



ECONOMICS AND MANAGEMENT SCIENCE

Macroeconomics

Macroeconomics 1

*Heterogeneity, Dynamics
and Welfare*

**Coordinated by
Rodolphe Dos Santos Ferreira
Thomas Seegmuller**

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SCIENCES

Economics and Management Science,
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Macroeconomics, Subject Heads –
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First published 2026 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

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Library of Congress Control Number: 2026943383

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-78945-248-8

ERC code:

SH1 Individuals, Markets and Organisations

SH1_1 Macroeconomics; monetary economics; economic growth, labour economics

The manufacturer's authorized representative according to the EU General Product Safety Regulation is Wiley-VCH GmbH, Boschstr. 12, 69469 Weinheim, Germany, e-mail: Product_Safety@wiley.com.

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Introduction

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Macroeconomics is about aggregates. Aggregates must be constructed so as not to sacrifice relevant information relevant to the specific problems being addressed. Although it is often useful to know the size of the cake, it may sometimes be more enlightening to know how it is partitioned. However, macroeconomic modelling has long bypassed the aggregation problem by resorting to the devices of the representative agent and of the one-sector production system, or by contracting time to a single point. In other words, it has long portrayed how Robinson Crusoe manages his coconut endowment, on a single day between the shipwreck and Friday's arrival.

Interestingly, these shortcuts have not been largely exploited at the beginning of macroeconomic theorising, when economists were not yet trying to keep everything within an all-encompassing model. Keynes, for instance, took into account interacting agents and diversified goods in the loose model sketched in the *General Theory*. Also, these shortcuts, omnipresent for several decades, are no longer found in the frontiers of contemporary macroeconomics.

Introducing multiple identical agents or identical sectors does not change much with respect to the representative agent and representative industry plain assumptions. What does change is taking agent and industry diversity into account. That is why it is natural to start with heterogeneity. The first three chapters of this volume deal with heterogeneity: (1) of agents, more specifically of households, as savers and spenders who differ in their time preferences over consumption and in their

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borrowing constraints; (2) of production sectors, which differ in their position in the production process (production of final, consumption, goods versus production of intermediate, investment, goods) and in their technological characteristics, namely their capital intensities; (3) of agents' information, with focus on the ensuing problems of coordination between differently informed and strategically interacting agents.

The other, already mentioned, major shortcut in macroeconomic theory is the contraction of time into a single point, resulting in static equilibrium analysis (which completely ignores the time dimension) or at best in quasi-static intertemporal equilibrium analysis (where time does not really go by). Present actions and plans for the future are then viewed as depending upon rational expectations of coming events, but are supposed to be all decided and coordinated today by futures and contingent markets. Genuine dynamics are however indispensable for comprehending fluctuations and growth, and are of course an essential ingredient of modern macroeconomic theory. Chapter 4 is a technical introduction to the theory of dynamical systems, in discrete and continuous time, providing the required tools for the analysis of business cycles and economic growth.

The switch from intertemporal equilibrium analysis, with all markets opening once and for all on the first date, to genuine dynamic analysis of sequence economies, where markets reopen over time and decisions are made on each date, has at first preserved the rational expectations hypothesis. According to this hypothesis, the agents have from the beginning a full understanding of their (possibly stochastic) environment, materialized in a well-specified model of the economy in which they live. A more realistic approach allows for misspecified subjective models, on the basis of which the agents form the best predictions they can, and which can be improved by adaptive learning. Chapter 5 overviews this approach of adaptive learning under restricted perceptions, while notwithstanding preserving the equilibrium condition of model consistency that is the hallmark of rational expectations models.

Macroeconomic theory also has a normative side: alternative attainable states must be evaluated in order to select the appropriate policy most likely to achieve the best outcome. When a representative consumer is available, this evaluation can readily be made by referring to the consumer's utility. However, once we introduce agent multiplicity and heterogeneity, the aggregation problem arises again. The simple efficiency criterion, which enables the selection of a set of states that satisfy the convergent interests of all individuals, so that no alternative attainable state would be unanimously preferred, is too weak. Conflicting interests must also be considered, bringing equity considerations into play. Chapter 6 discusses how social welfare criteria may come to rescue in this context, focusing on inequality measurement and distributional analysis.

Chapter 7, which concludes this volume, maintains the policy motivation while shifting to the positive side, by considering the implementation of social welfare options through electoral processes and, more generally, by inserting them into a political context. The chapter accordingly develops a political-economic approach to business cycle stabilization, distributional conflicts during the growth process, and the role and design of institutions.

1

Household Heterogeneity in Macrodynamics

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Heterogeneous household models in macroeconomic theory display many distinctive features compared to representative agent counterparts. Macro theory policy models may posit two classes of households, identified as savers and spenders. This essay presents an overview of one particular micro theoretic foundation for two class macroeconomic models based on Ramsey equilibrium theory, where households differ in their time preference over consumption, and they face borrowing constraints. Ramsey's "conjecture" where capital ownership is concentrated in the most patient household is explored in depth in this chapter. This work is grounded in his seminal 1928 paper on how much of its income should a nation save.

1.1. Introduction: saver–spender economies

Macrodynamic models based on representative agent theories are inadequate when differences in agents' objectives and constraints may matter for consumption-saving decisions and characterizing policy responses. Keynes (2015) recognized two types of economic agents in his writings during the Great Depression. For example, his article "The Means to Prosperity," published in 1933, proposed a calculation of the proportion of increased government expenditure designed to increase the national income. He argues that about 70% will be spent on consumption and the remainder on savings and investment. He makes a distinction between the wage-earning classes (and those on the dole) who save little, if any, of their increased income, and the

The author would like to thank Thomas Seegmuller for his helpful comments.

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remaining agents who save and invest. That is, Keynes implies there are roughly two classes of economic agents, which we might label savers and spenders following Mankiw's (2000) more contemporary savers–spenders theory of fiscal policy. The critical observation is that a representative agent model cannot, by its very construction of an aggregate consumer-investor, model the different behavioral response to a change in fiscal policy. Indeed, contemporary macrodynamic researchers are critical of the representative-based dynamic optimizing models that seem to underperform compared to heterogeneous agent models.

The important problem is to find a microeconomic theory foundation for savers–spenders theory. I propose the Ramsey equilibrium model with infinitely lived agents possessing different discount factors for future utilities provides such a theoretical foundation. In a steady-state equilibrium, the most impatient agent owns all the capital and has a larger consumption than the wage-earning only class achieves. This is “Ramsey’s conjecture,” and is formulated in Ramsey’s (1928) seminal paper. The purpose of this chapter is to survey this conjecture’s development and ramifications in a series of formal models. An explicit link is made with the savers–spenders macroeconomic theory. Connections to recent literature and interesting problems with roots in the basic examples and their corresponding theoretical studies are noted. This chapter draws from my survey (2006) as well as expands its coverage to subsequently published results. That survey also offers an extensive historical survey of the many agent models in neoclassical capital theory and the role of differing time preference attitudes as a force for income and wealth concentration in market economies.

1.2. Mankiw’s savers–spenders model

Mankiw’s (2000) savers–spenders theory of fiscal policy is based on his observation that the standard optimal growth model, suitably reinterpreted as a representative agent competitive equilibrium model, cannot serve as an adequate theory for the analysis of fiscal policy. Likewise, he argues that the standard overlapping generations model fails to provide a suitable foundation for fiscal policy analysis.¹ These models fail to supply theoretical explanations of three stylized facts. First, he notes that consumption smoothing is far from perfect. Aggregate consumption spending tracks current income far more than if households borrowed or lent capital to smooth their consumption streams. These people just hold enough capital to buffer their consumption against bad income shocks. He notes that these individuals possess high discount rates and often face borrowing constraints.² Second,

1. Thibault (2005) addresses savers–spenders foundations using overlapping generations modeling. His approach differs from the foundational path in this chapter. However, it is an interesting alternative way to describe how *rentiers* emerge in a market with heterogeneous overlapping generations.

2. See Mankiw (2000, p. 121).

a review of empirical income and wealth distributions in the United States implies, according to Mankiw, that wealth is more concentrated than income. Indeed, he argues that the lowest two quintals of the US income distribution account for 15% of the economy's income and hold 0.2% of the country's wealth. He infers from this observation that many households lack the financial resources to smooth their consumption streams over time, as assumed in the standard models of fiscal policy. Third, he observes that based on US data, only a small proportion of the population has sufficient wealth to provide their descendants with an inheritance. These individuals save beyond the levels necessary to smooth their consumption profiles. The vast majority of the population leaves minimal or no bequests. Their saving lacks an altruistic motive.

Mankiw concludes that the standard infinitely lived representative agent theory and corresponding overlapping generations model are inadequate to address fiscal policy in dynamic macroeconomic theory.³ His proposed alternative microeconomic foundation for fiscal policy analysis is based on a "particular sort of heterogeneity."⁴ He argues that an appropriate model should include both low-wealth households who fail to smooth their consumption over time and high-wealth families who smooth consumption not only from year-to-year, but also from generation to generation. Thus, the model should include consumers who plan ahead for themselves and their descendants (via bequests), while others live paycheck-to-paycheck.

Mankiw's theory divides the household sector into two populations. First, the *savers*, who have an operational bequest motive and infinite horizons, and second, the *spenders*, who consume their entire after-tax labor income in every period. Mankiw thinks of the savers as individual households solving infinite-horizon Ramsey-styled optimization problems and spenders as ones following simple rules-of-thumb consumption decisions that result in no savings.

The Ramsey equilibrium model surveyed here provides one possible microeconomic foundation for Mankiw's theory. The major difference is that the spenders in the Ramsey equilibrium model also solve infinite-horizon optimization problems, but *choose* to run their assets down to zero in the steady state and along at least some important realizations of the model's transition dynamics.⁵

3. Thibault (2005) offers a way to build saver–spenders overlapping generations models.

4. The basic arguments against the standard theories and Mankiw's alternative story are found in Mankiw (2000, p. 121).

5. I do not use the model to address fiscal policy analysis as my focus lies on the underlying properties of the economy's dynamics without an explicit government sector. Obviously, a full analysis of public policy within my version of the spenders–savers theory would be of some interest. Boyd (1990a) illustrates the possibilities for the Ramsey version of Mankiw's model in a capital tax incidence problem. Bosi and Seegmuller (2010b) in an elastic labor supply model explore progressive taxation and interesting complications in the tax-distorted economy's development.

Ramsey (1928, p. 559) addressed this problem in the last paragraph of his seminal paper. He claimed that in an economy with many agents and with one more patient than all others, a stationary equilibrium would emerge in which there are two classes. The first class owns all capital and consumes at *bliss*. The second group's capital ownership is zero. Moreover, their consumption is the *minimum to preserve life*. Ramsey's conjecture was first analyzed by Becker (1980) using a discrete time setup instead of Ramsey's continuous-time framework. Becker proved that the steady-state equilibrium exhibited a single capital owner, the most patient agent, who also supplied labor, while all the more impatient households supplied labor, lived on their wages and did not save. The broader "Ramsey Conjecture" subsequently explored in the literature focused attention on the model's dynamics, and whether or not the equilibrium converged to the steady state. This article reviews the main findings to date on "Ramsey's conjecture" as one foundational model for the two-class savers and spenders macroeconomic theory. The published literature has grown to the point that a complete survey is impossible within the page constraints. However, the reader should obtain a good overview of the issues covered.

1.3. The benchmark model: Ramsey's optimal growth model

Ramsey's discounted optimal growth model, cast in a discrete time infinite horizon, deterministic setting serves as the starting point for the heterogeneous agent theories reviewed in this survey. This *benchmark model* is presented along with an important example that eventually connects to the savers–spenders paradigm.

Assume a one-sector productive technology throughout where there is a single all-purpose aggregate capital-consumption good. Output at any time depends on the capital goods input from the previous time period combined with a fixed labor input. At each time the planner (or household) decides how much to consume and how much capital to be carried over to produce next period's goods. Depreciation is taken to be 100% per period, so the model is essentially a circulating capital model in the classical capital theory language. That is, capital is consumed within the period. Exponential depreciation can be modeled with some additional notation.

Time is taken in discrete intervals, $t = 1, 2, \dots$.⁶ Upper case letters denote real(vector)-valued sequences; the corresponding terms of the sequence are expressed as lower case letters, for example, $X = \{x_t\}_{t=1}^{\infty}$. The real-valued *sequence* X is *non-negative* if each component is non-negative, that is, $x_t \geq 0$ for each t .

6. Ramsey (1928) formulated his model in continuous time. The model presented here is cast in discrete time. This turns out to have some technical advantages over continuous time modeling when developing general existence of equilibrium theorems and analyzing the dynamics of particular equilibrium paths. Mitra and Sorger (2013) thoroughly analyze the continuous time analog of the discrete time Ramsey equilibrium theory.

Consumption and capital sequences are always non-negative in this survey article. Notation: $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

1.3.1. Preferences and technology: blanket assumptions

The optimal growth model is described by the central planner's intertemporal objective function, the technology as embodied in the production function, and the capital endowment. The one-sector model's neoclassical production function is $y = f(k)$, where k is the capital input and y is the total output of the single good one period later. Labor is a fixed factor, so the production function exhibits diminishing marginal returns to capital accumulation.

The following properties of f are maintained throughout this survey:

– $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is strictly increasing, strictly concave, twice continuously differentiable on $\mathbb{R}_{++}$ with $f(0) = 0$, $\lim_{k \rightarrow 0, k > 0} f'(k) \equiv f'(0+) = +\infty$, and $\lim_{k \rightarrow \infty} f'(k) < 1$.

These properties imply the existence of a *maximum sustainable capital stock*, $b = f(b) > 0$. The *Cobb–Douglas production function* $f(k) = Lk^\rho$ with $0 < \rho < 1$ and $L > 0$ satisfies these conditions.

The planner is an infinitely-lived agent that ranks alternative consumption sequences $\{c_t\}_{t=1}^\infty$ (write $\{c_t\}$ when the meaning is clear) according to a *discounted utility function*:

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t), \quad [1.1]$$

where the *discount factor* is δ with $0 < \delta < 1$ and *felicity (one-period return) function*, $u(c)$, satisfies the following conditions throughout this survey unless otherwise noted:

– $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing, strictly concave, twice continuously differentiable on $\mathbb{R}_{++}$ with $\lim_{c \rightarrow 0, c > 0} u'(c) \equiv u'(0+) = +\infty$.

Some theorems are demonstrated with the auxiliary assumption $u(0) = 0$, which implies $u(c) > 0$ whenever $c > 0$. The fundamental examples presented below posit the logarithmic utility function $u(c) = \ln c$, which is not bounded from below. In this case, the logarithmic structure may be exploited to prove results obtained with the assumption that u is bounded from below and $u(0) = 0$.

The above properties governing preferences and technology form the *blanket assumptions* that hold throughout the chapter unless otherwise noted.

1.3.2. The planner's problem

The planner's optimization problem expresses the *Ramsey optimal growth model*. It is the prototype for dynamic optimizing models cast in a deterministic, infinite-horizon, discrete time setup.

The planner's problem is: $\max \sum_{t=1}^{\infty} \delta^{t-1} u(c_t)$ by choice of $\{c_t, k_{t-1}\}$, non-negative for each t , and satisfying $c_t + k_t \leq f(k_{t-1})$ with $k_0 = k > 0$, given. A sequence $\{c_t, k_{t-1}\}$ satisfying the problem's constraints is said to be *feasible*. Given the concavity of the felicity and production functions there are two conditions which are necessary and sufficient for an optimal program $\{c_t, k_{t-1}\}$:

$$u'(c_t) = \delta f'(k_t) u'(c_{t+1}) \text{ for each } t; \quad [1.2]$$

$$\lim_{t \rightarrow \infty} \delta^t u'(c_t) k_t = 0. \quad [1.3]$$

The first equation is the *Euler equation*, and the second condition is the *transversality condition*. Both the optimal $\{c_t\}$ and $\{k_{t-1}\}$ are bounded away from 0 at each t since $u'(0+) = \infty$, $f'(0+) = \infty$ and $k > 0$. This planner's optimization problem has a solution for each initial capital stock $k > 0$ and that program is unique given the assumed properties of the felicity and production functions.

1.3.3. Decentralization: Fisher competitive equilibrium

Ramsey's (1928) seminal article on optimal capital accumulation concentrated on a single planner's infinite-horizon optimization problem. Modern economic theorists interpret this planning problem's solution as a *Fisher competitive equilibrium* for an economy with a representative agent whose preferences coincide with the planner's preferences over future consumption streams.⁷

Fisher (1930) proposed that intertemporal choice and consumption-savings decisions take place when a consumption-oriented household has access to a productive investment opportunity and is that activity's sole owner. In a market setting allowing costless borrowing and lending, this consumer will first invest in the project to maximize its present value, then use that present value and any other asset values known at time zero, to optimize consumption by choosing the largest discounted utility payoff subject to a budget constraint where there are discounted (or, present value) prices given for each time period's unit consumption. This idea – separating the consumption decision from the investment decision is known as *Fisher's separation*

7. See Becker and Boyd (1997) for more on this interpretation of the planner's problem and related equivalence theorems. The Fisherian model summarized here is borrowed from their monograph.

principle. The decentralized model sketched below captures this structure in the basic Ramsey model context.

The household-investor is a representative agent. The underlying assumption is a continuum of identical household-investors; each is named by a real number in $[0, 1]$. Identical means they have the same investment opportunity, initial capital and the same objective function as [1.1]. Each agent also supplies the same quantity of labor inelastically and all labor services are identical. So, a total one unit of labor (the measure of $[0, 1]$ being one unit) are ultimately supplied in the labor market. The time index zero, or *time 0*, is the *focal date* for all present value calculations. Imagine a numeraire, or unit of account, exists as an anchor for all prices and present value computations when private agents are self-regarding and forward-looking at time's start, namely $t = 0$. The utility function is discounted utility with focal date time zero.

This representative agent acts as a price-taker and owns the initial capital stock, so that agent acts both as an investor in the project and a supplier of capital at each time. Interpret the labor factor payment at each time, $w_{t+1} = f(k_t) - (1 + r_{t+1})k_t$, as the residual claim on profits within a time period. That is, labor services are identified as entrepreneurial services in a “worker-owned firm.” This household also earns interest (or, rents) from its capital ownership, k_t , at time t and paid at time $t + 1$.

The key price facing a consumer-investor can be thought of as a rental rate for capital goods in a single period of time. Denote this rate as r_t , which is the amount of some numeraire that must be paid to the capital stock's owner to rent one unit of capital inherited from time $t - 1$ and used to produce goods available when period t begins. It is assumed that the competitive market rental rate is taken as given by the consumer-investor. That is, the investor basically pays principal plus interest for each period's capital rental.

An important special feature of the one-sector model is the rental rate and interest rate may be identified as one and the same price. One-sector models have two price systems: one is a *current value price system* where the current price of consumption is the numeraire at that time and has price one, implying the capital goods current price is also unity. This Fisherian market model takes the rental price of capital as the basic building block for all the other prices since it captures the productive margins of additional capital rental in the production of tomorrow's goods. In particular, the sequence of rental rates reckoned in consumption units at each time yield the *present value price system* according to the recurrence relation: $p_t = (1 + r_{t+1})p_{t+1}$ where $p_0 = 1$ and $\{1 + r_t\}$ are given for each time ($t = 0, 1, 2, \dots$). Note that $p_0 = 1$ implies $p_1 = 1/(1 + r_1)$, etc. There is a numeraire commodity at time zero that is used as an accounting device to normalize present values, so $p_0 = 1$. The present value price system is used in the FCE's setup. The necessary conditions can be written in terms of current value prices and reappear in the heterogeneous agent Ramsey model.

Given $R = \{r_t\}$, the market prices of consumption in each period are specified according to the formula: $p_t = \prod_{\tau=1}^t (1 + r_\tau)^{-1}$. Interpret p_t as the present value price of the capital good at time t with focal date 0. Assume this consumer-investor is endowed with the initial capital stock $k > 0$ and owns the one-sector productive function technology.

The household is endowed with a positive capital stock, k , at time zero. Given R and the induced sequence $\{p_t\}$, the investor's optimization problem is to choose a sequence $\{k_t\}$ to achieve the maximum present value:

$$\pi(k, R) = \max \left\{ \sum_{t=1}^{\infty} p_t [f(k_{t-1})] - (1 + r_t) k_{t-1} : k_0 = k \right\}. \quad [1.4]$$

Note for this investor an optimal sequence is also one that maximizes *intertemporal (period-wise) profits* at each time: $\{k_t\}$ solves, for each t ,

$$\max_{k_{t-1}} [f(k_{t-1}) - (1 + r_t) k_{t-1}].$$

The optimal capital sequence satisfies $f'(k_t) = (1 + r_{t+1})$ at each t and $k_0 = k$.

The infinitely-lived household/consumer solves the optimization problem given $\{p_t\}$ and the optimized investment value $\pi(k, R)$:

$$\max_{\{c_t\}} \sum_{t=1}^{\infty} \delta^{t-1} u(c_t) \quad \text{subject to} \quad \sum_{t=1}^{\infty} p_t c_t \leq \pi(k, R) + k.$$

$\{r_t, c_t, k_t\}$ constitute a *Fisher Competitive Equilibrium (FCE)* if and only if $\{c_t, k_{t-1}\}$ with $k_0 = k$ solves the investor and household's problem subject to the market clearing (feasibility) condition $c_t + k_t = f(k_{t-1})$ for each t .

The Euler and transversality conditions for the FCE (optimal growth) model are necessary and sufficient. Consequently, the necessary conditions for the optimal growth problem match the sufficient conditions for an FCE and vice versa. Therefore, a solution to the FCE problem is feasible for the optimal growth problem and satisfies the sufficient conditions for an optimal growth program, and vice versa. Qualitative features of the optimal growth problem's solution carry over to the FCE. For example, the optimal capital sequence is monotonic. Hence, the FCE equilibrium capital sequence is monotonic.

This decentralization result, the equivalence of an optimum and the FCE consumption and capital sequences is a form of the first and second welfare theorems for this model. The introduction of heterogeneous agents in the *Ramsey equilibrium model* breaks this equivalence property.

1.3.3.1. *Malinvaud efficiency*

A fundamental problem in capital theory concerns whether or not the dynamic economic model is providing as much consumption as it can in a competitive equilibrium, such as the FCE. For optimal growth models and their FCE counterparts the answer is affirmative. The optimal capital accumulation program invests neither too much, nor too little, over time. The same cannot be said for the heterogeneous Ramsey equilibrium theory. In that setup, the equilibrium path may *not* solve a social welfare optimization problem. A Ramsey equilibrium may not be a Pareto optimal consumption allocation across the many households or across time. However, an efficiency criterion for aggregate consumption is available even though it is silent on the efficiency of its distribution across households. A brief review of Malinvaud's (1953) efficiency criterion applies to many-agent Ramsey model (as noted in Becker and Mitra (2012)).

Efficiency theory makes use of the productive technology and capital endowment only. It is assumed that more consumption at any time is always preferred to less, all other things equal. The one-sector model's blanket production function assumption is assumed, and the capital endowment satisfies $k > 0$. A *capital stock sequence* $\{K_t\}$ is a *capital stock program* provided it is non-negative at each time and $f(K_{t-1}) - K_t \geq 0$ for each $t \geq 1$. The *corresponding consumption program* is $\{C_t\}$ with $C_t = f(K_{t-1}) - K_t$. Upper case notation is used to remind us that these are aggregate variables. A capital stock program and its corresponding consumption program is *feasible* provided $K_0 = k$.

A feasible capital stock program $\{K'_t\}$ *dominates* the feasible capital stock program $\{K_t\}$, with $K_t \neq K'_t$ for some t , if the corresponding consumption program $\{C'_t\}$ has the property $C'_t \geq C_t$ for each t , with strict inequality for some t . A feasible capital program that is dominated is called *inefficient*; otherwise, it is said to be *efficient*.

Associated to any feasible capital program $\{K_t\}$, where $K_t > 0$ for all $t \geq 1$, is a sequence of *shadow prices* $\{p_t\}$, or *competitive prices*, which are recursively defined by $p_0 = 1, p_{t+1}f'(K_t) = p_t, t \geq 0$. These prices are *implied* or *imputed from* $\{K_t\}$. Note that such a price sequence satisfies the intertemporal (or period-wise) profit maximizing condition (given f is concave):

$$p_{t+1}f(K_t) - p_tK_t \geq p_{t+1}f(x) - p_tx \text{ for each } x \geq 0 \text{ and each } t \geq 0. \quad [1.5]$$

Shadow prices defined in this manner are strictly positive as $K_t > 0$ for each t .

In general, a sequence $\{K_t, p_t\}$ is *intertemporal profit maximizing* if $\{K_t\}$ is a feasible capital program starting from $K_0 = k > 0$, $\{p_t\}$ is a non-null, non-negative price sequence, and [1.5] obtains for each $t \geq 0$.

Starting with Malinvaud, many authors (e.g. Cass (1972)) have shown a close connection between shadow prices and ascertaining whether or not the underlying feasible program is efficient. The *Malinvaud sufficiency theorem* (Malinvaud (1953)) is:

THEOREM 1.1.—*If a sequence $\{K_t, p_t\}$ is intertemporal profit maximizing, with $p_t > 0$ for each $t \geq 0$, and:*

$$\lim_{t \rightarrow \infty} p_t K_t = 0, \quad [1.6]$$

then $\{K_t\}$ is efficient.

It suffices to verify $p_t \rightarrow 0$ as $t \rightarrow \infty$ for the models appearing in this chapter as $\{K_t\}$ is bounded.

Shadow prices satisfying [1.5] implicitly define a capital goods rental rate in each period by letting $f'(K_t) = 1 + r_{t+1}$. We turn this around and impute the market price sequence $\{p_t\}$ for an equilibrium from this profit condition with $p_0 = 1$ defining the numeraire.

1.3.4. The canonical representative agent example

The *logarithmic utility, Cobb–Douglas production economy* is an important example. Let consumption and capital at time t be denoted by c_t and x_t , respectively. The planner solves the Ramsey optimal growth model with $u(c) = \ln c$ and $f(k) = Lk^\rho$ when $0 < \delta, \rho < 1$ are this economy's deep taste and technology parameters; L is a factor which reflects the quantity of fixed labor. For the representative agent case, $L = 1$, and for the case of two agents supplying identical labor services analyzed below, $L = 2^{1-\rho}$, as the underlying production function in capital and labor is taken to be Cobb–Douglas.

This Ramsey problem is explicitly solved by a variety of methods.⁸ The solution is described by the *consumption policy function* $H(k) = (1 - \delta\rho)Lk^\rho$ and the *capital policy function* $h(k) = \delta\rho Lk^\rho$. At each date, the policy functions tell the decision maker how much to consume and how much to save given the current level of the capital stock, k . The optimal decision taken at any date depends only upon the amount of capital the planner starts the period with and not on the particular moment in calendar time. This is the solution's *time consistency* property.

8. See Boyd (1986, 1990a) and Becker and Boyd (1997) for the symmetry technique solution that underlies the example developed below. Other techniques are based on value function iteration using Bellman's optimality equation and Howard's policy improvement algorithm.

The optimal capital and consumption sequences are computed by iterating the policy functions. Carrying out that iteration yields the optimal capital sequence:

$$k_t(k) = (\delta \rho L)^{\rho^{t-1} + \dots + 1} k^{\rho^t}, \quad [1.7]$$

where $k_0(k) = k$ is the given initial capital stock.

The optimal capital sequence is monotonic and converges to the unique positive fixed point of the capital policy function. That fixed point, $k(\delta)$, is called the *modified golden-rule level of capital* and satisfies the equation $h(k(\delta)) = k(\delta)$, which implies that

$$k(\delta) = (\delta \rho L)^{\frac{1}{1-\rho}}. \quad [1.8]$$

Note that the partial sums of the geometric series $\rho^{t-1} + \rho^{t-2} + \dots + 1$ in [1.7] converge to $1/(1 - \rho)$ in equation [1.8].

If the positive initial capital is below the modified golden rule, then the economy accumulates capital and the sequence of optimal capital stocks increases and converges to the modified golden-rule capital stock. Similarly, the optimal capital stocks decrease and converge to the modified golden rule when the starting stock is larger than the positive fixed point. If the initial capital happens to equal the modified golden-rule stocks, then it will be optimal to maintain those stocks in every period. Thus, the modified golden-rule is a non-zero steady state of the dynamical system

$$k_{t+1} = h(k_t) = \delta \rho L k_t^\rho, \text{ with } k_0 = k.$$

The corresponding consumption sequence is also monotonic since the consumption policy function is increasing in capital. The resulting consumption sequence converges to the *modified golden-rule consumption* level defined by

$$c(\delta) = (1 - \delta \rho) (k(\delta))^\rho.$$

The optimal capital sequence, $\{k_t\} \nearrow k(\delta)$ (with $k_t \equiv k_t(\delta)$) when $k < k(\delta)$. Hence, the interest/rental rate sequence $\{1 + r_{t+1}\}$ is decreasing (by f strictly concave) and the corresponding single period profit (wage) sequence, $\{w_{t+1} = (1 - \rho) L k_t^\rho\}$, is increasing as capital accumulates. The monotonic convergence of the optimal capital and consumption sequences is known as the *turnpike theorem*. The *capital income* sequence for the Cobb–Douglas production function, $\{(1 + r_{t+1}) k_t = \rho L k_t^\rho\}$, is increasing as well as capital increases. Clearly $w_{t+1} + (1 + r_{t+1}) k_t = L k_t^\rho$. This relation implies the *gross (equals net here) national income equals the gross domestic product*, aggregate consumption plus investment, $c_t + k_{t+1} = L k_t^\rho$, at each time in an equilibrium allocation sequence.

Bliss (1975) defines the *orthodox vision of capital theory* as the idea that, as capital accumulation progresses, the interest rate falls and wage rate rises over time. Clearly,

this example illustrates the Orthodox Vision for the optimal growth problem, and by its equivalence to an FCE, it applies to the decentralized economy as well.

The observation that the capital income sequence is increasing may not hold in the heterogeneous household Ramsey equilibrium model. Indeed, for certain choices of the production technology, the monotonicity properties fail altogether as periodic or even chaotic Ramsey equilibrium sequences may arise. This scrambles the benchmark model's orthodox vision properties when the model extends to the Ramsey equilibrium story.

1.4. A two-agent Ramsey model

An optimal growth model with two or more agents whose discount factors differ (one agent is more patient than all the others) has the qualitative solution property that the most patient (dominant) household's consumption sequence in the optimal allocation converges to the economy's aggregate consumption. This means that agent's consumption approaches the aggregate consumption output and the more impatient household's consumption tends to zero as time progresses. Rader (1971, 1972, 1981) formalized this result for exchange economies. His latter paper emphasizes the class of Bernoulli (iso-elastic) one-period return functions, which include the logarithmic case. Capital theoretic versions are found in Bewley (1982), Coles (1985, 1986) and Le Van and Vailakis (2003). The following example, based on Becker (2012), illustrates the dominant household's emergence in an optimal allocation.

1.4.1. A log utility – Cobb–Douglas example

Assume for simplicity that there are only two households, denoted by $h = 1, 2$, with lifetime utility given by $\sum_{t=1}^{\infty} \delta_h^{t-1} \ln c_t^h$. Here $\{c_t^h\}_{t=1}^{\infty}$ is a given (non-negative) consumption sequence and $0 < \delta_h < 1$ is the agent's discount factor and $\delta_2 < \delta_1$. Agent 1 is more patient than agent 2. Let $\lambda \geq 0$ denote the welfare weight assigned to agent 1 and $(1 - \lambda)$ the welfare weight assigned to the second agent. Assume further that $0 \leq \lambda \leq 1$ until otherwise noted. The cases where λ equals 0–1 reduce to representative agent problems of the form found above. The planner's *weighted welfare maximization problem* (shortened to welfare maximization problem) is

$$\sup \lambda \sum_{t=1}^{\infty} \delta_1^{t-1} \ln c_t^1 + (1 - \lambda) \sum_{t=1}^{\infty} \delta_2^{t-1} \ln c_t^2 \quad [1.9]$$

by choice of non-negative sequences $\{c_t^1, c_t^2, k_{t-1}\}_{t=1}^{\infty}$ subject to

$$c_t^1 + c_t^2 + k_t \leq Lk_{t-1}^\rho, \text{ for } t = 1, 2, \dots$$

and $k_0 \leq k$, where $k > 0$ is given, $1 > \delta_1 > \delta_2 > 0$, and $L = 2^{1-\rho}$. The parameter ρ satisfies $0 < \rho < 1$. The parameter L is the labor input to the production process. Note

that both households inelastically supply one unit of identical labor services at each time to a Cobb–Douglas production function $F(\ell, k) = \ell^{1-\rho} k^\rho$ with $Lk^\rho = F(2, k)$.

This weighted welfare optimization problem finds a Pareto optimal allocation. Hence, it does not track individual capital holdings or differences in agent's endowments. The model's focus is each agent's consumption sequence and the aggregate optimal capital sequence's pattern.

The welfare maximization problem's objective can be rewritten as

$$\sup \sum_{t=1}^{\infty} \delta_1^{t-1} \left[\lambda \ln c_t^1 + (1 - \lambda) \left(\frac{\delta_2}{\delta_1} \right)^{t-1} \ln c_t^2 \right].$$

Define *aggregate consumption at time t* as c_t with $c_t^1 + c_t^2 = c_t$. The distribution of aggregate consumption within each period can be separated from the problem of calculating the optimal aggregate consumption over time by solving in every period the auxiliary problem

$$u(t, \lambda, c_t) = \sup \lambda \ln c_t^1 + (1 - \lambda) \left(\frac{\delta_2}{\delta_1} \right)^{t-1} \ln c_t^2$$

by choice of non-negative consumption levels c_t^1 and c_t^2 and given $c_t > 0$ subject to

$$c_t^1 + c_t^2 \leq c_t,$$

where λ is given and $0 < \lambda < 1$. The function $u(t, \lambda, c_t)$ is this program's value function. This auxiliary problem's first-order conditions imply that

$$\frac{c_t^2}{c_t^1} = \left(\frac{1 - \lambda}{\lambda} \right) \left(\frac{\delta_2}{\delta_1} \right)^{t-1}.$$

Using the constraint, it is easy to show that each agent's optimal consumption *share* can be written as

$$\begin{aligned} \frac{c_t^1}{c_t} &= \left(1 + \left(\frac{1 - \lambda}{\lambda} \right) \left(\frac{\delta_2}{\delta_1} \right)^{t-1} \right)^{-1}, \text{ and} \\ \frac{c_t^2}{c_t} &= \left(1 + \left(\frac{1 - \lambda}{\lambda} \right) \left(\frac{\delta_2}{\delta_1} \right)^{t-1} \right)^{-1} \left(\frac{1 - \lambda}{\lambda} \right) \left(\frac{\delta_2}{\delta_1} \right)^{t-1}. \end{aligned}$$

These equations yield an interesting result. Note that $(\delta_2/\delta_1)^{t-1} \searrow 0$ as $t \rightarrow \infty$ implies

$$\frac{c_t^1}{c_t} \nearrow 1 \text{ and } \frac{c_t^2}{c_t} \searrow 0 \quad [1.10]$$

provided the aggregate consumption path is bounded away from zero along a welfare maximizing path. This will hold as long as the initial capital stocks are positive. Hence, the first household emerges as the dominant consumer; its consumption approaches 100% of the economy's aggregate consumption and the second household's consumption shrinks towards zero.

The calculation of each agent's consumption share yields the explicit form of the planner's weighted single-period utility/value function by substitution and simplifying, is $u(t, \lambda, c_t) = \Delta_t \ln c_t$. The planner's discount factor at time t , focal date time 0, is $\Delta_t = (\lambda \delta_1^{t-1} + (1 - \lambda) \delta_2^{t-1})$. The dependence of Δ_t on the choice of the welfare weight λ is suppressed in this notation. Information stored in the value function u is sufficient to decompose the aggregate consumption into the optimal consumption allocations for each agent given the preassigned welfare weights. So, the welfare maximization problem is solved if and only if the Ramsey problem with a *time variable discount factor* defined below is solved. The latter problem is expressed as the *welfare optimization problem*:

$$\sup \sum_{t=1}^{\infty} \Delta_t \ln c_t \mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{L})$$

by choice of non-negative sequences $\{c_t, k_{t-1}\}_{t=1}^{\infty}$ with $k_0 \leq k$ ($k > 0$ given), subject to

$$c_t + k_t \leq Lk_{t-1}^{\rho} \text{ for } t = 1, 2, \dots$$

Let $y = Lk^{\rho}$ denote the equivalent goods endowment as viewed at $t = 1$.

This time-varying discount factor is clearly the weighted average of the two agents' discount factors where the weights are given by the preassigned welfare weights. Problem $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{L})$ is an example of the time varying discounted optimal growth model studied by Mitra (1979). The sequence $\mathbf{\Delta} \equiv \{\Delta_t\}_{t=1}^{\infty}$ does *not* form a geometric sequence, unlike the sequences $\{\delta_h^{t-1}\}_{t=1}^{\infty}$ for $h = 1, 2$.

The principal findings are:

THEOREM 1.2 (TWISTED TURNPIKE).—If $\{k_t(y)\}$ and $\{k_t(y^{\#})\}$ are optimal capital sequences starting from the endowments y and $y^{\#}$, respectively, then $|k_t(y) - k_t(y^{\#})| \rightarrow 0$ as $t \rightarrow \infty$.

THEOREM 1.3 (CONVERGENCE).—If $\{k_t(y)\}$ is the optimal capital sequence for the problem $\mathbf{P}(\mathbf{y}, \mathbf{\Delta}, \mathbf{1})$, then

$$\lim_{t \rightarrow \infty} k_t(y) = \bar{k},$$

where $\delta_1 \rho \bar{k}^{\rho-1} = 1$. Moreover, $\{k_t(y)\}$ is eventually a monotonic sequence.

Note that the capital stock $\bar{k}$ is the same as the steady-state capital stock in the optimal growth example when the planner's discount factor is δ_1 . One important difference: in the two-agent example, the capital stock $\bar{k}$ is *not* a stationary optimal capital stock! Indeed, with heterogeneous agents, there is *no* modified golden-rule *steady-state* capital stock in an optimal solution even though $\bar{k}$ serves as a long-run attractor for the model's optimal capital sequence. The reason $\mathbf{L} = \mathbf{1}$ in $\mathbf{P}(\mathbf{y}, \Delta, \mathbf{1})$ is due to a symmetry proof technique where the case of two units of labor in inelastic supply model may be transformed to an equivalent one unit of labor aggregate for purposes of constructing the proofs of theorems 1 and 2. Detailed proofs of these theorems are available in Becker (2012).

1.4.2. Time consistency and weighted welfare optimization

The example summarized above results in a time-inconsistent optimal program based on the planner's decision taken at time 1 since the sequence $\{\Delta_t\}$ is not a geometric sequence as long as $0 < \lambda < 1$. If the planner re-optimizes at some future date T , given the economy's capital stock at that time, the planner will not continue the optimal path initiated at time one. This is a Strotz (1955) type time inconsistency. Given the impatient household's consumption converges to zero in the long run, it is natural to ask about alternative specifications of the model's behavioral assumptions that *might* result in a more equal distribution of consumption and be time consistent as the planning horizon tends to infinity.

Drugeon and Wigniolle (2016, 2021) model the time consistency problem as a Nash equilibrium problem following the ideas in Phelps and Pollak (1968), where the planner's optimal decision is calculated assuming all future incarnations or selves follow the same decision rules. Each generation lives for one time period and is altruistic towards future generations. Successive generations play a strategic game between themselves and the planner when decision rules are identified as strategies. A Nash equilibrium results in a time-consistent equilibrium. The interested reader is referred to their papers, and in particular, their examples which illuminate their findings.

These two papers are also examples of an increasingly developing literature on collective choice models when private agents have differing discount factors. Pakhnin (2023) published an interesting survey article on this literature focusing on its connections with voting models and the issues raised for collective choice when time inconsistent optimal consumption sequences arise in Pareto optimal capital accumulation problems. This is a crucial problem. A resolution of the time inconsistency issue is as crucial for aggregating intertemporal preferences, as in the dynastic setting, in order to give a foundation for recommending a social discount factor for public goods and major long-term externalities (e.g. climate change) cost-benefit calculations.

1.5. The Ramsey equilibrium model

The basic Ramsey equilibrium model first appears in Becker (1980). The structure set out below is designed to support both stationary and non-stationary equilibrium results.

1.5.1. The basic model and blanket assumptions

This model adds several heterogeneous infinitely lived households to the basic one-sector model. The blanket assumptions on felicities and production carry over. The novelty is that agents are assumed to have different discount factors. If $H = 1$, then the Ramsey equilibrium model coincides with the standard optimal growth problem and its equivalent FCE counterpart.

There are $H \geq 1$ households indexed by $h = 1, \dots, H$. Households are endowed at time zero with capital stocks $k^h \geq 0$. Put $\mathbf{k} = \sum_h k^h$ and assume $\mathbf{k} > 0$. Let c_t^h, x_t^h denote the consumption and capital stock of household h at time t . Household h has *felicity function* u_h ; c_t^h is its argument. Household h discounts future utilities by the factor δ_h with $0 < \delta_h < 1$. Hence, the household's lifetime utility function is specified by $U_h(C_h) = \sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h)$. Households have *heterogeneous discount factors*. The results only require two types – one household is the most patient, and the others are less patient. This is expressed by assuming the first household's discount factor is larger than all the other households' discount factors. The first household is the *most patient* agent and the others are said to be *less patient* or *more impatient* than the first one. The *Heterogeneity Assumption* orders households from the most patient to the least patient.

Heterogeneity Assumption: $1 > \delta_1 > \delta_2 \geq \dots \geq \delta_H > 0$.

Production takes place in a one-sector, single capital good, model as described earlier and subject to the production function blanket assumption. The initial aggregate capital stock $\mathbf{k}$ is smaller than the maximum sustainable stock $b = f(b) > 0$.

1.5.2. The households' problems

Each infinitely lived household is supposed to perfectly anticipate the future course of rental and wage rates. Competitive markets operate at each date, so households' consumption-savings decisions do not, by themselves, influence prices. However, prices will be set in equilibrium to clear all markets. Households inelastically supply one unit of identical labor service at each time. Labor supply is exogenously fixed.

Let $\{1 + r_t, w_t\}$ be a sequence of one-period rental factors and wage rates, respectively. The sequences $\{1 + r_t, w_t\}$ are always taken to be non-negative and

non-zero. Households are competitive agents and perfectly anticipate the profile of factor returns $\{1 + r_t, w_t\}$. Given $\{1 + r_t, w_t\}$, h solves

$$P(h) : \sup \sum_{t=1}^{\infty} \delta_h^{t-1} u_h(c_t^h)$$

by choice of non-negative sequences $\{c_t^h, x_t^h\}$ satisfying $x_0^h = k^h$ and

$$c_t^h + x_t^h = w_t + (1 + r_t)x_{t-1}^h, \quad (t = 1, 2, \dots). \quad [1.11]$$

The market structure of this model requires capital assets to be non-negative at each moment of time and that agents without capital cannot borrow against the discounted value of their future wage income. The sequential budget constraint captures this structure. The borrowing constraint binds for the more impatient consumers in a stationary equilibrium. These individuals might be tempted to borrow against their future labor income if they were allowed to do so. If this occurred, their consumption would approach zero as time converges to infinity. Indeed, this is the type of outcome for equilibria in the exchange economy modeled by Rader (1971, 1972, 1981) and the general equilibrium capital theoretic models appearing in Bewley (1982), Coles (1985), Durán and Le Van (2003), Le Van and Vailakis (2003) and Becker (2012).

The *Euler inequality* necessary conditions for $\{c_t^h, x_t^h\}$ to solve $P(h)$ are $c_t^h > 0$ and

$$\delta_h(1 + r_{t+1})u'_h(c_{t+1}^h) \leq u'_h(c_t^h) \quad [1.12]$$

with equality whenever $x_t^h > 0$ so that the *Euler equation* holds:

$$\delta_h(1 + r_{t+1})u'_h(c_{t+1}^h) = u'_h(c_t^h). \quad [1.13]$$

The corresponding *transversality condition* is $\lim_{t \rightarrow \infty} \delta_h^{t-1} u'_h(c_t^h) x_{t-1}^h = 0$.

1.5.3. The production sector's objective

The production model derives from the FCE setup and focuses on myopic profit maximization at each time. Producers take the rental rate as given and solve $P(F)$ at each t :

$$P(F) : \sup[f(x_{t-1}) - (1 + r_t)x_{t-1}]$$

by choice of $x_{t-1} \geq 0$. The residual profit is treated as the total wage bill. It is shared equally by the identical households as wages – production is worker-owned.

If $0 < 1 + r_t < \infty$, then $f'' < 0$ implies there is a unique K_{t-1} which solves $P(F)$ at each t ; clearly

$$f'(K_{t-1}) = 1 + r_t. \quad [1.14]$$

The corresponding $\{w_t\}$ defined by

$$Hw_t = f(K_{t-1}) - (1 + r_t)K_{t-1} \quad [1.15]$$

is positive.

The expression for the wage bill can also be viewed as a function of the rental rate – invert [1.14] to solve for the capital stock in terms of the rental rate and substitute this into [1.15]. The result can be summarized with the factor price frontier function $w_t = \phi(1 + r_t)$, which expresses the *per-capital wage* as a function of the prevailing rental factor, $1 + r_t$ when the capital stock optimizes myopic profits.

1.5.4. The Ramsey economy

A collection $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2, \dots, H)$ satisfying the blanket and heterogeneity assumptions, and for which $k^h \geq 0$ for each h with $\mathbf{k} = \sum_{h=1}^H k^h > 0$, $\mathbf{k} \leq \mathbf{b}$, is said to be a *Ramsey economy*, or simply, an *economy*.

1.5.5. The equilibrium concept

The equilibrium concept is *perfect foresight*. Households perfectly anticipate the sequences of rental and wage rates. They solve their optimization problems for their planned consumption demand and capital supply sequences. The production sector calculates the capital demand at each time and the corresponding total output supply. Rentals are paid to the households for capital supplied and the residual profits are paid out as the total wage bill. An equilibrium occurs when the households' capital supply equals the production sector's capital demand at every point of time. Walras' law implies that the total consumption demand and supply of capital for the next period equals current output. Thus, in equilibrium, every agent is maximizing its objective function and planned supplies equal planned demands in every market at each t .

DEFINITION 1.1.– Sequences $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ constitute a Ramsey equilibrium for a given economy $\mathcal{E}$ provided:

- E1. For each h , $\{c_t^h, x_{t-1}^h\}$ solves $P(h)$ given $\{1 + r_t, w_t\}$.
- E2. For each t , K_{t-1} solves $P(F)$ given $1 + r_t$.
- E3. $Hw_t = f(K_{t-1}) - (1 + r_t)K_{t-1}$ ($t = 1, 2, \dots$).
- E4. $\sum_{h=1}^H x_{t-1}^h = K_{t-1}$ ($t = 1, 2, \dots$), $0 < \mathbf{k} = K_0 \leq \mathbf{b}^m$.

Thus, consumers maximize utility (E1) and producers maximize profits (E2). The labor market clearing condition is expressed in (E3). The capital market clearing condition is (E4). The output market balance follows by combining (E1)– (E4). This implies Walras' law holds at each time. Hence

$$\sum_{h=1}^H (c_t^h + x_t^h) = f(K_{t-1}). \quad [1.16]$$

Note that equilibrium consumption and capital sequences are bounded from above by the maximum sustainable stock.

1.6. Stationary Ramsey equilibrium models

A stationary Ramsey equilibrium is one in which all prices and quantities remain constant over time. This is the simplest form of equilibrium, and it is interpreted as the model's long-run stationary (steady state) solution. Ramsey's (1928) conjectured two-class solution is verified in the Ramsey model developed here. Note that he did not specify the model's formal structure, in contrast to his optimal accumulation studies. The model proposed here assumes there is a single most patient household. The steady-state income distribution is shown to be determined by the largest discount factor, or equivalently, the lowest discount rate. The household with the highest (lowest) discount factor (rate) owns all the capital and earns a wage income; all other households receive a wage income.

The most patient consumer in the stationary Ramsey equilibrium has two sources of income – rental income from capital and wage income from its supply of labor to the production sector. In Ramsey's original conjecture, the most patient consumer's consumption equaled "bliss"; the more impatient households consumed at a level just sufficient to sustain their lives. In the version of the model presented here, the more impatient consumers have a wage income, consumed within the period, and there is no presumption that it is at the minimal level that just supports their lives. Likewise, the most patient agent's consumption is not identified as "bliss," which is a formalization of consumption satiation.

1.6.1. Heterogeneous households and differing rates of impatience

A Ramsey equilibrium program is *stationary* for the economy $\mathcal{E}$ provided the equilibrium wage rate, rental rate, the aggregate capital stock and the allocations of capital and consumption are constant over time. Becker (1980) proves the basic theorem of Ramsey equilibrium theory. It proclaims the existence of a unique stationary equilibrium in which only the most patient household has capital – all other households have none and live off their wage incomes. Of course, this most patient

consumer also has a wage income, so that person achieves a higher consumption level than the others. Hence, there is an unequal distribution of income and wealth in this steady-state solution. The model is telling us that this inequality in stationary equilibrium consumption levels and capital holdings is the direct consequence of a single household being the most patient one, and its impatience rate is independent of the underlying consumption sequence.

Let k^{δ_1} be the unique solution to the equation $f'(k) = \delta_1^{-1}$.⁹ This capital stock will be the first household's capital and therefore the aggregate capital stock in the stationary equilibrium solution. Let $1 + \varrho_h = \delta_h^{-1}$ define agent h 's *pure rate of time preference (or, discount rate)*, ϱ_h . The assumed ordering of household's discount factors is equivalent to $0 < \varrho_1 < \varrho_2 \leq \dots \leq \varrho_H$.

THEOREM 1.4 (RAMSEY'S THEOREM).— *Let $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2, \dots, H)$ be a given economy. Then, there is a unique stationary Ramsey equilibrium given by*

- (S1) $(1 + \bar{r}) = \delta_1^{-1}$ or equivalently $\bar{r} = \varrho_1$;
- (S2) $\bar{w} = \phi(\delta_1^{-1})$;
- (S3) $\bar{k}_0^1 = k^{\delta_1}$ is the first household's initial allocation of capital;
- (S4) $\bar{k}_0^h = 0$ is the initial allocation of capital for $h = 2, \dots, H$;
- (S5) $\bar{c}^1 = \phi(\delta_1^{-1}) + \varrho_1 k^{\delta_1}$;
- (S6) $\bar{c}^h = \phi(\delta_1^{-1})$ for $h = 2, \dots, H$.

The proof's basic idea is to conjecture (S1)–(S6) identify an equilibrium and verify that for the given wage rate, rental rate and assignment of initial capital stocks, the resulting optimal solutions for the households' optimization problems are the constant consumption and capital profiles specified while the production sector profit maximization problem's solution is k^{δ_1} . Moreover, the household's capital supplies sum to the aggregate stocks demanded by the production sector, k^{δ_1} . The sufficiency conditions for the solution of each agent's optimization problems given the values of the prices and initial conditions show that the conjectured solutions solve the agent's problems given the wage rate, rental rate and capital endowments. The first household's sufficient condition holds as the conjectured solution satisfies that agent's no arbitrage and transversality conditions. The other households are handled separately in the chapter.

The theorem follows once this equilibrium is shown to be the only stationary solution to the model.

9. Otherwise, overbars are used to indicate values for endogenously determined variables which are stationary in a Ramsey equilibrium.

PROPOSITION 1.1.— *The stationary Ramsey equilibrium (S1)–(S6) is the only stationary Ramsey equilibrium.*

The proof is partly given below as it illustrates the nature of heterogeneous agent formal arguments. This lemma uses an argument from Radner's (1961) original turnpike theory. A comparison path is trapped between a “floor” and a “ceiling” by the establishment of some inequalities assuming the comparison path is a stationary equilibrium different from the one already established. The trick is to observe that the comparison path's ceiling must fall below its floor according to the bounding inequalities in the proof. Hence, this alleged stationary equilibrium is not one.

LEMMA 1.1.— *If $\delta_1(1 + \bar{r}) > 1$, then this $\bar{r}$ cannot be a stationary Ramsey equilibrium rental/interest rate.*

PROOF.— Suppose not, that is, there is a stationary Ramsey equilibrium corresponding to this rental rate. I show that this implies the first household's consumption sequence must be monotonically increasing, a contradiction to the stationarity of the equilibrium.

If this is an equilibrium configuration, then $P(1)$ has an optimal solution when the wage $\bar{w} = \phi(1 + \bar{r})$, $\delta_1(1 + \bar{r}) > 1$, and $\bar{k} = \bar{k}^1 \geq 0$. Call this solution $\{(c_t, x_{t-1})\}_{t=1}^\infty$, where the superscript and subscript labeling this agent has been dropped to simplify the notation for the felicity function and the consumption–capital sequences.

Claim: $x_t > 0$ for infinitely many t .

If, on the contrary, x_t is eventually zero, then for all $t \geq T$ for some T , the no arbitrage inequality holds:

$$u'(c_t) \geq \delta_1(1 + \bar{r})u'(c_{t+1}).$$

Since $x_t = 0$ for $t \geq T$, this and $\delta_1(1 + \bar{r}) > 1$ imply

$$u'(c_t) > u'(c_{t+1}), \text{ or} \\ c_t < c_{t+1} \text{ by } u'' < 0.$$

But $x_t = 0$ implies from the budget constraint that $c_t = \bar{w}$, which contradicts the eventual monotonicity of consumption shown above. Therefore, $x_t > 0$ for infinitely many t must hold.

This argument can be repeated in the case where capital is positive infinitely often to show that the consumption sequence must be strictly increasing. Furthermore, iteration of the no arbitrage inequality for periods $2, 3, \dots, T$ yields

$$u'(c_1) \geq [\delta_1(1 + \bar{r})]^{T-1} u'(c_T).$$

The RHS is the *floor* or lower bound on marginal utility at time 1. Note that $u'_1(c_1)$ acts as an upper bound, or *ceiling* on the RHS expression.

Since the consumption sequence is strictly increasing, only two possible cases need consideration:

Case (a): $\sup_t c_t = +\infty$.

Case (b): $\sup_t c_t = \eta < +\infty$.

The first case cannot occur since it contradicts diminishing returns and could never constitute an equilibrium.

In the second case the consumption sequence converges monotonically to η : that is, $c_t \nearrow \eta$ and by continuity of u' and $u'' < 0$, $u'(c_t) \searrow u'(\eta)$. Thus $u'(c_t) \geq u'(\eta)$ for all t implies

$$u'(c_1) \geq [\delta_1(1 + \bar{r})]^{T-1} u'(c_T) \geq [\delta_1(1 + \bar{r})]^{T-1} u'(\eta).$$

However, $\delta_1(1 + \bar{r}) > 1$ yields a contradiction as the right-hand side exceeds the left-hand side for T a very large natural number. That is, the ceiling falls below the floor.

Therefore the sequence $\{(c_t, x_{t-1})\}_{t=1}^{\infty}$ cannot be an equilibrium. □

If $\delta_1(1 + \bar{r}) < 1$, then this $\bar{r}$ cannot be a stationary Ramsey equilibrium rental/interest rate either by following a similar logic using a ceiling-floor argument.

This version of Ramsey's model uses a perfect foresight expectations hypothesis to close the model. We can think of the *long-run* as the steady-state solution. The *short-run* represents the transitional period from the start of the model at time zero until it converges to the steady state. Of course, this presumes the economy actually converges to the steady state – a presumption that will turn out to be true only with additional qualifications. Even so, it is useful to maintain the distinction between long-run solutions and short-run dynamics. This is so since the model will generate other types of periodic solutions besides the trivial periodic motion embodied in the stationary state. These non-trivial periodic solutions also have legitimate claims to be considered long-run solutions.

The exogenous labor supply assumption yields a stark, stationary income inequality. Bosi and Seegmuller (2010a) extend the basic steady-state theory to model variable labor supply where labor is a normal good. The most patient agent remains the only one with capital in the long run, but income inequality may be ameliorated because agents choose different labor supplies. Their steady-state analysis yields two interesting special cases. One, the most patient agent, is a pure capitalist since that

agent chooses to supply *no* labor. That means it must save out of its capital income to sustain the steady-state equilibrium. The other agents spend their labor incomes and do not save. In the second equilibrium type, the capitalist also supplies labor as in the exogenous labor supply story. However, the other agents again may choose different labor supplies. Both variants are suitable for savers–spenders theories, but the first is one where there is a pure capitalist who saves and spends while foregoing labor supply, while the others merely spend their wage incomes.

Irving Fisher (1930) noted that with unequal rates of impatience, a market economy starting from an equal distribution of wealth would evolve towards one that is unequal. Ramsey's steady state offers the most extreme interpretation of this long-run outcome. The long-run economy is populated by savers and spenders. The problem is to work out the model's explicit equilibrium dynamics to show that while some impatient agents may live on their capital for some time period, they will eventually enter a no capital state. However, once in a no-capital state, they may *not* stay there – a subtle point impacting the model's overall dynamic features. Indeed, will the long-run solution conjectured by Ramsey emerge from the self-interested consumption-saving decisions taken in the household sector, or will equilibrium paths have other characteristics? Answers to these basic questions are reviewed in the next section.

1.7. Ramsey equilibrium dynamics

The stationary Ramsey equilibrium assigns all the economy's capital to the most patient household. Does this capital stock ownership represent the economy's long-run equilibrium? Put differently, is it a stable equilibrium? For example, if society's initial capital stock is equally distributed, will it evolve towards the unequal distribution found in the steady-state equilibrium? More generally, will any equilibrium program starting from arbitrary initial allocations of capital converge to the steady state? The Ramsey equilibrium model must be thoroughly analyzed to solve this stability problem. The analysis of the model includes proving that a Ramsey equilibrium exists for an arbitrary initial distribution of capital. Ideally, the uniqueness of the equilibrium path should also be established. Necessary and sufficient conditions characterizing an equilibrium are also required in order to develop an equilibrium's dynamic properties. The stability question is addressable once the economy's equations of motion are firmly established.

The stability problem is the central question in Ramsey equilibrium theory. Does the introduction of additional agents, each a distinct individual with different tastes and endowments, sustain the capital monotonicity characteristic of the representative agent case? Or, does the introduction of heterogeneous agents complexify the model's dynamics? Do the increased number of agents and their interrelationships generate equilibrium fluctuations – either regular cycles or chaos, or do their joint actions dampen oscillations and promote convergence to the stationary Ramsey equilibrium?

If Ramsey's equilibrium steady state is stable, and all households initially hold capital, then it must be the case that all but the most patient consumer run their capital to zero. This could, in principle, happen either *asymptotically* or *eventually* (i.e. in finite time). In either event, the more impatient households' capital holdings are *transient*. However, a household always earns its wage payment at each time. Even if that household completely runs its capital to zero, it *always* has the option of saving and reacquiring capital. This point cannot be overemphasized since it lies at the heart of all the major difficulties in analyzing the model's dynamics.

The purpose of this section is to survey the model's dynamics. The existence of a Ramsey equilibrium is briefly noted. Sufficient conditions for a Ramsey equilibrium program are given that are particularly useful for constructing examples with particular dynamic features.

The main result on transient capital holdings by the impatient agents is the *recurrence theorem*. Sufficient conditions for showing once a household achieves a zero capital state it remains in that state forever are given. This property is the *turnpike property*. An example shows that in general, the turnpike property does not hold. Moreover, the steady state is unstable (another example). The economy can generate a path with undamped oscillations in the form of a two-period cycle. Sufficient conditions are then developed for convergence of an equilibrium to the steady state. In the end, the stability of Ramsey equilibrium paths is only shown for a class of economies. The potential for more complex equilibrium dynamics than convergence is subsequently reviewed.

1.7.1. Existence of Ramsey equilibrium and sufficient conditions for a Ramsey equilibrium path

Becker et al. (1991) gave sufficient conditions for the existence of a Ramsey equilibrium. Their theorem included recursive utility specifications allowing variable rates of impatience depending on stationary consumption plans, within a one-sector technology framework. Their general result applies to the time additive discounted utility case presented here. Their theorem, specialized to that framework, is stated below for reference.

THEOREM 1.5.—Let $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2, \dots, H)$ be a given economy. Then a Ramsey equilibrium exists.

Subsequent research by Becker et al. (2015) relaxes some restrictions embedded in the existence theorem (above). A by-product of their approach shows that no Ramsey equilibrium can exhibit a rational bubble. They use the sequence of rental rates to define present value prices and a market fundamental connection to the price of capital

goods at time zero. That is, that price is the present value of future rental payments (i.e. interest plus principal net of replacement for capital's depreciation). Intuitively, with a "high" interest rate, the present value of capital satisfies a transversality style condition, and there is no bubble (by definition, a component of the price different from the so-called fundamental value). In fact, in general, there are no equilibrium bubbles. This is of interest since the underlying markets are incomplete – not all intertemporal trades are admissible given the borrowing constraints. Rational bubbles have been associated with incomplete markets in the overlapping generations literature, Tirole ((1982), (1985)), and heterogeneous infinitely lived agent models in exchange economies, Kocherlakota (1992) and Huang and Werner (2000). The case of productive capital accumulation prevents interest rates from converging to zero opens the possibility that market fundamentals may differ from the time zero capital price in an equilibrium.

Necessary and sufficient conditions for a Ramsey equilibrium are useful for identifying suspected solutions and demonstrating the conjectured solution are, in fact, equilibria. Following the statement of general sufficient conditions, focus on a result formalized by Sorger (1994) applicable to a two-household economy in which the more impatient household never owns capital. His result provides a ready way to manufacture examples.¹⁰ This is illustrated by constructing an example of Mankiw's (2000) savers–spenders theory.

1.7.1.1. General sufficient conditions for Ramsey equilibrium

The necessary conditions for equilibrium turn out to be sufficient as well. This result turns on the assumed concavity of f and the u_h functions, as well as the satisfaction of the no arbitrage inequalities for each agent and their respective transversality conditions in combination with a market clearing equation.

THEOREM 1.6 (SORGER (1994)).—*Let $\mathcal{E} = (f, \{u_h, \delta_h, k_h\}, h = 1, 2, \dots, H)$ be a given economy. Assume that there exists a sequence $\{K_{t-1}, c_t^h, x_{t-1}^h\}$ such that the following conditions are satisfied for all $h = 1, 2, \dots, H$ and all $t \geq 1$:*

- 1) $c_t^h > 0, x_{t-1}^h \geq 0, K_{t-1} > 0$;
- 2) $x_0^h = k^h$;
- 3) $\delta_h f'(K_t) u'_h(c_{t+1}^h) \leq u'_h(c_t^h)$ with equality whenever $x_t^h > 0$;
- 4) $c_t^h + x_t^h = \frac{1}{H} [f(K_{t-1}) - f'(K_{t-1})K_{t-1}] + f'(K_{t-1})x_{t-1}^h$;
- 5) $\sum_{h=1}^H x_{t-1}^h = K_{t-1}$;
- 6) $\lim_{t \rightarrow \infty} \delta_h^{t-1} u'_h(c_t^h) x_t^h = 0$.

10. This lemma is already implicit in Becker and Foias's (1987) construction of a periodic Ramsey equilibrium path.

Then it follows that $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a Ramsey equilibrium for the economy $\mathcal{E}$, where the sequence of factor prices $\{1 + r_t, w_t\}$ is given by

$$1 + r_t = f'(K_{t-1}) \text{ and } Hw_t = f(K_{t-1}) - f'(K_{t-1})K_{t-1}.$$

This theorem's proof is a straightforward adaptation of standard arguments for sufficiency of the Euler equation and transversality condition in the one-sector Ramsey optimal growth problem.

1.7.1.2. Sufficient conditions for a class of two-agent economies

The construction of Ramsey equilibrium examples is greatly facilitated by the next result which gives sufficient conditions in a two-agent economy for sequences of aggregate capital, factor prices, consumption allocations and individual capital allocations to be an equilibrium where *only the most patient consumer holds capital*. This is a special type of equilibrium, and it is typically a dynamic one where the sequences changes over time. Of course, if the initial conditions are set just right, the equilibrium will be the stationary one. The importance of the lemma is that it asserts that given the most patient agent's function satisfying the blanket condition, u_1 , and discount factor, δ_1 , and for any function u_2 satisfying the blanket assumption, there is a discount factor, δ_2 , such that the heterogeneity assumption holds. Moreover, the given sequences of aggregate capital, factor prices, consumption and individual capital form a Ramsey equilibrium. In this equilibrium, the first household owns the entire capital stock for all time and the second household's initial capital stock is zero.

LEMMA 1.2 (SORGER (1994)).—Let u and f be functions satisfying the blanket assumptions, respectively, and let δ_1, ε_1 and ε_2 be positive real numbers with $\delta_1 \in (0, 1)$. Assume there exists a sequence $\{K_{t-1}, c_t\}$ such that the following conditions hold true for all $t \geq 1$:

- 1) $\varepsilon_1 < c_t, \varepsilon_1 < K_{t-1} < \varepsilon_2$;
- 2) $\delta_1 f'(K_t) u'(c_{t+1}) = u'(c_t)$;
- 3) $c_t = \frac{1}{2}[f(K_{t-1}) + f'(K_{t-1})K_{t-1}] - K_t$.

For any function u_2 satisfying the blanket conditions, and for any sufficiently small discount factor $\delta_2 > 0$ with $\delta_2 < \delta_1$, it holds that for the economy $\mathcal{E} = (f, \{u_h, \delta_h, k^h\}, h = 1, 2)$ defined by $u_1 = u, \delta_1, k^1 = K_0$ and $k^2 = 0$ has a Ramsey equilibrium $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ given by

$$\begin{aligned} 1 + r_t &= f'(K_{t-1}), c_t^1 = c_t, c_t^2 = w_t \\ &= \frac{1}{2}[f(K_{t-1}) - f'(K_{t-1})K_{t-1}], x_{t-1}^1 = K_{t-1}, \text{ and } x_{t-1}^2 = 0. \end{aligned} \tag{1.17}$$

1.7.1.3. *Savers and spenders redux: an example*

Mankiw's (2000) savers–spenders model can be formulated as a Ramsey equilibrium for a two-agent economy. Household 1 is the most patient consumer, acts as Mankiw's bequest motivated individual, and saves. Household 2 is the most impatient and acts as the economy's spender. Household 2 never saves and sets consumption equal to wage income in every period. This outcome represents a choice by Household 2 since the option to save is always present so long as the wage rate is positive. The choice of saving or spending for the more impatient Household 2 is the only departure of the model from Mankiw's framework, although it is certainly consistent with a close reading of his interpretation of the spending household's behavior. The model's specification is completed by assuming special functional forms for the production function and the first household's u_1 function. The preceding lemma takes care of the second household's preferences. This example draws on an example in Boyd's thesis (1986, pp. 34–37), reinterpreted for Mankiw's model.

Let production be Cobb–Douglas with $f(K) = AK^\alpha$, where $0 < \alpha < 1$ and $A = 2^{1-\alpha}$. The total factor productivity index, A , reflects the inelastic supply of two units of labor from the household sector. The first household's function $u_1(c_t^1) = \ln c_t^1$. Its discount factor is any δ_1 , with $\delta_1 \in (0, 1)$. Along an equilibrium profile, Household 1 solves

$$\sup_{\{c_t^1, x_{t-1}^1\}} \sum_{t=1}^{\infty} \delta_1^{t-1} \ln c_t^1$$

subject to:

$$c_t^1 + x_t^1 = \frac{1}{2}[f(K_{t-1}) - f'(K_{t-1})K_{t-1}] + f'(K_{t-1})x_{t-1}^1,$$

with $k^1 = \mathbf{k} = K_0$ given and the usual non-negativity constraints on consumption and individual capital holdings. The equilibrium is constructed so that $x_t^1 = K_t$ and $x_t^2 = 0$ for all t . Thus, the constraint facing the first household takes the form:

$$c_t^1 + K_t = \frac{1}{2}[f(K_{t-1}) - f'(K_{t-1})K_{t-1}] + f'(K_{t-1})K_{t-1}.$$

Given the Cobb–Douglas technology specification, this constraint can be rewritten as

$$c_t^1 + K_t = BK_{t-1}^\alpha,$$

where $B = [\frac{1}{2}(1 - \alpha) + \alpha]A$ and $A = 2^{1-\alpha}$.¹¹ Note that the first household's optimization problem (given the second household never holds capital) is just like the

11. Note that $B < A$ holds. For example, let $\alpha = 0.5$. Then, $B = 3\sqrt{2}/4$ and $A = \sqrt{2}$.

well-known canonical logarithmic economy – Cobb–Douglas example of Ramsey’s optimum growth problem. The difference is that the constant B appears and it differs from the technology’s total factor productivity index, A , to reflect the fact that the second household is consuming fifty percent of the total wage bill at each time while never saving. The optimal policy functions for this model can be found using Boyd’s symmetry technique (1990b) and are expressed by:

$$c_t^1 = (1 - \delta_1 \alpha) B K_{t-1}^\alpha;$$

$$K_t = \delta_1 \alpha B K_{t-1}^\alpha.$$

The aggregate capital sequence is found by iteration of the capital policy function with initial seed K_0 . The stationary Ramsey equilibrium capital stocks are easily seen to be $\bar{K} = [1/(\delta_1 \alpha B)]^{1/(1-\alpha)}$. If $K_0 < \bar{K}$, then the equilibrium capital stock sequence $K_t \uparrow \bar{K}$ and if $K_0 > \bar{K}$, then $K_t \downarrow \bar{K}$. The first household’s consumption shares the same monotonicity property. The second household’s consumption is given by

$$c_t^2 = w_t = \frac{(1 - \alpha)}{2} A K_{t-1}^\alpha.$$

Hence, its consumption is also monotonic in the same manner as the aggregate stock’s monotonicity pattern. The transversality condition holds for each household as their consumption levels converge to the stationary equilibrium levels and the aggregate stocks converge as well. Therefore, Sorger’s Lemma implies for any u_2 satisfying the blanket conditions, there is a $\delta_2 > 0$, with $\delta_2 < \delta_1$, such that the sequences so defined form a Ramsey equilibrium when the economy’s endowments are distributed as $k^1 = K_0$ and $k^2 = 0$. The Orthodox Vision applies to this specification of Mankiw’s theory. Thus, there are economies for which the Ramsey equilibria are stable and converge to the stationary equilibrium allocation.

The monotonicity of the aggregate capital sequence and its convergence to the stationary Ramsey equilibrium are interesting results in this context. How general are these properties? Is it true for arbitrary economies that the aggregate capital sequence is monotonic and convergent to the steady state? Does a household with no capital initially remain in the zero capital state forever?

1.7.2. The recurrence and turnpike properties

The relatively impatient households have no physical assets in the stationary equilibrium. So, if that equilibrium is stable, it must be the case that those households’ capital holdings converge to zero. This is *not* always true in a Ramsey equilibrium. It is of fundamental importance to understand why this is so and if there are conditions under which the relatively impatient consumers drive their capital holdings towards zero.

1.7.2.1. Recurrence property

There are a number of properties that an individual's capital sequence might satisfy. The non-negative sequence $\{x_{t-1}\}$ is said to have a *positive capital state* at time t provided $x_t > 0$. Likewise, the *zero capital state* occurs at time t whenever $x_t = 0$.

DEFINITION 1.2.— *The capital stock x is a recurrent state for the sequence $\{x_{t-1}\}$ if there exists $\{t_n\}_{n=1}^\infty$, $t_n < t_{n+1}$ and $x = x_{t_n}$ ($n = 1, 2, \dots$).*

The most general property of Ramsey equilibrium paths turns out to be the *recurrence property* – the zero capital state is recurrent for each $h \geq 2$. That is, the relatively impatient individuals achieve a zero capital state infinitely often. Moreover, an example will show that it is possible for an individual to have a positive capital state infinitely often along an equilibrium path as well as realize the zero capital state infinitely often in the same sequence. The economic point is that as long as the economy has a positive capital stock, the total wage bill is positive. This implies any agent achieving a zero capital state can always save some of its wage and earn a rental return on that investment next period. This implies there is a hidden option value embedded in wage-rental sequences in this borrowing constrained setup. Individuals with zero capital have the option to acquire capital at any time, but may choose not to exercise that choice as well.

Fix the economy $\mathcal{E}$ satisfying the blanket assumptions. The Recurrence Theorem is due to Becker and Foias (1987). Additional details are found in their paper.

THEOREM 1.7 (RECURRENCE THEOREM).— *If $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ is a Ramsey equilibrium for $\mathcal{E}$, then the no capital state is recurrent for each $h \geq 2$.*

PROOF.— It can be shown $\limsup_t c_t^h > 0$ for each h and $\limsup_T \prod_{t=1}^T \delta_1(1+r_{t+1}) < \infty$.

Suppose the no capital state is not recurrent for some $h \geq 2$. Then, there is a t_0 such that $x_t > 0$ for all $t > t_0$ (drop the h superscript for this $h \geq 2$). Iterating [1.12] for $t_0, \dots, T$, gives

$$\prod_{t=t_0}^T \delta(1+r_{t+1}) = \frac{u'(c_{t_0})}{u'(c_{T+1})}$$

or

$$\left(\frac{\delta}{\delta_1}\right)^{T-t_0} \prod_{t=t_0}^T \delta_1(1+r_{t+1}) = \frac{u'(c_{t_0})}{u'(c_{T+1})}$$

with $0 < \delta < \delta_1 < 1$.

But $(\delta/\delta_1)^{T-t_0} \rightarrow 0$ as $T \rightarrow \infty$ implies that $c_T \rightarrow 0$ contrary to $\limsup_T c_T > 0$.

□

The recurrence theorem is the most general result in the literature on the properties enjoyed in a dynamic Ramsey equilibrium.

An interesting consequence of the Recurrence Theorem is $\liminf_{t \rightarrow \infty} c_t^h > 0$ ($h = 1, 2, \dots, H$), so $c_t^h > 0$ for all t . This result distinguishes the Ramsey model with borrowing constraints from its complete market general equilibrium counterparts as found in Rader ((1971), (1972) and (1981)), Bewley (1982), Coles ((1985), (1986)), Durán and Le Van (2003), Le Van and Vailakis (2003) and Becker (2012). In their settings, the more impatient household's consumption converges to zero. The most patient household's consumption has a positive limit.

The recurrence theorem tells us households $h \geq 2$ achieve the zero capital state infinitely often. It would be nice from an analytical view if once a household achieved a zero capital state, it maintained that state for all remaining times. Unfortunately, that is not the case in general. This fact is illustrated by Michael Stern's (1998) example; details appear in Becker (2006). This example is constructed using the ideas in Becker and Foias's (1987) example of an economy with an equilibrium whose aggregate capital stock repeated every other period in a two-cycle. The Becker–Foias example was constructed so that only the first household owned capital. Stern's example shows more. It describes a periodic equilibrium where the most patient household holds capital all the time, and the most impatient household has capital infinitely often.

EXAMPLE 1.1 (STERN (1998)).— *There is a two household economy $\mathcal{E}$ and a corresponding Ramsey equilibrium in which the first household's capital stock alternates between two positive levels, x_H and x_L with $0 < x_L < x_H < b$, and the second household's capital stock alternates between zero capital and a positive capital stock, x . Moreover, the aggregate stock $K_H = x_H$ and $K_L = x + x_L$ with $K_L < K_H$. This implies that the relatively impatient households can hold positive capital stocks in infinitely many periods and these positive stocks are bounded away from zero.*

This example shows something more. A Ramsey equilibrium's aggregate capital stock need not converge to the stationary equilibrium capital stock. In particular, an equilibrium path of aggregate capital can oscillate in a two-period cycle, also called a 2-cycle. The example of a two-person savers–spenders model has monotonic paths of aggregate capital accumulation and clearly converges to the long-run steady-state solution in the limit. Moreover, in that case, the more impatient individual, starting from the zero capital state, remains there for all time. What economic condition within the model is the source of fluctuations in one case, but absent in the other, stable case?

1.7.2.2. The turnpike property

The *turnpike property* obtains if every $h \geq 2$ eventually reaches a no capital position and maintains that state thereafter. This property expresses the steady-state capital position of the relatively impatient households. Stern's example shows that without additional assumptions on technology and/or preferences, the turnpike property does not obtain. Yet, the savers–spenders example shows it does hold for some economies. Therefore, it is of some interest to work out sufficient conditions for it. A particularly useful condition is: if the steady state-stock is stable (in the sense that it is the limit of the economy's equilibrium capital stock sequence), then it must be true that each household $h \geq 2$ necessarily achieves a zero capital state in finite time and maintains it thereafter. The option to return to a positive capital state is never exercised by those agents.

PROPOSITION 1.2.—*Let $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ be a Ramsey equilibrium for an economy $\mathcal{E}$. If $\lim_{t \rightarrow \infty} K_t = K_\infty$ exists, then $K_\infty = k^{\delta_1}$ and $x_t^h = 0$ for each $h \geq 2$ and all t large enough. Alternatively, if $K_t \leq k^{\delta_1}$ for all t large enough, then $\lim_{t \rightarrow \infty} K_t = K_\infty$ exists and the turnpike property obtains.*

The savers–spenders example satisfies the conditions of the latter proposition. Hence, it illustrates how convergence of the aggregate stocks to the steady-state stock is linked with the turnpike property. Stern's example of a two-cycle fails this sufficient condition for the turnpike property. This takes us closer to answering when the economy converges and when convergence might fail.

1.7.2.3. On the source of periodic equilibria

Stern's example, and the one found in Becker and Foias (1987), show how cycles emerge as equilibria without further restrictions on the production function. Becker and Foias's example shows that the aggregate stock can fluctuate in a two-cycle while the second household consumes its wage and never saves. The turnpike property holds in their example. The first household owns the aggregate capital stock. Yet, the equilibrium fails to converge to the steady-state solution. It is important to note that in both examples, the cyclic equilibrium stocks oscillate around the steady-state stocks. In the case where capital increases from one period to the next, the marginal product of capital falls, capital income declines, the wage rate rises and the total income received by the first household declines. The case where capital falls from one period to the next has a similar interpretation.

In the familiar representative household Ramsey optimal growth model, total income, $f(K)$, is monotonically increasing in capital; this income is the sum of the wage bill, $f(K) - f'(K)K$, and capital income, $f'(K)K$. In contrast, with many households and the situation of the Becker–Foias example, the total income of the first household differs from the economy-wide income by an amount equal to the wage payments made to the non-capital holding agents. In this case, the total income of

Household 1 need not be increasing in capital. Indeed, as the first household increased its capital, the total income received by it fell: capital income declined by more than the increase in the wage received by the household. The wage increased with capital accumulation due to the negative slope of the factor-price frontier. In the next period, this household naturally saves less than before, thereby inducing a subsequent increase in its income. The fluctuations in income are perfectly foreseen together with the fluctuations in factor prices. The decline in capital income as capital increases is the key to their example. This decline is technologically based; it depends on f exhibiting a (variable) elasticity of substitution less than one. The first household is also sufficiently impatient so that incentives to smooth the fluctuations in consumption by arbitrage across periods do not exist. Thus, a decline in capital income as capital increases combines with the relative impatience of the first household to provide a cyclic equilibrium capital sequence.¹²

1.7.3. Equilibrium dynamics with capital income monotonicity

A sufficient condition for capital income monotonicity is that the production function's elasticity of substitution is greater than or equal to one. This condition is satisfied by the Cobb–Douglas production function. It turns out that this technological property combines with the other assumptions on the tastes and technology to yield a convergence theorem. The formal assumption is stated as:

Capital income monotonicity (CIM): For all $K > 0$, $(d/dK)(f'(K)K) > 0$.

Becker and Foias's (1987) convergence theorem is formally stated below. Its proof rests on showing that either the aggregate capital sequence is non-increasing, or it is eventually smaller than the steady-state stocks. In either case, it is then possible to show convergence. Formal details are found in their paper.

THEOREM 1.8 (CONVERGENCE THEOREM).—*Suppose CIM holds. For an economy $\mathcal{E}$ let $\{1 + r_t, w_t, K_{t-1}, c_t^h, x_{t-1}^h\}$ be a Ramsey equilibrium. Then the sequence of aggregate capital stocks, $\{K_{t-1}\}_{t=1}^\infty$, converges (eventually monotonically) to the stationary Ramsey equilibrium aggregate stocks and the turnpike property obtains.*

12. Woodford (1989) considers a two class borrowing constrained model consisting of capitalists and workers. The former optimize their consumption-saving decision over an infinite horizon. They save, but do not work. The latter just consume their wages. The representative capitalist's income is the economy's capital income. It need not be a monotonic function of the aggregate capital stock. Woodford shows periodic and even chaotic trajectories are possible equilibria. Also, see the presentation in Guesnerie and Woodford (1992, pp. 301–311). The Ramsey equilibrium model has one fundamental difference from Woodford's model. The households all work and even the most impatient can always choose to become a capitalist by saving in the Ramsey equilibrium model.

This theorem applies to the savers–spenders theory example. More generally, for any economy with a Cobb–Douglas production function, every Ramsey equilibrium converges to the steady state. Note that the theorem does not tell us equilibrium paths are unique, or even determinate. Moreover, the convergence is eventually monotonic.¹³ Hence, the Orthodox Vision eventually holds. The theorem gives reasonable conditions for Ramsey’s long-run capital ownership conjecture to hold as the economy’s short-run dynamics take it towards that stationary equilibrium, even when all agents start off with positive capital. In particular, just as Irving Fisher thought, an economy starting with an equal distribution of capital would eventually diverge from equal wealth as more impatient agents seek ever more current consumption at the expense of future consumption.

Becker et al. (2014) proposed *maximum income monotonicity (MIM)* as a weakening of CIM. The maximum income any household can achieve given the economy’s capital stock is:

$$M(K) = f'(K)K + \frac{[f(K) - f'(K)K]}{H}.$$

This happens if the household own all the capital stock. So, MIM trivially holds in the FCE model. The formal *(MIM) Condition* is: $M(K)$ increases in K for all $K > 0$. They go on to prove MIM implies the convergence theorem obtains. They provided an example of an approximate CES example that satisfies MIM. It is

$$f_{\theta}(K) = \begin{cases} \theta g(K) + (1 - \theta)BK^{\beta} & \text{for each } K > 0, \\ 0 & \text{if } K = 0, \end{cases},$$

where $0 < \theta < 1$, $B > 0$, $0 < \beta < 1$, and

$$g(K) = A [\alpha K^{-\rho} + (1 - \alpha)H^{-\rho}]^{\frac{-1}{\rho}},$$

is a CES production function with parameters $A > 0$, $0 < \alpha < 1$, $\rho > 0$, and the constant elasticity of substitution is smaller than 1.

1.7.3.1. Periodic and chaotic Ramsey equilibria

Special equilibria have already been noted to exist where the aggregate capital stock sequences exhibit two-cycles. CIM fails in these situations. Becker and Foias (1994) considered a dynamical system governing the first player’s capital stock evolution under the assumption that all other agents began without capital and maintained that position thereafter. They showed that without the capital monotonicity property, the stationary equilibrium could fail to be a saddle point. Indeed, they found a flip bifurcation. As the production function’s elasticity of substitution falls below one, the stationary state’s saddle point property is lost. A locally stable two-cycle is created. In hindsight, MIM must also fail in this case.

13. Of course, the theorem includes other production functions so long as (CIM) holds.

Sorger (1994) constructs examples of two-person economies where there are equilibria with periodic solutions of every odd period. He also shows there is a two-agent economy in which there are multiple equilibria. His example shows that without capital income monotonicity, the steady state is not unique in the class of *all* Ramsey equilibria starting from that initial state. In his example, there is another equilibrium sequence of capital stocks from the same initial distribution of capital as in the stationary equilibrium. This alternative equilibrium is necessarily periodic. This non-stationary equilibrium has an odd period at least as great as three. His proof basically follows the type of constructions used in Becker and Foias's (1987) cycle proof.

Equilibria are locally unique if there are no other equilibria in any sufficiently close neighborhood of any given equilibrium. Local uniqueness represents determinacy. Given the predetermined value of the aggregate capital stock (when the turnpike property holds from the beginning), there is a locally unique choice of the endogenously determined value of next period's stock which gives rise to an equilibrium aggregate capital sequence. Otherwise, equilibrium is said to be indeterminate. Sorger (1994) shows that there is an economy that has an indeterminate Ramsey equilibrium. This economy fails to satisfy CIM and the turnpike property applies to agents $h \geq 2$. His proof is based on constructing a locally asymptotically stable two-cycle. That is, there is an economy with a two-cycle Ramsey equilibrium such that for any initial nearby aggregate capital stock assigned to the first household, there is an open set of choices for the next period's stock such that the resulting equilibrium aggregate stocks converge asymptotically to this two-cycle Ramsey equilibrium.

The indeterminacy of equilibrium opens the possibility that sunspot equilibria might also emerge in a Ramsey equilibrium for an economy without the capital income monotonicity property. Sorger (1994) constructs an example of a rational expectations equilibrium which is a non-trivial stochastic process. That is, the deterministic Ramsey equilibrium model can have, in some economies, stochastic equilibria based on agents' expectations that the forecast wage and rental sequences form a stochastic process. These expectations are based on something other than the model's deep taste and technology parameters. Hence, the term sunspots – uncertainty that lies outside the economy's fundamentals. As in all sunspot models, agents merely have to believe there are random influences, from whatever source, and there can be an equilibrium where those beliefs are self-justifying. He argues that this feature makes the Ramsey equilibrium model similar to overlapping generations models that exhibit sunspots.

A chaotic Ramsey equilibrium has also been constructed by Sorger (1995). This result is troubling. A perfect foresight chaotic equilibrium's sensitive dependence on initial conditions implies that real households, restricted to making approximate calculations and forecasts, would be taken by the famous unseen hand to a time path of capital holdings, consumption, wage rates and rental rates, that none had

foreseen. Ultimately, whether the given Ramsey economy exhibits a stable solution (the Convergence Theorem holds), or one of the cyclic, or even chaotic equilibria emerges, turns on the economy's elasticity of substitution in production. This is, at least, a technological parameter. However, there is at least some empirical evidence (e.g. Berndt (1976)) that it is smaller than 1. So, it is possible that the complicated dynamics found in the Ramsey equilibrium literature may yet prove to be the more interesting feature of the model than the convergence theory.

1.7.4. *Malinvaud efficiency*

Becker and Mitra (2012) ask whether or not a Ramsey equilibrium's aggregate capital sequence (and, hence its aggregate consumption sequence) is Malinvaud efficient. They showed that any Ramsey equilibrium in which the most patient household's capital is always positive for each time larger than some period, S , then that Ramsey equilibrium is Malinvaud efficient. They also proved any two-cycle Ramsey equilibrium is Malinvaud efficient. In particular, Stern's example is efficient. They left open the question of whether or not every Ramsey equilibrium is Malinvaud efficient. Becker et al. (2014) showed by counterexample an inefficient equilibrium exists. They provided a three-period cycle Ramsey equilibrium that is inefficient. Their example verifies the most patient household holds no capital infinitely often (so, Sorger's Lemma is inapplicable). This is consistent with the recurrence theorem, but not consistent with Becker and Mitra's (2012) efficiency results. In particular, inefficiency can arise in a case when the most patient agent is frequently in complete ownership of the capital stock, as well as frequently in a zero capital state. The sufficient condition that the most patient agent eventually holds all capital from some time on fails. So, from this perspective, the savers–spenders model is not emergent in this equilibrium. Becker et al. (2014) also used MIM to expand the class of models for which the resulting equilibrium capital sequence is efficient when MIM holds. In this case, the two-class society eventually emerges as the turnpike property obtains.

The possibility of an inefficient Ramsey equilibrium is a problem. The first welfare theorem can fail when borrowing constraints are present. More importantly, any reasonable notion of second-best Pareto optimality must be consistent with conditions ensuring the efficient accumulation of capital.

1.7.5. *Continuous time*

All the modeling shown here uses discrete time. Yet, Ramsey's basic framework is cast in a continuous time setup. From a modeling perspective, it makes sense to ask if this matters. The answer is YES!

Mitra and Sorger (2013) recovered the existence of Becker's (1980) steady-state result, the convergence of each Ramsey equilibrium to that steady state, the

turnpike property and the efficiency of Ramsey equilibria in a continuous time Ramsey equilibrium. They achieve this without ANY specialization of the productive technology (CIM or MIM)! The continuous time model sharpens the results and eliminates all the periodic orbits, etc. found in the discrete time theory. Note that this is also a possible foundation for savers–spenders theory since there is a clear emergence of a two-class society divided along those lines as the turnpike property always holds. They interpret their results as a cautionary tale about economic intuition when results are highly sensitive to the modeler’s time index choice. Both modeling choices have advantages and disadvantages.

1.8. Conclusion

A theoretical foundation for Mankiw’s savers–spenders theory based on the discrete time Ramsey equilibrium is covered in this chapter. This is not the ONLY possible foundation, but it is one with a developed literature engaged with finding implications from heterogeneity in discounting and whether or not the equilibria exhibit savers–spenders properties.

Important findings include that the orthodox vision of capital theory can fail in one-sector heterogeneous agent models. The first welfare theorem fails since there exist inefficient equilibria. This casts doubt on the possibility of finding second-best optima with an appropriate notion of second best. Income and wealth are concentrated, and the existence of savers and spenders is reasonably robust to alterations in the model’s details.

There are many areas left uncovered, including extensions to household objectives with recursive utility functions admitting variable time preference rates, strategic equilibria where the capitalist acts like a monopolist as it is the sole capital supplier, and modifications when there are stochastic technology or stochastic discounting shocks and a rational expectations equilibrium must be studied.

1.9. References

- Becker, R.A. (1980). On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households. *Quarterly Journal of Economics*, 95, 375–382.
- Becker, R.A. (2006). Equilibrium dynamics with many agents. In *Handbook of Optimal Growth Theory, Vol. I: Discrete Time Theory*, Dana, R.-A., Le Van, C., Mitra, T., Nishimura, K. (eds.). Springer-Verlag, Berlin, Heidelberg, New York, 385–442.
- Becker, R.A. (2012). Optimal growth with heterogeneous agents and the twisted turnpike: An example. *International Journal of Economic Theory*, 8, 27–47.
- Becker, R. and Boyd III, J. (1997). *Capital Theory, Equilibrium Analysis and Recursive Utility*. Blackwell, Oxford.

- Becker, R.A. and Foias, C. (1987). A characterization of Ramsey equilibrium. *Journal of Economic Theory*, 41, 173–184.
- Becker, R.A. and Foias, C. (1994). The local bifurcation of Ramsey equilibrium. *Economic Theory*, 4, 719–744.
- Becker, R.A. and Mitra, T. (2012). Efficient Ramsey equilibria. *Macroeconomic Dynamics*, 16(Supplement 1), 18–32.
- Becker, R.A., Boyd, J.H.I., Foias, C. (1991). The existence of Ramsey equilibrium. *Econometrica*, 59, 441–460.
- Becker, R.A., Dubey, R.S., Mitra, T. (2014). On Ramsey equilibrium: Capital ownership pattern and inefficiency. *Economic Theory*, 55, 565–600.
- Becker, R.A., Bosi, S., Le Van, C., Seegmuller, T. (2015). On existence and bubbles of Ramsey equilibrium with borrowing constraints. *Economic Theory*, 58, 329–353.
- Berndt, E.R. (1976). Reconciling alternative estimates of the elasticity of substitution. *Review of Economics and Statistics*, 58, 59–68.
- Bewley, T. (1982). An integration of equilibrium theory and turnpike theory. *Journal of Mathematical Economics*, 10, 233–267.
- Bliss, C. (1975). *Capital theory and the distribution of income*. North-Holland. Amsterdam.
- Bosi, S. and Seegmuller, T. (2010a). On the ramsey equilibrium with heterogeneous consumers and endogenous labor supply. *Journal of Mathematical Economics*, 46, 475–492.
- Bosi, S. and Seegmuller, T. (2010b). On the role of progressive taxation in a Ramsey model with heterogeneous households. *Journal of Mathematical Economics*, 46, 977–996.
- Boyd, J.H.I. (1986). Preferences, technology and dynamic equilibria. PhD thesis. Indiana University, Bloomington.
- Boyd, J.H.I. (1990a). Dynamic tax incidence with heterogeneous agents. Working Paper, University of Rochester.
- Boyd, J.H.I. (1990b). Symmetries, dynamic equilibria, and the value function. In *Conservation Laws and Symmetry: Applications to Economics and Finance*, Sato, R. and Ramachandran, R.V. (eds). Kluwer Academic Publishers. Amsterdam. 225–259.
- Cass, D. (1972). On capital accumulation in the aggregative, neoclassical model of economic growth: A complete characterization. *Journal of Economic Theory*, 4, 200–223.
- Coles, J. (1985). Equilibrium turnpike theory with constant returns to scale and possibly heterogeneous discount factors. *International Economic Review*, 26, 671–679.
- Coles, J. (1986). Equilibrium turnpike theory with time-separable utility. *Journal of Economic Dynamics and Control*, 10, 367–394.
- Drugeon, J.-P. and Wigniolle, B. (2016). On time-consistent policy rules for heterogeneous discounting programs. *Journal of Mathematical Economics*, 63, 174–187.

- Drugeon, J.-P. and Wigniolle, B. (2021). On markovian collective choice with heterogeneous quasi-hyperbolic discounting. *Economic Theory*, 72, 1257–1296.
- Durán, J. and Le Van, C. (2003). Simple proof of existence of equilibrium in a one-sector growth model with bounded or unbounded returns from below. *Macroeconomic Dynamics*, 7, 317–332.
- Fisher, I. (1930). *Theory of Interest*. MacMillan Company, New York. (Reprinted by Augustus M. Kelley, 1970).
- Guesnerie, R. and Woodford, M. (1992). Endogenous fluctuations. In J.-J. Laffont. (ed.), *Advances in Economic Theory: Sixth World Congress*, Volume II. Cambridge University Press.
- Huang, K.X.D. and Werner, J. (2000). Asset price bubbles in Arrow–Debreu and sequential equilibrium. *Economic Theory*, 15, 253–278.
- Keynes, J.M. (2015). The means to prosperity. In R. Skidelsky. (ed.), *John Maynard Keynes: The Essential Keynes*. Penguin Classics, New York. [Originally published 1933].
- Kocherlakota, N.R. (1992). Bubbles and constraints on debt accumulation. *Journal of Economic Theory*, 57, 245–256.
- Le Van, C. and Vailakis, Y. (2003). Existence of a competitive equilibrium in a one-sector growth model with heterogeneous agents and irreversible investment. *Economic Theory*, 22, 743–771.
- Malinvaud, E. (1953). Capital accumulation and efficient allocation of resources. *Econometrica*, 21, 233–368.
- Mankiw, N.G. (2000). The savers-spenders theory of fiscal policy. *American Economic Review Papers and Proceedings*, 90, 120–125.
- Mitra, T. (1979). On optimal economic growth with variable discount rates: Existence and stability results. *International Economic Review*, 20, 133–145.
- Mitra, T. and Sorger, G. (2013). On Ramsey’s conjecture. *Journal of Economic Theory*, 148, 1953–1976.
- Pakhnin, M. (2023). Collective choice with heterogeneous time preferences. *Journal of Economic Surveys*, 37.
- Phelps, E.S. and Pollak, R.A. (1968). On second-best national saving and game equilibrium growth. *Review of Economic Studies*, 35, 186–199.
- Rader, T. (1971). *The Economics of Feudalism*. Gordon and Breach, New York.
- Rader, T. (1972). *Theory of General Economic Equilibrium*. Academic Press, New York.
- Rader, T. (1981). Utility over time: The homothetic case. *Journal of Economic Theory*, 25, 219–236.
- Radner, R. (1961). Paths of economic growth that are optimal with regard only to final state. *Review of Economic Studies*, 28, 98–104.

- Ramsey, F.P. (1928). A mathematical theory of saving. *Economic Journal*, 38, 543–559.
- Sorger, G. (1994). On the structure of Ramsey equilibrium: Cycles, indeterminacy, and sunspots. *Economic Theory*, 4, 745–764.
- Sorger, G. (1995). Chaotic Ramsey equilibrium. *International Journal of Bifurcation and Chaos*, 5, 373–380.
- Stern, M. L. (1998). Personal communication.
- Thibault, E. (2005). Existence and specific characters of rentiers: A savers-spenders theory approach. *Economic Theory*, 25, 401–419.
- Tirole, J. (1982). On the possibility of speculation under rational expectations. *Econometrica*, 50, 1163–1181.
- Tirole, J. (1985). Asset bubbles and overlapping generations. *Econometrica*, 53, 1071–1100.
- Woodford, M. (1989). Imperfect financial intermediation and complex dynamics. In *Economic Complexity: Chaos, Sunspots, Bubbles, and Nonlinearity*, Barnett, W.A., Geweke, J., Shell, K. (eds). Cambridge University Press, Cambridge.

2

Aggregation of the Production System: Aggregate and Multisectoral Models of Optimal Growth

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2.1. Introduction

Kydland and Prescott's pioneering contribution (1982), which was the origin of the Real Business Cycles theory, placed optimal growth in an aggregate model (initially proposed by Ramsey (1928)) at the heart of contemporary macroeconomic analysis. Real business cycles see productivity shocks at the center of the analysis of economic fluctuations, and the Ramsey model makes it possible to guarantee that in a perfectly deterministic environment, there is a single optimal trajectory with monotone convergence towards equilibrium in the long term, corresponding to the modified Golden Rule. Consequently, economic fluctuations are simply optimal adaptations by agents, consumers and producers with respect to stochastic shocks on technology, which modify the global productivity of factors.

However, it appears that this vision is largely dependent on the assumption of an aggregate representation of the economy. Although practical and easy to manipulate,

this formulation does not allow us to accurately account for the complex arbitrations between consumption and investment. In a real economy, these arbitrations depend on choices of allocation of capital and labor between different sectors of production. Indeed, in the aggregate model, it is implicitly assumed that there is infinite elasticity of substitution between current capital stock and investment. Formally, this is the same as considering a consumption function, generally called a social production function, characterized by a linear relationship between current capital stock and investment.

Multi-sector models were largely studied in the 1970s; however, their apparent complexity seemed contradictory to the simplicity of the aggregate model, and consequently, the large majority of theoretical work in macroeconomics prioritized an aggregate model of economics. Furthermore, in the multi-sector models, with at least one distinct consumption good and one distinct investment good, we will see that, unlike the Ramsey model, the convergence and monotone properties of the optimal trajectory are no longer automatically verified. The choice between consumption and investment is more complex and depends, notably, on the relative intensities of capital in the various sectors of production. Formally, the social production function is characterized here by a nonlinear relationship between current capital stock and investment. The objective then becomes to precisely understand the theoretical mechanism guaranteeing the convergence and monotone nature of the optimal trajectory in the aggregate model and thus to see how the consideration of at least two distinct production sectors can affect this mechanism. In order to do this, we will introduce an aggregation procedure that will make it possible to represent aggregate models and multi-sector models in a similar way through the use of a reduced form. This will then allow us to study both formulations in parallel, in order to precisely identify the distinctive elements.

This chapter is organized as follows. Section 2.2 gives a detailed introduction to the aggregation procedure mentioned above, which allows us to obtain an integrated representation of the growth model irrespective of the number of sectors being considered. We will begin, of course, with the aggregate model, by demonstrating the specific features that limit the scope of its properties. We will then offer a detailed description of the aggregation method for the two-sector model, which has proven to be easily generalized to any dimension.

Section 2.3 will introduce the standard aggregate optimal growth model. We will establish the turnpike theorem in a detailed manner. This theorem guarantees that, in any generality, there exists a single optimal trajectory that converges in a monotone manner towards the Modified Golden Rule. Based on the aggregation technique, the demonstration procedure relies on the consideration of a reduced formulation of the model, which is much more general than the aggregate model. We thus identify the key element at the heart of the monotone convergence result.

Section 2.4 offers a very precise introduction to the two-sector model of optimal growth. By again using the results from the aggregation procedure, we will give

a detailed overview of each step in the construction of the model starting from production technologies and the utility function for the representative consumer, which allows us to also construct a reduced formulation, which is shown to be characterized by a property that is significantly different from that of the aggregate model. This allows us to clearly make explicit all the sufficient conditions for the monotone convergence of the optimal trajectory. We will thus establish a conditional turnpike theorem.

Finally, in section 2.5, we will discuss the general case of an economy comprising a consumption good and n investment goods. We will demonstrate a generalization of the results obtained earlier. Section 2.6 concludes this chapter.

2.2. Aggregation of the production system

A large part of macroeconomic literature is focused on models that offer an entirely aggregate view of the economy, highlighting aggregates such as the Gross Domestic Product (GDP), investment or savings. However, this level of aggregation prevents an accurate representation of all the flows of manpower or capital that are constantly acting between different sectors. All too often, the chief argument for using an aggregate representation is its simple and intuitive character. In this section, we will show that it is also possible to take into account a disaggregated description of the economy while still preserving a level of simplicity that is comparable to the aggregated version. We will first look at the specific features of the standard aggregate model and then successively introduce two-sector and multi-sector versions of the economy, using an aggregation procedure. We will adopt a discrete time formulation ($t = 0, 1, 2, \dots, \infty$).

2.2.1. The specific case of the aggregate production sector

We will consider an aggregate economy in which the global production level Y_t is obtained from the capital K_t and the labor L_t through a neo-classical standard production function $F(K_t, L_t)$, which is homogeneous in the first degree and concave. The homogeneity in the first degree clearly makes it possible to guarantee constant returns to scale, while the concavity guarantees decreasing returns of capital and labor. This economy consists of a large number of identical firms operating in pure and perfect competition.

Let C_t denote the aggregate consumption, and $I_t = K_{t+1} - K_t$ denote the aggregate investment. If it is assumed that the capital depreciates over every period at a rate $\mu \in [0, 1]$, then in a closed economy, we will obtain the following equation for the goods market equilibrium:

$$F(K_t, L_t) = C_t + I_t + \mu K_t \Leftrightarrow K_{t+1} = F(K_t, L_t) + (1 - \mu)K_t - C_t$$

Using the assumption of an active population growing at the rate $n \geq 0$, or $L_{t+1} = (1+n)L_t$, we can rewrite the preceding equation in intensive form from the definition of per-capita capital $k_t = K_t/L_t$ and per-capita consumption $c_t = C_t/L_t$:

$$(1+n)k_{t+1} = f(k_t) + (1-\mu)k_t - c_t \quad [2.1]$$

where $f(k) = F(k, 1)$ satisfies the following standard properties:

ASSUMPTION 2.1.— $f(k)$ is C^2 and such that $\forall k > 0$, $f'(k) > 0$, $f''(k) < 0$. Furthermore, $f(0) = 0$, $f'(0) = +\infty$ and $f'(+\infty) = 0$.

With i_t denoting the per-capita investment so that

$$i_t = (1+n)k_{t+1} - k_t$$

from equation [2.1], we obtain a consumption function that is expressed as a *social production function* relating current capital stock and investment such that

$$c_t = f(k_t) - i_t - \mu k_t = f(k_t) + (1-\mu)k_t - (1+n)k_{t+1} \equiv T(k_t, k_{t+1}) \quad [2.2]$$

It must be noted that the cross derivative of the social production function is zero, i.e.

$$T_{12}(k_t, k_{t+1}) \equiv \frac{\partial^2 T(k_t, k_{t+1})}{\partial k_t \partial k_{t+1}} = 0$$

which means that there is infinite elasticity of substitution between the current capital stock and investment. As we will see later, such a feature will significantly impact the properties of the aggregate model.

2.2.2. A two-sector production structure

Let us now consider an economy comprising two sectors, one producing a consumption good y_0 and the other producing an investment good y . In order to consider per-capita variables, total labor is normalized to 1. Each good is produced using a standard technology with constant returns:

$$y_0 = f^0(k_0, l_0)$$

$$y = f^1(k_1, l_1)$$

with $k_0 + k_1 \leq k$, k being the total capital stock, and $l_0 + l_1 \leq 1$ being the normalized labor.

ASSUMPTION 2.2.— Each production function $f^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $i = 0, 1$, is C^2 , increasing with each argument, concave, homogeneous in the first degree and such that $\forall x > 0$, $f_1^i(0, x) = f_2^i(x, 0) = +\infty$, $f_1^i(+\infty, x) = f_2^i(x, +\infty) = 0$.

By assuming, as in the aggregate model, that the total population increases at the constant rate $n \geq 0$ and that the capital depreciates over each period at the rate $\mu \in [0, 1]$, we will formulate the capital accumulation equation that describes the investment good market equilibrium as follows:

$$(1 + n)k_{t+1} = y_t + (1 - \mu)k_t. \quad [2.3]$$

Indeed, the per-capita production of the investment good y_t must make it possible to replace the capital depreciation over the period μk_t , and to satisfy the net investment requirements knowing that the active population increases, $(1 + n)k_{t+1} - k_t$. Using i_t as the notation introduced in the aggregate model for the per-capita investment, it comes that

$$y_t = i_t + \mu k_t$$

2.2.2.1. Temporary equilibrium and the social production function

If we take time t , and consider that the capital stock accumulated up to this time, k_t , and the production level of the investment good y_t are known, the question is to determine the best allocation of the capital and labor between the two sectors so as to obtain the largest production level of the consumption good. The standard method used to solve this kind of optimal allocation problem is to consider that the representative firm in each sector must determine its optimal demands for capital and labor so as to maximize its profits. This is therefore the same as considering the following two programs:

$$\begin{aligned} \max_{\{k_0, l_0\}} \pi_0 &= p_0 f^0(k_0, l_0) - w_1 k_0 - w_0 l_0 \\ \max_{\{k_1, l_1\}} \pi_1 &= p_1 f^1(k_1, l_1) - w_1 k_1 - w_0 l_1 \end{aligned} \quad [2.4]$$

where p_0 represents the price of a consumption good, p_1 represents the price of an investment good, w_1 represents the capital remuneration rate, and w_0 represents the wage rate. It would clearly be helpful to choose a *numéraire* good. This choice has traditionally related to the consumption good, such that we introduce the relative price of the investment good $q \equiv p_1/p_0$, and we can use the normalization $p_0 = 1$ without any loss of generality.

In a context of pure and perfect competition, it can be easily deduced that the solution to the two profit-maximization problems [2.4] is perfectly equivalent to the solution of the following intersectoral allocation problem:

$$\begin{aligned}
 T(k, y) = \max_{k_0, l_0, k_1, l_1} & \quad f^0(k_0, l_0) \\
 \text{s.t.} & \quad y \leq f^1(k_1, l_1) \\
 & \quad k_0 + k_1 \leq k \\
 & \quad l_0 + l_1 \leq 1 \\
 & \quad k_0, k_1, l_0, l_1 \geq 0
 \end{aligned} \tag{2.5}$$

The value function of this program $T(k, y)$ is called the *social production function* and describes the boundary of the set of production possibilities. Indeed, $T(k, y)$ gives the maximum level of production of the consumption good that can thus be consumed by the representative consumer. It is therefore presented as a generalization of the function [2.2] considered in the aggregate model. In this context, it provides a way to aggregate the two-sector production structure and a formulation that is comparable to the formulation of the aggregate model. Furthermore, the function $T(k, y)$ describes a *temporary equilibrium* as it is obtained from the optimal choices for the use of capital and labor in each sector, given the capital stock k and the level of production of the investment good y . The social production function thus makes it possible to characterize the equilibrium over one period.

2.2.2.2. Characterization of the social production function

Let us consider the optimization program [2.5]. The associated Lagrangian is

$$\mathcal{L} = f^0(k_0, l_0) + q[f^1(k_1, l_1) - y] + w_1[k - k_0 - k_1] + w_0[1 - l_0 - l_1] \tag{2.6}$$

where, as we saw earlier, $q \equiv p_1/p_0$ is the relative price of the investment good, w_1 is the capital remuneration rate, and w_0 is the wage rate. All these prices are thus expressed in terms of the price of the normalized price of the consumption good $p_0 = 1$. The first-order conditions for a maximum are as follows:

$$\begin{aligned}
 \frac{\partial \mathcal{L}}{\partial k_0} &= f_1^0(k_0, l_0) - w_1 = 0 \\
 \frac{\partial \mathcal{L}}{\partial l_0} &= f_2^0(k_0, l_0) - w_0 = 0 \\
 \frac{\partial \mathcal{L}}{\partial k_1} &= qf_1^1(k_1, l_1) - w_1 = 0 \\
 \frac{\partial \mathcal{L}}{\partial l_1} &= qf_2^1(k_1, l_1) - w_0 = 0
 \end{aligned} \tag{2.7}$$

From these, we can easily derive the following standard relationships:

$$\begin{aligned}
 w_1 &= f_1^0(k_0, l_0) = q f_1^1(k_1, l_1) \\
 w_0 &= f_2^0(k_0, l_0) = q f_2^1(k_1, l_1) \\
 q &= \frac{f_1^0(k_0, l_0)}{f_1^1(k_1, l_1)} = \frac{f_2^0(k_0, l_0)}{f_2^1(k_1, l_1)}
 \end{aligned} \tag{2.8}$$

which associate the prices with the marginal productivities of the factors. It must be noted that the first-order conditions for the profit maximization programs [2.4] are perfectly identical to those where $q = p_1/p_0$ and $p_0 = 1$.

The solution to these conditions yields the optimal functions for the demand for capital and labor in each sector, i.e. $k_0(k, y)$, $l_0(k, y)$, $k_1(k, y)$ and $l_1(k, y)$. Using Assumption 2.2, the constraints on program [2.5] are necessarily saturated, and from this, it can therefore be derived that:

$$\begin{aligned}
 T(k, y) &= f^0(k_0(k, y), l_0(k, y)) \\
 y &= f^1(k_1(k, y), l_1(k, y)) \\
 k &= k_0(k, y) + k_1(k, y) \\
 1 &= l_0(k, y) + l_1(k, y)
 \end{aligned} \tag{2.9}$$

Based on the first-order conditions, and using Assumption 2.2, the basic properties of the function $T(k, y)$ were established by Benhabib and Nishimura (1979; 1981):

LEMMA 2.1.— *Under Assumption 2.2, the function $T(k, y)$ is C^2 and concave non-strictly, with $T_{11}(k, y) < 0$.*

The non-strict concavity of $T(k, y)$ implies that its Hessian matrix

$$H_T = \begin{pmatrix} T_{11}(k, y) & T_{12}(k, y) \\ T_{12}(k, y) & T_{22}(k, y) \end{pmatrix}$$

is singular, which means that $|H_T| = T_{11}(k, y)T_{22}(k, y) - T_{12}(k, y)^2 = 0$. Based on these general properties, we then need to make explicit the properties of the first and second derivatives of the social production function, $T(k, y)$.

A first proposition makes it possible to precisely determine the significance of the first derivatives of $T(k, y)$.

PROPOSITION 2.1.— *Using Assumption 2.2, $T(k, y)$ is an increasing function in k and a decreasing function in y with*

$$T_1(k, y) = w_1(k, y) \text{ et } -T_2(k, y) = q(k, y).$$

PROOF.— Starting from [2.9], we obtain the first derivatives of $T(k, y)$:

$$\begin{aligned} T_1(k, y) &= f_1^0(k_0(k, y), l_0(k, y)) \frac{\partial k_0(k, y)}{\partial k} + f_2^0(k_0(k, y), l_0(k, y)) \frac{\partial l_0(k, y)}{\partial k} \\ T_2(k, y) &= f_1^0(k_0(k, y), l_0(k, y)) \frac{\partial k_0(k, y)}{\partial y} + f_2^0(k_0(k, y), l_0(k, y)) \frac{\partial l_0(k, y)}{\partial y} \end{aligned}$$

as well as the following expressions:

$$\begin{aligned} \frac{\partial k_0(k, y)}{\partial k} &= 1 - \frac{\partial k_1(k, y)}{\partial k}, \quad \frac{\partial k_0(k, y)}{\partial y} = -\frac{\partial k_1(k, y)}{\partial y} \\ \frac{\partial l_0(k, y)}{\partial k} &= -\frac{\partial l_1(k, y)}{\partial k}, \quad \frac{\partial l_0(k, y)}{\partial y} = -\frac{\partial l_1(k, y)}{\partial y} \\ 1 &= f_1^1(k_1(k, y), l_1(k, y)) \frac{\partial k_1(k, y)}{\partial y} + f_2^1(k_1(k, y), l_1(k, y)) \frac{\partial l_1(k, y)}{\partial y} \\ 0 &= f_1^1(k_1(k, y), l_1(k, y)) \frac{\partial k_1(k, y)}{\partial k} + f_2^1(k_1(k, y), l_1(k, y)) \frac{\partial l_1(k, y)}{\partial k} \end{aligned}$$

By substituting these expressions in $T_1(k, y)$ and $T_2(k, y)$, and by using [2.8], it comes that

$$\begin{aligned} T_1(k, y) &= f_1^0 - \left(f_1^0 \frac{\partial k_1(k, y)}{\partial k} + f_2^0 \frac{\partial l_1(k, y)}{\partial k} \right) \\ &= f_1^0 - q \left(f_1^1 \frac{\partial k_1(k, y)}{\partial k} + f_2^1 \frac{\partial l_1(k, y)}{\partial k} \right) \\ &= f_1^0(k_0(k, y), l_0(k, y)) = q f_1^1(k_1(k, y), l_1(k, y)) = w_1(k, y) \\ T_2(k, y) &= -f_1^0 \frac{\partial k_1(k, y)}{\partial y} - f_2^0 \frac{\partial l_1(k, y)}{\partial y} = -q \left(f_1^1 \frac{\partial k_1(k, y)}{\partial y} + f_2^1 \frac{\partial l_1(k, y)}{\partial y} \right) \\ &= -\frac{f_1^0(k_0(k, y), l_0(k, y))}{f_1^1(k_1(k, y), l_1(k, y))} = -\frac{f_2^0(k_0(k, y), l_0(k, y))}{f_2^1(k_1(k, y), l_1(k, y))} = -q(k, y) \end{aligned} \quad \square$$

This proposition shows that $T_1(k, y)$ corresponds to the capital remuneration rate, while $-T_2(k, y)$ corresponds to the price of the investment good. It also shows that $T(k, y)$ describes the conventional arbitration between investment/saving and consumption. Indeed, at a constant investment level, i.e. with the level of production y being constant, any increase in the capital stock k makes it possible to increase the consumption $c = T(k, y)$. On the contrary, with a constant capital rate, any increase in the level of investment must be compensated by decreased consumption.

What remains now is to specify the properties of the second derivatives of $T(k, y)$. From Lemma 2.1, it is known that $T(k, y)$ is a concave function with

$$T_{11}(k, y) = \frac{\partial w_1}{\partial k}(k, y) < 0 \text{ and } T_{22}(k, y) = -\frac{\partial q}{\partial y}(k, y) \leq 0. \quad [2.10]$$

The first equality shows that the capital remuneration rate $w_1(k, y)$ is a strictly decreasing function of the capital. This corresponds to the conventional property of diminishing returns on capital. The second equation shows that the price of the

investment good $q(k, y)$ is an increasing function of its production level y . However, the concavity of $T(k, y)$ does not allow us to determine the sign of the cross-derivative:

$$T_{12}(k, y) = \frac{\partial w}{\partial y}(k, y) = T_{21}(k, y) = -\frac{\partial q}{\partial k}(k, y). \quad [2.11]$$

This point will be examined in the next proposition. We must initially introduce the technical coefficients of capital and labor in each sector:

$$a_{00}(k, y) = \frac{l_0}{y_0}, \quad a_{10}(k, y) = \frac{k_0}{y_0}, \quad a_{01}(k, y) = \frac{l_1}{y}, \quad a_{11}(k, y) = \frac{k_1}{y}. \quad [2.12]$$

Therefore, a_{0j} measures the quantity of labor used to produce one unit of the good $j = 0, 1$, while a_{1j} measures the quantity of capital used to produce one unit of the good $j = 0, 1$. This allows us to define a fundamental parameter for analyzing the two-sector model: the *relative difference in capital intensity* between the two sectors:

$$b(k, y) \equiv a_{01}(k, y) \left(\frac{a_{11}(k, y)}{a_{01}(k, y)} - \frac{a_{10}(k, y)}{a_{00}(k, y)} \right). \quad [2.13]$$

The sign of $b(k, y)$ is therefore positive (negative) if and only if the investment good sector is relatively more (less) capital intensive than the consumption good sector. Based on this parameter, we obtain the following proposition:

PROPOSITION 2.2.— *Under Assumption 2.2, the second derivatives of $T(k, y)$ are such that*

$$\begin{aligned} T_{11}(k, y) &< 0, \quad T_{22}(k, y) = T_{11}(k, y) [b(k, y)]^2 \leq 0, \\ T_{12}(k, y) &= T_{21}(k, y) = -T_{11}(k, y)b(k, y). \end{aligned}$$

PROOF.— Starting from the property of homogeneity of the production functions, we obtain:¹

$$\begin{aligned} y_0 &= k_0 f_1^0 + l_0 f_2^0 & \Leftrightarrow & \quad 1 = \frac{k_0}{y_0} w_1 + \frac{l_0}{y_0} w_0 \\ y &= k_1 f_1^1 + l_1 f_2^1 & \quad 1 &= \frac{k_1}{y} \frac{w_1}{q} + \frac{l_1}{y} \frac{w_0}{q} \end{aligned} \quad [2.14]$$

By using the definition of the technical coefficients [2.12], it is possible to rewrite the equations on the right-hand side in matrix form:

$$\begin{pmatrix} w_0 & w_1 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & q \end{pmatrix}. \quad [2.15]$$

1. In order to simplify the notations, there is no mention of the dependence of the different functions with respect to (k, y) .

This equation describes the factor price frontier and corresponds to the equality between the prices and production costs. The total differentiation of this equation yields:

$$(dw_0 \ dw) \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} + (w_0 \ w) \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = (0 \ dq) \quad [2.16]$$

From [2.12], it is also possible to totally differentiate the technical coefficients to get the following:

$$da_{00} = \frac{y_0 dl_0 - l_0 dy_0}{(y_0)^2}, \quad da_{10} = \frac{y_0 dk_0 - k_0 dy_0}{(y_0)^2}, \\ da_{01} = \frac{y dl_1 - l_1 dy}{(y)^2}, \quad da_{11} = \frac{y dk_1 - k_1 dy}{(y)^2}$$

Upon substituting these expressions in the second term on the left-hand side of equation [2.16], it comes that:

$$(w_0 \ w_1) \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = \begin{pmatrix} \frac{y_0(dw_0 dl_0 + w_1 dk_0) - dy_0(w_0 l_0 + w_1 k_0)}{(y_0)^2} \\ \frac{y(dw_0 dl_1 + w_1 dk_1) - dy(w_0 l_1 + w_1 k_1)}{(y)^2} \end{pmatrix}^t \quad [2.17]$$

where t denotes the transpose of the vector. From expressions [2.14] and the price equations [2.8], we find

$$y_0 = w_1 k_0 + w_0 l_0 \\ qy = w_1 k_1 + w_0 l_1$$

Upon substituting these expressions in [2.17] it comes that

$$(w_0 \ w_1) \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = \begin{pmatrix} \frac{w_0 dl_0 + w_1 dk_0 - dy_0}{y_0} \\ \frac{w_0 dl_1 + w_1 dk_1 - qdy}{y} \end{pmatrix}^t$$

Finally, the total derivative of $y_0 = f^0(k_0, l_0)$ and $y = f^1(k_1, l_1)$, considering the first-order conditions [2.7] yields

$$dy_0 = w_1 dk_0 + w_0 dl_0 \\ qdy = w_1 dk_1 + w_0 dl_1$$

such that²

$$(w_0 \ w_1) \begin{pmatrix} da_{00} & da_{01} \\ da_{10} & da_{11} \end{pmatrix} = 0.$$

2. This result is a direct consequence of the envelope theorem.

It can therefore be derived that

$$(dw_0 \ dw_1) \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = (0 \ dq).$$

Solving this system by eliminating dw_0 therefore yields

$$\frac{dq}{dw_1} = b.$$

Let us now consider the cross derivative $T_{12}(k, y)$. Starting from system [2.15], we can use the fact that at the temporary equilibrium, the price relative to the investment good q can also be written as a function of the capital remuneration rate w_1 , and from [2.10] and [2.11], it then follows that:

$$T_{12} = -\frac{\partial q}{\partial w_1} \frac{\partial w_1}{\partial k} = -T_{11}b.$$

By proceeding similarly for $T_{22}(k, y)$, it finally comes that

$$T_{22} = -\frac{\partial q}{\partial w_1} \frac{\partial w_1}{\partial y} = \left(\frac{\partial q}{\partial w_1}\right)^2 \frac{\partial w_1}{\partial k} = T_{11}b^2 \leq 0. \quad \square$$

Proposition 2.2 therefore shows that, unlike in an aggregate model, the cross derivative $T_{12}(k, y)$ is generally non-zero. Moreover, its sign is determined by the sign of the relative difference in capital intensity between the two sectors $b(k, y)$. Such a property will be crucial in interpreting later results on the local stability of the equilibrium in the two-sector model of optimal growth. Finally, it can be noted that in the case of an aggregate model, which can be identified here if both sectors have identical technologies, we have $b = 0$, and we will once again get a value of zero for the cross-derivative of the social production function, which represents infinite elasticity of substitution between the current capital stock and investment. It must be noted that in this case, we also have $T_{22} = 0$.

2.2.3. Generalization to an economy with $n + 1$ sectors

Literature on multi-sector models generally consider an economy with $n + 1$ sectors, made up of a pure consumption good y_0 and n investment goods y_j , $j = 1, \dots, n$. Since the labor supply is assumed to be inelastic, the total labor is normalized to 1. Each good is produced using a standard technology with constant returns:

$$y_0 = f^0(k_{10}, \dots, k_{n0}, l_0)$$

$$y_j = f^j(k_{1j}, \dots, k_{nj}, l_j)$$

where k_{ij} is the quantity of the investment good i used in the production of the good j , $\sum_{j=0}^n k_{ij} \leq k_i$, k_i being the total stock of capital i , and $\sum_{j=0}^n l_j \leq 1$.

ASSUMPTION 2.3.— *Each production function $f^i: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $i = 0, 1, \dots, n$, is C^2 , increasing with each argument, concave and homogeneous in the first degree.*

As in the two-sector case studied earlier, for any given vector $(k, y) = (k_1, \dots, k_n, y_1, \dots, y_n)$, we will consider the temporary equilibrium obtained from the optimal allocation of production factors between the $n + 1$ sectors:

$$\begin{aligned}
 T(k, y) &= \max_{k_{ij}, l_j} f^0(k_{10}, \dots, k_{n0}, l_0) \\
 \text{s.c. } y_j &\leq f^j(k_{1j}, \dots, k_{nj}, l_j) \quad j = 1, \dots, n \\
 \sum_{i=0}^n k_{ji} &\leq k_j \quad j = 1, \dots, n \\
 \sum_{i=0}^n l_i &\leq 1 \\
 k_{ij}, l_i &\geq 0 \quad i = 0, \dots, n, \quad j = 1, \dots, n
 \end{aligned} \tag{2.18}$$

As Benhabib and Nishimura (1979; 1981) show, the social production function $T(k, y)$ is C^2 and non-strictly concave.

By using the same techniques as in section 2.2.2, it is also possible to precisely characterize the first and secondary derivatives of the social production function. First, it is easy to demonstrate, from the first-order conditions in program [2.18], that the first derivatives correspond to the remuneration rates of the various capital stocks and to the prices of the investment goods:

$$\begin{aligned}
 \frac{\partial T}{\partial k_j}(k, y) &= w_j(k, y), \quad j = 1, \dots, n \\
 \frac{\partial T}{\partial y_j}(k, y) &= -q_j(k, y), \quad j = 1, \dots, n
 \end{aligned} \tag{2.19}$$

For the second cross-derivatives, we obtain:

$$\begin{aligned}
 T_{12}(k, y) &= -T_{11}(k, y)B \\
 &= -T_{11}(k, y) \begin{pmatrix} b_{110} & \cdots & b_{1n0} \\ \vdots & \ddots & \vdots \\ b_{n10} & \cdots & b_{nn0} \end{pmatrix} \begin{pmatrix} a_{01} & & 0 \\ & \ddots & \\ 0 & & a_{0n} \end{pmatrix}
 \end{aligned} \tag{2.20}$$

where

$$b_{ji0} = \frac{a_{ji}}{a_{0i}} - \frac{a_{j0}}{a_{00}}$$

is the relative difference in capital intensity j between the investment good sector i and the consumption good sector. It can then be noted that if $n = 1$, we indeed have $B = b$ with b defined in [2.13]. Similarly, we have

$$T_{22}(k, y) = B^t T_{11}(k, y) B$$

Based on this aggregate procedure, we will now see that the different properties of the social production function, based on whether the model is aggregate or multi-sector, significantly affect the characteristics of the growth path.

2.3. The aggregate model of optimal growth

Let us first consider the aggregate model of optimal growth, better known as *Ramsey's model* (1928).

2.3.1. The production sector

The production sector in the aggregate model is described in section 2.2.1. The classical conditions for maximizing the aggregate profit

$$\Pi_t = F(K_t, L_t) - r_t K_t - w_t L_t$$

where r_t is the real interest rate and w_t is the wage rate, obviously yield

$$r_t = \frac{\partial F(K_t, L_t)}{\partial K_t} \text{ and } w_t = \frac{\partial F(K_t, L_t)}{\partial L_t}$$

Using the reduced form of the production function, $f(k)$, respecting Assumption 2.1, we easily get

$$r_t = f'(k_t) \text{ et } w_t = f(k_t) - k_t f'(k_t) \quad [2.21]$$

The real interest rate thus corresponds to the marginal capital productivity, and the wage rate corresponds to the marginal labor productivity. In pure and perfect competition, these prices are considered to be given for producers and consumers.

2.3.2. Consumers

Consider an economy populated by N_t agents with identically infinite lifespans. To be consistent with the preceding section, it is assumed that the total population also increases at the rate $n \geq 0$, i.e. $N_{t+1} = (1+n)N_t$. The representative consumer offers an (inelastic) unit of work over a period and derives utility from consumption based on the following function:

ASSUMPTION 2.4.– $u(c)$ is C^2 and such that $\forall c > 0$, $u'(c) > 0$, $u''(c) < 0$. Furthermore, $u(0) = 0$, $u'(0) = +\infty$ and $u'(+\infty) = 0$.

Equilibrium on the labor market necessarily implies that $N_t = L_t$. The equation for per-capita capital accumulation [2.1] can thus be interpreted as an individual constraint. Using the perfect foresight assumption, and considering the production sector presented earlier, the representative consumer then maximizes the following intertemporal utility function:

$$\begin{aligned} \max_{\{c_t\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t u(c_t) \\ \text{s.c.} \quad & (1+n)k_{t+1} = f(k_t) + (1-\mu)k_t - c_t \\ & k_0 \text{ given} \end{aligned} \quad [2.22]$$

where $\delta \in (0, 1]$ represents the discount factor, which allows the measurement of its preference rate for the present. As δ decreases, the representative agent gets more impatient and refuses to postpone consumption to the future.

2.3.3. The reduced form of the model

In the literature, it is common to reformulate the intertemporal optimization program for the representative agent in a reduced form that is much more general in reality. Starting from the fundamental elements of the model, and especially the social production function [2.2], the following *indirect utility function* can be defined:

$$V(k_t, k_{t+1}) \equiv u(T(k_t, k_{t+1})) = u(f(k_t) + (1-\mu)k_t - (1+n)k_{t+1}) \quad [2.23]$$

Using Assumption 2.1–2.4, $V(x, y)$ is increasing in x , decreasing in y and strictly concave. The maximization program [2.22] can be rewritten as follows:

$$\begin{aligned} \max_{\{k_t\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \\ \text{s.c.} \quad & (k_t, k_{t+1}) \in \mathcal{D} \\ & k_0 \text{ given} \end{aligned}$$

with

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^2 / \frac{(1-\mu)k_t}{1+n} \leq k_{t+1} \leq \frac{f(k_t) + (1-\mu)k_t}{1+n} \right\}$$

being the set of admissible capital accumulation paths that are obtained simply from the capital accumulation equation [2.1] and the standard positivity conditions $k_t \geq 0$, $c_t \geq 0$, $\forall t \geq 0$.

Although the usefulness is relatively limited for the presentation of the aggregate model for optimal growth, this formulation is very general in reality and allows the synthesis of a large number of different models, especially the multi-sector formulations, as we will see in the following section. Furthermore, this formulation makes it possible to precisely clarify the central element that determines the fundamental properties of the optimal growth pathways, as we will see in this chapter and in the chapter entitled “Multisector models of optimal growth and complex dynamics” contained in Volume 4 of this series on *Fluctuations and Stabilization Policies*.

2.3.4. Intertemporal equilibrium

The intertemporal equilibrium for our model is obtained from the first-order condition for an internal maximum that corresponds to the standard Euler equation:

$$V_2(k_t, k_{t+1}) + \delta V_1(k_{t+1}, k_{t+2}) = 0 \quad [2.24]$$

where $V_i(x, y)$, $i = 1, 2$, corresponds to the partial derivative of the indirect utility function $V(x, y)$ with respect to its i -th argument.

It can easily be seen that the Euler equation corresponds to a second-order difference equation. Using Assumptions 2.1 and 2.4, it can also be deduced that the indirect utility function $V(k_t, k_{t+1})$ is concave (also see the following section where this property is discussed) such that the second-order condition is necessarily satisfied. Since the problem is formulated with an infinite horizon, it is also necessary to consider the following transversality condition:

$$\lim_{t \rightarrow +\infty} \delta^t k_t V_1(k_t, k_{t+1}) = 0 \quad [2.25]$$

The Euler equation and the transversality condition thus provide a set of conditions that are sufficient for an optimal solution to program [2.35].³ Thus, any capital path $\{k_t\}_{t=0}^{+\infty}$ starting from a given k_0 that satisfies conditions [2.24]–[2.25] is an *intertemporal equilibrium*.

2.3.5. Stationary state and linearized dynamic system

Starting from the definition of the indirect utility function [2.23], we can easily obtain the two partial derivatives

$$\begin{aligned} V_1(k_t, k_{t+1}) &= u'(c_t)[f'(k_t) + 1 - \mu] \\ V_2(k_t, k_{t+1}) &= -(1 + n)u'(c_t) \end{aligned} \quad [2.26]$$

3. See Michel (1990).

which allow us to rewrite the Euler equation [2.24] as follows:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{\delta}{(1+n)} [f'(k_{t+1}) + 1 - \mu] = 0 \quad [2.27]$$

This represents the classical intertemporal arbitrage between consumption and saving: the left-hand side represents the marginal substitution rate of consumption between two periods, and the right-hand side represents the returns from saving weighted by the degree of impatience of the representative agent. For a given real interest rate, the more impatient the agent (i.e. the smaller δ is), the greater the agent's tendency to limit their consumption today in order to save more and thus increase their future consumption. It must also be noted that the intertemporal substitution capacities of the representative agent's consumption also depend on the degree of concavity of the utility function, which affects the marginal utility ratios between two periods. Thus, the more concave the utility, the greater the agent's preference to consume in the present and thus limit variations in utility.

Let us note that we can also rewrite the transversality condition [2.25] as follows:

$$\lim_{t \rightarrow +\infty} \delta^t u'(c_t) k_{t+1} = 0$$

which then means that the discounted value of the capital in terms of utility must be zero at infinity. In other words, it is not optimal to save and to keep a capital stock with positive value till the end of time. It is preferable for the consumer to transform this capital into a consumption good and to consume this to enhance their utility level.

The stationary state is obtained by considering that $k_{t+1} = k_t = k_\delta^*$ and $c_{t+1} = c_t = c_\delta^*$ in the preceding Euler equation. We thus obtain the Modified Golden Rule k_δ^* defined as follows:

$$f'(k_\delta^*) = \frac{1+n}{\delta} - (1 - \mu) \quad [2.28]$$

It can be noted that if the discount factor is such that $\delta = 1$ and the representative agent is totally patient, we obtain the standard Golden Rule $f'(k_\delta^*) = \mu + n$. In all cases, the associated level of consumption is $c_\delta^* = f(k_\delta^*) - (\mu + n)k_\delta^* > 0$.

From the conditions imposed on the fundamentals of the model, it can easily be derived that:

PROPOSITION 2.3.— *Under Assumptions 2.1 and 2.4, there exists one unique stationary state, the modified Golden Rule k_δ^* , solution to equation [2.28].*

In order to study the local stability properties of the stationary state, it is helpful to linearize the Euler equation around the Modified Golden Rule. By using the reduced form, we obtain the following equation:

$$\delta V_{12}^* X_{t+2} + (\delta V_{11}^* + V_{22}^*) X_{t+1} + V_{12}^* X_t = 0$$

where $X_t = k_t - k_\delta^*$ represents the variation of k_t around the stationary state, and $V_{ij}^* = V_{ij}(k_\delta^*, k_\delta^*)$ represent the second derivatives of the indirect utility function evaluated at the stationary state. By considering a generic linear solution of the form $X_t = \lambda^t X_0$, we obtain the characteristic second-degree polynomial

$$P(\lambda) = \delta V_{12}^* \lambda^2 + (\delta V_{11}^* + V_{22}^*) \lambda + V_{12}^* = 0$$

As Levhari and Liviatan (1972) have demonstrated, from this expression, it can be easily derived that if λ is a root of the polynomial, then $1/(\delta\lambda)$ is also a root. This property is also verified if λ is a complex root.

2.3.6. Turnpike theorem

To study the stability of the stationary state, the roots of $P(\lambda)$ must be localized with respect to a unit circle. To do this, we must specify the expressions for the second derivatives of the indirect utility function $V(x, y)$. Starting from [2.26], the following cross-derivative can be immediately calculated:

$$V_{12}(k_t, k_{t+1}) = -(1+n)u''(c_t)[f'(k_t) + 1 - \mu] \quad [2.29]$$

When assessed using the Modified Golden Rule, it easily comes that

$$V_{12}^* = -(1+n)^2 u''(c_\delta^*) \delta^{-1} > 0$$

The positivity of this derivative is actually a consequence of the fact that the cross-derivative, $T_{12}(k_t, k_{t+1})$, of the social production function is zero. This allows us to rewrite the characteristic polynomial as follows:

$$\tilde{P}(\lambda) = \lambda^2 + \lambda \frac{\delta V_{11}^* + V_{22}^*}{\delta V_{12}^*} + \delta^{-1} = 0 \quad [2.30]$$

From this, it can be derived that the two characteristic roots λ_1 and λ_2 are such that

$$\begin{aligned} \lambda_1 \lambda_2 &= \delta^{-1} \geq 1 \\ \lambda_1 + \lambda_2 &= -\frac{\delta V_{11}^* + V_{22}^*}{\delta V_{12}^*} \end{aligned}$$

Starting from [2.26], it is also possible to calculate the other two second derivatives:⁴

$$\begin{aligned} V_{11}(k_t, k_{t+1}) &= u''(c_t)[f'(k_t) + 1 - \mu]^2 + u'(c_t)f''(k_t) < 0 \\ V_{22}(k_t, k_{t+1}) &= (1 + n)^2 u''(c_t) < 0 \end{aligned} \quad [2.31]$$

Evaluated at the stationary state, it finally comes that

$$\begin{aligned} V_{11}^* &= (1 + n)^2 u''(c_\delta^*) \delta^{-2} + u'(c_\delta^*) f''(k_\delta^*) < 0 \\ V_{22}^* &= (1 + n)^2 u''(c_\delta^*) < 0 \end{aligned}$$

From this, it can be immediately derived that the two characteristic roots are either positive, with at least one of the two being greater than 1, or are complex with a modulus greater than 1. It can also be observed that $\tilde{P}(0) = \delta^{-1} > 0$ and $\lim_{\lambda \rightarrow \pm\infty} \tilde{P}(\lambda) = +\infty$, while

$$\tilde{P}(1) = -\frac{\delta u'(c_\delta^*) f''(k_\delta^*)}{(1+n)^2 u''(c_\delta^*)} < 0$$

This allows us to conclude that the two roots are necessarily real and such that $\lambda_1 \in (0, 1)$ and $\lambda_2 > 1$. The stationary state is therefore a saddle point. Knowing that any unstable trajectory, i.e. outside of the invariant subspace associated with the stable root smaller than 1, cannot be optimal as it does not satisfy the transversality condition [2.25], we obtain the following general property of stability:

Turnpike theorem⁵.— Under Assumptions 2.1–2.4, for any initial given condition k_0 and for any $\delta \in (0, 1]$, there exists a unique optimal trajectory that converges monotonously towards the Modified Golden Rule k_δ^* .

This theorem is therefore clearly associated with the saddle point property for the stationary state. Indeed, in this framework, the two-dimensional space over which the Euler equation is defined can be decomposed into two one-dimensional subspaces: a *stable* subspace associated with the characteristic root that is smaller than 1, and an *unstable* subspace associated with the root that is greater than 1. Any trajectory that falls outside of the stable subspace cannot converge towards the stationary state and therefore cannot satisfy the transversality condition. The unique optimal trajectory is therefore that which, starting from the initial condition k_0 , consists of choosing the unique value of k_1 such that the couple (k_0, k_1) is located in the stable subspace. It

4. Starting from the second derivatives [2.29] and [2.31], it can easily be concluded that the Hessian matrix of the indirect utility function $V(x, y)$ is negative definite, which implies that $V(x, y)$ is concave.

5. See Michel (1993) for a detailed introduction to a decentralized equilibrium and for a discussion on how debt and public spending influence equilibrium.

therefore comes that any couple (k_t, k_{t+1}) , $\forall t \geq 0$ is still situated in this subspace such that $\lim_{t \rightarrow +\infty} k_t = k_\delta^*$. Graphically, we have Figure 2.1.

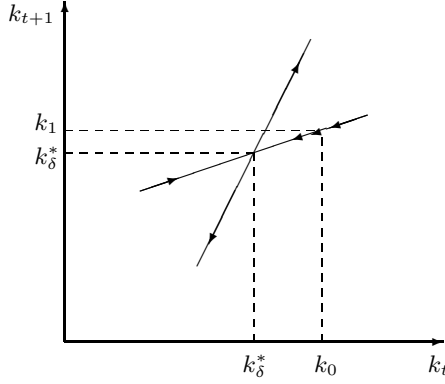


Figure 2.1. Saddle point property

Given that the characteristic roots are positive, all possibility of endogenous fluctuations is excluded from the aggregate model of optimal growth. It is also important to note that the monotone nature and convergence of the optimal trajectory are conserved for all values of the discount factor $\delta \in (0, 1]$. We will see that this property is no longer necessarily a guarantee in the two-sector model.

2.4. The two-sector model of optimal growth

Let us now consider the discrete time ($t = 0, 1, 2, \dots, \infty$) two-sector model of optimal growth. The production sphere is described in section 2.2.2. We will show that using the aggregative procedure for the technological sphere, it is possible to propose a reduced formulation that is entirely comparable with the formulation for the aggregate model.

2.4.1. The consumers

We will consider here an economy populated by a continuum of mass 1 of agents with identically infinite lifespans. The representative consumer offers an (inelastic) unit of labor over each period and derives utility from their consumption from the following function:

ASSUMPTION 2.5.— $u(c)$ is C^2 and such that $\forall c > 0$, $u'(c) > 0$, $u''(c) \leq 0$, $u(0) = 0$.

With such an assumption, which is slightly different from that used for the aggregate model, we conserve the possibility of having a linear utility function.

Using the perfect foresight assumption, and considering the production sector introduced in section 2.2.2, the representative consumer maximizes the following intertemporal utility function

$$\begin{aligned}
 \max_{\{c_t, y_t, k_{t+1}, k_{0t}, l_{0t}, k_{1t}, l_{1t}\}_{t=0}^{+\infty}} & \sum_{t=0}^{+\infty} \delta^t u(c_t) \\
 \text{s.c.} & c_t \leq y_{0t} = f^0(k_{0t}, l_{0t}) \\
 & (1+n)k_{t+1} = f^1(k_{1t}, l_{1t}) + (1-\mu)k_t \\
 & k_{0t} + k_{1t} \leq k_t \\
 & l_{0t} + l_{1t} \leq 1 \\
 & k_{0t}, l_{0t}, k_{1t}, l_{1t} \geq 0 \\
 & k_0 \text{ given}
 \end{aligned} \tag{2.32}$$

where $\delta \in (0, 1]$ still represents the discount factor that allows the measurement of their rate of preference for the present.

This optimization program describes both the optimal allocation of capital and labor between the two sectors and the optimal arbitrage between consumption and saving. Both these problems of choice differ in nature. The first is a static problem, which simply describes the maximization of the profit by the representative company in each sector and leads to a temporary equilibrium. This is the equilibrium that is analyzed using the aggregation technique presented in section 2.2.2. The second problem is that of an intertemporal choice.

2.4.2. The reduced form of the model

Considering its definition, it is very clear that the social production function defined in section 2.2.2 provides the level of consumption of the representative consumer. Indeed, we have $c_t = T(k_t, y_t)$, regardless of the date $t \geq 0$. The maximization program [2.32] can therefore be rewritten as follows:

$$\begin{aligned}
 \max_{\{y_t, k_{t+1}\}_{t=0}^{+\infty}} & \sum_{t=0}^{+\infty} \delta^t u(T(k_t, y_t)) \\
 \text{s.c.} & (1+n)k_{t+1} = y_t + (1-\mu)k_t \\
 & k_0 \text{ given}
 \end{aligned} \tag{2.33}$$

Thus, as with the aggregate model, we can introduce the *indirect utility function*:

$$V(k_t, k_{t+1}) = u(T(k_t, (1+n)k_{t+1} - (1-\mu)k_t)) \tag{2.34}$$

Furthermore, using Assumption 2.2, there exists a maximum capital stock $\bar{k} > 0$ such that $f^1(\bar{k}, 1) = \mu\bar{k}$, and $f^1(k, 1) > \mu k$ when $k < \bar{k}$, while $f^1(k, 1) < \mu k$ when $k > \bar{k}$. It is therefore natural to restrict the set of possible values of the capital stock to the compact set $K \equiv [0, \bar{k}]$ inasmuch as a capital stock exceeding $\bar{k}$ cannot be sustained over the long term.

The maximization program [2.33] can thus be expressed in reduced form as follows:

$$\begin{aligned} \max_{\{k_{t+1}\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \\ \text{s.c.} \quad & (k_t, k_{t+1}) \in \mathcal{D} \\ & k_0 \in K \text{ given} \end{aligned} \quad [2.35]$$

where

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^2 / k_t \in K \text{ and } \frac{(1-\mu)k_t}{1+n} \leq k_{t+1} \leq \frac{f^1(k_t, 1) + (1-\mu)k_t}{1+n} \right\}$$

once again represents the set of admissible capital paths. This formulation, in its reduced form, is traditionally used in the literature to establish very general results (see, for example, Boldrin and Montrucchio (1986)). Notably, it also allows us to include the case of Ramsey's aggregate model (1928), which was introduced in the preceding section.

2.4.3. Intertemporal equilibrium

The first-order condition for an internal maximum in problem [2.35] is given by the following Euler equation:

$$V_2(k_t, k_{t+1}) + \delta V_1(k_{t+1}, k_{t+2}) = 0 \quad [2.36]$$

which corresponds to an implicit, second-order difference equation. Using Assumptions 2.2 and 2.5, it is also easy to conclude that the indirect utility function $V(k_t, k_{t+1})$ is concave (see also the next section where this property is discussed) such that the second-order condition is necessarily fulfilled. In such a context, like in the aggregate model, the Euler equation and the following transversality condition

$$\lim_{t \rightarrow +\infty} \delta^t k_t V_1(k_t, k_{t+1}) = 0 \quad [2.37]$$

provide a set of conditions that are sufficient for an optimal solution to program [2.35].⁶ Thus, any capital path $\{k_t\}_{t=0}^{+\infty}$ which satisfies conditions [2.36]–[2.37] is an *intertemporal equilibrium*.

6. See Michel (1990).

From the complete characterization of the social production function $T(k, y)$ proposed in section 2.2.2, we can now explore the dynamic properties of the two-sector model of optimal growth.

2.4.4. Stationary state and linearized dynamic system

Starting from the definition for the indirect utility function [2.34], it is easy to calculate

$$\begin{aligned} V_1(k_t, k_{t+1}) &= u'(c_t)[T_1(k_t, y_t) - (1 - \mu)T_2(k_t, y_t)], \\ V_2(k_t, k_{t+1}) &= (1 + n)u'(c_t)T_2(k_t, y_t). \end{aligned}$$

Based on Proposition 2.1, the Euler equation [2.36] can thus be rewritten as follows:

$$\frac{(1+n)u'(c_t)}{\delta u'(c_{t+1})} = \frac{w_1(k_{t+1}, y_{t+1}) + (1-\mu)q(k_{t+1}, y_{t+1})}{q(k_t, y_t)}. \quad [2.38]$$

The left-hand side of this equation can be interpreted as an implicit interest factor $1 + i_{t+1}$, which would correspond to the consumption good being loaned on date t and being recovered on date $t + 1$ against remuneration, knowing, of course, that the total population increases at the rate of n over every period and that each individual has a preference for the present, measured by δ . The right-hand side of [2.38] can be rewritten as follows:

$$1 + \frac{w_1(k_{t+1}, y_{t+1})}{q(k_t, y_t)} + \frac{q(k_{t+1}, y_{t+1}) - q(k_t, y_t)}{q(k_t, y_t)} - \mu \frac{q(k_{t+1}, y_{t+1})}{q(k_t, y_t)}$$

where $w_1(k_{t+1}, y_{t+1})/q(k_t, y_t)$ represents the return on the investment good and $[q(k_{t+1}, y_{t+1}) - q(k_t, y_t)]/q(k_t, y_t) - \mu q(k_{t+1}, y_{t+1})/q(k_t, y_t)$ is the net capital gain on the depreciation. From this, it can be derived that the Euler equation is equivalent to the following equality:

$$i_{t+1} = \frac{w_1(k_{t+1}, y_{t+1})}{q(k_t, y_t)} + \frac{q(k_{t+1}, y_{t+1}) - q(k_t, y_t)}{q(k_t, y_t)} - \mu \frac{q(k_{t+1}, y_{t+1})}{q(k_t, y_t)}$$

which can be interpreted as a portfolio equilibrium condition.⁷ The transversality condition [2.37] can also be rewritten as follows:

$$\lim_{t \rightarrow +\infty} \delta^{t-1} u'(c_{t-1}) q_{t-1} k_t = 0$$

and like in the aggregate model, this signifies that the discounted value of the capital, in terms of utility, must be zero at infinity. Once again, it is not optimal to keep capital

7. For a more in-depth discussion on this, see Becker and Boyd (1997).

stock with a positive value until the end of time. It is preferable for the consumer to transform this capital into a consumption good and consume it to increase its utility level.

A stationary state is obtained by writing $k_{t+1} = k_t = k_\delta^*$, $y_{t+1} = y_t = y_\delta^* = (\mu + n)k_\delta^*$ in the Euler equation. From [2.8] and [2.38], it comes that

$$f_1^1(k_1(k_\delta^*, (\mu + n)k_\delta^*), l_1(k_\delta^*, (\mu + n)k_\delta^*)) = \frac{1+n}{\delta} - (1 - \mu). \quad [2.39]$$

This expression corresponds to a generalization of the Modified Golden Rule, defined earlier in the aggregate model of optimal growth. The left-hand side is the marginal productivity of the capital in the investment good sector evaluated at the temporary equilibrium. An obvious application of the demonstration of Theorem 3.1 given in the article by Becker and Tsyganov (2002) for the specific case of homogeneous agents enables us to establish the same result as in the standard Ramsey model (1928):

PROPOSITION 2.4.— *Under Assumptions 2.2 and 2.5, there exists a unique stationary state, the Modified Golden Rule, k_δ^* , which is a solution to equation [2.39].*

We can now study the local dynamic properties of the equilibrium in the neighborhood of a stationary state. Using a procedure that is identical to the case of the aggregate model, the linearization of the Euler equation around the Modified Golden Rule leads us to the following characteristic second-degree polynomial:

$$P(\lambda) = \lambda^2 \delta V_{12}^* + \lambda(\delta V_{11}^* + V_{22}^*) + V_{12}^* = 0 \quad [2.40]$$

where $V_{ij}^* = V_{ij}(k_\delta^*, k_\delta^*)$ represent the second derivatives of the indirect utility function evaluated at the stationary state. Let us assume for now that $V_{12}^* \neq 0$. The two characteristic roots, λ_1 and λ_2 , are such that

$$\lambda_1 \lambda_2 = \delta^{-1} \geq 1 \text{ et } \lambda_1 + \lambda_2 = -\frac{\delta V_{11}^* + V_{22}^*}{\delta V_{12}^*}. \quad [2.41]$$

We initially obtain the same properties as in the aggregate model. Indeed, it is known that $T(k, y)$ is a concave function, and using Assumption 2.5, this implies that the indirect utility function $V(k, k')$ is also concave. Thus, from [2.41], it can be derived that the two characteristic roots are either complex conjugate or real with the same sign. The sign of the roots or of their real part is determined from the sign of the cross-derivative V_{12}^* .

In the next two sections, a detailed analysis of the two characteristic roots will allow us to accurately determine the local stability properties of the stationary state. However, before proceeding to this analysis, it can be immediately shown that the two

characteristic roots are necessarily real. To establish such a property, it is necessary to calculate the discriminant of the characteristic polynomial:

$$\Delta = (\delta V_{11}^* + V_{22}^*)^2 - 4\delta(V_{12}^*)^2. \quad [2.42]$$

The following result can then be easily demonstrated:

LEMMA 2.2.— *Under Assumptions 2.2 and 2.5, $\Delta \geq 0$ so that $\lambda_1, \lambda_2 \in \mathbb{R}$.*

PROOF.— Starting from expression [2.42], it can be easily derived that:

$$\begin{aligned} \Delta &= (\delta V_{11}^* + V_{22}^* + 2\sqrt{\delta}V_{12}^*)(\delta V_{11}^* + V_{22}^* - 2\sqrt{\delta}V_{12}^*) \\ &= (1 \ \sqrt{\delta}) \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{12}^* & V_{22}^* \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{\delta} \end{pmatrix} \times (1 - \sqrt{\delta}) \begin{pmatrix} V_{11}^* & V_{12}^* \\ V_{12}^* & V_{22}^* \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{\delta} \end{pmatrix} \end{aligned}$$

Knowing that the Hessian matrix of V is negative quasi-definite, it can be derived that $\Delta \geq 0$ such that the characteristic roots are real. $\square$

Based on these results, we can now accurately explore the local stability properties of the equilibrium.

2.4.5. A conditional turnpike theorem

Let us begin by precisely studying the expressions for the second derivatives of the indirect utility function V , evaluated at the Modified Golden Rule. First, it is useful to note that the Euler equation at the stationary state can be expressed as follows:

$$-\frac{(1+n)}{\delta}T_2^* = T_1^* - (1 - \mu)T_2^*$$

or again

$$-T_2^* = \frac{\delta}{1+n-\delta(1-\mu)}T_1^* \equiv \theta T_1^*.$$

Using these two expressions coupled with Proposition 2.2 allows us to get

$$\begin{aligned} V_{11}^* &= \left(\frac{(1+n)\theta}{\delta} \right)^2 u''(c_\delta^*)(T_1^*)^2 + u'(c_\delta^*)T_{11}^*[1 + (1 - \mu)b]^2 \leq 0, \\ V_{22}^* &= (1 + n)^2 \theta^2 u''(c_\delta^*)(T_1^*)^2 + (1 + n)^2 u'(c_\delta^*)T_{11}^* b^2 \leq 0, \\ V_{12}^* &= -\frac{(1+n)^2 \theta^2}{\delta} u''(c_\delta^*)(T_1^*)^2 - (1 + n)u'(c_\delta^*)T_{11}^* b[1 + (1 - \mu)b]. \end{aligned} \quad [2.43]$$

It is immediately seen that unlike in the aggregate model, the sign of V_{12}^* is not necessarily positive. More precisely, if in the stationary state the consumption good is

relatively more capital-intensive than the investment good, then $b < 0$ and V_{12}^* could eventually also be negative. In this context, the two characteristic roots could therefore be negative, and a non-monotone dynamic may emerge. The question then is to find out whether the local stability property is still fulfilled.

Let us start by studying the analogous case of the aggregate model in which $V_{12}^* > 0$. To do this, we must first introduce several elasticities, which will later facilitate the economic interpretations of the results. As can be clearly seen in the expressions for the second derivatives of V , the curvature properties of the utility function and the social production function are very important. For example, it can be noted that if $u(c)$ is highly concave, then the second derivative of V_{12}^* has a high chance of being positive. This then eliminates any possibility of endogenous fluctuations as in the aggregate model. Let us thus consider the *intertemporal elasticity of substitution of consumption*, evaluated at the stationary state

$$\epsilon_c = - (u''(c_\delta^*)c_\delta^*/u'(c_\delta^*))^{-1} > 0. \quad [2.44]$$

The more concave the utility, the smaller this elasticity. It measures the increase in future consumption that is necessary to compensate for a drop of 1% in present consumption.

Similarly, the social production function can be characterized using two useful indicators: the *production elasticity of the consumption good* $y_0 = T(k, y)$ with respect to capital stock

$$\epsilon_{ck} = T_1^*(k_\delta^*, (\mu + n)k_\delta^*)k_\delta^*/T^*(k_\delta^*, (\mu + n)k_\delta^*) > 0 \quad [2.45]$$

which measures the increase in y_0 when k increases by 1%, and the *capital remuneration rate elasticity* w_1 with respect to capital stock

$$\epsilon_{w_1k} = -T_{11}^*(k_\delta^*, (\mu + n)k_\delta^*)k_\delta^*/T_1^*(k_\delta^*, (\mu + n)k_\delta^*) > 0 \quad [2.46]$$

which measures the decrease in w_1 when k increases by 1%, both these elasticities being evaluated at the stationary state.

Using equations [2.45] and [2.46], the derivatives [2.43] can be rewritten as follows:

$$\begin{aligned} V_{11}^* &= -u''(c_\delta^*)c_\delta^*T_{11}^* \left[\frac{\epsilon_{ck}}{\epsilon_{w_1k}} \left(\frac{(1+n)\theta}{\delta} \right)^2 + \epsilon_c[1 + (1-\mu)b]^2 \right] \\ V_{22}^* &= -u''(c_\delta^*)c_\delta^*T_{11}^*(1+n)^2 \left[\frac{\epsilon_{ck}}{\epsilon_{w_1k}}\theta^2 + \epsilon_cb^2 \right] \\ V_{12}^* &= u''(c_\delta^*)c_\delta^*T_{11}^*(1+n) \left[\frac{\epsilon_{ck}}{\epsilon_{w_1k}} \frac{(1+n)\theta^2}{\delta} + \epsilon_cb[1 + (1-\mu)b] \right] \end{aligned} \quad [2.47]$$

It can be immediately derived that the sign of the cross-derivative V_{12}^* depends exclusively on the sign of the expression in square brackets, which is positive,

especially if $b \geq 0$, i.e. in the case of the aggregate model with $b = 0$ or in the case where the investment good sector is more capital intensive than the consumption good sector.

Recalling that $\theta \equiv \delta/[1 + n - \delta(1 - \mu)]$, let us note that if the investment good is relatively more capital intensive, then $b > 0$ evaluated at the stationary state necessarily has an upper limit of θ . Such a result comes from system [2.15], which represents the equality between prices and production costs. Resolving this system by eliminating w_0 leads to the equality

$$\frac{a_{01}}{a_{00}} = w_1 \left(\frac{q}{w_1} - b \right) > 0$$

with, at the stationary state, $q/w_1 = \theta$.

More generally, we can now establish the following result.

THEOREM 2.1.— *Under Assumptions 2.2 and 2.5, consider the critical value*

$$\underline{\epsilon}_c \equiv -\frac{\epsilon_c k(1+n)\theta^2}{\epsilon_{w_1 k} \delta b [1 + (1-\mu)b]}.$$

If one of the following two conditions is fulfilled:

- i) $b \in \left(-\infty, -\frac{1}{1-\mu}\right] \cup [0, \theta)$;
- ii) $b \in \left(-\frac{1}{1-\mu}, 0\right)$ and $\epsilon_c \in (0, \underline{\epsilon}_c)$;

then for any given initial condition k_0 and for any $\delta \in (0, 1]$, there is a unique optimal trajectory that converges monotonously towards the Modified Golden Rule k_δ^* .

PROOF.— Starting from its expression in [2.47], it can be easily derived that $V_{12}^* > 0$ if and only if one of the conditions (i) or (ii) is fulfilled. Condition (i) guarantees the positivity of V_{12}^* either in the obvious case where $b \in [0, \theta)$, or in the case where $b < 0$ is such that $b[1 + (1 - \mu)b] \geq 0$. Condition (ii), on the contrary, leads to $b[1 + (1 - \mu)b] < 0$ but V_{12}^* remains positive because the intertemporal elasticity of substitution of consumption ϵ_c is sufficiently small. Since $V_{12}^* > 0$, it can be derived that the characteristic polynomial [2.40] is such that $P(0) > 0$ and $\lim_{\lambda \rightarrow +\infty} P(\lambda) = +\infty$. Furthermore, it comes that

$$P(1) = u'(c_\delta^*) T_{11}^* [1 + n - \delta(1 - \mu)] (\mu + n) \left(\frac{1}{\mu + n} - b \right) (\theta - b)$$

Knowing that $\theta \leq 1/(\mu + n)$ and $b < \theta$, it can be immediately concluded that $P(1) < 0$, which implies the existence of one characteristic root between 0 and 1, while the other is strictly greater than 1. As in the aggregate model, the stationary point is thus a saddle point. Knowing that any unstable trajectory, i.e. outside of the

invariant subspace associated with the stable root that is smaller than 1, cannot be optimal as it does not fulfill the transversality condition [2.37], we obtain the desired result. $\square$

This turnpike theorem obviously includes the particular case of an aggregate model that can be obtained in our framework by considering $b = 0$. In other words, it means that any two-sector model in which $V_{12}^* > 0$ is characterized by the existence of a unique optimal trajectory that converges monotonously towards the stationary state, and this is regardless of the value of the discount factor $\delta \in (0, 1]$. However, it is essential to note that from now on, in a two-sector economy, the property of the stationary state being a saddle point is no longer guaranteed. It is conditionally obtained depending on conditions that are related to both the utility function and the sectoral capital intensities.

2.5. Extension to an economy with $n + 1$ sectors

Using the aggregation procedure based on the social production function, as presented in section 2.2.3, it is entirely possible to generalize the above results to multi-sector models of any dimension.

2.5.1. The general reduced form

Formulated in discrete time, the maximization program for the representative agent is:

$$\begin{aligned} \max_{\{y_{jt}, k_{jt+1}\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t u(T(k_t, y_t)) \\ \text{s.c.} \quad & (1+n)k_{jt+1} = y_{jt} + (1-\mu)k_{jt} \quad j = 1, \dots, n \\ & k_{j0} \text{ given} \quad j = 1, \dots, n \end{aligned} \quad [2.48]$$

with $k_t = (k_{1t}, \dots, k_{nt})$ and $y_t = (y_{1t}, \dots, y_{nt})$. Without loss of generality, it can be assumed that the rate of depreciation of the capital $\mu \in [0, 1]$ is the same for each sector. As done earlier, by again using Assumption 2.5 to describe the preferences of the representative consumer, the following indirect utility function is obtained:

$$V(k_t, k_{t+1}) = u(T(k_t, (1+n)k_{t+1} - (1-\mu)k_t))$$

and program [2.48] can be rewritten in the following general reduced form:

$$\begin{aligned} \max_{\{k_t\}_{t=0}^{+\infty}} \quad & \sum_{t=0}^{+\infty} \delta^t V(k_t, k_{t+1}) \\ \text{s.c.} \quad & (k_t, k_{t+1}) \in \mathcal{D} \\ & k_0 \in K \text{ given} \end{aligned} \quad [2.49]$$

where $K \subset \mathbb{R}_+^n$ is a compact set,⁸

$$\mathcal{D} = \left\{ (k_t, k_{t+1}) \in \mathbb{R}_+^{2n} / k_t \in K \text{ and } \frac{(1-\mu)}{1+n} k_t \leq k_{t+1} \leq \frac{g(k_t) + (1-\mu)k_t}{1+n} \right\}$$

is the set of admissible capital paths and $g(k_t)$ is the solution to $T(k_t, g(k_t)) = 0$. The first-order conditions for an internal maximum are given by the Euler equations

$$\frac{\partial V}{\partial k_{jt+1}}(k_t, k_{t+1}) + \delta \frac{\partial V}{\partial k_{jt+1}}(k_{t+1}, k_{t+2}) = 0, \quad j = 1, \dots, n \quad [2.50]$$

which represents a set of n implicit, second-order difference equations. Using Assumptions 2.3 and 2.5, it can be easily concluded that the indirect utility function $V(k_t, k_{t+1})$ is concave, such that the second-order conditions are necessarily fulfilled. In this context, the Euler equations and following transversality conditions:

$$\lim_{t \rightarrow +\infty} \delta^t k_{jt} \frac{\partial V}{\partial k_{jt}}(k_t, k_{t+1}) = 0, \quad j = 1, \dots, n \quad [2.51]$$

yield a set of sufficient conditions for an optimal solution to program [2.49]. Thus, any capital path $\{k_t\}_{t=0}^{+\infty}$ which fulfills conditions [2.50]–[2.51] is an *intertemporal equilibrium*.

2.5.2. A generalization of the conditional turnpike theorem

It is clearly more difficult to study the existence and uniqueness of the stationary state as well as its stability properties.⁹ This kind of an analysis would go far beyond the objective of this chapter. However, in order to simplify, if we assume that the utility is linear $u(c) = c$, the population is constant $n = 0$ and each investment good completely depreciates over each period, i.e. $\mu = 1$, it is then possible to calculate the following matrix expressions:

$$\begin{aligned} V_{12}(k, k) &= -T_{11}(k, \mu k)B \\ V_{22}(k, k) - V_{12}(k, k) &= B^t T_{11}(k, \mu k)B + T_{11}(k, \mu k) \end{aligned}$$

Based on this, Cartigny and Venditti (1995) propose a generalization of Theorem 2.1. If it is assumed that matrix $V_{12}(k, k)$ is symmetric, i.e. $V_{12}(k, k) = V_{12}^t(k, k)$, and that matrix B is a positive quasi-definite matrix, then taking into account the concavity of $V(k_t, k_{t+1})$, it can be derived that the matrix

8. In the multi-sector case, this compact set is defined from the same argument as that used in section 2.4: thanks to the concavity properties of the technologies, there exists a maximum capital stock for each sector, beyond which the accumulation of capital cannot be sustained.

9. See Brock (1973) for results on the existence of a stationary state in multi-sector models.

$V_{22}(k, k) - V_{12}(k, k)$ is a negative quasi-definite matrix, and consequently, from Cartigny and Venditti's Theorem 3.1 (1995), it is possible to conclude that there is a unique optimal trajectory that converges monotonously on the stationary state.¹⁰ We thus obtain a generalization of Theorem 2.1 to an economy of $n + 1$ sectors, where the positive quasi-definite matrix B can be interpreted as a consumption good that is less capital intensive than the set of n investment goods.

2.6. Conclusion

This chapter was dedicated to the analysis of conditions that guarantee the existence of a unique equilibrium trajectory that converges monotonously in optimal growth models. Such a property, which represents one of the fundamental prerequisites in the Real Business Cycles Theory, has proven to be perfectly natural in Ramsey's aggregate model. However, it is not necessarily verified when the economy under consideration has multiple sectors. The objective of this chapter, therefore, was to precisely clarify the reasons for these different results.

In order to consider a homogeneous framework that was compatible with the aggregative and multi-sector formulations of the model, we first presented a general aggregation technique for the production system, which made it possible to get a general expression for the level of consumption starting from the current capital stocks and the investment using a *social production function*, regardless of the number of sectors being considered. We then proposed the complete characterization of this function. This allowed us to demonstrate that there is a fundamental difference between the aggregate model and multi-sector formulations. Indeed, in the aggregate model, the social production function is expressed as a linear function of the investment, which implies infinite elasticity between the current capital stock and the investment. On the contrary, when the economy consists of at least two sectors, with a consumption good and an investment good, the social production function becomes nonlinear. Its cross-derivative, which is generically non-zero, has proven to be crucial in determining the properties of the optimal trajectory. Furthermore, the sign of this derivative depends on the difference in capital intensity between the consumption good sector and the investment good sector. Furthermore, such a property is also generalizable to economies composed of a consumption good and n investment goods.

Based on this aggregation technique, we were able to demonstrate that in the aggregate model, the specificity of the social production function guaranteed the existence of a general turnpike theorem. On the contrary, in multi-sector formulations, only a conditional Turnpike theorem could be formulated, in which restrictions on the difference in capital intensity between the consumption good sector and the investment

10. See also Scheinkman (1976) for results on the local stability in a general multi-sector framework.

good sector had to be considered to guarantee the existence of a monotonously convergent dynamic. Based on these conditions, we will now examine in detail the conditions for the existence of optimal endogenous fluctuations in the chapter entitled “Multisector models of optimal growth and complex dynamics” contained in Volume 4 of this series on *Fluctuations and Stabilization Policies*.

2.7. References

- Becker, R. and Boyd III, J. (1997). *Capital Theory, Equilibrium analysis and Recursive Utility*. Blackwell, Oxford.
- Becker, R. and Tsyganov, E. (2002). Ramsey equilibrium in a two-sector model with heterogeneous households. *Journal of Economic Theory*, 105, 188–225.
- Benhabib, J. and Nishimura, K. (1979). The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth. *Journal of Economic Theory*, 21, 421–444.
- Benhabib, J. and Nishimura, K. (1981). Stability of equilibrium in dynamic models of capital theory. *International Economic Review*, 22, 275–293.
- Boldrin, M. and Montrucchio, L. (1986). On the indeterminacy of capital accumulation paths. *Journal of Economic Theory*, 40, 26–39.
- Brock, W. (1973). Some results on the uniqueness of the steady states in multisector models of optimum growth when future utilities are discounted. *International Economic Review*, 14, 535–559.
- Brock, W. and Scheinkman, J. (1976). Global asymptotic stability of optimal control systems with applications to the theory of economic growth. *Journal of Economic Theory*, 12, 164–190.
- Cartigny, P. and Venditti, A. (1995). Endogenous cycles in discrete symmetric multisector optimal growth models. *Journal of Optimization Theory and Applications*, 86, 17–36.
- Cass, D. and Shell, K. (1976). The structure and stability of competitive dynamical systems. *Journal of Economic Theory*, 12, 31–70.
- Kurz, M. (1968). The general instability of a class of competitive growth processes. *Review of Economic Studies*, 22, 275–293.
- Kydland, F. and Prescott, E. (1982). Time to build and aggregate fluctuations. *Econometrica*, 50, 1345–1370.
- Levhari, D. and Liviatan, N. (1972). On stability in the saddle point sense. *Journal of Economic Theory*, 4, 88–93.
- Magill, J. (1977). Some new results on the local stability of the process of capital accumulation. *Journal of Economic Theory*, 15, 174–210.

- McKenzie, L. (1986). Optimal economic growth, turnpike theorems and comparative dynamics. In *Handbook of Mathematical Economics, Vol. III*, Arrow, K. and Intriligator, M. (eds). North Holland, Amsterdam, 1281–1355.
- Michel, P. (1990). Some clarifications on the transversality condition. *Econometrica*, 58, 705–723.
- Michel, P. (1993). Croissance et équilibre intertemporel : une présentation simple d'un modèle de base. *Macroéconomie : développements récents*, 68(1–2), 127–139.
- Ramsey, F. (1928). A mathematical theory of savings. *Economic Journal*, 38, 543–559.
- Rockafellar, R. (1976). Saddle points of Hamiltonian systems in convex Lagrange problems having a nonzero discount rate. *Journal of Economic Theory*, 12, 368–390.
- Samuelson, P. (1973). Optimality of profit, including prices under ideal planning. *Proceedings of the National Academy of Sciences of the USA*, 70, 2109–2111.
- Scheinkman, J. (1976). On optimal steady states of N-sector growth models when utility is discounted. *Journal of Economic Theory*, 12, 11–30.
- Sutherland, W. (1970). On optimal development in a multi-sectoral economy: The discounted case. *Review of Economic Studies*, 46, 585–589.

3

Business Cycle Fluctuations and Monetary Policy under Heterogeneous Information

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3.1. Introduction

Modeling business cycle fluctuations and the non-neutrality of money is an ongoing challenge for macroeconomists. There is still a debate among them as to the most plausible frictional hypothesis to generate a model consistent with the empirical evidence. While the imperfect-information hypothesis developed by Phelps and Lucas in the 1970s had been abandoned in favor of the sticky-price hypothesis, a recent trend in the literature has put this hypothesis back on the agenda to explain business fluctuations and monetary non-neutrality. The success of this revival lies in the introduction of strategic uncertainty into a framework of heterogeneous information. Business cycle theories relying on this assumption state that economic agents face fundamental uncertainty about economic conditions as well as strategic uncertainty about how other market participants perceive these economic conditions. As a result, decisions tend to adjust slowly to changes in economic fundamentals, and small or temporary shocks can have large, lasting effects on macroeconomic aggregates, which is in line with empirical observations.

Macroeconomics 1,

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In another vein, recent advances in game theory under heterogeneous information have led to a renewed interest in incomplete information models. These models typically give a predominant role to higher-order beliefs in the decision-making process of agents, which leads to persistence in their reaction to new information. While the new information provides private agents with a view on exogenous variables, it only provides them with an ambiguous view on how other agents change their expectations about these exogenous variables and an even more ambiguous view on how other agents change their expectations about the expectations of others. The conjunction of strategic complementarities and heterogeneous information thus results in a powerful macroeconomic framework.

The current chapter presents this macroeconomic framework and provides a brief overview of recent advances in the literature. Section 3.2 illustrates how higher-order beliefs lead to an overreaction to public information, a mechanism that was introduced by Morris and Shin (2002). It provides technical details to solve a simple class of models based on beauty-contest games. Section 3.3 relates to monetary non-neutrality under heterogeneous information and explains how Woodford (2003a) initiated the revival of the heterogeneous-information hypothesis by adding strategic complementarities to the model of Lucas (1972). It further discusses the contribution of Maćkowiak and Wiederholt (2009) and Pasten and Schoenle (2016) in rationalizing the limited attention that firms pay to economic shocks. Section 3.4 explains how dispersed information impinges on the transmission of real shocks and business cycle fluctuations. For instance, Angeletos and La'O (2010) show that the nature of signals observed by economic agents is critical in the propagation of these shocks, and Angeletos and La'O (2013) highlight that non-fundamental shocks, such as sentiment shocks, can have significant economic effects through higher-order beliefs. Finally, section 3.5 deals with communication and stabilization policies. It exposes the debate about the welfare effect of public information between Morris and Shin (2002), Svensson (2006) and Angeletos and Pavan (2007) before discussing how information can be disclosed in order to reduce the overreaction of market participants. The section then discusses two extensions, which have important implications on the optimal communication policy. The first is to consider that information is endogenous, either because the agents' source of information is an economic variable that itself depends on the agents' actions (e.g. Amador and Weill (2010)) or because the agents freely choose the attention they give to different information sources (e.g. Hellwig and Veldkamp (2009)). The second extension, following James and Lawler (2011), is to add a policy action to the framework and to solve jointly for the optimal communication and stabilization policy.

Although brief, this overview of the literature reveals the potential that the conjunction of strategic complementarities and heterogeneous information offers to explain macroeconomic phenomena and suggests a promising future for this field of economics.

3.2. A beauty-contest model

Strategic complementarities are widespread in the economy: often agents tend to align their action with that of others. For instance, an investor may decide to invest in a company based on what other investors think about that company; a depositor may trust her bank to the extent that other depositors trust it as well; or a firm may set its price by anticipating the price set by its competitors. Such games are often referred to as Keynesian beauty contests. Keynes (1973 [1936]) compared the cognitive exercise of an investor in the financial markets to that of a participant in a newspaper beauty contest of the 1930s, where participants were rewarded for selecting the most popular faces among all participants, rather than those they might personally find the most attractive. In these contests, Keynes stressed that “it is not a case of choosing those [faces] that, to the best of one’s judgment, are really the prettiest, nor even those that average opinion genuinely thinks the prettiest. We have reached the third degree, where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees.”

Bearing in mind the very general scope of the beauty-contest analogy, we illustrate the mechanism by referring to the price-setting problem of firms in monopolistic competition. In a micro-founded economy¹ populated by a representative household and a continuum of monopolistically competitive firms, the demand side – expressed in logarithms – is summarized by the following equation:

$$q = p + y$$

where q is the level of aggregate nominal spending, p is the price level, and y is the level of aggregate real output (or output gap). The supply side is determined by the optimal price set by firm j

$$p_j = \mathbb{E}_j [(1 - \xi)y + p] = \mathbb{E}_j [(1 - \xi)q + \xi p] \quad [3.1]$$

where $\mathbb{E}_j$ represents the expectation operator of firm j conditional on its information and $\int_j p_j = p$. Each firm sets its price as a function of its own expectation of the nominal aggregate demand and its expectation about the price level. The parameter ξ determines to what extent the optimal price responds to the output gap and is increasing in the risk aversion of the household. As is generally the case in macroeconomic models, we assume that prices are strategic complements, so that $0 < \xi < 1$. The parameter ξ is the degree of strategic complementarities.

3.2.1. Information structure

In an environment characterized by strategic complementarities, a dual motive guides the economic agents in their action. An agent chooses his action to be close

1. See, for example, Adam (2007) or Clerc and Dos Santos Ferreira (2022).

to the economic fundamental and, at the same time, close to the average action of other agents. Equation [3.1] says that each firm sets its price based on its expectation of nominal aggregate demand q and of the price level p . This dual motive gives rise to a dual role for information because information helps not only to forecast the economic fundamental, but also to forecast the action of others. Whereas the best forecast of economic fundamental relies on *accurate* information, the best forecast of others' action relies on *common* information.

Assume that nominal aggregate demand follows an improper prior distribution on the reals and that each firm j receives two signals on nominal demand. The first signal is centered on the true value of q and contains an error term that is normally distributed and that is identically but independently drawn for each firm:

$$q_j = q + \varepsilon_j, \quad \text{with} \quad \varepsilon_j \sim N(0, \sigma_\varepsilon^2) \quad [3.2]$$

The first signal is private, as its error term is entirely specific to each firm. This private signal can be interpreted as an estimate of the demand that each firm makes from its own business activity.

The second signal is also centered on the true value of q and contains an error term that is normally distributed. However, all firms observe the same second signal:

$$D = q + \eta, \quad \text{with} \quad \eta \sim N(0, \sigma_\eta^2) \quad [3.3]$$

The second signal is public and common knowledge among firms. This public signal is typically disclosed by a public institution such as the central bank, and the error term η captures its perception error about the fundamental.

The distribution of error terms ε_j and η is common knowledge among firms.

3.2.2. Equilibrium

To derive the equilibrium behavior of firms, we need to compute the expectations of firm j about demand and the price level conditional on its information set. The expectation of demand $\mathbb{E}_j(q)$ writes

$$\mathbb{E}(q|q_j, D) = \frac{\sigma_\eta^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} q_j + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} D = (1 - \Omega) q_j + \Omega \cdot D \quad [3.4]$$

The best estimate of firm j about demand is an average of its signals, each weighted with Ω depending on their relative precision.

Deriving the conditional expectation of firm j about the price level is less straightforward because one has to form expectations about the behavior of other firms. There are two ways to derive the equilibrium: the *recursive method* and the *method of indeterminate coefficients*. Although equivalent, we present both methods for pedagogical purposes.

3.2.2.1. The recursive method

The recursive method consists of computing the weighted average of first- and higher-order expectations of firm j about demand. The pricing rule [3.1] can be rewritten by recursively substituting the average price by higher-order expectations on demand:

$$\begin{aligned}
 p_j &= \mathbb{E}_j \left[(1 - \xi)q + \xi \underbrace{\bar{\mathbb{E}} \left[(1 - \xi)q + \xi p \right]}_p \right] \\
 &= \mathbb{E}_j \left[(1 - \xi)q + \xi \bar{\mathbb{E}} \left[(1 - \xi)q + \xi \bar{\mathbb{E}} \left[(1 - \xi)q + \xi \bar{\mathbb{E}} \left[(1 - \xi)q + \dots \right] \right] \right] \right] \\
 &= (1 - \xi) \sum_{k=0}^{\infty} \xi^k \mathbb{E}_j \left[\bar{\mathbb{E}}^k(q) \right]
 \end{aligned} \tag{3.5}$$

where k is the degree of higher-order iteration, $\bar{\mathbb{E}}$ is the average expectation operator in the population such that $\bar{\mathbb{E}}(\cdot) = \int_j \mathbb{E}_j(\cdot) dj$ and we use the following notation for higher-order beliefs: $\bar{\mathbb{E}}^0(x) = x$ is the variable x , $\bar{\mathbb{E}}^1(x) = \bar{\mathbb{E}}(x)$ is the average expectation of x , $\bar{\mathbb{E}}^2(x) = \bar{\mathbb{E}}\bar{\mathbb{E}}^1(x) = \bar{\mathbb{E}}\bar{\mathbb{E}}(x)$ is the average expectation of the average expectation of x and so on.

Based on the first-order expectation of demand [3.4], expectations of higher orders are

$$\begin{aligned}
 \mathbb{E}_j \bar{\mathbb{E}}(q) &= (1 - \Omega)[(1 - \Omega)q_j + \Omega D] + \Omega D \\
 &= (1 - \Omega)^2 q_j + [1 - (1 - \Omega)^2] D \\
 \mathbb{E}_j \bar{\mathbb{E}}^{(2)}(q) &= (1 - \Omega)^2 [(1 - \Omega)q_j + \Omega D] + [1 - (1 - \Omega)^2] D \\
 &= (1 - \Omega)^3 q_j + [1 - (1 - \Omega)^3] D \\
 \mathbb{E}_j \bar{\mathbb{E}}^{(k-1)}(q) &= (1 - \Omega)^k q_j + [1 - (1 - \Omega)^k] D
 \end{aligned} \tag{3.6}$$

When information is heterogeneous across agents, an agent assigns more weight to common information in his expectation of others' expectation about fundamentals than in his own expectation about fundamentals: the law of iterated expectations fails and the weight assigned to common information increases with the degree of strategic complementarities. The weight assigned to the public signal, D , increases in expectations of higher orders because this signal better helps to predict others' expectation as its error term is common to all firms. These higher-order expectations receive themselves an increasing weight in price setting with higher degrees of strategic complementarities. Plugging higher-order expectations into equation [3.5], the pricing rule yields

$$\begin{aligned}
p_j &= (1 - \xi) \sum_{k=0}^{\infty} \xi^k [(1 - \Omega)^{k+1} q_j + [1 - (1 - \Omega)^{k+1}] D] \\
&= \frac{(1 - \xi)(1 - \Omega)}{(1 - \xi) + \xi \Omega} q_j + \frac{\Omega}{(1 - \xi) + \xi \Omega} D \\
&= \frac{(1 - \xi) \sigma_{\eta}^2}{\sigma_{\varepsilon}^2 + (1 - \xi) \sigma_{\eta}^2} q_j + \frac{\sigma_{\varepsilon}^2}{\sigma_{\varepsilon}^2 + (1 - \xi) \sigma_{\eta}^2} D = (1 - \omega) q_j + \omega D \quad [3.7]
\end{aligned}$$

The firm assigns a larger weight on the public signal D in its price setting than in its first-order expectation about the fundamental: $\omega > \Omega$. The firm is said to *overreact* to common information in the sense that, because of the coordination motive, it gives more weight to it than its accuracy would actually justify. This overreaction increases with the degree of strategic complementarities. Rational though from the agent's point of view, this overreaction can cause welfare losses, as emphasized by Morris and Shin (2002) and explained in section 3.5.

3.2.2.2. *The method of indeterminate coefficients*

The method of indeterminate coefficients consists of stating that the equilibrium price is a linear combination of the private and public signals with indeterminate weights, $p_j = (1 - \omega) q_j + \omega D$, and solving for this guess in the pricing rule [3.1]:

$$\begin{aligned}
p_j &= \mathbb{E}_j [(1 - \xi) q + \xi [(1 - \omega) q + \omega D]] \\
&= \underbrace{[(1 - \xi \omega)(1 - \Omega)]}_{1 - \omega} q_j + \underbrace{[\Omega + \xi \omega(1 - \Omega)]}_{\omega} D \\
&= \frac{(1 - \xi)(1 - \Omega)}{(1 - \xi) + \xi \Omega} q_j + \frac{\Omega}{(1 - \xi) + \xi \Omega} D \\
&= \frac{(1 - \xi) \sigma_{\eta}^2}{\sigma_{\varepsilon}^2 + (1 - \xi) \sigma_{\eta}^2} q_j + \frac{\sigma_{\varepsilon}^2}{\sigma_{\varepsilon}^2 + (1 - \xi) \sigma_{\eta}^2} D
\end{aligned}$$

The method of indeterminate coefficients yields the same solution as the recursive method in equation [3.7], indicating that the linear equilibrium is the unique equilibrium. While the method of indeterminate coefficients is shorter to derive, the recursive method explicitly exposes the increasing weight of public information in higher-order expectations, which helps to understand the role of common information in strategic complementarities.

With this simple illustration in mind, we can now look at how the introduction of strategic complementarities in macroeconomic models has led to a revival of the imperfect-information hypothesis as the source of the monetary non-neutrality.

3.3. Monetary non-neutrality

The “islands” model proposed by Lucas (1972) was the first model attempting to provide an explanation of monetary non-neutrality based on heterogeneous information. This model, however, had a major drawback: it was not able to generate persistent effects of monetary shocks on real variables. Lucas’ model indeed assumed that firms became fully aware of aggregate demand shocks one period after their occurrence. As a result, all nominal magnitudes were fully adjusted after a single period, so that the effects on real variables (however large) completely vanished at that point. By contrast, a well-known fact about monetary shocks – extensively documented in the vector autoregressive models (VAR) literature – is that they have highly persistent real effects.² Lucas’ model was therefore unable to account for a crucial dimension of monetary non-neutrality.

Michael Woodford, in a paper prepared for a conference held in 2001 in honor of Edmund Phelps (and published as Woodford 2003a), proposed to reconsider what he called the “Phelps-Lucas hypothesis” – according to which the real effects from nominal shocks stem from dispersed information. Woodford argued that two important departures from Lucas’ model would make this hypothesis consistent with persistent monetary non-neutralities. The first departure deals with the structure of information. In Lucas’ model, firms become aware of the current level of aggregate demand as soon as this level becomes public information (which is assumed to happen after one period). This means that firms are (implicitly) supposed to fully absorb and process all available information. Woodford, instead, follows a line of argumentation initially developed by Sims (1998, 2003), according to which firms would have limited capacity for processing information. In Woodford’s model, even though information about the current state of aggregate demand is immediately published, firms cannot pay more than limited attention to this information. Hence, it takes time for firms to process available information and to integrate it into their information set.

The second departure deals with the structure of markets. In Lucas’ model, firms operate in perfectly competitive markets. Woodford’s model instead assumes a monopolistic competition framework featuring strategic complementarities in price setting. Individual firms thus play a “beauty-contest game” with their competitors, in which their “action” consists of setting the price of the item they sell. In such an environment, firms find it optimal to set their price by reacting not only to their own expectations about current aggregate demand but also (since they are uncertain about the expectations of other firms) to their “higher-order expectations” – i.e. their expectations about other firms’ expectations regarding current aggregate demand, their expectations about other firms’ expectations about other firms’ expectations and so on. Therefore, besides their incapacity to know the current level of aggregate demand

2. Christiano et al. (2005), for instance, find that aggregate output returns to its initial level no sooner than three years after the occurrence of an expansionary monetary shock.

(due to their limited capacity for processing information), firms do not know either what other firms know about this level. In other words, Woodford's model involves what game theory calls imperfect common knowledge. The fact that higher-order expectations matter for price setting is a central aspect of Woodford's strategy to explain the persistence of real effects of monetary shocks, because higher-order expectations display even more inertia in response to aggregate demand shocks than firms' own expectations about these shocks.³

3.3.1. Imperfect common knowledge

Following Clerc and Dos Santos Ferreira (2022), we can formalize the main insights of Woodford's model as follows. Expressing all variables in logarithms and introducing a time subscript t , the demand side is summarized by the following equation:

$$q_t = p_t + y_t = q_{t-1} + \mu + \varepsilon_{q,t} \quad [3.8]$$

where q_t denotes the level of aggregate nominal spending (in period t), p_t is the price level, y_t is the level of aggregate output, μ is the drift in aggregate nominal spending, and $\varepsilon_{q,t}$ is the shocks on aggregate demand. These shocks follow a Gaussian white noise process with variance σ_q^2 . We assume that monetary shocks are the single source of aggregate demand disturbances, so that $\varepsilon_{q,t}$ actually denotes monetary shocks.

The supply side is composed of a continuum of firms in monopolistic competition. If there were no information frictions, firm j would set its period t price, $p_{j,t}^*$, according to:

$$p_{j,t}^* = \xi p_t + (1 - \xi)(q_t - y^n) \quad [3.9]$$

where $\xi < 1$ denotes a reduced-form parameter measuring the degree of strategic interactions in price setting, and y^n is the level of potential output (assumed to be constant).⁴ There is strategic complementarity in price setting when $\xi > 0$ (since the optimal price of a given firm positively depends on the prices set by the other firms), while there is strategic substitutability in price setting when $\xi < 0$.

All information about the aggregate state of the economy is publicly available to every firm. However, because of finite information-processing capacity, firms observe aggregate variables only imperfectly. The price actually set by firm j in period t , $p_{j,t}$, is thus given by:

$$p_{j,t} = \xi \mathbb{E}_{j,t} p_t + (1 - \xi)(\mathbb{E}_{j,t} q_t - y^n) \quad [3.10]$$

3. The importance of higher-order expectations was initially raised by Phelps (1983) and Townsend (1983) but not for the study of the non-neutrality of money.

4. Equation [3.9] is actually the combination of firm j 's log-linear approximation of the first-order condition, namely $p_{j,t}^* = p_t + (1 - \xi)(y_t - y^n)$, and $q_t = p_t + y_t$.

where $\mathbb{E}_{j,t}$ denotes the expectations operator conditional upon the information available to firm j in period t . Firms can solely observe a noisy private signal about current aggregate demand. The signal observed by firm j in period t , denoted by $s_{q,j,t}$, is represented as follows:

$$s_{q,j,t} = q_t + e_{q,j,t} \quad [3.11]$$

with $e_{q,j,t}$ following a Gaussian white noise process with variance $\tilde{\sigma}_q^2$, independent of $\varepsilon_{q,j,t}$, and independently distributed across firms. The precision of the signal is therefore given by $1/\tilde{\sigma}_q^2$: the higher $1/\tilde{\sigma}_q^2$, the more informative is the signal.

Since firms receive different signals (and ignore the signals received by others), information about current aggregate demand is dispersed.

Firms never know exactly the true value of aggregate demand, even after many periods. As a result, they can only use the history of their private signals up to period t in order to form expectations about q_t . In the case of firm j , we have:

$$\mathbb{E}_{j,t} q_t = \mathbb{E}[q_t | s_{q,j,t}, s_{q,j,t-1}, s_{q,j,t-2}, s_{q,j,t-3}, \dots] \quad [3.12]$$

However, equation [3.10] implies that the price set by a given firm also depends on this firm's expectation of the average level of prices charged by the other firms. The combination of dispersed information and strategic interactions in price setting thus induces firms to play beauty-contest games with their competitors.

Since all firms set prices in a similar way, each firm has to form an expectation of the average expectation among the other firms about q_t , an expectation of the average expectation of that average expectation and so on. Hence, on top of its own expectation of q_t , every firm has to form higher-order expectations about q_t .

Using the same notation as in equation [3.6], equation [3.10] can then be rewritten as:

$$p_{j,t} = (1 - \xi) \left[\sum_{k=0}^{\infty} \xi^k \mathbb{E}_{j,t}^{(k+1)} q_t \right] - (1 - \xi) y^n \quad [3.13]$$

The importance of higher-order expectations for pricing decisions critically depends on the value of ξ . When $\xi = 0$, there are no strategic interactions in price setting since (according to equation [3.10]) the price set by a given firm is independent of the prices set by the other firms. In this case, equation [3.13] makes it clear that pricing decisions only depend on firms' own expectations of q_t (i.e. on their first-order expectations of q_t), $\mathbb{E}_{j,t} q_t$. By contrast, when $\xi > 0$, there are strategic complementarities in price setting. In this case – which is the one usually retained in the pricing literature – equation [3.13] shows that higher-order expectations not only

matter for pricing decisions but also increasingly influence those decisions.⁵ Indeed, the stronger the strategic complementarities (i.e. the closer ξ is to 1), the higher the order of expectations firms have to consider, and the larger the weights firms have to attach to expectations of higher orders. Intuitively, this stems from the fact that the stronger the strategic elements between firms, the more important become the expectations held by the other firms in the decisions taken by a given firm.

Moreover, it can be shown that expectations of higher order react less strongly to private signals than expectations of lower order. For instance, when a positive monetary shock occurs, the resulting rise in the private signal observed by each individual firm induces a rise in third-order expectations, which is more sluggish than the rise in second-order expectations, itself more sluggish than the rise in first-order expectations.

Finally, applying Kalman-filtering techniques and averaging the various equations over j yields:

$$y_t - y^n = \kappa(y_{t-1} + \varepsilon_{q,t}) \quad \text{with } \kappa = 1 - 1/2 - (1 - \xi) \frac{\sigma_q^2}{\tilde{\sigma}_q^2} + \left[\left((1 - \xi) \frac{\sigma_q^2}{\tilde{\sigma}_q^2} \right)^2 + 4(1 - \xi) \frac{\sigma_q^2}{\tilde{\sigma}_q^2} \right]^{1/2} \quad [3.14]$$

From equation [3.14], we can see that y_t depends on both $\varepsilon_{q,t}$ and y_{t-1} . As a result (and contrary to what happens in Lucas' model, where it could be shown that y_t depends solely on $\varepsilon_{q,t}$), aggregate output responds not only to current realizations of monetary shocks, but also – and crucially – to past realizations of these shocks. This means that in Woodford's model, monetary shocks have persistent real effects.

The quantitative importance of this persistence is driven by κ . This coefficient is made of two components, each encapsulating a key mechanism at work. The first component is the precision of the signal, $1/\tilde{\sigma}_q^2$, illustrating the role played by the informativeness of the signal observed: when this signal is poorly informative (i.e. when $1/\tilde{\sigma}_q^2$ is low), it takes a very long time for firms to learn that a monetary shock has occurred (even though the information on this occurrence is available from the start) and to adjust their price accordingly.⁶ Hence, aggregate output displays

5. Several sources of strategic complementarities in price setting have been proposed, among which stand out decreasing returns to scale, firm-specific input markets, input-output linkages across sectors and real wage rigidities at the aggregate level. See Woodford (2003b, Chapter 3) and Leahy (2011) for a thorough discussion on these sources.

6. Firms, however, will never be certain that such a shock has occurred since they never know exactly the true value of aggregate demand.

persistent deviations from its potential level. The second component is the parameter ξ , illustrating the role played by strategic complementarities in price setting. When these complementarities are strong (i.e. when ξ is close to 1), firms need to form expectations of very high orders and to attach large weights to them in their pricing decisions. At the same time, expectations of higher orders react more sluggishly to information than expectations of lower orders. Strong strategic complementarities therefore slow down the response of prices to monetary shocks, leading to persistent deviations of aggregate output from potential.

Furthermore, and quite importantly, κ can be made arbitrarily large for values of ξ sufficiently close to 1, whatever the value of $1/\tilde{\sigma}_q^2$. This implies that when strategic complementarities in price setting are very strong, monetary shocks can have highly persistent effects on aggregate output even though firms are individually very well informed about the occurrence of these shocks. When prices are very strong strategic complements, indeed, the price set by any given firm is almost entirely determined by its higher-order expectations (and especially by those of very high orders) and only marginally by its first-order expectations. Individual firms thus find it optimal to adjust their price only sluggishly (chiefly reflecting the dynamics of expectations of very high orders), even though they are well aware that a monetary shock has occurred (i.e. even though first-order expectations adjust fairly quickly). Hence, provided a sufficiently high strategic uncertainty, highly persistent monetary non-neutralities can be obtained for a relatively low fundamental uncertainty.

3.3.2. *Rational inattention*

We have just seen that in Woodford's model, a poorly informative private signal – embodied in a low value of the signal precision $1/\tilde{\sigma}_q^2$ – is a central mechanism through which monetary shocks produce persistent real effects. A poorly informative signal is meant to represent the limited attention firms pay to the current state of aggregate demand when they have limited capacity for processing information. In this model, however, $1/\tilde{\sigma}_q^2$ is a free parameter whose value can be arbitrarily selected by the modeler.⁷ This means that retaining a low value for $1/\tilde{\sigma}_q^2$ amounts, in fact, to imposing that firms pay only limited attention to the state of aggregate demand. Therefore, an important source of persistent monetary non-neutralities in Woodford's model turns out to be nothing more than postulated.

The main contribution of Maćkowiak and Wiederholt (2009) is to explain the persistent real effects of monetary shocks by the rational decision of firms to pay limited attention to the state of aggregate demand. In the framework set out in that

7. By contrast, the parameter ξ – which encapsulates the second source of persistence in Woodford's framework – is a reduced-form parameter determined by the structural parameters of the model.

paper, firms allocate their attention to track both idiosyncratic and aggregate demand shocks. Maćkowiak and Wiederholt show that the allocation chosen by firms depends on the relative impact of these shocks on the prices firms would set absent information frictions. In this context, the signal precision $1/\tilde{\sigma}_q^2$ is no longer a free parameter: it is a variable chosen by firms as a function of the structural parameters driving the volatility of shocks and the responses of firms' frictionless optimal prices. For realistic values of these parameters, the resulting $1/\tilde{\sigma}_q^2$ is found to be very low, which means that it is actually optimal for firms to pay very limited attention to information about aggregate demand shocks. This rational inattention with respect to the state of aggregate demand implies that firms adjust very sluggishly their prices in response to monetary shocks, inducing large and persistent fluctuations in aggregate output.

The framework proposed by Maćkowiak and Wiederholt is basically the same as Woodford's. There are, however, three important differences. First, the price that would be set absent information frictions also depends on idiosyncratic (cost or demand) conditions. Firm j 's frictionless optimal price in period t becomes:

$$p_{j,t}^* = \xi p_t + (1 - \xi)(q_t - y^n) + \varepsilon_{z,j,t} \quad [3.15]$$

where $\varepsilon_{z,j,t}$ denotes an idiosyncratic state variable reflecting conditions specific to firm j . Firm-specific conditions follow a Gaussian white noise process with variance σ_z^2 , independent of $\varepsilon_{q,t}$, and independently distributed across firms.

Second, the flow of information that firms can effectively process in a given period is now explicitly introduced. This flow, denoted by κ , reflects firms' limited capacity for processing information (the lower firms' capacity, the lower κ). Moreover, firms can allocate this information flow – i.e. their attention – to track both idiosyncratic and aggregate demand shocks. The flow of information regarding idiosyncratic shocks is denoted by κ_z , while the flow of information regarding aggregate demand shocks is denoted by κ_q . Firms choose the values of κ_q and κ_z so as to maximize their expected profits, under the constraint:⁸

$$\kappa = \kappa_q + \kappa_z \quad [3.16]$$

As in Woodford's model, information processing limitations prevent firms from observing the precise values of aggregate variables. In particular, firms solely observe a noisy private signal about current aggregate demand. Firm j 's signal about aggregate demand in period t is still denoted by $s_{q,j,t}$, and still given by equation [3.11]. In addition, Maćkowiak and Wiederholt assume that information processing limitations also prevent firms from observing the precise values of idiosyncratic variables: firms solely observe a separate noisy private signal about their current idiosyncratic

8. In their baseline model, Maćkowiak and Wiederholt assume that firms choose the optimal values of κ_q and κ_z once and for all, in period $t = 0$.

conditions. This signal is modeled in the same fashion as the signal about aggregate demand. Hence, firm j 's signal about its own conditions in period t , denoted by $s_{z,j,t}$, is:

$$s_{z,j,t} = \varepsilon_{z,j,t} + e_{z,j,t} \quad [3.17]$$

with $e_{z,j,t}$ following a Gaussian white noise process with variance $\tilde{\sigma}_z^2$, independent of $\varepsilon_{z,j,t}$, and independently distributed across firms. The precision of this signal is therefore $1/\tilde{\sigma}_z^2$.

Third, the information structure is now endogenous. Firms, indeed, can choose the precision of the signals received. The latter depends on firms' attention: the more firms pay attention to a particular type of shocks, the less noisy the associated signal. For instance, the more firms pay attention to aggregate demand shocks (i.e. the higher κ_q), the more precise the signal $s_{q,j,t}$ (i.e. the higher is $1/\tilde{\sigma}_q^2$).

While more complicated to derive analytically, the resulting dynamics of aggregate output are actually very similar to those induced by Woodford's model and can thus be approximated by equation [3.14]. This means that, exactly as in that model, monetary shocks generate real effects whose persistence is determined by two mechanisms – respectively encapsulated by ξ and $1/\tilde{\sigma}_q^2$. In Maćkowiak and Wiederholt's model, however, $1/\tilde{\sigma}_q^2$ is no longer a free parameter: it is a variable whose value results from the amount of attention firms find it optimal to pay to aggregate demand shocks. Hence, the persistence produced by this model crucially depends on how firms allocate their attention between idiosyncratic and aggregate demand shocks.

Firms' allocation of attention is driven by the relative impact of idiosyncratic and aggregate demand shocks on frictionless optimal prices. This relative impact is, in turn, essentially determined by two factors. The first factor is the relative volatility of aggregate-demand and idiosyncratic shocks, i.e. σ_q^2/σ_z^2 . This is intuitive: if, for instance, aggregate demand shocks are more volatile than idiosyncratic shocks (i.e. if $\sigma_q^2/\sigma_z^2 > 1$), aggregate demand shocks will represent a relatively more important source of fluctuations for frictionless optimal prices, creating an incentive for firms to track these shocks with more accuracy. The second factor is the degree of strategic interactions in price setting, i.e. ξ . Let us assume that $\sigma_q^2/\sigma_z^2 > 1$, so that firms allocate most of their attention to track aggregate demand shocks. In this case, firms quickly adjust their prices in response to these shocks, since they are quite aware of their occurrence. As a result, the price level displays large fluctuations in response to aggregate demand shocks. If there are strong strategic complementarities (i.e. if ξ is close to 1), it can be seen from equation [3.15] that there will be an important feedback effect for firm j : the large movements in the price level imply that firm j 's frictionless optimal price will react even more to aggregate demand shocks. Firm j 's incentive to allocate most of its attention to this type of shocks is therefore reinforced. Hence, strategic complementarities in price setting result in strategic complementarities in

attention allocation: when ξ is close to 1, it is optimal for firms to allocate more attention to the shocks for which the other firms allocate most of their attention.

Maćkowiak and Wiederholt argue that, empirically, aggregate demand shocks are much less volatile than idiosyncratic shocks (whose large size is necessary to replicate the large average size of price changes observed in the micro-price data). This means that the actual value of σ_q^2/σ_z^2 is much lower than one, suggesting that firms would have a strong incentive to allocate most of their attention to track idiosyncratic shocks. Maćkowiak and Wiederholt also recall that, in the literature on price setting, ξ is usually set between 0.8 and 0.9. This large degree of strategic complementarities in price setting would entail a large degree of strategic complementarities in attention allocation, reinforcing firms' incentive to allocate attention mostly to idiosyncratic shocks. Under their baseline calibration, Maćkowiak and Wiederholt find that firms choose to allocate 96% of their attention to idiosyncratic shocks and, consequently, only 4% to aggregate demand shocks (i.e. $\kappa_q^* = 0.04\kappa$). Since the signal precision $1/\sigma_q^2$ positively depends on the attention paid to aggregate demand shocks, and since the fluctuations in aggregate output generated by monetary shocks are all the more persistent as $1/\tilde{\sigma}_q^2$ is low, the very limited attention firms rationally choose to pay to aggregate demand shocks induces highly persistent monetary non-neutralities.

3.3.3. *Rational inattention with multi-product firms*

In Maćkowiak and Wiederholt's model, firms produce and set the price of only one good. This assumption is standard in the literature investigating the real effects of monetary shocks – and especially in dynamic stochastic general equilibrium (DSGE) models. However, the menu cost literature has recently argued (in the wake of Midrigan 2011) that this assumption would actually have important implications for monetary non-neutrality. In particular, considering multi-product firms would allow us to weaken the selection effect, which is inherent in state-dependent pricing models and which explains why these models usually deliver much less monetary non-neutrality than their time-dependent counterparts. Alvarez and Lippi (2014) notably show that when the number of different goods priced by each firm tends to infinity, the selection effect vanishes and the amount of monetary non-neutrality generated by state-dependent models is as high as that generated by the popular model of Taylor (1980).⁹

Pasten and Schoenle (2016) embed multi-product firms into Maćkowiak and Wiederholt's model. They assume that firms have to allocate their limited attention to track three types of disturbances: aggregate demand shocks, firm-specific shocks and good-specific shocks (whose existence is documented by the authors). At first

9. For more details regarding the recent literature investigating monetary non-neutrality through menu cost models, see the chapter written by Clerc in this volume.

sight, it seems that a firm pricing a larger number of goods needs to spread thin its fixed amount of attention over a larger number of shocks. The attention allocated to tracking aggregate demand shocks is therefore mechanically reduced as the number of goods priced increases. As we have seen in the previous section, a lower amount of attention allocated to aggregate demand shocks turns into a lower signal precision $1/\tilde{\sigma}_q^2$, implying larger and more persistent real effects of monetary shocks. However, Pasten and Schoenle show that this spread-thin effect is dominated by the existence of economies of scope in information processing. These economies of scope stem from the fact that tracking aggregate-demand and firm-specific shocks, while requiring the same attention as tracking good-specific shocks, provides information that can be used to price all goods. The benefits of tracking aggregate-demand and firm-specific shocks thus scale up with the number of different goods priced, while the benefits of tracking good-specific shocks do not. Firms accordingly find it optimal to allocate more attention to aggregate demand shocks when they price more goods. The signal precision $1/\tilde{\sigma}_q^2$ therefore increases with the number of goods priced, which implies – in sharp contrast to what happens in menu cost models – that the amount of monetary non-neutrality decreases with this number.¹⁰

Pasten and Schoenle find that the amount of monetary non-neutrality declines quite substantially with the number of goods priced. When firms price two goods, for instance, this amount is cut by three with respect to the single-good (i.e. Maćkowiak and Wiederholt's) case. When firms price eight goods or more, monetary shocks have almost no effect on aggregate output. This decline reflects the increasing attention allocated to aggregate demand shocks as the number of goods priced increases: while firms allocate 3% of their total attention to track aggregate demand shocks in the single-good case, this figure rises to 11% and 24% in the two-good and eight-good cases, respectively.

Finally, it is worth emphasizing that strategic complementarities in price setting play an important role in Pasten and Schoenle's results. We have seen in the previous section that in rational inattention models, strategic complementarities in price setting induce strategic complementarities in attention allocation. The latter are particularly strong in the multi-product firm model. To illustrate this strength, Pasten and Schoenle develop an extension of their baseline model where different firms price different numbers of goods. It turns out that in this economy with heterogeneous firms, the firms that price a single good allocate much more attention to aggregate demand shocks than they do in the baseline economy (with homogenous firms). Hence, the interaction with firms allocating more attention to aggregate demand shocks leads to more attention

10. Pasten and Schoenle nevertheless make it clear that the allocation of attention, as well as the resulting amount of monetary non-neutrality, is independent of the number of goods firms price when those goods are not subject to shocks (or when good-specific shocks have no impact on firms' profits). Hence, what ultimately matters is the number of shocks that do require paying attention to, not the number of goods per se.

allocated to these shocks by firms that would have otherwise chosen to remain largely uninformed about them.

3.4. Business cycle fluctuations

The recent literature involving informational imperfections has also investigated how dispersed information would impinge on the transmission of fundamental shocks and extrinsic disturbances such as sentiment shocks. An important conclusion emerging from this work is that the exogenous or endogenous nature of the private signals observed by economic agents seems to be critical in the propagation of fundamental shocks: exogenous signals tend to dampen business cycle fluctuations, while endogenous signals tend to amplify them. Furthermore, sentiment shocks generate demand-driven fluctuations through self-fulfilling variations in expected demand.

3.4.1. *The propagation of fundamental shocks*

In order to illustrate these ideas, let us consider a standard Real Business Cycle model, in which we introduce the three following assumptions (taken from Angeletos and La'O 2010).

First, capital is assumed away in order to simplify the analysis.

Second, the economy is split into a large number of islands, indexed by $j \in (0, 1)$, and a mainland. Each island is populated by a self-employed worker (hereafter farmer), who produces a differentiated, island-specific consumption good. This introduces strategic interactions across islands: the optimal production on each island will depend on other islands' production. The goods produced on the different islands are subsequently traded on the mainland.

Third, each period is divided into two stages: production takes place in stage 1, while trading and consumption take place in stage 2. In stage 1, the farmer of any given island is uncertain about the productivity and output of other islands. Information is therefore dispersed at the time when production decisions are made. In stage 2, all information that was previously dispersed becomes publicly known, and markets open. Quantities are predetermined by the production decisions made during stage 1, but prices adjust so as to clear markets.

Expressing all variables in logarithms, production on island j is given by:

$$y_{j,t} = a_{j,t} + n_{j,t}$$

where $y_{j,t}$ denotes the level of output produced in period t , $n_{j,t}$ is the level of employment, and $a_{j,t}$ is the level of the exogenous local productivity.

Local productivity is the sum of an aggregate and an idiosyncratic component: $a_{j,t} = a_t + \varepsilon_{j,t}$, where a_t is the aggregate technology shock, and $\varepsilon_{j,t}$ is an idiosyncratic one. The former follows a random walk: $a_t = a_{t-1} + \varepsilon_{a,t}$, with $\varepsilon_{a,t}$ following a Gaussian white noise process with variance σ_a^2 . The latter follows a Gaussian white noise process with variance σ_j^2 , independent of $\varepsilon_{a,t}$, and independently distributed across islands.

The local productivity of island j is perfectly observed by the farmer located on that island. However, she is unable to distinguish between the aggregate and island-specific components at the time when production decisions are made. We assume that in stage 1 of each period, the only information the farmer has about the current-period aggregate shock, a_t , is summarized in a private signal of the form:

$$s_{a,j,t} = a_t + e_{a,j,t} \quad [3.18]$$

with $e_{a,j,t}$ following a Gaussian white noise process with variance $\tilde{\sigma}_a^2$, independent of $\varepsilon_{a,t}$, and independently distributed across islands.

Angeletos and La'O (2010) show that the general equilibrium of the resulting model reduces to the solution of the following fixed-point equation:

$$y_{j,t} = \alpha \mathbb{E}_{i,t} y_t + (1 - \alpha) \chi a_{j,t} \quad [3.19]$$

where $\mathbb{E}_{i,t}$ denotes the expectations operator conditional upon the information available to island j in period t , y_t is the level of aggregate output, and $\alpha < 1$ and $\chi > 0$ are the reduced-form parameters that depend on the underlying preferences and technology.

The parameter α measures the degree of strategic interactions across islands. As stressed by Angeletos and Lian (2016), two opposing mechanisms determine the level of α : a demand-side effect that contributes towards strategic complementarity ($\alpha > 0$), and a supply-side effect contributing towards strategic substitutability ($\alpha < 0$). Indeed, when a farmer expects aggregate output to go up, the demand for the local good and its relative price are expected to go up as well. This effect motivates the farmer to work and produce more. This is the demand-side effect, which induces local output to increase with expectations of aggregate output. The opposing supply-side effect stems from an income effect: when a farmer expects aggregate output to go up, income is also expected to go up, which tends to discourage labor supply and production. Whether α is positive or negative therefore depends on which of the two effects dominates. In what follows, we assume that the demand-side effect dominates, so that there are strategic complementarities across islands ($\alpha > 0$).

There is no specific intuition for the parameter χ , since this (reduced-form) parameter is a composite of structural (stemming from both technology and preferences) ones.

Had information been complete rather than dispersed, we would have had $\mathbb{E}_{j,t}y_t = y_t$ for all j , and, consequently:

$$y_t = \chi a_t$$

Hence, as predicted by the standard RBC model, aggregate output would have been: (i) exclusively pinned down by the technology shock and (ii) completely independent of the degree of strategic interactions α .

As equation [3.19] makes it clear, the combination of dispersed information and strategic complementarities across islands implies that individual farmers have to play beauty-contest games with the other farmers. They thus need to form higher-order expectations. Using the same notation as in equation [3.6], Angeletos and La'O (2010) show that the aggregate level of output can be expressed as follows:

$$y_t = \chi(1 - \alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{E}_{j,t}^{(k+1)} a_t \quad [3.20]$$

The reliance on higher-order expectations dampens the responses of aggregate output to aggregate technology shocks. Indeed, since expectations of higher order react less strongly to private signals than expectations of lower order, the fluctuations of aggregate output are milder than in the complete information case. Moreover, the strength of this dampening effect increases with the strength of strategic complementarities: the stronger strategic complementarities across islands (i.e. the closer α is to 1), the higher the order of expectations farmers have to consider, and the larger the weights they have to attach to expectations of higher orders.

This dampening effect can be more easily seen by noting that equation [3.20] can be rewritten as:

$$y_t = \chi a_{t-1} + \Phi \varepsilon_{a,t}$$

where Φ denotes a reduced-form parameter whose value satisfies $0 < \Phi < \chi$. Furthermore, Φ converges to 0 from above as α converges to 1 from below. Therefore, the impact of an aggregate technology shock on aggregate output, driven by Φ , is lower than its counterpart in the complete information case, which is driven by χ . In addition, this impact declines with the degree of strategic complementarities, whereas it is independent of this degree in the complete information case.

Chahrour and Gaballo (2021) argue that the existence of such a dampening effect ultimately stems from the fact that the signal received by each individual farmer is purely exogenous. They consider a more complicated RBC economy, comprising a housing sector. More specifically, households are located on different islands and consume both a traded consumption good and local housing. The consumption good is produced using labor from all islands, while local housing is produced using land,

local labor and a traded productive factor (the commodity good) whose supply is fixed. Local house prices can move either because of an island-specific shock affecting the future productivity of local labor, or because of a current aggregate shock to housing production. Given that most fluctuations in those prices are driven by local labor productivity, observing high house prices induces households to become optimistic about their own income prospects. However, an increase in local house prices can also result from a negative aggregate shock to housing production, since this kind of shock generates a rise in house prices across islands. When such a shock occurs, the observed increase in local house prices is misinterpreted by households as good news about future wages. They thus raise their demand for both consumption and housing. The resulting increase in aggregate demand further increases house prices (as well as the price of the commodity factor), reinforcing the initial price increase.

Hence, what started as a small change in housing supply leads, in equilibrium, to an economy-wide increase in house prices and to a boom in consumption and housing. Chahrour and Gaballo point out that the logic of their model extends to other sorts of fundamental shocks and to learning from the observation of any local price. In each case, the crucial feature is a price-optimism feedback channel: higher local prices begets economic optimism, which begets even higher local prices and so on. Economic fluctuations are therefore amplified, rather than dampened, and this amplification originates from the fact that agents receive information through the observation of signals taking the form of endogenous variables.

3.4.2. *Sentiments as a source of disturbance*

The literature investigating the role of heterogeneous information in business cycle fluctuations has not solely focused on fundamental shocks. Indeed, beauty-contest games – and the associated higher-order uncertainty – open the door to the emergence of shocks affecting the expectations of each agent regarding the expectations and actions of all other agents. These extrinsic disturbances have been commonly referred to as sentiment shocks after Angeletos and La’O (2013) and Benhabib et al. (2015).¹¹ Sentiment shocks are assumed to directly affect higher-order expectations: each agent, while being well aware that no fundamental shock has occurred, suddenly believes that the other agents expect such a shock to have actually taken place. Sentiment shocks therefore take the form of waves of optimism or pessimism and, as such, can be associated with forces akin to animal spirits.

Let us consider the RBC economy introduced in the previous section, with two modifications.

11. For alternative work on extrinsic disturbances, see the chapter by Kazuo Nishimura and Alain Venditti in Volume 4 of this series devoted to short-term macroeconomics.

First, we drop the idiosyncratic component of local productivity, so that the latter is only made of the aggregate component: $a_{j,t} = a_t$. Individual farmers are assumed to be unable to observe local productivity when they make their production decisions in stage 1 of each period. They can only observe the signal given by equation [3.18].

Second, we assume that each farmer believes that the private signals observed by the other farmers are biased: the prior of the farmer located on island j is that the noise of her signal follows a Gaussian white noise process with variance $\tilde{\sigma}_a^2$, and that the noise of the signals received by the other farmers follows a Gaussian process with mean μ_t and variance $\tilde{\sigma}_a^2$, where μ_t is a random variable that represents the perceived bias in one another's signals. We assume that μ_t follows an AR(1) process:

$$\mu_t = \rho\mu_{t-1} + \nu_t$$

where $0 \leq \rho < 1$ and ν_t follows a Gaussian white noise process with variance σ_μ^2 . Innovations in μ_t are called sentiment shocks.

It can be shown that the general equilibrium of the resulting model reduces to the solution of the following fixed-point equation:

$$y_{j,t} = \alpha \mathbb{E}_{j,t} y_t + (1 - \alpha) \phi \mathbb{E}_{j,t} a_t \quad [3.21]$$

where $\phi > 0$ is a reduced-form parameter that depends on the underlying preferences and technology. Let us now assume a positive innovation in μ_t . Such a positive sentiment shock means that, while they are observing a signal indicating no shock on aggregate productivity, individual farmers suddenly believe that the other farmers are observing wrong signals indicating a positive shock on aggregate productivity. As a result, they keep their own expectations of aggregate productivity unchanged (i.e. $\mathbb{E}_{j,t} a_t$ is not affected) but adjust their higher-order expectations upwards. This upward adjustment leads individual farmers to become optimistic about the level of aggregate output (i.e. to raise $\mathbb{E}_{j,t} y_t$). According to equation [3.21], the optimal response of each farmer is to raise her own level of output – since an increase in the expected level of aggregate activity means an increase in the expected demand for her product. Sentiment shocks thus generate demand-driven fluctuations through self-fulfilling variations in expected demand: a positive shock induces individual farmers to expect a higher level of aggregate demand; each farmer accordingly raises production and spending, pushing up aggregate demand and validating the initial expectation. However, given that $\rho < 1$, this wave of optimism is only short-lived. Consequently, farmers find it optimal to consume only a fraction of the increase in their income and to save the rest. These additional savings would have induced an increase in investment had capital not been assumed away.

3.5. Communication and stabilization policies

Sections 3.3 and 3.4 have shown how the conjunction of strategic complementarities and heterogeneous information allows macroeconomic models to

generate monetary non-neutrality and realistic business fluctuations. This framework also allows us to address economic policy issues. When imperfect information causes money non-neutrality and leads to business fluctuations, disclosure of information is likely to have a significant impact on the economy. An extensive literature has discussed the welfare effects of disclosing more information to market participants. Furthermore, as information disclosure shapes the behavior of economic agents, communication and stabilization policies need to be assessed in a comprehensive framework.

3.5.1. *Welfare effect of public disclosure*

The simple beauty-contest model derived in section 3.2 shows that – due to strategic complementarities – firms overreact to public information in order to take action close to that of others. Yet, how does overreaction to public information affect welfare? This question fueled a lively debate in the literature. It turns out that the answer depends on the analytical framework, particularly on the welfare function. It also depends on the type of fundamental about which imperfect information relates. The analytical framework developed in section 3.2 refers to an economy where the fundamental is the aggregate nominal demand and where imperfectly known variations of it are equivalent to demand shocks. Under complete information, demand shocks do not yield inefficiencies. Indeed, to the extent that a demand shock is perfect common knowledge among agents, no welfare loss follows in this framework. Welfare losses occur when information about aggregate demand is incomplete, and agents make inefficient use of their information. We summarize below the main lines of argument regarding the welfare effect of public information about demand shocks and then discuss how information can be disclosed to control the overreaction of agents.

The conclusions raised below about optimal communication policy do not hold when shocks yield inefficiencies under complete information, as is the case of supply or cost-push shocks. Because these shocks create inefficiencies even when they are perfect common knowledge, more information is typically socially costly since it strengthens firms' responses to inefficient shocks (e.g. Angeletos and Pavan (2007), Angeletos et al. (2016)).

3.5.1.1. *Is Morris and Shin (2002) pro or contra transparency?*

Morris and Shin (2002) show that, due to the overreaction mechanism, the disclosure of public information can be detrimental to welfare. In their model, the welfare loss takes the form

$$L_{MS} = \int_j (p_j - q)^2 dj = \int_j (p_j - p)^2 dj + (q - p)^2 \quad [3.22]$$

which yields, using the information structure of section 3.2 and the equilibrium pricing (equation [3.7]), the unconditional expected social loss

$$\mathbb{E}(L_{MS}) = (1 - \omega)^2 \sigma_\varepsilon^2 + \omega^2 \sigma_\eta^2$$

Let us assume that the central bank, or another public agency, has the choice between disclosing the public signal with precision σ_η^2 (transparency) and withholding this signal (opacity). What is the optimal communication strategy? When the central bank withholds the public signal, the price of each firm equals its private signal (i.e. $\omega = 0$) and the corresponding expected loss is determined by the price dispersion across agents σ_ε^2 . It turns out that withholding the public signal is preferable to disclosing it when the following condition holds: $2\xi - 1 > \sigma_\varepsilon^2/\sigma_\eta^2$, indicating that the disclosure of public information can be detrimental to welfare under strategic complementarities. This important result undermined the general presumption that more information generally improves the efficiency of market outcomes by showing that more information can decrease welfare even though welfare would be maximal under complete information.

However, the presumed anti-transparency result of Morris and Shin (2002) has been challenged by two main criticisms. First, Svensson (2006) observes that withholding public information improves welfare in the model of Morris and Shin only under unrealistic parameter values. Transparency is indeed detrimental to welfare when the information of the central bank on the fundamental is less accurate than the private information of firms, i.e. when $\sigma_\varepsilon^2 < \sigma_\eta^2$. This seems rather unrealistic because a central bank usually has a more accurate view of economic fundamentals than any individual firm, as shown by Romer and Romer (2000). Svensson concludes that the model of Morris and Shin actually speaks in favor of rather than against transparency.

Second, Woodford (2005) and Hellwig (2005) show that the anti-transparency result of Morris and Shin rests on the specificity of their utility and welfare functions and does not hold in a micro-founded framework. Because the utility of the representative household decreases significantly with the price dispersion across goods in these models, public information improves a household's utility more through its coordinating role on prices than it worsens utility through its distortive effect on the average price. In micro-founded models, the second-order approximation of the loss of the representative household increases with the dispersion of prices across firms and the distortion of the price level from demand:

$$L_{MF} = \frac{\theta}{1-\xi} \int_j (p_j - p)^2 dj + (q - p)^2 \quad [3.23]$$

where $\theta > 1$ is the parameter of price elasticity of demand in the Dixit–Stiglitz composite good. This micro-founded loss function assigned a much higher weight to price dispersion than the loss function [3.22] of Morris and Shin. As a result, the unconditional expected loss with the public signal, $\mathbb{E}(L_{MF}) = \theta(1-\omega)^2\sigma_\varepsilon^2/(1-\xi) + \omega^2\sigma_\eta^2$, is always smaller than the expected loss without the signal, $\theta\sigma_\varepsilon^2/(1-\xi)$, regardless of the variance of the error terms. Transparency thus improves welfare in micro-founded models.

Angeletos and Pavan (2007) generalize the analysis and show that the welfare effect of public information depends on the relation between the *equilibrium* degree of coordination in firms' action and the *efficient* degree of coordination from a social welfare perspective. The equilibrium degree of coordination corresponds to the degree of strategic complementarities which drives firms' coordination in equilibrium, i.e. ξ in equation [3.1]. The efficient degree of coordination is the degree of coordination of firms that would maximize welfare. When the equilibrium degree of coordination coincides with the efficient degree, transparency improves welfare because firms make an efficient use in equilibrium of their private and public information. By contrast, when the equilibrium degree of coordination exceeds the efficient degree, firms react too strongly to public information compared to what is socially desirable, and the disclosure of public information may be detrimental to welfare. This is the case in the model of Morris and Shin: differentiating the loss (equation [3.22]) with respect to ξ shows that the efficient degree of coordination is 0, which is smaller than the equilibrium degree ξ . Inversely, when the equilibrium degree of coordination is lower than the efficient degree, firms react too weakly to public information compared to what is socially desirable, and the disclosure of public information improves coordination and welfare. This is the case in the micro-founded model: differentiating the loss (equation [3.23]) with respect to ξ shows that the efficient degree of coordination is $1 - (1 - \xi)/\theta$, which is larger than the equilibrium degree ξ .

The welfare effect of public information thus depends on the gap between the equilibrium degree of coordination of firms and its efficient degree. The central bank has no influence on the equilibrium degree of coordination of firms and therefore cannot reduce its gap to the efficient degree to maximize welfare. But it can adjust its communication policy to control the overreaction to public information and reduce the welfare loss.

3.5.1.2. *Reducing overreaction*

How can the central bank reduce overreaction to its disclosures? The central bank can either disclose public information to a fraction of market participants only (partial publicity) or disclose public information with some ambiguity to all market participants (partial transparency). Both strategies are equivalent in reducing overreaction and improving welfare.

In the partial-publicity strategy, proposed by Cornand and Heinemann (2008), each firm j observes a private signal q_j and only a fraction of firms observes a public signal D . As previously, these signals deviate from the fundamental q by some error terms normally distributed. The private signal is defined as in equation [3.2]. However, only the fraction Q of firms observes the common signal D defined as in equation [3.3]. This signal is thus semi-public. Q is the degree of publicity.

Choosing a communication channel that does not reach all market participants reduces overreaction to the disclosure because uninformed participants cannot react

to it, whereas informed participants react less strongly because they know that some of their peers are uninformed. The equilibrium action of informed firms that get both the private and the semi-public signal is a linear combination of their signals and is given by

$$p_{j,Q} = \frac{(1 - \xi Q) \sigma_\eta^2}{\sigma_\varepsilon^2 + (1 - \xi Q) \sigma_\eta^2} q_j + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + (1 - \xi Q) \sigma_\eta^2} D \quad [3.24]$$

If $Q = 1$, all firms observe the public signal D and the equilibrium price (equation [3.24]) is equivalent to equation [3.7] in the previous section. However, when the fraction of informed firms falls, informed firms assign a smaller weight on the semi-public signal D because they anticipate in their higher-order expectations that the firms that do not observe the semi-public signal set their price equal to their private signal q_j .

What is the optimal share of firms Q to which the semi-public signal should be disclosed? The loss functions [3.22] and [3.23] discussed above can be rewritten as

$$L = \lambda \int_j (p_j - p)^2 dj + (q - p)^2 \quad [3.25]$$

with $\lambda = 1$ and $\lambda = \theta/(1 - \xi)$, respectively. Differentiating the unconditional expected loss with respect to Q gives the optimal degree of publicity that maximizes welfare: $Q^* = \min \left[\max \left[0, \frac{\lambda(\sigma_\varepsilon^2 + \sigma_\eta^2)}{(2 - 2\lambda + 3\xi\lambda)\sigma_\eta^2} \right], 1 \right]$. Interestingly, when $\lambda = 1$ as in Morris and Shin (2002), Q^* can be lower than 1 even though the central bank has more accurate information on the fundamental than firms, i.e. even though $\sigma_\eta^2 < \sigma_\varepsilon^2$. This means that the critique of Svensson is not robust to the introduction of partial publicity, as reducing publicity may improve welfare even though the central bank has more accurate information than firms.

The second strategy for reducing overreaction consists of disclosing public information to all market participants, but with ambiguity.¹² The degree of transparency is determined by the idiosyncratic noise of the public signal disclosed to all market participants. Each firm j observes a private signal q_j and a semi-public signal with an idiosyncratic noise D_j . As previously, these signals deviate from the fundamental q by some error terms normally distributed. The private signal is defined as in equation [3.2]. The semi-public signal is defined as $D_j = D + \phi_j$ with D defined as in equation [3.3] and $\phi_j \sim N(0, \sigma_\phi^2)$. The signal D_j is semi-public as it contains an error term η that is common to all firms and an error term ϕ_j that is private to each firm. The idiosyncratic error term captures the ambiguity of the public disclosure and the fact that each firm may interpret it differently. Central bankers are indeed known to

12. Heinemann and Illing (2002) were the first to introduce this concept in a speculative attack game.

speak with ambiguity, making their speeches equivocal and subject to interpretation.¹³ With this information setup, the equilibrium price of firm j is

$$p_j = \frac{(1 - \xi)\sigma_\eta^2 + \sigma_\phi^2}{\sigma_\varepsilon^2 + (1 - \xi)\sigma_\eta^2 + \sigma_\phi^2} q_j + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + (1 - \xi)\sigma_\eta^2 + \sigma_\phi^2} D_j$$

Communicating with ambiguity reduces overreaction to public information because ambiguity mitigates its focal role as it entails uncertainty about how other market participants interpret the disclosure. If $\sigma_\phi^2 = 0$, the central bank is perfectly transparent, all firms interpret its signal identically, and the equilibrium price is equivalent to equation [3.7] in the previous section.

Differentiating the unconditional expected loss (equation [3.25]) with respect to σ_ϕ^2 gives the optimal degree of transparency that maximizes welfare: $\sigma_\phi^{2*} = \max \left[0, \frac{2-3\lambda+3\xi\lambda}{\lambda} \sigma_\eta^2 - \sigma_\varepsilon^2 \right]$. Similarly to the strategy of partial publicity, the critique of Svensson is not robust to the introduction of partial transparency, as σ_ϕ^{2*} can be higher than 0 when $\lambda = 1$ even though the central bank has more accurate information than firms.

Baeriswyl and Cornand (2014) show that partial publicity and partial transparency are equivalent in reducing overreaction to public information and in improving welfare. Moreover, a given equilibrium weight in price setting yields the same dispersion of prices across firms $\int_j (p_j - p)^2 dj$ and the same distortion of the average price from the fundamental $(y - p)^2$ in both partial-publicity and partial-transparency strategies. They also run a laboratory experiment to test this equivalence result and show that partial publicity and partial transparency succeed in reducing overreaction to public information, though not as much as theory predicts.

3.5.2. Endogenous information

An important extension of the analytical framework described above is to relax the assumption that agents' information is exogenously given to make the acquisition of information or its content dependent on the actions of agents. In a nutshell, strategic complementarities in actions carry over to endogenous information, which strengthens the effect of complementarities on welfare. We review some of these extensions and their implications for welfare.

13. To illustrate the ambiguity of central bankers' speeches, let us recall the bon mot of Alan Greenspan in front of the American Congress: "Since I have become a central banker, I have learned to mumble with great incoherence. If I seem unduly clear to you, you must have misunderstood what I said" (1987). More recently (in December 2002), Mike Moskow, president of the Chicago Fed, commenting about the way the Fed communicates, says that it is: "a language in which it is possible to speak, without ever saying anything".

3.5.2.1. *Learning from prices*

As already underlined by Hayek (1945), market prices are not only exchange ratios between goods but also aggregators of information: prices play an informational role by aggregating agents' knowledge and belief about the state of the economy. The literature has analyzed firms' learning from market prices, on the one hand, and the central bank's learning from market outcomes, on the other hand. Relying on an island model à la Lucas (1972), Amador and Weill (2010) study the welfare effects of public information disclosure about productivity and/or monetary shocks in a microfounded model when firms learn from prices. The economy is subject to an aggregate monetary shock, an aggregate productivity shock and idiosyncratic real demand shocks. Economic agents cannot tell whether a high nominal price in their own sector is due to a high real demand shock or to a high monetary shock. However, they learn (partially) about the monetary shock by observing the price level. The informativeness of the latter is endogenous: agents use the price level as a signal to improve their forecast of the monetary shock. Because the public forecast is the same for everyone, all there is to learn about monetary shocks comes from other agents' private forecasts. The informativeness of the price level is determined by the weight agents assign to their private forecasts. If agents put more weight on their private forecasts, inducing that more private information feeds into prices, then the price level becomes more informative, improving in turn the quality of agents' forecasts. In this context, releasing more precise public information about the monetary shock, the productivity shock or both will induce agents to put more weight on these signals and thus less on private information, reducing the endogenous informational content of prices. This negative effect is amplified by strategic complementarities, which increase in equilibrium agents' uncertainty about the monetary shock and their real demands, and reduce welfare.

While learning from prices is exerted only via a public channel in Amador and Weill (2010), Amador and Weill (2012) envisage the possibility that agents also learn via a private channel. They consider that dispersed private knowledge about their own local markets (that also informs agents about aggregate economic conditions) diffuses in the population over time as households and firms learn from each other valuable bits of information that are not yet known to everyone. While the latter dissemination operates via a private channel, on the opposite, the information aggregated in prices or in the macroeconomic figures published by agencies is typically public. Amador and Weill show that as long as some of the learning is due to a private channel, public information may increase the overall level of information in the short run, but this comes at the cost of lowering the overall level of information in the long run. Therefore, when agents also learn from private channels, a release of public information can reduce welfare.

In both cases, by inducing private agents to rely less on their private information, the disclosure of public information reduces the informational efficiency of the price system, giving rise to a detrimental welfare effect.

Another branch of the literature highlights that policymakers themselves – and not only private agents – do rely on market outcomes to make their decision: central banks themselves assess the state of the economy by observing market outcomes, such as prices or the level of economic activity. Rather than directly observing fundamental shocks, the central bank collects information about economic fundamentals from observing the economic outcome. Indeed, economic shocks cannot be observed independently of the behavior of economic agents. At the same time, the central bank shapes the economy through the conduct of monetary policy. This gives a dual role to the central bank as both observer and shaper of the economy, posing a dilemma documented by Amato and Shin (2006) and Morris and Shin (2005): the more a central bank influences economic outcomes, the less reliable those outcomes are as indicators of the fundamental of the economy. Baeriswyl (2011) shows that taking this mirror effect into account worsens the potential detrimental effect of overreaction to public information highlighted by Morris and Shin (2002) and calls for a lower degree of central bank transparency than in the case where the central bank's information is exogenously given.

3.5.2.2. *Information acquisition*

Whereas the above papers focus on the endogeneity of information content, there is a literature that considers firms' strategic information acquisition choice. The main idea developed by Hellwig and Veldkamp (2009) and Myatt and Wallace (2012) is that when firms face the choice of acquiring different information sources (at some cost), they give more attention to the most public (and least private) signals. Indeed, strategic complementarities in actions give rise to strategic complementarities in information acquisition. The more agents' actions are complementary, the more the information acquired becomes public. Intuitively, public signals act as effective focal points for players' coordination. As the desire for coordination strengthens, agents pay more attention to such signals.

These theoretical predictions do not seem to be fully borne out in reality. For example, while theory predicts forward guidance by central banks about their intended future policy to have a significant impact on financial markets, their reaction seems surprisingly weak. Shaping market expectations seems more challenging in practice than in theory. One branch of the literature has set out to explain the possible causes of this so-called forward guidance puzzle, putting forward, like Angeletos and Lian (2018), the lack of common knowledge among market participants, or, like Garcia-Schmidt and Woodford (2019), the limited level of reasoning as possible causes of the agents' weak reaction. Both hypotheses are supported in the laboratory experiment by Baeriswyl et al. (2021). This study shows that participants in the lab pay less attention to public disclosure than theoretically predicted, which prevents the emergence of common knowledge, and operate only at a limited level of reasoning in choosing their action, both contributing to a weaker effect of public disclosure.

3.5.3. Stabilization policy

The models presented in the previous sections examined the effect of information disclosure on agents' actions and welfare but ignored the fact that information disclosure by a policymaker often goes hand in hand with the implementation of a policy action. In reality, central banks play a direct role in shaping macroeconomic outcomes through active policy interventions. James and Lawler (2011) make an important contribution by analyzing how the inclusion of policy action affects the welfare effect of information disclosure.

Suppose that the central bank stabilizes the economy affected by demand shocks q owing to its monetary instrument I . The nominal aggregate demand is composed of the demand shock and the monetary instrument. Firms set their price as a function of their first- and higher-order beliefs on the demand shock and the monetary instrument:

$$p_j = \mathbb{E}_j [(1 - \xi)(q + I) + \xi p] \quad [3.26]$$

The central bank sets its monetary instrument as a function of its information on the demand shock, $I = \rho \cdot D$. This policy function is common knowledge among firms.

As in the set-up above, each firm j receives a private signal as defined in equation [3.2]. The central bank also discloses its own information about the fundamental with some degree of transparency, D_j , as defined in section 3.5.1.2.

The equilibrium action of agent j yields

$$\begin{aligned} p_j &= \frac{(1 - \xi)\sigma_\eta^2 + (1 + \rho)\sigma_\phi^2}{\sigma_\varepsilon^2 + (1 - \xi)\sigma_\eta^2 + \sigma_\phi^2} q_j \\ &+ \frac{(1 + \rho)\sigma_\varepsilon^2 + \rho(1 - \xi)\sigma_\eta^2}{\sigma_\varepsilon^2 + (1 - \xi)\sigma_\eta^2 + \sigma_\phi^2} D_j = \omega_q \cdot q_j + \omega_D \cdot D_j \end{aligned} \quad [3.27]$$

with $\omega_q + \omega_D = 1 + \rho$. Given the equilibrium action of agents, the policy action coefficient ρ that maximizes the loss function

$$L = \lambda \int_j (p_j - p)^2 dj + (q + I - p)^2 = \lambda(\omega_q^2 \sigma_\varepsilon^2 + \omega_D^2 \sigma_\eta^2) + (\rho - \omega_D)^2 \sigma_\eta^2 \quad [3.28]$$

is given by

$$\rho^* = - \frac{(\lambda \sigma_\varepsilon^2 + (2\lambda - 2\lambda\xi - 1)\sigma_\eta^2 + \lambda \sigma_\phi^2)\sigma_\varepsilon^2}{(\lambda \sigma_\varepsilon^2 + \sigma_\eta^2)\sigma_\phi^2 + \lambda(\sigma_\varepsilon^2 + (1 - \xi)\sigma_\eta^2)^2} \quad [3.29]$$

The central bank counteracts the demand shock with its instrument, and its response increases in absolute value with the accuracy of its information ($\partial \rho / \partial \sigma_\eta^2 > 0$) to reach -1 when its information is perfect. Firms, of course, anticipate the stabilization policy of the central bank when setting their price.

How should then the central bank communicate when it implements an optimal instrument? Plugging the equilibrium action [3.27] and optimal instrument [3.29] into the loss function [3.28] and differentiating it with respect to σ_ϕ^2 shows that unconditional expected loss decreases with the opacity of the central bank: $\partial \mathbb{E}(L|_{\rho^*}) / \partial \sigma_\phi^2 < 0$. In other words, the central bank that stabilizes the economy with its monetary instrument should remain completely opaque about its information on the demand shock. Moreover, not only is zero disclosure desirable as part of an optimal stabilization policy, but it also achieves the first best-possible outcome, i.e. the outcome that would be achieved if firms made efficient use of their information.

The analysis by Morris and Shin (2002) points out that strategic complementarities lead firms to make socially inefficient use of their information, as public information is more predictive of the behavior of others than private information. In the James and Lawler (2011) framework, where the central bank stabilizes the economy with its instrument, the central bank uses its information efficiently as it aims to minimize the social loss rather than optimize the beauty contest of firms. The central bank's information is thus put to better use by implementing the monetary instrument than by disclosing it to firms that would overreact to it. Opacity is therefore beneficial when the central bank implements a policy action: a higher welfare can be reached when the central bank stabilizes shocks without communicating since it avoids firms' overreaction while ensuring all information is exploited optimally. Nevertheless, the question arises as to whether it is realistic to assume that the central bank's action can remain unknown to market participants. Baeriswyl and Cornand (2010) or Melosi (2017) state that the implementation of monetary policy is necessarily observed by market participants and, therefore, signals the central bank's information. The central bank must therefore take this signaling channel into account when making its monetary policy decision.

As in the case of pure communication policy, the analytical framework with stabilization policy has been extended to endogenous information structures.¹⁴ Paciello and Wiederholt (2014) study the combined effect of communication and stabilization policies when the amount of attention that firms devote to aggregate conditions is endogenous. There are two types of aggregate shocks: shocks that cause efficient fluctuations under perfect information (as we have assumed so far) and shocks that cause inefficient fluctuations under perfect information (e.g. markup shocks).

14. See, for example, Morris and Shin (2018) and Baeriswyl et al. (2020) for the study of the optimal stabilization and communication policies when the central bank observes the market outcome.

Owing to its policy, the central bank can discourage firms from paying attention to markup shocks. The optimal monetary policy in response to the first type of shocks is to fully stabilize the profit-maximizing price, implying that firms do not have to change prices in response to the shock. In response to the second type of shock, the central bank would like to avoid the allocation under perfect information. The optimal monetary policy is therefore to reduce inefficient price dispersion by stabilizing the profit-maximizing price in response to markup shocks, such that firms pay no attention to the latter. In addition, the monetary policy that stabilizes the profit-maximizing price in response to markup shocks has a direct negative effect on consumption, but this effect is weakened by the fact that price setters pay less attention to markup shocks since this implies that the price level increases less after a positive markup shock. Therefore, a monetary policy that reduces the response of profit-maximizing price to markup shocks reduces both inefficient price dispersion and consumption variance.

3.6. References

- Adam, K. (2007). Optimal monetary policy with imperfect common knowledge. *Journal of Monetary Economics*, 54, 267–301.
- Alvarez, F. and Lippi, F. (2014). Price setting with menu cost for multiproduct firms. *Econometrica*, 82(1), 89–135.
- Amador, M. and Weill, P.-O. (2010). Learning from prices: Public communication and welfare. *Journal of Political Economy*, 118, 866–907.
- Amador, M. and Weill, P.-O. (2012). Learning from private and public observations of others' actions. *Journal of Economic Theory*, 147, 910–940.
- Amato, J.D. and Shin, H.S. (2006). Imperfect common knowledge and the information value of prices. *Economic Theory*, 27, 213–241.
- Angeletos, G.-M. and La'O, J. (2010). Noisy business cycles. *NBER Macroeconomics Annual* 2009, 24, 319–378.
- Angeletos, G.-M. and La'O, J. (2013). Sentiments. *Econometrica*, 81(2), 739–779.
- Angeletos, G.-M. and Lian, C. (2016). Incomplete information in macroeconomics. *Handbook of Macroeconomics*, Vol.2, 1065–1240.
- Angeletos, G.-M. and Lian, C. (2018). Forward guidance without common knowledge. *American Economic Review*, 108, 2477–2512.
- Angeletos, G.-M. and Pavan, A. (2007). Efficient use of information and social value of information. *Econometrica*, 75, 1103–1142.
- Angeletos, G.-M., Iovino, L., La'O, J. (2016). Real rigidity, nominal rigidity, and the social value of information. *American Economic Review*, 106(1), 200–227.
- Baeriswyl, R. (2011). Endogenous central bank information and the optimal degree of transparency. *International Journal of Central Banking*, 7(2), 85–111.

- Baeriswyl, R. and Cornand, C. (2010). The signaling role of policy actions. *Journal of Monetary Economics*, 57, 682–695.
- Baeriswyl, R. and Cornand, C. (2014). Reducing overreaction to central banks disclosure: Theory and experiment. *Journal of the European Economic Association*, 12, 1087–1126.
- Baeriswyl, R., Cornand, C., Ziliotto, B. (2020). Observing and shaping the market: The dilemma of central banks. *Journal of Money, Credit and Banking*, 52, 1973–2005.
- Baeriswyl, R., Boun My, K., Cornand, C. (2021). Double overreaction in beauty-contests with information acquisition: Theory and experiment. *Journal of Monetary Economics*, 118(C), 432–445.
- Benhabib, J., Wang, P., Wen, Y. (2015). Sentiments and aggregate demand fluctuations. *Econometrica*, 83(2), 549–585.
- Chahrour, R. and Gaballo, G. (2021). Learning from house prices: Amplification and business fluctuations. *The Review of Economic Studies*, 88(4), 1720–1759.
- Christiano, L.J., Eichenbaum, M., Evans, C.L. (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy*, 113(1), 1–45.
- Clerc, P. and Dos Santos Ferreira, R. (2022). Dispersed information and the non-neutrality of money: Fifty years after Lucas 1972. *Journal of Economic Methodology*, 29(1), 86–104.
- Cornand, C. and Heinemann, F. (2008). Optimal degree of public information dissemination. *Economic Journal*, 118, 718–742.
- Garcia-Schmidt, M. and Woodford, M. (2019). Are low interest rates deflationary? A paradox of perfect-foresight analysis. *American Economic Review*, 109(1), 86–120.
- Hayek, F.A. (1945). The use of knowledge in society. *American Economic Review*, 35(4), 519–530.
- Heinemann F. and Illing G. (2002). Speculative attacks: Unique equilibrium and transparency. *Journal of International Economics*, 58, 429–450.
- Hellwig, C. (2005). Heterogeneous information and the welfare effects of public information disclosures. Paper, UCLA, Los Angeles, CA.
- Hellwig, C. and Veldkamp, L. (2009). Knowing what others know: Coordination motives in information acquisition. *Review of Economic Studies*, 76, 223–251.
- James, J.G. and Lawler, P. (2011). Optimal policy intervention and the social value of public information. *American Economic Review*, 101, 1561–1574.
- Keynes, J.M. (1973). *The General Theory of Employment, Interest, and Money*. McMillan, London [Original published 1936].
- Leahy, J. (2011). A survey of New Keynesian theories of aggregate supply and their relation to Industrial Organization. *Journal of Money, Credit and Banking*, 43(Supplement), 87–110.
- Lucas Jr, R. E. (1972). Expectations and the neutrality of money. *Journal of Economic Theory*, 4, 103–124.

- Maćkowiak, B. and Wiederholt, M. (2009). Optimal sticky prices under rational inattention. *American Economic Review*, 99(3), 769–803.
- Melosi, L. (2017). Signaling effect of monetary policy. *The Review of Economic Studies*, 84, 853–884.
- Midrigan, V. (2011). Menu costs, multiproduct firms, and aggregate fluctuations. *Econometrica*, 79(4), 1139–1180.
- Morris, S. and Shin, H.S. (2002). Social value of public information. *American Economic Review*, 92, 1521–1534.
- Morris, S. and Shin, H.S. (2005). Central bank transparency and the signal value of prices. *Brookings Papers on Economic Activity*, 2, 1–66.
- Morris, S. and Shin, H.S. (2018). Central bank forward guidance and the signal value of market prices. *AEA Papers and Proceedings*, 108, 572–577.
- Myatt, D.P. and Wallace, C. (2012). Endogenous information acquisition in coordination games. *Review of Economic Studies*, 79, 340–374.
- Paciello, L. and Wiederholt, M. (2014). Exogenous information, endogenous information, and optimal monetary policy. *Review of Economic Studies*, 81, 356–388.
- Pasten, E. and Schoenle, R. (2016). Rational inattention, multi-product firms and the neutrality of money. *Journal of Monetary Economics*, 80(1), 1–16.
- Phelps, E.S. (1983). The trouble with rational Expectations and the problem of inflation stabilization. In *Individual Forecasting and Aggregate Outcomes: “Rational Expectations” Examined*, Frydman, R. and Phelps, E.S. (eds). Cambridge University Press, Cambridge.
- Romer, C.D. and Romer, D.H. (2000). Federal reserve information and the behavior of interest rates. *American Economic Review*, 90(3), 429–457.
- Sims, C. A. (1998). Stickiness. *Carnegie-Rochester Conference Series on Public Policy*, 49, 317–356.
- Sims, C. A. (2003). Implications of rational inattention. *Journal of Monetary Economics*, 50, 665–690.
- Svensson, L.E.O. (2006). Social value of public information: Comment: Morris and Shin (2002) Is actually pro-transparency, not con. *American Economic Review*, 96, 448–452.
- Taylor, J.B. (1980). Aggregate dynamics and staggered contracts. *Journal of Political Economy*, 88(1), 1–23.
- Townsend, R. M. (1983). Forecasting the forecasts of others. *Journal of Political Economy*, 91(4), 546–588.
- Woodford, M. (2003a). Imperfect common knowledge and the effects of monetary policy. In *Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps*, Aghion, P., Frydman, R., Stiglitz, J., Woodford, M. (eds). Princeton University Press, Princeton.

Woodford, M. (2003b). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press, Princeton.

Woodford, M. (2005). Central Bank Communication and Policy Effectiveness. Working Paper No. 11898, NBER Working Paper Series.

4

Dynamical Systems

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4.1. Introduction

Economics is a quantitative science. An economy can be represented by a vector of variables $x(t)$ changing over time. Time t can be an integer or a real variable. Time is said to be discrete when $t \in \mathbb{Z}$, the set of integers, while it is said to be continuous when $t \in \mathbb{R}$, the set of real numbers. Here, as in the classical mechanics, time is an independent variable, while a vector $x \in X$ depends on time. The continuous-time function $x : \mathbb{R} \rightarrow X$ is denoted by $x(t)$ while the discrete-time function $x : \mathbb{Z} \rightarrow X$ is a sequence (x_t) with $x_t \equiv x(t)$. The reduced form of a model representing the economy is a system of dynamical equations.

Fluctuations and growth are prominent topics in macroeconomics. Theorists are used to represent them in infinite-horizon and overlapping-generations models.¹ Ramsey (1928) is arguably the most popular growth model and the baseline of the dynamic general-equilibrium theory. Capital and consumption are the variables of a system of two equations: capital accumulation and consumption

The author would like to thank Thomas Seegmuller for his helpful comments.

1. Seminal general-equilibrium models with capital accumulation are Ramsey (1928), Cass (1965) and Koopmans (1965) (infinite horizon), and Diamond (1965) (overlapping generations). Barro and Sala-i-Martin (1995) and Acemoglu (2009) cover a large spectrum of current macrodynamic issues.

smoothing. Economic transactions take place in discrete time, but a continuous-time representation is also suitable depending on the model extensions. Market imperfections (such as market power, financial frictions, pollution externalities) and agents' heterogeneity are often considered. Both can increase the number of equations making dynamics richer but solutions harder. The real world is complex and we need some mathematical tools to study these higher-dimensional systems, to draw a more realistic economic picture.

This chapter is an introduction to the theory of dynamical system. The first part focuses on dynamics in discrete time, represented by system of difference equations (better known as iterated maps), more used in economics, while the second one, on dynamics in continuous time, represented by systems of differential equations. The part in discrete time is largely inspired by Bosi and Ragot (2011), whereas the continuous-time part draws on Bosi and Desmarchelier (2019).²

Difference and differential equations can be ordinary or partial. In the first case, time is the only independent variable: $x = x(t)$ while, in the second case, there are other independent variables such as space: $x = x(s, t)$. In this chapter, we will consider only ordinary difference or differential equations (ODE).

4.2. Discrete-time dynamical systems

Discrete-time dynamics can be represented by matrix powers. The next section shows how to compute these powers. However, some elements of linear algebra are needed (see Lang (2008) among other popular textbooks).

4.3. Matrix powers

Just as a curve is approximated by its tangent, nonlinear dynamics can be approximated by a linear system that behaves locally like the original one. Consider a simple transition $x_{t+1} = f(x_t)$ from time t to $t + 1$, where x_t is a scalar and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a derivable function with, without loss of generality, $f(0) = 0$. f is approximated around zero by a linear form: $x_{t+1} = ax_t$, where $a \equiv f'(0)$. Given the initial condition x_0 , the solution to equation $x_{t+1} = ax_t$ is a power: $x_t = a^t x_0$. The sequence of points $(a^t x_0)_{t=0}^{+\infty}$ is said to be an orbit or a trajectory. In the case of a linear equation, the solution is an exponential function. In the general case of a linear n -dimensional system, the solution is the power of an n -by- n matrix.

2. See also Iooss (1979) and Hirsch and Smale (1974) for excellent introductions to the systems of difference and differential equations, respectively.

Now, let us compute the power of an n -by- n matrix A with components in a field $\mathbb{K}$ of scalars. A can be real or complex when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Define $A^2 = AA$ and the t th power by induction: $A^t = A^{t-1}A$ for any natural number $t > 1$. Set also $A^0 = I$, where I denotes the n -by- n identity matrix, and $A^1 = A$. If A is invertible and $t < 0$, define $A^t = (A^{-1})^{-t}$. Thus, the power A^t of an invertible matrix A makes sense even when the integer t is negative.

To compute the t th power of a square matrix A , we decompose A^t in a product of elementary matrices. The following problem looks very different from the original one. Solve the equation: $Av = \lambda v$, where λ is an unknown scalar and v is an unknown column of a vector space over a field $\mathbb{K}$. We are interested in non-trivial solutions $v \neq 0$. In fact, equation $Av = \lambda v$ is key in power decomposition. Write it differently: $B(\lambda)v = 0$, where $B(\lambda) \equiv \lambda I - A$ and I denotes the n -by- n identity matrix. The notion of linear independence is needed to solve $Av = \lambda v$.

Vectors $b_1, \dots, b_n$ are linearly independent if none of them is a linear combination of the others. Conversely, they are linearly dependent if there is a combination $\sum_{i=1}^n v_i b_i = 0$ with some nonzero coefficient v_i .

The determinant is a function from the set of square matrices to the field of their components. The determinant of a matrix A is denoted by $|A|$ or $\det A$. Laplace's formula provides a recursive definition. In the case of a 2-by-2 matrix:

$$A = (a_{ij}) \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad [4.1]$$

this rule gives: $\det A = a_{11}a_{22} - a_{12}a_{21}$.

An ordered set of n -dimensional columns $b_1, \dots, b_n \in \mathbb{K}^n$ form a square matrix $B \equiv [b_1, \dots, b_n]$. These vectors are linearly dependent if and only if the determinant of B is null: $\det B = 0$.

System $B(\lambda)v = 0$ becomes: $v_1 b_1 + \dots + v_n b_n = 0$, where some v_i is required to be nonzero. A nontrivial solution $v \neq 0$ exists if and only if $b_1, \dots, b_n$ are linearly dependent, i.e. $P(\lambda) = 0$, where $P(\lambda) \equiv \det B(\lambda) = \sum_{i=0}^n a_i \lambda^i$ with $a_n = 1$ is the characteristic polynomial of A .

The Fundamental Theorem of Algebra states that a non-constant single-variable polynomial of degree n with complex coefficients has at least one complex root. This theorem equivalently says that the polynomial has exactly as many complex roots as its degree, if each root is counted up to its multiplicity. Moreover, nonreal roots, if any, are always pairwise conjugate.

The eigenvalues λ are the scalar solutions to equation $P(\lambda) = 0$. For each eigenvalue λ , an eigenvector is a nonzero vector v , solution to system $B(\lambda)v = 0$.

According to the Fundamental Theorem of Algebra, an n -by- n real matrix has n eigenvalues in $\mathbb{C}$. The same holds for a complex matrix. These complex eigenvalues can be either real or nonreal and conjugate, either distinct or multiple. We will consider mainly real matrices and, hence, real characteristic polynomials with, possibly, nonreal roots.

The case of a two-dimensional matrix is rich enough to introduce and understand the general n -dimensional case.

4.3.1. Dimension two

There is no great loss of generality with dimension two. The characteristic polynomial of matrix [4.1] is quadratic: $P(\lambda) = \det(\lambda I_2 - A) = \lambda^2 - T\lambda + D = 0$, where $T \equiv a_{11} + a_{22}$ and $D \equiv a_{11}a_{22} - a_{12}a_{21}$ are the trace (sum of elements on the main diagonal) and the determinant³ with $a_{ij} \in \mathbb{C}$.

The trinomial equation $\lambda^2 - T\lambda + D = 0$ has two complex (real or nonreal) solutions: $\lambda_1 \equiv (T - \sqrt{T^2 - 4D})/2$ and $\lambda_2 \equiv (T + \sqrt{T^2 - 4D})/2$. In the case of a real matrix A , there are only three possibilities: (1) real and distinct eigenvalues ($T^2 > 4D$); (2) multiple (real) eigenvalues ($T^2 = 4D$); and (3) nonreal and conjugate eigenvalues ($T^2 < 4D$).

Now, consider a solution λ to $P(\lambda) = 0$. The corresponding eigenvector v is solution to the original system $Av = \lambda v$. Let $k \in \mathbb{K}$ be an arbitrary scalar. If v is solution to $Av = \lambda v$, the vector kv is solution as well: $A(kv) = kAv = k\lambda v = \lambda(kv)$.

How can we find n linearly independent eigenvectors in the case of multiple eigenvalues? Indeed, the same set of collinear eigenvectors kv , solutions to $Av = \lambda v$, associated with these identical eigenvalues. How to find a real eigenvector in the case of a nonreal eigenvalue of a real matrix? Indeed, the solution v to $Av = \lambda v$, with a nonreal eigenvalue λ , is a nonreal eigenvector.

4.3.1.1. Real and distinct eigenvalues

Focus on the simplest case. Rewrite the equations $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$ as $[Av_1, Av_2] = [\lambda_1 v_1, \lambda_2 v_2]$ where Av_i and $\lambda_i v_i$ are the n -dimensional columns. In compact terms, $AV = V\Lambda$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad [4.2]$$

3. Trace and determinant are also the sum and the product of eigenvalues, whatever the order of a square matrix.

is the diagonal matrix of eigenvalues and $V = [v_1, v_2]$ is the matrix of eigenvectors. V is an invertible matrix because the eigenvalues are real and distinct (see Bosi and Ragot (2011, p. 18, footnote 5)) and A can be written as a product of simple real matrices: $A = V\Lambda V^{-1}$. Λ is said to be a real normal (or canonical) form. V diagonalizes A through a change of basis in the vector space and the linear map represented by A in the original basis becomes $\Lambda = V^{-1}AV$ in the new one.

Now, it is clear why the solution to system $Av = \lambda v$ is key to compute the power of a matrix: $A^t = V\Lambda^t V^{-1}$, and the matrix power boils down to a simple scalar power:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^t = \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} \quad [4.3]$$

Rescaling the eigenvectors, we write their matrix $V \equiv [v_1, v_2]$ in its simplest form:

$$V = \begin{bmatrix} \lambda_1 - a_{22} & a_{12} \\ a_{21} & \lambda_2 - a_{11} \end{bmatrix} \quad [4.4]$$

(see Bosi and Ragot (2011, p. 19) for more details). You can quickly check that $V\Lambda V^{-1} = A$. However, V is not the unique solution: any other matrix $W \equiv [k_1 v_1, k_2 v_2]$ with nonzero scalars k_1 and k_2 suits. Indeed, the normal form is invariant: $\Lambda = V^{-1}AV = W^{-1}AW$.

4.3.1.2. Multiple eigenvalues

The eigenvalues coincide when the discriminant of the characteristic polynomial is null ($T^2 = 4D$), and their common value is necessarily real: $\lambda \equiv \lambda_1 = \lambda_2 = (a_{11} + a_{22})/2$. This nongeneric case is more difficult to handle: indeed, given λ , system $Av = \lambda v$ yields only a single eigenvector. How to write A as a product of matrices simple enough to compute its power? We are still interested in a real form. The one we obtain is triangular instead of diagonal, but its power remains easy to compute.

Let v be the representative of a family of collinear eigenvector $\{kv\}$ corresponding to a multiple eigenvalue λ . We would like to find another vector w to construct a new matrix V such that A can be written as a product of three real matrices, i.e. to change the basis of the vector space to transform A into a real triangular matrix.

We look for a generalized eigenvector w such that $(A - \lambda I)w = v$ and $A = V\Lambda V^{-1}$ with $V = [v, w]$ and

$$\Lambda \equiv \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad [4.5]$$

The powers of Λ are straightforward enough:

$$\Lambda^t = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^t = \begin{bmatrix} \lambda^t & t\lambda^{t-1} \\ 0 & \lambda^t \end{bmatrix} \quad [4.6]$$

We find the simplest matrix of eigenvectors by rescaling them suitably:

$$V = [v, w] = \begin{bmatrix} (a_{11} - a_{22})/2 & 1 \\ a_{21} & 0 \end{bmatrix} \quad [4.7]$$

(see Bosi and Ragot (2011, p. 21) for details). A quick check shows the rightness of this real decomposition: $A = V\Lambda V^{-1}$ (the inverse V^{-1} exists if and only if $\det V \neq 0$, i.e. $a_{21} \neq 0$; the case $a_{21} = 0$ is also straightforward (see Bosi and Ragot (2011, p. 21, footnote 8)). According to [4.6], matrix powers come down to scalar powers: $A^t = V\Lambda^t V^{-1}$.

4.3.1.3. *Nonreal eigenvalues*

Let the discriminant of the characteristic polynomial of real matrix A be negative: $T^2 < 4D$. In this case, the eigenvalues are nonreal and conjugate: $\lambda_1 \equiv a - ib$, $\lambda_2 \equiv a + ib$ with $a \equiv T/2$ and $b \equiv \sqrt{D - (T/2)^2} = \sqrt{D - a^2} > 0$. Moreover, $\lambda_1 \neq \lambda_2$ because $b > 0$. Solving $Av = \lambda v$, we also obtain nonreal and conjugate eigenvectors: if $u - iv$, with $u, v \in \mathbb{R}^2$, is the eigenvector of a nonreal eigenvalue λ_1 , then $u + iv$ is the conjugate eigenvector of the conjugate eigenvalue λ_2 . A decomposition $A = WMW^{-1}$ with

$$M = \begin{bmatrix} a - ib & 0 \\ 0 & a + ib \end{bmatrix}$$

and $W = [u - iv, u + iv]$ is possible but useless. We want instead to decompose the real matrix A in a product of three real matrices. Indeed, a real normal form generates a sequence of real points in the original vector space and the change of basis implemented by this form simplifies the orbit to a large extent. A decomposition $A = V\Lambda V^{-1}$ is suitable if Λ and V are real and the powers of Λ are easy to compute.

Let λ_1 be solution to $P(\lambda) = 0$: $A(u - iv) = (a - ib)(u - iv)$ implies $Au - iAv = au - bv - i(bu + av)$. Real and imaginary parts have to be equal. Thus, $Au = au - bv$ and $Av = bu + av$. We get $AV = [Au, Av] = [au - bv, bu + av] = [u, v]\Lambda = V\Lambda$, where $V \equiv [u, v]$ and

$$\Lambda \equiv \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad [4.8]$$

turns out to be the real normal form we were looking for in the nonreal case. Considering λ_2 instead of λ_1 , we would obtain the same matrices Λ and V . The decomposition $A = V\Lambda V^{-1}$ allows us to compute any power $A^t = V\Lambda^t V^{-1}$.

Rescaling suitably u and v , we can simplify the matrix of eigenvectors:

$$V = \begin{bmatrix} b(a_{22} - a_{11})/2 \\ 0 & -a_{21} \end{bmatrix} \quad [4.9]$$

V invertible because $ba_{21} \neq 0$.⁴ Using [4.8] and [4.9], we check easily that $A = V\Lambda V^{-1}$. Bosi and Ragot (2011, p. 23–26) provide arguments for this choice of V .

Now, let us compute the power Λ^t at the core of decomposition $A^t = V\Lambda^t V^{-1}$. Introduce the polar coordinates $\rho \equiv \sqrt{\det \Lambda} = \sqrt{a^2 + b^2} > 0$ and $\alpha \equiv \arctan(b/a)$ with $a = \rho \cos \alpha$ and $b = \rho \sin \alpha$. The real normal form and its powers become:

$$\Lambda = \rho \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \text{ and } \Lambda^t = \rho^t \begin{bmatrix} \cos(t\alpha) & \sin(t\alpha) \\ -\sin(t\alpha) & \cos(t\alpha) \end{bmatrix} \quad [4.10]$$

See Bosi and Ragot (2011, p. 26–27) for a proof of [4.10] with trigonometric identities.

EXAMPLE 4.1.– Consider the orbit $(A^t x_0)_{t=0}^{+\infty}$ where

$$A^t = \begin{bmatrix} 1 & z \\ -z & 1 \end{bmatrix}^t = \left(\sqrt{1+z^2}\right)^t \begin{bmatrix} \cos(t \arctan z) & \sin(t \arctan z) \\ -\sin(t \arctan z) & \cos(t \arctan z) \end{bmatrix}$$

is given by [4.10] with $\rho \equiv \sqrt{1+z^2}$ and $\alpha = \arctan z$. Setting the parameter value $z = 1/2$ and the initial condition $x_0^T = (x_{10}, x_{20}) = (1, 0)$, we obtain a sequence of points: $(A^t x_0)_{t=0}^{+\infty} = \left((1, 0)^T, (1, -1/2)^T, \dots\right)$ and a clockwise spiral-shaped trajectory in the x_t -plane. $\square$

The real normal forms [4.2], [4.5] and [4.10] corresponding to the three possible cases: (1) real and distinct, (2) multiple and (3) nonreal eigenvalues, are also called Jordan forms.

4.3.2. Higher dimensions

Real normal forms simplify the computation of matrix powers. In the case of higher dimension, the application of the Jordan method requires additional elements of linear algebra. Let us recall some definitions and results.⁵

4. $a_{21} = 0$ violates the assumption of nonreal eigenvalues: $T^2 < 4D$. Thus, $a_{21} \neq 0$ and $b > 0$.

5. The reader is referred to Lang (2008) among others for a classical introduction to linear algebra.

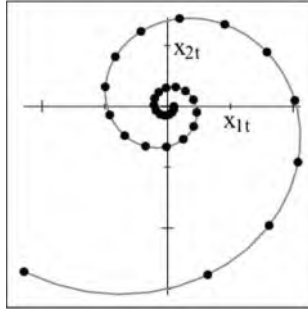


Figure 4.1. *Orbit*

4.3.2.1. *Invariant spaces*

A basis B is a set of linearly independent vectors whose linear combinations span the whole space. The number of bases of a given vector space is infinite, but the number of vectors in each basis of this space is constant. Such invariant number is the dimension of the space. The coordinates of a vector are the coefficients of the linear combination of elements in B . Given B , these coordinates are unique. For a proof, see Bosi and Ragot (2011, p. 29, Proposition 6) among others.

A subspace is a subset of the main space, which is closed under the operations of addition and multiplication by scalars. A vector space V is said to be a direct sum of the linear subspaces $V_1, \dots, V_m$ if every vector $v \in V$ has a unique representation $v = \sum_{i=1}^m v_i$ with $v_i \in V_i$ for $i = 1, \dots, m$. Consider two vector spaces V and W over the same field $\mathbb{K}$. A function $f : V \rightarrow W$ is said to be a linear map if $f(au + bv) = af(u) + bf(v)$ for any $a, b \in \mathbb{K}$ and $u, v \in V$. Consider subspaces $V_1, \dots, V_m$ of V such that V is their direct sum, and a linear map f such that $f(V_i) \subseteq V_i$ for any i . Such a family of subspaces always exists if f is an endomorphism.⁶

Let n_i be the dimension of the subspace V_i . Then, $\sum_{i=1}^m n_i = n$. If V is a direct sum of the subspaces V_i and B_i is a basis of the subspace V_i , then $B = [B_1, \dots, B_m]$ is also a basis of the main space V . For a proof, see Bosi and Ragot (2011, p. 36, Proposition 11) among others. The linear map f is represented in the basis $B = [B_1, \dots, B_m]$ by a block-diagonal square matrix M :

$$M = \begin{bmatrix} M_1 & \cdots & 0_{n_1 \times n_m} \\ \vdots & & \vdots \\ 0_{n_m \times n_1} & \cdots & M_m \end{bmatrix} \quad [4.11]$$

6. The trivial set $\{V\}$ (with $m = 1$) satisfies the conditions of direct sum and $f(V) \subseteq V$.

Each block M_i is an n_i -by- n_i submatrix, where n_i is the number of vectors in the sub-basis B_i . All the blocks outside the diagonal are null. The next theorem characterizes the structure of this vector space.

THEOREM 4.1.— *The linear map f is represented by the block-diagonal matrix [4.11] if and only if (1) V is a direct sum of the subspaces $V_1, \dots, V_m$ and (2) each subspace is invariant under the mapping f .*

For a simple proof in the particular case of a vector space $V \equiv \mathbb{K}^n$ over a field $\mathbb{K}$, and a system of bases (I_n, I_n) , the reader is referred to Bosi and Ragot (2011, p. 36–37).

Let $\mathbb{K}$ be a field (for instance, $\mathbb{R}$ or $\mathbb{C}$) and $y_{t+1} = Ay_t$ be a dynamical system with $(a_{ij}) \in \mathbb{K}$, $y_t \in \mathbb{K}^n$, $i, j = 1, \dots, n$, $t = 0, 1, \dots$. Given the initial condition y_0 , we compute the trajectory $y_t = A^t y_0$. Let

$$M \equiv \begin{bmatrix} \lambda_1 & \mu_2 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ & & \ddots & \\ \vdots & & & \lambda_{n-1} & \mu_n \\ 0 & \cdots & & 0 & \lambda_n \end{bmatrix} \quad [4.12]$$

be a matrix where $\lambda_1, \dots, \lambda_n$ are the n solutions (either distinct or multiple) of the polynomial equation $\det(\lambda I - A) = 0$ with $\mu_j = 0$ if $\lambda_j \neq \lambda_{j-1}$, or $\mu_j = 1$ if $\lambda_j = \lambda_{j-1}$ for $j = 2, \dots, n$.

Could we change the basis of the space $\mathbb{K}^n$: $z_t \equiv V^{-1}y_t$ with $V^{-1} \in \mathbb{K}^{n \times n}$, and transform the system $y_{t+1} = Ay_t$ in $z_{t+1} = Mz_t$? We are interested in a decomposition $A = VM^tV^{-1}$ because we know how to compute the powers of a block-diagonal matrix M and, hence, the trajectory $y_t = VM^tV^{-1}y_0$. The matrix M can be nonreal, even if $\mathbb{K} = \mathbb{R}$.⁷ In many applications, $\mathbb{K} = \mathbb{R}$ and we need to compute real trajectories. In these cases, if the matrix A has real and distinct eigenvalues, the n eigenvalues form a real diagonal matrix Λ : we can find another real matrix V of the same order and, hence, an explicit solution: $y_t = V\Lambda^tV^{-1}y_0$. V is a matrix of real eigenvectors, a Jordan basis of the whole space $\mathbb{R}^n$. If the matrix A has multiple real eigenvalues, the matrix M remains real, but no longer diagonal: there will be at least a nonzero term μ_j (as a matter of fact, equal to one). It is not always possible to diagonalize a square matrix⁸ but it is always possible to find a basis to transform A in a square matrix [4.12] of the same order.

7. Indeed, the eigenvalues λ_i on the main diagonal of [4.12] may be outside the field $\mathbb{K}$.

8. With $\mu_j = 0$ for $j = 2, \dots, n$.

4.3.2.2. Real spaces

If all the eigenvalues (either distinct or multiple) in [4.12] are real, M is yet the real Jordan form we are looking for to compute matrix powers. If the eigenvalues on the main diagonal are nonreal, computations become cumbersome and useless in the end. In this case, does a real Jordan form exist?

We know that an eigenvalue of A is a scalar λ solution to the linear system $Av = \lambda v$. The eigenspace of λ is the set of solutions v to the system $Av = \lambda v$. The null vector always belongs to the eigenspace. Every eigenspace is also a vector subspace. The eigenvalues λ of a real matrix (solution to a polynomial equation with real coefficients) and the eigenvectors v may be real or nonreal. An eigenvector associated with a nonreal eigenvalue is nonreal.

(1) Let the eigenvalue λ be real. Then, there is a nonzero eigenvector v solution to $Av = \lambda v$. Let $R_1(\lambda)$ be the subspace of real eigenvectors corresponding to λ . This subspace is said to be a real eigenspace. The *complexification* of $R_1(\lambda)$ is the subspace $C_1(\lambda) \equiv \{u + iv : u, v \in R_1(\lambda)\}$. $u + iv$ is also an eigenvector of the real eigenvalues λ and $R_1(\lambda) \subset C_1(\lambda)$.

(2) Let the eigenvalue λ be nonreal. Let $C_2(\lambda)$ be the set of eigenvectors associated with λ , which are necessarily nonreal. If $\lambda \equiv a + ib$ with $b \neq 0$ and $w \equiv u + iv \in C_2(\lambda)$, then the conjugate number $\bar{\lambda} \equiv a - ib$ is also an eigenvalue and the conjugate vector $\bar{w} \equiv u - iv \in C_2(\bar{\lambda})$ also an eigenvector.⁹ The sum of subspaces $C_2(\lambda) + C_2(\bar{\lambda})$ is a vector space. We call *realification* of $C_2(\lambda)$ or $C_2(\bar{\lambda})$ the subspace $R_2(\lambda)$ of real vectors in $C_2(\lambda) + C_2(\bar{\lambda})$. $R_2(\lambda) \subset C_2(\lambda) + C_2(\bar{\lambda})$ and $R_2(\lambda)$ contains no eigenvectors. With some definitional misuse, we can say that $R_2(\lambda)$ is a real eigenspace. It is easy to show that, if $u + iv \in C_2(\lambda)$ (or $u + iv \in C_2(\bar{\lambda})$), then $u, v \in R_2(\lambda) = R_2(\bar{\lambda})$ (see Bosi and Ragot (2011, p. 48, Proposition 30)). In addition, $R_1(\lambda)$ with a real eigenvalue λ is the realification of $C_1(\lambda)$, while $C_2(\lambda) + C_2(\bar{\lambda})$ is the complexification of $R_2(\lambda)$. See also Bosi and Ragot (2011, p. 48–49, Propositions 31 and 32) for proofs. From now on, let us omit the index $n = 1, 2$ in $R_n(\lambda)$ and $C_n(\lambda)$: $R(\lambda)$ and $C(\lambda)$ will denote $R_1(\lambda)$ and $C_1(\lambda)$ if λ is real, and $R_2(\lambda)$ and $C_2(\lambda)$ if λ is nonreal. Consider an eigenvalue λ of a linear map f , either real or nonreal. The next theorem highlights the invariance of eigenspaces $R(\lambda)$ and $C(\lambda)$ under f .

THEOREM 4.2.—*If a linear map f have distinct eigenvalues $\lambda_1, \dots, \lambda_r$ (real) and $(\mu_1, \bar{\mu}_1), \dots, (\mu_s, \bar{\mu}_s)$ with $\mu_j = a_j + ib_j$ (nonreal), then, $\mathbb{R}^n$ is a direct sum of real eigenspaces: $\sum_{i=1}^r R(\lambda_i) + \sum_{j=1}^s R(\mu_j)$, where each eigenspace is invariant under f . Hence, there exists a basis of $\mathbb{R}^n$ such that f is represented by a block-diagonal matrix:*

9. Indeed, since the matrix A is real, $\lambda w = Aw$ implies $\bar{\lambda} \bar{w} = \overline{\lambda w} = \overline{Aw} = \bar{A} \bar{w} = A \bar{w}$.

$$\begin{bmatrix} \lambda_1 & & & \cdots & 0 & 0 \\ & \ddots & & & \vdots & \vdots \\ & & \lambda_r & & & \\ & & & a_1 & b_1 & \\ & & & -b_1 & a_1 & \\ \vdots & & & & \ddots & \\ 0 & \cdots & & & & a_s & b_s \\ 0 & \cdots & & & & -b_s & a_s \end{bmatrix}$$

See Grandmont (2008, page 124) and, for a proof, Hirsch and Smale (1974, p. 67).

The number of eigenvalues is equal to $n = r + 2s$. The direct sum involves only $r + s$ eigenspaces because $R(\mu_j) = R(\bar{\mu}_j)$ for every j . Each $R(\lambda_i)$ is one-dimensional since all the eigenvalues are distinct and the eigenvectors of each λ_i are collinear, while each $R(\mu_j)$ is two-dimensional because, if $u_j + iv_j$ is an eigenvector of μ_j , the real eigenspace is spanned by u_j and v_j , two linearly independent vectors. Thus, the sum of the dimensions of these real eigenspaces is precisely $r + 2s = n$, the dimension of the main space $\mathbb{R}^n$.

In the general case of multiple eigenvectors, we can factorize the characteristic polynomial: $P(\lambda) = \prod_{i=1}^p (\lambda - \lambda_i)^{n_i} \prod_{j=1}^q [(\lambda - \mu_j)(\lambda - \bar{\mu}_j)]^{n_j}$, where n_i denotes the multiplicity of the real eigenvalue λ_i , while n_j is the multiplicity of the nonreal conjugate pair $(\mu_j, \bar{\mu}_j)$. We recover the dimension of the whole space: $\sum_{i=1}^p n_i + 2 \sum_{j=1}^q n_j = n$.

A spectral vector associated with a given eigenvalue λ of multiplicity m , either real or non-real, is a nonzero solution v to the system $(\lambda I_n - A)^m v = 0$. The eigenvector is simple if $(\lambda I_n - A)v = 0$; it is generalized if $(\lambda I_n - A)^l v = 0$ with $1 < l \leq m$.

(1) λ_i is a real value. We call real spectral subspace, denoted by $R^*(\lambda_i)$, the set of all the real vector solutions to $(\lambda I_n - A)^m v = 0$. Any nonzero vector belonging to $R^*(\lambda_i)$ is an eigenvector or a generalized eigenvector. The dimension of $R^*(\lambda_i)$ is n_i (see Hirsch and Smale (1974, p. 331–333) for a proof).

(2) μ_j is a nonreal value. We call the real spectral subspace, denoted by $R^*(\mu_j)$, the set of all the real vectors in $C^*(\mu_j) + C^*(\bar{\mu}_j)$, where $C^*(\mu_j)$ represents the set of solutions to $(\lambda I_n - A)^m v = 0$, i.e. the spectral subspace. Of course, $R^*(\mu_j) = R^*(\bar{\mu}_j)$ because $C^*(\mu_j) + C^*(\bar{\mu}_j) = C^*(\bar{\mu}_j) + C^*(\mu_j)$. The dimension of $R^*(\mu_j)$ is $2n_j$ (see Hirsch and Smale (1974, p. 331–333) for a proof).

Theorem 4.2 can be generalized to the case of multiple eigenvalues.

THEOREM 4.3.— Let $\lambda_1, \dots, \lambda_p, (\mu_1, \bar{\mu}_1), \dots, (\mu_q, \bar{\mu}_q)$ be the eigenvalues of the linear map f , where each value λ_i is real with multiplicity n_i and the pairs $(\mu_j, \bar{\mu}_j)$, with $\mu_j = a_j + ib_j$, are nonreal and conjugate with multiplicity n_j . Then, $\mathbb{R}^n$ is the direct sum of the real spectral subspaces: $\sum_{i=1}^p R^*(\lambda_i) + \sum_{j=1}^q R^*(\mu_j)$, where each real spectral subspace is invariant under f . Hence, there is a basis of $\mathbb{R}^n$ such that f is represented by a block-diagonal matrix:

$$\begin{bmatrix} \Lambda_1 & & \cdots & 0_{n_1 \times 2n_q} \\ & \ddots & & \vdots \\ & & \Lambda_p & \\ & & & M_1 \\ \vdots & & & \ddots & \\ 0_{2n_q \times n_1} & \cdots & & & M_q \end{bmatrix} \quad [4.13]$$

with blocks

$$\Lambda_i \equiv \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ & & \ddots & \\ \vdots & & & 1 \\ 0 & \cdots & & \lambda_i \end{bmatrix} \quad \text{and} \quad M_j \equiv \begin{bmatrix} a_j & b_j & 1 & 0 & \cdots & 0 & 0 \\ -b_j & a_j & 0 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \vdots & \vdots \\ & & & & & 1 & 0 \\ \vdots & \vdots & & & & 0 & 1 \\ 0 & 0 & \cdots & & & a_j & b_j \\ 0 & 0 & \cdots & & & -b_j & a_j \end{bmatrix}$$

of order $n_i \times n_i$ and $2n_j \times 2n_j$, respectively.

See Grandmont (2008, pages 125–126) and, for a proof, Hirsch and Smale (1974, pp. 126–132).

The number of eigenvalues is $\sum_{i=1}^p n_i + 2 \sum_{j=1}^q n_j = n$, but the direct sum involves only $p + q$ real spectral spaces. The dimension of each space $R^*(\lambda_i)$ is n_i because, given the value λ_i , n_i linearly independent real spectral vectors are solutions to $(\lambda I_n - A)^m v = 0$. Let $u_{jk} + iv_{jk}$ be the k th spectral vector associated with μ_j . The dimension of $R^*(\mu_j)$ is $2n_j$ because $2n_j$ real vectors: u_{jk} and v_{jk} with $k = 1, \dots, n_j$, are linearly independent. Since $R^*(\mu_j) = R^*(\bar{\mu}_j)$ for any j , $R^*(\bar{\mu}_j)$ is not required to appear in the direct sum $\sum_{i=1}^p R^*(\lambda_i) + \sum_{j=1}^q R^*(\mu_j)$. The dimension of the direct sum $\sum_{i=1}^p n_i + \sum_{j=1}^q 2n_j = n$ is precisely the dimension of the main space $\mathbb{R}^n$. Matrix [4.13] is the real normal form¹⁰ of the linear map f suitable to compute the matrix powers in the most general case: the t th power of a block-diagonal matrix remains a block-diagonal matrix where each block is just the t th power of the corresponding original block.

10. Also said real canonical (or Jordan) form.

To compute the powers of Λ_i , we write it as a binomial: $\Lambda_i = \lambda_i I_{n_i} + N_i$, where

$$N_i \equiv \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & & \ddots & \vdots \\ \vdots & & & 1 \\ 0 & \cdots & & 0 \end{bmatrix}$$

is an n_i -by- n_i matrix. The powers are given by: $\Lambda_i^t = \sum_{\tau=0}^t \binom{t}{\tau} \lambda_i^{t-\tau} N_i^\tau$, where

$$N^{n_i} = 0_{n_i}, N^{n_i-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ & & 0 & 0 \\ \vdots & & \vdots & \\ 0 & \cdots & 0 \end{bmatrix}, N^{n_i-2} = \begin{bmatrix} 0 & \cdots & 1 & 0 \\ & & 0 & 1 \\ \vdots & & \vdots & \\ 0 & \cdots & 0 \end{bmatrix}, \dots, N^0 = I_{n_i}$$

Along the same lines, we handle the case of a nonreal eigenvalue $\mu_j = a_j + ib_j$ of multiplicity n_j corresponding to a $2n_j$ -by- $2n_j$ block of matrix [4.13]: we write the block

$$M_j \equiv \begin{bmatrix} B_j & I_2 & \cdots & 0_2 \\ 0_2 & B_j & & \vdots \\ & & \ddots & \\ \vdots & & & B_j & I_2 \\ 0_2 & \cdots & & 0_2 & B_j \end{bmatrix} \quad \text{where } B_j \equiv \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$$

as $M_j = A_j + N_j$ with

$$A_j \equiv \begin{bmatrix} B_j & \cdots & 0_2 \\ \vdots & & \vdots \\ 0_2 & \cdots & B_j \end{bmatrix} \quad \text{and } N_j \equiv \begin{bmatrix} 0_2 & I_2 & \cdots & 0_2 \\ & & \ddots & \vdots \\ \vdots & & & I_2 \\ 0_2 & \cdots & & 0_2 \end{bmatrix}$$

The binomial trick allows us to compute the block powers: $M_j^t = \sum_{\tau=0}^t \binom{t}{\tau} A_j^{t-\tau} N_j^\tau$ with

$$A_j^{t-\tau} = \begin{bmatrix} B_j^{t-\tau} & \cdots & 0_2 \\ \vdots & & \vdots \\ 0_2 & \cdots & B_j^{t-\tau} \end{bmatrix} \quad \text{and } B_j^{t-\tau} = |\mu_j|^{t-\tau} \begin{bmatrix} \cos(t-\tau) \alpha_j & \sin(t-\tau) \alpha_j \\ -\sin(t-\tau) \alpha_j & \cos(t-\tau) \alpha_j \end{bmatrix}$$

where $\alpha_j \equiv \arctan b_j/a_j$ and $|\mu_j| \equiv \sqrt{a_j^2 + b_j^2}$, and

$$N^{n_j} = 0_{2n_j}, N^{n_j-1} = \begin{bmatrix} 0_2 & \cdots & 0_2 & I_2 \\ & & 0_2 & 0_2 \\ \vdots & & \vdots & \\ 0_2 & \cdots & 0_2 & \end{bmatrix}, N^{n_j-2} = \begin{bmatrix} 0_2 & \cdots & I_2 & 0_2 \\ & & 0_2 & I_2 \\ \vdots & & \vdots & \\ 0_2 & \cdots & 0_2 & \end{bmatrix}, \dots, N^0 = I_{2n_j}$$

Computation of $\Lambda_i^t = \sum_{\tau=0}^t \binom{t}{\tau} \lambda_i^{t-\tau} N_i^\tau$ and $M_j^t = \sum_{\tau=0}^t \binom{t}{\tau} A_j^{t-\tau} N_j^\tau$ rests on the application of Newton's binomial to the case of square matrices A and B of the same order that commute ($AB = BA$): $(A + B)^t = \sum_{\tau=0}^t \binom{t}{\tau} A^\tau B^{t-\tau}$ (a simple proof by induction is provided in Bosi and Ragot (2011, p. 53–55)).

4.4. Systems of ordinary difference equations

In this section, we will apply the theory of matrix powers to solve linear and nonlinear systems of ordinary difference equations (ODE).

A general system of ordinary difference equations is an implicit system of m equations: $G(t, y_t, y_{t+1}, \dots, y_{t+s}) = 0$, where t is an integer and s is a positive integer, and $G, y_{t+i} \in \mathbb{K}^m$, with $i = 0, \dots, s$, belong to a vector space over a field $\mathbb{K}$. It is a system of s th-order difference equations because the variables are lagged up to the order s . Setting $x_{0,t} \equiv t$ and $x_{i,t} \equiv y_{t+i-1}$ for $i = 1, \dots, s$, we obtain $x_{0,t+1} = x_{0,t} + 1$ and $x_{i,t+1} = x_{i+1,t}$ for $i = 1, \dots, s-1$ and the implicit system becomes $G(x_{0,t}, x_{1,t}, \dots, x_{s,t}, x_{s,t+1}) = 0$. Gathering equation $x_{0,t+1} = x_{0,t} + 1$, the $s-1$ m -dimensional systems $x_{i,t+1} = x_{i+1,t}$ and the m -dimensional implicit system $G(x_{0,t}, x_{1,t}, \dots, x_{s,t}, x_{s,t+1}) = 0$, we obtain a simpler $(1 + ms)$ -dimensional system $F(x_t, x_{t+1}) = 0$, where $F : (\mathbb{Z} \times \mathbb{K}^{ms})^2 \rightarrow \mathbb{Z} \times \mathbb{K}^{ms}$ is a vector of implicit functions and

$$x_t \equiv \begin{bmatrix} x_{0t} \\ x_{1t} \\ \vdots \\ x_{st} \end{bmatrix} \in \mathbb{Z} \times \mathbb{K}^{ms}$$

is a column vector (we omit the comma between the subscripts in the following).

A dynamical system is autonomous when the independent variable (here, time t) is no longer a direct argument G , i.e. $G(y_t, y_{t+1}, \dots, y_{t+s}) = 0$. As seen above, an autonomous s th-order difference equation is equivalent to a system of s first-order difference equations. Thus, an autonomous system of m s th-order equations becomes a system of ms first-order equations.

If, for any given x_t in the domain, a unique solution x_{t+1} to $F(x_t, x_{t+1}) = 0$ exists, a function $x_{t+1} = f(x_t)$ is defined and is called a map. Given the initial

condition $x_0 = x(0)$, the map f determines the dynamic sequence (x_t) . This sequence (x_t) is known as the orbit (or trajectory) of F starting from x_0 .

4.4.1. Autonomous systems

A steady (or stationary) state (also said a fixed point) x is solution to the system: $F(x, x) = 0$. Non-autonomous systems admit no fixed point because time goes on forever. To study dynamics with steady states and local dynamics around, from now on, we will focus only on systems of autonomous difference equations $F(x_t, x_{t+1}) = 0$.

4.4.2. Topological equivalence

Consider a real vector space $\mathbb{R}^m \ni y_{t+i}$ and a C^1 implicit system of autonomous ODE $G \in \mathbb{R}^m$. What could we say about the orbits around a steady state? The simplest approximation of the equivalent system $F(x_t, x_{t+1}) = 0$ is a linearization around the steady state, i.e. the first-order polynomial (Taylor) expansion: $F(x_t, x_{t+1}) \approx F(x, x) + D_{x_t} F(x, x)(x_t - x) + D_{x_{t+1}} F(x, x)(x_{t+1} - x)$, where

$$x_t \equiv \begin{bmatrix} x_{1t} \\ \vdots \\ x_{st} \end{bmatrix} \in \mathbb{R}^{ms}$$

By definition, a steady state satisfies $F(x, x) = 0$. Thus,

$$x_{t+1} - x \approx J(x, x)(x_t - x) \quad [4.14]$$

where $J(x, x) \equiv -[D_{x_{t+1}} F(x, x)]^{-1} D_{x_t} F(x, x)$ is the Jacobian matrix evaluated at the steady state if $D_{x_{t+1}} F(x, x)$ is invertible. The ms -by- ms Jacobian $J(x, x)$ can be explicitly obtained in terms of the Jacobian of the original system $G(X_t, x_{st+1}) = 0$ with $X_t \equiv (x_{1t}, \dots, x_{st})$:

$$J(x, x) = \begin{bmatrix} 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & & \vdots \\ 0_m & 0_m & \cdots & I_m \\ J_1(X, x_s) & J_2(X, x_s) & \cdots & J_s(X, x_s) \end{bmatrix}$$

where 0_m and I_m are the null matrix and the identity of order m , and $J_i(X, x_s) \equiv -[D_{x_{st+1}} G(X, x_s)]^{-1} D_{x_{it}} G(X, x_s)$ is an m -by- m Jacobian matrix. The existence of $J(x, x)$ is ensured by $\det D_{x_{st+1}} G(X, x_s) \neq 0$. From now on, for simplicity, we set $n \equiv ms$.

Is the linear system [4.14] topologically equivalent to the original system $F(x_t, x_{t+1}) = 0$ around the steady state? Two dynamical systems are topologically equivalent if they have similar orbits, i.e. if there is a homeomorphism (continuous function with continuous inverse) mapping the orbits of the one into the orbits of the other and preserving the sense of direction. Hence, this equivalence preserves the stability properties. Under the assumptions of the implicit function theorem, the system $F(x_t, x_{t+1}) = 0$ can be represented by an explicit function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in a neighborhood of x such that $x_{t+1} = f(x_t)$. f is the local map of F .

Consider a vector space V and an endomorphism $f : V \rightarrow V$. An isomorphism is a bijective morphism. A homeomorphism is an isomorphism represented by a continuous map with a continuous inverse. A diffeomorphism is an isomorphism mapping one differentiable manifold to another such that the map and its inverse are smooth (C^1).

Now, we generalize the notion of eigenvalue. A scalar λ is an eigenvalue of the linear endomorphism $f : V \rightarrow V$ if there exists a nonzero solution $v \in V$ to equation $f(v) = \lambda i(v)$, i.e. to $(f - \lambda i)(v) = 0$, where i is the identity and $f - \lambda i$ is a linear combination of endomorphisms. Let V be a vector space over the complex field $\mathbb{C}$. The spectrum Z of a linear map $f : V \rightarrow V$ is the set $Z \equiv \{\mu \in \mathbb{C} : f - \mu i \text{ non-invertible}\}$. The scalar μ is called a spectral value. When V is finite-dimensional, the spectrum of f coincides with the set of its eigenvalues λ . If V is a Banach space¹¹, a continuous linear isomorphism $f : V \rightarrow V$ is said to be hyperbolic if its spectrum Z has an empty intersection with the unit circle: $Z \cap \{c \in \mathbb{C} : |c| = 1\} = \emptyset$. If V is finite-dimensional, this means that f has no eigenvalues on the unit circle.

Orbits take place on manifolds. The notion of invariant manifold is key to study nonlinearity. A subset $M \subseteq \mathbb{R}^n$ is a smooth (C^1) m -dimensional manifold if any point in M has a neighborhood $N \subseteq \mathbb{R}^n$ such that $M \cap N$ is diffeomorphic to an open set in $\mathbb{R}^m$. The notion of a hyperbolic fixed point is also key in the theory of nonlinear dynamics. A fixed point x of a diffeomorphism $f : M \rightarrow M$ from a smooth manifold M in itself is called a hyperbolic fixed point if $Df(x)$ is a linear hyperbolic isomorphism.

In the case of a finite-dimensional system of difference equations $F(x_t, x_{t+1}) = 0$, a fixed point x is hyperbolic if the Jacobian matrix $J(x, x)$ evaluated at the steady state has no eigenvalues with modulus equal to one. Around a hyperbolic steady state, the linear system [4.14] is topologically equivalent to the nonlinear system

11. A metric space X is said to be complete if every Cauchy sequence in X converges in X . A Banach space $(V, |||)$ is a normed vector space such that V is complete under the metric induced by the norm $|||$.

$F(x_t, x_{t+1}) = 0$ and the stability properties are preserved. The Grobman–Hartman theorem paves the way to exploit such topological equivalence.

THEOREM 4.4 (GROBMAN–HARTMAN).— *If x is a hyperbolic fixed point of a diffeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, a homeomorphism h is defined in a neighborhood N of x such that $h(f(z)) = J(x, x)h(z)$ for any $z \in N$.*

For a proof, the reader is referred to Guckenheimer and Holmes (1983, p. 18).

4.4.2.1. Periodic orbits

A steady state x is a fixed point of the map f , i.e. solution to system $x = f(x)$. A fixed point can also be viewed as a particular case of periodic orbit. Consider a map $f : X \rightarrow X \subseteq \mathbb{R}^m$ and define the iterated map: $x_t = f^t(x_0) \equiv f(f^{t-1}(x_0))$. An orbit is a solution $(f^t(x_0))_{t=0}^\infty$ where f^0 means the identity. A point $x \in X$ is said to be periodic of period $p > 1$, if $x = f^p(x)$ and $x \neq f^i(x)$ for any $0 < i < p$. A steady state is a periodic orbit of period $p = 1$. A hyperbolic fixed point of the iterate f^p is said to be a hyperbolic periodic point.

4.4.2.2. Stability

A metric space (X, d) is a set X where a notion of distance d between its elements is defined.

Liapunov and asymptotic stability are the main notions of stability for metric spaces. A fixed point $x \in X$ of the map $f : X \rightarrow X$ is Lyapunov stable if, for any $\varepsilon > 0$, there is $\delta > 0$ such that $d(x, y) < \delta$ implies $d(x, f^t(y)) < \varepsilon$ for any $t \in \mathbb{N}$. A fixed point $x \in X$ of the map $f : X \rightarrow X$ is asymptotically stable if there is a neighborhood N of x such that $\lim_{t \rightarrow \infty} d(x, f^t(y)) = 0$ for any $y \in N$. A stationary state x is asymptotically stable, if all nearby orbits tend to it. Thus, an asymptotically stable steady state is also Lyapunov stable. The converse is not true.

The basin of attraction of a steady state x is the largest neighborhood such that any entering orbit converges asymptotically to x . When the basin of attraction coincides with the whole space X , the steady state is said to be globally asymptotically stable.

As the fixed point is a particular case of periodic orbit, the stable fixed point is a particular case of attractor. A set A is said to be invariant under the map f if $A \subseteq X$ and $f(A) \subseteq A$. In this case, any orbit starting in A remains inside: $(f^t(y)) \subseteq A$ if $y \in A$. Consider a metric space (X, d) : an attractor is (1) an invariant set A (2) included in an open neighborhood $N \subseteq X$ such that $x \in N$ implies $\lim_{t \rightarrow \infty} d(A, f^t(x)) = 0$, which is (3) minimal. Minimal means that no proper subset of A satisfies conditions (1) and (2). The largest N with property (2) is called the basin of attraction.

Chaotic orbits are aperiodic. They are neither fixed points, nor periodic orbits, nor trajectories repelled to infinity that never return. A strange attractor contains chaotic solutions. Stable fixed points, limit cycles, limit tori and strange attractors are prominent examples of attractors.

4.4.2.3. Local stability

The Grobman–Hartman theorem allows us to approximate nonlinear dynamics around a hyperbolic steady state by a linear system, while preserving the stability properties: the stability results for linear systems can apply to linearized systems. Consider a simple one-dimensional example, a C^1 map $f : X \rightarrow \mathbb{R}$ with $X \subseteq \mathbb{R}$ and hyperbolic steady state x . The difference equation $x_{t+1} = f(x_t)$ is locally stable around the steady state if $|f'(x)| < 1$. Now, let us generalize this result to higher dimensions.

Focus on the Jacobian matrix $J(x, x)$ of system [4.14]. Its eigenvectors span eigenspaces through the steady state x . System [4.14] drives linear dynamics in differences $x_t - x$ around x , the new origin of the space. Under the assumptions of Theorem 4.4, topologically equivalent linear dynamics take place along the eigenspaces spanned by the (stable and unstable) eigenvectors of $J(x, x)$. Orbits along nonlinear manifolds are replaced by orbits along these linear subspaces.

Consider a C^1 map $F : X \rightarrow X \subseteq \mathbb{R}^n$ with hyperbolic steady state x and $D_{x_{t+1}}F(x, x) \neq 0$, such that the local map $x_{t+1} = f(x_t)$ is a diffeomorphism defined in an open ball $B(x, r)$ centered in x of radius r . The local stable manifold S is the set of $y \in B(x, r)$ such that $\lim_{t \rightarrow +\infty} d(x, f^t(y)) = 0$ and $f^t(y) \in B(x, r)$ for any $t \geq 0$. S is a subset of the ball such that, for any $y \in S$, the solution to $x_{t+1} = f(x_t)$ starting from y is a trajectory converging to the steady state and included in the ball. The local unstable manifold U is the set of $y \in B(x, r)$ such that $\lim_{t \rightarrow +\infty} d(x, f^{-t}(y)) = 0$ and $f^{-t}(y) \in B(x, r)$ for any $t \geq 0$. U is a subset of the ball such that, for any $y \in S$, the solution to $x_t = f^{-1}(x_{t+1})$ starting from y is a trajectory converging to the steady state and included in the ball.

THEOREM 4.5 (SPECTRAL THEOREM).— *In the case of a hyperbolic isomorphism $f : V \rightarrow V$, V is the direct sum of two invariant subspaces E_s and E_u , and f is a direct sum of two linear endomorphisms $f_s : E_s \rightarrow E_s$ and $f_u : E_u \rightarrow E_u$ with spectra $Z_s = Z \cap \{c : |c| < 1\}$ and $Z_u = Z \cap \{c : |c| > 1\}$.*

If the Jacobian matrix $J(x, x)$ represents the hyperbolic isomorphism $x_{t+1} = f(x_t)$ around the steady state x , the stable (unstable) eigenspace E_s (E_u) is spanned by the set of eigenvectors of $J(x, x)$ whose eigenvalues have modulus less (greater) than one.

THEOREM 4.6 (STABLE AND UNSTABLE MANIFOLDS).— *Consider the local diffeomorphism $x_{t+1} = f(x_t)$ of a map $F(x_t, x_{t+1}) = 0$ around a hyperbolic steady state x . Assume that the Jacobian matrix $J(x, x)$ in [4.14] has m eigenvalues inside and $n - m$ eigenvalues outside the unit circle. Then, (1) there is a unique local m -dimensional C^1 manifold S , tangent in x to the stable space E_s of system [4.14] and invariant under the map f ; (2) there is a unique local $(n - m)$ -dimensional C^1*

manifold U , tangent in x to the unstable space E_u of system [4.14] and invariant under the map f^{-1} . If the map f is C^p , these local manifolds are also C^p .

The local manifold has the dimension of its tangent space. When the dimension of the local stable (unstable) manifold is zero, the fixed point is a source (sink). Otherwise, it is a saddle. Thus, a hyperbolic steady state is (1) a sink if all the eigenvalues lie inside the unit circle; (2) a source if all the eigenvalues lie outside the unit circle; and (3) a saddle otherwise (at least one eigenvalue lies inside and another outside the unit circle).¹²

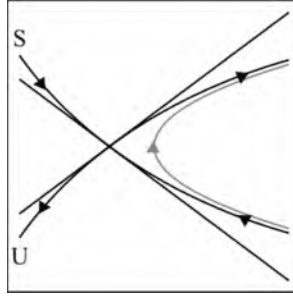


Figure 4.2. Local manifolds

4.4.3. Linear systems

A finite-dimensional autonomous linear system can be written

$$x_{t+1} = Ax_t + b \quad [4.15]$$

where x_t , x_{t+1} and b are the n -dimensional vectors while A is a real square matrix of order n . When $b = 0$, the linear system is said to be homogeneous. We know that a nonlinear hyperbolic autonomous ordinary system can be approximated around the steady state by [4.15].¹³

4.4.3.1. Steady state

If $\det(I - A) \neq 0$, the unique solution to [4.15] with $x = x_t = x_{t+1}$ is the steady state $x = (I - A)^{-1}b$.

12. The global stability of nonlinear systems is an advanced topic we leave aside in this chapter. The interested readers are referred to Bosi and Ragot (2011, p. 65–67) for an introduction based on the notion of contraction mapping.

13. Indeed, [4.14] becomes $x_{t+1} = J(x, x)x_t + [I - J(x, x)]x$. Simply set $A \equiv J(x, x)$ and $b \equiv (I - A)x$ to recover [4.15].

EXAMPLE 4.2.– *The steady state of a system*

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \end{bmatrix} = s \begin{bmatrix} 0 & 1 \\ -1 & r \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} 1 + s(s-r) \\ 1 + s(s-r) \end{bmatrix} \quad [4.16]$$

is given by $x = (I - A)^{-1}b = (1 + s(1-r), 1-s)^T$. $\square$

4.4.3.2. Dynamics

Written differently [4.15]: $x_{t+1} - x = A(x_t - x)$, with $x = (I - A)^{-1}b$. Setting $y_t \equiv x_t - x$, we obtain the system $y_{t+1} = Ay_t$ with steady state $y = 0$, and, recursively, the solution $y_t = A^t y_0$ with initial condition $y_0 = x_0 - x$. To find the orbit, as seen above, we need to decompose A^t in a product of computable powers: $y_t = V\Lambda^t V^{-1}y_0$, where Λ is the real normal form and V is the matrix of (generalized) eigenvectors. Thus, we obtain the explicit solution to the original system [4.15]: $x_t = (I - A)^{-1}b + V\Lambda^t V^{-1} [x_0 - (I - A)^{-1}b]$. There is no great loss of generality in considering two-dimensional orbits, i.e. planar phase diagrams: in the case of higher-dimensional systems and distinct eigenvalues, the real normal form is block-diagonal with blocks of orders one and two and the main space becomes a direct sum of real eigenspaces of the same order.

There are three possible cases: (1) two real and distinct eigenvalues, (2) a unique real eigenvalue of multiplicity two, (3) two nonreal and conjugate eigenvalues. $P(\lambda) = \lambda^2 - T\lambda + D$ is the characteristic polynomial of matrix [4.1]. Let $a \equiv T/2$, $b \equiv \sqrt{D - a^2}$, $c \equiv \sqrt{a^2 - D}$. The three cases become: (1) $D < a^2$ and $(\lambda_1, \lambda_2) = (a - c, a + c)$, with Λ^t and V given by [4.3] and [4.4]; (2) $D = a^2$ and $(\lambda_1, \lambda_2) = (a, a)$ with Λ^t and V given by [4.6] and [4.7]; (3) $D > a^2$ and $(\lambda_1, \lambda_2) = (a - ib, a + ib)$ with Λ^t and V given by [4.10] and [4.9].

EXAMPLE 4.3.– *Reconsider [4.16] with the initial condition $x_0 = (0, 0)^T$. The eigenvalues are given by $(\lambda_1, \lambda_2) = s \left(r/2 - \sqrt{(r/2)^2 - 1}, r/2 + \sqrt{(r/2)^2 - 1} \right)$. Focus on three cases: (1) $r = 4$, (2) $r = 2$, (3) $r = 0$.*

(1) In the case $r = 4$, the eigenvalues are real and distinct: $(\lambda_1, \lambda_2) = s(2 - \sqrt{3}, 2 + \sqrt{3})$. According to [4.3] and [4.4], we obtain

$$A^t = \frac{1}{6 + 4\sqrt{3}} \begin{bmatrix} -2 - \sqrt{3} & 1 \\ -1 & 2 + \sqrt{3} \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} \begin{bmatrix} -2 - \sqrt{3} & 1 \\ -1 & 2 + \sqrt{3} \end{bmatrix} \quad [4.17]$$

Consider two subcases: (1.1) $s = 1/4$ and (1.2) $s = 1$.

(1.1) The eigenvalues $(\lambda_1, \lambda_2) = (2 - \sqrt{3}, 2 + \sqrt{3})/4$ belong to the interval $(0, 1)$ and the orbit converges monotonically from the initial condition $x_0 = (0, 0)^T$ to the steady state $x = (1/4, 3/4)^T$ according to [4.17]: $x_t = x - A^t x \approx (0.25, 0.75)^T - \lambda_1^t (0.0528, 0.0142)^T - \lambda_2^t (0.1972, 0.7358)^T$ (see Figure 4.3).

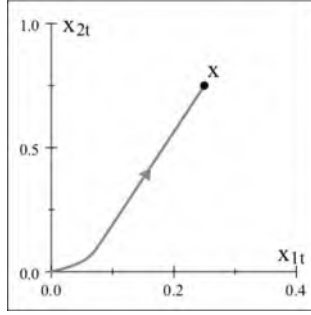


Figure 4.3. Monotonic convergence to the steady state

(1.2) The eigenvalues become $(\lambda_1, \lambda_2) = (2 - \sqrt{3}, 2 + \sqrt{3})$. $|\lambda_2|$ lies outside the interval $(0, 1)$, and dynamics no longer converge to the steady state $x = (-2, 0)^T$. According to [4.17] and the initial condition $x_0 = (0, 0)^T$, we obtain: $x_t = x - A^t x \approx (-2, 0)^T + \lambda_1^t (2.1547, 0.5774)^T - \lambda_2^t (0.1547, 0.5774)^T$.

(2) In the case $r = 2$, there is a single eigenvalue of multiplicity two: $\lambda_1 = \lambda_2 = s$. The matrix power A^t is decomposed according to [4.6] and [4.7]:

$$A^t = V \Lambda^t V^{-1} = \begin{bmatrix} 1 & -1/s \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s^t & t s^{t-1} \\ 0 & s^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -s & s \end{bmatrix} \quad [4.18]$$

Consider two subcases: (2.1) $s = 1/2$ and (2.2) $s = 2$.

(2.1) The eigenvalues $\lambda_1, \lambda_2 = 1/2$ are in the interval $(0, 1)$. Dynamics converge monotonically according to [4.18] along the segment connecting the initial condition $(0, 0)$ to the steady state $x = (1/2, 1/2)^T$: $x_t = x - A^t x = (1 - s^t)(1/2, 1/2)^T$.

(2.2) If $s = 2$, $|\lambda| = 2$ and the modulus no longer belongs to the interval $(0, 1)$. Dynamics diverge according to [4.18] along the bisector of the first orthant from the initial condition $(0, 0)$ far away from the steady state $x = (-1, -1)^T$: $x_t = x - A^t x = (s^t - 1)(1, 1)^T$.

(3) In the case $r = 0$, the eigenvalues are nonreal and conjugate with $a = 0$ and $b = s$: $(\lambda_1, \lambda_2) = s(-i, i)$. A^t is written as a product of computable matrices according to [4.10] and [4.9]:

$$A^t = V\Lambda^t V^{-1} = |s|^t \begin{bmatrix} \cos(t\pi/2) & \sin(t\pi/2) \\ -\sin(t\pi/2) & \cos(t\pi/2) \end{bmatrix} \quad [4.19]$$

since $\rho \equiv \sqrt{a^2 + b^2} = |s| > 0$ and $\alpha \equiv \arctan(b/a) = \pi/2$.

Consider two subcases: (3.1) $s = 1/2$ and (3.2) $s = 2$.

(3.1) The eigenvalues become $(\lambda_1, \lambda_2) = (-i/2, i/2)$ with modulus $|\lambda_i| = 1/2$ in the interval $(0, 1)$. Dynamics converge according to [4.19] from the initial condition $x_0 = (0, 0)^T$ to the steady state $x = (3/2, 1/2)^T$ along a clockwise spiral:

$$x_t = x - A^t x = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} - \frac{1}{2^t} \begin{bmatrix} \cos(t\pi/2) & \sin(t\pi/2) \\ -\sin(t\pi/2) & \cos(t\pi/2) \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

(see Figure 4.4).

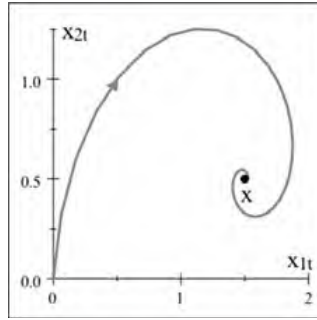


Figure 4.4. Spiral convergence to the steady state

(3.2) The eigenvalues $(\lambda_1, \lambda_2) = (-2i, 2i)$ have modulus $|\lambda_i| = 2$ outside the interval $(0, 1)$. The orbit diverges from the initial condition $x_0 = (0, 0)^T$ along a clockwise spiral far away from the steady state $x = (3, -1)^T$. [4.19] yields:

$$x_t = x - A^t x = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - 2^t \begin{bmatrix} \cos(t\pi/2) & \sin(t\pi/2) \\ -\sin(t\pi/2) & \cos(t\pi/2) \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(see Figure 4.5).

□

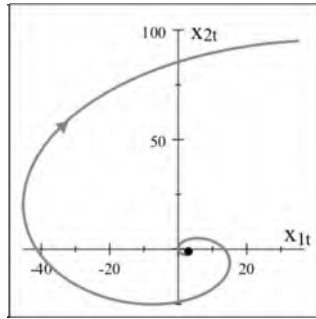


Figure 4.5. *Divergence away from the steady state*

4.5. Local bifurcations

A linearization preserves the stability properties around a hyperbolic steady state (Grobman–Hartman). In the case of a finite-dimensional space, these properties depend on the eigenvalues of the Jacobian matrix evaluated at this steady state. A bifurcation is a sudden qualitative change in the behavior of a dynamical system under a small smooth parameter change. The bifurcation is said to be local if the change in the orbit structure happens in an arbitrarily small neighborhood of a steady state; global, if this change cannot be confined to an arbitrarily small neighborhood in the phase space.

In discrete time, a local bifurcation generically occurs when one eigenvalue $\lambda(p)$ of the Jacobian matrix evaluated at the steady state, crosses the unit circle in response to a change of parameter p .¹⁴ We normalize to zero the critical parameter value of bifurcation ($p = 0$). Three classes of elementary bifurcations exist: (1) saddle node: $\lambda(0) = +1$; (2) flip: $\lambda(0) = -1$; (3) Neimark–Sacker: $|\lambda(0)| = |a(0) \pm ib(0)| = 1$ with $b(p) \neq 0$ in a neighborhood of $p = 0$.

According to Theorem 4.3, $\mathbb{R}^n$ is the direct sum of the real spectral subspaces invariant under f : $\sum_{i=1}^p R^*(\lambda_i) + \sum_{j=1}^q R^*(\mu_j)$. The stable E_s (unstable E_u) is the subspace spanned by all the real spectral subspaces $R^*(\lambda_i)$ and $R^*(\mu_j)$ corresponding to the eigenvalues λ_i and μ_j of modulus less (greater) than one. The stable S (unstable U) is the invariant manifold tangent to the stable (unstable) subspace. The center subspace E_c is the subspace spanned by all the real spectral subspaces $R^*(\lambda_i)$ and $R^*(\mu_j)$ corresponding to the eigenvalues λ_i and μ_j of modulus equal to one. The center manifold C is an invariant manifold tangent to the center subspace.

14. The orientation can change when the real part of an eigenvalue crosses zero (see Grandmont (2008) for details).

The codimension is the number of parameters to vary to get a bifurcation. Focus on codimension one, i.e. on elementary bifurcations when only one parameter changes. Generically: (1) only one eigenvalue and the associated one-dimensional invariant manifold are involved in a saddle-node or a flip bifurcation; (2) two nonreal conjugate eigenvalues and the associated two-dimensional invariant manifold are involved in the Neimark–Sacker bifurcation. Generically means that, when an eigenvalue (or a conjugate pair of eigenvalues in the case of Neimark–Sacker bifurcation) crosses the unit circle, no other eigenvalue simultaneously crosses the circle. Then, a higher-dimensional dynamics reduces to a single equation in the case of a saddle or a flip, or to a two-dimensional system in the case of a Neimark–Sacker bifurcation: the change in the other eigenvalues does not affect the qualitative dynamical properties. Thus, generically, only a one- or a two-dimensional center manifold is concerned with the bifurcation, while the other manifolds preserve their stability properties.

THEOREM 4.7 (CENTER MANIFOLD THEOREM FOR MAPS).— *Consider a C^m map $f : X \rightarrow X$ with $X \subseteq \mathbb{R}^n$ and a steady state $x = f(x)$. Then, there are C^m stable and unstable manifolds S and U tangent at x to the stable and unstable subspaces E_s and E_u , and a C^{m-1} center manifold C tangent at x to the center subspace E_c . The manifolds S , U and C are all invariant under the map f . In addition, S and U are unique while C might not be unique.*

For a proof, see Kelley (1967).

This theorem allows us to bring the bifurcation analysis of higher-dimensional systems to the one of a single difference equations (in the case of saddle node and flip) or a two-dimensional system (in the case of Neimark–Sacker) along the center manifold (see also Carr (1981)).

A dynamical system involves endogenous variables and parameters. The steady states and the eigenvalues depend on these parameters. The bifurcation value is the critical parameter value at which an eigenvalue crosses the unit circle. Linear systems experience cycles when a real eigenvalue is equal to -1 or nonreal and conjugate eigenvalues have unit modulus. However, a slight change in the matrix representing the linear application rules them out. By contrast, nonlinear maps may generate persistent cycles under small perturbations.

Consider a transition map g from the current state $y_t \in \mathbb{R}^n$ to the future state y_{t+1} , which depends also, without loss of generality, on a single parameter¹⁵ $q \in \mathbb{R}$: $y_{t+1} = g(y_t, q)$. We assume that a fixed point $y(q)$ of the map as solution to the n -dimensional system $y = g(y, q)$ exists and that the system undergoes a bifurcation

15. If $q \in \mathbb{R}^m$ is a vector, let only a single component q_i vary. For simplicity, we consider only codimension-one bifurcations in discrete time. At the end of the chapter, we will address the issue of codimension-two bifurcations in continuous time.

at $q = q^*$. We normalize the parameter and the endogenous variables as follows: $p \equiv q - q^*$ and $x_t \equiv y_t - y(q^*)$. The dynamical system becomes $x_{t+1} = y_{t+1} - y(q^*) = g(x_t + y(q^*), p + q^*) - y(q^*)$, i.e. $x_{t+1} = f(x_t, p)$ where $f(x_t, p) \equiv g(x_t + y(q^*), p + q^*) - y(q^*)$ with $p = 0$ as bifurcation value and $x = 0$ as fixed point at $p = 0$. f undergoes a local bifurcation at $p = 0$ if the stability properties of the steady state change.¹⁶ In this case, $|\lambda(0)| = 1$.

In the following, for simplicity, we will denote by f_x and f_p the first-order partial derivatives of the function $f(x_t, p)$: $f_x \equiv \partial f / \partial x_t$ and $f_p \equiv \partial f / \partial p$. Second-order derivatives are similarly denoted: $f_{xx} \equiv \partial^2 f / \partial x_t^2$ and $f_{px} \equiv \partial^2 f / (\partial p \partial x_t)$.

4.5.1. Saddle-node bifurcations

In the spirit of the Center Manifold Theorem, we consider the bifurcations of a scalar x_t . A dynamic diagram in the (x_t, x_{t+1}) -plane (see Figure 4.6) and a bifurcation diagram in the (p, x) -plane are convenient to represent these bifurcations. Any intersection of the map $x_{t+1} = f(x_t, p)$ with the 45-degree line $x_{t+1} = x_t$ given p is a fixed point.

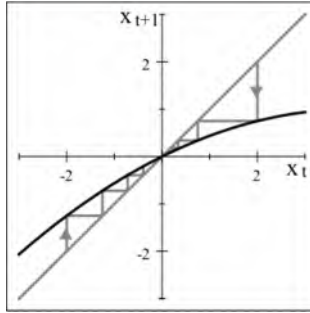


Figure 4.6. Local stability

One-dimensional dynamics have a unique and real eigenvalue $\lambda(p) = f_x(x, p)$. The steady state is stable if $|f_x(x, p)| < 1$, while a saddle-node bifurcation occurs at $p = 0$ if $f_x(0, 0) = 1$.

EXAMPLE 4.4.—The map $x_{t+1} = f(x_t, p) = -x_t^2/16 + (1-p)x_t$ with $p \in (0, 1)$ has two steady states: $x = 0$ and $x = -16p$. The real eigenvalue at $x = 0$ is

16. We do not consider bifurcations arising with multiple eigenvalues. These are nongeneric cases.

$1 - p \in (0, 1)$. The dynamic diagram in Figure 4.6 shows the local stability around $x = 0$ at $p = 1/2$ with $f_x(0, 1/2) = 1/2 \in (0, 1)$. $\square$

4.5.1.1. Generic saddle-node bifurcation

THEOREM 4.8 (GENERIC SADDLE-NODE BIFURCATION).— If

$$f(0, 0) = 0, f_x(0, 0) = 1, f_p(0, 0) \neq 0 \text{ and } f_{xx}(0, 0) \neq 0 \quad [4.20]$$

the system undergoes a saddle-node bifurcation at $p = 0$. Two fixed points $x_1 < 0 < x_2$ arise as soon as p crosses zero: one is asymptotically stable, the other unstable. There are four cases.

(1) $f_p(0, 0) < 0$ and $f_{xx}(0, 0) < 0$. f has two fixed points for small negative p and no fixed point for small positive p . x_1 is unstable and x_2 asymptotically stable.

(2) $f_p(0, 0) < 0$ and $f_{xx}(0, 0) > 0$. f has no fixed point for small negative p and two fixed points for small positive p . x_1 is asymptotically stable and x_2 unstable.

(3) $f_p(0, 0) > 0$, $f_{xx}(0, 0) < 0$. f has no fixed point for small negative p and two fixed points for small positive p . x_1 is unstable and x_2 asymptotically stable.

(4) $f_p(0, 0) > 0$, $f_{xx}(0, 0) > 0$. f has two fixed points for small negative p and no fixed point for small positive p . x_1 is asymptotically stable and x_2 unstable.

In sequence, conditions [4.20] mean that $x = 0$ is a fixed point at $p = 0$ ($f(0, 0) = 0$); the bifurcation is a saddle node ($f_x(0, 0) = 1$); the curve moves upward (downward) from the origin as soon as p increases (decreases) ($f_p(0, 0) > 0$ (< 0)); the curve is convex (concave) around the origin ($f_{xx}(0, 0) > 0$ (< 0)). If $f_p(0, 0) > 0$, as soon as p decreases (increases) from zero, two steady states appear: the lower x_1 being asymptotically stable (unstable); the higher x_2 unstable (asymptotically stable). As soon as p becomes positive (negative), the steady state disappears. The converse happens if $f_p(0, 0) < 0$. Find more details on the existence of these steady states in Bosi and Ragot (2011, p. 80–81).

In the following, let us call normal form the most parsimonious polynomial representation. The normal form of a generic saddle node satisfies all the restrictions in [4.20]: $f(x_t, p) = ap + x_t + cx_t^2$ with $a, c = \pm 1$. Indeed, $f(0, 0) = 0$, $f_x(0, 0) = 1$, $f_p(0, 0) = a \neq 0$ and $f_{xx}(0, 0) = 2c \neq 0$. The four cases of Theorem 4.8 correspond to the following normal forms: (1) $x_{t+1} = -p + x_t - x_t^2$, (2) $x_{t+1} = -p + x_t + x_t^2$, (3) $x_{t+1} = p + x_t - x_t^2$, (4) $x_{t+1} = p + x_t + x_t^2$.

EXAMPLE 4.5.— Focus on the fourth case with $p > -1/4$. The solutions to the equation $x = p + x + x^2$ are the fixed points $x_1 = -\sqrt{-p}$ and $x_2 = \sqrt{-p}$ with $p < 0$. If $p < 0$, the eigenvalue $f_x(x_t, p) = 1 + 2x_t$ can be evaluated at both the steady states: $f_x(x_1, p) = 1 - 2\sqrt{-p} \in (0, 1)$ and $f_x(x_2, p) = 1 + 2\sqrt{-p} > 1$. Thus,

x_1 is asymptotically stable while x_2 is unstable. The dynamic diagram in Figure 4.7 shows the fourth case: $x_{t+1} = p + x_t + x_t^2$. The dashed branch in the bifurcation diagram of Figure 4.8 represents the unstable steady state. $\square$

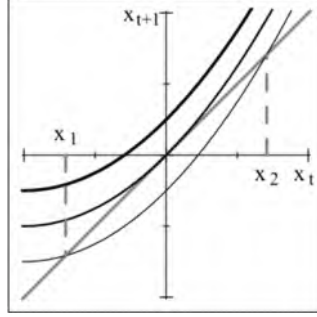


Figure 4.7. Saddle node

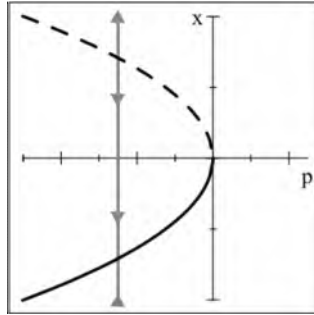


Figure 4.8. Bifurcation diagram

4.5.1.2. Transcritical bifurcation

In order to obtain a new type of saddle node, we replace the third condition in [4.20] with a non-generic restriction: $f_p(0, 0) = 0$, and we introduce a new generic condition: $f_{px}(0, 0) \neq 0$.

THEOREM 4.9 (TRANSCRITICAL BIFURCATION).— If

$$f(0, 0) = 0, f_x(0, 0) = 1, f_p(0, 0) = 0, f_{xx}(0, 0) \neq 0, f_{px}(0, 0) \neq 0 \quad [4.21]$$

then f has two fixed points: $x = 0$ and $x = x^*$. The zero one is invariant in a neighborhood of $p = 0$. The stability properties of these steady states are reversed as soon as p goes through 0. There are four cases.

(1) $f_{xx}(0,0) < 0$ and $f_{px}(0,0) < 0$. Then, $x^* > 0$ for $p < 0$ and $x^* < 0$ for $p > 0$. 0 is unstable and $x^* > 0$ asymptotically stable for $p < 0$. Conversely, 0 becomes asymptotically stable and $x^* < 0$ unstable for $p > 0$.

(2) $f_{xx}(0,0) < 0$ and $f_{px}(0,0) > 0$. Then, $x^* < 0$ for $p < 0$ and $x^* > 0$ for $p > 0$. 0 is asymptotically stable and $x^* < 0$ unstable for $p < 0$. Conversely, 0 becomes unstable and $x^* > 0$ asymptotically stable for $p > 0$.

(3) $f_{xx}(0,0) > 0$ and $f_{px}(0,0) < 0$. Then, $x^* < 0$ for $p < 0$ and $x^* > 0$ for $p > 0$. 0 is unstable and $x^* < 0$ asymptotically stable for $p < 0$. Conversely, 0 becomes asymptotically stable and $x^* > 0$ unstable for $p > 0$.

(4) $f_{xx}(0,0) > 0$ and $f_{px}(0,0) > 0$. Then, $x^* > 0$ for $p < 0$ and $x^* < 0$ for $p > 0$. 0 is asymptotically stable and $x^* > 0$ unstable for $p < 0$. Conversely, 0 becomes unstable and $x^* < 0$ asymptotically stable for $p > 0$.

Let us explain conditions [4.21]. Take for instance case (4): the curve goes through the origin ($f(0,0) = 0$); the bifurcation belongs to the family of saddle node ($f_x(0,0) = 1$); 0 remains a steady state in a neighborhood of $p = 0$ ($f_p(0,0) = 0$); the curve is convex around the origin ($f_{xx}(0,0) > 0$); the rotation around the origin $x = 0$ is counterclockwise ($f_{px}(0,0) > 0$). The curve is tangent to the 45-degree line when $p = 0$ and intersects this line at $x^* > 0$ for $p < 0$ and at $x^* < 0$ for $p > 0$. Moreover, for $p < 0$, the slope at $x^* > 0$ is greater than one and therefore x^* is unstable while 0 is asymptotically stable. Conversely, for $p > 0$, the negative steady state is asymptotically stable while 0 becomes unstable. The analysis of the other three cases is left to the reader.

The normal form of this bifurcation is $f(x_t, p) = x_t + bp x_t + cx_t^2$ with $b, c = \pm 1$. We note that $f(0,0) = 0$, $f_x(0,0) = 1$, $f_p(0,0) = 0$, $f_{xx}(0,0) = 2c \neq 0$ and $f_{px}(0,0) = b \neq 0$. Thus, the four cases of Theorem 4.9 correspond to the following normal forms: (1) $x_{t+1} = x_t - px_t - x_t^2$, (2) $x_{t+1} = x_t + px_t - x_t^2$, (3) $x_{t+1} = x_t - px_t + x_t^2$, (4) $x_{t+1} = x_t + px_t + x_t^2$.

EXAMPLE 4.6.— Focus on the fourth case with $-1 < p < 1$. The solutions to the equation $x = x + px + x^2$ are the fixed points $x = 0$ and $x = x^* = -p$. The eigenvalue $f_x(x_t, p) = 1 + p + 2x_t$ can be evaluated at both the steady states: $f_x(0, p) = 1 + p \in (0, 2)$ and $f_x(x^*, p) = 1 - p \in (0, 2)$. If $-1 < p < 0$, then $f_x(0, p) \in (0, 1)$ and 0 is asymptotically stable, while $f_x(x^*, p) \in (1, 2)$ and x^* is unstable. If $0 < p < 1$, then $f_x(0, p) \in (1, 2)$ and 0 becomes unstable, while $f_x(x^*, p) \in (0, 1)$ and x^* is asymptotically stable. The dynamic diagram in Figure 4.9 shows the fourth case: $x_{t+1} = x_t + px_t + x_t^2$. As above, the dashed lines in the bifurcation diagram of Figure 4.10 represent the unstable steady state. $\square$

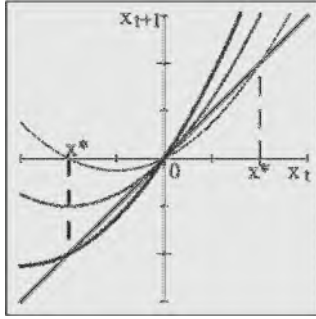


Figure 4.9. *Transcritical*

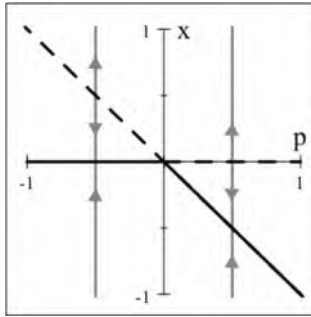


Figure 4.10. *Bifurcation diagram*

4.5.1.3. Pitchfork bifurcation

Consider another nongeneric case. We replace the third and the fourth conditions in [4.20] with the non-generic restriction: $f_p(0,0) = 0$ (as in the transcritical case) and $f_{xx}(0,0) = 0$. We introduce two generic conditions: $f_{px}(0,0) \neq 0$ (as in the transcritical case) and $f_{xxx}(0,0) \neq 0$, in order to obtain a new type of saddle-node bifurcation: the transition curve exhibits at 0 a change in concavity.

THEOREM 4.10 (PITCHFORK BIFURCATION).— *If*

$$\begin{aligned} f(0,0) = 0, f_x(0,0) = 1, f_p(0,0) = 0, f_{xx}(0,0) = 0, f_{px}(0,0) \neq 0, \\ f_{xxx}(0,0) \neq 0 \end{aligned} \quad [4.22]$$

then the system undergoes a pitchfork bifurcation, i.e. an unstable (asymptotically stable) fixed point becomes asymptotically stable (unstable) and two other unstable (asymptotically stable) fixed points arise around. There are four cases:

(1) $f_{px}(0,0) < 0$, $f_{xxx}(0,0) < 0$ (supercritical pitchfork bifurcation). For $p < 0$, f has three fixed points $x_1 < x_0 = 0 < x_2$ with x_0 unstable and x_1, x_2 asymptotically stable. For $p > 0$, f has a unique asymptotically stable fixed point ($x_0 = 0$).

(2) $f_{px}(0,0) < 0$, $f_{xxx}(0,0) > 0$ (subcritical pitchfork bifurcation). For $p < 0$, f has a unique unstable fixed point ($x_0 = 0$). For $p > 0$, f has three fixed points $x_1 < x_0 = 0 < x_2$ with now x_0 asymptotically stable and x_1, x_2 unstable.

(3) $f_{px}(0,0) > 0$, $f_{xxx}(0,0) < 0$ (supercritical pitchfork bifurcation). For $p < 0$, f has a unique asymptotically stable fixed point ($x_0 = 0$). For $p > 0$, f has three fixed points $x_1 < x_0 = 0 < x_2$ with x_0 now unstable and x_1, x_2 asymptotically stable.

(4) $f_{px}(0,0) > 0$, $f_{xxx}(0,0) > 0$ (subcritical pitchfork bifurcation). For $p < 0$, f has three fixed points $x_1 < x_0 = 0 < x_2$ with x_0 now asymptotically stable and x_1, x_2 unstable. For $p > 0$, f has a unique unstable fixed point ($x_0 = 0$).

Let us explain conditions [4.22]. Take for instance case (3): the curve goes through the origin ($f(0,0) = 0$); the bifurcation belongs to the saddle-node family ($f_x(0,0) = 1$); 0 remains a steady state in a neighborhood of $p = 0$ ($f_p(0,0) = 0$); 0 is an inflexion point ($f_{xx}(0,0) = 0$); the curve rotates in counterclockwise sense ($f_{px}(0,0) > 0$); the curve f is strictly convex for slightly negative x_t and strictly concave for slightly positive x_t ($f_{xxx}(0,0) < 0$). Thus, for slightly negative p , only 0 is a steady state while, as p goes through 0, two new steady states appear around 0, the lower being asymptotically stable (because of the convexity) as well as the higher (because of the concavity). Due to the stability of the new fixed point x_1 and x_2 , the bifurcation is said to be supercritical. Case (4) differs from (3) because $f_{xxx}(0,0) > 0$. This means that the curve f is strictly concave for slightly negative x_t and strictly convex for slightly positive x_t . This entails the reversion of the stability properties of the steady states. Because of the instability of the fixed points x_1 and x_2 , the bifurcation is said to be subcritical. The analysis of cases (1) and (2) with $f_{px}(0,0) < 0$ (clockwise rotation) is left to the reader.

The normal form of this bifurcation is $f(x_t, p) = x_t + bpx_t + dx_t^3$ with $b, d = \pm 1$. We note that $f(0,0) = 0$, $f_x(0,0) = 1$, $f_p(0,0) = 0$, $f_{xx}(0,0) = 0$, $f_{px}(0,0) = b \neq 0$, $f_{xxx}(0,0) = 6d \neq 0$. Thus, the four cases of Theorem 4.10 correspond to the following normal forms: (1) $x_{t+1} = x_t - px_t - x_t^3$, (2) $x_{t+1} = x_t - px_t + x_t^3$, (3) $x_{t+1} = x_t + px_t - x_t^3$, (4) $x_{t+1} = x_t + px_t + x_t^3$.

EXAMPLE 4.7.— Focus on the fourth case: $x_{t+1} = x_t + px_t + x_t^3$, with $p > -1$. The solutions to the equation $x = x + px + x^3$ are the fixed points $x_1 = -\sqrt{-p} < x_0 = 0 < x_2 = \sqrt{-p}$. The eigenvalue $f_x(x_t, p) = 1 + p + 3x_t^2$ can be evaluated at these three steady states: $f_x(0, p) = 1 + p > 0$ and $f_x(x_1, p) = f_x(x_2, p) = 1 - 2p < 3$. If $-1 < p < 0$, then $f_x(0, p) \in (0, 1)$ and $f_x(x_1, p) = f_x(x_2, p) \in (1, 3)$: 0 is asymptotically stable while x_1 and x_2 are unstable. If $p > 0$, then $f_x(0, p) > 1$ while

the fixed points x_1 and x_2 disappear (indeed $\sqrt{-p}$ is nonreal). 0 becomes unstable. Thus, the pitchfork bifurcation is subcritical. The dynamic diagram in Figure 4.11 shows the fourth case: $x_{t+1} = x_t + px_t + x_t^3$. The dashed curves in the bifurcation diagram of Figure 4.12 show the unstable steady state. $\square$

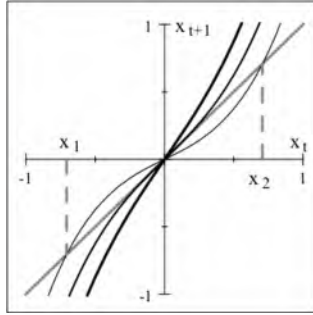


Figure 4.11. Pitchfork

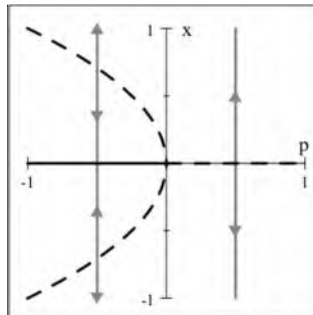


Figure 4.12. Bifurcation diagram

4.5.2. Flip bifurcation

A flip bifurcation occurs when a real eigenvalue goes through -1 . The flip bifurcation is related to the existence of oscillatory orbits and two-period cycles. In order to study the emergence of two-period cycles, we look at the second iterate of the map: $f^2 \equiv f \circ f$. The flip bifurcation of f is equivalent to a pitchfork for f^2 . We determine the conditions of a flip for f through the conditions of a pitchfork for f^2 : a cycle of period two for f is a fixed point for f^2 . A steady state x_0 for f remains a steady state for f^2 , but a cycle of period two for f requires the existence of two new steady states x_1 and x_2 for f^2 . Since $\lambda_{f^2} = f_x f_x = \lambda_f^2 = (-1)^2 = 1$, we are interested in a saddle-node bifurcation for f^2 and, since three steady states for f^2 are

involved (x_0 and the two new stationary points x_1 and x_2 around), in a pitchfork for f^2 . The following conditions are required for a pitchfork of the second iterate:

$$\begin{aligned} f^2(0, 0) = 0, f_x^2(0, 0) = 1, f_p^2(0, 0) = 0, f_{xx}^2(0, 0) = 0, \\ f_{px}^2(0, 0) \neq 0, f_{xxx}^2(0, 0) \neq 0 \end{aligned} \quad [4.23]$$

The following Theorem finds equivalent conditions in terms of the derivatives of f . A bifurcation generating a stable (unstable) periodic cycle is said to be supercritical (subcritical).

THEOREM 4.11 (FLIP BIFURCATION).— *If*

$$\begin{aligned} f(0, 0) = 0, f_x(0, 0) = -1, f_p(0, 0) = 0, f_{px}(0, 0) \neq 0, \\ 3f_{xx}(0, 0)^2 + 2f_{xxx}(0, 0) \neq 0 \end{aligned} \quad [4.24]$$

then a cycle of period two emerges around the steady state 0 when p goes through 0. There are four cases.

(1) $f_{px}(0, 0) < 0$ and $3[f_{xx}(0, 0)]^2 + 2f_{xxx}(0, 0) < 0$ (subcritical flip bifurcation). For $p < 0$ (slightly negative), there is a repulsive cycle of period two and $x_0 = 0$ is an asymptotically stable steady state. For $p > 0$ (slightly positive), the steady state 0 becomes unstable and the cycle of period two disappears.

(2) $f_{px}(0, 0) < 0$ and $3[f_{xx}(0, 0)]^2 + 2f_{xxx}(0, 0) > 0$ (supercritical flip bifurcation). For $p < 0$ (slightly negative), there is no cycle of period two and $x_0 = 0$ is an asymptotically stable steady state. For $p > 0$ (slightly positive), the steady state 0 becomes unstable and an attractive cycle of period two appears around.

(3) $f_{px}(0, 0) > 0$ and $3[f_{xx}(0, 0)]^2 + 2f_{xxx}(0, 0) < 0$ (subcritical flip bifurcation). For $p < 0$ (slightly negative), there is no cycle of period two and $x_0 = 0$ is an unstable steady state. For $p > 0$ (slightly positive), the steady state 0 becomes asymptotically stable and a repulsive cycle of period two appears around.

(4) $f_{px}(0, 0) > 0$ and $3[f_{xx}(0, 0)]^2 + 2f_{xxx}(0, 0) > 0$ (supercritical flip bifurcation). For $p < 0$ (slightly negative), there is an attractive cycle of period two and $x_0 = 0$ is an unstable steady state. For $p > 0$ (slightly positive), the steady state 0 becomes asymptotically stable and the cycle of period two disappears.

The proof in Bosi and Ragot (2011, p. 86–89) applies the chain rule to the second iterate $f^2(x_t, p) \equiv f(f(x_t, p), p)$ and transforms conditions [4.23] on f^2 in conditions [4.24] on the original function f . We can interpret conditions [4.24] as the corresponding conditions for a pitchfork bifurcation.

The normal form of this bifurcation is $f(x_t, p) = -x_t + bpx_t + dx_t^3$ with $b, d = \pm 1$. We note that $f(0, 0) = 0$, $f_x(0, 0) = -1$, $f_p(0, 0) = 0$, $f_{px}(0, 0) =$

$b \neq 0$, $3[f_{xx}(0,0)]^2 + 2f_{xxx}(0,0) = 12d \neq 0$. Thus, the four cases of Theorem 4.11 correspond to the following normal forms: (1) $x_{t+1} = -x_t - px_t - x_t^3$, (2) $x_{t+1} = -x_t - px_t + x_t^3$, (3) $x_{t+1} = -x_t + px_t - x_t^3$, (4) $x_{t+1} = -x_t + px_t + x_t^3$.

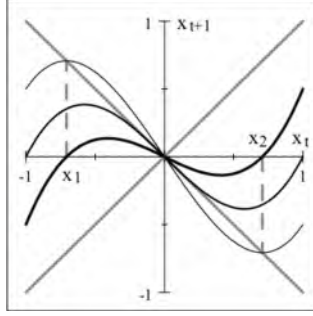


Figure 4.13. Flip

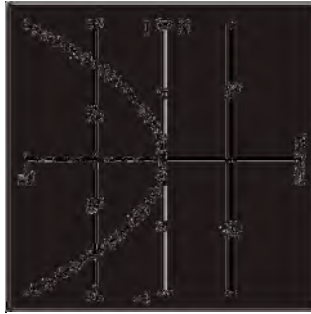


Figure 4.14. Bifurcation diagram

EXAMPLE 4.8.— Focus on the fourth case: $x_{t+1} = x_t(-1 + p + x_t^2)$, with $-1 < p < 1$. The steady state and the cycle of period two are fixed points of the second iterate. The second iterate is given by

$$x_{t+2} = x_t(-1 + p + x_t^2) \left(-1 + p + [x_t(-1 + p + x_t^2)]^2 \right)$$

Its fixed points are 3 among the 9 solutions to the equation:

$$x = x(-1 + p + x^2) \left(-1 + p + [x(-1 + p + x^2)]^2 \right)$$

that is the intersections of the map with the bisector of the second orthant: $x_{t+1} = -x_t$: $-x = -x + px + x^3$. We obtain the steady state: $x_0 = 0$, and the two points

of the cycle: $x_1 = -\sqrt{-p}$ and $x_2 = \sqrt{-p}$. The eigenvalue of the map: $f_x(x_t, p) = -1 + p + 3x_t^2$, can be evaluated at the steady state: $f_x(0, p) = -1 + p \in (-2, 0)$ while the eigenvalue of the second iterate: $\lambda_{f^2} = (f_x)^2$, can be evaluated at the points of the cycle: $\lambda_{f^2} = [f_x(x_1, p)]^2 = [f_x(x_2, p)]^2 = (-1 - 2p)^2$. If $p < 0$, then $f_x(0, p) \in (-2, -1)$ and $\lambda_{f^2} \in (0, 1)$: 0 is unstable while the cycle is asymptotically stable (attractive). If $p > 0$, then $f_x(0, p) \in (-1, 0)$: 0 becomes asymptotically stable while the cycle disappears (indeed $\sqrt{-p}$ becomes nonreal). Because of the stability of the cycle, the flip bifurcation is supercritical. The dynamic diagram in Figure 4.13 shows the fourth case: $x_{t+1} = -x_t + px_t + x_t^3$. The dashed line in the bifurcation diagram of Figure 4.14 represents the unstable steady state. $\square$

4.5.3. Neimark–Sacker bifurcation

In order to study the occurrence of a flip bifurcation for a map f , we have studied that of a pitchfork for f^2 , the second iterate. Now, we introduce a polar representation of f to study the occurrence of a Neimark–Sacker bifurcation for f as a pitchfork of its polar transformation. This bifurcation gives rise to two-dimensional limit cycles. The Neimark–Sacker is also known as Hopf bifurcation for maps.¹⁷ A Neimark–Sacker bifurcation occurs when two nonreal and conjugate eigenvalues μ and $\bar{\mu}$ cross the unit circle: $|a(0) \pm ib(0)| = 1$ with $b(p) \neq 0$ in a neighborhood of $p = 0$ (normalized value). The complex number $\mu(p) = a(p) + ib(p)$ is represented in polar coordinates by $\mu(p) = \rho(p) [\cos \alpha(p) + i \sin \alpha(p)]$, where $\alpha \equiv \arctan(b/a)$ and $\rho \equiv \sqrt{a^2 + b^2}$ represent the angle and the modulus in the Argand plane.

As in the case of pitchfork bifurcation, assume the nongeneric condition $f_p(0, 0) = 0$, i.e. $f(0, p) = 0$ in a neighborhood of $p = 0$. The Neimark–Sacker bifurcation arises for $\rho(0) = 1$ and $\rho'(0) \neq 0$. After centering the origin in the steady state, we obtain the approximated orbit: $y_t = \Lambda^t y_0$ where $y_t = V^{-1}x_t$ is vector in the new basis. The Center Manifold Theorem allows us to consider a two-dimensional subvector z_t of y_t corresponding to the center manifold and its reduced dynamics: $z_t \approx M^t z_0$, where M^t is given by [4.10]:

$$M^t = \rho^t \begin{bmatrix} \cos(t\alpha) & \sin(t\alpha) \\ -\sin(t\alpha) & \cos(t\alpha) \end{bmatrix}$$

The orbits locally converge (diverge) to the steady state if $\rho < 1$ (> 1). The composition of a circular motion driven by α (clockwise or counterclockwise) with a radial contraction (expansion) driven by ρ generates a convergent (divergent) spiral. In the spirit of Theorem [4.7], we can analyze the bifurcation on the two-dimensional eigenspace associated with the center manifold disregarding the other directions.

17. See the Hopf in continuous time at the end of the chapter.

Write z_t in polar coordinates: $z_t = \rho_t (\cos \alpha_t, \sin \alpha_t)^T$, where $\rho_t \equiv \sqrt{z_{1t}^2 + z_{2t}^2}$ and $\alpha_t \equiv \arctan(z_{2t}/z_{1t})$. The transition is approximated by

$$\rho_{t+1} \begin{bmatrix} \cos \alpha_{t+1} \\ \sin \alpha_{t+1} \end{bmatrix} = z_{t+1} \approx \rho \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} z_t \approx \rho \rho_t \begin{bmatrix} \cos(\alpha_t - \alpha) \\ \sin(\alpha_t - \alpha) \end{bmatrix} \quad [4.25]$$

i.e. by $\rho_{t+1} \approx \rho(p) \rho_t$ and $\alpha_{t+1} \approx \alpha_t - \alpha(p)$ (see Bosi and Ragot (2011, p. 90–92) for more details). The nonlinear dynamics $\rho_{t+1} = \psi(\rho_t, p)$ can be approximated by a Taylor polynomial $\rho_{t+1} \approx \sum_{i=0}^n a_i(p) \rho_t^i$. The first-order approximation gives $\rho_{t+1} \approx \rho(p) \rho_t$ and entails $a_0(p) = 0$ and $a_1(p) = \rho(p)$. However, the Neimark–Sacker bifurcation involves the cubic term of the polynomial, corresponding to a pitchfork bifurcation of the modulus ρ_t in polar coordinates: $\rho_{t+1} \approx \rho(p) \rho_t + a_2(p) \rho_t^2 + a_3(p) \rho_t^3 \equiv \varphi(\rho_t, p)$. A pitchfork bifurcation for φ around the steady state $\rho^* = 0$ takes place at $p = 0$ if

$$\begin{aligned} \varphi(0, 0) &= 0, \varphi_\rho(0, 0) = 1, \varphi_p(0, 0) = 0, \varphi_{\rho\rho}(0, 0) \\ &= 0, \varphi_{\rho p}(0, 0) \neq 0, \varphi_{\rho\rho\rho}(0, 0) \neq 0 \end{aligned} \quad [4.26]$$

In particular, $\varphi_{\rho\rho}(0, 0) = 0$ implies $a_2(0) = 0$ in a neighborhood of $p = 0$, and $\varphi_{\rho\rho\rho}(0, 0) \neq 0$ entails $a_3(0) \neq 0$ (see the expression for $a_3(p)$ in Grandmont (2008, p. 149)). We obtain $\rho_{t+1} \approx \rho(p) \rho_t + a_3(p) \rho_t^3$. Since the modulus (radius) is non-negative, the other steady state, solution to $\rho \approx \varphi(\rho, p)$, is given by $\rho_+^*(p) = \sqrt{[1 - \rho(p)]/a_3(p)}$ with $\rho_+^*(0) = 0$. When the steady state becomes positive ($\rho_+^*(p) > 0$), an invariant closed curve emerges in a neighborhood of $p = 0$ (see Bosi and Ragot (2011, p. 92–94) for an interpretation of pitchfork conditions in [4.26]). The next theorem sums up these results.

THEOREM 4.12 (NEIMARK–SACKER BIFURCATION).—*Let the 2D map $x_{t+1} = f(x_t, p)$ have two nonreal (conjugate) eigenvalues $\rho(p) [\cos \alpha(p) \pm i \sin \alpha(p)]$ at the steady state $x = 0$ such that $\rho(0) = 1$, $\rho'(0) \neq 0$ and $a_3(p) \neq 0$, where $a_3(p)$ is the third-order coefficient of the polynomial approximation $\rho_{t+1} \approx \varphi(\rho_t, p)$. Then, generically, the system undergoes a Neimark–Sacker bifurcation at $p = 0$, i.e. an invariant closed curve arises around the steady state $x = 0$. There are four cases:*

(1) $\rho'(0) < 0$, $a_3(p) < 0$ (supercritical Neimark-Sacker bifurcation). For $p < 0$ (slightly negative), we have $\rho(p) > 1$ and the steady state $x = 0$ is unstable. In addition, $\rho_+^*(p) > 0$: there is an asymptotically stable invariant curve around $x = 0$. For $p > 0$ (slightly positive), we have $\rho(p) < 1$ and the steady state $x = 0$ becomes asymptotically stable. In addition, $\rho_+^*(p)$ becomes nonreal: the invariant curve disappears.

(2) $\rho'(0) < 0$, $a_3(p) > 0$ (subcritical Neimark-Sacker bifurcation). For $p > 0$ (slightly positive), we have $\rho(p) < 1$ and the steady state $x = 0$ is asymptotically

stable. In addition, $\rho_+^*(p) > 0$: there is an unstable invariant curve around $x = 0$. For $p < 0$ (slightly negative), we have $\rho(p) > 1$ and the steady state $x = 0$ becomes unstable. In addition, $\rho_+^*(p)$ becomes nonreal: the invariant curve disappears.

(3) $\rho'(0) > 0$, $a_3(p) < 0$ (supercritical Neimark-Sacker bifurcation). For $p > 0$ (slightly positive), we have $\rho(p) > 1$ and the steady state $x = 0$ is unstable. In addition, $\rho_+^*(p) > 0$: there is an asymptotically stable invariant curve around $x = 0$. For $p < 0$ (slightly negative), we have $\rho(p) < 1$ and the steady state $x = 0$ becomes asymptotically stable. In addition, $\rho_+^*(p)$ becomes nonreal: the invariant curve disappears.

(4) $\rho'(0) > 0$, $a_3(p) > 0$ (subcritical Neimark-Sacker bifurcation). For $p < 0$ (slightly negative), we have $\rho(p) < 1$ and the steady state $x = 0$ is asymptotically stable. In addition, $\rho_+^*(p) > 0$: there is an unstable invariant curve around $x = 0$. For $p > 0$ (slightly positive), we have $\rho(p) > 1$ and the steady state $x = 0$ becomes unstable. In addition, $\rho_+^*(p)$ becomes nonreal: the invariant curve disappears.

The normal form of this bifurcation is given by $\rho_{t+1} = \rho_t + b\rho_t + d\rho_t^3 = \varphi(\rho_t, p)$ and $\alpha_{t+1} = \alpha_t - q$ with $b, d = \pm 1$. q is a constant. $q < 0$ ($q > 0$) entails a counterclockwise (clockwise) rotation. Conditions [4.26] are all satisfied by this normal form. The four cases of Theorem 4.12 correspond to the following subcases: (1) $\rho_{t+1} = \rho_t - p\rho_t - \rho_t^3$, (2) $\rho_{t+1} = \rho_t - p\rho_t + \rho_t^3$, (3) $\rho_{t+1} = \rho_t + p\rho_t - \rho_t^3$, (4) $\rho_{t+1} = \rho_t + p\rho_t + \rho_t^3$.

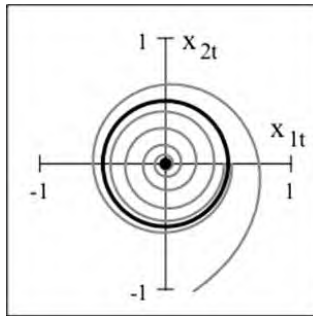


Figure 4.15. $p = -1/4$

EXAMPLE 4.9.— Focus on the fourth case: $\rho_{t+1} = \rho_t + p\rho_t + \rho_t^3$, with $0 < \rho(p) < 2$, i.e. $-1 < p < 1$, and $q > 0$. The non-negative solutions to the equation $\rho = \rho + p\rho + \rho^3$ are the fixed points $\rho^*(p) = 0$ and $\rho_+^*(p) = \sqrt{-p}$ (indeed $\rho(p) = 1 + p$ and $a_3(p) = 1$). The steady state $\rho_+^*(p)$ exists if and only if $p \leq 0$. The eigenvalue is given by $\varphi_\rho(\rho_t, p) = 1 + p + 3\rho_t^2$. We evaluate this eigenvalue at the two non-negative steady states: $\varphi_\rho(0, p) = 1 + p$ and $\varphi_\rho(\sqrt{-p}, p) = 1 - 2p$. Thus, the steady state

$\rho^*(p)$ is unstable if and only if $p > 0$ while, complementarily, the steady state $\rho_+^*(p)$ is unstable if and only if $p < 0$. In terms of the original two-dimensional dynamics $x_{t+1} = f(x_t, p)$, we conclude as follows. If $-1 < p < 0$ (slightly negative), then the steady state $x = 0$ is asymptotically stable, surrounded by an unstable invariant closed curve of radius $\rho_+^*(p) = \sqrt{-p}$. Instability means that, from a starting point inside the invariant closed curve, a clockwise spiral ($q > 0$) converges to the steady state $x = 0$. From a starting point outside the invariant closed curve, a clockwise spiral ($q > 0$) diverges far away from the limit circle. Because of the instability of the invariant closed curve, this Neimark-Sacker bifurcation is said to be subcritical. Using a continuous-time approximation (see Bosi and Ragot (2011, p. 96–97) for more details), Figures 4.15–4.17 show the fourth case: the subcritical limit cycle disappears as soon as p goes through 0. $\square$

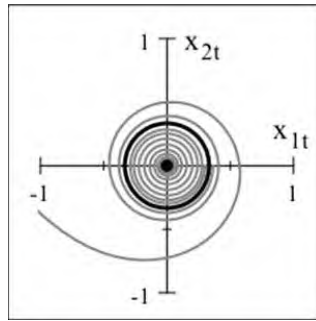


Figure 4.16. $p = -1/9$

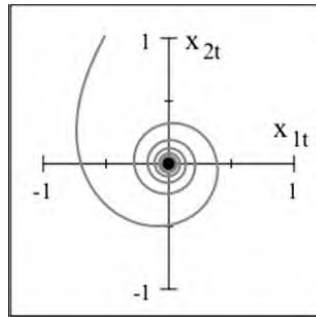


Figure 4.17. $p = 1/9$

In this chapter, we leave aside global bifurcations and chaos. The interested readers are referred to a short introduction in Bosi and Ragot (2011, p. 97–99) and more advanced textbooks such as Devaney (2003).

4.5.4. A geometrical approach

In the spirit of Samuelson (1941), we can represent the stability properties in the (T, D) -plane instead of the Argand plane (see Figure 4.18).

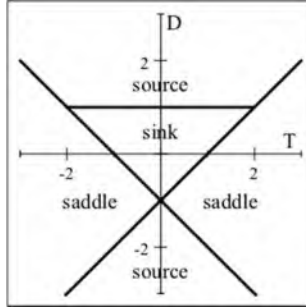


Figure 4.18. *Stability triangle*

We evaluate the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D$ at -1 and 1 : along the line $D = T - 1$, one eigenvalue is equal to 1 because $P(1) = 1 - T + D = 0$; along the line $D = -T - 1$, one eigenvalue is equal to -1 because $P(-1) = 1 + T + D = 0$. Along the segment $D = 1$ with $|T| < 2$, the eigenvalues are nonreal and conjugate with unit modulus. Focus on points outside these lines and the segment. Inside the triangle in Figure 4.18, the steady state is a sink ($D < 1$ and $|T| < 1 + D$). It is a saddle point if (T, D) lies in the left cone of the lines $D = T - 1$ and $D = -T - 1$, or in the right cone of these lines ($|1 + D| < |T|$). It is a source otherwise. A local bifurcation arises when one eigenvalue crosses the unit circle in the Argand plane or, equivalently, the pair (T, D) goes through one of the lines $D = T - 1$ or $D = -T - 1$, or the segment $D = 1$ with $|T| \leq 2$. When a bifurcation parameter changes, (T, D) moves in the (T, D) -plane. Generically, a generic saddle-node occurs when (T, D) crosses $D = T - 1$, a flip when (T, D) goes through the line $D = -T - 1$, whereas a Neimark–Sacker when (T, D) crosses the segment $D = 1$ with $|T| \leq 2$.

4.6. Continuous-time systems

Consider a finite-dimensional system of autonomous ordinary differential equation (ODE): $\dot{x} = f(x, p)$, where f is a smooth function, $x \in \mathbb{R}^n$ a vector of variables and $p \in \mathbb{R}^m$ a vector of parameters. We assume that f has a steady state $x^*(p)$, solution to $\dot{x} = f(x, p) = 0$.

In continuous time, a steady state (fixed point) is said to be hyperbolic if the Jacobian matrix evaluated at the point has no eigenvalues with zero real parts. A hyperbolic fixed point of a continuous-time system is a sink (source) if all the

eigenvalues of its Jacobian matrix have a negative (positive) real part, and it is a saddle otherwise. In the case of a hyperbolic steady state, the Grobman–Hartman Theorem applies and the original nonlinear system is locally topologically equivalent to its linearization around the steady state. In continuous time, a local bifurcation generically arises at the critical value $p = p^*$ when the real part of an eigenvalue $\lambda(p)$ of the Jacobian matrix crosses zero when p goes through p^* . The codimension is the number of parameters that must be varied to observe the bifurcation. We observe, generically, two kinds of bifurcations of codimension one: (1) zero bifurcation when a real eigenvalue crosses zero: $\lambda(p^*) = 0$, (2) Andronov–Hopf–Poincaré bifurcation (Hopf thereafter) when the real part of two complex and conjugate eigenvalues $\lambda(p) = a(p) \pm ib(p)$ crosses zero. More precisely, we require $a(p^*) = 0$ and $b(p) \neq 0$ in a neighborhood of $p = p^*$. The class of zero bifurcations in continuous time correspond to the class of saddle-node bifurcations in discrete time, while the Hopf bifurcation in continuous time to the Neimark–Sacker bifurcation in discrete time.

Codimension-two bifurcations involve instead two parameters. Let two bifurcations occur at $p_2 = \varphi(p_1)$ and $p_2 = \psi(p_1)$, where φ and ψ are two smooth functions, and the other parameters be constant. If the bifurcation curves φ and ψ (obtained varying p_1) cross at p^* in the (p_1, p_2) -plane with $p_2^* = \varphi(p_1^*) = \psi(p_1^*)$, then the dynamical system generically experiences a codimension-two bifurcation. In the following, for simplicity, we will consider only simple codimension-two bifurcations we can detect by checking the eigenvalues of the Jacobian matrix. There are other codimension-two bifurcations that involve higher-order terms of the Taylor expansion of the original system.¹⁸

Simple codimension-two bifurcations of autonomous ODE depend on the dimension. A double-zero (also said Bogdanov–Takens) bifurcation arises when the critical steady state has a zero eigenvalue of multiplicity two. Thus, as the Hopf one, it requires a dimension $n \geq 2$. The zero-Hopf (also said Gavrilov–Guckenheimer) bifurcation occurs when the critical steady state has a zero eigenvalue and a pair of purely imaginary eigenvalues. It needs a dimension $n \geq 3$. The double-Hopf bifurcation occurs at the intersection of two Hopf bifurcation curves in the parameter plane. It requires a dimension $n \geq 4$.

In the following, we will consider bifurcations for systems of dimension one to four, often considered in macroeconomic theory (see the two-dimensional Ramsey

18. For instance, the cusp bifurcation generically occurs when the Jacobian matrix has a single zero real eigenvalue and the quadratic coefficient of the normal form for the zero bifurcation is also zero (see Kuznetsov (1998, p. 301)). A generalized Hopf bifurcation generically occurs when the Jacobian matrix possesses a pair of purely imaginary eigenvalues (Hopf bifurcation) and the first Lyapunov coefficient (expression [4.31] below) is zero (see Kuznetsov (1998, p. 307)).

model (1928) and its higher-dimensional extension with market imperfections or heterogeneous agents). Thus, we will consider, in sequence, zero, Hopf, double-zero, zero-Hopf, double-Hopf bifurcations, and we will leave aside more sophisticated bifurcations arising in dimension $n \geq 5$, combining simultaneously more zero and Hopf bifurcations.

4.7. Local bifurcations in dimension one

Let us begin with dimension and codimension one: $n = m = 1$. In this case, dynamics are driven by a single differential equation and local dynamics by a single eigenvalue, the partial derivative of f with respect to x evaluated at the steady state $x^*(p)$: $\lambda(p) = f_x(x^*(p), p)$. A zero bifurcation arises when this (real) eigenvalue crosses zero: $\lambda(p^*) = 0$. Because of dimension one, there is no room for Hopf or codimension-two bifurcations.

CONDITION 4.1 (NON-DEGENERACY).— *Under the non-degeneracy conditions:*

$$f_p(x^*(p^*), p^*) \neq 0 \text{ and } f_{xx}(x^*(p^*), p^*) \neq 0 \quad [4.27]$$

a generic saddle-node bifurcation arises if and only if $f_x(x^*(p^*), p^*) = 0$. The transcritical bifurcation violates the first condition in [4.27] with $f_p(x^*(p^*), p^*) = 0$, $f_{xx}(x^*(p^*), p^*) \neq 0$, $f_{xp}(x^*(p^*), p^*) \neq 0$. The pitchfork bifurcation violates both the conditions in [4.27] with $f_p(x^*(p^*), p^*) = 0$, $f_{xx}(x^*(p^*), p^*) = 0$, $f_{xp}(x^*(p^*), p^*) \neq 0$, $f_{xxx}(x^*(p^*), p^*) \neq 0$.

The normal forms for the generic saddle node, the transcritical and the pitchfork that respect all the conditions above with $p^* = 0$ and $x^*(0) = 0$ are, respectively: (S) $\dot{x} = pa_0 + a_2x^2$ with $a_0, a_2 \neq 0$; (T) $\dot{x} = pa_1x + a_2x^2$ with $a_1, a_2 \neq 0$; (P) $\dot{x} = pa_1x + a_3x^3$ with $a_1, a_3 \neq 0$.

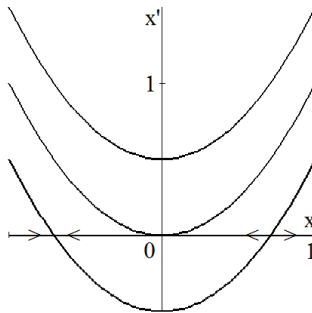


Figure 4.19. Saddle node

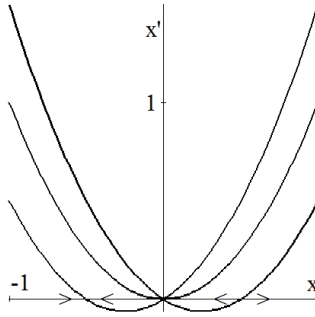


Figure 4.20. *Transcritical*

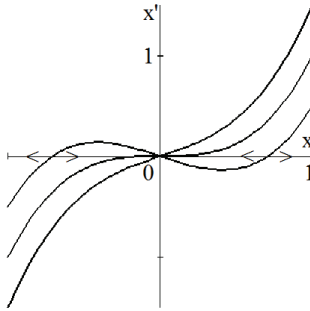


Figure 4.21. *Pitchfork*

EXAMPLE 4.10.— In Figures 4.19–4.21, we represent at $p = -1/2, 0, 1/2$, respectively, the normal forms: (S) $\dot{x} = p + x^2$ (a stable and an unstable steady state coalesce and disappear); (T) $\dot{x} = px + x^2$ (the stability properties of the two steady states are reversed); (P) $\dot{x} = px + x^3$ (two unstable steady states coalesce and disappear around a stable steady state which becomes unstable). $\square$

4.8. Local bifurcations in higher dimensions

Consider a system of ordinary differential equations (ODEs) $\dot{x} = f(x, p)$ with $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$. As in the one-dimensional case, local bifurcations of higher-dimensional systems require non-degeneracy conditions to be satisfied. These conditions depend on the bifurcation we are focusing on, and involve multilinear forms of the system $\dot{x} = f(x, p)$:

$$\begin{aligned}
B_h(u, v) &\equiv \sum_{i,j=1}^n \frac{\partial^2 f_h}{\partial x_i \partial x_j} \Big|_{(x^*(p), p)} u_i v_j \text{ and} \\
C_h(u, v, w) &\equiv \sum_{i,j,k=1}^n \frac{\partial^3 f_h}{\partial x_i \partial x_j \partial x_k} \Big|_{(x^*(p), p)} u_i v_j w_k
\end{aligned} \tag{4.28}$$

Compute the third-order Taylor expansion around x^* given p : $f(x, p) \approx f(x^*, p) + Jv + B(v, v)/2 + C(v, v, v)/6 = Jv + B(v, v)/2 + C(v, v, v)/6$, where $J = (\partial f_i / \partial x_j) \equiv (a_{ij})$ represents the n -by- n Jacobian matrix of partial derivatives evaluated at the steady state. Moreover,

$$v \equiv \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \equiv \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}, B(v, v) \equiv \begin{bmatrix} v^T H_1 v \\ \vdots \\ v^T H_n v \end{bmatrix} \text{ and } C(v, v, v) \equiv \begin{bmatrix} C_1(v, v, v) \\ \vdots \\ C_n(v, v, v) \end{bmatrix}$$

where $H_h = (\partial^2 f_h / (\partial x_i \partial x_j)) \equiv (b_{ij}^h)$ denotes the n -by- n Hessian matrix of f_h . All the derivatives are evaluated at (x^*, p) .

In the following, we will consider bifurcations for system of dimension $n \leq 4$. Explicit expressions for $C_h(v, v, v)$ when $n = 3, 4$ are provided in Bosi and Desmarchelier (2019, p. 39 and p. 43). Let S_i be the sums of principal minors of order i of the Jacobian matrix J . The value of each S_i depends on the steady state x^* as well as on the bifurcation parameter p we are focusing on. The eigenvalues λ_i are the roots of the characteristic polynomial: $P(\lambda) \equiv \prod_{i=1}^n (\lambda - \lambda_n) = \sum_{i=0}^n (-1)^i S_i \lambda^{n-i}$ with $S_0 \equiv 1$, and the vector $(S_1, \dots, S_n)$ is informative about the vector of eigenvalues $(\lambda_1, \dots, \lambda_n)$. Indeed, $T = S_1 = \lambda_1 + \dots + \lambda_n$ and $D = S_n = \lambda_1 \dots \lambda_n$ and the other S_i are also sums of products of eigenvalues. For instance, in the case $n = 3$, we have: $S_1 = \lambda_1 + \lambda_2 + \lambda_3 = T$, $S_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3$, $S_3 = \lambda_1 \lambda_2 \lambda_3 = D \equiv |J|$. The expressions S_i for $n = 3, 4$ are also provided in Bosi and Desmarchelier (2019, p. 39 and 44).

4.8.1. Zero bifurcation

When a real eigenvalue crosses zero, a so-called zero bifurcation takes place. Since only one parameter moves, we are in codimension one. In the case of a generic saddle node, two steady states (one stable, the other unstable) coalesce and disappear. In the case of a transcritical bifurcation, two steady states (one stable, the other unstable) coalesce and then separate again while exchanging their stability properties. Finally, in the case of a pitchfork bifurcation, three steady states coalesce into one: a stable (unstable) steady state surrounded by two unstable (stable) becomes unstable (stable).

Let the system $\dot{x} = f(x, p)$ have a steady state $x^*(p)$ and a single zero eigenvalue at $p = p^*$, i.e. $\lambda_1(p^*) = 0$ with $\operatorname{Re} \lambda_i(p^*) \neq 0$ for $i \geq 2$. Non-degeneracy conditions for the class of zero bifurcations in codimension one are given by

$$\left(\frac{\partial f_1}{\partial p}, \dots, \frac{\partial f_n}{\partial p} \right)_{(x^*(p^*), p^*)} \tilde{v} \neq 0 \text{ and } \tilde{v}^T B(v, v) \neq 0 \quad [4.29]$$

where B is given by [4.28], $v \in \mathbb{R}^n$ is a null-vector of J with a unit norm: $Jv = 0$, $v^T v = 1$, and $\tilde{v} \in \mathbb{R}^n$ is the adjoint null-vector of J : $J^T \tilde{v} = 0$ with $\tilde{v}^T v = 1$ (more details on the non-degeneracy conditions in Guckenheimer and Holmes (1983, p. 148) and Kuznetsov (1998, p. 172–174)).

PROPOSITION 4.1 (DIMENSION TWO).— *Under the non-degeneracy conditions [4.29], a generic zero bifurcation generically arises when a real eigenvalue crosses zero, i.e. if and only if $D = 0$.*

The proof is straightforward. If a real eigenvalue crosses zero, we have $D = 0$. Conversely, let $D = 0$. If one eigenvalue is nonreal, both are nonreal and conjugate. Thus, $0 = D = \lambda_1 \lambda_2 = \alpha^2 + \beta^2$ implies $\alpha = \beta = 0$ and, therefore, $\lambda_1 = \lambda_2 = 0 \in \mathbb{R}$, a contradiction: both the eigenvalues are real. In this case, $D = 0$ implies that one of them is zero.

Non-degeneracy conditions [4.29] apply to dimensions $n \geq 2$, and Proposition 4.1 also holds for $n = 3, 4$.

PROPOSITION 4.2 (DIMENSIONS THREE AND FOUR).— *Let the non-degeneracy conditions [4.29] hold. A generic saddle-node bifurcation generically arises when a real eigenvalue crosses zero, i.e. if and only if $D = 0$.*

For proofs in dimensions three and four, see Bosi and Desmarchelier (2019, p. 47, Proposition 2, and p. 48, Proposition 15, respectively).

As seen in the case of dimension one, conditions for transcritical and pitchfork bifurcations of systems in dimension two, three and four are less generic, but Propositions 4.1 and 4.2 on the generic zero bifurcation also hold for these bifurcations under the respective non-degeneracy conditions we obtain applying Condition 4.1 along the one-dimensional center manifold.

4.8.2. Hopf bifurcation

A Hopf bifurcation generates limit cycles. It is supercritical (subcritical), and the cycle is stable (unstable) when the first Lyapunov coefficient (see below [4.31]) is negative (positive).

Focus on the non-degeneracy conditions for the Hopf bifurcation in codimension one. Let J have two complex conjugate eigenvalues $\lambda_{1,2}(p) = \alpha(p) \pm i\beta(p)$ such that α is a smooth function of p . A Hopf bifurcation occurs at $p = p^*$ when the real part is zero: $\alpha(p^*) = 0$, i.e. when $\lambda_1(p^*) = -i\beta(p^*)$, $\lambda_2(p^*) = i\beta(p^*)$ with $\beta(p^*) > 0$ and nonzero real parts of the other eigenvalues, and the following non-degeneracy conditions hold:

$$\alpha'(p^*) \neq 0 \text{ and } l_1(p^*) \neq 0 \quad [4.30]$$

where

$$l_1(p^*) \equiv$$

$$\frac{\operatorname{Re} \left[\bar{v}^T C(v, v, \bar{v}) - 2\bar{v}^T B(v, J^{-1}B(v, \bar{v})) + \bar{v}^T B(\bar{v}, [2i\beta(p^*)I_n - J]^{-1}B(v, v)) \right]}{2\beta(p^*)} \quad [4.31]$$

is the first Lyapunov coefficient and C is given by [4.28], $v \in \mathbb{C}^n$ is a complex eigenvector of J such that $Jv = i\beta(p^*)v$ and $\bar{v}^T v = 1$, and $\tilde{v} \in \mathbb{C}^n$ is the adjoint eigenvector of J such that $J^T \tilde{v} = -i\beta(p^*)\tilde{v}$ with $\tilde{v}^T v = 1$.¹⁹ Conditions for a Hopf bifurcations are different in dimensions two, three and four as we will see.

PROPOSITION 4.3 (DIMENSION TWO).—*Let genericity conditions [4.30] hold. A Hopf bifurcation arises when the real part of conjugate nonreal eigenvalues becomes zero, i.e. if and only if $T = 0$ and $D > 0$.*

In the case of dimension two, the proof is straightforward. In a two-dimensional dynamical system, we require at the bifurcation value: $\lambda_1 = i\beta = -\lambda_2$. Thus, $T = \lambda_1 + \lambda_2 = 0$ and $D = \lambda_1 \lambda_2 = \beta^2 > 0$. Conversely, let $T = 0$ and $D > 0$. In the case of a two-dimensional system, the eigenvalues are both real or both nonreal and conjugate. If they are real, $T = 0$ implies $\lambda_1 = -\lambda_2 \in \mathbb{R}$. Thus, $D = \lambda_1 \lambda_2 = -\lambda_1^2 \leq 0$, a contradiction. Therefore, the eigenvalues are both nonreal and conjugate with $\beta > 0$. In this case, $0 = T = \lambda_1 + \lambda_2 = 2\alpha$. Then, $\lambda_1(p^*) = -i\beta(p^*)$, $\lambda_2(p^*) = i\beta(p^*)$ with $\beta(p^*) > 0$ and a Hopf bifurcation occurs at $p = p^*$.

In the case of dimension three, while genericity conditions still apply, the proof involves not only D and T , as above, but also S_2 (sum of principal minors of order two) and turns out to be more difficult.

PROPOSITION 4.4 (DIMENSION THREE).—*Let genericity conditions [4.30] hold. A Hopf bifurcation generically arises when the real part of two conjugate nonreal eigenvalues becomes zero, i.e. if and only if $D = S_2 T$ and $S_2 > 0$.*

19. $\bar{v}$ denotes the complex conjugate of v . $\bar{u}^T v$ is the usual scalar product $\langle u, v \rangle$ in the complex space $\mathbb{C}^n \ni u, v$. Moreover, $\tilde{v}$ denotes the complex conjugate of $\bar{v}$. For the Hopf genericity conditions, see Kuznetsov (1998, p. 100 and p. 175–178).

For a proof, see Asada and Yoshida (2003) or Bosi and Desmarchelier (2019, p. 47, Proposition 4).

Finally, in dimension four, the proof involves also S_3 and becomes even more difficult.

PROPOSITION 4.5 (DIMENSION FOUR).— *Let genericity conditions [4.30] hold. A Hopf bifurcation generically arises when the real part of two conjugate nonreal eigenvalues becomes zero, i.e. if and only if $S_2 = S_3/T + DT/S_3$ and T and S_3 have the same sign.*

For a proof, see Asada and Yoshida (2003) or Bosi and Desmarchelier (2019, p. 48, Proposition 16).

Focus now on codimension-two bifurcations.

4.8.3. Double-zero bifurcation

A double-zero (so-called Bogdanov–Takens) involves a steady state with a zero eigenvalue of multiplicity two. Consider a real eigenvalue λ_1 and the curve $\lambda_1(p) = 0$ in the (p_1, p_2) -plane. Let another real eigenvalue λ_2 becomes zero at p^* along this curve. In this case, the center manifold becomes two-dimensional and a double-zero bifurcation arises at p^* .²⁰ More precisely, the system $\dot{x} = f(x, p)$ has two steady states: the first one is a saddle point while the second is surrounded by a limit cycle. A double-zero bifurcation arises when the two steady states and the limit cycle shrink in one point. When a double-zero bifurcation takes place, the limit cycle disappears because an orbit homoclinic to the saddle point appears and destroys the limit cycle (Kuznetsov (1998, p. 382) for more details).

Let the system $\dot{x} = f(x, p)$ have a steady state $x^*(p)$. A double-zero bifurcation occurs at $p = p^*$ when J has two zero eigenvalues, say $\lambda_1 = \lambda_2 = 0$, and the following genericity conditions hold:

$$\tilde{v}^T B(v, v) [\tilde{w}^T B(v, v) + \tilde{v}^T B(v, w)] \neq 0 \quad [4.32]$$

and the map $(x, p) \rightarrow (f, T, D)$ is regular²¹ at $(x, p) = (x^*(p^*), p^*)$, with $v, w, \tilde{v}, \tilde{w} \in \mathbb{R}^n$ such that v and w are an eigenvector and a generalized eigenvector

20. The center manifold is the union of the orbits tangent to the center subspace spanned by the generalized eigenvectors corresponding to $\lambda_1 = 0$ and $\lambda_2 = 0$. Center manifold and center subspace have the same dimension.

21. A regular map between algebraic varieties is a function between these varieties locally represented by polynomials.

of J corresponding to $\lambda_1 = 0$, i.e. $Jv = 0$ and $Jw = v$; $\tilde{v}$ is an adjoint eigenvector of v and $\tilde{w}$ a corresponding generalized eigenvector, i.e. $J^T \tilde{v} = 0$ and $J^T \tilde{w} = \tilde{v}$. Moreover, these eigenvectors satisfy the orthonormality properties: $v^T \tilde{v} = w^T \tilde{w} = 0$ and $v^T \tilde{w} = \tilde{v}^T w = 1$.²²

PROPOSITION 4.6 (DIMENSION TWO).— *Under the genericity conditions [4.32], a double-zero bifurcation occurs when the Jacobian matrix has a zero eigenvalue of multiplicity two, i.e. if and only if $T = D = 0$.*

The proof is straightforward. A double-zero bifurcation arises if and only if two real eigenvalues cross zero: $\lambda_1 = \lambda_2 = 0$. Therefore, $D = \lambda_1 \lambda_2 = 0$ and $T = \lambda_1 + \lambda_2 = 0$. Conversely, if $D = 0$, at least one eigenvalue is zero, say $\lambda_1 = 0$. $T = \lambda_1 + \lambda_2 = 0$ implies that the other eigenvalue is also zero: $\lambda_2 = 0$.

In dimension three and four, under the genericity conditions [4.32], a double zero bifurcation occurs when J has two zero eigenvalues, say $\lambda_1 = \lambda_2 = 0$, and the real parts of the other eigenvalues are nonzero.

PROPOSITION 4.7 (DIMENSIONS THREE AND FOUR).— *Let the genericity conditions [4.32] hold. A double-zero bifurcation occurs in dimension three when the Jacobian matrix has a zero eigenvalue of multiplicity two and the real parts of the other eigenvalues are nonzero, i.e. generically, if and only if $S_2 = D = 0$ and, in dimension four if and only if $S_3 = D = 0$.*

For proofs in dimensions three and four, see Bosi and Desmarchelier (2019, p. 47, Proposition 6, and p. 48, Proposition 17, respectively).

4.8.4. Zero-Hopf bifurcation

The zero-Hopf (so-called Gavrilov–Guckenheimer) arises when the critical steady state has a zero eigenvalue and a pair of purely imaginary eigenvalues, and corresponds generically to the intersection of two bifurcations curves: Hopf and saddle node. It is possible only in dimension $n \geq 3$. Differently from the double-zero bifurcation, the limit cycle is preserved and its interaction with the saddle-node bifurcation can lead to richer dynamics including the cases of invariant torus and local chaos (Kuznetsov, 1998).

Let the system $\dot{x} = f(x, p)$ have a steady state $x^*(p)$. Let $v \in \mathbb{R}^n$ and $w \in \mathbb{C}^n$ be two eigenvectors such that $Jv = 0$ and $Jw = i\beta(p^*)w$, and $\tilde{v} \in \mathbb{R}^n$ and $\tilde{w} \in \mathbb{C}^n$ be two adjoint eigenvectors such that $J^T \tilde{v} = 0$ and $J^T \tilde{w} = -i\beta(p^*)\tilde{w}$ with the following orthonormality properties: $v^T \tilde{v} = w^T \tilde{w} = 1$. A zero-Hopf bifurcation

22. See Kuznetsov (1998, p. 314–330) and Guckenheimer and Kuznetsov (2007a).

arises when $\lambda_1(p^*) = 0$, $\lambda_2(p^*) = i\beta(p^*) = -\lambda_3(p^*)$ and $\beta(p^*) > 0$, and the following non-degeneracy conditions hold:

$$\operatorname{Re} \left[\bar{w}^T B(v, w) \right] \neq 0, X(p^*) Y(p^*) Z(p^*) \neq 0 \quad [4.33]$$

where $X(p^*) \equiv \tilde{v}^T B(v, v)$, $Y(p^*) \equiv \tilde{v}^T B(w, \bar{w})$, and the map $(x, p) \rightarrow (f, T, D)$ is regular at $(x, p) = (x(p^*), p^*)$. For the cumbersome computation of $Z(p^*)$, see Bosi and Desmarchelier (2019, p. 47), Kuznetsov (1998, p. 330–342) or Guckenheimer and Kuznetsov (2007b).

PROPOSITION 4.8 (DIMENSION THREE).— *Let genericity conditions [4.33] hold. A zero-Hopf bifurcation occurs when the Jacobian matrix has a zero eigenvalue and a pair of purely imaginary eigenvalues, i.e. if and only if $D = T = 0$ jointly with $S_2 > 0$.*

For a proof, see Bosi and Desmarchelier (2019, p. 47–48, Proposition 8).

In the case of a four-dimensional system, the characterization involves also the sum of principal minors of order three.

PROPOSITION 4.9 (DIMENSION FOUR).— *Let genericity conditions [4.33] hold. A zero-Hopf bifurcation generically arises when the Jacobian matrix has a zero eigenvalue and a pair of purely imaginary eigenvalues, i.e. if and only if $D = 0$ and $S_3 = TS_2$ with $S_2 > 0$.*

For a proof, see Bosi and Desmarchelier (2019, p. 48, Proposition 18).

4.8.5. Double-Hopf bifurcation

The double-Hopf bifurcation occurs at the intersection of two Hopf bifurcation curves in the parameter plane. This bifurcation leads generically to a torus. The double-Hopf bifurcation can imply richer dynamics such as Smale horseshoes and local chaos (see Kuznetsov (1998) among others). Since two coalescing Hopf bifurcations are involved, the double-Hopf bifurcation is possible only if $n \geq 4$.

Let the system $\dot{x} = f(x, p)$ have a steady state $x^*(p)$. A double-Hopf bifurcation occurs when $\lambda_1 = i\beta(p^*)$, $\lambda_2 = -i\beta(p^*)$, $\lambda_3 = i\delta(p^*)$ and $\lambda_4 = -i\delta(p^*)$ with $\beta(p^*) > \delta(p^*) > 0$ and the following genericity conditions hold:

$$m\beta(p^*) \neq n\delta(p^*) \quad [4.34]$$

where $m, n \in \mathbb{N}$ are positive natural numbers such that $m + n \leq 3$, and the map $p \rightarrow (\operatorname{Re} \lambda_1(p), \operatorname{Re} \lambda_3(p))$ is regular at $p = p^*$.²³

23. See Kuznetsov (1998, p. 349–366) and Guckenheimer and Kuznetsov (2008) among others.

PROPOSITION 4.10 (DIMENSION FOUR).— *Let genericity conditions [4.34] hold. A double-Hopf bifurcation arises when the Jacobian matrix has two pairs of purely imaginary eigenvalues, i.e. if and only if $T = S_3 = 0$ with $D > 0$, $S_2 > 0$ and $S_2^2 \geq 4D$.*

For a proof, see Bosi and Desmarchelier (2019, p. 48–49, Proposition 20).

4.9. References

- Acemoglu, D. (2009). *Introduction to Modern Economic Growth*. Princeton University Press, Princeton, NJ.
- Asada, T. and Yoshida, H. (2003). Coefficient criterion for four-dimensional Hopf bifurcations: A complete mathematical characterization and applications to economic dynamics. *Chaos, Solitons and Fractals*, 18, 525–536.
- Barro, R.J. and Sala-i-Martin, X. (1995). *Economic Growth*. McGraw-Hill, New York.
- Bosi, S. and Desmarchelier, D. (2019). Local bifurcations of three and four-dimensional systems: A tractable characterization with economic applications. *Mathematical Social Sciences*, 97, 38–50.
- Bosi, S. and Ragot, L. (2011). *Introduction to Discrete-Time Dynamics*. CLUEB, Bologna.
- Carr, J. (1981). *Applications of Center Manifold Theory*. Springer Verlag, New York.
- Cass, D. (1965). Optimum growth in an aggregative model of capital accumulation. *Review of Economic Studies*, 32, 233–240.
- Devaney, R. (2003). *An Introduction to Chaotic Dynamical Systems*. Westview Press, Boulder, CO.
- Diamond, P. (1965). National debt in a neoclassical growth model. *American Economic Review*, 55, 1126–1150.
- Grandmont, J.-M. (2008). Nonlinear difference equations, bifurcations and chaos: An introduction. *Research in Economics*, 62, 122–177.
- Guckenheimer, J. and Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, Berlin.
- Guckenheimer, J. and Kuznetsov, Y. (2007a). Bogdanov-Takens bifurcation. *Scholarpedia*, 2(1), 1854.
- Guckenheimer, J. and Kuznetsov, Y. (2007b). Fold-Hopf bifurcation. *Scholarpedia*, 2(10), 1855.
- Guckenheimer, J. and Kuznetsov, Y. (2008). Hopf-Hopf bifurcation. *Scholarpedia*, 3(8), 1856.
- Hale, J.K. and Koçak, H. (1991). *Dynamics and Bifurcations*. Springer-Verlag, New York.
- Haragus, M. and Iooss, G. (2011). *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems*. Springer-Verlag, London.

- Hirsch, M.W. and Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York.
- Iooss, G. (1979). *Bifurcations of Maps and Applications*. North-Holland, New York.
- Kelley, A. (1967). The stable, center stable, center, center unstable and unstable manifolds. *Journal of Differential Equations*, 3, 546–570.
- Koopmans, T.C. (1965). On the concept of optimal economic growth. *Pontificae Academiae Scientiarum Scripta Varia*, 28, 225–300.
- Kuznetsov, Y. (1998). *Elements of Applied Bifurcation Theory*. Springer, New York, 112.
- Lang, S. (2008). *Introduction to Linear Algebra*. Springer-Verlag, Berlin.
- Ramsey, F. (1928). A mathematical theory of saving. *Economic Journal*, 38, 543–559.
- Samuelson, P.A. (1941). The stability of equilibrium: Comparative statics and dynamics. *Econometrica*, 9, 97–120.

5

Adaptive Learning and the Restricted Perceptions Approach

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Rational expectations makes overly strong cognitive and informational assumptions while being empirically unrealistic. Adaptive learning is an alternative modeling paradigm that assumes agents generate their expectations like a good econometrician who specifies and estimates forecasting models. In many models, though, people will identify and estimate, using available data, a parametric family of (linear) forecast models that are almost always misspecified. Nevertheless, despite the misspecification, they do not make systematic forecast errors. In a restricted perceptions equilibrium (RPE), each agent uses an optimal forecast model among the candidates under consideration. Notably, like a rational expectations equilibrium, an RPE is a Nash equilibrium concept: the optimal forecast model for an individual depends on the behaviors and thus the models used by other agents. The restricted perceptions approach brings realism and cognitive consistency into models of expectation formation while preserving the model consistency that is the hallmark of rational expectations models; this chapter overviews the adaptive learning and restricted perceptions approach.

5.1. Introduction

Modern macroeconomic models place people and firms in environments where they make repeated decisions over time and under uncertainty. Consequently, macroeconomic behavior and outcomes depend on people's expectations about future events. For instance, when people expect prices to rise, they act in ways that make

Macroeconomics 1,

coordinated by Rodolphe DOS SANTOS FERREIRA and Thomas SEEGMULLER.

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prices rise. Businesses set higher prices in anticipation of higher costs, workers demand higher wages and consumers buy now rather than later. One of the most critical and open issues is: How do people formulate their expectations?

Since the 1970s, the prevailing macroeconomic paradigm is rational expectations. The rational expectations hypothesis aligns subjective expectations with actual outcomes, a strong assumption for a theory of expectation formation that requires people to make correct predictions given the information available to them. At the same time, people's subjective beliefs could, by pure chance, equate to the actual probability distribution, which depends on those beliefs. A theory of purposeful behavior would hold that agents could calculate the correct expectations taking into account all information. However, calculating rational beliefs requires agents to correctly understand the complete economic structure, the exogenous state variables, and the preferences and beliefs of all other individuals.

Instead, many models depart from rational expectations through bounded rationality in beliefs or decision-making. Sargent (1993) warned researchers about the dangers of the "wilderness of bounded rationality". The appeal of rational expectations is that they are model consistent. Equating subjective expectations and actual outcomes within a model environment removes beliefs as an independent macroeconomic force. Once departing from model consistency, it is difficult to know where to add modeling discipline. Evans and Honkapohja (2001) argue in favor of a *cognitive consistency principle*: agents should be modeled like a good economist who specifies, estimates and revises models. The cognitive consistency principle says that the model can still discipline beliefs by determining the state variables agents include in their forecasting models. An estimation process, though, will lead to real-time learning that may or may not converge to rational expectations. This adaptive learning approach encompasses a broad range of schemes for how people might forecast variables based on observed data.

Adopting the cognitive consistency principle naturally leads us down a path where boundedly rational forecasting models do not nest rational expectations as a particular case. White (1994) begins his text on econometric inference by noting that all models are misspecified. The restricted perceptions approach is a burgeoning field of bounded rationality that studies the consequences of misspecified beliefs in macroeconomic models. Restricted perceptions are similar to rational expectations except the agents' own subjective model misses the truth in some dimension. With restricted perceptions, agents restrict attention to a (linear) parametric family of forecast models that are misspecified in some dimension, for example, the number of lags, variables and linearity, among others. However, it is possible to discipline beliefs by asking the agents to do their best, given their misspecification. In a restricted perceptions equilibrium, agents' perceived model is the projection of the endogenous variable on their restricted approximating model – i.e. it is the best forecast given the agent's information, knowledge and abilities. Thus, restricted perceptions preserve

the discipline of rational expectations, while allowing for substantial departures. This chapter overviews the restricted perceptions approach and its applications.

Restricted perceptions grant a significant and plausible role to beliefs in driving macroeconomic fluctuations, exemplified notably in the context of sunspot phenomena. It is well established that within the rational expectations framework, economic outcomes can become self-fulfilling prophecies, known as sunspot equilibria (Grandmont 1985). Here, the core idea is that individuals might alter their expectations based on purely extrinsic events, termed “sunspots”, leading to increased consumption and production. However, these sunspot equilibria are often criticized for their fragility and instability, particularly when subject to adaptive learning processes. In contrast, under the lens of restricted perceptions, individuals’ reactions to sunspot events are robust. These reactions might be influenced by sunspots because they act as proxies for overlooked real-world elements in their forecasting models. Such elements could include critical but subtle factors such as nonlinear economic relationships or the concealed policy preferences of central banks. This nuanced approach allows for a more stable and realistic representation of how beliefs, even when imperfectly formed, can significantly impact macroeconomic dynamics.

Section 5.2 introduces the model, while section 5.3 studies the rational expectations solution and introduces the idea of adaptive learning and expectational stability (“E-stability”). In section 5.4, there is an overview of the restricted perceptions approach, and section 5.5 illustrates several economic applications. Section 5.6 surveys the literature and concludes.

5.2. Model

To provide a clear foundation, we start with a simple reduced-form model. Consider the relationship of an economic variable y_t to its subjective future expectations $\hat{E}_t y_{t+1}$, and an exogenous variable z_t , representing uncertain events. We will delve into concrete economic applications later in this chapter.

5.2.1. *Temporary equilibrium and adaptive learning*

In a celebrated contribution, Grandmont (1977) focused on the concept of a “temporary equilibrium”. Here, economic transactions occur over time, and decisions are based on the expectations held at each moment. The idea of adaptive learning posits that these expectations, and consequent decisions, evolve in response to accumulating data. Our model encapsulates this in a linear expectational framework:

$$y_t = \alpha + \beta \hat{E}_t y_{t+1} + \gamma' z_t \quad [5.1]$$

where $\alpha, \beta \in \mathbb{R}$, $\hat{E}_t$ represents the expectations operator (details below), and z_t is an $(n \times 1)$ vector of stationary exogenous variables.¹ Throughout, assume that

$$z_t = \rho z_{t-1} + \varepsilon_t$$

is stationary with eigenvalues of ρ inside the unit circle and $\varepsilon_t \sim N(0, \Sigma_\varepsilon)$.

5.2.2. Rational versus boundedly rational expectations

When $\hat{E}_t = E_t$, agents have rational expectations, perfectly predicting y_{t+1} except for unforeseeable shocks such as ε_{t+1} . In this scenario, equation [5.1] becomes an expectational difference equation. A rational expectations equilibrium is a stable solution to this equation, where long time series for y_t are determined partly by its one-step-ahead expected value $E_t y_{t+1}$. This self-referential nature can lead to self-fulfilling phenomena such as asset-price bubbles and sunspot shocks.

This chapter, however, focuses on bounded rationality models. These models allow the expectation operator to deviate from conditional mathematical expectations, mapping unobservable random variable to functions of observables. Rational expectations equate this operator with the orthogonal projection onto the space of measurable functions generated by observables, i.e. the conditional expectations operator. The adaptive learning approach, as detailed in Evans and Honkapohja (2001), maintains model consistency without assuming perfect alignment between expectations and outcomes. Agents in this framework possess a linear forecasting model, akin to rational agents, but with an evolving understanding of relationships within the model, akin to an econometrician regressing y_t on known variables over time.

5.2.3. Restricted perceptions and model misspecification

The restricted perceptions approach extends beyond adaptive learning and rational expectations, accommodating situations where agents' forecasting model differs from those posited by rational expectations. This approach acknowledges that the actual data-generating process might remain perpetually unknown to agents. Let the actual data-generating process produce a probability distribution over outcomes, say $f(y^t|\phi)$, where ϕ is a parameter vector in the actual data-generating process. Then, an approximating model can also be described by a probability distribution over outcomes, but with an alternative parameterization: $f(y^t|\theta_j)$. We can index different

1. More general reduced form systems include lagged endogenous variables and nonlinearities. Restricting attention to models in the form of [5.1] is for expositional ease. The restricted perceptions approach has been applied in fully general environments.

approximating models by the parameterization θ_j . As a concrete example, restricted perceptions might underparameterize a model by omitting some variable(s) in z_t .

The rational expectations hypothesis equates the approximating, or subjective model, with the true model: $f(y^t|\phi) = f(y^t|\theta_j)$. Theoretically, there are things to like about rational expectations. There is model consistency in agents' expectations by equating outcomes with subjective beliefs. A well-specified learning process could lead the agents' approximating model to converge to the rational expectations equilibrium. The stability of rational expectations equilibria is the subject of Evans and Honkapohja (2001). However, the assumption is strong, and it may not be reasonable to expect alignment between subjective beliefs and actual outcomes in many settings. The learning models described in this chapter preserve many of the salient features of rational expectations by imposing a set of moment conditions on the approximating model so that, at least for long periods, the agent within the model would be unable to detect their misspecification. The case for these moment conditions, and their equilibrium implications, is the focus of this chapter.

Sargent (1993) highlighted the dilemma faced by research into bounded rationality. On the one hand, rational expectations are an a priori strong and unreasonable assumption. On the other hand, the model disciplines rational expectations, and departing from it may land us in a "wilderness of bounded rationality". Evans and Honkapohja (2001) advocate for disciplining bounded rationality via the *cognitive consistency principle*, which holds that we should model agents' forecasting behavior as if they were a good econometrician. Applied econometricians specify, estimate and revise econometric models. Thus, the early literature on adaptive learning assumed that agents forecast based on linear econometric models that nest rational expectations. The learning process occurs as agents revise their coefficient estimates by least squares as the economy generates new data over time. In a wide class of models, least-squares learning converges to rational expectations.

However, it was White (1994) who opened his classic econometrics book by observing that all models are misspecified. Econometricians often face degrees of freedom problems that force them to specify parsimonious models. The actual data-generating process may be nonlinear, but econometricians typically estimate linear models. A wide variety of shocks impact the economy, and only a subset of those exogenous variables may be observable by the econometrician.

5.2.4. An example environment

The environment is a mean-variance linear asset pricing model similar to De Long et al. (1990) as developed in Branch and Evans (2011a). There is a continuum of agents born each time t , indexed by $\omega_t \in \Omega$. Each agent lives for two periods, and discounts at the rate $0 < \beta < 1$. The number n_t of young agents is iid with $En_t^{-1} = 1$.

Each agent receives an endowment of y when young and consumes only when old. The endowment is non-storable, but agents save using one of two assets: a riskless storage technology with gross return $R = \beta^{-1} > 1$ payable when old; a risky asset, in the form of a Lucas tree, which is in fixed outside supply s_0 , with claims to the tree traded competitively at a price p_t . An agent of type ω holds s_{dt} shares in the risky asset. The risky asset pays a stochastic dividend q_{t+1} . Because the population of young agents is random, the per-capita supply of the asset $z_t = s_0/n_t$ is also random. Finally, young agents also face risk in the form of an idiosyncratic asset float shock $f_t(\omega)$ that randomly redistributes holdings of the asset among old agents. The asset float shock proxies for idiosyncratic variations in asset float because of, for instance, lock-up expirations. The focus here is on a small noise limit.

5.2.4.1. *Temporary equilibrium*

Preferences are of the CARA form

$$U(c_{t+1}) = -\exp\{-ac_{t+1}\}$$

with $a > 0$ being the coefficient of absolute risk aversion. Agents assume a normal distribution for $p_{t+1} + q_{t+1}$, which leads to the household portfolio decision for young agent ω in the form of a mean-variance optimization problem:

$$\max_{s_{dt}} -\exp\left\{-a\hat{E}_t c_{t+1} + (a^2/2) \text{Var}_t^* c_{t+1}\right\}$$

subject to

$$c_{t+1} = (y - p_t s_{dt}(\omega)) \beta^{-1} + f_t(\omega) s_{dt}(\omega) (p_{t+1} + q_{t+1})$$

Here, $\hat{E}_t$ denotes the subjective conditional expectation and Var_t^* denotes the subjective conditional variance. Taking conditional subjective expectations leads to the first-order condition

$$-\beta^{-1} p_t + f_t(\omega) \hat{E}_t (p_{t+1} + q_{t+1}) - a\sigma^2 s_{dt} = 0$$

Here, we assume that subjective conditional variance expectations are homogeneous and, without a loss of generality, we take σ^2 to be exogenous and time-invariant.²

Market equilibrium requires that $s_0 = \int s_{dt}(\omega) d\omega$. Integrating the first-order condition across agents leads to

$$\frac{a\sigma^2 s_0}{n_t} = \int f_t(\omega) \hat{E}_t^\omega (p_{t+1} + q_{t+1}) d\omega - \beta^{-1} p_t$$

2. Branch and Evans (2011a) focus on an environment where the endogeneity of σ^2 plays a central role in generating asset price bubbles and crashes.

Now, we focus on the small noise limit $f_t(\omega) \rightarrow 1$, so that the temporary equilibrium³ asset-pricing condition becomes

$$p_t = \beta \hat{E}_t (p_{t+1} + q_{t+1}) - a\sigma^2 \beta s_t$$

or, equivalently,

$$p_t = \beta \hat{E}_t (p_{t+1} + q_{t+1}) + \gamma s_t \quad [5.2]$$

where $\hat{E}(x) = \int E^\omega(x) d\omega$ is the aggregate (linear) subjective expectations operator and s_t is the stochastic share supply s_0/n_t . Finally, we assume that the share supply follows a stationary AR(1) process

$$s_t = \rho s_{t-1} + \varepsilon_t \quad [5.3]$$

Equations [5.1]–[5.3] provide the (temporary) equilibrium value for price p_t given the aggregate expectations operator $\hat{E}_t$. The structural model takes the same form as the expectational difference equation in [5.1] with $z'_t = (\hat{E}_t q_{t+1}, s_t)$. The next section describes a variety of behavioral assumptions that lead to misspecification and restricted perceptions equilibria.

5.3. Rational expectations equilibrium and E-stability

In order to solve the temporary equilibrium equation [5.1], we need to make specific assumptions about the expectations operator. This section assumes rational expectations, $\hat{E}_t = E_t$, solves for the rational expectations equilibria, and then uses the adaptive learning approach to assess the plausibility of the equilibrium.

5.3.1. A unique rational expectations equilibrium

Let the expectational feedback parameter $0 < \beta < 1$ in [5.1]. In this case, there is a unique rational expectations equilibrium of the form

$$y_t = \bar{a} + \bar{b}' z_t$$

This is a minimal state variable solution as y_t is an exact linear function of the state variables in [5.1]. We can find the rational expectations equilibrium values $(\bar{a}, \bar{b})$ by the method of undetermined coefficients. From the linear form,

$$E_t y_{t+1} = \bar{a} + \bar{b}' \rho z_t$$

3. A temporary equilibrium is a time t market equilibrium in which agents solve their optimization problem taking as given own subjective expectations over payoff-relevant variables whose determination is treated as exogenous by each of the agents.

in which case, the true data-generating process is

$$y_t = \alpha + \beta \bar{a} + [\beta \bar{b}' \rho + \gamma'] z_t$$

Equating the linear form and the true data-generating process leads to

$$\bar{a} = \alpha + \beta \bar{a}$$

$$\bar{b} = [\beta \bar{b}' \rho + \gamma']$$

or, after solving,

$$\bar{a} = \frac{\alpha}{1 - \beta}$$

$$\bar{b} = (I - \beta \rho)^{-1} \gamma$$

The method of undetermined coefficients illustrates how rational expectations align beliefs and outcomes. If agents believe that y_t is linearly related to the variables in z_t , then their expectations will be $E_t y_{t+1} = \bar{a} + \bar{b}' \rho z_t$. Then, the temporary equilibrium [5.1], given those expectations, says that outcomes are

$$y_t = \alpha + \beta \bar{a} + [\beta \bar{b}' \rho + \gamma'] z_t$$

So, rational agents think that y_t is linearly related only to z_t – this is the minimal set of state variables – and, consequently, y_t is linearly related only to z_t in actuality. Beliefs and outcomes are aligned.

One way in which the adaptive learning approach is useful is to ask, if the agents did not know $\bar{a}$ and $\bar{b}$, but formed estimates of it through least-squares regressions of y_t on a constant and z_t , would those estimated values a_t, b_t eventually converge to $\bar{a}, \bar{b}$? Evans and Honkapohja (2001) develop the E-stability principle and demonstrate how an “E-stability condition” can determine whether a rational expectations equilibrium is learnable under a reasonable learning rule, for example, least-squares learning. The approach is similar to the method of undetermined coefficients. Assume agents’ beliefs come from an approximating model, called a *perceived law of motion*, of the form

$$y_t = a + b' z_t \Rightarrow \hat{E}_t y_{t+1} = a + b' \rho z_t \quad [5.4]$$

The perceived law of motion is a linear forecasting rule of the same form as the unique rational expectations equilibrium but for an arbitrary parameterization (a, b) . Given those expectations, plugging into [5.1] leads to the *actual law of motion* implied by the perceived law of motion:

$$y_t = \alpha + \beta a + [\beta b' \rho + \gamma'] z_t \quad [5.5]$$

Equating the perceived and actual laws of motion leads to the same parameter values $(\bar{a}, \bar{b})$ uncovered by the method of undetermined coefficients.

The E-stability approach, though, notes that [5.5] can be rewritten as

$$y_t = T(a, b)' X_t$$

where $X_t' = (1, z_t)$ and

$$T(a, b)' = [\alpha + \beta a, \beta b' \rho + \gamma']$$

A rational expectations equilibrium is a fixed point to the “T-map”: $(\bar{a}, \bar{b}) = T(\bar{a}, \bar{b})$. Up to this point, we are exactly describing the method of undetermined coefficients.

It turns out, though, that for a wide range of learning algorithms, convergence of real-time learning – in the sense that $(\bar{a}_t, \bar{b}_t) \rightarrow (\bar{a}, \bar{b})$ – is governed by the local asymptotic stability of the E-stability ordinary differential equation:

$$\frac{d(a, b)}{d\tau} = T(a, b) - (a, b) \quad [5.6]$$

A rational expectations equilibrium is expectationally stable (E-stable) provided that $(\bar{a}, \bar{b})$ is a locally stable resting point to [5.6]. That is, for (a, b) initially near $(\bar{a}, \bar{b})$, we have $d(a, b)/d\tau \rightarrow 0$ as $\tau \rightarrow \infty$. The variable τ is notional time as [5.6] delivers learnability conditions but is not, strictly speaking, a real-time learning algorithm: see below. Note that the rational expectations equilibrium $(\bar{a}, \bar{b})$ is a resting point of the E-stability equation. The T-map, intuitively, tells us what coefficients an econometrician would recover with a long data history if (a, b) remained constant. Thus, the E-stability o.d.e. says that a good learning algorithm will adjust the approximating model whenever the actual coefficients $T(a, b)$ differ from the perceived coefficients (a, b) . If that adjustment moves towards the rational expectations equilibrium, it is E-stable.

We can compute those E-stability conditions by examining the eigenvalues of the Jacobian to the right-hand side of [5.6], evaluated at the rational expectations equilibrium. E-stability then gives us conditions under which a small deviation from $\bar{a}, \bar{b}$ would converge back to its rational expectations equilibrium values. The minimal state variable solution has eigenvalues:

$$DT_a(\bar{a}, \bar{b}) = \beta$$

$$DT_b(\bar{a}, \bar{b}) = \beta \rho$$

Recalling that ρ is a diagonal matrix whose values are less than one, it follows that the key condition is that $DT_a(\bar{a}, \bar{b}) = \beta < 1$. We conclude, when $0 < \beta < 1$ and there is a unique rational expectations equilibrium, it is also learnable.

5.3.2. Multiple rational expectations equilibria

Suppose instead that $|\beta| > 1$. The model is now indeterminate with a continuum of possible rational expectations equilibria. It turns out that the full class of solutions is of the form

$$y_t = \bar{a} + \bar{b}'z_t + \bar{c}y_{t-1} + \bar{d}\eta_t$$

where η_t is an extrinsic random variable satisfying $E_{t-1}\eta_t = 0$, i.e. a sunspot variable. Sunspot equilibria give agents' beliefs an independent role in the economy as they allow dependence on a self-fulfilling sunspot variable. In many cases, however, sunspot variables are not stable under learning.

The full class of solutions now extends beyond the minimal number of variables and includes lags of y and the extrinsic sunspot shock η_t . To see that this is an equilibrium, consider a simplified version of [5.1] where $\alpha = 0, \gamma = 0$. Then,

$$y_t = \beta \hat{E}_t y_{t+1}$$

Note that $y_t = 0$ is a rational expectations equilibrium in this simplified setting: if $y_t = 0$, then $E_t y_{t+1} = 0$ and $y_t = \beta \times 0 = 0$. Suppose instead that the agents' perceive that

$$y_t = cy_{t-1} \Rightarrow \hat{E}_t y_{t+1} = c\hat{E}_t y_t = c^2 y_{t-1}$$

The minimal state variable solution $y_t = 0$ arises with $c = 0$. Without imposing anything on c , though, the actual law of motion becomes

$$y_t = \beta c^2 y_{t-1} \equiv T(c)y_{t-1}$$

The method of undetermined coefficients solves for the rational expectations equilibrium by computing $\bar{c} = T(\bar{c})$. In this simplified case, the rational expectations equilibrium values for c are either 0 or $1/\beta$. So, we have two different types of rational expectations equilibria. Only one – the minimal state variable solution – is learnable. Take the equilibrium with $\bar{c} = 1/\beta$, and note that $T'(\bar{c}) = T'(\beta^{-1}) = 2\beta\beta^{-1} = 2 > 1$. So, if agents believe that $y_t = cy_{t-1}$ with c near $1/\beta$, their learning will push them away from $1/\beta$. Sunspot rational expectations equilibria are not E-stable.

The remainder of the chapter examines the implications of instances where the approximating model does not nest the actual model. The focus is on reasonable restrictions on the approximating models that lead to a restricted perceptions equilibrium. Novel economic phenomena will arise, including the existence of E-stable sunspot equilibria.

5.3.3. Why E-stability?

Under least-squares learning, agents update (a_t, b_t, c_t) each period using all data up to that point. That is, coefficient estimates come from a regression model

$$y_t = \theta' x_t + e_t$$

where $x_t = (1, z'_t, y_{t-1})'$ and $\theta' = (a, b', c)$. The full-sample least-squares estimate based on data from $t = 1, \dots, T$ is

$$\theta_T = \left(\sum_{t=1}^T x_t x_t' \right)^{-1} \left(\sum_{t=1}^T x_t y_t \right)$$

It turns out, though, that with the right initialization, it is possible to write the least-squares estimator recursively as

$$\theta_t = \theta_{t-1} + t^{-1} R_t^{-1} x_t (y_t - x_t' \theta_{t-1})$$

$$R_t = R_{t-1} + t^{-1} (x_t x_t' - R_{t-1})$$

where R_t is the sample second-moment matrix of the regressors. Using induction, it is possible to prove the equivalence of these formulas when R is initialized using the first k observations as $R_k = k^{-1} \sum_{i=1}^k x_i x_i'$.

The E-stability principle draws a connection between the conditions that govern how θ_t in recursive least-squares behaves asymptotically with the conditions of the E-stability ordinary differential equation. Readers are referred to Evans and Honkapohja (2001) for technical details. Here, we can infer the connection.

The previous analysis showed how we can write the actual law of motion implied by agents' beliefs using the T-map:

$$y_t = T(\theta)' x_t$$

now using our new notation for θ and x . Plug this into the recursive updating equation for θ_t :

$$\theta_t = \theta_{t-1} + t^{-1} R_t^{-1} x_t [x_t' T(\theta_{t-1}) - x_t' \theta_{t-1}]$$

Now, let us rewrite again,

$$\frac{\theta_t - \theta_{t-1}}{t^{-1}} = R_t^{-1} x_t x_t' [T(\theta_{t-1}) - \theta_{t-1}]$$

while noticing that the term in the brackets is now the right-hand side of the E-stability ordinary differential equation. The remainder of the derivation takes a continuous time

limit and observes that the recursive second moment updating equation implies that $R_t^{-1}x_t x_t' \rightarrow I$.

The E-stability concept is important with other forms of learning besides adaptive and least-squares. Guesnerie (1992) introduced the idea of educative, or rationalizable, reasoning into macroeconomic models by assuming that agents form their expectations by considering the range of possible actions and outcomes that other rational agents in the market might choose. This process involves a sophisticated level of strategic thinking, where each agent's expectations and actions are contingent on their beliefs about other agents' beliefs and behaviors. The agents, through this interactive process, coordinate their beliefs within the model, leading to outcomes that are consistent with rational behavior under specific common knowledge assumptions on higher-order beliefs. Building upon this concept, Evans and Guesnerie (2003) and Evans and Guesnerie (2005) introduce educative E-stability to assess the stability of educative reasoning. Their approach emphasizes the role of logical, forward-looking reasoning and the coordination among agents in achieving equilibrium outcomes. Educative E-stability delves into the conditions under which the economy's equilibrium is not just learned adaptively based on historical data, but is also "educated" or logically inferred by agents who are actively engaged in deducing the model's implications. This approach bridges the gap between the theoretical underpinnings of rational expectations and the practical aspects of how agents might realistically form and coordinate their expectations, highlighting a more dynamic and interactive process of equilibrium attainment in economic models.

E-stability is a simple way to calculate learnability conditions for least-squares learning. Yet, it is built on powerful machinery.

5.4. Restricted perceptions equilibrium

A restricted perceptions equilibrium (RPE) allows for misspecification in the approximating models entertained by agents. An RPE maintains cross-equation restrictions such as the rational expectations hypothesis by requiring that the approximating model is the optimal linear projection within its class. This section describes a few common examples of misspecification and restricted perceptions equilibria. The first example is the case where the approximating models are under-parameterized. In the second example, the actual model is nonlinear, but agents forecast with a linear model. The third example assumes that the actual model contains hidden state variables unobserved by agents.

5.4.1. Under-parameterization

As a first example, consider the case of [5.1] with bivariate exogenous shocks:

$$y_t = \beta \hat{E}_t y_{t+1} + \gamma_1 z_{1t} + \gamma_2 z_{2t}$$

and

$$z_{jt} = \rho_j z_{jt-1} + \varepsilon_{jt}, j = 1, 2$$

The innovations ε_{jt} are mean-zero with variances σ_j^2 and $E\varepsilon_{1t}\varepsilon_{2t} = \sigma_{12}$.

In a complete information environment, the rational expectations equilibrium is

$$y_t = \frac{\gamma_1}{1 - \beta\rho_1} z_{1t} + \frac{\gamma_2}{1 - \beta\rho_2} z_{2t}$$

The effects of the shocks have a direct effect parameterized by γ_1 . The shocks also have an indirect effect that arises through the self-referential features of the model: y_t depends on expectations about its future values. The rational expectations equilibrium is the expected present value of the direct effects.

Branch and Evans (2006) note that in some situations, the cognitive consistency principle would lead agents to specify parsimonious forecasting models. In many environments, econometricians face a degree of freedom limitation that leads them to pare down the number of lags or exogenous variables. Forecasters often find, in environments with a structural change of unknown form, that simple, parsimonious models perform better. In this simple univariate example, complexity is not an issue, but it provides an analytical example of how under-parameterization can alter the equilibrium dynamics.

Suppose that agents have a perceived law of motion (approximating model) that conditions only on z_{1t} :

$$y_t = b_1 z_{1t} + \epsilon_t \Rightarrow \hat{E}_t y_{t+1} = b_1 \rho_1 z_{1t} \quad [5.7]$$

where ϵ_t is a perceived noise variable. The actual law of motion is, then,

$$y_t = [\beta b_1 \rho_1 + \gamma_1] z_{1t} + \gamma_2 z_{2t} \quad [5.8]$$

The method of undetermined coefficients does not help pin down the value of b_1 . Because z_{2t} is serially correlated and possibly correlated with z_{1t} , the rational expectations value of $b_1 = \gamma_1 / (1 - \beta\rho_1)$ will not give the best forecast of y_t , in a least-squares sense. Instead, we need to compute b_1 from the linear projection of y_t onto the restricted space of variables, z_{1t} .

Beliefs in a restricted perceptions equilibrium will satisfy the least-squares orthogonality condition that delivers the approximating model as the best linear model in its restricted class. In the current example,

$$E z_{1t} (y_t - b_1 z_{1t}) = 0$$

Solving for b_1 :

$$b_1 = \frac{E y_t z_{1t}}{E z_{1t}^2}$$

Alternatively, after plugging in the actual law of motion [5.8]:

$$b_1 = [\beta b_1 \rho_1 + \gamma_1] + \gamma_2 \frac{E z_{1t} z_{2t}}{E z_{1t}^2} \equiv T(b_1)$$

A few comments:

– Note that the least-squares projection of y_t on z_{1t} implies that the coefficient b_1 consists of two terms. The first is the actual coefficient in the data-generating process [5.8]. The second term is the omitted variable bias that emerges when z_{1t}, z_{2t} are correlated.

– The least-squares orthogonality condition implies that $b_1 = T(b_1)$, an expression similar to the previous section. However, there is no one-to-one mapping from the perceived law of motion to the actual law of motion; since the approximating model omits the z_{2t} term, it is under-parameterized. Instead, the T-map here is a “projected T-map”. The interpretation of this T-map is that if agents held a perceived law of motion of the form [5.7], with fixed b_1 , then with a sufficiently long sample, the regression of y_t on z_{1t} would estimate a coefficient $T(b_1)$.

Thus, a *restricted perceptions equilibrium* b_1^* is a fixed point to the T-map:

$$b_1^* = T(b_1^*)$$

Simple calculations show that

$$b_1^* = \frac{\gamma_1}{1 - \beta \rho_1} + \frac{\gamma_2}{1 - \beta \rho_1} \frac{\sigma_{12}}{\sigma_1^2} \frac{1 - \rho_1^2}{1 - \rho_1 \rho_2}$$

The RPE value for the belief parameter b_1^* equals its REE value plus a term that captures the omitted variable bias. If the omitted variable is uncorrelated with the regressor, i.e. $\sigma_{12} = 0$, then there is no bias. The bias is increasing in the size of the omitted variable’s direct effect, γ_2 , the strength of the correlation (relative to the variance of z_{1t}) and the persistence of the omitted variable ρ_2 .

Branch and Evans (2006) asked whether underparameterization or parsimony could lead to an equilibrium with heterogeneous expectations. The approach was to give agents a choice between all possible under-parameterized forecasting models. Agents could, for instance, select models based on their mean-squared forecast errors. Defining a *misspecification equilibrium* as a restricted perceptions equilibrium where agents only choose the best-performing models, heterogeneity will arise when each model delivers equivalent mean-squared errors. Branch and Evans (2006) call this *intrinsic heterogeneity*.

Let n denote the fraction of agents who forecast with [5.7]. Then, $1 - n$ agents have expectations $\hat{E}^2 y_{t+1} = b_2 \rho_2 z_{2t}$. The actual law of motion is

$$y_t = [n\beta b_1 \rho_1 + \gamma_1] z_{1t} + [(1 - n)\beta b_2 \rho_2 + \gamma_2] z_{2t}$$

The indirect effect depends on the population distribution across the two models, i.e. n . The idea of a misspecification equilibrium is that n, b_1, b_2 are all equilibrium objects. So, there are now a pair of orthogonality conditions for $j = 1, 2$:

$$E z_{jt} (y_t - b_j z_{jt}) = 0$$

and a selection rule:

$$n = \begin{cases} 1 & \text{if } F(1) > 0 \\ 0 & \text{if } F(0) < 0 \\ \hat{n} \in (0, 1) & \text{if } F(\hat{n}) = 0 \end{cases}$$

where

$$F(n) = E(y_t - b_2 z_{2t})^2 - E(y_t - b_1 z_{1t})^2$$

is the relative forecast accuracy of the z_1 model vis à vis the z_2 model.

It turns out that when $0 < \beta < 1$, then in equilibrium $n = 0$, $n = 1$, or both⁴. When there is negative feedback in the model with $-1 < \beta < 0$, there is the possibility of intrinsic heterogeneity. This case requires that the indirect effect is strong enough – i.e. sufficiently negative β – and that the two exogenous variables are correlated and sufficiently volatile.

5.4.2. Linear beliefs in a nonlinear model

Applied forecasting models are typically linear. Macroeconomic models, particularly their DSGE variants, are solved as log-linear approximations around a steady state. However, most macroeconomic environments produce a nonlinear relationship between equilibrium outcomes and state variables, including expectations.

Another restricted perceptions approach assumes that the economy's agents form expectations via a linear forecasting model and that the equilibrium law of motion is nonlinear (c.f. Branch and McGough (2005); Hommes et al. (2013)). Consider a nonlinear version of [5.1]:

$$y_t = G(y_{t+1}^e) + v_t \quad [5.9]$$

4. More concretely, in the case of multiple equilibria, there is a knife-edge equilibrium with $0 < n < 1$, but this equilibrium is unstable under learning.

where y is univariate, G is continuous and real-valued, and v_t is white noise with compact support. The earlier OLG example, with a more general preference structure, could lead to a pricing relationship such as [5.9]. The increasing returns model of Evans and Honkapohja (2001) is another example environment. Implicitly assumed in [5.9] is that agents hold point expectations, i.e. $\hat{E}_t G(y_{t+1}) = G(y_{t+1}^e)$. Among other technical assumptions, Branch and McGough (2005) impose that the function G is symmetric about a steady-state α .

Solving stochastic nonlinear rational expectations models such as [5.9] is complicated. Most researchers approximate the solution by solving for a solution to a first- or second-order expansion around α . The restricted perceptions approach, on the other hand, assumes that the data-generating process [5.9] is nonlinear, but agents hold a linear perceived law of motion:

$$y_t = a + b(y_{t-1} - a) \Rightarrow y_{t+1}^e = a + b^2(y_{t-1} - a) \quad [5.10]$$

A restricted perceptions equilibrium determines the coefficients a, b as the optimal least-squares projection of y_t onto the space of linear models of the form [5.10]. In this case, the coefficient for a will reflect the unconditional mean of y , and b will equal the unconditional first-order autocorrelation, where the former is taken with respect to the asymptotic distribution for y_t and computing the latter is taken from the joint asymptotic distribution over (y_t, y_{t-1}) . Branch and McGough (2005) provide a general existence and uniqueness result for symmetric G .

As an example, consider the function $G(y) = F(y - \alpha) + \alpha$, where

$$F(y) = \begin{cases} y^\beta & \text{if } y \geq 0 \\ -(-y)^\beta & \text{else} \end{cases}$$

where $0 < \beta < 1$. The actual law of motion, then, is

$$y_t = G[\alpha + b^2(y_{t-1} - \alpha)] + v_t \quad [5.11]$$

Suppose that $\beta = 1/3, \alpha = a = 1.1, v_t \in [-1, 1]$. Branch and McGough (2005) compute that $b^* \approx 0.74$.

Figure 5.1 plots the restricted perceptions equilibrium and outcomes. The solid line is the function G plotted in phase space. The dashed line is the perceived law of motion [5.10]. The circles represent the outcomes from a typical 1,000-period simulation. Given linear beliefs [5.10] with $(a, b) = (1.1, 0.74)$ and resulting data-generating process [5.11] leads to a stochastic process with a linear trend consistent with the linear beliefs. Although the agents mistakenly believe that the underlying process is linear, the actual nonlinear process produces realizations that self-confirm those beliefs within the restricted perceptions equilibrium.

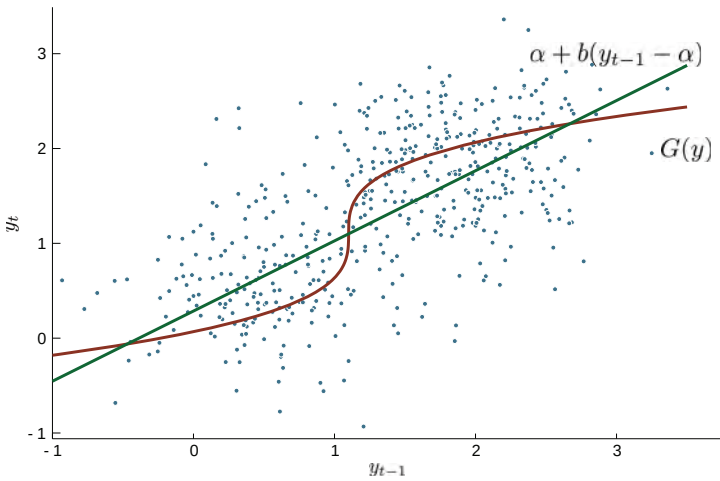


Figure 5.1. *Restricted perceptions equilibrium: linear forecasting model in a nonlinear world. For a color version of this figure, see www.iste.co.uk/dossantosferreira/macroeconomics.zip*

The applications section will consider restricted perceptions where linear beliefs in a nonlinear world produce novel economic phenomena near-rational sunspot equilibria. However, the restricted perceptions approach also provides a computationally and economically intuitive method for solving nonlinear stochastic models.

5.4.3. Hidden state variables

In part, random shocks drive macroeconomic and asset-pricing models. A model is a simple formalization of a complex economic process disturbed by a significant number of exogenous forces. It is unlikely that people will observe all of those shocks. What are the consequences of hidden variables to a model?

5.4.3.1. Unobservable variables and rational expectations

With rational expectations, agents' beliefs come from expectations conditional on all available information. But, what if some information is not available? This is not just theoretical curiosity: central banks set policy interest rates given their own views on the inflation-unemployment trade-off. These views and preferences of central bankers change over time. We do not observe them. Interestingly, whether hidden variables matter under rational expectations depends on the timing of when y_t is part

of the information used by agents when forecasting $E_t y_{t+1}$. If we think economies work by people taking actions given their beliefs and then markets clear and determine y_t , then we would not expect even rational agents to observe y_t at the time they predict $E_t y_{t+1}$.

Now, let us return to the linear model [5.1] with $0 < \beta < 1$ and z_t univariate. The unique rational expectations equilibrium is

$$y_t = (1 - \beta\rho)^{-1} z_t$$

But, using the Wold decomposition on the assumed process for $z_t = \rho z_{t-1} + \varepsilon_t$, we have

$$z_t = (1 - \rho L)^{-1} \varepsilon_t$$

where L is the lag operator ($Lz_t = z_{t-1}$), and so substituting for z_t

$$\begin{aligned} y_t &= (1 - \beta\rho)^{-1} (1 - \rho L)^{-1} \varepsilon_t \\ \Leftrightarrow (1 - \rho L)y_t &= (1 - \beta\rho)^{-1} \varepsilon_t \\ \Leftrightarrow y_t &= \rho y_{t-1} + (1 - \beta\rho)^{-1} \varepsilon_t \end{aligned}$$

So, an equivalent representation of the rational expectations equilibrium takes the form of a first-order autoregressive process (AR(1)).

The question asked by Marcet and Sargent (1989) is what does a rational expectations equilibrium look like if z_t is unobservable or a hidden variable to economic agents? The answer, it turns out, depends on whether y_t is contemporaneously observable.

Suppose that agents perceive the process to follow an AR(1)

$$y_t = by_{t-1} + d\varepsilon_t$$

If agents observe y_t contemporaneously, then

$$E_t y_{t+1} = by_t$$

and the actual law of motion becomes

$$y_t = \beta by_t + \gamma z_t$$

or,

$$y_t = by_{t-1} + \frac{\gamma}{1 - \beta b} \varepsilon_t$$

So, the rational expectations equilibrium takes the usual form when y_t is observable and $b = \rho$. After all, when $b = \rho$, then our actual law of motion is identical to the rewritten version of the unique equilibrium.

Now suppose that y_t is observed with a lag: $E_{t-1}y_{t+1} = b^2y_{t-1}$. In this case, the rational expectations equilibrium follows an $AR(\infty)$; it depends on an infinite number of lags. To see this, note that with the $AR(1)$ perceived law of motion and where y_t is not currently observable, the actual law of motion is

$$y_t = \beta b^2 y_{t-1} + \gamma z_t$$

or again substituting with $z_t = (1 - \rho L)^{-1} \varepsilon_t$,

$$(1 - \rho L) (1 - \beta b^2 L) y_t = \gamma \varepsilon_t$$

When agents cannot observe z_t or contemporaneous y_t and guess an $AR(1)$ equilibrium, the actual data-generating process is an $AR(2)$, with 2 lags of y_t . More generally, if agents have an $AR(p)$ perceived law of motion, then

$$E_{t-1}y_{t+1} = b_1 (b_1 y_{t-1} + \dots + b_p y_{t-p}) + b_2 y_{t-1} + \dots + b_p y_{t-p+1}$$

and the true data-generating process is an $AR(p+1)$:

$$(1 - \rho L) \left[1 - \beta \sum_{j=1}^p (b_1 b_j + b_{j+1}) L^j \right] y_t = \gamma \varepsilon_t$$

It follows that with some components of z_t hidden and y_t observed with a lag, the rational expectations equilibrium is an $AR(\infty)$. In this case, if the agents had an infinitely long history of y_t , they could reverse engineer the equation with infinite lags of y_t to filter and recover the hidden z_t shocks.

5.4.3.2. Unobservable variables and restricted perceptions

Of course, it is not practical to estimate the coefficients of an $AR(\infty)$ with finite data histories. Hommes and Zhu (2014) find a restricted perceptions equilibrium with an $AR(1)$ forecasting equation. Following the generalization in Branch et al. (2022), this section traces the implications of an under-parameterized $AR(1)$ perceived law of motion.

Assume that agents hold beliefs consistent with an $AR(1)$ perceived law of motion:

$$y_t = b y_{t-1} + \varepsilon_t \Rightarrow \hat{E}_t y_{t+1} = b^2 y_{t-1} \quad [5.12]$$

Then, the actual law of motion is

$$y_t = \beta b^2 y_{t-1} + \gamma z_t$$

In a restricted perceptions equilibrium, the coefficient b needs to be such that the approximating model delivers the best statistical fit in an ordinary least-squares sense. In other words, b will satisfy the least-squares orthogonality condition:

$$Ey_{t-1}(y_t - by_{t-1}) = 0$$

Or solving for b ,

$$b = \frac{Ey_t y_{t-1}}{Ey_{t-1}^2} \equiv T(b)$$

The right-hand side is the first-order autocorrelation coefficient for y_t . Note that because the model is self-referential, the right-hand side depends on b : if agents believe that y_t has an autocorrelation coefficient b , then $\hat{E}_t y_{t+1} = b^2 y_{t-1}$, and the resulting actual autocorrelation coefficient is b . The restricted perceptions equilibrium is a weaker version of rational expectations: rather than trying to match the full distribution, agents' beliefs just need to be consistent with this one empirical moment in the data.

To calculate the T-map, we need to compute various moments based off of the actual law of motion:

$$\begin{aligned} Ey_t y_{t-1} &= E(\beta b^2 y_{t-1} + \gamma z_t) y_{t-1} \\ &= \beta b^2 Ey^2 + \gamma^2 Ez^2 + 2\beta b^2 \gamma \rho Eyz \end{aligned}$$

using that $Ez_t y_{t-1} = E(\rho z_{t-1} + \varepsilon_t) y_{t-1} = \rho Eyz$. Then, similarly calculating Eyz and Ey^2 , straightforward computations show that the T-map is

$$b \rightarrow \frac{\beta b^2 + \rho}{1 + \beta b^2 \rho} = T(b)$$

Hommes and Zhu (2014) and Branch et al. (2022) prove the existence of an RPE $\hat{b}$ that is a fixed point to the T-map. This equilibrium value for $\hat{b}$ is a complicated expression of β and ρ .

Interestingly, the restricted perceptions equilibrium identified here may not be unique, and other equilibria can depend on sunspots. The applications portion of this chapter studies the complete set of solutions.

5.5. Applications

This section turns to a few novel applications that arise from the restricted perceptions approach. These are non-Ricardian beliefs, sunspot equilibria that are stable under learning, and random-walk beliefs that lead to asset price bubbles and inflation scares.

5.5.1. (Non-)Ricardian beliefs

Several adaptive learning papers develop non-Ricardian beliefs in New Keynesian-type models (c.f. Evans et al. (2009); Eusepi and Preston (2018); Woodford (2013)). A boundedly rational consumption function combines the Euler equations with a household intertemporal budget constraint without assuming that the household correctly understands the government's budget constraint and debt solvency. Instead, the household also must forecast the government's debt and primary surplus evolution. In these environments, the government's level of debt and surplus are state variables. Branch and Gasteiger (2022) study the consequences of under-parameterization in forecasting in this environment without directly imposing Ricardian beliefs. The under-parameterization here reflects that most applied analyses of the effects of fiscal policy typically only include a single fiscal variable. Here, though, it is a convenient formalization for how non-Ricardian beliefs could emerge endogenously.

Woodford (2013) introduces a special case of a purely real New Keynesian model without imposing Ricardian beliefs. The key equations are:

$$b_{t+1} = \beta^{-1} (b_t - s_t) \quad [5.13]$$

$$y_t = v_t + (1 - \beta) b_t \quad [5.14]$$

$$v_t = (1 - \beta) (b_{t+1} - b_t) + \hat{E}_t v_{t+1} \quad [5.15]$$

$$s_t = \phi_b b_t + z_t \quad [5.16]$$

The variables in the linearized equations denote log deviations from steady state. Equation [5.13] is the government's flow budget constraint, and it relates the stock of one-period government debt issued at time t , b_{t+1} , to the difference between the beginning of period debt and the primary surplus s_t . The parameter $0 < \beta < 1$ is the discount rate, and β^{-1} is equal to the steady-state gross real interest rate. The second equation [5.14] is the consumption function after imposing the goods market clearing condition that $c_t = y_t$. The variable v_t is a continuation value that reflects the annuitized value of future tax liabilities and returns on debt holdings that enter the household consumption function. So, aggregate output depends on this forward-looking variable as well as the existing stock of debt. Consumption, therefore, has a standard permanent income form, here written recursively. Equation [5.15] provides the recursion that determines the continuation value v_t . The question of the Ricardian equivalence is whether the household's expectations $\hat{E}_t v_{t+1}$ correctly anticipate future surpluses and debt issuances. If so, then y_t will not depend on government debt b_t , and Ricardian equivalence holds. The final equation is the fiscal rule with the policy coefficient $1 - \beta < \phi_b < 1$ and white noise policy shocks z_t .

Since b_{t+1} , depending on b_t, s_t , is the relevant state variable, the full information equilibrium will depend on b_{t+1} if it is observable or separately on b_t, s_t . Suppose that agents do not know b_{t+1} and the flow government budget constraint. Moreover, the

agents prefer parsimonious models and include a single fiscal variable in their forecast equation. Then, there are two potential forecasting models:

$$v_t = \psi^s s_{t-1} + \epsilon_t \quad [5.17]$$

$$v_t = \psi^b b_{t-1} + \epsilon_t \quad [5.18]$$

The first model [5.17] depends only on the flow budget surplus, while the model [5.18] depends on the previous period's government debt level. Loosely speaking, n parameterizes the extent of non-Ricardian beliefs. A restricted perceptions equilibrium jointly pins down the belief parameters in [5.17]-[5.18] and the aggregate variables [5.13]-[5.16].

Rather than imposing that all agents forecast with the same model, let

$$\hat{E}_t v_{t+1} = n\psi^s s_t + (1-n)\psi^b b_t$$

where n is the fraction of households that forecast with the surplus model [5.17]. In a restricted perceptions equilibrium, there are a pair of least-squares orthogonality conditions:

$$E s_{t-1} [v_t - \psi^s s_{t-1}] = 0$$

$$E b_{t-1} [v_t - \psi^b b_{t-1}] = 0$$

Branch and Gasteiger (2022) prove the existence, given n , of a unique RPE.

Consider the special case of $n = 0$; all households forecast with the debt model. Then, it turns out that the RPE is

$$y_t = -(\beta^{-1} - 1) z_t$$

$$\psi^b = -(\beta^{-1} - 1)(1 - \phi_b)$$

Note that the aggregate output does not depend on the debt stock b_t . It does depend on the innovation to the primary surplus z_t , but since $\beta \approx 1$, the effect is negligible. Branch and Gasteiger (2022) call this a form of weak Ricardian equivalence.

Now, let $n = 1$; all households forecast with the surplus model. Then, Woodford (2013) finds that

$$y_t = \left[\frac{(1-\beta)(1+\beta-\phi_b)}{\beta(1+\beta)+\phi_b} \right] b_t - \left[\frac{\beta(1-\beta^{-1})}{\beta(1+\beta)+\phi_b} \right] z_t$$

$$\phi^s = -\frac{\beta^{-1}(1-\beta)(1-\beta^2-\phi_b)}{\beta(1+\beta)+\phi_b} < \beta^{-1} - 1$$

Generalizing, Branch and Gasteiger (2022) show that the unique RPE, given n is of the form

$$y_t = \xi_1(n)b_t + \xi_2(n)z_t$$

with

$$\xi_1(n) \neq 0 \Leftrightarrow n > 0$$

So, the Ricardian equivalence is fragile. Only if *every* agent forecasts with the stock of debt will they correctly understand the consequences of fiscal policy. For any $n > 0$ – including $n \rightarrow 0$ – then *neither* type of agent holds Ricardian beliefs. In this sense, Ricardian equivalence is fragile and occurs in a highly restrictive self-confirming equilibrium.

5.5.2. Sunspots

Section 5.3 showed how belief-driven fluctuations could arise through indeterminacy and sunspot equilibria. There are two drawbacks to rational sunspots as a model of self-confirming fluctuations: first, often, the sunspot equilibria are unstable under learning; second, the indeterminacy regions in the DSGE model often coincide with empirically unrealistic parameterizations. The restricted perceptions approach, however, can overcome both limitations of rational sunspots.

5.5.2.1. Statistical sunspots

Branch et al. (2022) show that with unobservable variables, the restricted perceptions approach can generate sunspot equilibria in determinate models that are stable under learning.

The approach is to extend [5.12] to include dependence on an extrinsic shock:

$$y_t = by_{t-1} + d\xi_t \Leftrightarrow \hat{E}_t y_{t+1} = b^2 y_{t-1} + d(b + \phi)\xi_t \quad [5.19]$$

where

$$\xi_t = \phi\xi_{t-1} + v_t$$

is an extrinsic noise term uncorrelated with the hidden variable z_t . To ease exposition, assume that $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$, $v_t \sim N(0, \sigma_v^2)$, $\sigma_{v,\varepsilon} = 0$. If $d \neq 0$, then the economy depends on this extrinsic noise. The question is whether ξ_t can matter in a self-confirming way, and we can interpret it as a sunspot. Branch et al. (2022) study how ξ_t can matter in a restricted perceptions equilibrium.

The least-squares orthogonality condition is

$$E(y_t - by_{t-1} - d\xi_t)(y_{t-1}, \xi_t)' = 0$$

The sunspot will matter for y_t when there is a non-zero correlation between y_t and ξ_t :

$$d = (1 - b\phi) \frac{E y_t \xi_t}{E \xi_t^2}$$

But, y_t depends in turn on (b, d) . For this reason, Branch et al. (2022) call ξ_t a statistical sunspot; its presence arises because of a self-confirming statistical correlation.

Proceeding as in section 5.4.3.2, the d -component of the T-map is

$$d \rightarrow \frac{d\beta(b + \phi)(1 - b\phi)}{1 - \beta b^2 \phi}$$

There are two fixed points to the T_d component of the T-map: $d^* = 0$ and

$$b^* = \frac{1 - \beta\phi}{\beta(1 - \phi^2)}$$

When $d^* = 0$, the PLM [5.19] is identical to [5.12] and the earlier RPE results. When $d^* \neq 0$, then the RPE value can be found as a solution to the T_b component (see Branch et al. (2022) for details):

$$(d^*)^2 = \xi(b^*, \beta, \rho, \phi)$$

Branch et al. (2022) show that for fixed ϕ – the serial correlation of the sunspot variable:

- 1) There exists a unique RPE with $(b, d) = (\hat{b}, 0)$, and $\hat{b}$ is the equilibrium identified by Hommes and Zhu (2014).
- 2) There exist threshold values $\tilde{\beta}(\phi), \tilde{\rho}(\beta, \phi)$ so that sunspot RPE $(b, d) = (b^*, \pm d^*)$ exists $\Leftrightarrow \tilde{\beta} < \beta < 1$ and $\tilde{\rho} < \rho < 1$.
- 3) When sunspot RPE exists, the sunspot RPE is E-stable, while the fundamentals RPE with $d = 0$ is E-unstable.

Figure 5.2 shows the results. The solid lines correspond to the fixed points of the two T-map components. The vector field indicates the E-stability dynamics. An RPE occurs where the contours intersect. There are three RPE. There is the fundamental RPE with $d = 0$. There are also two symmetric sunspot RPE. Note, in particular, that the sunspot RPE is E-stable. Thus, with hidden variables in an environment with a unique rational expectations equilibrium, we expect the economy to exhibit dependence on self-confirming sunspots.

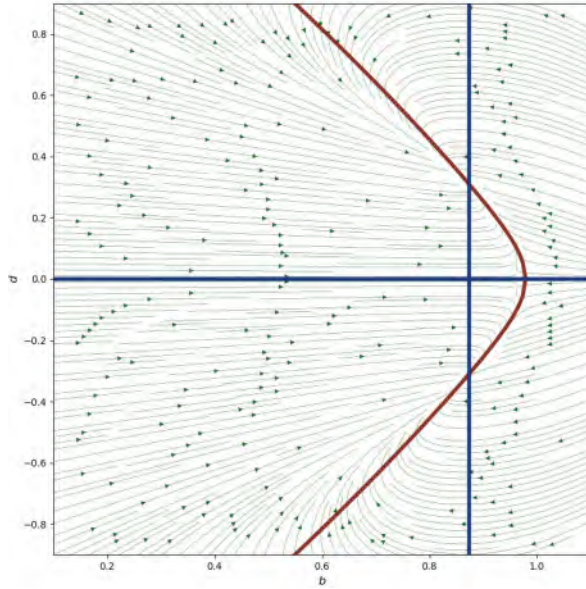


Figure 5.2. Statistical sunspots. For a color version of this figure, see www.iste.co.uk/dossantosferreira/macroeconomics.zip

5.5.2.2. Near-rational sunspots

Evans and McGough (2020b) find existence of E-stable *near-rational sunspot equilibria* in nonlinear models where the steady state is locally indeterminate. Their approach, like section 5.4.2, assumes that agents forecast with a linear model while the data-generating process is nonlinear.

Suppose, as in Evans and McGough (2020b) that

$$y_t = F(y_{t+1}^e)$$

with

$$F(y) = \theta y + \mu y^3$$

Agents have a linear perceived law of motion:

$$y_t = a + d\eta_t$$

where η_t is a serially correlated process

$$\eta_t = \lambda\eta_{t-1} + e_t$$

The coefficient λ satisfies a “resonant frequency condition” so that a sunspot equilibrium in the linearized version corresponds to the sunspot equilibrium found in section 5.3

Now in a restricted perceptions equilibrium, the coefficients (a, d) are found from the least-squares projection of the actual nonlinear law of motion onto the space spanned by $(1, \eta_t)$. The main results in Evans and McGough (2020b) can be illustrated in their simple cubic example with $\theta = -5$ and $\mu = 2$. The case $\theta < 0$ is equivalent to the negative feedback case in [5.1]. Figure 5.3 plots the existence and E-stability of near-rational sunspots.

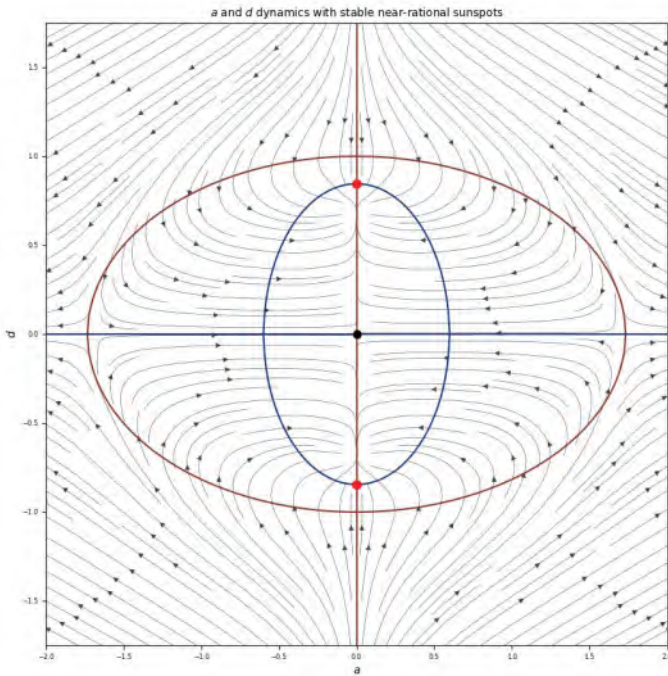


Figure 5.3. *Near-rational sunspot equilibria. For a color version of this figure, see www.iste.co.uk/dossantosferreira/macroeconomics.zip*

Figure 5.3 shows the determination of near-rational sunspots. The outer circle and vertical line correspond of fixed points to the T_a component of the T-map: given a value for the coefficient d on the sunspot, these are all the optimal values for a from the least-squares orthogonality condition. The inner circle and horizontal axis are the fixed points to T_d : given a value for the intercept (or, mean) a , these are all the

optimal values for the coefficient d . The intersection of the contours is RPE: $(\bar{a}, \bar{d}) = (T_a(\bar{a}, \bar{d}), T_d(\bar{a}, \bar{d}))$. A near-rational sunspot equilibrium simply requires that $d \neq 0$. Therefore, intersections in Figure 5.3 with $d \neq 0$ are near-rational sunspot equilibria. In Figure 5.3 these near-rational sunspot equilibria are denoted intersections along the vertical axis and marked with circles. There are two symmetric near-rational sunspot equilibria.

Towards the beginning of this chapter, we saw that sunspot equilibria are generally unstable in linear models. Figure 5.3 also shows the E-stability. The vector field lines denote, for initial coefficient values, the expected learning path. The steady state is saddle-path stable: if agents believe there are no sunspots, $d = 0$, then their adaptive learning process will bring them to an equilibrium without sunspots $(\bar{a}, \bar{d}) = (0, 0)$. However, it is the near-rational sunspots that are stable under learning. All of the paths (besides the aforementioned saddle path) point towards the near-rational sunspots.

Forecasting with a linear model – as most economists do – in a nonlinear setting – as most models are – produces learnable sunspots that would not arise in a fully rational setting.

5.5.3. Random-walk beliefs

In the presence of hidden variables, the restricted perceptions approach finds equilibria that are more serially correlated than under rational expectations. Moreover, these serially correlated and volatile equilibria are stable under learning. In other settings, restricted perceptions equilibria can exist that are self-confirming but not stable under learning. Nonetheless, learning dynamics can still be drawn towards the RPE and exert influence over dynamics for a finite period. In linear self-referential models, these RPE take the form of random-walk beliefs.

Continuing with the general model [5.1], suppose that z_t is univariate white noise and that agents perceive that y follows a random walk without drift:

$$y_t = y_{t-1} + \epsilon_t \Rightarrow \hat{E}_t y_{t+1} = y_{t-1}$$

The actual law of motion, then, is

$$y_t = \alpha + \beta y_{t-1} + \gamma z_t$$

It is convenient instead to rewrite the perceived and actual laws of motion as moving average processes (MA). Note that the perceived law of motion is also

$$(1 - L)y_t = \epsilon_t \Rightarrow y_t = (1 - L)^{-1}\epsilon_t \equiv g(L)\epsilon_t$$

and $g(L)$ is a lag polynomial with an infinite number of lags of ϵ_t .⁵ Similarly, from the actual law of motion

$$(1 - \beta L)y_t = \alpha + \gamma z_t$$

$$\Leftrightarrow y_t = \frac{\alpha}{1 - \beta} + \gamma(1 - \beta L)^{-1} z_t \equiv \mu + f(L)z_t$$

Putting it altogether, we have infinitely ordered moving average $MA(\infty)$ processes for both perceived and actual laws of motion:

$$\text{PLM: } y_t = g(L)\epsilon_t \quad [5.20]$$

$$\text{ALM: } y_t = \mu + f(L)z_t \quad [5.21]$$

where $g(L) = (1 - L)^{-1}$, $\mu = \alpha/(1 - \beta)$, $f(L) = \gamma(1 - \beta L)^{-1}$. In [5.21], the unconditional mean of y_t is $\alpha/(1 - \beta)$, the same value as in the rational expectations equilibrium. However, when z_t is iid (as in this example), y_t is not serially correlated. Here, the random-walk beliefs induce serial correlation that would not exist under rational expectations.

For large values of $0 < \beta < 1$, the serial correlation induced by random-walk beliefs is nearly self-fulfilling. In fact, as $\beta \rightarrow 1$, the moving average structure is identical under the perceived law of motion [5.20] and the actual law of motion [5.21]. In a similar setting, Sargent (1999) notes that random-walk beliefs can track constants well. In the rational expectations equilibrium, y_t equals μ plus iid innovations. The perceived law of motion [5.20] does not have a constant and uses higher-order moments to track low-frequency movements in the unconditional mean. This latter point provides key intuition for why random-walk beliefs, in models with strong expectational feedback, can be *expected* to arise under learning.

Section 5.3 showed that the iid rational expectations equilibrium is E-stable. It is natural to ask, what kind of real-time learning dynamic could generate nearly self-fulfilling random-walk beliefs? The answer is a variant of recursive least-squares called “constant gain learning”.

Suppose that agents have an AR(1) perceived law of motion:

$$y_t = a + by_{t-1} + \epsilon_t \Rightarrow \hat{E}_t y_{t+1} = a(1 + b) + b^2 y_{t-1} \quad [5.22]$$

Note that this PLM nests both the rational expectations equilibrium and random-walk beliefs. Let $\theta_t = (a_t, b_t)'$, $x_t = (1, y_{t-1})'$ and R_t be the sample

5. The Wold decomposition rewrites an AR(1) as an $MA(\infty)$. Take $z_t = \rho z_{t-1} + \varepsilon_t = \rho(\rho z_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$, etc. Then, $z_t = \sum_{j=0}^{\infty} \rho^j z_{t-j} = z_t [1 + \rho L + \rho L^2 + \dots] = (1 - \rho L)^{-1} z_t$.

estimate of the unconditional covariance matrix $Ex_t x_t'$. Then, recursive estimates of a_t, b_t are updated via the stochastic recursive algorithm:

$$\theta_t = \theta_{t-1} + \phi_t R_t^{-1} x_t (y_t - \theta_{t-1}' x_t)$$

$$R_t = R_{t-1} + \phi_t (x_t x_t' - R_{t-1})$$

At $t = 0$, agents have priors over a_0, b_0 . They form expectations via [5.22], shocks occur and new data are determined by [5.1]. Then, θ_t, R_t are updated, and the process repeats. The variable ϕ_t is called a gain sequence. Recursive least-squares arises when $\phi_t = 1/t$. In this case, the learning estimates place equal weight on all data points. When $\phi_t = \phi$, it is called constant gain learning. With a constant gain, the learning algorithm places geometrically declining weights on past data. Constant gain learning is advisable when agents are concerned about structural change of an unknown form.

The E-stability principle says that the recursive least-squares estimates $\theta_t \rightarrow \theta^*$, the REE values for (a, b) , with probability 1 as $t \rightarrow \infty$. Under constant gain learning, however, the estimates θ_t do not settle down to the REE values. Because of the time-invariant gain, it turns out that θ_t can converge in distribution: for large t and as $\phi \rightarrow 0$, $\theta_t \sim N(\mu, \phi V)$. So, constant gain learning beliefs, for large samples, are distributed around the rational expectations equilibrium with a variance proportional to the constant gain ϕ .

Even more insightful is a result in Evans and Honkapohja (2001), which as $\phi \rightarrow 0$ and t is large, the random learning path $\theta(\tau)$ converges weakly to the solution $\bar{\theta}(\tau, \theta_0)$ – for any θ_0 in a neighborhood of the rational expectations equilibrium – to the mean-dynamics:

$$\dot{\theta} = R^{-1} M(\theta) (T(\theta) - \theta)$$

$$\dot{R} = M(\theta) - R$$

and where τ maps the discrete time sequence for θ_t into a continuous time path $\theta(\tau)$. The mean-dynamics o.d.e. are found via a continuous-time interpolation of the recursive least-squares algorithm and appealing to a law of large numbers. The result tells us that the solution to the mean-dynamics o.d.e. delivers the *expected path* for the real-time learning estimates θ_t following a sequence of shocks that drive learning to θ_0 . Those sequences of shocks that “initialize” the mean dynamics are called escape dynamics; see Williams (2019).

A numerical example demonstrates the possibility for learning to be drawn towards random-walk beliefs. Set $\beta = 0.95, \alpha = 0.1, \sigma_z = 0.1$. For $\theta_0 \approx (2, 0)$, E-stability tells us that learning will converge to the rational expectations equilibrium $\theta^* = (2, 0)$. It is during the *transition path* that random-walk beliefs can arise. Suppose that $\theta_0 = (2, 0.5)$. Figure 5.4 shows the mean dynamic paths for $a(\tau), b(\tau)$. Note that eventually,

learning converges to the rational expectations equilibrium. The transition path, however, features $(a, b) \approx (0, 1)$ for a finite stretch of time: these are random-walk beliefs as in [5.20]. Eventually, the E-stability of the rational expectations equilibrium restores beliefs to the rational expectations equilibrium values.

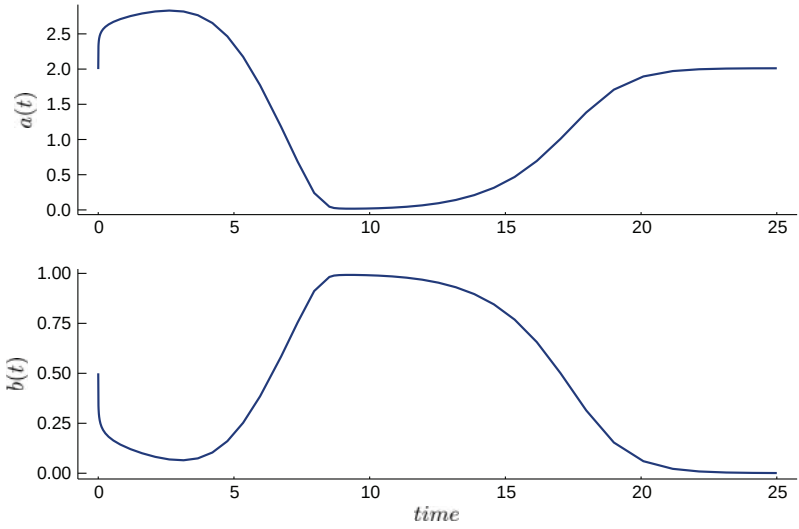


Figure 5.4. Mean dynamics: random-walk beliefs

What drives beliefs towards random-walk beliefs? It is the same intuition described above that random-walk beliefs can introduce self-fulfilling serial correlation and approximate low-frequency drift. Imagine a positive sequence of shocks to z_t . The agents' econometric model detects the serial correlation when regressing y_t on a constant and a lag. The self-referential nature of the model [5.1] then induces more serial correlation, which the agents' model again detects. The random-walk beliefs are nearly self-confirming. Eventually, a new sequence of shocks counters these beliefs, and agents learn the actual process again. This type of sequence of shocks – Sargent calls it the “most likely unlikely” sequence – triggers an escape from the rational expectations equilibrium. Then, the mean dynamics show a temporary form of restricted perceptions.

Random-walk beliefs that arise endogenously have novel applications that arise by creating additional persistence and volatility:

1) Branch and Evans (2011a) incorporated adaptive learning into a mean-variance asset-pricing model that takes the form of [5.1]. The shock z_t is a shock to share supply, i.e. asset float. In that framework, the right sequence of shocks to share supply

– say because of lock-up expirations – led to a constant gain learning process drawn towards random-walk beliefs. At that moment, agents mistakenly perceive all stock price innovations as permanent, increasing their demand and increasing prices further. The result is self-fulfilling bubbles or crashes in stock prices.

2) Branch and Evans (2017) studied constant gain learning in various New Keynesian monetary models with non-zero long-run inflation targets. If agents imperfectly understand the central bank's inflation target, then to forecast inflation, the agents need estimates of both the conditional mean and persistence of inflation. That gives rise to the possibility of endogenous random-walk learning dynamics. The key result in that study is that higher long-run inflation targets increase the likelihood of random-walk beliefs. Thus, increasing the target can trigger random-walk beliefs and an inflation scare. Once at the higher target, the emergence of random-walk beliefs can lead to an inflation scare or even a collapse to the zero lower bound. Lower values for the inflation target produce stable learning dynamics.

5.6. Literature review and conclusion

Early contributions to the restricted perceptions approach include Marcet and Sargent (1989), Evans et al. (1993) and Marcet and Sargent (1995). The papers Marcet and Sargent (1989, 1995) explore hidden variables and their implications for expectations. Evans et al. (1993) assume agents mistakenly fit an AR(1) regression model to data in an economy with complex, deterministic dynamics.

Evans and Honkapohja (2001) offered the first example of an under-forecasting model in a cobweb model and a forward-looking model such as [5.1] that includes a lag. The E-stability properties of the latter are different with restricted perceptions than with rational expectations. Branch and Evans (2006) extended the under-parameterization framework to allow the agents to select their model optimally in a random-utility setting, generalizing Brock and Hommes (1997) to a stochastic environment. The key result of that paper is that heterogeneous expectations can arise endogenously. Multiple equilibria can arise in a monetary model, as shown by Branch and Evans (2007), similar to the Ricardian beliefs example discussed here, and can generate time-varying inflation volatility. Similarly, Branch and Evans (2010) find that endogenous choice of under-parameterized models generates empirically realistic shifts in the means and variances of stock returns. Other approaches to predictor selection appear in Markiewicz (2012) and Cho and Kasa (2015).

The restricted perceptions approach sometimes goes under alternative names. For instance, the consistent expectations equilibrium of Hommes and Sorger (1998) asks forecasts to be consistent with a finite number of sample autocorrelations. Hommes and Zhu (2014) describe a first-order consistent expectations equilibrium that they call a behavioral learning equilibrium. Finally, Cho and Kasa (2015) have an under-parameterization example that they label a self-confirming equilibrium. The

restricted perceptions approach is encompassing, and these alternative definitions are refinements.

Recent work incorporates restricted perceptions into empirically realistic business cycle and monetary models. The key distinction of these classes of models is the assumption of infinitely lived agents that solve complicated intertemporal optimization problems. Evans et al. (2022) present a general existence result for restricted perceptions equilibria when they make specific bounded optimality assumptions. Branch and McGough (2011) find that restricted perceptions and heterogeneous beliefs can amplify volatility in a real business cycle model. Branch and Evans (2011b) find hysteresis effects in a New Keynesian model in which agents choose from among under-parameterized models.

Besides the nonlinear models referenced earlier, other papers have developed the restricted perceptions approach in nonlinear models. Evans and McGough (2020a) study learning with linear under-parameterized forecasts in a nonlinear cobweb model. Similarly, Shin (2022) develops an asset-pricing model with costly market participation and under-parameterized forecasting equations.

5.7. References

- Branch, W.A. and Evans, G.W. (2006). Intrinsic heterogeneity in expectation formation. *Journal of Economic Theory*, 127, 264–295.
- Branch, W.A. and Evans, G.W. (2007). Model uncertainty and endogenous volatility. *Review of Economic Dynamics*, 10, 207–237.
- Branch, W.A. and Evans, G.W. (2010). Asset return dynamics and learning. *Review of Financial Studies*, 23(4), 1651–1680.
- Branch, W.A. and Evans, G.W. (2011a). Learning about risk and return: A simple model of bubbles and crashes. *American Economic Journal: Macroeconomics*, 3(3), 159–191.
- Branch, W.A. and Evans, G.W. (2011b). Monetary policy and heterogeneous expectations. *Economic Theory*, 47(2–3), 395–421.
- Branch, W.A. and Evans, G.W. (2017). Unstable inflation targets. *Journal of Money, Credit and Banking*, 49(4), 767–806.
- Branch, W.A. and Gasteiger, E. (2022). Endogenously (non-)ricardian beliefs. Technical Report, TU Wien, Vienna.
- Branch, W.A. and McGough, B. (2005). Consistent expectations and misspecification in stochastic non-linear economies. *Journal of Economic Dynamics and Control*, 29, 659–676.
- Branch, W.A. and McGough, B. (2011). Business cycle amplification with heterogeneous expectations. *Economic Theory*, 47(2–3), 395–421.

- Branch, W.A., McGough, B., Zhu, M. (2022). Statistical sunspots. *Theoretical Economics*, 17(1), 291–329.
- Brock, W.A. and Hommes, C.H. (1997). A rational route to randomness. *Econometrica*, 65, 1059–1095.
- Cho, I.-K. and Kasa, K. (2015). Learning and model validation. *Review of Economic Studies*, 82, 45–82.
- De Long, B., Shleifer, A., Summers, L., Waldmann, R. (1990). Noise trader risk in financial markets. *Journal of Political Economy*, 98(4), 703–738.
- Eusepi, S. and Preston, B. (2018). Fiscal foundations of inflation: Imperfect knowledge. *American Economic Review*, 108(9), 2551–2589.
- Evans, G.W. and Guesnerie, R. (2003). Coordination on saddle path solutions: The eductive viewpoint – linear univariate models. *Macroeconomic Dynamics*, 7, 42–62.
- Evans, G.W. and Guesnerie, R. (2005). Coordination on saddle path solutions: The eductive viewpoint – linear multivariate models. *Journal of Economic Theory*, 124(2), 202–229.
- Evans, G.W. and Honkapohja, S. (2001). *Learning and Expectations in Macroeconomics*. Princeton University Press. Princeton, NJ.
- Evans, G.W. and McGough, B. (2020a). Equilibrium stability in a nonlinear cobweb model. *Economics Letters*, 193, 109130.
- Evans, G.W. and McGough, B. (2020b). Stable near-rational sunspot equilibria. *Journal of Economic Theory*, 186, 104987.
- Evans, G.W., Honkapohja, S., Sargent, T.J. (1993). On the preservation of deterministic cycles when some agents perceive them to be random fluctuations. *Journal of Economic Dynamics and Control*, 17, 705–721.
- Evans, G.W., Honkapohja, S., Mitra, K. (2009). Anticipated fiscal policy and learning. *Journal of Monetary Economics*, 56, 930–953.
- Evans, D., Evans, G.W., McGough, B. (2022). The RPEs of RBCs and other DSGEs. *Journal of Economic Dynamics and Control*, forthcoming.
- Grandmont, J.-M. (1977). Temporary general equilibrium theory. *Econometrica*, 45(3), 535–572.
- Grandmont, J.-M. (1985). On endogenous competitive business cycles. *Econometrica*, 53, 995–1045.
- Guesnerie, R. (1992). An exploration of the eductive justifications of the rational-expectations hypothesis. *American Economic Review*, 82, 1254–1278.
- Hommes, C.H. and Sorger, G. (1998). Consistent expectations equilibria. *Macroeconomic Dynamics*, 2, 287–321.
- Hommes, C.H. and Zhu, M. (2014). Behavioral learning equilibria. *Journal of Economic Theory*, 150, 778–814.

- Hommes, C.H., Sorger, G., Wagener, F. (2013). Consistency of linear forecasts in a nonlinear stochastic economy. In *Global analysis of Dynamic Models in Economics and Finance: Essays in Honour of Laura Gardini*, Bischi, G., Chiarella, C., Sushko, I. (eds). Springer, Berlin.
- Marcet, A. and Sargent, T.J. (1989). Convergence of least-squares learning in environments with hidden state variables and private information. *Journal of Political Economy*, 97, 1306–1322.
- Marcet, A. and Sargent, T.J. (1995). *Speed of Convergence of Recursive Least Squares: Learning with Autoregressive Moving-Average Perceptions*, Basil Blackwell, Oxford, 179–215.
- Markiewicz, A. (2012). Model uncertainty and exchange rate volatility. *International Economic Review*, 53(3), 815–844.
- Sargent, T.J. (1993). *Bounded Rationality in Macroeconomics*. Oxford University Press, Oxford.
- Sargent, T.J. (1999). *The Conquest of American Inflation*. Princeton University Press, Princeton NJ.
- Shin, M. (2022). Endogenous participation, risk, and learning in the stock market. *Macroeconomic Dynamics*, 26(3), 649–681.
- White, H. (1994). *Estimation, Inference and Specification Analysis*. Econometric Society Monograph. Cambridge University Press, Cambridge.
- Williams, N. (2019). Escape dynamics in learning models. *Review of Economic Studies*, 86(2), 882–912.
- Woodford, M. (2013). Macroeconomic analysis without the rational expectations hypothesis. *Annual Review of Economics*, 5, 303–346.

6

Welfarist Social Justice and Distributional Analysis

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The chapter provides an introduction to the theory of inequality measurement and distributional analysis in connection with welfarist theories of social justice. The chapter then presents the main methodologies for comparing distributions of a single observable attribute – for example, income – on the basis of these theories of justices. The presentation focuses mainly, but not exclusively, on dominance approaches that have the merit of being agreed upon by a large spectrum of ethical points of view.

“Justice is the first virtue of social institutions, as truth is of systems of thought. A theory however elegant and economical must be rejected if it is untrue; likewise, laws and institutions no matter how efficient and well-arranged must be reformed and abolished if they are unjust” (John Rawls, *A theory of Justice*, 1971)

“The importance of the formal results lies ultimately in their relevance to normal communication and to things that people argue about and fight for”

(Amartya K. Sen, *On Economic Inequality*, 1973).

6.1. Introduction

In an interview at the BBC in September 2022, the (very short-serving) British prime minister Liz Truss said: “To look at everything from the light of distribution is

Macroeconomics 1,

coordinated by Rodolphe DOS SANTOS FERREIRA and Thomas SEEGMULLER.

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wrong; what I'm about is growing the economy." This strong preference for "growing the economy" over distributing its benefits among its beneficiaries is a value judgment, just like the opposite preference for "distribution" over "growth". These two value judgments obviously ride on the premise that there could be a conflict between the two objectives, a premise that is not a value judgment but a positive statement about the causal effect of "growth" on "distribution" (or the other way around). The two value judgments are also based on the postulate that growth and distribution are both worth-pursuing objectives, which is also a value judgment that has been recently questioned by the environmentalist so-called "degrowth" movement. Any value judgment rides – implicitly or explicitly – on a theory of justice or, as indicated by John Rawls' quote above, a theory of the "good" arrangement of social institutions, and, more generally, human life in society.

This chapter provides an introduction to the economic theory of value judgments, sometimes called "normative economics". In some sense, normative economics can be viewed as the economists' theory of justice. This theory, which emerged in the second half of the 20th century through the seminal – but somewhat nihilistic – work of Kenneth Arrow (1950) (see also the book Arrow (1963)), and developed – more constructively – through the work of Amartya Sen (1970) (see also Sen 1973; Sen 1986; and Sen 1992) and many others, rigorously examines the criteria by which we can compare alternative *social states* from an "ethical goodness" view point. The term "social state" is interpreted, in this approach, as a complete description of all the relevant features of a situation involving a collection of individuals. In addition to providing criteria for comparing social states, normative economics is interested in the implementation of those criteria on empirical data. The focus of normative economics on ranking social states already indicates an important departure from the perspective on justice provided by Rawls (1971) quote above. In the quote, Rawls compares justice – as a virtue of social institutions – with truth – as a virtue of systems of thoughts. This comparison suggests that Rawls views justice as a binary 0-1 affair, just like truth is commonly believed to be. Economists do not view justice in such a binary way. Justice is a matter of degree. A social state is more, or less, just than another. But a social state is rarely totally just or, for that matter, unjust.¹

The chapter focuses on the *welfarist* approach to normative evaluation which, despite some challenges in the last 30 years, has remained the leading approach to the matter. Welfarism views the welfare or well-being achieved by individuals in the different social states as a necessary and sufficient information for comparing those states. Put differently, two social states that differ in many ways but that provide every individual with the same level of well-being, are indistinguishable from a welfarist perspective. Hence, ranking social states amounts to ranking the lists

1. Further arguments in favor of thinking in a comparative rather than a binary way are provided in the Introduction of Broome (1999).

of welfare levels achieved by individuals in those states. While theoretically clear, this welfarist approach is somewhat abstract. It is not easy – if even possible – to observe individual welfare. How can we formulate concrete value judgments such as Liz Truss’s preference for growth over redistribution on the basis of a magnitude as difficultly observable as individual welfare?

This chapter examines the answer given to this question by the theory of inequality measurement and, more generally, distributional analysis. As its name indicates, distributional analysis is concerned with comparing distributions of specific empirically observable attributes among agents, which could be individuals (as will often be assumed here), households or even larger communities. The best known attributes considered in distributional analysis are of a *pecuniary* nature. Examples of such pecuniary attribute include consumption spending (amount of money spent per time period), income (amount of money earned by time period) and wealth (the market value of all owned assets). Liz Truss, for example, was clearly referring to the British national income and its distribution among British individuals when she was expressing her strong preference for growth over distribution. Income is a (relatively) easily observable attribute that is related, but not equivalent, to individual welfare. Income is related to individual welfare because in a market economy, income is spent freely by people over goods and services that provide welfare. Hence, welfare can be considered to be a function of income. This functional link, combined with reasonable assumptions about the properties of the linking function, enables the conversion of abstract theoretical comparisons of distributions of welfare into concrete empirical comparisons of distributions of income.

When focusing on distributions of only *one observable attribute* such as income, there are *two* features that enter into consideration. One, highly valued by Liz Truss, concerns the “size of the cake” and the other concerns the “distribution of the cake”. While each feature in isolation can be defined somewhat precisely – as shown in this chapter – a difficulty of distributional analysis concerns policies that affect the two features in different directions, for example, by increasing the size of the cake while reducing the equality of its distribution. The advantage of comparing observable income distributions through theoretical welfarist lenses is to enable the trade-off between the two features – growth and distribution – to be expressed in a clear ethical way.

Focusing on distributions of only one observable attribute – be it as important as income – is limiting, however. While income is converted into many goods and services that contribute to welfare, it does not buy everything. Besides income, individuals differ by education, health, gender, religion, caste, declared happiness (not to be confused with ethically relevant welfare), etc. Even when interested in income only, a monthly income of \$4,000 earned by an adult couple is hardly comparable to a monthly income of \$3,500 earned by a single adult. In the last 30 years or so, distributional analysis has therefore increasingly focused on the

comparisons of distributions of *multiple observable attributes*. Enlarging the number of attributes significantly raises the complexity of the analysis by enlarging the number of features of the distributions that can be traded-off against others. Indeed, with several attributes, we need to trade-off concerns for the size and the distribution of each attribute. But we need to also handle concerns for the extent of correlation between the different attributes. Is it the case that those who earn the low incomes tend to also be those who have poor health, or who come from discriminated castes? While an ethical case can be made *against* correlation of attributes and cumulation of handicaps/advantages, the plausibility of this case depends a great deal upon the considered attributes, and their assumed impact on individual welfare. The issue becomes even more complex when we consider more than two attributes. In that case, we must also be concerned with how the correlation between two attributes is affected by a third attribute, etc. For consideration of space, this chapter will not cover distributions of more than one observable attribute.

When performing normative or distributional analysis, it is important to be clear about the measurability and the interpersonal comparability of the relevant features of the social states. In the welfarist paradigm, for example, is the normatively relevant individual welfare comparable across individuals? Is it a cardinally meaningful magnitude, such as temperature, or a mere ordinal variable that only conveys information on its level, but not on its difference? Similar questions can be addressed for more observable attributes such as income or health. While income is commonly considered to be cardinally meaningful and comparable across individuals, it is far from clear that the current measurement of health (say in a self-declared way) is sufficiently precise to generate a cardinal scale comparable across individuals. Perhaps not surprisingly, the criteria that we have at our disposal to compare social states depend crucially upon the available information about the relevant feature of those states. As shown by the famous Arrow's (1950) impossibility theorem, welfarism is an almost empty theory if individual welfare is not assumed to be interpersonally comparable. In the same vein, the very notion of preference (or distaste in the case of Liz Truss) for income equality is very difficult to define if income is not assumed to be cardinally measurable.

The remainder of this chapter is organized as follows. Section 6.2 recalls the welfarist underpinnings of distributional analysis and indicates some of their difficulties. Section 6.3 is devoted to the distributional analysis that involves one observable attribute.

6.2. Welfarist normative evaluation

6.2.1. Preliminaries

As mentioned above, normative evaluation is concerned with ranking the *social states* that could result from various policy interventions and that are of concern for

a community of individuals. In this section, we denote by N the finite set of $n \geq 2$ individuals who form the considered community and by $\mathbf{X}$ the set of all conceivable social states. While we do not make any assumption about the size of the set $\mathbf{X}$ – which could be finite or not – we will always assume that it contains at least three elements. We also observe that we treat the community N of individuals as given here. Yet, in many practical instances of policy or normative evaluation, we may want to compare different communities (say France and China) involving a different number of individuals. In the two last sections, we will see how we can adapt the theoretical apparatus of this section to comparisons of social states involving different communities of people.

Welfarism is a particular approach to normative evaluation for which the list of welfare levels achieved by individuals in the various social states is a sufficient information for ranking those states. At the very heart of welfarism lies, therefore, a definition of individual welfare. A well-known – at least since Epicurus² – and rather coherent definition of welfare is that of pleasure and happiness. This definition is at the heart of the well-known utilitarian tradition that originates from the work of Beccaria, Hume and, of course, Bentham who began his famous book *An Introduction to the Principles of Morals and Legislation* by writing:

“Nature has placed mankind under the governance of two sovereign masters, pain and pleasure.(...)They govern us in all we do, in all we say, in all we think: every effort we can make to throw off our subjection will serve but to demonstrate and confirm it”.(...) The *principle of utility* recognizes this subjection, and assumes it for the foundation of that system, the object of which is to rear the fabric of felicity by the hands of reason and of law. By the principle of utility is meant that principle which approves or disapproves of every action whatsoever, according to the tendency it appears to have to augment or diminish the happiness of the party whose interest is in question. If that party be the community in general, then the happiness of the community: if a particular individual, then the happiness of that individual.

But this happiness/pleasure definition of welfare – which describes it as a “mental state” – is by no means the only one. An alternative definition, favored to some

2. “For this reason we call pleasure the alpha and omega of a happy life. Pleasure is our first and kindred good. It is the starting-point of every choice and of every aversion, and to it we come back, inasmuch as we make feeling the rule by which to judge of every good thing. And since pleasure is our first and native good, for that reason we do not choose every pleasure whatever, but often pass over many pleasures when a greater annoyance ensues from them. And often we consider pains superior to pleasures when submission to the pains for a long time brings us as a consequence a greater pleasure” (Epicurus, Letter to Menoeceus).

extent by economists, and forcefully defended by modern philosophers such as Griffin (1986), defines well-being as the extent to which informed desire is satisfied. This chapter will not provide detailed interpretative accounts of what individual well-being really is (Griffin (1986) is a must-read by anyone interested in such matters). I will simply assume that every individual achieves some well-being (say happiness) in every state in which they are involved.

6.2.2. Measurability and comparability of individual well-being

Suppose that the well-being of agent i is measured by a function $U_i : \mathbf{X} \rightarrow \mathbb{R}$ that associates with every social state x , the “well-being” $U_i(x)$ experienced by i in x . How precise is the measurement of i ’s well-being provided by U_i ?

The lowest possible such precision is the *ordinal* one that underlies the *ranking* of social states provided by the numbers assigned to them by the function U_i . If U_i measures well-being in an ordinal fashion, then the *only* information that it conveys about the social states lies in their ranking. Hence, the number $U_i(x)$ is no smaller than the number $U_i(y)$ if and only if the level of well-being achieved by i in x is no smaller in x than in y . The imprecision of the information conveyed by an ordinal measure of well-being is reflected in the large number of *different* functions that convey the *same* information about the ranking of the states. If U_i is a function that measures i ’s well-being ordinally, and if $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function (of one variable), then the function $V_i : \mathbf{X} \rightarrow \mathbb{R}$ defined, for every state $x \in \mathbf{X}$, by $V_i(x) = \Phi(U_i(x))$ also provides a numerical measure of i ’s well-being that is just as legitimate as that provided by U_i . Indeed, the statement $U_i(x) \geq U_i(y)$ is true if and only if the statement $\Phi(U_i(x)) \geq \Phi(U_i(y))$ is true whatever is the increasing function Φ . Formally, we can thus define the relation of ordinal equivalence between a function U_i and a function V_i as follows.

DEFINITION 6.1.—A function $V_i : \mathbf{X} \rightarrow \mathbb{R}$ is *ordinally equivalent* to a function $U_i : \mathbf{X} \rightarrow \mathbb{R}$ if there is an increasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $V_i(x) = \Phi(U_i(x))$ for all x in $\mathbf{X}$.

The reader should note that this notion of ordinal equivalence is *symmetric* (V_i is ordinally equivalent to U_i if and only if U_i is ordinally equivalent to V_i), *transitive* and, of course, *reflexive*. Hence, it defines what is called in mathematics an *equivalence relation* among numerical functions. The lack of precision of ordinal measurement is thus reflected by the large size of the set of “ordinally equivalent” functions. This size is so large that it does not allow the formulation of comparative statements about *differences* in well-being. For example, a statement such as “the first beer increases individual i ’s well-being more than the second beer” cannot be formulated with an ordinal notion of well-being. Indeed, suppose that the social states x , y and z describe situations where i drinks, respectively, two beers, one beer and zero beer, everything

else being the same for everyone else. When expressed with the function U_i , the statement “the first beer increases i ’s well-being more than the second” writes:

$$U_i(x) - U_i(y) < U_i(y) - U_i(z)$$

or, equivalently:

$$\frac{U_i(x) + U_i(z)}{2} < U_i(y)$$

The truth value of such a statement is not preserved by all functions V_i ordinarily equivalent to U_i . Indeed, the fact of having:

$$\frac{U_i(x) + U_i(z)}{2} < U_i(y)$$

does not imply that:

$$\frac{\Phi(U_i(x)) + \Phi(U_i(z))}{2} < \Phi(U_i(y))$$

holds for any increasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$. For example,

$$\frac{1 + 4}{2} < 3$$

can be true while

$$\frac{1^3 + 4^3}{2} < 3^3$$

is not true.

In order for a numerical measure of i ’s well-being to allow for comparisons of well-being *differences* and *levels*, we need that statements of the form:

$$U_i(w) - U_i(x) \leq U_i(y) - U_i(z) \quad [6.1]$$

be preserved by all functions U_i that provide an equivalent measurement of well-being. The set of such functions would need to be smaller than the set of all functions that can be obtained from each other by means of an increasing function. It is not hard to see that inequality [6.1] would be preserved if instead of measuring i ’s well-being with the function U_i , we would use the function $V_i : \mathbf{X} \rightarrow \mathbb{R}$ defined, for any x in $\mathbf{X}$, by:

$$V_i(x) = a + bU_i(x)$$

for a real number a and a *strictly positive* real number b . A function $f : A \rightarrow \mathbb{R}$ (with $A \subset \mathbb{R}$) that writes $f(x) = a + bx$ for some real numbers a and b is said to be *affine*. An affine function is increasing in its argument if and only if b is strictly positive. Hence,

if V_i is another function that measures i 's well-being and which results from applying to U_i an increasing and affine transformation, then V_i will also satisfy inequality [6.1] since, for any strictly positive real number b :

$$\begin{aligned}
 U_i(w) - U_i(x) &\leq U_i(y) - U_i(z) \\
 &\iff \\
 b[U_i(w) - U_i(x)] &\leq b[U_i(y) - U_i(z)] \text{ (if } b > 0) \\
 &\iff \\
 [a + bU_i(w) - (a + bU_i(x))] &\leq [a + bU_i(y) - (a + bU_i(z))] \\
 &\iff \\
 V_i(w) - V_i(x) &\leq V_i(y) - V_i(z)
 \end{aligned}$$

regardless of the number a .

A measurement of well-being that enables the comparisons of well-being differences is said to be *cardinal*. The numerical functions that measure well-being in a cardinally equivalent way are thus all functions that result from applying an affine and increasing function from each to another. The formal definition of this notion of cardinal equivalence is as follows.

DEFINITION 6.2.—A function $V_i : \mathbf{X} \rightarrow \mathbb{R}$ is *cardinally equivalent* to a function $U_i : \mathbf{X} \rightarrow \mathbb{R}$ if there exists a real number a and a strictly positive real number b such that $V_i(x) = a + bU_i(x)$.

If two functions are cardinally equivalent, they are also ordinally equivalent, because an increasing affine function is an increasing function. The converse, however, does not hold. Cardinal measurement is more precise than ordinal measurement because the class of cardinally equivalent function is a proper subset of the class of ordinally equivalent functions.

A good example of a cardinally measured phenomenon is temperature. In order to measure temperature – at least since the discovery of the thermometer in the 17th century – we put a sealed tube of liquid (alcohol or mercury) within melting ice and, after a few minutes, indicates with a line the level reached by the liquid in the tube. We then put the same tube of liquid in boiling water and, after letting the liquid in the tube stabilize at the (higher) level associated with boiling water, we draw a new line at this new level. We then choose the two numbers associated with the freezing and boiling points of water under the sole constraint than the second number be larger than the first. We then provide a measurement of temperature by subdividing in some way the gap between these two numbers and by calling “degrees” the elementary such subdivisions. For example, in the Celsius scale, the numbers assigned to the freezing and boiling points of water are 0 and 100, respectively. In the Fahrenheit scale used in

the US, the numbers assigned to those two points are 32 and 212. Hence, if $V_i(x)$ is the temperature of x measured on the Fahrenheit scale, and $U_i(x)$ is the temperature of x measured on the Celsius scale, we have:

$$V_i(x) = 32 + \frac{9U_i(x)}{5}$$

Physics provides examples of phenomena measured even more precisely than temperature. Temperature measurement gives indeed two degrees of freedom. The choice of the “zero” (freezing point of water), and the choice of the water’s boiling point (whose difference from the freezing point, which can be arbitrarily subdivided in equally spaced sub-units, provides the “unit of measurement”). But certain physical magnitudes do not provide the freedom to choose the zero. The age of an object or a living individual, for example, is measured in the number of time units (seconds, minutes, months, years) since the birth of the individual. There is no freedom here to choose the zero, albeit there is freedom to choose the unit. For example, if $V_i(x)$ is the age of x in months, and $U_i(x)$ is the age of x in years, then $V_i(x) = 12U_i(x)$ is the age of x in months.

The age of a living individual is an example of a phenomenon whose numerical measurement is made through a *ratio scale*. This terminology arises from the fact that this measurement enables the formulation of statements about ratios (e.g. the age of Bob is twice that of Alice) that cannot be performed with only a cardinal scale (one cannot meaningfully say that the temperature of this water is half the temperature of this coffee). The formal notion of ratio scale equivalence is as follows.

DEFINITION 6.3.— *A function $V_i : \mathbf{X} \rightarrow \mathbb{R}$ is the ratio-scale equivalent to a function $U_i : \mathbf{X} \rightarrow \mathbb{R}$ if there is a strictly positive real number b such that $V_i(x) = bU_i(x)$.*

As is clear, the ratio-scale equivalence of two functions implies their cardinal equivalence but not the other way around. Hence, ratio-scale measurement of well-being is more precise than its cardinal measurement which is itself more precise than its ordinal measurement.

What kind of measurement of individual well-being can we dispose of in practice? Ordinal measurement is somewhat easy to obtain, notably by running a questionnaire asking people to rank the social states from their point of view. We can also adopt a “revealed preferences” approach *à la* Afriat (1967), Samuelson (1948), Houthakker (1950) or Varian (2006) (among many other references) and obtain information about the choices made by the individual in various circumstances. Under the (testable) assumption that these choices are compatible with the maximization of a complete, reflexive and transitive ranking of the objects of choices, and the (non-testable) assumption that this ranking is connected to the individual’s well-being, we can recover the ranking revealed by the choices, and use this ranking to assign numbers to the social states in a way consistent with the ranking. Any system of numbers thus constructed provides an ordinal measurement of the individual’s well-being.

It is admittedly more difficult to obtain a cardinal measurement of subjective individual well-being. How could we obtain, from empirical data, a numerical measurement of well-being difference that would correspond to, say, the difference of height of a column of liquid in a tube. Some approaches by questionnaire survey, somewhat popular nowadays within “happiness economics” (Clark et al. 2008) proceed by asking sampled individuals to locate their level of subjective happiness on a scale ranging from, say, 1 (unhappy) to 4 (very happy). The fact of providing the subjects with the scale suggests that the difference between 1 and 2 is the same than that between 3 and 4. Yet, there is no evidence that the individuals who agree to locate themselves on such a scale also perceives the difference between 1 and 2 to be the same – or even comparable – to the difference between 3 and 4.

However, the practical difficulties that we may encounter in measuring cardinally individual well-being in specific circumstances do not imply that such cardinal measurement is impossible or inconceivable. We can conceptually conceive to cardinally measure a phenomenon even if we do not have a widely accepted measuring device available for doing so. After all, the humans have spoken and discussed about temperature and its variation long before having invented the thermometer. It is not therefore unreasonable to believe that individuals do experience more or less “pleasant” or “happy” sensations in the various circumstances of their life and that these sensations could be the object of cardinal measurement. For example, the impression that the first beer that we drink on a particular Friday night increase more the well-being than the second seems to correspond rather well to a subjective reality experienced by many of us!

Beside well-being, other numerical indicators of the individuals’ situations are “naturally” measured in a cardinal or even a more precise fashion. Income, for example – measured in a *numéraire* with respect to which the prices of the goods that this income can purchase – is a numerical indicator that has a ratio-scale significance. An income of zero has a meaning, as does a difference of 100 euros, or a doubling of our income. Other indicators of health – for example, the body mass index, the rate of cholesterol or the level of blood pressure – are also often interpreted cardinally, at least by physicians.

6.2.3. Formulating welfarist criteria by means of social welfare functionals

The basic ingredient of welfarism is, for every individual $i \in N$, a (utility) function $U_i : \mathbf{X} \rightarrow \mathbb{R}$ that assigns to every state x the well-being level $U_i(x)$ obtained by i in x . A *normative criterion* is a binary relation $\succsim$ on $\mathbf{X}$ that compares social states on the basis of their “ethical goodness”. The statement $x \succsim y$ is accordingly interpreted as saying that state x is at least as good socially as state y . For any binary relation $\succsim$ on $\mathbf{X}$, we denote by $\succ$ its asymmetric factor (“strictly better than”) defined by

$x \succ y$ if and only if $x \succsim y$ holds but $y \succsim x$ does not and by $\sim$ its symmetric factor (“equivalent to”) defined by $x \sim y$ if and only if both $x \succsim y$ and $y \succsim x$ holds. A binary relation is *reflexive* if it satisfies $x \succsim x$ for every state x in $\mathbf{X}$, is *complete* if for any two distinct social states x and y , the fact of not observing $x \succsim y$ implies $y \succsim x$ (and therefore $y \succ x$) and is transitive if for any three social states x , y and z , the statement $x \succsim z$ always follows the statements $x \succsim y$ and $y \succsim z$. A reflexive and transitive binary relation is called a *quasi-ordering* and a reflexive, transitive and complete binary relation is called an *ordering*.

Given the list $U_1, \dots, U_n$ of real valued utility functions having $\mathbf{X}$ as a domain that measure individuals’ well-beings, a normative criterion is said to be welfarist if there exists a binary relation $\succeq^U$ on $\mathbb{R}^n$ such that $x \succsim y$ if and only if $(U_1(x), \dots, U_n(x)) \succeq^U (U_1(y), \dots, U_n(y))$. This is just a formal way to say that welfarism compares social states *only* on the basis of the distribution of well-being levels achieved by the individuals in those states.

There are many conceivable welfarist criteria for comparing social states. Probably, the best known of them is the *Utilitarian* criterion promoted by Bentham more than 200 years ago that compares states on the basis of the sum of the well-being levels achieved by individuals in those states (the larger the sum, the higher the state). This Utilitarian criterion – denoted $\succsim^{UTIL}$ – is therefore defined by:

$$x \succsim^{UTIL} y \Leftrightarrow \sum_{i \in N} U_i(x) \geq \sum_{i \in N} U_i(y) \quad [6.2]$$

The Utilitarian criterion aggregates individuals’ well-beings by summing them. As a result, this criterion does not care about how the sum of well-beings is distributed among the individuals, and would consider as equivalent a state where a given level of total happiness is attributed to a single individual, the others receiving nothing and a state where this happiness is equally split among all. Yet, the Utilitarian criterion is only one among many conceivable ways of ranking distributions of well-being. Another often used criterion for this purpose focuses exclusively on the well-being of the least well-off person in the society (the higher this well-being, the better the state). This *Maximin* criterion is sometimes called Rawlsian because of its formal resemblance to the difference principle formulated by John Rawls in his famous book *A theory of Justice*.³ Formally, the Maximin criterion is defined by:

$$x \succsim^{Maxi \min} y \Leftrightarrow \min[U_1(x), \dots, U_n(x)] \geq \min[U_1(y), \dots, U_n(y)] \quad [6.3]$$

In the case where the utility functions that measure individuals’ well-beings are positively valued – they need not be – the Utilitarian and Maximin criteria can be

3. The reference to Rawls is, however, abusive because Rawls developed his theory of justice – and the difference principle – to a large extent against welfarism. Hence, Rawls considered that his difference principle should apply to some index of “primary goods”, and not to well-being.

viewed as special members of a larger family of welfarist criteria for comparing social states: the *mean-of-order r* family, for some real number $r \leq 1$. This family of criteria – denoted $\succsim^{M(r)}$ for any given r – is defined by:

$$x \succsim^{M(r)} y \iff \left[\sum_{i=1}^n U_i(x)^r \right]^{1/r} \geq \left[\sum_{i=1}^n U_i(y)^r \right]^{1/r} \quad [6.4]$$

if $r \neq 0$ and by:

$$x \succsim^{M(0)} y \iff \sum_{i=1}^n \ln U_i(x) \geq \sum_{i=1}^n \ln U_i(y) \quad [6.5]$$

otherwise. Utilitarianism obviously corresponds to the case where $r = 1$. We can show that the mean-of-order- r criterion gets as close as we wish to the Maximin criterion by taking a sufficiently negative r . We can therefore interpret r as measuring the love for well-being inequality (the higher the r , the greater the love).

All mean-of-order r criteria are positively sensitive to individual welfare: strictly increasing the well-being of everyone is considered to be strictly good, and weakly increasing the well-being of everyone is never considered to be harmful. It is of course possible to think of criteria that would be negatively sensitive to individual welfare, and that would consider favorably policies that hurt some people without bringing any benefit to others. The positive sensitivity of the normative criteria to individuals' well-being is very commonly expressed by the requirement that they be compatible with the Pareto⁴ criterion, $\succsim^{PAR}$, defined by:

$$x \succsim^{PAR} y \iff U_i(x) \geq U_i(y) \text{ for all } i \quad [6.6]$$

While any mean of order r criterion is capable of ranking any two social states – they are complete – this is not so for the Pareto criterion that cannot compare social states between which two (or more) individuals have conflicting welfare interest. However, when individual welfares are all (weakly or strictly) increased in a move from one social state to another – what economists call a Pareto improvement – then the move is considered worth doing by the Pareto criterion. Economists are so fond of this criterion that they have made it the ethical pillar of their definition of efficiency. Economists call indeed efficient any state that is not Pareto-dominated by any other in the set of feasible social states.

Any mean-of-order r criterion provides a reflexive, complete and transitive ranking of social states. The Pareto criterion provides a reflexive and transitive but not complete ranking of those states. There are other welfarist criteria around that are reflexive and complete but that may not consistently (transitively) rank all social states.

4. After the Italian sociologist and economist Vilfredo Pareto (1841–1923).

One of them is the widely used *majoritarian criterion* that compares social states on the basis of the majority of individuals who obtain a larger utility in one state than in another. The formal definition of the majoritarian criterion, denoted $\succsim^{MAJ}$ is as follows:

$$x \succsim^{MAJ} y \iff \#\{i : U_i(x) \geq U_i(y)\} \geq \#\{i : U_i(y) \geq U_i(x)\} \quad [6.7]$$

This criterion enables the comparison of any two social states (possibly by considering them equivalent). However, as shown by Condorcet more than 200 years ago, the majoritarian criterion does not always provide a transitive ranking of social states. An illustration of this “Condorcet paradox” is provided in the following example involving three social states x , y and z and three individuals whose well-being obtained in the three states are as in the following table:

social state	1's well-being	2's well-being	3's well-being
x	5	4	3
y	2	2	6
z	1	7	4

Table 6.1. Illustration of the Condorcet Paradox

As can be seen, $x \succsim^{MAJ} y$ because there are two individuals (1 and 2) who obtain a larger well-being in x than in y and only one individual (3) who is happier in state y than in state x . Similarly, we can see that $y \succsim^{MAJ} z$ because two individuals (1 and 3) are happier in y than in z , while only individual 2 prefers z to y . If the majoritarian criterion were consistent (transitive), we would expect that x be socially better than z . Yet, a majority of people (2 and 3) are happier in z than in x .

Obtaining “reasonable” welfarist criteria for comparing social states is not so easy. One of the main lessons that can be derived from the famous Arrow’s (1950) impossibility theorem, formulated precisely in the seventies through the work of d’Aspremont and Gevers (1977), Hammond (1976), Strasnick (1975) and Sen (1977), is that unless individual well-being levels and/or differences can be interpersonally compared, there are indeed no such “reasonable” welfarist criteria. Moreover, even if we accept the possibility of comparing well-being interpersonally, the number of such “reasonable” welfarist criteria is rather small.

In the classical social choice approach, the formulation of a welfarist criterion is assimilated to the problem of constructing a *social welfare functional*, defined to be a function from a set of admissible profiles of utility functions (one such function for every individual) into the set of binary relations over $\mathbf{X}$. Let $\mathcal{U}$ be the set of all functions from $\mathbf{X}$ to $\mathbb{R}$, and let $\mathbb{D}^U \subset \mathcal{U}^n$ be a domain of admissible profiles of utility functions. Let also $\mathcal{B}$ denote the set of all binary relations on $\mathbf{X}$ and $\mathcal{O} \subset \mathcal{B}$ denote the subset of those binary relations that are orderings. A social welfare functional

$W : \mathbb{D}^U \longrightarrow \mathcal{B}$ maps any profile of utility functions $(U_1, \dots, U_n)$ in the domain into a social ranking $\succsim = W(U_1, \dots, U_n)$ of $\mathbf{X}$. We say that W is a functional – rather than a function – because its arguments are themselves (utility) functions. The Utilitarian, Maximin, Mean of order r , Pareto and Majoritarian criteria defined, respectively, by [6.2], [6.3], [6.4], [6.5], [6.6] and [6.7] are all examples of social welfare functionals. Observe that the Pareto and the Majoritarian criteria are binary relations, but not orderings, on $\mathbf{X}$.

When defining a social welfare functional, it is important to be precise about the measurability of the individual's well-being that the functional supposes. Is the functional assuming, for example, that individuals' well-being is cardinally measured? Or does it simply assume that it is ordinally measured? More importantly – as it will turn out – is this well-being assumed to be comparable among different individuals? Can we say meaningfully that the well-being of Bob is larger (say) than that of Ann? Or can we say that Bob has been hurt more severely by the policy than Ann has?

A starting point, implicitly considered by Kenneth Arrow (1950) in his famous impossibility theorem and often assumed in standard economic theory, is to view individual well-being as ordinally measured and non-comparable between different individuals. The formal definition of this requirement is as follows.

DEFINITION 6.4. – *Individuals' well-beings are considered to be ordinally measurable and interpersonally non-comparable by the social welfare functional $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ if $W(U_1, \dots, U_n) = W(V_1, \dots, V_n)$ for any pair of profiles of utility functions $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ in $\mathbb{D}^U$ such that $U_i(x) = \Phi_i(V_i(x))$ for all $i \in N$ and all $x \in \mathbf{X}$ for some increasing function $\Phi_i : \mathbb{R} \longrightarrow \mathbb{R}$.*

In plain English, a social welfare functional considers individual well-being to be ordinally measurable and interpersonally non-comparable if the ranking of social states produced by the functional is *invariant* to the replacements of the individual utility functions by ordinally equivalent functions as per Definition 6.1. The logic of this definition should be clear. Replacing one individual utility function by another that is ordinally equivalent to it should not be seen as changing anything substantial in the individual's well-being. Hence, the social ranking should not be affected by such a change. Since well-being is not interpersonally comparable, those replacements of individuals' utility functions by others that are ordinally equivalent can be done independently across individuals.

In the same spirit, but with the restriction of the invariance requirement to the replacement of utility functions by cardinally equivalent functions as per Definition 6.2, we can define in the following way the requirement for the social welfare functional to consider individual well being to be cardinally measurable and interpersonally non-comparable.

DEFINITION 6.5.— *Individuals' well-beings are considered to be cardinally measurable and interpersonally non-comparable by the functional $W : \mathbb{D}^U \rightarrow \mathcal{B}$ if $W(U_1, \dots, U_n) = W(V_1, \dots, V_n)$ for any pair of profiles of utility functions $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ in $\mathbb{D}^U$ such that $U_i(x) = a_i + b_i V_i(x)$ for all $i \in \{1, \dots, n\}$ and $x \in \mathbf{X}$, for some real numbers a_i and strictly positive real numbers b_i .*

We may, however, admit the possibility that the indicators of individuals' well-being be comparable across different individuals. For example, the body mass index is often used as an indicator of health and is typically believed to be comparable across individuals. If Bob has a body mass index of 42 and Ann has a body mass index of 22, then many doctors and health specialist would not hesitate in considering that Bob is worse off than Ann.

We can then define as follows the requirement that the social welfare functional consider well-being to be ordinally measurable and interpersonally comparable.

DEFINITION 6.6.— *Individuals' well-beings are considered to be ordinally measurable and interpersonally comparable by the social welfare functional $W : \mathbb{D}^U \rightarrow \mathcal{B}$ if $W(U_1, \dots, U_n) = W(V_1, \dots, V_n)$ for any pair of profiles of utility functions $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ in $\mathbb{D}^U$ such that $U_i(x) = \Phi(V_i(x))$ for all $i \in N$ and all $x \in \mathbf{X}$ for some increasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$.*

Individuals' well-beings are thus considered ordinally measurable and interpersonally comparable if the ranking of social states is unaffected when the same increasing transformation is applied to everyone's utility function. The limitation of this invariance requirement to changes in the numerical scale of well-being measurement that are common to all individuals precisely captures the interpersonal comparability of individual welfare.

Concerning cardinal measurability, which enables us to make meaningful comparative statements about changes in well-being, there are at least two ways by which we may define interpersonal comparability. One, partial, consists of allowing changes in individual well-being to be interpersonally comparable, while not necessarily requiring the levels of well-being to be. For example, we could be able to say that Ann has been hurt more badly by some policy than Bob, even though we may be reluctant in comparing the levels of well-being achieved by Bob and Ann before and after the policy. The other, complete, form of comparability of cardinal individual well-being amounts to requiring that both levels and changes in well-being be interpersonally comparable. These two notions of cardinal interpersonal comparability are defined as follows.

DEFINITION 6.7.— *Individuals' well-beings are considered cardinally measurable and unit comparable across individuals by the social welfare functional $W : \mathbb{D}^U$*

$\longrightarrow \mathcal{B}$ if $W(U_1, \dots, U_n) = W(V_1, \dots, V_n)$ for any pair of profiles of utility functions $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ in $\mathbb{D}^U$ for which $U_i(x) = a_i + bV_i(x)$ for all $i \in \{1, \dots, n\}$ and $x \in \mathbf{X}$, for some real numbers $a_1, \dots, a_n$ and some strictly positive real number b .

DEFINITION 6.8.— *Individuals' well-beings are considered cardinally measurable and fully comparable across individuals by the social welfare functional $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ if $W(U_1, \dots, U_n) = W(V_1, \dots, V_n)$. For any pair of profiles of utility functions $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ in $\mathbb{D}^U$ for which $U_i(x) = a + bV_i(x)$ for all $i \in \{1, \dots, n\}$ and $x \in \mathbf{X}$, for some real number a and strictly positive real number b .*

The last considered notion of measurability and interpersonal comparability of individual's well-being is the following one of *ratio-scale measurability* and *interpersonal comparability*. Hence, we do not consider the logical possibility of having ratio-scale measurability but interpersonal non-comparability.

DEFINITION 6.9.— *Individuals' well-beings are considered to be cardinally measurable and fully comparable across individuals by the social welfare functional $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ if $W(U_1, \dots, U_n) = W(V_1, \dots, V_n)$ for any pair of profiles of utility functions $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ in $\mathbb{D}^U$ for which $U_i(x) = bV_i(x)$ for all $i \in \{1, \dots, n\}$ and $x \in \mathbf{X}$, for strictly positive real number b .*

There are logical connections between these various notions of measurability and interpersonal comparability that are summarized in Figure 6.1. For sure ordinal measurability and interpersonal non-comparability implies cardinal measurability and interpersonal non-comparability. We also observe that for any given measurability – ordinal or cardinal as it might be – interpersonal non-comparability implies interpersonal comparability (unit or full for cardinal measurability). Similarly, cardinal measurability and interpersonal unit comparability implies cardinal measurability and interpersonal full comparability which implies itself ratio-scale measurability and interpersonal comparability. However, there are no implications that connect ordinal measurability and interpersonal comparability, on the one hand, and cardinal measurability and interpersonal unit comparability, on the other.

Observe that the Majoritarian social welfare functional defined by [6.7] satisfies the invariance requirement underlying ordinal and non-comparable measurability. Indeed:

$$\begin{aligned} x \succsim^{MAJ} y &\iff \#\{i : U_i(x) \geq U_i(y)\} \geq \#\{i : U_i(y) \geq U_i(x)\} \\ &\iff \\ \#\{i : \Phi_i(U_i(x)) \geq \Phi_i(U_i(y))\} &\geq \#\{i : \Phi_i(U_i(y)) \geq \Phi_i(U_i(x))\} \end{aligned}$$

for any increasing functions $\Phi_i : \mathbb{R} \rightarrow \mathbb{R}$ for $i = 1, \dots, n$. A similar observation can be made about the Pareto criterion defined by [6.6].

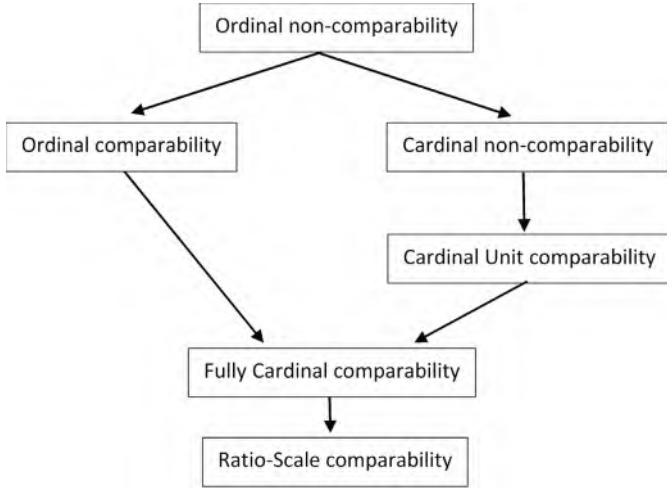


Figure 6.1. Logical implications between measurability and comparability notions

The Utilitarian social welfare functional defined by [6.2] relies on the cardinal measurability and unit comparability of individuals' well-being. Indeed:

$$\begin{aligned}
 x \succsim^{UTIL} y &\iff \sum_{i=1}^n U_i(x) \geq \sum_{i=1}^n U_i(y) \\
 &\iff \\
 \sum_{i=1}^n [a_i + bU_i(x)] &\geq \sum_{i=1}^n [a_i + bU_i(y)]
 \end{aligned}$$

for any strictly positive real number b and any list of n real numbers $(a_1, \dots, a_n)$. The reader may check that any other measurability assumption that is not implied – as per Figure 6.1 – by cardinal measurability and unit comparability is not satisfied by the Utilitarian functional.

The Maximin functional defined by [6.3] assumes an ordinal and comparable measurability of individuals' well-being since:

$$\begin{aligned}
 x \succsim^{MaxMin} y &\iff \text{Min}[U_1(x), \dots, U_n(x)] \geq \text{Min}[U_1(y), \dots, U_n(y)] \\
 &\iff \text{Min}[\Phi(U_1(x)), \dots, \Phi(U_n(x))] \geq \text{Min}[\Phi(U_1(y)), \dots, \Phi(U_n(y))]
 \end{aligned}$$

for every increasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$. Here again, we can check that any other form of measurability that is not implied by ordinal comparable measurability (notably

ordinal non-comparability and cardinal unit comparability) is not satisfied by the Maximin functional.

Finally, we can observe that the whole family of mean-of-order r social welfare functional supposes a ratio-scale comparability of individual well-being because if $r \neq 0$ (the case $r = 0$ is left to the reader):

$$\begin{aligned}
 x \succsim^{M(r)} y &\iff [\sum_{i=1}^n U_i(x)^r]^{1/r} \geq [\sum_{i=1}^n U_i(y)^r]^{1/r} \\
 &\iff \\
 b[\sum_{i=1}^n U_i(x)^r]^{1/r} &\geq b[\sum_{i=1}^n U_i(y)^r]^{1/r} \text{ (for any strictly positive real } b) \\
 &\iff \\
 [\sum_{i=1}^n (bU_i(x))^r]^{1/r} &\geq [\sum_{i=1}^n (bU_i(y))^r]^{1/r}
 \end{aligned}$$

Again, we can check that any invariance requirement that is stronger than that underlying ratio-scale measurability and interpersonal comparability is violated by a mean of order r (if we except of course the case $r = 1$).

6.2.4. What is a welfarist functional?

There are a great many ways by which we can produce normative criteria as a function(al) of the way by which individuals transform social states into well-being. Certainly not all of them are appealing from an ethical point of view, and certainly not all of them are welfarist.

Since the foundational work of Arrow (1963), it has become (reasonably) widely accepted that the following five properties are natural requirements to impose on a functional $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ that maps profiles of individual utility functions in the domain $\mathbb{D}^U$ into a normative ranking of social states.

The first property – referred to as “collective rationality” – requires the normative ranking of social states to be an ordering. Hence, it should lead to a transitive ranking of all social states, and should never leave two social states “non compared”. The formal statement of this requirement is as follows.

AXIOM 6.1 (COLLECTIVE RATIONALITY).– *For every profile $(U_1, \dots, U_n)$ of utility functions in the domain $\mathbb{D}^U$, $W(U_1, \dots, U_n) \in \mathcal{O}$.*

This requirement is indeed somewhat natural. It would be rather strange that the normative criterion used to compare social states exhibit logical cycles. Imagine a community of individuals being initially at some state z who decides, on the basis of such a criterion, to implement a policy leading them to y . Imagine then that this same community later recognizes that a further move from y to x would also be normatively worth doing as per the very same criterion. They would then get to x . Clearly, this community would find itself rather disappointed to end up, by a sequence of two moves that were considered good by the criterion, into a social state that the very same criterion finds worse than the initial state. The requirement of completeness is also somewhat appealing. Incomplete criteria make the choice between incomparable options undecidable. Yet, as will be seen in the next section, there are some interesting robust normative criteria that are incomplete. Hence, the requirement of completeness is probably less appealing than that of transitivity.

The next property requires the domain of the social welfare functional to be unrestricted and, as a result, that the functional produces a ranking of social states for any logically conceivable list of individual utility functions. The precise statement of this requirement is as follows.

AXIOM 6.2 (UNRESTRICTED DOMAIN).— $\mathbb{D}^U = \mathcal{U}^n$.

This requirement is undoubtedly strong, because it requires that a welfarist criterion be generated for every specification of the individuals' well beings. Yet, some of these specifications may be rather peculiar. They may, for instance, involve people deriving pleasure from the suffering of others or, less extremely, expressing concerns for matters that are "not of their business". Unrestricted domain also allows people to develop "expensive tastes" (see, for example, Cohen (1990)) and well-being functions that require large quantities of expensive yachts, cars, wines and caviar in order to achieve minimal well-being level. The unrestricted domain axiom requires indeed that a normative criterion be provided in all those cases that many may consider "outlandish" or "morally irrelevant". The difficulty of course is to identify a priori what is a "normal" or "morally acceptable" individual utility function.

Welfarism rides to a varying extent upon three variants of the Pareto – or the unanimity – principle. These three variants all say – with important different nuances – that when unanimity arises, it should be respected. These principles seem, therefore, hardly disputable within a theory that intends to define social welfare as an increasing function(al) of individuals' welfares. The three variants of the Pareto principle are the following.

AXIOM 6.3 (WEAK PARETO PRINCIPLE).— $\forall (U_1, \dots, U_n) \in \mathbb{D}^U, \forall x, y \in X, (U_i(x) > U_i(y) \forall i \in N) \implies x \succ y$ where $\succ$ is the asymmetric factor of the binary relation $\succsim = W(U_1, \dots, U_n)$.

AXIOM 6.4 (PARETO-INDIFFERENCE).— $\forall (U_1, \dots, U_n) \in \mathbb{D}^U, \forall x, y \in X, (U_i(x) = U_i(y) \forall i \in N) \implies x \sim y$ where $\sim$ is the symmetric factor of the binary relation $\succsim$ defined by $\succsim = W(U_1, \dots, U_n)$.

AXIOM 6.5 (STRONG PARETO PRINCIPLE).— $\forall (U_1, \dots, U_n) \in \mathbb{D}^U, \forall x, y \in X, (U_i(x) \geq U_i(y) \forall i \in N) \implies x \succsim y$ and $(U_i(x) \geq U_i(y) \forall i \in N$ and $U_h(x) > U_h(y)$ for at least one individual $h \in N \implies x \succ y$ where $\succsim = W(U_1, \dots, U_n)$.

The strong Pareto principle says that a policy that hurts no one should never be rejected and says also – somewhat strongly – that a policy that hurts no one and that benefits at least to some should be preferred to the status quo. The weak Pareto principle only says that a policy that strictly benefits everybody (without a single exception) should be adopted (at least over the status quo). Pareto-indifference says that the normative criterion should be indifferent between any two states that provide exactly the same level of welfare to everyone. This axiom is at the heart of welfarism as we will soon see. We could oppose to Pareto-indifference on the basis that if two states really are a matter of indifference to everybody, then we could use other criteria than well-being – for example, the “French tradition” – to rank them strictly. Pareto-indifference rules out that we do so. If two states are a matter of indifference to everyone, then they should be a matter of social indifference.

A non-controversial non-dictatorship condition is also imposed on the social welfare functional. After all, selecting one member of the society and taking their well-being as the standard by which to compare social state is a clear – if not appealing – possibility. Non-dictatorship rules that out.

AXIOM 6.6 (NON-DICTATORSHIP).— $\forall i \in N, \exists (U_1, \dots, U_n) \in \mathbb{D}^U$ and $x, y \in X$ such that $U_i(x) > U_i(y)$ and $y W(U_1, \dots, U_n) x$.

The last property, which also plays a key role in the welfarist approach to normative evaluation, is the following independence with respect to irrelevant alternatives. The axiom is formulated in the current social welfare functional theoretic context as follows.

AXIOM 6.7 (INDEPENDENCE WITH RESPECT TO IRRELEVANT ALTERNATIVES).— $\forall x, y \in X, \forall (U_1, \dots, U_n), (V_1, \dots, V_n) \in \mathbb{D}^U$ such that $U_i(x) = V_i(x)$ and $U_i(y) = V_i(y) \forall i \in \{1, \dots, n\}$, we must have $x W(U_1, \dots, U_n) y \Leftrightarrow x W(V_1, \dots, V_n) y$.

In plain English, a social welfare functional satisfies Independence with respect to irrelevant alternatives if the ranking of any two social states that it produces is unaffected when the numerical scales used to measure individuals’ well-being are changed in a way that does not modify the number assigned to the two states for every individual. Since a welfarist social welfare functional is supposed to depend only upon the well-being achieved by the individuals in the states, any change in the

measurement of well-being that leaves unchanged the level of well-being achieved by every one in two states should not affect the social ranking of those two states.

As it turns out, these axioms are very closely related to the welfarist approach discussed above. Specifically, we will now show that if a social welfare functional satisfies Axioms 6.1, 6.2, 6.4 and 6.7, then the only information it can use to compare any two social states is the distribution of well-beings achieved by the individuals in those states. Any other information about those states will thus be ignored by the social welfare functional. This important result – unproved for space consideration – is formally stated as follows.

LEMMA 6.1 (WELFARISM).— *If a Social Welfare Functional $W : \mathbb{D}^U \rightarrow \mathcal{B}$ satisfies Axioms 6.1, 6.2, 6.4 and 6.7, we can find an ordering $\succsim^U$ on $\mathbb{R}^n$ such that, for every profile of utility functions $(U_1, \dots, U_n)$ in the domain $\mathbb{D}^U$, we have, for any two social states x and y :*

$$x \ W(U_1, \dots, U_n) \ y \iff (U_1(x), \dots, U_n(x)) \succsim^U (U_1(y), \dots, U_n(y))$$

Welfarism has been criticized by some (Roemer 1996; or Sen 1979). A given distribution of well-being levels – the only relevant ingredient for normative appraisal according to welfarism – can be obtained in many different ways. Is it that clear that all those ways are ethically equivalent? Amartya Sen (1979) has often answered negatively to this question on the basis of examples of the following type. Suppose that Ann has more income than Bob, and that this income difference produces a distribution of happiness in which Ann receives 10 and Bob 1. Imagine now that a distribution of income between Ann and Bob leads to a distribution of happiness in which Ann gets 6 and Bob gets 5. Many welfarist ethical principles (including Maximin and all mean-of-order r criteria) would recommend such a redistribution of income on the basis of the comparison of the distribution of happiness to which it leads as compared to the initial one. Imagine then that the distribution of happiness that provides Ann with 6 and Bob with 5 could also be obtained by allowing Bob to rape Ann. Welfarism would then imply that if we have preferred the income distribution to the *status quo* on the basis of the two distributions of happiness associated with these two situations, we should also prefer the situation in which Bob rapes Ann to that in which he does not. Welfarism indeed does not care if a particular distribution of well-being has been achieved by a particular distribution of income or by the rape of one individual by another. It only cares about the distributions of well-being, irrespective of how these distributions have come about.

Not all economists and philosophers are convinced by the devastating character, for welfarism, of examples of this kind. Indeed, the strong ethical opposition against rape in many human communities often comes from the impression that the victims of rape suffer to an extent that cannot be compared with the displeasure of paying large income taxes. Hence, the opposition to rape as compared to redistributive taxation

comes precisely from a welfarist observation that the negative impact of rapes on the victims' well-being is much larger than the impact of taxes on tax payers. This observation is absent from the example.

6.2.5. Possibilities of welfarist functionals

Somewhat surprisingly, there are not so many welfarist functionals – in the sense of Lemma 6.1 – that can be defined. And somewhat surprisingly also, the types of welfarist criteria that can be formulated depend crucially upon the measurability and, especially, the interpersonal comparability of individual well-being.

6.2.5.1. Interpersonal non-comparability

We first state the result that, without interpersonal comparability, it is impossible to find a social welfare functional that satisfy Axioms 6.1, 6.2, 6.3, 6.4, 6.6 and 6.7. This can be seen as a restatement, in the social welfare functional setting, of Arrow (1950) theorem. However, we emphasize in this theorem that the cardinal measurability of individual's well-being is of no help for coming up with a social welfare functional. The theorem that we state and prove, exploiting the welfarist lemma, is the following.

THEOREM 6.1.– *There does not exist any social welfare functional $W : \mathbb{D}^U \rightarrow \mathcal{B}$ that considers individual well-being to be cardinally measurable and interpersonally non-comparable that satisfy Axioms 6.1, 6.2, 6.3, 6.4, 6.6 and 6.7.*

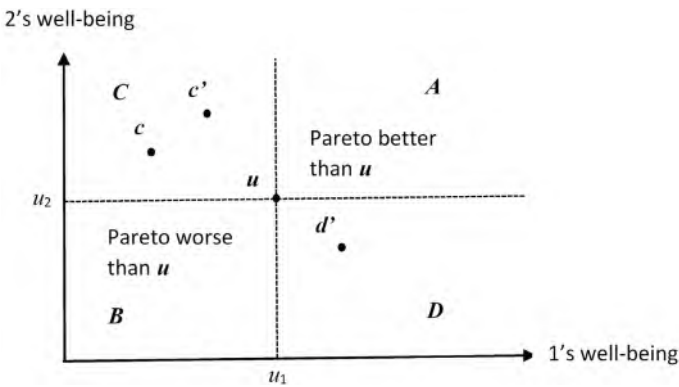


Figure 6.2. Social preference for two-person distributions of utility in the ordinal or cardinal non-comparable case

PROOF.– We only provide a diagrammatic proof of this result for the case where $n = 2$ (two individuals), using the welfarist Lemma and the nice diagrammatic arguments proposed in Blackorby et al. (1984). If a social welfare functional satisfies

Axioms 6.1, 6.2, 6.4 and 6.7, we can use the welfarist Lemma and map the problem of comparing social states by means of some ordering $W(U_1, \dots, U_n)$ into the problem of comparing distributions of well-being levels by means of the ordering $\succsim^U$. Let us therefore consider an arbitrary distribution $u = (u_1, u_2)$ of two utility levels depicted in Figure 6.2. As shown on the picture, utility distributions in the B zone (south-west of u) are worse than u by the Pareto Axiom 6.3, while utility distributions in the A zone are better than u by virtue of the same axiom. Let us now show that any two utility distributions such as c and c' in the C zone of the picture are ranked in the same way vis-à-vis u . Since the social welfare functional considers individuals' well-beings to be cardinally measurable and interpersonally non-comparable (Definition 6.5), we have:

$$\begin{aligned} c = (c_1, c_2) &\succsim^U (u_1, u_2) = \mathbf{u} \iff \\ (\alpha_1 c_1 + \beta_1, \alpha_2 c_2 + \beta_2) &\succsim^U (\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) \end{aligned} \quad [6.8]$$

for any two strictly positive real numbers α_1 and α_2 and any two real numbers β_1 and β_2 . Hence, if we set:

$$\alpha_i = \frac{u_i - c'_i}{u_i - c_i} > 0$$

and

$$\beta_i = \frac{u_i(b_i - a_i)}{(u_i - a_i)}$$

for $i = 1, 2$, we can then write the equivalence [6.8] as:

$$\begin{aligned} a = (c_1, c_2) &\succsim^U (u_1, u_2) = u \iff \\ &(\frac{(u_1 - c'_1)c_1}{u_1 - c_1} + \frac{u_1(c'_1 - c_1)}{u_1 - c_1}, \frac{(u_2 - c'_2)c_2}{u_2 - c_2} + \frac{u_2(c'_2 - c_2)}{u_2 - c_2}) \\ &= (c'_1, c'_2) \succsim^U (\frac{(u_1 - c'_1)u_1}{u_1 - c_1} + \frac{u_1(c'_1 - c_1)}{u_1 - c_1}, \frac{(u_2 - c'_2)u_2}{u_2 - c_2} + \frac{u_2(c'_2 - c_2)}{u_2 - c_2}) \\ &= (u_1, u_2) \end{aligned}$$

Hence, the ranking of c vis-à-vis $\mathbf{u}$ is the same as the ranking of c' vis-à-vis $\mathbf{u}$. By the weak Pareto Axiom 6.3 and Axiom 6.1 (transitivity) c and c' cannot be both indifferent to $\mathbf{u}$ (and therefore to themselves by transitivity). Hence, c and c' are both either strictly preferred to $\mathbf{u}$ or strictly worse than $\mathbf{u}$. Using an analogous reasoning, we can also show that all utility distributions in the D zone are also all ranked strictly vis-à-vis $\mathbf{u}$ in the same way. Hence, the distributions in the D zone are all either strictly worse, or all strictly better than $\mathbf{u}$. Let us now show that the ranking of any distribution in the C zone vis-à-vis $\mathbf{u}$ is the opposite of the ranking of any distribution in the D zone vis-à-vis $\mathbf{u}$. More precisely, the ranking of a distribution in the C zone such as c

vis-à-vis $\mathbf{u}$ is the same as the ranking of $\mathbf{u}$ vis-à-vis any point in the D zones such as $\mathbf{d}$. Again, we will use the fact that the social welfare functional considers individual's well-being to be cardinally measurable and interpersonally comparable and, because of this (implication [6.8]):

$$\begin{aligned}\mathbf{c} = (c_1, c_2) &\succsim^U (u_1, u_2) = \mathbf{u} \iff \\ (\alpha_1 c_1 + \beta_1, \alpha_2 c_2 + \beta_2) &\succsim^U (\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2)\end{aligned}$$

which can be written, by setting:

$$\alpha_i = \frac{d_i - u_i}{u_i - c_i} > 0$$

and

$$\beta_i = \frac{u_i^2 - c_i d_i}{(u_i - c_i)}$$

for $i = 1, 2$ as:

$$\begin{aligned}(c_1, c_2) &\succsim^U (u_1, u_2) \iff \\ &\left(\frac{(d_1 - u_1)c_1 + u_1^2 - c_1 d_1}{u_1 - c_1}, \frac{(d_2 - u_2)c_2 + u_2^2 - c_2 d_2}{u_2 - c_2} \right) \\ &= (u_1, u_2) \succsim^U \left(\frac{(d_1 - u_1)u_1 + u_1^2 - c_1 d_1}{u_1 - c_1}, \frac{(d_2 - u_2)u_2 + u_2^2 - c_2 d_2}{u_2 - c_2} \right) \\ &= (d_1, d_2)\end{aligned}$$

Hence, it is indeed the case that the ranking of any distribution in the C zone vis-à-vis $\mathbf{u}$ must be the opposite of the ranking of any distribution in the D zone vis-à-vis $\mathbf{u}$. Hence, if all distributions in C are better (worse) than $\mathbf{u}$, all distributions in D must be worse (better) than $\mathbf{u}$. We are then left with two possibilities:

(1) All distributions of utilities in the zones A and C that provide individual 2 with a larger well-being than u_2 are preferred to $u = (u_1, u_2)$ and all distributions of utilities in the zones B and D that provide 2 with a lower well-being than u_2 are worse than u .

(2) Alternatively, all distributions of utilities in the zones A and D that provide individual 1 with a better well-being than u_1 are preferred to $u = (u_1, u_2)$ and all distributions of utilities in the zones B and C that provide individual 1 with a lower well-being than u_1 are worse than u .

Case (1) means that the ranking of all distributions of utility is based on the utility of individual 2, who is then a dictator. Case (2) implies on its end that those

distributions of utility are ranked solely on the basis of the utility of individual 1, who is then the dictator. Hence, dictatorship of one of the two individuals is the only social welfare functional that considers well-being to be cardinally measurable and interpersonally non-comparable that satisfy Axioms 6.1, 6.2, 6.3, 6.4 and 6.7. Hence, it is impossible for such a functional to also satisfy Axiom 6.6. $\square$

The general proof of this theorem – for any number of individuals – can be found in d’Aspremont and Gevers (1977). Hence, the main lesson of this theorem is that welfarism is not a useful normative theory if individual’s well-being cannot be compared across individuals. It is important to note that the cardinal measurability of individual well-being – assumed in this theorem – does not provide any help on that front.

6.2.5.2. Ordinal measurability and interpersonal comparability

Many ethical theories are based on the principle of a *symmetrical* treatment of individuals. This principle states that only the *ethically relevant* characteristics of individuals should matter for appraising alternative social states. Ethically irrelevant characteristics of individuals such as their names, skin color, length of feet, etc., should have no bearing on the ethical appraisal of those states. Of course, the application of this principle requires an accepted partition of the set of individual characteristics between those that are ethically relevant, and those that are not.

In the welfarist approach discussed here, this partition seems clear enough. The utility function of an individual summarizes all the ethically relevant information on the individual. Hence, all other characteristics of the individual that do not affect their utility function, and notably those who determine the individual’s name or index are ethically irrelevant. A very natural way to express this irrelevance is to require the social welfare functional to be insensitive to any permutation of individual indices. Hence, it should not matter for normative appraisal whether the utility function U describes Ann’s well-being or that it describes Bob’s. The formal definition of this symmetry property is the following.

DEFINITION 6.10 (Symmetry). – A social welfare functional $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ is said to be symmetric if $W(U_1, \dots, U_n) = W(V_1, \dots, V_n)$ for any two profiles of individual utility functions $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ in the domain $\mathbb{D}^U$ such that $(U_1, \dots, U_n)$ is a permutation of $(V_1, \dots, V_n)$.

The dictatorship of a particular individual – say Ann – is a good example of a non-symmetric social welfare functional. If we permute the utility function of Ann with that of Bob, we would change the social ranking of social states, i.e. under Ann’s dictatorship, entirely determined by the comparison of Ann’s utility levels in the states that are obviously modified by the permutation. The failure of Ann’s dictatorship to satisfy the symmetry property may actually be one of the reasons underlying the

aversion that most of us have towards dictatorship. After all, why should Ann be the dictator instead of Bob?

Since symmetry implies non-dictatorship, we can conclude from Theorem 6.1 that there are no symmetric social welfare functionals that consider individual well-being to be cardinally measurable and interpersonally non-comparable and that satisfy Axioms 6.1, 6.2, 6.3, 6.4 and 6.7. We will now show that this impossibility vanishes if individual well-being is considered to be ordinally measurable and interpersonally comparable. Indeed, in that case, we can find symmetric social welfare functionals satisfying Axioms 6.1, 6.2, 6.3, 6.4 and 6.7. These functionals are known as “*positional dictatorships*”. They compare social states based on the well-being level of the individual who occupies a specific *position* in the ranking of well-being levels achieved by individuals in those states. Since well-being is interpersonally comparable, we can indeed compare, in every state, individual well-being levels and order those individuals in terms of their well-being. The dictatorship of position k (for any $k \in N$) ranks the social states based on the well-being of the individual who occupies, in those states, the k th position (from the lowest to the highest) in the ordering of individual well-beings. Observe that the Maximin criterion defined by Equation [6.3] is a particular element of this family of criteria for which $k = 1$ (the worst off individuals is the “dictator”). It is important to note the symmetric character of such a positional dictatorship social welfare functional. It is the position of the individual in the distribution of well-being levels – and not the name – that determines the “dictatorship” status of the individual. The actual identity of the individual who is in that position may change from one state to another (Ann may be the worst-off in state x , but Bob will be state y).

For any social state x in $\mathbf{X}$ and list of utility functions $(U_1, \dots, U_n)$ in the domain $\mathbb{D}^U$, we denote by $(U_{(1)}(x), \dots, U_{(n)}(x))$ the (increasingly) ordered permutation of the ordered list $(U_1(x), \dots, U_n(x))$. We can then formally define a social welfare functional that implies a dictatorship of a position k (for some integer $k \leq n$) as follows.

DEFINITION 6.11.— *A social welfare functional $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ implies a dictatorship of position k (for some $k \in \{1, \dots, n\}$) if, for every profile of utility functions $(U_1, \dots, U_n)$ in $\mathbb{D}^U$ and every social states x and y in $\mathbf{X}$, having $U_{(k)}(x) > U_{(k)}(y)$ implies that $x \succ y$ where $\succ$ is the asymmetric factor of the binary relation $\succsim = W(U_1, \dots, U_n)$.*

The following theorem says that social welfare functionals that imply a dictatorship of a position k (for a certain k) are the only symmetric such functionals which consider that individual well-being is ordinally measurable and interpersonally comparable and which satisfy Axioms 6.1, 6.2, 6.3, 6.4 and 6.7.

THEOREM 6.2.— *If $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ is a symmetric social welfare functional that considers individual well-being to be ordinally measurable and interpersonally comparable and that satisfies Axioms 6.1, 6.2, 6.3, 6.4 and 6.7, then W implies a dictatorship of some position $k \in \{1, \dots, n\}$.*

PROOF.— Just like for Theorem 6.1, we geometrically illustrate the proof for the case $n = 2$ using Blackorby et al. (1984) approach. In that case, the only two possible positions for the dictatorship are $k = 1$ (dictatorship of the worst-off) and $k = 2$ (dictatorship of the best-off). The fact that the functional W satisfies Axioms 6.1, 6.2, 6.4 and 6.7 enables us to resort to the welfarist Lemma 6.1, and therefore to replace any social state by the distribution of well-being levels assigned to it by the profile of utility functions and to replace the ranking $W(U_1, \dots, U_n)$ of social states with the ranking $\succsim^U$ of those distributions of well-being. In order to show that the two types of positional dictatorships are the only symmetric social welfare functionals that consider well-being to be ordinally measurable and interpersonally comparable and that satisfy Axioms 6.1, 6.2, 6.3, 6.4 and 6.7, we consider an arbitrary distribution $u = (u_1, u_2)$ of individual well-being and we use Figure 6.3 to indicate how all other conceivable distributions of well-being compare with u . Since W is symmetric, the ordered permutation $u_{(\cdot)}$ of u must be socially indifferent to u . By the Pareto Axiom 6.3 (Pareto), all distributions of individual well-being in the zones A , B and C are socially better than u and all distributions in the D zone are socially better than $u_{(\cdot)}$. Since distribution $u_{(\cdot)}$ is indifferent to u , it follows by transitivity (Axiom 6.1) that distributions in the D zone are also socially better than u . Applying the same reasoning in the reverse order, we are led to the conclusion that all distributions of well-being in the zones F , G , H and I are strictly worse than u (and $u_{(\cdot)}$). We now show that any two distributions of well-being such as a in b in the K zone (and by symmetry in the L zone) must be ranked in the same way vis-à-vis u (and vis-à-vis $u_{(\cdot)}$ by symmetry). Since W considers individual well-being to be ordinally measurable and interpersonally comparable, it must lead to a ranking of (a_1, a_2) with respect to (u_1, u_2) by $\succsim^U$ which is the same as the ranking of $(g(a_1), g(a_2))$ with respect to $(g(u_1), g(u_2))$ for any increasing function $g : \mathbb{R} \rightarrow \mathbb{R}$. Consider then the increasing function g whose graph is as shown in Figure 6.4 (where the numbers u_2 , a_2 , a_1 , b_2 , b_1 and u_1 are just as they are in Figure 6.3). This function g – that can obviously not be affine – maps u_i to u_i and a_i to b_i (for $i = 1, 2$). We can therefore conclude that the ranking of (a_1, a_2) with respect to (u_1, u_2) must be the same as the ranking of $(g(a_1), g(a_2)) = (b_1, b_2)$ with respect to $(g(u_1), g(u_2)) = (u_1, u_2)$. Since by the Pareto axiom, a and b cannot be equivalent to each other, they cannot – by transitivity – be both equivalent to u . Hence, they are either both strictly better than u or both strictly worse than u . By an analogous reasoning, we may also conclude that any two distributions of well-being in the J zone of Figure 6.3 must also be ranked in the same way with respect to u (and therefore by symmetry to $u_{(\cdot)}$). By transitivity and symmetry, the same thing can be said of any two distributions in the E zone of Figure 6.3 vis-à-vis both $u_{(\cdot)}$ and u . Let us now show that the ranking of a distribution of well-being in the J zone with respect to u must be the opposite of the ranking of

any distribution in the K zone vis-à-vis u . Precisely, and referring again to Figure 6.3, let us show that the ranking of a point such as a in the K zone vis-à-vis u is the same as the ranking of u vis-à-vis a point such as c in the J zone. Since the functional W considers well-being to be ordinally measurable and interpersonally comparable, it must consider that the ranking of the distribution of well-being (a_1, a_2) with respect to (u_1, u_2) by the binary relation $\succsim^U$ is the same as the ranking of $(g(a_1), g(a_2))$ with respect to $(g(u_1), g(u_2))$ for any increasing function g , including the increasing function g whose graph is depicted in Figure 6.5. This function, which cannot be affine or linear, maps u_j into c_j and a_j into u_j (for $j = 1, 2$). We therefore have $(a_1, a_2) \succsim^U (u_1, u_2) \iff (g(a_1), g(a_2)) = (u_1, u_2) \succsim^U (g(u_1), g(u_2)) = (c_1, c_2)$. Hence, if $\mathbf{a}$ is better than $\mathbf{u}$, $\mathbf{c}$ is worse than $\mathbf{u}$ and if $\mathbf{a}$ is worse than $\mathbf{u}$, then $\mathbf{c}$ is better than $\mathbf{u}$. Referring again to Figure 6.3, we thus end up in one of the following two situations:

(1) The distributions of well-being in the zones $E-F-G-H-I-J$ are all worse than $\mathbf{u}$ and the distributions in the zones $A-B-C-D-L-K$ are all better than $\mathbf{u}$.

(2) The distributions of well-being in the zones $F-G-H-I-L-K$ are all worse than $\mathbf{u}$ and the distributions of well-being in the zones $A-B-C-D-E-J$ are all better than $\mathbf{u}$.

In situation (1), position 1 (min) is the dictator while in situation (2), it is position 2 (max) that dictates. Hence, being a positional dictatorship is the only way for a symmetric social welfare functional that considers well-being to be ordinally measurable and interpersonally comparable to satisfy Axioms 6.1, 6.2, 6.4, 6.3 and 6.7. $\square$

The generalization of this theorem to any number of individuals is somewhat cumbersome. Proofs can be found notably in Gevers (1979) (Theorem 4). We should note that positional dictatorships are the only *symmetric* social welfare functionals that consider individuals' well-beings to be ordinally measurable and interpersonally comparable and that satisfy Axioms 6.1, 6.2, 6.4, 6.3 and 6.7. There are, however, other non-symmetric dictatorships that satisfy those axioms, including the dictatorship of, say, individual 1. Hence, allowing well-being to be comparable across individuals significantly enlarges the class of welfarist criteria that we can consider, even though the enlargement is still limited to rules that have "dictatorial power" of some kind.

The family of positional dictatorship rules characterized by Theorem 6.2 is large. It contains the Maximin criterion that compares social states based on the well-being level of the worst-off agent. It also contains the opposite Maximax criterion, which favors the view point of the best-off, as well as other criteria such as those, for example, would base the ranking of the states on the *median* well-being level, associated with the position $(n+1)/2$ if n is odd, and one of the two positions $n/2$, $n/2+1$ if n is even.

The notion of positional dictatorship does not say anything about the ranking of social states that provide the positional dictator with the same well-being. For

example, the Maximin criterion considers equivalent any two social states that provide the same level of well-being to the worst-off, irrespective of what happens to better-off people. We may consider this feature of the Maximin criterion a bit excessive. Indeed, if the well-being of the worst-off is the same in two social states, shouldn't we look then at what happens to better-off individuals? At least the strong Pareto Axiom 6.5 – violated by the Maximin criterion – which expresses a strict preference for a policy that does not hurt anyone and that strictly benefits some, would answer this question positively. There is a criterion which, like the Maximin, gives priority to the worst-off but, when the well-being of the worst-off is preserved, could attach importance to the well-being of better-off individuals. This criterion, proposed by Sen (1970) (p. 138, note 11) and Kolm (1972), is the *Leximin*. It ranks social state on the basis of the well-being of the worst-off but, if this well-being is unaffected, next considers the well-being of the second worst-off and, if the well-being of the second worst-off is also unaffected, looks at the well-being of the third worst-off, etc. The formal definition of the Leximin social welfare functional is the following.

DEFINITION 6.12 (LEXIMIN).— We call *Leximin* the social welfare functional $W^{LexMin} : \mathbb{D}^U \longrightarrow \mathcal{B}$ defined by: $x W^{LexMin}(U_1, \dots, U_n) y$ if and only if there is a position $k \in \{1, \dots, n\}$ such that $U_{(i)}(x) = U_{(i)}(y)$ for all positions $i < k$ and $U_{(k)}(x) \geq U_{(k)}(y)$.

Many positional dictatorship rankings are not ethically appealing. For example, the Maximax criterion – or a lexicographic extension thereof that would ride on the dictatorship of the best-off – is heavily elitist, and gives infinite weight to agents who are already quite favored. Hence, it hurts a widespread *prioritarian* ethical intuition (see Anderson (2022) for a recent account) according to which the “balance of justice” should be biased towards less well-off people. Couldn't we eliminate positional dictatorship rules that do not satisfy the prioritarian requirement? Peter J. Hammond (1976) has proposed almost 50 years ago an “equity-principle” – very egalitarian in nature – that leads indeed to a drastic reduction of the set of positional dictatorships. The Hammond equity principle is the following.

DEFINITION 6.13.— A social welfare functional $W : \mathbb{D}^U \longrightarrow \mathcal{B}$ satisfies *Hammond equity* if, for every profile of utility functions $(U_1, \dots, U_n)$ in the domain $\mathbb{D}^U$, it is such that $x \succ y$ for any pair of social states x and y in $\mathbf{X}$ and any two individuals h and i such that $U_h(y) < U_h(x) \leq U_i(x) < U_i(y)$ and $U_g(x) = U_g(y)$ for all individuals $g \neq h, i$ and all other individuals g .

Hence, Hammond equity considers that any reduction of a welfare discrepancy between two individuals is a good thing *ceteris paribus*. The following theorem shows how drastic is the reduction of the class of positional dictatorships that results from adding Hammond equity to the list of requirements imposed on a social welfare functional.

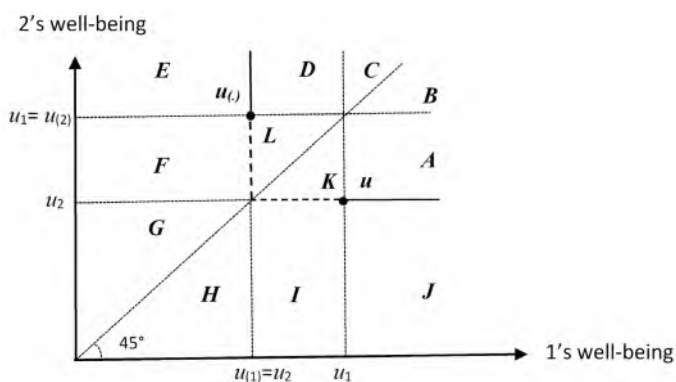


Figure 6.6. Social preference for two-person distributions of utility in the ordinal and fully comparable case with Hammond equity

THEOREM 6.3.— *The functional W^{LexMin} of Definition 6.12 is the only symmetric social welfare functional that considers individual well-being to be ordinally measurable and interpersonally comparable and that satisfies Axioms 6.1, 6.2, 6.4, 6.5 and 6.7 as well as the Hammond equity principle.*

PROOF.— The geometric illustration of the proof for two individuals is very simple and consists, starting from Figure 6.3 in the proof of Theorem 6.2, of showing that the possibility of a max (situation (2) at the end of the proof of Theorem 6.2 above) is ruled out by Hammond equity. Indeed, Hammond equity requires that any distribution of well-being in the *K* zone (which gives more well-being to the worst-off individual 2 and less well-being to the best-off individual 1 without modifying the welfare rank of the two individuals) is better than *u*. Observe also that by virtue of the strong Pareto Axiom 6.5 used here, the distributions of individual well-being that are strictly preferred to *u* (and to $u_{(.)}$ by symmetry) are those contained in the zones *A*, *B*, *C*, *D*, *L* and *K* of Figure 6.6 constructed from Figure 6.3 with the exception of the dotted bold lines that are separating the zones *K* and *L* from the zones *F*, *G*, *H* and *I* (as well as from the two distributions *u* and $u_{(.)}$). The distributions of well-being that are strictly worse than *u* (and $u_{(.)}$) are those contained in the zones *E*, *F*, *G*, *H*, *I* and *J* of Figure 6.6 with the exception of the full bold lines that are separating the zones *A* and *J*, on the one hand, and the zones *D* and *E*, on the other. Only the distributions *u* and $u_{(.)}$ are equivalent from the viewpoint of the Leximin criterion. $\square$

The proof for the general *n*-individual cases is more complex (Hammond 1976; Gevers 1979). The crux of the matter is that under Hammond equity, Leximin is the only “acceptable” (in the sense of Axioms 6.1, 6.2, 6.4, 6.5 and 6.7) welfarist ordering of social states that considers well-being to be ordinally measurable and interpersonally comparable.

6.2.5.3. Cardinal measurability and interpersonal unit comparability

We have seen above that cardinal measurability of individual well-being along with interpersonal comparability of differences in those well-being (units) is a possibility that is logically independent of the ordinal measurability and full interpersonal comparability considered in the preceding section. Ordinal measurability and interpersonal comparability requires that we can compare levels of well-being across individuals but that we cannot make sense of any statement about changes in well-being (be it for a single individuals or among different individuals). Cardinal measurability and interpersonal unit comparability requires that we can make sense and compare across individuals changes in well-being (and say meaningfully things such as “Ann has more suffered from the policy than Bob did”) but that we cannot compare welfare levels across different individuals. The Utilitarian social welfare functional associated with the name of Jeremy Bentham defined by [6.2] supposes, as noted above, a cardinal measurability and interpersonal unit comparability. The following theorem states that this Utilitarian functional is, as a matter of fact, the only symmetric social welfare functional that considers individual well-being to be cardinally measurable and unit comparable across individuals that satisfy Axioms 6.1, 6.2, 6.3, 6.4 and 6.7.

THEOREM 6.4.— *The Utilitarian functional defined by [6.2] is the only symmetric social welfare functional that considers individual well-being to be cardinally measurable and interpersonally unit comparable and that satisfies Axioms 6.1, 6.2, 6.4, 6.5 and 6.7.*

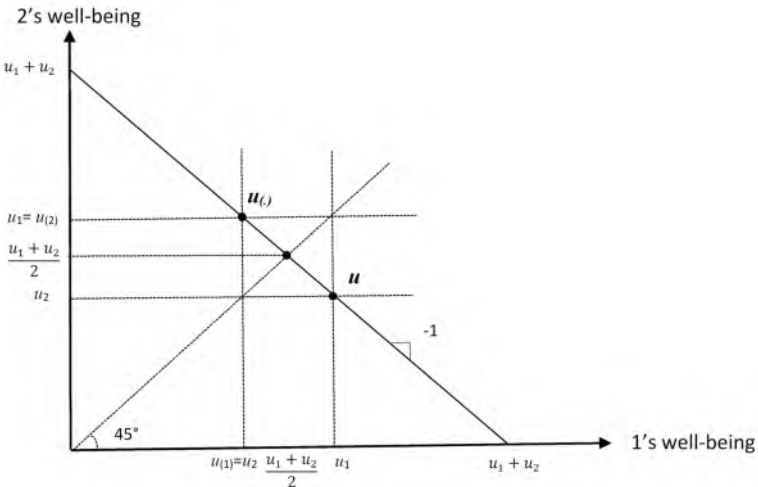


Figure 6.7. Social preference for two-person distributions of utility in the cardinal and interpersonally unit comparable case

PROOF.— Like for the other theorems of this section, we illustrate the argument (see Figure 6.7) for two individuals using the welfarist Lemma 6.1 and making the reasoning on the ranking $\succsim^U$ of distributions of well-being rather than on the ranking $W(U_1, \dots, U_n)$ of social states. Consider any distribution u and its ordered permutation $u_{(\cdot)}$ that must be socially equivalent to it symmetry. Consider now the egalitarian distribution of well-being $((u_1 + u_2)/2, (u_1 + u_2)/2)$ that provides each individual with half of the total well-being $u_1 + u_2$ in u (or in $u_{(\cdot)}$). This egalitarian distribution is at the intersection of the line segment connecting u to $u_{(\cdot)}$ and the egalitarian 45° line that describes all egalitarian distributions of well-being. Since the social welfare functional considers individual well-being to be cardinally measurable and interpersonally unit comparable, the social ranking of the distribution u with respect to $((u_1 + u_2)/2, (u_1 + u_2)/2)$ must be the same as the ranking of $(a_1 + bu_1, a_2 + bu_2)$ vis-à-vis $(a_1 + b(u_1 + u_2)/2, a_2 + b(u_1 + u_2)/2)$ for any real numbers a_1 and a_2 and every strictly positive real number b , including therefore the numbers:

$$a_1 = (u_2 - u_1)/2,$$

$$a_2 = (u_1 - u_2)/2$$

and

$$b = 1.$$

Using these particular numbers, we conclude that the ranking of u with respect to $((u_1 + u_2)/2, (u_1 + u_2)/2)$ is the same as the ranking of $((u_2 - u_1)/2 + u_1, (u_1 - u_2)/2 + u_2) = ((u_1 + u_2)/2, (u_1 + u_2)/2)$ with respect to $((u_2 - u_1)/2 + (u_1 + u_2)/2, (u_1 - u_2)/2 + (u_1 + u_2)/2) = (u_2, u_1)$. Hence, u must be ranked with respect to the egalitarian distribution $((u_1 + u_2)/2, (u_1 + u_2)/2)$ in the same way than this egalitarian distribution must be ranked with respect to the ordered permutation $u_{(\cdot)}$ of u . Since u and $u_{(\cdot)}$ are equivalent thanks to symmetry thus lead us to conclude – using the transitivity of $\succsim^U$, that the egalitarian distribution $((u_1 + u_2)/2, (u_1 + u_2)/2)$ be equivalent to u and $u_{(\cdot)}$. Since the argument here does not depend upon the particular choice of u and its ordered permutation, it shows that any two distributions of well-being on the straight line passing through u and $u_{(\cdot)}$ are socially equivalent. By the weak Pareto axiom, all distributions at the north-east of u (and of $u_{(\cdot)}$) are better than u and all distributions in the south-west of u (and of $u_{(\cdot)}$) are socially worse than u . Hence, the ranking of all distributions in $\mathbb{R}^2$ is solely based on the ranking of the straight lines of slope -1 passing through any point and its symmetric, with the points at the north-east being better than those in the north-west. That is, all points in $\mathbb{R}^2$ are ranked on the basis of the sum of their coordinates and this completes the proof. $\square$

Had symmetry not been imposed, Theorem 6.4 would have exhibited the larger class of *weighted Utilitarian* functionals that consider individual well-being to be

cardinally measurable and interpersonally unit comparable and that satisfy Axioms 6.1, 6.2, 6.4, 6.5 and 6.7. Weighted Utilitarian functionals are those defined by:

$$x W(U_1, \dots, U_n) y \iff \sum_{i=1}^n \alpha_i U_i(x) \geq \sum_{i=1}^n \alpha_i U_i(y)$$

for some non-negative (by the weak Pareto axiom) weights $(\alpha_1, \dots, \alpha_n)$ not all equal to zero. The Utilitarian case corresponds to $\alpha_i = \alpha_{i+1} = 1$ (or any other strictly positive number) for all $i = 1, \dots, n - 1$. Observe that this family contains the dictatorship of individual h as particular case (by setting $\alpha_h = 1$ and $\alpha_i = 0$ for all other i). The case $\alpha_i = \alpha_{i+1}$ for all $i = 1, \dots, n - 1$ is the only one that satisfies symmetry.

6.2.5.4. *Ratio-scale measurability and interpersonal comparability*

The strongest measurability and comparability requirement imposed on a social welfare functional is to consider well-being to be ratio-scale measurable and fully comparable across individuals. Ratio-scale measurability and interpersonal full comparability requires the existence of a common “zero level” of well-being. Within this setting, it can be shown, if utility functions are restricted to be positively valued, that the family of mean-of-order r social welfare functionals are the only symmetric social welfare functionals that consider individual well-being to be ratio-scale measurable and interpersonally comparable that satisfy Axioms 6.1, 6.2, 6.4, 6.5 and 6.7 as well as an additional axiom of *independence with respect to unconcerned individuals*.⁵ We refer the reader to Blackorby and Donaldson (1982) for a complete statement and proof of this result.

6.3. Ranking distributions of a single observable attribute

6.3.1. *Preliminaries*

We now apply the welfarist approach to social states for which the features that are visible to the evaluator are lists of non-negative numbers⁶, one such number for every concerned agent. These numbers represent the quantities of an observable attribute (income say) received by the agents. Note the distinction made between what is visible to the analyst (income) and what is normatively relevant (welfare). We are thus comparing by means of a binary relation $\succsim$ (typically assumed to be quasi-ordering) states such as $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$ where x_i and y_j

5. This axiom requires the ranking of two social states to be independent of the well-being levels of the individuals who do not experience welfare changes between the two states. Such individuals are considered “unconcerned” by the choice between the two states.

6. All of what is written here could apply to the possibility of some (or all) the numbers to be negative modulo minor changes in some of the definitions.

for $i = 1, \dots, n$ and $j = 1, \dots, m$ denote the income of individuals i and j in states $\mathbf{x}$ and $\mathbf{y}$ (respectively) and n and m are two (possibly distinct) integers. Hence, we do not assume that the same community of individuals is involved in $\mathbf{x}$ and in $\mathbf{y}$. For example, $\mathbf{x}$ may be the list of yearly incomes earned by French adults in 2022 and $\mathbf{y}$ the list of those income earned by Polish adults in the same year. Because of this, the binary relation $\succsim$ is a priori defined on the set $\bigcup_{n \in \mathbb{N}} \mathbb{R}^n$ of all conceivable lists of numbers of any length. Any two such lists will differ by the identity and number of individuals who receive the attribute, the total quantity of attribute that is distributed among the individuals and the way this total quantity is shared among those individuals. The two latter aspects deal, respectively, with efficiency (“size of the cake”, favored by Liz Truss) and equity or equality (“portioning the cake”, disfavored by Liz Truss). We will examine these two notions in turn. First, however, we address the possibility that the identities and numbers of individuals who receive income may differ across social states.

6.3.2. Different identities and numbers of individuals

The potentially different identities of income recipients across social states are handled by applying the symmetry principle to lists of income rather than, as in Definition 6.10, to lists of well-being levels. We specifically formulate the following axiom on $\succsim$.

AXIOM 6.8 (SYMMETRY).— *For any $n \in \mathbb{N}$, any $\mathbf{x} \in \mathbb{R}^n$ and any $n \times n$ permutation matrix $\mathbf{A}$, $\mathbf{x} \sim \mathbf{A} \cdot \mathbf{x}'$.⁷*

As its welfarist cousin, the Symmetry axiom requires that the names (index, identities) of the individuals do not matter for evaluating the distribution of income among them. Hence, permuting Bob and Ann’s incomes should be a matter of social indifference, even though it is unlikely to be so for Bob and Ann themselves. It should be noted that this symmetry principle, which is quite natural when applied to a ranking of individual well-being levels, is a more demanding requirement when applied to lists of quantities of an attribute that are far less comprehensive than well-being. Indeed, we could argue that permuting Bob and Ann’s income is not a matter of indifference if Bob is healthy and coming from a favored caste while Ann is severely impaired and coming from a discriminated caste. This points to the limitation of summarizing all aspects of a person’s situation by a single observable number, and calls for enlarging the list of observable attributes to – in this example – health and caste status.

7. A permutation matrix is a square matrix whose entries are either 0 or 1 and whose rows and columns all sum to 1. The symbols “ $\cdot$ ” and “ $'$ ” denote, respectively, scalar product and transposition.

If we nonetheless subscribe to symmetry, along with all practitioners who apply distributional analysis to a single attribute, we may find it convenient to define, for every list $\mathbf{x} = (x_1, \dots, x_n)$ of n real numbers, its ordered permutation $\mathbf{x}_{(\cdot)} = (x_{(1)}, \dots, x_{(n)})$ defined by $\mathbf{x}_{(\cdot)} = A \cdot \mathbf{x}'$ for some permutation matrix A such that $x_{(i)} \leq x_{(i+1)}$ for all $i = 1, \dots, n-1$. The ordered permutation of $\mathbf{x}$ just relabels the individuals in such a way that 1 is the poorest, 2 the second poorest, etc.

We handle the potentially differing numbers of individuals in the social states using the *Dalton principle of population*. This principle states that replicating any list of n numbers k times – regardless of k – is a matter of social indifference.

AXIOM 6.9 (DALTON POPULATION PRINCIPLE).– *For any k and $n \in \mathbb{N}$, any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$:*

$$(x_1, \dots, x_n) \sim (\underbrace{x_1, \dots, x_1}_{k\text{-times}}, \underbrace{x_2, \dots, x_2}_{k\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{k\text{-times}})$$

If we leave aside demographic considerations – that may lead us to favor or dislike population growth in itself – it is difficult to find arguments against this principle, which immediately enables the comparison of any two lists of numbers as if these lists were involving the same number of individuals. Indeed, consider two lists of income levels $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ assigned, respectively, to n and m individuals, with $n \neq m$. To compare those lists, we first use the Dalton population principle and conclude that:

$$(x_1, \dots, x_n) \sim (\underbrace{x_1, \dots, x_1}_{m\text{-times}}, \underbrace{x_2, \dots, x_2}_{m\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{m\text{-times}})$$

We then use the same principle again to conclude that:

$$(y_1, \dots, y_m) \sim (\underbrace{y_1, \dots, y_1}_{n\text{-times}}, \underbrace{y_2, \dots, y_2}_{n\text{-times}}, \dots, \underbrace{y_m, \dots, y_m}_{n\text{-times}})$$

We finally compare the lists $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ involving different number of individuals in exactly the same way as we compare the m -replicate of $(x_1, \dots, x_n)$ with the n -replicate of $(y_1, \dots, y_m)$, two lists of income levels assigned to the same number nm of individuals.

Hence, from now on, we will always assume that the number of income recipients is fixed.

6.3.3. Different sizes of the cake

Consider any list $\mathbf{x} = (x_1, \dots, x_n)$ of n numbers. The mean of $\mathbf{x}$, denoted as $\mu(\mathbf{x})$, and defined by:

$$\mu(x) = \frac{\sum_{i=1}^n x_i}{n}$$

is a very common measure of the “size of the cake” or “aggregate efficiency”. Per capita income and its evolution over time – called *economic growth* when positive – is indeed one of the most scrutinized indicators of the overall economic performance, which was favored so much by Liz Truss in her BBC interview. Surprisingly enough, there are not many justifications for using the mean μ as a measure of aggregate efficiency. For one thing, the ranking of lists of numbers induced by the comparison of their means rides crucially upon the cardinal measurability and interpersonal unit comparability of the attribute. Indeed, the ranking of lists of numbers induced by the comparisons of their mean is invariant to a multiplication of all the numbers by the same positive number, but is *not* invariant to the transformation of those numbers by the same increasing function. Hence, the mean would not be an appropriate indicator of “the size of cake” for an attribute only measured in an ordinal way.⁸

Another indicator of the size of the cake that is suitable to attributes whose measurement is only ordinal is the median of $\mathbf{x}$, denoted as $m(\mathbf{x})$, and defined by:

$$m(\mathbf{x}) = x_{((n+1)/2)}$$

if n is odd and by:

$$m(\mathbf{x}) = x_{(n/2)}$$

if n is even.⁹ Indeed, as shown before for well-being, a ranking of lists of numbers based on their median is a positional dictatorship rule that considers the attribute to be ordinally measurable and interpersonally comparable. Yet, while the median is sometimes also used as a measure of aggregate efficiency – labor economists often refer to the median salary, for example – there does not seem to be particularly good reasons for doing so.

While there are no solid justifications for measuring the “size of the cake” by the mean or, sometimes, by the median, there are a few well-accepted definitions of what an elementary efficiency gain or, equivalently, an undisputable increase in the size of the cake is. The first are the analogues, to ranking distributions

8. See Allison and Foster (2004) for an additional discussion of this point.

9. If n is even, both $n/2$ and $n/2 + 1$ could be used as “medians”.

of an observable attribute, of the Pareto principles discussed earlier when ranking distributions of well-being. If we focus on the strong version of the Pareto principle, it says that increasing the income of at least one individual without decreasing the income of anyone else is a clear gain in efficiency. There is, however, a stronger notion of efficiency gain that we may want to consider in the current context given the anonymity postulate made above. This notion, associated with Patrick Suppes (1966), says that a Pareto improvement applied to the ordered permutation of the considered list of income levels is also a clear efficiency gain.¹⁰ This notion of Suppes improvement – or domination – is formally defined as follows.

DEFINITION 6.14.— We say that the list of numbers $y = (y_1, \dots, y_n)$ has been obtained from the list $z = (z_1, \dots, z_n)$ by Suppes improvements if $y_{(.)} > z_{(.)}$.¹¹

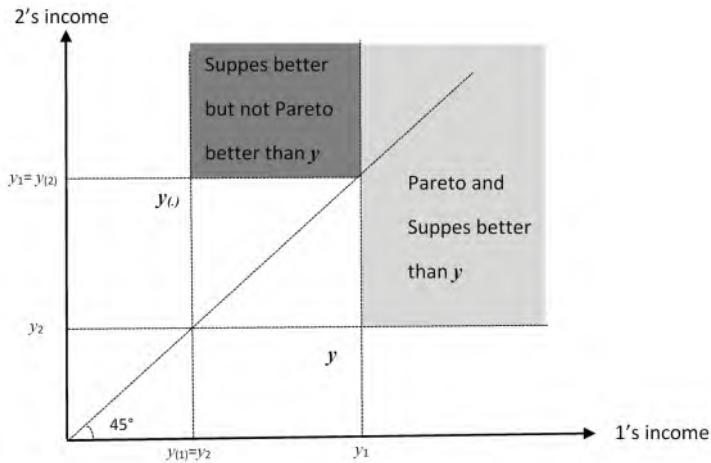


Figure 6.8. Pareto and Suppes dominance for two-person distributions of utility

The difference between Pareto and Suppes domination is illustrated for $n = 2$ in Figure 6.8. As can be seen, the lists of numbers that Suppes-dominate any given list such as (y_1, y_2) in Figure 6.8 contains those that Pareto dominate, as well as many others. Suppes dominance, therefore, provides more complete ranking than Pareto dominance. Observe that both Suppes and Pareto improvements lead to an increase in the mean income (and a non-decrease in the median income¹²). Hence, both indicators

10. A very good discussion of the Suppes criterion is provided in Blackorby and Donaldson (1977).

11. The notations for vector inequalities are $\geq$, $>$ and $>>$.

12. Median income, however, strictly increases with respect to Pareto (Suppes) improvements that affect all components of the vector y ($y_{(.)}$).

of overall efficiency are positively sensitive to the elementary notions of efficiency gains conveyed by Suppes or Pareto improvements, which is the very least that we can require from an overall efficiency indicator.

Suppes improvements alone are sometimes capable of leading to clear-cut comparisons of distributions of attribute. For example, if we take disposable income (in 2018) as the attribute, and look at distributions of this income among deciles of the population – which amounts to assuming that all individuals in a decile are the same and can be treated as a single person. We can see in Figure 6.9 that the list of disposable incomes in Australia and Canada both result from Suppes improvements over the French list. Indeed, for every decile, the average disposable income of the decile is higher in Australia and Canada than in France. However, Suppes improvements alone do not enable us to compare Australia and Canada. Indeed, the average income is higher in Canada than in Australia for the first 9 deciles, but the reverse ranking is observed for the 10th decile.

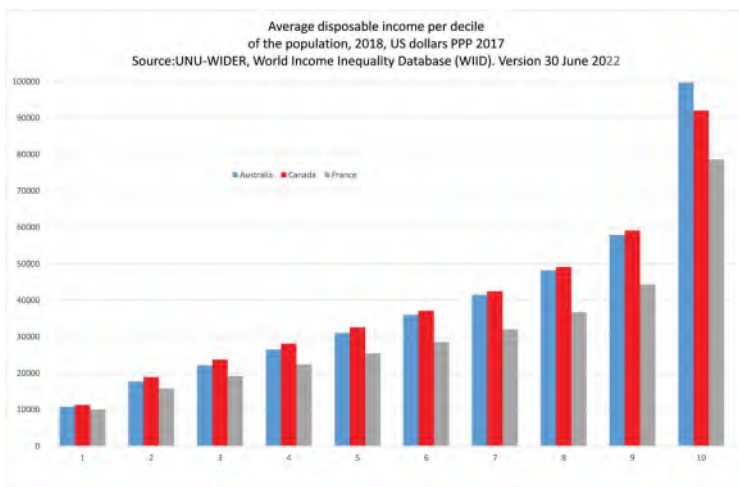


Figure 6.9. *Average disposable income per decile of the population, 2018, US dollars PPP 2017* Source: UNU-WIDER, World Income Inequality Database (WIID). Version 30 June 2022. For a color version of this figure, see www.iste.co.uk/dossantosferreira/macroeconomics.zip

Under the assumption that income never decreases individual well-being, the possibility of moving from one list of attribute quantities to another by Suppes improvements alone is recognized as worth doing by an extremely large range of welfarist judgments. Indeed, it is not difficult to prove the following (Lehmann 1955; Quirk and Saposnik 1962).

THEOREM 6.5.— Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$ be two lists of quantities of attribute received by n agents. Then, the following four statements are equivalent:

- (i) $(x_1, \dots, x_n)$ Suppes dominates $(y_1, \dots, y_n)$.
- (ii) For every $a \in \mathbb{R}$, $\#\{i : x_i \leq a\} \leq \#\{i : y_i \leq a\}$.
- (iii) $\sum_{i=1}^n U(x_i) \geq \sum_{i=1}^n U(y_i)$ for any non-decreasing (utility) function $U : \mathbb{R} \rightarrow \mathbb{R}$.
- (iv) $W(U(x_1), \dots, U(x_n)) \geq W(U(y_1), \dots, U(y_n))$ for any non-decreasing (utility) function $U : \mathbb{R} \rightarrow \mathbb{R}$ and any anonymous social welfare functional W defined on the domain of all such utility functions that satisfy the strong Pareto principle.

Statement (ii) of Theorem 6.5 is the requirement that the cumulative distribution associated with $(x_1, \dots, x_n)$ 1st-order stochastically dominates (Hadar and Russell 1974) the distribution associated with $(y_1, \dots, y_n)$. Hence, Suppes dominance of $(x_1, \dots, x_n)$ over $(y_1, \dots, y_n)$ is equivalent to observing that the fraction of the population earning no more than any quantity of the attribute is smaller in $(x_1, \dots, x_n)$ than in $(y_1, \dots, y_n)$. Statement (iii) says that all utilitarian ethical observers who assume that individuals convert attribute into well-being by the same weakly increasing utility function consider $(x_1, \dots, x_n)$ to be better than $(y_1, \dots, y_n)$. Statement (iv) says that the unanimity over the ranking of $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ is actually larger since it covers all anonymous social welfare functionals defined over profiles of increasing individual utility functions (provided they are the same for all individuals). Utilitarian functionals are only one subclass of this larger family. It is noteworthy that when we consider unanimity over a large class of individual utility functions, there is no difference between seeking unanimity over all anonymous and Paretian welfarist functions and limiting unanimity to the utilitarian subclass of those functions (Gravel and Moyes 2013 for further discussion). The requirement that all individuals convert attributes into well-being using the same utility function may be considered a strong assumption. This reflects the “fundamental preference” view point expressed by Serge-Christophe Kolm (1972) according to which “all individuals have the same needs, the same tastes and the same desires” once we introduce, as arguments of their utility functions, all factors that could differentiate them. Indeed, there is a sense by which, when all potential sources of individuals’ differentiation are taken into account (including gender, age, synaptic connections, endorphins, etc.), individuals can indeed be viewed as identical “machines” capable of transforming these sources into well-being.¹³ Of course, the one-dimensional setting considered here in which individuals differ only by only one observable attribute makes Kolm reasoning a bit hard to swallow. Yet, if we accept it, be it simply because we do not really know how we could differentiate between individuals beyond the fact that they differ in the

13. A mathematical formulation of Kolm’s intuition has been proposed by Howe (1987).

quantity of attributes that they receive, Theorem 6.5 shows that an extremely wide class of welfarist principles recognize Suppes improvement as indisputably worth doing.

A critique that can be made to Suppes dominance as an efficiency principle is its lack of sensitivity to inequality. For example, doubling the yearly income of Elon Musk while leaving that of anyone else unchanged is a Pareto (and Suppes) improvement, even though such a change is likely to increase income inequality (a concept that we have not yet defined). We can formulate, however, weaker notions of efficiency gains that apply only to lists of incomes that are “equivalent” in terms of equality. One commonly held view about inequality is the “relativist” view according to which increasing (decreasing) the income of everyone in the same proportion does not change inequalities. If this view is held, then we can define the following notion of *Relative-Inequality-Preserving Suppes improvement*.

DEFINITION 6.15.— *We say that the list of numbers $\mathbf{y} = (y_1, \dots, y_n)$ is a Relative-Inequality-Preserving Suppes improvement over the list $\mathbf{z} = (z_1, \dots, z_n)$ if there exists a real number $t > 1$ such that $y_{(\cdot)} = tz_{(\cdot)}$.*

Hence, a Relative-Inequality-Preserving Suppes improvement is an increase in the same proportion of the income associated with every rank in the income distribution. This kind of improvement, sometimes referred to as “inclusive growth”, is obviously recorded favorably by aggregate indicators of efficiency such as the mean or the median. Not everyone, however, subscribes to the relativist view according to which a multiplication of everyone’s income by the same positive factor does not change income inequality. After all, increasing the incomes of both Elon Musk and an individual who earns zero by 10% can hardly be said to leave the income inequality between these two persons unchanged.

We find also (Kolm 1969; Kolm 1976a; Kolm 1976b; Blackorby and Donaldson 1980; and Moyes 1987) an absolutist view of inequality according to which it is the addition of the same amount of income to everyone (up to a permutation) that leaves inequalities unchanged. If we hold this view, then we can define the following notion of *Absolute-Inequality-Preserving-Suppes improvement*, which is also recorded favorably by the mean or the median.

DEFINITION 6.16.— *We say that the list of numbers $\mathbf{y}$ is an Absolute-Inequality-Preserving Suppes improvement over the list $\mathbf{z}$ if a real number $a > 0$ exists such that $y_{(i)} = z_{(i)} + a$ for every i .*

Having defined elementary notions of efficiency gains that leave unchanged inequalities in the distribution of income – depending upon what this latter requirement means – we now turn to the definitions of elementary notions of inequality reduction and of synthetic numerical indicators – inequality indices – that are consistent with those notions.

6.3.4. Different portions of the cake

While the ideal of perfect equality – for example, the situation where everyone gets the same quantity of the attribute – is clear enough, the notion of increasing or reducing inequality is less so. The most widely accepted notion is that underlying the *Pigou–Dalton principle of transfer*. According to this principle, any transfer of a given quantity of the attribute from a relatively well-endowed individual to a less endowed individual constitutes a clear inequality reduction. The formal definition of a Pigou–Dalton transfer is the following.

DEFINITION 6.17 (PIGOU–DALTON TRANSFER).– *We say that the list of numbers $\mathbf{y} = (y_1, \dots, y_n)$ has been obtained from the list $\mathbf{z} = (z_1, \dots, z_n)$ by means of a Pigou–Dalton transfer if there are two individuals g and h and a real number $\Delta > 0$ such that:*

- (i) $y_i = z_i$ for all $i \neq g, h$ and,
- (ii) $y_h = z_h - \Delta \geq z_g + \Delta = y_g$.

Since the transferred quantity Δ is fixed (what is given by the donator is exactly what is obtained by the recipient), a Pigou–Dalton transfer has no effect on the mean of the list of numbers. Because of this, a Pigou–Dalton transfer can be seen as a pure equalization operation that preserves the “size of the cake” when the latter is measured by the mean. It should be noted that the cardinal measurability of the attribute is crucial for the definition of a Pigou–Dalton transfer, and its underlying requirement that the quantities given and received be the same.

There is a stronger notion of equalization than Pigou–Dalton that does not require any preservation of the “size of the cake” and that does not ride on the cardinal measurability of the attribute. This notion, which underlies Definition 6.13 of Hammond equity, is sometimes referred to as a Hammond transfer (Gravel et al. 2019; Gravel et al. 2021; and Jenkins 2021). It is defined as follows.

DEFINITION 6.18 (HAMMOND TRANSFER).– *We say that the list of numbers $\mathbf{y} = (y_1, \dots, y_n)$ has been obtained from the list $\mathbf{z} = (z_1, \dots, z_n)$ by means of a Hammond transfer if there are two individuals g and h and two strictly positive real numbers Δ_g and Δ_h such that:*

- (i) $y_i = z_i$ for all $i \neq g, h$ and,
- (ii) $y_h = z_h - \Delta_h \geq z_g + \Delta_g = y_g$.

Precisely because it does not require the quantities given and received to be the same, the notion of inequality reduction captured by a Hammond transfer is stronger than that associated with a Pigou–Dalton transfer, which is nothing but a specific kind of Hammond transfer. Indeed, while Pigou–Dalton transfers do not affect the aggregate quantity of the distributed attribute, Hammond transfers may be inefficient

(e.g. the donator may give more than what the recipient receives) or, to the contrary, efficient (the recipient receives more than what the donator gives). Pigou–Dalton and two other kinds of Hammond transfers are shown in Figure 6.10. If performing a Pigou–Dalton transfer between two individuals is considered to be an indisputably equalizing operation, then performing such Pigou–Dalton transfers any finite number of times should also be considered to be equalizing. Hence, we can define as follows the Pigou–Dalton reduction of inequality.

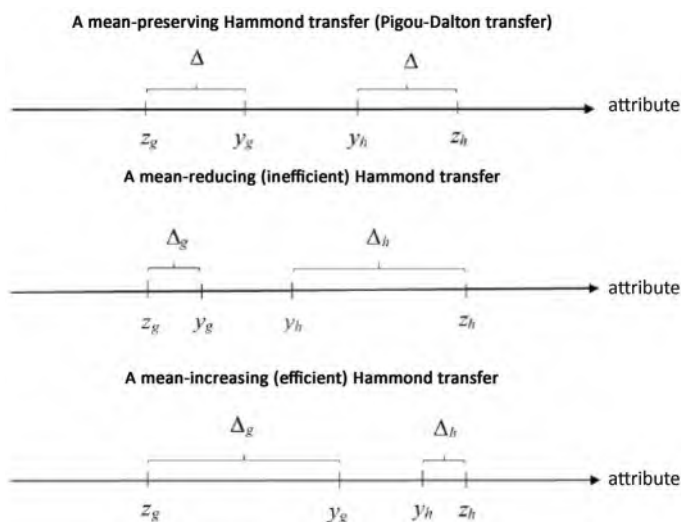


Figure 6.10. Various types of Hammond transfers

DEFINITION 6.19 (PIGOU–DALTON EQUALIZATION).— We say that the list of numbers $\mathbf{y} = (y_1, \dots, y_n)$ is more equal than the list $\mathbf{z} = (z_1, \dots, z_n)$ for the Pigou–Dalton criterion if there exists a sequence of lists of numbers $\{x^t = (x_1^t, \dots, x_n^t)\}_{t=0}^l$ for some integer $l \geq 1$ such that:

- (i) $x^0 = \mathbf{y}$,
- (ii) $x^l = \mathbf{z}$ and,
- (iii) x^t has been obtained from x^{t+1} by means of a Pigou–Dalton transfer as per Definition 6.17 for $t = 0, \dots, l - 1$.

For example, the list of numbers (2, 4, 8, 11, 16) is more equal than the list (1, 5, 7, 10, 18) for the Pigou–Dalton criterion because it is possible to go from (1, 5, 7, 10, 18) to (2, 4, 8, 11, 16) by a sequence of three Pigou–Dalton transfers in the sense of Definition 6.17 (between individuals 1 and 2, between individuals 3 and 5, and between individuals 4 and 5).

Similarly, we can define Hammond equalization as performing Hammond transfers any finite number of times, as follows.

DEFINITION 6.20 (HAMMOND EQUALIZATION).— *We say that the list of numbers $\mathbf{y} = (y_1, \dots, y_n)$ is more equal than the list $\mathbf{z} = (z_1, \dots, z_n)$ for the Hammond criterion if there exists a sequence of lists of numbers $\{x^t = (x_1^t, \dots, x_n^t)\}_{t=0}^l$ for some integer $l \geq 1$ such that:*

$$(i) x^0 = y,$$

$$(ii) x^l = z \text{ and,}$$

(iii) x^t has been obtained from x^{t+1} by means of a Hammond transfer as per Definition 6.18 for $t = 0, \dots, l - 1$.

Since Pigou–Dalton transfers are particular kinds of Hammond transfers, it is clear that Pigou–Dalton equalization implies Hammond equalization. The converse, however, does not hold. For example, numbers $(2, 3, 6, 7, 16)$ are more equal in the Hammond sense – but not in the Pigou–Dalton sense – than numbers $(1, 5, 5, 5, 18)$. A sequence of three Hammond transfers that enables us to go from $(1, 5, 5, 5, 18)$ to $(2, 3, 6, 7, 16)$ is the following:

1 : $(1, 5, 5, 5, 18)$ to $(1, 5, 5, 8, 16)$ (Hammond transfer from 5 to 4)

2 : $(1, 5, 5, 8, 16)$ to $(2, 3, 5, 8, 16)$ (Hammond transfer from 2 to 1)

3 : $(2, 3, 5, 8, 16)$ to $(2, 3, 6, 7, 12)$ (Hammond transfer from 4 to 3)

Observe that the first Hammond transfer in the sequence is somewhat efficient (individual 5 loses 2 in exchange of individual 3 gaining 3). The second transfer is, however, inefficient since individual 2 gives away 2 while individual 1 only gets 1 in exchange. The last transfer is, however, a Pigou–Dalton transfer of 1. Overall, the sequence of Hammond transfers involves no change in the size of cake (the efficiency of the first transfer being exactly offset by the inefficiency of the second).

There is an easy test for checking if one list of numbers is more equal than another in terms of the Pigou–Dalton criterion. The test is the well-known Lorenz dominance test that is defined as follows.

DEFINITION 6.21 (LORENZ DOMINANCE).— *The list of numbers $\mathbf{y} = (y_1, \dots, y_n)$ Lorenz-dominates the list $\mathbf{z} = (z_1, \dots, z_n)$ if the inequality*

$$\sum_{i=1}^k y_{(i)} \geq \sum_{i=1}^k z_{(i)}$$

holds for every $k \in \{1, \dots, n\}$.

In words, $\mathbf{y}$ Lorenz-dominates $\mathbf{z}$ if the total attribute held by the k poorest individuals is larger in $\mathbf{y}$ than in $\mathbf{z}$ irrespective of what k is. The Lorenz dominance test is implemented very simply by plotting points of the form $(k, \sum_{i=1}^k y_{(i)})$ and $(k, \sum_{i=1}^k z_{(i)})$ (and possibly the line segments connecting them, forming the “Lorenz curve”) and checking that the points (line segments) of the dominating distribution be nowhere below those of the dominated distribution.

The equivalence between Pigou–Dalton equalization as per Definition 6.19 and Lorenz dominance, when applied to lists of numbers with the same mean, is a corollary of the Hardy–Littlewood–Polya theorem¹⁴. This corollary, which also establishes how appealing the reduction of inequality is from a welfarist or utilitarian standpoint, is formally stated as follows.

THEOREM 6.6 (HARDY–LITTLEWOOD–POLYA).— *For any two lists of numbers $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$ such that $\mu(\mathbf{y}) = \mu(\mathbf{z})$, the following five statements are equivalent:*

- (i) $\mathbf{y}_{(\cdot)}$ is more equal than $\mathbf{z}_{(\cdot)}$ for the Pigou–Dalton criterion;
- (ii) $\mathbf{y}$ Lorenz-dominates $\mathbf{z}$;
- (iii) $\sum_{i=1}^n \max[t - y_i, 0] \leq \sum_{i=1}^n \max[t - z_i, 0]$ for every real number t ;
- (iv) $\sum_{i=1}^n U(x_i) \geq \sum_{i=1}^n U(y_i)$ for any concave¹⁵ (utility) function $U : \mathbb{R} \rightarrow \mathbb{R}$;
- (v) $W(U(x_1), \dots, U(x_n)) \geq W(U(y_1), \dots, U(y_n))$ for any concave (utility) function $U : \mathbb{R} \rightarrow \mathbb{R}$ and any anonymous social welfare functional W defined on the domain of all such utility functions satisfying the strong Pareto principle and sensitivity to Pigou–Dalton transfers of utility.

Theorem 6.6 thus shows the equivalence between Pigou–Dalton equalization and Lorenz dominance. It also shows, through statement (iii), that Pigou–Dalton equalization is equivalent to poverty reduction for any poverty line, when poverty is measured by the poverty gap, defined to be the minimal quantity of attribute required to eliminate all poverty relative to the poverty line. This requirement of poverty reduction for every poverty line is conceptually similar to the first-order stochastic dominance of Theorem 6.5 (statement (ii)), which requires the fraction of poor to be smaller in the dominating than in the dominated list of numbers for all poverty lines. Statement (iii) of Theorem 6.6 is actually often referred to as second-order

14. After Hardy et al. (1952) who formulate the first version of this theorem. A beautiful proof can be found in Berge (1959) (p. 191–196) (see also Dasgupta et al. (1973) and Kolm (1969)).

15. A function $f : A \rightarrow \mathbb{R}$ where A is a convex subset of $\mathbb{R}^l$ (for some integer l) is *concave* if it satisfies, for any $a, a' \in A$ and any $\lambda \in [0, 1] : f(\lambda a + (1 - \lambda)a') \geq \lambda f(a) + (1 - \lambda)f(a')$.

stochastic dominance, shown by this theorem to be equivalent, when applied to a list of numbers with the same mean, to Pigou–Dalton equalization. Statements (iv) and (v) of Theorem 6.6 show that Pigou–Dalton equalization is favorably valued by a large class of utilitarian ethical observers, actually by all such observers who assume that individuals convert the attribute into utility by the same *concave* utility function. Observe, by contrast to Theorem 6.5, that it is the concavity of the utility function, and not its increasingness, which is important for the utilitarian ethical observer to favor Pigou–Dalton equalization. An utilitarian ethical observer may thus value favorably Pigou–Dalton transfers even if they do not view the attribute as increasing utility – provided that they believe individual well-being to be a concave function of attribute quantity. Observe that the requirement of concavity of the utility function only makes sense if the observable attribute is cardinally measurable. Indeed, concavity of the utility function is not a property that is invariant to any monotonically increasing transformation of the attribute over which the function is defined. Statement (v) of Theorem 6.6 is the analogue to Statement (iv) in Theorem 6.5 (with the increasingness requirement on the individual utility function replaced by concavity) but with an additional restriction imposed on the general social welfare functional W . Indeed, in addition to satisfying anonymity and the strong Pareto principle, Statement (v) of Theorem 6.6 requires W to be weakly (and positively) sensitive to the Pigou–Dalton transfer of utility.¹⁶

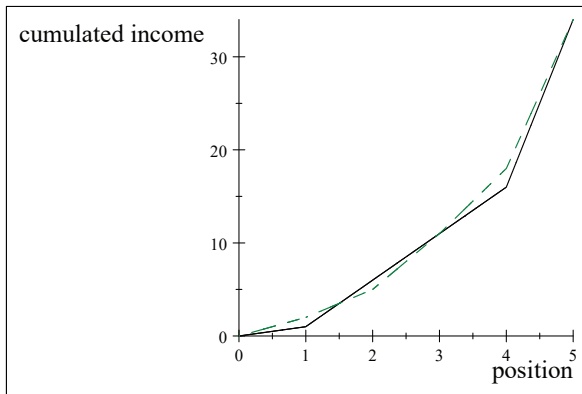


Figure 6.11. Lorenz curves for the lists (1, 5, 5, 5, 18) and (2, 3, 6, 7, 16)

To see the usefulness of the Lorenz test for checking if Pigou–Dalton equalization has taken place or not, reconsider the lists (1, 5, 5, 5, 18) and (2, 3, 6, 7, 16) with the

16. A function of n real variables that is symmetric (as W is by anonymity) and weakly positively sensitive to Pigou–Dalton transfers among any two of its arguments is often called Schur-concave (Eichorn and Gehrig 1982).

latter being more equal than the former by the Hammond criterion but – as we claimed – not by the Pigou–Dalton criterion. To show this last statement, the Lorenz curves of (1, 5, 5, 5, 18) (in black) and (2, 3, 6, 7, 16) (in dotted-green) have been plotted in Figure 6.11. The fact that the two curves cross twice implies by virtue of Theorem 6.6 that none of the two lists is more equal than the other by the Pigou–Dalton criterion.

Is there an empirical test similar to Lorenz dominance to verify whether or not there is Hammond equalization as per Definition 6.20? It turns out that there is, and that the test is extremely simple. The test amounts indeed to verify if one list of numbers stands higher than the other for both the Leximin criterion – as per Definition 6.12 but applied to lists of numbers instead of lists of utility levels – and the (anti) Leximax criterion. The anti-Leximax criterion seeks to reduce the quantity of the attribute received by the individual who is the best endowed in each of the two lists and, in the case where the largest quantity of the attribute received is the same, seeks to reduce the second highest endowment of the attribute, etc. Hence, while Leximin expresses an infinite love for the poor, the anti-Leximax reflects an infinite hate for the rich. Hammond equalization turns out to be the intersection of these two (infinite) sentiments, as formally stated in the following (a proof of this result is given by Gravel et al. (2019)).

THEOREM 6.7. – *For any two lists of numbers $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$ such that $\mu(\mathbf{y}) = \mu(\mathbf{z})$, the following two statements are equivalent:*

- (i) $\mathbf{y}_{(\cdot)}$ is more equal than $\mathbf{z}_{(\cdot)}$ for the Hammond criterion.
- (ii) *There are indices i and j with $1 \leq i < j \leq n$ such that $y_{(i)} > z_{(i)}$, $y_{(g)} = z_{(g)}$ for all $1 \leq g < i$ (if there are such g), $y_{(j)} < z_{(j)}$, $y_{(l)} = z_{(l)}$ for all $j < g \leq n$ (if there are such l).*

While there are limited numerical indicators used to measure the overall efficiency of a list of numbers – the most prominent of them being the mean and the median – there are many numerical indices that measure the inequalities of these lists. Just like the mean or the median – as measures of efficiency – are not sensitive to inequality, inequality indices are not sensitive to the efficiency of the list of numbers. As mentioned above, the most commonly used inequality indices ride either on the relative view, where multiplying everyone’s attribute quantity by the same strictly positive factor leaves inequality unchanged, or on the absolute view, where adding the same quantity of the attribute to everyone leaves inequality unchanged.¹⁷ Both of these views, it should be noted, assumes that the attribute is cardinally measurable (so that a multiplication or a translation by a common factor is meaningful). Since the relative view of inequality is – for some reason – more popular than the absolute one,

17. See Bossert and Pfingsten (1990) for an “intermediate” view.

this chapter focuses on relative inequality indices, specifically defining a (continuous) relative inequality index.

DEFINITION 6.22.— *A continuous relative inequality index is any continuous function $I : \mathbb{R}_+^n \rightarrow \mathbb{R}$ that verifies, for every $(x_1, \dots, x_n) \in \mathbb{R}_+^n$:*

- (i) $I(x_1, \dots, x_n) = I(\lambda x_1, \dots, \lambda x_n)$ for any strictly positive number λ ;
- (ii) $I(x_1, \dots, x_n) = I(x_{(1)}, \dots, x_{(n)})$; and,
- (iii) $I(z_1, \dots, z_n) < I(x_1, \dots, x_n)$ for any $(z_1, \dots, z_n) \in \mathbb{R}_+^n$ obtained from $(x_1, \dots, x_n)$ by means of a Pigou–Dalton transfer.

Hence, a relative inequality index is any (continuous) function that respects the relative view of inequality (ii), the anonymity principle (iii) and the notion of equalization underlying the Pigou–Dalton principle of transfer (iii).

The most widely used and commented upon relative inequality index is the (relative) Gini coefficient, denoted by I^G which can be defined, for any list $(x_1, \dots, x_n)$ of numbers, using one of the three following equivalent ways:

$$I^G(x_1, \dots, x_n) = \frac{2 \sum_{i=1}^n [i\mu(x_1, \dots, x_n) - \sum_{h=1}^i x_{(h)}]}{\mu(x_1, \dots, x_n)n^2} \quad [6.9]$$

$$= 1 - \frac{\sum_{i=1}^n (2(n-i) + 1)x_{(i)}}{\mu(x_1, \dots, x_n)n^2} \quad [6.10]$$

$$= \frac{2 \sum_{i,j} |x_i - x_j|}{\mu(x_1, \dots, x_n)n^2} \quad [6.11]$$

The first definition [6.9] of the Gini coefficient views it as (twice) the surface area between the *relative* Lorenz curve associated with the list $(x_1, \dots, x_n)$ and the relative Lorenz curve associated with the perfectly egalitarian distribution in which everyone gets the average quantity $\mu(x_1, \dots, x_n)$ of the attribute. The relative Lorenz curve of any list $(z_1, \dots, z_n)$ of numbers is nothing else than the Lorenz curve defined above applied to the list of *shares* $(\frac{z_1}{n\mu(z_1, \dots, z_n)}, \dots, \frac{z_n}{n\mu(z_1, \dots, z_n)})$ of the sum of the numbers rather than on the list of numbers itself. It is obviously the relative view of inequality that leads us to focus on the list of shares of total income rather than on the list of incomes themselves. If income inequality is indeed unaffected when everyone's income is multiplied by any positive number, then only the share of the total income held by individuals matters for appraising inequality. Obviously, the distribution of shares associated with the perfectly egalitarian list is $(\frac{1}{n}, \dots, \frac{1}{n})$ so the relative Lorenz curve associated with this egalitarian distribution is the 45° line in the two-dimensional $[0, 1] \times [0, 1]$ space.

Definition [6.10] of the Gini coefficient views it as (the additive inverse of) a weighted average of the shares of total attribute of the population ordered from the poorest to the richest, with the weight being decreasing with respect to the position of the individual in the distribution. Definition [6.11] views the Gini coefficient as the sum, over all pairs of individuals, of the absolute values of their income (shares) differences.

The popularity of the Gini coefficient as an index of inequality is probably explained by both its ease of calculation and its intuitive justification underlying either [6.9] (the area between the Lorenz curve associated with the perfectly equal distribution and the current one) or [6.11] (e.g. it is “simply” the sum of all pairwise income (share) differences observed in the population). When defined by [6.10], the Gini coefficient is easily shown to belong to the larger S -Gini family of indices analyzed by Donaldson and Weymark (1980) and axiomatically characterized by Bossert (1990). This family of indices is summarized by a single parameter $\delta \geq 1$ that reflects the relative weight assigned by the index to low numbers in the list. For any value of δ , the S -Gini index $I_\delta^{SG}(x_1, \dots, x_n)$ associated with $(x_1, \dots, x_n)$ is defined by:

$$I_\delta^{SG}(x_1, \dots, x_n) = 1 - \frac{\sum_{i=1}^n [(n-i+1)^\delta - (n-i)^\delta] x_{(i)}}{\mu(x_1, \dots, x_n) n^2}$$

The case $\delta = 1$ corresponds to a total insensitivity to inequality (all lists of numbers have inequality $((n-1)/n)$, the case $\delta = 2$ corresponds to the Gini coefficient and the S Gini index can get as close as we wish to $1 - \frac{x_{(1)}}{\mu(x_1, \dots, x_n) n^2}$ (inequality is negatively measured by the ratio of the lowest numbers over the sum of those numbers) by taking δ sufficiently large. While there is no axiomatic characterization of the particular choice of $\delta = 2$ in this family, Aaberge (2001) has provided a different axiomatic characterization of the Gini coefficient by taking as starting point the fact that any list of incomes can be summarized by its Lorenz curve.

The other widely used relative inequality indices are those belonging to the Generalized Entropy family. This family of indices is also associated with a single non-negative parameter c . For any given value of c , the Generalized Entropy index $I_c^{GE}(x_1, \dots, x_n)$ associated with $(x_1, \dots, x_n)$ is defined by:

$$\begin{aligned} I_c^{GE}(x_1, \dots, x_n) &= \frac{1}{nc(c-1)} \sum_{i=1}^n \left(\frac{x_i}{\mu(x_1, \dots, x_n)} \right)^c - 1 \text{ if } c \neq 0, 1 \\ &= -\frac{\sum_{i=1}^n \ln \frac{x_i}{\mu(x_1, \dots, x_n)}}{n} \text{ if } c = 0 \text{ and,} \\ &= \sum_{i=1}^n \frac{x_i}{n\mu(x_1, \dots, x_n)} \ln \frac{x_i}{\mu(x_1, \dots, x_n)} \text{ if } c = 1 \end{aligned}$$

Many specific indices of inequality are contained within this family. Examples of them are the mean-log deviation ($c = 0$), the Theil index ($c = 1$), the Atkinson indices (Atkinson (1970)) with varying inequality aversion ($c \in]0, 1[$) and the coefficient of variation ($c = 2$).

Generalized entropy indices are convenient to use in applied work because of their *decomposability*. It is indeed common for researchers to decompose the total population of attribute holders into various subpopulations and to see how this decomposition affects inequality. For example, we may want to see how the evolution of the overall income inequality in Europe results from the evolution of inequality within every European country as well as the evolution of inequality across European countries. Similarly, we may want to see how inequalities among men, among women and between men and women are affecting the overall inequality observed in a society. For the purpose of performing these inequality decompositions, it is convenient for the inequality measured at the population level by the index to be written as the sum of the inequalities within groups, measured by the same index, and the inequality between the group means, also measured by the same index. As it turns out, requiring an inequality index to have this additive decomposability property imposes severe restrictions on the index. Shorrocks (1984) has actually shown that generalized entropy indices are the only relative inequality indices that are decomposable in this sense. Hence, it is not in general possible to express the total income inequality (say) measured by the Gini coefficient in France as a whole as the sum of the (possibly different) Gini coefficients observed in the subpopulations of men and women and the Gini coefficient applied to the two average incomes of men and women.

6.3.5. *Different portions and sizes of the cake*

Suppes improvements – either in the prudent relative or absolute inequality preserving forms of Definitions 6.15 and 6.16 or in the more demanding form of Definition 6.14 – are clear instances of efficiency gains. The commonly used numerical indicators of efficiency such as the mean and the median all recognize Suppes improvements to be worth doing. Pigou–Dalton transfers, on the other hand, are clear instances of inequality reduction that are considered such – actually by definition – by all inequality indices that preserve the overall efficiency of the income distribution measured by the mean. Hammond transfers are also clear instances of inequality reduction but they do not carry the same appeal because they do not preserve the mean. In this section, we examine how we can design criteria for comparing distributions of an observable attribute that are sensitive to both clear efficiency gains and clear equality reductions and, actually, only to them. Hence, we do not propose any principle other than the sensitivity to both efficiency gain and equality reduction which would enable us, say, to express Liz Truss’ preference for “growing the economy” over distributing its benefit to its members.

We start with Suppes improvements and Pigou–Dalton transfers. Since a Pigou–Dalton transfer preserves the mean – and thus the efficiency – of any list of numbers on which it is performed, it is natural to consider that any list of numbers obtained from another as a result of a sequence of Pigou–Dalton transfers and/or Suppes improvements is better than the latter. Indeed, and as shown in the next theorem which combines Theorems 6.5 and 6.6, a rather large spectrum of welfarist functionals would subscribe to this point of view. And as shown also in this theorem, Lorenz dominance applied to lists of numbers with possibly different means (a criterion called by Shorrocks (1983) “generalized Lorenz dominance”) enables us to easily verify if we can go from a list to another by a finite sequence of Suppes and/or Pigou–Dalton transfers.

THEOREM 6.8.—*For any two lists of numbers $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$, the following five statements are equivalent:*

(i) $\mathbf{y}_{(\cdot)}$ has been obtained from $\mathbf{z}_{(\cdot)}$ by a finite sequence of Pigou–Dalton transfers (Definition 6.17) and/or Suppes improvements (Definition 6.14);

(ii) $\mathbf{y}$ Lorenz-dominates $\mathbf{z}$;

(iii) $\sum_{i=1}^n \max[t - y_i, 0] \leq \sum_{i=1}^n \max[t - z_i, 0]$ for every real number t ;

(iv) $\sum_{i=1}^n U(x_i) \geq \sum_{i=1}^n U(y_i)$ for any increasing and concave (utility) function $U : \mathbb{R} \rightarrow \mathbb{R}$;

(v) $W(U(x_1), \dots, U(x_n)) \geq W(U(y_1), \dots, U(y_n))$ for any increasing and concave (utility) function $U : \mathbb{R} \rightarrow \mathbb{R}$ and any anonymous social welfare functional W defined on the domain of all such utility functions satisfying the strong Pareto principle and sensitivity to Pigou–Dalton transfers of utility.

Observe that the utilitarian (or welfarist) unanimity obtained in this theorem is smaller than those obtained in Theorems 6.5 and 6.6 because it is actually the intersection of those two. Indeed, the welfarist or utilitarian unanimity of Theorem 6.8 rides on the requirement that the individual utility function is both increasing and concave with respect to the attribute. Yet, since this double requirement is rather natural (well-being is increasing at a decreasing rate with respect to the attribute), we can say that (generalized) Lorenz dominance, when observed, provides an extremely good reason for preferring the dominating list of number to the dominated one. In the comparison of the distributions of disposable incomes per decile in Australia and Canada performed above, the two countries could not be compared by the criterion of Suppes dominance alone. Hence, neither of the two countries could be seen as indisputably more efficient than the other. However, when we add Pigou–Dalton equalization considerations to efficiency ones, we obtain the clear verdict that Canada dominates Australia, as shown in Figure 6.12.

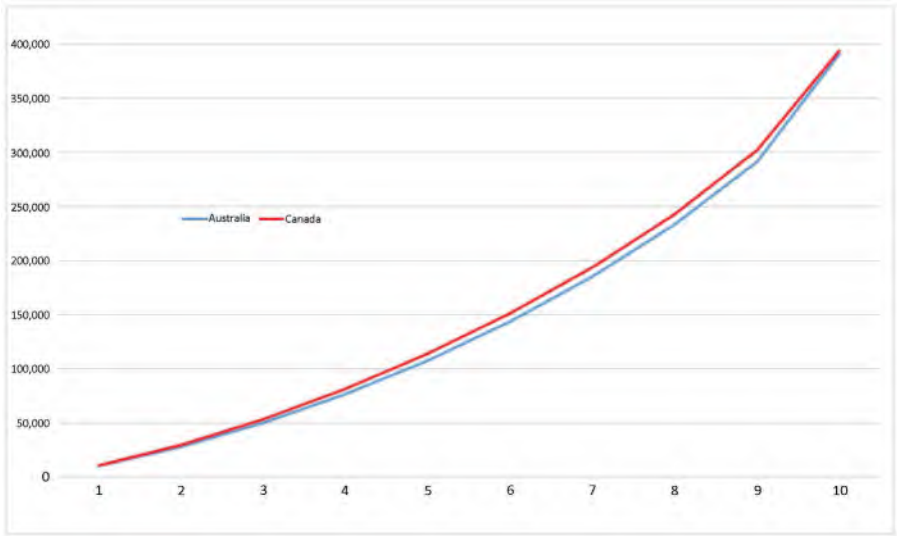


Figure 6.12. Generalized Lorenz curves of disposable income per decile of the population 2018, US dollars PPP 2017. Source: UNU-WIDER, World Income Inequality Database (WIID). Version 30 June 2022. For a color version of this figure, see www.iste.co.uk/dossantosferreira/macroeconomics.zip

As indicated above, Suppes dominance does not preserve relative – or absolute – inequality. Because of this, it can be said that the combination of Suppes improvements (that do not preserve inequality) and Pigou–Dalton transfers (that preserve efficiency) underlying (generalized) Lorenz dominance is giving too much importance to efficiency as compared to equality. Indeed, all efficiency gains – even those who benefit only to Elon Musk – are considered worth doing by generalized Lorenz dominance, while only equalizations that leave the mean unchanged have the same attractiveness. While the rather large welfarist unanimity that supports this view – including the most inequality averse Maximin criterion – makes it somewhat appealing, strongly egalitarian people may nonetheless consider the weight given to efficiency considerations relative to egalitarian ones in Generalized Lorenz dominance to be excessive. A nice result, due to Shorrocks (1983), provides a (much less discriminatory) dominance test that gives more weight to equality by valuing only Relative-inequality-preserving Suppes improvements along with Pigou–Dalton transfers. Shorrocks (1983) has indeed proved the following.

THEOREM 6.9.– For any two lists of numbers $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$, the following two statements are equivalent:

(i) $y_{(\cdot)}$ has been obtained from $z_{(\cdot)}$ by a finite sequence of Pigou–Dalton transfers (Definition 6.17) and/or Relative-inequality-preserving Suppes improvements (Definition 6.15).

(ii) The list of shares $(\frac{y_1}{n\mu(y_1, \dots, y_n)}, \dots, \frac{y_n}{n\mu(y_1, \dots, y_n)})$ Lorenz-dominates the list of shares $(\frac{z_1}{n\mu(z_1, \dots, z_n)}, \dots, \frac{z_n}{n\mu(z_1, \dots, z_n)})$ and $\mu(y_1, \dots, y_n) \geq \mu(z_1, \dots, z_n)$.

Observe that this theorem does not connect statements (i) and/or (ii) to any welfarist or even utilitarian unanimity. Indeed, it would be quite difficult to justify relative-inequality preserving Suppes improvements by welfarist considerations alone. Suppes improvements, irrespective of whether or not they preserve relative inequality, will have a definite effect on the well-being levels of the individuals who benefit from them. Hence, it is very difficult to require that these Suppes improvement leave individuals' well-being unaffected when they are not relative-inequality-preserving but they increase individuals' well-being when they are relative-inequality-preserving. Yet, if we are willing to give up welfarist concerns and to subscribe to the (strongly egalitarian) view that equalization is of paramount importance and that only efficiency gains that do not affect relative inequality are worth considering, then we are led to a very simple operational test: simply compare income (say) distributions in terms of both *per capita* income and relative Lorenz curve, and conclude in undisputable improvements when both per capita income growth and relative Lorenz dominance are observed. In the Australia, Canada, France example given above, the per capita disposable incomes of these three countries in 2018 were, respectively, 39 152, 39 441 and 31 321 US dollars of 2017 in Purchasing Power Parity. The relative Lorenz curves of these three countries were as depicted in Figure 6.13. As can be seen, the relative Lorenz curves of both Canada and France lie everywhere above Australia's curve. Since per capita income is slightly higher in Canada than in Australia, this indicates that we can go from Australia to Canada by a finite sequence of relative-inequality-preserving Suppes improvements and Pigou–Dalton transfers. Hence, the generalized Lorenz dominance of Canada over Australia is also agreed upon by the more egalitarian criterion underlying Theorem 6.9. However, the (first-order dominance) of both Australia and Canada over France – based on a very large unanimity of welfarist principles – is *not* agreed upon by the egalitarian criterion of Theorem 6.9. Indeed, the relative Lorenz curves of Canada and France cross. Hence, despite the fact that per capita disposable income is higher in Canada than in France, we cannot say that we can go from France to Canada by a finite sequence of relative inequality preserving Suppes improvements and Pigou–Dalton transfers. While France's relative Lorenz curve lies everywhere above that of Australia, Australia has a higher per capita disposable income than France. Because of this, the two countries are not comparable. It is the (relatively) good performance of France in terms of equality – as reflected by its relative Lorenz curve – that prevents this country from being dominated by the other two.

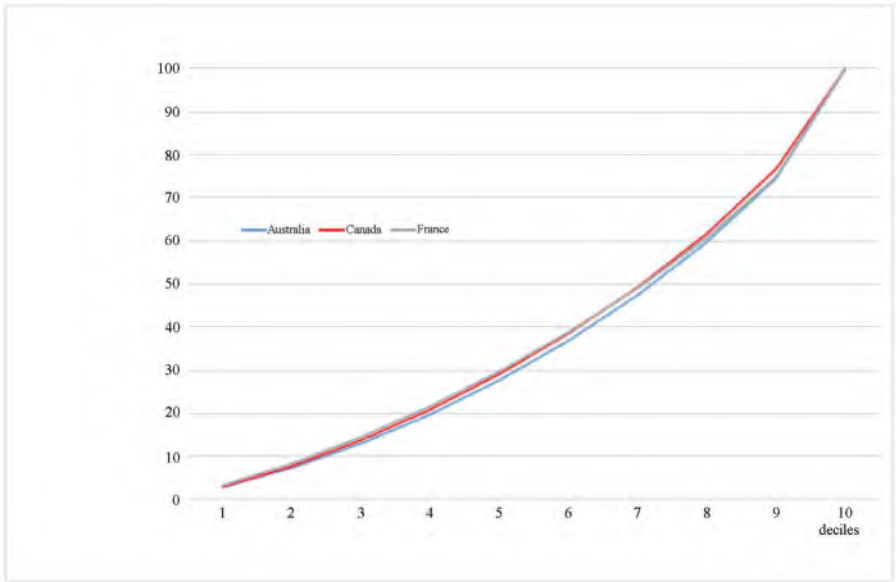


Figure 6.13. *Relative Lorenz curves of per capita disposable incomes in 2018 in Australia, Canada and France. For a color version of this figure, see www.iste.co.uk/dossantosferreira/macroeconomics.zip*

Another way to incorporate stronger egalitarian concerns other than those underlying generalized Lorenz dominance in the comparisons of income (say) distributions is to combine Hammond transfers and Suppes (or Pareto) improvements. Hammond transfers are redistributive transfers that do not require the preservation of any indicator of overall efficiency. Hence, they already incorporate a strong egalitarian flavor. If we subscribe to this strong egalitarian view, as well as to the idea that performing Suppes improvement is also worth doing, then we are led – as an almost immediate consequence of Theorem 6.3 of the preceding section – to compare lists of numbers on the basis of the Leximin criterion. Note that this criterion, when applied to Australia, Canada and France, would rank the three countries in exactly the same way as the generalized Lorenz criterion.

6.4. Conclusion

This chapter has summarized the main theoretical justifications for a welfarist theory of social justice and indicated the usefulness of the approach for evaluating policies affecting the distribution of one attribute – typically income – among individuals. The possibility of obtaining sound welfarist criteria crucially rides on the possibility of comparing well-being across individuals. And even if this

interpersonal comparability is secured, the available welfarist criteria happen to be limited to utilitarianism (if interpersonal comparability of well-being differences is possible) or positional dictatorships (under interpersonal comparability of well-being levels). While well-being may be difficult to observe in practice, welfarism leads to simple dominance tests that can be applied to distributions of an observable attribute if minimal assumptions are made on the way the observable attribute influences the unobservable well-being. Two such assumptions discussed in the chapter are increasingness and concavity. Welfarism can therefore provide foundations to one-dimensional distributional analysis by exhibiting clear notions of efficiency gains, equality reductions and trade-offs among those considerations that can be connected easily to commonly used indices that aim at measuring those features. In the last 30 years, the literature on distributional analysis has increasingly focused on distributions involving more than one attribute. This raises analytical challenges that are beyond the scope of this chapter.

6.5. References

- Aaberge, R. (2001). Axiomatic characterization of the Gini coefficient Lorenz curve orderings. *Journal of Economic Theory*, 101, 115–132.
- Afriat, S. (1967). The construction of utility functions from expenditure data. *International Economic Review*, 8, 67–77.
- Allison, R.A. and Foster, J.E. (2004). Measuring health inequality using qualitative data. *Journal of Health Economics*, 23, 505–524.
- Anderson, R.J. (2022). *Prioritarianism*. Cambridge University Press, Cambridge.
- Arrow, K.J. (1950). A difficulty in the concept of welfare. *Journal of Political Economy*, 58, 328–346.
- Arrow, K.J. (1963). *Social Choice and Individual Values*. Cowles Foundation Monograph. Yale University Press, New Haven.
- d'Aspremont, C. and Gevers, L. (1977). Equity and the informational basis of collective choice. *Review of Economic Studies*, 44, 199–209.
- Atkinson, A.B. (1970). On the measurement of inequality. *Journal of Economic Theory*, 2, 244–263.
- Berge, C. (1959). *Espaces Topologiques et Fonctions Multivoques*. Dunod, Paris.
- Blackorby, C. and Donaldson, D. (1977). Utility vs equity: Some plausible quasi-orderings. *Journal of Public Economics*, 7, 365–382.
- Blackorby, C. and Donaldson, D. (1980). A theoretical treatment of indices of absolute inequality. *International Economic Review*, 21, 107–136.
- Blackorby, C. and Donaldson, D. (1982). Ratio-scale and translation-scale full interpersonal comparability without domain restrictions: Admissible social evaluation functions. *International Economic Review*, 23, 249–268.

- Blackorby, C., Donaldson, D., Weymark, J.A. (1984). Social choice with interpersonal utility comparisons: A diagrammatic introduction. *International Economic Review*, 25, 327–356.
- Bossert, W. (1990). An axiomatization of the single-series Ginis. *Journal of Economic Theory*, 50, 82–92.
- Bossert, W. and Pfingsten, A. (1990). Intermediate inequality: Concepts, indices and welfare implications. *Mathematical Social Sciences*, pp. 117–134.
- Broome, J. (1999). *Ethics out of Economics*. Cambridge University Press, Cambridge.
- Clark, A.E., Fritjer, P., Shields, M.A. (2008). Relative income, happiness and utility: An explanation for the easterlin paradox and other puzzles. *Journal of Economic Literature*, 46, 95–144.
- Cohen, G.A. (1990). Equality of what? On welfare, goods and capabilities. *Recherches économiques de Louvain*, 56, 357–382.
- Dasgupta, P., Sen, A.K., Starrett, D. (1973). Notes on the measurement of inequality. *Journal of Economic Theory*, 6, 180–187.
- Donaldson, D. and Weymark, J.A. (1980). A single-parameter generalization of the Gini indices of inequality. *Journal of Economic Theory*, 22, 67–86.
- Eichorn, W. and Gehrig, W. (1982). Measurement of inequality in economics. In *Applied Mathematics — Optimization and Operations Research*, Korte, B. (ed.). North Holland, Amsterdam, 657–693.
- Gevers, L. (1979). On interpersonal comparability and social welfare orderings. *Econometrica*, 47, 75–90.
- Gravel, N. and Moyes, P. (2013). Utilitarianism or welfarism: Does it make a difference? *Social Choice and Welfare*, 40, 529–551.
- Gravel, N., Magdalou, B., Moyes, P. (2019). Inequality measurement with an ordinal and continuous variable. *Social Choice and Welfare*, 52, 453–475.
- Gravel, N., Magdalou, B., Moyes, P. (2021). Ranking distributions of an ordinal variable. *Economic Theory*, 71, 33–80.
- Griffin, J. (1986). *Well-Being: Its Meaning, Measurement and Moral Importance*. Clarendon Press, Oxford.
- Hadar, J. and Russell, W. (1974). Stochastic dominance in choice under uncertainty. In *Essays on Economics Behavior under Uncertainty*, Balch, M.S., McFadden, D.L., Wu, S. (eds). North Holland, Amsterdam.
- Hammond, P.J. (1976). Equity, Arrow's conditions and Rawls's difference principle. *Econometrica*, 44, 793–803.
- Hardy, G.H., Littlewood, J.E., Polya, G. (1952). *Inequalities*, 2nd edition. Cambridge University Press, Cambridge.
- Houthakker, H. (1950). Revealed preference and the utility function. *Economica*, 17, 159–174.

- Howe, R. (1987). Sections and extensions of concave functions. *Journal of Mathematical Economics*, 16, 53–64.
- Jenkins, S.P. (2021). Inequality comparisons with ordinal data. *Review of Income and Wealth*, 67, 547–563.
- Kolm, S.C. (1969). The optimal production of social justice. In *Public Economics*, Guitton, H. and Margolis, J. (eds). Macmillan, London.
- Kolm, S.C. (1972). *Justice et Équité*. Editions du CNRS, Paris.
- Kolm, S.C. (1976a). Unequal inequalities I. *Journal of Economic Theory*, 12, 416–442.
- Kolm, S.C. (1976b). Unequal inequalities II. *Journal of Economic Theory*, 13, 82–111.
- Lehmann, E.L. (1955). Ordered family of distributions. *The Annals of Mathematics and Statistics*, 26, 399–419.
- Moyes, P. (1987). A new concept of Lorenz domination. *Economics Letter*, 23, 203–207.
- Quirk, J.D. and Saposnik, R. (1962). Admissibility and measurable utility functions. *Review of Economic Studies*, 29, 140–146.
- Rawls, J. (1971). *A Theory of Justice*. Berknap Press of Harvard University Press, Cambridge, MA.
- Roemer, J.E. (1996). *Theories of Justice*. Harvard University Press, Cambridge, MA.
- Samuelson, P.A. (1948). Consumption theory in terms of revealed preferences. *Econometrica*, 15, 243–253.
- Sen, A.K. (1970). *Collective Choice and Social Welfare*. Holden Day, San Francisco.
- Sen, A.K. (1973). *On Economic Inequality*. Clarendon, Oxford.
- Sen, A.K. (1977). Social choice theory: A re-examination. *Econometrica*, 45, 53–89.
- Sen, A.K. (1979). Personal utilities and public judgements: or what's wrong with welfare economics. *Economic Journal*, 89.
- Sen, A.K. (1986). Social choice theory. In *Handbook of Mathematical Economics: Volume III*, Arrow, K. and Intriligator, M.D. (eds). North-Holland, Amsterdam, 1073–1181.
- Sen, A.K. (1992). *Inequality Reexamined*. Harvard University Press, Cambridge, MA.
- Shorrocks, A.F. (1983). Ranking income distributions. *Economica*, 50, 3–17.
- Shorrocks, A.F. (1984). Inequality decomposition by population subgroups. *Econometrica*, 52, 1369–1386.
- Strasnick, S.L. (1975). Preference priority and the maximization of social welfare. PhD Thesis, Harvard University.
- Suppes, P. (1966). Some formal models of grading principles. *Synthese*, 16, 284–306.
- Varian, H. (2006). Revealed preferences. In *Samuelsonian Economics and the Twenty First Century*, Szenberg, M., Ramrattan, L., Gottesman, A.A. (eds). Oxford University Press, Oxford, 99–115.

7

Political Economy of Macroeconomic Outcomes

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7.1. Introduction

7.1.1. *What do we mean by political economy?*

When analyzing policies, economists generally make a distinction between positive and normative statements. A normative statement refers to what ought to be done, from the point of view of some social welfare criterion, while a positive statement refers to what is supposed to happen. The field of political economy, which would be better labeled as political economics, is the application of positive economics to the actual policies being implemented by governments and other public policymakers. Rather than taking policies as exogenous, political economy attempts to explain them.

The frontier between normative and positive economics is more tenuous than it looks. Any normative statement about policy may be reinterpreted as a positive one. If a policy maximizes some social welfare function, then it is a positive prediction that a policymaker whose objective coincides with that social welfare function will implement that policy. Moreover, a policymaker who represents any interests, no matter how special they are, will elect policies that are in the Pareto set, provided the latter is properly defined, i.e. taking into account the distortionary effects of the

Macroeconomics 1,

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available tax instruments as well as other institutions. Deviations from this basic insight come from strategic interactions between conflicting interests who fail to coordinate. Even then, many positive models of social choice predict that the outcome maximizes some specific instance of a social welfare function. Let us give three examples of this:

1) Median voter theorems predict that the outcome of an election is the median voter's preferred one.¹ That is, it maximizes a social welfare function which aggregates individual welfare with a weight equal to zero on all agents except the median voter.

2) Probabilistic voting theories predict that the policy outcome maximizes a weighted average of the individual utilities of various social groups, with the weight of group i being proportional to the marginal density of swing voters in group i .²

3) The menu auction approach assumes that interest groups, or lobbies, compete with one another by offering reward structures to the policy maker.³ That is, each interest group i offers a reward $z_i(y)$ to the government, where y is the policy elected by the government, which, if purely selfish, will therefore pick the value of y that maximizes $\sum_i z_i(y)$. In this literature, it has been shown that the government will choose a policy that maximizes a utilitarian aggregate of the interest groups' payoffs. This does not mean that the policy coincides with the preferred one of a utilitarian social planner. This is because only a fraction of society may be organized as a lobby. Rather, the prediction is that policy will maximize the utilitarian aggregate welfare of the organized groups, i.e. a social welfare aggregate with a weight of 1 on each organized group and 0 on everyone else. If the entire society is organized, then policy does indeed maximize utilitarian social welfare.

For these reasons, the political economy literature in macroeconomics, which is discussed in this chapter, has been rather casual in its representation of the political process, often assuming right away some objective function which stood as a metaphor for the power structure. Of course, coordination failures may lead to suboptimal outcomes, and some institutional structures are more likely to lead to coordination failures than others. Yet, even if policy is not the outcome of a struggle between interest groups, coordination failures may deliver suboptimal policies, despite the agents that fail to coordinate actually pursuing the same goals. One example is the famous Barro–Gordon model of inflation bias discussed below. It should be pointed out that, as far as macroeconomics are concerned, this part of the literature that emphasizes coordination failures draw much more from standard game theoretic tools than from an accurate description of the political institutions. These metaphorical features will be obvious from the models that are reviewed below.

1. See Roberts (1977), Grandmont (1978), Gans and Smart (1996).

2. See Lindbeck and Weibull (1987).

3. See Bernheim and Whinston (1986), Grossman and Helpman (1994).

7.1.2. Scope of this chapter

Given the size of the literature on political economics, and given space constraints, this chapter focuses on those models that are most relevant for the understanding of macroeconomic policies and outcomes, such as inflation, growth, unemployment, crises, or trade and budget deficits.⁴ It does not discuss the social choice foundations of those models. It also ignores all the literature that has no direct macro-implications, such as that on, say, coups, civil wars or corruption. Finally, its focus is analytical, so that the large, and more recent, purely empirical literature will not be surveyed either.

The chapter is organized into three parts. The first part deals with the positive theories of short-term fiscal and monetary policy, those policy instruments that lie at the core of business cycle theories. The second part deals with the literature that studies the role of distributive conflict in growth models. The third part deals with the positive analysis of macro-relevant institutions, mostly fiscal institutions and labor market institutions, including the issue of reform dynamics.

7.2. The political economy of short-term fiscal and monetary policy

7.2.1. Opportunistic governments

7.2.1.1. Discretionary money creation in the Cagan hyperinflation model

A proto-political economy literature explains monetary and/or fiscal policies based on the view that governments behave opportunistically. Perhaps, we should return to the Cagan (1956) model of hyperinflation. Surely, hyperinflation is a highly undesirable outcome. Furthermore, as the government controls money creation, it could refuse to accommodate accelerating inflation and thus avoid a hyperinflationary episode. Consequently, hyperinflation is to a large extent a consequence of government policies that allow for it. How could we explain this, since, intuitively, a benevolent government should try to avoid it? In the Cagan model, the government must finance a given level of real public spending G through money creation:

$$\dot{M}_t = P_t G,$$

where M is the money stock and P is the price level at date t . There is a demand for real balances which, through its impact on nominal interest rates, is a decreasing function of expected inflation. We follow Cagan and use an exponential specification:

$$\frac{M_t}{P_t} = a e^{-b\pi^e}, \quad [7.1]$$

4. The focus is therefore similar to the Drazen (2002) textbook.

where π^e is the expected inflation. Finally, expectations are formed adaptively, through an error-correction process:

$$\dot{\pi}^e = \lambda(\pi - \pi^e).$$

The model can be reduced to a first-order law of motion for expected inflation:

$$\dot{\pi}^e = \frac{\lambda}{1 + \lambda b} \left(G e^{b\pi^e} / a - \pi^e \right).$$

Clearly, if G is too high, the term in brackets is strictly positive for any π^e . In this case, expected inflation and, by way of consequence, inflation itself, keep accelerating. Thus, the root cause of hyperinflation is excess monetization of budget deficits. A similar conclusion holds in the Calvo (1978) rational expectations version of the model.

In this economy, the government is opportunistic in two ways. First, its objective is to obtain an amount G of seigniorage. This makes it somewhat similar to a Leviathan government – i.e. a government that aims at the highest possible level of revenues – in that it must be controlled by some faction which gains from getting G . Second, it satisfies its objective by setting the inflation rate at each date, through money creation, irrespective of any precommitment or institutional constraint.

In order for the above analysis to be valid, three questions must be answered:

- 1) Why is the government unable to reduce the fiscal deficit by raising taxes?
- 2) Why is the government unable to reduce the fiscal deficit by lowering spending?
- 3) Why is the government unable to finance its deficit by issuing debt instead of creating money?

The political economy literature has provided some elements of answers to those questions. The ability to raise taxes depends on “state capacity”, which may be defined as the general level of productivity in government. Economies with a low state capacity have a low efficiency in collecting taxes, and are therefore more likely to monetize deficits. Lowering spending may be prevented because interest groups compete to poach a common pool of fiscal resources or resist bearing the burden of fiscal stabilization (see sections 7.2.3.3 and 7.2.4.1). Finally, it may be that issuing debt is too costly because markets would purchase government bonds only at a prohibitive risk premium. This brings about the issue of the *credibility* of fiscal and monetary policy, to which we now turn.

7.2.1.2. *Credibility of government policies*

The abundant literature on credibility aims at explaining apparently inefficient outcomes, such as excess inflation or excess deficits, or an inefficient tax structure, as equilibria of dynamic games played between policymakers and the public. The general

point, here, is that it is optimal to precommit on future policies, yet a government that can act in a discretionary fashion will be tempted to renege on the preannounced policy course when it is time to implement it.

To analyze these issues, let us start with a few general, and rather obvious, observations.

First, the issue should not arise outside an interactive framework. Consider, for example, an agent who lives for two periods and maximizes an indirect utility function

$$U = u(a_1, a_2),$$

where the agent takes action a_i at date i .⁵ Assume the agent decides a_1 and a_2 at $t = 1$. Denoting derivatives by subscripts and assuming differentiability, the optimal course of action, denoted by a_1^*, a_2^* , is characterized by the two first-order conditions (FOC):

$$u_1(a_1^*, a_2^*) = 0$$

$$u_2(a_1^*, a_2^*) = 0.$$

Now, assume that, at $t = 2$, the agent can freely *reoptimize*, i.e. pick a_2 independently of the plan for a_2 that was made at $t = 1$. This *discretionary* choice for a_2 generally depends on past decisions a_1 . Denoting it by $\tilde{a}_2(a_1)$, this function satisfies

$$u_2(a_1, \tilde{a}_2(a_1)) = 0.$$

It is clear that

$$\tilde{a}_2(a_1^*) = a_2^*,$$

meaning the agent will proceed according to plan.

Second, in any situation, an agent can always get a higher pay-off under commitment than under discretion. This is because it is always possible for the agent to compute their future choices, contingent on the history of shocks and the actions of other agents, and to commit on a rule, contingent on the same variables, that replicates those choices. In other words, the choice set of the agent is larger under commitment than under discretion.

For the record, let us note that deviations from rationality may invalidate these two principles: an agent with behavioral biases may, at date $t = 2$, have preferences

5. Here, a_i can be interpreted as any type of economic decision: investment, saving, labor supply, allocation of consumption and so forth.

for a_2 that differ from $u(a_1, \cdot)$, in which case we will have $\tilde{a}_2(a_1^*) \neq a_2^*$.⁶ In such a situation, everything takes place as if a_2 is chosen by a different individual from the “incarnation” of the agent at $t = 1$. Also, an agent with bounded rationality may be unable to commit on a rule that is too complex, in which case discretionary behavior may lead to superior outcomes.

To understand the key issue of credibility, assume now that the above agent is the policymaker and only chooses a_2 , while a_1 is chosen by the public whose payoff is given by some function $v(a_1, a_2)$. The choice of a_1 by the public generates a reaction function

$$a_1 = f(a_2^e),$$

where a_2^e is the policy decision at $t = 2$ expected by the public at $t = 1$.

Suppose the policymaker decides on a_2 at $t = 1$. Then, the resulting policy $\hat{a}_2$ satisfies

$$u_1(f(\hat{a}_2), \hat{a}_2)f'(\hat{a}_2) + u_2(f(\hat{a}_2), \hat{a}_2) = 0. \quad [7.2]$$

Assume again that the policymaker can reoptimize at $t = 2$ and choose a_2 irrespective of what was planned at $t = 1$. The novelty here is that a_1 has already been undertaken. The choice of a_2 can ignore its feedback effect on a_1 , which was not the case when setting the plan at $t = 1$. Consequently, the policy choice $\tilde{a}_2(a_1)$ simply satisfies

$$u_2(a_1, \tilde{a}_2) = 0, \quad [7.3]$$

and, as long as $u_1 f' \neq 0$, $\tilde{a}_2 \neq \hat{a}_2$, the actual policy will differ from the planned one. The ex ante optimal plan is *time-inconsistent*. The policymaker has an incentive to deviate from the preannounced plan, because the public can act upon expectations of a_2 at $t = 0$, but is “stuck” with its past decisions at $t = 1$. Deviating allows the policymaker to improve its welfare, relative to implementing the plan, by disconnecting expectations from realizations. The optimal solution described by [7.2] will actually be implemented only if the government fully commits to its preset plan; that is, it refrains from deviating. This is what we mean by a credible government.

In the limit, a policymaker which knows it will deviate while being able to make announcements that are believed by the public, will not pick a_2^e according to [7.2] but, instead, so that $u_1 f' = 0$. That is, it will manipulate expectations so as to get its most favorable value of a_1 , while picking the corresponding optimal choice for a_2 . If f spans the whole space, the policymaker will be able to replicate the first-order conditions $u_1 = u_2 = 0$ that correspond to the case where it could directly impose

6. See Elster (1986).

a_1 upon the public. Of course, neither this outcome nor the one characterized by [7.2–7.3] are rational expectations equilibria. In such an equilibrium, also called a *time-consistent* solution, the public must correctly anticipate the future actions of the government. Hence, the REE, denoted with superscript R , is characterized by

$$\begin{aligned} a_1^R &= f(a_2^R) \\ u_2(a_1^R, a_2^R) &= 0. \end{aligned}$$

The preceding discussion is summarized in Figure 7.1.

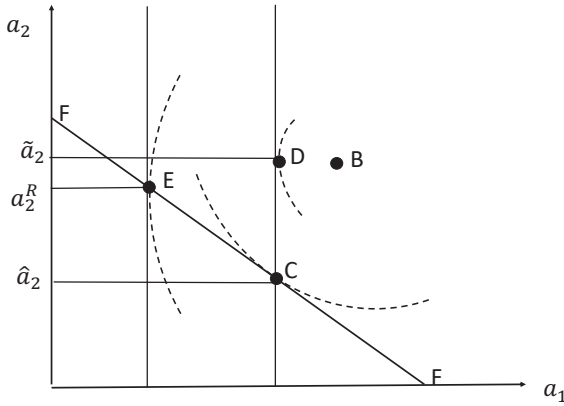


Figure 7.1. Commitment, deviation, and time-consistent outcomes

I have assumed that f is an affine decreasing relationship between a_1 and a_2 , represented by the FF schedule, and that the preferences of the policymaker are associated with some bliss point B whose coordinates are (a_1^*, a_2^*) .⁷ This point, drawn above FF and such that $u_1 = u_2 = 0$, is precisely the outcome the government would get if agents naively believe its promises, while it will deviate from them. Point D , characterized by a value of $a_1 = f(\hat{a}_2)$ and $a_2 = \tilde{a}_2$, is the outcome picked by a policymaker who planned to implement the optimal plan determined by [7.2] but eventually chose a_2 discretionarily. Point C is the full commitment outcome, corresponding to a plan which is implemented, against the incentives to deviate, i.e. characterized by [7.2]. Point E is the rational expectations equilibrium: it must lie on FF , otherwise expectations would not be rational, and the indifference

7. To avoid confusion, though, observe that preferences are not necessarily circular – and are not so in Figure 7.1. Otherwise, the policy chosen when deviating, $\tilde{a}_2$, would coincide with the time consistent equilibrium policy a_2^R , since all points with a vertical tangent to the indifference curve are then aligned along a horizontal one going through B . This is not the case, of course, for more general preferences.

curve must have a vertical tangent, capturing the fact that a_2 is chosen discretionarily once a_1 has been set. It is clear from Figure 7.1 that, among the two potential rational expectations outcomes – the time-consistent one and the full commitment, time-inconsistent, one – point C yields a higher welfare to the policymaker. This is an instance of the above-stated principle that commitment always dominates discretion.

An important application of this model, discussed by Kydland and Prescott (1977), is capital taxation. We can interpret a_1 as the capital stock, resulting from investment decisions by the public, and a_2 as the tax rate on capital. Clearly, it makes economic sense to assume that there is a decreasing relationship between capital taxation and capital. Also, it makes sense to assume that B be above FF , which rules out the unpalatable case that the economy delivers “too much capital” if the government picks its bliss level tax rate. Therefore, we are in the configuration of Figure 7.1. We can then see that when it is able to commit on future capital taxation, the government will choose a lower tax rate and get a higher capital stock, as in point C , than when it sets it discretionarily, as in point E . The explanation is simple: once productive investment has been made, it is irreversible, and increasing the tax rate on installed capital is much less distortionary than when such an increase is decided before the investment decision. Consequently, the government has an incentive to renege on any pre-announced optimal tax policy which takes into account the negative effect of taxes on investment, and if it does, it will pick a higher tax rate than the preset one. Therefore, the rational expectations equilibrium reflects the understanding by private agents that the government has an incentive to deviate, and it involves a higher tax rate and a lower capital stock than the time-inconsistent, full commitment outcome.

In practical terms, this means that countries where the government lacks credibility, perhaps because they do not have built-in institutional mechanisms that allow us to precommit future policies, will have higher interest rates, a higher marginal product of capital, lower wages, lower GDP per capita, less FDI, all else being equal, than other countries. This may help explain a number of phenomena, like, for example, the Lucas (1990) paradox, which states that capital fails to move from (typically rich) countries where its marginal product is low to (presumably) poor countries where its marginal product is high.

The same logic can be applied to public debt: as default is a tax on holders of public debt, a government that can commit not to default will be able to issue more public debt and to enjoy a lower risk premium on its bonds. Governments that lack an ability to commit may be able to issue debt only at a very high cost and are more likely to monetize budget deficits instead.

7.2.1.3. *Inflation bias: the Barro–Gordon model*

The Barro–Gordon (1983) model predicts that lack of credibility leads to an inflationary bias. It rests on two key assumptions. First, there is a “Lucas” aggregate supply curve relating output to the difference between actual and expected inflation.

Therefore, only inflationary surprises matter, and consequently the government cannot systematically exploit an inflation-output trade-off so as to manipulate aggregate demand in order to stimulate output. Second, the economy is plagued by structural distortions, such as union power, monopoly power, inefficient bureaucracy, low state capacity or corruption, that lead to the natural rate of output being lower than its optimal counterpart. This creates a temptation for the government to boost output by creating an inflationary surprise. Yet, this temptation is understood by private agents, who factor it into their (rational) expectations of inflation. In equilibrium, the government just creates too much inflation, while being unable to actually stimulate output, since that excess inflation is perfectly anticipated by the public. If the government could commit on a future course of action, it would credibly pick a zero inflation rate, achieving the same output with a lower cost of inflation.

At this stage, it is worth spelling out the model's key equations. The aggregate supply curve is given by

$$y = \alpha(\pi - \pi^e) + \bar{y}, \quad [7.4]$$

where $y = \log$ output, $\pi = \text{inflation}$, $\pi^e = \text{expected inflation}$ and $\bar{y}$ is a parameter that pins down the *natural* rate of output, which prevails when expectations are correct. This specification for the aggregate supply curve is convenient for a static model and reflects the prevalence of a “Lucas” supply curve, i.e. a Phillips curve relating y_t to $\pi_t - E_{t-1}\pi_t$, at the time this literature was written. Predictions would change somewhat if instead the aggregate supply curve were given by the “New Keynesian Phillips Curve”

$$y_t = \bar{y} + \alpha(\pi_t - \beta E_t \pi_{t+1})^8. \quad [7.5]$$

8. See Clarida et al. (1999). In such a world, perfectly expected inflation raises output as long as $E_t \pi_{t+1}$ does not go up by too much. Furthermore, there exists a long-run trade-off between output and inflation as long as $\beta < 1$. Adapting [7.7] to the NK Phillips curve yields

$$\pi_t = \frac{\alpha\varphi}{1 + \alpha^2\varphi} [\alpha\beta E_t \pi_{t+1} + y^* - \bar{y}]$$

and

$$y_t - \bar{y} = \frac{\alpha^2\varphi(y^* - \bar{y}) - \alpha\beta E_t \pi_{t+1}}{1 + \alpha^2\varphi}$$

In a steady state, π is constant over time and we get

$$\pi = \frac{\alpha\varphi}{1 + \alpha^2\varphi(1 - \beta)}(y^* - \bar{y})$$

$$y - \bar{y} = \frac{\alpha^2\varphi(1 - \beta)}{1 + \alpha^2\varphi(1 - \beta)}(y^* - \bar{y}).$$

The inflationary bias is now associated with a positive effect on output, unless $\beta = 1$, in which case the predictions resemble that of the standard Barro–Gordon model.

The government sets the inflation rate π directly, by “fine-tuning” the economy, i.e. setting aggregate demand, through fiscal and monetary instruments, so as to deliver its preferred outcome along the output-inflation trade-off. Its objective function reflects losses from unemployment (or suboptimal activity) and inflation and is given by

$$U(\pi, y) = -\pi^2 - \varphi(y - y^*)^2, \quad [7.6]$$

where y^* is a parameter that pins down the *optimal* level of output, and φ is the weight put on output stabilization around that optimal level, relative to price stability, in the objective function.⁹ Following Barro and Gordon, we assume that $y^* > \bar{y}$, meaning that structural distortions deliver a suboptimally low natural rate of output.

Inflationary expectations are set prior to the choice of π by the government. This opens up the possibility of acting in a discretionary fashion.

The analysis of the previous section applies. We can again use Figure 7.1 by setting $a_1 = -\pi^e$ and $a_2 = \pi$. The FF schedule is defined by $a_1 + a_2 = 0$ and goes through the origin. The bliss point for the government is obtained at $\pi = 0$ and $y = y^*$, i.e. $a_1^* = \frac{y^* - \bar{y}}{\alpha} > 0$, $a_2^* = 0$. This defines a point B which is clearly above FF , so since in Figure 7.1 point E is above and on the left of C , the equilibrium without commitment has more inflation than the equilibrium with commitment. The latter is easy to compute: since the government cannot affect output in a predictable way, the best inflation rate it can precommit on is $\hat{\pi} = 0$. Yet, once expectations are set at $\pi = 0$, the government is tempted to deviate by setting a higher inflation rate. As a function of expected inflation, the rate set by a discretionary government is such that

$$U_1 + \alpha U_2 = 0 \iff \pi = \frac{\alpha \varphi}{1 + \alpha^2 \varphi} [\alpha \pi^e + y^* - \bar{y}]. \quad [7.7]$$

Thus, in particular, at $\pi^e = 0$, $\pi > 0$. The discretionary government will renege on the inflation rate it would have precommitted to. The REE equilibrium corresponding to point E can then be computed by equating π and π^e in the preceding formula, yielding

$$\tilde{\pi} = \alpha \varphi (y^* - \bar{y}) > 0. \quad [7.8]$$

This simple formula encompasses the three main predictions of the model. First, the inflationary bias is larger, the greater the structural distortions in the economy, i.e. the greater the gap between y^* and $\bar{y}$. In fact, absent these distortions, i.e. if $y^* = \bar{y}$, there is no inflationary bias: $\tilde{\pi} = \hat{\pi} = 0$. There is no credibility problem either, as

9. We assume the optimal inflation rate is zero, but nothing would be changed if it were nonzero instead: the inflation bias would then be measured as the difference between actual inflation and optimal inflation.

the optimal plan yields $\pi = 0$ and $y = y^* = \bar{y}$. This coincides with the special case where, in Figure 7.1, the bliss point B lies on FF . The indifference curve is then degenerate and B coincides with C, D , and E . Second, the inflationary bias is larger, the greater the weight on output in the government's objective function. In other words, the more concerned the government is about the inadequacy of the natural rate of output, the more it will harm society by creating excess inflation to no avail. Third, the inflationary bias is larger, the greater the magnitude of the response of output to an inflationary surprise, α . The inefficient inflation bias does not arise from any conflict of interest between the government and the public: nothing is changed in the analysis if U coincides with the actual social welfare of that economy. The inflation bias is due to the government's inability to commit to its future policy course. The attempt to create an inflationary surprise is *well intended*: the government uses monetary policy to bring y closer to y^* , a socially desirable objective. The attempt is defeated in equilibrium precisely because it is expected by the public: inflationary expectations go up, in response to the government's incentives to deviate, to the point where the marginal benefit of an inflationary surprise just equates its marginal cost. For this, the actual inflation must be high enough – at $\pi = \pi^e = 0$, the marginal welfare cost of raising inflation above π^e is second order compared to its marginal benefit due to greater output; therefore, $\pi = 0$ cannot be an equilibrium absent commitment. Ex ante, the public would agree to become less rational, so as to allow the government to actually implement a welfare-improving inflationary surprise. For example, society would benefit if people were indoctrinated to expect a fixed inflation rate lower than $\tilde{\pi}$ rather than to form their expectations rationally, with the best possible π^e being equal to $-a_1^* < 0$. This is because, since $\partial U / \partial \pi^e = -\varphi \alpha (y^* - y) < 0$, lowering inflationary expectations is a “free lunch” as long as $y < y^*$, since it delivers a more favorable output-inflation trade-off.

The relevance of the analysis rests on the assumption that the government is unable to raise $\bar{y}$ through structural reforms. This opens up another research issue, which is that these reforms may be blocked by special interests. We return to it below.

7.2.1.4. *Solutions to the time consistency problem*

Following the seminal contributions of Kydland and Prescott and Barro and Gordon, the subsequent literature has analyzed the extent to which a commitment outcome (point C) could be enforced, i.e. in the case of monetary policy, the central bank would be credible in its pledge to pursue a low inflation target. Let us mention some approaches:

- Trigger strategies: in a repeated game, we can show that, under some parameter values, an equilibrium exists by which the central bank chooses $\pi = 0$ each period. If the central bank deviates (choosing point D which yields a higher intra-period payoff), then the public punishes it by coordinating its expectations on $\tilde{\pi}$ every period, during some amount of time. It is then rational for the bank to indeed pick $\pi = \tilde{\pi}$; the punishment comes from the fact that each period, the economy is at the static

discretionary equilibrium E instead of C . If the loss from punishment is high enough, the central bank will refrain from deviating and will indeed choose C each period. This equilibrium is subgame perfect in the sense that, given beliefs, each agent acts rationally following a deviation. That is, if it is common knowledge that $\pi = \tilde{\pi}$ for, say, K periods after a deviation, then during that punishment spell, it is optimal for the public to set $\pi^e = \tilde{\pi}$ and then for the policymaker to indeed fulfill those expectations by picking $\pi = \tilde{\pi}$. This approach has a number of flaws. First, it rests on the agents having an infinite horizon. Otherwise, the central bank would deviate in the last period, and, by induction would always deviate. Then, the only equilibrium will be the time-consistent outcome E at each date, called “repeated Nash”. Second, there exist a large number of subgame-perfect equilibria, including repeated Nash, which is supported by the simple self-confirming beliefs that the central bank will always pick $\pi = \tilde{\pi}$. Nevertheless, this approach has a rather robust prediction, namely that a commitment outcome is more likely to be supported as a subgame perfect equilibrium, the lower the discount rate of the policymaker. This is because the gains of deviating accrue in the current period, while the costs of deviation are experienced after that.

– Reputation: in this approach, proposed by Backus and Driffill (1985), following Kreps and Wilson (1982), policymakers are of different types. A “strong type” has preferences such that $\varphi = 0$ and therefore always implements $\pi = 0$. A weak type has $\varphi > 0$. At any date t , inflationary expectations are given by $\pi_t^e = (1 - p_t)\tilde{\pi}_t$, where p_t is the probability that the policymaker is strong and $\tilde{\pi}_t$ is the average inflation rate implemented by the weak policymaker. Assume there are two periods. Then, at $t = 2$, a weak government will act discretionarily and pick π according to [7.7], yielding

$$\begin{aligned}\tilde{\pi}_2 &= \frac{\alpha\varphi(y^* - \bar{y})}{1 + \alpha^2\varphi p_2} \\ y_2 &= \frac{\alpha^2\varphi p_2 y^* + \bar{y}}{1 + \alpha^2\varphi p_2}.\end{aligned}$$

Clearly, the outcome is closer to the bliss point $\pi = 0$ and $y = y^*$, the greater p_2 , i.e. the greater the “reputation” of the policymaker for being tough. This, in turn, creates an incentive for the policymaker to choose $\pi_1 = 0$ so as to preserve its reputation, since under any other choice, the public, which is Bayesian, would update its beliefs to $p_2 = 0$, while we have $p_2 = p_1$ if the policymaker acts as “tough” at $t = 1$. One weakness of this approach, in the case of the Barro–Gordon model, is that if the authorities are benevolent and U is social welfare, it is unclear why the public would put any positive probability on the likelihood that they would be “tough”.

– Strategic delegation (Rogoff, 1985): the idea, here, is that the central bank is made independent of government pressure to act discretionarily, while a conservative central banker (CCB) is appointed. The CCB has preferences such that $\varphi = 0$ and therefore picks $\pi = 0$ no matter what. As shown by Rogoff, this entails costs if there are shocks, because the CCB will not stabilize those shocks appropriately, as they

would put too much weight on price stability relative to output stability, compared to society. Therefore, delegation to a CCB is optimal provided the volatility of aggregate shocks is not too large relative to the magnitude of the inflationary bias, i.e. the RHS of [7.8].¹⁰

– It has been argued that institutions could be implemented that would change the structure of the economy so as to reduce the equilibrium inflationary bias. For example, Giavazzi and Pagano (1988) argue that in an open economy, joining a fixed exchange rate regime would make inflation more costly to society, due to the associated loss of competitiveness, thereby reducing the inflationary bias. It remains to be explained why it is easier to credibly commit to the exchange rate regime than to credibly commit to a low inflation target, and why such an institutional change can be implemented, contrary to structural reforms that would raise $\bar{y}$ and therefore lower π .

7.2.2. Elections and business cycles

The credibility literature recognizes opportunistic behavior in policymakers and predicts that such opportunism will lead to inefficient biases in fiscal and monetary policy. Yet, it falls short of relating these issues to actual political institutions. This is what the literature on political business cycles tries to achieve. We can classify it into two areas: *non-partisan theories* relate fiscal and monetary biases to the electoral cycle; *partisan theories* relate them to the ideology of the party in power.

7.2.2.1. Non-partisan theory

An early model of political business cycle is Nordhaus (1975). Its key point is that, prior to an election, the government will stimulate the economy so as to boost its chances of being reelected. Therefore, we should expect expansions to occur shortly before election years, and recessions to occur when the next election is remote into the future. Despite its common sense, this theory is plagued by a number of conceptual as well as modeling issues. Nordhaus considers a government which optimizes over two periods, indexed by $t = 1, 2$, and whose objective function is $\gamma U(\pi_1, y_1) + U(\pi_2, y_2)$, where U is defined by [7.6]. The government is in power over these two periods, and there is an election at the end of period 2. The election process is not modeled as such, but captured by the fact the government's horizon does not go beyond its current mandate and by the weight $\gamma < 1$ on the first period, which conveys the idea that election years are more important to politicians than other periods in their mandate. Yet, in each period, it is optimal to set π according to [7.7]: the expression is the same at each date and does not depend on γ . The two periods differ, however, because there

10. In principle, though, if the ecosystem of preferences is large enough, it should be possible to pick a central banker whose preferences are such that $y^* = \bar{y}$ while the weight on output in their utility function remains equal to φ . Such a policymaker will not have any inflationary bias and at the same time will appropriately stabilize shocks.

is a dynamic linkage between date 1 and date 2, due to the fact that expectations are adaptive and revised according to

$$\pi_{t+1}^e = \theta \pi_t + (1 - \theta) \pi_t^e.$$

While π_1^e is inherited from the past, the government can manipulate π_2^e through its choice of π_1 . Also, its choice of π_2 will affect expectations for the subsequent date, but that is beyond its mandate and not taken into account in its objective. Hence, the marginal benefit of inflation at $t = 2$ is $U_1(\pi_2, y_2)$, while the marginal benefit of inflation at $t = 1$ is $\gamma U_1(\pi_1, y_1) + U_2(\pi_2, y_2) \frac{\partial y_2}{\partial \pi_2^e} \frac{\partial \pi_2^e}{\partial \pi_1} = \gamma U_1(\pi_1, y_1) - \alpha \theta U_2(\pi_2, y_2)$. Because of the negative term $-\alpha \theta U_2(\pi_2, y_2)$, the government will tend to inflate less at $t = 1$ than at $t = 2$, and this effect is stronger, the lower γ , i.e. the bigger the relative weight of the election year in the government's objective. Thus, the model predicts a business cycle that will be synchronized with the election cycle, with each mandate starting with a recession and ending with an expansion. But the driving force for this asymmetry rests on the fact that expectations are adaptive and that the government strategically manipulates them to profit from lower expected inflation at date 2. If, for example, inflation expectations were fixed, i.e. $\theta = 0$, the asymmetry between the two dates would disappear, as the FOCs for inflation would be $U_1(\pi_1, y_1) = U_1(\pi_2, y_2) = 0$. If expectations were rational, π_2^e would be forward looking and independent of π_1 , and inflation would be the same in each period and given by [7.8]. In both cases, policy would not depend on γ , because the governments, having two instruments π_2 and π_1 , can de facto target $U(\pi_1, y_1)$ and $U(\pi_2, y_2)$ independently.

Two additional caveats apply. First, under rational expectations, the government cannot generate a systematic inflationary surprise, and output will stick to its natural rate $\bar{y}$ in each period. Any attempt to stimulate the economy at the end of the mandate would then only deliver higher inflation and paradoxically reduce welfare, which, intuitively, should make it more likely that the election is lost. The economy can be in a boom at $t = 2$ only to the extent that the greater inflation level is partly unexpected, due to the adaptive character of expectations. Second, the model implicitly assumes that voters reward the government for policies that deliver a higher level of U , i.e. that voting takes place retrospectively. Yet, rational agents should base their voting decisions on expectations of future policies, not on past outcomes.

The Rogoff—Sibert (1988) model addresses all these issues. The incumbent government has an unobservable type, called competence. More competent governments are more efficient at levying taxes, i.e. they have a greater state capacity. Prior to an election, the government sets public spending, which will be financed after the election. Rogoff and Sibert construct a “separating” equilibrium where a competent government picks an inefficiently high spending level, which an incompetent government would have avoided because it is too costly for it. The public then rationally infers that the government is competent, which raises its reelection probability. If we supplement their model with Keynesian effects, for

example, by introducing sticky prices in the private sector, then we would get the Nordhaus prediction that there is a boom, and therefore an inflationary burst prior to an election.¹¹ Finally, the reason why the incumbent government has a greater incentive to signal its type in the election year rather than prior to it is that there are unobservable competence shocks, so that its latest actions are the most informative for voters.

7.2.2.2. *Partisan theories*

Partisan theories relate the fiscal and monetary policy stance to the ideology of the party in power. Hibbs (1987) puts it simply by assuming that, on an output-inflation trade-off, a left-wing government should pick a greater level of output and, therefore, of inflation, than a right-wing one (see Figure 7.2). Why is that so? Presumably, the Left cares about the interests of the working class, who are collectively better-off, the higher the employment rate, while the Right cares about the interests of savers, who want to protect their nominal assets against inflation.

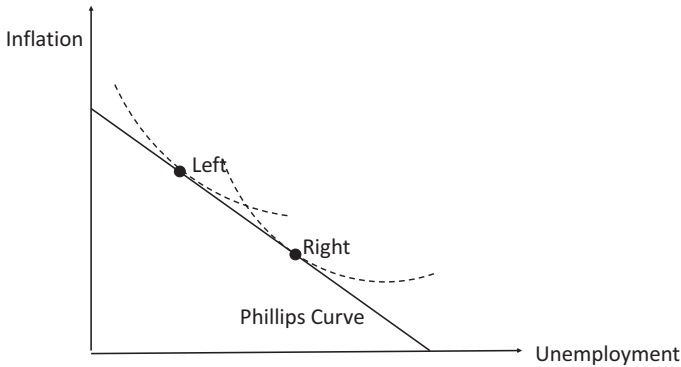


Figure 7.2. *Partisan determination of policy choices in the Hibbs model*

These views are somewhat naive: should we think of the Left as representing the interests of employed workers or unemployed workers? While the latter benefit from a boom as they have a greater chance of finding a job, the former, in many models,

11. With a Lucas supply curve, the stimulus exerted by the government in an election year will typically lead to an expansion, because the associated inflation is in part a surprise, due to the fact that ex ante the type of the government is unknown, and only revealed ex post by its policy choice, which reveals its type due to the fact that this is a separating equilibrium. These effects are similar to those observed in the Alesina (1988) model discussed below. With a New Keynesian supply curve such as [7.5], it is natural that $E_t \pi_{t+1} < \pi_t$ in election years, since date $t + 1$ is not an election year and therefore the economy will experience a smaller stimulus at $t + 1$ than at t . Consequently, even with $\beta = 1$, the model predicts a boom in the election year.

lose from a boom to the extent that their nominal wages are sticky and the boom is inflationary. Conversely, savers may shield their portfolio against inflation by investing in real or indexed assets. To the extent that it is harder to index wages against inflation than purchasing indexed bonds or stocks, we may even argue that the working class is more vulnerable to inflation than the middle class. Finally, whose interests are actually represented by the so-called Left and Right remains an open question.

Nevertheless, let us abstract from these issues and consider that there are two parties that differ in their preferred point on the Phillips curve, and label the most inflationary one as “Left”, and the other as “Right”. This can be formalized by assuming that each party is endowed with preferences over macroeconomic outcomes given by [7.6], with the weight on output φ being greater for the Left, i.e. $\varphi_L > \varphi_R$. In the Hibbs model, an upward sloping trade-off between output and inflation exists, and it follows that the economy has both more output and more inflation under the Left than under the Right.

Yet, under rational expectations and a supply curve given by [7.4], nobody should vote for the Left, since its perfectly anticipated attempt to stimulate output will only generate more inflation. That is, unless the Left can credibly commit to a low inflation rate, in which case it hardly deserves to be labeled “Left”. Indeed, as the full commitment equilibrium is such that $\pi = \hat{\pi} = 0$, it is independent of φ : the Left and the Right, if they can precommit, will pursue the same macro-policies, despite their different preferences. Such policy convergence has nothing to do with the one obtained under Downsian competition, which would rather be interpreted as convergence in platforms (i.e. in our case, the value of φ) so as to cater to the median voter in order to prevail in an election. It simply reflects the fact that the equilibrium output level, if expectations are correct, is independent of macro-policy.

Alesina (1988) brings the partisan theory of political business cycles in line with the rational expectations hypothesis. It is now assumed that expectations are formed prior to the election whose result is uncertain. The electoral business cycle stems from the fact that the inflation rate picked by the election winner after the election is partly a surprise, because it was not certain that this party would win the election.

Let p be the probability that the Left wins the election. By [7.7], it will pick an inflation rate equal to

$$\pi_L = k(\varphi_L) [\alpha\pi^e + y^* - \bar{y}],$$

where $k(\varphi)$ is an increasing function of φ defined by $k(\varphi) = \frac{\alpha\varphi}{1+\alpha^2\varphi}$. Similarly, the inflation rate chosen by the Right, if it wins, is

$$\pi_R = k(\varphi_R) [\alpha\pi^e + y^* - \bar{y}] < \pi_L.$$

Ex ante, the expected inflation is formed rationally and therefore given by

$$\pi^e = p\pi_L + (1 - p)\pi_R.$$

Solving for π^e , we get

$$\pi^e = \frac{\bar{k}(y^* - \bar{y})}{1 - \alpha\bar{k}},$$

where $\bar{k} = pk(\varphi_L) + (1-p)k(\varphi_R)$. From there, it is straightforward to compute output and inflation depending on the election winner:

$$\pi_i = \frac{k(\varphi_i)(y^* - \bar{y})}{1 - \alpha k};$$

$$y_L = \bar{y} + (1-p)\frac{\alpha(y^* - \bar{y})}{1 - \alpha\bar{k}}(k(\varphi_R) - k(\varphi_L))$$

$$y_R = \bar{y} - p\frac{\alpha(y^* - \bar{y})}{1 - \alpha\bar{k}}(k(\varphi_R) - k(\varphi_L))$$

The magnitude of inflation and output fluctuations after the election is greater, the greater the gap between the optimal and natural rates of output, $y^* - \bar{y}$. As in Barro and Gordon, the inflationary bias would disappear absent distortions to real output. Here, however, the inflationary bias affects output, because part of monetary policy is unanticipated due to electoral uncertainty. A victory of the Right leads to a recession, because then the actual inflation is lower than expected. Conversely, a victory of the Left leads to an expansion.

Observe that, on average, output remains equal to $\bar{y}$: $Ey = py_L + (1-p)y_R = \bar{y}$. It is also informative to compute the variance of GDP

$$\begin{aligned} V = E(y - \bar{y})^2 &= [p(1-p)^2 + (1-p)p^2] \left(\frac{\alpha(y^* - \bar{y})}{1 - \alpha\bar{k}} (k(\varphi_R) - k(\varphi_L)) \right)^2 \\ &= p(1-p) \left(\frac{\alpha(y^* - \bar{y})}{1 - \alpha\bar{k}} (k(\varphi_R) - k(\varphi_L)) \right)^2 \end{aligned}$$

GDP volatility is larger, (i) the greater the ideological differences between the two parties, as captured by $(k(\varphi_R) - k(\varphi_L))$, (ii) the greater the gap between the optimal and natural outputs, $y^* - \bar{y}$, which magnifies the divergence in actual policies between the two parties, and (iii) the greater the electoral uncertainty, as measured by $p(1-p)$, which is maximum at $p = 0.5$ and is in fact the variance of a random dummy variable equal to 1 if the Left wins.

The model predicts that in years that are not preceded by an election, output reverts to the natural rate, regardless of the party in power. This is because all uncertainty regarding the identity of the policymaker has disappeared. On the other hand, the difference in inflation rates is magnified: inflation is now determined by [7.8], i.e. $\hat{\pi}_i = \alpha\varphi_i(y^* - \bar{y})$, and it is straightforward to check that $\hat{\pi}_L > \pi_L > \pi_R > \hat{\pi}_R$.

After an election, the Left enjoys an expectational benefit due to the fact that the Right might have won with some probability, while the Right suffers an expectational penalty for the opposite reason. These effects are no longer present in subsequent years of the political mandate.

7.2.3. Strategic fiscal policies

The political business cycle literature analyzes how electoral and partisan considerations may relate to macroeconomic fluctuations. But such considerations also have implications for fiscal and monetary policies on average. The general result here is that partisan conflicts of interest, as well as electoral uncertainty and political instability, tend to lead to excess deficits and spending.

7.2.3.1. The benchmark: tax smoothing (Barro 1979)

Our starting point is a simple two-period model, which will turn out to be a useful template for discussing a number of contributions in the literature. We consider some social group, and assume its utility is

$$U = (a_1 g_1 - \theta \frac{g_1^2}{2}) - \tau_1 - \gamma \frac{\tau_1^2}{2} + \beta \left((a_2 g_2 - \theta \frac{g_2^2}{2}) - \tau_2 - \gamma \frac{\tau_2^2}{2} \right). \quad [7.9]$$

Here, g_i is public spending at date i on some publicly provided group that supposedly benefits this social group, and τ_i is the tax rate at date i . The term $a_i g_i$ reflects the benefit to the group of public spending at date i , and we allow for the marginal benefit of spending, captured by a_i , to vary over time. The term $-\tau_i$ captures the direct resource cost of publicly provided goods, while $\gamma \frac{\tau_i^2}{2}$ is the distortionary cost of taxation.

The government can issue a possibly negative debt at $t = 1$, so that budget constraints are

$$g_1 = \tau_1 + d \quad [7.10]$$

$$g_2 = \tau_2 - Rd, \quad [7.11]$$

where $R \geq 1$ is the gross interest rate between the two dates. From there, we can solve for the optimal tax policy by substituting these two equations into [7.9] and maximize with respect to τ_i and d , getting the following first-order conditions:

$$\begin{aligned} a_1 - \theta g_1 - 1 - \gamma \tau_1 &= 0 \\ a_2 - \theta g_2 - (1 + \gamma \tau_2) &= 0 \\ a_1 - \theta g_1 &= \beta R(a_2 - \theta g_2) \end{aligned}$$

The above equations can be solved; in particular, we get

$$d = \frac{\gamma(a_1 - \beta R a_2) + \theta(1 - \beta R)}{\theta\gamma(1 + \beta R^2)}. \quad [7.12]$$

The determination of public debt reflects the incentives to smooth taxes over time in order to minimize their overall distortions, so that time fluctuations in the marginal benefit of public goods must be met with changes in public debt. Also, public debt reflects imbalances between the rate of time preference in the objective function and the real interest rate. In the benchmark case where the two are equal, i.e. $\beta R = 1$, the numerator would be equal to $\gamma(a_1 - a_2)$, and debt would be positive if and only if $a_1 > a_2$, i.e. spending is transitorily high. If impatience exceeds the interest rate, then $\beta R < 1$, and the constant term in the numerator $1 - \beta R$ is positive. Debt is then positive for $a_1 = a_2$, reflecting the value of postponing taxation, as at the margin of $d = 0$, the interest losses of doing so are smaller than the utility gains. As long as $d > 0$, a higher β reduces d : more patient governments choose a lower level of public debt.

The corresponding tax rates are

$$\begin{aligned} \tau_1 &= \frac{\beta R(Ra_1 + a_2) - (1 + \beta R^2) + \frac{\theta}{\gamma}(1 - \beta R)}{(1 + \beta R^2)(\theta + \gamma)}; \\ \tau_2 &= \frac{(Ra_1 + a_2) - (1 + \beta R^2) + \frac{\theta}{\gamma}R(1 - \beta R)}{(1 + \beta R^2)(\theta + \gamma)}, \end{aligned}$$

implying perfect tax smoothing, i.e. $\tau_1 = \tau_2$ if $\beta R = 1$, but, since $\tau_1 - \tau_2 \propto (\beta R - 1) \left((Ra_1 + a_2) + \frac{\theta}{\gamma}(R - 1) \right)$, governments that are more impatient relative to the interest rate, i.e. such that $\beta R < 1$, will pick a greater tax rate at $t = 2$ since it is beneficial to postpone the associated burden.

7.2.3.2. Political instability and excess deficits

Alesina and Tabellini (1990) and Tabellini and Alesina (1990) show that political instability is likely to lead to excess deficits relative to the tax smoothing benchmark. Their idea is that debt is used strategically by incumbent governments so as to bind the hands of future governments with a different political stance: the greater the level of debt, the more future governments will be prevented from implementing their preferred policies.

Suppose that there are two parties, denoted by L and R , and that they differ in the nature of the public good they favor. That is, their preferences are the same as in [7.9], but the L group only derives utility from public good g_L , while the R group derives utility from public good g_R . Let us assume that L is in power at date $t = 1$

and that at the end of this period there is an election, won by L with probability p . Let $i \in \{R, L\}$ be the party in power at $t = 2$, and $-i$ denote the other one. Let us assume $\beta = R = 1$, and $a_1 = a_2 = a$, so that, from the above, without elections we would have $d = 0$, and

$$g_1 = g_2 = \tau_1 = \tau_2 = \frac{a - 1}{\theta + \gamma}.$$

We want to know the budget deficit picked by the Left at $t = 1$, i.e. the value of d . Any positive value of d would be suboptimally high relative to the no election, tax smoothing benchmark.

We solve for the equilibrium by backward induction. The government at $t = 2$ picks g_2 so as to maximize period 2 utility, i.e.

$$U_2^{ii} = ag_2 - \theta \frac{g_2^2}{2} - \tau_2 - \gamma \frac{\tau_2^2}{2}, \quad [7.13]$$

subject to [7.11], where d is treated as fixed and exogenous since it is inherited from the choices made by the government prior to the election. Solving, we get that

$$a - 1 = \theta g_2 + \gamma \tau_2,$$

hence

$$g_2 = \frac{a - 1 - \gamma d}{\theta + \gamma}; \quad [7.14]$$

$$\tau_2 = \frac{a - 1 + \theta d}{\theta + \gamma}. \quad [7.15]$$

Not surprisingly, a higher level of debt reduces spending and raises taxes, as both margins are used to pay it back.

Since g_2 entirely consists of public good g_i , the costs of reimbursing the public debt that are associated with lower spending are entirely borne by group i . The crux of the argument is that party $-i$, if in power at $t = 1$, would ignore those costs, considering it would suffer only from the higher taxes, and would therefore pick too high a level of debt.

To proceed, let us compute the resulting marginal utility of public debt for each party at $t = 2$. From [7.13], we get that

$$\frac{dU_2^{ii}}{dd} = (a - \theta g_2) \frac{dg_2}{dd} - (1 + \gamma \tau_2) \frac{d\tau_2}{dd} = -\frac{a\gamma + \theta}{\gamma + \theta} - \frac{\gamma\theta}{\gamma + \theta} d < 0. \quad [7.16]$$

However, this only applies to party i . Party $-i$ does not value the public good produced by the government in power, so that its utility is given by

$$U_2^{i,-i} = -\tau_2 - \gamma \frac{\tau_2^2}{2}. \quad [7.17]$$

Consequently, for group $-i$, we get that

$$\begin{aligned} \frac{dU_2^{i,-i}}{dd} &= -(1 + \gamma\tau_2) \frac{d\tau_2}{dd} = -\frac{\theta(a\gamma + \theta)}{(\gamma + \theta)^2} - \frac{\gamma\theta^2}{(\gamma + \theta)^2} d \\ &= \frac{\theta}{\gamma + \theta} \frac{dU_2^{ii}}{dd} > \frac{dU_2^{ii}}{dd}. \end{aligned}$$

Clearly, the marginal disutility of debt is smaller for $-i$, who only bears the tax cost of fiscal adjustment, not the cost of reduced public spending.

Let us now analyze the choice of d made by L at $t = 1$. It picks τ_1 , g , and d so as to maximize its expected utility, given by

$$EU^L = ag_1 - \theta \frac{g_1^2}{2} - \tau_1 - \gamma \frac{\tau_1^2}{2} + pU_2^{LL} + (1 - p)U_2^{RL}$$

subject to [7.10]. The FOCs are

$$a - \theta g_1 + p \frac{dU_2^{LL}}{dd} + (1 - p) \frac{dU_2^{RL}}{dd} = 0 \quad [7.18]$$

$$-1 - \gamma\tau_1 - p \frac{dU_2^{LL}}{dd} - (1 - p) \frac{dU_2^{RL}}{dd} = 0. \quad [7.19]$$

Solving, we get

$$d = \frac{(1 - p)(\gamma a + \theta)}{\theta(\gamma(1 + p) + 2\theta)}. \quad [7.20]$$

This expression is strictly positive unless $p = 1$, and larger, the smaller the reelection probability of the incumbent government. Expectations of losing power lead to excess deficits. By comparing [7.20] with [7.12] for $R = 1$, we can see that the debt level is equivalent to the one picked by a more impatient government such that

$$\beta = \frac{\gamma p + \theta}{\gamma + \theta},$$

which can be interpreted as a weighted average of the discount factor applying to the future disutility from tax adjustments, equal to 1, and the discount factor applying to future disutility from spending adjustments, equal to p , since the latter is experienced only if L is reelected.

Hence, one interpretation of political instability is that it reduces the incumbent government's discount factor, which in turn leads to more myopic behavior, implying, here, suboptimally high deficits and debt.

Similarly, in the trigger strategy approach to the credibility problem, we have argued that the more patient the policymaker is, the more likely the commitment solution is to exist. Again, greater political instability would reduce the policymaker's discount factor and make it more likely that the only subgame perfect equilibrium is repeated Nash, i.e. the suboptimally high discretionary inflation rate.

The model by Persson and Svensson (1989) establishes the possibility of even more paradoxical results, compared to that of Alesina–Tabellini: a conservative government at $t = 1$, i.e. a government that wants a low level of public spending could run large deficits, despite its own conservative stance, in order to leave a high level of debt that would force subsequent, more “progressive” governments, to reduce their spending level, thus bringing it more in line with the preferences of the conservative government.

7.2.3.3. *Excess deficits as a common property problem*

A line of research considers lax fiscal policies as the result of divided government, as opposed to political uncertainty. That is, different branches of government control different expenditure items, while any fiscal imbalances are mutualized at a higher level of government. The ability of the central government to issue debt is then treated by each branch as a common pool of resources, from which it can tap, while bearing only a fraction of the future fiscal adjustment costs associated with the additional common debt generated by its own spending decisions.

This issue has been studied in a number of papers, including Tornell and Velasco (1992) and Tornell and Lane (1999). It may apply to several contexts: imperfect agency of several government bureaucratic branches, fiscal federation with local spending autonomy, fiscal indiscipline in monetary unions, and so forth.

Let us use the preceding model and assume there are n regions indexed by $j = 1, \dots, n$. Each region sets its own tax rate τ_{ij} and its own expenditure level g_{ij} at date i , in a unilateral fashion, while deficits are mutualized in a common debt d . Hence, we have that

$$d = \sum_{j=1}^n (g_{1j} - \tau_{1j}); \quad [7.21]$$

$$-d = \sum_{j=1}^n (g_{2j} - \tau_{2j}). \quad [7.22]$$

At $t = 2$, each region maximizes its period 2 utility, $U_{2j} = ag_{2j} - \theta \frac{g_{2j}^2}{2} - \tau_{2j} - \gamma \frac{\tau_{2j}^2}{2}$, subject to [7.22]. For any region j and any region-specific variable x , we denote by $x_{-j} = \sum_{j' \neq j} x_{j'}$. The FOCs for date 2 are

$$\tau_{2i} = \frac{a - 1 + \theta(g_{2,-i} - \tau_{2,-i} + d)}{\theta + \gamma},$$

$$g_{2i} = \frac{a - 1 - \gamma(g_{2,-i} - \tau_{2,-i} + d)}{\theta + \gamma}.$$

In any symmetrical Nash equilibrium, we must have $g_{2i} = g_2$ and $\tau_{2i} = \tau_2$, for all i , so that the preceding expressions yield

$$g_2 = \frac{a - 1 - \gamma d/n}{\theta + \gamma}$$

$$\tau_2 = \frac{a - 1 + \theta d/n}{\theta + \gamma}.$$

From there, taking the same steps as for [7.16], we can compute the marginal utility of debt at $t = 2$ for any region and we get

$$\frac{dU_2}{dd} = -\frac{a\gamma + \theta}{n(\gamma + \theta)} - \frac{\gamma\theta}{n^2(\gamma + \theta)}d.$$

We can observe that this is smaller in absolute value, the greater n . This reflects the fact that any region incurs only a fraction n of the necessary fiscal adjustment at $t = 2$.

From there, we can compute the FOCs for the choice of τ_{1i} and g_{1i} by each region. Again, in a symmetrical equilibrium, these quantities do not depend on i . The FOCs are similar to [7.16] and [7.18] with $p = 1$ and we get

$$a - \theta g_1 + \frac{dU_2}{dd} = 0 \tag{7.23}$$

$$-1 - \gamma \tau_1 - \frac{dU_2}{dd} = 0, \tag{7.24}$$

while [7.21] boils down to

$$d = n(g_1 - \tau_1).$$

We can then solve for the three fiscal variables g_1 , τ_1 , and d , getting

$$\begin{aligned} g_1 &= \frac{a[\gamma(n-1) + \theta(n+1)] - 2\theta}{\theta(\gamma + \theta)(n+1)} \\ \tau_1 &= \frac{2a\gamma - [\gamma(n+1) + \theta(n-1)]}{\gamma(\gamma + \theta)(n+1)} \\ d &= \frac{(a\gamma + \theta)n(n-1)}{\theta\gamma(n+1)}. \end{aligned} \quad [7.25]$$

Clearly, as n goes up, i.e. as the common pool problem gets worse, g_1 goes up, τ_1 falls and d goes up, as well as the debt level per region, d/n .

It is again fruitful to interpret those results in terms of the tragedy of the commons making each region more impatient. Comparing [7.25] with [7.12] for $R = 1$, we can see that the debt level per region is the same as if each region were on its own but had a discount factor equal to

$$\beta = \frac{1}{n}.$$

7.2.4. Stabilizations and crises

The literature on strategic fiscal policies also sheds some lights on why fiscal crises occur. The general theme, here, is that conflict between special interest groups is likely to reinforce the likelihood of crises and to make them more severe.

7.2.4.1. Delayed stabilization

An influential paper by Alesina and Drazen (1991) shows that crises may result from a war of attrition between interest groups about the sharing of the burden of stabilization. They borrow from the analysis of wars of attrition in an oligopoly, which shows that for such wars to take place at all, it must be that participants are not fully informed about the type of other participants; otherwise, the usual Coasian argument applies, i.e. the outcome of the war, which is perfectly foreseeable, would be frontloaded, which would thus spare the costs of the war to all parties.

Let us analyze the Alesina–Drazen argument in the context of the preceding model. Assume there is an initial level of debt d inherited from the past, and, for simplicity, that there is no public spending in either period, and also that $\beta = R = \gamma = 1$. Taxes must be levied in order to reimburse that debt. It may be that at $t = 1$ fiscal stabilization is delayed, implying that taxes are zero at $t = 1$. In such a situation, taxes must be higher, and therefore more distortionary at $t = 2$. That is, tax smoothing is violated. Thus, one cost of delayed stabilization is an increase in the deadweight loss of taxation.

There are two groups who differ from their vulnerability to the crisis. That is, by assumption, each group bears a cost of delay at $t = 1$ equal to c_i , where $i \in \{H, L\}$ is the group's type, and $c_L < c_H$. Thus, the fiscal crisis is costly in addition to its direct effect on tax distortions. The L type suffers less from the crisis and, most importantly, types are not observable.

The groups must decide how to share the burden of reimbursing the debt between themselves. At the beginning of $t = 1$, and before observing the other player's strategy, each group chooses between conceding or delaying. If both groups delay, then at $t = 1$ taxes are zero and at $t = 2$ taxes are equally split between the two groups, implying that each group pays a tax equal to $\tau/2$. Note that this is independent of the actual group type. Consequently, the second period welfare of each group is, from [7.9],

$$U_{D2} = -\frac{d}{2} - \frac{d^2}{8}. \quad [7.26]$$

If one group delays and the other concedes, then taxes are smoothed and an amount $d/2$ of the debt is reimbursed in each period. Furthermore, the conceding group bears a disproportionate share of the burden, paying a tax equal to $\alpha \frac{d}{2}$ in each period, where $\alpha > 1/2$. Hence, over the two periods, utility is

$$U_{SC} = -\alpha d - \frac{d^2 \alpha^2}{4} \quad [7.27]$$

for the conceding group and

$$U_{SD} = -(1 - \alpha)d - \frac{d^2(1 - \alpha)^2}{4}$$

for the other group.

If both groups concede, taxes are smoothed and the burden is shared equally. The total payoff U_{SE} then coincides with U_{SC} for $\alpha = 0.5$, thus

$$U_{SE} = -\frac{d}{2} - \frac{d^2}{16}.$$

We will assume that the crisis is severe enough that even ignoring delay costs, the conceding party is better-off under stabilization than delay, i.e. $U_{SC} > U_{D2}$. Otherwise, there would be an intrinsic incentive to delay just to get a more even sharing of the burden.¹² Comparing [7.27] and [7.26], we see that this holds if and only if

$$d > \frac{8\alpha - 4}{\gamma(1 - 2\alpha^2)} \text{ and } \alpha < \frac{1}{\sqrt{2}}.$$

12. In a standard, infinite-horizon model of attrition war, after delay, both parties are in the same situation; thus, this bias is not present.

Let us assume that the type of each player is randomly drawn in a population in which there is a proportion p of the L type. Let us construct a Nash equilibrium where type L delays while type H concedes. In such a situation, delays occur with probability p^2 . Note that the above assumption that $U_{SC} > U_{D2}$ rules out a situation where all types would choose to delay.

The total value of delaying for type i is

$$V_{Di} = -c_i p + (1 - p)U_{SD} + pU_{D2},$$

while the value of conceding is

$$V_{C,i} = pU_{SC} + (1 - p)U_{SE}.$$

It is optimal to delay if and only if $V_{Di} \geq V_{C,i}$. That is,

$$c_i p \leq (1 - p)(U_{SD} - U_{SE}) + p(U_{D2} - U_{SC}) = X(p). \quad [7.28]$$

Observe that while $U_{D2} - U_{SC} < 0$, on the other hand

$$U_{SD} - U_{SE} = (\alpha - 1/2) d \left(1 + \frac{d}{4} \left(\frac{3}{2} - \alpha \right) \right) > 0. \quad [7.29]$$

Therefore, there exists a critical

$$p^* = \frac{U_{SD} - U_{SE}}{U_{SD} - U_{SE} + U_{SC} - U_{D2}}$$

such that the RHS [7.28] is positive if and only if $p \leq p^*$. Then, the assumed outcome is indeed a Nash equilibrium provided

$$c_L \leq X(p)/p \leq c_H. \quad [7.30]$$

Since $X/p \sim (U_{SD} - U_{SE})/p$, $\lim_{p \rightarrow 0} X/p = +\infty$. Therefore, for [7.30] to hold, p must be neither close to 1 nor close to 0. The delay equilibrium may occur only if there is enough uncertainty. Furthermore, [7.30] suggests that such an equilibrium is more likely, the greater the difference between c_H and c_L . Finally, equation [7.29] implies that if $\alpha = 1/2$, $p^* = 0$, so that [7.30] never holds: the unevenness of the distribution of adjustment must be large enough for delay to occur.

7.2.4.2. Strategic fiscal indiscipline

In the same vein, Saint-Paul et al. (2021) asked a different question: why is it that society would pick unsustainable macroeconomic policies that may be conducive to a fiscal crisis? Their idea is that the fiscal adjustments that are made in a crisis, in the form of reneging on previous commitments on fiscal spending, are unevenly distributed, perhaps through the delayed stabilization mechanisms emphasized by

Alesina and Drazen. If some interest groups are relatively spared by the adjustment process, ex ante they benefit from excess, unsustainable, spending plans, because they can leverage a subsequent crisis to their benefit by shifting the costs to the other groups.

There are two groups denoted by $i = 1, 2$. Assume that public spending only occurs at $t = 2$. Furthermore, the good is publicly provided rather than public. Each group gets an amount g_i . Taxes must be such that

$$g_1 + g_2 = \tau$$

Utility is the same as in [7.9], except that the economy only exists at $t = 2$. For simplicity, we assume $\beta = 1$ and $\theta = 0$. At date $t = 1$, people decide on an *entitlement* level of publicly provided goods $\bar{g}$, which defines each citizen's access right to those goods. Importantly, for cultural or institutional reasons, this entitlement is set uniformly for all citizens. Also, by assumption, group 1 has relatively more power than group 2 in setting the entitlement. For simplicity, we shall assume that $\bar{g}$ maximizes the expected utility of group 1.

At date $t = 2$, there is a macroeconomic shock which affects the ability of the government to raise revenue. Specifically, τ cannot exceed $\bar{\tau}$, where $\bar{\tau}$ is drawn randomly over some interval $[\tau_{\min}, \tau_{\max}]$, with c.d.f. F and density $f = F'$. If $\bar{\tau} < 2\bar{g}$, there is a fiscal crisis: the government is not able to provide each citizen with their entitlement; therefore, people have to be rationed. In the Saint-Paul et al. paper (2021), people differ by their probability of getting their entitlement, and that probability is higher for the favored group. At the same time, it is also lower for everybody, the more severe the crisis, i.e. the bigger the gap between total commitments $2\bar{g}$ and actual fiscal resources $\bar{\tau}$. Over some range, an increase in the entitlement level raises the expected consumption of the publicly provided good for the favored group in a fiscal crisis, because the higher level of the good consumed more than compensates for the reduced probability of getting the good. At the same time, due to the crisis, at the margin this is not financed by higher taxes but by a reduction in the access to the publicly provided good by the less favored group. To capture this in a simple way here, I assume that under a crisis the rationing scheme relatively favors group one, i.e.

$$g_1 = \alpha\bar{g} + \frac{1 - \alpha}{2}\bar{\tau},$$

$$g_2 = -\alpha\bar{g} + \frac{1 + \alpha}{2}\bar{\tau}.$$

Note that if $\bar{\tau} = 2\bar{g}$, $g_1 = g_2 = \bar{g}$. However, as $\bar{\tau}$ falls, g_1 falls less fast than g_2 . The difference is equal to α . Therefore, α captures the degree of favoritism in this economy.

The expected utility of group 1, at date $t = 1$ is

$$Eu_1 = (1 - F(2\bar{g})) (a\bar{g} - 2\bar{g} - 2\gamma\bar{g}^2) \\ + \int_{\tau_{\min}}^{2\bar{g}} \left[a \left(\alpha\bar{g} + \frac{1-\alpha}{2}\bar{\tau} \right) - \bar{\tau} - \gamma\frac{\bar{\tau}^2}{2} \right] f(\bar{\tau})d\bar{\tau}.$$

If crises never occurred, group 1 would set $\bar{g}$ so as to maximize its (certain) utility, given by $a\bar{g} - 2\bar{g} - 2\gamma\bar{g}^2$, and we would have

$$\bar{g} = \frac{a-2}{4\gamma} = \bar{g}^*.$$

However, with crises, we have that

$$\frac{d(Eu_1)}{d\bar{g}} = (1 - F(2\bar{g})) (a - 2 - 4\gamma\bar{g}) + F(2\bar{g})\alpha a.$$

This expression is strictly positive for any $\bar{g} \leq \bar{g}^*$, implying that the value of $\bar{g}$ that maximizes group 1's utility is strictly greater than $\bar{g}^*$. From this expression, the effect is stronger, the greater α , i.e. the stronger the degree of favoritism.

The fiscal crisis allows the favored group to bypass the neutrality of the State and indirectly shift the burden of taxation towards the unfavored group. In a crisis, the total public spending is constrained by fiscal capacity. The rationing of access to publicly provided goods is such that the favored group gets more, the higher the entitlement. Thus, the favored group has an interest in artificially boosting entitlements, even knowing those promises will not be fulfilled. This effect is stronger, the greater the likelihood of a crisis, and at the same time, it also makes the crisis more likely.

7.2.4.3. *Financial crises and bailouts*

Since the 2008 financial crisis and the associated controversies about “socialism for the rich” and the social desirability of policies such as financial deregulation, loans to risky households and bailing out financial intermediaries, a literature has emerged which studies the political economy aspects of financial crises.

Glaeser (2012) discusses how various methods of handling financial intermediaries affect systemic risk. Regulating the intermediary entails the risk of capture, i.e. corruption by the regulated industry. Direct public management is associated with the risk of populism, broadly defined as managing the firm for the benefit of some special segment of society who will reward the politician with votes. Finally, laissez-faire fails to correct for market failures such as monopoly power or excess risk-taking due to limited liability. For example, in the case of a public utility, capture may lead to a policy favoring the firm, i.e. a regulated price that is too high and might even match the monopoly price, while populism may lead to excess losses and too low prices in order to transfer resources to the poor constituency of the populist. Interestingly, though,

in the case of financial intermediaries, both populism and capture may lead to excess risk-taking. Under capture, the intermediary manages to manipulate the regulator so as to reduce its own equity, which raises the return on equity in good times as well as the risk of bankruptcy. In turn, the populist may pursue the same lax policies so as to increase the amount of cheap loans available to its poor constituency.

Farhi and Tirole (2012) show that *ex ante* strategic risk-taking by financial institutions may compel the government to bail them out *ex post* in case of a crisis, due to its systemic nature. In turn, expectations of such a bailout validate the decision to overexpose oneself to risk *ex ante*. The key idea is that there is a strategic complementarity across financial actors: the greater the proportion of them that takes excess risks, the more systemic the crisis will be, and the greater the incentives of the government to intervene. As is generally the case under strategic complementarities, there are multiple equilibria: if all intermediaries are sound, a given one will not take excess risks because its failure has small aggregate effects and the government will not bail it out. Hence, there exists an equilibrium where all intermediaries are sound. Conversely, if all intermediaries are over-exposed, a massive bailout program is expected in case of a crisis, and a prudent intermediary will simply have a lower expected return on equity by foregoing the option to profit from the bailout under a crisis scenario. Thus, that is an equilibrium situation as well.

7.2.4.4. *Macroeconomic populism*

Dornbusch and Edwards (1992) coined the term “macroeconomic populism”, referring to policies, in particular in Latin America, that appeal to the working class in the short run but eventually lead to macroeconomic crises: high inflation, capital flight, sovereign default, massive exchange rate depreciation, and so forth. These policies may consist of unsustainable increases in transfers, pensions, public employment, minimum wages, unemployment benefits, food and energy subsidies, and so forth.

A key question is: why would people vote for such policies if they expect them to lead to a crisis?

A general class of answers to that question relies on the view that common people lack the cognitive ability to think through the long-term consequences and the general equilibrium consequences of populist policies, taking into account only the direct, short-term benefits of those policies for themselves (see, for example, Gersbach and Schniewind (2001)).

That being said, can we think of rational explanations for macroeconomic populism?

Acemoglu et al. (2013) develop a signaling theory of populist policies, which, however, does not capture the specifically unsustainable feature of macroeconomic populism in the sense of Dornbusch and Edwards. They define populism as policies

that are skewed to the left relative to the preferences of the median voter, i.e. entail more redistribution and social protection than what people would normally choose. According to their theory, people rationally vote for policies that are left of the median-voter, because they want to rule out a politician that would cater to the wealthy elites and implement right-wing policies *ex post*, contrary to their electoral platform. To signal that they are not corruptible by the elites, successful candidates must have an electoral platform that is skewed to the left. This signaling mechanism is somewhat similar to that of Rogoff and Siebert, whose model is indeed also a potential explanation for macroeconomic populism.

Clearly, the above-mentioned Saint-Paul et al. (2021) theory of strategic fiscal indiscipline, by relating the likelihood of fiscal crisis to the degree of favoritism in society, also provides foundations for the phenomenon of macroeconomic populism in the sense of Dornbusch and Edwards.¹³

7.3. Distributive conflict and long-term growth

7.3.1. *Democracy, inequality and redistribution: the canonical Meltzer–Richard model*

A substantial body of literature on growth and inequality builds on the basic model of Meltzer and Richard (1981). These authors aim to explain the size of government in a democracy as the outcome of redistributive conflict between rich and poor.

People differ by their ability, denoted by a . Their pre-tax income is $y = al$, where l is their labor supply. They have the same utility, which depends on consumption, labor, as well as public goods:

$$U(c, l, G) = c - \gamma \frac{l^2}{2} + G \quad [7.31]$$

They vote on the level of the public good G , which is financed by proportional taxation on income. Let τ be the tax rate, then the government's budget constraint is

$$G = \tau Y, \quad [7.32]$$

where Y is GDP. An agent with ability a supplies $l(a)$ in equilibrium and earns, and consumes, $al(1 - \tau)$. Maximization of [7.31] then implies that

$$l = l(a) = \frac{a(1 - \tau)}{\gamma}, \quad [7.33]$$

13. See Saint-Paul (2020) for empirical evidence linking micro-indicators of favoritism with aggregate fiscal stability.

so that the agent's after tax income is

$$y(a) = \frac{a^2(1 - \tau)^2}{\gamma}.$$

Furthermore

$$Y = (1 - \tau)\bar{Y}, \quad [7.34]$$

where

$$\bar{Y} = E \frac{a^2}{\gamma}.$$

Using [7.31], [7.32] and [7.34], we can compute the indirect utility of an agent with ability a :

$$U(\tau, a) = \frac{a^2(1 - \tau)^2}{2\gamma} + \tau(1 - \tau)\bar{Y}.$$

Clearly, it is concave and therefore single-peaked, and, at the same time $\partial^2 U / \partial \tau \partial a < 0$. Therefore, the standard voting theorems, based on single-peakedness and/or single-crossing preferences, apply: the collectively decided tax rate is the one preferred by the median voter, whose type is denoted by a_m . The equilibrium tax rate is determined by that agent's first-order condition, yielding

$$\begin{aligned} a_m^2(1 - \tau) &= (1 - 2\tau)\gamma\bar{Y} \\ \tau &= \max \left(\frac{Ea^2 - a_m^2}{|2Ea^2 - a_m^2|}, 0 \right) \end{aligned}$$

The equilibrium tax rate depends on income inequality in the following way: it is an increasing function of the ratio between mean and median (squared) ability, Ea^2/a_m^2 which, since income is proportional to a^2 is also equal to the ratio between mean and median income. It is generally considered that, empirically, the median is poorer than the mean because the distribution of income is skewed to the right. From there, the two key predictions are that (i) $\tau > 0$, despite that here, by assumption, G is a perfect substitute for c and has the same marginal utility per euro spent, and (ii) an increase in inequality raises τ , implying that more unequal societies have bigger governments, as the outcome of redistributive conflict between poorer and richer agents. Furthermore, because taxation distorts labor supply, all else being equal, more unequal societies will also be poorer. To see this, observe that, absent taxation, income is proportional to a^2 . Hence, two economies that would have the same GDP per capita absent redistribution would have the same $\bar{Y}$. Clearly, then, from [7.34], the more unequal economy, which has a higher value of τ , has a lower GDP.

7.3.2. Growth and inequality

Following these insights, Persson and Tabellini (1994) and Alesina and Rodrik (1994) have developed endogenous growth models that typically predict that inequality is harmful for long-run growth, because it leads to higher redistributive taxation, through the political decision-making system. Endogenous growth models generally rest on the existence of constant aggregate returns to accumulable factors of production. Assume the aggregate production function is $Y = Kf(X)$, where K is capital and X denotes a vector of other inputs. Constant aggregate returns to K may in part be due to externalities and spillovers, so that the formulation is not incompatible with constant returns to scale (i.e. to all inputs) at the level of an individual firm. Also, the analysis could be generalized to several accumulable inputs. There are various ways of modeling the accumulation process. In the Ramsey (1928) growth model, in a balanced growth path, the growth rate is related to the return to capital through the consumer's Euler equation. In continuous time, this reads as

$$g = \sigma(r - \rho),$$

where σ is the intertemporal elasticity of substitution in consumption, ρ is the subjective discount rate, and r is the net private return to capital. Here, $r = \alpha f(X)$, where α is the fraction of the social return to capital that is appropriated by private agents, so that $g = \sigma(\alpha f(X) - \rho)$. Any policy that reduces inputs also reduces the economy's long-run growth rate. This is the case, in particular, for the redistributive tax in the Meltzer–Richard model, which reduces the aggregate labor input. Alternatively, if the public good is financed by a tax on capital rather than labor, we have that $r = (1 - \tau)\alpha f(X)$, and again there is a negative effect of τ on g , although through a different channel. In the Diamond (1965) overlapping generations model, capital accumulation comes from the savings of the young, who save for retirement, out of their labor income. In discrete time, this reads as $K_{t+1} = s(r)\theta Y_t$, where θ is the share of labor income in total GDP and $s(r)$ is the fraction of wages saved by the young and expressed as a function of the return to savings r . In a balanced growth path, the growth rate of the capital stock is therefore equal to

$$1 + g = s(r)\theta f(X).$$

Again, any reduction in inputs will reduce the growth rate, unless it raises θ by enough. The effect of a tax on capital is more complex, and may run in the opposite direction. If the substitution effect in savings dominates, then $s' > 0$, and an increase in capital taxation reduces r and therefore g . But the opposite prevails if the wealth effect dominates and $s' < 0$.

7.3.2.1. Caveats and qualifications

The Meltzer–Richard–Alesina–Rodrik–Persson–Tabellini theory is a simple, elegant and powerful analysis of how distributive conflicts lead to reduced output and growth. However, it has a number of shortcomings.

First, in order to be able to apply median voter theorems, we need to put considerable structure upon the redistribution mechanism, in effect constraining it to be governed by a single parameter, so as to avoid Condorcet paradox configurations. For example, the assumption of a lump-sum transfer explains that the median voter willingly supports redistribution to the poor because they benefit from the same transfer as well. Under a more complex tax system, the median voter would want to calibrate fiscal transfers so as to get more than the poor, yet the latter would then side with the rich in blocking such a policy. In general, this would lead to Condorcet cycles, yet we do observe transfers that benefit a minority of poor people in reality.

The dimension of inequality that matters for the predictions critically depends on the assumptions we make about the structure of the tax system. If wealth is taxed, for example, then the tax rate should depend on the mean–median wealth ratio, instead of the mean–median labor income ratio.

Second, the argument applies to democracies. The mechanism should not be present in non-democracies. In that respect, one interesting prediction is that democratization, understood as a reduction in the franchise that raises the voting rights of poorer people, should lead to greater redistributive taxation and lower growth. The former prediction seems to be borne out by the experience of Western countries since the beginning of the 20th century, yet growth has not slowed down over this period, compared to the 19th century. Perotti (1996) has shown in an econometric work that while inequality tends to be negatively associated with growth, there is no evidence in the data of a positive association between the former and taxation, nor of a negative association between taxation and growth.

Third, the prediction depends on the nature of spending. It can be that G is a growth-enhancing public good. In Barro (1990), public spending enters the production function and, up to a point, an increase in spending raises the growth rate. A representative agent would choose a welfare maximizing level of public spending which, due to discounting, would fall short of the growth-maximizing level. In an unequal society, the median voter would choose a greater level than this, due to the redistributive component of spending. If that effect is not too large, growth would be enhanced; thus, there would be a positive association between growth and inequality. In Saint-Paul and Verdier (1993), redistribution is in the form of public education, which raises aggregate human capital. The latter is intergenerationally transmitted, so that education also tends to reduce inequality. All else being equal, greater inequality is good for growth, and at the same time tends to fall over time, due to the very redistribution it induces.

Fourth, the effect of an increase in inequality depends on the nature of that increase. If an increase in inequality is concentrated at the bottom of the distribution

of income, the median may well become richer relative to the mean, and redistribution may go down. This will be the case in a simple three-class model where there is a proportion $p < 1/2$ of poor, and $q < 1/2$ of rich people, the rest being accounted for by the middle class. Holding the income of each member of a given class constant, an increase in p has no effect on median income, while it reduces mean income, thus also reducing the equilibrium tax rate according to the Meltzer–Richard model. Such mechanisms are studied by Saint-Paul (2001) and Bénabou (2000). In Saint-Paul, the poor are credit constrained and transfers raise their opportunities to join the middle class. An increase in the number of poor people makes society more fiscally conservative, which further raises future poverty. Through this complementarity between lower redistribution and greater poverty, a low poverty, high redistribution steady state may coexist with a high poverty, low redistribution one.

7.4. Institutions and reform

I now turn to the analysis of institutions, limiting myself to those that are relevant to important macroeconomic outcomes.

Institutions belong to two categories. First, *constitutions* that do not specify any policy, but prescribe the process by which policies are set. Second, *entrenched policies* that commit future policy courses by raising the cost of reneging on that commitment. In practice, legally, a number of entrenched policies are written in constitutions, just because changing constitutions is more costly than changing laws. This is the case, for example, for the famous Roe versus Wade amendment, which was eventually overturned by the US supreme court precisely because constitutions are not supposed to be used to entrench positive rights.

The literature has dealt with two separate outcomes: first, the effect of given political constitutions on macroeconomic (notably, fiscal) outcomes, regardless of whether the objective of said constitutions at the time they were set had anything to do with macroeconomic objectives – for example, a constitution might specify a certain degree of separation of power so as to preclude tyranny, without any regard for the consequences of those provisions for government debt. Second, the optimal design of monetary or fiscal rules, which may in particular help society solve the time consistency problem discussed in section 7.2.1.

The literature leaves many research questions open, in particular because it is difficult to come up with a canonical modeling strategy which delivers results that are not sensitive to special assumptions. This will be clear from the two contributions we now discuss.

7.4.1. Fiscal constitutions

7.4.1.1. The effect of political systems on fiscal outcomes

In a series of contributions, Persson, Roland and Tabellini have analyzed the effect of political systems on fiscal policy outcomes.¹⁴

Their model can be sketched as follows. There are three equal sized regions, indexed by $i = 1, 2, 3$. They appoint representatives who decide on policy. People in each region derive utility from consuming a private consumption good, a public good common to all regions g , and a region-specific transfer r_i . Taxes, denoted by τ , are mutualized, and have a maximum level equal to $\bar{\tau}$. The utility of an agent in region i is

$$U(\tau, r_i, g) = r_i - \tau + \alpha g - \beta g^2/2. \quad [7.35]$$

Elected representatives only care about two things: the private rents derived from holding office, and being reelected. Each region appoints only one representative. Within each region, voters can somehow coordinate so as to commit on a punishment strategy consisting of not reelecting their representative if they underperform relative to the standard set in that strategy. However, voters do not coordinate across regions.

Hence, the utility of the politician who represents region i is $s_i + V_R$, if reelected, and s_i if not reelected, where s_i is the private rent extracted from society's resources, and V_R stands for the value of being reelected.

The central government's budget constraint is

$$\sum_{i=1}^3 r_i + \sum_{i=1}^3 s_i + g = 3\tau.$$

The authors first consider the case of a "pure legislature", which is characterized by the fact that there is a unique agenda-setter and no separation of powers. An agenda setter a is selected at the beginning of the game, and to implement their preferred policies they must build a coalition, i.e. convince the representative of another region to vote in favor of their platform. If the agenda setter fails to build a coalition, a default policy is implemented such that there are no transfers and public goods provided to the people, and $s_i = \tau = \sigma$. Given this default policy, it is rational for voters to pick a strategy such that their representative is not reelected if a coalition is not built. This way, the people can enforce a better outcome for themselves through the threat of electoral punishment.

A first result is that the two regions, denoted by m and n , that are not the agenda-setter compete with each other to be part of the coalition. The lower the

14. See Persson and Tabellini (1996, 2004) and Persson et al. (2000).

transfers r_m and r_n , the greater the resources left for politicians to appropriate for themselves. Therefore, all else being equal, the politician will pick the cheapest coalition partner, i.e. the one, say m , which asks for the lowest r . Assume $r_m > 0$. Since n is left out of the coalition, the platform will then be such that $r_n = 0$. Then, however, it is rational for voters in n to underbid m by conditioning reelection on any positive r_n smaller than r_m , in which case its representative will replace m as a coalition member. Consequently, in equilibrium, $r_m = r_n = 0$.

On the other hand, region a knows that its representative remains the agenda-setter no matter what. It will pick the reelection strategy that maximizes its utility, subject to the constraint that the a representative is able to build a coalition. For this to be the case, a must offer m a private rent that leaves them indifferent between being part of the coalition and the default payoff. That is, we must have $s_m = \sigma - V_R$. While the threat point of m is pinned down by the no coalition payoff σ , the threat point of a is higher: it can always build a coalition that would not be reelected. For this, it has to transfer $s_m = \sigma$ to its partner. It can then set taxes at their maximum level $\bar{\tau}$ and appropriate all remaining resources, getting $s_a = 3\bar{\tau} - \sigma$. Therefore, for the no reelection threat to be effective, it must be that

$$s_a + V_R \geq 3\bar{\tau} - \sigma, \quad [7.36]$$

implying that the total rents to the politicians in the coalition must be equal to $s = s_a + s_m = 3\bar{\tau} - 2V_R$. The punishment strategy of the electorate in region a must then solve the following problem:

$$\begin{aligned} & \max r_a - \tau + \alpha g - \beta g^2 / 2 \\ & s.t. r_a + g \leq 3\tau - s. \end{aligned}$$

That is, the people in a can somewhat dictate the values of r_a and g through the no reelection threat, but for this threat to be credible, they must lose an amount s of resources transferred to the politicians. To avoid dealing with too many cases, let us assume that $2V_R \geq \frac{\alpha-1}{\beta}$ and that $\bar{\tau} \geq \frac{2V_R}{3}$. The solution is then $\tau = \bar{\tau}$, $g = \frac{\alpha-1}{\beta}$, and $r_a = 2V_R - \frac{\alpha-1}{\beta}$.

The authors then compare those results to a “presidential” system, characterized by greater separation of power. They assume that there are two agenda-setters. The first agenda-setter, denoted by a_τ , picks the tax rate; the second one, denoted by a_g , then chooses r_i , s_i and g . At each stage of the game, the agenda-setter’s choice must be validated by a majority of the parliament, i.e. it must find another representative to support the proposal. Furthermore, again, voters can precommit on a reelection strategy that punishes the politicians for deviating from their interests.

The second stage of the game yields identical outcomes as a pure legislature, where the maximum tax rate $\bar{\tau}$ is replaced by the actual value of τ decided in the first stage of

the game. Therefore, from the preceding analysis, voters of region a_τ do not expect to have any transfer, and the level of the public good g is independent of τ . Furthermore, the first agenda-setter will be selected as a junior partner with some probability, in which case they get a rent s_m which is independent of τ . Therefore, their incentives are independent of τ ; consequently, voters in region a_τ will set their reelection strategy so as to pick the lowest possible value of τ , provided they get the correct value of g . From the above discussion, we note that if it were true that $\bar{\tau} \leq \frac{\sigma + V_R}{3}$, then, from [7.36], the no reelection threat is enforced with strict inequality at $s_a = 0$. That is, the economy is then in a zone where the threat of appropriation of all tax revenues by the second agenda-setter and its junior partner is not strong enough for them to extract strictly positive rents, due to a low fiscal capacity. We then have, instead of the above expressions, $s = s_m = \sigma - V_R$ and $r_a = 3\bar{\tau} - \frac{\alpha-1}{\beta} + V_R - \sigma$. This expression should be nonnegative; otherwise, g will be constrained to be below the region optimum $\frac{\alpha-1}{\beta}$, which harms group a_τ . Consequently, this group will set a reelection strategy such that $r_a = 0$, i.e. they will set $\tau = \frac{1}{3} \left(\sigma - V_R + \frac{\alpha-1}{\beta} \right)$.

Separation of powers entails more fiscal discipline because the agenda-setter that decides upon τ will compute it so as to curb the incentives of the “spenders” to transfer resources to their constituency and to appropriate private rents. In some sense, however, this conclusion ignores the Persson and Svensson (1989) insight that tying the hands of future policymakers is likely to lead to high deficits in addition to lower public spending.

7.4.1.2. *Endogenous fiscal constitutions*

The determination of fiscal constitutions remains an open question. While existing research is substantial, much remains to be explored. In particular, from a historical and modeling perspective, two hard questions must be answered. First, in what sense is an institution different from a policy? Second, how are institutions set, and how, and why, does the process for setting institutions differ from the process for setting policies?

The two questions are linked. For example, Starr (2019), from a historical perspective, discusses a number of institutions that really are entrenched policies. Under that view, an institution is a peculiar legal state variable that is being set by the incumbent power so as to precommit future policies. In that respect, it is not much different from any other state variable that would have the same effect, such as public debt in the literature discussed in section 7.2.3.2. This view of institutions is consistent with assuming that institutions are set discretionarily by the current government. Conversely, institutions might be set *ex ante* under a veil of ignorance about preferences, by some founding fathers that maximize some social welfare function.

An illustration of the first approach is a series of papers by Besley, Ilzetzki and Persson, who discuss the endogenous determination of state capacity.¹⁵ To illustrate their results, let us build upon the preceding section's model. There are two periods and two groups, whose welfare at $t = 2$ is defined by [7.35]. At $t = 1$, for simplicity, we assume there are no public goods, so that utility is just $2r_i - \tau$. At each date, there is an incumbent government, who represents the interests of one of the two groups. This government may change from $t = 1$ to $t = 2$. At date $t = 1$, the government decides on the tax rate, the level of transfers and future tax capacity at date $t = 2$, which is denoted by $\bar{\tau}$. Building tax capacity entails a cost $\delta\bar{\tau}$. Hence, the incumbent government's budget constraint at $t = 1$ is

$$r_1 + r_2 + \delta\bar{\tau} = 2\tau.$$

We assume that there is an exogenous level of fiscal capacity $\bar{\tau}$ at $t = 1$, which is large enough to be non-binding, and that the incumbent can freely set r_1 and r_2 . Clearly, then it will pick $\tau = \bar{\tau}$, $r_2 = 0$ and $r_1 = 2\bar{\tau} - \delta\bar{\tau}$.

With probability p , the incumbent at $t = 1$ is in power at $t = 2$. With probability $1 - p$, the representative of the other group is in power. The party in power at $t = 2$ chooses taxes, transfers and public goods, i.e. faces the budget constraint

$$r_1 + r_2 + g = 2\tau.$$

The value of τ it chooses cannot exceed the level of fiscal capacity $\bar{\tau}$. Also, the structure of transfers is subjected to a "cohesion" constraint which puts a bound on the level of inter-group discrimination. That is, if i is the party in power and $-i$ is the other party, we must have $r_{-i} \geq \sigma r_i$, where $\sigma \in [0, 1]$ is an index of social cohesion.

Unless $\sigma = 1$, one extra euro of taxation at date 2 allows us to finance $\frac{2}{1+\sigma} > 1$ euros of transfers for the group in power. Consequently, this group will always choose $\tau = \bar{\tau}$. Therefore, $r_i = \frac{2\bar{\tau}-g}{1+\sigma}$, and the government picks g so as to maximize [7.35], or equivalently

$$V_{2P}(i) = \frac{1-\sigma}{1+\sigma}\bar{\tau} + \left(\alpha - \frac{1}{1+\sigma}\right)g - \beta g^2/2$$

subject to

$$g \leq 2\bar{\tau}.$$

The unconstrained optimum is

$$g_2^* = \frac{1}{\beta} \left(\alpha - \frac{1}{1+\sigma} \right).$$

15. See Besley and Persson (2009, 2010), Besley et al. (2013).

Therefore:

– If $\bar{\tau} > \frac{g_2^*}{2}$, $g = g_2^*$ and $r_i = \frac{2\bar{\tau} - g_2^*}{1 + \sigma}$, $r_{-i} = \sigma \frac{2\bar{\tau} - g_2^*}{1 + \sigma}$. The resulting utilities are, for the party in power and the opposition:

$$V_{2P} = \frac{1 - \sigma}{1 + \sigma} \bar{\tau} + \frac{1}{2\beta} \left(\alpha - \frac{1}{1 + \sigma} \right)^2 \quad [7.37]$$

$$V_{2O} = -\frac{1 - \sigma}{1 + \sigma} \bar{\tau} + \frac{1}{2\beta} \left(\alpha - \frac{1}{1 + \sigma} \right) \left(\alpha + \frac{1 - 2\sigma}{1 + \sigma} \right) \quad [7.38]$$

– If $\bar{\tau} \leq \frac{g_2^*}{2}$, $g_2^* = 2\bar{\tau}$, $r_i = r_{-i} = 0$, and

$$V_{2P} = V_{2O} = (2\alpha - 1) \bar{\tau} - 2\beta \bar{\tau}^2. \quad [7.39]$$

At $t = 2$, the marginal cost of public spending for the incumbent is $\frac{1}{1 + \sigma}$, the foregone transfers to the incumbent associated with a unit increase in spending. This is smaller, the greater the level of social cohesion, since greater social cohesion raises the share of spending financed by the opposition group through lower transfers. Hence, the optimal spending level g_2^* is greater when σ is higher. It may be, however, that fiscal capacity is too low to finance that level, in which case the incumbent group will opt for zero transfers and spend all tax revenues on public goods. This is more likely to be so, the greater σ ; the key role played by social cohesion here is to raise the fiscal cost of transfers relative to public spending, because it makes it necessary that transfers be paid to everybody, which is inefficient from a pure redistributive aspect.

In period 1, the party in power maximizes

$$2\bar{\tau} - \delta \bar{\tau} + pV_{2P} + (1 - p)V_{2O}.$$

It is easy to show that this defines a concave function of $\bar{\tau}$. From [7.37-7.38], in the zone where $\bar{\tau} > \frac{g_2^*}{2}$, the expected future marginal utility of fiscal capacity is equal to $\frac{(2p-1)(1-\sigma)}{1+\sigma}$. Therefore, if

$$\frac{(2p-1)(1-\sigma)}{1+\sigma} > \delta, \quad [7.40]$$

the incumbent will pick the highest level of fiscal capacity $\bar{\tau} = 1$ and, at $t = 2$, the economy will be in a “redistributive regime” where $g_2 = g_2^*$ and $r_i > 0$. This is more likely to happen, the greater the degree of political stability p and the smaller the level of social cohesion σ . The incumbent party chooses high tax revenues because it has a high probability of remaining in power and can redistribute most of them beyond public goods to its own constituency. Note that if $p < 1/2$, this regime never holds: the incumbent at $t = 1$, in expectations, always loses from transfers at $t = 2$.

Conversely, if $\frac{(2p-1)(1-\sigma)}{1+\sigma} < \delta$, the net benefit of fiscal capacity in this region is always negative, and the incumbent will pick $\bar{\tau} \leq \frac{g_2^*}{2}$. In this regime, the net marginal benefit of fiscal capacity, from [7.39], is equal to

$$m(\bar{\tau}) = -(1 + \delta) + 2\alpha - 4\beta\bar{\tau}. \quad [7.41]$$

There are two possible solutions:

– If

$$\frac{1 - \sigma}{1 + \sigma} \leq \delta, \quad [7.42]$$

we can check that $m(\frac{g_2^*}{2}) \leq 0$. The incumbent then picks an interior solution for $\bar{\tau}$, i.e.

$$\bar{\tau} = \frac{2\alpha - (1 + \delta)}{4\beta}.$$

The resulting level of spending,

$$g_2^{**} = \frac{2\alpha - (1 + \delta)}{2\beta},$$

is the one that maximizes utilitarian social welfare, taking into account the cost of fiscal capacity. Thus, despite conflicts of interests, the economy is in a “common interest” equilibrium where the level of public spending is socially optimal.

– If [7.42] is violated, then the incumbent picks $\bar{\tau} = \frac{g_2^*}{2}$, i.e. the kink in its expected welfare function. Spending is again equal to g_2^* . Furthermore,

$$g_2^{**} - g_2^* = \frac{\frac{1-\sigma}{1+\sigma} - \delta}{2\beta} > 0. \quad [7.43]$$

In this regime, called a “weak state”, fiscal capacity is insufficient to deliver the socially optimal public goods provision. Conversely, in the redistributive regime, [7.40] implies that [7.43] is reversed: in addition to spending on transfers, society spends too much on public goods. This is because, ex post, having set a fiscal capacity high enough to pay transfers, the party in power incurs a marginal cost of public goods equal to the net transfer loss to itself, which is at most equal to 1, whereas the true social cost of public goods is equal to $1 + \delta$, as shown in [7.41]. For the optimal outcome to be possible, it must be that fiscal capacity is binding at $t = 2$.

Political stability is a mixed blessing when it delivers the redistributive outcome: the incumbent at $t = 1$ has a high probability of remaining in power, which makes it eager to raise fiscal capacity so as to redistribute in its favor and at the expense of the other group. As a result, the state is too large and spends too much on public goods

and redistribution. Political instability delivers weak states if social cohesion is low, but “common interest” outcomes if it is high enough. If social cohesion is low, the incumbent favors a weak state to prevent being partially expropriated at $t = 2$ should it lose power, but this comes at the cost of an underprovision of public goods. If social cohesion is high, it is not worth the group in power reducing public good provision at $t = 2$ below g_2^{**} to increase transfers, because transfers are not very redistributive. The incumbent at $t = 1$ then naturally selects the socially optimal public spending.

Greater social cohesion makes it more likely that the common interest outcome prevails. It raises the threshold of p beyond which the redistributive state emerges, and, below that level, makes it less likely that a weak state prevails. Given δ and p , for σ high enough, [7.40] is violated and [7.42] holds.

7.4.2. Labor market institutions

A key long-term economic outcome is equilibrium unemployment. For decades, it has been considered as abnormally high in some countries, in particular in continental Europe compared to the United States. The root causes of such a phenomenon have been ascribed to the labor market rigidities associated with the “European social model”: collective bargaining, minimum wages, employment protection, unemployment benefits and labor taxes. It is then natural to ask where the political support for those rigidities comes from, and why they are more pervasive in some countries than others.

7.4.2.1. Unemployment benefits

An early contribution is Wright (1986), who studies the political determinants of unemployment benefits. He applies the Meltzer–Richard model to an economy where workers differ in employment risk. To analyze this, we modify the model of section 7.3.1 as follows.

All agents have the same income y and pay a tax t to the unemployment benefit system. People differ by their unemployment risk, denoted by u . On average, the utility of an agent with risk u is the average of their utility between the employment state and the unemployment state

$$U(u, t, b) = (1 - u)v(y - t) + uv(b),$$

where $v(\cdot)$ is a concave utility function and b is the level of unemployment benefits. Normalizing population to 1, the budget constraint of the unemployment benefit system is

$$b\bar{u} = t(1 - \bar{u}),$$

where $\bar{u}$ is the average unemployment rate. Consequently, the preferred benefit level of an individual with risk u is the one that maximizes $(1 - u)v(y - \frac{b\bar{u}}{1-\bar{u}}) + uv(b)$. The corresponding first-order condition is

$$v'(b) = \frac{1-u}{u} \frac{\bar{u}}{1-\bar{u}} v'(y - \frac{b\bar{u}}{1-\bar{u}}).$$

Clearly, as u goes up, the RHS falls. To compensate, the optimal b must go up. People favor greater unemployment benefits, the more exposed they are to unemployment. Furthermore, for the average individual such that $u = \bar{u}$, marginal utility is equalized between the two states, i.e. the individual favors full insurance.

As in the Meltzer–Richard model, the equilibrium value of b is the one preferred by the median voter, i.e. the person with the median value of u , denoted by u_m . It is generally considered that unemployment is concentrated among a minority of the population, like, say, high-school dropouts or new entrants in the labor market. Thus, we expect that $u_m < \bar{u}$, implying that $v'(b) < v'(y - t)$. In equilibrium, there is less than full insurance. This is because the UB system redistributes away from workers with low unemployment risk and in favor of people with high unemployment risk. Since the decisive voter is less exposed to unemployment than the average, they will favor a level of unemployment benefits lower than full insurance, all the more so, the more they are insulated from unemployment.

7.4.2.2. *Employment protection*

Following Saint-Paul (1997, 2002), let us now discuss the determinants of the political support for employment protection.

Consider an agent who may be in two alternative states, denoted by $i = 1, 2$. He earns an income flow equal to y_i in state i . Let p_{ij} be the transition probability, per unit of time, from state i to state j . Let r be the interest rate at which future income flows are discounted. It can be shown that the net expected present discounted value of being in state i is

$$V_i = \frac{(r + p_{ji})y_i + p_{ij}y_j}{r(r + p_{ij} + p_{ji})}. \quad [7.44]$$

Saint-Paul then considers an economy in the fashion of Shapiro and Stiglitz (1984) and Mortensen and Pissarides (1994). Firms freely enter the market at a high idiosyncratic productivity state where a worker produces m_H per unit of time. With probability λ per unit of time, the firm falls into a low productivity state where output is $m_L < m_H$. With probability s per unit of time, the job is destroyed. In the low productivity state, the firm never returns to the high productivity state, and, again, the job is destroyed with probability s per unit of time.

Workers are either employed or unemployed. When employed, they earn the market wage w , which is endogenous. When unemployed, they earn zero. Wage

formation is non-competitive, so that in equilibrium employed workers are able to extract a rent above their opportunity cost of labor. More precisely, absent rigidities, the wage would be equal to the opportunity cost of labor, given by the annuity equivalent of the net present value of being unemployed V_u , i.e. rV_u . We assume that employed workers are able to extract a fixed rent q per unit of time above that opportunity cost of work, so that in equilibrium

$$w = rV_u + q. \quad [7.45]$$

The greater q , the more wage formation is noncompetitive, i.e. further away from a Walrasian benchmark whereby the unemployed could perfectly underbid the employed. Furthermore, in this simple world, wages are the same regardless of the firm's productivity state.

7.4.2.2.1. The flexible society

In the flexible society, there is no employment protection legislation. Firms dismiss workers as soon as the net value of a job falls below zero. If a firm in the low productivity state had a nonnegative net present value, then, since wages are the same throughout the economy, it would have a strictly positive value in the high productivity state. But this contradicts free entry. Thus, it must be that (i) a job is destroyed as soon as the firm falls into the low state and (ii) wages are equal to m_H , which ensures that the net value of a high productivity firm is zero. The job loss rate is $s + \lambda$, while the job finding rate is denoted by a . From [7.44], we have that

$$V_e = \frac{(r + a)m_H}{r(r + s + \lambda + a)}$$

and

$$V_u = \frac{am_H}{r(r + s + \lambda + a)}.$$

This determines a job finding rate that is consistent with the wage formation process defined by [7.45]:

$$a = (m_H/q - 1)(r + s + \lambda) = a_F$$

Substituting, we have that

$$\begin{aligned} V_{uF} &= \frac{m_H - q}{r}, \\ V_{eF} &= \frac{m_H(r + s + \lambda) - q(s + \lambda)}{r(r + s + \lambda)}. \end{aligned} \quad [7.46]$$

In equilibrium, wages cannot differ from productivity, equal to m_H . An increase in the rent extraction power of the insiders, q , is met by a fall in the opportunity cost

of work through a lower a , i.e. a higher unemployment duration. For this reason, both V_u and V_e are decreasing functions of q , although the former is more sensitive than the latter. As q goes to zero, the economy converges to a Walrasian benchmark where a becomes infinite, meaning that a job is found instantaneously. In this economy, $V_e = V_u = \frac{m_H}{r}$. A worker who loses their job because their employer falls in the low state immediately finds another job in a firm that is in the high state.

7.4.2.2.2. The rigid economy

In the rigid economy, it is prohibited to dismiss a worker when falling into the low state. On the other hand, jobs are still destroyed for strictly exogenous reasons, with probability s per unit of time.

To compute the equilibrium wage, we apply [7.44] to the value of a firm in the high state, denoted by J_H . We note that the transition probability between that state and the low state is λ , the reverse is zero, and that future income flows must be discounted at a rate equal to $r + s$, the sum of the interest rate and the flow probability of job destruction. Thus, we get

$$J_H = \frac{(r + s)(m_H - w) + \lambda(m_L - w)}{(r + s)(r + s + \lambda)}.$$

By free entry, we must have $J_H = 0$. Consequently, wages are a discounted weighted average of the job's productivity over its lifetime

$$w = \frac{(r + s)m_H + \lambda m_L}{r + s + \lambda} = \bar{m}. \quad [7.47]$$

Rigid institutions reduce the exposure of employed workers to unemployment, but they reduce wages, as low productivity jobs are maintained. Firms make positive profits out of high productivity jobs but losses out of low productivity jobs.

The welfare of employed and unemployed workers is now given by

$$V_e = \frac{(r + a)\bar{m}}{r(r + s + a)}$$

and

$$V_u = \frac{a\bar{m}}{r(r + s + a)}.$$

The equilibrium job finding rate is again the one consistent with the wage setting process [7.45] in equilibrium:

$$a = (r + s) \left(\frac{\bar{m}}{q} - 1 \right) = a_R$$

From there, we can recompute V_e and V_u :

$$V_{eR} = \frac{(r + s)\bar{m} - sq}{r(r + s)} \quad [7.48]$$

and

$$V_{uR} = \frac{\bar{m} - q}{r}.$$

7.4.2.2.3. Comparing the two economies

Clearly, wages are higher in the flexible economy, and so is productivity. Employment protection generates misallocation by maintaining people in low productivity jobs, which in turn reduces wages. How about the job finding rate? We have that

$$\begin{aligned} a_F - a_R &= (m_H/q - 1)(r + s + \lambda) - (r + s) \left(\frac{\bar{m}}{q} - 1 \right) \\ &= (r + s) \frac{m_H - \bar{m}}{q} + \lambda(m_H/q - 1) > 0, \end{aligned}$$

since q cannot exceed m_H , otherwise, wages would also exceed the maximum productivity level m_H . Thus, the job finding rate is higher in the flexible society. All else being equal, eliminating firing costs, or indeed any tax on profits, raises the incentives to enter the market, eventually leading to a higher job finding rate in equilibrium.

Turning now to the welfare of the unemployed, we observe that $V_{uF} > V_{uR}$. The unemployed always favor the flexible economy, since wages are higher (due to the elimination of labor misallocation to low productivity jobs) and unemployment duration is lower. This is a special case of a more general result proved in Saint-Paul (2000), namely that under free entry, the interests of the unemployed are aligned with those of capital. This is because any regulatory shift that, in partial equilibrium, raises profits, is eventually dissipated, through free entry, by an increase in the opportunity cost of work, i.e. an increase in the value of being unemployed.

7.4.2.2.4. The support for rigidity

When will “society” decide to implement employment protection? Saint-Paul assumes that most of the political power rests with the employed. Therefore, the model predicts that rigidity prevails provided $V_{eR} > V_{eF}$, or equivalently, comparing [7.48] with [7.46] and rearranging

$$(r + s + \lambda)(r + s)(m_H - \bar{m}) < qr\lambda.$$

Using [7.47], we find that this is equivalent to

$$(r + s)(m_H - m_L) < qr.$$

This expression suggests that rigidity is more likely to prevail under the following conditions:

- A greater q : the rents earned by the employed over their opportunity cost of work are greater. This makes sense since they lose that rent upon job loss. Thus, rigidity “spreads”: institutions that allow incumbent workers (the “insiders”) to raise wages above the market clearing level also generate greater support for another form of rigidity, namely employment protection.
- A smaller s : the less the insiders are exposed to unemployment through the exogenous process of job destruction, the greater the support for employment protection, as the insiders consider themselves less likely to become outsiders.
- A smaller $m_H - m_L$: the lower the productivity loss when falling into the low state, the lower the negative effect on wages of preserving low productivity jobs through employment protection and, in turn, the greater the support for employment protection by the insiders.

7.4.2.2.5. Status quo bias

The model also predicts status quo bias: the support for rigidity is greater if society is initially rigid. Rigidity creates its own support by maintaining a mass of workers in low productivity jobs. These people would lose their jobs if society became rigid. Therefore, they support rigidity by comparing V_{eR} with V_{uF} , which, since $V_{uF} < V_{eF}$, implies that they are more likely to support the rigid status quo than insiders in high productivity jobs. In contrast, this class of low productivity workers does not exist if the status quo is initially flexible.

7.4.3. Reform dynamics

I now discuss some of the literature on reform dynamics. This literature is concerned about how to implement a given reform facing potential opposition. It typically leaves aside an important conceptual issue, which is why an incumbent government, who aggregates social preferences in a certain way, would want to make a reform that is “politically difficult”. The concept of reform has no place in any theory that delivers a political equilibrium as an outcome. By construction, then, this outcome is not “difficult”, while any alternative policy (a “reform”) will not happen, again by construction. The most rigorous way to view the literature on reform is as an analysis of the process by which a political equilibrium is obtained, whenever it differs from some inherited status quo, which determines the relevant initial state variables of the economy.

7.4.3.1. Status quo bias

Status quo bias is one reason why reform dynamics matter: a policy change may be difficult to implement, but get enough support to persist once implemented. In such a world, the political equilibrium depends on initial conditions.

Adjustment costs are an obvious source of status quo bias, even in a non-political world with a single decision-maker. As discussed in section 7.4.2.2.5, people's policy preferences depend on their current economic state, which is itself contingent upon policies and institutions. The latter may therefore persist to the extent that they generate their own constituency. Finally, uncertainty about the outcome of a reform will push in favor of the status quo to the extent that agents are risk-averse, although again this would also be true in a Robinson Crusoe economy and therefore hardly qualifies as a political bias.

One intriguing source of status quo bias is associated with an uneven distribution of information about gains and losses from reform. Consider the following example, borrowed from Fernandez and Rodrik (1991): a reform brings a payoff equal to +1 to 60% of the population, and equal to -1 to the remaining 40%, who therefore lose from it. Clearly, a utilitarian social planner would opt for the reform. Also, if all agents knew their gains, a majority of 60% would vote in favor of the reform. Under a total veil of ignorance, each agent would consider they have a 60% chance of being a winner for the reform, and thus the reform would pass with unanimity. Now consider the following situation: 40% of the winners are known, while the remaining 60% of the population do not know whether they will be winners or losers. Consequently, they expect to be winners with a probability equal to $\frac{20}{60} = \frac{1}{3}$. While the 40% of winners who are identified undoubtedly support the reform, the net expected gain for the others is now $\frac{1}{3} \times 1 + \frac{2}{3} \times (-1) = -\frac{1}{3} < 0$. Therefore, the reform is now blocked by 60% of the population who have a net expected loss. Identification of part but not all of the winners generates a structure of uncertainty such that the status quo prevails, despite that nobody would vote in favor of reverting to it should the reform be imposed, once all agents have observed their net gains from the reform.

7.4.3.2. *Reform design: gradualism, etc.*

Dewatripont and Roland (1992, 1995) have analyzed how to design a given reform so as to make it politically successful. They focus on an agenda-setting government which tries to phase out an economic sector, for example, a publicly owned sector in the context of a privatization program. Workers need to be compensated so as to accept to leave the sector. Furthermore, workers are heterogeneous and differ by their willingness to relocate. The authors show that a complete shutdown of the sector involves the transfer of informational rents to the workers for whom relocation is easy. A gradual phasing out of the sector may be preferable, under some circumstances. This is because it allows the low relocation cost workers to self-select by relocating earlier. This makes it possible for the government to save on informational rents by first offering a low compensation to movers and, then, later, a higher one for those who did not accept the first offer. The initial compensation can be made incentive compatible, i.e. computed to be high enough so as to deter the low relocation cost workers from waiting for the future more generous offers, while, despite that constraint, gradualism remains less costly to the budget than a complete reform.

Saint-Paul (1993) understands the two-tier labor market of many European countries, by which temporary contracts coexist with permanent ones, as a gradual reform strategy that can be used by governments so as to implement a transition towards a fully flexible labor market. Temporary contracts raise the employment prospects of the unemployed while preserving the interests of incumbent workers who keep their initial permanent ones. As the share of temporary workers grows in the economy, a critical date is reached after which the government could use the temporary workers as a constituency in favor of a complete reform that would eliminate the remaining stock of permanent contracts. This is a mixed blessing for the reform, though, because, in expectation of this, incumbent workers may block the reform *ex ante*. To avoid this, the government may slow down the rate at which the share of temporary contracts goes up over time, by designing them so as to force the employer to convert them into permanent ones after a while. Thus, political considerations can explain both the dual nature of European labor markets and the restrictions put on the use of temporary contracts in Europe.

Bertola and Ichino (1995) discuss how labor market reforms, in particular reforms of employment protection, may backfire if they lack credibility. Specifically, a reduction in employment protection is initially met by a mass of job destruction, i.e. for example, in the context of the model of section 7.4.2.2, all the workers who hold low productivity jobs in the rigid economy. The gains from reform, in terms of employment, are only obtained gradually, as this excess unemployment is gradually absorbed into new, more profitable, economic sectors. If the reform lacks credibility, it may be reversed once these employment losses have been experienced, and the economy may suffer from both the high job destruction rate of a flexible economy and the low job creation rate of a rigid economy. The argument highlights the high cost of policy reversals and the importance of commitment in the design of labor market reforms.

7.5. Conclusion

Despite its length, this chapter only covers a fraction of the literature on political economy, ignoring many theoretical extensions of the basic models discussed here, as well as the burgeoning empirical literature on various subjects such as comparative economic development, the role of the media and social network platforms, the determinants of populism, trust, and so forth. These are examples of the many issues upon which the more recent literature focuses on, mostly from an empirical point of view. It is to be hoped that the many theoretical research questions left open by the works discussed above will again be given proper attention.

7.6. References

Acemoglu, D., Egorov, G., Sonin, K. (2013). A political theory of populism. *Quarterly Journal of Economics*, 128(2), 771–805.

- Alesina, A. (1988). Macroeconomic policy in a two-party system as a repeated game. *Quarterly Journal of Economics*, 102(3) (Aug., 1987), 651–678.
- Alesina, A. and Drazen, A. (1991). Why are stabilizations delayed? *American Economic Review*, 81(5), 1170–1188.
- Alesina, A. and Rodrik, D. (1994). Distributive politics and economic growth. *Quarterly Journal of Economics*, 109(2), 465–490.
- Alesina, A. and Tabellini, G. (1990). A positive theory of fiscal deficits and government debt. *Review of Economic Studies*, 57(3), 403–414.
- Barro, R.J. (1979). On the determination of the public debt. *Journal of Political Economy*, 87(5), 940–971.
- Barro, R.J. (1990). Government spending in a simple model of endogenous growth. *Journal of Political Economy*, 98(5), S103–S125.
- Barro, R. and Gordon, D. (1983). A positive theory of monetary policy in a natural rate model. *Journal of Political Economy*, 91(4) (Aug., 1983), 589–610.
- Benabou, R. (n.d.). Unequal societies: Income distribution and the social contract. *American Economic Review*, 90(1), 96–129.
- Bernheim, B.D and Whinston, M.D. (1986). Menu auctions, resource allocation, and economic influence. *Quarterly Journal of Economics*, 101(1), 1–31.
- Bertola, G. and Ichino, A. (1995). Crossing the river: A comparative perspective on Italian employment dynamics. *Economic Policy*, 10(21), 359–420.
- Besley, T. and Persson, T. (2009). The origins of state capacity: Property rights, taxation, and politics. *American Economic Review*, 99(4).
- Besley, T. and Persson, T. (2010). State capacity, conflict and development. *Econometrica*, 78(1), 1–34.
- Besley, T., Ilzetzki, E., Persson, T. (2013). Weak states and steady states: The dynamics of fiscal capacity. *American Economic Journal: Macroeconomics*, 5(4), 205–235.
- Cagan, P. (1956). The monetary dynamics of hyperinflation. In *Studies in the Quantity Theory of Money*, Friedman, M. (ed.). The University of Chicago Press, Chicago, 25–117.
- Calvo, G. (1978). On the time consistency of optimal policy in a monetary economy. *Econometrica*, 46(6), 1411–1428.
- David, B. and Driffill, J. (1985). Inflation and reputation. *American Economic Review*, 75(3) (Jun., 1985), 530–538.
- Dewatripont, M. and Roland, G. (1992). Economic reform and dynamic political constraints. *Review of Economic Studies*, 59(4), 703–730.
- Dewatripont, M. and Roland, G. (1995). The design of reform packages under uncertainty. *American Economic Review*, 85(5), 1207–1223.

- Diamond, P.A. (1965). National debt in a neoclassical growth model. *American Economic Review*, 55, 1126–1150.
- Dornbusch, R. and Edwards, S. (1992). *The Macroeconomics of Populism in Latin America*. University of Chicago Press, Chicago.
- Drazen, A. (2002). *Political Economy in Macroeconomics*. Princeton University Press, Princeton.
- Drazen, A. and Grilli, V. (1993). The benefit of crises for economic reforms. *American Economic Review*, 83(3), 598–607.
- Elster, J. (ed.) (1986). *The Multiple Self*. Cambridge University Press, Cambridge.
- Farhi, E. and Tirole, J. (2012). Collective moral hazard, maturity mismatch, and systemic bailouts. *American Economic Review*, 102(1), 60–93.
- Fernandez, R. and Rodrik, D. (1991). Resistance to reform: Status quo bias in the presence of individual-specific uncertainty. *American Economic Review*, 81(5), 1146–1155.
- Finn, K. and Prescott, E. (1977). Rules rather than discretion: The inconsistency of optimal plans. *Journal of Political Economy*, 85(3), 473–492.
- Gans, J. and Smart, M. (1996). Majority voting with single-crossing preferences. *Journal of Public Economics*, 59, 219–237.
- Gersbach, H. and Schniewind, A. (2001). Awareness of general equilibrium effects and unemployment. IZA Discussion Papers 394, Institute of Labor Economics (IZA).
- Giavazzi, F. and Pagano, M. (1988). The advantage of tying one's hands: EMS discipline and central bank credibility. *European Economic Review*, 32, 303–330.
- Glaeser, E.L. (2012). The political risks of fighting market failures: Subversion, populism and the government sponsored enterprises. NBER Working Papers 18112, National Bureau of Economic Research, Inc.
- Grandmont, J.-M. (1978). Intermediate preferences and the majority rule. *Econometrica*, 46(2), 317–330.
- Grossman, G. and Helpman, E. (1994). Protection for sale. *American Economic Review*, 84(4), 833–850.
- Hibbs, D.A. (1987). *The American Political Economy: Macroeconomics and Electoral Politics*. Harvard University Press, Cambridge, MA, and London.
- Kreps, D. and Wilson, R. (1982). Reputation and Imperfect Information. *Journal of Economic Theory*, 27, 253–279.
- Lindbeck, A. and Weibull, J.W. (1987). Balanced-budget redistribution as the outcome of political competition. *Public Choice*, 52, 273–297.
- Lucas, R. (1990). Why doesn't capital flow from rich to poor countries? *American Economic Review*, 80(2), 92–96.

- Meltzer, A.H. and Richard, S. (1981). A rational theory of the size of government. *Journal of Political Economy*, 89(5), 914–927.
- Mortensen, D.T. and Pissarides, C.A. (1994). Job creation and Job destruction in the theory of unemployment. *Review of Economic Studies*, 61(3), 397–415.
- Nordhaus, W. (1975). The political business cycle. *Review of Economic Studies*, 42(2), 169–190.
- Perotti, R. (1996). Growth, income distribution, and democracy: What the data say. *Journal of Economic Growth*, Springer, 1(2), June, 149–187.
- Persson, T. and Svensson, L.E.O. (1989). Why a stubborn conservative would run a deficit: Policy with time-inconsistent preferences. *Quarterly Journal of Economics*, 104(2), 325–345.
- Persson, T. and Tabellini, G. (1996). Federal fiscal constitutions: Risk sharing and redistribution. *Journal of Political Economy*, 104(5), 979–1009.
- Persson, T. and Tabellini, G. (1994). Is inequality harmful for growth? *American Economic Review*, 84(3), 600–621.
- Persson, T. and Tabellini, G. (2004). Constitutional rules and fiscal policy outcomes. *American Economic Review*, 94(1), 25–45.
- Persson, T., Roland, G., Tabellini, G. (2000). Comparative politics and public finance. *Journal of Political Economy*, 108(6), 1121–1161.
- Ramsey, F. (1928). A mathematical theory of saving. *Economic Journal*, 38(152), 543–559.
- Richard, C., Gali, J., Gertler, M. (1999). The science of monetary policy: A new Keynesian perspective. *Journal of Economic Literature*, 37(4), 1661–1707.
- Roberts, K. (1977). Voting over income tax schedules. *Journal of Public Economics*, 8, 329–340.
- Rogoff, K. (1985). The optimal degree of commitment to an intermediate monetary target. *Quarterly Journal of Economics*, 100(4), 1169–1189.
- Rogoff, K. and Sibert, A. (1988). Elections and macroeconomic policy cycles. *Review of Economic Studies*, LV(55)(181), 1–16.
- Saint-Paul, G. (1993). On the political economy of labor market flexibility. *NBER Macroeconomics Annual 1993*, 8, 151–196.
- Saint-Paul, G. (1997). The rise and persistence of rigidities. *American Economic Review Papers and Proceedings*, 87(2), 290–294.
- Saint-Paul, G. (2000). *The Political Economy of Labor Market Institutions*. Oxford University Press, Oxford.
- Saint-Paul, G. (2001). The dynamics of exclusion and fiscal conservatism. *Review of Economic Dynamics*, 4(2), 275–302.

- Saint-Paul, G. (2002). The political economy of employment protection. *Journal of Political Economy*, 110(3), 672–704.
- Saint-Paul, G. (2020). From microeconomic favoritism to macroeconomic populism. *Journal of the Finnish Economic Association*.
- Saint-Paul, G. and Verdier, T. (1993). Education, democracy and growth. *Journal of Development Economics*, 42(2), 399–407.
- Saint-Paul, G., Ticchi, D., Vindigni, A. (2021). Engineering crises: Favoritism and strategic fiscal indiscipline. *Economics and Politics*, 33(3).
- Shapiro, C. and Stiglitz, J.E. (1984). Equilibrium unemployment as a worker discipline device. *American Economic Review*, 74(3), 433–444.
- Starr, P. (2019). *Entrenchment: Wealth, Power, and the Constitution of Democratic Societies*. Yale University Press, New Haven, CT.
- Tabellini, G. and Alesina, A. (1990). Voting on the budget deficit. *American Economic Review*, 80(1), 37–49.
- Tornell, A. and Lane, P. (1999). The voracity effect. *American Economic Review*, 89(1), 22–46.
- Tornell, A. and Velasco, A. (1992). The tragedy of the commons and economic growth: Why does capital flow from poor to rich countries? *Journal of Political Economy*, 100(6), 1208–1231.
- Wright, R. (1986). The redistributive roles of unemployment insurance and the dynamics of voting. *Journal of Public Economics*, 31(3), 377–399.

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