atmosphere-hydrosphere-cryosphere system, the GIA effects must be removed from the data. The responses of the solid Earth and the oceans to the GIA induced polar motion are included in the GIA model. Hence, the GIA induced polar motion should not be included in the polar motion data used to correct the pole tide in the study of the linear trend of current mass changes.

A GIA model is derived from two other models: a model of ice history and a model of Maxwell viscoelasticity of the Earth's interior. As compared to the PREM, the Maxwell viscoelastic Earth model includes the profile of one more parameter, i.e., the viscosity for thousands-of-years-long forcings. Neither the ice history nor the viscosity profile is well constrained by data. The readers are referred to Peltier et al. (2022) for detail. Here we mention that any GIA model should be used with care (e.g. Guo et al., 2012).

Two approximate inherent relations of a GIA model are relevant to the interpretation of the data of gravity variation, crustal displacement, and sea level change. We denote the GIA rates of crustal uplift, geoidal height, and gravity due to mass redistribution as  $\dot{u}_r(\text{GA})$ ,  $\dot{N}(\text{GA})$ , and  $\dot{g}_m(\text{GA})$ , respectively. These notations are

for later convenience when other effects are included. We express the first two as

$$\dot{u}_r(\text{GA}) = \sum_{n=2}^N \sum_{k=-n}^n \dot{u}_n^k(\text{GA}) \bar{\mathcal{H}}_n^k(\theta, \lambda), \quad (9.1)$$

$$\dot{N}(\text{GA}) = R \sum_{n=2}^N \sum_{k=-n}^n \dot{\Phi}_n^k(\text{GA}) \bar{\mathcal{H}}_n^k(\theta, \lambda), \quad (9.2)$$

where  $\bar{\mathcal{H}}_n^k(\theta, \lambda)$  is the fully normalized spherical harmonics defined in (4.340) and (4.341). The two approximate inherent relations are (Wahr et al., 1995)

$$\dot{u}_n^k(\text{GA}) \approx \frac{2n+1}{2R} \dot{\Phi}_n^k(\text{GA}), \quad (9.3)$$

$$\dot{u}(\text{GA}) \approx 6.5 \dot{g}_m(\text{GA}), \quad (9.4)$$

where the units of  $u_r(\text{GA})$  and  $g_m(\text{GA})$  are mm and  $\mu\text{Gal}$ , respectively. We note that  $\Phi_n^k(\text{GA})$  is actually the Stokes coefficient. Purcell et al. (2011) found an alternative of (9.3), with  $1.1677n - 0.5233$  replacing  $(2n+1)/2$ .

The gravity-uplift relation (9.4) can be expressed as

$$\dot{g}_m(\text{GA}) \approx 0.15 \dot{u}_r(\text{GA}). \quad (9.5)$$

Note that, if the uplift  $\dot{u}_r(\text{GA})$  is purely caused by a thickening of the crust by filling the same crustal material of density  $2.67 \text{ g}\cdot\text{cm}^{-3}$ , the opposite of the Bouguer plate correction applies, which is  $\dot{g}_{\text{BP}} = 0.11 \dot{u}_r(\text{GA})$ . The relation (9.5) implies that the material filling the void induced by the uplift has a density of about  $3.64 \text{ g}\cdot\text{cm}^{-3}$ . This is the density of about 400 km deep according to PREM. The rate of gravity variation due to mass redistribution,  $\dot{g}_m(\text{GA})$ , is that as measured using spaceborne or airborne (positioned by GNSS) technologies. The opposite of the free air correction,  $-0.31 \dot{u}_r(\text{GA})$ , should be added for the rate of gravity change measured by a ground gravimeter. We mention that (9.5) has been confirmed with observations (e.g., Bagherbandi & Sjöberg, 2025).

When the rotational feedback is included in a GIA model, the potential on the ocean surface defined as an equipotential surface includes the variation of the centrifugal potential associated with the GIA induced polar drift. The variation of the centrifugal potential, which affects the spherical harmonic coefficients  $\dot{\Phi}_2^{-1}(\text{GA})$  and  $\dot{\Phi}_2^1(\text{GA})$  in (9.2), is not observed by GRACE. Therefore, when a GIA model is used to interpret GRACE data, the variation of the centrifugal potential must not be included in the coefficients  $\dot{\Phi}_2^{-1}(\text{GA})$  and  $\dot{\Phi}_2^1(\text{GA})$ . In addition, the degree 1 gravitational potential is not observed by GRACE. Hence, two sets of spherical harmonic coefficients are provided in contemporary GIA models, one is derived from the rate of the absolute sea level of the GIA model, i.e., (9.2), and the other is for interpreting GRACE data (e.g., Peltier et al., 2018). We always assume

hereafter that the effect of the polar drift is not present in  $\dot{\Phi}_2^{-1}(\text{GA})$  and  $\dot{\Phi}_2^1(\text{GA})$  when they are used as a part of the GRACE observed variation of the gravitational potential. However, it is the rate of the absolute sea level of the GIA model, i.e., (9.2), that is observed by satellite altimetry.

### 9.2.7 GNSS Station Displacement

The GNSS network serving the ITRF is managed by the International GNSS service (IGS). Stations and their coordinates as time series are available from the IGS website,<sup>11</sup> which can also be accessed through the GGOS website. There are numerous national GNSS networks for national surveying and mapping services, and local GNSS networks for various applications such as managing transportation and monitoring displacement. In the U.S., these networks can be accessed through the National Geodetic Survey (NGS) of the National Oceanic and Atmospheric Administration (NOAA).<sup>12</sup> These networks are composed of permanent stations, which are referred to as Continuously Operating Reference Stations (CORS).

The time series of the coordinates of a station can be understood as the record of the displacement of the station. We know that the plate tectonic motion is irrelevant to elastic loading, and should be removed from the data. However, there are numerous other causes. Relevant to the elastic loading is only the displacement of the bedrock, which moves with the crust below as a whole. Displacements of ground stations caused by the thickening or thinning of the soil layer or the poroelastic deformation as a function of underground water level are not elastic loading effects. Therefore, in order for the data to be interpreted in terms of elastic loading, the displacement of the station must represent that of the bedrock.

### 9.2.8 Satellite Altimetry Sea Level Data

Enormous data collected by the satellite altimetry missions thus far are available. These data are also combined by several institutions to provide grided sea level data, including the mean sea surface height (MSSH) above a reference Earth ellipsoid and the variable sea level anomaly (SLA), the latter being the difference between the actual sea surface height (SSH) and the former. The institutions and data of various levels can be accessed through the Physical Oceanography Distributed Active Archive Center (PO.DAAC) hosted by JPL, NASA.<sup>13</sup> We mention particularly the AVISO, the French Space Agency (CNES: Centre National d'Etudes Spatiales)

<sup>11</sup> <https://igs.org/itrf>

<sup>12</sup> <https://geodesy.noaa.gov/CORS/>

<sup>13</sup> [https://podaac.jpl.nasa.gov/Altimetric\\_Data\\_Information](https://podaac.jpl.nasa.gov/Altimetric_Data_Information)

altimetry data processing center,<sup>14</sup> whose SLA data gridded uniformly in both space and time derived by combining data from several missions are widely used.

The satellite altimetry measured SLA is a geometric height. Relevant to the elastic loading theory is the SLA induced by the flow of ocean water, called eustatic SLA, which is associated with mass transport. To convert the altimetry measured SLA to the eustatic SLA, the ocean water volume change due to the variations of temperature (thermosteric) and salinity (halosteric) must be corrected. The correction, called steric (thermosteric and halosteric) sea level change, is also provided by the AVISO in the format of uniform space-time grid. To the depth of this book, we assume that the term *steric* includes also other minor causes of water volume change not mentioned.

### 9.2.9 Atmospheric and Oceanic Models

The tidal-generating force as well as the Earth tide and ocean tide it induces is much better known than the non-tidal variations in the atmosphere and oceans. The former is the largest, which is driven by harmonic forcings with frequencies known exactly, thus being much easier to determine using observation data. The latter are the next largest, which is the concern here. Their driven mechanisms are too complicated to be known with sufficient accuracy/precision. Hence, they are modeled using numerical simulations.

All valuable data are assimilated to derive the models. Even though satellite data are immense, their coverage in both space and time are still sparse and irregular as compared to the need of continuity in both space and time. Furthermore, not all types of data can be measured from satellites. Therefore, all the data are used as inputs to the physical equations of the problem to solve for all the quantities concerned, which are continuous functions in both space and time. The functions are discretized over a space-time grid based on an interpolation approach, meaning that the continuous functions can be expressed using their discrete values over the grid. The physical equations are solved for the discrete values of the functions.

For satellite gravimetry missions like the GRACE and GOCE, the gravitational and loading effects of the mass transport in the atmosphere and oceans, called atmospheric and ocean de-aliasing (AOD) model, should be subtracted from the data before deriving the model of the gravitational potential. An AOD model is derived from an atmospheric- and an ocean-circulation model.

The most used atmospheric model is that of the European Centre for Medium-Range Weather Forecasts (ECMWF).<sup>15</sup> Besides the Forecast model for real-time applications, more accurate/precise re-analysis models of the past, named ERAs, are available for post-processing applications. Relevant to the content of this book is that the location-dependent density and velocity of the atmosphere are provided. The

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<sup>14</sup> <https://www.aviso.altimetry.fr/>

<sup>15</sup> <https://www.ecmwf.int/>

variations are mostly diurnal but different from day to day. Therefore, the shorter the time step the better. The latest re-analysis model, ERA5, has a time step of 1 hour, a 31 km grid on the Earth's surface, and 137 layers over a height of 80 km. As the speed of a gravimetry satellite is on the order of  $\sim 8 \text{ km}\cdot\text{s}^{-1}$ , interpolation to any time is required to compute the gravitational attraction at the satellite position.

The most recent data of GRACE AOD and EOP excitation function provided by the Deutsches GeoForschungsZentrum (GFZ) (Dobslaw et al., 2017; Dobslaw & Dill, 2018) are computed using an ocean model derived by the Max Planck Institute for Meteorology (MPIOM), which is a component of a more comprehensive Earth system model (MPI-ESM). An ocean model derived by a project of the National Centers for Environmental Prediction (NCEP)—the Estimating the Circulation and Climate of the Ocean (ECCO)<sup>16</sup>—is also widely used. An alternative ocean modeling effort is the multi-institutional HYbrid Coordinate Ocean Model consortium (HYCOM) sponsored by the National Ocean Partnership Program (NOPP) of the U.S. Global Ocean Data Assimilation Experiment (GODAE) funded by several agencies.<sup>17</sup>

The AOD model is excluded in the GRACE Level-2 product, i.e., the monthly models of the gravitational potential, since it is removed from the data beforehand. As supplements, the monthly averages of the atmospheric and oceanic contributions to the AOD model are provided separately in case a user may need to add one or both of them back to the Level-2 data.

### 9.2.10 Land Water Content Models

The surface and soil moisture water content is modeled using relevant data and models, such as precipitation, runoff, evaporation, and etc (Rodell et al., 2004). Several models are made available through the NASA's Global Land Data Assimilation System (GLDAS) website.<sup>18</sup> These models provide the water contents only down to a depth of  $\sim 2$  to  $\sim 3.5$  m. Therefore, the recharge and extraction of the underground water below the depth reached by a model are not accounted for. Besides the GLDAS website, the readers are also referred to Fang et al. (2009) for data descriptions and usages.

### 9.2.11 Cryosphere Data and Models

Ice sheets and glaciers accounts for 99.5% of the permanent ice volume. The volumes of the Antarctic Ice Sheet (AIS), the Greenland Ice Sheet (GrIS), and mountain glaciers are, in terms of sea level equivalent (the global sea level rises they

<sup>16</sup> <https://podaac.jpl.nasa.gov/ECCO>, <https://ecco-group.org/>

<sup>17</sup> <https://www.hycom.org>

<sup>18</sup> <https://ldas.gsfc.nasa.gov/gldas>

would cause if fully melted), 57.9, 7.42, and 0.32 m, respectively. Ample knowledge about ice sheets and glaciers can be found through professional websites.<sup>19</sup> For the concerns of this book, the complexity of ice sheets and glaciers lies in their layering from snow on the top to glacier ice below a certain depth with a varying depth-dependent density. The intermediate layer between the snow and glacier ice is in granular state transformed from snow by compression and recrystallization over time, called firn. The transition from snow to glacier ice is referred to as firn compaction. In order to evaluate the mass change using data of surface elevation change, a model of density throughout the depth is required (e.g., Li & Zwally, 2011).

Various cryosphere and related geophysical data have been collected through satellite, airborne, and field observations. The largest relevant data centers are the U.S. National Snow and Ice Data Center (NSIDC)<sup>20</sup> and the ESA Open Science Catalog.<sup>21</sup> Modelling of ice sheets and glaciers, particularly the AIS and GrIS, has been an on-going effort of the glaciological community. The readers are referred to the recent paper by Medley et al. (2022).

### 9.3 Applications of the Data and Models

We focus on the study of the present-day mass transport in the atmosphere-hydrosphere-cryosphere system using geodetic data, which is a problem of elastic loading. We first analyze the sources of the gravity field and the causes of the Earth surface displacement. We then discuss how they are related to the data. Finally, we show some examples of real world applications of the data.

#### 9.3.1 Decomposition of the Gravity and Surface Displacement

We consider the solar system as an isolated system, i.e., the effects of extrasolar celestial bodies are ignored in the physical equations of the Earth system. According to (3.52), we express the equation of motion of a unit point mass (or each term is divided by the mass) in the ITRS as

$$\vec{a}_{\text{ITRS}} = \vec{f}_{\text{NG}} + \vec{f}_{\text{G}} - 2\vec{\omega} \times \vec{v}_{\text{ITRS}} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - \frac{d\vec{\omega}}{dt} \times \vec{r} - \vec{a}_{\text{BCRS}}^{\text{CM}}. \quad (9.6)$$

In this equation:

- $\vec{a}_{\text{ITRS}}$  is the acceleration of the unit point mass in the ITRS;
- $\vec{f}_{\text{NG}}$  is the total non-gravitational force;

<sup>19</sup> e.g., <https://www.antarcticglaciers.org>

<sup>20</sup> <https://nsidc.org/data>

<sup>21</sup> <https://opendata.esa.int/themes/cryosphere/catalog>

- $\vec{f}_G$  is the gravitational attraction of the masses in the entire solar system;
- $\vec{\omega}$  is the angular velocity of the ITRS with respect to the GCRS;
- $\vec{v}_{\text{ITRS}}$  is the velocity of the unit point mass in the ITRS;
- $d\vec{\omega}/dt$  is the rate of variation of  $\vec{\omega}$  (can be expressed in either ITRS or GCRS);
- $\vec{a}_{\text{BCRS}}^{\text{CM}}$  is the acceleration of the center of mass of the Earth system in the barycentric celestial reference system (BCRS), which is, as comparison with the geocentric celestial reference system, an inertial reference system with the origin on the barycenter of the solar system;
- $-2\vec{\omega} \times \vec{v}_{\text{ITRS}}$  is the Coriolis force (which is the negative of the Coriolis acceleration);
- $-\vec{\omega} \times (\vec{\omega} \times \vec{r})$  is the centrifugal force (or centrifugal acceleration, which is the negative of the centripetal acceleration);
- $-(d\vec{\omega}/dt) \times \vec{r}$  is the Euler force (also referred to as Poincaré force), which is extremely small and is usually ignored.

For a moving vector gravimeter (that measures all the three components of the gravity), the equation of motion of the sensor is (9.6). The reading of this type of gravimeter is in fact  $\vec{f}_{\text{NG}}$  exerted on the sensor, which partly balances the other terms at the right-hand side of (9.6) and partly induces the acceleration at the left-hand side. The gravity as to be defined in the next paragraph is obtained by correcting this reading. Evidently, the correction needs the independent measurements of the acceleration  $\vec{a}_{\text{ITRS}}$  and the velocity  $\vec{v}_{\text{ITRS}}$ .

The term *gravity* is defined differently in different contexts. In the ITRS, we define the gravity  $\vec{g}_{\text{ITRS}}$  as the free fall (meaning  $\vec{f}_{\text{NG}} = 0$ ) acceleration of a test mass when its velocity vanishes. According to (9.6), we have

$$\vec{g}_{\text{ITRS}} = \vec{f}_G - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - \vec{a}_{\text{BCRS}}^{\text{CM}}, \quad (9.7)$$

where the extremely small Euler force is ignored as usual. We decompose  $\vec{f}_G$  as

$$\vec{f}_G = \vec{f}_{\text{ES}} + \vec{f}_{\text{OCB}}, \quad (9.8)$$

where  $\vec{f}_{\text{ES}}$  is the gravitational attraction of the Earth system, and  $\vec{f}_{\text{OCB}}$  that of other celestial bodies in the solar system. We know that

$$\vec{f}_{\text{TGF}} = \vec{f}_{\text{OCB}} - \vec{a}_{\text{BCRS}}^{\text{CM}} \quad (9.9)$$

is the tidal-generating force. We can thus write (9.7) as

$$\vec{g}_{\text{ITRS}} = \vec{f}_{\text{ES}} + \vec{f}_C + \vec{f}_{\text{TGF}}, \quad (9.10)$$

where  $\vec{f}_C = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$  is the centrifugal acceleration, and  $\vec{f}_{\text{ES}} + \vec{f}_C$  the total gravity of the Earth system. Note that both  $\vec{f}_{\text{ES}}$  and  $\vec{f}_C$  vary with time.

We decompose the centrifugal acceleration as

$$\vec{f}_C = \vec{f}_{C0} + \vec{f}_{PM}, \quad (9.11)$$

where  $\vec{f}_{C0} = -\vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r})$  is the nominal mean defined such that the Earth rotates around the pole of the ITRS  $Ox_1x_2x_3$  with the mean angular rate  $\omega_0$ , i.e.,  $\vec{\omega}_0 = \omega_0 \hat{e}_3$ , and  $\vec{f}_{PM}$  is the increment due to the offset of the rotation pole from the ITRS pole referred to as polar motion. Note that this decomposition is incomplete; the effect of the variation of the rotation rate is ignored. We decompose the gravitational attraction of the Earth system as

$$\vec{f}_{ES} = \vec{f}_{SES} + \vec{f}_{ET} + \vec{f}_{AOT} + \vec{f}_{AOTL} + \vec{f}_{NTAO} + \vec{f}_{NTAOL} + \vec{f}_{PT} + \vec{f}_{GIA} + \vec{f}_{OESP}, \quad (9.12)$$

where  $\vec{f}_{SES}$  is the static gravitational attraction of the Earth system, which can be understood as the average during the time span of interest, and the other terms are temporally variable. The sources represented by the time-dependent terms are as follows: Earth tide (ET), atmospheric and ocean tides (AOT), loading of the atmospheric and ocean tides (AOTL), non-tidal atmospheric and oceanic processes (NTAO), loading of non-tidal atmospheric and oceanic processes (NTAOL), pole tide (PT), glacial isostatic adjustment (GIA), and other Earth system processes not mentioned (OESP). All the effects of the GIA on the gravitational attraction is included in  $\vec{f}_{GIA}$ , including that of the deformation of the Earth induced by the Earth rotation variation it causes. Therefore,  $\vec{f}_{PT}$  is related to the deformation of the Earth and sea level change (as well as its loading effect) in response to  $\vec{f}_{PM}$  without the GIA effect. Lastly, we write (9.10), (9.11), and (9.12) together as

$$\begin{aligned} \vec{g}_{ITRS} = & \vec{f}_{SES} + \vec{f}_{C0} + \vec{f}_{PM} + \vec{f}_{TGF} + \vec{f}_{ET} \\ & + \vec{f}_{AOT} + \vec{f}_{AOTL} + \vec{f}_{NTAO} + \vec{f}_{NTAOL} + \vec{f}_{PT} + \vec{f}_{GIA} + \vec{f}_{OESP}. \end{aligned} \quad (9.13)$$

This is what a vector gravimeter at standstill in the ITRS measures, which is the negative of  $\vec{f}_{NG}$  in (9.6) with vanishing  $\vec{a}_{ITRS}$  and  $\vec{v}_{ITRS}$ . We note that  $\vec{f}_{SES} + \vec{f}_{C0}$  is the static gravity field of the Earth discussed in Chap. 5.

In the GCRS, we defined the gravity as the free fall acceleration of a test mass (it is independent of the velocity, and hence, the vanishing of velocity is no longer required). We have

$$\begin{aligned} \vec{g}_{GCRS} = & \vec{f}_{SES} + \vec{f}_{TGF} + \vec{f}_{ET} + \vec{f}_{AOT} + \vec{f}_{AOTL} \\ & + \vec{f}_{NTAO} + \vec{f}_{NTAOL} + \vec{f}_{PT} + \vec{f}_{GIA} + \vec{f}_{OESP}. \end{aligned} \quad (9.14)$$

As the angular velocity of the ITRS with respect to the GCRS does not enter the dynamic equation in the GCRS, the centrifugal force  $\vec{f}_C$  is absent. However, as the pole tide is related to a deformation of the Earth and a sea level change (including



its loading effect),  $\vec{f}_{PT}$  is a real physical force that does cause an acceleration in the GCRS. Hence, it is included. When the orbit of a satellite is computed in the GCRS,  $\vec{g}_{GCRS}$  is the total gravity that the satellite is subject to.

All terms in (9.13) and (9.14) are conservative forces. Therefore, every one of them can be expressed as the gradient of its potential.

Finally, we provide some comments on the term  $\vec{f}_{OESP}$ . It includes the contributions of all other Earth system processes not included in the terms proceeding it, and is supposed to be smaller. These processes could be, to name a few, the convection in the mantle and the core, the plate tectonic motion, the growth and depletion of glaciers as well as the accompanying elastic loading effects, the variation of land water storage as well as the accompanying elastic loading effects, etc.

Following the decomposition of the sources of the gravity, the causes of the Earth surface displacement can be readily decomposed as

$$\vec{u} = \vec{u}_{ET} + \vec{u}_{AOTL} + \vec{u}_{NTAOL} + \vec{u}_{PT} + \vec{u}_{GIA} + \vec{u}_{OESP} \quad (9.15)$$

in the ITRS. Of particular is that  $\vec{u}_{OESP}$  could be (1) much larger than the proceeding terms, such as the contributions from the plate tectonic motion, Earthquakes, and the change of the thickness of the soil layer, and (2) opposite to the elastic loading, such as the poroelastic deformation. Therefore, for studying the elastic loading, it must be assured that  $\vec{u}$  is the displacement of the bed rock, or the crustal rock, with the plate tectonic motion removed.

### 9.3.2 Fundamental Equations for Using the Data and Models

We suppose to study the non-tidal present-day mass transport in the atmosphere-hydrosphere-cryosphere system, briefly referred to as present-day mass change (PMC), and the GIA using the following data:

1. The GRACE determined monthly gravitational potential of the Earth in the form of spherical harmonic series.
2. The SLR determined monthly coordinates of the CM in the ITRF.
3. The monthly SLA over a uniform space grid derived by combining the data of multiple satellite altimetry missions.
4. The steric (i.e., thermosteric+halosteric+other minor causes) corrections to the monthly sea level mentioned.
5. The monthly coordinates of a network of GNSS stations situated on the bed rock with the plate tectonic motion removed.

The data set (1) is independent of the ITRF, as the degree 1 spherical harmonic term, which is related to the coordinates of the CM, is not included. The data set (4) does not concern the ITRF at all. We assume that the data sets (2), (3), and (5) are provided in the same ITRF realization. Data sets in different ITRF realizations are inconsistent, and should be first assimilated into the same ITRF realization.

The ITRF origin is determined empirically. It can not be exactly on the CM. We do not know the degree 1 load Love numbers for it. It is irrelevant to use the degree 1 load Love numbers defined for the CM, CF, CH, or CL frame to interpret the data sets (3) and (4) for surface mass changes. The data sets (1) and (2) combined provide the spherical harmonic series of the gravitational potential in the ITRF, whose degree 1 term is from the data set (2) and is the only data on the position of the CM. But this is the position of the CM with respect to the ITRF origin, and no evident information about the position of the CM in the CF, CH, or CL frame can be drawn.

We have to use the CM, CF, CH, or CL frame for correctly interpreting the data sets for the degree 1 surface mass changes based on the elastic loading theory. The CF frame is usually adopted in the literature, which we follow in the general narratives. This means that the degree 1 load Love numbers in the CF frame are to be used in the computation of the loading effects.

For studying the PMC and GIA, the first task is to assimilate the data, which is to keep the terms NTAO, NTAOL, and GIA in (9.14) and (9.15) by removing all the other terms. The term PT includes the ocean pole tide, whose formulation is beyond the scope of this book. The readers are referred to the IERS convention 2010 (Petit & Luzum, 2010) from the perspective of users. The assimilated data should have the following properties:

- The effects of the tidal-generating potential, Earth tide, and the solid Earth pole tide are removed from the GRACE Level-2 and surface displacement data.
- The effects of the ocean tide, atmospheric tide, and ocean pole tide are removed from the GRACE Level-2, geocenter, and surface displacement data.
- The AOD model (and other de-aliasing models related to the PMC if applied) is added back to the GRACE Level-2 data. If a GIA model is removed for de-aliasing, it is also added back. We ignore the gravitational effects of the plate tectonic motion and other forcings from the deep interior such as the convection in the mantle and the core.
- In the SLA data derived from satellite altimetry observations, all tidal effects (i.e., the combination of the vertical displacement of the ocean floor and the change of ocean depth) caused by the Earth tide, ocean tide, atmospheric tide (including the response of the ocean to it), and ocean pole tide are removed. The wind effect is also removed. Other de-aliasing models related to the PMC, if applied, are added back.
- The plate tectonic motion and other significant constituents of the displacement not related to the elastic loading of the PMC and GIA are removed from the surface displacement data. If some of these constituents at a station are known to be significant, but they are not known or modeled with sufficient accuracy/precision, this station is discarded.

We now relate the assimilated data to the PMC and GIA. We add an argument to indicate the type or source of the quantities: (PC) for PMC, (GA) for GIA, (GR) for GRACE, (GC) for geocenter, (SD) for GNSS measured surface displacement, (SA)

for satellite altimetry, (EU) for eustatic, (ST) for steric, etc. We use only two letters in the arguments as a compromise so that most equations are not too long to be fitted in a line.

We are concerned with changes, either the difference between two monthly or yearly averages, the linear trend, or the seasonal variations. Therefore, we use increments of the quantities in the formulation for the meanings to be apparent; the linear trend can be understood as the increment in a year. We approximate the PMC as a thin layer on the Earth's surface assumed to be spherical. Following (7.204), we express the PMC as a series of the real fully normalized spherical harmonics defined in (4.340) and (4.341),

$$\delta s(\text{PC}) = \sum_{k=-1}^1 \delta s_1^k(\text{PC}) \bar{\mathcal{H}}_1^k(\theta, \lambda) + \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta s_n^k(\text{PC}) \bar{\mathcal{H}}_n^k(\theta, \lambda), \quad (9.16)$$

which is independent of the origin of the coordinate system. The degree 0 term is dropped since it always vanishes. The degree 1 term is written separately for emphasizing that it is particular. Similarly, following (7.218) and (7.219), we express the change of the geoidal height related to the gravitational effect of  $\delta s(\text{PC})$  as well as its loading effect in the CF frame as

$$\delta N(\text{PC}) = R \sum_{k=-1}^1 \delta \Phi_1^k(\text{PC}) \bar{\mathcal{H}}_1^k(\theta, \lambda) + R \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta \Phi_n^k(\text{PC}) \bar{\mathcal{H}}_n^k(\theta, \lambda). \quad (9.17)$$

For brevity, we omit the argument ( $R$ ) in the components of the surface displacement and write them as  $\delta u_r$ ,  $\delta u_\theta$ , and  $\delta u_\lambda$ . According to (7.209), the loading effect of  $\delta s(\text{PC})$  is

$$\begin{cases} \delta u_r(\text{PC}) = \sum_{k=-1}^1 \delta u_1^k(\text{PC}) \bar{\mathcal{H}}_1^k(\theta, \lambda) + \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta u_n^k(\text{PC}) \bar{\mathcal{H}}_n^k(\theta, \lambda), \\ \delta u_\theta(\text{PC}) = \sum_{k=-1}^1 \delta t_1^k(\text{PC}) \frac{\partial \bar{\mathcal{H}}_1^k(\theta, \lambda)}{\partial \theta} + \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta t_n^k(\text{PC}) \frac{\partial \bar{\mathcal{H}}_n^k(\theta, \lambda)}{\partial \theta}, \\ \delta u_\lambda(\text{PC}) = \sum_{k=-1}^1 \delta t_1^k(\text{PC}) \frac{\partial \bar{\mathcal{H}}_1^k(\theta, \lambda)}{\sin \theta \partial \lambda} + \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta t_n^k(\text{PC}) \frac{\partial \bar{\mathcal{H}}_n^k(\theta, \lambda)}{\sin \theta \partial \lambda}, \end{cases} \quad (9.18)$$

where  $\delta u_n^k(\text{PC})$  and  $\delta t_n^k(\text{PC})$  implies *upward* and *tangential*, respectively. The relations among the spherical harmonic coefficients are, according to (7.215)–(7.219),

$$\delta \Phi_n^k(\text{PC}) = \frac{3}{R\bar{\rho}} \frac{1+k'_n}{2n+1} \delta s_n^k(\text{PC}), \quad \delta s_n^k(\text{PC}) = \frac{R\bar{\rho}}{3} \frac{2n+1}{1+k'_n} \delta \Phi_n^k(\text{PC}), \quad (9.19)$$

$$\delta u_n^k(\text{PC}) = \frac{3}{\bar{\rho}} \frac{h'_n}{2n+1} \delta s_n^k(\text{PC}), \quad \delta s_n^k(\text{PC}) = \frac{\bar{\rho}}{3} \frac{2n+1}{h'_n} \delta u_n^k(\text{PC}), \quad (9.20)$$

$$\delta t_n^k(\text{PC}) = \frac{3}{\bar{\rho}} \frac{l'_n}{2n+1} \delta s_n^k(\text{PC}), \quad \delta s_n^k(\text{PC}) = \frac{\bar{\rho}}{3} \frac{2n+1}{l'_n} \delta t_n^k(\text{PC}), \quad (9.21)$$

which also apply to the case  $n = 1$ . The definition of the CF frame implies that

$$\left\{ \delta \Phi_1^1(\text{PC}), \delta \Phi_1^{-1}(\text{PC}), \delta \Phi_1^0(\text{PC}) \right\} = \frac{1}{\sqrt{3}R} \left\{ \delta X_{\text{CF}}^{\text{CM}}(\text{PC}), \delta Y_{\text{CF}}^{\text{CM}}(\text{PC}), \delta Z_{\text{CF}}^{\text{CM}}(\text{PC}) \right\}, \quad (9.22)$$

$$\int_{\omega} [\delta u_r(\text{PC}) \hat{e}_r + \delta u_{\theta}(\text{PC}) \hat{e}_{\theta} + \delta u_{\lambda}(\text{PC}) \hat{e}_{\lambda}] d\omega = 0, \quad (9.23)$$

where  $\delta X_{\text{CF}}^{\text{CM}}(\text{PC})$ ,  $\delta Y_{\text{CF}}^{\text{CM}}(\text{PC})$ , and  $\delta Z_{\text{CF}}^{\text{CM}}(\text{PC})$  are the  $x_1$ -,  $x_2$ -, and  $x_3$ -coordinates of the CM in the CF frame induced by the PMC, respectively. In (9.23), only the degree 1 terms are nontrivial, i.e., the identity holds for the  $n > 1$  terms whatever the origin is chosen.

We implicitly understand that the GIA is expressed in the CF frame. This is not guaranteed in GIA modeling, where other frames, e.g., the CE frame, may be adopted. However, we can transform a GIA model developed in any frame to the CF frame, as formulated in the Appendix of this chapter. We express the GIA induced surface displacement (which is a spheroidal field) and change of the geoidal height as

$$\begin{cases} \delta u_r(\text{GA}) = \sum_{k=-1}^1 \delta u_1^k(\text{GA}) \bar{\mathcal{H}}_1^k(\theta, \lambda) + \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta u_n^k(\text{GA}) \bar{\mathcal{H}}_n^k(\theta, \lambda), \\ \delta u_{\theta}(\text{GA}) = \sum_{k=-1}^1 \delta t_1^k(\text{GA}) \frac{\partial \bar{\mathcal{H}}_1^k(\theta, \lambda)}{\partial \theta} + \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta t_n^k(\text{GA}) \frac{\partial \bar{\mathcal{H}}_n^k(\theta, \lambda)}{\partial \theta}, \\ \delta u_{\lambda}(\text{GA}) = \sum_{k=-1}^1 \delta t_1^k(\text{GA}) \frac{\partial \bar{\mathcal{H}}_1^k(\theta, \lambda)}{\sin \theta \partial \lambda} + \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta t_n^k(\text{GA}) \frac{\partial \bar{\mathcal{H}}_n^k(\theta, \lambda)}{\sin \theta \partial \lambda}; \end{cases} \quad (9.24)$$

$$\delta N(\text{GA}) = R \delta \Phi_0^0 + R \sum_{k=-1}^1 \delta \Phi_1^k(\text{GA}) \bar{\mathcal{H}}_1^k(\theta, \lambda) + R \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta \Phi_n^k(\text{GA}) \bar{\mathcal{H}}_n^k(\theta, \lambda). \quad (9.25)$$

Note that a degree 0 term appears in  $\delta N(\text{GA})$ . It is not due to a change of the total mass of the Earth system. The term *geoid* in the GIA modeling community is understood as the equipotential surface of gravity that coincides with the ocean surface. Therefore, the degree 0 term represents the GIA induced global mean Sea level change with respect to the reference system (noting that the degree 0 term is

independent of the choice of the origin). Therefore, the GIA induced ocean depth change is

$$\delta D(\text{GA}) = \delta N(\text{GA}) - \delta u_r(\text{GA}). \quad (9.26)$$

Assuming that the coastlines do not change and that the total water volume is conserved, the global mean ocean depth would not change, and hence, the GIA induced global mean Sea level change with respect to the reference system is in fact the average of  $\delta u_r(\text{GA})$  over the oceans. Therefore, the term  $R\delta\Phi_0^0$  does not contribute to the PMC.

The change of the geoidal height based on the GRACE Level-2 data is

$$\delta N(\text{GR}) = R \sum_{n=2}^{\infty} \sum_{k=-n}^n \delta\Phi_n^k(\text{GR}) \bar{\mathcal{H}}_n^k(\theta, \lambda), \quad (9.27)$$

where the degree 1 term is not determined. The relation with PMC and GIA is

$$\delta\Phi_n^k(\text{GR}) = \delta\Phi_n^k(\text{PC}) + \delta\Phi_n^k(\text{GA}), \quad n > 1. \quad (9.28)$$

For evident reason, the spherical harmonic coefficients  $\dot{\Phi}_2^{-1}(\text{GA})$  and  $\dot{\Phi}_2^1(\text{GA})$  include the variation of the centrifugal potential associated with the GIA induced polar drift in (9.26), but not in (9.28).

We assume that two kinds of geocenter data are available, one is the coordinates of the CM in the ITRF, and the other the coordinates of the CF in the ITRF. We thus have

$$\begin{Bmatrix} \delta X_{\text{CF}}^{\text{CM}}(\text{GC}) \\ \delta Y_{\text{CF}}^{\text{CM}}(\text{GC}) \\ \delta Z_{\text{CF}}^{\text{CM}}(\text{GC}) \end{Bmatrix} = \begin{Bmatrix} \delta X_{\text{ITRF}}^{\text{CM}}(\text{GC}) \\ \delta Y_{\text{ITRF}}^{\text{CM}}(\text{GC}) \\ \delta Z_{\text{ITRF}}^{\text{CM}}(\text{GC}) \end{Bmatrix} - \begin{Bmatrix} \delta X_{\text{ITRF}}^{\text{CF}}(\text{GC}) \\ \delta Y_{\text{ITRF}}^{\text{CF}}(\text{GC}) \\ \delta Z_{\text{ITRF}}^{\text{CF}}(\text{GC}) \end{Bmatrix}, \quad (9.29)$$

where, as in (9.22),  $X$ ,  $Y$ , and  $Z$  are the  $x_1$ -,  $x_2$ -, and  $x_3$  coordinates, respectively, the subscript indicates the reference system, and the superscript the point of interest. The degree 1 spherical harmonic coefficients of the gravitational potential in the CF frame determined using the geocenter data are

$$\begin{Bmatrix} \delta\Phi_1^1(\text{GC}) \\ \delta\Phi_1^{-1}(\text{GC}) \\ \delta\Phi_1^0(\text{GC}) \end{Bmatrix} = \frac{1}{\sqrt{3}R} \begin{Bmatrix} \delta X_{\text{CF}}^{\text{CM}}(\text{GC}) \\ \delta Y_{\text{CF}}^{\text{CM}}(\text{GC}) \\ \delta Z_{\text{CF}}^{\text{CM}}(\text{GC}) \end{Bmatrix}. \quad (9.30)$$

They are related to the PMC and GIA according to

$$\begin{Bmatrix} \delta\Phi_1^1(\text{GC}) \\ \delta\Phi_1^{-1}(\text{GC}) \\ \delta\Phi_1^0(\text{GC}) \end{Bmatrix} = \begin{Bmatrix} \delta\Phi_1^1(\text{PC}) \\ \delta\Phi_1^{-1}(\text{PC}) \\ \delta\Phi_1^0(\text{PC}) \end{Bmatrix} + \begin{Bmatrix} \delta\Phi_1^1(\text{GA}) \\ \delta\Phi_1^{-1}(\text{GA}) \\ \delta\Phi_1^0(\text{GA}) \end{Bmatrix}. \quad (9.31)$$

We see that the data of the coordinates of the CM in the ITRF alone cannot be used to study mass changes, but the data of the coordinates of the CF in the ITRF are not available as a mature product.

We use  $\delta u_r(\text{SD})$ ,  $\delta u_\theta(\text{SD})$ , and  $\delta u_\lambda(\text{SD})$  to denote the components of the surface displacement as measured by GNSS, and  $\delta u_r(\text{CF})$ ,  $\delta u_\theta(\text{CF})$ , and  $\delta u_\lambda(\text{CF})$  the components in the CF frame. As the GNSS data are interpreted in the ITRF, the components of the total surface displacement in the CF frame is

$$\begin{Bmatrix} \delta u_r(\text{CF}) \\ \delta u_\theta(\text{CF}) \\ \delta u_\lambda(\text{CF}) \end{Bmatrix} = \begin{Bmatrix} \delta u_r(\text{SD}) \\ \delta u_\theta(\text{SD}) \\ \delta u_\lambda(\text{SD}) \end{Bmatrix} - \begin{Bmatrix} \delta u_r^{\text{GC}} \\ \delta u_\theta^{\text{GC}} \\ \delta u_\lambda^{\text{GC}} \end{Bmatrix}, \quad (9.32)$$

where the second term at the right-hand side is the surface displacement corresponding to the offset of the CF with respect to the ITRF. This identity states that the displacement vector with respect to the ITRF is the sum of the displacement vector with respect to the CF frame and the position vector of the CF in the ITRF. We have, in vector form,

$$\begin{cases} \delta u_r^{\text{GC}} = \hat{e}_r \cdot [\delta X_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_1 + \delta Y_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_2 + \delta Z_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_3], \\ \delta u_\theta^{\text{GC}} = \hat{e}_\theta \cdot [\delta X_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_1 + \delta Y_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_2 + \delta Z_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_3], \\ \delta u_\lambda^{\text{GC}} = \hat{e}_\lambda \cdot [\delta X_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_1 + \delta Y_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_2 + \delta Z_{\text{ITRF}}^{\text{CF}}(\text{GC})\hat{e}_3]. \end{cases} \quad (9.33)$$

According to (2.448), we obtain

$$\begin{cases} \delta u_r^{\text{GC}} = \delta X_{\text{ITRF}}^{\text{CF}}(\text{GC}) \sin \theta \cos \lambda + \delta Y_{\text{ITRF}}^{\text{CF}}(\text{GC}) \sin \theta \sin \lambda + \delta Z_{\text{ITRF}}^{\text{CF}}(\text{GC}) \cos \theta, \\ \delta u_\theta^{\text{GC}} = \delta X_{\text{ITRF}}^{\text{CF}}(\text{GC}) \cos \theta \cos \lambda + \delta Y_{\text{ITRF}}^{\text{CF}}(\text{GC}) \cos \theta \sin \lambda - \delta Z_{\text{ITRF}}^{\text{CF}}(\text{GC}) \sin \theta, \\ \delta u_\lambda^{\text{GC}} = -\delta X_{\text{ITRF}}^{\text{CF}}(\text{GC}) \sin \lambda + \delta Y_{\text{ITRF}}^{\text{CF}}(\text{GC}) \cos \lambda. \end{cases} \quad (9.34)$$

The relation with PMC and GIA is

$$\begin{Bmatrix} \delta u_r(\text{CF}) \\ \delta u_\theta(\text{CF}) \\ \delta u_\lambda(\text{CF}) \end{Bmatrix} = \begin{Bmatrix} \delta u_r(\text{PC}) \\ \delta u_\theta(\text{PC}) \\ \delta u_\lambda(\text{PC}) \end{Bmatrix} + \begin{Bmatrix} \delta u_r(\text{GA}) \\ \delta u_\theta(\text{GA}) \\ \delta u_\lambda(\text{GA}) \end{Bmatrix}, \quad (9.35)$$

We see that, in order to use the GNSS data to study the degree 1 mass change, the coordinates of the CF in the ITRF is required.

Last, we examine the satellite altimetry data. The change of the ocean surface height in the CF frame,  $\delta H(\text{CF})$ , can be computed from that measured by satellite altimetry,  $\delta H(\text{SA})$ , by taking away  $\delta u_r^{\text{GC}}$ ,

$$\delta H(\text{CF}) = \delta H(\text{SA}) - \delta u_r^{\text{GC}}. \quad (9.36)$$

We further define the eustatic change of the ocean surface height as

$$\delta H(\text{EU}) = \delta H(\text{CF}) - \delta D(\text{ST}), \quad (9.37)$$

where  $\delta D(\text{ST})$  is the steric contribution to the change of ocean depth. The eustatic change of the ocean depth is then

$$\delta D(\text{EU}) = \delta H(\text{EU}) - [\delta u_r(\text{PC}) + \delta u_r(\text{GA})] . \quad (9.38)$$

Note that the change of height  $\delta H(\text{EU})$  is defined to be with respect to the CF frame, and the change of depth  $\delta D(\text{EU})$  is with respect to the ocean bottom. The density of the total water mass change is

$$\delta s_w(\text{TT}) = \rho_w \delta D(\text{EU}) , \quad (9.39)$$

where the argument (TT) stands for total, as  $\delta D(\text{EU})$  includes both the contributions of the PMC and GIA. With

$$\delta D(\text{EU}) = \delta D(\text{PC}) + \delta D(\text{GA}) , \quad (9.40)$$

we can decompose  $\delta s_w(\text{TT})$  according to

$$\delta s_w(\text{TT}) = \delta s_w(\text{PC}) + \delta s_w(\text{GA}) ; \quad (9.41)$$

$$\delta s_w(\text{PC}) = \rho_w \delta D(\text{PC}) , \quad (9.42)$$

$$\delta s_w(\text{GA}) = \rho_w \delta D(\text{GA}) . \quad (9.43)$$

The total PMC is over the oceans is

$$\delta s(\text{PC}) = \delta s_a(\text{PC}) + \delta s_w(\text{PC}) , \quad (9.44)$$

where  $\delta s_a(\text{PC})$  is the atmospheric part.

Now the physical equations relating the data to the PMC and GIA is complete.

### 9.3.3 Principles of Some Applications

We use some examples to demonstrate how the data can be used to investigate the mass transport in the Earth system. Certainly, more types of data are used together, more geophysical processes can be investigated. The physical principles are emphasized assuming the errors in the data are of smaller order of magnitudes than the signals sought.

#### 9.3.3.1 Use of the GRACE Data Alone

We start with the simplest case of using the GRACE Level-2 data alone. Based on the physical equations provided in the previous subsection, no information on the degree 1 spherical harmonic coefficients is available in the monthly models of the gravitational potential. Therefore, the degree 1 term is usually ignored. We emphasize that it is physically wrong to use the degree 1 term of the gravitational potential computed from the monthly coordinates of the CM in the ITRF to investigate the PMC based on the elastic loading theory. For the accuracy/precision

required in the investigation of the PMC, the ITRF is an empirical reference frame whose physical meaning is not known, and hence, no degree 1 load Love numbers can be formulated for it.

We have used the argument (GR) to denote the GRACE result with all the de-aliasing models added back. Here we assume that the only de-aliasing model applied in the processing of GRACE data is an AOD model composed of an atmospheric (AD) and an oceanic de-aliasing (OD) model, which are denoted using the arguments (AD) and (OD), respectively. We further use the argument (G2), meaning GRACE Level-2, to denote the GRACE monthly gravitational model without the de-aliasing model, i.e., without adding the AOD model back. We have

$$\delta\Phi_n^k(\text{GR}) = \delta\Phi_n^k(\text{G2}) + \delta\Phi_n^k(\text{AD}) + \delta\Phi_n^k(\text{OD}). \quad (9.45)$$

By applying the elastic loading theory represented by (9.16), (9.17), and (9.19) to the (G2) portion, we obtain

$$\delta s(\text{G2}) = \frac{R\bar{\rho}}{3} \sum_{n=2}^{n_{\text{max}}^{\text{G2}}} \sum_{k=-n}^n \frac{2n+1}{1+k'_n} \left[ \delta\Phi_n^k(\text{G2}) - \delta\Phi_n^k(\text{GA}) \right] \bar{\mathcal{H}}_n^k(\theta, \lambda). \quad (9.46)$$

where the spherical harmonic series is truncated up to the maximum degree and order of the GRACE Level-2 product denoted as  $n_{\text{max}}^{\text{G2}}$ . Noting that the coefficient  $\delta\Phi_n^k(\text{G2})$  defined here includes  $\delta\Phi_n^k(\text{GA})$ , therefore, the latter is removed from the former in (9.46) based on a GIA model before applying the elastic loading theory. As implied by the notation,  $\delta s(\text{G2})$  is the (G2) portion of the PMC. The total PMC is decomposed according to

$$\delta s(\text{PC}) = \delta s(\text{G2}) + \delta s(\text{AD}) + \delta s(\text{OD}). \quad (9.47)$$

Over the land,  $\delta s(\text{G2})$  includes the changes of the glacial, hydrological, and deeper under ground water masses. Over the oceans,  $\delta s(\text{G2}) + \delta s(\text{OD})$  is the change of the ocean water mass. Both of them are corrupted by the error in the AD model. Since the GIA signal is a linear trend, the error of the GIA model affects only the estimate of the linear trend in the PMC.

There are two problems in the application of (9.46). First, if we do not know anything about the higher degree and order terms beyond  $n_{\text{max}}^{\text{G2}}$ , which are neglected, the result of  $\delta s(\text{G2})$  is usually smoothed to eliminate their effects. This is customarily done by performing a Gaussian smoothing with an angular smoothing radius of  $\pi/n_{\text{max}}^{\text{G2}}$ , e.g.,  $3^\circ$  for  $n_{\text{max}}^{\text{G2}} = 60$ . The computation consists of only including, in each term at the right-hand side of (9.46), the degree-only dependent factor  $\beta_n$  formulated in Sect. 4.6.3 and shown in Fig. 4.7. Smoothing averages the signal, causing leakages of signal from larger magnitude regions to adjacent lower magnitude regions. If the higher magnitude regions and lower magnitude regions can be separated, the leakage can be very much reduced by inverting the smoothed result (Guo et al., 2010). If the coefficients  $\delta\Phi_n^k(\text{G2}) - \delta\Phi_n^k(\text{GA})$  can be forward modeled based on the knowledge



of  $\delta s(G2)$ , the application of (9.46) is then for improving the prior knowledge of  $\delta s(G2)$  using the lower degree and order gravitational field. Hence, a smoothing is not really necessary (e.g., Chen et al., 2009).

The second problem is that the data of  $\delta\Phi_n^k(G2)$  include a correlation error, resulting in the presence of fictitious large south-north stripes in the result of  $\delta s(G2)$ . Although a Gaussian smoothing can significantly reduce the stripes, the application of a decorrelation filter may still be necessary before computing  $\delta s(G2)$  (e.g., Duan et al., 2009).

### 9.3.3.2 Use of the GRACE and Satellite Altimetry Data Together: Ice Sheets

Since we have two types of observations, we may, to the best, use them to study both the PMC and the GIA. The satellite altimetry data do include the degree 1 term, but no other relevant information is available for us to take further action. Therefore, the effect the degree 1 term in the satellite altimetry data has to be ignored.

Over an ice sheet, we suppose that the satellite altimetry data are reduced to the height change of a representative firm surface based on a firm compaction model. We use the argument (RA) to denote this surface, meaning reduced from the satellite altimetry data. The mass change of the ice sheet,  $\delta s(IS)$ , and the height change of this surface,  $\delta H(RA)$ , are related according to

$$\delta s(IS) = \rho_f [\delta H(RA) - \delta u_r(GA) - \delta u_r(PC)] , \quad (9.48)$$

where  $\rho_f$  is a known nominal firm density, and  $\delta u_r(GA) + \delta u_r(PC)$  the total uplift of the ice sheet bottom. Note that the loading effect of the PMC all over the Earth's surface contributes to  $\delta u_r(PC)$ . For consistency with  $\delta\Phi_n^k(G2)$ , the loading effect of the AOD model in the satellite altimetry data is removed beforehand. Therefore, within the context of the present problem, the term PMC will be used in a restricted sense; it is what we obtain from  $\delta\Phi_n^k(G2)$ , i.e., the AOD model is excluded. As  $\rho_f$  and  $\delta H(RA)$  rely on the firm compaction model adopted, significant uncertainties are expected in them, which would bring errors in the result.

We use the change of gravity at positions fixed in the ITRF right above the ice sheet such that it can be interpreted as

$$\delta g_m(INV) = 2\pi G\delta s(IS) + 0.15\delta u_r(GA) , \quad (9.49)$$

where  $2\pi G\delta s(IS)$  is the gravity of  $\delta s(IS)$  obtained according to (5.117), and  $0.15\delta u_r(GA)$  the mass contribution of the GIA to the gravity obtained according to (9.5). In this way,  $\delta g_m(INV)$  includes only the direct effect of  $\delta s(IS)$  and the GIA effect.

An approximation of the change of gravity  $\delta g_m(INV)$  can be obtained using the GRACE data  $\delta\Phi_n^k(G2)$ . Note that the term PMC here is what we obtain from  $\delta\Phi_n^k(G2)$ , i.e., the AOD model is excluded. This consists of first removing, from  $\delta\Phi_n^k(G2)$ , the indirect effect of the elastic loading of the PMC over all the Earth's surface and the direct effect of the PMC outside the region of interest, and then

using the result to compute the change of the gravity. A GIA model is require for this purpose, whose error affects only what are removed. Therefore, for the result of  $\delta s(\text{IS})$  and  $\delta u_r(\text{GA})$  to be meaningful, their contribution to the change of the gravity obtained should be reasonably larger than the errors in the computation of what are removed. The procedure is as follows:

1. It can be readily inferred that the indirect effect of elastic loading of the PMC over all the Earth's surface in  $\delta\Phi_n^k(\text{G2})$  is  $[k'_n/(1 + k'_n)][\delta\Phi_n^k(\text{G2}) - \delta\Phi_n^k(\text{GA})]$ . We therefore subtract it from  $\delta\Phi_n^k(\text{G2})$ .
2. First, compute  $\delta s(\text{G2})$  according to (9.46) over a global grid. Second, set the values at the grid points inside the region of interest to zero, and then perform a discrete spherical harmonic transform (DSHT) to obtain the changes of the spherical harmonic coefficients of the PMC outside the region of interest. Third, compute the corresponding changes of the spherical harmonic coefficients of the gravitational potential according to (9.19) by setting  $k'_n = 0$ , i.e., without the indirect effect. Finally, subtract the result from that of the step 1, and denote the result as  $\delta\Phi_n^k(\text{INV})$ , which is to be used to compute  $\delta g_m(\text{INV})$ .
3. The expression of the change of the gravitational potential used for the inversion outside the Earth is

$$\delta\phi(\text{INV}) = Rg_R \sum_{n=2}^{n_{\text{max}}^{\text{G2}}} \sum_{k=-n}^n \left(\frac{R}{r}\right)^{n+1} \delta\Phi_n^k(\text{INV}) \bar{\mathcal{H}}_n^k(\theta, \lambda). \quad (9.50)$$

The gravity  $\delta g_m(\text{INV})$  is defined positive downward. We thus have

$$\begin{aligned} \delta g_m(\text{INV}) &= - \left\{ \frac{\partial}{\partial r} [\delta\phi(\text{INV})] \right\}_{r=R} \\ &= g_R \sum_{n=2}^{n_{\text{max}}^{\text{G2}}} \sum_{k=-n}^n (n+1) \delta\Phi_n^k(\text{INV}) \bar{\mathcal{H}}_n^k(\theta, \lambda). \end{aligned} \quad (9.51)$$

We know  $\rho_f$ ,  $\delta H(\text{RA})$ , and  $\delta g_m(\text{INV})$  in (9.48) and (9.49), and we are determining  $\delta s(\text{IS})$  and  $\delta u_r(\text{GA})$ . By substituting (9.48) into (9.49), we obtain

$$\delta u_r(\text{GA}) = \frac{\delta g_m(\text{INV}) - 2\pi G\rho_f [\delta H(\text{RA}) - \delta u_r(\text{PC})]}{0.15 - 2\pi G\rho_f}. \quad (9.52)$$

We then solve (9.48) and this equation iteratively. The first step is to solve for  $\delta u_r(\text{GA})$  and  $\delta s(\text{IS})$  by dropping the term  $\delta u_r(\text{PC})$ , which is assumed to be the smallest. We then update the solution iteratively by computing  $\delta u_r(\text{PC})$  using the solution of  $\delta s(\text{IS})$ . This requires to (1) compute  $\delta s(\text{G2})$  according to (9.46) over a global grid, (2) replace the values at the grid points inside the region of interest by the result of  $\delta s(\text{IS})$ , and (3) using the result to compute  $\delta u_r(\text{PC})$  according to (9.18) and (9.20). The models adopted in the literature may be simpler, e.g., Riva

et al. (2009) dropped  $\delta u_r$ (PC) and used  $2\pi G\rho_c$  and  $2\pi G\delta s$ (G2) instead of 0.15 and  $\delta g_m$ (INV), respectively, where  $\rho_c$  is the density of the crust.

We did not mention the assimilation between the GRACE data and the satellite altimetry data. If the change of ice sheet elevation is extremely smooth, such that the signals in the GRACE-like and satellite altimetry-like data beyond the degree and order  $n_{\max}^{G2}$  are negligible, no data assimilation or smoothing is required. Otherwise, smoothing is required for both sets of data. The smoothing of GRACE data is straightforward in the spectral domain. However, the smoothing of the satellite altimetry data has to be performed in the space domain where the data are provided, and should be performed during the data preprocessing stage. The GRACE data are less affected by the correlation error at higher latitude, such as over the Antarctica and Greenland (e.g. Guo et al., 2010). Nevertheless, the application of a relatively weak decorrelation filter may still be required.

A remark is that the negative of the error in the AD model is included in the result of  $\delta s$ (IS), which is evident. However, the dominant error should be from the firm compaction model leading to the data of  $\rho_f$  and  $\delta H$ (RA). It can hardly be believed that the small GIA signal be reasonably beyond the error level in these data, except probably a few places where huge ice domes were present during the last glacial maximum and the present GIA signal is large. Even though attempts have been made to investigate both  $\delta s$ (IS) and  $\delta u_r$ (GA) by combining GRACE and satellite altimetry data (e.g., Riva et al., 2009), it may be more realistic to use the GRACE data to constrain  $\rho_f$  and  $\delta H$ (RA) of the firm compaction model. We mention that the GIA signal at the places of large enough ice domes during the last glacial maximum can be identified in the GRACE data (e.g., Guo et al., 2012).

### 9.3.3.3 Use of the GRACE and Satellite Altimetry Data Together: Oceans

The satellite altimetry data are available only over the oceans. Hence, they can only be combined with the GRACE data to determine the ocean water mass change  $\delta s_w$ (PC) as expressed in (9.42) and the GIA crustal uplift  $\delta u_r$ (GA) in the same region; an AD model has to be adopted for the atmospheric part of the PMC. However, the surface mass change around the coastlines derived from the change of the GRACE potential coefficients is corrupted by a leakage of signal between the ocean and the continent, which is caused by the limitation of the attainable  $n_{\max}^{G2}$  and the need of smoothing. This leakage, even it can be remarkably reduced (e.g. Guo et al., 2010), would still cause a large error that makes the problem invalid. Furthermore, the satellite altimetry data over coastal oceans are also corrupted by larger errors. Therefore, besides the regions where satellite altimetry data are not available, a region along the coastlines should also be excluded. In this regard, an OD model should still be used for the ocean regions excluded in the study. We use the change of the GRACE potential coefficients with the AOD model added back, i.e.,  $\delta\Phi_n^k$ (GR). The change of the surface mass density of the OD model,  $\delta s_w$ (OD), can be computed independently, and is left to the readers.

As shown in (9.44), the total PMC over the ocean,  $\delta s$ (PC), is composed of the ocean mass change  $\delta s_w$ (PC) and the atmospheric mass change  $\delta s_a$ (PC). In reality,

we cannot determine  $\delta s_w(\text{PC})$  using only the satellite altimetry data and the known corrections. What we can obtain from these data is the eustatic change of the ocean surface height,  $\delta H(\text{EU})$ , which is related to the eustatic ocean depth change  $\delta D(\text{EU})$  through (9.38), and then to the density of the total water mass change  $\delta s_w(\text{TT})$  through (9.39). Since no data set is available to decompose  $\delta s_w(\text{TT})$  into  $\delta s_w(\text{PC})$  and  $\delta s_w(\text{GA})$  as defined in (9.41)–(9.43), what we can actually study is to determine  $\delta s_w(\text{TT})$  and  $\delta u_r(\text{GA})$  using the data sets  $\delta H(\text{EU})$ ,  $\delta \Phi_n^k(\text{GR})$ , and  $\delta s_w(\text{OD})$ . In this way,  $\delta s_w(\text{GA})$  is included in  $\delta s_w(\text{TT})$ , and hence, (9.38) should be reinterpreted as, according to (9.40),

$$\delta D(\text{PC}) = \delta H(\text{EU}) - [\delta u_r(\text{PC}) + \delta D(\text{GA}) + \delta u_r(\text{GA})]. \quad (9.53)$$

We remind that  $\delta u_r(\text{PC})$  is the elastic loading effect of all the PMC, and  $\delta u_r(\text{GA})$  is the GIA crustal uplift. According to (9.26), we can write this formula as

$$\delta D(\text{PC}) = \delta H(\text{EU}) - \delta u_r(\text{PC}) - \delta N(\text{GA}). \quad (9.54)$$

Note that  $\delta s_w(\text{PC})$  is related to  $\delta D(\text{PC})$  according to (9.42), i.e.,  $\delta s_w(\text{PC}) = \rho_w \delta D(\text{PC})$ . For the same reason as in the case of ice sheet, the degree 1 effect in the satellite altimetry data is ignored.

We know that  $\delta D(\text{GA})$  and  $\delta u_r(\text{GA})$  are on the same order of magnitude with opposite signs, and  $\delta N(\text{GA})$  is an order of magnitude smaller than them (e.g., Guo et al., 2012). Therefore, we make the approximation  $\delta N(\text{GA}) = 0$  in (9.54) to obtain

$$\delta D(\text{PC}) = \delta H(\text{EU}) - \delta u_r(\text{PC}). \quad (9.55)$$

The error due to neglecting  $\delta N(\text{GA})$  is to be quantified later.

To determine  $\delta s_w(\text{PC})$  and  $\delta u_r(\text{GA})$  in the region studied, we use the changes of the gravitational potential coefficients,  $\delta \Phi_n^k(\text{INV})$ , which include only the direct effect of  $\delta s_w(\text{PC})$  and the GIA effect. This is equivalent to remove all effects of the PMC from  $\delta \Phi_n^k(\text{GR})$  except the direct effect of  $\delta s_w(\text{PC})$  in the region studied. Noting that  $\delta s_w(\text{PC})$  is the sum of  $\delta s_w(\text{OD})$  and  $\delta s(\text{G2})$  with the errors in the AD model ignored, we express  $\delta \Phi_n^k(\text{INV})$  as

$$\delta \Phi_n^k(\text{INV}) = \delta \Phi_n^k(\text{GR}) - \delta \Phi_n^k(\text{AD}) - \delta \Phi_n^k(\text{OD}_e) - \delta \Phi_n^k(\text{IND}_w^{\text{PC}}). \quad (9.56)$$

The last two terms at the right-hand side are newly defined. The third term includes the direct and indirect effects of  $\delta s_w(\text{OD})$  over the ocean regions excluded in the study. The last term is the indirect effect of  $\delta s_w(\text{PC})$  within the region studied, which can be computed according to (9.19) by setting  $1 + k'_n$  to  $k'_n$ . In accordance with the satellite altimetry data provided as point values, we also use the point values of the gravity. Similar to (9.51), the gravity to be use in the inversion is

$$\delta g_m(\text{INV}) = g_R \sum_{n=2}^{n_{\text{G2}}^{\text{max}}} \sum_{k=-n}^n (n+1) \delta \Phi_n^k(\text{INV}) \bar{\mathcal{H}}_n^k(\theta, \lambda). \quad (9.57)$$

It is related to  $\delta s_w(\text{PC})$  and  $\delta u_r(\text{GA})$  according to

$$\delta g_m(\text{INV}) = 2\pi G \delta s_w(\text{PC}) + 0.15 \delta u_r(\text{GA}) . \quad (9.58)$$

We can now quantify the error caused by neglecting  $\delta N(\text{GA})$  in (9.54). It induces an error of  $\delta N(\text{GA})$  in  $\delta D(\text{PC})$ , or an error of  $\rho_w \delta N(\text{GA})$  in  $\delta s_w(\text{PC})$ , which corresponds to an error of  $2\pi G \rho_w \delta N(\text{GA})$  in the first term at the right-hand side of (9.58). First,  $\delta N(\text{GA})$  is an order of magnitude smaller than  $\delta u_r(\text{GA})$ . Second,  $2\pi G \rho_w = 0.043 = 0.29 \times 0.15$ . Therefore, the error in the first term is far below the magnitude of the second term at the right-hand side of (9.58). However, this approximation does not mean that the term neglected is below the error level of the satellite altimetry data, e.g., in the study of mean global mean sea level change, the GIA effect of  $R\delta\Phi_0^0 = -0.3 \text{ mm} \cdot \text{yr}^{-1}$  must be considered (Peltier, 2009).

We still need the formula for computing  $\delta u_r(\text{PC})$ , which includes the effects of the PMC all over the Earth. The PMC obtained using  $\delta\Phi_n^k(\text{GR})$  is

$$\delta s(\text{GR}) = \frac{R\bar{\rho}}{3} \sum_{n=2}^{n_{\text{max}}^{\text{G2}}} \sum_{k=-n}^n \frac{2n+1}{1+k'_n} \left[ \delta\Phi_n^k(\text{GR}) - \delta\Phi_n^k(\text{GA}) \right] , \quad (9.59)$$

where  $\delta\Phi_n^k(\text{GA})$  is from a GIA model. Within the region studied, we replace  $\delta s(\text{GR})$  by the more accurate estimate  $\delta s_w(\text{PC}) + \delta s_a(\text{AD})$ , which is not affected by the GIA model. We then compute the loading effect  $\delta u_r(\text{PC})$  according to (9.20).

We have all the relations require for determining  $\delta s_w(\text{PC})$  and  $\delta u_r(\text{GA})$  in the region studied. We see that once we know  $\delta s_w(\text{PC})$ , we can compute all the quantities in (9.56)–(9.59). Therefore, the equations (9.55) and (9.58) can be solved iteratively. The initial value of  $\delta s_w(\text{PC})$  can be obtained in two ways: one way is to set  $\delta u_r(\text{PC}) = 0$  in (9.55) with (9.42) taken into account, i.e.,  $\delta s_w(\text{PC}) = \rho_w \delta D(\text{PC})$ ; the other way is by taking away  $\delta s_a(\text{AD})$  from  $\delta s(\text{GR})$  computed according to (9.59). We note one more time that the negative of the error in the AD model is included in the result of  $\delta s_w(\text{PC})$ , which is evident.

If the GRACE and satellite altimetry data are accurate enough, we can use the result of  $\delta u_r(\text{GA})$  as data in GIA modeling. The GIA model in turn provides  $\delta N(\text{GA})$  in (9.54). Therefore, an iterative solution procedure including GIA modeling can be envisaged with the approximate formula (9.55) replaced by the accurate formula (9.54), together with other data customarily used in GIA modeling. This is an expectation purely based on theory. Its practical realisability depends certainly on the accuracy/precision of all the data involved.

### 9.3.3.4 Use of the GRACE, GNSS and Geocenter Data Together

Lastly, we demonstrate a use of the geocenter data. For the problem to be physically meaningful with minimal complexity, we combine the geocenter data with only the GRACE and GNSS crustal displacement data to solve for both the PMC and GIA

globally. Problems of this kind using more comprehensively the data available are referred to Wu et al. (2010, 2012).

The GNSS crustal displacement data alone are also used for inverting regional surface mass changes, where the elastic loading theory is applied straightforwardly (e.g., Fu et al., 2015). This kind of problem is not to be further addressed here.

The GRACE data are global, and are smoothed due to the truncation to  $n_{\max}^{G2}$  and a further Gaussian smoothing, while the GNSS data are point values. The GRACE data are provided as monthly average, while the GNSS data are practically continuous in time. Therefore, there is a need of data assimilation. The feasible approach is to preprocess the GNSS data, which consists of removing the loading effect of the local short spacial wavelength surface mass changes, and then average over time. This certainly requires the knowledge of the surface mass changes beyond the resolution of the GRACE data. Models as summarized in Sect. 9.2, if available, can be used. It is beyond the scope of this book to explore this topic further.

The GRACE data of the changes of the spherical harmonic coefficients of the gravitational potential include both the PMC and GIA contributions,

$$\delta\Phi_n^k(\text{GR}) = \delta\Phi_n^k(\text{PC}) + \delta\Phi_n^k(\text{GA}), \quad n > 1. \quad (9.60)$$

We express the vertical crustal displacement as, according to (9.32)–(9.35),

$$\delta u_r(\text{SD}) = \sum_{k=-1}^1 \delta u_1^k(\text{SD}) \bar{\mathcal{H}}_1^k(\theta, \lambda) + \sum_{n=2}^{n_{\max}^{G2}} \sum_{k=-n}^n \left[ \delta u_n^k(\text{PC}) + \delta u_n^k(\text{GA}) \right] \bar{\mathcal{H}}_n^k(\theta, \lambda), \quad (9.61)$$

where the first term at the right-hand side is the degree 1 term of  $\delta u_r(\text{SD})$  in the ITRF. According to (9.3),  $\delta u_n^k(\text{GA})$  and  $\delta\Phi_n^k(\text{GA})$  are related according to

$$\delta u_n^k(\text{GA}) = \alpha_n \delta\Phi_n^k(\text{GA}), \quad n > 1. \quad (9.62)$$

We have written  $(2n+1)/(2R)$  as  $\alpha_n$  here, so that other estimates can be used, such as  $(1.1677n - 0.5233)/R$ , as mentioned below (9.4). In fact, the original model output of the ratios  $\alpha_n$ , which are proven to hold for a variety of Earth rheology profiles and ice histories, could be used instead of the analytic expression obtained by fitting them (Purcell et al., 2011). In addition,  $\delta\Phi_n^k(\text{PC})$  and  $\delta u_n^k(\text{PC})$  are related to the spherical harmonic coefficients of the PMC,  $\delta s_n^k(\text{PC})$ . According to (9.16)–(9.20), we have, for  $n > 1$ ,

$$\delta\Phi_n^k(\text{PC}) = \frac{3}{R\bar{\rho}} \frac{1+k'_n}{2n+1} \delta s_n^k(\text{PC}), \quad \delta u_n^k(\text{PC}) = \frac{3}{\bar{\rho}} \frac{h'_n}{2n+1} \delta s_n^k(\text{PC}). \quad (9.63)$$

We can finally write (9.60) and (9.61) as a system of linear equation,

$$\left\{ \begin{array}{l} \delta\Phi_n^k(\text{GR}) = \frac{3}{R\bar{\rho}} \frac{1+k'_n}{2n+1} \delta s_n^k(\text{PC}) + \frac{1}{\alpha_n} \delta u_n^k(\text{GA}), \quad n > 1; \\ \delta u_r(\text{SD}) = \sum_{k=-1}^1 \delta u_1^k(\text{SD}) \bar{\mathcal{H}}_1^k(\theta, \lambda) \\ \quad + \sum_{n=2}^{n_{\text{max}}^{\text{G2}}} \sum_{k=-n}^n \left[ \frac{3}{\bar{\rho}} \frac{h'_n}{2n+1} \delta s_n^k(\text{PC}) + \delta u_n^k(\text{GA}) \right] \bar{\mathcal{H}}_n^k(\theta, \lambda). \end{array} \right. \quad (9.64)$$

The problem is thus to solve for  $\delta s_n^k(\text{PC})$ ,  $\delta u_n^k(\text{GA})$ , and  $\delta u_1^k(\text{SD})$  using data of  $\delta\Phi_n^k(\text{GR})$  and  $\delta u_1^k(\text{SD})$ . Therefore, there is a need of enough data of  $\delta u_r(\text{SD})$ , so that there are more data than unknowns. In this way, an error term should be included in each of the equations (9.64), and their best fit solution can be obtained using the least squares method.

What we obtain as the solution of (9.64) are the spherical harmonic coefficients  $\delta s_n^k(\text{PC})$  and  $\delta u_n^k(\text{GA})$  beyond degree 1 and the degree 1 spherical harmonic coefficients  $\delta u_1^k(\text{SD})$ . The coefficients  $\delta s_n^k(\text{PC})$  can be used to compute all the three components of the surface displacement beyond degree 1 due to the elastic loading of the PMC,  $\delta u_n^k(\text{GA})$  provide the vertical displacement beyond degree 1 caused by GIA, which can be used as data in GIA modeling, and  $\delta u_1^k(\text{SD})$  provide the total degree 1 vertical displacement, which can be used to infer the changes of the coordinates of the CH in the ITRF. The horizontal components of the surface displacement due to GIA beyond degree 1 can be obtained by subtracting the contribution of the elastic loading of the PMC from the GNSS data of horizontal surface displacement, which can also be used as data in GIA modeling. Note that we do not have enough information to infer the changes of the coordinates of the CF in the ITRF.

In the following, we discuss the use of the geocenter data, which provide the change of position of the CM in the ITRF.

We first address the problem in the CH frame. The changes of the coordinates of the CH in the ITRF can be inferred from the coefficients  $\delta u_1^k(\text{SD})$ . The total degree 1 vertical displacement in the ITRF is

$$\begin{aligned} \delta u_{r1}(\text{SD}) &= u_1^1(\text{SD}) \bar{\mathcal{H}}_1^1(\theta, \lambda) + \delta u_1^{-1}(\text{SD}) \bar{\mathcal{H}}_1^{-1}(\theta, \lambda) + \delta u_1^0(\text{SD}) \bar{\mathcal{H}}_1^0(\theta, \lambda) \\ &= \sqrt{3} [\delta u_1^1(\text{SD}) \sin \theta \cos \lambda + \delta u_1^{-1}(\text{SD}) \sin \theta \sin \lambda + \delta u_1^0(\text{SD}) \cos \theta]. \end{aligned} \quad (9.65)$$

By definition, the changes of the coordinates of the CH in the ITRF are

$$\begin{cases} \delta X_{\text{ITRF}}^{\text{CH}} = \frac{1}{4\pi} \int_{\omega} \hat{e}_1 \cdot [\delta u_{r1}(\text{SD}) \hat{e}_r] d\omega = \frac{1}{4\pi} \int_{\omega} \delta u_{r1}(\text{SD}) \sin \theta \cos \lambda d\omega, \\ \delta Y_{\text{ITRF}}^{\text{CH}} = \frac{1}{4\pi} \int_{\omega} \hat{e}_2 \cdot [\delta u_{r1}(\text{SD}) \hat{e}_r] d\omega = \frac{1}{4\pi} \int_{\omega} \delta u_{r1}(\text{SD}) \sin \theta \sin \lambda d\omega, \\ \delta Z_{\text{ITRF}}^{\text{CH}} = \frac{1}{4\pi} \int_{\omega} \hat{e}_3 \cdot [\delta u_{r1}(\text{SD}) \hat{e}_r] d\omega = \frac{1}{4\pi} \int_{\omega} \delta u_{r1}(\text{SD}) \cos \theta d\omega. \end{cases} \quad (9.66)$$

These formulae are analogous to the case of GIA indicated in the last paragraph of the Appendix of this chapter. Substitute  $\delta u_{r1}(\text{SD})$  by its expression given by (9.65), and then evaluate the integrals of the sine and cosine functions analytically. We obtain

$$\left\{ \delta X_{\text{ITRF}}^{\text{CH}}, \delta Y_{\text{ITRF}}^{\text{CH}}, \delta Z_{\text{ITRF}}^{\text{CH}} \right\} = \frac{1}{\sqrt{3}} \left\{ \delta u_1^1(\text{SD}), \delta u_1^{-1}(\text{SD}), \delta u_1^0(\text{SD}) \right\}. \quad (9.67)$$

This formula can be understood straightforwardly by comparing it with (9.65) and the first formula of (9.76) (where  $\delta u_r^{\text{FO}}$ ,  $\delta X_{\text{MF}}^{\text{CF}}$ ,  $\delta Y_{\text{MF}}^{\text{CF}}$ , and  $\delta Z_{\text{MF}}^{\text{CF}}$  correspond to  $\delta u_{r1}(\text{SD})$ ,  $\delta X_{\text{ITRF}}^{\text{CH}}$ ,  $\delta Y_{\text{ITRF}}^{\text{CH}}$ , and  $\delta Z_{\text{ITRF}}^{\text{CH}}$ , respectively). The changes of the coordinates of the CM in the CH frame can be computed according to

$$\begin{Bmatrix} \delta X_{\text{CH}}^{\text{CM}} \\ \delta Y_{\text{CH}}^{\text{CM}} \\ \delta Z_{\text{CH}}^{\text{CM}} \end{Bmatrix} = \begin{Bmatrix} \delta X_{\text{ITRF}}^{\text{CM}}(\text{GC}) \\ \delta Y_{\text{ITRF}}^{\text{CM}}(\text{GC}) \\ \delta Z_{\text{ITRF}}^{\text{CM}}(\text{GC}) \end{Bmatrix} - \begin{Bmatrix} \delta X_{\text{ITRF}}^{\text{CH}} \\ \delta Y_{\text{ITRF}}^{\text{CH}} \\ \delta Z_{\text{ITRF}}^{\text{CH}} \end{Bmatrix}, \quad (9.68)$$

where the argument (GC) indicates the geocenter data. These changes of the coordinates are composed of the contribution of the elastic loading of the PMC and that of the GIA,

$$\begin{Bmatrix} \delta X_{\text{CH}}^{\text{CM}} \\ \delta Y_{\text{CH}}^{\text{CM}} \\ \delta Z_{\text{CH}}^{\text{CM}} \end{Bmatrix} = \begin{Bmatrix} \delta X_{\text{CH}}^{\text{CM}}(\text{PC}) \\ \delta Y_{\text{CH}}^{\text{CM}}(\text{PC}) \\ \delta Z_{\text{CH}}^{\text{CM}}(\text{PC}) \end{Bmatrix} + \begin{Bmatrix} \delta X_{\text{CH}}^{\text{CM}}(\text{GA}) \\ \delta Y_{\text{CH}}^{\text{CM}}(\text{GA}) \\ \delta Z_{\text{CH}}^{\text{CM}}(\text{GA}) \end{Bmatrix}. \quad (9.69)$$

We have no data to decompose the left-hand side into the two terms at the right-hand side. Therefore, the second term at the right-hand side has to be taken from a GIA model. Its computation using the surface displacement of the GIA model is provided in the Appendix of this chapter. The rest is to use the result of the first term at the right-hand side to study the degree 1 PMC, which is straightforward, and is left to the readers.

We now address the problem in the CF frame. We need to determine the changes of the coordinates of the CF in the ITRF. As already indicated, we do not have



enough information to do so. Therefore, further assumptions or approximations are required. An obvious approach is to remove all the degree 2 and above terms in the surface displacement based on all the useful data and models available, and assume the residuals to be dominated by the degree 1 term. For our present problem, all the three components of the surface displacement of degree 2 and above due to both the elastic loading of the PMC and the GIA are estimated. We use the superscript RD to indicate the residuals of the surface displacement obtained after removing the degree 2 and above terms. The relations between these residuals and the changes of the coordinates of the CF in the ITRF are, analogous to (9.76),

$$\begin{cases} \delta u_r^{\text{RD}} = \delta X_{\text{ITRF}}^{\text{CF}} \sin \theta \cos \lambda + \delta Y_{\text{ITRF}}^{\text{CF}} \sin \theta \sin \lambda + \delta Z_{\text{ITRF}}^{\text{CF}} \cos \theta + e_r, \\ \delta u_\theta^{\text{RD}} = \delta X_{\text{ITRF}}^{\text{CF}} \cos \theta \cos \lambda + \delta Y_{\text{ITRF}}^{\text{CF}} \cos \theta \sin \lambda - \delta Z_{\text{ITRF}}^{\text{CF}} \sin \theta + e_\theta, \\ \delta u_\lambda^{\text{RD}} = -\delta X_{\text{ITRF}}^{\text{CF}} \sin \lambda + \delta Y_{\text{ITRF}}^{\text{CF}} \cos \lambda + e_\lambda, \end{cases} \quad (9.70)$$

where  $e_r$ ,  $e_\theta$ , and  $e_\lambda$  are the errors in the respective residuals, including the errors incurred in the removal of the degree 2 and above terms. We have one set of these equations for each station. Therefore, the changes of the coordinates can be computed by a least squares fit. An easier approach is to convert the residual surface displacement into the rectangular coordinate system. We can then write (9.70) as

$$\delta x_1^{\text{RD}} = \delta X_{\text{ITRF}}^{\text{CF}} + e_1, \quad \delta x_2^{\text{RD}} = \delta Y_{\text{ITRF}}^{\text{CF}} + e_2, \quad \delta x_3^{\text{RD}} = \delta Z_{\text{ITRF}}^{\text{CF}} + e_3, \quad (9.71)$$

where  $e_1$ ,  $e_2$ , and  $e_3$  are the errors in the respective residuals in the rectangular coordinate system. Evidently, the result of the least square fit is that  $\delta X_{\text{ITRF}}^{\text{CF}}$ ,  $\delta Y_{\text{ITRF}}^{\text{CF}}$ , and  $\delta Z_{\text{ITRF}}^{\text{CF}}$  are the averages of  $\delta x_1^{\text{RD}}$ ,  $\delta x_2^{\text{RD}}$  and  $\delta x_3^{\text{RD}}$ , respectively, of all the stations. The CF thus approximated is also referred to as center of network (e.g., Wu et al., 2010). The rest is similar to the case of the CH frame. We do not have the data to separate the contribution of the elastic loading of the PMC and that of the GIA. Therefore, we have to adopt a model for the contribution of the GIA. After taking away the contribution of the GIA, we can use the rest to investigate the degree 1 PMC. The formulation is left to the readers.

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## Appendix: Frame Transformation of a GIA Model

A GIA model, whatever the reference system, or model frame, it is based on, can always be reduced to the CF, CH, CL, CM, or CE frame. Our concern here is the CF and CH frames. We first formulate the problem for the CF frame. The small difference between the CH and CF frames will be mentioned afterward.

We use the superscript MF or the argument (MF) to denote quantities in the model frame, and implicitly understand the argument (GA) as based on the CF

frame. The coordinates of the CF in the MF are, by definition,

$$\begin{cases} \delta X_{\text{MF}}^{\text{CF}} = \frac{1}{4\pi} \int_{\omega} \hat{e}_1 \cdot [\delta u_r^{\text{MF}} \hat{e}_r + \delta u_{\theta}^{\text{MF}} \hat{e}_{\theta} + \delta u_{\lambda}^{\text{MF}} \hat{e}_{\lambda}] d\omega, \\ \delta Y_{\text{MF}}^{\text{CF}} = \frac{1}{4\pi} \int_{\omega} \hat{e}_2 \cdot [\delta u_r^{\text{MF}} \hat{e}_r + \delta u_{\theta}^{\text{MF}} \hat{e}_{\theta} + \delta u_{\lambda}^{\text{MF}} \hat{e}_{\lambda}] d\omega, \\ \delta Z_{\text{MF}}^{\text{CF}} = \frac{1}{4\pi} \int_{\omega} \hat{e}_3 \cdot [\delta u_r^{\text{MF}} \hat{e}_r + \delta u_{\theta}^{\text{MF}} \hat{e}_{\theta} + \delta u_{\lambda}^{\text{MF}} \hat{e}_{\lambda}] d\omega. \end{cases} \quad (9.72)$$

According to (2.335), we can write these as

$$\begin{cases} \delta X_{\text{MF}}^{\text{CF}} = \frac{1}{4\pi} \int_{\omega} [\delta u_r^{\text{MF}} \sin \theta \cos \lambda + \delta u_{\theta}^{\text{MF}} \cos \theta \cos \lambda - \delta u_{\lambda}^{\text{MF}} \sin \lambda] d\omega, \\ \delta Y_{\text{MF}}^{\text{CF}} = \frac{1}{4\pi} \int_{\omega} [\delta u_r^{\text{MF}} \sin \theta \sin \lambda + \delta u_{\theta}^{\text{MF}} \cos \theta \sin \lambda + \delta u_{\lambda}^{\text{MF}} \cos \lambda] d\omega, \\ \delta Z_{\text{MF}}^{\text{CF}} = \frac{1}{4\pi} \int_{\omega} [\delta u_r^{\text{MF}} \cos \theta - \delta u_{\theta}^{\text{MF}} \sin \theta] d\omega. \end{cases} \quad (9.73)$$

The integrals can be evaluated numerically using the model values of  $\delta u_r^{\text{MF}}$ ,  $\delta u_{\theta}^{\text{MF}}$ , and  $\delta u_{\lambda}^{\text{MF}}$  usually provided over a regular  $\theta - \lambda$  grid.

The components of the surface displacement in the CF frame are

$$\begin{Bmatrix} \delta u_r(\text{GA}) \\ \delta u_{\theta}(\text{GA}) \\ \delta u_{\lambda}(\text{GA}) \end{Bmatrix} = \begin{Bmatrix} \delta u_r^{\text{MF}} \\ \delta u_{\theta}^{\text{MF}} \\ \delta u_{\lambda}^{\text{MF}} \end{Bmatrix} - \begin{Bmatrix} \delta u_r^{\text{FO}} \\ \delta u_{\theta}^{\text{FO}} \\ \delta u_{\lambda}^{\text{FO}} \end{Bmatrix}, \quad (9.74)$$

where the superscript FO in the second term at the right-hand side stands for frame offset, indicating that this term is the surface displacement corresponding to the offset of the CF with respect to the MF. The expression of the frame offset term is similar to (9.33) and (9.34). We have

$$\begin{cases} \delta u_r^{\text{FO}} = \hat{e}_r \cdot [\delta X_{\text{MF}}^{\text{CF}} \hat{e}_1 + \delta Y_{\text{MF}}^{\text{CF}} \hat{e}_2 + \delta Z_{\text{MF}}^{\text{CF}} \hat{e}_3], \\ \delta u_{\theta}^{\text{FO}} = \hat{e}_{\theta} \cdot [\delta X_{\text{MF}}^{\text{CF}} \hat{e}_1 + \delta Y_{\text{MF}}^{\text{CF}} \hat{e}_2 + \delta Z_{\text{MF}}^{\text{CF}} \hat{e}_3], \\ \delta u_{\lambda}^{\text{FO}} = \hat{e}_{\lambda} \cdot [\delta X_{\text{MF}}^{\text{CF}} \hat{e}_1 + \delta Y_{\text{MF}}^{\text{CF}} \hat{e}_2 + \delta Z_{\text{MF}}^{\text{CF}} \hat{e}_3]; \end{cases} \quad (9.75)$$

$$\begin{cases} \delta u_r^{\text{FO}} = \delta X_{\text{MF}}^{\text{CF}} \sin \theta \cos \lambda + \delta Y_{\text{MF}}^{\text{CF}} \sin \theta \sin \lambda + \delta Z_{\text{MF}}^{\text{CF}} \cos \theta, \\ \delta u_{\theta}^{\text{FO}} = \delta X_{\text{MF}}^{\text{CF}} \cos \theta \cos \lambda + \delta Y_{\text{MF}}^{\text{CF}} \cos \theta \sin \lambda - \delta Z_{\text{MF}}^{\text{CF}} \sin \theta, \\ \delta u_{\lambda}^{\text{FO}} = -\delta X_{\text{MF}}^{\text{CF}} \sin \lambda + \delta Y_{\text{MF}}^{\text{CF}} \cos \lambda. \end{cases} \quad (9.76)$$

These represent a degree 1 spheroidal term, which and can be written as

$$\begin{cases} \delta u_r^{\text{FO}} = \frac{1}{\sqrt{3}} \left[ \delta X_{\text{MF}}^{\text{CF}} \bar{\mathcal{H}}_1^1(\theta, \lambda) + \delta Y_{\text{MF}}^{\text{CF}} \bar{\mathcal{H}}_1^{-1}(\theta, \lambda) + \delta Z_{\text{MF}}^{\text{CF}} \bar{\mathcal{H}}_1^0(\theta, \lambda) \right], \\ \delta u_\theta^{\text{FO}} = \frac{1}{\sqrt{3}} \left[ \delta X_{\text{MF}}^{\text{CF}} \frac{\partial \bar{\mathcal{H}}_1^1(\theta, \lambda)}{\partial \theta} + \delta Y_{\text{MF}}^{\text{CF}} \frac{\partial \bar{\mathcal{H}}_1^{-1}(\theta, \lambda)}{\partial \theta} + \delta Z_{\text{MF}}^{\text{CF}} \frac{\partial \bar{\mathcal{H}}_1^0(\theta, \lambda)}{\partial \theta} \right], \\ \delta u_\lambda^{\text{FO}} = \frac{1}{\sqrt{3}} \left[ \delta X_{\text{MF}}^{\text{CF}} \frac{\partial \bar{\mathcal{H}}_1^1(\theta, \lambda)}{\sin \theta \partial \lambda} + \delta Y_{\text{MF}}^{\text{CF}} \frac{\partial \bar{\mathcal{H}}_1^{-1}(\theta, \lambda)}{\sin \theta \partial \lambda} \right], \end{cases} \quad (9.77)$$

The equivalence between (9.76) and (9.77) can be easily verified using the expression of  $\bar{\mathcal{H}}_1^k(\theta, \lambda)$ .

There are two ways to convert the geoidal height from the MF to the CF frame:

$$\delta N(\text{GA}) = \delta N^{\text{MF}} - \delta u_r^{\text{FO}}; \quad (9.78)$$

$$\begin{Bmatrix} \delta \Phi_1^1(\text{GA}) \\ \delta \Phi_1^{-1}(\text{GA}) \\ \delta \Phi_1^0(\text{GA}) \end{Bmatrix} = \begin{Bmatrix} \delta \Phi_1^1(\text{MF}) \\ \delta \Phi_1^{-1}(\text{MF}) \\ \delta \Phi_1^0(\text{MF}) \end{Bmatrix} - \frac{1}{\sqrt{3}R} \begin{Bmatrix} \delta X_{\text{MF}}^{\text{CF}} \\ \delta Y_{\text{MF}}^{\text{CF}} \\ \delta Z_{\text{MF}}^{\text{CF}} \end{Bmatrix}. \quad (9.79)$$

It is left to the readers to verify their equivalence based on (9.25).

For the CH frame, we only need to change CF to CH, and keep only the first term in the brackets in the integrands of (9.72) and (9.73).

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