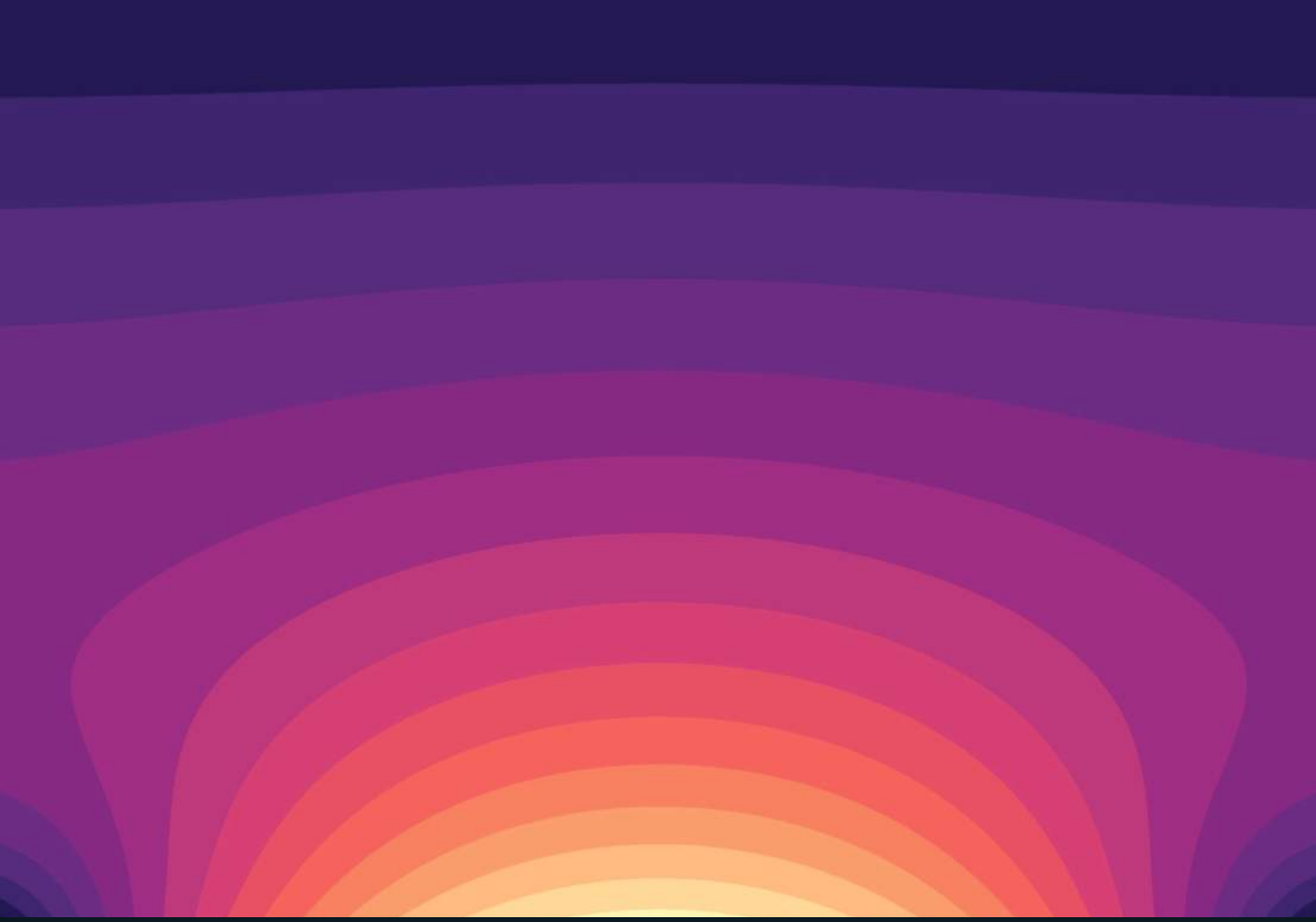

APPLIED PARTIAL DIFFERENTIAL EQUATIONS

with Fourier Series and Boundary Value Problems

SIXTH EDITION

RICHARD HABERMAN | PAUL CHOBOTER

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APPLIED PARTIAL DIFFERENTIAL EQUATIONS

Sixth Edition

with Fourier Series and Boundary Value Problems

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To Liz, Ken, and Vicki.

To my parents, Ivan and Verna Choboter.

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PREFACE

For many years, Richard Haberman's *Applied Partial Differential Equations* has been a favorite of math instructors because it focuses on problem solving without neglecting the theoretical foundations important for student learning. This textbook discusses partial differential equations of applied mathematics, the physical sciences, and engineering. Partial differential equations can be used to model phenomena such as heat flow, the propagation of light and sound waves, fluid dynamics, and traffic flow. This book approaches the subject from an applied mathematics perspective. The equations are motivated and derived with simple models. Solution techniques are developed patiently, and mathematical results are frequently given physical interpretations.

At the outset of this sixth edition, I wanted to sincerely acknowledge the late great Prof. Haberman. I myself have learned so much from his textbooks, and it is humbling to now have the opportunity to work on this edition.

New to This Edition

We very much wanted to keep Richard Haberman's voice and tone intact, while we updated and clarified the text wherever possible. Within those constraints, we have made edits to every single chapter. Some of the more significant edits are in Chapter 2, where the discussion of the method of separation of variables is first discussed. We added a clarifying summary of the method, and completely rewrote the Appendix to Chapter 2 that reviews the solutions to linear homogeneous ordinary differential equations.

This edition includes 57 new or edited exercises. Some of the exercises are introduced at a foundational level, intended to give the students a concrete example of the ideas discussed, and some are more advanced. For example, in Chapter 11 where Green's functions for the wave equation are discussed, one particularly novel exercise (supported by a similar new in-text example) has students work through an application of Green's functions to derive the classic change of observed frequency due to motion relative to the source, known as the Doppler effect.

Perhaps the most exciting new feature is the addition of 18 MATLAB programs, available for download. Some of these routines animate certain static figures found in the printed text. Some demonstrate numerical solution techniques of the partial differential equations. Most are designed for students (and teachers!) to dig in and play around with parameters to get a deeper intuition for how the solutions respond. For instance, one program dynamically illustrates the difference between the phase speed and the group speed of the superposition of several waves of similar wavelengths.

Topics Discussed in This Text

Standard topics such as the method of separation of variables, Fourier series, orthogonal functions, and Fourier transforms are developed with considerable detail. Finite difference numerical methods for partial differential equations are clearly presented with considerable depth. A briefer presentation is made of the finite element method. This text also has an extensive presentation of the method of characteristics for linear and nonlinear wave equations, including discussion of the dynamics of shock waves for traffic flow. Nonhomogeneous problems are carefully introduced, including Green's functions for Laplace's, heat, and wave equations. Numerous topics are included, such as differentiation and integration of Fourier series, Sturm–Liouville and multidimensional eigenfunctions, Rayleigh quotient, Bessel functions for a vibrating circular membrane, and Legendre polynomials for spherical problems. Some optional advanced material is included (for example, asymptotic expansion of large eigenvalues, calculation of perturbed frequencies using the Fredholm alternative, stability conditions for finite difference methods, and direct and inverse scattering).

Applications briefly discussed include the lift and drag associated with fluid flow past a circular cylinder, Snell's law of refraction for light and sound waves, the derivation of the eikonal equation from the wave equation, dispersion relations for water waves, wave guides, and fiber optics.

Organization of Topics by Chapter

The core material for a typical course consists of most of Chapters 1–5 and 7. This will usually be supplemented by a few other topics. The text is somewhat flexible for an instructor, since most of Chapters 6–13 depend only on Chapters 1–5. Chapter 11 on Green's functions for the heat and wave equation is an exception, since it requires Chapters 9 and 10.

Chapter 14 is more advanced, discussing linear and nonlinear dispersive waves, stability, and perturbation methods. It is self-contained and accessible to strong undergraduates. Group velocity and envelope equations for linear dispersive waves, whose applications include the rainbow caustic of optics, are analyzed. Nonlinear dispersive waves are discussed, including an introductory presentation of solitons for the weakly nonlinear long wave equation (Korteweg–de Vries) and the weakly nonlinear wave envelope equation (nonlinear Schrödinger). In addition, instability and bifurcation phenomena for partial differential equations are discussed, along with perturbation methods (multiple scale and boundary layer problems).

The book also includes diffusion of a chemical pollutant, Galerkin numerical approximation for the frequencies, similarity solution for the heat equation, two-dimensional Green's function for the wave equation, nonuniqueness of shock velocity and its resolution, spatial structure of traveling shock wave, stability and bifurcation theory for systems of ordinary differential equations, two-spatial-dimensional wave envelope equations, analysis of modulational instability, long wave instabilities, pattern formation for reaction diffusion equations, and the Turing instability.

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Finally, I would like to sincerely thank the team at Pearson for their help and guidance through the process of editing the text. In particular, thank you to Jon Krebs for initially welcoming me to the team, and to Allison DeVille and Lauren Morse for their expertise and patience in guiding me through the editing process.

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1
Heat Equation

1.1 Introduction

We wish to discuss the solution of elementary problems involving partial differential equations, the kinds of problems that arise in various fields of science and engineering. A **partial differential equation** (PDE) is an equation containing partial derivatives, for example,

$$\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0. \tag{1.1.1}$$

We could begin our study by determining what functions $u(x, t)$ satisfy (1.1.1). However, we prefer to start by investigating a physical problem. We do this for two reasons. First, our mathematical techniques probably will be of greater interest to you when it becomes clear that these methods analyze physical problems. Second, we will actually find that physical considerations will motivate many of our mathematical developments.

Many diverse subject areas in engineering and the physical sciences are dominated by the study of partial differential equations. No list could be all-inclusive. However, the following examples should give you a feeling for the type of areas that are highly dependent on the study of partial differential equations: acoustics, aerodynamics, elasticity, electrodynamics, fluid dynamics, geophysics (seismic wave propagation), heat transfer, meteorology, oceanography, optics, petroleum engineering, plasma physics (ionized liquids and gases), and quantum mechanics.

We will follow a certain philosophy of applied mathematics in which the analysis of a problem will have three stages:

1. Formulation
2. Solution
3. Interpretation

We begin by formulating the equations of heat flow describing the transfer of thermal energy. Heat energy is caused by the agitation of molecular matter. Two basic processes take place in order for thermal energy to move: conduction and convection. **Conduction** results from the collisions of neighboring molecules in which the kinetic energy of vibration of one molecule is transferred to its nearest neighbor. Thermal energy is thus spread by conduction even if the molecules themselves do not move their location appreciably. In addition, if a vibrating molecule moves from one region to another, it takes its thermal energy with it. This type of movement of thermal energy is called **convection**. In order to begin our study with relatively simple problems, we will study heat flow only in cases in which the conduction of heat energy is much more significant than its convection. We will thus think of heat flow primarily in the case of solids, although heat transfer in fluids (liquids and gases) is also primarily by conduction if the fluid velocity is sufficiently small.

1.2 Derivation of the Conduction of Heat in a One-Dimensional Rod

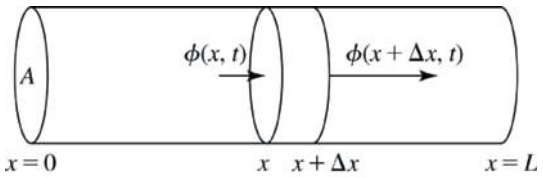
THERMAL ENERGY DENSITY

We begin by considering a rod of constant cross-sectional area A oriented in the x-direction (from $x = 0$ to $x = L$) as illustrated in Fig. 1.2.1. We temporarily introduce the amount of thermal energy per unit volume as an unknown variable and call it the **thermal energy density**:

$$e(x, t) \equiv \text{thermal energy density.}$$

We assume that all thermal quantities are constant across a section; the rod is one-dimensional. The simplest way this may be accomplished is to insulate perfectly the lateral surface area of the rod. Then no thermal energy can pass through the lateral surface. The dependence on x and t corresponds to a situation in which the rod is not uniformly heated; the thermal energy density varies from one cross section to another.

Figure 1.2.1 One-dimensional rod with heat energy flowing into and out of a thin slice.



Heat energy

We consider a thin slice of the rod contained between x and $x + \Delta x$ as illustrated in Fig. 1.2.1. If the thermal energy density is constant throughout the volume, then the total energy in the slice is the product of the thermal energy density and the volume. In general, the energy density is not constant. However, if Δx is exceedingly small, then $e(x, t)$ may be approximated as a constant within the slice so that

$$\text{heat energy} = e(x, t)A\,\Delta x,$$

since the volume of the thin slice is $A\,\Delta x$.

Conservation of heat energy

The heat energy between x and $x + \Delta x$ changes in time due only to heat energy flowing across the edges (x and $x + \Delta x$) and heat energy generated inside (due to positive or negative sources of heat energy). No heat energy changes are due to flow across the lateral surface, since we have assumed that the lateral surface is insulated. The fundamental heat flow process is described by the word equation

rate of change of heat energy in time	=	heat energy flowing across boundaries per unit time	+	heat energy generated inside per unit time.

This is called **conservation of heat energy**. For the small slice, the rate of change of heat energy is

$$\frac{\partial}{\partial t}[e(x, t)A\,\Delta x],$$

where the partial derivative $\frac{\partial}{\partial t}$ is used because x is being held fixed.

Heat flux

Thermal energy flows to the right or left in a one-dimensional rod. We introduce the **heat flux**:

$$\phi(x, t) \equiv \begin{array}{l}\text{heat flux (the amount of thermal energy } \textit{per unit} \\ \textit{time} \text{ flowing to the right } \textit{per unit surface area}).\end{array}$$

If $\phi(x, t) < 0$, it means that heat energy is flowing to the left. Heat energy flowing per unit time across the boundaries of the slice is $\phi(x, t)A - \phi(x + \Delta x, t)A$, since the heat flux is the flow per unit surface area and it must be multiplied by the surface area. If $\phi(x, t) > 0$ and $\phi(x + \Delta x, t) > 0$, as illustrated in Fig. 1.2.1, then the heat energy flowing per unit time at x contributes to an increase of the heat energy in the slice, whereas the heat flow at $x + \Delta x$ decreases the heat energy.

Heat sources

We also allow for internal sources of thermal energy:

$$Q(x, t) = \text{heat energy per unit volume generated per unit time,}$$

perhaps due to chemical reactions or electrical heating. $Q(x, t)$ is approximately constant in space for a thin slice, and thus the total thermal energy generated per unit time in the thin slice is approximately $Q(x, t)A\,\Delta x$.

Conservation of heat energy (thin slice)

The rate of change of heat energy is due to thermal energy flowing across the boundaries and internal sources:

$$\frac{\partial}{\partial t}[e(x, t)A\,\Delta x] \approx \phi(x, t)A - \phi(x + \Delta x, t)A + Q(x, t)A\,\Delta x. \tag{1.2.1}$$

Equation (1.2.1) is not precise because various quantities were assumed approximately constant for the small cross-sectional slice. We claim that (1.2.1) becomes increasingly accurate as $\Delta x \to 0$. Before giving a careful (and mathematically rigorous) derivation, we will just attempt to explain the basic ideas of the limit process, $\Delta x \to 0$. In the limit as $\Delta x \to 0$, (1.2.1) gives no interesting information, namely, $0 = 0$. However, if we first divide by Δx and then take the limit as $\Delta x \to 0$, we obtain

$$\frac{\partial e}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x, t) - \phi(x + \Delta x, t)}{\Delta x} + Q(x, t), \tag{1.2.2}$$

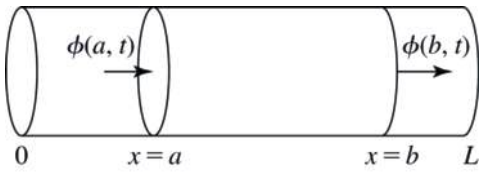
where the constant cross-sectional area has been cancelled. We claim that this result is exact (with no small errors), and hence we replace the $\approx$ in (1.2.1) by $=$ in (1.2.2). In this limiting process, $\Delta x \to 0$, t is being held fixed. Consequently, from the definition of a partial derivative,

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \tag{1.2.3}$$

Conservation of heat energy (exact)

An alternative derivation of conservation of heat energy has the advantage of our not being restricted to small slices. The resulting approximate calculation of the limiting process ($\Delta x \to 0$) is avoided. We consider any *finite* segment (from $x = a$ to $x = b$) of the original one-dimensional rod (see Fig. 1.2.2). We will investigate the conservation of heat energy in this region. The total heat energy is $\int_a^b e(x, t)A\,dx$, the sum of the contributions of the infinitesimal slices. Again it changes only due to heat energy flowing through the side edges ($x = a$ and $x = b$) and heat energy generated inside the region, and thus (after canceling the constant A)

Figure 1.2.2 Heat energy flowing into and out of a finite segment of a rod.



$$\boxed{\frac{d}{dt}\int_a^b e\,dx = \phi(a, t) - \phi(b, t) + \int_a^b Q\,dx.} \tag{1.2.4}$$

Technically, an ordinary derivative d/dt appears in (1.2.4), since $\int_a^b e\,dx$ depends only on t, not also on x. However,

$$\frac{d}{dt}\int_a^b e\,dx = \int_a^b \frac{\partial e}{\partial t}\,dx$$

if a and b are constants (and if e is continuous). This holds since inside the integral the ordinary derivative now is taken keeping x fixed, and hence it must be replaced by a partial derivative. Every term in (1.2.4) is now an ordinary integral if we notice that

$$\phi(a, t) - \phi(b, t) = -\int_a^b \frac{\partial \phi}{\partial x}\, dx$$

(this[1] being valid if ϕ is continuously differentiable). Consequently,

$$\int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q \right) dx = 0.$$

This integral must be zero *for arbitrary aand b*; the area under the curve must be zero for arbitrary limits. This is possible only if the integrand itself is identically zero.[2] Thus, we rederive (1.2.3) as

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \tag{1.2.5}$$

Equation (1.2.4), the **integral conservation law**, is more fundamental than the differential form (1.2.5). Equation (1.2.5) is valid in the usual case in which the physical variables are continuous.

A further explanation of the minus sign preceding $\partial\phi/\partial x$ is in order. For example, if $\partial\phi/\partial x > 0$ for $a \le x \le b$, then the heat flux ϕ is an increasing function of x. The heat is flowing greater to the right at $x = b$ than at $x = a$ (assuming that $b > a$). Thus (neglecting any effects of sources Q), the heat energy must decrease between $x = a$ and $x = b$, resulting in the minus sign in (1.2.5).

Temperature and specific heat

We usually describe materials by their temperature,

$$u(x, t) \equiv \text{temperature},$$

not their thermal energy density. Distinguishing between the concepts of temperature and thermal energy is not necessarily a trivial task. Only in the mid-1700s did the existence of accurate experimental apparatus enable physicists to recognize that it may take different amounts of thermal energy to raise two different materials from one temperature to another larger temperature. This necessitates the introduction of the **specific heat** (or heat capacity):

$$c \equiv \begin{array}{l}\text{specific heat (the heat energy that must be supplied to a unit}\\ \text{mass of a substance to raise its temperature one unit).}\end{array}$$

In general, from experiments (and our definition) the specific heat c of a material depends on the temperature u. For example, the thermal energy necessary to raise a unit mass from 0°C to 1°C could be different from that needed to raise the mass from 85°C to 86°C for the same substance. Heat flow problems with the specific heat depending on the temperature are mathematically quite complicated. (Exercise 1.2.7 briefly discusses this situation.) Often for restricted temperature intervals, the specific heat is approximately independent of the temperature. However, experiments suggest that different materials require different amounts of thermal energy to heat up. Since we would like to formulate the correct equation in situations in which the composition of our one-dimensional rod might vary from position to position, the specific heat will depend on x, $c = c(x)$. In many problems, the rod is made of one material (a uniform rod), in which case we will let the specific heat c be a constant. In fact, most of the solved problems in this text (as well as other books) correspond to this approximation, c constant.

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[1] This is one of the fundamental theorems of calculus.

[2] Most proofs of this result are inelegant. Suppose that $f(x)$ is continuous and $\int_a^b f(x)\, dx = 0$ for arbitrary a and b. We wish to prove $f(x) = 0$ for all x. We can prove this by assuming that there exists a point x_0 such that $f(x_0) \neq 0$ and demonstrating a contradiction. If $f(x_0) \neq 0$ and $f(x)$ is continuous, then there exists some region near x_0 in which $f(x)$ is of one sign. Pick a and b to be in this region, and hence $\int_a^b f(x)\, dx \neq 0$ since $f(x)$ is of one sign throughout. This contradicts the statement that $\int_a^b f(x)\, dx = 0$, and hence it is impossible for $f(x_0) \neq 0$. Equation (1.2.5) follows.

Thermal energy

The thermal energy in a thin slice is $e(x, t)A\,\Delta x$. However, it is also defined as the energy it takes to raise the temperature from a reference temperature 0° to its actual temperature $u(x, t)$. Since the specific heat is independent of temperature, the heat energy per unit mass is just $c(x)u(x, t)$. We thus need to introduce the **mass density** $\rho(x)$:

$$\rho(x) \equiv \text{mass density}(\text{mass per unit volume}),$$

allowing it to vary with x, possibly due to the rod being composed of nonuniform material. The total mass of the thin slice is $\rho A\,\Delta x$. The total thermal energy in any thin slice is thus $c(x)u(x, t) \bullet \rho A\,\Delta x$, so that

$$e(x, t)A\,\Delta x = c(x)u(x, t)\rho A\,\Delta x.$$

In this way we have explained the basic relationship between thermal energy and temperature:

$$\boxed{e(x, t) = c(x)\rho(x)u(x, t).} \tag{1.2.6}$$

This states that the thermal energy per unit volume equals the thermal energy per unit mass per unit degree times the temperature times the mass density (mass per unit volume). When the thermal energy density is eliminated using (1.2.6), conservation of thermal energy [(1.2.3) or (1.2.5)] becomes

$$\boxed{c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q.} \tag{1.2.7}$$

Fourier's law

Usually (1.2.7) is regarded as one equation in two unknowns, the temperature $u(x, t)$ and the heat flux (flow per unit surface area per unit time) $\phi(x, t)$. How and why does heat energy flow? In other words, we need an expression for the dependence of the flow of heat energy on the temperature field. First, we summarize certain qualitative properties of heat flow with which we are all familiar:

1. If the temperature is constant in a region, no heat energy flows.
2. If there are temperature differences, the heat energy flows from the hotter region to the colder region.
3. The greater the temperature differences (for the same material), the greater is the flow of heat energy.
4. The flow of heat energy will vary for different materials, even with the same temperature differences.

Fourier (1768–1830) recognized properties 1 through 4 and summarized them (as well as numerous experiments) by the formula

$$\boxed{\phi = -K_0\frac{\partial u}{\partial x},} \tag{1.2.8}$$

known as **Fourier's law of heat conduction**. Here, $\partial u/\partial x$ is the derivative of the temperature; it is the slope of the temperature (as a function of x for fixed t); it represents temperature differences (per unit length). Equation (1.2.8) states that the heat flux is proportional to the temperature difference (per unit length). If the temperature u increases as x increases (i.e., the temperature is hotter to the right), $\partial u/\partial x > 0$, then we know (property 2) that heat energy flows to the left. This explains the minus sign in (1.2.8).

We designate the coefficient of proportionality K_0. It measures the ability of the material to conduct heat and is called the **thermal conductivity**. Experiments indicate that different materials conduct heat differently; K_0 depends on the particular material. The larger K_0 is, the greater the flow of heat energy with the same temperature differences. A material with a low value of K_0 would be a poor conductor of heat energy (and ideally suited for home insulation). For a rod composed of different materials, K_0 will be a function of x. Furthermore, experiments show that the ability to conduct heat for most materials is different at different temperatures, $K_0(x, u)$. However, just as with the specific heat c, the dependence on the temperature is often not important in particular problems.

Thus, throughout this text, we will assume that the thermal conductivity K_0 depends only on x, $K_0(x)$. Usually, in fact, we will discuss uniform rods in which K_0 is a constant.

Heat equation

If Fourier's law, (1.2.8), is substituted into the conservation of heat energy equation, (1.2.7), a partial differential equation results:

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q.} \tag{1.2.9}$$

We usually think of the sources of heat energy Q as being given, and the only unknown being the temperature $u(x, t)$. The thermal coefficients c, ρ, K_0 all depend on the material and hence may be functions of x. In the special case of a uniform rod, in which c, ρ, K_0 are all constants, the partial differential equation (1.2.9) becomes

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q.$$

If, in addition, there are no sources, $Q = 0$, then after dividing by the constant $c\rho$, the partial differential equation becomes

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},} \tag{1.2.10}$$

where the constant k,

$$k = \frac{K_0}{c\rho},$$

is called the **thermal diffusivity**, the thermal conductivity divided by the product of the specific heat and mass density. Equation (1.2.10) is often called the **heat equation**; it corresponds to no sources and constant thermal properties. If heat energy is initially concentrated in one place, (1.2.10) will describe how the heat energy spreads out, a physical process known as **diffusion**. Other physical quantities besides temperature smooth out in much the same manner, satisfying the same partial differential equation (1.2.10). For this reason, (1.2.10) is also known as the **diffusion equation**. For example, the concentration $u(x, t)$ of chemicals (such as perfumes and pollutants) satisfies the diffusion equation (1.2.8) in certain one-dimensional situations.

Diffusion of a chemical pollutant

Let $u(x, t)$ be the **density** or **concentration** of the chemical per unit volume. Consider a one-dimensional region (Fig. 1.2.2) between $x = a$ and $x = b$ with constant cross-sectional area A. The total amount of the chemical in the region is $\int_a^b u(x, t)A\,dx$. We introduce the **flux** $\phi(x, t)$ of the chemical, the amount of the chemical per unit surface flowing to the right per unit time. The rate of change with respect to time of the total amount of chemical in the region equals the amount of chemical flowing in per unit time minus the amount of chemical flowing out per unit time. Thus, after canceling the constant cross-sectional area A, we obtain the **integral conservation law** for the chemical concentration:

$$\boxed{\frac{d}{dt}\int_a^b u(x, t)\,dx = \phi(a, t) - \phi(b, t).} \tag{1.2.11}$$

Since $\frac{d}{dt}\int_a^b u(x, t)\,dx = \int_a^b \frac{\partial u}{\partial t}\,dx$ and $\phi(a, t) - \phi(b, t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$, it follows that $\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x}\right)dx = 0$. Since the integral is zero for arbitrary regions, the integrand must be zero, and in this way we derive the differential **conservation law** for the chemical concentration:

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0.} \tag{1.2.12}$$

Chemicals spread out from regions of high concentration to regions of low concentration. According to **Fick's law of diffusion**, the flux is proportional to $\frac{\partial u}{\partial x}$, the spatial derivative of the chemical concentration:

$$\boxed{\phi = -k\frac{\partial u}{\partial x}.} \tag{1.2.13}$$

If the concentration $u(x, t)$ is constant in space, there is no flow of the chemical. If the chemical concentration is increasing to the right $\left(\frac{\partial u}{\partial x} > 0\right)$, then atoms of chemicals migrate to the left, and vice versa. The proportionality constant k is called the chemical **diffusivity**, and it can be measured experimentally. When Fick's law (1.2.13) is used in the basic conservation law (1.2.12), we see that the chemical concentration satisfies the **diffusion equation**:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},} \tag{1.2.14}$$

since we are assuming as an approximation that the diffusivity is constant. Fick's law of diffusion for chemical concentration is analogous to Fourier's law for heat diffusion. Our derivations are quite similar.

Exercises 1.2

1.2.1. Briefly explain the minus sign:

(a) in conservation law (1.2.3) or (1.2.5) if $Q = 0$

(b) in Fourier's law (1.2.8)

(c) in conservation law (1.2.12), $\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$

(d) in Fick's law (1.2.13)

1.2.2. If $u(x, t) = \cos x \sin t$, $\phi(x, t) = \sin x \sin t$, $a = 0$, and $b = 2\pi$, then (1.2.11) is satisfied but (1.2.12) is not. Explain how this doesn't contradict the claim that (1.2.12) follows from (1.2.11).

1.2.3. Derive the heat equation for a rod assuming constant thermal properties and no sources.

(a) Consider the total thermal energy between x and $x + \Delta x$.

(b) Consider the total thermal energy between $x = a$ and $x = b$.

1.2.4. Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources by considering the total thermal energy between $x = a$ and $x = b$.

1.2.5. Derive the diffusion equation for a chemical pollutant.

(a) Consider the total amount of the chemical in a thin region between x and $x + \Delta x$.

(b) Consider the total amount of the chemical between $x = a$ and $x = b$.

1.2.6. Derive an equation for the concentration $u(x, t)$ of a chemical pollutant if the chemical is produced due to chemical reaction at the rate of $\alpha u(\beta - u)$ per unit volume.

1.2.7. Suppose that the specific heat is a function of position and temperature, $c(x, u)$.

(a) Show that the heat energy per unit mass necessary to raise the temperature of a thin slice of thickness Δx from 0° to $u(x, t)$ is not $c(x)u(x, t)$, but instead $\int_0^u c(x, \overline{u})\, d\overline{u}$.

(b) Rederive the heat equation in this case. Show that (1.2.3) remains unchanged.

1.2.8. Consider conservation of thermal energy (1.2.4) for any segment of a one-dimensional rod $a \le x \le b$. By using the fundamental theorem of calculus,

$$\frac{\partial}{\partial b}\int_a^b f(x)\, dx = f(b),$$

derive the heat equation (1.2.9).

***1.2.9.** If $u(x, t)$ is known, give an expression for the total thermal energy contained in a rod $(0 < x < L)$.

1.2.10. Consider a thin one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x, t)$. Derive the partial differential equation for the temperature $u(x, t)$.

(b) Assume that $w(x, t)$ is proportional to the temperature difference between the rod $u(x, t)$ and a known outside temperature $\gamma(x, t)$. Derive

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x, t) - \gamma(x, t)]h(x), \tag{1.2.15}$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

(c) Compare (1.2.15) with the equation for a one-dimensional rod whose lateral surfaces are insulated, but with heat sources.

(d) Specialize (1.2.15) to a rod of circular cross section with constant thermal properties and 0° outside temperature.

***(e)** Consider the assumptions in part (d). Suppose that the temperature in the rod is uniform [i.e., $u(x, t) = u(t)$]. Determine $u(t)$ if initially $u(0) = u_0$.

1.3 Initial and Boundary Conditions

1.3.1 INITIAL CONDITIONS

The partial differential equations describing the flow of heat energy, (1.2.9) or (1.2.10), have one time derivative. When an ordinary differential equation has one derivative, the initial value problem consists of solving the differential equation with one initial condition. Newton's law of motion for the position x of a particle yields a second-order ordinary differential equation, $md^2x/dt^2 = \text{forces}$. It involves second derivatives. The initial value problem consists of solving the differential equation with two initial conditions, the initial position x and the initial velocity dx/dt. From these pieces of information (including the knowledge of the forces), by solving the differential equation with the initial conditions, we can predict the future motion of a particle in the x-direction. We wish to do the same process for our partial differential equation, that is, predict the future temperature. Since the heat equations have one time derivative, we must be given one **initial condition** (IC) (usually at $t = 0$), the initial temperature. It is possible that the initial temperature is not constant, but depends on x. Thus, we must be given the initial temperature distribution,

$$u(x, 0) = f(x).$$

Is this enough information to predict the future temperature? We know the initial temperature distribution and know that the temperature changes according to the partial differential equation (1.2.9) or (1.2.10). However, we need to know what happens at the two boundaries, $x = 0$ and $x = L$. Without knowing this information, we cannot predict the future. Two conditions are needed corresponding to the second spatial derivatives present in (1.2.9) or (1.2.10), usually one condition at each end.

1.3.2 BOUNDARY CONDITIONS

In solving the heat equation, either (1.2.9) or (1.2.10), one **boundary condition** (BC) is needed at each end of the rod. The appropriate condition depends on the physical mechanism in effect at each end. Often the condition at the boundary depends on both the material inside and outside the rod. To avoid a more difficult mathematical problem, we will assume that the outside environment is known, not significantly altered by the rod.

Prescribed temperature

In certain situations, the temperature of the end of the rod, for example, $x = 0$, may be approximated by a **prescribed temperature**,

$$u(0, t) = u_B(t), \tag{1.3.1}$$

where $u_B(t)$ is the temperature of a fluid bath (or reservoir) with which the rod is in contact. The kind of boundary condition where the unknown function u is specified on the boundary is called a **Dirichlet** condition.

Insulated boundary

In other situations, it is possible to prescribe the heat flow rather than the temperature,

$$-K_0(0)\frac{\partial u}{\partial x}(0,\ t) = \phi(t), \tag{1.3.2}$$

where $\phi(t)$ is given. This is equivalent to giving one condition for the first derivative, $\partial u/\partial x$, at $x = 0$. The slope is given at $x = 0$. Equation (1.3.2) *cannot* be integrated in x because the slope is known only at one value of x. The simplest example of the prescribed heat flow boundary condition is when an end is perfectly **insulated**. In this case, there is no heat flow at the boundary. If $x = 0$ is insulated, then

$$\frac{\partial u}{\partial x}(0,\ t) = 0. \tag{1.3.3}$$

This kind of boundary condition, where the derivative of the unknown function $\partial u/\partial x$ is specified on the boundary, is called a **Neumann** condition.

Newton's law of cooling

When a one-dimensional rod is in contact at the boundary with a moving fluid (e.g., air), then neither the prescribed temperature nor the prescribed heat flow may be appropriate. For example, let us imagine a very warm rod in contact with cooler moving air. Heat will leave the rod, heating up the air. The air will then carry the heat away. This process of heat transfer is called **convection**. However, the air will be hotter near the rod. Again, this is a complicated problem; the air temperature will actually vary with distance from the rod (ranging between the bath and rod temperatures). Experiments show that, as a good approximation, the heat flow leaving the rod is proportional to the temperature difference between the bar and the prescribed external temperature. This boundary condition is called **Newton's law of cooling**. If it is valid at $x = 0$, then

$$-K_0(0)\frac{\partial u}{\partial x}(0,\ t) = -H[u(0,\ t) - u_B(t)], \tag{1.3.4}$$

where the proportionality constant H is called the **heat transfer coefficient** (or the convection coefficient). This boundary condition[3] involves a linear combination of u and $\partial u/\partial x$. We must be careful with the sign of proportionality. If the rod is hotter than the bath $[u(0,\ t) > u_B(t)]$, then usually heat flows out of the rod at $x = 0$. Thus, heat is flowing to the left, and in this case, the heat flow would be negative. That is why we introduced a minus sign in (1.3.4) (with $H > 0$). The same conclusion would have been reached had we assumed that $u(0,\ t) < u_B(t)$. Another way to understand the signs in (1.3.4) is to again assume that $u(0,\ t) > u_B(t)$. The temperature is hotter to the right at $x = 0$, and we should expect the temperature to continue to increase to the right. Thus, $\partial u/\partial x$ should be positive at $x = 0$. Equation (1.3.4) is consistent with this argument. In Exercise 1.3.1, you are asked to derive, in the same manner, that the equation for Newton's law of cooling at a right endpoint $x = L$ is

$$-K_0(L)\frac{\partial u}{\partial x}(L,\ t) = H[u(L,\ t) - u_B(t)], \tag{1.3.5}$$

where $u_B(t)$ is the external temperature at $x = L$. We immediately note the significant sign difference between the left boundary (1.3.4) and the right boundary (1.3.5).

The coefficient H in Newton's law of cooling is experimentally determined. It depends on properties of the rod, as well as fluid properties (including the fluid velocity). If the coefficient is very small, then very little heat energy flows across the boundary. In the limit as $H \to 0$, Newton's law of cooling approaches the insulated boundary condition. We can think of Newton's law of cooling for $H \neq 0$ as representing an imperfectly insulated boundary. If $H \to \infty$, the boundary

[3]For another situation in which (1.3.4) is valid, see Berg and McGregor (1966).

condition approaches the one for prescribed temperature, $u(0, t) = u_B(t)$. This is most easily seen by dividing (1.3.4), for example, by H:

$$-\frac{K_0(0)}{H}\frac{\partial u}{\partial x}(0, t) = -[u(0, t) - u_B(t)].$$

Thus, $H \to \infty$ corresponds to no insulation at all.

Summary

We have described three different kinds of boundary conditions. For example, at $x = 0$,

$$u(0, t) = u_B(t) \qquad \text{prescribed temperature}$$
$$-K_0(0)\frac{\partial u}{\partial x}(0, t) = \phi(t) \qquad \text{prescribed heat flux}$$
$$-K_0(0)\frac{\partial u}{\partial x}(0, t) = -H[u(0, t) - u_B(t)]. \qquad \text{Newton's law of cooling}$$

These same conditions could hold at $x = L$, noting that the change of sign ($-H$ becoming H) is necessary for Newton's law of cooling. One boundary condition occurs at each boundary. It is not necessary that both boundaries satisfy the same kind of boundary condition. For example, it is possible for $x = 0$ to have a prescribed oscillating temperature

$$u(0, t) = 100 - 25\cos t,$$

and for the right end, $x = L$, to be insulated,

$$\frac{\partial u}{\partial x}(L, t) = 0.$$

Exercises 1.3

1.3.1. Consider a one-dimensional rod, $0 \le x \le L$. Assume that the heat energy flowing out of the rod at $x = L$ is proportional to the temperature difference between the end temperature of the bar and the known external temperature. Derive (1.3.5); briefly, physically explain why $H > 0$.

***1.3.2.** Two one-dimensional rods of different materials joined at $x = x_0$ are said to be in **perfect thermal contact** if the temperature is continuous at $x = x_0$:

$$u(x_0-, t) = u(x_0+, t)$$

and no heat energy is lost at $x = x_0$ (i.e., the heat energy flowing out of one flows into the other). What mathematical equation represents the latter condition at $x = x_0$? Under what special condition is $\partial u/\partial x$ continuous at $x = x_0$?

***1.3.3.** Consider a bath containing a fluid of specific heat c_f and mass density ρ_f that surrounds the end $x = L$ of a one-dimensional rod. Suppose that the bath is rapidly stirred in a manner such that the bath temperature is approximately uniform throughout, equaling the temperature at $x = L$, $u(L, t)$. Assume that the bath is thermally insulated except at its perfect thermal contact with the rod, where the bath may be heated or cooled by the rod. Determine an equation for the temperature in the bath. (This will be a boundary condition at the end $x = L$.) (*Hint*: See Exercise 1.3.2.)

1.4 Equilibrium Temperature Distribution

1.4.1 PRESCRIBED TEMPERATURE

Let us now formulate a simple, but typical, problem of heat flow. If the thermal coefficients are constant and there are no sources of thermal energy, then the temperature $u(x, t)$ in a one-dimensional rod $0 \le x \le L$ satisfies

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{1.4.1}$$

The solution of this partial differential equation must satisfy the initial condition

$$u(x, 0) = f(x) \tag{1.4.2}$$

and one boundary condition at each end. For example, each end might be in contact with different large baths, such that the temperature at each end is prescribed:

$$\begin{aligned} u(0, t) &= T_1(t) \\ u(L, t) &= T_2(t). \end{aligned} \tag{1.4.3}$$

One aim of this text is to enable the reader to solve the problem specified by (1.4.1)–(1.4.3).

Equilibrium temperature distribution

Before we begin to attack such an initial and boundary value problem for partial differential equations, we discuss a physically related question for ordinary differential equations. Suppose that the boundary conditions at $x = 0$ and $x = L$ were **steady** (i.e., independent of time),

$$u(0, t) = T_1 \quad \text{and} \quad u(L, t) = T_2,$$

where T_1 and T_2 are given constants. We define an **equilibrium** or **steady-state** solution to be a temperature distribution that does not depend on time, that is, $u(x, t) = u(x)$. Since $\partial/\partial t\, u(x) = 0$, the partial differential equation becomes $k(\partial^2 u/\partial x^2) = 0$, but partial derivatives are not necessary, and thus

$$\boxed{\frac{d^2u}{dx^2} = 0.} \tag{1.4.4}$$

The boundary conditions are

$$\boxed{\begin{aligned} u(0) &= T_1 \\ u(L) &= T_2. \end{aligned}} \tag{1.4.5}$$

In doing steady-state calculations, the initial conditions are usually ignored. Equation (1.4.4) is a rather trivial second-order ordinary differential equation (ODE). Its general solution may be obtained by integrating twice. Integrating (1.4.4) yields $du/dx = C_1$, and integrating a second time shows that

$$u(x) = C_1 x + C_2. \tag{1.4.6}$$

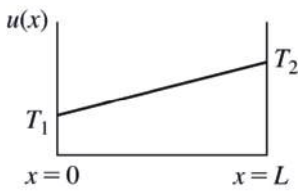
We recognize (1.4.6) as the general equation of a straight line. Thus, from the boundary conditions (1.4.5), the equilibrium temperature distribution is the straight line that equals T_1 at $x = 0$ and T_2 at $x = L$, as sketched in Fig. 1.4.1. Geometrically there is a unique equilibrium solution for this problem. Algebraically, we can determine the two arbitrary constants, C_1 and C_2, by applying the boundary conditions, $u(0) = T_1$ and $u(L) = T_2$:

$$\begin{aligned} u(0) = T \quad &\text{implies} \quad T_1 = C_2 \\ u(L) = T_2 \quad &\text{implies} \quad T_2 = C_1 L + C_2. \end{aligned} \tag{1.4.7}$$

It is easy to solve (1.4.7) for the constants $C_2 = T_1$ and $C_1 = (T_2 - T_1)/L$. Thus, the unique equilibrium solution for the steady-state heat equation with these fixed boundary conditions is

$$\boxed{u(x) = T_1 + \frac{T_2 - T_1}{L} x.} \tag{1.4.8}$$

Figure 1.4.1 Equilibrium temperature distribution.



Approach to equilibrium

For the time-dependent problem, (1.4.1) and (1.4.2), with steady boundary conditions (1.4.5), we expect the temperature distribution $u(x, t)$ to change in time; it will not remain equal to its initial distribution $f(x)$. If we wait a very, very long time, we would imagine that the influence of the two ends should dominate. The initial conditions are usually forgotten. Eventually, the temperature is physically expected to approach the equilibrium temperature distribution, since the boundary conditions are independent of time:

$$\lim_{t\to\infty} u(x, t) = u(x) = T_1 + \frac{T_2 - T_1}{L}x. \tag{1.4.9}$$

In Section 8.2, we will solve the time-dependent problem and show that (1.4.9) is satisfied. However, if a steady state is approached, it is more easily obtained by directly solving the equilibrium problem.

1.4.2 INSULATED BOUNDARIES

As a second example of a steady-state calculation, we consider a one-dimensional rod, again with no sources and with constant thermal properties, but this time with insulated boundaries at $x = 0$ and $x = L$. The formulation of the time-dependent problem is

$$\text{PDE:} \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{1.4.10}$$

$$\text{IC:} \qquad u(x, 0) = f(x) \tag{1.4.11}$$

$$\text{BC1:} \qquad \frac{\partial u}{\partial x}(0, t) = 0 \tag{1.4.12}$$

$$\text{BC2:} \qquad \frac{\partial u}{\partial x}(L, t) = 0. \tag{1.4.13}$$

The equilibrium problem is derived by setting $\partial u/\partial t = 0$. The equilibrium temperature distribution satisfies

$$\text{ODE:} \quad \boxed{\frac{d^2u}{dx^2} = 0} \tag{1.4.14}$$

$$\text{BC1:} \quad \boxed{\frac{du}{dx}(0) = 0} \tag{1.4.15}$$

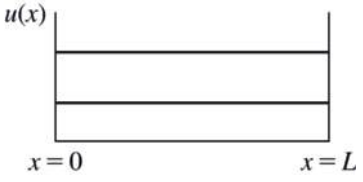
$$\text{BC2:} \quad \boxed{\frac{du}{dx}(L) = 0,} \tag{1.4.16}$$

where the initial condition is neglected (for the moment). The general solution of $d^2u/dx^2 = 0$ is again an arbitrary straight line,

$$u = C_1x + C_2. \tag{1.4.17}$$

The boundary conditions imply that the slope must be zero at both ends. Geometrically, any straight line that is flat (zero slope) will satisfy (1.4.15) and (1.4.16), as illustrated in Fig. 1.4.2.

Figure 1.4.2 Various constant equilibrium temperature distributions (with insulated ends).



The solution is any constant temperature. Algebraically, from (1.4.17), $du/dx = C_1$ and both boundary conditions imply $C_1 = 0$. Thus,

$$u(x) = C_2 \tag{1.4.18}$$

for any constant C_2. Unlike the first example (with fixed temperatures at both ends), here there is not a unique equilibrium temperature. Any constant temperature is an equilibrium temperature distribution for insulated boundary conditions. Thus, for the time-dependent initial value problem, we expect

$$\lim_{t\to\infty} u(x, t) = C_2;$$

if we wait long enough, a rod with insulated ends should approach a constant temperature. This seems physically quite reasonable. However, it does not make sense that the solution should approach an arbitrary constant; we ought to know what constant it approaches. In this case, the lack of uniqueness was caused by the complete neglect of the initial condition. In general, the equilibrium solution will not satisfy the initial condition. However, the particular constant equilibrium solution is determined by considering the initial condition for the time-dependent problem (1.4.11). Since both ends are insulated, the total thermal energy is constant. This follows from the integral conservation of thermal energy of the entire rod [see (1.2.4)] or by just integrating (1.4.10) from $x = 0$ to $x = L$:

$$\frac{d}{dt}\int_0^L c\rho u\, dx = -K_0\frac{\partial u}{\partial x}(0, t) + K_0\frac{\partial u}{\partial x}(L, t). \tag{1.4.19}$$

Since both ends are insulated,

$$\int_0^L c\rho u\, dx = \text{constant}. \tag{1.4.20}$$

One implication of (1.4.20) is that the initial thermal energy must equal the final ($\lim_{t\to\infty}$) thermal energy. The initial thermal energy is $c\rho\int_0^L f(x)\, dx$, since $u(x, 0) = f(x)$, while the equilibrium thermal energy is $c\rho\int_0^L C_2\, dx = c\rho C_2\, L$, since the equilibrium temperature distribution is a constant $u(x, t) = C_2$. The constant C_2 is determined by equating these two expressions for the constant total thermal energy, $c\rho\int_0^L f(x)\, dx = c\rho C_2 L$. Solving for C_2 shows that the desired unique steady-state solution should be

$$\boxed{u(x) = C_2 = \frac{1}{L}\int_0^L f(x)\, dx,} \tag{1.4.21}$$

the **average** of the initial temperature distribution. It is as though the initial condition is not entirely forgotten. Later we will find a $u(x, t)$ that satisfies (1.4.10)–(1.4.13) and show that $\lim_{t\to\infty} u(x, t)$ is given by (1.4.21).

Exercises 1.4

1.4.1. Find the equilibrium temperature distribution, in the case where there is uniform heating along the length of the rod,

$$u_t = u_{xx} + 1, \quad u(x, 0) = f(x), \quad u_x(0, t) = 4, \quad u(2, t) = 5$$

for some insignificant initial temperature distribution $f(x)$.

1.4.2. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

***(a)**	$Q = 0$,	$u(0) = 0$,	$u(L) = T$
(b)	$Q = 0$,	$u(0) = T$,	$u(L) = 0$
(c)	$Q = 0$,	$\frac{\partial u}{\partial x}(0) = 0$,	$u(L) = T$
***(d)**	$Q = 0$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) = \alpha$
(e)	$\frac{Q}{K_0} = 1$,	$u(0) = T_1$,	$u(L) = T_2$
***(f)**	$\frac{Q}{K_0} = x^2$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) = 0$

(g) $Q = 0, \qquad u(0) = T, \qquad \dfrac{\partial u}{\partial x}(L) + u(L) = 0$

***(h)** $Q = 0, \qquad \dfrac{\partial u}{\partial x}(0) - [u(0) - T] = 0, \qquad \dfrac{\partial u}{\partial x}(L) = \alpha$

In these you may assume that $u(x, 0) = f(x)$.

1.4.3. Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $Q/K_0 = x$ of thermal energy, subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

***(a)** Determine the heat energy generated per unit time inside the entire rod.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$.

(c) What relationships should exist between the answers in parts (a) and (b)?

1.4.4. Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in perfect thermal contact at $x = 1$. For $0 < x < 1$, there is one material ($c\rho = 1$, $K_0 = 1$) with a constant source ($Q = 1$), whereas for the other $1 < x < 2$, there are no sources ($Q = 0$, $c\rho = 2$, $K_0 = 2$) (see Exercise 1.3.2) with $u(0) = 0$ and $u(2) = 0$.

1.4.5. If both ends of a rod are insulated, derive *from the partial differential equation* that the total thermal energy in the rod is constant.

1.4.6. Consider a one-dimensional rod $0 \le x \le L$ of known length and known constant thermal properties without sources. Suppose that the temperature is an *unknown* constant T at $x = L$. Determine T if we know (in the steady state) both the temperature and the heat flow at $x = 0$.

1.4.7. The two ends of a uniform rod of length L are insulated. There is a constant source of thermal energy $Q_0 \neq 0$, and the temperature is initially $u(x, 0) = f(x)$.

(a) Show mathematically that there does not exist any equilibrium temperature distribution. Briefly explain physically.

(b) Calculate the total thermal energy in the entire rod.

1.4.8. For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

***(a)** $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, \qquad u(x, 0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0, t) = 1, \qquad \dfrac{\partial u}{\partial x}(L, t) = \beta$

(b) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \qquad u(x, 0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0, t) = 1, \qquad \dfrac{\partial u}{\partial x}(L, t) = \beta$

(c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + x - \beta, \qquad u(x, 0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0, t) = 0, \qquad \dfrac{\partial u}{\partial x}(L, t) = 0$

1.4.9. Express the integral conservation law for the entire rod with constant thermal properties. Assume the heat flow is known to be different constants at both ends. By integrating with respect to time, determine the total thermal energy in the rod. (*Hint*: Use the initial condition.)

(a) Assume there are no sources.

(b) Assume the sources of thermal energy are constant.

1.4.10. Derive the integral conservation law for the entire rod with constant thermal properties by integrating the heat equation (1.2.10) (assuming no sources). Show the result is equivalent to (1.2.4).

1.4.11. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x, 0) = f(x)$, $\frac{\partial u}{\partial x}(0, t) = 5$, $\frac{\partial u}{\partial x}(L, t) = 6$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

1.4.12. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$, $u(x, 0) = f(x)$, $\frac{\partial u}{\partial x}(0, t) = \beta$, $\frac{\partial u}{\partial x}(L, t) = 7$.

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).

(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t \to \infty} u(x, t)$.

1.4.13. Suppose the concentration $u(x, t)$ of a chemical satisfies Fick's law (1.2.13), and the initial concentration is given as $u(x, 0) = f(x)$. Consider a region $0 < x < L$ in which

the flow is specified at both ends $-k\frac{\partial u}{\partial x}(0,\ t) = \alpha$ and $-k\frac{\partial u}{\partial x}(L,\ t) = \beta$. Assume α and β are constants.

(a) Express the conservation law for the entire region.

(b) Determine the total amount of chemical in the region as a function of time (using the initial condition).

(c) Under what conditions is there an equilibrium chemical concentration, and what is it?

1.4.14. Do Exercise 1.4.13 if α and β are functions of time.

1.5 Derivation of the Heat Equation in Two or Three Dimensions

INTRODUCTION

In Section 1.2, we showed that for the conduction of heat in a one-dimensional rod the temperature $u(x,\ t)$ satisfies

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q.$$

In cases in which there are no sources ($Q = 0$) and the thermal properties are constant, the partial differential equation becomes

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where $k = K_0/c\rho$. Before we solve problems involving these partial differential equations, we will formulate partial differential equations corresponding to heat flow problems in two or three spatial dimensions. We will find the derivation to be similar to the one used for one-dimensional problems, although important differences will emerge. We propose to derive new and more complex equations (before solving the simpler ones) so that, when we do discuss *techniques* for the solutions of PDEs, we will have more than one example to work with.

Heat energy

We begin our derivation by considering any arbitrary three-dimensional subregion R. As in the one-dimensional case, conservation of heat energy is summarized by the following word equation:

Rate of change of heat energy	=	heat energy flowing across the boundaries per unit time	+	heat energy generated inside per unit time,

where the heat energy within an arbitrary subregion R is

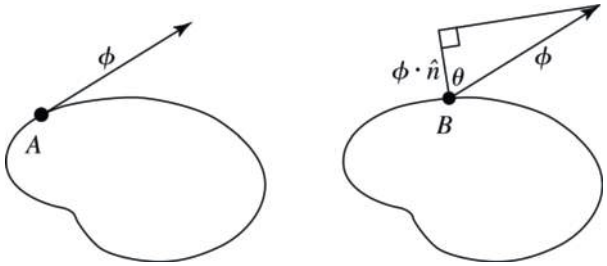
$$\text{heat energy} = \iiint_R c\rho u\ dV,$$

instead of the one-dimensional integral used in Section 1.2.

Heat flux vector and normal vectors

We need an expression for the flow of heat energy. In a one-dimensional problem, the heat flux ϕ is defined to the right ($\phi < 0$ means flowing to the left). In a three-dimensional problem, the heat flows in some direction, and hence the **heat flux is a vector** $\boldsymbol{\phi}$. The magnitude of $\boldsymbol{\phi}$ is the amount of heat energy flowing per unit time per unit surface area. However, in considering conservation of heat energy, it is only the heat flowing *across the boundaries* per unit time that is important. If, as at point A in Fig. 1.5.1, the heat flow is parallel to the boundary, then there is no heat energy *crossing* the boundary at that point. In fact, it is only the normal component of the heat flow that contributes (as illustrated by point B in Fig. 1.5.1). At any point, there are two normal vectors, an inward and an outward normal $\boldsymbol{n}$. We will use the convention of utilizing only the **unit outward normal vector $\hat{\mathbf{n}}$** (where the $\hat{}$ stands for a unit vector).

Figure 1.5.1 The heat flux vector $\boldsymbol{\phi}$ is tangent to the boundary at point A but it's not tangent to the boundary at point B. The normal component of $\boldsymbol{\phi}$ is $\boldsymbol{\phi} \cdot \hat{\mathbf{n}}$.



Conservation of heat energy

At each point the amount of heat energy flowing *out* of the region R per unit time per unit surface area is the outward normal component of the heat flux vector. From Fig. 1.5.1 at point B the outward normal component of the heat flux vector is $|\phi|\cos\theta = \phi \cdot n/|n| = \phi \cdot \hat{\mathbf{n}}$. If the heat flux vector ϕ is directed inward, then $\phi \cdot \hat{\mathbf{n}} < 0$ and the outward flow of heat energy is negative. To calculate the total heat energy flowing out of R per unit time, we must multiply $\phi \cdot \hat{\mathbf{n}}$ by the differential surface area dS and "sum" over the entire surface that encloses the region R. This[4] is indicated by the closed surface integral $\oiint \phi \cdot \hat{\mathbf{n}}\, ds$. This is the amount of heat energy (per unit time) leaving the region R and (if positive) results in a decreasing of the total heat energy within R. If Q is the rate of heat energy generated per unit volume, then the total heat energy generated per unit time is $\iiint_R Q\, dV$. Consequently, conservation of heat energy for an arbitrary three-dimensional region R becomes

$$\boxed{\frac{d}{dt}\iiint_R c\rho u\, dV = -\oiint \phi \cdot \hat{\mathbf{n}}\, dS + \iiint_R Q\, dV.} \tag{1.5.1}$$

Divergence theorem

In one dimension, a way in which we derived a partial differential relationship from the integral conservation law was to notice (via the fundamental theorem of calculus) that

$$\phi(a) - \phi(b) = -\int_a^b \frac{\partial \phi}{\partial x}\, dx;$$

that is, the flow through the boundaries can be expressed as an integral over the entire region for one-dimensional problems. We claim that the divergence theorem is an analogous procedure for functions of three variables. The divergence theorem deals with a vector $\boldsymbol{A}$ (with components A_1, A_2 and A_3)

$$\boldsymbol{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}},$$

and its divergence defined as follows:

$$\nabla \cdot \boldsymbol{A} \equiv \frac{\partial}{\partial x}A_1 + \frac{\partial}{\partial y}A_2 + \frac{\partial}{\partial z}A_3. \tag{1.5.2}$$

Note that the divergence of a vector is a scalar. **The divergence theorem states that the volume integral of the divergence of any continuously differentiable vector $\boldsymbol{A}$ is the closed surface integral of the outward normal component of $\boldsymbol{A}$**:

$$\boxed{\iiint_R \nabla \cdot \boldsymbol{A}\, dV = \oiint \boldsymbol{A} \cdot \hat{\mathbf{n}}\, dS.} \tag{1.5.3}$$

This is also known as Gauss's theorem. It can be used to relate certain surface integrals to volume integrals, and vice versa. It is very important and very useful (both immediately and later in this text).

[4]Sometimes the notation ϕ_n is used instead of $\phi \cdot \hat{n}$, meaning the outward normal component of ϕ.

We omit a derivation, which may be based on repeating the one-dimensional fundamental theorem in all three dimensions.

Application of the divergence theorem to heat flow

In particular, the closed surface integral that arises in the conservation of heat energy (1.5.1), corresponding to the heat energy flowing across the boundary per unit time, can be written as a volume integral according to the divergence theorem, (1.5.3). Thus, (1.5.1) becomes

$$\frac{d}{dt}\iiint_R c\rho u \, dV = -\iiint_R \nabla \cdot \phi \, dV + \iiint_R Q \, dV. \tag{1.5.4}$$

We note that the time derivative in (1.5.4) can be put inside the integral (since R is fixed in space) if the time derivative is changed to a partial derivative. Thus, all the expressions in (1.5.4) are volume integrals over the same volume, and they can be combined into one integral:

$$\iiint_R \left[c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q \right] dV = 0. \tag{1.5.5}$$

Since this integral is zero for all regions R, it follows (as it did for one-dimensional integrals) that the integrand itself must be zero:

$$c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q = 0$$

or, equivalently,

$$\boxed{c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \phi + Q.} \tag{1.5.6}$$

Equation (1.5.6) reduces to (1.2.3) in the one-dimensional case.

Fourier's law of heat conduction

In one-dimensional problems, from experiments according to Fourier's law, the heat flux ϕ is proportional to the derivative of the temperature, $\phi = -K_0 \, \partial u/\partial x$. The minus sign is related to the fact that thermal energy flows from hot to cold. $\partial u/\partial x$ is the change in temperature per unit length. These same ideas are valid in three dimensions. In the appendix, we derive that the heat flux vector ϕ is proportional to the temperature gradient $\left(\nabla u \equiv \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}\right)$:

$$\boxed{\phi = -K_0 \nabla u,} \tag{1.5.7}$$

known as **Fourier's law of heat conduction**, where again K_0 is called the thermal conductivity. Thus, in three dimensions the gradient ∇u replaces $\partial u/\partial x$.

Heat equation

When the heat flux vector, (1.5.7), is substituted into the conservation of heat energy equation, (1.5.6), a partial differential equation for the temperature results:

$$\boxed{c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.} \tag{1.5.8}$$

In the cases in which there are no sources of heat energy ($Q = 0$) and the thermal coefficients are constant, (1.5.8) becomes

$$\frac{\partial u}{\partial t} = k \nabla \cdot (\nabla u), \tag{1.5.9}$$

where $k = K_0/c\rho$ is again called the thermal diffusivity. From their definitions, we calculate the divergence of the gradient of u:

$$\nabla \cdot (\nabla u) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \equiv \nabla^2 u.} \tag{1.5.10}$$

This expression $\nabla^2 u$ is defined to be the **Laplacian** of u. Thus, in this case

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u.} \tag{1.5.11}$$

Equation (1.5.11) is often known as the **heat** or **diffusion equation** in three spatial dimensions. The notation $\nabla^2 u$ is often used to emphasize the role of the del operator ∇:

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}.$$

Note that ∇u is ∇ operating on u, while $\nabla \cdot \boldsymbol{A}$ is the vector dot product of del with $\boldsymbol{A}$.

Furthermore, ∇^2 is the dot product of the del operator with itself or

$$\nabla \cdot \nabla = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right)$$

operating on u, hence the notation del squared, ∇^2.

Initial boundary value problem

In addition to (1.5.8) or (1.5.11), the temperature satisfies a given initial distribution,

$$u(x, y, z, 0) = f(x, y, z).$$

The temperature also satisfies a boundary condition at every point on the surface that encloses the region of interest. The boundary condition can be of various types (as in the one-dimensional problem). The temperature could be prescribed,

$$u(x, y, z, t) = T(x, y, z, t),$$

everywhere *on the boundary* where T is a known function of t at each point of the boundary. It is also possible that the flow across the boundary is prescribed. Frequently, we might have the boundary (or part of the boundary) **insulated**. This means that there is no heat flow across that portion of the boundary. Since the heat flux vector is $-K_0\nabla u$, the heat flowing out will be the unit outward normal component of the heat flow vector, $-K_0\nabla u \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit outward normal to the boundary surface. Thus, at an insulated surface,

$$\nabla u \cdot \hat{\mathbf{n}} = 0.$$

Recall that $\nabla u \cdot \hat{\mathbf{n}}$ is the directional derivative of u in the outward normal direction; it is also called the normal derivative.[5]

Often Newton's law of cooling is a more realistic condition at the boundary. It states that the heat energy flowing out per unit time per unit surface area is proportional to the difference between the temperature at the surface u and the temperature outside the surface u_b. Thus, if Newton's law of cooling is valid, then at the boundary

$$-K_0\nabla u \cdot \hat{\mathbf{n}} = H(u - u_b). \tag{1.5.12}$$

Note that usually the proportionality constant $H > 0$, since if $u > u_b$, then we expect that heat energy will flow out and $-K_0\nabla u \cdot \hat{\mathbf{n}}$ will be greater than zero. Equation (1.5.12) verifies the two forms of Newton's law of cooling for one-dimensional problems. In particular, at $x = 0$, $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$ and the left-hand side (l.h.s.) of (1.5.12) becomes $K_0\,\partial u/\partial x$, while at $x = L$, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ and the l.h.s. of (1.5.12) becomes $-K_0\,\partial u/\partial x$ [see (1.3.4) and (1.3.5)].

Steady state

If the boundary conditions and any sources of thermal energy are independent of time, it is possible that there exist steady-state solutions to the heat equation satisfying the given steady boundary condition:

$$0 = \nabla \cdot (K_0\nabla u) + Q.$$

[5]Sometimes (in other books and references) the notation $\partial u/\partial n$ is used. However, to calculate $\partial u/\partial n$ we usually calculate the dot product of the two vectors, ∇u and $\hat{\mathbf{n}}$, $\nabla u \cdot \hat{\mathbf{n}}$, so we will not use the notation $\partial u/\partial n$ in this text.

Note that an equilibrium temperature distribution $u(x, y, z)$ satisfies a partial differential equation when more than one spatial dimension is involved. In the case with constant thermal properties, the equilibrium temperature distribution will satisfy

$$\nabla^2 u = -\frac{Q}{K_0}, \tag{1.5.13}$$

known as **Poisson's equation**.

If, in addition, there are no sources ($Q = 0$), then

$$\boxed{\nabla^2 u = 0;} \tag{1.5.14}$$

the Laplacian of the temperature distribution is zero. Equation (1.5.14) is known as **Laplace's equation**. It is also known as the **potential equation**, since the gravitational and electrostatic potentials satisfy (1.5.14) if there are no sources. We will solve a number of problems involving Laplace's equation in later sections.

Two-dimensional problems

All the previous remarks about three-dimensional problems are valid if the geometry is such that the temperature depends only on x, y, and t. For example, Laplace's equation in two dimensions, x and y, corresponding to equilibrium heat flow with no sources (and constant thermal properties) is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

since $\partial^2 u/\partial z^2 = 0$. Two-dimensional results can be derived directly (without taking a limit of three-dimensional problems), by using fundamental principles in two dimensions. We will not repeat the derivation. However, we can easily outline the results. Every time a volume integral $\left(\iiint_R \cdots dV\right)$ appears, it must be replaced by a surface integral over the entire two-dimensional plane region $\left(\iint_R \cdots dS\right)$. Similarly, the boundary contribution for three-dimensional problems, which is the closed surface integral $\oiint \cdots dS$, must be replaced by the closed line integral $\oint \cdots d\tau$, an integration over the boundary of the two-dimensional plane surface. These results are not difficult to derive since the divergence theorem in three dimensions,

$$\iiint_R \nabla \cdot A \, dV = \oiint A \cdot \hat{\mathbf{n}} \, dS, \tag{1.5.15}$$

is valid in two dimensions, taking the form

$$\iint_R \nabla \cdot \boldsymbol{A} \, dS = \oint \boldsymbol{A} \cdot \hat{\mathbf{n}} \, d\tau. \tag{1.5.16}$$

Sometimes (1.5.16) is called *Green's theorem*, but we prefer to refer to it as the two-dimensional divergence theorem. In this way only one equation need be familiar to the reader, namely (1.5.15); *the conversion to two-dimensional form involves changing only the number of integral signs*.

Polar and cylindrical coordinates

The Laplacian,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{1.5.17}$$

is important for the heat equation (1.5.11) and its steady-state version (1.5.14), as well as for other significant problems in science and engineering. Equation (1.5.17), written in Cartesian coordinates, is most useful when the geometrical region under investigation is a rectangle or a rectangular box. Other coordinate systems are frequently useful. In practical applications, we may need the formula that expresses the Laplacian in the appropriate coordinate system. In circular cylindrical coordinates, with r the radial distance from the z-axis and θ the angle

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z, \end{aligned}} \tag{1.5.18}$$

the Laplacian can be shown to equal the following formula:

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.} \tag{1.5.19}$$

There may be no need to memorize this formula, as it can often be looked up in a reference book. As an aid in minimizing errors, it should be noted that every term in the Laplacian has the dimension of u divided by two spatial dimensions [just as in Cartesian coordinates, (1.5.17)]. Since θ is measured in radians, which have no dimensions, this remark aids in remembering to divide $\partial^2 u/\partial\theta^2$ by r^2. In polar coordinates (by which we mean a two-dimensional coordinate system with z fixed, usually $z = 0$), the Laplacian is the same as (1.5.19) with $\partial^2 u/\partial z^2 = 0$ since there is no dependence on z. Equation (1.5.19) can be derived (see the Exercises) using the chain rule for partial derivatives, applicable for changes of variables.

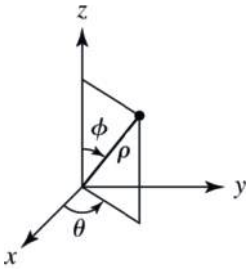
In some physical situations it is known that the temperature does *not* depend on the polar angle θ; it is said to be circularly or axially symmetric. In that case

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}. \tag{1.5.20}$$

Spherical coordinates

Geophysical problems as well as electrical problems with spherical conductors are best solved using spherical coordinates (ρ, θ, ϕ). The radial distance is ρ, the angle from the pole (z-axis) is ϕ, and the cylindrical (or azimuthal) angle is θ. Note that if ρ is constant and the angle ϕ is a constant, a circle is generated with radius $\rho \sin\phi$ (as shown in Fig. 1.5.2) so that

Figure 1.5.2 Spherical coordinates.



$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi. \end{aligned}} \tag{1.5.21}$$

The angle from the pole ranges from 0 to π (while the usual cylindrical angle ranges from 0 to 2π). It can be shown that the Laplacian satisfies

$$\boxed{\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}.} \tag{1.5.22}$$

Exercises 1.5

1.5.1. Let $c(x, y, z, t)$ denote the concentration of a pollutant (the amount per unit volume).

(a) What is an expression for the total amount of pollutant in the region R?

(b) Suppose that the flow $\boldsymbol{J}$ of the pollutant is proportional to the gradient of the concentration. (Is this reasonable?) Express conservation of the pollutant.

(c) Derive the partial differential equation governing the diffusion of the pollutant.

***1.5.2.** For conduction of thermal energy, the heat flux vector is $\phi = -K_0\nabla u$. If in addition the molecules move at an average velocity $\boldsymbol{V}$, a process called **convection**, then briefly explain why $\phi = -K_0\nabla u + c\rho u\boldsymbol{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

1.5.3. Consider the polar coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

(a) Since $r^2 = x^2 + y^2$, show that $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$, and $\frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{r}$.

(b) Show that $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$.

(c) Using the chain rule, show that $\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}$ and hence $\nabla u = \frac{\partial u}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial u}{\partial\theta}\hat{\boldsymbol{\theta}}$.

(d) If $\boldsymbol{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}}$, show that $\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(A_\theta)$, since $\partial\hat{\mathbf{r}}/\partial\theta = \hat{\theta}$ and $\partial\hat{\theta}/\partial\theta = -\hat{\mathbf{r}}$ follows from part (b).

(e) Show that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$.

1.5.4. Using Exercise 1.5.3(a) and the chain rule for partial derivatives, derive the special case of Exercise 1.5.3(e) if $u(r)$ only.

1.5.5. Assume that the temperature is circularly symmetric: $u = u(r, t)$, where $r^2 = x^2 + y^2$. We will derive the heat equation for this problem. Consider any circular annulus $a \le r \le b$.

(a) Show that the total heat energy is $2\pi \int_a^b c\rho u r \, dr$.

(b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is $-2\pi b K_0 \, \partial u/\partial r|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

1.5.6. Modify Exercise 1.5.5 if the thermal properties depend on r.

1.5.7. Derive the heat equation in two dimensions by using Green's theorem, (1.5.16), the two-dimensional form of the divergence theorem.

1.5.8. If Laplace's equation is satisfied in three dimensions, show that

$$\oiint \nabla u \cdot \hat{\mathbf{n}} \, dS = 0$$

for any closed surface. (*Hint*: Use the divergence theorem.) Give a physical interpretation of this result (in the context of heat flow).

1.5.9. Determine the equilibrium temperature distribution inside a circular annulus $(r_1 \le r \le r_2)$:

***(a)** if the outer radius is at temperature T_2 and the inner at T_1

(b) if the outer radius is insulated and the inner radius is at temperature T_1

1.5.10. Determine the equilibrium temperature distribution inside a circle $(r \le r_0)$ if the boundary is fixed at temperature T_0.

***1.5.11.** Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad a < r < b,$$

subject to

$$u(r, 0) = f(r), \ \frac{\partial u}{\partial r}(a, t) = \beta, \text{ and } \frac{\partial u}{\partial r}(b, t) = 1.$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

1.5.12. Assume that the temperature is spherically symmetric, $u = u(r, t)$, where r is the distance from a fixed point $(r^2 = x^2 + y^2 + z^2)$. Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi \int_a^b c\rho u r^2 \, dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0 \, \partial u/(\partial r)_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

***1.5.13.** Determine the *steady-state* temperature distribution between two concentric *spheres* with radii 1 and 4, respectively, if the temperature of the outer sphere is maintained at 80° and the inner sphere at 0° (see Exercise 1.5.12).

1.5.14. Isobars are lines of constant temperature. Show that isobars are perpendicular to any part of the boundary that is insulated.

1.5.15. Derive the heat equation in three dimensions assuming constant thermal properties and no sources.

1.5.16. Express the integral conservation law for any three-dimensional object. Assume there are no sources. Also assume the heat flow is specified, $\nabla u \cdot \hat{\mathbf{n}} = g(x, y, z)$, on the entire boundary and does not depend on time. By integrating with respect to time, determine the total thermal energy. (*Hint*: Use the initial condition.)

1.5.17. Derive the integral conservation law for any three-dimensional object (with constant thermal properties) by integrating the heat equation (1.5.11) (assuming no sources). Show that the result is equivalent to (1.5.1).

Orthogonal curvilinear coordinates

A coordinate system (u, v, w) may be introduced and defined by $x = x(u, v, w)$, $y = y(u, v, w)$, and $z = z(u, v, w)$. The radial vector $\boldsymbol{r} \equiv x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Partial derivatives of $\boldsymbol{r}$ with respect to a coordinate are in the direction of the coordinate. Thus, for example, a vector in the u-direction $\partial r/\partial u$ can be made a unit vector $\hat{\boldsymbol{e}}_u$ in the u-direction by dividing by its length $h_u = |\partial r/\partial u|$ called the **scale factor**: $\hat{\boldsymbol{e}}_u = \frac{1}{h_u}\,\partial \boldsymbol{r}/\partial u$.

1.5.18. Determine the scale factors for cylindrical coordinates.

1.5.19. Determine the scale factors for spherical coordinates.

1.5.20. The gradient of a scalar can be expressed in terms of the new coordinate system $\nabla g = a\,\partial \boldsymbol{r}/\partial u + b\,\partial \boldsymbol{r}/\partial v + c\,\partial \boldsymbol{r}/\partial w$, where you will determine the scalars a, b, c. Using $dg = \nabla g \cdot d\boldsymbol{r}$, derive that the **gradient** in an orthogonal curvilinear coordinate system is given by

$$\nabla g = \frac{1}{h_u}\frac{\partial g}{\partial u}\hat{\boldsymbol{e}}_u + \frac{1}{h_v}\frac{\partial g}{\partial v}\hat{\boldsymbol{e}}_v + \frac{1}{h_w}\frac{\partial g}{\partial w}\hat{\boldsymbol{e}}_w. \tag{1.5.23}$$

An expression for the divergence is more difficult to derive, and we will just state that if a vector $\boldsymbol{p}$ is expressed in terms of this new coordinate system $\boldsymbol{p} = p_u\hat{\boldsymbol{e}}_u + p_v\hat{\boldsymbol{e}}_v + p_w\hat{\boldsymbol{e}}_w$, then the **divergence** satisfies

$$\nabla \cdot \boldsymbol{p} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w p_u) + \frac{\partial}{\partial v}(h_u h_w p_v) + \frac{\partial}{\partial w}(h_u h_v p_w)\right]. \tag{1.5.24}$$

1.5.21. Using (1.5.23) and (1.5.24), derive the **Laplacian** in an orthogonal curvilinear coordinate system:

$$\nabla^2 T = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial T}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial T}{\partial w}\right)\right]. \tag{1.5.25}$$

1.5.22. Using (1.5.25), derive the Laplacian for cylindrical coordinates.

1.5.23. Using (1.5.25), derive the Laplacian for spherical coordinates.

1.5.24. Derive the three-dimensional heat equation assuming constant thermal properties and assuming there are no sources of thermal energy. (*Hint*: Use an arbitrary three-dimensional region, and use the divergence theorem.)

1.5.25. Suppose a sphere of radius 2 satisfies $\frac{\partial u}{\partial t} = \nabla^2 u + 5$ with $u(x, y, z, 0) = f(x, y, z)$ and on the surface of the sphere it is given that $\nabla u \cdot \hat{\mathbf{n}} = 6$, where $\hat{\mathbf{n}}$ is a unit outward normal vector. Calculate the total thermal energy for this sphere as a function of time. (*Hint*: Use the divergence theorem.)

Appendix to 1.5: Review of Gradient and a Derivation of Fourier's Law of Heat Conduction

Experimentally, for isotropic[6] materials (i.e., without preferential directions) **heat flows from hot to cold in the direction in which temperature differences are greatest**. The heat flow is proportional (with proportionality constant K_0, the thermal conductivity) to the rate of change of temperature in this direction.

The change in the temperature Δu is

$$\Delta u = u(\boldsymbol{x} + \Delta \boldsymbol{x}, t) - u(\boldsymbol{x}, t) \approx \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z.$$

[6]Examples of nonisotropic materials are certain crystals and grainy woods.

In the direction $\hat{\boldsymbol{\alpha}} = \alpha_1\hat{\mathbf{i}} + \alpha_2\hat{\mathbf{j}} + \alpha_3\hat{\mathbf{k}}$, $\Delta\boldsymbol{x} = \Delta s\hat{\boldsymbol{\alpha}}$, where Δs is the distance between $\boldsymbol{x}$ and $\boldsymbol{x} + \Delta\boldsymbol{x}$. Thus, the rate of change of the temperature in the direction $\hat{\boldsymbol{\alpha}}$ is the **directional derivative**:

$$\frac{\partial u}{\partial s} = \lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = \alpha_1 \frac{\partial u}{\partial x} + \alpha_2 \frac{\partial u}{\partial y} + \alpha_3 \frac{\partial u}{\partial z} = \alpha \bullet \nabla u,$$

where it has been convenient to define the following *vector*:

$$\boxed{\nabla u \equiv \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}},} \tag{1.5.26}$$

called the **gradient** of the temperature. From the property of dot products, if θ is the angle between $\hat{\boldsymbol{\alpha}}$ and ∇u, then the directional derivative is $|\nabla u| \cos\theta$ since $|\hat{\boldsymbol{\alpha}}| = 1$. The largest rate of change of u (the largest directional derivative) is $|\nabla u| > 0$, and it occurs if $\theta = 0$ (i.e., in the direction of the gradient). Since this derivative is positive, the temperature increase is greatest in the direction of the gradient. Since heat energy flows in the direction of decreasing temperatures, **the heat flow vector is in the opposite direction to the heat gradient**. It follows that

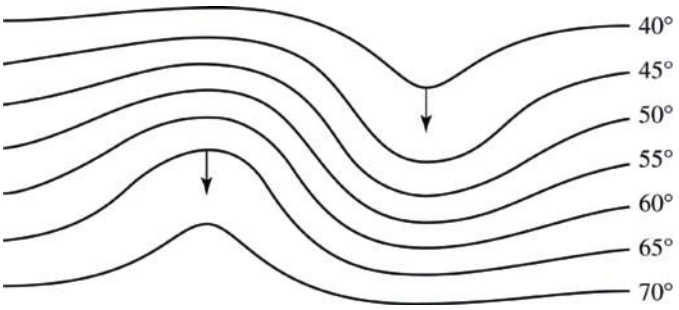
$$\boxed{\phi = -K_0 \nabla u,} \tag{1.5.27}$$

since $|\nabla u|$ equals the magnitude of the rate of change of u (in the direction of the gradient). This again is called Fourier's law of heat conduction. Thus, in three dimensions, the gradient ∇u replaces $\partial u/\partial x$.

Another fundamental property of the gradient is that it is normal (perpendicular) to the level surfaces. It is easier to illustrate this in a two-dimensional problem (see Fig. 1.5.3) in which the temperature is constant along level curves (rather than level surfaces). To show that the gradient is perpendicular, consider the surface on which the temperature is the constant T_0, $u(x, y, z, t) = T_0$. We calculate the differential of both sides (at a fixed time) along the surface. Since T_0 is constant, $dT_0 = 0$. Therefore, using the chain rule of partial derivatives,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0. \tag{1.5.28}$$

Figure 1.5.3 The gradient is perpendicular to level surfaces of the temperature.



Equation (1.5.28) can be written as

$$\left(\frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}\right) \bullet \left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}\right) = 0$$

or

$$\nabla u \bullet \left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}\right) = 0. \tag{1.5.29}$$

$dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$ represents any vector in the tangent plane of the level surface. From (1.5.29), its dot product with ∇u is zero; that is, ∇u is perpendicular to the tangent plane. Thus, ∇u is perpendicular to the surface $u =$ constant.

We have thus learned two properties of the gradient, ∇u:

1. Direction: ∇u is perpendicular to the surface $u =$ constant. ∇u is also in the direction of the largest directional derivative. (u increases in the direction of the gradient.)
2. Magnitude: $|\nabla u|$ is the largest value of the directional derivative.

2
Method of Separation of Variables

2.1 Introduction

In Chapter 1 we developed from physical principles an understanding of the heat equation and its corresponding initial and boundary conditions. We are ready to pursue the mathematical solution of some typical problems involving partial differential equations. We will use a technique called the **method of separation of variables**. You will have to become an expert in this method, and so we will discuss quite a few examples. We will emphasize problem-solving techniques, but we must also understand how not to misuse the technique.

A relatively simple, but typical, problem for the equation of heat conduction occurs for a one-dimensional rod ($0 \le x \le L$) when all the thermal coefficients are constant. Then the PDE,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}, \qquad \begin{matrix} t > 0 \\ 0 < x < L, \end{matrix} \tag{2.1.1}$$

must be solved subject to the initial condition,

$$u(x,0) = f(x), \qquad 0 < x < L, \tag{2.1.2}$$

and two boundary conditions. For example, if both ends of the rod have prescribed temperature, then

$$\begin{matrix} u(0,t) = T_1(t) \\ u(L,t) = T_2(t), \end{matrix} \qquad t > 0. \tag{2.1.3}$$

The method of separation of variables is used when the partial differential equation and the boundary conditions are linear and homogeneous, concepts we now explain.

2.2 Linearity

As in the study of ordinary differential equations, the concept of linearity will be very important for us. A **linear operator** L satisfies the **linearity property**

$$\boxed{L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)} \tag{2.2.1}$$

for any two functions u_1 and u_2, where c_1 and c_2 are arbitrary constants. We introduce the **heat operator**:

$$L_H(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}. \tag{2.2.2}$$

We first show the heat operator is also a linear operator and satisfies the linearity property

$$\begin{aligned} L_H(c_1u_1 + c_2u_2) &= \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) \\ &= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} - kc_2\frac{\partial^2 u_2}{\partial x^2} \\ &= c_1\left(\frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2}\right) + c_2\left(\frac{\partial u_2}{\partial t} - k\frac{\partial^2 u_2}{\partial x^2}\right) \\ &= c_1L_H(u_1) + c_2L_H(u_2). \end{aligned}$$

A **linear equation** for the unknown u is of the form

$$\boxed{L(u) = f,} \tag{2.2.3}$$

where L is a linear operator and f is known. The heat equation is a linear partial differential equation:

$$L_H(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f(x, t), \tag{2.2.4}$$

or, in the more common form,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f(x, t). \tag{2.2.5}$$

If $f = 0$, then (2.2.5) becomes $L(u) = 0$, called a **linear homogeneous** equation. An example of a linear homogeneous partial differential equation is the heat equation,

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0. \tag{2.2.6}$$

From (2.2.1) it follows that $L(0) = 0$ (let $c_1 = c_2 = 0$). Therefore, $u = 0$ is always a solution of a linear homogeneous equation. For example, $u = 0$ satisfies the heat equation (2.2.6). We call $u = 0$ the *trivial* solution of a linear homogeneous equation. The simplest way to test whether an equation is homogeneous is to substitute the function u identically equal to zero. If $u \equiv 0$ satisfies a linear equation, then it must be that $f = 0$ and hence the linear equation is homogeneous. Otherwise, if $f \neq 0$, the equation is said to be **nonhomogeneous**.

The fundamental property of linear operators (2.2.1) allows solutions of linear equations to be added together in the following sense:

Principle of Superposition

If u_1 and u_2 satisfy a linear *homogeneous* equation, then an arbitrary linear combination of them, $c_1u_1 + c_2u_2$, also satisfies the same linear homogeneous equation.

The proof of this relies on the definition of a linear operator. Suppose that u_1 and u_2 are two solutions of a linear homogeneous equation. That means that $L(u_1) = 0$ and $L(u_2) = 0$. Let us calculate $L(c_1u_1 + c_2u_2)$. From the definition of a linear operator,

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2).$$

Since u_1 and u_2 are homogeneous solutions, it follows that $L(c_1u_1 + c_2u_2) = 0$. This means that $c_1u_1 + c_2u_2$ satisfies the linear homogeneous equation $L(u) = 0$ if u_1 and u_2 satisfy the *same* linear homogeneous equation.

The concepts of linearity and homogeneity also apply to boundary conditions, in which case the variables are evaluated at specific points. Examples of *linear* boundary conditions are the conditions we have discussed:

$$u(0, t) = f(t) \tag{2.2.7}$$

$$\frac{\partial u}{\partial x}(L, t) = g(t) \tag{2.2.8}$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \tag{2.2.9}$$

$$-K_0\frac{\partial u}{\partial x}(L, t) = h[u(L, t) - g(t)]. \tag{2.2.10}$$

Only (2.2.9) is satisfied by $u \equiv 0$ (of the linear conditions) and hence is homogeneous. It is not necessary that a boundary condition be $u(0, t) = 0$ for $u \equiv 0$ to satisfy it.

Exercises 2.2

2.2.1. Which of the following partial differential equations are linear and which ones are nonlinear? Of the linear ones, which ones are homogeneous?

(a) $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = f(x, t)$

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u$

(c) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + \beta(x, t)$

(d) $\dfrac{\partial u}{\partial t} - k\dfrac{\partial u}{\partial x} = u$

(e) $\dfrac{\partial^2 u}{\partial t^2} - k\dfrac{\partial u}{\partial x} = \sin u$

2.2.2. Show that any linear combination of linear operators is a linear operator.

2.2.3. **(a)** Show that $L(u) = \frac{\partial}{\partial x}[K_0(x)\frac{\partial u}{\partial x}]$ is a linear operator.

(b) Show that $L(u) = \frac{\partial}{\partial x}[K_0(x, u)\frac{\partial u}{\partial x}]$ is not, in general, a linear operator.

2.2.4. Show that $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(u, x, t)$ is linear if $Q = \alpha(x, t)u + \beta(x, t)$ and, in addition, homogeneous if $\beta(x, t) = 0$.

2.2.5. In this exercise, we derive superposition principles for nonhomogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i, what is a particular solution for $f_1 + f_2$?

2.2.6. If L is a linear operator, show that $L\left(\sum_{n=1}^{M} c_n u_n\right) = \sum_{n=1}^{M} c_n L(u_n)$. Use this result to show that the principle of superposition may be extended to any finite number of homogeneous solutions.

2.3 Heat Equation with Zero Temperatures at Finite Ends

2.3.1 INTRODUCTION

Partial differential equation (2.1.1) is linear, but it is homogeneous only if there are no sources, $Q(x, t) = 0$. The boundary conditions (2.1.3) are also linear, and they too are homogeneous only if $T_1(t) = 0$ and $T_2(t) = 0$. We thus first propose to study

$$\text{PDE: } \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \begin{matrix} 0 < x < L \\ t > 0 \end{matrix} \tag{2.3.1}$$

$$\text{BC: } \begin{matrix} u(0, t) = 0 \\ u(L, t) = 0 \end{matrix} \tag{2.3.2}$$

$$\text{IC: } u(x, 0) = f(x). \tag{2.3.3}$$

The problem consists of a linear homogeneous partial differential equation with linear homogeneous boundary conditions. There are two reasons for our investigating this type of problem, (2.3.1)–(2.3.3), besides the fact that we claim it can be solved by the method of separation of variables. First, this problem is a relevant physical problem corresponding to a one-dimensional rod $(0 < x < L)$ with no sources and both ends immersed in a 0° temperature bath. We are very interested in predicting how the initial thermal energy (represented by the initial condition) changes in this relatively simple physical situation. Second, it will turn out that in order to solve the nonhomogeneous problem (2.1.1)–(2.1.3), we will need to know how to solve the homogeneous problem, (2.3.1)–(2.3.3).

The **method of separation of variables** solves (2.3.1)–(2.3.3) through several steps. First, we seek **product solutions** of the partial differential equation (2.3.1), which reduces the PDE to a pair of coupled ODEs. One of these ODEs is a time-dependent differential equation and the other is a **boundary value (eigenvalue) problem** that ensures the BC (2.3.2) is satisfied. Next, we find the infinitely many solutions of the boundary value problem, and sum all the solutions together using the **principle of superposition**. Finally, we use the **orthogonality** of the solutions to find the correct coefficients so that the sum satisfies the IC (2.3.3). In Sections 2.3.2 through 2.3.6, we carefully work through the details of each of these steps before summarizing the entire method in Section 2.3.7.

2.3.2 PRODUCT SOLUTIONS

In the **method of separation of variables**, the first step is to determine solutions in the product form

$$\boxed{u(x,\,t) = \phi(x)G(t),} \tag{2.3.4}$$

where $\phi(x)$ is only a function of x and $G(t)$ only a function of t. The **product solution** (2.3.4) must satisfy the *linear homogeneous* partial differential equation (2.3.1) and boundary conditions (2.3.2), but for the moment we ignore the initial condition. Later we will explain how to satisfy the initial conditions.

Let us be clear from the beginning—we do not give any reasons why we chose the form (2.3.4). (Daniel Bernoulli invented this technique in the 1700s. It works because it reduces a PDE to ODEs, as we shall see.) We substitute the assumed product form, (2.3.4), into the partial differential equation (2.3.1):

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t),$$

and consequently the heat equation (2.3.1) implies that

$$\phi(x)\frac{dG}{dt} = k\frac{d^2\phi}{dx^2}G(t). \tag{2.3.5}$$

We note that we can "separate variables" by dividing both sides of (2.3.5) by $\phi(x)G(t)$:

$$\frac{1}{G}\frac{dG}{dt} = k\frac{1}{\phi}\frac{d^2\phi}{dx^2}.$$

Now the variables have been "separated" in the sense that the left-hand side is only a function of t and the right-hand side only a function of x. We can continue in this way, but it is convenient (but not necessary) also to divide by the constant k, and thus

$$\underbrace{\frac{1}{KG}\frac{dG}{dt}}_{\substack{\text{function}\\ \text{of } t \text{ only}}} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{\substack{\text{function}\\ \text{of } x \text{ only}}}. \tag{2.3.6}$$

This could be obtained directly from (2.3.5) by dividing by $k\phi(x)G(t)$. How is it possible for a function of time to equal a function of space? If x and t are both to be arbitrary independent variables, then x cannot be a function of t (or t a function of x) as seems to be specified by (2.3.6). The important idea is that we claim it is necessary that both sides of (2.3.6) must equal the same constant:

$$\boxed{\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda,} \tag{2.3.7}$$

where λ is an arbitrary *constant* known as the **separation constant**.[1] We will explain momentarily the mysterious minus sign, which was introduced only for convenience.

Equation (2.3.7) yields two ordinary differential equations, one for $G(t)$ and one for $\phi(x)$:

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi} \tag{2.3.8}$$

$$\boxed{\frac{dG}{dt} = -\lambda kG.} \tag{2.3.9}$$

We reiterate that λ is a constant, and it is the same constant that appears in both (2.3.8) and (2.3.9). The product solutions, $u(x,\,t) = \phi(x)G(t)$, must also satisfy the two homogeneous

[1]As further explanation for the constant in (2.3.7), let us say the following. Suppose that the left-hand side of (2.3.7) is some function of t, $(1/kG)\ dG/dt = w(t)$. If we differentiate with respect to x, we get zero: $0 = d/dx(1/\phi\ d^2\phi/dx^2)$. Since $1/\phi\ d^2\phi/dx^2$ is only a function of x, this implies that $1/\phi\ d^2\phi/dx^2$ must be a constant, its derivative equaling zero. In this way (2.3.7) follows.

boundary conditions. For example, $u(0, t) = 0$ implies that $\phi(0)G(t) = 0$. There are two possibilities. Either $G(t) \equiv 0$ (the meaning of $\equiv$ is identically zero, for all t) or $\phi(0) = 0$. If $G(t) \equiv 0$, then from (2.3.4), the assumed product solution is identically zero, $u(x, t) \equiv 0$. This is not very interesting. [$u(x, t) \equiv 0$ is called the **trivial solution** since $u(x, t) \equiv 0$ automatically satisfies any homogeneous PDE and any homogeneous BC.] Instead, we look for nontrivial solutions. For nontrivial solutions, we must have

$$\phi(0) = 0. \tag{2.3.10}$$

By applying the other boundary condition, $u(L, t) = 0$, we obtain in a similar way that

$$\phi(L) = 0. \tag{2.3.11}$$

Product solutions, in addition to satisfying two ordinary differential equations, (2.3.8) and (2.3.9), must also satisfy boundary conditions (2.3.10) and (2.3.11).

2.3.3 TIME-DEPENDENT ORDINARY DIFFERENTIAL EQUATION

The advantage of the product method is that it transforms a partial differential equation, which we do not know how to solve, into two ordinary differential equations. The boundary conditions impose two conditions on the x-dependent ordinary differential equation (ODE). The time-dependent equation has no additional conditions, just

$$\frac{dG}{dt} = -\lambda k G. \tag{2.3.12}$$

Let us solve (2.3.12) first before we discuss solving the x-dependent ODE with its two homogeneous boundary conditions. Equation (2.3.12) is a first-order linear homogeneous differential equation *with constant coefficients*. We can obtain its general solution quite easily. Nearly all constant-coefficient (linear and homogeneous) ODEs can be solved by seeking exponential solutions, $G = e^{rt}$, where in this case by substitution the characteristic polynomial is $r = -\lambda k$. Therefore, the general solution of (2.3.12) is

$$G(t) = ce^{-\lambda k t}. \tag{2.3.13}$$

Recall that for linear homogeneous equations, if $e^{-\lambda k t}$ is a solution, then $ce^{-\lambda k t}$ is a solution for any arbitrary multiplicative constant c. The solution of the time-dependent ODE is a simple exponential.

Let us take a moment to discuss the parameter λ and anticipate its effect on the solution. The parameter λ is the separation constant and, for the moment, it is arbitrary. However, eventually we will discover that only certain values of λ are allowable. If $\lambda > 0$, the solution $G(t)$ exponentially decays as t increases (since $k > 0$). If $\lambda < 0$, the solution exponentially increases, and if $\lambda = 0$, the solution remains constant in time. In the heat conduction problem, we do not intuitively expect the temperature $u(x, t)$ to grow exponentially in time. Since $u(x, t)$ is proportional to $G(t)$, we thus do not expect the solution $G(t)$ to grow exponentially in time. For these reasons, we anticipate $\lambda \geq 0$; we have not proved that statement, but we will do so later. (When we see any parameter we often automatically assume it is positive, even though we shouldn't!) In fact, that is why we introduced the expression $-\lambda$ when we first separated variables [see (2.3.7)]. If we had named the separation constant, say, μ (instead of $-\lambda$), then our previous arguments would have suggested that $\mu \leq 0$. In general, when separating variables in (2.3.7), we anticipate the solution of the time-dependent ODE to be an exponential, and, anticipating that $G(t)$ does not exponentially grow, we expect the separation constant to be negative or zero. We then introduce $-\lambda$ as the separation constant for convenience, since we would now *expect* $\lambda \geq 0$. We next show how we actually determine all allowable separation constants. We will verify mathematically that $\lambda \geq 0$, as we expect by the physical arguments presented previously.

2.3.4 BOUNDARY VALUE (EIGENVALUE) PROBLEM

The x-dependent part of the assumed product solution, $\phi(x)$, satisfies a second-order ODE with two homogeneous boundary conditions:

$$\boxed{\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned}} \tag{2.3.14}$$

We call (2.3.14) a **boundary value problem** for ordinary differential equations. In the usual first course in ordinary differential equations, only initial value problems are specified. For example (think of Newton's law of motion for a particle), we solve second-order differential equations ($md^2y/d^2t = F$) subject to two initial conditions [$y(0)$ and $dy/dt(0)$ given] *both at the same time*. Initial value problems are quite nice, as *usually* there *exist unique* solutions to initial value problems. However, (2.3.14) is quite different. It is a boundary value problem, since the two conditions are not given at the same place (e.g., $x = 0$) but at two different places, $x = 0$ and $x = L$. There is no simple theory that guarantees that the solution exists or is unique to this type of problem. In particular, we note that $\phi(x) = 0$ satisfies the ODE and both homogeneous boundary conditions, no matter what the separation constant λ is, even if $\lambda < 0$; it is referred to as the **trivial solution** of the boundary value problem. It corresponds to $u(x, t) \equiv 0$, since $u(x, t) = \phi(x)G(t)$. If solutions of (2.3.14) had been unique, then $\phi(x) \equiv 0$ would be the only solution; we would not be able to obtain nontrivial solutions of a linear homogeneous PDE by the product (separation of variables) method. Fortunately, there are other solutions of (2.3.14). However, there do not exist nontrivial solutions of (2.3.14) for all values of λ. Instead, we will show that there are certain special values of λ, called **eigenvalues**[2] of the boundary value problem (2.3.14), for which there are nontrivial solutions, $\phi(x)$. A nontrivial $\phi(x)$, which exists only for certain values of λ, is called an **eigenfunction** corresponding to the eigenvalue λ.

Let us try to determine the eigenvalues λ. In other words, for what values of λ are there nontrivial solutions of (2.3.14)? We solve (2.3.14) directly. There are three cases with fundamentally different kinds of solutions, depending on the value of λ: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

Eigenvalues and eigenfunctions ($\lambda > 0$)

Let us first consider the case in which $\lambda > 0$. The boundary value problem is

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{2.3.15}$$

$$\phi(0) = 0 \tag{2.3.16}$$

$$\phi(L) = 0. \tag{2.3.17}$$

If $\lambda > 0$, the general solution of (2.3.15) is

$$\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, \tag{2.3.18}$$

for details on finding this general solution, see the appendix to this section.[3]

We now apply the boundary conditions. $\phi(0) = 0$ implies that

$$0 = c_1.$$

The cosine term vanishes, since the solution must be zero at $x = 0$. Thus, $\phi(x) = c_2 \sin\sqrt{\lambda}x$. Only the boundary condition at $x = L$ has not been satisfied. $\phi(L) = 0$ implies that

$$0 = c_2 \sin\sqrt{\lambda}L.$$

Either $c_2 = 0$ or $\sin\sqrt{\lambda}L = 0$. If $c_2 = 0$, then $\phi(x) \equiv 0$ since we already determined that $c_1 = 0$. This is the trivial solution, and we are searching for those values of λ that have nontrivial solutions. The eigenvalues λ must satisfy

$$\sin\sqrt{\lambda}L = 0. \tag{2.3.19}$$

$\sqrt{\lambda}L$ must be a zero of the sine function. A sketch of $\sin z$ (see Fig. 2.3.1) or our knowledge of the sine function shows that $\sqrt{\lambda}L = n\pi$. $\sqrt{\lambda}L$ must equal an integral multiple of π, where n is a positive integer since $\sqrt{\lambda} > 0$ ($n = 0$ is not appropriate since we assumed that $\lambda > 0$ in this derivation). The **eigenvalues** λ are

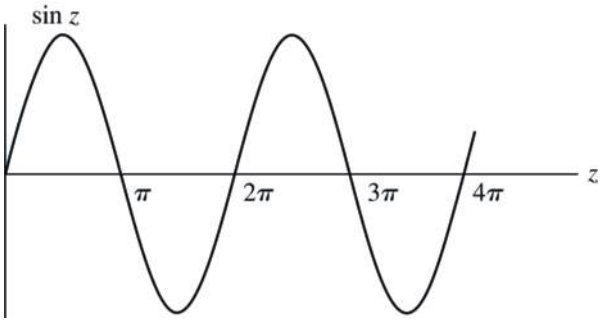
$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots. \tag{2.3.20}$$

---

[2]The word *eigenvalue* comes from the German word *eigenwert*, meaning characteristic value.

[3]In some examples, other independent solutions are chosen. $\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$ are usually the most convenient, but $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$ can be used. Exercise 2.3.4(c) illustrates the advantage of sometimes choosing $\cos\sqrt{\lambda}(x - a)$ and $\sin\sqrt{\lambda}(x - a)$ as independent solutions.

Figure 2.3.1 Zeros of sin z.



The eigenfunction corresponding to the eigenvalue $\lambda = (n\pi/L)^2$ is

$$\phi(x) = c_2 \sin\sqrt{\lambda}x = c_2 \sin\frac{n\pi x}{L}, \tag{2.3.21}$$

where c_2 is an arbitrary multiplicative constant. Often we pick a convenient value for c_2; for example, $c_2 = 1$. We should remember, though, that any specific eigenfunction can always be multiplied by an arbitrary constant, since the PDE and BCs are linear and homogeneous.

Eigenvalue ($\lambda = 0$)

Now we will determine if $\lambda = 0$ is an eigenvalue for (2.3.15) subject to the boundary conditions (2.3.16) and (2.3.17). If $\lambda = 0$, (2.3.15) implies that

$$\phi = c_1 + c_2 x,$$

which can be found by integrating the differential equation $d^2\phi/dx^2 = 0$ twice. To determine whether $\lambda = 0$ is an eigenvalue, the homogeneous boundary conditions must be applied. $\phi(0) = 0$ implies that $0 = c_1$, and thus $\phi = c_2 x$. In addition, $\phi(L) = 0$ implies that $0 = c_2 L$. Since the length L of the rod is positive ($L \neq 0$), $c_2 = 0$ and thus $\phi(x) \equiv 0$. This is the trivial solution, so we say that $\lambda = 0$ is *not* an eigenvalue for *this* problem [(2.3.15) and (2.3.16), (2.3.17)]. Be wary, though; $\lambda = 0$ is an eigenvalue for other problems and should be looked at individually for any new problem you may encounter.

Eigenvalues ($\lambda < 0$)

Are there any negative eigenvalues? If $\lambda < 0$, the solution of

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{2.3.22}$$

is not difficult, but you may have to be careful because $-\lambda$ is positive. It is convenient in the $\lambda < 0$ case to let

$$\lambda = -s.$$

Then $s > 0$ and the differential equation (2.3.22) becomes

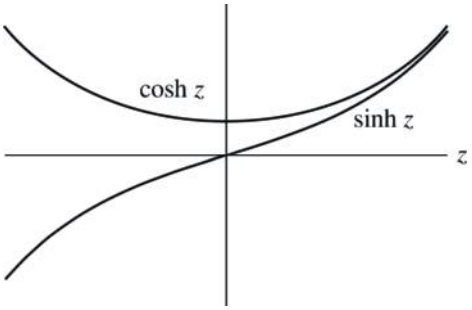
$$\frac{d^2\phi}{dx^2} = s\phi. \tag{2.3.23}$$

Two independent solutions are $e^{\sqrt{s}x}$ and $e^{-\sqrt{s}x}$, or $\cosh\sqrt{s}x$ and $\sinh\sqrt{s}x$ (see the appendix to this section). The hyperbolic functions are more convenient in this case. The general solution is

$$\phi = c_3 \cosh\sqrt{s}x + c_4 \sinh\sqrt{s}x, \tag{2.3.24}$$

To determine if there are any negative eigenvalues ($\lambda < 0$, but $s > 0$ since $\lambda = -s$), we again apply the boundary conditions. From (2.3.24), $\phi(0) = 0$ implies that $0 = c_3$, and hence $\phi = c_4 \sinh\sqrt{s}x$. The other boundary condition, $\phi(L) = 0$, implies that $c_4 \sinh\sqrt{s}L = 0$. Since $\sqrt{s}L > 0$ and since $\sinh(z)$ is never zero for a positive argument (see Fig. 2.3.2), it follows that $c_4 = 0$. Thus, $\phi(x) \equiv 0$. The only solution of (2.3.23) for $\lambda < 0$ that solves the homogeneous boundary conditions is the trivial solution. Thus, there are no negative eigenvalues. For this example, the existence of negative eigenvalues would have corresponded to exponential growth in time. We did not expect such solutions on physical grounds, and here we have verified

Figure 2.3.2 Hyperbolic functions.



mathematically in an explicit manner that there cannot be any negative eigenvalues for this problem. In some other problems there can be negative eigenvalues. Later (Section 5.3) we will formulate a theory, involving the Rayleigh quotient, in which we will know before we start many problems that there cannot be negative eigenvalues. This will at times eliminate calculations such as the ones just performed.

Eigenfunctions—summary

We summarize our results for the boundary value problem resulting from separation of variables:

$$\boxed{\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned}}$$

This boundary value problem will arise many times in the text. It is helpful to become familiar with the result that the eigenvalues λ are all positive (not zero or negative),

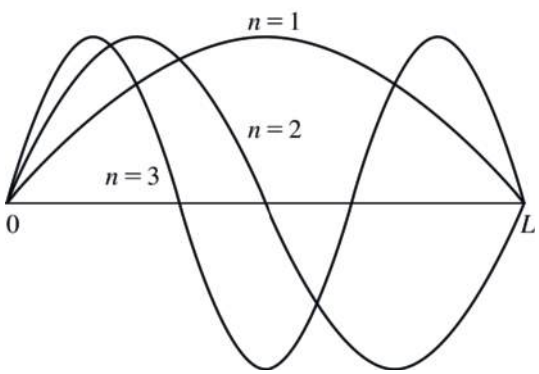
$$\boxed{\lambda = \left(\frac{n\pi}{L}\right)^2,}$$

where n is any positive integer, $n = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are

$$\boxed{\phi(x) = \sin\frac{n\pi x}{L}.}$$

If we introduce the notation λ_1 for the first (or lowest) eigenvalue, λ_2 for the next, and so on, we see that $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$. The corresponding eigenfunctions are sometimes denoted $\phi_n(x)$, the first few of which are sketched in Fig. 2.3.3. All eigenfunctions are (of course) zero at both $x = 0$ and $x = L$.

Figure 2.3.3 Eigenfunctions $\sin n\pi x/L$ and their zeros.



Spring-mass analog

We have obtained solutions of $d^2\phi/dx^2 = -\lambda\phi$. Here we present the analog of this to a spring-mass system, which some of you may find helpful. A spring-mass system subject to Hooke's law satisfies $md^2y/dt^2 = -ky$, where $k > 0$ is the spring constant. Thus, if $\lambda > 0$, the ODE (2.3.15) may be thought of as a spring-mass system with a restoring force. Thus, if $\lambda > 0$, the solution should oscillate. We should not be surprised that the BCs (2.3.16), (2.3.17) can be satisfied for $\lambda > 0$; a nontrivial solution of the ODE, which is zero at $x = 0$, has a chance of being zero again at $x = L$ since there is a restoring force and the solution of the ODE oscillates. We have shown that this can happen for specific values of $\lambda > 0$. However, if $\lambda < 0$, then the force is not restoring. It would seem less likely that a nontrivial solution that is zero at $x = 0$ could possibly be zero again at $x = L$. We must not always trust our intuition entirely, so we have verified these facts mathematically.

2.3.5 PRODUCT SOLUTIONS AND THE PRINCIPLE OF SUPERPOSITION

In summary, we obtained product solutions of the heat equation, $\partial u/\partial t = k\,\partial^2 u/\partial x^2$, satisfying the specific homogeneous boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$ only corresponding to $\lambda > 0$. These solutions, $u(x, t) = \phi(x)G(t)$, have $G(t) = ce^{-\lambda kt}$ and $\phi(x) = c_2 \sin\sqrt{\lambda}x$, where we determined from the boundary conditions [$\phi(0) = 0$ and $\phi(L) = 0$] the allowable values of the separation constant λ, $\lambda = (n\pi/L)^2$. Here n is a positive integer. Thus, product solutions of the heat equation are

$$\boxed{u(x, t) = B\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t},} \qquad n = 1, 2, \ldots, \tag{2.3.25}$$

where B is an arbitrary constant ($B = cc_2$). This is a different solution for each n. Note that as t increases, these special solutions exponentially decay, in particular, for these solutions, $\lim_{t\to\infty} u(x, t) = 0$. In addition, $u(x, t)$ satisfies a special initial condition, $u(x, 0) = B\sin n\pi x/L$.

Initial value problems

We can use the simple product solutions (2.3.25) to satisfy an initial value problem if the initial condition happens to be just right. For example, suppose that we wish to solve the following initial value problem:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ \text{BC:} \quad & u(0, t) = 0 \\ & u(L, t) = 0 \\ \text{IC:} \quad & u(x, 0) = 4\sin\frac{3\pi x}{L}. \end{aligned}$$

Our product solution $u(x, t) = B\sin n\pi x/L \cdot e^{-k(n\pi/L)^2 t}$ satisfies the initial condition $u(x, 0) = B\sin n\pi x/L$. Thus, by picking $n = 3$ and $B = 4$, we will have satisfied the initial condition. Our solution of this example is thus

$$u(x, t) = 4\sin\frac{3\pi x}{L}e^{-k(3\pi/L)^2 t}.$$

It can be proved that this physical problem (as well as most we consider) has a unique solution. Thus, it does not matter what procedure we used to obtain the solution.

Principle of superposition

The product solutions appear to be very special, since they may be used directly only if the initial condition happens to be of the appropriate form. However, we wish to show that these solutions are useful in many other situations; in fact, in all situations. Consider the same PDE and BCs, but instead subject to the initial condition

$$u(x, 0) = 4\sin\frac{3\pi x}{L} + 7\sin\frac{8\pi x}{L}.$$

The solution of this problem can be obtained by adding together two simpler solutions obtained by the product method:

$$u(x,\ t)\ =\ 4\sin\frac{3\pi x}{L}e^{-k(3\pi/L)^2 t}\ +\ 7\sin\frac{8\pi x}{L}e^{-k(8\pi/L)^2 t}.$$

We immediately see that this solves the initial condition (substitute $t\ =\ 0$) as well as the boundary conditions (substitute $x\ =\ 0$ and $x\ =\ L$). Only slightly more work shows that the partial differential equation has been satisfied. This is an illustration of the principle of superposition.

Superposition (extended)

The principle of superposition can be extended to show that if $u_1,\ u_2,\ u_3,\ \ldots,\ u_M$ are solutions of a linear homogeneous problem, then any linear combination of these is also a solution, $c_1u_1\ +\ c_2u_2\ +\ c_3u_3\ +\ \ldots\ +c_Mu_M\ =\ \sum_{n=1}^{M}c_nu_n$, where c_n are arbitrary constants. Since we know from the method of separation of variables that $\sin n\pi x/L\ \bullet\ e^{-k(n\pi/L)^2 t}$ is a solution of the heat equation (solving zero boundary conditions) for all positive n, it follows that any linear combination of these solutions is also a solution of the linear homogeneous heat equation. Thus,

$$u(x,\ t)\ =\ \sum_{n=1}^{M}B_n\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t} \tag{2.3.26}$$

solves the heat equation (with zero boundary conditions) for any finite M. We have added solutions to the heat equation, keeping in mind that the "amplitude" B could be different for each solution, yielding the subscript B_n. Equation (2.3.26) shows that we can solve the heat equation if initially

$$u(x,\ 0)\ =\ f(x)\ =\ \sum_{n=1}^{M}B_n\sin\frac{n\pi x}{L}, \tag{2.3.27}$$

that is, if the initial condition equals a finite sum of the appropriate sine functions. What should we do in the usual situation in which $f(x)$ is *not* a finite linear combination of the appropriate sine functions? The theory of Fourier series (to be described with considerable detail in Chapter 3) states that

1. Any function $f(x)$ (with certain very reasonable restrictions, to be discussed later) can be approximated (in some sense) by a finite linear combination of $\sin n\pi x/L$.
2. The approximation may not be very good for small M, but gets to be a better and better approximation as M is increased (see Section 5.10).
3. If we consider the limit as $M\ \rightarrow\ \infty$, then not only is (2.3.27) the best approximation to $f(x)$ using combinations of the eigenfunctions, but (again in some sense) the resulting infinite series will converge to $f(x)$ [with some restrictions on $f(x)$, to be discussed].

We thus claim (and clarify and make precise in Chapter 3) that nearly any initial condition $f(x)$ can be written as an infinite linear combination of $\sin n\pi x/L$, known as a type of **Fourier series**:

$$f(x)\ =\ \sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}. \tag{2.3.28}$$

What is more important is that we also claim that the corresponding infinite series is the solution of our heat conduction problem:

$$\boxed{u(x,\ t)\ =\ \sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}.} \tag{2.3.29}$$

Analyzing infinite series such as (2.3.28) and (2.3.29) is not easy. We must discuss the convergence of these series as well as briefly discuss the validity of an infinite series solution of our entire problem. For the moment, let us ignore these somewhat theoretical issues and concentrate on the construction of these infinite series solutions.

2.3.6 ORTHOGONALITY OF SINES

One very important practical point has been neglected. Equation (2.3.29) is our solution with the coefficients B_n satisfying (2.3.28) (from the initial conditions), but how do we determine the coefficients B_n? We assume it is possible that

$$\boxed{f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L},} \tag{2.3.30}$$

where this is to hold over the region of the one-dimensional rod, $0 \le x \le L$. We will assume that standard mathematical operations are also valid for infinite series. Equation (2.3.30) represents one equation in an infinite number of unknowns, but it should be valid at every value of x. If we substitute a thousand different values of x into (2.3.30), each of the thousand equations would hold, but there would still be an infinite number of unknowns. This is not an efficient way to determine the B_n. Instead, we frequently will employ an extremely important technique based on noticing (perhaps from a table of integrals) that the eigenfunctions $\sin n\pi x/L$ satisfy the following integral property:

$$\boxed{\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n, \end{cases}} \tag{2.3.31}$$

where m and n are positive integers. This can be derived (Exercise 2.3.7) using trigonometric addition formulas. However, perhaps we can suggest just to memorize (2.3.31).

To use these conditions, (2.3.31), to determine B_n, we multiply both sides of (2.3.30) by $\sin m\pi x/L$ (for any fixed integer m, independent of the "dummy" index n):

$$f(x)\,\sin\frac{m\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\,\sin\frac{m\pi x}{L}. \tag{2.3.32}$$

Next we integrate (2.3.32) from $x = 0$ to $x = L$:

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx. \tag{2.3.33}$$

For finite series, the integral of a sum of terms equals the sum of the integrals. We assume that this is valid for this infinite series. Now we evaluate the infinite sum. From the integral property (2.3.31), we see that each term of the sum is zero whenever $n \neq m$. In summing over n, eventually n equals m. It is only for that one value of n (i.e., $n = m$) that there is a contribution to the infinite sum. The only term that appears on the right-hand side of (2.3.33) occurs when n is replaced by m:

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = B_m \int_0^L \sin^2\frac{m\pi x}{L}\,dx.$$

Since the integral on the right equals $L/2$, we can solve for B_m:

$$\boxed{B_m = \frac{\displaystyle\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx}{\displaystyle\int_0^L \sin^2\frac{m\pi x}{L}\,dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx.} \tag{2.3.34}$$

This result is very important, and so is the method by which it was obtained. Try to learn both. The integral in (2.3.34) is considered to be known since $f(x)$ is the given initial condition. The integral cannot usually be evaluated, in which case numerical integrations (on a computer) may need to be performed to get explicit numbers for B_m, $m = 1, 2, 3, \ldots$.

You will find that the formula (2.3.31), $\int_0^L \sin^2 n\pi x/L\,dx = L/2$, is quite useful in many different circumstances, including applications having nothing to do with the material of this text. One reason for its applicability is that there are many periodic phenomena in nature ($\sin\omega t$), and usually energy or power is proportional to the square ($\sin^2\omega t$). The average energy is then proportional to $\int_0^{2\pi/\omega} \sin^2\omega t\,dt$ divided by the period $2\pi/\omega$. It is worthwhile to memorize that the average over a full period of sine or cosine squared is $\frac{1}{2}$. Thus, *the integral over any number of complete periods of the*

square of a sine or cosine is one half the length of the interval. In this way $\int_0^L \sin^2 n\pi x/L\, dx = L/2$, since the interval 0 to L is either a complete or a half-period of $\sin n\pi x/L$.

Orthogonality

Two vectors $\boldsymbol{A} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\boldsymbol{B} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ are orthogonal (perpendicular) if $\boldsymbol{A} \cdot \boldsymbol{B} = a_1b_1 + a_2b_2 + a_3b_3 = 0$. In a similar manner, we say that the functions $A(x)$ and $B(x)$ are **orthogonal** over the interval $0 \le x \le L$ whenever

$$\int_0^L A(x)B(x)\, dx = 0.$$

We borrow the terminology "orthogonal" from perpendicular vectors because $\int_0^L A(x)B(x)\, dx = 0$ is analogous to a zero dot product, as is explained further in the appendix to this section. A set of functions each member of which is orthogonal to every *other* member is called an **orthogonal set of functions**. An example is that of the functions $\sin n\pi x/L$, the eigenfunctions of the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

They are mutually orthogonal because of (2.3.31). Therefore, we call (2.3.31) an orthogonality condition.

In fact, we will discover that for most other boundary value problems, the eigenfunctions will form an orthogonal set of functions (with certain modifications discussed in Chapter 5 with respect to Sturm–Liouville eigenvalue problems).

2.3.7 SEPARATION OF VARIABLES SUMMARY

We are finally ready to completely state the technique of separation of variables.

The **separation of variables** solution of a linear and homogeneous PDE with homogeneous boundary conditions is a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} B_n\phi_n(x)G_n(t)$$

where

1. **each term** $\phi_n(x)G_n(t)$ solves the PDE;
2. **each function** $\phi_n(x)$ satisfies the homogeneous boundary conditions at $x = 0$ and $x = L$; and
3. the **complete sum** $\sum_{n=1}^{\infty} B_n\phi_n(x)G_n(t)$ satisfies the initial condition.

The principle of superposition implies that if each term satisfies the PDE, then so will the sum $\sum_{n=1}^{\infty} B_n\phi_n(x)G_n(t)$. This requires the PDE to be linear. Since the boundary conditions are homogeneous, requiring each $\phi_n(x)$ to satisfy the boundary conditions is sufficient to guarantee that each term $\phi_n(x)G_n(t)$ satisfies the homogeneous boundary conditions, and therefore $u(x, t)$ also does. Thus, this technique may be applied only when the problem consists of a linear PDE with homogeneous BCs.

The steps required to apply the technique of separation of variables to solve a linear and homogeneous PDE with homogeneous boundary conditions are as follows:

1. Substitute the product solutions $u(x, t) = \phi(x)G(t)$ into the PDE to determine the ordinary differential equations for $\phi(x)$ and $G(t)$. Introduce a separation constant.
2. Solve the eigenvalue problem for $\phi(x)$, which results in an infinite sequence of functions $\phi_n(x)$ and eigenvalues λ_n. This determines the separation constants.
3. Solve the time-dependent ODE for $G(t)$. Each separation constant λ_n results in a different function $G_n(t)$.
4. Assemble the sum $u(x, t) = \sum_{n=1}^{\infty} B_n\phi_n(x)G_n(t)$. Use the orthogonality of the eigenfunctions to determine the coefficients B_n that allow $u(x, t)$ to satisfy the initial conditions.

2.3.8 FORMULATION, SOLUTION, AND INTERPRETATION OF AN EXAMPLE

As an example, let us analyze our solution in the case in which the initial temperature is constant, 100°C. This corresponds to a physical problem that is easy to reproduce in the laboratory. Take a one-dimensional rod and place the entire rod in a large tub of boiling water (100°C). Let it sit there for a long time. After a while (we expect) the rod will be 100°C throughout. Now insulate the lateral sides (if that had not been done earlier) and suddenly (at $t = 0$) immerse the two ends in large well-stirred baths of ice water, 0°C. The mathematical problem is

$$\text{PDE:}\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad t > 0, \qquad 0 < x < L \tag{2.3.35}$$

$$\text{BC:}\quad \begin{matrix} u(0, t) = 0 \\ u(L, t) = 0, \end{matrix} \qquad t > 0 \tag{2.3.36}$$

$$\text{IC:}\quad u(x, 0) = 100, \qquad 0 < x < L. \tag{2.3.37}$$

According to (2.3.29) and (2.3.34), the solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}, \tag{2.3.38}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{2.3.39}$$

and $f(x) = 100$. Recall that the coefficient B_n was determined by having (2.3.38) satisfy the initial condition,

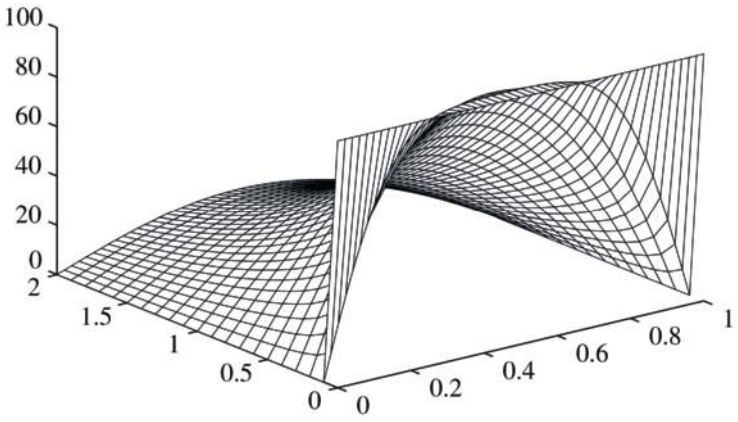
$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}. \tag{2.3.40}$$

We calculate the coefficients B_n from (2.3.39):

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L 100\sin\frac{n\pi x}{L}\,dx = \frac{200}{L}\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\Big|_0^L \\ &= \frac{200}{n\pi}(1 - \cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \dfrac{400}{n\pi} & n \text{ odd} \end{cases} \end{aligned} \tag{2.3.41}$$

since $\cos n\pi = (-1)^n$, which equals 1 for n even and -1 for n odd. The solution (2.3.38) is graphed in Fig. 2.3.4. The series (2.3.40) will be studied further in Chapter 3. In particular, we must explain the intriguing situation that the initial temperature equals 100 everywhere, but the series (2.3.40) equals 0 at $x = 0$ and $x = L$ (due to the boundary conditions). Carefully note that the initial condition $u = 100$ (2.3.37) applies only on the interval $0 < x < L$, and not at the endpoints, where the boundary conditions (2.3.36) require $u = 0$. It will turn out (see Chapter 3) that initial and boundary conditions can even have jump discontinuities, and the resulting Fourier series will converge!

Figure 2.3.4 Time dependence of temperature $u(x, t)$.



MATLAB Files

Download Animation of Figure 2.3.4 (.zip)

https://bit.ly/3QcdkBj

These files are provided as zipped .m files for use in MATLAB on desktop platform.

Approximations to the initial value problem

We have now obtained the solution to the initial value problem (2.3.35)–(2.3.37) for the heat equation with zero boundary conditions ($x = 0$ and $x = L$) and initial temperature distribution equaling 100. The solution is (2.3.38), with B_n given by (2.3.41). The solution is quite complicated, involving an infinite series. What can we say about it? First, we notice that $\lim_{t\to\infty} u(x, t) = 0$. The temperature distribution approaches a steady state, $u(x, t) = 0$. This is not surprising physically since both ends are at 0°; we expect all the initial heat energy contained in the rod to flow out the ends. The equilibrium problem, $d^2u/dx^2 = 0$ with $u(0) = 0$ and $u(L) = 0$, has a unique solution, $u \equiv 0$, agreeing with the limit as t tends to infinity of the time-dependent problem.

One question of importance that we can answer is the manner in which the solution approaches steady state. If t is large, what is the approximate temperature distribution, and how does it differ from the steady state 0°? We note that each term in (2.3.38) decays at a different rate. The more oscillations in space, the faster the decay. If t is such that $kt(\pi/L)^2$ is large, then each succeeding term is much smaller than the first. We can then approximate the infinite series by only the first term:

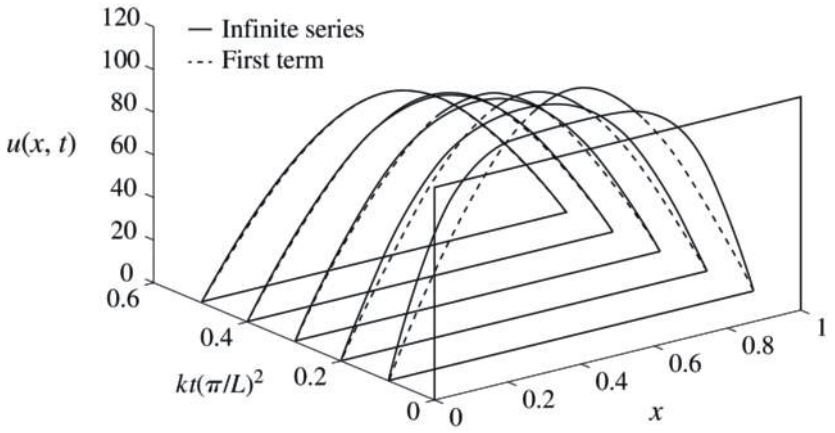
$$u(x, t) \approx \frac{400}{\pi} \sin\frac{\pi x}{L} e^{-k(\pi/L)^2 t}. \tag{2.3.42}$$

The larger t is, the better this is as an approximation. Even if $kt(\pi/L)^2 = \frac{1}{2}$, this is not a bad approximation since

$$\frac{e^{-k(3\pi/L)^2 t}}{e^{-k(\pi/L)^2 t}} = e^{-8(\pi/L)^2 kt} = e^{-4} = 0.018\ldots.$$

Thus, if $kt(\pi/L)^2 \geq \frac{1}{2}$, we can use the simple approximation. We see that for these times the spatial dependence of the temperature is just the simple rise and fall of $\sin \pi x/L$, as illustrated in Fig. 2.3.5. The peak amplitude, occurring in the middle $x = L/2$, decays exponentially in time. For $kt(\pi/L)^2$ less than $\frac{1}{2}$, the spatial dependence cannot be approximated by one simple sinusoidal function; more terms are necessary in the series. The solution can be easily computed, using a finite number of terms. In some cases, many terms may be necessary, and there would be better ways to calculate $u(x, t)$.

Figure 2.3.5 Time dependence of temperature (using the infinite series) compared to the first term. Note the first term is a good approximation if the time is not too small.



Exercises 2.3

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

***(a)** $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$

(b) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} - v_0\dfrac{\partial u}{\partial x}$

***(c)** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(d) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$

***(e)** $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^4 u}{\partial x^4}$

***(f)** $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$

2.3.2. Find the general solution of each boundary value problem, or show that no solution exists. Note that there may be no solution, a unique solution, or infinitely many solutions.

(a) $y'' + 4y = 0, \quad y(0) = 3, \quad y(1) = 5.$

(b) $y'' + 4y = 0, \quad y(0) = 3, \quad y(\pi) = 5.$

(c) $y'' + 4y = 0, \quad y(0) = 3, \quad y(\pi) = 3.$

2.3.3. **(a)** A copper rod is 100 cm long and held at 0°C. Suppose that the point exactly at the middle of the bar is heated to 100°C by an external heat source and the ends of the rod are held at 0°C. This situation is maintained until a steady state results. Find this steady-state temperature distribution. (See Section 1.4.)

(b) At time $t = 0$ (which occurs some time after the steady-state of part (a) has been reached), the heat source at the center is removed. Find the temperature as a function of position and time.

2.3.4. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, $\lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

***(b)** $\phi(0) = 0$ and $\phi(1) = 0$

***(c)** $\phi(a) = 0$ and $\phi(b) = 0$

2.3.5. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0, t) = 0 \qquad \text{and} \qquad u(L, t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x, 0) = 6\sin\frac{9\pi x}{L}$

(b) $u(x, 0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

***(c)** $u(x, 0) = 2\cos\frac{3\pi x}{L}$

(d) $u(x, 0) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$

(e) $u(x, 0) = x^2 - 7x$

[Your answer in part (c) may involve certain integrals that do not need to be evaluated.]

2.3.6. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0, t) = 0$, $u(L, t) = 0$, and $u(x, 0) = f(x)$.

***(a)** What is the total heat energy in the rod as a function of time?

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

***(c)** What relationship should exist between parts (a) and (b)?

2.3.7. Evaluate (be careful if $n = m$)

$$\int_0^L \sin\frac{n\pi x}{L}\,\sin\frac{m\pi x}{L}\,dx \qquad \text{for } n > 0,\ m > 0.$$

Use the trigonometric identity

$$\sin a \sin b = \frac{1}{2}\,[\cos(a - b) - \cos(a + b)].$$

***2.3.8.** Evaluate

$$\int_0^L \cos\frac{n\pi x}{L}\,\cos\frac{m\pi x}{L}\,dx \qquad \text{for } n \ge 0,\ m \ge 0.$$

Use the trigonometric identity

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)].$$

(Be careful if $a - b = 0$ or $a + b = 0$.)

2.3.9. Consider the following boundary value problem (if necessary, see Section 2.4.1):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ with } \frac{\partial u}{\partial x}(0, t) = 0, \ \frac{\partial u}{\partial x}(L, t) = 0, \text{ and } u(x, 0) = f(x).$$

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint*: The answer is

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t} \cos\frac{n\pi x}{L}.\Big]$$

What is λ_n?

(c) Show that the initial condition, $u(x, 0) = f(x)$, is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

(d) Using Exercise 2.3.8, solve for A_0 and $A_n (n \geq 1)$.

(e) What happens to the temperature distribution as $t \to \infty$? Show that it approaches the steady-state temperature distribution (see Section 1.4).

***2.3.10.** Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise 1.2.5) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem $[u(x, 0) = f(x)]$ if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

***2.3.11.** Redo Exercise 2.3.10 if $\alpha < 0$. [Be especially careful if $-\alpha/k = (n\pi/L)^2$.]

2.3.12. Use the technique of separation of variables to solve the modified heat equation

$$t\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 5, \ t > 1,$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u(5, t) = 0, \ t > 1,$$

and the initial condition

$$u(1, x) = 3x, \quad 0 < x < 5.$$

(*Hint*: The ordinary differential equation you find for $G(t)$ will be different than for the usual heat equation problem. To solve it, consider trying $G(t) = t^p$, and solve for the constant p.)

2.3.13. For two- and three-dimensional vectors, the fundamental property of dot products, $\boldsymbol{A} \cdot \boldsymbol{B} = |A||B|\cos\theta$, implies that

$$|\boldsymbol{A} \cdot \boldsymbol{B}| \leq |\boldsymbol{A}||\boldsymbol{B}|. \tag{2.3.43}$$

In this exercise, we generalize this to n-dimensional vectors and functions, in which case (2.3.43) is known as **Schwarz's inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.43).]

(a) Show that $|\boldsymbol{A} - \gamma\boldsymbol{B}|^2 > 0$ implies (2.3.43), where $\gamma = \boldsymbol{A} \cdot \boldsymbol{B}/\boldsymbol{B} \cdot \boldsymbol{B}$.

(b) Express the inequality using both

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

***(c)** Generalize (2.3.43) to functions. [*Hint*: Let $\boldsymbol{A} \cdot \boldsymbol{B}$ mean the integral $\int_0^L A(x)B(x)\,dx$.]

Appendix to 2.3: Orthogonality of Functions

Two vectors A and B are orthogonal if $A \cdot B = 0$. In component form, $A = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $B = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$; A and $\boldsymbol{B}$ are orthogonal if $\sum_i a_i b_i = 0$. A function $A(x)$ can be thought of as a vector. If only three values of x are important, x_1, x_2, and x_3, then the components of the function $A(x)$ (thought of as a vector) are $A(x_1) \equiv a_1$, $A(x_2) \equiv a_2$, and $A(x_3) \equiv a_3$. The function $A(x)$ is orthogonal to the function $B(x)$ (by definition) if $\sum_i a_i b_i = 0$. However, in our problems, all values of x between 0 and L are important. *The function $A(x)$ can be thought of as an infinite-dimensional vector*, whose components are $A(x_i)$ for all x_i on some interval. In this manner, the function $A(x)$ would be said to be orthogonal to $B(x)$ if $\sum_i A(x_i)B(x_i) = 0$, where the summation was to include all points between 0 and L. It is thus natural to *define* the function $A(x)$ to be orthogonal to $B(x)$ if $\int_0^L A(x)B(x)\,dx = 0$. The integral replaces the vector dot product; both are examples of "inner products."

In vectors, we have the three mutually perpendicular (orthogonal) unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, known as the standard basis vectors. In component form,

$$\boldsymbol{A} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}.$$

a_1 is the projection of $\boldsymbol{A}$ in the $\hat{\mathbf{i}}$ direction, and so on. Sometimes we wish to represent $\boldsymbol{A}$ in terms of other mutually orthogonal vectors (which may not be unit vectors) $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, called an *orthogonal set of vectors*. Then

$$\boldsymbol{A} = \alpha_u \boldsymbol{u} + \alpha_v \boldsymbol{v} + \alpha_w \boldsymbol{w}.$$

To determine the coordinates α_u, α_v, α_w with respect to this orthogonal set, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, we can form certain dot products. For example,

$$\boldsymbol{A} \cdot \boldsymbol{u} = \alpha_u \boldsymbol{u} \cdot \boldsymbol{u} + \alpha_v \boldsymbol{v} \cdot \boldsymbol{u} + \alpha_w \boldsymbol{w} \cdot \boldsymbol{u}.$$

Note that $\boldsymbol{v} \cdot \boldsymbol{u} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{u} = 0$, since we assumed that this new set was mutually orthogonal. Thus, we can easily solve for the coordinate α_u, of $\boldsymbol{A}$ in the $\boldsymbol{u}$-direction,

$$\alpha_u = \frac{\boldsymbol{A} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}}. \tag{2.3.44}$$

($\alpha_u \boldsymbol{u}$ is the vector projection of $\boldsymbol{A}$ in the $\boldsymbol{u}$-direction.)

For functions, we can do a similar thing. If $f(x)$ can be represented by a linear combination of the orthogonal set, $\sin n\pi x/L$, then

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

where the B_n may be interpreted as the coordinates of $f(x)$ with respect to the "direction" (or basis vector) $\sin n\pi x/L$. To determine these coordinates, we take the inner product with an arbitrary basis function (vector) $\sin n\pi x/L$, where the inner product of two functions is the integral of their product. Thus, as before,

$$\int_0^L f(x) \sin \frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\,dx.$$

Since $\sin n\pi x/L$ is an orthogonal set of functions, $\int_0^L \sin n\pi x/L \sin m\pi x/L\,dx = 0$ for $n \neq m$. Hence, we solve for the coordinate (coefficient) B_n:

$$B_n = \frac{\int_0^L f(x) \sin n\pi x/L\,dx}{\int_0^L \sin^2 n\pi x/L\,dx}. \tag{2.3.45}$$

This is seen to be the same idea as the projection formula (2.3.44). Our standard formula (2.3.32), $\int_0^L \sin^2 n\pi x/L\, dx = L/2$, returns (2.3.45) to the more familiar form,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{2.3.46}$$

Both formulas (2.3.44) and (2.3.45) are divided by something. In (2.3.44) it is $\boldsymbol{u}\cdot\boldsymbol{u}$, or the length of the vector $\boldsymbol{u}$ squared. Thus, $\int_0^L \sin^2 n\pi x/L dx$ may be thought of as the length squared of $\sin n\pi x/L$ (although here *length* means nothing other than the square root of the integral). In this manner, the length squared of the function $\sin n\pi x/L$ is $L/2$, which is an explanation of the appearance of the term $2/L$ in (2.3.46).

Appendix to 2.3: Solutions of Linear Homogeneous ODEs

Since second-order linear homogeneous ODEs come up so frequently, especially in the technique of separation of variables and boundary-value problems, we provide a brief review here.

To compute the solution to a homogeneous, linear ODE with constant coefficients, substitute a trial solution $\phi = e^{rx}$ into the ODE and solve for the constant r. For example, to find the general solution to

$$\frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx} + 15\phi = 0,$$

let $\phi = e^{rx}$. Then

$$r^2e^{rx} - 2re^{rx} + 15e^{rx} = 0,$$

which factors to

$$e^{rx}(r^2 - 2r + 15) = 0.$$

Since $e^{rx} \neq 0$ for any finite values of r and x, we have

$$(r^2 - 2r + 15) = 0.$$

This polynomial is called the **characteristic polynomial** of the ODE. This polynomial factors to

$$(r - 5)(r + 3) = 0,$$

so $r = 5$ and $r = -3$ are the roots of the characteristic polynomial. Thus, two solutions of the ODE are $\phi = e^{5x}$ and $\phi = e^{-3x}$. Any linear combination of solutions of a linear homogeneous ODE will be another solution of the same ODE, so in this example the general solution is the set of all functions of the form

$$\phi(x) = c_1e^{5x} + c_2e^{-3x}.$$

In general, after substituting $\phi = e^{rx}$ into the second-order ODE, the resulting characteristic polynomial is quadratic. There are three cases:

1. **Distinct real roots**. Write them as $r = r_1$ and $r = r_2$. Then the general solution is
$$\phi = c_1e^{r_1x} + c_2e^{r_2x}.$$
2. **Repeated roots**. Call it $r = r_1$. The general solution is
$$\phi = c_1e^{r_1x} + c_2xe^{r_1x}.$$
3. **Complex roots**, $r = a \pm bi$. The general solution is
$$\phi = c_1e^{ax}\cos(bx) + c_2e^{ax}\sin(bx).$$

To gain some insight why the repeated-roots case has that curious extra factor of x in the second solution, consider solving

$$\frac{d^2\phi}{dx^2} + 10\frac{d\phi}{dx} + 25\phi = 0.$$

This is a case where the characteristic polynomial $p(r) = r^2 + 10r + 25$ has a repeated root $r = -5$, and thus the above rule tells us the general solution is $\phi = c_1 e^{-5x} + c_2 x e^{-5x}$, but let's see how that result comes about. This ODE may be written in the form

$$\left(\frac{d}{dx} + 5\right)\left(\frac{d}{dx} + 5\right)\phi = 0.$$

In this form, we can break it up and solve it in two steps. Define the second parenthesis acting on ϕ to be a new function $u(x)$: $(\frac{d}{dx} + 5)\phi = u(x)$. Then the equation we want to solve simplifies to

$$\left(\frac{d}{dx} + 5\right) u = 0.$$

However, we know the solution to this differential equation: $u(x) = Ce^{-5x}$. With $u(x)$ known, we can turn to solving ϕ itself,

$$\left(\frac{d}{dx} + 5\right) \phi = u(x) \quad \Rightarrow \quad \frac{d\phi}{dx} + 5\phi = Ce^{-5x}.$$

Finally, using the integrating factor technique with $I(x) = e^{5x}$, we can solve the ODE

$$\frac{d}{dx}(e^{5x}\phi) = C \quad \Rightarrow \quad e^{5x}\phi = Cx + D \quad \Rightarrow \quad \phi(x) = Cxe^{-5x} + De^{-5x}.$$

Now let's discuss the complex-roots case. Suppose the characteristic polynomial has roots $r_1 = a + bi$ and $r_2 = a - bi$. Then, two complex-valued solutions are

$$\phi_1(x) = e^{(a+bi)x} \quad \text{and} \quad \phi_2(x) = e^{(a-bi)x}. \tag{2.3.47}$$

So, one way to express the general solution is

$$\phi(x) = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}.$$

However, there are other ways to represent the general solution. Recall that, to find the general solution to a **second-order** linear homogeneous ODE, we only need to find **two** linearly independent solutions, which will be our basis functions. The general solution is the set of all linear combinations of those basis functions. With a different set of basis functions, the general solution may be expressed in another, possibly more useful, way. We will rely on the idea that any linear combination of solutions to a linear homogeneous ODE is another solution.

Using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can rewrite the functions (2.3.48) in the form

$$\phi_1(x) = e^{ax}(\cos(bx) + i\sin(bx)) \quad \text{and} \quad \phi_2(x) = e^{ax}(\cos(bx) - i\sin(bx)).$$

One useful linear combination of these functions is

$$\frac{1}{2}(\phi_1(x) + \phi_2(x)) = e^{ax}\cos(bx)$$

and another nice combination is

$$\frac{1}{2i}(\phi_1(x) - \phi_2(x)) = e^{ax}\sin(bx).$$

Thus, $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are two real-valued functions that solve the ODE, giving the general solution

$$\phi = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx).$$

Special case: trigonometric solutions

One important special case of the linear homogenous ODE is

$$\frac{d^2\phi}{dx^2} + b^2\phi = 0 \qquad (b \neq 0).$$

This falls into the complex-roots case, with $r = \pm bi$, so we can find the general solution by setting $a = 0$ in the appropriate formula above. However, this case is so common that we strongly recommend students do not start with $\phi = e^{rx}$, but rather familiarize themselves with the result:

The general solution of the ODE

$$\frac{d^2\phi}{dx^2} + b^2\phi = 0 \qquad (b \neq 0)$$

may be written

$$\phi = c_1 \cos(bx) + c_2 \sin(bx).$$

Another valid expression of the general solution in this case is with a phase shift in the arguments,

$$\phi = c_3 \cos(b(x - x_0)) + c_4 \sin(b(x - x_0))$$

For example, the general solution of

$$\frac{d^2\phi}{dx^2} + 25\phi = 0$$

is

$$\phi = c_1 \cos 5x + c_2 \sin 5x$$

but the general solution could instead be written

$$\phi = c_3 \cos 5(x - 2) + c_4 \sin 5(x - 2).$$

This form of the general solution would be particularly useful if one of the boundary conditions were given at $x = 2$.

Special case: exponential solutions

A second important special case is

$$\frac{d^2\phi}{dx^2} - a^2\phi = 0 \qquad (a \neq 0). \tag{2.3.48}$$

This is part of the distinct real roots case, with $r = \pm a$, so the general solution is

$$\phi = c_1 e^{ax} + c_2 e^{-ax}. \tag{2.3.49}$$

Frequently, we instead use the hyperbolic functions. The definitions of the hyperbolic functions are

$$\cosh z \equiv \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z \equiv \frac{e^z - e^{-z}}{2},$$

simple linear combinations of exponentials. These are sketched in Fig. 2.3.2. Note that $\sinh 0 = 0$ and $\cosh 0 = 1$ (the results analogous to those for trigonometric functions).

Also note that $d/dz \cosh z = \sinh z$ and $d/dz \sinh z = \cosh z$, quite similar to trigonometric functions, but easier to remember because of the lack of the annoying appearance of any minus signs in the differentiation formulas. If hyperbolic functions are used instead of exponentials, the general solution of (2.3.48) can be written as

$$\phi = c_3 \cosh ax + c_4 \sinh ax, \tag{2.3.50}$$

a form equivalent to (2.3.49). We summarize this case:

The general solution of the ODE

$$\frac{d^2\phi}{dx^2} - a^2\phi = 0 \qquad (a \neq 0)$$

may be written

$$\phi = c_1 e^{ax} + c_2 e^{-ax}$$

or in the equivalent form

$$\phi = c_3 \cosh ax + c_4 \sinh ax.$$

The hyperbolic function version of the solution is especially useful in boundary value problems where one of the boundary conditions is given at $x = 0$.

2.4 Worked Examples with the Heat Equation (Other Boundary Value Problems)

2.4.1 HEAT CONDUCTION IN A ROD WITH INSULATED ENDS

Let us work out in detail the solution (and its interpretation) of the following problem defined for $0 \leq x \leq L$ and $t \geq 0$:

$$\text{PDE:} \quad \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \tag{2.4.1}$$

$$\text{BC1:} \quad \boxed{\frac{\partial u}{\partial x}(0, t) = 0} \tag{2.4.2}$$

$$\text{BC2:} \quad \boxed{\frac{\partial u}{\partial x}(L, t) = 0} \tag{2.4.3}$$

$$\text{IC:} \quad \boxed{u(x, 0) = f(x).} \tag{2.4.4}$$

As a review, this is a heat conduction problem in a one-dimensional rod with constant thermal properties and no sources. This problem is quite similar to the problem treated in Section 2.3, the only difference being the boundary conditions. Here the ends are insulated, whereas in Section 2.3, the ends were fixed at 0°. Both the partial differential equation and the boundary conditions are linear and homogeneous. Consequently, we apply the method of separation of variables. We may follow the general procedure described in Section 2.3.7. The assumption of product solutions,

$$u(x, t) = \phi(x)G(t), \tag{2.4.5}$$

implies from the PDE as before that

$$\frac{dG}{dt} = -\lambda kG \tag{2.4.6}$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \tag{2.4.7}$$

where λ is the separation constant. Again,

$$G(t) = ce^{-\lambda kt}. \tag{2.4.8}$$

The insulated boundary conditions, (2.4.2) and (2.4.3), imply that the separated solutions must satisfy $d\phi/dx(0) = 0$ and $d\phi/dx(L) = 0$. The separation constant λ is then determined by finding those λ for which nontrivial solutions exist for the following boundary value problem:

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi} \tag{2.4.9}$$

$$\boxed{\frac{d\phi}{dx}(0) = 0} \tag{2.4.10}$$

$$\boxed{\frac{d\phi}{dx}(L) = 0.} \tag{2.4.11}$$

Although the ordinary differential equation for the boundary value problem is the same one as previously analyzed, the boundary conditions are different. We must repeat some of the analysis. Once again, three cases should be discussed: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ (since we will assume the eigenvalues are real).

For $\lambda > 0$, the general solution of (2.4.9) is again

$$\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x. \tag{2.4.12}$$

We need to calculate $d\phi/dx$ to satisfy the boundary conditions:

$$\frac{d\phi}{dx} = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x\right). \tag{2.4.13}$$

The boundary condition $d\phi/dx(0) = 0$ implies that $0 = c_2\sqrt{\lambda}$, and hence $c_2 = 0$, since $\lambda > 0$. Thus, $\phi = c_1 \cos\sqrt{\lambda}x$ and $d\phi/dx = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x$. The eigenvalues λ and their corresponding eigenfunctions are determined from the remaining boundary condition, $d\phi/dx(L) = 0$:

$$0 = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}L.$$

As before, for nontrivial solutions, $c_1 \neq 0$, and hence $\sin\sqrt{\lambda}L = 0$. The eigenvalues for $\lambda > 0$ are the same as the previous problem, $\sqrt{\lambda}L = n\pi$ or

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3 \ldots, \tag{2.4.14}$$

but the corresponding eigenfunctions are cosines (not sines),

$$\phi(x) = c_1 \cos\frac{n\pi x}{L}, \quad n = 1, 2, 3 \ldots. \tag{2.4.15}$$

The resulting product solutions of the PDE are

$$u(x, t) = A\cos\frac{n\pi x}{L}e^{-(n\pi/L)^2 kt}, \quad n = 1, 2, 3 \ldots, \tag{2.4.16}$$

where A is an arbitrary multiplicative constant.

Before applying the principle of superposition, we must see if there are any other eigenvalues. If $\lambda = 0$, then

$$\phi = c_1 + c_2 x, \tag{2.4.17}$$

where c_1 and c_2 are arbitrary constants. The derivative of ϕ is

$$\frac{d\phi}{dx} = c_2.$$

Both boundary conditions, $d\phi/dx(0) = 0$ and $d\phi/dx(L) = 0$, give the same condition, $c_2 = 0$. Thus, there are nontrivial solutions of the boundary value problem for $\lambda = 0$, namely, $\phi(x)$ equaling any constant

$$\phi(x) = c_1. \tag{2.4.18}$$

The time-dependent part is also a constant, since $e^{-\lambda kt}$ for $\lambda = 0$ equals 1. Thus, another product solution of both the linear homogeneous PDE and BCs is $u(x, t) = A$, where A is any constant.

We do not expect there to be any eigenvalues for $\lambda < 0$, since in this case the time-dependent part grows exponentially. In addition, it seems unlikely that we would find a nontrivial linear combination of exponentials that would have a zero slope at both $x = 0$ and $x = L$. In Exercise 2.4.11, you are asked to show that there are no negative eigenvalues for $\lambda < 0$.

In order to satisfy the initial condition, we use the principle of superposition. We should take a linear combination of *all* product solutions of the PDE (not just those corresponding to $\lambda > 0$). Thus,

$$\boxed{u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}e^{-(n\pi/L)^2 kt}.} \tag{2.4.19}$$

It is interesting to note that this is equivalent to

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}e^{-(n\pi/L)^2 kt} \tag{2.4.20}$$

since $\cos 0 = 1$ and $e^0 = 1$. In fact, (2.4.20) is often easier to use in practice. We prefer the form (2.4.19) *in the beginning stages of the learning process*, since it more clearly shows that the solution consists of terms arising from the analysis of two somewhat distinct cases, $\lambda = 0$ and $\lambda > 0$.

The initial condition $u(x, 0) = f(x)$ is satisfied if

$$\boxed{f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L},} \tag{2.4.21}$$

for $0 \leq x \leq L$. The validity of (2.4.21) will also follow from the theory of Fourier series. Let us note that in the previous problem, $f(x)$ was represented by a series of sines. Here $f(x)$ consists of a series of cosines and the constant term. The two cases are different due to the different boundary

conditions. To complete the solution, we need to determine the arbitrary coefficients A_0 and $A_n (n \geq 1)$. Fortunately, it is known that $\cos n\pi x/L$ satisfies the following **orthogonality relation**:

$$\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases} \tag{2.4.22}$$

for n and m nonnegative integers. Note that $n = 0$ or $m = 0$ corresponds to a constant 1 contained in the integrand. The constant $L/2$ is another application of the statement that the average of the square of a sine or cosine function is $\frac{1}{2}$. The constant L in (2.4.22) is quite simple since for $n = m = 0$, (2.4.22) becomes $\int_0^L dx = L$. Equation (2.4.22) states that the **cosine functions** (including the constant function) **form an orthogonal set of functions**. We can use that idea, in the same way as before, to determine the coefficients. Multiplying (2.4.21) by $\cos m\pi x/L$ and integrating from 0 to L yields

$$\int_0^L f(x) \cos\frac{m\pi x}{L}\, dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx.$$

This holds for all $m = 0, 1, 2 \ldots$. The case in which $m = 0$ corresponds just to integrating (2.4.21) directly. Using the orthogonality results, it follows that only the mth term in the infinite sum contributes:

$$\int_0^L f(x) \cos\frac{m\pi x}{L}\, dx = A_m \int_0^L \cos^2\frac{m\pi x}{L}\, dx.$$

The factor $\int_0^L \cos^2 m\pi x/L\, dx$ has two different cases: $m = 0$ and $m \neq 0$. Solving for A_m yields

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx \tag{2.4.23}$$

$$(m \geq 1) \quad A_m = \frac{2}{L}\int_0^L f(x) \cos\frac{m\pi x}{L}\, dx. \tag{2.4.24}$$

The two different formulas are a somewhat annoying feature of this series of cosines. They are simply caused by the factors $L/2$ and L in (2.4.22).

There is a significant difference between the solutions of the PDE for $\lambda > 0$ and the solution for $\lambda = 0$. All the solutions for $\lambda > 0$ decay exponentially in time, whereas the solution for $\lambda = 0$ remains constant in time. Thus, as $t \to \infty$, the complicated infinite series solution (2.4.19) approaches steady state:

$$\lim_{t \to \infty} u(x, t) = A_0 = \frac{1}{L}\int_0^L f(x)\, dx.$$

Not only is the steady-state temperature constant, A_0, but we recognize the constant A_0 as the average of the initial temperature distribution. This agrees with information obtained previously. Recall from Section 1.4 that the equilibrium temperature distribution for the problem with insulated boundaries is not unique. Any constant temperature is an equilibrium solution, but using the ideas of conservation of total thermal energy, we know that the constant must be the average of the initial temperature.

Example with insulated ends

Suppose the initial temperature of a one-dimensional rod is proportional to the distance from one end of the rod, say $u(x, 0) = 5x$. As we will see in Chapter 8, this may be interpreted as the steady-state temperature resulting from holding one end of the rod at a temperature of 0, and the other end at a temperature of $5L$ for a long time. Then, at $t = 0$, suppose the ends of the rod are insulated. The problem is

$$\text{PDE:} \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad t > 0, \qquad 0 < x < L \tag{2.4.25}$$

$$\text{BC:} \quad \frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0, \qquad t > 0, \tag{2.4.26}$$

$$\text{IC:} \quad u(x, 0) = 5x, \qquad 0 < x < L. \tag{2.4.27}$$

The solution in the insulated-end case is given by (2.4.19)

$$u(x,\,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}, \tag{2.4.28}$$

where A_0 and A_n (for $n \geq 1$) are determined by (2.4.23) and (2.4.24) with the initial condition $f(x) = 5x$,

$$A_0 = \frac{1}{L}\int_0^L 5x\,dx = \frac{5}{L}\frac{1}{2}x^2\Big|_0^L = \frac{5L}{2},$$

and

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L 5x\cos\frac{n\pi x}{L}\,dx \\ &= \frac{10}{L}\left(\frac{Lx}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L - \int_0^L \frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx\right) \\ &= \frac{10}{L}\frac{L^2}{n^2\pi^2}(\cos n\pi - 1) = \frac{10L}{n^2\pi^2}\left((-1)^n - 1\right). \end{aligned}$$

We used integration by parts and $\sin n\pi = 0$ in the evaluation of A_n. Thus, the solution to this heat conduction problem is

$$u(x,\,t) = \frac{5L}{2} + \sum_{n=1}^{\infty}\frac{10L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\frac{n\pi x}{L}e^{-(n\pi/L)^2 kt}.$$

It is now easy to confirm that the steady-state temperature is

$$\lim_{t\to\infty} u(x,\,t) = \frac{5L}{2},$$

which is the average of the initial temperature $f(x) = 5x$ over $0 < x < L$. The solution is plotted in Fig. 2.4.1 for a few different times with parameters $L = 1$ and $k = 1$. It is apparent for $t > 0$ that $\partial u/\partial x = 0$ at $x = 0$ and $x = 1$, and that the temperature tends towards the average value of 5/2.

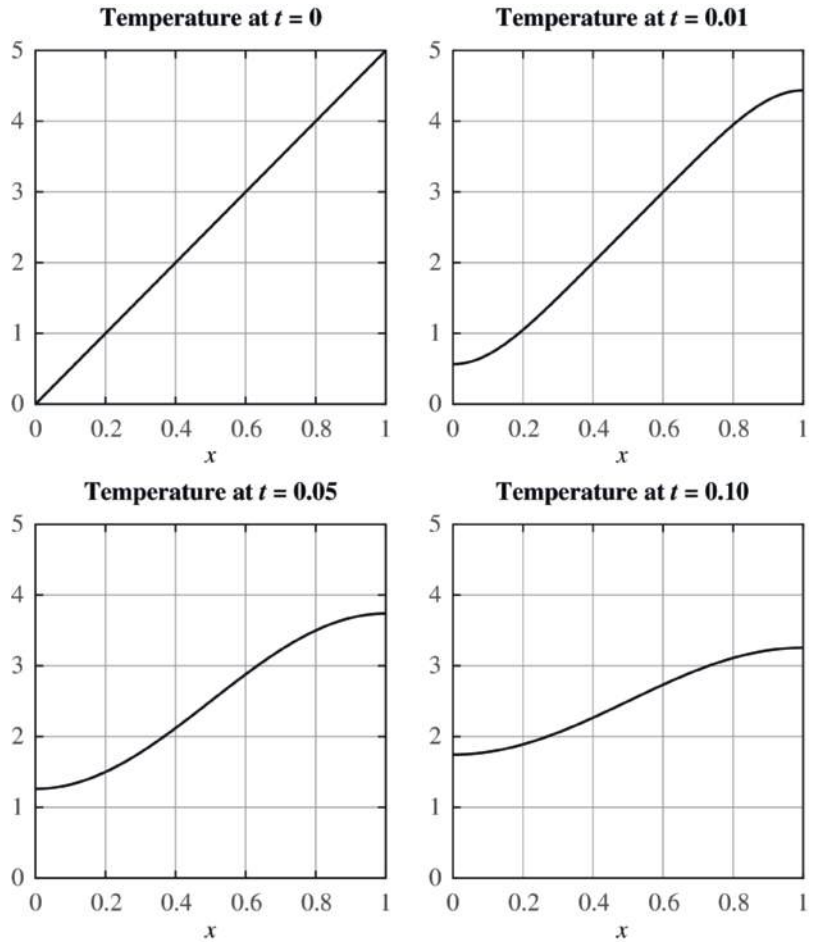


Figure 2.4.1 Solution of the heat equation with insulated ends.

MATLAB Files

Download Animation of Figure 2.4.1 (.zip)

https://bit.ly/447ZjKL

These files are provided as zipped .m files for use in MATLAB on desktop platform.

2.4.2 HEAT CONDUCTION IN A THIN INSULATED CIRCULAR RING

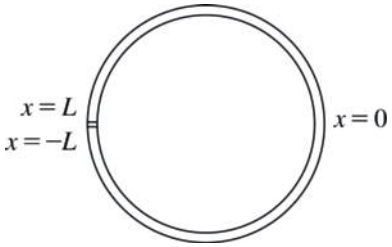
We have investigated a heat flow problem whose eigenfunctions are sines and one whose eigenfunctions are cosines. In this subsection, we illustrate a heat flow problem whose eigenfunctions are both sines *and* cosines.

Let us formulate the appropriate initial boundary value problem if a thin wire (with lateral sides insulated) is bent into the shape of a circle, as illustrated in Fig. 2.4.1. For reasons that will not be apparent for a while, we let the wire have length $2L$ (rather than L, as for the two previous heat conduction problems). Since the circumference of a circle is $2\pi r$, the radius is $r = 2L/2\pi = L/\pi$. If the wire is thin enough, it is reasonable to assume that the temperature in the wire is constant along cross sections of the bent wire. In this situation, the wire should satisfy a one-dimensional heat equation, where the distance is actually the arc length x along the wire:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.} \tag{2.4.29}$$

We have assumed that the wire has constant thermal properties and no sources. It is convenient in this problem to measure the arc length x, such that x ranges from $-L$ to $+L$ (instead of the more usual $+0$ to $2L$).

Figure 2.4.2 Thin circular ring.



Let us assume that the wire is very tightly connected to itself at the ends ($x = -L$ to $x = +L$). The conditions of perfect thermal contact should hold there (see Exercise 1.3.2). The temperature $u(x, t)$ is continuous there:

$$\boxed{u(-L, t) = u(L, t).} \tag{2.4.30}$$

Also, since the heat flux must be continuous there (and the thermal conductivity is constant everywhere), the derivative of the temperature is also continuous:

$$\boxed{\frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t).} \tag{2.4.31}$$

The two boundary conditions for the partial differential equation are (2.4.30) and (2.4.31). The initial condition is that the initial temperature is a given function of the position along the wire:

$$\boxed{u(x, 0) = f(x).} \tag{2.4.32}$$

The mathematical problem consists of the linear homogeneous PDE (2.4.29) subject to linear homogeneous BCs (2.4.30), (2.4.31). As such, we will proceed in the usual way to apply the method of separation of variables. Product solutions $u(x, t) = \phi(x)G(t)$ for the heat equation have been obtained previously, where $G(t) = ce^{-\lambda kt}$. The corresponding boundary value problem is

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi} \tag{2.4.33}$$

$$\boxed{\phi(-L) = \phi(L)} \tag{2.4.34}$$

$$\boxed{\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L).} \tag{2.4.35}$$

The boundary conditions (2.4.34) and (2.4.35) each involve both boundaries (sometimes called the **mixed** type). The specific boundary conditions (2.4.34) and (2.4.35) are referred to as **periodic boundary conditions** since although the problem can be thought of physically as being defined only for $-L < x < L$, it is often thought of as being defined periodically for all x; the temperature will be periodic ($x = x_0$ is the same physical point as $x = x_0 + 2L$, and hence must have the same temperature).

If $\lambda > 0$, the general solution of (2.4.33) is again

$$\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x.$$

The boundary condition $\phi(-L) = \phi(L)$ implies that

$$c_1 \cos\sqrt{\lambda}(-L) + c_2 \sin\sqrt{\lambda}(-L) = c_1 \cos\sqrt{\lambda}L + c_2 \sin\sqrt{\lambda}L.$$

Since cosine is an even function, $\cos\sqrt{\lambda}(-L) = \cos\sqrt{\lambda}L$, and since sine is an odd function, $\sin\sqrt{\lambda}(-L) = -\sin\sqrt{\lambda}L$, it follows that $\phi(-L) = \phi(L)$ is satisfied only if

$$c_2 \sin\sqrt{\lambda}L = 0. \tag{2.4.36}$$

Before solving (2.4.36), we analyze the second boundary condition, which involves the derivative,

$$\frac{d\phi}{dx} = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x\right).$$

Thus, $d\phi/dx(-L) = d\phi/dx(L)$ is satisfied only if

$$c_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0, \tag{2.4.37}$$

where the evenness of cosines and the oddness of sines have again been used. Conditions (2.4.36) and (2.4.37) are easily solved. If $\sin\sqrt{\lambda}L \neq 0$, then $c_1 = 0$ and $c_2 = 0$, which is just the trivial solution. Thus, for nontrivial solutions,

$$\sin\sqrt{\lambda}L = 0,$$

which determines the eigenvalues λ. We find (as before) that $\sqrt{\lambda}L = n\pi$ or, equivalently, that

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3 \ldots . \tag{2.4.38}$$

We chose the wire to have length $2L$ so that the eigenvalues have the same formula as before. (This will mean less to remember, as all our problems have a similar answer.) However, in this problem (unlike the others), there are no additional constraints that c_1 and c_2 must satisfy. Both are arbitrary. We say that both $\sin n\pi x/L$ and $\cos n\pi x/L$ are eigenfunctions corresponding to the eigenvalue $\lambda = (n\pi/L)^2$.

In fact, any linear combination of $\cos n\pi x/L$ and $\sin n\pi x/L$ is an eigenfunction:

$$\phi(x) = c_1 \cos\frac{n\pi x}{L} + c_2 \sin\frac{n\pi x}{L}, \tag{2.4.39}$$

but this is always to be understood when the statement is made that both are eigenfunctions. There are thus two infinite families of product solutions of the partial differential equation, $n = 1, 2, 3 \ldots$,

$$u(x, t) = \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \quad \text{and } u(x, t) = \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}. \tag{2.4.40}$$

All of these correspond to $\lambda > 0$.

If $\lambda = 0$, the general solution of (2.4.33) is

$$\phi = c_1 + c_2 x.$$

The boundary condition $\phi(-L) = \phi(L)$ implies that

$$c_1 - c_2 L = c_1 + c_2 L.$$

Thus, $c_2 = 0$, $\phi(x) = c_1$ and $d\phi/dx = 0$. The remaining boundary condition, (2.4.34), is automatically satisfied. We see that

$$\phi(x) = c_1.$$

Any constant is an eigenfunction, corresponding to the eigenvalue zero. Sometimes we say that $\phi(x) = 1$ is the eigenfunction, since it is known that any multiple of an eigenfunction is always an eigenfunction. Product solutions $u(x, t)$ are also constants in this case. Note that there is only one

independent eigenfunction corresponding to $\lambda = 0$, while for each positive eigenvalue in this problem, $\lambda = (n\pi/L)^2$, there are two independent eigenfunctions, $\sin n\pi x/L$ and $\cos n\pi x/L$. Not surprisingly, it can be shown that there are no eigenvalues in which $\lambda < 0$.

The principle of superposition must be used before applying the initial condition. The most general solution obtainable by the method of separation of variables consists of an arbitrary linear combination of all product solutions:

$$\boxed{u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}.} \tag{2.4.41}$$

The constant a_0 is the product solution corresponding to $\lambda = 0$, whereas two families of arbitrary coefficients, a_n and b_n, are needed for $\lambda > 0$. The initial condition $u(x, 0) = f(x)$ is satisfied if

$$\boxed{f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.} \tag{2.4.42}$$

Here the function $f(x)$ is a linear combination of both sines and cosines (and a constant), unlike the previous problems, where either sines or cosines (including the constant term) were used. Another crucial difference is that (2.4.42) should be valid for the entire ring, which means that $-L \le x \le L$, whereas the series of just sines or cosines was valid for $0 \le x \le L$. The theory of Fourier series will show that (2.4.42) is valid and, more important, that the previous series of just sines or cosines are but special cases of the series in (2.4.42).

For now, we wish just to determine the coefficients a_0, a_n, and b_n from (2.4.42). Again the eigenfunctions form an orthogonal set since integral tables verify the following orthogonality conditions:

$$\boxed{\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}} \tag{2.4.43}$$

$$\boxed{\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}} \tag{2.4.44}$$

$$\boxed{\int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = 0,} \tag{2.4.45}$$

where n and m are arbitrary (nonnegative) integers. The constant eigenfunction corresponds to $n = 0$ or $m = 0$. Integrals of the square of sines or cosines ($n = m$) are evaluated again by the "half the length of the interval" rule. The last of these formulas, (2.4.45), is particularly simple to derive, since sine is an odd function and cosine is an even function.[4] Note that, for example, $\cos n\pi x/L$ is orthogonal to every *other* eigenfunction [sines from (2.4.45), cosines and the constant eigenfunction from (2.4.43)].

The coefficients are derived in the same manner as before. A few steps are saved by noting (2.4.42) is equivalent to

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

If we multiply this by both $\cos m\pi x/L$ and $\sin m\pi x/L$ and then integrate from $x = -L$ to $x = L$, we obtain

$$\int_{-L}^{L} f(x) \begin{Bmatrix} \cos\dfrac{m\pi x}{L} \\ \sin\dfrac{m\pi x}{L} \end{Bmatrix} dx = \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos\frac{n\pi x}{L} \begin{Bmatrix} \cos\dfrac{m\pi x}{L} \\ \sin\dfrac{m\pi x}{L} \end{Bmatrix} dx + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\frac{n\pi x}{L} \begin{Bmatrix} \cos\dfrac{m\pi x}{L} \\ \sin\dfrac{m\pi x}{L} \end{Bmatrix} dx.$$

[4]The product of an odd and an even function is odd. By antisymmetry, the integral of an odd function over a symmetric interval is zero.

If we utilize (2.4.43)–(2.4.45), we find that

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = a_m\int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,dx$$

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = b_m\int_{-L}^{L} f(x)\sin^2\frac{m\pi x}{L}\,dx.$$

Solving for the coefficients in a manner that we are now familiar with yields

$$(m \geq 1)\quad \boxed{\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx. \end{aligned}} \tag{2.4.46}$$

The solution to the problem is (2.4.41), where the coefficients are given by (2.4.46).

2.4.3 SUMMARY OF BOUNDARY VALUE PROBLEMS

In many problems, including the ones we have just discussed, the specific simple constant-coefficient differential equation,

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

forms the fundamental part of the boundary value problem. We collect in the table in one place the relevant formulas for the eigenvalues and eigenfunctions for the typical boundary conditions already discussed. You will find it helpful to understand these results because of their enormous applicability throughout this text. It is important to note that, in these cases, whenever $\lambda = 0$ is an eigenvalue, the eigenfunction is a constant (corresponding to $n = 0$ in $\cos n\pi x/L$). For closed-book examinations, instructors might find it useful to provide Table 2.4.1, though it may be helpful for students just to memorize all cases.

TABLE 2.4.1 Boundary Value Problems for $\dfrac{d^2\phi}{dx^2} = -\lambda\phi$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\dfrac{d\phi}{dx}(0) = 0$ $\dfrac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\dfrac{n\pi x}{L}$	$\cos\dfrac{n\pi x}{L}$	$\sin\dfrac{n\pi x}{L}$ and $\cos\dfrac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\dfrac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\dfrac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\dfrac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{L}$
Coefficients	$B_n = \dfrac{2}{L}\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx$	$A_0 = \dfrac{1}{L}\int_0^L f(x)\,dx$ $A_n = \dfrac{2}{L}\int_0^L f(x)\cos\dfrac{n\pi x}{L}\,dx$	$a_0 = \dfrac{1}{2L}\int_{-L}^{L} f(x)\,dx$ $a_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\cos\dfrac{n\pi x}{L}\,dx$ $b_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\sin\dfrac{n\pi x}{L}\,dx$

Exercises 2.4

2.4.1. Determine the eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$ for the problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \phi(0) = 0, \qquad \frac{d\phi}{dx}(5) = 0.$$

2.4.2. Determine the eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$ for the problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \frac{d\phi}{dx}(0) = 0, \qquad \phi(7) = 0.$$

2.4.3. Determine the eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$ for the problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \qquad \frac{d\phi}{dx}(0) = 0, \qquad \frac{d\phi}{dx}(3) = 0.$$

2.4.4. Consider the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad 0 < x < L,$$

$$\text{subject to } \phi(0) = 0, \text{ and } \frac{d\phi}{dx}(L) = 0.$$

(a) Show the eigenfunctions are of the form

$$\phi_n(x) = \cos\left(\frac{(2n-1)\pi x}{2L}\right) \text{ for } n = 1, 2, 3, \ldots.$$

(b) Use the identity $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$ to show that these eigenfunctions satisfy

$$\int_0^L \phi_n(x)\,\phi_m(x)\,dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

2.4.5. Consider the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad 0 < x < L,$$

$$\text{subject to } \frac{d\phi}{dx}(0) = 0, \text{ and } \phi(L) = 0.$$

(a) Show the eigenfunctions are of the form

$$\phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right) \text{ for } n = 1, 2, 3, \ldots.$$

(b) Use the identity $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ to show that these eigenfunctions satisfy

$$\int_0^L \phi_n(x)\phi_m(x)\,dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

2.4.6. Solve the heat conduction problem using separation of variables.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 9,\ t > 0,$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \qquad \frac{\partial u}{\partial x}(9, t) = 0, \quad u(x, 0) = 8 + 5x.$$

***2.4.7.** Solve the heat equation $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, subject to

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad t > 0$$

$$\frac{\partial u}{\partial x}(L, t) = 0 \quad t > 0.$$

(a) $u(x, 0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$

(b) $u(x, 0) = 6 + 4\cos\dfrac{3\pi x}{L}$

(c) $u(x, 0) = -2\sin\dfrac{\pi x}{L}$

(d) $u(x, 0) = -3\cos\dfrac{8\pi x}{L}$

(e) $u(x, 0) = 4x$

(f) $u(x, 0) = 8x^2$

2.4.8. Use the result of Exercise 2.4.2 to solve

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, & 0 < x < 3, \quad t > 0, \\ \frac{\partial u}{\partial x}(0, t) &= 0, & u(3, t) = 0, \quad t > 0, \\ u(x, 0) &= 5\cos(\pi x/2) - 7\cos(5\pi x/6), & 0 < x < 3. \end{aligned}$$

2.4.9. Use the result of Exercise 2.4.1 to solve

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, & 0 < x < 1/2, \quad t > 0, \\ u(0, t) &= 0, & \frac{\partial u}{\partial x}(1/2, t) = 0, \quad t > 0, \\ u(x, 0) &= 7x, & 0 < x < 1/2. \end{aligned}$$

***2.4.10.** Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

subject to

$$\phi(0) = \phi(2\pi) \quad \text{and} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

2.4.11. Explicitly show that there are no negative eigenvalues for $\frac{d^2\phi}{dx^2} = -\lambda\phi$ subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$.

2.4.12. This problem presents an alternative derivation of the heat equation for a thin wire. The equation for a circular wire of finite thickness is the two-dimensional heat equation (in polar coordinates). Show that this reduces to (2.4.25) if the temperature does not depend on r and if the wire is very thin.

2.4.13. Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:

(a) directly from the equilibrium problem (see Section 1.4)

(b) by computing the limit as $t \to \infty$ of the time-dependent problem

2.4.14. Use Table 2.4.1 and the results of Exercises 2.4.1 and 2.4.2 to solve the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ subject to the following conditions:

(a) $u(0, t) = 0, u(L, t) = 0, u(x, 0) = f(x)$

(b) $u(0, t) = 0, \frac{\partial u}{\partial x}(L, t) = 0, u(x, 0) = f(x)$

(c) $\frac{\partial u}{\partial x}(0, t) = 0, u(L, t) = 0, u(x, 0) = f(x)$

(d) $\frac{\partial u}{\partial x}(0, t) = 0, \frac{\partial u}{\partial x}(L, t) = 0, u(x, 0) = f(x)$

2.5 Laplace's Equation: Solutions and Qualitative Properties

2.5.1 LAPLACE'S EQUATION INSIDE A RECTANGLE

In order to obtain more practice, we consider a different kind of problem that can be analyzed by the method of separation of variables. We consider steady-state heat conduction in a two-dimensional region. To be specific, consider the equilibrium temperature inside a rectangle $(0 \le x \le L, 0 \le y \le H)$ when the temperature is a prescribed function of position (independent of time) on the boundary. The equilibrium temperature $u(x, y)$ satisfies Laplace's equation with the following boundary conditions:

$$\text{PDE:} \quad \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \tag{2.5.1}$$

$$\text{BC1:} \quad \boxed{u(0, y) = g_1(y)} \tag{2.5.2}$$

$$\text{BC2:} \quad \boxed{u(L, y) = g_2(y)} \tag{2.5.3}$$

$$\text{BC3:} \quad \boxed{u(x, 0) = f_1(x)} \tag{2.5.4}$$

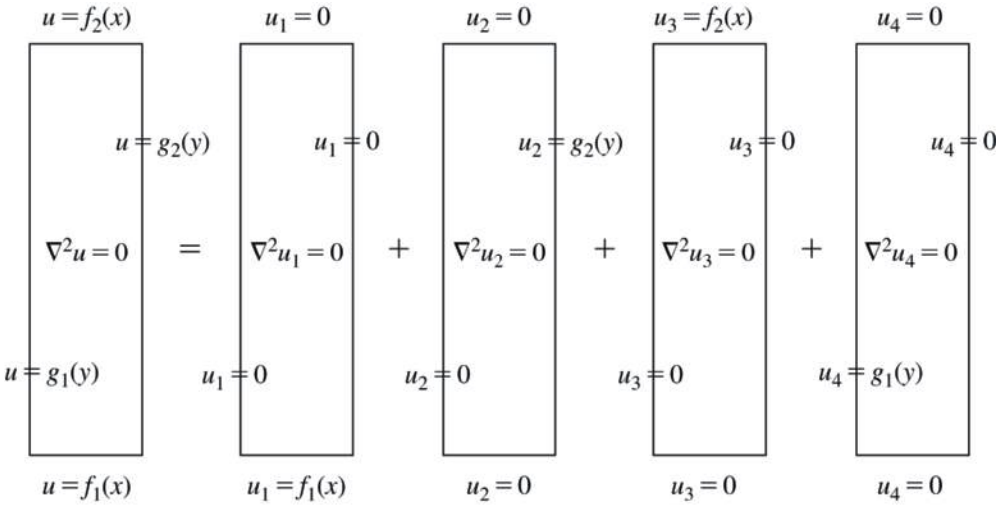
$$\text{BC4:} \quad \boxed{u(x, H) = f_2(x),} \tag{2.5.5}$$

where $f_1(x)$, $f_2(x)$, $g_1(y)$, and $g_2(y)$ are given functions of x and y, respectively. Here the partial differential equation is linear and homogeneous, but the boundary conditions, although linear, are not homogeneous. We will not be able to apply the method of separation of variables to this problem in its present form, because when we separate variables, the boundary value problem (determining the separation constant) must have homogeneous boundary conditions. In this example, all the boundary conditions are nonhomogeneous. We can get around this difficulty by noting that the original problem is nonhomogeneous due to the four nonhomogeneous boundary conditions. The idea behind the principle of superposition can be used sometimes for nonhomogeneous problems (see Exercise 2.2.5). We break our problem into four problems, each having one nonhomogeneous condition. We let

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y), \tag{2.5.6}$$

where each $u_i(x, y)$ satisfies Laplace's equation with one nonhomogeneous boundary condition and the related three homogeneous boundary conditions, as diagrammed in Fig. 2.5.1. Instead of directly solving for u, we will indicate how to solve for u_1, u_2, u_3, and u_4. Why does the sum satisfy our problem? We check to see that the PDE and the four nonhomogeneous BCs will be satisfied. Since u_1, u_2, u_3, and u_4 satisfy Laplace's equation, which is linear and homogeneous, $u \equiv u_1 + u_2 + u_3 + u_4$ also satisfies the same linear and homogeneous PDE by the principle of superposition. At $x = 0$: $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, and $u_4 = g_1(y)$. Therefore, at $x = 0$: $u = u_1 + u_2 + u_3 + u_4 = g_1(y)$, the desired non-homogeneous condition. In a similar manner we can check that all four nonhomogeneous conditions have been satisfied.

Figure 2.5.1 Laplace's equation inside a rectangle.



The method to solve for any of the $u_i(x, y)$ is the same; only certain details differ. We will solve only for $u_4(x, y)$ and will leave the rest for the Exercises:

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0} \tag{2.5.7}$$

$$\text{BC1:} \quad \boxed{u_4(0, y) = g_1(y)} \tag{2.5.8}$$

$$\text{BC2:} \quad \boxed{u_4(L, y) = 0} \tag{2.5.9}$$

$$\text{BC3:} \quad \boxed{u_4(x, 0) = 0} \tag{2.5.10}$$

$$\text{BC4:} \quad \boxed{u_4(x, H) = 0.} \tag{2.5.11}$$

We propose to solve this problem by the method of separation of variables. We begin by ignoring the nonhomogeneous condition $u_4(0, y) = g_1(y)$. Eventually, we will add together product solutions to synthesize $g_1(y)$. We look for product solutions

$$u_4(x, y) = h(x)\,\phi(y). \tag{2.5.12}$$

From the three homogeneous boundary conditions, we see that

$$h(L) = 0 \tag{2.5.13}$$

$$\phi(0) = 0 \tag{2.5.14}$$

$$\phi(H) = 0. \tag{2.5.15}$$

The y-dependent solution $\phi(y)$ has two homogeneous boundary conditions and will become an eigenvalue problem in y, whereas the x-dependent solution $h(x)$ has only one homogeneous boundary condition. If (2.5.12) is substituted into Laplace's equation, we obtain

$$\phi(y)\frac{d^2h}{dx^2} + h(x)\frac{d^2\phi}{dy^2} = 0.$$

The variables can be separated by dividing by $h(x)\,\phi(y)$, so that

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2}. \tag{2.5.16}$$

The left-hand side is a function only of x, while the right-hand side is a function only of y. Both must equal a separation constant. Do we want to use $-\lambda$ or λ? One will be more convenient. If the separation constant is negative (as it was before), (2.5.16) implies that $h(x)$ oscillates and $\phi(y)$ is composed of exponentials. This seems doubtful, since the homogeneous boundary conditions (2.5.13)–(2.5.15) show that the y-dependent solution satisfies two homogeneous conditions: $\phi(y)$ must be zero at $y = 0$ and at $y = H$. Exponentials in y are not expected to work. On the other hand, if the separation constant is positive, (2.5.16) implies that $h(x)$ is composed of exponentials and $\phi(y)$ oscillates. This seems more reasonable, and we thus introduce the separation constant λ (but we do *not* assume $\lambda \geq 0$):

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \lambda. \tag{2.5.17}$$

This results in two ordinary differential equations:

$$\frac{d^2h}{dx^2} = \lambda h$$

$$\frac{d^2\phi}{dy^2} = -\lambda\phi.$$

The x-dependent problem is *not* a boundary value problem, since it does not have two homogeneous boundary conditions:

$$\boxed{\frac{d^2h}{dx^2} = \lambda h} \tag{2.5.18}$$

$$\boxed{h(L) = 0.} \tag{2.5.19}$$

However, the y-dependent problem is a boundary value problem and will be used to determine the eigenvalues λ (separation constants):

$$\boxed{\frac{d^2\phi}{dy^2} = -\lambda\phi} \tag{2.5.20}$$

$$\boxed{\phi(0) = 0} \tag{2.5.21}$$

$$\boxed{\phi(H) = 0.} \tag{2.5.22}$$

This boundary value problem is one that has arisen before, but here the length of the interval is H. All the eigenvalues are positive, $\lambda > 0$. The eigenfunctions are clearly sines, since $\phi(0) = 0$. Furthermore, the condition $\phi(H) = 0$ implies that

$$\boxed{\begin{aligned}\lambda &= \left(\frac{n\pi}{H}\right)^2 \\ \phi(y) &= \sin\frac{n\pi y}{H}\end{aligned}} \qquad n = 1, 2, 3, \ldots . \tag{2.5.23}$$

To obtain product solutions we now must solve (2.5.18) with (2.5.19). Since $\lambda = (n\pi/H)^2$,

$$\frac{d^2h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 h. \tag{2.5.24}$$

The general solution is a linear combination of exponentials or a linear combination of hyperbolic functions. Either can be used, but neither is particularly suited for solving the homogeneous boundary condition $h(L) = 0$. We can obtain our solution more expeditiously if we note that both $\cosh n\pi(x - L)/H$ and $\sinh n\pi(x - L)/H$ are linearly independent solutions of (2.5.24). The general solution can be written as a linear combination of these two:

$$h(x) = a_1 \cosh\frac{n\pi}{H}(x - L) + a_2 \sinh\frac{n\pi}{H}(x - L), \tag{2.5.25}$$

although it should now be clear that $h(L) = 0$ implies that $a_1 = 0$ (since $\cosh 0 = 1$ and $\sinh 0 = 0$). As we could have guessed originally,

$$h(x) = a_2 \sinh\frac{n\pi}{H}(x - L). \tag{2.5.26}$$

The reason (2.5.25) is the solution (besides the fact that it solves the DE) is that it is a simple translation of the more familiar solution, $\cosh n\pi x/L$ and $\sinh n\pi x/L$. We are allowed to translate solutions of differential equations only if the differential equation does not change (said to be *invariant*) upon translation. Since (2.5.24) has constant coefficients, thinking of the origin being at $x = L$ (namely, $x' = x - L$) does not affect the differential equation, since $d^2h/dx'^2 = (n\pi/H)^2 h$ according to the chain rule. For example, $\cosh n\pi x'/H = \cosh n\pi(x - L)/H$ is a solution.

Product solutions are

$$u_4(x, y) = A \sin\frac{n\pi y}{H} \sin\mathrm{h}\frac{n\pi}{H}(x - L). \tag{2.5.27}$$

You might now check that Laplace's equation is satisfied as well as the three required homogeneous conditions. It is interesting to note that one part (the y) oscillates and the other (the x) does not. This is a general property of Laplace's equation, not restricted to this geometry (rectangle) or to these boundary conditions.

We want to use these product solutions to satisfy the remaining condition, the nonhomogeneous boundary condition $u_4(0, y) = g_1(y)$. Product solutions do *not* satisfy nonhomogeneous conditions. Instead, we again use the principle of superposition. If (2.5.27) is a solution, so is

$$\boxed{u_4(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{H} \sin\mathrm{h}\frac{n\pi}{H}(x - L).} \tag{2.5.28}$$

Evaluating at $x = 0$ will determine the coefficients A_n from the nonhomogeneous boundary condition:

$$g_1(y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi}{H}(-L).$$

This is the same kind of series of sine functions we have already briefly discussed, if we associate $A_n \sinh n\pi(-L)/H$ as its coefficients. Thus (by the orthogonality of $\sin n\pi y/H$ for y between 0 and H),

$$A_n \sinh\frac{n\pi}{H}(-L) = \frac{2}{H}\int_0^H g_1(y) \sin\frac{n\pi y}{H}\, dy.$$

Since $\sinh n\pi(-L)/H$ is never zero, we can divide by it and obtain finally a formula for the coefficients:

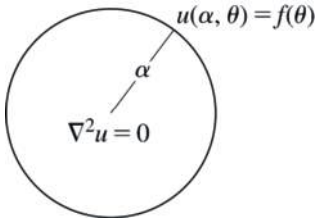
$$\boxed{A_n = \frac{2}{H \sinh n\pi(-L)/H} \int_0^H g_1(y) \sin \frac{n\pi y}{H}\, dy.} \tag{2.5.29}$$

Equation (2.5.28) with coefficients determined by (2.5.29) is the solution only for $u_4(x, y)$. The original $u(x, y)$ is obtained by adding together four such solutions.

2.5.2 LAPLACE'S EQUATION INSIDE A CIRCULAR DISK

Suppose that we had a thin circular disk of radius a (with constant thermal properties and no sources) with the temperature prescribed on the boundary, as illustrated in Fig. 2.5.2. If the temperature on the boundary is independent of time, then it is reasonable to determine the equilibrium temperature distribution. The temperature satisfies Laplace's equation, $\nabla^2 u = 0$. The geometry of this problem suggests that we use polar coordinates, so that $u = u(r, \theta)$. In particular, on the circle, $r = a$, the temperature distribution is a prescribed function of θ, $u(a, \theta) = f(\theta)$. The problem we want to solve is

Figure 2.5.2 Laplace's equation inside a circular disk.



$$\text{PDE:} \quad \boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0} \tag{2.5.30}$$

$$\text{BC:} \quad \boxed{u(a, \theta) = f(\theta).} \tag{2.5.31}$$

At first glance it would appear that we cannot use separation of variables because there are no homogeneous subsidiary conditions. However, the introduction of polar coordinates requires some discussion that will illuminate the use of the method of separation of variables. If we solve Laplace's equation on a rectangle (see Section 2.5.1), $0 \le x \le L$, $0 \le y \le H$, then conditions are necessary at the endpoints of definition of the variables: $x = 0, L$ and $y = 0, H$. Fortunately, these coincide with the physical boundaries. However, for polar coordinates, $0 \le r \le a$ and $-\pi \le \theta \le \pi$ (where there is some freedom in our definition of the angle θ). Mathematically, we need conditions at the endpoints of the coordinate system, $r = 0, a$ and $\theta = -\pi, \pi$. Here, only $r = a$ corresponds to a physical boundary. Thus, we need conditions motivated by considerations of the physical problem at $r = 0$ and at $\theta = \pm\pi$. Polar coordinates are singular at $r = 0$; for physical reasons, we will prescribe that the temperature is finite or, equivalently, **bounded** there:

$$\text{boundedness at origin} \quad \boxed{|u(0, \theta)| < \infty.} \tag{2.5.32}$$

Conditions are needed at $\theta = \pm\pi$ for mathematical reasons. It is similar to the circular wire situation. $\theta = -\pi$ corresponds to the same points as $\theta = \pi$. Although there really is not a boundary, we say that the temperature is continuous there and the heat flow in the θ-direction is continuous, which imply

$$\text{periodicity} \quad \boxed{\begin{array}{l} u(r, -\pi) = u(r, \pi) \\ \dfrac{\partial u}{\partial \theta}(r, -\pi) = \dfrac{\partial u}{\partial \theta}(r, \pi), \end{array}} \tag{2.5.33}$$

as though the two regions were in perfect thermal contact there (see Exercise 1.3.2). Equations (2.5.33) are called **periodicity conditions**; they are equivalent to $u(r, \theta) = u(r, \theta + 2\pi)$. We note that subsidiary conditions (2.5.32) and (2.5.33) are all linear and homogeneous (it's easy to check that $u \equiv 0$ satisfies these three conditions). In this form, the mathematical problem appears somewhat similar to Laplace's equation inside a rectangle. There are four conditions. Here, fortunately, only one is nonhomogeneous, $u(a, \theta) = f(\theta)$. This problem is thus suited for the method of separation of variables.

We look for special product solutions,

$$u(r, \theta) = \phi(\theta)G(r), \tag{2.5.34}$$

that satisfy the PDE (2.5.30) and the three homogeneous conditions (2.5.32) and (2.5.33). Note that (2.5.34) does *not* satisfy the nonhomogeneous boundary condition (2.5.31). Substituting (2.5.34) into the periodicity conditions shows that

$$\begin{aligned} \phi(-\pi) &= \phi(\pi) \\ \frac{d\phi}{d\theta}(-\pi) &= \frac{d\phi}{d\theta}(\pi); \end{aligned} \tag{2.5.35}$$

the θ-dependent part also satisfies the **periodic boundary conditions**. Thus, $\phi(\theta)$ will satisfy an eigenvalue problem, where θ is the angular variable. The product form will satisfy Laplace's equation if

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0.$$

The variables are *not* separated by dividing by $G(r)\phi(\theta)$ since $1/r^2$ remains multiplying the θ-dependent terms. Instead, divide by $(1/r^2)G(r)\phi(\theta)$, in which case

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda. \tag{2.5.36}$$

The separation constant is introduced as λ (rather than $-\lambda$) since there are two homogeneous conditions in θ, (2.5.35), and we therefore expect oscillations in θ. Equation (2.5.36) yields two ordinary differential equations. The boundary value problem to determine the separation constant is

$$\boxed{\begin{array}{c} \dfrac{d^2\phi}{d\theta^2} = -\lambda\phi \\ \hline \phi(-\pi) = \phi(\pi) \\ \dfrac{d\phi}{d\theta}(-\pi) = \dfrac{d\phi}{d\theta}(\pi). \end{array}} \tag{2.5.37}$$

The eigenvalues λ are determined in the usual way. In fact, this is one of the three standard problems, the identical problem as for the circular wire (with $L = \pi$). Thus, the eigenvalues are

$$\lambda = \left(\frac{n\pi}{L}\right)^2 = n^2, \tag{2.5.38}$$

with the corresponding eigenfunctions being both

$$\sin n\theta \quad \text{and} \quad \cos n\theta. \tag{2.5.39}$$

The case $n = 0$ must be included (with only a constant being the eigenfunction).

The r-dependent problem is

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda = n^2, \tag{2.5.40}$$

which when written in the more usual form becomes

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2G = 0. \tag{2.5.41}$$

Here, the condition at $r = 0$ has already been discussed. We have prescribed $|u(0, \theta)| < \infty$. For the product solutions, $u(r, \theta) = \phi(\theta)G(r)$, it follows that the condition at the origin is that $G(r)$ must be bounded there,

$$|G(0)| < \infty. \tag{2.5.42}$$

Equation (2.5.41) is linear and homogeneous but has nonconstant coefficients. There are exceedingly few second-order linear equations with nonconstant coefficients that we can solve easily. Equation (2.5.41) is one such case, an example of an equation known by a number of different names: **equidimensional** or **Cauchy** or **Euler**. The simplest way to solve (2.5.41) is to note that for the linear differential operator in (2.5.41), any power $G = r^p$ reproduces itself.[5] On substituting $G = r^p$ into (2.5.41), we determine that $[p(p-1) + p - n^2]r^p = 0$. Thus, there usually are two distinct solutions

$$p = \pm n,$$

[5]For constant-coefficient linear differential operators, exponentials reproduce themselves.

except when $n = 0$, in which case there is only one independent solution in the form r^p. For $n \neq 0$, the general solution of (2.5.41) is

$$G = c_1 r^n + c_2 r^{-n}. \tag{2.5.43}$$

For $n = 0$ (and $n = 0$ is important since $\lambda = 0$ is an eigenvalue in this problem), one solution is $r^0 = 1$ or any constant. A second solution for $n = 0$ is most easily obtained from (2.5.40). If $n = 0$, $\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0$. By integration, $r dG/dr$ is constant, or, equivalently, dG/dr is proportional to $1/r$. The second independent solution is thus $\ln r$. Thus, for $n = 0$, the general solution of (2.5.41) is

$$G = \overline{c}_1 + \overline{c}_2 \ln r. \tag{2.5.44}$$

Equation (2.5.41) has only one homogeneous condition to be imposed, $|G(0)| < \infty$, so it is not an eigenvalue problem. The boundedness condition would not have imposed any restrictions on the problems we have studied previously. However, here (2.5.43) or (2.5.44) shows that solutions may approach ∞ as $r \to 0$. Thus, for $|G(0)| < \infty$, $c_2 = 0$ in (2.5.43) and $\overline{c}_2 = 0$ in (2.5.44). The r-dependent solution (which is bounded at $r = 0$) is

$$G(r) = c_1 r^n, \quad n \geq 0,$$

where for $n = 0$ this reduces to just an arbitrary constant.

Product solutions by the method of separation of variables, which satisfy the three homogeneous conditions, are

$$r^n \cos n\theta \ (n \geq 0) \text{ and } r^n \sin n\theta \ (n \geq 1).$$

Note that as in rectangular coordinates for Laplace's equation, oscillations occur in one variable (here θ) and do not occur in the other variable (r). By the principle of superposition, the following solves Laplace's equation inside a circle:

$$\boxed{u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta,} \qquad \begin{array}{l} 0 \leq r < a \\ -\pi < \theta \leq \pi. \end{array} \tag{2.5.45}$$

In order to solve the nonhomogeneous condition, $u(a, \theta) = f(\theta)$,

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta, \qquad -\pi < \theta \leq \pi. \tag{2.5.46}$$

The prescribed temperature is a linear combination of all sines and cosines (including a constant term, $n = 0$). This is exactly the same question that we answered in Section 2.4.2 with $L = \pi$ if we let $A_n a^n$ be the coefficient of $\cos n\theta$ and $B_n a^n$ be the coefficient of $\sin n\theta$. Using the orthogonality formulas, it follows that

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \\ (n \geq 1) \quad A_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \\ B_n a^n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta. \end{aligned} \tag{2.5.47}$$

Since $a^n \neq 0$, the coefficients A_n and B_n can be uniquely solved for from (2.5.47).

Equation (2.5.45) with coefficients given by (2.5.47) determines the steady-state temperature distribution inside a circle. The solution is relatively complicated, often requiring the numerical evaluation of two infinite series. For additional interpretations of this solution, see Chapter 9, on Green's functions.

2.5.3 FLUID FLOW OUTSIDE A CIRCULAR CYLINDER (LIFT)

In heat flow, conservation of thermal energy can be used to derive Laplace's equation $\nabla^2 u = 0$ under certain assumptions. In fluid dynamics, conservation of mass and conservation of momentum can be used to also derive Laplace's equation:

$$\nabla^2 \psi = 0,$$

in the following way. In the Exercises, it is shown that **conservation of mass** for a fluid along with the assumption of a constant mass density ρ yields

$$\nabla \cdot \boldsymbol{u} = 0, \text{ or in two dimensions } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.5.48}$$

where the **velocity** has x- and y-components $\boldsymbol{u} = (u, v)$. A **stream function** ψ is often introduced that automatically satisfies (2.5.48):

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}. \tag{2.5.49}$$

Often streamlines ($\psi =$ constant) are graphed that will be parallel to the fluid flow. It can be shown that in some circumstances the fluid is irrotational ($\nabla \times \boldsymbol{u} = 0$) so that the stream function satisfies Laplace's equation:

$$\boxed{\nabla^2 \psi = 0.} \tag{2.5.50}$$

The simplest example is a constant flow in the x-direction $\boldsymbol{u} = (U, 0)$, in which case the stream function is $\psi = Uy$, clearly satisfying Laplace's equation.

As a first step in designing airplane wings, scientists have considered the flow around a circular cylinder of radius a. For more details we refer the interested reader to Acheson [1990]. The velocity potential must satisfy Laplace's equation, which as before in polar coordinates is (2.5.30). We will assume that far from the cylinder the flow is uniform, so that as an approximation for large r,

$$\psi \approx Uy = Ur \sin\theta, \tag{2.5.51}$$

since we will use polar coordinates. The boundary condition is that the radial component of the fluid flow must be zero at $r = a$. The fluid flow must be parallel to the boundary, and hence we can assume

$$\psi(a, \theta) = 0. \tag{2.5.52}$$

By separation of variables, including the $n = 0$ case given by (2.5.44),

$$\psi(r, \theta) = c_2 + c_1 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \sin n\theta, \tag{2.5.53}$$

where the $\cos n\theta$ terms could be included (but would vanish). By applying the boundary condition at $r = a$, we find

$$\begin{aligned} c_2 + c_1 \ln a &= 0 \\ A_n a^n + B_n a^{-n} &= 0, \end{aligned}$$

so that

$$\psi(r, \theta) = c_1 \ln \frac{r}{a} + \sum_{n=1}^{\infty} A_n \left(r^n - \frac{a^{2n}}{r^n} \right) \sin n\theta. \tag{2.5.54}$$

In order for the fluid velocity to be approximately a constant at infinity with $\psi \approx Uy = Ur \sin\theta$ for large r, $A_n = 0$ for $n \geq 2$ and $A_1 = U$. Thus,

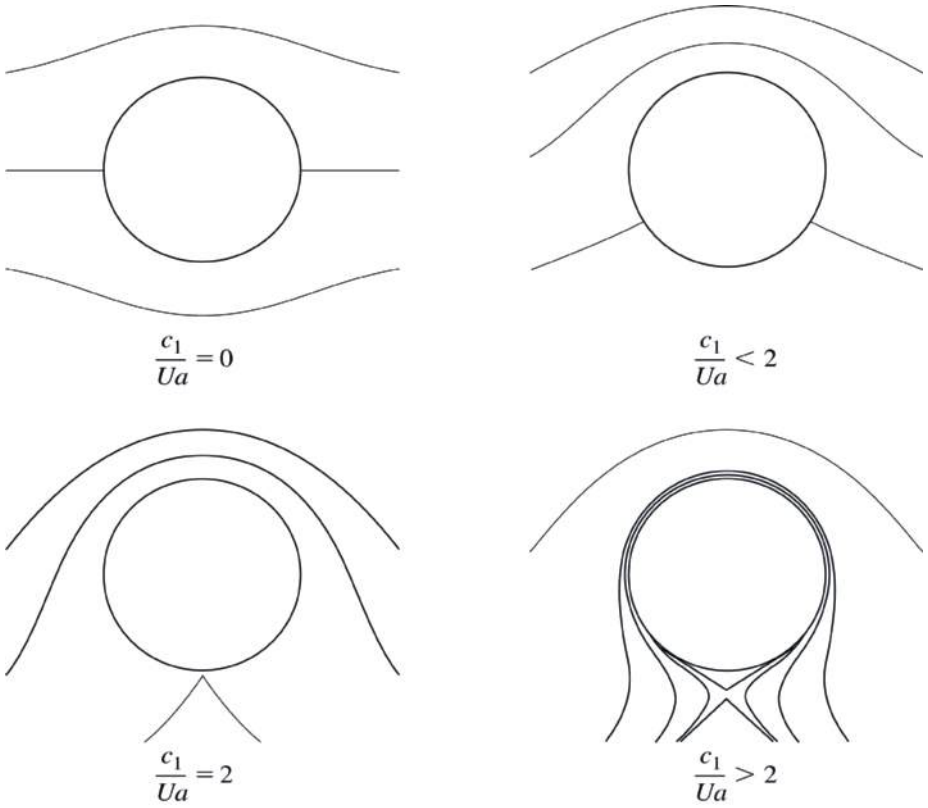
$$\boxed{\psi(r, \theta) = c_1 \ln \frac{r}{a} + U \left(r - \frac{a^2}{r} \right) \sin\theta.} \tag{2.5.55}$$

It can be shown in general that the fluid velocity in polar coordinates can be obtained from the stream function: $u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$, $u_\theta = -\frac{\partial \psi}{\partial r}$. Thus, the θ-component of the fluid velocity is $u_\theta = -\frac{c_1}{r} - U\left(1 + \frac{a^2}{r^2}\right) \sin\theta$. The **circulation** is defined to be $\int_0^{2\pi} u_\theta r \, d\theta = -2\pi c_1$. For a given velocity at infinity, different flows depending on the circulation around a cylinder are illustrated in Fig. 2.5.3.

The **pressure** p of the fluid exerts a force in the direction opposite to the outward normal to the cylinder $\left(\frac{x}{a}, \frac{y}{a}\right) = (\cos\theta, \sin\theta)$. The **drag** ($x$-direction) and **lift** (y-direction) forces (per unit length in the z direction) exerted by the fluid on the cylinder are

$$\boldsymbol{F} = -\int_0^{2\pi} p(\cos\theta, \sin\theta) \, a \, d\theta. \tag{2.5.56}$$

Figure 2.5.3 Flow past cylinder and lift $= 2\pi\rho c_1 U$.



For steady flows such as this one, the pressure is determined from **Bernoulli's condition**

$$p + \frac{1}{2}\rho|u|^2 = \text{constant}. \tag{2.5.57}$$

Thus, the pressure is lower where the velocity is higher. If the circulation is clockwise around the cylinder (a negative circulation), then intuitively (which can be verified) the velocity will be higher above the cylinder than below, and the pressure will be lower on the top of the cylinder, and hence lift (a positive force in the y-direction) will be generated. At the cylinder $u_r = 0$, so that there $|u|^2 = u_\theta^2$. It can be shown that the x-component of the force, the drag, is zero, but the y-component, the lift, is given by (since the integral involving the constant vanishes)

$$F_y = \frac{1}{2}\rho \int_0^{2\pi} \left[-\frac{c_1}{r} - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right]^2 \sin\theta \; a \; d\theta \tag{2.5.58}$$

$$F_y = \rho \frac{c_1}{a} U 2 \int_0^{2\pi} \sin^2\theta \; a \; d\theta = \rho 2\pi c_1 U, \tag{2.5.59}$$

which has been simplified since $\int_0^{2\pi} \sin\theta \, d\theta = \int_0^{2\pi} \sin^3\theta \, d\theta = 0$ due to the oddness of the sine function. The lift vanishes if the circulation is zero. A negative circulation (positive c_1) results in a lift force on the cylinder by the fluid.

In the real world, the drag is more complicated. Boundary layers exist due to the viscous nature of the fluid. The pressure is continuous across the boundary layer so that the preceding analysis is still often valid. However, things get much more complicated when the boundary layer separates from the cylinder, in which case a more substantial drag force occurs (which has been ignored in this elementary treatment). A plane will fly if the lift is greater than the weight of the plane. However, to fly fast, a powerful engine is necessary to apply a force in the x-direction to overcome the drag.

2.5.4 QUALITATIVE PROPERTIES OF LAPLACE'S EQUATION

Sometimes the method of separation of variables will not be appropriate. If quantitative information is desired, numerical methods (see Chapter 6) may be necessary. In this subsection we briefly describe some qualitative properties that may be derived for Laplace's equation.

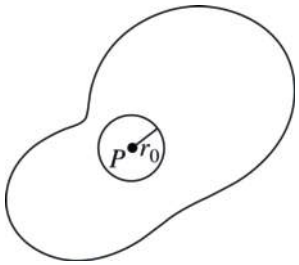
Mean value theorem

Our solution of Laplace's equation inside a circle, obtained in Section 2.5.2 by the method of separation of variables, yields an important result. If we evaluate the temperature at the origin, $r = 0$, we discover from (2.5.45) that

$$u(0, \theta) = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta;$$

the temperature there equals the average value of the temperature at the edges of the circle. This is called the **mean value property** for Laplace's equation. It holds in general in the following specific sense. Suppose that we wish to solve Laplace's equation in any region R (see Fig. 2.5.4). Consider *any* point P inside R and a circle of any radius r_0 (such that the circle is inside R). Let the temperature on the circle be $f(\theta)$, using polar coordinates centered at P. Our previous analysis still holds, and thus **the temperature at any point is the average of the temperature along any circle of radius r_0 (*lying inside R*) centered at that point**.

Figure 2.5.4 Circle within any region.



Maximum principles

We can use this to prove the **maximum principle** for Laplace's equation: **In steady state, assuming no sources the temperature cannot attain its maximum in the interior** (unless the temperature is a constant everywhere). The proof is by contradiction. Suppose that the maximum was at point P, as illustrated in Fig. 2.5.4. However, this should be the average of all points on any circle (consider the circle drawn). It is impossible for the temperature at P to be larger. This contradicts the original assumption, which thus cannot hold. We should not be surprised by the maximum principle. If the temperature was largest at point P, then in time the concentration of heat energy would diffuse and in steady state the maximum could not be in the interior. By letting $\psi = -u$, we can also show that the temperature cannot attain its minimum in the interior. It follows that in **steady state the maximum and minimum temperatures occur on the boundary**.

Well-posedness and uniqueness

The maximum principle is a very important tool for further analysis of partial differential equations, especially in establishing qualitative properties (see, e.g., Protter and Weinberger [1967]). We say that a problem is **well posed** if there exists a unique solution that depends continuously on the nonhomogeneous data (i.e., the solution varies a small amount if the data are slightly changed). This is an important concept for physical problems. If the solution changed dramatically with only a small change in the data, then any physical measurement would have to be exact in order for the solution to be reliable. Fortunately, most standard problems in partial differential equations are well posed. For example, the maximum principle can be used to prove that Laplace's equation $\nabla^2 u = 0$ with u specified as $u = f(\boldsymbol{x})$ on the boundary is well posed.

Suppose that we vary the boundary data a small amount such that

$$\nabla^2 v = 0 \text{ with } v = g(\boldsymbol{x})$$

on the boundary, where $g(\boldsymbol{x})$ is nearly the same as $f(\boldsymbol{x})$ everywhere on the boundary. We consider the difference between these two solutions, $w = u - v$. Due to the linearity,

$$\nabla^2 w = 0 \text{ with } w = f(\boldsymbol{x}) - g(\boldsymbol{x})$$

on the boundary. The maximum (and minimum) principles for Laplace's equation imply that the maximum and minimum occur on the boundary. Thus, at any point inside,

$$\min(f(\boldsymbol{x}) - g(\boldsymbol{x})) \le w \le \max(f(\boldsymbol{x}) - g(\boldsymbol{x})). \tag{2.5.60}$$

Since $g(\boldsymbol{x})$ is nearly the same as $f(\boldsymbol{x})$ everywhere, w is small, and thus the solution v is nearly the same as u; the solution of Laplace's equation slightly varies if the boundary data are slightly altered.

We can also prove that the solution of Laplace's equation is unique. We prove this by contradiction. Suppose that there are two solutions, u and v satisfying Laplace's equation that satisfy the same boundary condition [i.e., let $(f(\boldsymbol{x}) = g(\boldsymbol{x}))$]. If we again consider the difference $(w = u - v)$, then the maximum and minimum principles imply [see (2.5.60)] that inside the region

$$0 \le w \le 0.$$

We conclude that $w = 0$ everywhere inside, and thus $u = v$, proving that if a solution exists, it must be unique. These properties (uniqueness and continuous dependence on the data) show that Laplace's equation with u specified on the boundary is a well-posed problem.

Solvability condition

If on the boundary the heat flow $-K_0 \nabla u \cdot \hat{\mathbf{n}}$ is specified instead of the temperature, Laplace's equation may have no solutions [for a one-dimensional example, see Exercise 1.4.8(b)]. To show this, we integrate $\nabla^2 u = 0$ over the entire two-dimensional region

$$0 = \iint \nabla^2 u \, dx \, dy = \iint \nabla \cdot (\nabla u) \, dx \, dy.$$

Using the (two-dimensional) divergence theorem, we conclude that (see Exercise 1.5.8)

$$0 = \oint \nabla u \cdot \hat{\mathbf{n}} \, ds. \tag{2.5.61}$$

Since $\nabla u \cdot \hat{\mathbf{n}}$ is proportional to the heat flow through the boundary, (2.5.61) implies that the *net* heat flow through the boundary must be zero in order for a steady state to exist. This is clear physically, because otherwise there would be a change (in time) of the thermal energy inside, violating the steady-state assumption. Equation (2.5.61) is called the **solvability condition** or **compatibility condition** for Laplace's equation.

Exercises 2.5

2.5.1. Solve $u(x, y)$ using separation of variables,

$$\nabla^2 u = 0, \qquad 0 < x < 2, \qquad 0 < y < 7,$$

with boundary conditions

$$u(0, y) = 0, \qquad u(2, y) = \sin\left(\frac{4\pi y}{7}\right), \qquad u(x, 0) = 0, \qquad u(x, 7) = 0.$$

2.5.2. Consider Laplace's equation inside a rectangle,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 3, \qquad 0 \le y \le 5,$$

with the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(3, y) = 7, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 5) = A,$$

where A is a constant.

(a) What does the constant A need to be so that a solution to this problem exists?

(b) Assuming A is that constant, find the solution.

2.5.3. Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions [*Hint*: Separate variables. If there are two homogeneous boundary conditions in y, let $u(x, y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in x, let $u(x, y) = \phi(x)h(y)$.]:

***(a)** $\dfrac{\partial u}{\partial x}(0, y) = 0, \quad \dfrac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = f(x)$

(b) $\dfrac{\partial u}{\partial x}(0, y) = g(y), \quad \dfrac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = 0$

***(c)** $\dfrac{\partial u}{\partial x}(0, y) = 0, \quad u(L, y) = g(y), \quad u(x, 0) = 0, \quad u(x, H) = 0$

(d) $u(0, y) = g(y), \quad u(L, y) = 0, \quad \dfrac{\partial u}{\partial y}(x, 0) = 0, \quad u(x, H) = 0$

***(e)** $u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) - \dfrac{\partial u}{\partial y}(x, 0) = 0, \quad u(x, H) = f(x)$

(f) $u(0, y) = f(y), \quad u(L, y) = 0, \quad \dfrac{\partial u}{\partial y}(x, 0) = 0, \quad \dfrac{\partial u}{\partial y}(x, H) = 0$

(g) $\dfrac{\partial u}{\partial x}(0, y) = 0, \quad \dfrac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}, \quad \dfrac{\partial u}{\partial y}(x, H) = 0$

(h) $u(0, y) = 0, \quad u(L, y) = g(y), \quad u(x, 0) = 0, \quad u(x, H) = 0$

2.5.4. Consider $u(x, y)$ satisfying Laplace's equation inside a rectangle $(0 < x < L, 0 < y < H)$ subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0$$
$$\frac{\partial u}{\partial x}(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, H) = f(x).$$

***(a)** *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a). [*Hint*: You may use (2.5.16) without derivation.]

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition

$$u(x, y, 0) = g(x, y).$$

***2.5.5.** Solve Laplace's equation *outside* a circular disk $(r \geq a)$ subject to the boundary condition [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \cdot$]:

(a) $u(a, \theta) = \ln 2 + 4\cos 3\theta$

(b) $u(a, \theta) = f(\theta)$

You may assume that $u(r, \theta)$ remains finite as $r \to \infty$.

***2.5.6.** For Laplace's equation inside a circular disk $(r \leq a)$, using (2.5.45) and (2.5.47), show that

$$u(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\left[-\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \bar{\theta})\right] d\bar{\theta}.$$

Using $\cos z = \mathrm{Re}(e^{iz})$, sum the resulting geometric series to obtain Poisson's integral formula.

2.5.7. Solve Laplace's equation inside the quarter-circle of radius $1 (0 \leq \theta \leq \pi/2,\ 0 \leq r \leq 1)$ subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \cdot$]:

***(a)** $\frac{\partial u}{\partial \theta}(r, 0) = 0,\ u\left(r, \frac{\pi}{2}\right) = 0,\ u(1, \theta) = f(\theta)$

(b) $\frac{\partial u}{\partial \theta}(r, 0) = 0,\ \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{2}\right) = 0,\ u(1, \theta) = f(\theta)$

***(c)** $u(r, 0) = 0,\ u\left(r, \frac{\pi}{2}\right) = 0,\ \frac{\partial u}{\partial r}(1, \theta) = f(\theta)$

(d) $\frac{\partial u}{\partial \theta}(r, 0) = 0,\ \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{2}\right) = 0,\ \frac{\partial u}{\partial r}(1, \theta) = g(\theta)$

Show that the solution [part (d)] exists only if $\int_0^{\pi/2} g(\theta)\, d\theta = 0$. Explain this condition physically.

2.5.8. Solve Laplace's equation inside a semicircle of radius $a (0 < r < a,\ 0 < \theta < \pi)$ subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2}{d\theta^2} = \cdot$]:

***(a)** $u = 0$ on the diameter and $u(a, \theta) = g(\theta)$

(b) the diameter is insulated and $u(a, \theta) = g(\theta)$

2.5.9. Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u(r, 0) = 0, \quad u\left(r, \frac{\pi}{3}\right) = 0, \quad u(a, \theta) = f(\theta)$

***(b)** $\frac{\partial u}{\partial y}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{3}\right) = 0, \quad u(a, \theta) = f(\theta)$

2.5.10. Solve Laplace's equation inside a circular annulus $(a < r < b)$ subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

***(a)** $u(a, \theta) = f(\theta), \quad u(b, \theta) = g(\theta)$

(b) $\frac{\partial u}{\partial r}(a, \theta) = 0, \quad u(b, \theta) = g(\theta)$

(c) $\frac{\partial u}{\partial r}(a, \theta) = f(\theta), \quad \frac{\partial u}{\partial r}(b, \theta) = g(\theta)$

If there is a solvability condition, state it and explain it physically.

***2.5.11.** Solve Laplace's equation inside a 90° sector of a circular annulus $(a < r < b, 0 < \theta < \pi/2)$ subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)\, G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u(r, 0) = 0, \quad u(r, \pi/2) = 0, \quad u(a, \theta) = 0, \quad u(b, \theta) = f(\theta)$

(b) $u(r, 0) = 0, \quad u(r, \pi/2) = f(r), \quad u(a, \theta) = 0, \quad u(b, \theta) = 0$

2.5.12. Using the maximum principles for Laplace's equation, prove that the solution of Poisson's equation, $\nabla^2 u = g(\boldsymbol{x})$, subject to $u = f(\boldsymbol{x})$ on the boundary, is unique.

2.5.13. Do Exercise 1.5.8.

2.5.14. **(a)** Using the divergence theorem, determine an alternative expression for $\iiint u\nabla^2 u \, dx\, dy\, dz$.

(b) Using part (a), prove that the solution of Laplace's equation $\nabla^2 u = 0$ (with u given on the boundary) is unique.

(c) Modify part (b) if $\nabla u \cdot \hat{\mathbf{n}} = 0$ on the boundary.

(d) Modify part (b) if $\nabla u \cdot \hat{\mathbf{n}} + hu = 0$ on the boundary. Show that Newton's law of cooling corresponds to $h > 0$.

2.5.15. Prove that the temperature satisfying Laplace's equation cannot attain its minimum in the interior.

2.5.16. Show that the "backward" heat equation

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0, t) = u(L, t) = 0$ and $u(x, 0) = f(x)$, is *not* well posed. [*Hint*: Show that if the data are changed an arbitrarily small amount, for example,

$$f(x) \to f(x) + \frac{1}{n}\sin\frac{n\pi x}{L}$$

for large n, then the solution $u(x, t)$ changes by a large amount.]

2.5.17. Solve Laplace's equation inside a semi-infinite strip $(0 < x < \infty, 0 < y < H)$ subject to the boundary conditions [*Hint*: In Cartesian coordinates, $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$,

inside a semi-infinite strip ($0 \le y \le H$ and $0 \le x < \infty$), it is known that if $u(x, y) = F(x)\,G(y)$, then $\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2}$.]:

(a) $\dfrac{\partial u}{\partial y}(x, 0) = 0, \qquad \dfrac{\partial u}{\partial y}(x, H) = 0, \qquad u(0, y) = f(y)$

(b) $u(x, 0) = 0, \qquad u(x, H) = 0, \qquad u(0, y) = f(y)$

(c) $u(x, 0) = 0, \qquad u(x, H) = 0, \qquad \dfrac{\partial u}{\partial x}(0, y) = f(y)$

(d) $\dfrac{\partial u}{\partial y}(x, 0) = 0, \qquad \dfrac{\partial u}{\partial y}(x, H) = 0, \qquad \dfrac{\partial u}{\partial x}(0, y) = f(y)$

Show that the solution [part (d)] exists only if $\int_0^H f(y)\,dy = 0$.

2.5.18. Consider Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = g(y), \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = f(x).$$

(a) What is the solvability condition and its physical interpretation?

(b) Show that $u(x, y) = A(x^2 - y^2)$ is a solution if $f(x)$ and $g(y)$ are constants [under the conditions of part (a)].

(c) Under the conditions of part (a), solve the general case [nonconstant $f(x)$ and $g(y)$]. [*Hints*: Use part (b) and the fact that $f(x) = f_{\text{av}} + [f(x) - f_{\text{av}}]$, where $f_{\text{av}} = \dfrac{1}{L}\int_0^L f(x)\,dx$.]

2.5.19. Show that the mass density $\rho(x, t)$ satisfies $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$ due to conservation of mass.

2.5.20. If the mass density is constant, using the result of Exercise 2.5.19, show that $\nabla \cdot \boldsymbol{u} = 0$.

2.5.21. Show that the streamlines are parallel to the fluid velocity.

2.5.22. Show that anytime there is a stream function, $\nabla \times \boldsymbol{u} = -\hat{\mathbf{k}}\nabla^2\psi$.

2.5.23. From $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, derive $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $u_\theta = -\frac{\partial \psi}{\partial r}$.

2.5.24. Show the drag force is zero for a uniform flow past a cylinder including circulation.

2.5.25. Consider the velocity u_θ at the cylinder. Where do the maximum and minimum occur?

2.5.26. Consider the velocity u_θ at the cylinder. If the circulation is negative, show that the velocity will be larger above the cylinder than below.

2.5.27. A stagnation point is a place where $\boldsymbol{u} = 0$. For what values of the circulation does a stagnation point exist on the cylinder?

2.5.28. For what values of θ will $u_r = 0$ off the cylinder? For these θ, where (for what values of r) will $u_\theta = 0$ also?

2.5.29. Show that $\psi = \alpha\frac{\sin\theta}{r}$ satisfies Laplace's equation. Show that the streamlines are circles. Graph the streamlines.

2.5.30. Solve Laplace's equation inside a rectangle:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions

$$u(0, y) = g(y) \qquad u(x, 0) = 0$$
$$u(L, y) = 0 \qquad u(x, H) = 0.$$

2.5.31. Solve Laplace's equation inside a circle of radius a,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

subject to the boundary condition

$$u(a, \theta) = f(\theta).$$

3
Fourier Series

3.1 Introduction

In solving partial differential equations by the method of separation of variables, we have discovered that important conditions [e.g., the initial condition, $u(x,0) = f(x)$] could be satisfied only if $f(x)$ could be equated to an infinite linear combination of eigenfunctions of a given boundary value problem. Three specific cases have been investigated. One yielded a series involving sine functions, one yielded a series of cosines only (including a constant term), and the third yielded a series that included all of these previous terms.

We will begin by investigating series with both sines and cosines, because we will show that the others are just special cases of this more general series. For problems with the periodic boundary conditions on the interval $-L \le x \le L$, we asked whether the following infinite series (known as a **Fourier series**) makes sense:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{3.1.1}$$

Does the infinite series converge? Does it converge to $f(x)$? Is the resulting infinite series really a solution of the partial differential equation (and does it also satisfy all the other subsidiary conditions)? Mathematicians tell us that none of these questions have simple answers. Nonetheless, *Fourier series usually work quite well* (especially in situations where they arise naturally from physical problems). Joseph Fourier developed this type of series in his famous treatise on heat flow in the early 1800s.

The first difficulty that arises is that we claim (3.1.1) will not be valid for *all* functions $f(x)$. However, (3.1.1) will hold for some kinds of functions and will need only a small modification for other kinds of functions. In order to communicate various concepts easily, we will discuss only functions $f(x)$ that are piecewise smooth.

A function $f(x)$ is defined to be **piecewise smooth** on some interval if the interval can be divided into a finite number of subintervals such that

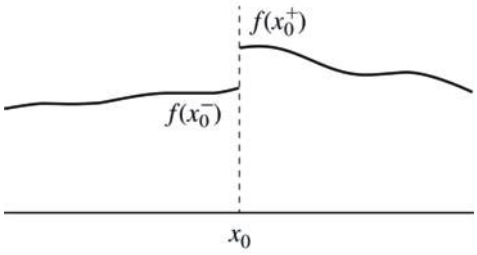
1. Both $f(x)$ and $f'(x)$ are continuous at all points on the interior of the subintervals; and
2. The one-sided limits of $f(x)$ and $f'(x)$ both exist (as finite values) at the boundaries of the subintervals.

Loosely speaking, a function $f(x)$ is piecewise smooth on some interval if both $f(x)$ and its derivative $f'(x)$ are continuous[1] at all but possibly a few points, where $f(x)$ and $f'(x)$ may have jump discontinuities. A function $f(x)$ has a **jump discontinuity** at a point $x = x_0$ if the limit from the left and the limit from the right both exist and are *unequal*, as illustrated in Fig. 3.1.1. Here, we introduce the standard shorthand notation for one-sided limits,

$$f(x_0^+) = \lim_{x \to x_0^+} f(x), f(x_0^-) = \lim_{x \to x_0^-} f(x),$$

[1]We do not give a definition of a continuous function here. However, one known useful fact is that if a function approaches ∞ at some point, then it is *not* continuous in any interval including that point.

Figure 3.1.1 Jump discontinuity at $x = x_0$.



An example of a piecewise smooth function is sketched in Fig. 3.1.2. Note that $f(x)$ has two jump discontinuities at $x = x_1$ and at $x = x_3$. Also, $f(x)$ is continuous[2] for $x_1 \le x \le x_3$, but df/dx is not continuous for $x_1 \le x \le x_3$. Instead, df/dx is continuous for $x_1 \le x \le x_2$ and $x_2 \le x \le x_3$. The interval can be broken up into pieces in which both $f(x)$ and df/dx are continuous. In this case there are four pieces, $x \le x_1$, $x_1 \le x \le x_2$, $x_2 \le x \le x_3$, and $x \ge x_3$. Almost all functions occurring in practice (and certainly most that we discuss in this book) will be piecewise smooth. Let us briefly give an example of a function that is not piecewise smooth. Consider $f(x) = x^{1/3}$, as sketched in Fig. 3.1.3. It is *not* piecewise smooth on any interval that includes $x = 0$, because $df/dx = 1/3x^{-2/3}$ goes to ∞ as $x \to 0$. In other words, any region including $x = 0$ cannot be broken up into pieces such that df/dx is continuous.

Figure 3.1.2 Example of a piecewise smooth function.

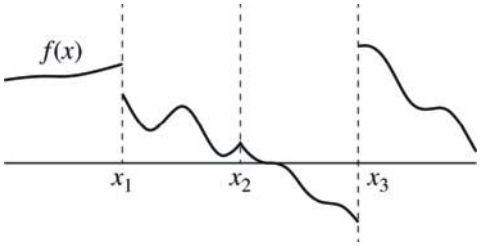
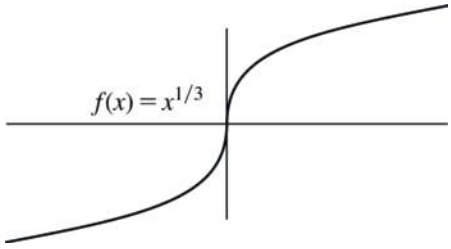
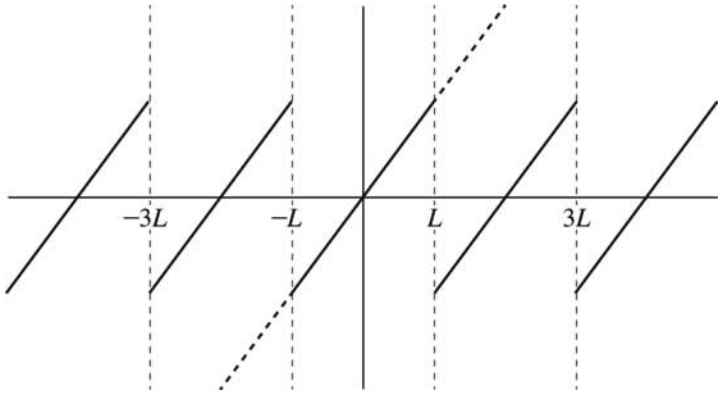


Figure 3.1.3 Example of a function that is not piecewise smooth.



Each function in the Fourier series is periodic with period $2L$. Thus, the **Fourier series of $f(x)$ on the interval $-L \le x \le L$ is periodic with period $2L$**. The function $f(x)$ does not need to be periodic. We need the **periodic extension** of $f(x)$. To sketch the periodic extension of $f(x)$, simply sketch $f(x)$ for $-L \le x \le L$ and then continually repeat the same pattern with period $2L$ by translating the original sketch for $-L \le x \le L$. For example, let us sketch in Fig. 3.1.4 the periodic extension of $f(x) = \frac{3}{2}x$ [the function $f(x) = \frac{3}{2}x$ is sketched in dotted lines for $|x| > L$]. Note the difference between $f(x)$ and its periodic extension.

Figure 3.1.4 Periodic extension of $f(x) = \frac{3}{2}x$.



[2] $f(x)$ is continuous on a closed interval $a \le x \le b$ means it is continuous at every point on the interior of the interval, continuous from the right at a, and continuous from the left at b. Thus, $f(x)$ can be continuous on $a \le x \le b$, but not continuous at a or b.

Exercises 3.1

3.1.1. What is the period of the function

$$f(x) = 3 + 2\cos(\pi x/5) - 4\sin(2\pi x/5) + 7\cos(\pi x)?$$

3.1.2. Sketch the periodic extension of $f(x) = x^2$, $-4 \le x \le 4$. Show at least 3 periods.

3.2 Statement of Convergence Theorem

DEFINITIONS OF FOURIER COEFFICIENTS AND A FOURIER SERIES

We will be forced to distinguish carefully between a function $f(x)$ and its Fourier series over the interval $-L \le x \le L$:

$$\text{Fourier series} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{3.2.1}$$

The infinite series may not even converge, and if it converges, it may not converge to $f(x)$. However, if the series converges, we learned in Chapter 2 how to determine the **Fourier coefficients** a_0, a_n, then b_n using certain orthogonality integrals, (2.3.32). We will use those results as the definition of the Fourier coefficients:

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned}} \tag{3.2.2}$$

The Fourier series of $f(x)$ over the interval $-L \le x \le L$ is defined to be the infinite series (3.2.1), where the Fourier coefficients are given by (3.2.2). We immediately note that a Fourier series does not exist unless, for example, a_0 exists [i.e., unless $\left|\int_{-L}^{L} f(x)\,dx\right| < \infty$]. This eliminates certain functions from our consideration. For example, we do not ask what is the Fourier series of $f(x) = 1/x^2$.

Even in situations in which $\int_{-L}^{L} f(x)\,dx$ exists, the infinite series may not converge; furthermore, if it converges, it may not converge to $f(x)$. We use the notation

$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},}$$

where $\sim$ means that $f(x)$ is on the left-hand side and the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) is on the right-hand side (even if the series diverges), but the two functions may be completely different. The symbol $\sim$ is read as "has the Fourier series (on a given interval)."

At first, we state a theorem summarizing certain properties of Fourier series:

Convergence theorem for Fourier series

If $f(x)$ is *piecewise smooth* on the interval $-L \le x \le L$, then the Fourier series of $f(x)$ converges for all $-\infty < x < \infty$.

Let $f_E(x)$ denote the *periodic extension* of $f(x)$.

1. At any point x where $f_E(x)$ is continuous, the Fourier series of $f(x)$ converges to $f_E(x)$.
2. At any point x where $f_E(x)$ has a jump discontinuity, the Fourier series of $f(x)$ converges to the average of the left- and right-hand limits.

$$\frac{1}{2}(f_E(x+) + f_E(x-)).$$

We refer to this as **Fourier's theorem**. It is a remarkable theorem.

Mathematically, if $f(x)$ is piecewise smooth, then for $-L < x < L$ (excluding the endpoints),

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{3.2.3}$$

where the Fourier coefficients are given by (3.2.2). *At points where $f(x)$ is continuous*, $f(x+) = f(x-)$, and hence (3.2.3) implies that for $-L < x < L$,

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.} \tag{3.2.4}$$

The Fourier series actually converges to $f(x)$ at points between $-L$ and $+L$, where $f(x)$ is continuous. At the endpoints, $x = L$ or $x = -L$, the infinite series converges to the average of the two values of the periodic extension. Outside the range $-L \le x \le L$, the Fourier series converges to a value easily determined using the known periodicity (with period $2L$) of the Fourier series.

We have derived the Fourier series using the orthogonality of the eigenfunctions. However, we want to remember the result, not the detailed calculation of the trigonometric integrals. We may want to just memorize (3.2.4), and the coefficients of the series by practice or tables. In Chapter 5, we will show that the orthogonality of the eigenfuctions follows automatically from the differential equation and the homogeneous boundary conditions.

Sketching Fourier series

Now we are ready to apply Fourier's theorem. To sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$), we

1. Sketch $f(x)$ for $-L < x < L$.
2. Sketch the periodic extension of $f(x)$.

According to Fourier's theorem, the Fourier series converges (here *converge* means "equals") to the periodic extension, where the periodic extension is continuous (which will be almost everywhere). However, at points of jump discontinuity of the periodic extension, the Fourier series converges to the average. Therefore, there is a third step:

3. Mark an "×" at the average of the two values at any jump discontinuity of the periodic extension.

EXAMPLE

Consider

$$f(x) = \begin{cases} 0 & -L \le x < \dfrac{L}{2} \\ 1 & \dfrac{L}{2} < x \le L. \end{cases} \tag{3.2.5}$$

We would like to determine the Fourier series of $f(x)$ on $-L \le x \le L$. We begin by sketching $f(x)$ for $-L \le x \le L$ in Fig. 3.2.1. Note that $f(x)$ is piecewise smooth, so we can apply Fourier's theorem. The periodic extension of $f(x)$ is sketched in Fig. 3.2.2. Often the understanding of the process is made clearer by sketching at least three full periods, $-3L \le x \le 3L$, even though in the applications to partial differential equations, only the interval $-L \le x \le L$ is absolutely needed. The Fourier series of $f(x)$ equals the periodic extension of $f(x)$, wherever the periodic extension is continuous (i.e., at all x except the points of jump discontinuity, which are $x = L/2$, L, $L/2 + 2L$, $-L$, $L/2 - 2L$, etc.). According to Fourier's theorem, at these points of jump discontinuity, the Fourier series of $f(x)$ must converge to the average. These points should be marked, perhaps with an ×, as in Fig. 3.2.2. At $x = L/2$ and $x = L$ (as well as $x = L/2 \pm 2nL$ and $x = L \pm 2nL$), the Fourier series converges to the average, $\frac{1}{2}$. In summary, for this example,

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} \frac{1}{2} & x = -L \\ 0 & -L < x < L/2 \\ \frac{1}{2} & x = L/2 \\ 1 & L/2 < x < L \\ \frac{1}{2} & x = L. \end{cases}$$

Figure 3.2.1 Sketch of $f(x)$.

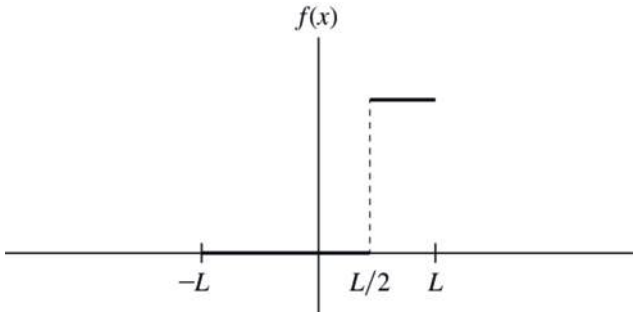
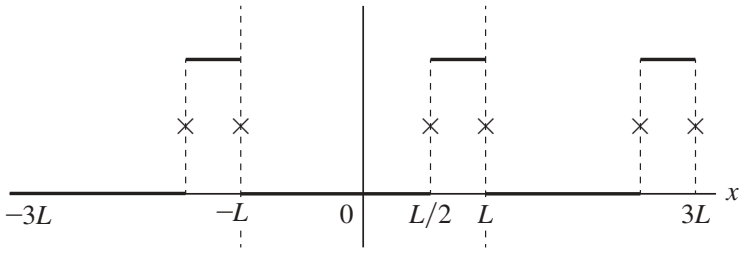


Figure 3.2.2 Fourier series of $f(x)$.



Fourier series converge to functions that are closely related to the original function, but are different in specific ways. Note for example that $f(L) = 1$, but the Fourier series of $f(x)$ evaluated at $x = L$ is equal to $\frac{1}{2}$. ■

Fourier coefficients

For a given $f(x)$, it is *not* necessary to calculate the Fourier coefficients in order to sketch the Fourier series of $f(x)$. However, it is important to know how to calculate the Fourier coefficients, given by (3.2.2). The calculation of Fourier coefficients can be an algebraically involved process. Sometimes it is an exercise in the method of integration by parts. Often, calculations can be simplified by judiciously using integral tables or computer algebra systems. In any event, we can always use a computer to approximate the coefficients numerically. As an overly simple example but one that illustrates some important points, consider $f(x)$ given by (3.2.5). From (3.2.2), the coefficients are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{L/2}^{L} dx = \frac{1}{4} \tag{3.2.6}$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L} \cos\frac{n\pi x}{L}\,dx = \frac{1}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L} \\ &= \frac{1}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L} \sin\frac{n\pi x}{L}\,dx = \frac{-1}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L} \\ &= \frac{1}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right). \end{aligned} \tag{3.2.8}$$

Some common simplifications that arise in these calculations are $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

Exercises 3.2

3.2.1. Let $f(x) = 2x - 1$ on $-3 \le x \le 3$. (So, in this problem, $L = 3$.) Sketch the Fourier series of $f(x)$ on $-9 \le x \le 9$.

3.2.2. For the same function as in the previous exercise, state the value that the Fourier series converges to at the following values of x:

(a) $x = 2$ **(b)** $x = 3$ **(c)** $x = 6$

(d) $x = 9$ **(e)** $x = 16$ **(f)** $x = 60$

3.2.3. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$). Compare $f(x)$ to its Fourier series:

(a) $f(x) = 1$ ***(b)** $f(x) = x^2$

(c) $f(x) = 1 + x$ ***(d)** $f(x) = e^x$

(e) $f(x) = \begin{cases} x & -L < x < 0 \\ 2x & 0 < x < L \end{cases}$ ***(f)** $f(x) = \begin{cases} 0 & -L < x \le 0 \\ 1 + x & 0 < x \le L \end{cases}$

(g) $f(x) = \begin{cases} x & -L \le x < L/2 \\ 0 & L/2 \le x \le L \end{cases}$

3.2.4. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) and determine the Fourier coefficients:

***(a)** $f(x) = x$ **(b)** $f(x) = e^{-x}$

***(c)** $f(x) = \sin\dfrac{\pi x}{L}$ **(d)** $f(x) = \begin{cases} 0 & -L \le x < 0 \\ x & 0 \le x < L \end{cases}$

(e) $f(x) = \begin{cases} 1 & |x| \le L/2 \\ 0 & L/2 < |x| \le L \end{cases}$ ***(f)** $f(x) = \begin{cases} 0 & -L < x < 0 \\ 1 & 0 < x < L \end{cases}$

(g) $f(x) = \begin{cases} 1 & -L < x \le 0 \\ 2 & 0 < x < L \end{cases}$

3.2.5. Show that the Fourier series operation is linear: that is, show that the Fourier series of $c_1 f(x) + c_2 g(x)$ is the sum of c_1 times the Fourier series of $f(x)$ and c_2 times the Fourier series of $g(x)$.

3.2.6. Suppose that $f(x)$ is piecewise smooth. What value does the Fourier series of $f(x)$ converge to at the endpoint $x = -L$? at $x = L$?

3.3 Fourier Cosine and Sine Series

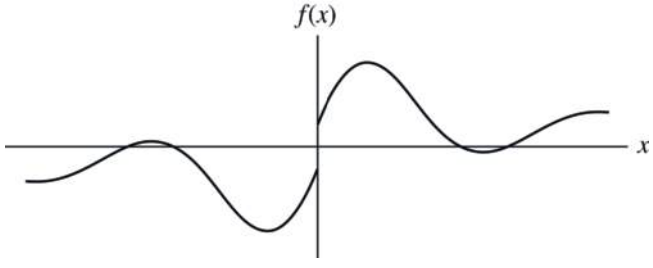
In this section, we show that the series of only sines, and the series of only cosines, are special cases of a Fourier series.

3.3.1 FOURIER SINE SERIES

Odd functions

An odd function is a function with the property $f(-x) = -f(x)$. The sketch of an odd function for $x < 0$ will be minus the mirror image of $f(x)$ for $x > 0$, as illustrated in Fig. 3.3.1. Examples

Figure 3.3.1 An odd function.



of odd functions are $f(x) = x^3$ (in fact, any odd power) and $f(x) = \sin 4x$. The integral of an odd function over a symmetric interval is zero (any contribution from $x > 0$ will be cancelled by a contribution from $x < 0$).

Fourier series of odd functions

Let us calculate the Fourier coefficients of an odd function:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0.$$

Both are zero because the integrand, $f(x)\cos n\pi x/L$, is odd [being the product of an even function $\cos n\pi x/L$ and an odd function $f(x)$]. Since $a_n = 0$, all the cosine functions (which are even) will not appear in the Fourier series of an odd function. The Fourier series of an odd function is an infinite series of odd functions (sines):

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{3.3.1}$$

if $f(x)$ is odd. In this case formulas for the Fourier coefficients b_n may be simplified:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \tag{3.3.2}$$

since the integral of an even function over the symmetric interval $-L$ to L is twice the integral from 0 to L. For odd functions, information about $f(x)$ is needed only for $0 \le x \le L$.

Fourier sine series

However, only occasionally are we given an odd function and asked to compute its Fourier series. Instead, frequently series of only sines arise in the context of separation of variables. Recall that the temperature in a one-dimensional rod $0 < x < L$ with zero-temperature ends $[u(0, t) = u(L, t) = 0]$ satisfies

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}, \tag{3.3.3}$$

where the initial condition $u(x, 0) = f(x)$ is satisfied if

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}. \tag{3.3.4}$$

$f(x)$ must be represented as a series of sines; (3.3.4) appears in the same form as (3.3.1). However, there is a significant difference. In (3.3.1) $f(x)$ is given as an odd function and is defined for $-L \le x \le L$. In (3.3.4), $f(x)$ is defined only for $0 \le x \le L$ (it is just the initial temperature distribution); of course, $f(x)$ is not necessarily odd. If $f(x)$ is given only for $0 \le x \le L$, then it can be *extended* as an odd function; see Fig. 3.3.2, called the **odd extension of $f(x)$**. The odd extension of $f(x)$ is defined for $-L \le x \le L$. Fourier's theorem will apply [if the odd extension of $f(x)$ is piecewise smooth, which just requires that $f(x)$ is piecewise smooth for $0 \le x \le L$]. Moreover, since the odd extension of $f(x)$ is certainly odd, its Fourier series involves only sines:

$$\text{the odd extension of } f(x) \sim \sum_{n=1}^{\infty} B_n \sin n\pi x/L,\ -L \le x \le L,$$

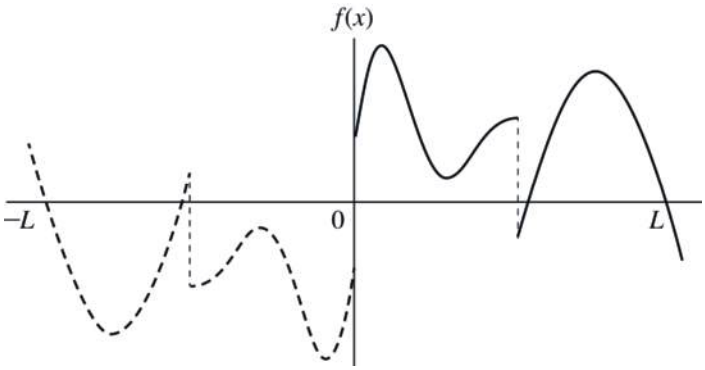
where B_n are given by (3.3.2). However, we are interested only in what happens between $x = 0$ and $x = L$. In that region, $f(x)$ is identical to its odd extension:

$$\boxed{f(x) \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L},} \qquad 0 \le x \le L, \tag{3.3.5}$$

where

$$\boxed{B_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx.} \tag{3.3.6}$$

Figure 3.3.2 Odd extension of $f(x)$.



We call this the **Fourier sine series of $f(x)$** (on the interval $0 \le x \le L$). This series (3.3.5) is nothing but an example of a Fourier series. As such, we can simply apply Fourier's theorem; just remember that $f(x)$ is defined only for $0 \le x \le L$. We may think of $f(x)$ as being odd (although it is not necessarily) by extending $f(x)$ as an odd function. Formula (3.3.6) is very important but does not need to be memorized. It can be *derived* from the formulas for a Fourier series simply by assuming that $f(x)$ is odd. [It is more accurate to say that we consider the odd extension of $f(x)$.] Formula (3.3.6) has a factor of 2 and the integrals are only from $x = 0$ to $x = L$ since the integrand is even.

According to Fourier's theorem, sketching the Fourier sine series of $f(x)$ is easy:

1. Sketch $f(x)$ for $0 < x < L$.
2. Sketch the odd extension of $f(x)$ for $-L < x < 0$.
3. Extend as a periodic function (with period $2L$).
4. Mark an $\times$ at the average at points where the odd periodic extension of $f(x)$ has a jump discontinuity.

EXAMPLE

As an example, we show how to sketch the Fourier sine series of $f(x) = 100$. We consider $f(x) = 100$ only for $0 \le x \le L$. We begin by sketching in Fig. 3.3.3 its odd extension. The Fourier sine series of $f(x)$ equals the Fourier series of the odd extension of $f(x)$. In Fig. 3.3.4, we repeat periodically the odd extension (with period $2L$). At points of discontinuity, the average is marked with an $\times$. According to Fourier's theorem (as illustrated in Fig. 3.3.4), the Fourier sine series of 100 actually equals 100 for $0 < x < L$, but the infinite series does not equal 100 at $x = 0$ and $x = L$:

$$100 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L. \tag{3.3.7}$$

Figure 3.3.3 Odd extension of $f(x) = 100$.

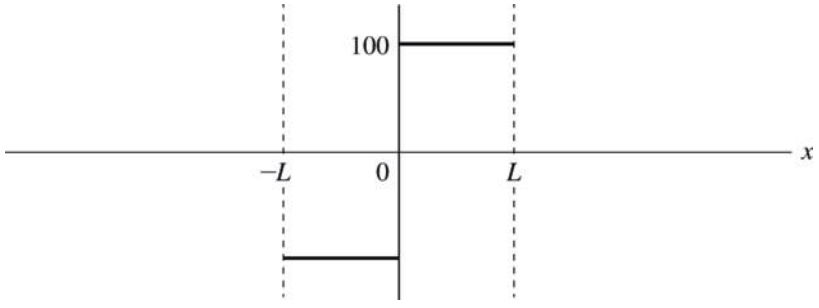
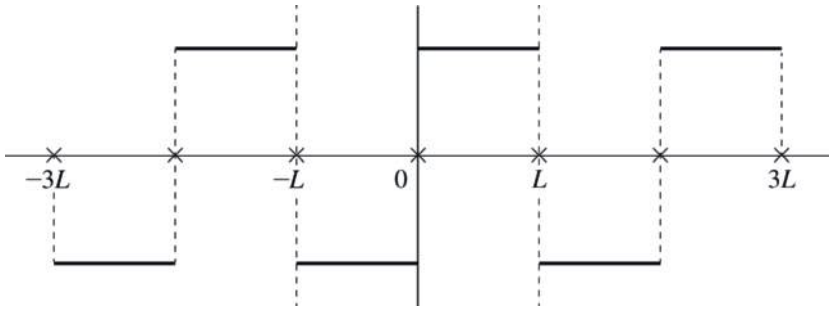


Figure 3.3.4 Fourier sine series of $f(x) = 100$.



At $x = 0$, Fig. 3.3.4 shows that the Fourier sine series converges to 0, because at $x = 0$ the odd property of the sine series yields the average of 100 and –100, which is 0. For similar reasons, the Fourier sine series also converges to 0 at $x = L$. These observations agree with the result of substituting $x = 0$ (and $x = L$) into the infinite series of sines. The Fourier coefficients are determined from (3.3.6) as before [see (2.3.42)]:

$$B_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx = \frac{200}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & n \text{ even} \\ \dfrac{400}{n\pi} & n \text{ odd.} \end{cases} \tag{3.3.8}$$

PHYSICAL EXAMPLE

One of the simplest examples is the Fourier sine series of a constant. This problem arose in trying to solve the one-dimensional heat equation with zero boundary conditions and constant initial temperature, 100°:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, && 0 < x < L, t > 0 \\ \text{BC1:} \quad & u(0, t) = 0 && t > 0 \\ \text{BC2:} \quad & u(L, t) = 0 && t > 0 \\ \text{IC:} \quad & u(x, 0) = f(x) = 100°. && 0 < x < L \end{aligned}$$

We recall from Section 2.3 that the method of separation of variables implied that

$$u(x, t) = \sum_{n=1}^{\infty} B_n\,\sin\frac{n\pi x}{L}e^{-(n\pi/L)^2 kt}. \tag{3.3.9}$$

The initial conditions are satisfied if

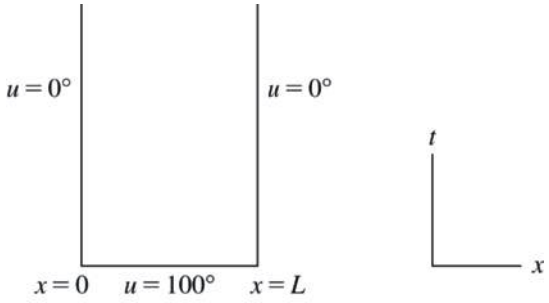
$$100 = f(x) = \sum_{n=1}^{\infty} B_n\,\sin\frac{n\pi x}{L},\ 0 < x < L.$$

This may be interpreted as the Fourier sine series of $f(x) = 100$ [see (3.3.8)]. Equivalently, B_n may be determined from the orthogonality of $\sin n\pi x/L$ [see (2.3.42)].

Mathematically, the Fourier series of the initial condition has a rather bizarre behavior at $x = 0$ (and at $x = L$). In fact, for this problem, the physical situation is not very well defined at $x = 0$ (at $t = 0$). This might be illustrated in a space-time diagram, Fig. 3.3.5. We note that Fig. 3.3.5 shows that the domain of our problem is $t \geq 0$ and $0 \leq x \leq L$. However, there is a conflict that occurs at $x = 0$, $t = 0$ between the initial condition and the boundary condition. The initial condition ($t = 0$) prescribes the temperature to be 100° even as $x \to 0$, whereas the boundary condition ($x = 0$) prescribes the temperature to be 0° even as $t \to 0$. Thus, the physical problem has a discontinuity at $x = 0, t = 0$. In the actual physical world, the temperature cannot be discontinuous. We introduced a discontinuity into our mathematical model by "instantaneously" transporting (at $t = 0$) the rod from a 100° bath to a 0° bath at $x = 0$. It actually takes a finite time, and the temperature would be continuous. Nevertheless, the transition from 0° to 100° would occur over an exceedingly small distance and time. We introduce the temperature discontinuity to approximate the more complicated real physical situation. Fourier's theorem thus illustrates how the physical discontinuity at $x = 0$ (initially, at $t = 0$) is reproduced mathematically. The Fourier sine series of 100° (which represents the physical solution at $t = 0$) has the nice property that it equals

100° for all x inside the rod, $0 < x < L$ (thus satisfying the initial condition there), but it equals 0° at the boundaries, $x = 0$ and $x = L$ (thus also satisfying the boundary conditions). The Fourier sine series of 100° is a strange mathematical function, but so is the physical approximation for which it is needed.

Figure 3.3.5 Boundary and initial conditions.



Finite number of terms in Fourier series computations and the Gibbs phenomenon

Let us gain some confidence in the validity of Fourier series. The Fourier sine series of $f(x) = 100$ states that

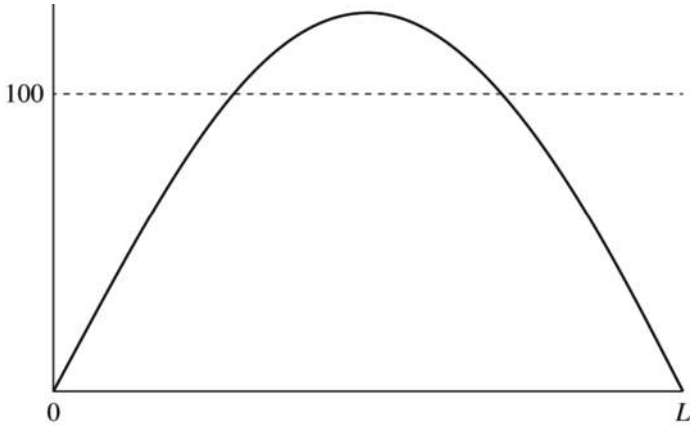
$$100 = \frac{400}{\pi}\left(\frac{\sin \pi x/L}{1} + \frac{\sin 3\pi x/L}{3} + \frac{\sin 5\pi x/L}{5} + \cdots\right). \tag{3.3.10}$$

Do we believe (3.3.10)? Certainly, it is not valid at $x = 0$ (as well as the other boundary $x = L$), since at $x = 0$ every term in the infinite series is zero (they cannot add to 100). However, the theory of Fourier series claims that (3.3.10) is valid everywhere except the two ends. For example, we claim it is valid at $x = L/2$. Substituting $x = L/2$ into (3.3.10) shows that

$$100 = \frac{400}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) \text{ or } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

At first this may seem strange. However, it is Euler's formula for π. It can be used to *compute* π (although very inefficiently); it can also be shown to be true without relying on the theory of infinite trigonometric series (see Exercise 3.3.22). The validity of (3.3.10) for other values of x, $0 < x < L$, may also surprise you. We will sketch the left- and right-hand sides of (3.3.10), hopefully convincing you of their equality. We will sketch the r.h.s. by adding up the contribution of each term of the series. Of course, we cannot add up the required infinite number of terms; we will settle for a finite number of terms. In fact, we will sketch the sum of the first few terms to see how the series approaches the constant 100 as the number of terms increases. It is helpful to know that $400/\pi = 127.32395\ldots$ (although for rough sketching 125 or 130 will do). The first term $(400/\pi)\sin \pi x/L$ by itself is the basic first rise and fall of a sine function; it is not a good approximation to the constant 100, as illustrated in Fig. 3.3.6. On the other hand, for just one term in an

Figure 3.3.6 First term of Fourier sine series of $f(x) = 100$.



MATLAB Files

Download Animation of Figure 3.3.8 (.zip)

https://bit.ly/3Una6gW

These files are provided as zipped .m files for use in MATLAB on desktop platform.

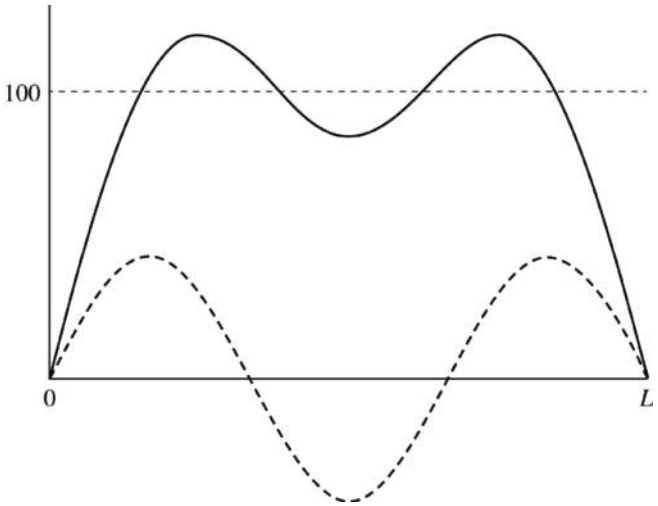


Figure 3.3.7 First two nonzero terms of Fourier sine series of $f(x) = 100$.

infinite series, it is not such a bad approximation. The next term to be added is $(400/3\pi)\sin 3\pi x/L$. This is a sinusoidal oscillation, with one third the amplitude and one third the period of the first term. It is positive near $x = 0$ and $x = L$, where the approximation needs to be increased, and it is negative near $x = L/2$, where the approximation needs to be decreased. It is sketched in dashed lines and then added to the first term in Fig. 3.3.7. Note that the sum of the two nonzero terms already seems to be a considerable improvement over the first term. Computer plots of some partial sums are given in Fig. 3.3.8.

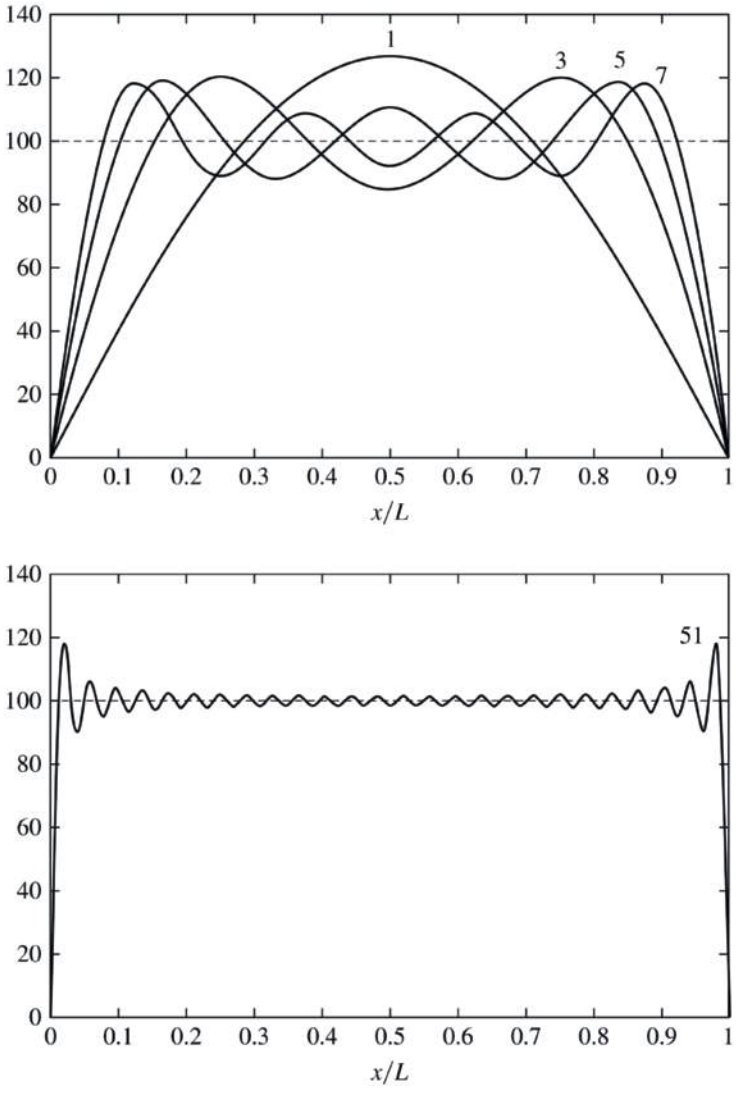


Figure 3.3.8 Various partial sums of Fourier sine series of $f(x) = 100$. Using 51 terms (including $n = 51$), the finite series is a good approximation to $f(x) = 100$ away from the endpoints. Near the endpoints (where there is a jump discontinuity of 200), there is a 9% overshoot (Gibbs phenomenon).

Actually, a lot can be learned from Fig. 3.3.8. Perhaps now it does seem reasonable that the infinite series converges to 100 for $0 < x < L$. The worst places (where the finite series differs most from 100) are getting closer and closer to $x = 0$ and $x = L$ as the number of terms increases. *For a finite number of terms in the series*, the solution starts from zero at $x = 0$ and shoots up beyond 100, what we call the primary **overshoot**. Fig. 3.3.8 illustrates the overshoot. We can even extrapolate to guess what happens for 1000 terms. The series should become more and more accurate as the number of terms increases. We might expect the overshoot to vanish as $n \to \infty$, but put a straight edge on the points of maximum overshoot. It just does not seem to approach 100. Instead, it is far away from that, closer to 118. This overshoot is an example of the **Gibbs phenomenon**. In general (for large n), there is an overshoot (and corresponding undershoot) of approximately 9% of the jump discontinuity. In this case (see Fig. 3.3.4), the Fourier *sine* series of $f(x) = 100$ jumps from -100 to $+100$ at $x = 0$. Thus, the finite series will overshoot by about 9% of 200, or approximately 18. The Gibbs phenomenon occurs only when a *finite* series of eigenfunctions approximates a *discontinuous* function.

Further example of a Fourier sine series

We consider the Fourier sine series of $f(x) = x$. $f(x) = x$ is sketched on the interval $0 \le x \le L$ in Fig. 3.3.9a. The odd-periodic extension of $f(x)$ is sketched in Fig. 3.3.9b. The jump discontinuity of the odd-periodic extension at $x = (2n-1)L$ shows that, for example, the Fourier sine series of $f(x) = x$ converges to zero at $x = L$. In contrast, $f(L) \neq 0$. We note that the Fourier sine series of $f(x) = x$ actually equals x for $-L < x < L$,

$$x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \qquad -L < x < L. \tag{3.3.11}$$

The Fourier coefficients are determined from (3.3.6):

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\, dx = \frac{2L}{n\pi}(-1)^{n+1}, \tag{3.3.12}$$

where the integral can be evaluated by integration by parts (or by a table).

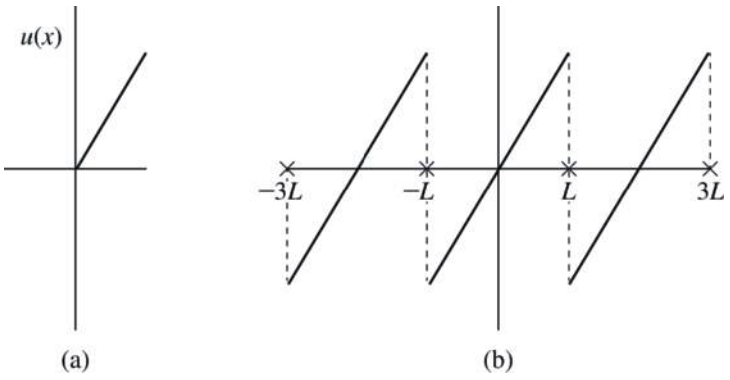
MATLAB Files

Download Animation of Figure 3.3.9 (.zip)

https://bit.ly/4d264Sp

These files are provided as zipped .m files for use in MATLAB on desktop platform.

Figure 3.3.9 (a) $f(x) = x$ and (b) its Fourier sine series.



EXAMPLE

We now consider the Fourier sine series of $f(x) = \cos \pi x/L$. This may seem strange to ask for a sine series expansion of an even function, but in applications often the function is given only from $0 \le x \le L$ and *must* be expanded in a series of sines due to the boundary conditions. $\cos \pi x/L$ is sketched in Fig. 3.3.10a. It is an even function when considered to be on a domain symmetric about the origin $-L \le x \le L$. However, if we first restrict its domain to $0 \le x \le L$, we can then build its odd extension, which is sketched in Fig. 3.3.10b. The Fourier sine series of $f(x)$ equals the Fourier series of the odd extension of $f(x)$. Thus, we repeat the sketch in Fig. 3.3.10b periodically (see Fig. 3.3.11), placing an $\times$ at the average of the two values at the jump discontinuities. The Fourier sine series representation of $\cos \pi x/L$ is

$$\cos\frac{\pi x}{L} \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \qquad 0 \le x \le L,$$

where with some effort we obtain

$$B_n = \frac{2}{L}\int_0^L \cos\frac{\pi x}{L}\,\sin\frac{n\pi x}{L}dx = \begin{cases} 0 & n \text{ odd} \\ \dfrac{4n}{\pi(n^2-1)} & n \text{ even.} \end{cases} \tag{3.3.13}$$

Figure 3.3.10 (a) $f(x) = \cos \pi x/L$ and (b) its odd extension.

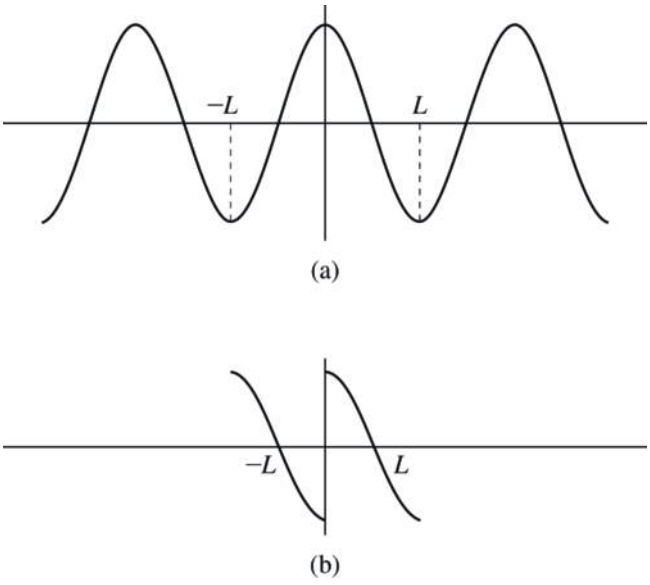
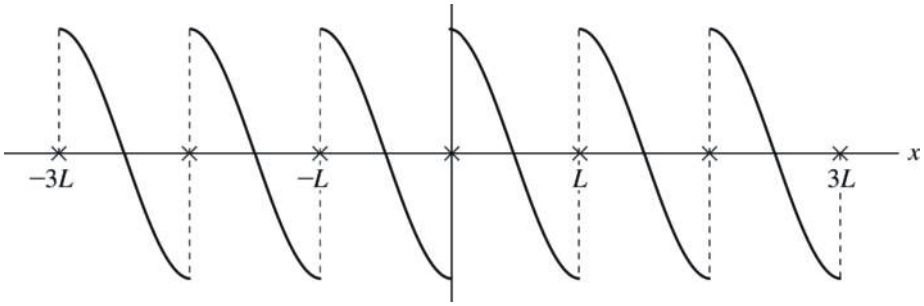


Figure 3.3.11 Fourier sine series of $f(x) = \cos \pi x/L$.



According to Fig. 3.3.11 (based on Fourier's theorem), equality holds for $0 < x < L$, but not at $x = 0$ and not at $x = L$:

$$\cos\frac{\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L.$$

At $x = 0$ and at $x = L$, the infinite series must converge to 0, since all terms in the series are 0 there. Figure 3.3.11 agrees with this. You may be a bit puzzled by an aspect of this problem. You may have recalled that $\sin n\pi x/L$ is orthogonal to $\cos m\pi x/L$, and thus expected all the B_n in (3.3.12) to be 0. However, $B_n \neq 0$. The subtle point is that you should remember that $\cos m\pi x/L$ and $\sin n\pi x/L$ are orthogonal *on the interval* $-L \leq x \leq L$, $\int_{-L}^{L} \cos m\pi x/L \sin n\pi x/L\, dx = 0$; they are not orthogonal on $0 \leq x \leq L$. ■

3.3.2 FOURIER COSINE SERIES

Even functions

Similar ideas are valid for even functions, in which $f(-x) = f(x)$. Let us develop the basic results. The sine coefficients of a Fourier series will be zero for an even function,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\frac{n\pi x}{L}\,dx = 0,$$

since $f(x)$ is even. The Fourier series of an even function is a representation of $f(x)$ involving an infinite sum of only even functions (cosines):

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}, \tag{3.3.14}$$

if $f(x)$ is even. The coefficients of the cosines may be evaluated using information about $f(x)$ only between $x = 0$ and $x = L$, since

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx \tag{3.3.15}$$

$$(n \geq 1) \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \tag{3.3.16}$$

using the fact that for $f(x)$ even, $f(x)\cos n\pi x/L$ is even.

Often, $f(x)$ is not given as an even function. Instead, in trying to represent an arbitrary function $f(x)$ using an infinite series of $\cos n\pi x/L$, the eigenfunctions of the boundary value problem $d^2\phi/dx^2 = -\lambda\phi$ with $d\phi/dx(0) = 0$ and $d\phi/dx(L) = 0$, we wanted

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}, \tag{3.3.17}$$

only for $0 < x < L$. We had previously determined the coefficients A_n to be the same as given by (3.3.15) and (3.3.16), but our reason was because of the orthogonality of $\cos n\pi x/L$. To relate (3.3.17) to a Fourier series, we simply introduce the **even extension of $f(x)$**, an example being illustrated in Fig. 3.3.12. If $f(x)$ is piecewise smooth for $0 \leq x \leq L$, then its even extension will also be piecewise smooth, and hence Fourier's theorem can be applied to the even extension of $f(x)$. Since the even extension of $f(x)$ is an even function, the Fourier series of the even extension of $f(x)$ will have only cosines:

$$\text{even extension of } f(x) \sim \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad -L \leq x \leq L,$$

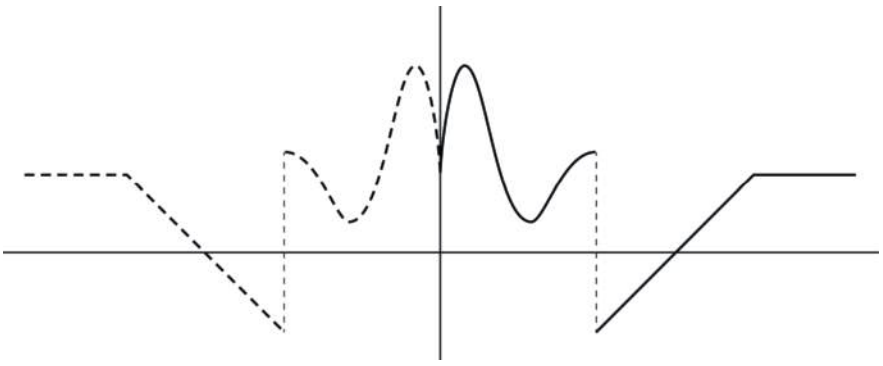
where a_n is given by (3.3.15) and (3.3.16). In the region of interest, $0 \leq x \leq L$, $f(x)$ is identical to the even extension. The resulting series in that region is called the **Fourier cosine series of $f(x)$** (on the interval $0 \leq x \leq L$):

$$\boxed{f(x) \sim \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}, \qquad 0 \leq x \leq L} \tag{3.3.18}$$

$$\boxed{A_0 = \frac{1}{L}\int_{0}^{L} f(x)dx} \tag{3.3.19}$$

$$\boxed{A_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx.} \tag{3.3.20}$$

Figure 3.3.12 Even extension of $f(x)$.



The Fourier cosine series of $f(x)$ is exactly the Fourier series of the even extension of $f(x)$. Since we can apply Fourier's theorem, we have an algorithm to sketch the Fourier cosine series of $f(x)$:

1. Sketch $f(x)$ for $0 < x < L$.
2. Sketch the even extension of $f(x)$ for $-L < x < 0$.
3. Extend as a periodic function (with period $2L$).
4. Mark $\times$ at points of discontinuity at the average.

EXAMPLE

We consider the Fourier cosine series of $f(x) = x$. $f(x)$ is sketched in Fig. 3.3.13a [note that $f(x)$ is odd!]. We consider $f(x)$ only from $x = 0$ to $x = L$ and then extend it in Fig. 3.3.13b as an even function. Next, we sketch the Fourier series of the even extension, by periodically extending the even extension (see Fig. 3.3.14). Note that between $x = 0$ and $x = L$, the Fourier cosine series has no jump discontinuities. The Fourier cosine series of $f(x) = x$ actually equals x, so that

$$x = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}, \qquad 0 \le x \le L. \tag{3.3.21}$$

Figure 3.3.13 (a) $f(x) = x$, (b) its even extension.

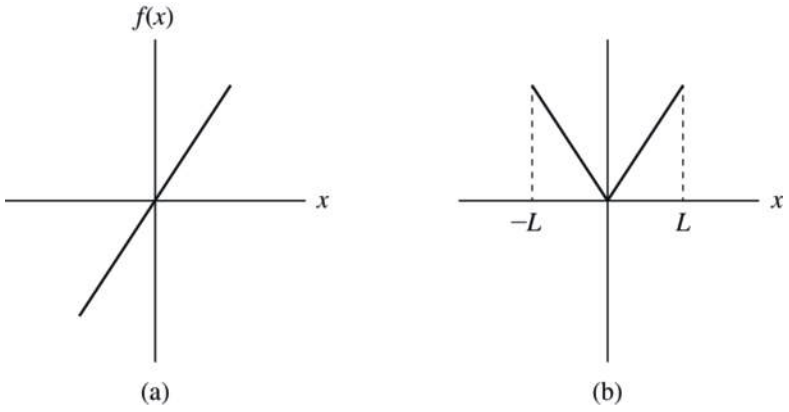
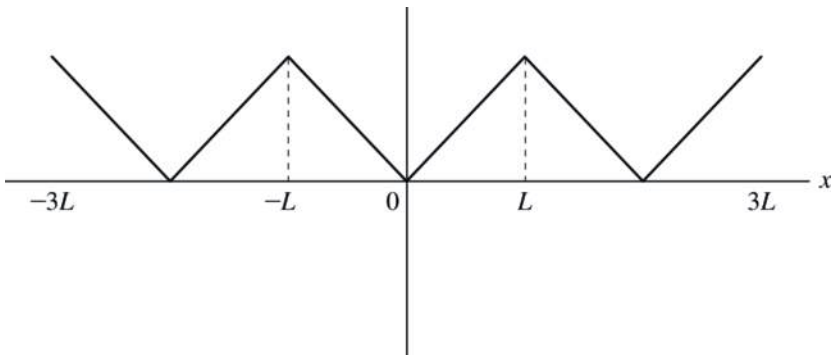


Figure 3.3.14 Fourier cosine series of the even extension of $f(x)$.



The coefficients are given by the following integrals:

$$A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\frac{1}{2}x^2\Big|_0^L = \frac{L}{2} \tag{3.3.22}$$

$$A_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx = \frac{2}{L}\frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi x}{L} - \frac{n\pi x}{L}\sin\frac{n\pi x}{L}\right)\Big|_0^L = \frac{2L}{(n\pi)^2}(\cos n\pi - 1). \tag{3.3.23}$$

The latter integral was integrated by parts $\int u\,dv = uv - \int u\,dv$, where $u = x$ and $dv = \cos\frac{n\pi x}{L}\,dx$, so that $du = dx$ and $v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$. Integrals can also be evaluated by tables or a symbolic computation program. ■

MATLAB Files

Download Animation of Figure 3.3.14 (.zip)

https://bit.ly/3UmzjYO

These files are provided as zipped .m files for use in MATLAB on desktop platform.

3.3.3 REPRESENTING f(x) BY BOTH A SINE AND COSINE SERIES

It may be apparent that any function $f(x)$ (which is piecewise smooth) defined on $-L < x < L$ may be represented both as a Fourier sine series and as a Fourier cosine series. The one you would use is dictated by the boundary conditions (if the problem arose in the context of a solution to a partial differential equation using the method of separation of variables). It is also possible to use a Fourier series (including both sines and cosines). As an example, we consider the sketches of the Fourier, Fourier sine, and Fourier cosine series of

$$f(x) = \begin{cases} -\frac{L}{2}\sin\frac{\pi x}{L} & x < 0 \\ x & 0 < x < \frac{L}{2} \\ L - x & x > \frac{L}{2}. \end{cases}$$

The graph of $f(x)$ is sketched for $-L < x < L$ in Fig. 3.3.15. The Fourier series of $f(x)$ is sketched by repeating this pattern with period $2L$. On the other hand, for the Fourier sine (cosine) series, first sketch the odd (even) extension of the function $f(x)$ before repeating the pattern. These three are sketched in Fig. 3.3.16. Note that for $-L \le x \le L$, only the Fourier series of $f(x)$ actually equals $f(x)$. However, for all three cases, the series equals $f(x)$ over the region $0 \le x \le L$.

Figure 3.3.15 The graph of $f(x)$ for $-L < x < L$.

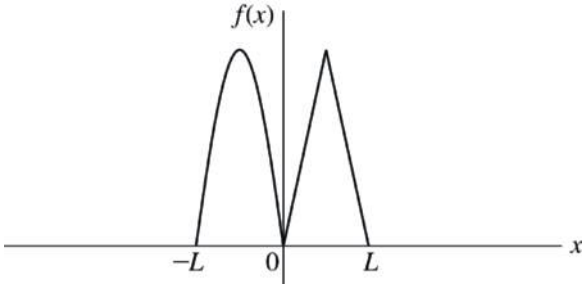
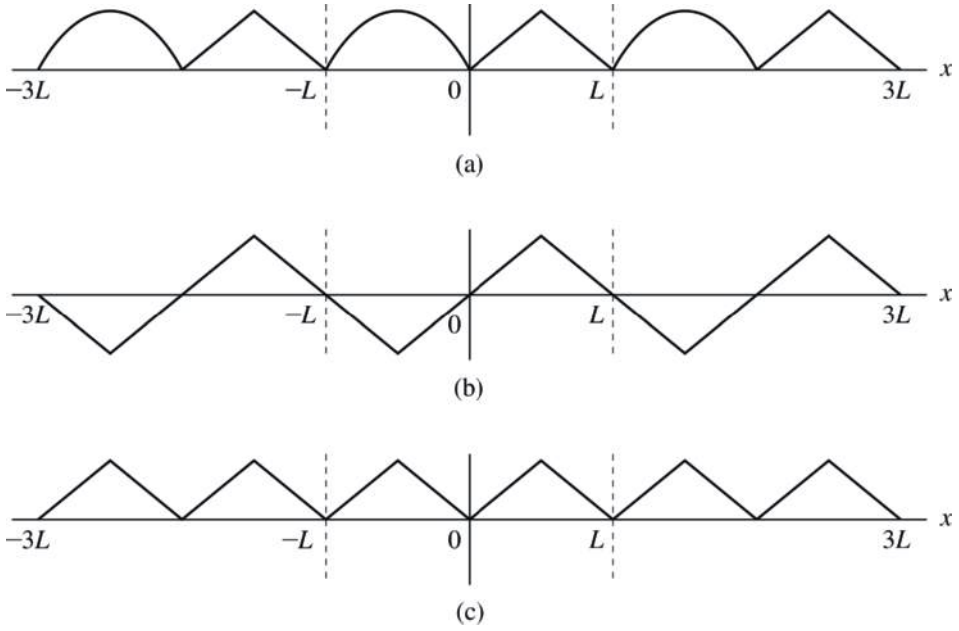


Figure 3.3.16 (a) Fourier series of $f(x)$; (b) Fourier sine series of $f(x)$; (c) Fourier cosine series of $f(x)$.



3.3.4 EVEN AND ODD PARTS

Let us consider the Fourier series of a function $f(x)$ that is not necessarily even or odd:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{3.3.24}$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

It is interesting to see that a Fourier series is the sum of a series of cosines and a series of sines. For example, $\sum_{n=1}^{\infty} b_n \sin n\pi x/L$ is not, in general, the Fourier sine series of $f(x)$, because the coefficients, $b_n = 1/L\int_{-L}^{L} f(x)\sin n\pi x/L\,dx$, are *not*, in general, the same as the coefficients of a Fourier sine series $\left[2/L\int_0^L f(x)\sin n\pi x/L\,dx\right]$. This series of sines by itself ought to be the Fourier sine series of some function; let us determine this function.

Equation (3.3.24) shows that $f(x)$ is represented as a sum of an even function (for the series of cosines must be an even function) and an odd function (similarly, the sine series must be odd). This is a general property of functions, since for any function it is rather obvious that

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]. \tag{3.3.25}$$

Note that the first bracketed term is an even function; we call it the **even part of $f(x)$**. The second bracketed term is an odd function, called the **odd part of $f(x)$**:

$$f_e(x) \equiv \frac{1}{2}[f(x) + f(-x)] \quad \text{and} \quad f_o(x) \equiv \frac{1}{2}[f(x) - f(-x)]. \tag{3.3.26}$$

In this way, any function is written as the sum of an odd function (the odd part) and an even function (the even part). For example, if $f(x) = 1/(1+x)$,

$$\frac{1}{1+x} = \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right] + \frac{1}{2}\left[\frac{1}{1+x} - \frac{1}{1-x}\right] = \frac{1}{1-x^2} - \frac{x}{1-x^2}.$$

This is the sum of an even function, $1/(1-x^2)$, and an odd function, $-x/(1-x^2)$. Consequently, the Fourier series of $f(x)$ equals the Fourier series of $f_e(x)$ [which is a cosine series since $f_e(x)$ is even] plus the Fourier series of $f_o(x)$ (which is a sine series since $f_o(x)$ is odd). This shows that the series of sines (cosines) that appears in (3.3.17) is the Fourier sine (cosine) series of $f_o(x)(f_e(x))$. We summarize our result with the statement:

The Fourier series of $f(x)$ equals the Fourier sine series of $f_o(x)$ plus the Fourier cosine series of $f_e(x)$, where $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$, and $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$.

Please do not confuse this result with even and odd *extensions*. For example, the even part of $f(x) = \frac{1}{2}[f(x) + f(-x)]$, while the

$$\text{even extension of } f(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0. \end{cases}$$

3.3.5 CONTINUOUS FOURIER SERIES

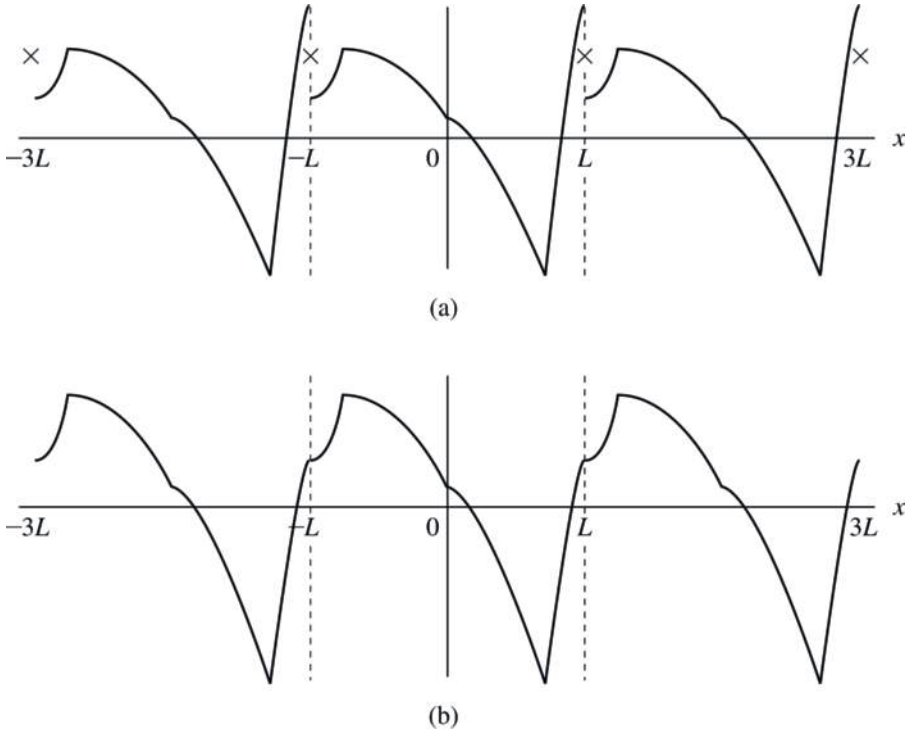
The convergence theorem for Fourier series shows that the Fourier series of $f(x)$ may be a different function than $f(x)$. Nevertheless, over the interval of interest, they are the same except at those few points where the periodic extension of $f(x)$ has a jump discontinuity. Sine (cosine) series are analyzed in the same way, where instead the odd (even) periodic extension must be considered. In addition to points of jump discontinuity of $f(x)$ itself, the various extensions of $f(x)$ may introduce a jump discontinuity. From the examples in the preceding section, we observe that *sometimes* the resulting series does not have any jump discontinuities. In these cases the Fourier series of $f(x)$ will actually equal $f(x)$ in the range of interest. Also, the Fourier series itself will be a continuous function.

It is worthwhile to summarize the conditions under which a Fourier series is continuous:

For piecewise smooth $f(x)$, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ for $-L \le x \le L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.

It is necessary for $f(x)$ to be continuous; otherwise, there will be a jump discontinuity [and the Fourier series of $f(x)$ will converge to the average]. In Fig. 3.3.17, we illustrate the significance of the condition $f(-L) = f(L)$. We illustrate two *continuous* functions, only one of which satisfies $f(-L) = f(L)$. The condition $f(-L) = f(L)$ insists that the repeated pattern (with period $2L$) will be continuous at the endpoints. The preceding boxed statement is a fundamental result for all Fourier series. It explains the following similar theorems for Fourier sine and cosine series.

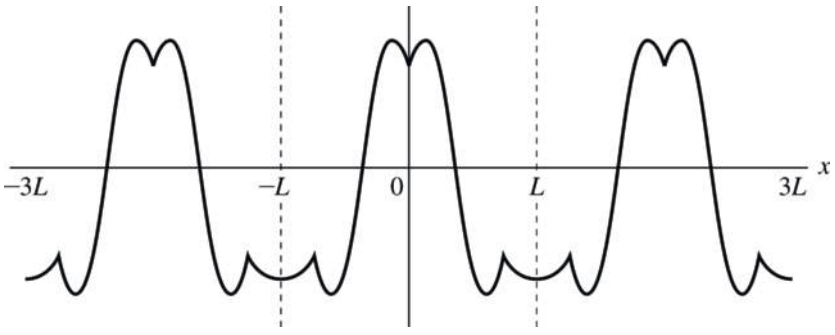
Figure 3.3.17 Fourier series of a continuous function with (a) $f(-L) \neq f(L)$ and (b) $f(-L) = f(L)$.



Consider the Fourier cosine series of $f(x)$ [$f(x)$ has been extended as an even function]. If $f(x)$ is continuous, is the Fourier cosine series continuous? An example that is continuous for $0 \le x \le L$ is sketched in Fig. 3.3.18. First, we extend $f(x)$ evenly and then periodically. It is easily seen that

For piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous.

Figure 3.3.18 Fourier cosine series of a continuous function.



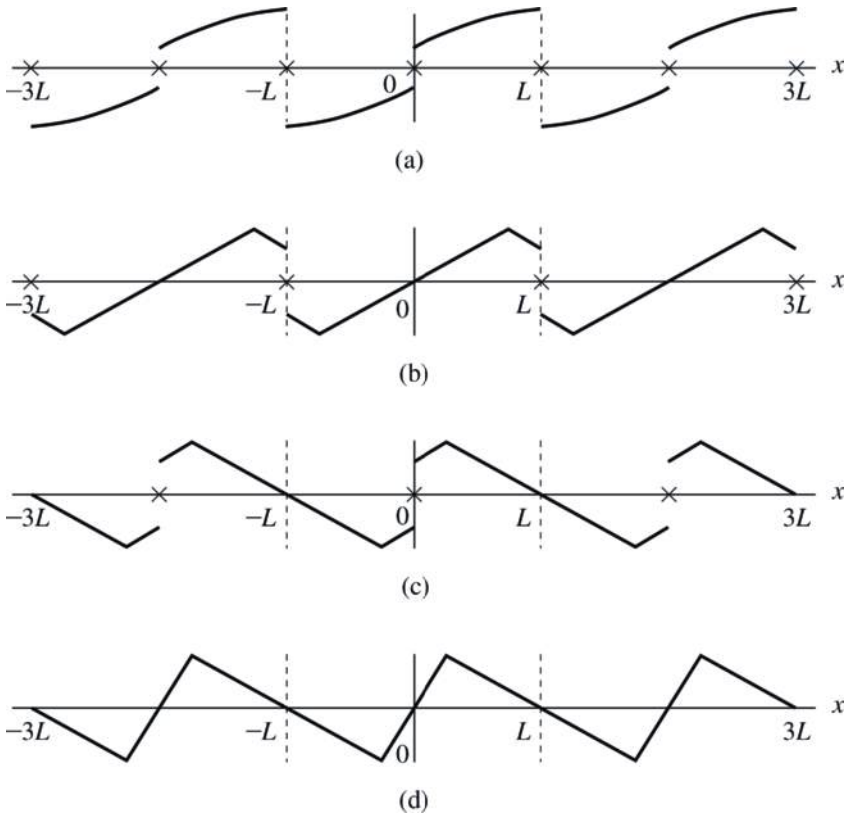
We note that no additional conditions on $f(x)$ are necessary for the cosine series to be continuous [besides $f(x)$ being continuous]. One reason for this result is that if $f(x)$ is continuous for $0 \le x \le L$, then the even extension will be continuous for $-L \le x \le L$. Also note that the even extension is the same at $\pm L$. Thus, the periodic extension will automatically be continuous at the endpoints.

Compare this result to what happens for a Fourier sine series. Four examples are considered in Fig. 3.3.19, all continuous functions for $0 \le x \le L$. From the first three examples, we see that it is possible for the Fourier sine series of a continuous function to be discontinuous. It is seen that

For piecewise smooth functions $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous and both $f(0) = 0$ *and* $f(L) = 0$.

If $f(0) \neq 0$, then the odd extension of $f(x)$ will have a jump discontinuity at $x = 0$, as illustrated in Figs. 3.3.19a and c. If $f(L) \neq 0$, then the odd extension at $x = -L$ will be of opposite sign from $f(L)$. Thus, the periodic extension will not be continuous at the endpoints if $f(L) \neq 0$ as in Figs. 3.3.19a and b.

Figure 3.3.19 Fourier sine series of a continuous function with (a) $f(0) \neq 0$ and $f(L) \neq 0$; (b) $f(0) = 0$ but $f(L) \neq 0$; (c) $f(L) = 0$ but $f(0) \neq 0$; and (d) $f(0) = 0$ and $f(L) = 0$.



Exercises 3.3

3.3.1. Let $f(x) = 1 + 3x$ on $0 \le x \le 5$.

(a) Sketch the Fourier sine series of $f(x)$ on $-15 \le x \le 15$.

(b) Sketch the Fourier cosine series of $f(x)$ on $-15 \le x \le 15$.

(c) To what values does the Fourier sine series of $f(x)$ converge at $x = 5$, 9, 22, and 101?

(d) To what values does the Fourier cosine series of $f(x)$ converge at $x = 5$, 9, 22, and 101?

3.3.2. For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$:

(a) $f(x) = 1$

(b) $f(x) = 1 + x$

(c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$

***(d)** $f(x) = e^x$

(e) $f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$

3.3.3. For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients:

(a) $f(x) = \cos \pi x/L$ [Verify formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

***(d)** $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

3.3.4. For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a *finite* number of nonzero terms (at least the first two) of the Fourier sine series:

(a) $f(x) = \cos \pi x/L$ [Use formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = x$ [Use formula (3.3.12).]

3.3.5. Sketch the Fourier cosine series of $f(x) = \sin \pi x/L$. Briefly discuss.

3.3.6. For the following functions, sketch the Fourier cosine series of $f(x)$ and determine its Fourier coefficients:

(a) $f(x) = x^2$

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

3.3.7. For the following functions, sketch the Fourier cosine series of $f(x)$. Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier cosine series:

(a) $f(x) = x$ [Use formulas (3.3.22) and (3.3.23).]

(b) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [Use carefully formulas (3.2.6) and (3.2.7).]

(c) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$ [*Hint*: Add the functions in parts (b) and (c).]

3.3.8. Show that e^x is the sum of an even and an odd function.

3.3.9. **(a)** Determine formulas for the even extension of any $f(x)$. Compare to the formula for the even part of $f(x)$.

(b) Do the same for the odd extension of $f(x)$ and the odd part of $f(x)$.

(c) Calculate and sketch the four functions of parts (a) and (b) if

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0. \end{cases}$$

Graphically add the even and odd parts of $f(x)$. What occurs? Similarly, add the even and odd extensions. What occurs then?

3.3.10. What is the sum of the Fourier sine series of $f(x)$ and the Fourier cosine series of $f(x)$? [What is the sum of the even and odd extensions of $f(x)$?]

***3.3.11.** If $f(x) = \begin{cases} x^2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$, what are the even and odd parts of $f(x)$?

3.3.12. Given a sketch of $f(x)$, describe a procedure to sketch the even and odd parts of $f(x)$.

3.3.13. **(a)** Graphically show that the even terms (n even) of the Fourier sine series of any function on $0 \leq x \leq L$ are odd (antisymmetric) around $x = L/2$.

(b) Consider a function $f(x)$ that is odd around $x = L/2$. Show that the odd coefficients (n odd) of the Fourier sine series of $f(x)$ on $0 \leq x \leq L$ are zero.

***3.3.14.** Consider a function $f(x)$ that is even around $x = L/2$. Show that the even coefficients (n even) of the Fourier sine series of $f(x)$ on $0 \leq x \leq L$ are zero.

3.3.15. **(a)** Consider a function $f(x)$ that is even around $x = L/2$. Show that the odd coefficients (n odd) of the Fourier cosine series of $f(x)$ on $0 \leq x \leq L$ are zero.

(b) Explain the result of part (a) by considering a Fourier cosine series of $f(x)$ on the interval $0 \leq x \leq L/2$.

3.3.16. Consider a function $f(x)$ that is odd around $x = L/2$. Show that the even coefficients (n even) of the Fourier cosine series of $f(x)$ on $0 \leq x \leq L$ are zero.

3.3.17. Fourier series can be defined on other intervals besides $-L \leq x \leq L$. Suppose that $g(y)$ is defined for $a \leq y \leq b$. Represent $g(y)$ using periodic trigonometric functions with period $b - a$. Determine formulas for the coefficients. [*Hint*: Use $L = (b - a)/2$ and the linear transformation

$$y = \frac{a + b}{2} + x.]$$

3.3.18. Use the strategy of Exercise 3.3.17 to find the Fourier series of $g(y) = y$ on $2 \leq y \leq 3$.

3.3.19. Fourier sine and cosine series can be defined on intervals besides $0 \leq x \leq L$. Suppose that $g(y)$ is defined for $a \leq y \leq b$. Use the transformation $y = a + x$ and $L = b - a$ to find formulas for the coefficients of the Fourier sine series and Fourier cosine series of $g(y)$.

3.3.20. Use the strategy of Exercise 3.3.19 to find the Fourier sine series of $g(y) = y$ on $2 \leq y \leq 3$.

3.3.21. Use the strategy of Exercise 3.3.19 to find the Fourier cosine series of $g(y) = y$ on $2 \leq y \leq 3$.

3.3.22. Consider

$$\int_0^1 \frac{dx}{1 + x^2}.$$

(a) Evaluate explicitly.

(b) Use the Taylor series of $1/(1 + x^2)$ (itself a geometric series) to obtain an infinite series for the integral.

(c) Equate part (a) to part (b) in order to derive a formula for π.

3.3.23. For continuous functions,

(a) Under what conditions does $f(x)$ equal its Fourier series for all x, $-L \leq x \leq L$?

(b) Under what conditions does $f(x)$ equal its Fourier sine series for all x, $0 \leq x \leq L$?

(c) Under what conditions does $f(x)$ equal its Fourier cosine series for all x, $0 \leq x \leq L$?

3.4 Term-By-Term Differentiation of Fourier Series

In solving partial differential equations by the method of separation of variables, the homogeneous boundary conditions sometimes suggest that the desired solution is an infinite series of either sines or cosines. For example, we consider one-dimensional heat conduction with zero boundary conditions. As before, we want to solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{3.4.1}$$

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad u(x, 0) = f(x). \tag{3.4.2}$$

By the method of separation of variables combined with the principle of superposition (taking a *finite* linear combination of solutions), we know that

$$u(x, t) = \sum_{n=1}^{N} B_n \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}$$

solves the partial differential equation and the two homogeneous boundary conditions. To satisfy the initial conditions, in general an infinite series is needed. Does the infinite series

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \tag{3.4.3}$$

satisfy our problem? The theory of Fourier sine series shows that the Fourier coefficients B_n can be determined to satisfy any (piecewise smooth) initial condition [i.e., $B_n = 2/L \int_0^L f(x) \sin n\pi x/L \, dx$]. To see if the infinite series actually satisfies the partial differential equation, we substitute (3.4.3) into (3.4.1). *If* the infinite Fourier series can be differentiated term by term, then

$$\frac{\partial u}{\partial t} = -\sum_{n=1}^{\infty} k \left(\frac{n\pi}{L}\right)^2 B_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}$$

and

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}.$$

Thus, the heat equation ($\partial u/\partial t = k \, \partial^2 u/\partial x^2$) is satisfied by the infinite Fourier series obtained by the method of separation of variables, if term-by-term differentiation of a Fourier series is valid.

TERM-BY-TERM DIFFERENTIATION OF INFINITE SERIES

Unfortunately, infinite series (even convergent infinite series) cannot always be differentiated term by term. It is not always true that

$$\frac{d}{dx}\sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n \frac{du_n}{dx};$$

the interchange of operations of differentiation and infinite summation is not always justified. However, we will find that in solving partial differential equations, all the procedures we have performed on the infinite Fourier series are valid. We will state and prove some needed theorems concerning the validity of term-by-term differentiation of just the type of Fourier series that arise in solving partial differential equations.

COUNTEREXAMPLE

Even for Fourier series, term-by-term differentiation is not always valid. To illustrate the difficulty in term-by-term differentiation, consider the Fourier sine series of x (on the interval $0 \le x \le L$) sketched in Fig. 3.4.1:

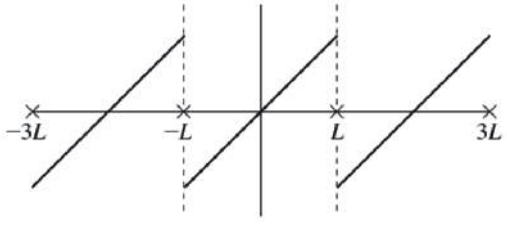
$$x = 2\sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L}, \text{ on } 0 \le x < L,$$

as obtained earlier [see (3.3.11) and (3.3.12)]. If we differentiate the function on the left-hand side, then we have the function 1. However, if we formally differentiate term by term the function on the right, then we arrive at

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \cos\frac{n\pi x}{L}.$$

This is a cosine series, but it is not the cosine series of $f(x) = 1$ (the cosine series of 1 is just 1). Thus, Fig. 3.4.1 is an example where we cannot differentiate term by term.[3]

Figure 3.4.1 Fourier sine series of $f(x) = x$.



[3]In addition, the resulting infinite series does not even converge anywhere, since the nth term does not approach zero.

Fourier series

We claim that this difficulty occurs any time the Fourier series of $f(x)$ has a jump discontinuity. Term-by-term differentiation is not justified in these situations. Instead, we claim (and prove in an exercise) that

A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth.

An alternative form of this theorem is written if we remember the condition for the Fourier series to be continuous:

If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

The result of term-by-term differentiation is the Fourier series of $f'(x)$, which may not be continuous. Similar results for sine and cosine series are of more frequent interest to the solution of our partial differential equations.

Fourier cosine series

For Fourier cosine series,

If $f'(x)$ is piecewise smooth, then a continuous Fourier cosine series of $f(x)$ can be differentiated term by term.

The result of term-by-term differentiation is the Fourier sine series of $f'(x)$, which may not be continuous. Recall that $f(x)$ needs only to be continuous for its Fourier cosine series to be continuous. Thus, this theorem can be stated in the following alternative form:

If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

These statements apply to the Fourier cosine series of $f(x)$:

$$\boxed{f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L},} \qquad 0 \le x \le L, \tag{3.4.4}$$

where the $=$ sign means that the infinite series converges to $f(x)$ for all x $(0 \le x \le L)$ since $f(x)$ is continuous. Mathematically, these theorems state that term-by-term differentiation is valid,

$$f'(x) \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right) A_n \sin\frac{n\pi x}{L}, \tag{3.4.5}$$

where $\sim$ means equality where the Fourier sine series of $f'(x)$ is continuous and means the series converges to the average where the Fourier sine series of $f'(x)$ is discontinuous.

EXAMPLE

Consider the Fourier cosine series of x [see (3.3.21), (3.3.22), and (3.3.23)],

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd only}} \frac{1}{n^2} \cos\frac{n\pi x}{L}, \qquad 0 \le x \le L, \tag{3.4.6}$$

as sketched in Fig. 3.4.2. Note the continuous nature of this series for $0 \le x \le L$, which results in the $=$ sign in (3.4.6). The derivative of this Fourier cosine series is sketched in Fig. 3.4.3: it is the Fourier sine series of $f(x) = 1$. The Fourier sine series of $f(x) = 1$ can be obtained by term-by-term differentiation of the Fourier cosine series of $f(x) = x$. Assuming that term-by-term differentiation of (3.4.6) is valid as claimed, it follows that

$$1 \sim \frac{4}{\pi} \sum_{n \text{ odd only}} \frac{1}{n} \sin\frac{n\pi x}{L}, \tag{3.4.7}$$

which is, in fact, correct [see (3.3.8)].

Figure 3.4.2 Fourier cosine series of $f(x) = x$.

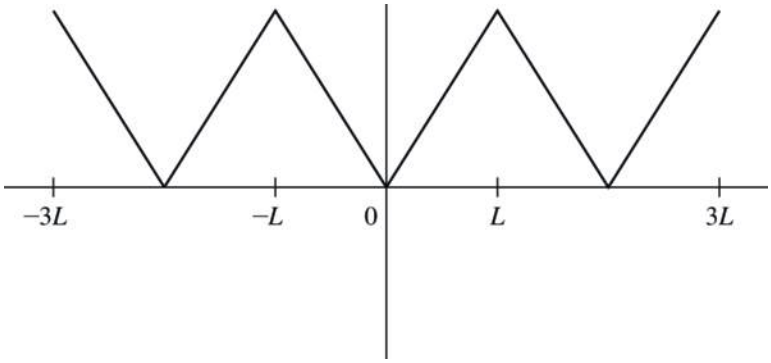
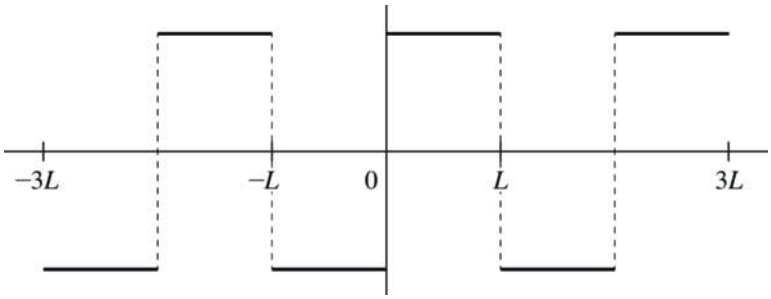


Figure 3.4.3 Fourier sine series of df/dx.



Fourier sine series

A similar result is valid for Fourier sine series:

If $f'(x)$ is piecewise smooth, then a continuous Fourier sine series of $f(x)$ can be differentiated term by term.

However, if $f(x)$ is continuous, then the Fourier sine series is continuous only if $f(0) = 0$ and $f(L) = 0$. Thus, we must be careful in differentiating term by term a Fourier sine series. In particular,

If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$.

Proofs

The proofs of these theorems are all quite similar. We include one since it provides a way to learn more about Fourier series and their differentiability. We will prove the validity of term-by-term differentiation of the Fourier sine series of a *continuous* function $f(x)$, in the case when $f'(x)$ is piecewise smooth and $f(0) = 0 = f(L)$:

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \tag{3.4.8}$$

where B_n are expressed below. An equality holds in (3.4.8) only if $f(0) = 0 = f(L)$. If $f'(x)$ is piecewise smooth, then $f'(x)$ has a Fourier cosine series

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}, \tag{3.4.9}$$

where A_0 and A_n are expressed in (3.4.10) and (3.4.11). This series will not converge to $f'(x)$ at points of discontinuity of $f'(x)$. We will have succeeded in showing a Fourier sine series may be term-by-term differentiated *if* we can verify that

$$f'(x) \sim \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) B_n \cos\frac{n\pi x}{L}$$

[i.e., if $A_0 = 0$ and $A_n = (n\pi/L)B_n$]. The Fourier cosine series coefficients are derived from (3.4.9). If we integrate by parts, we obtain

$$A_0 = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L}[f(L) - f(0)] \tag{3.4.10}$$

$$(n \neq 0) \qquad A_n = \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[f(x)\cos\frac{n\pi x}{L}\Big|_0^L + \frac{n\pi}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right]. \tag{3.4.11}$$

But from (3.4.8), B_n is the Fourier sine series coefficient of $f(x)$,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

and thus for $n \neq 0$,

$$A_n = \frac{n\pi}{L}B_n + \frac{2}{L}[(-1)^n f(L) - f(0)]. \tag{3.4.12}$$

We thus see by comparing Fourier cosine coefficients that the Fourier sine series can be term-by-term differentiated only if both $f(L) - f(0) = 0$ (so that $A_0 = 0$) and $(-1)^n f(L) - f(0) = 0$ [so that $A_n = (n\pi/L)B_n$]. Both of these conditions hold only if

$$f(0) = f(L) = 0,$$

exactly the conditions for a Fourier sine series of a continuous function to be continuous. Thus, we have completed the proof. However, this demonstration has given us more information. Namely, it gives the formula to differentiate the Fourier sine series of a continuous function when the series is not continuous. We have that

If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$,

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L},$$

cannot, in general, be differentiated term by term. However,

$$f'(x) \sim \frac{1}{L}[f(L) - f(0)] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}B_n + \frac{2}{L}\left((-1)^n f(L) - f(0)\right)\right]\cos\frac{n\pi x}{L}. \tag{3.4.13}$$

In this proof, it may appear that we never needed $f(x)$ to be continuous. However, we applied integration by parts in order to derive (3.4.9). In the usual presentation in calculus, integration by

parts is stated as being valid if both $u(x)$ and $v(x)$ *and their derivatives* are continuous. This is overly restrictive for our work. As is clarified somewhat in an exercise, we state that integration by parts is valid if only $u(x)$ and $v(x)$ are continuous. It is not necessary for their derivatives to be continuous. Thus, the result of integration by parts is valid only if $f(x)$ is continuous.

EXAMPLE

Let us reconsider the Fourier sine series of $f(x) = x$,

$$x \sim 2\sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L}. \tag{3.4.14}$$

We already know that $(d/dx)x = 1$ does not have a Fourier cosine series that results from term-by-term differentiation of (3.4.14) since $f(L) \neq 0$. However, (3.4.13) may be applied since $f(x)$ is continuous [and $f'(x)$ is piecewise smooth]. Noting that $f(0) = 0$, $f(L) = L$, and $(n\pi/L)B_n = 2(-1)^{n+1}$, it follows that the Fourier cosine series of df/dx is

$$\frac{df}{dx} \sim 1.$$

The constant function 1 is exactly the Fourier cosine series of df/dx since $f = x$ implies that $df/dx = 1$. Thus, the r.h.s. of (3.4.13) gives the correct expression for the Fourier cosine series of $f'(x)$ when the Fourier sine series of $f(x)$ is known, even if $f(0) \neq 0$ and/or $f(L) \neq 0$. ■

Method of eigenfunction expansion

Let us see how our results concerning the conditions under which a Fourier series may be differentiated term by term may be applied to our study of partial differential equations. We consider the heat equation (3.4.1) with zero boundary conditions at $x = 0$ and $x = L$. We will show that (3.4.3) is the correct infinite series representation of the solution of this problem. We will show this by utilizing an alternative scheme to obtain (3.4.3) known as the **method of eigenfunction expansion**, whose importance is that it may also be used when there are sources or the boundary conditions are not homogeneous (see Exercises 3.4.9–3.4.12 and Chapter 7). We begin by assuming that we have a solution $u(x, t)$ that is continuous such that $\partial u/\partial t, \partial u/\partial x$, and $\partial^2 u/\partial x^2$ are also continuous. Now we expand the unknown solution $u(x, t)$ in terms of the eigenfunctions of the problem (with homogeneous boundary conditions). In this example, the eigenfunctions are $\sin n\pi x/L$, suggesting a Fourier sine series for each time:

$$\boxed{u(x, t) \sim \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{L};} \tag{3.4.15}$$

the Fourier sine coefficients B_n will depend on time, $B_n(t)$.

The initial condition $[u(x, 0) = f(x)]$ is satisfied if

$$f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin\frac{n\pi x}{L}, \tag{3.4.16}$$

determining the Fourier sine coefficient initially,

$$B_n(0) = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx. \tag{3.4.17}$$

All that remains is to investigate whether the Fourier sine series representation of $u(x, t)$, (3.4.15), can satisfy the heat equation, $\partial u/\partial t = k\,\partial^2 u/\partial x^2$. To do that, we must differentiate the Fourier sine series. It is here that our results concerning term-by-term differentiation are useful.

First, we need to compute two derivatives with respect to x. If $u(x, t)$ is continuous, then the Fourier sine series of $u(x, t)$ can be differentiated term by term if $u(0, t) = 0$ and $u(L, t) = 0$. Since these are exactly the boundary conditions on $u(x, t)$, it follows from (3.4.15) that

$$\frac{\partial u}{\partial x} \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos\frac{n\pi x}{L}. \tag{3.4.18}$$

Since $\partial u/\partial x$ is also assumed to be continuous, an equality holds in (3.4.18). Furthermore, the Fourier cosine series of $\partial u/\partial x$ can now be term-by-term differentiated, yielding

$$\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 B_n(t)\,\sin\frac{n\pi x}{L}. \tag{3.4.19}$$

Note the importance of the separation of variables solution. Sines were differentiated at the stage in which the boundary conditions occurred that allowed sines to be differentiated. Cosines occurred with no boundary condition, consistent with the fact that a Fourier cosine series does not need any subsidiary conditions in order to be differentiated. To complete the substitution of the Fourier sine series into the partial differential equation, we need only to compute $\partial u/\partial t$. If we can also term-by-term differentiate with respect to t, then

$$\frac{\partial u}{\partial t} \sim \sum_{n=1}^{\infty}\frac{dB_n}{dt}\,\sin\frac{n\pi x}{L}. \tag{3.4.20}$$

If this last term-by-term differentiation is justified, we see that the Fourier sine series (3.4.15) solves the partial differential equation if

$$\frac{dB_n}{dt} = -k\left(\frac{n\pi}{L}\right)^2 B_n(t). \tag{3.4.21}$$

The Fourier sine coefficient $B_n(t)$ satisfies a first-order linear differential equation with constant coefficients. The solution of (3.4.21) is

$$B_n(t) = B_n(0)e^{-(n\pi/L)^2 kt},$$

where $B_n(0)$ is given by (3.4.17). Thus, we have derived that (3.4.3) is valid, justifying the method of separation of variables.

Can we justify term-by-term differentiation with respect to the parameter t? The following theorem states the conditions under which this operation is valid:

The Fourier series of a continuous function $u(x, t)$ (depending on a parameter t)

$$u(x, t) = a_0(t) + \sum_{n=1}^{\infty}\left[a_n(t)\,\cos\frac{n\pi x}{L} + b_n(t)\,\sin\frac{n\pi x}{L}\right]$$

can be differentiated term by term with respect to the parameter t, yielding

$$\frac{\partial}{\partial t}u(x, t) \sim a_0'(t) + \sum_{n=1}^{\infty}\left[a_n'(t)\,\cos\frac{n\pi x}{L} + b_n'(t)\,\sin\frac{n\pi x}{L}\right]$$

if $\partial u/\partial t$ is piecewise smooth.

We omit its proof (see Exercise 3.4.7), which depends on the fact that

$$\frac{\partial}{\partial t}\int_{-L}^{L} g(x, t)\,dx = \int_{-L}^{L}\frac{\partial g}{\partial t}\,dx$$

is valid if g is continuous.

In summary, we have verified that the Fourier sine series is actually a solution of the heat equation satisfying the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$. Now we have two reasons for choosing a Fourier sine series for this problem. First, the method of separation of variables implies that if $u(0, t) = 0$ and $u(L, t) = 0$, then the appropriate eigenfunctions are $\sin n\pi x/L$. Second, we now see that all the differentiations of the infinite sine series are justified, where we need to assume that $u(0, t) = 0$ and $u(L, t) = 0$, exactly the physical boundary conditions.

Exercises 3.4

3.4.1. The integration-by-parts formula

$$\int_a^b u\frac{dv}{dx}\,dx = uv\Big|_a^b - \int_a^b v\frac{du}{dx}\,dx$$

is known to be valid for functions $u(x)$ and $v(x)$, which are continuous and have continuous first derivatives. However, we will assume that u, v, du/dx, and dv/dx are continuous only for $a \le x \le c$ and $c \le x \le b$; we assume that all quantities may have a jump discontinuity at $x = c$.

***(a)** Derive an expression for $\int_a^b u\, dv/dx\, dx$ in term of $\int_a^b v\, du/dx\, dx$.

(b) Show that this reduces to the integration-by-parts formula if u and v are continuous across $x = c$. It is *not* necessary for du/dx and dv/dx to be continuous at $x = c$.

3.4.2. Suppose that $f(x)$ and df/dx are piecewise smooth. Prove that the Fourier series of $f(x)$ can be differentiated term by term if the Fourier series of $f(x)$ is continuous.

3.4.3. Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0$, $f(x_0^-) = \alpha$ and $f(x_0^+) = \beta$] and df/dx is piecewise smooth.

***(a)** Determine the Fourier sine series of df/dx in terms of the Fourier cosine series coefficients of $f(x)$.

(b) Determine the Fourier cosine series of df/dx in terms of the Fourier sine series coefficients of $f(x)$.

3.4.4. Suppose that $f(x)$ and df/dx are piecewise smooth.

(a) Prove that the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$.

(b) Prove that the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

3.4.5. Using (3.3.13) determine the Fourier cosine series of $\sin \pi x/L$.

3.4.6. There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of e^x:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}. \tag{3.4.22}$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\frac{n\pi x}{L},$$

the Fourier sine series of e^x. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos\frac{n\pi x}{L}. \tag{3.4.23}$$

Since Equations (3.4.22) and (3.4.23) give the Fourier cosine series of e^x, they must be identical. Thus,

$$\left.\begin{aligned} A_0 &= 0 \\ A_n &= 0 \end{aligned}\right\} \quad \text{(obviously wrong!)}.$$

By correcting the mistakes, you should be able to obtain A_0 and A_n *without* using the typical technique, that is, $A_n = 2/L \int_0^L e^x \cos n\pi x/L\, dx$.

3.4.7. Prove that the Fourier series of a continuous function $u(x, t)$ can be differentiated term by term with respect to the parameter t if $\partial u/\partial t$ is piecewise smooth.

3.4.8. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to

$$\partial u/\partial x(0, t) = 0, \quad \partial u/\partial x(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x).$$

Solve in the following way. Look for the solution as a Fourier cosine series. Assume that u and $\partial u/\partial x$ are continuous and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Justify all differentiations of infinite series.

***3.4.9.** Consider the heat equation with a known source $q(x, t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x, t) \quad \text{with} \quad u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

Assume that $q(x, t)$ (for each $t > 0$) is a piecewise smooth function of x. Also assume that u and $\partial u/\partial x$ are continuous functions of x (for $t > 0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Thus,

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}.$$

Justify spatial term-by-term differentiation. What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

3.4.10. Modify Exercise 3.4.9 if instead $\partial u/\partial x(0, t) = 0$ and $\partial u/\partial x(L, t) = 0$.

3.4.11. Consider the *nonhomogeneous* heat equation (with a steady heat source):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x).$$

Solve this equation with the initial condition

$$u(x, 0) = f(x)$$

and the boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

Assume that a continuous solution exists (with continuous derivatives). [*Hints*: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to x.]

***3.4.12.** Solve the following *nonhomogeneous* problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t}\cos\frac{3\pi x}{L} \quad [\text{assume that } 2 \neq k(3\pi/L)^2]$$

subject to

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x).$$

Use the following method. Look for the solution as a Fourier cosine series. Justify all differentiations of infinite series (assume appropriate continuity).

3.4.13. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to

$$u(0, t) = A(t), \quad u(L, t) = 0, \quad \text{and} \quad u(x, 0) = g(x).$$

Assume that $u(x, t)$ has a Fourier sine series. Determine a differential equation for the Fourier coefficients (assume appropriate continuity).

3.5 Term-By-Term Integration of Fourier Series

In doing mathematical manipulations with infinite series, we must remember that some properties of finite series do not hold for infinite series. In particular, Section 3.4 indicated that we must be especially careful differentiating term by term an infinite Fourier series. The following theorem, however, enables us to integrate Fourier series without caution:

A Fourier series of piecewise smooth $f(x)$ can always be integrated term by term, and the result is a *convergent* infinite series that always *converges* to the integral of $f(x)$ for $-L \leq x \leq L$ (even if the original Fourier series has jump discontinuities).

Remarkably, the new series formed by term-by-term integration is continuous. However, the new series may not be a Fourier series.

To quantify this statement, let us suppose that $f(x)$ is piecewise smooth and hence has a Fourier series in the range $-L \le x \le L$ (not necessarily continuous):

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{3.5.1}$$

We will prove our claim that we can just integrate this result term by term:

$$\int_{-L}^{x} f(t)dt \sim a_0(x + L) + \sum_{n=1}^{\infty}\left(a_n \int_{-L}^{x} \cos\frac{n\pi t}{L}dt + b_n \int_{-L}^{x} \sin\frac{n\pi t}{L}dt\right).$$

Performing the indicated integration yields

$$\int_{-L}^{x} f(t)dt \sim a_0(x + L) + \sum_{n=1}^{\infty}\left[\frac{a_n}{n\pi/L}\sin\frac{n\pi x}{L} + \frac{b_n}{n\pi/L}\left(\cos n\pi - \cos\frac{n\pi x}{L}\right)\right]. \tag{3.5.2}$$

We will actually show that the preceding statement is valid with an $=$ sign. If term-by-term integration from $-L$ to x of a Fourier series is valid, then any definite integration is also valid since

$$\int_a^b = \int_{-L}^{b} - \int_{-L}^{a}.$$

EXAMPLE

Term-by-term integration has some interesting applications. Recall that the Fourier sine series for $f(x) = 1$ is given by

$$1 \sim \frac{4}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right), \tag{3.5.3}$$

where $\sim$ is used since (3.5.3) is an equality only for $0 < x < L$. Integrating term by term from 0 to x results in

$$x \sim \frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) - \frac{4L}{\pi^2}\left(\cos\frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \cdots\right), \quad 0 \le x \le L, \tag{3.5.4}$$

where because of our theorem the $=$ sign can be used. We immediately recognize that (3.5.4) should be the Fourier cosine series of the function x. It was obtained by integrating the Fourier sine series of $f(x) = 1$. However, an infinite series of constants appears in (3.5.4); it is the constant term of the Fourier cosine series of x. In this way we can evaluate that infinite series,

$$\frac{4L}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = \frac{1}{L}\int_0^L x\,dx = \frac{1}{2}L.$$

Thus, we obtain the usual form for the Fourier cosine series for x,

$$x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \cdots\right), \quad 0 \le x \le L. \tag{3.5.5}$$

The process of deriving new series from old ones can be continued. Integrating (3.5.5) from 0 to x yields

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{\sin 3\pi x/L}{3^3} + \frac{\sin 5\pi x/L}{5^3} + \cdots\right). \tag{3.5.6}$$

This example illustrates that *integrating a Fourier series term by term does not necessarily yield another Fourier series*. However, (3.5.6) can be looked at as either yielding

1. the Fourier sine series of $x^2/2 - (L/2)x$, or
2. the Fourier sine series of $x^2/2$, where the Fourier sine series of x is needed first [see (3.3.11) and (3.3.12)].

An alternative procedure is to perform indefinite integration. In this case an arbitrary constant must be included and evaluated. For example, reconsider the Fourier sine series of $f(x) = 1$, (3.5.3). By term-by-term indefinite integration, we derive the Fourier cosine series of x,

$$x = c - \frac{4L}{\pi^2}\left(\cos\frac{\pi x}{L} + \frac{\cos 3\pi x/L}{3^2} + \frac{\cos 5\pi x/L}{5^2} + \cdots\right).$$

The constant of integration is not arbitrary; it must be evaluated. Here c is again the constant term of the Fourier cosine series of x, $c = (1/L)\int_0^L x\,dx = L/2$. ■

PROOF ON INTEGRATING FOURIER SERIES.

Consider

$$F(x) = \int_{-L}^{x} f(t)\,dt. \tag{3.5.7}$$

This integral is a continuous function of x since $f(x)$ is piecewise smooth. $F(x)$ has a continuous Fourier series only if $F(L) = F(-L)$ [otherwise, remember that the periodic nature of the Fourier series implies that the Fourier series does not converge to $F(x)$ at the endpoints $x = \pm L$]. However, note that from the definition (3.5.7),

$$F(-L) = 0 \quad \text{and} \quad F(L) = \int_{-L}^{L} f(t)dt = 2La_0.$$

Thus, in general, $F(x)$ does not have a continuous Fourier series. In Fig. 3.5.1, $F(x)$ is sketched, illustrating the fact that usually $F(-L) \neq F(L)$. However, consider the straight line connecting the point $F(-L)$ to $F(L)$, $y = a_0(x + L)$. $G(x)$, defined to be the difference between $F(x)$ and the straight line,

$$G(x) \equiv F(x) - a_0(x + L), \tag{3.5.8}$$

will be zero at both ends, $x = \pm L$,

$$G(-L) = G(L) = 0,$$

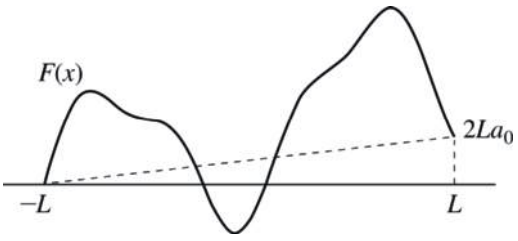
as illustrated in Fig. 3.5.1. $G(x)$ is also continuous. Thus, $G(x)$ satisfies the properties that enable the Fourier series of $G(x)$ actually to equal $G(x)$:

$$G(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right), \tag{3.5.9}$$

where the = sign is emphasized. These Fourier coefficients can be computed as

$$A_n = \frac{1}{L}\int_{-L}^{L}(F(x) - a_0(x + L))\cos\frac{n\pi x}{L}dx \quad (n \neq 0).$$

Figure 3.5.1 $F(x)$ with $F(-L) \neq F(L)$.



The x-term can be dropped since it is odd (i.e., $\int_{-L}^{L} x\cos n\pi x/L\,dx = 0$). The resulting expression can be integrated by parts as follows:

$$u = F(x) - a_0L, \qquad dv = \cos\frac{n\pi x}{L}dx$$

$$du = \frac{dF}{dx}dx = f(x)dx, \qquad v = \frac{L}{n\pi}\sin\frac{n\pi x}{L},$$

yielding

$$A_n = \frac{1}{L}\left[(F(x) - a_0L)\left.\frac{\sin n\pi x/L}{n\pi/L}\right|_{-L}^{L} - \frac{L}{n\pi}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\right] = -\frac{b_n}{n\pi/L}, \tag{3.5.10}$$

where we have recognized that b_n is the Fourier sine coefficient of $f(x)$. In a similar manner (which we leave as an exercise), it can be shown that

$$B_n = \frac{a_n}{n\pi/L},$$

where a_n is the Fourier cosine coefficient of $f(x)$. A_0 can be calculated in a different manner (the previous method will not work). Since $G(L) = 0$ and the Fourier series of $G(x)$ is pointwise convergent, from (3.5.9) it follows that

$$0 = A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi = A_0 - \sum_{n=1}^{\infty} \frac{b_n}{n\pi/L} \cos n\pi$$

since $A_n = -b_n/(n\pi/L)$. Thus, we have shown from (3.5.9) that

$$F(x) = a_0(x + L) + \sum_{n=1}^{\infty} \left[\frac{a_n}{n\pi/L} \sin \frac{n\pi x}{L} + \frac{b_n}{n\pi/L} \left(\cos n\pi - \cos \frac{n\pi x}{L} \right) \right], \tag{3.5.11}$$

exactly the result of simple term-by-term integration. However, notice that (3.5.11) is *not* the Fourier series of $F(x)$, since $a_0 x$ appears. Nonetheless, (3.5.11) is valid. We have now justified term-by-term integration of Fourier series.

Exercises 3.5

3.5.1. Consider

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{3.5.12}$$

(a) Determine b_n from (3.3.11), (3.3.12), and (3.5.6).

(b) For what values of x is (3.5.12) an equality?

***(c)** Derive the Fourier cosine series for x^3 from (3.5.12). For what values of x will this be an equality?

3.5.2. **(a)** Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of x^2.

(b) From part (a), determine the Fourier sine series of x^3.

3.5.3. Generalize Exercise 3.5.2, in order to derive the Fourier sine series of x^m, m odd.

***3.5.4.** Suppose that $\cosh x \sim \sum_{n=1}^{\infty} b_n \sin n\pi x/L$.

(a) Determine b_n by correctly differentiating this series twice.

(b) Determine b_n by integrating this series twice.

3.5.5. Show that B_n in (3.5.9) satisfies $B_n = a_n/(n\pi/L)$, where a_n is defined by (3.5.1).

3.5.6. Evaluate

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

by evaluating (3.5.5) at $x = 0$.

***3.5.7.** Evaluate

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

using (3.5.6).

3.6 Complex form of Fourier Series

With periodic boundary conditions, we have found the theory of Fourier series to be quite useful:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{3.6.1}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx \tag{3.6.2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx \tag{3.6.3}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx. \tag{3.6.4}$$

To introduce complex exponentials instead of sines and cosines, we use Euler's formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

It follows that

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{in\pi x/L} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)e^{-in\pi x/L}. \tag{3.6.5}$$

In order to have only $e^{-in\pi x/L}$, we change the dummy index in the first summation, replacing n by $-n$. Thus,

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}[a_{(-n)} - ib_{(-n)}]e^{-in\pi x/L} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)e^{-in\pi x/L}.$$

From the definition of a_n and b_n, (3.6.3) and (3.6.4), $a_{(-n)} = a_n$ and $b_{(-n)} = -b_n$. Thus, if we define

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{a_n + ib_n}{2}, \end{aligned}$$

then $f(x)$ becomes simply

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}.} \tag{3.6.6}$$

Equation (3.6.6) is known as the **complex form of the Fourier series of $f(x)$**.[4] It is equivalent to the usual form. It is more compact to write, but it is used only infrequently. In this form the complex Fourier coefficients are

$$c_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right)dx \quad (n \neq 0).$$

We immediately recognize a simplification, using Euler's formula. Thus, we derive a formula for the complex Fourier coefficients

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx} \quad (\text{all } n). \tag{3.6.7}$$

Notice that the complex Fourier series representation of $f(x)$ has $e^{-in\pi x/L}$ and is summed over the discrete integers corresponding to the sum over the discrete eigenvalues. The complex Fourier coefficients, on the other hand, involve $e^{+in\pi x/L}$ and are integrated over the region of definition of $f(x)$ (with periodic boundary conditions), namely, $-L \leq x \leq L$. If $f(x)$ is real, $c_{-n} = \overline{c}_n$ (see Exercise 3.6.2).

COMPLEX ORTHOGONALITY

There is an alternative way to derive the formula for the complex Fourier coefficients. Always, in the past, we have determined Fourier coefficients using the orthogonality of the eigenfunctions. A similar idea holds here. However, here the eigenfunctions $e^{-in\pi x/L}$ are complex. For complex functions the concept of orthogonality must be slightly modified. A complex function ϕ is said to be orthogonal to a complex function ψ (over an interval $a \leq x \leq b$) if $\int_a^b \overline{\phi}\psi\, dx = 0$, where $\overline{\phi}$ is the complex conjugate of ϕ. This guarantees that the length squared of a complex function f, defined by $\int_a^b \overline{f}f\, dx$, is positive (this would not have been valid for $\int_a^b ff\, dx$ since f is complex).

Using this notion of orthogonality, the eigenfunctions $e^{-in\pi x/L}$, $-\infty < n < \infty$, can be verified to form an orthogonal set because by simple integration

$$\int_{-L}^{L} \overline{(e^{-im\pi x/L})}e^{-in\pi x/L}\, dx = \begin{cases} 0 & n \neq m \\ 2L & n = m, \end{cases}$$

[4] As before, an equal sign appears if $f(x)$ is continuous [and periodic, $f(-L) = f(L)$]. At a jump discontinuity of $f(x)$ in the interior, the series converges to $[f(x+) + f(x-)]/2$.

since

$$\overline{(e^{-im\pi x/L})} = e^{im\pi x/L}.$$

Now to determine the complex Fourier coefficients c_n, we multiply (3.6.6) by $e^{im\pi x/L}$ and integrate from $-L$ to $+L$ (assuming that the term-by-term use of these operations is valid). In this way

$$\int_{-L}^{L} f(x)e^{im\pi x/L}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} e^{im\pi x/L}e^{-in\pi x/L}\,dx.$$

Using the orthogonality condition, the sum reduces to one term, $n = m$. Thus,

$$\int_{-L}^{L} f(x)e^{im\pi x/L}\,dx = 2Lc_m,$$

which explains the $1/2L$ in (3.6.7) as well as the switch of signs in the exponent.

Exercises 3.6

***3.6.1.** Consider

$$f(x) = \begin{cases} 0 & x < x_0 \\ 1/\Delta & x_0 < x < x_0 + \Delta \\ 0 & x > x_0 + \Delta. \end{cases}$$

Assume that $x_0 > -L$ and $x_0 + \Delta < L$. Determine the complex Fourier coefficients c_n.

3.6.2. If $f(x)$ is real, show that $c_{-n} = \overline{c}_n$.

4

Wave Equation: Vibrating Strings and Membranes

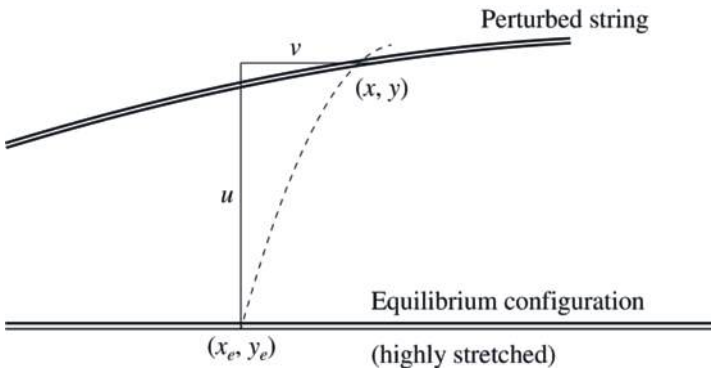
4.1 Introduction

At this point in our study of partial differential equations, the only physical problem we have introduced is the conduction of heat. To broaden the scope of our discussions, we now investigate the vibrations of perfectly elastic strings and membranes. We begin by formulating the governing equations for a vibrating string from physical principles. The appropriate boundary conditions will be shown to be similar in a mathematical sense to those boundary conditions for the heat equations. Examples will be solved by the method of separation of variables.

4.2 Derivation of a Vertically Vibrating String

A vibrating string is a complicated physical system. We would like to present a simple derivation. A string vibrates only if it is tightly stretched. Consider a horizontally stretched string in its equilibrium configuration, as illustrated in Fig. 4.2.1. We imagine that the ends are tied down in some way (to be described in Section 4.3), maintaining the tightly stretched nature of the string. You may wish to think of stringed musical instruments as examples. We begin by tracking the motion of each particle that comprises the string. We let x_e be the x-coordinate of a particle when the string is in the equilibrium position. The string moves in time; it is located somewhere other than the equilibrium position at time t, as illustrated in Fig. 4.2.1. The trajectory of a particle x_e is indicated with both horizontal and vertical components.

Figure 4.2.1 Vertical and horizontal displacements of a particle on a highly stretched string.



We will assume the slope of the string is small, in which case it can be shown that the horizontal displacement v can be neglected. As an approximation, **the motion is entirely vertical**, $x = x_e$ for all t. In this situation, the vertical displacement u depends on x and t:

$$y = u(x, t). \tag{4.2.1}$$

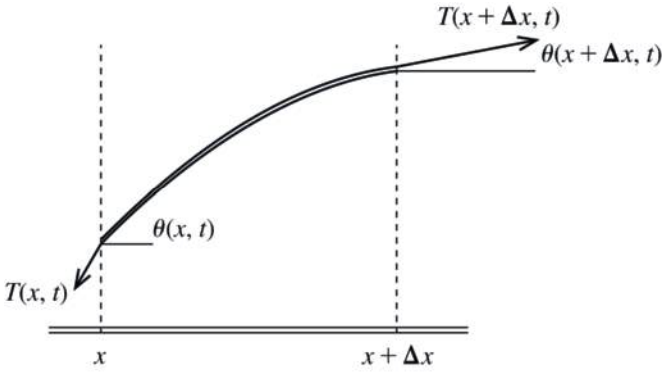
Derivations including the effect of a horizontal displacement are necessarily complicated (see Weinberger [1965] and Antman [1980]). In general ($x \neq x_e$), it is best to let $y = u(x_e, t)$.

Newton's law

We consider an infinitesimally thin segment of the string contained between x and $x + \Delta x$ (as illustrated in Fig. 4.2.2). In the unperturbed (yet stretched) horizontal position, we assume that the **mass density** $\rho_0(x)$ is known. For the thin segment, the total mass is approximately $\rho_0(x)\ \Delta x$. Our object is to derive a partial differential equation describing how the displacement u changes in time. Accelerations are due to forces; we must use Newton's law. For simplicity we will analyze Newton's law for a point mass:

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{4.2.2}$$

Figure 4.2.2 Stretching of a finite segment of string, illustrating the tensile forces.



We must discuss the forces acting on this segment of the string. There are body forces, which we assume act only in the vertical direction (e.g., the gravitational force), as well as forces acting on the ends of the segment of string. We assume that the string is **perfectly flexible**; it offers no resistance to bending. This means that the force exerted by the rest of the string on the endpoints of the segment of the string is in the direction tangent to the string. This tangential force is known as the **tension** in the string, and we denote its magnitude by $T(x, t)$. In Fig. 4.2.2, we show that the force due to the tension (exerted by the rest of the string) pulls at both ends in the direction of the tangent, trying to stretch the small segment. To obtain components of the tensile force, the angle θ between the horizon and the string is introduced. The angle depends on both the position x and time t. Furthermore, the slope of the string may be represented as either dy/dx or $\tan\theta$:

$$\text{slope} = \frac{dy}{dx} = \tan\theta(x, t) = \frac{\partial u}{\partial x}. \tag{4.2.3}$$

The horizontal component of Newton's law prescribes the horizontal motion, which we claim is small and can be neglected. The vertical equation of motion states that the mass $\rho_0(x)\ \Delta x$ times the vertical component of acceleration ($\partial^2 u/\partial t^2$, where $\partial/\partial t$ is used since x is fixed for this motion) equals the vertical component of the tensile forces plus the vertical component of the body forces:

$$\begin{aligned}\rho_0(x)\ \Delta x\frac{\partial^2 u}{\partial t^2} = {} & T(x + \Delta x, t)\ \sin\theta(x + \Delta x, t) \\ & - T(x, t)\ \sin\theta(x, t) + \rho_0(x)\ \Delta x\ Q(x, t),\end{aligned} \tag{4.2.4}$$

where $T(x, t)$ is the magnitude of the tensile force and $Q(x, t)$ is the vertical component of the body force per unit mass. Dividing (4.2.4) by Δx and taking the limit as $\Delta x \to 0$ yields

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}[T(x, t)\ \sin\theta(x, t)] + \rho_0(x)Q(x, t). \tag{4.2.5}$$

For small angles θ,

$$\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta,$$

and hence (4.2.5) becomes

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) + \rho_0(x)Q(x, t). \tag{4.2.6}$$

Perfectly elastic strings

The tension of a string is determined by experiments. Real strings are nearly **perfectly elastic**, by which we mean that the magnitude of the tensile force $T(x, t)$ depends only on the local stretching of the string. Since the angle θ is assumed to be small, the stretching of the string is nearly the same as for the unperturbed highly stretched horizontal string, where the tension is constant, T_0 (to be in equilibrium). Thus, **the tension $T(x, t)$ may be approximated by a constant T_0**. Consequently, the small vertical vibrations of a highly stretched string are governed by

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x, t)\rho_0(x). \tag{4.2.7}$$

One-dimensional wave equation

If the only body force per unit mass is gravity, then $Q(x, t) = -g$ in (4.2.7). In many such situations, this force is small (relative to the tensile force $\rho_0 g \ll |T_0\, \partial^2 u/\partial x^2|$) and can be neglected. Alternatively, gravity sags the string, and we can calculate the vibrations with respect to the sagged equilibrium position. In either way we are often led to investigate (4.2.7) in the case in which $Q(x, t) = 0$,

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} \tag{4.2.8}$$

or

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},} \tag{4.2.9}$$

where $c^2 = T_0/\rho_0(x)$. Equation (4.2.9) is called the **one-dimensional wave equation**. The notation c^2 is introduced because $T_0/\rho_0(x)$ has the dimension of velocity squared. We will show that c is a very important velocity. For a uniform string, c is constant.

Exercises 4.2

4.2.1. **(a)** How small does θ need to be so that replacing $\sin\theta$ with $\tan\theta$ results in a relative error of less than 1%?

(b) The length of a guitar string is 65 cm and it is displaced upward in the very center by 1 cm. At one end of the string, what angle does it make with the horizontal direction?

4.2.2. **(a)** Using Equation (4.2.7), compute the sagged equilibrium position $u_E(x)$ if $Q(x, t) = -g$. The boundary conditions are $u(0) = 0$ and $u(L) = 0$.

(b) Show that $v(x, t) = u(x, t) - u_E(x)$ satisfies (4.2.9).

4.2.3. Show that c^2 has the dimensions of velocity squared.

4.2.4. Consider a particle whose x-coordinate (in horizontal equilibrium) is designated by x_e. If its vertical and horizontal displacements are u and v, respectively, determine its position x and y. Then show that

$$\frac{dy}{dx} = \frac{\partial u/\partial x_e}{1 + \partial v/\partial x_e}.$$

4.2.5. Derive equations for horizontal and vertical displacements without ignoring v. Assume that the string is perfectly flexible and that the tension is determined by an experimental law.

4.2.6. Derive the partial differential equation for a vibrating string in the simplest possible manner. You may assume the string has constant mass density ρ_0, you may assume the tension T_0 is constant, and you may assume small displacements (with small slopes).

4.3 Boundary Conditions

The partial differential equation for a vibrating string, (4.2.7) or (4.2.8), has a second-order spatial partial derivative. We will apply one boundary condition at each end, just as we did for the one-dimensional heat equation.

The simplest boundary condition is that of a fixed end, usually fixed with zero displacement. For example, if a string is fixed (with zero displacement) at $x = L$, then

$$u(L, t) = 0. \tag{4.3.1}$$

Alternatively, we might vary an end of the string in a prescribed way:

$$u(L, t) = f(t). \tag{4.3.2}$$

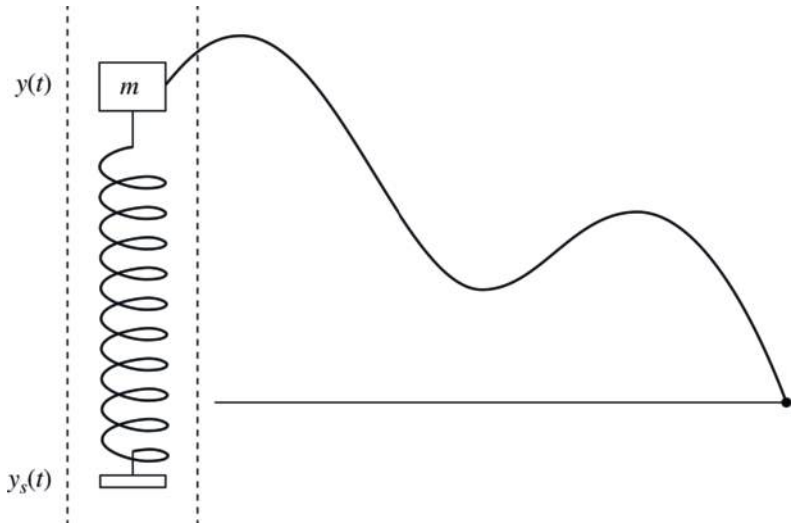
Both (4.3.1) and (4.3.2) are linear boundary conditions; (4.3.2) is nonhomogeneous, while (4.3.1) is homogeneous.

Variable support boundary conditions (optional)

A more unusual boundary condition occurs if one end of the string is attached to a dynamical system. Let us suppose that the left end, $x = 0$, of a string is attached to a spring-mass system, as illustrated in Fig. 4.3.1. We will insist that the motion be entirely vertical. To accomplish this, we must envision the mass to be on a vertical track (possibly frictionless). The track applies a horizontal force to the mass when necessary to prevent the large horizontal component of the tensile force from turning over the spring-mass system. The string is attached to the mass so that if the position of the mass is $y(t)$, so is the position of the left end:

$$u(0, t) = y(t). \tag{4.3.3}$$

Figure 4.3.1 Spring-mass system with a variable support attached to a stretched string.



However, $y(t)$ is unknown and itself satisfies an ordinary differential equation determined from Newton's laws. We assume that the spring has unstretched length l and obeys Hooke's law with spring constant k. To make the problem even more interesting, we let the support of the spring move in some prescribed way, $y_s(t)$. Thus, the length of the spring is $y(t) - y_s(t)$ and the stretching of the spring is $y(t) - y_s(t) - l$. According to Newton's law (using Hooke's law with spring constant k),

$$m\frac{d^2y}{dt^2} = -k(y(t) - y_s(t) - l) + \text{other forces on mass.}$$

The other vertical forces on the mass are a tensile force applied by the string $T(0, t)\sin\theta(0, t)$ and a force $g(t)$ representing any other external forces on the mass. Recall that we must be restricted to small angles, such that the tension is nearly constant, T_0. In that case, the vertical component is approximately $T_0 \partial u/\partial x$:

$$T(0, t)\sin\theta(0, t) \approx T(0, t)\frac{\sin\theta(0, t)}{\cos\theta(0, t)} = T(0, t)\frac{\partial u}{\partial x}(0, t) \approx T_0\frac{\partial u}{\partial x}(0, t),$$

since for small angles $\cos\theta \approx 1$. In this way, the boundary condition at $x = 0$ for a vibrating string attached to a spring-mass system [with a variable support $y_s(t)$ and an external force $g(t)$] is

$$\boxed{m\frac{d^2}{dt^2}u(0, t) = -k(u(0, t) - y_s(t) - l) + T_0\frac{\partial u}{\partial x}(0, t) + g(t).} \tag{4.3.4}$$

Let us consider some special cases in which there are no external forces on the mass, $g(t) = 0$. If, in addition, the mass is sufficiently small so that the forces on the mass are in balance, then

$$T_0\frac{\partial u}{\partial x}(0, t) = k(u(0, t) - u_E(t)), \tag{4.3.5}$$

where $u_E(t)$ is the equilibrium position of the mass, $u_E(t) = y_s(t) + l$. This form, known as the nonhomogeneous **elastic** boundary condition, is exactly analogous to Newton's law of cooling [with an external temperature of $u_E(t)$] for the heat equation. If the equilibrium position of the mass coincides with the equilibrium position of the string, $u_E(t) = 0$, the homogeneous version of the elastic boundary condition results:

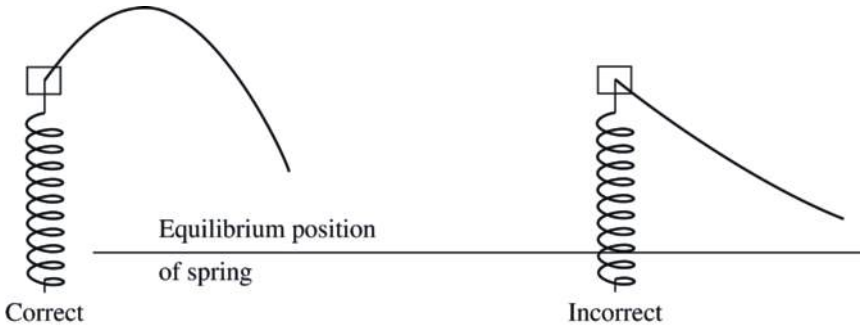
$$T_0\frac{\partial u}{\partial x}(0, t) = ku(0, t). \tag{4.3.6}$$

$\partial u/\partial x$ is proportional to u. Since for physical reasons $T_0 > 0$ and $k > 0$, the signs in (4.3.6) are prescribed. This is the same choice of signs that occurs for Newton's law of cooling. A diagram (Fig. 4.3.2) illustrates both the correct and incorrect choices of signs. This figure shows that (assuming $u = 0$ is an equilibrium position for both string and mass) if $u > 0$ at $x = 0$, then $\partial u/\partial x > 0$, in order to get a balance of vertical forces on the *massless* spring-mass system. A similar argument shows that there is an important sign change if the elastic boundary condition occurs at $x = L$:

$$T_0\frac{\partial u}{\partial x}(L, t) = -k(u(L, t) - u_E(t)), \tag{4.3.7}$$

the same sign change we obtained for Newton's law of cooling.

Figure 4.3.2 Boundary conditions for massless spring-mass system.



For a vibrating string, another boundary condition that can be discussed is the **free end**. It is not literally free. Instead, the end is attached to a frictionless vertically moving track as before and is free to move up and down. There is no spring-mass system, nor external forces. However, we can obtain this boundary condition by taking the limit as $k \to 0$ of either (4.3.6) or (4.3.7):

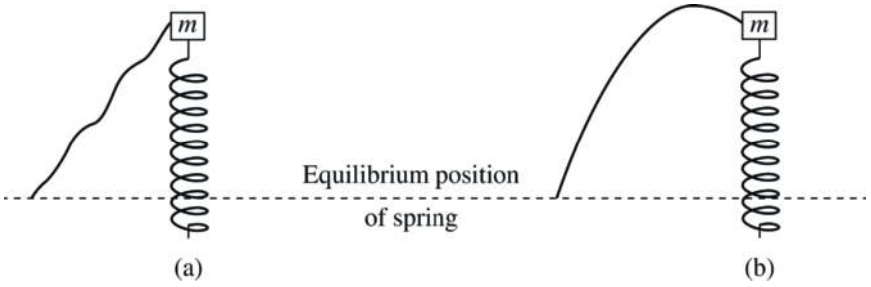
$$T_0\frac{\partial u}{\partial x}(L, t) = 0. \tag{4.3.8}$$

This says that the vertical component of the tensile force must vanish at the end since there are no other vertical forces at the end. If the vertical component did not vanish, the end would have an infinite vertical acceleration. Boundary condition (4.3.8) is exactly analogous to the insulated boundary condition for the one-dimensional heat equation.

Exercises 4.3

4.3.1. If $m = 0$, which of the diagrams for the right end shown in Fig. 4.3.3 is possibly correct? *Briefly* explain. Assume that the mass can move only vertically.

Figure 4.3.3

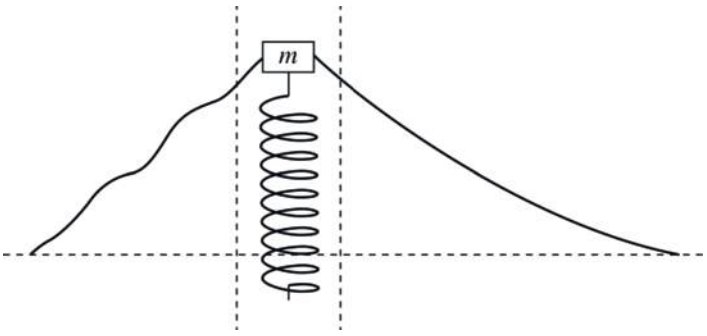


4.3.2. Consider two vibrating strings connected at $x = L$ to a spring-mass system on a vertical frictionless track as in Fig. 4.3.4. Assume that the spring is unstretched when the string is horizontal (the spring has a fixed support). Also suppose that there is an external force $f(t)$ acting on the mass m.

(a) What "jump" conditions apply at $x = L$ relating the string on the left to the string on the right?

(b) In what situations is this mathematically analogous to perfect thermal contact?

Figure 4.3.4



4.4 Vibrating String with Fixed Ends

In this section, we solve the one-dimensional wave equation, which represents a uniform vibrating string without external forces,

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},} \tag{4.4.1}$$

where $c^2 = T_0/\rho_0$, subject to the simplest homogeneous boundary conditions,

$$\text{BC1:} \quad \boxed{\begin{aligned} u(0, t) &= 0 \\ u(L, t) &= 0, \end{aligned}} \tag{4.4.2}$$

both ends being fixed with zero displacement. Since the partial differential equation (4.4.1) contains the second time derivative, two initial conditions are needed. We prescribe both $u(x, 0)$ and $\partial u/\partial t(x, 0)$:

$$\text{IC:} \quad \boxed{\begin{array}{l} u(x, 0) = f(x) \\ \dfrac{\partial u}{\partial t}(x, 0) = g(x), \end{array}} \tag{4.4.3}$$

corresponding to being given the initial position and the initial velocity of each segment of the string. These two initial conditions are not surprising, as the wave equation was derived from Newton's law by analyzing each segment of the string as a particle; ordinary differential equations for particles require both initial position and velocity.

Since both the partial differential equation and the boundary conditions are linear and homogeneous, the method of separation of variables is attempted. As with the heat equation, the nonhomogeneous initial conditions are put aside temporarily. We look for special product solutions of the form:

$$u(x, t) = \phi(x)h(t). \tag{4.4.4}$$

Substituting (4.4.4) into (4.4.1) yields

$$\phi(x)\frac{d^2h}{dt^2} = c^2h(t)\frac{d^2\phi}{dx^2}. \tag{4.4.5}$$

Dividing by $\phi(x)h(t)$ separates the variables, but it is more convenient to divide additionally by the constant c^2, since then the resulting eigenvalue problem will not contain the parameter c^2:

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda. \tag{4.4.6}$$

A separation constant is introduced since $(1/c^2)(1/h)(d^2h/dt^2)$ depends only on t and $(1/\phi)(d^2\phi/dx^2)$ depends only on x. The minus sign is inserted *purely for convenience*. With this minus sign, let us explain why we expect that $\lambda > 0$. We need the two ordinary differential equations that follow from (4.4.6):

$$\frac{d^2h}{dt^2} = -\lambda c^2h \tag{4.4.7}$$

and

$$\frac{d^2\phi}{dx^2} = -\lambda\phi. \tag{4.4.8}$$

The two homogeneous boundary conditions (4.4.2) show that

$$\phi(0) = \phi(L) = 0. \tag{4.4.9}$$

Thus, (4.4.8) and (4.4.9) form a boundary value problem. Instead of first reviewing the solution of (4.4.8) and (4.4.9), let us analyze the time-dependent ODE (4.4.7). If $\lambda > 0$, the general solution of (4.4.7) is a linear combination of sines and cosines,

$$h(t) = c_1 \cos c\sqrt{\lambda}t \ + c_2 \sin c\sqrt{\lambda}t. \tag{4.4.10}$$

If $\lambda = 0, h(t) = c_1 + c_2t$, and if $\lambda < 0$, $h(t)$ is a linear combination of exponentially growing and decaying solutions in time. Since we are solving a vibrating string, it should seem more reasonable that the time-dependent solutions oscillate. This does not prove that $\lambda > 0$. Instead, it serves as an immediate motivation for choosing the minus sign in (4.4.6). Now by analyzing the boundary value problem, we may indeed determine that the eigenvalues are nonnegative.

The boundary value problem is

$$\begin{aligned} &\frac{d^2\phi}{dx^2} = -\lambda\phi \\ &\phi(0) = 0 \\ &\phi(L) = 0. \end{aligned}$$

Although we could solve this by proceeding through three cases, we ought to recall that all the eigenvalues are positive. In fact,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3 \ldots,$$

and the corresponding eigenfunctions are $\sin n\pi x/L$. The time-dependent part of the solution has been obtained previously, (4.4.10). Thus, there are two families of product solutions: $\sin n\pi x/L \sin n\pi ct/L$ and $\sin n\pi x/L \cos n\pi ct/L$. The principle of superposition then implies that we should be able to solve the initial value problem by considering a linear combination of all product solutions:

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi x}{L} \sin\frac{n\pi ct}{L}\right). \tag{4.4.11}$$

The initial conditions (4.4.3) are satisfied if

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \\ g(x) &= \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\frac{n\pi x}{L}. \end{aligned} \tag{4.4.12}$$

The boundary conditions have implied that sine series are important. There are two initial conditions and two families of coefficients to be determined. From our previous work on Fourier sine series, we know that $\sin n\pi x/L$ forms an orthogonal set. A_n will be the coefficients of the Fourier sine series of $f(x)$, and $B_n n\pi c/L$ will be for the Fourier sine series of $g(x)$:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \\ B_n \frac{n\pi c}{L} &= \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx. \end{aligned} \tag{4.4.13}$$

EXAMPLE

Consider the one-dimensional wave equation (4.4.1) defined on $0 < x < 5$, with homogeneous boundary conditions (4.4.2) at $x = 0$ and $x = 5$, and with initial conditions

$$u(x, 0) = x, \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad (0 < x < 5).$$

At $t = 0$, the string is displaced away from its equilibrium position, and has no initial velocity. The solution is given by (4.4.11) with

$$A_n = \frac{2}{5}\int_0^5 x \sin\frac{n\pi x}{5}\, dx = \frac{10}{n\pi}(-1)^{n+1} \quad \text{and} \quad B_n = 0.$$

That is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{10}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{5} \cos\frac{n\pi ct}{5}.$$

■

EXAMPLE

In this problem, we solve the wave equation (4.4.1) defined on $0 < x < 3$, with homogeneous boundary conditions (4.4.2) at $x = 0$ and $x = 3$, and with initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = \sin 2\pi x, \quad (0 < x < 3).$$

Again, the solution is given by the infinite series (4.4.11), this time with $L = 3$, and coefficients

$$A_n = 0 \quad \text{and} \quad B_n = \frac{2}{n\pi c}\int_0^3 \sin 2\pi x \sin\frac{n\pi x}{3}\, dx.$$

To evaluate B_n, note that $\sin 2\pi x$ and $\sin \frac{n\pi x}{3}$ are orthogonal on $0 < x < 3$ for all $n \neq 6$, and when $n = 6$, the integral evaluates to $\int_0^3 \sin^2 2\pi x \, dx = 3/2$. Thus, the only nonzero coefficient is B_6 and the solution is

$$u(x, t) = \frac{1}{2\pi c} \sin 2\pi x \sin 2\pi c t \qquad \blacksquare$$

Let us interpret these results in the context of musical stringed instruments (with fixed ends). The vertical displacement is composed of a linear combination of simple product solutions,

$$\sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi c t}{L} \right).$$

These are called the **normal modes** of vibration. The intensity of the sound produced depends on the amplitude,[1] $\sqrt{A_n^2 + B_n^2}$. The time dependence is simple harmonic with **circular frequency** (the number of oscillations in 2π units of time) equaling $n\pi c/L$, where $c = \sqrt{T_0/\rho_0}$. The sound produced consists of the superposition of these infinite number of **natural frequencies** $(n = 1, 2, \ldots)$. The normal mode $n = 1$ is called the first harmonic or fundamental. In the case of a vibrating string, the fundamental mode has a circular frequency of $\pi c/L$.[2] The larger the natural frequency, the higher the pitch of the sound produced. To produce a desired fundamental frequency, $c = \sqrt{T_0/\rho_0}$ or L can be varied. Usually, the mass density is fixed. Thus, the instrument is tuned by varying the tension T_0; the larger T_0, the higher the fundamental frequency. While playing a stringed instrument, the musician can also vary the pitch by varying the effective length L, by clamping down the string. Shortening L makes the note higher. The nth normal mode is called the nth harmonic. For vibrating strings (with fixed ends), the frequencies of the higher harmonics are all integral multiples of the fundamental. It is not necessarily true for other types of musical instruments. This is thought to be pleasing to the ear.

Let us attempt to illustrate the motion associated with each normal mode. The fundamental and higher harmonics are sketched in Fig. 4.4.1. To indicate what these look like, we sketch for various values of t. At each t, each mode looks like a simple oscillation in x. The amplitude varies periodically in time. These are called **standing waves**. In all cases there is no displacement at both ends due to the boundary conditions. For the second harmonic $(n = 2)$, the displacement is also zero for all time in the middle $x = L/2$. $x = L/2$ is called a **node** for the second harmonic. Similarly, there are two nodes for the third harmonic. This can be generalized: The nth harmonic has $n - 1$ nodes.[3]

It is interesting to note that the vibration corresponding to the second harmonic looks like two identical strings each with length $L/2$ vibrating in the fundamental mode, since $x = L/2$ is a node. We should find that the frequencies of vibration are identical; that is, the frequency for the fundamental $(n = 1)$ with length $L/2$ should equal the frequency for the second harmonic $(n = 2)$ with length L. The formula for the frequency $\omega = n\pi c/L$ verifies this observation.

Each standing wave can be shown to be composed of two traveling waves. For example, consider the term $\sin n\pi x/L \sin n\pi c t/L$. From trigonometric identities,

$$\sin \frac{n\pi x}{L} \sin \frac{n\pi c t}{L} = \underbrace{\frac{1}{2} \cos \frac{n\pi}{L}(x - ct)}_{\substack{\text{wave traveling} \\ \text{to the right} \\ (\text{with velocity } c)}} - \underbrace{\frac{1}{2} \cos \frac{n\pi}{L}(x + ct)}_{\substack{\text{wave traveling} \\ \text{to the left} \\ (\text{with velocity } -c)}} . \tag{4.4.14}$$

In fact, since the solution (4.4.11) to the wave equation consists of a superposition of standing waves, it can be shown that this solution is a combination of just two waves (each rather complicated)—one traveling to the left at velocity $-c$ with fixed shape and the other to the right

MATLAB Files

Download Animation of solution (.zip)

https://bit.ly/4d7AYss

These files are provided as zipped .m files for use in MATLAB on desktop platform.

[1] $A_n \cos \omega t + B_n \sin \omega t = \sqrt{A_n^2 + B_n^2} \sin(\omega t + \theta)$, where $\theta = \tan^{-1} A_n/B_n$.

[2] Frequencies are usually measured in cycles per second, not cycles per 2π units of time. The fundamental thus has a frequency of $c/2L$, cycles per second.

[3] You can visualize experimentally this result by rapidly oscillating at the appropriate frequency one end of a long rope that is tightly held at the other end. The result appears more easily for an expandable spiral telephone cord or a "slinky."

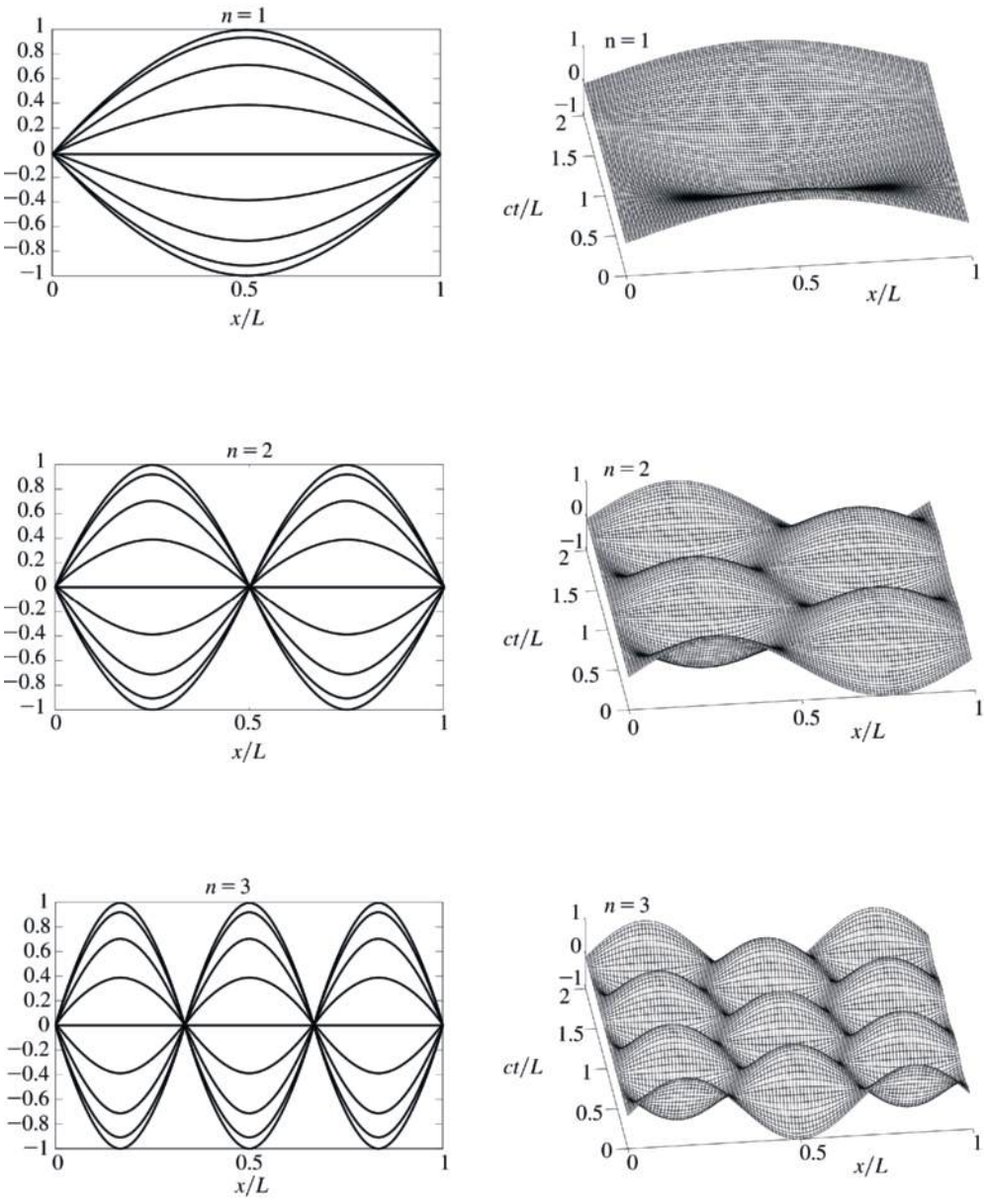
MATLAB Files

Download Animation of Figure 4.4.1 (.zip)

https://bit.ly/3Q7SiDK

These files are provided as zipped .m files for use in MATLAB on desktop platform.

Figure 4.4.1 Normal modes of vibration for a string.



at velocity c with a different fixed shape. We are claiming that the solution to the one-dimensional wave equation can be written as

$$u(x, t) = R(x - ct) + (x + ct),$$

even if the boundary conditions are not fixed at $x = 0$ and $x = L$. We will show and discuss this further in the Exercises and in Chapter 12.

Exercises 4.4

4.4.1. Solve the wave equation (4.4.1) defined on $0 < x < 4$, with homogeneous boundary conditions (4.4.2) at $x = 0$ and $x = 4$, and with initial conditions

$$u(x, 0) = \begin{cases} 0 & 0 \le x \le 1 \\ 1 & 1 < x \le 2 \\ 0 & 2 < x \le 4 \end{cases}, \quad \frac{\partial u}{\partial t}(x, 0) = 0.$$

4.4.2. Solve the wave equation (4.4.1) defined on $0 < x < 2$, with homogeneous boundary conditions (4.4.2) at $x = 0$ and $x = 2$, and with initial conditions

$$u(x, 0) = \sin \pi x, \quad \frac{\partial u}{\partial t}(x, 0) = 0.$$

4.4.3. Solve the wave equation (4.4.1) defined on $0 < x < 10$, with homogeneous boundary conditions (4.4.2) at $x = 0$ and $x = 10$, and with initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = x(10 - x).$$

4.4.4. Consider vibrating strings of uniform density ρ_0 and tension T_0.

***(a)** What are the natural frequencies of a vibrating string of length L fixed at both ends?

***(b)** What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H, t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.

(c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

4.4.5. In Section 4.2, it was shown that the displacement u of a nonuniform string satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q,$$

where Q represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

(a) Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.

(b) Separate variables if $\rho_0(x)$ and $\alpha(x)$ but T_0 is constant for physical reasons. Analyze the time-dependent ordinary differential equation.

***(c)** Specialize part (b) to the constant coefficient case. Solve the initial value problem if $\alpha < 0$:

$$u(0, t) = 0, \quad u(x, 0) = 0$$
$$u(L, t) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = f(x).$$

What are the frequencies of vibration?

4.4.6. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

***(b)** Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0$$

and the initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

4.4.7. Redo Exercise 4.4.6(b) if $4\pi^2 \rho_0 T_0 / L^2 < \beta^2 < 16\pi^2 \rho_0 T_0 / L^2$.

4.4.8. For (4.4.1)–(4.4.3), from (4.4.11) show that

$$u(x, t) = R(x - ct) + S(x + ct),$$

where R and S are some functions.

4.4.9. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially at rest, $g(x) = 0$, show that

$$u(x, t) = \frac{1}{2}[F(x - ct) + F(x + ct)],$$

where $F(x)$ is the odd-periodic extension of $f(x)$. [*Hints*:

1. For all x, $F(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$.

2. $\sin a \cos b = \frac{1}{2}[\sin(a + b) + \sin(a - b)].$]

Comment: This result shows that *the practical difficulty of summing an infinite number of terms of a Fourier series may be avoided for the one-dimensional wave equation.*

4.4.10. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially unperturbed, $f(x) = 0$, with the initial velocity given, show that

$$u(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\overline{x})\, d\overline{x},$$

where $G(x)$ is the odd-periodic extension of $g(x)$. [*Hints*:

1. For all x, $G(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}$.

2. $\sin a \sin b = \frac{1}{2}[\cos(a - b) - \cos(a + b)].$]

See the comment after Exercise 4.4.9.

4.4.11. From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L, \tag{4.4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

4.4.12. What happens to the total energy E of a vibrating string (see Exercise 4.4.11)

(a) If $u(0, T) = 0$ and $u(L, t) = 0$?

(b) If $\frac{\partial u}{\partial x}(0, t) = 0$ and $u(L, t) = 0$?

(c) If $u(0, t) = 0$ and $\frac{\partial u}{\partial x}(L, t) = -\gamma u(L, t)$ with $\gamma > 0$?

(d) If $\gamma < 0$ in part (c)?

4.4.13. Show that the potential and kinetic energies (defined in Exercise 4.4.11) are equal for a traveling wave, $u = R(x - ct)$.

4.4.14. Using (4.4.15), prove that the solution of (4.4.1)–(4.4.3) is unique.

4.4.15. **(a)** Using (4.4.15), calculate the energy of one normal mode.

(b) Show that the total energy, when $u(x, t)$ satisfies (4.4.11), is the sum of the energies contained in each mode.

4.5 Vibrating Membrane

The heat equation in one spatial dimension is $\partial u/\partial t = k\partial^2 u/\partial x^2$. In two or three dimensions, the temperature satisfies $\partial u/\partial t = k\nabla^2 u$. In a similar way, the vibration of a string (one dimension) can be extended to the vibration of a membrane (two dimensions).

The vertical displacement of a vibrating string satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

There are important physical problems that solve

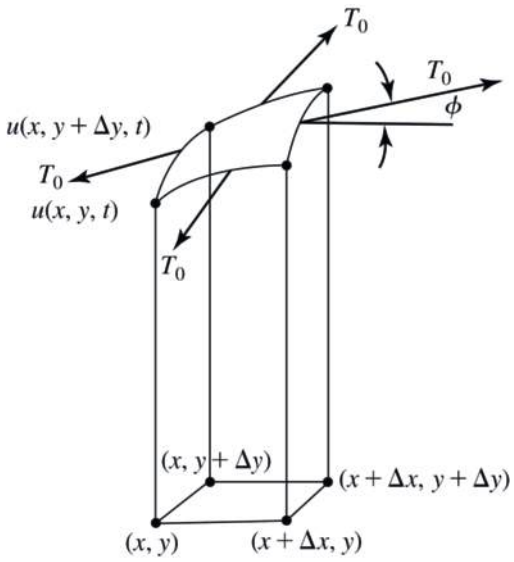
$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4.5.1}$$

known as the two-dimensional wave equation. An example of a physical problem that satisfies a two-dimensional wave equation is the vibration of a highly-stretched membrane.

This can be thought of as a two-dimensional vibrating string. We again introduce the displacement $z = u(x, y, t)$, which depends on x, y, and t (as illustrated in Fig. 4.5.1). If all slopes

(i.e., $\partial u/\partial x$ and $\partial u/\partial y$) are small, then, as an approximation, we may assume that the vibrations are entirely vertical and the tension is approximately constant. Then the mass density (mass per unit surface area), $\rho_0(x, y)$, of the membrane in the unperturbed position does not change appreciably when the membrane is perturbed.

Figure 4.5.1 Slightly stretched membrane with approximately constant tension T_0.



We assume all slopes ($\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$) are small, so that, as an approximation, the vibrations are entirely vertical and the tension is approximately constant. We consider any small rectangular segment of the membrane between x and $x + \Delta x$ and y and $y + \Delta y$, as illustrated in Fig 4.5.1. Here, T_0 is the constant force per unit length exerted by the stretched membrane on any edge of this section of the membrane. Newton's law for the vertical motion of the small segment at (x, y) is

$$m\frac{\partial^2 u}{\partial t^2}(x, y, t) = F, \tag{4.5.2}$$

where F is the sum of all the vertical forces and m is the mass of this section of the membrane:

$$m = \rho_0 \Delta x \Delta y,$$

where ρ_0 is the mass density of the membrane (mass per unit area). The vertical forces are the vertical components of the tensile forces in effect along all four sides. For example, on the side where x is fixed at $x + \Delta x$, y varies from y to $y + \Delta y$, the vertical component of tension (per unit length) is $\sin\phi$, where ϕ is the angle of inclination *for fixed* y. Since ϕ is assumed small, $\sin\phi \approx \tan\phi = \partial u/\partial x$, where this is to be evaluated at each value of y. To determine the entire contribution at this one edge, we must sum up (integrate) from y to $y + \Delta y$ for fixed x. However, ϕ will not vary by much, and we claim that we may approximate $\sin\phi$ by $\partial u/\partial x(x + \Delta x, y)$ for all y from y to $y + \Delta y$. Thus, the total contribution to the vertical component of tension for this one side is obtained by multiplying the value at each point by the length of the side Δy:

$$T_0 \Delta y \frac{\partial u}{\partial x}(x + \Delta x, y).$$

Furthermore, we should check the sign. If $\partial u/\partial x > 0$ (as illustrated in Fig. 4.5.1), then the vertical component of the tensile force is upward at $x + \Delta x$ (but downward at x). In a similar way, we can analyze the vertical components along the other sides. Thus,

$$\begin{aligned}\rho_0 \Delta x\, \Delta y \frac{\partial^2 u}{\partial t^2}(x, y, t) = T_0\Big[&\Delta y \frac{\partial u}{\partial x}(x + \Delta x, y) - \Delta y \frac{\partial u}{\partial x}(x, y) \\ &+ \Delta x \frac{\partial u}{\partial y}(x, y + \Delta y) - \Delta x \frac{\partial u}{\partial y}(x, y)\Big],\end{aligned} \tag{4.5.3}$$

using the Taylor series (linearization), $f(x + \Delta x, y) \approx f(x, y) + \Delta x\, \partial f/\partial x$ and $g(x, y + \Delta y) \approx g(x, y) + \Delta y\, \partial g/\partial y$, where $f = \partial u/\partial x$ and $g = \partial u/\partial y$, we obtain

$$\rho_0 \Delta x \Delta y \frac{\partial^2 u}{\partial t^2}(x, y, t) = T_0\, \Delta x \Delta y \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\right] = T_0\, \Delta x \Delta y \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]. \quad (4.5.4)$$

Dividing (4.5.4) by $\rho_0 \Delta x \Delta y$ and taking the limit as both $\Delta x \to 0$ and $\Delta y \to 0$ yield the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (4.5.5)$$

where again $c^2 = T_0/\rho_0$. We postpone the solution of problems involving a vibrating membrane until Chapter 7.

Exercises 4.5

4.5.1. If a membrane satisfies an "elastic" boundary condition, show that

$$T_0 \nabla u \cdot \hat{\mathbf{n}} = -ku \quad (4.5.6)$$

if there is a restoring force per unit length proportional to the displacement, where $\hat{\mathbf{n}}$ is a unit normal of the two-dimensional boundary.

4.5.2. Derive the two-dimensional wave equation for a vibrating membrane.

4.6 Reflection and Refraction of Electromagnetic (Light) and Acoustic (Sound) Waves

Disturbances in a uniform media frequently satisfy the **three-dimensional wave equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \quad (4.6.1)$$

In a fluid, small displacements u to a uniform mass density ρ and pressure p satisfy (4.6.1), where the coefficient c satisfies $c^2 = \frac{\partial p}{\partial \rho}$. In electrodynamics, each component of a system of vector fields satisfies (4.6.1), where $c^2 = c_{\text{light}}^2/\mu\varepsilon$ can be related to the speed of light in a vacuum c_{light} and the permeability μ and the dielectric constant ε.

Special **plane traveling wave** solutions of (4.6.1) exist of the form

$$u = Ae^{i(k_1 x + k_2 y + k_3 z - wt)} = Ae^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}. \quad (4.6.2)$$

(The real or imaginary part has physical significance.) A is the constant amplitude. The vector $\boldsymbol{k}$, called the **wave vector**, is in the direction of the wave (perpendicular to the **wave fronts** $\boldsymbol{k} \cdot \boldsymbol{x} - \omega t =$ constant). The magnitude of the wave vector $k \equiv |\boldsymbol{k}|$ is the **wave number** since it can be shown (see the Exercises) to equal the number of waves in 2π distance in the direction of the wave (in the $\boldsymbol{k}$-direction). (The **wave length** equals $\frac{2\pi}{k}$.)

The temporal frequency ω for plane wave solutions of the wave equations is determined by substituting (4.6.2) into (4.6.1):

$$\omega^2 = c^2 k^2, \quad (4.6.3)$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$. It is important that ω is a function of k satisfying (4.6.3). As in one dimension, it can be shown that

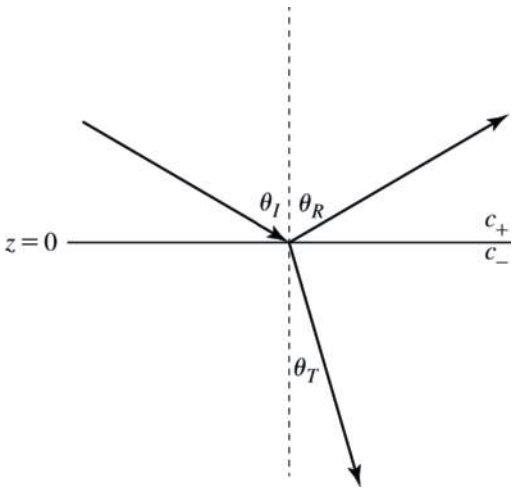
$$\text{traveling wave or phase velocity} = \frac{\omega}{k} = \pm c. \quad (4.6.4)$$

The plane wave solution corresponds to one component of a multidimensional Fourier series (transform—see Chapter 10) if the wave equation is defined in a finite (infinite) region. As we show in the next subsection, frequently plane waves are considered to be generated at infinity.

4.6.1 SNELL'S LAW OF REFRACTION

We assume that we have two different materials (different mass densities for sound waves or different dielectric constants for electromagnetic waves) extending to infinity with a plane boundary between them. We assume the wave speed is c_+ for $z > 0$ and $c_$ for $z < 0$, as shown in Fig. 4.6.1. We assume that there is an **incident** plane wave satisfying (4.6.2) [with wave vector $\boldsymbol{k}_I$ and frequency $\omega = \omega_+(k_I) = c_+k_I$] propagating from infinity with $z > 0$ with amplitude $\boldsymbol{A} = 1$, which we normalize to 1. We assume that the incident wave makes an angle θ_I with the normal.

Figure 4.6.1 Reflected and refracted (transmitted) waves.



We assume that there is a **reflected** wave in the upper media satisfying (4.6.2) [with unknown wave vector $\boldsymbol{k}_R$ and frequency $\omega = \omega_+(k_R) = c_+k_R$] with unknown complex amplitude R. Due to the wave equation being linear, the solution in the upper media is the sum of the incident and reflected waves:

$$u = e^{i(\boldsymbol{k}_I \cdot \boldsymbol{x} - c_+k_It)} + Re^{i(\boldsymbol{k}_R \cdot \boldsymbol{x} - c_+k_Rt)} \text{ for } z > 0. \tag{4.6.5}$$

We will show that the wave vector of the reflected wave is determined by the familiar relationship that the angle of reflection equals the angle of incidence.

In the lower media $z < 0$, we assume that a **refracted wave** exists, which we call a **transmitted wave** and introduce the subscript T. We assume that the transmitted wave satisfies (4.6.2) [with unknown wave number vector $\boldsymbol{k}_T$ and frequency $\omega = \omega_-(k_T) = c_-k_T$] with unknown complex amplitude T:

$$u = Te^{i(\boldsymbol{k}_T \cdot \boldsymbol{x} - c_-k_Tt)} \text{ for } z < 0. \tag{4.6.6}$$

In addition, we will show that this refracted wave may or may not exist as prescribed by Snell's law of refraction.

There are two boundary conditions that must be satisfied at the interface between the two materials $z = 0$. One of the conditions is that u must be continuous. At $z = 0$,

$$e^{i(\boldsymbol{k}_I \cdot \boldsymbol{x} - c_+k_It)} + Re^{i(\boldsymbol{k}_R \cdot \boldsymbol{x} - c_+k_Rt)} = Te^{i(\boldsymbol{k}_T \cdot \boldsymbol{x} - c_-k_Tt)}. \tag{4.6.7}$$

Since this must hold for all time, the frequencies of the three waves must be the same:

$$\omega_+(k_I) = \omega_+(k_R) = \omega_-(k_T). \tag{4.6.8}$$

From the frequency equation (4.6.3), we conclude that the reflected wave has the same wavelength as the incident wave, but the refracted wave has a different wavelength:

$$c_+k_I = c_+k_R = c_-k_T. \tag{4.6.9}$$

From (4.6.7), k_1 and k_2 (the projection of $\boldsymbol{k}$ in the x- and y-directions) must be the same for all three waves. Since $k_I = k_R$, the z-component of the reflected wave must be minus the z-component of the incident wave. Thus, the **angle of reflection equals the angle of incidence**,

$$\theta_R = \theta_I, \tag{4.6.10}$$

where the angles are measured with respect to the normal to the surface. Note that $\boldsymbol{k} \cdot \boldsymbol{x} = k|\boldsymbol{x}|\cos\phi$ is the same for all three waves, where ϕ is the angle between $\boldsymbol{k}$ and $\boldsymbol{x}$ with $z = 0$. Thus, for the transmitted wave, the angle of transmission (refraction) satisfies

$$k_I \sin\theta_I = k_T \sin\theta_T.$$

Using (4.6.9), **Snell's law** follows:

$$\boxed{\frac{\sin\theta_T}{\sin\theta_I} = \frac{k_I}{k_T} = \frac{c_-}{c_+}.} \tag{4.6.11}$$

Many important and well-known results from optics follow from Snell's law. For example, in the usual case of the upper media (c_+) being air and the lower media (c_-) water, then it is known that $c_+ > c_-$. In this case, from Snell's law (4.6.11) $\sin\theta_T < \sin\theta_I$, so that the transmitted wave is refracted toward the normal (as shown in Fig. 4.6.1).

If $c_+ < c_-$, then Snell's law (4.6.11) predicts in some cases that $\sin\theta_T > 1$, which is impossible. There is a critical angle of incidence $\sin\theta_I = \frac{c_+}{c_-}$ at which total internal reflection first occurs. For larger angles, the transmitted solution is not a plane wave but is an evanescent wave exponentially decaying, as we describe in a later subsection. The refracted plane wave does not exist.

4.6.2 INTENSITY (AMPLITUDE) OF REFLECTED AND REFRACTED WAVES

Here, we will assume that a refracted plane wave exists. With these laws for the reflected and refracted waves, the one boundary condition (4.6.7) for the continuity of u becomes simply

$$1 + R = T. \tag{4.6.12}$$

We cannot solve for either amplitude R or T without the second boundary condition. The second boundary condition can be slightly different in different physical applications. Thus, the results of this subsection do not apply to all physical problems, but the method we use may be applied in all cases and the results obtained may be slightly different.

We assume the second boundary condition is $\frac{\partial u}{\partial z} = 0$ at $z = 0$. From (4.6.5) and (4.6.6), it follows that

$$k_{3_I} + k_{3_R} R = k_{3_T} T. \tag{4.6.13}$$

From Fig. 4.6.1, the z-components of the wave number of three waves satisfy

$$\begin{aligned} k_{3_I} &= -k_I \cos\theta_I \\ k_{3_R} &= k_R \cos\theta_R = k_I \cos\theta_I \\ k_{3_T} &= -k_T \cos\theta_T = -k_I \frac{\sin\theta_I}{\sin\theta_T}\cos\theta_T, \end{aligned} \tag{4.6.14}$$

where we have recalled that the reflected wave satisfies $k_{3_R} = -k_{3_I}$ and we have used Snell's law (4.6.11) to simplify the z-component of the wave number of the transmitted wave. Using (4.6.14), the second boundary condition (4.6.13) becomes (after dividing by $-k_I \cos\theta_I$)

$$1 - R = \frac{\sin\theta_I}{\sin\theta_T}\frac{\cos\theta_T}{\cos\theta_I} T. \tag{4.6.15}$$

The complex amplitudes of reflection and transmission can be determined by adding the two linear equations, (4.6.12) and (4.6.15),

$$\begin{aligned} T &= \frac{2}{1 + \dfrac{\sin\theta_I}{\sin\theta_T}\dfrac{\cos\theta_T}{\cos\theta_I}} = \frac{2\sin\theta_T\cos\theta_I}{\sin(\theta_T + \theta_I)} \\ R &= \frac{2\sin\theta_T\cos\theta_I - \sin(\theta_T + \theta_I)}{\sin(\theta_T + \theta_I)} = \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)}. \end{aligned}$$

4.6.3 TOTAL INTERNAL REFLECTION

If $\sin\theta_I \frac{c_-}{c_+} > 1$, then Snell's law cannot be satisfied for a plane transmitted (refracted) wave $u = Te^{i(\boldsymbol{k}_T \cdot \boldsymbol{x} - c_- k_T t)}$ in the lower media. Because of the boundary condition at $z = 0$, the x- and

y-components of the wave number of a solution must be the same as for the incident wave. Thus, the transmitted wave number should satisfy $\boldsymbol{k_T} = \left(k_{1_I}, k_{2_I}, k_{3_I}\right)$. If we apply Snell's law (4.6.11) and solve for k_{3_T},

$$k_{3_T} = \pm\sqrt{k_I^2\left(\frac{c_+^2}{c_-^2} - \sin^2\theta_I\right)}, \tag{4.6.16}$$

since $k_1^2 + k_2^2 = k^2\sin^2\theta_I$. We find that k_{3_T} is imaginary, suggesting that there are solutions of the wave equation (in the lower media),

$$\frac{\partial^2 u}{\partial t^2} = c_-^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{4.6.17}$$

which exponentially grow and decay in z. We look for a product solution of the form

$$u(x, y, z, t) = w(z)e^{i(k_1x + k_2y - \omega t)}, \tag{4.6.18}$$

where k_1 and k_2 are the wave numbers associated with the incident wave and ω is the frequency of the incident wave. We insist that (4.6.18) satisfy (4.6.17) so that

$$\frac{d^2w}{dz^2} = \left(k_1^2 + k_2^2 - \frac{\omega^2}{c_-^2}\right)w = k_I^2\left(\sin^2\theta_I - \frac{c_+^2}{c_-^2}\right)w.$$

Thus, $w(z)$ is a linear combination of exponentially growing and decaying terms in z. Since we want our solution to decay exponentially as $z \to -\infty$, we choose the solution to the wave equation in the lower media to be

$$u(x, y, z, t) = Te^{k_I\sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}z}e^{i(k_1x + k_2y - \omega t)}, \tag{4.6.19}$$

instead of the plane wave. This is a horizontal two-dimensional plane wave whose amplitude exponentially decays in the $-z$-direction. It is called an **evanescent** wave (exponentially decaying in the $-z$-direction).

The continuity of u and $\frac{\partial u}{\partial z}$ at $z = 0$ (using (4.6.5) for $z > 0$ and (4.6.19) for $z < 0$) is satisfied if

$$1 + R = T \tag{4.6.20}$$

$$ik_{3_I}(1 - R) = Tk_I\sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}. \tag{4.6.21}$$

These equations can be simplified and solved for the reflection coefficient and T (the amplitude of the evanescent wave at $z = 0$). R (and T) will be complex, corresponding to phase shifts of the reflected (and evanescent) wave.

Exercises 4.6

4.6.1. Show that for a plane wave given by (4.6.2), the number of waves in 2π distance in the direction of the wave (the $\boldsymbol{k}$-direction) is $k \equiv |\boldsymbol{k}|$.

4.6.2. Show that the phase of a plane wave stays the same moving in the direction of the wave if the velocity is $\frac{\omega}{k}$.

4.6.3. In optics, the index of refraction is defined as $n = \frac{c_{\text{light}}}{c}$. Express Snell's law using the indices of refraction.

4.6.4. Find R and T for the evanescent wave by solving the simultaneous equations (4.6.20) and (4.6.21).

4.6.5. Find R and T by assuming that $k_{3_I} = \pm i\beta$, where β is defined by (4.6.16). Which sign do we use to obtain exponential decay as $z \to -\infty$?

5
Sturm–Liouville Eigenvalue Problems

5.1 Introduction

We have found the method of separation of variables to be quite successful in solving some homogeneous partial differential equations with homogeneous boundary conditions. In all examples we have analyzed so far, the boundary value problem that determines the needed eigenvalues (separation constants) has involved the simple ordinary differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0. \tag{5.1.1}$$

Explicit solutions of this equation determined the eigenvalues λ from the homogeneous boundary conditions. The principle of superposition resulted in our needing to analyze infinite series. We pursued three different cases (depending on the boundary conditions): Fourier sine series, Fourier cosine series, and Fourier series (both sines and cosines). Fortunately, we verified by explicit integration that the eigenfunctions were orthogonal. This enabled us to determine the coefficients of the infinite series from the remaining nonhomogeneous condition.

In this section, we further explain and generalize these results. We show that the orthogonality of the eigenfunctions can be derived even if we cannot solve the defining differential equation in terms of elementary functions [as in (5.1.1)]. Instead, orthogonality is a direct result of the differential equation. We investigate other boundary value problems resulting from separation of variables that yield other families of orthogonal functions. These generalizations of Fourier series will not always involve sines and cosines since (5.1.1) is not necessarily appropriate in every situation.

5.2 Examples

5.2.1 HEAT FLOW IN A NONUNIFORM ROD

In Section 1.2, we showed that the temperature u in a nonuniform rod solves the following partial differential equation:

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q,} \tag{5.2.1}$$

where Q represents any possible sources of heat energy. Here, in order to consider the case of a *nonuniform* rod, we allow the thermal coefficients c, ρ, and K_0 to depend on x. The method of separation of variables can be applied only if (5.2.1) is linear and *homogeneous*. Usually, to make (5.2.1) homogeneous, we consider only situations without sources, $Q = 0$. However, we will be slightly more general. We will allow the heat source Q to be proportional to the temperature u,

$$\boxed{Q = \alpha u,} \tag{5.2.2}$$

in which case

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u.} \tag{5.2.3}$$

We also allow α to depend on x (but not on t), as though the specific types of sources depend on the material. Although $Q \neq 0$, (5.2.3) is still a linear and homogeneous partial differential

equation. To understand the effect of this source Q, we present a plausible physical situation in which terms such as $Q = \alpha u$ might arise. Suppose that a chemical reaction generates heat (called an *exothermic* reaction) corresponding to $Q > 0$. Conceivably, this reaction could be more intense at higher temperatures. In this way the heat energy generated might be proportional to the temperature and thus $\alpha > 0$ (assuming that $u > 0$). Other types of chemical reactions (known as *endothermic*) would remove heat energy from the rod and also could be proportional to the temperature. For positive temperatures ($u > 0$), this corresponds to $\alpha < 0$. In our problem, $\alpha = \alpha(x)$, and hence it is possible that $\alpha > 0$ in some parts of the rod and $\alpha < 0$ in other parts. We summarize these results by noting that if $\alpha(x) < 0$ for all x, then heat energy is being taken out of the rod, and vice versa. Later in our mathematical analysis, we will correspondingly discuss the special case $\alpha(x) < 0$.

Equation (5.2.3) is suited for the method of separation of variables if, in addition, we assume that there is one homogeneous boundary condition (as yet unspecified) at each end, $x = 0$ and $x = L$. We have already analyzed cases in which $\alpha = 0$ and c, ρ, and K_0 are constant. In separating variables, we substitute the product form,

$$u(x, t) = \phi(x)h(t), \tag{5.2.4}$$

into (5.2.3), which yields

$$c\rho\phi(x)\frac{dh}{dt} = h(t)\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi(x)h(t).$$

Dividing by $\phi(x)h(t)$ does not necessarily separate variables since $c\rho$ may depend on x. However, dividing by $c\rho\phi(x)h(t)$ is always successful:

$$\frac{1}{h}\frac{dh}{dt} = \frac{1}{c\rho\phi}\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho} = -\lambda. \tag{5.2.5}$$

The separation constant $-\lambda$ has been introduced with a minus sign because in this form the time-dependent equation [following from (5.2.5)],

$$\frac{dh}{dt} = -\lambda h, \tag{5.2.6}$$

has exponentially decaying solutions if $\lambda > 0$. Solutions to (5.2.6) exponentially grow if $\lambda < 0$ (and are constant if $\lambda = 0$). Solutions exponentially growing in time are not usually encountered in physical problems. However, for problems in which $\alpha > 0$ for at least part of the rod, thermal energy is being put into the rod by the exothermic reaction, and hence it is possible for there to be some negative eigenvalues ($\lambda < 0$).

The spatial differential equation implied by separation of variables is

$$\boxed{\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0,} \tag{5.2.7}$$

which forms a boundary value problem when complemented by two homogeneous boundary conditions. This differential equation is not $d^2\phi/dx^2 + \lambda\phi = 0$. Neither does (5.2.7) have constant coefficients, because the thermal coefficients K_0, c, ρ, and α are not constant. *In general, one way in which nonconstant-coefficient differential equations occur is in situations where physical properties are nonuniform.*

EXAMPLE

For an example with specific functions, suppose an engineer was studying heat flow through a metal rod that smoothly transitions from one alloy at $x = 0$ to another one at $x = 6$. The density of the alloy is modeled with $\rho(x) = 2 + \tanh(x - 3)$, and the specific heat $c = 1$. Suppose also that the thermal conductivity $K_0(x)$ has a local maximum in the middle of the transition, such that $K_0(x) = 1 + e^{-(x-3)^2}$. The fuel helping to supply the heat is partially used up as it moves along the length of the rod, modeled by $\alpha(x) = e^{-x/6}$. The resulting differential equation (5.2.3) then becomes

$$(2 + \tanh(x - 3))\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left(1 + e^{-(x-3)^2}\right)\frac{\partial u}{\partial x}\right) + e^{-x/6}u.$$

Separating variables with the product form $u(x, t) = \phi(x)h(t)$, we find the time-dependent equation is (5.2.6) and the spatial differential equation is

$$\frac{d}{dx}\left(\left(1 + e^{-(x-3)^2}\right)\frac{\partial u}{\partial x}\right) + e^{-x/6}\phi + \lambda(2 + \tanh(x - 3))\phi = 0. \quad \blacksquare$$

Note that we cannot decide on the appropriate convenient sign for the separation constant by quickly analyzing the spatial ordinary differential equation (5.2.7) with its homogeneous boundary conditions. Usually we will be unable to solve (5.2.7) in the variable-coefficient case, other than by a numerical approximate solution on the computer. Consequently, we will describe in Section 5.3 certain important qualitative properties of the solution of (5.2.7). Later, with a greater understanding of (5.2.7), we will return to reinvestigate heat flow in a nonuniform rod. For now, let us describe another example that yields a boundary value problem with nonconstant coefficients.

5.2.2 CIRCULARLY SYMMETRIC HEAT FLOW

Nonconstant-coefficient differential equations can also arise if the physical parameters are constant. In Section 1.5, we showed that if the temperature u in some plane two-dimensional region is circularly symmetric (so that u depends only on time t and on the radial distance r from the origin), then u solves the linear and homogeneous partial differential equation

$$\boxed{\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),} \tag{5.2.8}$$

under the assumption that all the thermal coefficients are constant.

We apply the method of separation of variables by seeking solutions in the form of a product:

$$u(r, t) = \phi(r)h(t).$$

Equation (5.2.8) then implies that

$$\phi(r)\frac{dh}{dt} = \frac{kh(t)}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right).$$

Dividing by $\phi(r)h(t)$ separates the variables, but also dividing by the constant k is convenient since it eliminates this constant from the resulting boundary value problem:

$$\frac{1}{k}\frac{1}{h}\frac{dh}{dt} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda. \tag{5.2.9}$$

The two ordinary differential equations implied by (5.2.9) are

$$\frac{dh}{dt} = -\lambda kh \tag{5.2.10}$$

$$\boxed{\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0.} \tag{5.2.11}$$

The separation constant is denoted by $-\lambda$ since we expect solutions to exponentially decay in time, as is implied by (5.2.10) if $\lambda > 0$. The nonconstant coefficients in (5.2.11) are due to geometric factors introduced by the use of polar coordinates. Later in this text (Section 7.7), we will show that (5.2.11) can be solved using Bessel functions. However, the general discussions in the remainder of this chapter will be quite valuable in our understanding of this problem.

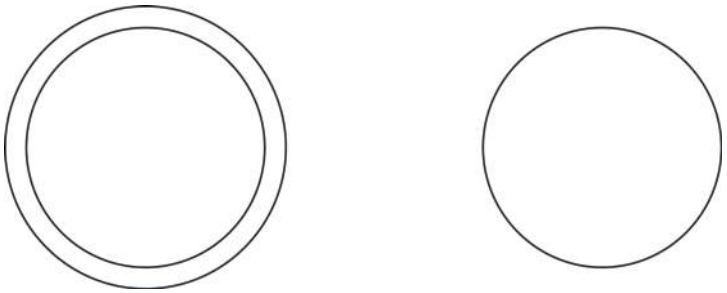
Let us consider the appropriate homogeneous boundary conditions for circularly symmetric heat flow in two different geometries: inside a circular annulus (as illustrated in Fig. 5.2.1a) and inside a circle (as illustrated in Fig. 5.2.1b). In both cases we assume that all boundaries are fixed at zero temperature. For the annulus, the boundary conditions for (5.2.11) are that the temperature should be zero at the inner ($r = a$) and outer ($r = b$) concentric circular walls:

$$u(a, t) = 0 \quad \text{and} \quad u(b, t) = 0.$$

Both of these boundary conditions are exactly of the type we have already studied. However, for the circle, the same second-order differential equation (5.2.11) has only one boundary condition, $u(b, t) = 0$. Since the physical variable r ranges from $r = 0$ to $r = b$, we need a homogeneous

boundary condition at $r = 0$ for mathematical reasons. (This is the same problem that occurred in studying Laplace's equation inside a cylinder. However, in that situation a nonhomogeneous condition was given at $r = b$.) On the basis of physical reasoning, we expect that the condition at $r = 0$ is that the temperature is bounded there, $|u(0, t)| < \infty$. This is an example of a **singularity condition**. It is homogeneous; it is the boundary condition that we apply at $r = 0$. Thus, we have homogeneous conditions at both $r = 0$ and $r = b$ for the circle.

Figure 5.2.1 (a) Circular annulus; (b) circle.



Exercises 5.2

5.2.1. Find the time-dependent and spatial ordinary differential equations implied by separation of variables $u(x, t) = \phi(x)h(t)$ applied to the partial differential equation

$$(x^2 + 5)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left((8 + \sin^2 x)\frac{\partial u}{\partial x}\right) + \sqrt{1 + x^2}u.$$

5.2.2. Consider equation (5.2.3) in the case where we allow α to be a function of t. Show that the product form $u(x, t) = \phi(x)h(t)$ successfully yields a time-dependent equation and a spatial differential equation, as long as c and ρ are assumed to be constant.

5.3 Sturm–Liouville Eigenvalue Problems

5.3.1 GENERAL CLASSIFICATION

Differential equation

A boundary value problem consists of a linear homogeneous differential equation and corresponding linear homogeneous boundary conditions. All of the differential equations for boundary value problems that have been formulated in this text can be put in the following form:

$$\boxed{\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda\sigma\phi = 0,} \tag{5.3.1}$$

where λ is the eigenvalue. Here, the variable x is defined on a finite interval $a < x < b$. Four examples are as follows:

1. *Simplest case*: $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$; in which case, $p = 1, q = 0, \sigma = 1$.
2. *Heat flow in a nonuniform rod*: $\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0$; in which case, $p = K_0, q = \alpha$, $\sigma = c\rho$.
3. *Vibrations of a nonuniform string*: $T_0\frac{d^2\phi}{dx^2} + \alpha\phi + \lambda\rho_0\phi = 0$; in which case, $p = T_0$ (constant), $q = \alpha, \sigma = \rho_0$ (see Exercise 5.3.1).
4. *Circularly symmetric heat flow*: $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0$; here, the independent variable $x = r$ and $p(x) = x$, $q(x) = 0$, $\sigma(x) = x$.

Many interesting results are known concerning any equation in the form (5.3.1). Equation (5.3.1) is known as a **Sturm–Liouville differential equation**, named after two famous mathematicians active in the mid-1800s who studied it.

Boundary conditions

The linear homogeneous boundary conditions that we have studied are of the form to follow. We also introduce some mathematical terminology:

	Heat flow	Vibrating string	Mathematical terminology
$\phi = 0$	Fixed (zero) temperature	Fixed (zero) displacement	First kind or Dirichlet condition
$\frac{d\phi}{dx} = 0$	Insulated	Free	Second kind or Neumann condition
$\frac{d\phi}{dx} = \pm h\phi$ $\begin{pmatrix}+\text{left end}\\ -\text{right end}\end{pmatrix}$	(Homogeneous) Newton's law of cooling 0° outside temperature, $h = H/K_0$, $h > 0$ (physical)	(Homogeneous) Elastic boundary condition $h = k/T_0$, $h > 0$ (physical)	Third kind or Robin condition
$\phi(-L) = \phi(L)$ $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$	Perfect thermal contact	—	Periodicity condition (example of mixed type)
$\lvert\phi(0)\rvert < \infty$	Bounded temperature	—	Singularity condition

5.3.2 REGULAR STURM–LIOUVILLE EIGENVALUE PROBLEM

A regular Sturm–Liouville eigenvalue problem consists of the Sturm–Liouville differential equation,

$$\boxed{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0,} \qquad a < x < b, \tag{5.3.2}$$

subject to the boundary conditions that we have discussed (excluding periodic and singular cases):

$$\boxed{\begin{aligned}\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0\\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0,\end{aligned}} \tag{5.3.3}$$

where β_i are real. In addition, to be called regular, the coefficients p, q, and σ must be real and continuous everywhere (including the endpoints) and $p > 0$ and $\sigma > 0$ everywhere (also including the endpoints). For the regular Sturm–Liouville eigenvalue problem, many important general theorems exist. In Section 5.5, we will prove these results, and in Sections 5.7 and 5.8, we will develop some more interesting examples that illustrate the significance of the general theorems.

Statement of theorems

At first let us just state (in one place) all the theorems we will discuss more fully later (and in some cases prove). For any regular Sturm–Liouville problem, all of the following theorems are valid:

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots .$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$
   Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \qquad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b \left[p(d\phi/dx)^2 - q\phi^2\right] dx}{\int_a^b \phi^2 \sigma\, dx},$$
   where the boundary conditions may somewhat simplify this expression.

It should be mentioned that for Sturm–Liouville eigenvalue problems that are not "regular," these theorems may be valid. An example of this is illustrated in Sections 7.7 and 7.8.

5.3.3 EXAMPLE AND ILLUSTRATION OF THEOREMS

We will individually illustrate the meaning of these theorems (before proving many of them in Section 5.5) by referring to the simplest example of a regular Sturm–Liouville problem:

$$\boxed{\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned}} \tag{5.3.4}$$

The constant-coefficient differential equation has zero boundary conditions at both ends. As we already know, the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \text{with} \qquad \phi_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots ,$$

giving rise to a Fourier sine series.

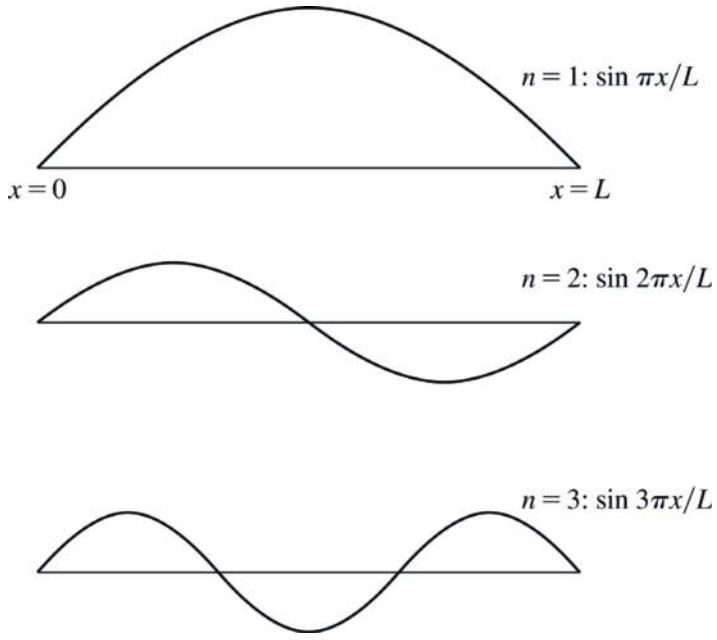
1. **Real eigenvalues**. Our theorem claims that all eigenvalues λ of a regular Sturm–Liouville problem are real. Thus, the eigenvalues of (5.3.4) should all be real. We know that the eigenvalues are $(n\pi/L)^2$, $n = 1, 2, \ldots$. However, in determining this result (see Section 2.3.4) we analyzed three cases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. We did not bother to look for complex eigenvalues because it is a relatively difficult task and we would have obtained no additional eigenvalues other than $(n\pi/L)^2$. This theorem (see Section 5.5 for its proof) is thus very useful. It guarantees that we do not even have to consider λ being complex.

2. **Ordering of eigenvalues**. There is an infinite number of eigenvalues for (5.3.4), namely, $\lambda = (n\pi/L)^2$ for $n = 1, 2, 3, \ldots$. Sometimes we use the notation $\lambda_n = (n\pi/L)^2$. Note that there is a smallest eigenvalue, $\lambda_1 = (\pi/L)^2$, but no largest eigenvalue, since $\lambda_n \to \infty$ as $n \to \infty$. Our theorem claims that this idea is valid for any regular Sturm–Liouville problem.

3. **Zeros of eigenfunctions**. For the eigenvalues of (5.3.4), $\lambda_n = (n\pi/L)^2$, the eigenfunctions are known to be $\sin n\pi x/L$. We use the notation $\phi_n(x) = \sin n\pi x/L$. The eigenfunction is unique (to within an arbitrary multiplicative constant).

   An important and interesting aspect of this theorem is that we claim that for all regular Sturm–Liouville problems, the nth eigenfunction has exactly $(n-1)$ zeros, not counting the endpoints. The eigenfunction ϕ_1 corresponding to the smallest eigenvalue (λ_1, $n = 1$) should have no zeros in the interior. The eigenfunction ϕ_2 corresponding to the next-smallest eigenvalue (λ_2, $n = 2$) should have exactly one zero in the interior, and so on. We use our eigenvalue problem (5.3.4) to illustrate these properties. The eigenfunctions $\phi_n(x) = \sin n\pi x/L$ are sketched in Fig. 5.3.1 for $n = 1, 2, 3$. Note that the theorem is verified (since we count zeros only at interior points); $\sin \pi x/L$ has no zeros between $x = 0$ and $x = L$, $\sin 2\pi x/L$ has one zero between $x = 0$ and $x = L$, and $\sin 3\pi x/L$ has two zeros between $x = 0$ and $x = L$.

Figure 5.3.1 Zeros of eigenfunctions $\sin n\pi x/L$.



4. **Series of eigenfunctions**. According to this theorem, the eigenfunctions can always be used to represent any piecewise smooth function $f(x)$,

$$\boxed{f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).} \tag{5.3.5}$$

   Thus, for our example (5.3.4),

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}.$$

   We recognize this as a Fourier sine series. We know that any piecewise smooth function can be represented by a Fourier sine series and the infinite series converges to $[f(x+) + f(x-)]/2$ for $0 < x < L$. It converges to $f(x)$ for $0 < x < L$ if $f(x)$ is continuous there. This theorem thus claims that the convergence properties of Fourier sine series are valid for all series of eigenfunctions of any regular Sturm–Liouville eigenvalue problem. Equation (5.3.5) is referred to as an expansion of $f(x)$ in terms of the eigenfunctions $\phi_n(x)$

or, more simply, as an eigenfunction expansion. It is also called a **generalized Fourier series of $f(x)$**. The coefficients a_n are called the coefficients of the eigenfunction expansion or the **generalized Fourier coefficients**. The fact that rather arbitrary functions may be represented in terms of an infinite series of eigenfunctions will enable us to solve partial differential equations by the method of separation of variables.

5. **Orthogonality of eigenfunctions**. The preceding theorem enables a function to be represented by a series of eigenfunctions, (5.3.5). Here, we will show how to determine the generalized Fourier coefficients, a_n. According to the important theorem we are now describing, the eigenfunctions of any regular Sturm–Liouville eigenvalue problem will always be orthogonal. The theorem states that a weight $\sigma(x)$ must be introduced into the orthogonality relation:

$$\boxed{\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0} \quad \text{if} \quad \lambda_n \neq \lambda_m. \tag{5.3.6}$$

Here, $\sigma(x)$ is the possibly variable coefficient that multiplies the eigenvalue λ in the differential equation defining the eigenvalue problem. Since corresponding to each eigenvalue there is only one eigenfunction, the statement "if $\lambda_n \neq \lambda_m$" in (5.3.6) may be replaced by "if $n \neq m$." For the Fourier sine series example, the defining differential equation is $d^2\phi/dx^2 + \lambda\phi = 0$, and hence a comparison with the form of the general Sturm–Liouville problem shows that $\sigma(x) = 1$. Thus, in this case the weight is 1, and the orthogonality condition, $\int_0^L \sin n\pi x/L \sin m\pi x/L\,dx = 0$, follows if $n \neq m$, as we already know.

As with Fourier sine series, we use the orthogonality condition to determine the generalized Fourier coefficients. In order to utilize the orthogonality condition (5.3.6), we must multiply (5.3.5) by $\phi_m(x)$ *and* $\sigma(x)$. Thus,

$$f(x)\phi_m(x)\sigma(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)\phi_m(x)\sigma(x),$$

where we assume these operations on infinite series are valid, and hence introduce equal signs. Integrating from $x = a$ to $x = b$ yields

$$\int_a^b f(x)\phi_m(x)\sigma(x)\,dx = \sum_{n=1}^{\infty} a_n \int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx.$$

Since the eigenfunctions are orthogonal [with weight $\sigma(x)$], all the integrals on the right-hand side vanish except when n reaches m:

$$\int_a^b f(x)\phi_m(x)\sigma(x)\,dx = a_m \int_a^b \phi_m^2(x)\sigma(x)\,dx.$$

The integral on the right is nonzero since the weight $\sigma(x)$ must be positive (from the definition of a regular Sturm–Liouville problem), and hence we may divide by it to determine the generalized Fourier coefficient a_m:

$$\boxed{a_m = \frac{\int_a^b f(x)\phi_m(x)\sigma(x)\,dx}{\int_a^b \phi_m^2(x)\sigma(x)\,dx}}. \tag{5.3.7}$$

In the example of a Fourier sine series, $a = 0$, $b = L$, $\phi_n = \sin n\pi x/L$, and $\sigma(x) = 1$. Thus, if we recall the known integral that $\int_0^L \sin^2 n\pi x/L\,dx = L/2$, (5.3.7) reduces to the well-known formula for the coefficients of the Fourier sine series. It is not always possible to evaluate the integral in the denominator of (5.3.7) in a simple way.

6. **Rayleigh quotient**. In Section 5.6, we will prove that the eigenvalue may be related to its eigenfunction in the following way:

$$\lambda = \frac{-p\phi\,d\phi/dx\big|_a^b + \int_a^b \left[p(d\phi/dx)^2 - q\phi^2\right]dx}{\int_a^b \phi^2\sigma\,dx}, \tag{5.3.8}$$

known as the **Rayleigh quotient**. The numerator contains integrated terms and terms evaluated at the boundaries. Since the eigenfunctions cannot be determined without knowing the eigenvalues, this expression is never used directly to determine the eigenvalues. However, interesting

and significant results can be obtained from the Rayleigh quotient without solving the differential equation. Consider the Fourier sine series example (5.3.4) that we have been analyzing: $a = 0$, $b = L$, $p(x) = 1$, $q(x) = 0$, and $\sigma(x) = 1$. Since $\phi(0) = 0$ and $\phi(L) = 0$, the Rayleigh quotient implies that

$$\lambda = \frac{\int_0^L (d\phi/dx)^2\,dx}{\int_0^L \phi^2\,dx}. \tag{5.3.9}$$

Although this does not determine λ, since ϕ is unknown, it gives useful information. Both the numerator and the denominator are ≥ 0. Since ϕ cannot be identically zero and be called an eigenfunction, the denominator cannot be zero. Thus, $\lambda \geq 0$ follows from (5.3.9). Without solving the differential equation, we immediately conclude that there cannot be any negative eigenvalues. When we first determined eigenvalues for this problem, we worked rather hard to show that there were no negative eigenvalues (see Section 2.3). Now we can simply apply the Rayleigh quotient to eliminate the possibility of negative eigenvalues *for this example*. Sometimes, as we shall see later, we can also show that $\lambda \geq 0$ in harder problems.

Furthermore, even the possibility of $\lambda = 0$ can sometimes be analyzed using the Rayleigh quotient. For the simple problem (5.3.4) with zero boundary conditions, $\phi(0) = 0$ and $\phi(L) = 0$, let us see if it is possible for $\lambda = 0$ directly from (5.3.9). $\lambda = 0$ only if $d\phi/dx = 0$ for all x. Then, by integration, ϕ must be a constant for all x. However, from the boundary conditions [either $\phi(0) = 0$ or $\phi(L) = 0$], that constant must be zero. Thus, $\lambda = 0$ only if $\phi = 0$ everywhere. But if $\phi = 0$ everywhere, we do not call ϕ an eigenfunction. Thus, $\lambda = 0$ is not an eigenvalue *in this case*, and we have further concluded that $\lambda > 0$; all the eigenvalues must be positive. This is concluded *without* using solutions of the differential equation. The known eigenvalues in this example, $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$, are clearly consistent with the conclusions from the Rayleigh quotient. Other applications of the Rayleigh quotient will appear in later sections.

Exercises 5.3

***5.3.1.** Do Exercise 4.4.5(b). Show that the spatial ordinary differential equation may be put into Sturm–Liouville form.

5.3.2. Consider

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u + \beta\frac{\partial u}{\partial t}.$$

(a) Give a brief physical interpretation. What signs must α and β have to be physical?

(b) Allow ρ, α, and β to be functions of x. Show that separation of variables works only if $\beta = c\rho$, where c is a constant.

(c) If $\beta = c\rho$, show that the spatial equation is a Sturm–Liouville differential equation. Solve the time equation.

***5.3.3.** Consider the non-Sturm–Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)\phi] = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm–Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)\phi] = 0.$$

Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $\sigma(x)$, and $q(x)$?

5.3.4. Consider heat flow with convection (see Exercise 1.5.2):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}.$$

(a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm–Liouville form.

***(b)** Solve the initial boundary value problem

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

(c) Solve the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

5.3.5. For the Sturm–Liouville eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0,$$

verify the following general properties:

(a) There is an infinite number of eigenvalues with a smallest but no largest.

(b) The *nth* eigenfunction has $n - 1$ zeros.

(c) The eigenfunctions are complete and orthogonal.

(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

5.3.6. Redo Exercise 5.3.5 for the Sturm–Liouville eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

5.3.7. Which of statements 1–5 of the theorems of this section are valid for the following eigenvalue problem?

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \quad \text{with} \\ \phi(-L) &= \phi(L) \\ \frac{d\phi}{dx}(-L) &= \frac{d\phi}{dx}(L). \end{aligned}$$

5.3.8. Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0, \ \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

5.3.9. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = 0 \quad \text{and} \quad \phi(b) = 0. \tag{5.3.10}$$

(a) Show that multiplying by $1/x$ puts this in the Sturm–Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

(b) Show that $\lambda \geq 0$.

***(c)** Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest but no largest.

(d) The eigenfunctions are orthogonal with what weight according to Sturm–Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the *nth* eigenfunction has $n - 1$ zeros.

5.3.10. Reconsider Exercise 5.3.9 with the boundary conditions

$$\frac{d\phi}{dx}(1) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(b) = 0.$$

5.4 Worked Example: Heat Flow in a Nonuniform Rod Without Sources

In this section, we illustrate the application to partial differential equations of some of the general theorems on regular Sturm–Liouville eigenvalue problems. Consider the heat flow in a nonuniform rod (with possibly nonconstant thermal properties c, ρ, K_0) *without sources*; see Section 1.2 or 5.2.1. At the left end, $x = 0$, the temperature is prescribed to be 0°, and the right end is insulated. The initial temperature distribution is given. The mathematical formulation of this problem is

$$\text{PDE}: \boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)} \tag{5.4.1}$$

$$\text{BC}: \boxed{\begin{aligned} u(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \end{aligned}} \tag{5.4.2}$$

$$\text{IC}: \boxed{u(x,0) = f(x).} \tag{5.4.3}$$

Since the partial differential equation and the boundary conditions are linear and homogeneous, we seek special solutions (ignoring the initial condition) in the product form:

$$u(x,t) = \phi(x)h(t). \tag{5.4.4}$$

After separation of variables (for details, see Section 5.2.1), we find that the time part satisfies the ordinary differential equation

$$\frac{dh}{dt} = -\lambda h, \tag{5.4.5}$$

while the spatial part solves the following regular Sturm–Liouville eigenvalue problem:

$$\boxed{\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \lambda c\rho\phi = 0} \tag{5.4.6}$$

$$\boxed{\phi(0) = 0} \tag{5.4.7}$$

$$\boxed{\frac{d\phi}{dx}(L) = 0.} \tag{5.4.8}$$

According to our theorems concerning Sturm–Liouville eigenvalue problems, there is an infinite sequence of eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$. *We assume* that $\phi_n(x)$ *are known* (it might be a difficult problem to determine approximately the first few using numerical methods, but nevertheless it can be done). The time-dependent part of the differential equation is easily solved,

$$h(t) = ce^{-\lambda_n t}. \tag{5.4.9}$$

In this way, we obtain an infinite sequence of product solutions of the partial differential equation

$$u(x,t) = \phi_n(x)e^{-\lambda_n t}. \tag{5.4.10}$$

According to the principle of superposition, we attempt to satisfy the initial condition with an infinite linear combination of these product solutions:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n\phi_n(x)e^{-\lambda_n t}.} \tag{5.4.11}$$

This infinite series has the property that it solves the PDE and the homogeneous BCs. We will show that we can determine the as-yet unknown constants a_n from the initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x). \tag{5.4.12}$$

Our theorems imply that any piecewise smooth $f(x)$ can be represented by this type of series of eigenfunctions. The coefficients a_n are the generalized Fourier coefficients of the initial condition. Furthermore, the eigenfunctions are orthogonal with a weight $\sigma(x) = c(x)\rho(x)$, determined from the physical properties of the rod:

$$\int_0^L \phi_n(x)\phi_m(x)c(x)\rho(x)\,dx = 0 \quad \text{for } n \neq m.$$

Using these orthogonality formulas, the generalized Fourier coefficients are

$$\boxed{a_n = \frac{\int_0^L f(x)\phi_n(x)c(x)\rho(x)\,dx}{\int_0^L \phi_n^2(x)c(x)\rho(x)\,dx}}. \tag{5.4.13}$$

We claim that (5.4.11) is the desired solution, with coefficients given by (5.4.13).

In order to give a minimal interpretation of the solution, we should ask what happens for large t. Since the eigenvalues form an increasing sequence, each succeeding term in (5.4.11) is exponentially smaller than the preceding term for large t. Thus, for large time the solution may be accurately approximated by

$$u(x, t) \approx a_1\phi_1(x)e^{-\lambda_1 t}. \tag{5.4.14}$$

This approximation is not very good if $a_1 = 0$, in which case (5.4.14) should begin with the first nonzero term. However, often the initial temperature $f(x)$ is nonnegative (and not identically zero). In this case, we will show from (5.4.13) that $a_1 \neq 0$:

$$a_1 = \frac{\int_0^L f(x)\phi_1(x)c(x)\rho(x)\,dx}{\int_0^L \phi_1^2(x)c(x)\rho(x)\,dx}. \tag{5.4.15}$$

It follows that $a_1 \neq 0$, because $\phi_1(x)$ is the eigenfunction corresponding to the lowest eigenvalue and has no zeros; $\phi_1(x)$ is of one sign. Thus, if $f(x) > 0$, it follows that $a_1 \neq 0$, since $c(x)$ and $\rho(x)$ are positive physical functions. In order to sketch the solution for large fixed t, (5.4.14) shows that all that is needed is the first eigenfunction. At the very least, a numerical calculation of the first eigenfunction is easier than the computation of the first hundred.

For large time, the "shape" of the temperature distribution in space stays approximately the same in time. Its amplitude grows or decays in time depending on whether $\lambda_1 > 0$ or $\lambda_1 < 0$ (it would be constant in time if $\lambda_1 = 0$). Since this is a heat flow problem with no sources and with zero temperature at $x = 0$, we certainly expect the temperature to be exponentially decaying toward 0° (i.e., we expect that $\lambda_1 > 0$). Although the right end is insulated, heat energy should flow out the left end since there $u = 0$. We now prove mathematically that all $\lambda > 0$. Since $p(x) = K_0(x)$, $q(x) = 0$, and $\sigma(x) = c(x)\rho(x)$, it follows from the Rayleigh quotient that

$$\lambda = \frac{\int_0^L K_0(x)(d\phi/dx)^2\,dx}{\int_0^L \phi^2 c(x)\rho(x)\,dx}, \tag{5.4.16}$$

where the boundary contribution to (5.4.16) vanished due to the specific homogeneous boundary conditions, (5.4.7) and (5.4.8). It immediately follows from (5.4.16) that all $\lambda \geq 0$, since the thermal coefficients are positive. Furthermore, $\lambda > 0$, since $\phi =$ constant is not an allowable eigenfunction [because $\phi(0) = 0$]. Thus, we have shown that $\lim_{t\to\infty} u(x, t) = 0$ for this example.

Exercises 5.4

5.4.1. Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$

where c, ρ, K_0, and α are functions of x, subject to

$$u(0, t) = 0$$
$$u(L, t) = 0$$
$$u(x, 0) = f(x).$$

Assume that the appropriate eigenfunctions are known.

(a) Show that the eigenvalues are positive if $\alpha < 0$ (see Section 5.2.1).

(b) Solve the initial value problem.

(c) Briefly discuss $\lim_{t\to\infty} u(x, t)$.

5.4.2. Consider the one-dimensional heat equation for nonconstant thermal properties

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$$

with the initial condition $u(x, 0) = f(x)$. [*Hint*: Suppose it is known that if $u(x, t) = \phi(x)h(t)$, then

$$\frac{1}{h}\frac{dh}{dt} = \frac{1}{c(x)\rho(x)\phi}\frac{d}{dx}\left[K_0(x)\frac{d\phi}{dx}\right] = -\lambda.$$

You may assume the eigenfuctions are known. Briefly discuss $\lim_{t\to\infty} u(x, t)$. Solve the initial value problem:

(a) with boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$

***(b)** with boundary conditions $\frac{\partial u}{\partial x}(0, t) = 0$ and $\frac{\partial u}{\partial x}(L, t) = 0$,

Derive coefficients using orthogonality.]

***5.4.3.** Solve

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

with $u(r, 0) = f(r)$, $u(0, t)$ bounded, and $u(a, t) = 0$. You may assume that the corresponding eigenfunctions, denoted $\phi_n(r)$, are known and are complete. [*Hint*: See Section 5.2.2.]

5.4.4. Consider the following boundary value problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

Solve such that $u(x, 0) = \sin \pi x/L$ (initial condition). [*Hint*: If necessary, use a table of integrals.]

5.4.5. Consider

$$\rho\frac{\partial u^2}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u,$$

where $\rho(x) > 0$, $\alpha(x) < 0$, and T_0 is constant, subject to

$$u(0, t) = 0, \quad u(x, 0) = f(x)$$
$$u(L, t) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = g(x).$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem.

***5.4.6.** Consider the vibrations of a *nonuniform* string of mass density $\rho_0(x)$. Suppose that the left end at $x = 0$ is fixed and the right end obeys the elastic boundary condition: $\partial u/\partial x = -(k/T_0)u$ at $x = L$. Suppose that the string is *initially at rest* with a known initial position $f(x)$. Solve this initial value problem. [*Hints*: Assume that the appropriate eigenvalues and corresponding eigenfunctions are known. What differential equations with what boundary conditions do they satisfy? The eigenfunctions are orthogonal with what weighting function?]

5.5 Self-Adjoint Operators and Sturm–Liouville Eigenvalue Problems

INTRODUCTION

In this section, we prove some of the properties of regular Sturm–Liouville eigenvalue problems:

$$\boxed{\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0} \tag{5.5.1}$$

$$\boxed{\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0} \tag{5.5.2}$$

$$\boxed{\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,} \tag{5.5.3}$$

where β_i are real and where, on the finite interval $(a \le x \le b)$, p, q, σ are real continuous functions and p, σ are positive $[p(x) > 0$ and $\sigma(x) > 0]$. At times we will make some comments on the validity of our results if some of these restrictions are removed.

The proofs of these statements are somewhat difficult. We will not prove that there are an infinite number of eigenvalues. We will have to rely for understanding on the examples already presented and on some further examples developed in later sections. For Sturm–Liouville eigenvalue problems that are *not* regular, there may be no eigenvalues at all. However, in most cases of physical interest (on finite intervals) there will still be an infinite number of discrete eigenvalues. We also will not attempt to prove that any piecewise smooth function can be expanded in terms of the eigenfunctions of a regular Sturm–Liouville problem (known as the completeness property). We will not attempt to prove that each succeeding eigenfunction has one additional zero (oscillates one more time).

Linear operators

The proofs we will investigate are made easier to follow by the introduction of **operator** *notation*. Let L stand for the linear differential operator $d/dx[\,p(x)\ d/dx] + q(x)$. An operator acts on a function and yields another function. The notation means that for this L acting on the function $y(x)$,

$$L(y) \equiv \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y. \tag{5.5.4}$$

Thus, $L(y)$ is just a shorthand notation. For example, if $L \equiv d^2/dx^2 + 6$, then $L(y) = d^2y/dx^2 + 6y$ or $L(e^{2x}) = 4e^{2x} + 6e^{2x} = 10e^{2x}$.

The Sturm–Liouville differential equation is rather cumbersome to write over and over again. The use of the linear operator notation is somewhat helpful. Using the operator notation

$$L(\phi) + \lambda\sigma(x)\phi = 0, \tag{5.5.5}$$

where λ is an eigenvalue and ϕ the corresponding eigenfunction, L can operate on any function, not just an eigenfunction.

Lagrange's identity

Most of the proofs we will present concerning Sturm–Liouville eigenvalue problems are immediate consequences of an interesting and fundamental formula known as **Lagrange's identity**. For convenience, we will use the operator notation. We calculate $uL(v) - vL(u)$, where u and v are *any two functions* (not necessarily eigenfunctions). Recall that

$$L(u) = \frac{d}{dx}\left(p\frac{du}{dx}\right) + qu \quad \text{and} \quad L(v) = \frac{d}{dx}\left(p\frac{dv}{dx}\right) + qv,$$

and hence

$$uL(v) - vL(u) = u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + uqv - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - vqu, \tag{5.5.6}$$

where a simple cancellation of $uqv - vqu$ should be noted. The right-hand side of (5.5.6) is manipulated to an exact differential:

$$\boxed{uL(v) - vL(u) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right],} \tag{5.5.7}$$

known as the **differential form of Lagrange's identity**. To derive (5.5.7), we note from the product rule that

$$u\frac{d}{dx}\left(p\frac{dv}{dx}\right) = \frac{d}{dx}\left[u\left(p\frac{dv}{dx}\right)\right] - \left(p\frac{dv}{dx}\right)\frac{du}{dx},$$

and similarly

$$v\frac{d}{dx}\left(p\frac{du}{dx}\right) = \frac{d}{dx}\left[v\left(p\frac{du}{dx}\right)\right] - \left(p\frac{du}{dx}\right)\frac{dv}{dx}.$$

Equation (5.5.7) follows by subtracting these two. Later [see (5.5.21)] we will use the differential form, (5.5.7).

Green's formula

The integral form of Lagrange's identity is also known as **Green's formula**. It follows by integrating (5.5.7):

$$\boxed{\int_a^b [uL(v) - vL(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b} \tag{5.5.8}$$

for any functions[1] u and v. This is a very useful formula.

EXAMPLE

If $p = 1$ and $q = 0$ (in which case $L = d^2/dx^2$), (5.5.7) simply states that

$$u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2} = \frac{d}{dx}\left(u\frac{dv}{dx} - v\frac{du}{dx}\right),$$

which is easily independently checked. For this example, Green's formula is

$$\int_a^b \left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right)dx = \left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b. \qquad \blacksquare$$

Self-adjointness

As an important case of Green's formula, suppose that u and v are any two functions, but with the additional restriction that the boundary terms happen to vanish,

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0.$$

Then, from (5.5.8), $\int_a^b [uL(v) - vL(u)]\,dx = 0$.

Let us show how it is possible for the boundary terms to vanish. Instead of being arbitrary functions, we restrict u and v to *both* satisfy the same set of homogeneous *boundary conditions*. For example, suppose that u and v are *any* two functions that satisfy the following set of boundary conditions:

$$\phi(a) = 0$$
$$\frac{d\phi}{dx}(b) + h\phi(b) = 0.$$

Since both u and v satisfy these conditions, it follows that

$$u(a) = 0 \qquad \text{and} \qquad v(a) = 0$$
$$\frac{du}{dx}(b) + hu(b) = 0 \qquad \frac{dv}{dx}(b) + hv(b) = 0;$$

otherwise, u and v are arbitrary. In this case, the boundary terms for Green's formula vanish:

$$\begin{aligned} p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left[u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right] \\ &= p(b)[-u(b)hv(b) + v(b)hu(b)] = 0. \end{aligned}$$

Thus, for any functions u and v both satisfying these homogeneous boundary conditions, we know that

$$\int_a^b [uL(v) - vL(u)]\,dx = 0.$$

In fact, we claim (see Exercise 5.5.1) that the boundary terms also vanish for any two functions u and v that both satisfy the same set of boundary conditions of the type that occur in the regular

[1]The integration requires du/dx and dv/dx to be continuous.

Sturm–Liouville eigenvalue problems (5.5.2) and (5.5.3). Thus, when discussing any regular Sturm–Liouville eigenvalue problem, we have the following theorem:

> If u and v are any two functions satisfying the same set of homogeneous boundary conditions (of the regular Sturm–Liouville type), then $\int_a^b [uL(v) - vL(u)]\,dx = 0$. (5.5.9)

When (5.5.9) is valid, we say that the operator L (with the corresponding boundary conditions) is **self-adjoint**.[2]

The boundary terms also vanish in circumstances other than for boundary conditions of the regular Sturm–Liouville type. Two important further examples will be discussed briefly. The **periodic boundary condition** can be generalized (for nonconstant-coefficient operators) to

$$\phi(a) = \phi(b) \quad \text{and} \quad p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b).$$

In this situation, (5.5.9) also can be shown (see Exercise 5.5.1) to be valid. Another example in which the boundary terms in Green's formula vanish is the "singular" case. The singular case occurs if the coefficient of the second derivative of the differential operator is zero at an endpoint; for example, if $p(x) = 0$ at $x = 0$ [i.e., $p(0) = 0$]. At a singular endpoint, a singularity condition is imposed. The usual singularity condition at $x = 0$ is that $\phi(0)$ is bounded. It can also be shown that (5.5.9) is valid (see Exercise 5.5.1) if both u and v satisfy this singularity condition at $x = 0$ and any regular Sturm–Liouville type of boundary condition at $x = b$.

Orthogonal eigenfunctions

We now will show the usefulness of Green's formula. We will begin by proving the important orthogonality relationship for Sturm–Liouville eigenvalue problems. For many types of boundary conditions, **eigenfunctions corresponding to different eigenvalues are orthogonal with weight** $\sigma(x)$. To prove that statement, let λ_n and λ_m be eigenvalues with corresponding eigenfunctions $\phi_n(x)$ and $\phi_m(x)$. Using the operator notation, the differential equations satisfied by these eigenfunctions are

$$L(\phi_n) + \lambda_n\sigma(x)\phi_n = 0 \tag{5.5.10}$$

$$L(\phi_m) + \lambda_m\sigma(x)\phi_m = 0. \tag{5.5.11}$$

In addition, both ϕ_n and ϕ_m satisfy the same set of homogeneous boundary conditions. Since u and v are arbitrary functions, we may let $u = \phi_m$ and $v = \phi_n$ in Green's formula:

$$\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)]\,dx = p(x)\left(\phi_m\frac{d\phi_n}{dx} - \phi_n\frac{d\phi_m}{dx}\right)\Bigg|_a^b.$$

$L(\phi_n)$ and $L(\phi_m)$ may be eliminated from (5.5.10) and (5.5.11). Thus,

$$(\lambda_m - \lambda_n)\int_a^b \phi_n\phi_m\sigma\,dx = p(x)\left(\phi_m\frac{d\phi_n}{dx} - \phi_n\frac{d\phi_m}{dx}\right)\Bigg|_a^b, \tag{5.5.12}$$

corresponding to multiplying (5.5.10) by ϕ_m, multiplying (5.5.11) by ϕ_n, subtracting the two, and then integrating. We avoided these steps (especially the integration) by applying Green's formula. For many different kinds of boundary conditions (i.e., regular Sturm–Liouville types, periodic case, and the singular case), the boundary terms vanish if u and v both satisfy the same set of homogeneous boundary conditions. Since u and v are eigenfunctions, they satisfy this condition, and thus (5.5.12) implies that

$$(\lambda_m - \lambda_n)\int_a^b \phi_n\phi_m\sigma\,dx = 0. \tag{5.5.13}$$

If $\lambda_m \neq \lambda_n$, then it immediately follows that

$$\boxed{\int_a^b \phi_n\phi_m\sigma\,dx = 0.} \tag{5.5.14}$$

[2]We usually avoid in this text an explanation of an **adjoint** operator. Here, L equals its adjoint and so is called *self-adjoint*.

In other words, eigenfunctions (ϕ_n and ϕ_m) corresponding to different eigenvalues ($\lambda_n \neq \lambda_m$) are orthogonal with weight $\sigma(x)$.

Real eigenvalues

We can use the orthogonality of eigenfunctions to prove that the eigenvalues are real. Suppose that λ is a complex eigenvalue and $\phi(x)$ the corresponding eigenfunction (also allowed to be complex since the differential equation defining the eigenfunction would be complex):

$$L(\phi) + \lambda\sigma\phi = 0. \tag{5.5.15}$$

We introduce the overbar notation for the complex conjugate (e.g., if $z = x + iy$, then $\overline{z} = x - iy$). Note that if $z = 0$, then $\overline{z} = 0$. Thus, the complex conjugate of (5.5.15) is also valid:

$$\overline{L(\phi)} + \overline{\lambda}\sigma\overline{\phi} = 0, \tag{5.5.16}$$

assuming that the coefficient σ is real and hence $\overline{\sigma} = \sigma$. The complex conjugate of $L(\phi)$ is exactly L operating on the complex conjugate of ϕ, $\overline{L(\phi)} = L(\overline{\phi})$, since the coefficients of the linear differential operator are also real (see Exercise 5.5.7). Thus,

$$L(\overline{\phi}) + \overline{\lambda}\sigma\overline{\phi} = 0. \tag{5.5.17}$$

If ϕ satisfies boundary conditions with real coefficients, then $\overline{\phi}$ satisfies the same boundary conditions. For example, if $d\phi/dx + h\phi = 0$ at $x = a$, then by taking complex conjugates, $d\overline{\phi}/dx + h\overline{\phi} = 0$ at $x = a$. Equation (5.5.17) and the boundary conditions show that $\overline{\phi}$ satisfies the Sturm–Liouville eigenvalue problem, but with the eigenvalue being $\overline{\lambda}$. We have thus proved the following theorem[3]: **If λ is a complex eigenvalue with corresponding eigenfunction ϕ, then $\overline{\lambda}$ is also an eigenvalue with corresponding eigenfunction $\overline{\phi}$.**

However, we will show that λ cannot be complex. As we have shown, if λ is an eigenvalue, then so too is $\overline{\lambda}$. According to our fundamental orthogonality theorem, the corresponding eigenfunctions (ϕ and $\overline{\phi}$) must be orthogonal (with weight σ). Thus, from (5.5.13),

$$\left(\lambda - \overline{\lambda}\right)\int_a^b \phi\overline{\phi}\sigma\, dx = 0. \tag{5.5.18}$$

Since $\phi\overline{\phi} = |\phi|^2 \geq 0$ (and $\sigma > 0$), the integral in (5.5.18) is ≥ 0. In fact, the integral can equal zero only if $\phi \equiv 0$, which is prohibited since ϕ is an eigenfunction. Thus, (5.5.18) implies that $\lambda = \overline{\lambda}$, and hence λ is real; **all the eigenvalues are real**. The eigenfunctions can always be chosen to be real.

Unique eigenfunctions (regular and singular cases)

We next prove that there is only one eigenfunction corresponding to an eigenvalue (except for the case of periodic boundary conditions). Suppose that there are two different eigenfunctions ϕ_1 and ϕ_2 corresponding to the same eigenvalue λ. We say λ is a "multiple" eigenvalue with multiplicity 2. In this case, both

$$\begin{aligned} L(\phi_1) + \lambda\sigma\phi_1 &= 0 \\ L(\phi_2) + \lambda\sigma\phi_2 &= 0. \end{aligned} \tag{5.5.19}$$

Since λ is the same in both expressions,

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = 0. \tag{5.5.20}$$

This can be integrated by some simple manipulations. However, we avoid this algebra by simply quoting the differential form of Lagrange's identity:

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx}\left[p\left(\phi_2\frac{d\phi_1}{dx} - \phi_1\frac{d\phi_2}{dx}\right)\right]. \tag{5.5.21}$$

From (5.5.20) it follows that

$$p\left(\phi_1\frac{d\phi_2}{dx} - \phi_2\frac{d\phi_1}{dx}\right) = \text{constant}. \tag{5.5.22}$$

[3]A "similar" type of theorem follows from the quadratic formula: For a quadratic equation with real coefficients, if λ is a complex root, then so is $\overline{\lambda}$. This also holds for any algebraic equation with real coefficients.

Often we can evaluate the constant from one of the boundary conditions. For example, if $d\phi/dx + h\phi = 0$ at $x = a$, a short calculation shows that the constant equals zero. In fact, we claim (Exercise 5.5.10) that the constant also equals zero if *at least one of the boundary conditions is of the regular Sturm–Liouville type (or of the singular type)*. For any of these boundary conditions it follows that

$$\phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx} = 0. \tag{5.5.23}$$

This is equivalent to $d/dx(\phi_2/\phi_1) = 0$, and hence *for these boundary conditions*

$$\phi_2 = c\phi_1. \tag{5.5.24}$$

This shows that any two eigenfunctions ϕ_1 and ϕ_2 corresponding to the same eigenvalue must be a constant multiple of each other *for the preceding boundary conditions*. The two eigenfunctions are dependent; there is only one linearly independent eigenfunction; the eigenfunction is unique.

Nonunique eigenfunctions (periodic case)

For periodic boundary conditions, we *cannot* conclude that the constant in (5.5.22) must be zero. Thus, it is possible that $\phi_2 \neq c\phi_1$ and that there *might* be two different eigenfunctions corresponding to the same eigenvalue.

For example, consider the simple eigenvalue problem with periodic boundary conditions,

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(-L) &= \phi(L) \\ \frac{d\phi}{dx}(-L) &= \frac{d\phi}{dx}(L). \end{aligned} \tag{5.5.25}$$

We know that the eigenvalue 0 has any constant as the unique eigenfunction. The other eigenvalues, $(n\pi/L)^2$, $n = 1, 2, \ldots$, each have two linearly independent eigenfunctions, $\sin n\pi x/L$ and $\cos n\pi x/L$. This, we know, gives rise to a Fourier series. However, (5.5.25) is not a regular Sturm–Liouville eigenvalue problem, since the boundary conditions are not of the prescribed form. Our theorem about unique eigenfunctions does not apply; we *may* have two[4] eigenfunctions corresponding to the same eigenvalue. Note that it is still *possible* to have only one eigenfunction, as occurs for $\lambda = 0$.

Nonunique eigenfunctions (Gram–Schmidt orthogonalization)

We can solve for generalized Fourier coefficients (and correspondingly we are able to solve some partial differential equations) because of the orthogonality of the eigenfunctions. However, our theorem states that eigenfunctions *corresponding to different eigenvalues* are automatically orthogonal [with weight $\sigma(x)$]. For the case of periodic (or mixed-type) boundary conditions, it is possible for there to be more than one independent eigenfunction corresponding to the same eigenvalue. For these multiple eigenvalues, the eigenfunctions are not automatically orthogonal to each other. In the appendix to Section 7.5, we will show that we always are able to construct the eigenfunctions such that they are orthogonal by a process called Gram–Schmidt orthogonalization.

Exercises 5.5

5.5.1. A Sturm–Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$$

because then $\int_a^b [uL(v) - vL(u)]\,dx = 0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems:

(a) $\phi(0) = 0$ and $\phi(L) = 0$

(b) $\dfrac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

[4]No more than two independent eigenfunctions are possible, since the differential equation is of second order.

(c) $\frac{d\phi}{dx}(0) - h\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(d) $\phi(a) = \phi(b)$ and $p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b)$

(e) $\phi(a) = \phi(b)$ and $\frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$ [self-adjoint only if $p(a) = p(b)$]

(f) $\phi(L) = 0$ and [in the situation in which $p(0) = 0$] $\phi(0)$ bounded and $\lim_{x\to 0} p(x)\frac{d\phi}{dx} = 0$

***(g)** Under what conditions is the following self-adjoint (if p is constant)?

$$\phi(L) + \alpha\phi(0) + \beta\frac{d\phi}{dx}(0) = 0$$
$$\frac{d\phi}{dx}(L) + \gamma\phi(0) + \delta\frac{d\phi}{dx}(0) = 0$$

5.5.2. Prove that the eigenfunctions corresponding to different eigenvalues (of the following eigenvalue problem) are orthogonal:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0$$

with the boundary conditions

$$\phi(1) = 0$$
$$\phi(2) - 2\frac{d\phi}{dx}(2) = 0.$$

What is the weighting function?

5.5.3. Consider the eigenvalue problem $L(\phi) = -\lambda\sigma(x)\phi$, subject to a given set of homogeneous boundary conditions. Suppose that

$$\int_a^b [uL(v) - vL(u)]\,dx = 0$$

for all functions u and v satisfying the same set of boundary conditions. Prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).

5.5.4. Give an example of an eigenvalue problem with more than one eigenfunction corresponding to an eigenvalue.

5.5.5. Consider

$$L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9.$$

(a) Show that $L(e^{rx}) = (r+3)^2 e^{rx}$.

(b) Use part (a) to obtain solutions of $L(y) = 0$ (a second-order constant-coefficient differential equation).

(c) If z depends on x and a parameter r, show that

$$\frac{\partial}{\partial r}L(z) = L\left(\frac{\partial z}{\partial r}\right).$$

(d) Using part (c), evaluate $L(\partial z/\partial r)$ if $z = e^{rx}$.

(e) Obtain a second solution of $L(y) = 0$, using part (d).

5.5.6. Prove that if x is a root of a sixth-order polynomial *with real coefficients*, then $\bar{x}$ is also a root.

5.5.7. For

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

with p and q real, carefully show that

$$\overline{L(\phi)} = L(\bar{\phi}).$$

5.5.8. Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $uL(v) - vL(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [uL(v) - vL(u)]\,dx$ in terms of the boundary data for any functions u and v.

(c) Show that $\int_0^1 [uL(v) - vL(u)]\,dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\phi(0) = 0, \qquad \phi(1) = 0$$
$$\frac{d\phi}{dx}(0) = 0, \quad \frac{d^2\phi}{dx^2}(1) = 0.$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [uL(v) - vL(u)]\,dx = 0.$$

(e) For the eigenvalue problem [using the boundary conditions in part (c)]

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

***5.5.9.** For the eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$$

subject to the boundary conditions

$$\phi(0) = 0, \qquad \phi(1) = 0$$
$$\frac{d\phi}{dx}(0) = 0, \qquad \frac{d^2\phi}{dx^2}(1) = 0,$$

show that the eigenvalues are less than or equal to zero ($\lambda \leq 0$). (Don't worry; in a physical context that is exactly what is expected.) Is $\lambda = 0$ an eigenvalue?

5.5.10. **(a)** Show that (5.5.22) yields (5.5.23) if at least one of the boundary conditions is of the regular Sturm–Liouville type.

(b) Do part (a) if one boundary condition is of the singular type.

5.5.11. ***(a)** Suppose that

$$L = p(x)\frac{d^2}{dx^2} + r(x)\frac{d}{dx} + q(x).$$

Consider

$$\int_a^b vL(u)\,dx.$$

By repeated integration by parts, determine the adjoint operator L^* such that

$$\int_a^b [uL^*(v) - vL(u)]dx = H(x)\Big|_a^b.$$

What is $H(x)$? Under what conditions does $L = L^*$, the **self-adjoint** case? [*Hint*: Show that

$$L^* = p\frac{d^2}{dx^2} + \left(2\frac{dp}{dx} - r\right)\frac{d}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right).\Big]$$

(b) If

$$u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) + u(L) = 0,$$

what boundary conditions should $v(x)$ satisfy for $H(x)|_0^L = 0$, called the adjoint boundary conditions?

5.5.12. Consider nonself-adjoint operators as in Exercise 5.5.11. The eigenvalues λ may be complex as well as their corresponding eigenfunctions ϕ.

(a) Show that if λ is a complex eigenvalue with corresponding eigenfunction ϕ, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue with eigenfunction $\overline{\phi}$.

(b) The eigenvalues of the adjoint L^* may be different from the eigenvalues of L. Using the result of Exercise 5.5.11, show that the eigenfunctions of $L(\phi) + \lambda\sigma\phi = 0$ are orthogonal with weight σ (in a complex sense) to eigenfunctions of $L^*(\psi) + \nu\sigma\psi = 0$ if the eigenvalues are different. Assume that ψ satisfies adjoint boundary conditions. You should also use part (a).

5.5.13. Using the result of Exercise 5.5.11, prove the following part of the **Fredholm alternative** (for operators that are not necessarily self-adjoint): A solution of $L(u) = f(x)$ subject to homogeneous boundary conditions may exist only if $f(x)$ is orthogonal to all solutions of the homogeneous adjoint problem.

5.5.14. If L is the following first-order linear differential operator

$$L = p(x)\frac{d}{dx},$$

then determine the adjoint operator L^* such that

$$\int_a^b [uL^*(v) - vL(u)]dx = H(x)\Big|_a^b.$$

What is $B(x)$? [*Hint*: Consider $\int_a^b vL(u)\,dx$ and integrate by parts.]

5.5.15. Consider the eigenvalue problem $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0$ $(0 < r < 1)$, subject to the boundary conditions $|\phi(0)| < \infty$ (you may also assume $\frac{d\phi}{dr}$ is bounded) and $\frac{d\phi}{dr}(1) = 0$. Prove that eigenfunctions corresponding to different eigenvalues are orthogonal. (With what weight?)

5.5.16. Consider the one-dimensional wave equation for nonconstant density so that

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$ and the initial conditions $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = f(x)$. [*Hint*: Suppose it is known that if $u(x, t) = \phi(x)G(t)$, then

$$\frac{1}{G}\frac{d^2G}{dt^2} = \frac{c^2(x)}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.]$$

(a) With the initial conditions $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = f(x)$, prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).

(b) With the initial conditions $u(x, 0) = f(x)$, $\frac{\partial u}{\partial t}(x, 0) = 0$, assume eigenfunctions are orthogonal (with what weight?). Derive coefficients using orthogonality.

5.5.17. Consider the one-dimensional wave equation with c constant,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + q(x, t),$$

subject to the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$, and the initial conditions $u(x, 0) = f(x)$, $\frac{\partial u}{\partial t}(x, 0) = 0$.

(a) Use the method of eigenfunction expansion (justify spatial term-by-term differentiation)

$$u(x, t) = \sum b_n(t)\phi_n(x).$$

What are the eigenfunctions $\phi_n(x)$? What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this ordinary differential equation.

(b) What is the general solution of the differential equation for $b_n(t)$ if $q(x, t) = 0$? Solve for $b_n(t)$ including initial conditions.

5.5.18. Prove Green's formula for the Sturm–Liouville operator $L(y) = \frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy$.

Appendix to 5.5: Matrix Eigenvalue Problem and Orthogonality of Eigenvectors

The matrix eigenvalue problem

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}, \tag{5.5.26}$$

where $\boldsymbol{A}$ is an $n \times n$ real matrix (with entries a_{ij}) and $\boldsymbol{x}$ is an n-dimensional column vector (with components x_i), has many properties similar to those of the Sturm–Liouville eigenvalue problem.

EIGENVALUES AND EIGENVECTORS

For all values of λ, $\boldsymbol{x} = 0$ is a "trivial" solution of the homogeneous linear system (5.5.26). We ask, for what values of λ are there nontrivial solutions? In general, (5.5.26) can be rewritten as

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}, \tag{5.5.27}$$

where $\boldsymbol{I}$ is the identity matrix. According to the theory of linear equations (elementary linear algebra), a nontrivial solution exists only if

$$\det[\boldsymbol{A} - \lambda \boldsymbol{I}] = 0. \tag{5.5.28}$$

Such values of λ are called **eigenvalues**, and the corresponding nonzero vectors $\boldsymbol{x}$ called **eigenvectors**.

In general, (5.5.28) yields an nth-degree polynomial (known as the **characteristic polynomial**) that determines the eigenvalues; there will be n eigenvalues (but they may not be distinct). Corresponding to each distinct eigenvalue, there will be an eigenvector.

EXAMPLE

If $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix}$, then the eigenvalues satisfy

$$0 = \det \begin{bmatrix} 2-\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1),$$

the characteristic polynomial. The eigenvalues are $\lambda = 4$ and $\lambda = -1$. For $\lambda = 4$, (5.5.26) becomes

$$2x_1 + x_2 = 4x_1 \quad \text{and} \quad 6x_1 + x_2 = 4x_2,$$

or, equivalently, $x_2 = 2x_1$. The eigenvector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an arbitrary multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda = 4$. For $\lambda = -1$,

$$2x_1 + x_2 = -x_1 \quad \text{and} \quad 6x_1 + x_2 = -x_2,$$

and thus the eigenvector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an arbitrary multiple of $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. ■

Green's formula

The matrix $\boldsymbol{A}$ may be thought of as a linear operator in the same way that

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

is a linear differential operator. $\boldsymbol{A}$ operates on n-dimensional vectors producing an n-dimensional vector, while L operates on functions and yields a function. In analyzing the Sturm–Liouville eigenvalue problem, Green's formula was important:

$$\int_a^b [uL(v) - vL(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b,$$

where u and v are arbitrary functions. Often, the boundary terms vanished. For vectors, the dot product is analogous to integration, $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_i a_i b_i$, where a_i and b_i are the ith components of,

respectively, $\boldsymbol{a}$ and $\boldsymbol{b}$ (see Section 2.3 Appendix). By direct analogy to Green's formula, we would be led to investigate $\boldsymbol{u} \cdot \boldsymbol{Av}$ and $\boldsymbol{v} \cdot \boldsymbol{Au}$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are arbitrary vectors. Instead, we analyze $\boldsymbol{u} \cdot \boldsymbol{Av}$ and $\boldsymbol{v} \cdot \boldsymbol{Bu}$, where $\boldsymbol{B}$ is any $n \times n$ matrix:

$$\boldsymbol{u} \cdot \boldsymbol{Av} = \sum_i \left(u_i \sum_j a_{ij} v_j \right) = \sum_i \sum_j a_{ij} u_i v_j$$
$$\boldsymbol{v} \cdot \boldsymbol{Bu} = \sum_i \left(v_i \sum_j b_{ij} u_j \right) = \sum_i \sum_j b_{ij} u_j v_i = \sum_i \sum_j b_{ji} u_i v_j,$$

where an alternative expression for $\boldsymbol{v} \cdot \boldsymbol{Bu}$ was derived by interchanging the roles of i and j. Thus,

$$\boldsymbol{u} \cdot \boldsymbol{Av} - \boldsymbol{v} \cdot \boldsymbol{Bu} = \sum_i \sum_j (a_{ij} - b_{ji}) u_i v_j.$$

If we let $\boldsymbol{B}$ equal the transpose of $\boldsymbol{A}$ (i.e., $b_{ji} = a_{ij}$), whose notation is $\boldsymbol{B} = \boldsymbol{A}^T$, then we have the following theorem:

$$\boxed{\boldsymbol{u} \cdot \boldsymbol{Av} - \boldsymbol{v} \cdot \boldsymbol{A}^T \boldsymbol{u} = 0,} \tag{5.5.29}$$

analogous to Green's formula.

Self-adjointness

The difference between $\boldsymbol{A}$ and its transpose, $\boldsymbol{A}^T$, in (5.5.29) causes insurmountable difficulties for us. We will thus restrict our attention to **symmetric** matrices, in which case $\boldsymbol{A} = \boldsymbol{A}^T$. For *symmetric matrices*,

$$\boxed{\boldsymbol{u} \cdot \boldsymbol{Av} - \boldsymbol{v} \cdot \boldsymbol{Au} = 0,} \tag{5.5.30}$$

and we will be able to use this result to prove the same theorems about eigenvalues and eigenvectors for matrices as we proved about Sturm–Liouville eigenvalue problems.

For symmetric matrices, eigenvectors corresponding to different eigenvalues are orthogonal

To prove this, suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are eigenvectors corresponding to λ_1 and λ_2, respectively:

$$\boldsymbol{Au} = \lambda_1 \boldsymbol{u} \quad \text{and} \quad \boldsymbol{Av} = \lambda_2 \boldsymbol{v}.$$

If we directly apply (5.5.30), then

$$(\lambda_2 - \lambda_1) \boldsymbol{u} \cdot \boldsymbol{v} = 0.$$

Thus, if $\lambda_1 \neq \lambda_2$ (different eigenvalues), the corresponding eigenvectors are orthogonal in the sense that

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0. \tag{5.5.31}$$

We leave as an exercise the proof that the eigenvalues of a symmetric real matrix are real.

EXAMPLE

The eigenvalues of the real symmetric matrix $\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ are determined from $(6 - \lambda)(3 - \lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2) = 0$. For $\lambda = 2$, the eigenvector satisfies

$$6x_1 + 2x_2 = 2x_1 \quad \text{and} \quad 2x_1 + 3x_2 = 2x_2,$$

and hence $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. For $\lambda = 7$, it follows that

$$6x_1 + 2x_2 = 7x_1 \quad \text{and} \quad 2x_1 + 3x_2 = 7x_2,$$

and the eigenvector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. As we have just proved for any real symmetric matrix, the eigenvectors are orthogonal, $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 - 2 = 0$. ■

Eigenvector expansions

For real symmetric matrices it can be shown that if an eigenvalue repeats R times, there will be R independent eigenvectors corresponding to that eigenvalue. These eigenvectors are automatically orthogonal to any eigenvectors corresponding to a different eigenvalue. The Gram–Schmidt procedure (see Section 7.5 Appendix) can be applied so that all R eigenvectors corresponding to the same eigenvalue can be constructed to be mutually orthogonal. In this manner, for real symmetric $n \times n$ matrices, n orthogonal eigenvectors can always be obtained. Since these vectors are orthogonal, they span the n-dimensional vector space and may be chosen as basis vectors. Any vector $\boldsymbol{v}$ may be represented in a series of the eigenvectors:

$$\boldsymbol{v} = \sum_{i=1}^{n} c_i \boldsymbol{\phi_i}, \tag{5.5.32}$$

where $\boldsymbol{\phi_i}$ is the ith eigenvector. For regular Sturm–Liouville eigenvalue problems, the eigenfunctions are complete, meaning that any (piecewise smooth) function can be represented in terms of an eigenfunction expansion

$$f(x) \sim \sum_{i=1}^{\infty} c_i \phi_i(x). \tag{5.5.33}$$

This is analogous to (5.5.32). In (5.5.33), the Fourier coefficients c_i are determined by the orthogonality of the eigenfunctions. Similarly, the coordinates c_i in (5.5.32) are determined by the orthogonality of the eigenvectors. We dot Equation (5.5.32) into $\boldsymbol{\phi_m}$:

$$\boldsymbol{v} \cdot \boldsymbol{\phi_m} = \sum_{i=1}^{n} c_i \boldsymbol{\phi_i} \cdot \boldsymbol{\phi_m} = c_m \boldsymbol{\phi_m} \cdot \boldsymbol{\phi_m},$$

since $\boldsymbol{\phi_i} \cdot \boldsymbol{\phi_m} = 0, i \neq m$, determining c_m.

Linear systems

Sturm–Liouville eigenvalue problems arise in separating variables for partial differential equations. One way in which the matrix eigenvalue problem occurs is in "separating" a linear homogeneous system of ordinary differential equations with constant coefficients. We will be *very* brief. A linear homogeneous first-order system of differential equations may be represented by

$$\boxed{\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}\boldsymbol{v},} \tag{5.5.34}$$

where $\boldsymbol{A}$ is an $n \times n$ matrix and $\boldsymbol{v}$ is the desired n-dimensional vector solution. $\boldsymbol{v}$ usually satisfies given initial conditions, $\boldsymbol{v}(0) = \boldsymbol{v}_0$. We seek special solutions of the form of simple exponentials:

$$\boxed{\boldsymbol{v}(t) = e^{\lambda t}\boldsymbol{\phi},} \tag{5.5.35}$$

where $\boldsymbol{\phi}$ is a constant vector. This is analogous to seeking product solutions by the method of separation of variables. Since $d\boldsymbol{v}/dt = \lambda e^{\lambda t}\boldsymbol{\phi}$, it follows that

$$\boldsymbol{A}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}. \tag{5.5.36}$$

Thus, there exist solutions to (5.5.34) of the form (5.5.35) if λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{\phi}$ is a corresponding eigenvector. We now restrict our attention to *real symmetric matrices* $\boldsymbol{A}$. There will always be n mutually orthogonal eigenvectors $\boldsymbol{\phi_i}$. We have obtained n special solutions to the linear homogeneous system (5.5.34). A *principle of superposition* exists, and hence a linear combination of these solutions also satisfies (5.5.34):

$$\boxed{\boldsymbol{v} = \sum_{i=1}^{n} c_i e^{\lambda_i t}\boldsymbol{\phi_i}.} \tag{5.5.37}$$

We attempt to determine c_i so that (5.5.37) satisfies the initial conditions, $\boldsymbol{v}(0) = \boldsymbol{v}_0$:

$$\boldsymbol{v}_0 = \sum_{i=1}^{n} c_i \boldsymbol{\phi}_i.$$

Here, the orthogonality of the eigenvectors is helpful, and thus, as before,

$$c_i = \frac{\boldsymbol{v}_0 \bullet \boldsymbol{\phi}_i}{\boldsymbol{\phi}_i \bullet \boldsymbol{\phi}_i}.$$

5.5 Appendix Exercises

5.5A.1. Prove that the eigenvalues of real symmetric matrices are real.

5.5A.2. **(a)** Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

has only one independent eigenvector.

(b) Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has two independent eigenvectors.

5.5A.3. Consider the eigenvectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}.$$

(a) Show that the eigenvectors are not orthogonal.

(b) If the "dot product" of two vectors is defined as follows,

$$\boldsymbol{a} \bullet \boldsymbol{b} = \frac{1}{4} a_1 b_1 + a_2 b_2,$$

show that the eigenvectors are orthogonal with this dot product.

5.5A.4. Solve $d\boldsymbol{v}/dt = \boldsymbol{A}\boldsymbol{v}$ using matrix methods if

***(a)** $\boldsymbol{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$, $\boldsymbol{v}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$, $\boldsymbol{v}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

5.5A.5. Show that the eigenvalues are real and the eigenvectors orthogonal:

(a) $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}$

***(b)** $\boldsymbol{A} = \begin{bmatrix} 3 & 1-i \\ 1+i & 1 \end{bmatrix}$ (see Exercise 5.5A.6)

5.5A.6. For a matrix $\boldsymbol{A}$ whose entries are complex numbers, the complex conjugate of the transpose is denoted by $\boldsymbol{A}^H$. For matrices in which $\boldsymbol{A}^H = \boldsymbol{A}$ (called **Hermitian**):

(a) Prove that the eigenvalues are real.

(b) Prove that eigenvectors corresponding to different eigenvalues are orthogonal (in the sense that $\boldsymbol{\phi}_i \bullet \overline{\boldsymbol{\phi}}_m = 0$, where the overbar denotes the complex conjugate).

5.6 Rayleigh Quotient

The Rayleigh quotient can be derived from the Sturm–Liouville differential equation,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0, \tag{5.6.1}$$

by multiplying (5.6.1) by ϕ and integrating:

$$\int_a^b \left[\phi\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi^2\right] dx + \lambda\int_a^b \phi^2\sigma \quad dx = 0.$$

Since $\int_a^b \phi^2 \sigma \; dx > 0$, we can solve for λ:

$$\lambda = \frac{-\int_a^b \left[\phi \frac{d}{dx}\left(p \frac{d\phi}{dx} \right) + q\phi^2 \right] dx}{\int_a^b \phi^2 \sigma \; dx}. \tag{5.6.2}$$

Integration by parts [$\int u\, dv = uv - \int v\, du$, where $u = \phi$, $dv = d/dx(p\, d\phi/dx)\, dx$ and hence $du = d\phi/dx\, dx$, $v = p\, d\phi/dx$] yields an expression involving the function ϕ evaluated at the boundary:

$$\boxed{\lambda = \frac{-p\phi \frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p \left(\frac{d\phi}{dx} \right)^2 - q\phi^2 \right] dx}{\int_a^b \phi^2 \sigma \, dx},} \tag{5.6.3}$$

known as the **Rayleigh quotient**. In Sections 5.3 and 5.4, we have indicated some applications of this result. Further discussion will be given in Section 5.7.

NONNEGATIVE EIGENVALUES

Often in physical problems, the sign of λ is quite important. As shown in Section 5.2.1, $dh/dt + \lambda h = 0$ in certain heat flow problems. Thus, positive λ corresponds to exponential decay in time, while negative λ corresponds to exponential growth. On the other hand, in certain vibration problems (see Section 5.7), $d^2h/dt^2 = -\lambda h$. There, only positive λ corresponds to the "usually" expected oscillations. Thus, in both types of problems we often expect $\lambda \geq 0$:

The Rayleigh quotient (5.6.3) directly proves that $\lambda \geq 0$ if

$$\text{(a)} \; -p\, \phi \frac{d\phi}{dx}\Big|_a^b \geq 0, \text{ and}$$
$$\text{(b)} \; q \leq 0. \tag{5.6.4}$$

We claim that both (a) and (b) are physically reasonable conditions for nonnegative λ. Consider the boundary constraint, $-p\phi \; d\phi/dx|_a^b \geq 0$. The simplest types of homogeneous boundary conditions, $\phi = 0$ and $d\phi/dx = 0$, do not contribute to this boundary term, satisfying (a). The condition $d\phi/dx = h\phi$ (for the physical cases of Newton's law of cooling or the elastic boundary condition) has $h > 0$ at the left end, $x = a$. Thus, it will have a positive contribution at $x = a$. The sign switch at the right end, which occurs for this type of boundary condition, will also cause a positive contribution. The periodic boundary condition [e.g., $\phi(a) = \phi(b)$ and $p(a)\, d\phi/dx(a) = p(b)\, d\phi/dx(b)$] as well as the singularity condition [$\phi(a)$ bounded, if $p(a) = 0$] also do not contribute. Thus, in all these cases $-p\phi \; d\phi/dx|_a^b \geq 0$.

The source constraint $q \leq 0$ also has a meaning in physical problems. For heat flow problems, $q \leq 0$ corresponds ($q = \alpha$, $Q = \alpha u$) to an energy-absorbing (endothermic) reaction, while for vibration problems, $q \leq 0$ corresponds ($q = \alpha$, $Q = \alpha u$) to a restoring force.

Minimization principle

The Rayleigh quotient cannot be used to determine explicitly the eigenvalue (since ϕ is unknown). Nonetheless, it can be quite useful in estimating the eigenvalues. This is because of the following theorem: **The minimum value of the Rayleigh quotient for all continuous functions satisfying the boundary conditions** (but not necessarily the differential equation) **is the lowest eigenvalue**:

$$\boxed{\lambda_1 = \min \frac{-pu \; du/dx|_a^b + \int_a^b \left[p(du/dx)^2 - qu^2 \right] dx}{\int_a^b u^2 \sigma \; dx},} \tag{5.6.5}$$

where λ_1 represents the smallest eigenvalue. The minimization includes all continuous functions that satisfy the boundary conditions. The minimum is obtained only for $u = \phi_1(x)$, the lowest eigenfunction. For example, the lowest eigenvalue is important in heat flow problems (see Section 5.4).

Trial functions

Before proving (5.6.5), we will indicate how (5.6.5) is applied to obtain bounds on the lowest eigenvalue. Equation (5.6.5) is difficult to apply directly since we do not know how to minimize over all functions. However, let u_T be *any* continuous function satisfying the boundary conditions; u_T is known as a **trial function**. We compute the Rayleigh quotient of this trial function, $RQ[u_T]$:

$$\lambda_1 \le RQ[u_T] = \frac{-pu_T\, du_T/dx\big|_a^b + \int_a^b [p\,(du_T/dx)^2 - qu_T^2]\,dx}{\int_a^b u_T^2\sigma\,dx}. \tag{5.6.6}$$

We have noted that λ_1 must be less than or equal to the quotient since λ_1 is the minimum of the ratio for all functions. Equation (5.6.6) gives an **upper bound** for the lowest eigenvalue.

EXAMPLE

Consider the well-known eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$
$$\phi(0) = 0$$
$$\phi(1) = 0.$$

We already know that $\lambda = n^2\pi^2 (L = 1)$, and hence the lowest eigenvalue is $\lambda_1 = \pi^2$. For this problem, the Rayleigh quotient simplifies, and (5.6.6) becomes

$$\lambda_1 \le \frac{\int_0^1 (du_T/dx)^2\,dx}{\int_0^1 u_T^2\,dx}. \tag{5.6.7}$$

Trial functions must be continuous and satisfy the homogeneous boundary conditions, in this case, $u_T(0) = 0$ and $u_T(1) = 0$. In addition, we claim that the closer the trial function is to the actual eigenfunction, the more accurate is the bound of the lowest eigenvalue. Thus, we also choose trial functions with no zeros in the interior, since we already know theoretically that the lowest eigenfunction does not have a zero. We will compute the Rayleigh quotient for the three trial functions sketched in Fig. 5.6.1. For

$$u_T = \begin{cases} x, & x < \frac{1}{2} \\ 1 - x, & x > \frac{1}{2}, \end{cases}$$

(5.6.7) becomes

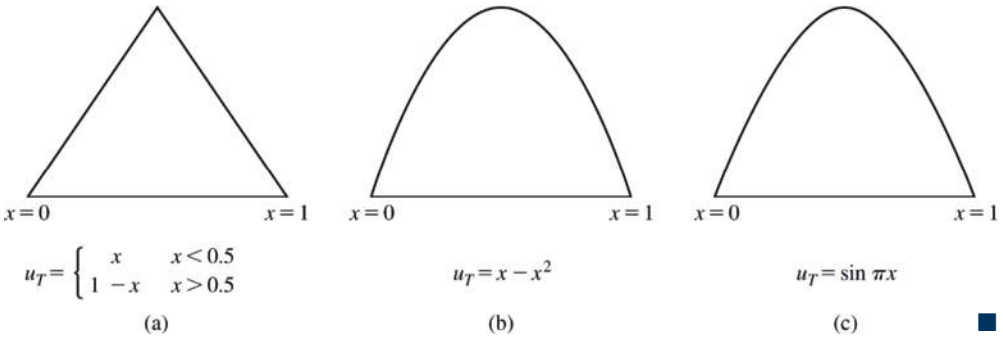
$$\lambda_1 \le \frac{\int_0^{1/2} dx + \int_{1/2}^1 dx}{\int_0^{1/2} x^2\,dx + \int_{1/2}^1 (1-x)^2\,dx} = \frac{1}{\frac{1}{24} + \frac{1}{24}} = 12,$$

a fair upper bound for the exact answer $\pi^2 (\pi^2 \approx 9.8696 \ldots)$. For $u_T = x - x^2$, (5.6.7) becomes

$$\lambda_1 \le \frac{\int_0^1 (1-2x)^2\,dx}{\int_0^1 (x-x^2)^2\,dx} = \frac{\int_0^1 (1 - 4x + 4x^2)\,dx}{\int_0^1 (x^2 - 2x^3 + x^4)\,dx} = \frac{1 - 2 + \frac{4}{3}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{5}} = 10,$$

a more accurate bound. Since $u_T = \sin\pi x$ is the actual lowest eigenfunction, the Rayleigh quotient for this trial function will exactly equal the lowest eigenvalue. Other applications of the Rayleigh quotient will be shown in later sections.

Figure 5.6.1 Trial functions: are continuous, satisfy the boundary conditions, and are of one sign.



Proof

It is usual to prove the minimization property of the Rayleigh quotient using a more advanced branch of applied mathematics known as the calculus of variations. We do not have the space here to develop that material properly. Instead, we will give a proof based on eigenfunction expansions. We again calculate the Rayleigh quotient (5.6.3) for any function u that is *continuous* and *satisfies the homogeneous boundary conditions*. In this derivation, the equivalent form of the Rayleigh quotient, (5.6.2), is more useful:

$$RQ[u] = \frac{-\int_a^b uL(u)\,dx}{\int_a^b u^2\sigma\,dx}, \tag{5.6.8}$$

where the operator notation is quite helpful. We expand the rather arbitrary function u in terms of the (usually unknown) eigenfunctions $\phi_n(x)$:

$$u = \sum_{n=1}^{\infty} a_n\phi_n(x). \tag{5.6.9}$$

L is a linear differential operator. We expect that

$$L(u) = \sum_{n=1}^{\infty} a_n L(\phi_n(x)), \tag{5.6.10}$$

since this is valid for finite series. In Chapter 7 we show that (5.6.10) is valid *if u is continuous and satisfies the same homogeneous boundary conditions as the eigenfunctions $\phi_n(x)$*. Here, ϕ_n are eigenfunctions, and hence $L(\phi_n) = -\lambda_n\sigma\phi_n$. Thus, (5.6.10) becomes

$$L(u) = -\sum_{n=1}^{\infty} a_n\lambda_n\sigma\phi_n, \tag{5.6.11}$$

which can be thought of as the eigenfunction expansion of $L(u)$. If (5.6.11) and (5.6.9) are substituted into (5.6.8) and different dummy summation indices are utilized for the product of two infinite series, we obtain

$$RQ[u] = \frac{\int_a^b \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m a_n\lambda_n\phi_n\phi_m\sigma\right) dx}{\int_a^b \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m a_n\phi_n\phi_m\sigma\right) dx}. \tag{5.6.12}$$

We now do the integration in (5.6.12) before the summation. We recall that the eigenfunctions are orthogonal $\left(\int_a^b \phi_n\phi_m\sigma\,dx = 0 \text{ if } n \neq m\right)$, which implies that (5.6.12) becomes

$$RQ[u] = \frac{\sum_{n=1}^{\infty} a_n^2\lambda_n\int_a^b \phi_n^2\sigma\,dx}{\sum_{n=1}^{\infty} a_n^2\int_a^b \phi_n^2\sigma\,dx}. \tag{5.6.13}$$

This is an exact expression for the Rayleigh quotient in terms of the generalized Fourier coefficients a_n of u. We denote λ_1 as the lowest eigenvalue ($\lambda_1 < \lambda_n$ for $n > 1$). Thus,

$$RQ[u] \geq \frac{\lambda_1 \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx}{\sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx} = \lambda_1. \tag{5.6.14}$$

Furthermore, the equality in (5.6.14) holds only if $a_n = 0$ for $n > 1$ (i.e., only if $u = a_1\phi_1$). We have shown that the smallest value of the Rayleigh quotient is the lowest eigenvalue λ_1. Moreover, the Rayleigh quotient is minimized only when $u = a_1\phi_1$ (i.e., when u is the lowest eigenfunction).

We thus have a minimization theorem for the lowest eigenvalue λ_1.We can ask if there are corresponding theorems for the higher eigenvalues. Interesting generalizations immediately follow from (5.6.13). If we insist that $a_1 = 0$, then

$$RQ[u] = \frac{\sum_{n=2}^{\infty} a_n^2 \lambda_n \int_a^b \phi_n^2 \sigma \, dx}{\sum_{n=2}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx}. \tag{5.6.15}$$

This means that in addition we are restricting our function u to be orthogonal to ϕ_1, since $a_1 = \int_a^b u\phi_1\sigma \, dx / \int_a^b \phi_1^2 \sigma \, dx$. We now proceed in a similar way. Since $\lambda_2 < \lambda_n$ for $n > 2$, it follows that

$$RQ[u] \geq \lambda_2,$$

and furthermore the equality holds only if $a_n = 0$ for $n > 2$ [i.e., $u = a_2\phi_2(x)$] since $a_1 = 0$ already. We have just proved the following theorem: The minimum value for all continuous functions *$u(x)$ that are orthogonal to the lowest eigenfunction* and satisfy the boundary conditions is the next-to-lowest eigenvalue. Further generalizations also follow directly from (5.6.13).

Exercises 5.6

5.6.1. Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of

(a) $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\phi(1) = 0$

(b) $\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0$ with $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) + 2\phi(1) = 0$

***(c)** $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ with $\phi(0) = 0$ and $\frac{d\phi}{dx}(1) + \phi(1) = 0$ (See Exercise 5.8.10.)

(d) $\frac{d}{dx}\left[(1 + x^2)\frac{d\phi}{dx}\right] + \lambda\phi = 0$ with $\phi(0) = 0$ and $\phi(1) = 0$

(e) $\frac{d}{dx}\left[(1 + x)\frac{d\phi}{dx}\right] + (\lambda - x)\phi = 0$ with $\phi(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$

5.6.2. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).

5.6.3. Prove that (5.6.10) is valid in the following way. Assume $L(u)/\sigma$ is piecewise smooth so that

$$\frac{L(u)}{\sigma} = \sum_{n=1}^{\infty} b_n \phi_n(x).$$

Determine b_n. [*Hint*: Using Green's formula (5.5.5), show that $b_n = -a_n\lambda_n$ if u and du/dx are continuous and if u satisfies the same homogeneous boundary conditions as the eigenfunctions $\phi_n(x)$.]

5.6.4. Consider the eigenvalue problem $\frac{d}{dr}\left(r\frac{d\phi}{dx}\right) + \lambda r\phi = 0$ $(0 < r < 1)$, subject to the boundary conditions $|\phi(0)| < \infty$ (you may also assume $\frac{d\phi}{dr}$ is bounded) and $\frac{d\phi}{dr}(1) = 0$.

(a) Prove that $\lambda > 0$.

(b) Solve the initial value problem. You may assume the eigenfunctions are known. Derive coefficients using orthogonality.

5.7 Worked Example: Vibrations of a Nonuniform String

Some additional applications of the Rayleigh quotient are best illustrated in a physical problem. Consider the vibrations of a nonuniform string [constant tension T_0, but variable mass density $\rho(x)$] without sources $(Q = 0)$: See Section 4.2. We assume that both ends are fixed with zero displacement. The mathematical equations for the initial value problem are

$$\text{PDE:} \quad \boxed{\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2}} \tag{5.7.1}$$

$$\text{BC:} \quad \boxed{\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}} \tag{5.7.2}$$

$$\text{IC:} \quad \boxed{\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x). \end{aligned}} \tag{5.7.3}$$

Again since the partial differential equation and the boundary conditions are linear and homogeneous, we are able to apply the method of separation of variables. We look for product solutions:

$$u(x,t) = \phi(x)h(t), \tag{5.7.4}$$

ignoring the nonzero initial conditions. It can be shown that $h(t)$ satisfies

$$\frac{d^2h}{dt^2} = -\lambda h, \tag{5.7.5}$$

while the spatial part solves the following regular Sturm–Liouville eigenvalue problem:

$$\begin{aligned} T_0\frac{d^2\phi}{dx^2} + \lambda\rho(x)\phi &= 0 \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned} \tag{5.7.6}$$

Usually, we presume that the infinite sequence of eigenvalues λ_n and corresponding eigenfunctions $\phi_n(x)$ are known. However, in order to analyze (5.7.5), it is necessary to know something about λ. From physical reasoning, we certainly expect $\lambda > 0$ since we expect oscillations, but we will show that the Rayleigh quotient easily guarantees that $\lambda > 0$. For (5.7.6), the Rayleigh quotient (5.6.3) becomes

$$\lambda = \frac{T_0\int_0^L (d\phi/dx)^2\,dx}{\int_0^L \phi^2\rho(x)\,dx}. \tag{5.7.7}$$

Clearly, $\lambda \geq 0$ (and as before it is impossible for $\lambda = 0$ in this case). Thus, $\lambda > 0$.

We now are assured that the solution of (5.7.5) is a linear combination of $\sin\sqrt{\lambda}t$ and $\cos\sqrt{\lambda}t$. There are two families of product solutions of the partial differential equation, $\sin\sqrt{\lambda_n}t\,\phi_n(x)$ and $\cos\sqrt{\lambda_n}t\,\phi_n(x)$. According to the principle of superposition, the solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\sqrt{\lambda_n}t\,\phi_n(x) + \sum_{n=1}^{\infty} b_n \cos\sqrt{\lambda_n}t\,\phi_n(x).} \tag{5.7.8}$$

We need to show only that the two families of coefficients can be obtained from the initial conditions:

$$f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} a_n \sqrt{\lambda_n}\, \phi_n(x). \tag{5.7.9}$$

Thus, b_n are the generalized Fourier coefficient of the initial position $f(x)$, while $a_n\sqrt{\lambda_n}$ are the generalized Fourier coefficients for the initial velocity $g(x)$. Thus, due to the orthogonality of the eigenfunction [with weight $\rho(x)$], we can easily determine a_n and b_n:

$$\boxed{b_n = \frac{\int_0^L f(x)\phi_n(x)\rho(x)\,dx}{\int_0^L \phi_n^2 \rho\;\, dx}} \tag{5.7.10}$$

$$\boxed{a_n\sqrt{\lambda_n} = \frac{\int_0^L g(x)\phi_n(x)\rho(x)\,dx}{\int_0^L \phi_n^2 \rho\, dx},} \tag{5.7.11}$$

The Rayleigh quotient can be used to obtain additional information about the lowest eigenvalue λ_1. (Note that the lowest frequency of vibration is $\sqrt{\lambda_1}$.) We know that

$$\lambda_1 = \min \frac{T_0 \int_0^L (du/dx)^2\, dx}{\int_0^L u^2 \rho(x)\, dx}. \tag{5.7.12}$$

We have already shown (see Section 5.6) how to use trial functions to obtain an upper bound on the lowest eigenvalue. This is not always convenient since the denominator in (5.7.12) depends on the mass density $\rho(x)$. Instead, we will develop another method for an upper bound. By this method we will also obtain a lower bound.

Let us suppose, as is usual, that the variable mass density has upper and lower bounds,

$$0 < \rho_{\min} \le \rho(x) \le \rho_{\max}.$$

For any $u(x)$ it follows that

$$\rho_{\min}\int_0^L u^2\, dx \le \int_0^L u^2\rho(x)\, dx \le \rho_{\max}\int_0^L u^2\, dx.$$

Consequently, from (5.7.12),

$$\frac{T_0}{\rho_{\max}} \min \frac{\int_0^L (du/dx)^2\, dx}{\int_0^L u^2\, dx} \le \lambda_1 \le \frac{T_0}{\rho_{\min}} \min \frac{\int_0^L (du/dx)^2\, dx}{\int_0^L u^2\, dx}. \tag{5.7.13}$$

We can evaluate the expressions in (5.7.13), since we recognize the minimum of $\int_0^L (du/dx)^2\, dx / \int_0^L u^2\, dx$ subject to $u(0) = 0$ and $u(L) = 0$ as the lowest eigenvalue of a different problem: namely, one with constant coefficients,

$$\frac{d^2\phi}{dx^2} + \bar{\lambda}\phi = 0$$

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

We already know that $\bar{\lambda} = (n\pi/L)^2$, and hence the lowest eigenvalue for this problem is $\bar{\lambda}_1 = (\pi/L)^2$. But the minimization property of the Rayleigh quotient implies that

$$\bar{\lambda}_1 = \min \frac{\int_0^L (du/dx)^2\, dx}{\int_0^L u^2\, dx}.$$

Finally, we have proved that the lowest eigenvalue of our problem with variable coefficients satisfies the following inequality:

$$\frac{T_0}{\rho_{\max}}\left(\frac{\pi}{L}\right)^2 \le \lambda_1 \le \frac{T_0}{\rho_{\min}}\left(\frac{\pi}{L}\right)^2.$$

We have obtained an upper and a lower bound for the smallest eigenvalue. By taking square roots,

$$\frac{\pi}{L}\sqrt{\frac{T_0}{\rho_{\max}}} \le \sqrt{\lambda_1} \le \frac{\pi}{L}\sqrt{\frac{T_0}{\rho_{\min}}}.$$

The physical meaning of this is clear: the lowest frequency of oscillation of a variable string lies in between the lowest frequencies of vibration of two constant-density strings, one with the minimum density and the other with the maximum. Similar results concerning the higher frequencies of vibration are also valid but are harder to prove (see Weinberger [1965] or Courant and Hilbert [1953]).

Exercises 5.7

5.7.1. Find an upper and lower bound for the smallest eigenvalue of the nonuniform vibrating string problem, where the string is fixed at $x = 0$ and $x = 3$, with $T_0 = 20$ and $\rho(x) = \sqrt{16 + x^2}$.

***5.7.2.** Determine an upper and a (nonzero) lower bound for the lowest frequency of vibration of a nonuniform string fixed at $x = 0$ and $x = 1$ with $c^2 = 1 + 4\alpha^2\left(x - \frac{1}{2}\right)^2$.

5.7.3. Consider heat flow in a one-dimensional rod without sources with nonconstant thermal properties. Assume that the temperature is zero at $x = 0$ and $x = L$. Suppose that $c\rho_{\min} \le c\rho \le c\rho_{\max}$, and $K_{\min} \le K_0(x) \le K_{\max}$. Obtain an upper and (nonzero) lower bound on the slowest exponential rate of decay of the product solution.

5.7.4. Assume (5.7.1)–(5.7.6) are valid. Consider the one-dimensional wave equation with nonconstant density so that c^2 depends on x,

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2}.$$

(a) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).

(b) Use the Rayleigh quotient to prove that $\lambda > 0$.

(c) Solve the initial value problem. You may assume the eigenfunctions are known. Derive coefficients using orthogonality.

5.8 Boundary Conditions of the Third Kind

INTRODUCTION

So far we have analyzed two general varieties of boundary value problems: very specific, easily solved ones (such as the ones that give rise to Fourier sine series, Fourier cosine series, or Fourier series), and somewhat abstract Sturm–Liouville eigenvalue problems, where our theorems guaranteed many needed properties. In one case the differential equation had constant coefficients (with simple boundary conditions), and in the other we discussed differential equations with variable coefficients.

In this section, we analyze problems with a boundary condition of the third kind. The boundary value problems will also be easily solved (since the differential equation will still have constant coefficients). However, due to its boundary conditions, it will illustrate more convincingly the general ideas of Sturm–Liouville eigenvalue problems.

PHYSICAL EXAMPLES

We consider some simple problems with constant physical parameters. Heat flow in a uniform rod satisfies

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \tag{5.8.1}$$

while a uniform vibrating string solves

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{5.8.2}$$

In either case we suppose that the left end is fixed, but the right end satisfies a homogeneous boundary condition of the third kind:

$$u(0,t) = 0 \tag{5.8.3}$$

$$\frac{\partial u}{\partial x}(L,t) = -hu(L,t). \tag{5.8.4}$$

Recall that, for heat conduction, (5.8.4) corresponds to Newton's law of cooling if $h > 0$, and for the vibrating string problem, (5.8.4) corresponds to a restoring force if $h > 0$, the so-called elastic boundary condition. We note that usually in physical problems $h \geq 0$. However, for mathematical reasons we will investigate both cases with $h < 0$ and $h \geq 0$. If $h < 0$, the vibrating string has a destabilizing force at the right end, while for the heat flow problem, thermal energy is being constantly put into the rod through the right end. ■

Sturm–Liouville eigenvalue problem

After separation of variables,

$$u(x,t) = G(t)\phi(x), \tag{5.8.5}$$

the time part satisfies the following ordinary differential equations:

$$\text{heat flow: } \frac{dG}{dt} = -\lambda kG \tag{5.8.6}$$

$$\text{vibrating string: } \frac{d^2G}{dt^2} = -\lambda c^2 G. \tag{5.8.7}$$

We wish to concentrate on the effect of the third type of boundary condition (5.8.4). For either physical problem, the spatial part, $\phi(x)$, satisfies the following regular Sturm–Liouville eigenvalue problem:

$$\boxed{\frac{d^2\phi}{dx^2} + \lambda\phi = 0} \tag{5.8.8}$$

$$\boxed{\phi(0) = 0} \tag{5.8.9}$$

$$\boxed{\frac{d\phi}{dx}(L) + h\phi(L) = 0,} \tag{5.8.10}$$

where h is a given fixed constant. If $h \geq 0$, this is what we call the "physical" case, while if $h < 0$, we call it the "nonphysical" case. Although the differential equation (5.8.8) has constant coefficients, the boundary conditions will give rise to some new ideas. For the moment we ignore certain aspects of our theory of Sturm–Liouville eigenvalue problems (except for the fact that the eigenvalues are real). In solving (5.8.8) we must consider three distinct cases: $\lambda > 0, \lambda < 0$, and $\lambda = 0$. This will be especially important when we analyze the nonphysical case $h < 0$.

Positive eigenvalues

If $\lambda > 0$, the solution of the differential equation is a linear combination of sines and cosines:

$$\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x. \tag{5.8.11}$$

The boundary condition $\phi(0) = 0$ implies that $0 = c_1$, and hence

$$\boxed{\phi(x) = c_2 \sin\sqrt{\lambda}x.} \tag{5.8.12}$$

Clearly, sine functions are needed to satisfy the zero condition at $x = 0$. We will need the first derivative,

$$\frac{d\phi}{dx} = c_2\sqrt{\lambda}\cos\sqrt{\lambda}x.$$

Thus, the boundary condition of the third kind, (5.8.10), implies that

$$c_2\left(\sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L\right) = 0. \tag{5.8.13}$$

If $c_2 = 0$, (5.8.12) shows that $\phi \equiv 0$, which cannot be an eigenfunction. Thus, eigenvalues exist for $\lambda > 0$ for all values of λ that satisfy

$$\sqrt{\lambda}\cos\sqrt{\lambda}L + h\sin\sqrt{\lambda}L = 0. \tag{5.8.14}$$

The more elementary case $h = 0$ will be analyzed later. Equation (5.8.14) is a transcendental equation for the positive eigenvalues λ (if $h \neq 0$). In order to solve (5.8.14), it is convenient to divide by $\cos\sqrt{\lambda}L$ to obtain an expression for $\tan\sqrt{\lambda}L$:

$$\boxed{\tan\sqrt{\lambda}L = -\frac{\sqrt{\lambda}}{h}.} \tag{5.8.15}$$

We are allowed to divide by $\cos\sqrt{\lambda}L$ because it is not zero [if $\cos\sqrt{\lambda}L = 0$, then $\sin\sqrt{\lambda}L \neq 0$ and (5.8.14) would not be satisfied]. We could have obtained an expression for cotangent rather than tangent by dividing (5.8.14) by $\sin\sqrt{\lambda}L$, but we are presuming that the reader feels more comfortable with the tangent function.

Graphical technique ($\lambda > 0$)

Equation (5.8.15) is a transcendental equation. We cannot solve it exactly. However, let us describe a *graphical technique* to obtain information about the eigenvalues. In order to graph the solution of a transcendental equation, we introduce an artificial coordinate z. Let

$$z = \tan\sqrt{\lambda}L \tag{5.8.16}$$

and thus also

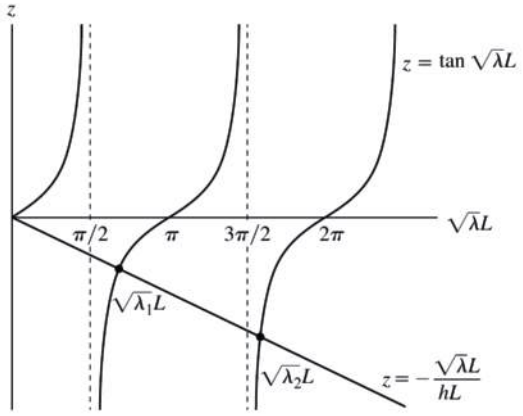
$$z = -\frac{\sqrt{\lambda}}{h}. \tag{5.8.17}$$

Now the simultaneous solution of (5.8.16) and (5.8.17) (i.e., their points of intersection) corresponds to solutions of (5.8.15). Equation (5.8.16) is a pure tangent function (not compressed) as a function of $\sqrt{\lambda}L$, where $\sqrt{\lambda}L > 0$ since $\lambda > 0$. We sketch (5.8.16) in Fig. 5.8.1. We note that the tangent function is periodic with period π; it is zero at $\sqrt{\lambda}L = 0, \pi, 2\pi$, and so on, and it approaches $\pm\infty$ as $\sqrt{\lambda}L$ approaches $\pi/2$, $3\pi/2$, $5\pi/2$, and so on. We will intersect the tangent function with (5.8.17). Since we are sketching our curves as functions of $\sqrt{\lambda}L$, we will express (5.8.17) as a function of $\sqrt{\lambda}L$. This is easily done by multiplying the numerator and denominator of (5.8.17) by L:

$$z = -\frac{\sqrt{\lambda}L}{hL}. \tag{5.8.18}$$

As a function of $\sqrt{\lambda}L$, (5.8.18) is a straight line with slope $-1/hL$. However, this line is sketched quite differently depending on whether $h > 0$ (physical case) or $h < 0$ (nonphysical case).

Figure 5.8.1 Graphical determination of positive eigenvalues ($h > 0$).



Positive eigenvalues (physical case, h > 0)

The intersection of the two curves is sketched in Fig. 5.8.1 for the physical case ($h > 0$). There are an infinite number of intersections; each corresponds to a positive eigenvalue. (We exclude

$\sqrt{\lambda}L = 0$ since we have assumed throughout that $\lambda > 0$.) The eigenfunctions are $\phi = \sin\sqrt{\lambda}x$, where the allowable eigenvalues are determined graphically.

We cannot determine these eigenvalues exactly. However, we know from Fig. 5.8.1 that

$$\frac{\pi}{2} < \sqrt{\lambda_1}L < \pi, \tag{5.8.19}$$

$$\frac{3\pi}{2} < \sqrt{\lambda_2}L < 2\pi, \tag{5.8.20}$$

and so on. It is interesting to note that as n increases, the intersecting points more closely approach the position of the vertical portions of the tangent function. We thus are able to obtain the following approximate (asymptotic) formula for the eigenvalues:

$$\sqrt{\lambda_n}L \sim \left(n - \frac{1}{2}\right)\pi \tag{5.8.21}$$

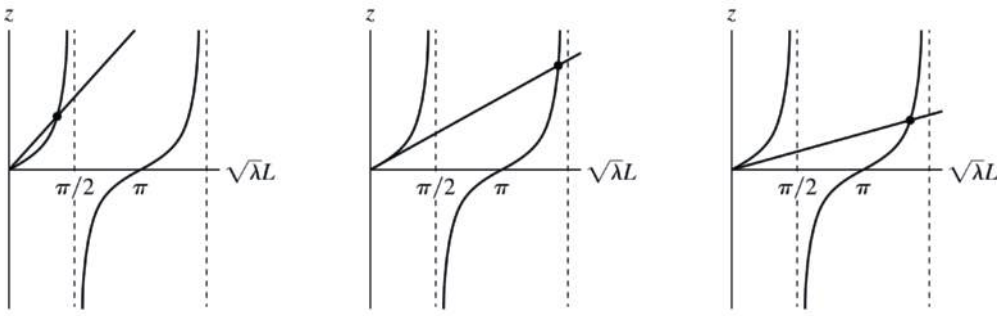
as $n \to \infty$. This becomes more and more accurate as $n \to \infty$. An asymptotic formula for the large eigenvalues similar to (5.8.21) exists even for cases where the differential equation cannot be explicitly solved. We will discuss this in Section 5.9.

To obtain accurate values, a numerical method such as Newton's method (as often described in elementary calculus texts) can be used. A practical scheme is to use Newton's numerical method for the first few roots, until you reach a root whose solution is reasonably close to the asymptotic formula, (5.8.21) (or improvements to this elementary asymptotic formula). Then, for larger roots, the asymptotic formula (5.8.21) is accurate enough.

Positive eigenvalues (nonphysical case, h $<$ 0)

The nonphysical case ($h < 0$) also will be a good illustration of various general ideas concerning Sturm–Liouville eigenvalue problems. If $h < 0$, positive eigenvalues again are determined by graphically sketching (5.8.15), $\tan\sqrt{\lambda}L = -\sqrt{\lambda}/h$. The straight line (here with positive slope) must intersect the tangent function. It intersects the "first branch" of the tangent function only if the slope of the straight line is greater than 1 (see Fig. 5.8.2a). We are using the property of the tangent function that its slope is 1 at $x = 0$ and its slope increases along the first branch. Thus, if $h < 0$ (the nonphysical case), there are two major subcases ($-1/hL > 1$ and $0 < -1/hL < 1$) and a minor subcase ($-1/hL = 1$). We sketch these three cases in Fig. 5.8.2. In each of these three figures, there are an infinite number of intersections, corresponding to an infinite number of positive eigenvalues. The eigenfunctions are again $\sin\sqrt{\lambda}x$.

Figure 5.8.2 Graphical determination of positive eigenvalues: (a) $0 > hL > -1$; (b) $hL = -1$; (c) $hL < -1$.



In these cases, the graphical solutions also show that the large eigenvalues are approximately located at the singularities of the tangent function. Equation (5.8.21) is again asymptotic; the larger n becomes, the more accurate (5.8.21) gets.

Zero eigenvalue

Is $\lambda = 0$ an eigenvalue for (5.8.8)–(5.8.10)? Equation (5.8.11) is not the general solution of (5.8.8) if $\lambda = 0$. Instead,

$$\phi = c_1 + c_2 x; \tag{5.8.22}$$

the eigenfunction must be a straight line. The boundary condition $\phi(0) = 0$ makes $c_1 = 0$, insisting that the straight line goes through the origin,

$$\phi = c_2 x. \tag{5.8.23}$$

Finally, $d\phi/dx(L) + h\phi(L) = 0$ implies that

$$c_2(1 + hL) = 0. \tag{5.8.24}$$

If $hL \neq -1$ (including all physical situations, $h > 0$), it follows that $c_2 = 0$, $\phi = 0$, and thus $\lambda = 0$ is not an eigenvalue. However, if $hL = -1$, then from (5.8.24) c_2 is arbitrary, and $\lambda = 0$ is an eigenvalue with eigenfunction x.

Negative eigenvalues

We do not expect any negative eigenvalues in the physical situations [see (5.8.6) and (5.8.7)]. If $\lambda < 0$, we introduce $s = -\lambda$, so that $s > 0$. Then (5.8.8) becomes

$$\frac{d^2\phi}{dx^2} = s\phi. \tag{5.8.25}$$

The zero boundary condition at $x = 0$ suggests that it is more convenient to express the general solution of (5.8.25) in terms of the hyperbolic functions:

$$\phi = c_1 \cosh \sqrt{s}x + c_2 \sinh \sqrt{s}x. \tag{5.8.26}$$

Only the hyperbolic sines are needed, since $\phi(0) = 0$ implies that $c_1 = 0$:

$$\begin{aligned} \phi &= c_2 \sinh \sqrt{s}x \\ \frac{d\phi}{dx} &= c_2 \sqrt{s} \cosh \sqrt{s}x. \end{aligned} \tag{5.8.27}$$

The boundary condition of the third kind, $d\phi/dx(L) + h\phi(L) = 0$, implies that

$$c_2\left(\sqrt{s}\cosh \sqrt{s}L + h\sinh \sqrt{s}L\right) = 0. \tag{5.8.28}$$

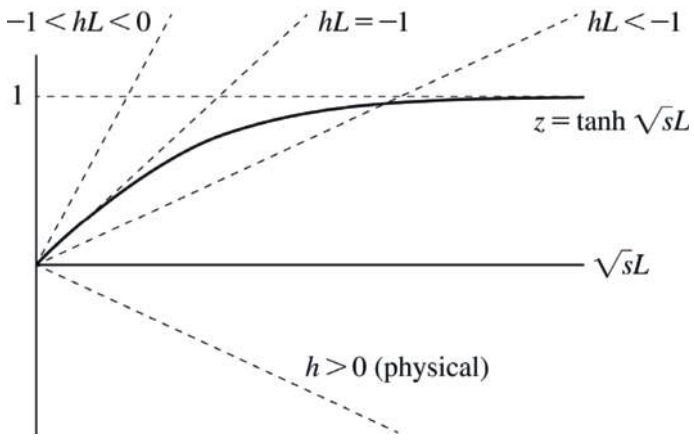
At this point it is apparent that the analysis for $\lambda < 0$directly parallels that which occurred for $\lambda > 0$ (with hyperbolic functions replacing the trigonometric functions). Thus, since $c_2 \neq 0$,

$$\boxed{\tanh \sqrt{s}L = -\frac{\sqrt{s}}{h} = -\frac{\sqrt{s}L}{hL}.} \tag{5.8.29}$$

Graphical solution for negative eigenvalues

Negative eigenvalues are determined by the graphical solution of transcendental equation (5.8.29). Here, properties of the hyperbolic tangent function are quite important. tanh is sketched as a function of $\sqrt{s}L$ in Fig. 5.8.3.

Figure 5.8.3 Graphical determination of negative eigenvalues.



Let us note some properties of the tanh function that follow from its definition:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

As $\sqrt{s}L \to \infty$, $\tanh \sqrt{s}L$ asymptotes to 1. We will also need to note that the slope[5] of tanh equals 1 at $\sqrt{s}L = 0$ and decreases toward zero as $\sqrt{s}L \to \infty$. This function must be intersected with the straight line implied by the r.h.s. of (5.8.29). The same four cases appear, as sketched in Fig. 5.8.3. In physical situations ($h > 0$), there are no intersections with $\sqrt{s}L > 0$; there are no negative eigenvalues in the physical situations ($h > 0$). All the eigenvalues are nonnegative. However, if $hL < -1$ (and only in these situations), then there is exactly one intersection; there is one negative eigenvalue (if $hL < -1$). If we denote the intersection by $s = s_1$, the negative eigenvalue is $\lambda = -s_1$, and the corresponding eigenfunction is $\phi = \sinh \sqrt{s_1}x$. In nonphysical situations, there are a finite number of negative eigenvalues (one if $hL < -1$, none otherwise).

Special case h = 0

Although if $h = 0$, the boundary conditions are not of the third kind, the eigenvalues and eigenfunctions are still of interest. If $h = 0$, then all eigenvalues are positive [see (5.8.24) and (5.8.28)] and easily explicitly determined from (5.8.14):

$$\lambda = \left[\frac{(n - 1/2)\pi}{L}\right]^2, n = 1, 2, 3 \ldots.$$

The eigenfunctions are $\sin \sqrt{\lambda}x$.

Summary

We have shown there to be five somewhat different cases depending on the value of the parameter h in the boundary condition. Table 5.8.1 summarizes the eigenvalues and eigenfunctions for these cases.

TABLE 5.8.1 Eigenfunctions for (5.8.8)–(5.8.10)

		$\lambda > 0$	$\lambda = 0$	$\lambda < 0$
Physical	$h > 0$	$\sin \sqrt{\lambda}x$		
	$h = 0$	$\sin \sqrt{\lambda}x$		
Nonphysical	$-1 < hL < 0$	$\sin \sqrt{\lambda}x$		
	$hL = -1$	$\sin \sqrt{\lambda}x$	x	
	$hL < -1$	$\sin \sqrt{\lambda}x$		$\sinh \sqrt{s_1}x$

[5] $d/dx \tanh x = \operatorname{sech}^2 x = 1/\cosh^2 x$.

In some sense there actually are only three cases: If $-1 < hL$, all the eigenvalues are positive; if $hL = -1$, there are no negative eigenvalues, but zero is an eigenvalue; and if $hL < -1$, there are still an infinite number of positive eigenvalues, but there is also one negative one.

Rayleigh quotient

We have shown by explicitly solving the eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \tag{5.8.30}$$

$$\phi(0) = 0 \tag{5.8.31}$$

$$\frac{d\phi}{dx}(L) + h\phi(L) = 0, \tag{5.8.32}$$

that in physical problems ($h \geq 0$) all the eigenvalues are positive, while in nonphysical problems ($h < 0$) there may or may not be negative eigenvalues. We will show that the Rayleigh quotient is consistent with this result:

$$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right] dx}{\int_a^b \phi^2\sigma\, dx} = \frac{h\phi^2(L) + \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^L \phi^2\, dx} \tag{5.8.33}$$

since from (5.8.30), $p(x) = 1$, $\sigma(x) = 1$, $q(x) = 0$, and $a = 0$, $b = L$, and where the boundary conditions (5.8.31) and (5.8.32) have been utilized to simplify the boundary terms in the Rayleigh quotient. If $h \geq 0$ (the physical cases), it readily follows from (5.8.33) that the eigenvalues must be positive, exactly what we concluded by doing the explicit calculations. However, if $h < 0$ (nonphysical case), the numerator of the Rayleigh quotient contains a negative term $h\phi^2(L)$ and a positive term $\int_0^L (d\phi/dx)^2\, dx$. It is impossible to make any conclusions concerning the sign of λ. Thus, it may be possible to have negative eigenvalues if $h < 0$. However, we are unable to conclude that there must be negative eigenvalues. A negative eigenvalue occurs only when $|h\phi^2(L)| > \int_0^L (d\phi/dx)^2\, dx$. From the Rayleigh quotient, we cannot determine when this happens. It is only from an explicit calculation that we know that a negative eigenvalue occurs only if $hL < -1$.

Zeros of eigenfunctions

The Sturm–Liouville eigenvalue problem that we have been discussing in this section,

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(0) &= 0 \\ \frac{d\phi}{dx}(L) + h\phi(L) &= 0, \end{aligned} \tag{5.8.34}$$

is a good example for illustrating the general theorem concerning the zeros of the eigenfunctions. The theorem states that the eigenfunction corresponding to the lowest eigenvalue has no zeros in the interior. More generally, the nth eigenfunction has $n - 1$ zeros.

There are five cases of (5.8.34) worthy of discussion: $h > 0$, $h = 0$, $-1 < hL < 0$, $hL = -1$, $hL < -1$. However, the line of reasoning used in investigating the zeros of the eigenfunctions is quite similar in all cases. For that reason we will analyze only one case ($hL < -1$) and leave the others for the exercises. In this case ($hL < -1$) there is one negative eigenvalue (with corresponding eigenfunction $\sinh\sqrt{s_1}x$) and an infinite number of positive eigenvalues (with corresponding eigenfunctions $\sin\sqrt{\lambda}x$). We will need to analyze carefully the positive eigenvalues and so we reproduce Fig. 5.8.2c (as Fig. 5.8.4), used for the graphical determination of the eigenvalues in $hL < -1$. We designate the intersections starting from λ_n, $n = 2$, since the lowest eigenvalue is negative, $\lambda_1 = -s_1$. Graphically, we are able to obtain bounds for these eigenvalues:

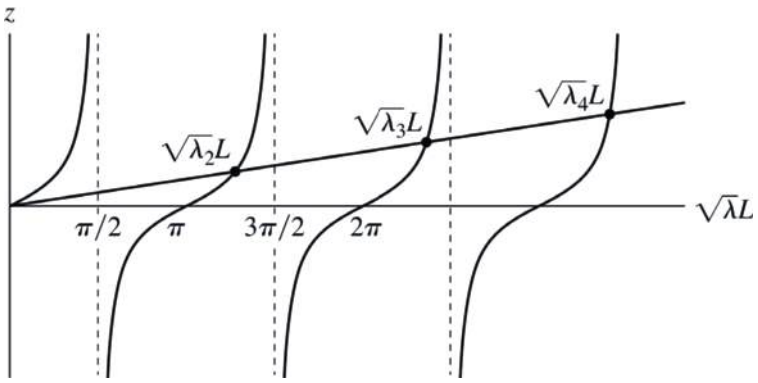
$$\pi < \sqrt{\lambda_2}L < \frac{3\pi}{2} \tag{5.8.35}$$

$$2\pi < \sqrt{\lambda_3}L < \frac{5\pi}{2}, \tag{5.8.36}$$

which is easily generalized as

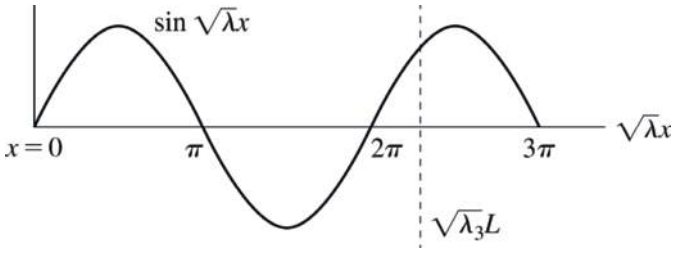
$$(n-1)\pi < \sqrt{\lambda_n}L < (n-1/2)\pi, n \geq 2. \tag{5.8.37}$$

Figure 5.8.4 Positive eigenvalues ($hL < -1$).



Let us investigate zeros of the eigenfunctions. The lowest eigenfunction is $\sinh\sqrt{s_1}x$. Since the hyperbolic sine function is never zero (except at the end $x = 0$), we have verified one part of the theorem. The eigenfunction corresponding to the lowest eigenvalue does not have a zero in the interior. The other eigenfunctions are $\sin\sqrt{\lambda_n}x$, sketched in Fig. 5.8.5. In this figure, the endpoint $x = 0$ is clearly marked, but $x = L$ depends on λ. For example, for λ_3, the endpoint $x = L$ occurs at $\sqrt{\lambda_3}L$, which is sketched in Fig. 5.8.5 due to (5.8.36). As x varies from 0 to L, the eigenfunction is sketched in Fig. 5.8.5 up to the dashed line. This eigenfunction has two zeros $\left(\sqrt{\lambda_3}x = \pi \text{ and } 2\pi\right)$. This reasoning can be used for any of these eigenfunctions. Thus, the number of zeros for the nth eigenfunction corresponding to λ_n is $n - 1$, exactly as the general theorem specifies. Our theorem does *not* state that the eigenfunction corresponding to the lowest *positive* eigenvalue has no zeros. Instead, **the eigenfunction corresponding to the lowest eigenvalue has no zeros**. To repeat, in this example the lowest eigenvalue is negative and *its* corresponding eigenfunction has no zeros.

Figure 5.8.5 Zeros of the eigenfunctions $\sin\sqrt{\lambda}x$.



Heat flow with a nonphysical boundary condition

To understand further the boundary condition of the third kind, let us complete the investigation of one example. We consider heat flow in a uniform rod:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ \text{BC1:} \quad & u(0,t) = 0 \\ \text{BC2:} \quad & \frac{\partial u}{\partial x}(L,t) = -hu(L,t) \\ \text{IC:} \quad & u(x,0) = f(x). \end{aligned} \tag{5.8.38}$$

We assume that the temperature is zero at $x = 0$ and that the "nonphysical" case ($h < 0$) of the boundary condition of the third kind is imposed at $x = L$. Thermal energy flows into the rod at $x = L$ [if $u(L,t) > 0$].

Separating variables,

$$u(x, t) = \phi(x)G(t), \tag{5.8.39}$$

yields

$$\frac{dG}{dt} = -\lambda kG \tag{5.8.40}$$

$$\boxed{\frac{d^2\phi}{dx^2} + \lambda\phi = 0} \tag{5.8.41}$$

$$\boxed{\phi(0) = 0} \tag{5.8.42}$$

$$\boxed{\frac{d\phi}{dx}(L) + h\phi(L) = 0.} \tag{5.8.43}$$

The time part is an exponential, $G = ce^{-\lambda kt}$. Here, we consider only the case in which

$$\boxed{hL < -1.}$$

Then there exists one negative eigenvalue ($\lambda_1 = -s_1$), with corresponding eigenfunction $\sinh\sqrt{s_1}x$, where s_1 is determined as the unique solution of $\tanh\sqrt{s}L = -\sqrt{s}/h$. *The time part exponentially grows.* All the other eigenvalues λ_n are positive. For these the eigenfunctions are $\sin\sqrt{\lambda}x$ (where $\tan\sqrt{\lambda}x = -\sqrt{\lambda}/h$ has an infinite number of solutions), while the corresponding time-dependent part exponentially decays, being proportional to $e^{-\lambda kt}$. The forms of the product solutions are $\sin\sqrt{\lambda}x\, e^{-\lambda kt}$ and $\sinh\sqrt{s_1}x\, e^{s_1 kt}$. Here, the somewhat "abstract" notation may be considered more convenient; the product solutions are $\phi_n(x)e^{-\lambda_n kt}$, where the eigenfunctions are

$$\phi_n(x) = \begin{cases} \sinh\sqrt{s_1}x & n = 1 \\ \sin\sqrt{\lambda_n}x & n > 1. \end{cases}$$

According to the principle of superposition, we attempt to satisfy the initial value problem with a linear combination of *all* possible product solutions:

$$u(x, t) = \sum_{n=1}^{\infty} a_n\phi_n(x)e^{-\lambda_n kt}.$$

The initial condition, $u(x, 0) = f(x)$, implies that

$$f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).$$

Since the coefficient $\sigma(x) = 1$ in (5.8.41), the eigenfunctions $\phi_n(x)$ are orthogonal with weight 1. Thus, we know that the generalized Fourier coefficients of the initial condition $f(x)$ are

$$a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx} = \begin{cases} \int_0^L f(x)\sinh\sqrt{s_1}x\,dx \Big/ \int_0^L \sinh^2\sqrt{s_1}x\,dx & n = 1 \\ \int_0^L f(x)\sin\sqrt{\lambda_n}x\,dx \Big/ \int_0^L \sin^2\sqrt{\lambda_n}x\,dx & n \geq 2. \end{cases}$$

In particular, we could show $\int_0^L \sin^2\sqrt{\lambda_n}x\,dx \neq L/2$. Perhaps we should emphasize one additional point. We have utilized the theorem that states that eigenfunctions corresponding to different eigenvalues are orthogonal; it is guaranteed that $\int_0^L \sin\sqrt{\lambda_n}x\,\sin\sqrt{\lambda_m}x\,dx = 0$ $(n \neq m)$ and $\int_0^L \sin\sqrt{\lambda_n}x\,\sinh\sqrt{s_1}x\,dx = 0$. We do not need to verify these by integration (although it can be done).

Other problems with boundary conditions of the third kind appear in the Exercises.

Exercises 5.8

5.8.1. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to $u(0, t) = 0$, $\frac{\partial u}{\partial x}(L, t) = -hu(L, t)$, and $u(x, 0) = f(x)$.

(a) Solve if $hL > -1$.

(b) Solve if $hL = -1$.

5.8.2. Consider the eigenvalue problem (5.8.8)–(5.8.10). Show that the nth eigenfunction has $n - 1$ zeros in the interior if

(a) $h > 0$

(b) $h = 0$

***(c)** $-1 < hL < 0$

(d) $hL = -1$

5.8.3. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) + h\phi(L) = 0$ with $h > 0$.

(a) Prove that $\lambda > 0$ (without solving the differential equation).

***(b)** Determine all eigenvalues graphically. Obtain upper and lower bounds. Estimate the large eigenvalues.

(c) Show that the nth eigenfunction has $n - 1$ zeros in the interior.

5.8.4. Redo Exercise 5.8.3 parts (b) and (c) only if $h < 0$.

5.8.5. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $\frac{\partial u}{\partial x}(0, t) = 0$, $\frac{\partial u}{\partial x}(L, t) = -hu(L, t)$, and $u(x, 0) = f(x)$.

(a) Solve if $h > 0$.

(b) Solve if $h < 0$.

5.8.6. Consider (with $h > 0$)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0, t) - hu(0, t) = 0, \qquad u(x, 0) = f(x)$$

$$\frac{\partial u}{\partial x}(L, t) = 0, \qquad \frac{\partial u}{\partial t}(x, 0) = g(x).$$

[*Hint*: Suppose it is known that if $u(x, t) = \phi(x)G(t)$, then

$$\frac{1}{c^2 G}\frac{d^2 G}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.]$$

(a) Show that there are an infinite number of different frequencies of oscillation.

(b) Estimate the large frequencies of oscillation.

(c) Solve the initial value problem.

5.8.7. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ subject to } \phi(0) = 0 \text{ and } \phi(\pi) - 2\frac{d\phi}{dx}(0) = 0.$$

(a) Show that usually

$$\int_0^\pi \left(u\frac{d^2 v}{dx^2} - v\frac{d^2 u}{dx^2}\right) dx \neq 0$$

for any two functions u and v satisfying these homogeneous boundary conditions.

***(b)** Determine all positive eigenvalues.

***(c)** Determine all negative eigenvalues.

***(d)** Is $\lambda = 0$ an eigenvalue?

***(e)** Is it possible that there are other eigenvalues besides those determined in parts (b)–(d)? *Briefly* explain.

5.8.8. Consider the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \phi(0) - \frac{d\phi}{dx}(0) = 0, \quad \phi(1) + \frac{d\phi}{dx}(1) = 0.$$

(a) Using the Rayleigh quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?

(b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

***(c)** Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

(d) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with

$$u(0, t) - \frac{\partial u}{\partial x}(0, t) = 0, \quad u(1, t) + \frac{\partial u}{\partial x}(1, t) = 0, \quad u(x, 0) = f(x).$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume that they are known.

5.8.9. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ with } \phi(0) = \frac{d\phi}{dx}(0) \text{ and } \phi(1) = \beta\frac{d\phi}{dx}(1).$$

For what values (if any) of β is $\lambda = 0$ an eigenvalue?

5.8.10. Consider the special case of the eigenvalue problem of Section 5.8:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ with } \phi(0) = 0 \text{ and } \frac{d\phi}{dx}(1) + \phi(1) = 0.$$

***(a)** Determine the lowest eigenvalue to at least two or three significant figures using tables or a calculator.

***(b)** Determine the lowest eigenvalue using a root-finding algorithm (e.g., Newton's method) on a computer.

(c) Compare either part (a) or (b) to the bound obtained using the Rayleigh quotient [see Exercise 5.6.1(c)].

5.8.11. Determine all negative eigenvalues for

$$\frac{d^2\phi}{dx^2} + 5\phi = -\lambda\phi \text{ with } \phi(0) = 0 \text{ and } \phi(\pi) = 0.$$

5.8.12. Consider $\partial^2 u/\partial t^2 = c^2\partial^2 u/\partial x^2$ with the boundary conditions

$$u = 0 \quad \text{at } x = 0$$

$$m\frac{\partial^2 u}{\partial t^2} = -T_0\frac{\partial u}{\partial x} - ku \quad \text{at } x = L.$$

(a) Give a brief physical interpretation of the boundary conditions.

(b) Show how to determine the frequencies of oscillation. Estimate the large frequencies of oscillation.

(c) *Without* attempting to use the Rayleigh quotient, explicitly determine if there are any separated solutions that do not oscillate in time. [*Hint*: There are none.]

(d) Show that the boundary condition is *not* self-adjoint: that is, show

$$\int_0^L \left(u_n \frac{d^2 u_n}{dx^2} - u_m \frac{d^2 u_n}{dx^2} \right) dx \neq 0$$

even when u_n and u_m are eigenfunctions corresponding to different eigenvalues.

***5.8.13.** Simplify $\int_0^L \sin^2 \sqrt{\lambda} x \, dx$ when λ is given by (5.8.15).

5.9 Large Eigenvalues (Asymptotic Behavior)

For the variable-coefficient case, the eigenvalues for the Sturm–Liouville differential equation,

$$\frac{d}{dx}\left[p(x) \frac{d\phi}{dx} \right] + [\lambda \sigma(x) + q(x)]\phi = 0, \tag{5.9.1}$$

usually must be calculated numerically. We know that there will be an infinite number of eigenvalues with no largest one. Thus, there will be an infinite sequence of large eigenvalues. In this section, we state and explain reasonably good approximations to these large eigenvalues and corresponding eigenfunctions. Thus, numerical solutions will be needed for only the first few eigenvalues and eigenfunctions.

A careful derivation with adequate explanations of the asymptotic method would be lengthy. Nonetheless, some motivation for our result will be presented. We begin by attempting to approximate solutions of the differential equation (5.9.1) if the unknown eigenvalue λ is large ($\lambda \gg 1$). Interpreting (5.9.1) as a spring-mass system (x is time, ϕ is position) with time-varying parameters is helpful. Then (5.9.1) has a large restoring force $[-\lambda \sigma(x)\phi]$ such that we expect the solution to have rapid oscillation in x. Alternatively, we know that eigenfunctions corresponding to large eigenvalues have many zeros. Since the solution oscillates rapidly, over a few periods (each small) the variable coefficients are approximately constant. Thus, near any point x_0, the differential equation may be approximated crudely by one with constant coefficients:

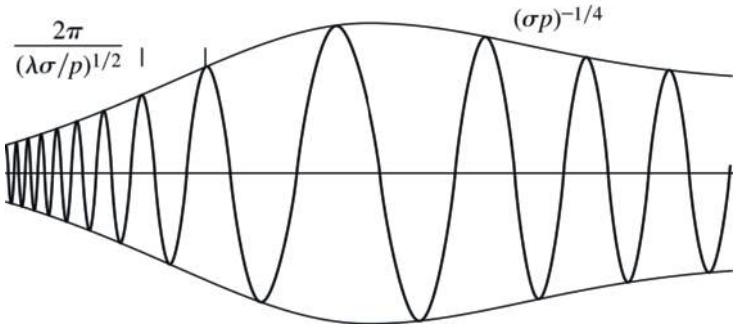
$$p(x_0)\frac{d^2\phi}{dx^2} + \lambda \sigma(x_0)\phi \approx 0, \tag{5.9.2}$$

since in addition $\lambda \sigma(x) \gg q(x)$. According to (5.9.2), the solution is expected to oscillate with "local" spatial (circular) frequency

$$\text{frequency} = \sqrt{\frac{\lambda \sigma(x_0)}{p(x_0)}}. \tag{5.9.3}$$

This frequency is large ($\lambda \gg 1$), and thus the period is small, as assumed. The frequency (and period) depends on x, but it varies slowly; that is, over a few periods (a short distance) the period hardly changes. After many periods, the frequency (and period) may change appreciably. This slowly varying period will be illustrated in Fig. 5.9.1.

Figure 5.9.1 Liouville–Green asymptotic solution of differential equation showing rapid oscillation (or, equivalently, relatively slowly varying amplitude).



From (5.9.2), one might expect the amplitude of oscillation to be constant. However, (5.9.2) is only an approximation. Instead, we should expect the amplitude to be approximately constant over each period. Thus, both the amplitude and frequency are slowly varying:

$$\phi(x) = A(x)\cos\psi(x), \tag{5.9.4}$$

where sines can also be used. The appropriate asymptotic formula for the phase $\psi(x)$ can be obtained using the ideas we have just outlined. Since the period is small, only the values of x near any x_0 are needed to understand the oscillation implied by (5.9.4). Using the Taylor series of $\psi(x)$, we obtain

$$\phi(x) = A(x)\cos[\psi(x_0) + (x - x_0)\psi'(x_0) + \cdots]. \tag{5.9.5}$$

This is an oscillation with local frequency $\psi'(x_0)$. Thus, **the derivative of the phase is the local frequency**. From (5.9.2), we have motivated that the local frequency should be $[\lambda\sigma(x_0)/p(x_0)]^{1/2}$. Thus, we expect

$$\psi'(x_0) = \lambda^{1/2}\left[\frac{\sigma(x_0)}{p(x_0)}\right]^{1/2}. \tag{5.9.6}$$

This reasoning turns out to determine the phase correctly:

$$\psi(x) = \lambda^{1/2}\int^x \left[\frac{\sigma(x_0)}{p(x_0)}\right]^{1/2} dx_0. \tag{5.9.7}$$

Note that the phase does not equal the frequency times x (unless the frequency is constant).

Precise asymptotic techniques[6] beyond the scope of this text determine the slowly varying amplitude. It is known that two independent solutions of the differential equation can be approximated accurately (if λ is large) by

$$\boxed{\phi(x) \approx (\sigma p)^{-1/4}\exp\left[\pm i\lambda^{1/2}\int^x \left(\frac{\sigma}{p}\right)^{1/2} dx_0\right],} \tag{5.9.8}$$

where sines and cosines may be used instead. A rough sketch of these solutions (using sines or cosines) is given in Fig. 5.9.1. The solution oscillates rapidly. The **envelope** of the wave is the slowly varying function $(\sigma p)^{-1/4}$, indicating the relatively slow amplitude variation. The local frequency is $(\lambda\sigma/p)^{1/2}$, corresponding to the period $2\pi/(\lambda\sigma/p)^{1/2}$.

To determine the large eigenvalues, we must apply the boundary conditions to the general solution (5.9.8). For example, if $\phi(0) = 0$, then

$$\phi(x) = (\sigma p)^{-1/4}\sin\left(\lambda^{1/2}\int_0^x \left(\frac{\sigma}{p}\right)^{1/2} dx_0\right) + \cdots. \tag{5.9.9}$$

The second boundary condition, for example, $\phi(L) = 0$, determines the eigenvalues:

$$0 = \sin\left(\lambda^{1/2}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx_0\right) + \cdots.$$

Thus, we derive the asymptotic formula for the large eigenvalues $\lambda^{1/2}\int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx_0 \approx n\pi$, or, equivalently,

$$\lambda \sim \left[n\pi \Big/ \int_0^L \left(\frac{\sigma}{p}\right)^{1/2} dx_0\right]^2, \tag{5.9.10}$$

[6]These results can be derived by various ways, such as the W.K.B.(J.) method (which should be called the Liouville–Green method) or the method of multiple scales. References for these asymptotic techniques include books by Bender and Orszag [1999], Kevorkian and Cole [1996], and Nayfeh [2002].

valid if n is large. Often, this formula is reasonably accurate even when n is not very large. The eigenfunctions are given approximately by (5.9.9), where (5.9.10) should be used. Note that $q(x)$ does not appear in these asymptotic formulas; $q(x)$ does not affect the eigenvalue to *leading* order. However, more accurate formulas exist that take $q(x)$ into account.

EXAMPLE

Consider the eigenvalue problem

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda(1+x)\phi &= 0 \\ \phi(0) &= 0 \\ \phi(1) &= 0. \end{aligned}$$

Here, $p(x) = 1, \sigma(x) = 1 + x, q(x) = 0$, and $L = 1$. Our asymptotic formula (5.9.10) for the eigenvalues is

$$\lambda \sim \left[\frac{n\pi}{\int_0^1 (1+x_0)^{1/2}\, dx_0}\right]^2 = \frac{n^2\pi^2}{\left[\frac{2}{3}(1+x_0)^{3/2}\Big|_0^1\right]^2} = \frac{n^2\pi^2}{\frac{4}{9}(2^{3/2}-1)^2}. \tag{5.9.11}$$

In Table 5.9.1 we compare numerical results (using an accurate numerical scheme on the computer) with the asymptotic formula. Equation (5.9.11) is even a reasonable approximation if $n = 1$. The percent or relative error of the asymptotic formula improves as n increases. However, the error stays about the same (though small). There are improvements to (5.9.10) that account for the approximately constant error.

TABLE 5.9.1 Eigenvalues λ_n

n	Numerical answer* (assumed accurate)	Asymptotic formula (5.9.11)	Error
1	6.548395	6.642429	0.094034
2	26.464937	26.569718	0.104781
3	59.674174	59.781865	0.107691
4	106.170023	106.278872	0.108849
5	165.951321	166.060737	0.109416
6	239.0177275	239.1274615	0.109734
7	325.369115	325.479045	0.109930

*Courtesy of E. C. Gartland, Jr.

Exercises 5.9

5.9.1. Estimate (to leading order) the large eigenvalues and corresponding eigenfunctions for

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + [\lambda\sigma(x) + q(x)]\phi = 0$$

if the boundary conditions are

(a) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

***(b)** $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(c) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + h\phi(L) = 0$

5.9.2. Consider

$$\frac{d^2\phi}{dx^2} + \lambda(1+x)\phi = 0$$

subject to $\phi(0) = 0$ and $\phi(1) = 0$. Roughly sketch the eigenfunctions for λ large. Take into account amplitude and period variations.

5.9.3. Consider for $\lambda \gg 1$

$$\frac{d^2\phi}{dx^2} + [\lambda\sigma(x) + q(x)]\phi = 0.$$

***(a)** Substitute

$$\phi = A(x) \exp\left[i\lambda^{1/2}\int_0^x \sigma^{1/2}(x_0)\,dx_0\right].$$

Determine a differential equation for $A(x)$.

(b) Let $A(x) = A_0(x) + \lambda^{-1/2}A_1(x) + \cdots$. Solve for $A_0(x)$ and $A_1(x)$. Verify (5.9.8).

(c) Suppose that $\phi(0) = 0$. Use $A_1(x)$ to improve (5.9.9).

(d) Use part (c) to improve (5.9.10) if $\phi(L) = 0$.

***(e)** Obtain a recursion formula for $A_n(x)$.

5.10 Approximation Properties

In many practical problems of solving partial differential equations by separation of variables, it is impossible to compute and work with an infinite number of terms of the infinite series. It is more usual to use a finite number of terms.[7] In this section, we briefly discuss the use of a finite number of terms of generalized Fourier series.

We have claimed that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions,

$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x). \tag{5.10.1}$$

Due to the orthogonality [with weight $\sigma(x)$] of the eigenfunctions, the generalized Fourier coefficients can be determined easily:

$$a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2\sigma\,dx}. \tag{5.10.2}$$

However, suppose that we can use only the first M eigenfunctions to approximate a function $f(x)$,

$$f(x) \approx \sum_{n=1}^{M} \alpha_n\phi_n(x). \tag{5.10.3}$$

What should the coefficients α_n be? Perhaps if we use a finite number of terms, there would be a better way to approximate $f(x)$ than by using the generalized Fourier coefficients, (5.10.2). We will pick these new coefficients α_n so that $\sum_{n=1}^{M}\alpha_n\phi_n(x)$ is the "best," approximation to $f(x)$. There are many ways to define the "best," but we will show a way that is particularly useful. In general, the coefficients α_n will depend on M. For example, suppose that we choose $M = 10$ and calculate at $\alpha_1, \cdots, \alpha_{10}$ so that (5.10.3) is "best" in some way. After this calculation, we may decide that the approximation in (5.10.3) is not good enough, so we may wish to include more terms, for example, $M = 11$. We would then have to recalculate all 11 coefficients that make (5.10.3) "best" with $M = 11$. We will show that there is a way to define "best" such that the coefficients α_n do *not* depend on M; that is, in going from $M = 10$ to $M = 11$ only one additional coefficient need be computed, namely, α_{11}.

MEAN-SQUARE DEVIATION

We define *best approximation* as the approximation with the least error. However, *error* can be defined in many different ways. The difference between $f(x)$ and its approximation $\sum_{n=1}^{M}\alpha_n\phi_n(x)$ depends on x. It is possible for $f(x) - \sum_{n=1}^{M}\alpha_n\phi_n(x)$ to be positive in some regions and negative

[7] Often, for numerical answers to problems in partial differential equations you may be better off using direct numerical methods.

in others. One possible measure of the error is the maximum of the deviation over the entire interval: $\max\left|f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)\right|$. This is a reasonable definition of the error, but it is rarely used, since it is very difficult to choose the α_n to minimize this maximum deviation. Instead, we usually define the **error** to be the **mean-square deviation**,

$$\boxed{E \equiv \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)\right]^2 \sigma(x)\, dx.} \tag{5.10.4}$$

Here, a large penalty is paid for the deviation being large on even a small interval. We introduce a weight factor $\sigma(x)$ in our definition of the error because we will show that it is easy to minimize this error *only with the weight* $\sigma(x)$. $\sigma(x)$ is the same function that appears in the differential equation defining the eigenfunctions $\phi_n(x)$; the weight appearing in the error is the same weight as needed for the orthogonality of the eigenfunctions.

The error, defined by (5.10.4), is a function of the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_M$. To minimize a function of M variables, we usually use the first-derivative condition. We insist that the first partial derivative with respect to each α_i is zero:

$$\frac{\partial E}{\partial \alpha_i} = 0, \quad i = 1, 2, \cdots, M.$$

We calculate each partial derivative and set it equal to zero:

$$0 = \frac{\partial E}{\partial \alpha_i} = -2 \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)\right] \phi_i(x) \sigma(x)\, dx, \quad i = 1, 2, \cdots, M, \tag{5.10.5}$$

where we have used the fact that $\partial/\partial \alpha_i \left(\sum_{n=1}^{M} \alpha_n \phi_n(x)\right) = \phi_i(x)$. There are M equations, (5.10.5), for the M unknowns. This would be rather difficult to solve, except for the fact that the eigenfunctions are orthogonal with the same weight $\sigma(x)$ that appears in (5.10.5). Thus, (5.10.5) becomes

$$\int_a^b f(x) \phi_i(x) \sigma(x)\, dx = \int_a^b \phi_i^2(x) \sigma(x)\, dx.$$

The ith equation can be solved easily for α_i. In fact, $\alpha_i = a_i$ [see (5.10.2)]; all first partial derivatives are zero if the coefficients are chosen to be the generalized Fourier coefficients. We should still show that this actually minimizes the error (not just a local critical point, where all first partial derivatives vanish). We in fact will show that the **best approximation** (*in the mean-square sense using the first M eigenfunctions*) **occurs when the coefficients are chosen to be the generalized Fourier coefficients**: In this way (1) the coefficients are easy to determine, and (2) the coefficients are independent of M.

Proof

To prove that the error E is actually minimized, we will not use partial derivatives. Instead, our derivation proceeds by expanding the square deviation in (5.10.4):

$$E = \int_a^b \left(f^2 - 2 \sum_{n=1}^{M} \alpha_n f \phi_n + \sum_{n=1}^{M} \sum_{m=1}^{M} \alpha_n \alpha_m \phi_n \phi_m \right) \sigma\, dx. \tag{5.10.6}$$

Some simplification again occurs due to the orthogonality of the eigenfunctions:

$$E = \int_a^b \left(f^2 - 2 \sum_{n=1}^{M} \alpha_n f \phi_n + \sum_{n=1}^{M} \alpha_n^2 \phi_n^2 \right) \sigma\, dx. \tag{5.10.7}$$

Each α_n appears at most quadratically:

$$E = \sum_{n=1}^{M} \left[\alpha_n^2 \int_a^b \phi_n^2 \sigma\, dx - 2\alpha_n \int_a^b f \phi_n \sigma\, dx \right] + \int_a^b f^2 \sigma\, dx, \tag{5.10.8}$$

and this suggests completing the square,

$$E = \sum_{n=1}^{M} \left[\int_a^b \phi_n^2 \sigma \, dx \left(\alpha_n - \frac{\int_a^b f\phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx} \right)^2 - \frac{\left(\int_a^b f\phi_n \sigma \, dx \right)^2}{\int_a^b \phi_n^2 \sigma \, dx} \right] + \int_a^b f^2 \sigma \, dx. \tag{5.10.9}$$

The only term that depends on the unknowns α_n appears in a nonnegative way. The minimum occurs only if that first term vanishes, determining the best coefficients,

$$\alpha_n = \frac{\int_a^b f\phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx}, \tag{5.10.10}$$

the same result as obtained using the simpler first derivative condition.

Error

In this way (5.10.9) shows that the minimal error is

$$\boxed{E = \int_a^b f^2 \sigma \, dx - \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \sigma \, dx,} \tag{5.10.11}$$

where (5.10.10) has been used. Equation (5.10.11) shows that as M increases, the error decreases. Thus, we can think of a generalized Fourier series as an approximation scheme. **The more terms in the truncated series that are used, the better the approximation**.

EXAMPLE

For a Fourier sine series, where $\sigma(x) = 1$, $\phi_n(x) = \sin n\pi x/L$ and $\int_0^L \sin^2 n\pi x/L \, dx = L/2$, it follows that

$$E = \int_0^L f^2 \, dx - \frac{L}{2} \sum_{n=1}^{M} \alpha_n^2, \tag{5.10.12}$$

where the α_n are the usual Fourier sine coefficients, $\alpha_n = (2/L) \int_a^b f(x) \sin n\pi x/L \, dx$. ■

Bessel's inequality and Parseval's equality

Since $E \geq 0$ [see (5.10.4)], it follows from (5.10.11) that

$$\boxed{\int_a^b f^2 \sigma \, dx \geq \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \sigma \, dx,} \tag{5.10.13}$$

known as **Bessel's inequality**. More importantly, we claim that for any Sturm–Liouville eigenvalue problem, the eigenfunction expansion of $f(x)$ converges in the mean to $f(x)$, by which we mean [see (5.10.4)] that

$$\lim_{M \to \infty} E = 0;$$

the mean-square deviation vanishes as $M \to \infty$. This shows **Parseval's equality**:

$$\boxed{\int_a^b f^2 \sigma \, dx = \sum_{n=1}^{\infty} \alpha_n^2 \int_a^b \phi_n^2 \sigma \, dx.} \tag{5.10.14}$$

Parseval's equality, (5.10.14), is a generalization of the Pythagorean theorem. For a *right* triangle, $c^2 = a^2 + b^2$. This has an interpretation for vectors. If $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$, then $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = a^2 + b^2$. Here, a and b are components of $\mathbf{v}$ in an *orthogonal* basis of *unit* vectors. Here, we represent the function $f(x)$ in terms of our orthogonal eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Parseval's equality takes on a particularly nice form in terms of eigenfunctions of *unit* length. An orthogonal basis of unit vectors is called an *orthonormal* basis. Let $\hat{\phi}_n(x) = \phi_n(x)/l$, where l is the length of $\phi_n(x)$,

$$l^2 = \int_a^b \phi_n^2 \sigma \, dx.$$

Then

$$f(x) = \sum_{n=1}^{\infty} \hat{a}_n \hat{\phi}_n,$$

where $\hat{a}_n = a_n l$, and Parseval's equality simply states

$$\int_a^b f^2 \sigma \, dx = \sum_{n=1}^{\infty} \hat{a}_n^2.$$

That is, the length of f^2 equals the sum of the squares of the components of f when using an orthonormal basis.

Exercises 5.10

5.10.1. Consider the Fourier sine series for $f(x) = 1$ on the interval $0 \le x \le L$. How many terms in the series should be kept so that the mean-square error is 1% of $\int_0^L f^2 \sigma \, dx$?

5.10.2. Obtain a formula for an infinite series using Parseval's equality applied to the

(a) Fourier sine series of $f(x) = 1$ on the interval $0 \le x \le L$

***(b)** Fourier cosine series of $f(x) = x$ on the interval $0 \le x \le L$

(c) Fourier sine series of $f(x) = x$ on the interval $0 \le x \le L$

5.10.3. Consider any function $f(x)$ defined for $a \le x \le b$. Approximate this function by a constant. Show that the best such constant (in the mean-square sense, i.e., minimizing the mean-square deviation) is the constant equal to the average of $f(x)$ over the interval $a \le x \le b$.

5.10.4. **(a)** Using Parseval's equality, express the error in terms of the tail of a series.

(b) Redo part (a) for a Fourier sine series on the interval $0 \le x \le L$.

(c) If $f(x)$ is piecewise smooth, estimate the tail in part (b). [*Hint*: Use integration by parts.]

5.10.5. Show that if

$$L(f) = \frac{d}{dx}\left(p\frac{df}{dx}\right) + qf,$$

then

$$-\int_a^b fL(f) \, dx = -pf\frac{df}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{df}{dx}\right)^2 - qf^2\right] dx$$

if f and df/dx are continuous.

5.10.6. Assuming that the operations of summation and integration can be interchanged, show that if

$$f = \sum \alpha_n \phi_n \text{ and } g = \sum \beta_n \phi_n,$$

then for normalized eigenfunctions

$$\int_a^b fg\sigma \, dx = \sum_{n=1}^{\infty} \alpha_n \beta_n,$$

a generalization of Parseval's equality.

5.10.7. Using Exercises 5.10.5 and 5.10.6, prove that

$$-\sum_{n=1}^{\infty}\lambda_n\alpha_n^2 = -pf\frac{df}{dx}\bigg|_a^b + \int_a^b\left[p\left(\frac{df}{dx}\right)^2 - qf^2\right]dx. \tag{5.10.15}$$

[*Hint*: Let $g = L(f)$, assuming that term-by-term differentiation is justified.]

5.10.8. According to Schwarz's inequality (proved in Exercise 2.3.13), the absolute value of the *pointwise error* satisfies

$$\left|f(x) - \sum_{n=1}^{M}\alpha_n\phi_n\right| = \left|\sum_{n=M+1}^{\infty}\alpha_n\phi_n\right| \le \left\{\sum_{n=M+1}^{\infty}|\lambda_n|\alpha_n^2\right\}^{1/2}\left\{\sum_{n=M+1}^{\infty}\frac{\phi_n^2}{|\lambda_n|}\right\}^{1/2}. \tag{5.10.16}$$

Furthermore, Chapter 9 introduces a Green's function $G(x, x_0)$, which is shown to satisfy

$$\sum_{n=1}^{\infty}\frac{\phi_n^2}{\lambda_n} = -G(x, x). \tag{5.10.17}$$

Using (5.10.15), (5.10.16), and (5.10.17), derive an upper bound for the pointwise error (in cases in which the generalized Fourier series is pointwise convergent). Examples and further discussion of this are given by Weinberger [1995].

6

Finite Difference Numerical Methods for Partial Differential Equations

6.1 Introduction

Partial differential equations are often classified. Equations with the same classification have qualitatively similar mathematical and physical properties. We have studied primarily the simplest prototypes. The heat equation ($\partial u/\partial t = k\partial^2 u/\partial x^2$) is an example of a **parabolic** partial differential equation. Solutions usually exponentially decay in time and approach an equilibrium solution. Information propagates at an infinite velocity and discontinuities are smoothed out. The wave equation ($\partial^2 u/\partial t^2 = c^2\partial^2 u/\partial x^2$) typifies **hyperbolic** partial differential equations. There are modes of vibration. Information propagates at a finite velocity, and thus discontinuities persist. Laplace's equation ($\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$) is an example of an **elliptic** partial differential equation. Solutions usually satisfy maximum principles. The terminology *parabolic, hyperbolic,* and *elliptic* result from transformation properties of the conic sections (e.g., see Weinberger, 1995).

In previous chapters, we have studied various methods to obtain explicit solutions of some partial differential equations of physical interest. Except for the one-dimensional wave equation, the solutions were rather complicated, involving an infinite series or an integral representation. In many current situations, detailed numerical calculations of solutions of partial differential equations are needed. Our previous analyses suggest computational methods (e.g., the first 100 terms of a Fourier series). However, usually there are more efficient methods to obtain numerical results. In this chapter, we develop finite difference methods to numerically approximate solutions of the different types of partial differential equations (i.e., parabolic, hyperbolic, and elliptic). We will describe only the heat, wave, and Laplace's equations, but algorithms for more complicated problems (including nonlinear ones) will become apparent.

6.2 Finite Differences and Truncated Taylor Series

POLYNOMIAL APPROXIMATIONS

The fundamental technique for finite difference numerical calculations is based on polynomial approximations to $f(x)$ near $x = x_0$. We let $x = x_0 + \Delta x$, so that $\Delta x = x - x_0$. If we approximate $f(x)$ by a constant near $x = x_0$, we choose $f(x_0)$. A better approximation (see Fig. 6.2.1) to $f(x)$ is its tangent line at $x = x_0$:

$$f(x) \approx f(x_0) + \underbrace{(x - x_0)}_{\Delta x}\frac{df}{dx}(x_0), \tag{6.2.1}$$

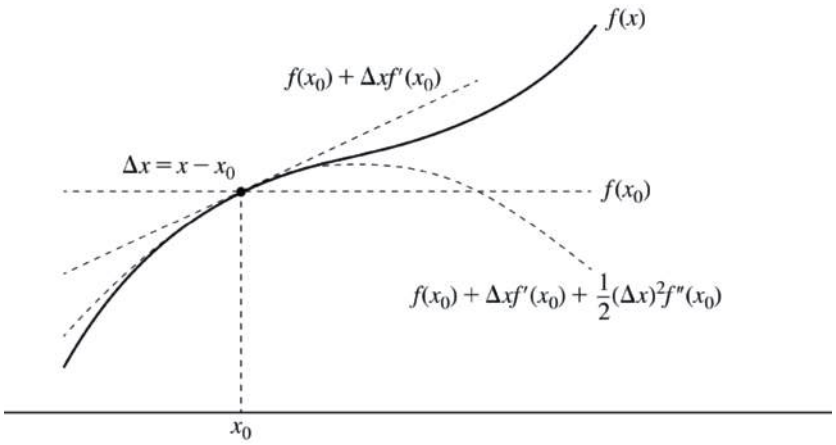
a linear approximation (a first-order polynomial). We can also consider a quadratic approximation to $f(x)$, $f(x) \approx f(x_0) + \Delta x f'(x_0) + (\Delta x)^2 f''(x_0)/2!$, whose value, first, and second derivatives at $x = x_0$ equal that for $f(x)$. Each such succeeding higher-degree polynomial approximation to $f(x)$ is more and more accurate if x is near enough to x_0 (i.e., if Δx is small).

Truncation error

A formula for the error in these polynomial approximations is obtained directly from

$$f(x) = f(x_0) + \Delta x f'(x_0) + \cdots + \frac{(\Delta x)^n}{n!}f^{(n)}(x_0) + R_n, \tag{6.2.2}$$

Figure 6.2.1 Taylor polynomials.



known as **the Taylor polynomial with remainder**. The remainder R_n (also called the **truncation error**) is known to be in the form of the next term of the series, but evaluated at a usually unknown intermediate point:

$$R_n = \frac{(\Delta x)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_{n+1}), \text{ where } x_0 < \xi_{n+1} < x = x_0 + \Delta x. \tag{6.2.3}$$

For this to be valid, $f(x)$ must have $n+1$ continuous derivatives.

EXAMPLE

The error in the tangent line approximation is given by (6.2.3) with $n = 1$:

$$f(x_0 + \Delta x) = f(x_0) + \Delta x \frac{df}{dx}(x_0) + \frac{(\Delta x)^2}{2!}\frac{d^2 f}{dx^2}(\xi_2), \tag{6.2.4}$$

called the **extended mean value theorem**. If Δx is small, then ξ_2 is contained in a small interval, and the truncation error is almost determined (provided that $d^2 f/dx^2$ is continuous),

$$R \approx \frac{(\Delta x)^2}{2}\frac{d^2 f}{dx^2}(x_0).$$

We say that the truncation error is $O(\Delta x)^2$, "order delta-x squared," meaning that

$$|R| \leq C(\Delta x)^2,$$

since we usually assume that $d^2 f/dx^2$ is bounded ($|d^2 f/dx^2| < M$). Thus, $C = M/2$. ■

First-derivative approximations

Through the use of Taylor series, we are able to approximate derivatives in various ways. For example, from (6.2.4),

$$\frac{df}{dx}(x_0) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - \frac{\Delta x}{2}\frac{d^2 f}{dx^2}(\xi_2). \tag{6.2.5}$$

We thus introduce a finite difference approximation, the **forward difference** approximation to df/dx:

$$\boxed{\frac{df}{dx}(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.} \tag{6.2.6}$$

This is nearly the definition of the derivative. Here, we use a forward difference (but do not take the limit as $\Delta x \to 0$). Since (6.2.5) is valid for all Δx, we can let Δx be replaced by $-\Delta x$ and derive the **backward difference** approximation to df/dx:

$$\frac{df}{dx}(x_0) = \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} + \frac{\Delta x}{2}\frac{d^2 f}{dx^2}(\bar{\xi}_2) \tag{6.2.7}$$

and hence

$$\boxed{\frac{df}{dx}(x_0) \approx \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} = \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}.} \tag{6.2.8}$$

By comparing (6.2.5) to (6.2.6) and (6.2.7) to (6.2.8), we observe that the truncation error is $O(\Delta x)$ and nearly identical for both forward and backward difference approximations of the first derivative.

To obtain a more accurate approximation for $df/dx(x_0)$, we can average the forward and backward approximations. By adding (6.2.5) and (6.2.7),

$$2\frac{df}{dx}(x_0) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{\Delta x} + \frac{\Delta x}{2}\left[\frac{d^2f}{dx^2}(\bar{\xi}_2) - \frac{d^2f}{dx^2}(\xi_2)\right]. \tag{6.2.9}$$

Since $\bar{\xi}_2$ is near ξ_2, we expect the error nearly to cancel and thus be much less than $O(\Delta x)$. To derive the error in this approximation, we return to the Taylor series for $f(x_0 - \Delta x)$ and $f(x_0 + \Delta x)$:

$$f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{(\Delta x)^2}{2!}f''(x_0) + \frac{(\Delta x)^3}{3!}f'''(x_0) + \dots. \tag{6.2.10}$$

$$f(x_0 - \Delta x) = f(x_0) - \Delta x f'(x_0) + \frac{(\Delta x)^2}{2!}f''(x_0) - \frac{(\Delta x)^3}{3!}f'''(x_0) + \dots. \tag{6.2.11}$$

Subtracting (6.2.11) from (6.2.10) yields

$$f(x_0 + \Delta x) - f(x_0 - \Delta x) = 2\Delta x f'(x_0) + \frac{2}{3!}(\Delta x)^3 f'''(x_0) + \dots.$$

We thus expect that

$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} - \frac{(\Delta x)^2}{6}f'''(\xi_3), \tag{6.2.12}$$

which is proved in an exercise. This leads to the **centered difference** approximation to $df/dx(x_0)$:

$$\boxed{f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x},} \tag{6.2.13}$$

Equation (6.2.13) is usually preferable since it is more accurate [the truncation error is $O(\Delta x)^2$] and involves the same number (2) of function evaluations as both the forward and backward difference formulas. However, as we will show later, it is not always better to use the centered difference formula.

These finite difference approximations to df/dx are **consistent**, meaning that the truncation error vanishes as $\Delta x \to 0$. More accurate finite difference formulas exist, but they are used less frequently.

EXAMPLE

Consider the numerical approximation of $df/dx(1)$ for $f(x) = \log x$ using $\Delta x = 0.1$. Unlike practical problems, here we know the exact answer, $df/dx(1) = 1$. Using the values $x_0 = 1$, $\Delta x = 0.1$, $f(x_0 + \Delta x) = f(1.1) = \log(1.1) = 0.0953102$ and $f(x_0 - \Delta x) = \log(0.9) = -0.1053605$ yields the numerical results reported in Table 6.2.1. Theoretically, the error should be an order of magnitude Δx smaller for the centered difference. We observe this phenomenon. To understand the error further, we calculate the expected error E using an estimate of the remainder. For forward or backward differences,

$$E \approx \left|\frac{\Delta x}{2}\frac{d^2f}{dx^2}(1)\right| = \frac{0.1}{2} = +0.05,$$

whereas for a centered difference,

$$E \approx \frac{(\Delta x)^2}{6}\frac{d^3f}{dx^3}(1) = \frac{(0.1)^2}{6}2 = 0.00333\dots.$$

These agree quite well with our actual tabulated errors. Estimating the errors this way is rarely possible to do in practice since usually the second and third derivatives are not known to begin with.

TABLE 6.2.1:

	Forward	Backward	Centered
Difference formula	0.953102	1.053605	1.00335
|Error|	4.6898%	5.3605%	0.335%

Second derivatives

By adding (6.2.10) and (6.2.11), we obtain

$$f(x_0 + \Delta x) + f(x_0 - \Delta x) = 2f(x_0) + (\Delta x)^2 f''(x_0) + \frac{2(\Delta x)^4}{4!} f^{(iv)}(x_0) + \dots .$$

We thus expect that

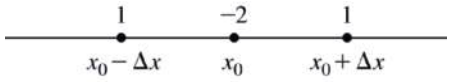
$$f''(x_0) = \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2} - \frac{(\Delta x)^2}{12} f^{(iv)}(\xi). \tag{6.2.14}$$

This yields a finite difference approximation for the second derivative with an $O(\Delta x)^2$ truncation error:

$$\boxed{\frac{d^2 f}{dx^2}(x_0) \approx \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2}.} \tag{6.2.15}$$

Equation (6.2.15) is called the **centered difference** approximation for the second derivative since it also can be obtained by repeated application of the centered difference formulas for first derivatives (see Exercise 6.2.2). The centered difference approximation for the second derivative involves three function evaluations, $f(x_0 - \Delta x)$, $f(x_0)$, and $f(x_0 + \Delta x)$. The respective "weights," $1/(\Delta x)^2$, $-2/(\Delta x)^2$, and $1/(\Delta x)^2$, are illustrated in Fig. 6.2.2. In fact, in general, *the weights must sum to zero for any finite difference approximation to any derivative.*

Figure 6.2.2 Weights for centered difference approximation of second derivative.



Partial derivatives

In solving partial differential equations, we analyze functions of two or more variables, for example, $u(x, y)$, $u(x, t)$, and $u(x, y, t)$. Numerical methods often use finite difference approximations. Some (but not all) partial derivatives may be obtained using our earlier results for functions of one variable. For example, if $u(x, y)$, then $\partial u/\partial x$ is an ordinary derivative du/dx, keeping y fixed. We may use the forward, backward, or centered difference formulas. Using the centered difference formula,

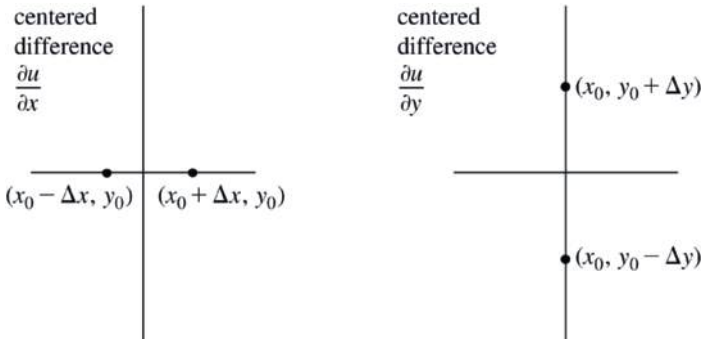
$$\frac{\partial u}{\partial x}(x_0, y_0) \approx \frac{u(x_0 + \Delta x, y_0) - u(x_0 - \Delta x, y_0)}{2\Delta x}.$$

For $\partial u/\partial y$, we keep x fixed and thus obtain

$$\frac{\partial u}{\partial y}(x_0, y_0) \approx \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0 - \Delta y)}{2\Delta y},$$

using the centered difference formula. These are both two-point formulas, which we illustrate in Fig. 6.2.3.

Figure 6.2.3 Points for first partial derivatives.



In physical problems, we often need the Laplacian $\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$. We use the centered difference formula for second derivatives (6.2.15), adding the formula for x fixed to the one for y fixed:

$$\nabla^2 u(x_0, y_0) \approx \frac{u(x_0 + \Delta x, y_0) - 2u(x_0, y_0) + u(x_0 - \Delta x, y_0)}{(\Delta x)^2} + \frac{u(x_0, y_0 + \Delta y) - 2u(x_0, y_0) + u(x_0, y_0 - \Delta y)}{(\Delta y)^2}. \tag{6.2.16}$$

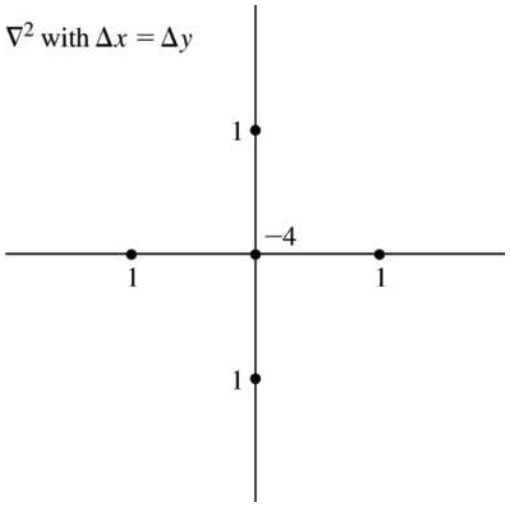
Here, the error is the larger of $O(\Delta x)^2$ and $O(\Delta y)^2$. We often let $\Delta x = \Delta y$, obtaining the standard five-point finite difference approximation to the Laplacian ∇^2,

$$\boxed{\nabla^2 u(x_0, y_0) \approx \frac{u(x_0 + \Delta x, y_0) + u(x_0 - \Delta x, y_0) + u(x_0, y_0 + \Delta y) + u(x_0, y_0 - \Delta y) - 4u(x_0, y_0)}{(\Delta x)^2},} \tag{6.2.17}$$

as illustrated in Fig. 6.2.4, where $\Delta x = \Delta y$. Note that the relative weights again sum to zero.

Other formulas for derivatives may be found in "Numerical Interpolation, Differentiation, and Integration" by P. J. Davis and I. Polonsky (Chapter 25 of Abramowitz and Stegun [1974]).

Figure 6.2.4 Weights for the Laplacian ($\Delta x = \Delta y$).



Exercises 6.2

6.2.1. **(a)** Show that the truncation error for the centered difference approximation of the first derivative (6.2.13) is $-(\Delta x)^2 f'''(\xi_3)/6$. [*Hint*: Consider the Taylor series of $g(\Delta x) = f(x + \Delta x) - f(x - \Delta x)$ as a function of Δx around $\Delta x = 0$.]

(b) Explicitly show that (6.2.13) is exact for any quadratic polynomial.

6.2.2. Derive (6.2.15) by twice using the centered difference approximation for first derivatives.

6.2.3. Derive the truncation error for the centered difference approximation of the second derivative.

6.2.4. Suppose that we did not know (6.2.15) but thought it possible to approximate $d^2f/dx^2(x_0)$ by an unknown linear combination of the three function values, $f(x_0 - \Delta x)$, $f(x_0)$, and $f(x_0 + \Delta x)$:

$$\frac{d^2f}{dx^2} \approx af(x_0 - \Delta x) + bf(x_0) + cf(x_0 + \Delta x).$$

Determine a, b, and c by expanding the right-hand side in a Taylor series around x_0 using (6.2.10) and (6.2.11) and equating coefficients through $d^2f/dx^2(x_0)$.

***6.2.5.** Derive the one-sided approximation to df/dx by assuming

$$\frac{df}{dx} \approx af(x_0) + bf(x_0 + \Delta x) + cf(x_0 + 2\Delta x),$$

expanding in a Taylor series about x_0, and equating coefficients of $f(x_0)$, df/dx, and d^2f/dx^2. What are the coefficients a, b, and c? What is the truncation error?

6.2.6. Derive the most accurate five-point approximation for $f'(x_0)$ involving $f(x_0)$, $f(x_0 \pm \Delta x)$, and $f(x_0 \pm 2\Delta x)$. What is the order of magnitude of the truncation error?

***6.2.7.** Derive an approximation for $\partial^2 u/\partial x\,\partial y$ whose truncation error is $O(\Delta x)^2$. [*Hint*: Twice apply the centered difference approximations for first-order partial derivatives.]

6.2.8. How well does $\frac{1}{2}[f(x) + f(x + \Delta x)]$ approximate $f(x + \Delta x/2)$ (i.e., what is the truncation error)?

6.3 Heat Equation

6.3.1 INTRODUCTION

In this subsection, we introduce a numerical finite difference method to solve the one-dimensional heat equation without sources on a finite interval $0 < x < L$:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \\ u(0, t) &= 0 \\ u(L, t) &= 0 \\ u(x, 0) &= f(x). \end{aligned}} \tag{6.3.1}$$

6.3.2 A PARTIAL DIFFERENCE EQUATION

We will begin by replacing the partial differential equation at the point $x = x_0$, $t = t_0$ by an approximation based on our finite difference formulas for the derivatives. We can do this in many ways. Eventually, we will learn why some ways are good and others bad. Somewhat arbitrarily we choose a **forward difference in time** for $\partial u/\partial t$:

$$\frac{\partial u}{\partial t}(x_0, t_0) = \frac{u(x_0, t_0 + \Delta t) - u(x_0, t_0)}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_0, \eta_1),$$

where $t_0 < \eta_1 < t_0 + \Delta t$. For spatial derivatives, we introduce our **spatial centered difference** scheme

$$\frac{\partial^2 u}{\partial x^2}(x_0, t_0) = \frac{u(x_0 + \Delta x, t_0) - 2u(x_0, t_0) + u(x_0 - \Delta x, t_0)}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_1, t_0),$$

where $x_0 < \xi_1 < x_0 + \Delta x$. The heat equation at any point $x = x_0$, $t = t_0$ becomes

$$\frac{u(x_0,t_0 + \Delta t) - u(x_0,t_0)}{\Delta t} = k\frac{u(x_0 + \Delta x,t_0) - 2u(x_0,t_0) + u(x_0 - \Delta x,t_0)}{(\Delta x)^2} + E, \tag{6.3.2}$$

exactly, where the discretization (or truncation) error is

$$E = \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(x_0,\eta_1) - \frac{k(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_1,t_0). \tag{6.3.3}$$

Since E is unknown, we cannot solve (6.3.2). Instead, we introduce the *approximation* that results by ignoring the truncation error:

$$\frac{u(x_0, t_0 + \Delta t) - u(x_0, t_0)}{\Delta t} \approx k\frac{u(x_0 + \Delta x, t_0) - 2u(x_0, t_0) + u(x_0 - \Delta x, t_0)}{(\Delta x)^2}. \quad (6.3.4)$$

To be more precise, we introduce $\tilde{u}(x_0, t_0)$, an approximation at the point $x = x_0, t = t_0$ of the exact solution $u(x_0, t_0)$. We let the approximation $\tilde{u}(x_0, t_0)$ solve (6.3.4) exactly,

$$\frac{\tilde{u}(x_0, t_0 + \Delta t) - \tilde{u}(x_0, t_0)}{\Delta t} = k\frac{\tilde{u}(x_0 + \Delta x, t_0) - 2\tilde{u}(x_0, t_0) + \tilde{u}(x_0 - \Delta x, t_0)}{(\Delta x)^2}. \quad (6.3.5)$$

$\tilde{u}(x_0, t_0)$ is the exact solution of an equation that is only approximately correct. We hope that the desired solution $u(x_0, t_0)$ is accurately approximated by $\tilde{u}(x_0, t_0)$.

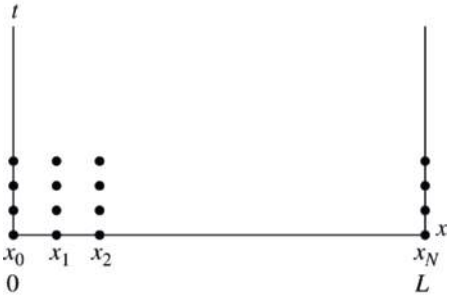
Equation (6.3.5) involves points separated a distance Δx in space and Δt in time. We thus introduce a uniform mesh Δx and a constant discretization time Δt. A space-time diagram (Fig. 6.3.1) illustrates our mesh and time discretization on the domain of our initial boundary value problem. We divide the rod of length L into N equal intervals, $\Delta x = L/N$. We have $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_N = N\,\Delta x = L$. In general,

$$\boxed{x_j = j\Delta x.} \quad (6.3.6)$$

Similarly, we introduce time step sizes Δt such that

$$\boxed{t_m = m\Delta t.} \quad (6.3.7)$$

Figure 6.3.1 Space-time discretization.



The exact temperature at the mesh point $u(x_j, t_m)$ is approximately $\tilde{u}(x_j, t_m)$, which satisfies (6.3.5). We introduce the following *notation*:

$$\boxed{\tilde{u}(x_j, t_m) \equiv u_j^{(m)},} \quad (6.3.8)$$

indicating the exact solution of (6.3.5) at the jth mesh point at time t_m. Equation (6.3.5) will be satisfied at each mesh point $x_0 = x_j$ at each time $t_0 = t_m$ (excluding the space-time boundaries). Note that $x_0 + \Delta x$ becomes $x_j + \Delta x = x_{j+1}$ and $t_0 + \Delta t$ becomes $t_m + \Delta t = t_{m+1}$. Thus,

$$\boxed{\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} = k\frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{(\Delta x)^2},} \quad (6.3.9)$$

for $j = 1, \ldots, N-1$ and m starting from 1. We call (6.3.9) a **partial difference equation**. The local truncation error is given by (6.3.3); it is the larger of $O(\Delta t)$ and $O(\Delta x)^2$. Since $E \to 0$ as $\Delta x \to 0$ and $\Delta t \to 0$, the approximation (6.3.9) is said to be **consistent** with the partial differential equation (6.3.1).

In addition, we insist that $u_j^{(m)}$ satisfies the initial conditions (at the mesh points)

$$u_j^{(0)} = u(x, 0) = f(x) = f(x_j), \quad (6.3.10)$$

where $x_j = j\Delta x$ for $j = 0, \ldots, N$. Similarly, $u_j^{(m)}$ satisfies the boundary conditions (at each time step)

$$u_0^{(m)} = u(0, t) = 0 \quad (6.3.11)$$

$$u_N^{(m)} = u(L, t) = 0. \quad (6.3.12)$$

If there is a physical (and thus mathematical) discontinuity at the initial time at any boundary point, then we can analyze $u_0^{(0)}$ or $u_N^{(0)}$ in different numerical ways.

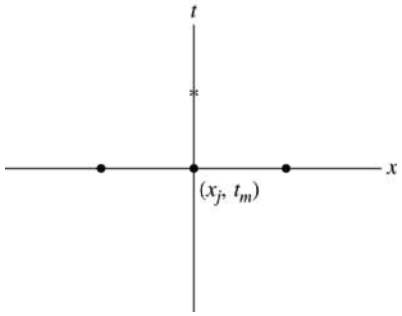
6.3.3 COMPUTATIONS

Our finite difference scheme (6.3.9) involves four points, three at the time t_m and one at the advanced time $t_{m+1} = t_m + \Delta t$, as illustrated by Fig. 6.3.2. We can "march forward in time" by solving for $u_j^{(m+1)}$, starred in Fig. 6.3.2:

$$\boxed{u_j^{(m+1)} = u_j^{(m)} + s\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right),} \tag{6.3.13}$$

where s is a dimensionless parameter.

Figure 6.3.2 Marching forward in time.



$$\boxed{s = k\frac{\Delta t}{(\Delta x)^2}.} \tag{6.3.14}$$

$u_j^{(m+1)}$ is a linear combination of the specified three earlier values. We begin our computation using the initial condition $u_j^{(0)} = f(x_j)$, for $j = 1, \ldots, N-1$. Then (6.3.13) specifies the solution $u_j^{(1)}$ at time Δt, and we continue the calculation. For mesh points adjacent to the boundary (i.e., $j = 1$ or $j = N-1$), (6.3.13) requires the solution on the boundary points ($j = 0$ or $j = N$). We obtain these values from the boundary conditions. In this way we can easily solve our discrete problem numerically. Our proposed scheme is easily programmed on a computer.

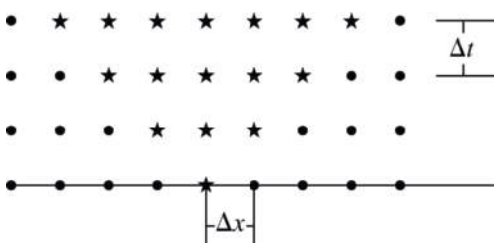
Propagation speed of disturbances

As a simple example, suppose that the initial conditions at the mesh points are zero except for 1 at some interior mesh point far from the boundary. At the first time step, (6.3.13) will imply that the solution is zero everywhere except at the original nonzero mesh point and its two immediate neighbors. This process continues as illustrated in Fig. 6.3.3, where stars represent nonzero values. The isolated initial nonzero value spreads out at a constant speed (until the boundary has been reached). This disturbance propagates at velocity $\Delta x/\Delta t$. However, for the heat equation, disturbances move at an infinite speed (see Chapter 10). In some sense our numerical scheme poorly approximates this property of the heat equation. However, if the parameter s is fixed, then the numerical propagation speed is

$$\frac{\Delta x}{\Delta t} = \frac{k\Delta x}{s(\Delta x)^2} = \frac{k}{s\Delta x}.$$

As $\Delta x \to 0$ (with s fixed), this speed approaches ∞ as is desired.

Figure 6.3.3 Propagation speed of disturbances.



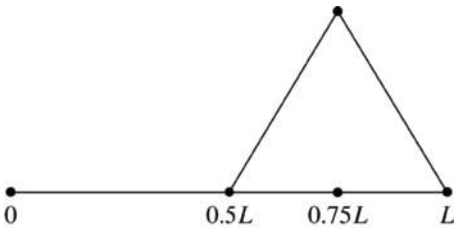
Computed example

To compute with (6.3.13), we must specify Δx and Δt. Presumably, our solution will be more accurate with smaller Δx and Δt. Certainly, the local truncation error will be reduced. An obvious disadvantage of decreasing Δx and Δt will be the resulting increased time necessary for any numerical computation. This trade-off usually occurs in numerical calculations. However, there is a more severe difficulty that we will need to analyze. To indicate the problem, we will compute using (6.3.13). First, we must choose Δx and Δt. In our calculations, we fix $\Delta x = L/10$ (nine interior points and two boundary points). Since our partial difference equation (6.3.13) depends on the parameter $s = k\Delta t/(\Delta x)^2$, we pick Δt so that, as examples, $s = 1/4$ and $s = 1$. In both cases we assume $u(x, 0) = f(x)$ is the specific initial condition sketched in Fig. 6.3.4, and the zero boundary conditions (6.3.11) and (6.3.12). The exact solution of the partial differential equation is

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \\ a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx, \end{aligned} \tag{6.3.15}$$

as described in Chapter 2. It shows that the solution decays exponentially in time and approaches a simple sinusoidal ($\sin \pi x/L$) shape in space for large t. Elementary computer calculations of our numerical scheme, (6.3.13), for $s = \frac{1}{4}$ and $s = 1$, are sketched in Fig. 6.3.5 (with smooth curves sketched through the nine interior points at fixed values of t). For $s = \frac{1}{4}$ these results seem quite reasonable, agreeing with our qualitative understanding of the exact solution. On the other hand, the solution of (6.3.13) for $s = 1$ is absurd. Its most obvious difficulty is the negative temperatures. The solution then grows wildly in time with rapid oscillations in space and time. None of these phenomena are associated with the heat equation. The finite difference approximation yields unusable results if $s = 1$. In the next subsection, we explain these results. We must understand how to pick $s = k(\Delta t)/(\Delta x)^2$ so that we are able to obtain reasonable numerical solutions.

Figure 6.3.4 Initial condition.



6.3.4 FOURIER–VON NEUMANN STABILITY ANALYSIS

Introduction

In this subsection, we analyze the finite difference scheme for the heat equation obtained by using a forward difference in time and a centered difference in space:

$$\text{pde}^1\text{:}\quad \boxed{u_j^{(m+1)} = u_j^{(m)} + s\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right)} \tag{6.3.16}$$

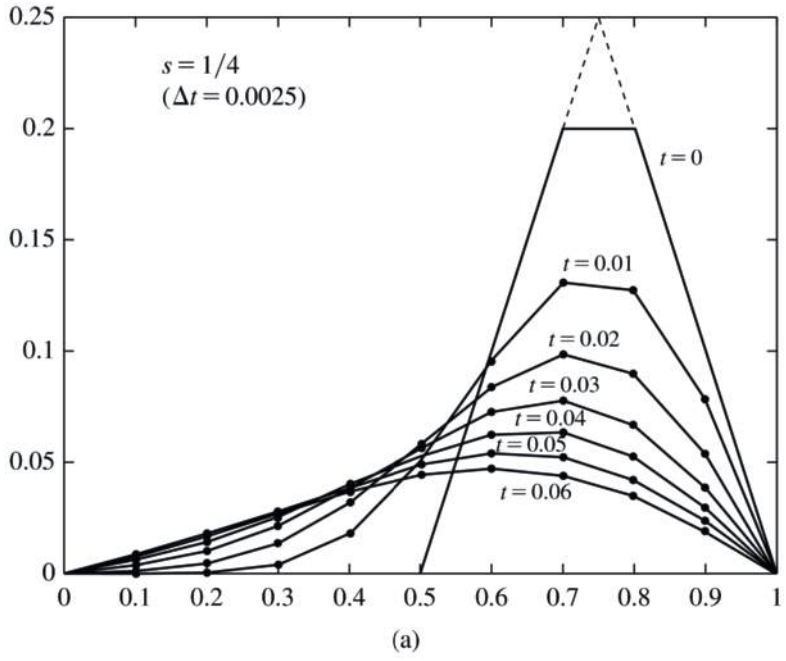
$$\text{IC:}\quad \boxed{u_j^{(0)} = f(x_j) = f_j} \tag{6.3.17}$$

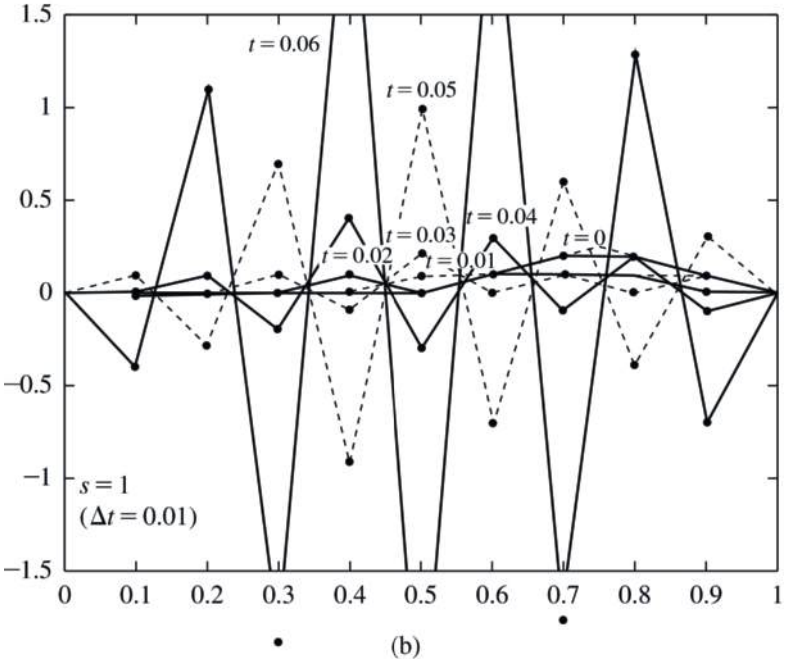
$$\text{BC:}\quad \boxed{\begin{aligned} u_0^{(m)} &= 0 \\ u_N^{(m)} &= 0, \end{aligned}} \tag{6.3.18}$$

where $s = k(\Delta t)/(\Delta x)^2$, $x_j = j\Delta x$, $t = m\Delta t$, and hopefully $u(x_j, t) \approx u_j^{(m)}$. We will develop von Neumann's ideas of the 1940s based on Fourier-type analysis.

[1]pde here means partial *difference* equation.

Figure 6.3.5 Computations for the heat equation $s = k(\Delta t)/(\Delta x)^2$: (a) $s = \frac{1}{4}$ stable and (b) $s = 1$ unstable.





MATLAB Files

Download Animations of Figure 6.3.5 (.zip)

https://bit.ly/3Wdhz3c

These files are provided as zipped .m files for use in MATLAB on desktop platform.

Eigenfunctions and product solutions

In Section 6.3.5, we show that the method of separation of variables can be applied to the partial difference equation. There are special product solutions with wave number α of the form

$$u_j^{(m)} = e^{i\alpha x} Q^{t/\Delta t} = e^{i\alpha j \Delta x} Q^m. \tag{6.3.19}$$

By substituting (6.3.19) into (6.3.16) and canceling $e^{i\alpha x} Q^m$, we obtain

$$Q = 1 + s(e^{i\alpha \Delta x} - 2 + e^{-i\alpha \Delta x}) = 1 - 2s[1 - \cos(\alpha \Delta x)]. \tag{6.3.20}$$

Q is the same for positive and negative α. Thus, a linear combination of $e^{\pm i\alpha x}$ may be used. The boundary condition $u_0^{(m)} = 0$ implies that $\sin \alpha x$ is appropriate, while $u_N^{(m)} = 0$ implies that $\alpha = n\pi/L$. Thus, there are solutions of (6.3.16) with (6.3.18) of the form

$$\boxed{u_j^{(m)} = \sin\frac{n\pi x}{L} Q^{t/\Delta t},} \tag{6.3.21}$$

where Q is determined from (6.3.20),

$$\boxed{Q = 1 - 2s\left[1 - \cos\left(\frac{n\pi\Delta x}{L}\right)\right],} \tag{6.3.22}$$

and $n = 1, 2, 3, \ldots, N-1$, as will be explained. For partial *differential* equations there are an infinite number of eigenfunctions ($\sin n\pi x/L$, $n = 1, 2, 3, \ldots$). However, we will show that for our partial *difference* equation there are only $N-1$ independent eigenfunctions ($\sin n\pi x/L$, $n = 1, 2, 3, \ldots, N-1$):

$$\phi_j = \sin\frac{n\pi x}{L} = \sin\frac{n\pi j\Delta x}{L} = \sin\frac{n\pi j}{N}, \tag{6.3.23}$$

the same eigenfunctions as for the partial differential equation (in this case). For example, for $n = N$, $\phi_j = \sin \pi j = 0$ (for all j). Furthermore, ϕ_j for $n = N+1$ is equivalent to ϕ_j for $n = N-1$ since

$$\sin\frac{(N+1)\pi j}{N} = \sin\left(\frac{\pi j}{N} + j\pi\right) = \sin\left(\frac{\pi j}{N} - j\pi\right) = -\sin\frac{(N-1)\pi j}{N}.$$

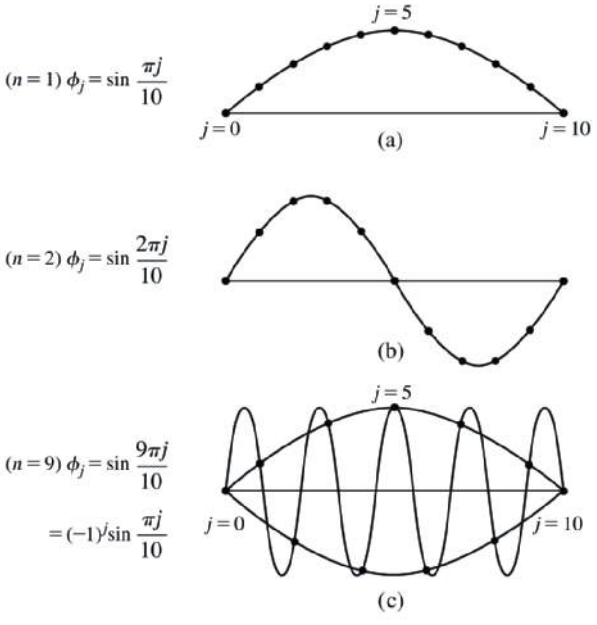
In Fig. 6.3.6, we sketch some of these "eigenfunctions" (for $N = 10$). For the partial difference equation due to the discretization, the solution is composed of only $N-1$ waves. This number of waves equals the number of independent mesh points (excluding the endpoints). The wave with the smallest wavelength is

$$\sin\frac{(N-1)\pi x}{L} = \sin\frac{(N-1)\pi j}{N} = (-1)^{j+1}\sin\frac{\pi j}{N},$$

which alternates signs at every point. The general solution is obtained by the principle of superposition, introducing $N-1$ constants β_n:

$$u_j^{(m)} = \sum_{n=1}^{N-1} \beta_n \sin\frac{n\pi x}{L}\left[1 - 2s\left(1 - \cos\frac{n\pi}{N}\right)\right]^{t/\Delta t}, \tag{6.3.24}$$

Figure 6.3.6 Eigenfunctions for the discrete problem.



where

$$s = \frac{k\Delta t}{(\Delta x)^2}.$$

These coefficients can be determined from the $N-1$ initial conditions, using the discrete orthogonality of the eigenfunctions $\sin n\pi j/N$. The analysis of this discrete Fourier series is described in Exercises 6.3.3 and 6.3.4.

Comparison to partial differential equation

The product solutions $u_j^{(m)}$ of the partial difference equation may be compared with the product solutions $u(x,\ t)$ of the partial differential equation:

$$\boxed{\begin{aligned} u_j^{(m)} &= \sin\frac{n\pi x}{L}\left[1-2s\left(1-\cos\frac{n\pi}{N}\right)\right]^{t/\Delta t}, \qquad n = 1,\ 2,\ \ldots,\ N-1 \\ u(x,\ t) &= \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}, \qquad\qquad\qquad\quad n = 1,\ 2,\ \ldots, \end{aligned}}$$

where $s = k\Delta t/(\Delta x)^2$. For the partial differential equation, each wave exponentially decays, $e^{-k(n\pi/L)^2 t}$. For the partial difference equation, the time dependence (corresponding to the spatial part $\sin n\pi x/L$) is

$$Q^m = \left[1-2s\left(1-\cos\frac{n\pi}{N}\right)\right]^{t/\Delta t}. \tag{6.3.25}$$

Stability

If $Q > 1$, there is exponential growth in time, while exponential decay occurs if $0 < Q < 1$. The solution is constant in time if $Q = 1$. However, in addition, it is possible for there to be a convergent oscillation in time ($-1 < Q < 0$), a pure oscillation ($Q = -1$), and a divergent oscillation ($Q < -1$). These possibilities are discussed and sketched in Section 6.3.5. The value of Q will determine stability. If $|Q| \le 1$ for all solutions, we say that the numerical scheme is **stable**. Otherwise, the scheme is **unstable**.

We return to analyze $Q^m = Q^{t/\Delta t}$, where $Q = 1-2s(1-\cos n\pi/N)$. Here, $Q \le 1$; the solution cannot be a purely growing exponential in time. However, the solution may be a convergent or divergent oscillation as well as being exponentially decaying. We do not want the numerical scheme to have divergent oscillations in time.[2] If s is too large, Q may become too negative.

Since $Q \le 1$, the solution will be "stable" if $Q \ge -1$. To be stable, $1-2s(1-\cos n\pi/N) \ge -1$ for $n = 1,\ 2,\ 3,\ \ldots\ N-1$, or, equivalently,

$$s \le \frac{1}{1-\cos n\pi/N} \text{ for } n = 1,\ 2,\ 3,\ \ldots,\ N-1.$$

For stability, s must be less than or equal to the smallest of these values, which occurs for $n = N-1$,

$$s \le \frac{1}{1-\cos(N-1)\pi/N}.$$

To simplify the criteria, we note that $1-\cos(N-1)\pi/N < 2$, and hence we are *guaranteed* that the numerical solution will be stable if $s \le \frac{1}{2}$:

$$s \le \frac{1}{2} < \frac{1}{1-\cos(N-1)\pi/N}. \tag{6.3.26}$$

In practice, we cannot be stable with s much larger than $\frac{1}{2}$, since $\cos(N-1)\pi/N = -\cos\pi/N$, and hence for N large, $1-\cos(N-1)\pi/N \approx 2$.

If $s > \frac{1}{2}$, usually $Q < -1$ (but not necessarily) for some n. Then the numerical solution will contain a divergent oscillation. We call this a **numerical instability**. If $s > \frac{1}{2}$, the most rapidly "growing" solution corresponds to a rapid oscillation ($n = N-1$) in space. **The numerical**

[2]Convergent oscillations do not duplicate the behavior of the partial differential equation. However, they at least decay. We tolerate oscillatory decaying terms.

instability is characterized by divergent oscillation in time ($Q < -1$) of a rapidly oscillatory ($n = N - 1$) solution in space. Generally, if we observe computer output of this form, we probably have a numerical scheme that is unstable, and hence not reliable. This is what we observed numerically when $s = 1$. For $s = \frac{1}{4}$, the solution behaved quite reasonably. However, for $s = 1$ a divergent oscillation was observed in time, rapidly varying in space.

Since $s = k\Delta t/(\Delta x)^2$, the restriction $s \leq \frac{1}{2}$ says that

$$\Delta t \leq \frac{\frac{1}{2}(\Delta x)^2}{k}. \tag{6.3.27}$$

This puts a practical constraint on numerical computations. The time steps Δt must not be too large (otherwise the scheme becomes unstable). In fact, since Δx must be small (for accurate computations), (6.3.27) shows that the time step must be exceedingly small. Thus, the forward time, centered spatial difference approximation for the heat equation is somewhat expensive to use.

To minimize calculations, we make Δt as large as possible (maintaining stability). Here, $s = \frac{1}{2}$ would be a good value. In this case, the partial difference equation becomes

$$u_j^{(m+1)} = \frac{1}{2}\left[u_{j+1}^{(m)} + u_{j-1}^{(m)}\right].$$

The temperature at time Δt later is the average of the temperatures to the left and right.

Convergence

As a further general comparison between the difference and differential equations, we consider the limits of the solution of the partial difference equation as $\Delta x \to 0 (N \to \infty)$ and $\Delta t \to 0$. We will show that the time dependence of the discretization converges to that of the heat equation if $n/N \ll 1$ (as though we fix n and let $N \to \infty$). If $n/N \ll 1$, then $\cos n\pi/N \approx 1 - \frac{1}{2}(n\pi/N)^2$ from its Taylor series, and hence

$$Q^{t/\Delta t} \approx \left[1 - s\left(\frac{n\pi}{N}\right)^2\right]^{t/\Delta t} = \left[1 - k\Delta t\left(\frac{n\pi}{L}\right)^2\right]^{t/\Delta t}, \tag{6.3.28}$$

where $N = L/\Delta x$. Thus, as $\Delta t \to 0$,

$$Q^{t/\Delta t} \to e^{-k(n\pi/L)^2 t} \tag{6.3.29}$$

since e may be defined as $e = \lim_{z\to 0}(1 + z)^{1/z}$. If $n/N \ll 1$, then by more careful analysis [taking logs of (6.3.28)], it can be shown that $Q^m - \exp\left(-k(n\pi/L)^2 t\right) = O(\Delta t)$. It is usually computed with fixed s. To improve a computation, we cut Δx in half (and thus Δt will be decreased by a factor of 4). Thus, if $n/N \ll 1$ with s fixed, numerical errors (at fixed x and t) are cut by a factor of 4 if the discretization step size Δx is halved (and time step Δt is cut in four). All computations should be done with s satisfying the stability condition.

However, difficulties may occur in practical computations if n/N is not small. *These are the highly spatially oscillatory solutions of the partial difference equation. For the heat equation these are the only waves that may cause difficulty.*

Lax equivalency theorem

The relationship between convergence and stability can be generalized. The **Lax equivalency theorem** states that for *consistent* finite difference approximations of time-dependent *linear* partial differential equations that are well posed, the numerical scheme converges if it is stable and it is stable if it converges.

A simplified determination of the stability condition

It is often convenient to analyze the stability of a numerical method quickly. From our analysis (based on the method of separation of variables), we have shown that there are special solutions to the difference equation that oscillate in x:

$$u_j^{(m)} = e^{i\alpha x} Q^{t/\Delta t}, \tag{6.3.30}$$

where

$$x = j\Delta x \quad \text{and} \quad t = m\Delta t.$$

From the boundary conditions α is restricted. Often, to simplify the stability analysis, we ignore the boundary conditions and allow α to be any value.[3] In this case, stability follows from (6.3.20) if $s \leq \frac{1}{2}$.

Random walk

The partial difference equation (6.3.16) may be put in the form

$$u_j^{(m+1)} = su_{j-1}^{(m)} + (1-2s)u_j^{(m)} + su_{j+1}^{(m)}. \tag{6.3.31}$$

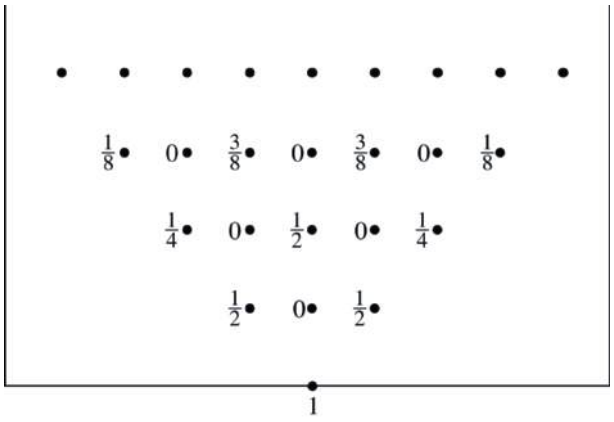
In the stable region, $s \leq \frac{1}{2}$; this may be interpreted as a probability problem known as a **random walk**. Consider a "drunkard" who in each unit of time Δt stands still or walks randomly to the left or to the right one step Δx. We do not know precisely where that person will be. We let $u_j^{(m)}$ be the probability that a drunkard is located at point j at time $m\Delta t$. We suppose that the person is equally likely to move one step Δx to the left or to the right in time Δt, with probability s each. Note that for this interpretation s must be less than or equal to $\frac{1}{2}$. The person cannot move further than Δx in time Δt; thus, the person stays still with probability $1-2s$. The probability that the person will be at $j\Delta x$ at the next time $(m+1)\Delta t$ is then given by (6.3.31): It is the sum of the probabilities of the three possible events. For example, the person might have been there at the previous time with probability $u_j^{(m)}$ and did not move with probability $1-2s$; the probability of this compound event is $(1-2s)u_j^{(m)}$. In addition, the person might have been one step to the left with probability $u_{j-1}^{(m)}$ [or right with probability $u_{j+1}^{(m)}$] and moved one step in the appropriate direction with probability s.

The largest time step for stable computations $s = \frac{1}{2}$ corresponds to a random walk problem with zero probability of standing still. If the initial position is known with certainty, then

$$u_j^{(0)} = \begin{cases} 1 & j = \text{initial known location} \\ 0 & j = \text{otherwise.} \end{cases}$$

Thereafter, the person moves to the left or right with probability $\frac{1}{2}$. This yields the binomial probability distribution as illustrated by Pascal's triangle (see Fig. 6.3.7).

Figure 6.3.7 Pascal's triangle.



6.3.5 SEPARATION OF VARIABLES FOR PARTIAL DIFFERENCE EQUATIONS AND ANALYTIC SOLUTIONS OF ORDINARY DIFFERENCE EQUATIONS

The partial difference equation can be analyzed by the same procedure we used for partial differential equations—we separate variables. We begin by assuming that (6.3.16) has special product solutions of the form

$$\boxed{u_j^{(m)} = \phi_j h_m.} \tag{6.3.32}$$

[3]Our more detailed stability analysis showed that the unstable waves occur only for very short wavelengths. For these waves, the boundary is perhaps expected to have the least effect.

By substituting (6.3.19) into (6.3.16), we obtain

$$\phi_j h_{m+1} = \phi_j h_m + s(\phi_{j+1} h_m - 2\phi_j h_m + \phi_{j-1} h_m).$$

Dividing this by $\phi_j h_m$ separates the variables:

$$\frac{h_{m+1}}{h_m} = 1 + s\left(\frac{\phi_{j+1} + \phi_{j-1}}{\phi_j} - 2\right) = +\lambda,$$

where λ is a separation constant.

The partial difference equation thus yields two ordinary difference equations. The difference equation in discrete time is of first order (meaning involving one difference):

$$\boxed{h_{m+1} = +\lambda h_m.} \tag{6.3.33}$$

The separation constant λ (as in partial differential equations) is determined by a boundary value problem, here, a second-order difference equation,

$$\boxed{\phi_{j+1} + \phi_{j-1} = -\left(\frac{-\lambda + 1 - 2s}{s}\right)\phi_j,} \tag{6.3.34}$$

with two homogeneous boundary conditions from (6.3.18):

$$\phi_0 = 0 \tag{6.3.35}$$

$$\phi_N = 0. \tag{6.3.36}$$

First-order difference equations

First-order linear homogeneous difference equations with constant coefficients, such as (6.3.33), are easy to analyze. Consider

$$h_{m+1} = \lambda h_m, \tag{6.3.37}$$

where λ is a constant. We simply note that

$$h_1 = \lambda h_0, \quad h_2 = \lambda h_1 = \lambda^2 h_0, \text{ and so on.}$$

Thus, the solution is

$$h_m = \lambda^m h_0, \tag{6.3.38}$$

where h_0 is an initial condition for the first-order difference equation.

An alternative way to obtain (6.3.38) is to assume that a homogeneous solution exists in the form $h_m = Q^m$. Substitution of this into (6.3.37) yields $Q^{m+1} = \lambda Q^m$ or $Q = \lambda$, rederiving (6.3.38). This latter technique is analogous to the substitution of e^{rt} into constant-coefficient homogeneous differential equations.

The solution (6.3.38) is sketched in Fig. 6.3.8 for various values of λ. Note that if $\lambda > 1$, then the solution exponentially grows [$\lambda^m = e^{m \log \lambda} = e^{(\log \lambda/\Delta t)t}$] since $m = t/\Delta t$]. If $0 < \lambda < 1$, the solution exponentially decays. Furthermore, if $-1 < \lambda < 0$, the solution has an oscillatory (and exponential) decay, known as a **convergent oscillation**. On the other hand, if $\lambda < -1$, the solution has a **divergent oscillation**.

In some situations we might even want to allow λ to be complex. Using the polar form of a complex number, $\lambda = re^{i\theta}$, $r = |\lambda|$ and $\theta = \arg \lambda$ (or angle), we obtain

$$\lambda^m = r^m e^{im\theta} = |\lambda|^m (\cos m\theta + i \sin m\theta). \tag{6.3.39}$$

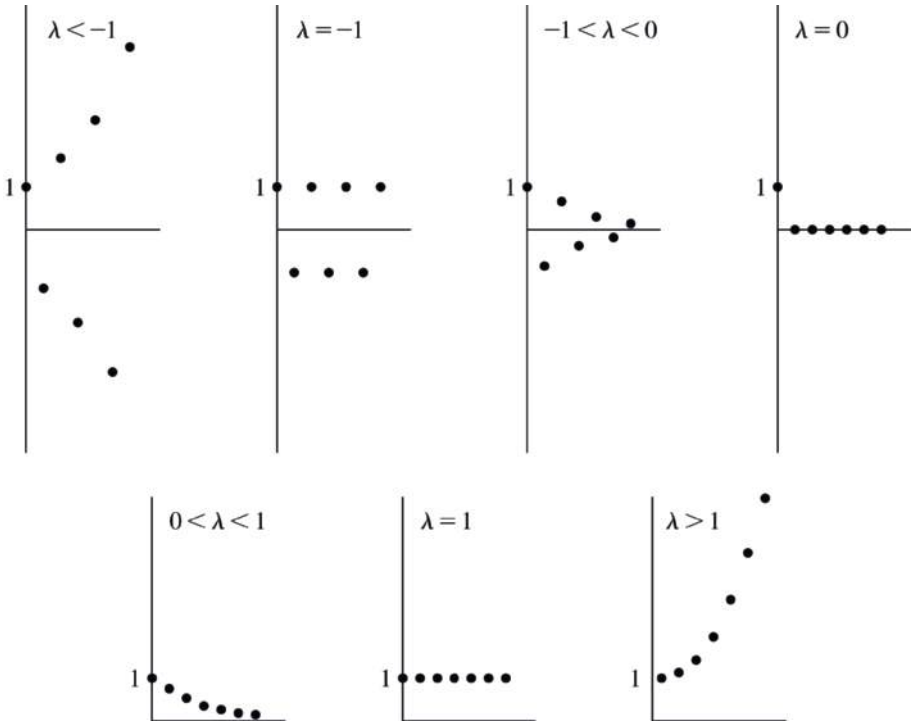
For example, the real part is $|\lambda|^m \cos m\theta$. As a function of m, λ^m oscillates (with period $m = 2\pi/\theta = 2\pi/\arg \lambda$). The solution grows in discrete time m if $|\lambda| > 1$ and decays if $|\lambda| < 1$.

We now can summarize (including the complex case). The solution λ^m of $h_{m+1} = \lambda h_m$ remains bounded as m increases (t increases) if $|\lambda| \leq 1$. It grows if $|\lambda| > 1$.

Second–order difference equations

Difference equation (6.3.34) has constant coefficients [since $(-\lambda + 1 - 2s)/s$ is independent of the step j]. An analytic solution can be obtained easily. For any constant-coefficient difference equation, homogeneous solutions may be obtained by substituting $\phi_j = Q^j$, as we could do for first-order difference equations [see (6.3.38)].

Figure 6.3.8 Solutions of first-order difference equations.



The boundary conditions, $\phi_0 = \phi_N = 0$, suggest that the solution may oscillate. This usually occurs if Q is complex with $|Q| = 1$, in which case an equivalent substitution is

$$\phi_j = (|Q|e^{i\theta})^j = e^{i\theta j} = e^{i\theta(x/\Delta x)} = e^{i\alpha x}, \tag{6.3.40}$$

since $j = x/\Delta x$, defining $\alpha = \theta/\Delta x = (\arg Q)/\Delta x$. In Exercise 6.3.2, it is shown that (6.3.34) implies that $|Q| = 1$, so that (6.3.40) may be used. Substituting (6.3.40) into (6.3.34) yields an equation for the wave number α:

$$e^{i\alpha\Delta x} + e^{-i\alpha\Delta x} = \frac{\lambda - 1 + 2s}{s}$$

or, equivalently,

$$\boxed{2\cos(\alpha\Delta x) = \frac{\lambda - 1 + 2s}{s}.} \tag{6.3.41}$$

This yields two values of α (one the negative of the other), and thus instead of $\phi_j = e^{i\alpha x}$ we use a linear combination of $e^{\pm i\alpha x}$, or

$$\phi_j = c_1 \sin\alpha x + c_2 \cos\alpha x. \tag{6.3.42}$$

The boundary conditions, $\phi_0 = \phi_N = 0$, imply that $c_2 = 0$ and $\alpha = n\pi/L$, where $n = 1, 2, 3, \ldots$. Thus,

$$\boxed{\phi_j = \sin\frac{n\pi x}{L} = \sin\frac{n\pi j\Delta x}{L} = \sin\frac{n\pi j}{N}.} \tag{6.3.43}$$

Further analysis follows that of the preceding subsection.

6.3.6 MATRIX NOTATION

Matrix[4] notation is often convenient for analyzing the discretization of partial differential equations. For fixed t, $u(x, t)$ is a function only of x. Its discretization $u_j^{(m)}$ is defined at each of the $N + 1$ mesh points (at every time step). We introduce a *vector* $\boldsymbol{u}$ of dimension $N + 1$ that changes

[4]This section requires some knowledge of linear algebra.

at each time step; it is a function of m, $\boldsymbol{u}^{(m)}$. The jth component of $\boldsymbol{u}^{(m)}$ is the value of $u(x, t)$ at the jth mesh point:

$$(\boldsymbol{u}^{(m)})_j = u_j^{(m)}. \tag{6.3.44}$$

The partial difference equation is

$$u_j^{(m+1)} = u_j^{(m)} + s\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right). \tag{6.3.45}$$

If we apply the boundary conditions, $u_0^{(m)} = u_N^{(m)} = 0$, then

$$u_1^{(m+1)} = u_1^{(m)} + s\left(u_2^{(m)} - 2u_1^{(m)} + u_0^{(m)}\right) = (1 - 2s)u_1^{(m)} + su_2^{(m)}.$$

A similar equation is valid for $u_{N-1}^{(m+1)}$. At each time step there are $N - 1$ unknowns. We introduce the $N - 1 \times N - 1$ **tridiagonal matrix** $\boldsymbol{A}$ with all entries zero except for the main diagonal (with entries $1 - 2s$) and neighboring diagonals (with entries s):

$$\boldsymbol{A} = \begin{bmatrix} 1-2s & s & 0 & 0 & 0 & 0 & 0 \\ s & 1-2s & s & 0 & 0 & 0 & 0 \\ 0 & s & 1-2s & s & 0 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & s & 1-2s & s & 0 \\ 0 & 0 & 0 & 0 & s & 1-2s & s \\ 0 & 0 & 0 & 0 & 0 & s & 1-2s \end{bmatrix}. \tag{6.3.46}$$

The partial difference equation becomes the following vector equation:

$$\boldsymbol{u}^{(m+1)} = \boldsymbol{A}\boldsymbol{u}^{(m)}. \tag{6.3.47}$$

The vector $\boldsymbol{u}$ changes in a straightforward way. We start with $\boldsymbol{u}^{(0)}$ representing the initial condition. By direct computation,

$$\begin{aligned} \boldsymbol{u}^{(1)} &= \boldsymbol{A}\boldsymbol{u}^{(0)} \\ \boldsymbol{u}^{(2)} &= \boldsymbol{A}\boldsymbol{u}^{(1)} = \boldsymbol{A}^2\boldsymbol{u}^{(0)}, \end{aligned}$$

and thus

$$\boldsymbol{u}^{(m)} = \boldsymbol{A}^m\boldsymbol{u}^{(0)}. \tag{6.3.48}$$

The matrix $\boldsymbol{A}$ raised to the mth power describes how the initial condition influences the solution at the mth time step ($t = m\Delta t$).

To understand this solution, we introduce the **eigenvalues** μ of the matrix $\boldsymbol{A}$, the values μ such that there are nontrivial vector solutions $\boldsymbol{\xi}$:

$$\boldsymbol{A}\boldsymbol{\xi} = \mu\boldsymbol{\xi}. \tag{6.3.49}$$

The eigenvalues satisfy

$$\det[\boldsymbol{A} - \mu\boldsymbol{I}] = 0, \tag{6.3.50}$$

where $\boldsymbol{I}$ is the identity matrix. Nontrivial vectors $\boldsymbol{\xi}$ that satisfy (6.3.49) are called **eigenvectors** corresponding to μ. Since $\boldsymbol{A}$ is an $(N - 1) \times (N - 1)$ matrix, $\boldsymbol{A}$ has $N - 1$ eigenvalues. However, some of the eigenvalues may not be distinct, there may be multiple eigenvalues (or **degeneracies**). For a distinct eigenvalue, there is a unique eigenvector (to within a multiplicative constant); in the case of a multiple eigenvalue (of multiplicity k), there may be at most k linearly independent eigenvectors. If for some eigenvalue there are less than k eigenvectors, we say the matrix is **defective**. If $\boldsymbol{A}$ is real and symmetric [as (6.3.46) is], it is known that any possible multiple eigenvalues are *not* defective. Thus, the matrix $\boldsymbol{A}$ has $N - 1$ eigenvectors (which can be shown to be linearly independent). Furthermore, if $\boldsymbol{A}$ is real and symmetric, the eigenvalues (and consequently the eigenvectors) are real and the eigenvectors are orthogonal (see Section 5.5 Appendix). We let μ_n be the nth eigenvalue and $\boldsymbol{\xi}_n$ the corresponding eigenvector.

We can solve vector equation (6.3.47) (equivalent to the partial difference equation) using the method of eigenvector expansion. (This technique is analogous to using an eigenfunction expansion to solve the partial differential equation.) Any vector can be expanded in a series of the eigenvectors:

$$\boldsymbol{u}^{(m)} = \sum_{n=1}^{N-1} c_n^{(m)}\boldsymbol{\xi}_n. \tag{6.3.51}$$

The vector $\boldsymbol{u}$ changes with m (time), and thus the constants $c_n^{(m)}$ depend on m:

$$\boldsymbol{u}^{(m+1)} = \sum_{n=1}^{N-1} c_n^{(m+1)} \boldsymbol{\xi}_n. \tag{6.3.52}$$

However, from (6.3.47),

$$\boldsymbol{u}^{(m+1)} = \boldsymbol{A}\boldsymbol{u}^{(m)} = \sum_{n=1}^{N-1} c_n^{(m)} \boldsymbol{A}\boldsymbol{\xi}_n = \sum_{n=1}^{N-1} c_n^{(m)} \mu_n \boldsymbol{\xi}_n, \tag{6.3.53}$$

where (6.3.51) and (6.3.49) have been used. By comparing (6.3.52) and (6.3.53), we determine a first-order difference equation with constant coefficients for $c_n^{(m)}$:

$$c_n^{(m+1)} = \mu_n c_n^{(m)}. \tag{6.3.54}$$

This is easily solved,

$$c_n^{(m)} = c_n^{(0)} (\mu_n)^m, \tag{6.3.55}$$

and thus

$$\boxed{\boldsymbol{u}^{(m)} = \sum_{n=1}^{N-1} c_n^{(0)} (\mu_n)^m \boldsymbol{\xi}_n.} \tag{6.3.56}$$

$c_n^{(0)}$ can be determined from the initial condition.

From (6.3.56), the growth of the solution as t increases (m increases) depends on $(\mu_n)^m$, where $m = t/\Delta t$. We recall that since μ_n is real,

$$(\mu_n)^m = \begin{cases} \text{exponential growth} & \mu_n > 1 \\ \text{exponential decay} & 0 < \mu_n < 1 \\ \text{convergent oscillation} & -1 < \mu_n < 0 \\ \text{divergent oscillation} & \mu_n < -1. \end{cases}$$

This numerical solution is unstable if any eigenvalue $\mu_n > 1$ or any $\mu_n < -1$.

We need to obtain the $N-1$ eigenvalues μ of $\boldsymbol{A}$:

$$\boldsymbol{A}\boldsymbol{\xi} = \mu\boldsymbol{\xi}. \tag{6.3.57}$$

We let ξ_j be the jth component of $\boldsymbol{\xi}$. Since $\boldsymbol{A}$ is given by (6.3.46), we can rewrite (6.3.57) as

$$s\xi_{j+1} + (1-2s)\xi_j + s\xi_{j-1} = \mu\xi_j \tag{6.3.58}$$

with

$$\xi_0 = 0 \text{ and } \xi_N = 0. \tag{6.3.59}$$

Equation (6.3.58) is equivalent to

$$\xi_{j+1} + \xi_{j-1} = \left(\frac{\mu + 2s - 1}{s}\right)\xi_j. \tag{6.3.60}$$

By comparing (6.3.60) with (6.3.34), we observe that the eigenvalues μ of $\boldsymbol{A}$ are the eigenvalues λ of the second-order difference equation obtained by separation of variables. Thus [see (6.3.20)],

$$\mu = 1 - 2s(1 - \cos(\alpha\Delta x)), \tag{6.3.61}$$

where $\alpha = n\pi/L$ for $n = 1, 2, \ldots, N-1$. As before, the scheme is usually unstable if $s > \frac{1}{2}$. To summarize this simple case, the eigenvalues can be explicitly determined using Fourier-type analysis.

In more difficult problems it is rare that the eigenvalues of large matrices can be obtained easily. Sometimes the **Gershgorin circle theorem** (see Strang [1993] for an elementary proof) is useful: **Every eigenvalue of $\boldsymbol{A}$ lies in at least one of the circles $\boldsymbol{c_1, \ldots, c_{N-1}}$ in the complex plane where $\boldsymbol{c_i}$ has its center at the $\boldsymbol{i}$th diagonal entry and its radius equal to the sum of the absolute values of the rest of that row**. If a_{ij} are the entries of $\boldsymbol{A}$, then all eigenvalues μ lie in at least one of the following circles:

$$|\mu - a_{ii}| \leq \sum_{\substack{j=1 \\ (j \neq i)}}^{N-1} |a_{ij}|. \tag{6.3.62}$$

For our matrix $\boldsymbol{A}$, the diagonal elements are all the same $1 - 2s$ and the rest of the row sums to $2s$ (except the first and last rows, which sum to s). Thus, two circles are $|\mu - (1 - 2s)| < s$ and the other $N - 3$ circles are

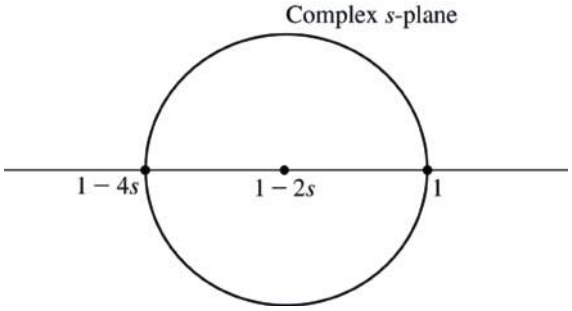
$$|\mu - (1 - 2s)| < 2s. \tag{6.3.63}$$

All eigenvalues lie in the resulting regions [the biggest of which is given by (6.3.63)], as sketched in Fig. 6.3.9. Since the eigenvalues μ are also known to be real, Fig. 6.3.9 shows that

$$1 - 4s \leq \mu \leq 1.$$

Stability is guaranteed if $-1 \leq \mu \leq 1$, and thus, the Gershgorin circle theorem implies that the numerical scheme is stable if $s \leq \frac{1}{2}$. If $s > \frac{1}{2}$, the Gershgorin circle theorem does *not* imply the scheme is unstable.

Figure 6.3.9 Gershgorin circles for A corresponding to discretization of the heat equation.



6.3.7 NONHOMOGENEOUS PROBLEMS

The heat equation with sources may be computed in the same way. Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x, t) \\ u(0, t) &= A(t) \\ u(L, t) &= B(t) \\ u(x, 0) &= f(x). \end{aligned}$$

As before, we use a forward difference in time and a centered difference in space. We obtain the following numerical approximation:

$$\begin{aligned} \frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} &= \frac{k}{(\Delta x)^2}\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right) + Q(j\,\Delta x, m\Delta t) \\ u_0^{(m)} &= A(m\Delta t) \\ u_N^{(m)} &= B(m\Delta t) \\ u_j^{(0)} &= f(j\Delta x). \end{aligned}$$

The solution is easily computed by solving for $u_j^{(m+1)}$. We claim that our stability analysis for homogeneous problems is valid for nonhomogeneous problems. Thus, we compute with $s = k\Delta t/(\Delta x)^2 \leq \frac{1}{2}$.

6.3.8 OTHER NUMERICAL SCHEMES

The numerical scheme for the heat equation, which uses the centered difference in space and forward difference in time, is stable if $s = k\Delta t/(\Delta x)^2 \leq \frac{1}{2}$. The time step is small [being proportional to $(\Delta x)^2$]. We might desire a less expensive scheme. The truncation error is the sum of terms, one proportional to Δt and the other to $(\Delta x)^2$. If s is fixed (for example, $s = \frac{1}{2}$), both errors are $O(\Delta x)^2$ since $\Delta t = s(\Delta x)^2/k$.

Richardson's scheme

For a less expensive scheme, we might try a more accurate time difference. Using centered differences in both space and time was first proposed by Richardson in 1927:

$$\frac{u_j^{(m+1)} - u_j^{(m-1)}}{\Delta t} = \frac{k}{(\Delta x)^2}\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right) \tag{6.3.64}$$

or

$$u_j^{(m+1)} = u_j^{(m-1)} + s\left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}\right), \tag{6.3.65}$$

where again $s = k\Delta t/(\Delta x)^2$. Here, the truncation error is the sum of a $(\Delta t)^2$ and $(\Delta x)^2$ terms. Although in some sense this scheme is more accurate than the previous one, (6.3.65) should never be used. Exercise 6.3.12(a) shows that this numerical method is always unstable.

Crank–Nicolson scheme

Crank and Nicolson (in 1947) suggested an alternative way to utilize centered differences. The forward difference in time

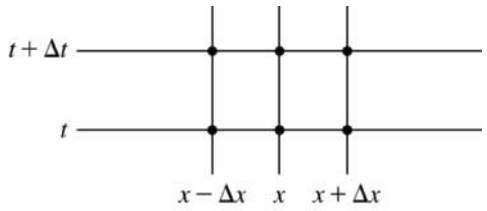
$$\frac{\partial u}{\partial t} \approx \frac{u(t + \Delta t) - u(t)}{\Delta t}$$

may be interpreted as the centered difference around $t + \Delta t/2$. The error in approximating $\partial u/\partial t(t + \Delta t/2)$ is $O(\Delta t)^2$. Thus, we discretize the second derivative at $t + \Delta t/2$ with a centered difference scheme. Since this involves functions evaluated at this in-between time, we take the average at t and $t + \Delta t$. This yields the Crank–Nicolson scheme,

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} = \frac{k}{2}\left[\frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{(\Delta x)^2} + \frac{u_{j+1}^{(m+1)} - 2u_j^{(m+1)} + u_{j-1}^{(m+1)}}{(\Delta x)^2}\right]. \tag{6.3.66}$$

It is not obvious, but nevertheless true (Exercise 6.3.13), that the truncation error remains the sum of two terms, one $(\Delta x)^2$ and the other $(\Delta t)^2$. The advantage of the Crank–Nicolson method is that the scheme is stable for all $s = k\Delta t/(\Delta x)^2$, as is shown in Exercise 6.3.12(b). Δt can be as large as desired. We can choose Δt to be proportional to Δx [rather than $(\Delta x)^2$]. The error is then $O(\Delta x)^2$, an equivalent accuracy as the earlier scheme with much less work (computing). Crank–Nicolson is a practical method. However, the Crank–Nicolson scheme (see Fig. 6.3.10) involves six points (rather than four for the simpler stable method), three of which are at the advanced time. We cannot directly march forward in time with (6.3.66). Instead, to advance one time step, (6.3.66) requires the solution of a linear system of $N - 1$ equations. The scheme (6.3.66) is called **implicit** [while (6.3.13) is called **explicit**]. The matrices involved are tridiagonal, and thus the linear system may be solved using Gauss elimination easily (and relatively inexpensively) even if N is large.

Figure 6.3.10 Implicit Crank–Nicolson scheme.



6.3.9 OTHER TYPES OF BOUNDARY CONDITIONS

If $\partial u/\partial x = g(t)$ at $x = 0$ (rather than u being given at $x = 0$), then we must introduce a numerical approximation for the boundary condition. Since the discretization of the partial differential equation has an $O(\Delta x)^2$ truncation error, we may introduce an equal error in the boundary condition by using a centered difference in space:

$$\frac{\partial u}{\partial x} \approx \frac{u(x + \Delta x, t) - u(x - \Delta x, t)}{2\Delta x}.$$

In this case the boundary condition $\partial u/\partial x = g(t)$ at $x = 0$ becomes

$$\frac{u_1^{(m)} - u_{-1}^{(m)}}{2\Delta x} = g(t) = g(m\Delta t) = g_m. \tag{6.3.67}$$

We use (6.3.67) to obtain an expression for the temperature at the *fictitious* point ($x_{-1} = -\Delta x$):

$$u_{-1}^{(m)} = u_1^{(m)} - 2\Delta x g_m. \tag{6.3.68}$$

In this way we determine the value at the fictitious point initially, $u_{-1}^{(0)}$. This fictitious point is needed to compute the boundary temperature at later times via the partial difference equation. If we use forward difference in time and centered difference in space, (6.3.16) can now be applied for $j = 0$ to $j = N - 1$. For example, at $x = 0$ ($j = 0$),

$$\begin{aligned} u_0^{(m+1)} &= u_0^{(m)} + s\left(u_1^{(m)} - 2u_0^{(m)} + u_{-1}^{(m)}\right) \\ &= u_0^{(m)} + s\left(u_1^{(m)} - 2u_0^{(m)} + u_1^{(m)} - 2\Delta x g_m\right), \end{aligned}$$

where (6.3.68) has been used. In this way a partial differential equation can be solved numerically with boundary conditions involving the derivative. (The fictitious point is eliminated between the boundary condition and the partial differential equation.)

Exercises 6.3

6.3.1. **(a)** Show that the truncation error for our numerical scheme, (6.3.3), becomes much smaller if $k(\Delta t)/(\Delta x)^2 = \frac{1}{6}$. [*Hint*: u satisfies the partial differential equation in (6.3.1).]

(b) If $k\Delta t/(\Delta x)^2 = \frac{1}{6}$, determine the order of magnitude of the truncation error.

6.3.2. By letting $\phi_j = Q^j$, show that (6.3.34) is satisfied only if $|Q| = 1$. [*Hint*: First show that

$$Q^2 + \left(\frac{-\lambda + 1 - 2s}{s}\right)Q + 1 = 0.\Big]$$

6.3.3. Define $L(\phi) = \phi_{j+1} + \phi_{j-1} + \gamma\phi_j$.

(a) Show that $uL(v) - vL(u) = w_{j+1} - w_j$, where $w_j = u_{j-1}v_j - v_{j-1}u_j$.

(b) Since summation is analogous to integration, derive the discrete version of Green's formula,

$$\sum_{i=1}^{N-1} [uL(v) - vL(u)] = w_N - w_0.$$

(c) Show that the right-hand side of part (b) vanishes if both u and v satisfy the homogeneous boundary conditions (6.3.18).

(d) Letting $\gamma = (1 - 2s)/s$, the eigenfunctions ϕ satisfy $L(\phi) = (\lambda/s)\phi$. Show that eigenfunctions corresponding to different eigenvalues are orthogonal in the sense that

$$\sum_{i=1}^{N-1} \phi_i \psi_i = 0.$$

***6.3.4.** **(a)** Using Exercise 6.3.3, determine β_n in (6.3.24) from the initial conditions $u_j^{(0)} = f_j$.

(b) Evaluate the normalization constant

$$\sum_{j=1}^{N-1} \sin^2 \frac{n\pi j}{N}$$

for each eigenfunction (i.e., fix n). (*Hint*: Use the double-angle formula and a geometric series.)

6.3.5. Show that at each successive mesh point the sign of the solution alternates for the most unstable mode (of our numerical scheme for the heat equation, $s > \frac{1}{2}$).

6.3.6. Evaluate $1/[1 - \cos(N - 1)\pi/N]$. What conclusions concerning stability do you reach?

(a) $N = 4$ **(b)** $N = 6$ **(c)** $N = 8$ ***(d)** $N = 10$

(e) Asymptotically for large N

6.3.7. Numerically compute solutions to the heat equation with the temperature initially given in Fig. 6.3.4. Use (6.3.16)–(6.3.18) with $N = 10$. Do for various s (discuss stability):

(a) $s = 0.49$ **(b)** $s = 0.50$ **(c)** $s = 0.51$ **(d)** $s = 0.52$

6.3.8. Under what condition will an initially positive solution $[u(x, 0) > 0]$ remain positive $[u(x, t) > 0]$ for our numerical scheme (6.3.9) for the heat equation?

6.3.9. Consider

$$\frac{d^2u}{dx^2} = f(x) \qquad \text{with} \qquad u(0) = 0 \qquad \text{and} \qquad u(L) = 0.$$

(a) Using the centered difference approximation for the second derivative and dividing the length L into three equal mesh lengths (see Section 6.3.2), derive a system of linear equations for an approximation to $u(x)$. Use the notation $x_i = i\Delta x, f_i = f(x_i)$, and $u_i = u(x_i)$. (*Note:* $x_0 = 0$, $x_1 = \frac{1}{3}L$, $x_2 = \frac{2}{3}L$, $x_3 = L$.)

***(b)** Write the system as a matrix system $\boldsymbol{Au} = \boldsymbol{f}$. What is $\boldsymbol{A}$?

(c) Solve for u_1 and u_2.

(d) Show that a "Green's function" matrix $\boldsymbol{G}$ can be defined:

$$u_i = \sum_j G_{ij} f_j \quad (\boldsymbol{u} = \boldsymbol{Gf}).$$

What is $\boldsymbol{G}$? Show that it is symmetric, $G_{ij} = G_{ji}$.

6.3.10. Suppose that in a random walk, at each Δt the probability of moving to the right Δx is a and the probability of moving to the left Δx is also a. The probability of staying in place is $b(2a + b = 1)$.

(a) Formulate the difference equation for this problem.

***(b)** Derive a partial differential equation governing this process $\Delta x \to 0$ and $\Delta t \to 0$ such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{\Delta t} = \frac{k}{s}.$$

(c) Suppose that there is a wall (or cliff) on the right at $x = L$ with the property that *after* the wall is reached, the probability of moving to the left is a, to the right c, and for staying in place $1 - a - c$. Assume that no one returns from $x > L$. What condition is satisfied at the wall? What is the resulting boundary condition for the partial differential equation? (Let $\Delta x \to 0$ and $\Delta t \to 0$ as before.) Consider the two cases $c = 0$ and $c \neq 0$.

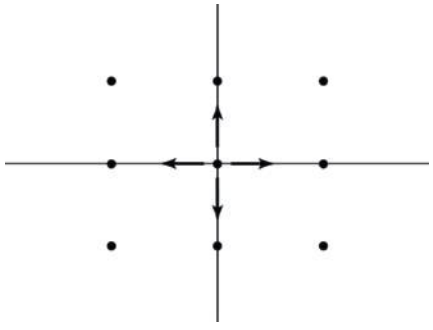
6.3.11. Suppose that, in a two-dimensional random walk, at each Δt it is equally likely to move right Δx or left or up Δy or down (as illustrated in Fig. 6.3.11).

(a) Formulate the difference equation for this problem.

(b) Derive a partial differential equation governing this process if $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta t \to 0$ such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{\Delta t} = \frac{k_1}{s} \qquad \text{and} \qquad \lim_{\substack{\Delta y \to 0 \\ \Delta t \to 0}} \frac{(\Delta y)^2}{\Delta t} = \frac{k_2}{s}.$$

Figure 6.3.11 Figure for Exercise 6.3.11.



6.3.12. Use a simplified determination of stability [i.e., substitute $u_j^{(m)} = e^{i\alpha x}Q^{t/\Delta t}$], to investigate the following:

(a) Richardson's centered difference in space and time scheme for the heat equation, (6.3.65)

(b) Crank–Nicolson scheme for the heat equation, (6.3.66)

6.3.13. Investigate the truncation error for the Crank–Nicolson method, (6.3.66).

6.3.14. For the following matrices,

1. Compute the eigenvalue.

2. Compute the Gershgorin row circles.

3. Compare (1) and (2) according to the theorem.

(a) $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ ***(c)** $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 0 & \frac{1}{3} & 2 \end{bmatrix}$

6.3.15. For the examples in Exercise 6.3.14, compute the Gershgorin (column) circles. Show that a corresponding theorem is valid for them.

6.3.16. Using forward differences in time and centered differences in space, analyze carefully the stability of the difference scheme if the boundary condition for the heat equation is

$$\frac{\partial u}{\partial x}(0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L) = 0.$$

[*Hint*: See Section 6.3.9.] Compare your result to the one for the boundary conditions $u(0) = 0$ and $u(L) = 0$.

6.3.17. Solve on a computer [using (6.3.9)] the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$ with $u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = \sin \pi x$ with $\Delta x = 1/100$. Compare to analytic solution at $x = 1/2$, $t = 1$ by computing the error there (the difference between the analytic solution and the numerical solution). Pick Δt so that

(a) $s = 0.4$

(b) $s = 0.6$

(c) to improve the calculation in part (a), let $\Delta x = 1/200$ but keep $s = 0.4$

(d) compare the errors in parts (a) and (c)

6.3.18. Write the Crank–Nicolson scheme (6.3.66) as a matrix system. Let $N = 5$, so that the unkowns are $u_1^{(m+1)}$, $u_2^{(m+1)}$, $u_3^{(m+1)}$, and $u_4^{(m+1)}$. Let $\boldsymbol{u}^{(m+1)}$ be the vector with those four unknowns as components. If $\boldsymbol{A}\boldsymbol{u}^{(m+1)} = \boldsymbol{f}(\boldsymbol{u}^{(m)})$, then what is the 4×4 matrix $\boldsymbol{A}$ and the 4-component vector $\boldsymbol{f}$? Express $\boldsymbol{A}$ and $\boldsymbol{f}$ in terms of s wherever possible. Assume homogeneous boundary conditions $u(0, t) = 0, u(L, t) = 0$.

6.4 Two-Dimensional Heat Equation

Similar ideas may be applied to numerically compute solutions of the two-dimensional heat equation

$$\boxed{\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).}$$

We introduce a two-dimensional mesh (or lattice), where for convenience we assume that $\Delta x = \Delta y$. Using a forward difference in time and the formula for the Laplacian based on a centered difference in both x and y [see (6.2.17)], we obtain

$$\boxed{\frac{u_{j,l}^{(m+1)} - u_{j,l}^{(m)}}{\Delta t} = \frac{k}{(\Delta x)^2}\left[u_{j+1,l}^{(m)} + u_{j-1,l}^{(m)} + u_{j,l+1}^{(m)} + u_{j,l-1}^{(m)} - 4u_{j,l}^{(m)}\right],} \tag{6.4.1}$$

where $u_{j,l}^{(m)} \approx u(j\Delta x, l\Delta y, m\Delta t)$. We march forward in time using (6.4.1).

STABILITY ANALYSIS

As before, the numerical scheme may be unstable. We perform a simplified stability analysis, and thus ignore the boundary conditions. We investigate possible growth of spatially periodic waves by substituting

$$u_{j,l}^{(m)} = Q^{t/\Delta t} e^{i(\alpha x + \beta y)} \tag{6.4.2}$$

into (6.4.1). We immediately obtain

$$\begin{aligned} Q &= 1 + s(e^{i\alpha\Delta x} + e^{-i\alpha\Delta x} + e^{i\beta\Delta y} + e^{-i\beta\Delta y} - 4) \\ &= 1 + 2s(\cos\alpha\Delta x + \cos\beta\Delta y - 2), \end{aligned}$$

where $s = k\Delta t/(\Delta x)^2$ and $\Delta x = \Delta y$. To ensure stability, $-1 < Q < 1$, and hence we derive the stability condition for the two-dimensional heat equation

$$s = \frac{k\Delta t}{(\Delta x)^2} \le \frac{1}{4}. \tag{6.4.3}$$

EXAMPLE

As an elementary example, yet one in which an exact solution is not available, we consider the heat equation on an L-shaped region sketched in Fig. 6.4.1. We assume that the temperature is initially zero. Also, on the boundary $u = 1000$ at $x = 0$, but $u = 0$ on the rest of the boundary. We compute with the largest stable time step, $s = \frac{1}{4}\left[\Delta t = (\Delta x)^2/4k\right]$, so that (6.4.1) becomes

$$u_{j,l}^{(m+1)} = \frac{\left[u_{j+1,l}^{(m)} + u_{j-1,l}^{(m)} + u_{j,l+1}^{(m)} + u_{j,l-1}^{(m)}\right]}{4}. \tag{6.4.4}$$

In this numerical scheme, the temperature at the next time is the average of the four neighboring mesh (or lattice) points at the present time. We choose $\Delta x = \frac{1}{10}(\Delta t = 1/400k)$ and sketch the numerical solution in Figs. 6.4.2 and 6.4.3. We draw contours of approximately equal temperature in order to observe the thermal energy invading the interior of this region.

The partial difference equation is straightforward to apply if the boundary is composed entirely of mesh points. Usually, this is unlikely, in which case some more complicated procedure must be applied for the boundary condition.

Figure 6.4.1 Initial condition.

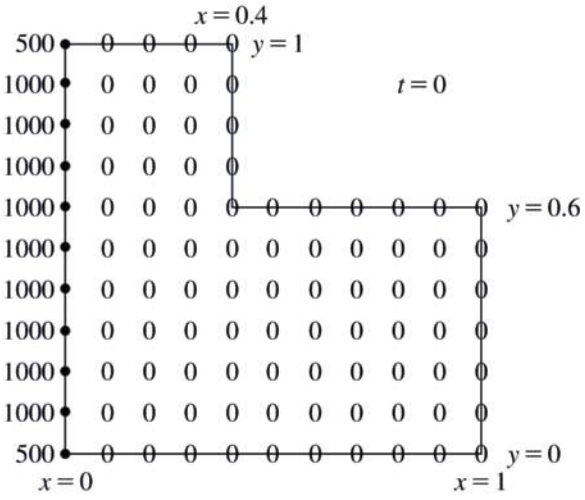


Figure 6.4.2 Numerical computation of temperature in an L-shaped region.

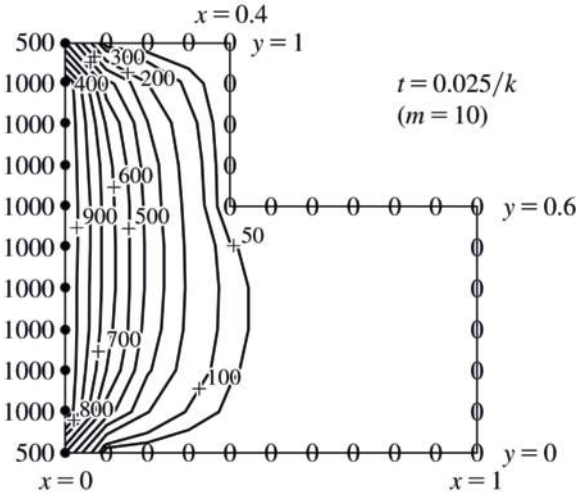
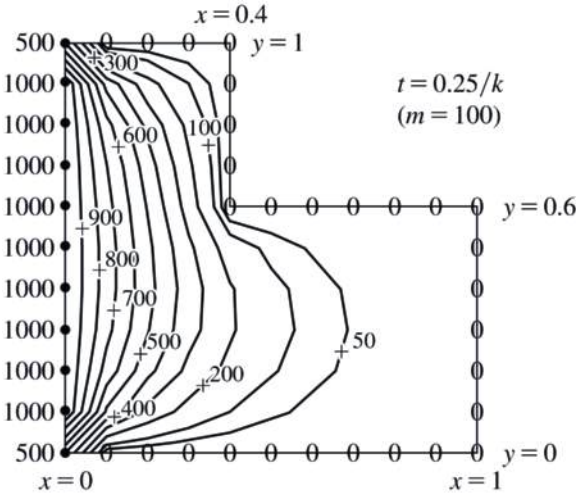


Figure 6.4.3 Numerical computation of temperature in an L-shaped region.



MATLAB Files

Download Animation of Figure 6.4.3 (.zip)

https://bit.ly/3xLxfAQ

These files are provided as zipped .m files for use in MATLAB on desktop platform.

Exercises 6.4

***6.4.1.** Derive the stability condition (ignoring the boundary condition) for the two-dimensional heat equation if $\Delta x \neq \Delta y$.

6.4.2. Derive the stability condition (including the effect of the boundary condition) for the two-dimensional heat equation with $u(x, y) = 0$ on the four sides of a square if $\Delta x = \Delta y$.

6.4.3. Derive the stability condition (ignoring the boundary condition) for the three-dimensional heat equation if $\Delta x = \Delta y = \Delta z$.

6.4.4. Solve numerically the heat equation on a rectangle $0 < x < 1$, $0 < y < 2$ with the temperature initially zero. Assume that the boundary conditions are zero on three sides, but $u = 1$ on one long side.

6.5 Wave Equation

We may also compute solutions to the one-dimensional wave equation by introducing finite difference approximations. Using **centered differences in space and time**, the wave equation,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},} \tag{6.5.1}$$

becomes the following partial difference equation:

$$\boxed{\frac{u_j^{(m+1)} - 2u_j^{(m)} + u_j^{(m-1)}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{(\Delta x)^2}.} \tag{6.5.2}$$

The truncation error is the sum of a term of $O(\Delta x)^2$ and one of $O(\Delta t)^2$. By solving (6.5.2) for $u_j^{(m+1)}$, the solution may be marched forward in time. Three levels of time are involved in (6.5.2), as indicated in Fig. 6.5.1. $u(x, t)$ is needed "initially" at two values of $t(0$ and $-\Delta t)$ to start the calculation. We use the two initial conditions for the wave equation, $u(x, 0) = f(x)$ and $\partial u/\partial t(x, 0) = g(x)$, to compute at $m = 0$ and at $m = -1$. Using a centered difference in time for $\partial u/\partial t$ [so as to maintain an $O(\Delta t)^2$ truncation error] yields

$$u_j^{(0)} = f(x_j) = f(j\Delta x) \tag{6.5.3}$$

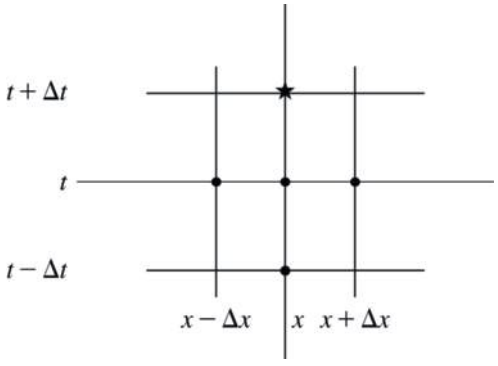
$$\frac{u_j^{(1)} - u_j^{(-1)}}{2\Delta t} = g(x_j) = g(j\Delta x). \tag{6.5.4}$$

To begin the calculation we must compute $u_j^{(-1)}$. The initial conditions, (6.5.3) and (6.5.4), are two equations in three unknowns, $u_j^{(-1)}$, $u_j^{(0)}$, and $u_j^{(1)}$. The partial difference equation at $t = 0$ provides a third equation:

$$u_j^{(1)} = 2u_j^{(0)} - u_j^{(-1)} + \frac{c^2}{(\Delta x/\Delta t)^2}\left(u_{j-1}^{(0)} - 2u_j^{(0)} + u_{j+1}^{(0)}\right). \tag{6.5.5}$$

$u_j^{(1)}$ may be eliminated from (6.5.4) and (6.5.5) since $u_j^{(0)}$ is known from (6.5.3). In this way we can solve for $u_j^{(-1)}$. Once $u_j^{(-1)}$ and $u_j^{(0)}$ are known, the later values of u may be computed via (6.5.2). Boundary conditions may be analyzed as before.

Figure 6.5.1 Marching forward in time for the wave equation.



STABILITY

Our limited experience should already suggest that a stability analysis is important. To determine if any spatially periodic waves grow, we substitute

$$u_j^{(m)} = Q^{t/\Delta t} e^{i\alpha x} \tag{6.5.6}$$

into (6.5.2), yielding

$$Q - 2 + \frac{1}{Q} = \sigma, \tag{6.5.7}$$

where

$$\sigma = \frac{c^2}{(\Delta x/\Delta t)^2}(e^{i\alpha\Delta x} - 2 + e^{-i\alpha\Delta x}) = \frac{2c^2}{(\Delta x/\Delta t)^2}[\cos(\alpha\Delta x) - 1]. \tag{6.5.8}$$

Equation (6.5.7) is a quadratic equation for Q since (6.5.2) involves three time levels and two time differences:

$$Q^2 - (\sigma + 2)\, Q + 1 = 0 \text{ and thus } Q = \frac{\sigma + 2 \pm \sqrt{(\sigma + 2)^2 - 4}}{2}. \tag{6.5.9}$$

The two roots correspond to two ways in which waves evolve in time. If $-2 < \sigma + 2 < 2$, the roots are complex conjugates of each other. In this case (as discussed earlier)

$$Q^m = (re^{i\theta})^m = r^m e^{im\theta},$$

where $r = |Q|$ and $\theta = \arg Q$. Since

$$|Q|^2 = \frac{(\sigma + 2)^2}{4} + \frac{4 - (\sigma + 2)^2}{4} = 1,$$

the solution oscillates for fixed x as m (time) increases if $-2 < \sigma + 2 < 2$. This is similar to the wave equation itself, which permits time-periodic solutions when the spatial part is periodic (e.g., $\sin n\pi x/L \cos n\pi ct/L$). If $\sigma + 2 > 2$ or $\sigma + 2 < -2$, the roots are real with product equal to 1 [see (6.5.9)]. Thus, one root will be greater than 1 in absolute value, giving rise to an unstable behavior.

The solution will be stable if $-2 < \sigma + 2 < 2$ or $-4 < \sigma < 0$. From (6.5.8) we conclude that our numerical scheme is stable if

$$\boxed{\frac{c}{\Delta x/\Delta t} \leq 1,} \tag{6.5.10}$$

known as the **Courant stability condition** (for the wave equation). Here, c is the speed of propagation of signals for the wave equation and $\Delta x/\Delta t$ is the speed of propagation of signals for the discretization of the wave equation. Thus, we conclude that for stability, **the numerical scheme must have a greater propagation speed than the wave equation itself**. In this way the numerical scheme will be able to account for the real signals being propagated. The stability condition again limits the size of the time step, in this case

$$\Delta t \leq \frac{\Delta x}{c}. \tag{6.5.11}$$

Convergence

The time-dependent part oscillates if $-2 < \sigma + 2 < 2$ (the criterion for stability):

$$Q^{t/\Delta t} = e^{i\frac{t}{\Delta t}\arg Q.} \tag{6.5.12}$$

We note that if $\alpha\Delta x$ is small (where α is the wave number, usually $n\pi/L$), then $\cos(\alpha\Delta x) \approx 1 - \frac{1}{2}(\alpha\Delta x)^2$ from its Taylor series. Thus, the parameter σ is small and negative:

$$\sigma \approx -\frac{1}{2}(\alpha\Delta x)^2 \frac{2c^2}{(\Delta x/\Delta t)^2}.$$

In this case the temporal frequency of the solutions of the partial difference equations is

$$\frac{\arg Q}{\Delta t} = \frac{\tan^{-1}\left(\dfrac{\sqrt{4 - (\sigma + 2)^2}}{\sigma + 2}\right)}{\Delta t} \approx \frac{\left(\dfrac{\sqrt{4 - (\sigma + 2)^2}}{\sigma + 2}\right)}{\Delta t} \approx \frac{\left(\sqrt{-\sigma}\right)}{\Delta t} \approx c\alpha, \tag{6.5.13}$$

since σ is very small and negative and $\tan\phi \approx \phi$ for small angles ϕ. Equation (6.5.13) shows that the temporal frequency of the partial difference equation approaches the frequency of the partial differential equation. Perhaps this is clearer by remembering that α is the wave number, usually $n\pi/L$. Thus, if $\alpha\Delta x$ is small, the partial difference equation has solutions that are approximately the same as the partial differential equation:

$$Q^{t/\Delta t}e^{i\alpha x} = e^{i\frac{1}{\Delta t}\arg Q}e^{i\alpha x} \approx e^{i\alpha x}e^{ic\alpha t}.$$

It can also be shown that the error (difference between the solutions of the partial difference and differential equations at fixed x and t) is $O(\Delta t)$ if $\alpha\Delta x$ is small. It is usual to compute with fixed $\Delta x/\Delta t$. To improve a computation, we cut Δx in half (and thus Δt will be decreased by a factor of 2). Thus, if $\alpha\Delta x \ll 1$ with $\Delta x/\Delta t$ fixed, numerical errors (at fixed x and t) are cut in half if the discretization step size Δx is halved (and time step Δt is halved). All computations should be done satisfying the Courant stability condition (6.5.10).

Exercises 6.5

6.5.1. Modify the Courant stability condition for the wave equation to account for the boundary condition $u(0) = 0$ and $u(L) = 0$.

6.5.2. Consider the wave equation subject to the initial conditions

$$u(x, 0) = \begin{cases} 1 & \frac{L}{4} < x < \frac{3L}{4} \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0$$

and the boundary conditions

$$u(0, t) = 0$$
$$u(L, t) = 0.$$

Use nine interior mesh points and compute using centered differences in space and time. Compare to the exact solution.

(a) $\Delta t = \Delta x/2c$ **(b)** $\Delta t = \Delta x/c$ **(c)** $\Delta t = 2\Delta x/c$

6.5.3. For the wave equation, $u(x, t) = f(x - ct)$ is a solution, where f is an arbitrary function. If $c = \Delta x/\Delta t$, show that $u_j^{(m)} = f(x_j - ct_m)$ is a solution of (6.5.2) for arbitrary f.

6.5.4. Show that the conclusion of Exercise 6.5.3 is not valid if $c \neq \Delta x/\Delta t$.

6.5.5. Consider the first-order wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

(a) Determine a partial difference equation by using a forward difference in time and a centered difference in space.

***(b)** Analyze the stability of this scheme (without boundary conditions).

***6.5.6.** Modify Exercise 6.5.5 for centered difference in space and time.

6.5.7. Solve on a computer [using (6.5.2)] the wave equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$ with $u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = \sin \pi x$ and $\partial u/\partial t(x, 0) = 0$ with $\Delta x = 1/100$. For the initial condition for the first derivative $\partial u/\partial t(x, 0) = 0$, use a forward difference instead of the centered difference (3.5.4). Compare to analytic solution at $x = 1/2$, $t = 1$ by computing the error there (the difference between the analytic solution and the numerical solution). Pick Δt so that

(a) $\Delta x/\Delta t = 1.5$ **(b)** $\Delta x/\Delta t = 0.5$

(c) To improve the calculation in part (a), let $\Delta x = 1/200$ but keep $\Delta x/\Delta t = 1.5$.

(d) Compare the errors in parts (a) and (c).

6.6 Laplace's Equation

INTRODUCTION

Laplace's equation

$$\boxed{\nabla^2 u = 0} \tag{6.6.1}$$

is usually formulated in some region such that the function u, or its derivative in the direction normal to the boundary, must be specified at each point along the boundary. A time variable does

not occur, so that a numerical finite difference method must proceed somewhat differently than for the heat or wave equations.

Using the standard centered difference discretization, Laplace's equation in two dimensions becomes the following partial difference equation (assuming that $\Delta x = \Delta y$):

$$\frac{u_{j+1,\,l} + u_{j-1,\,l} + u_{j,\,l+1} + u_{j,\,l-1} - 4u_{j,\,l}}{(\Delta x)^2} = 0, \tag{6.6.2}$$

where, hopefully, $u_{j,\,l} \approx u(j\,\Delta x,\, l\,\Delta y)$.

The boundary condition may be analyzed in the same way as for the heat and wave equations. In the simplest case it is specified along the boundary (composed of mesh points). The temperatures at the interior mesh points are the unknowns. Equation (6.6.2) is valid at each of these interior points. Some of the terms in (6.6.2) are determined from the boundary conditions, but most terms remain unknown. Equation (6.6.2) can be written as a linear system. Gaussian elimination can be used, but in many practical situations the number of equations and unknowns (equaling the number of interior mesh points) is too large for efficient numerical calculations. This is especially true in three dimensions, where even a coarse $20 \times 20 \times 20$ grid will generate 8000 linear equations in 8000 unknowns.

By rearranging (6.6.2), we derive

$$\boxed{u_{j,\,l} = \frac{u_{j+1,\,l} + u_{j-1,\,l} + u_{j,\,l+1} + u_{j,\,l-1}}{4}.} \tag{6.6.3}$$

The temperature $u_{j,\,l}$ must be the average of its four neighbors. Thus, the solution of the discretization of Laplace's equation satisfies a mean value property. Also, from (6.6.3), we can prove discrete maximum and minimum principles. These properties are analogous to similar results for Laplace's equation itself (see Section 2.5.4).

Jacobi iteration

Instead of solving (6.6.3) exactly, it is more usual to use an approximate iterative scheme. One should not worry very much about the errors in solving (6.6.3) *if they are small* since (6.6.3) already is an approximation of Laplace's equation.

We cannot solve (6.6.3) directly since the four neighboring temperatures are not known. However, the following procedure will yield the solution. We can make an initial guess of the solution and use the averaging principle (6.6.3) to "update" the solution:

$$u_{j,\,l}^{(\text{new})} = \frac{1}{4}\left(u_{j+1,\,l} + u_{j-1,\,l} + u_{j,\,l-1} + u_{j,\,l+1}\right)^{(\text{old})}.$$

We can continue to do this, a process called **Jacobi iteration**. We introduce the notation $u_{j,\,l}^{(0)}$ for the initial guess, $u_{j,\,l}^{(1)}$ for the first iterate (determined from $u_{j,\,l}^{(0)}$), $u_{j,\,l}^{(2)}$ for the second iterate (determined from $u_{j,\,l}^{(1)}$), and so on. Thus, the $(m+1)$st iterate satisfies

$$\boxed{u_{j,\,l}^{(m+1)} = \frac{1}{4}\left(u_{j+1,\,l}^{(m)} + u_{j-1,\,l}^{(m)} + u_{j,\,l+1}^{(m)} + u_{j,\,l-1}^{(m)}\right).} \tag{6.6.4}$$

If the iterates converge, by which we mean that if

$$\lim_{m\to\infty} u_{j,\,l}^{(m+1)} = v_{j,\,l},$$

then (6.6.4) shows that $v_{j,\,l}$ satisfies the discretization of Laplace's equation (6.6.3).

Equation (6.6.4) is well suited for a computer. We cannot actually compute an infinite number of iterations. In practice, we stop the iteration process when $u_{j,\,l}^{(m+1)} - u_{j,\,l}^{(m)}$ is small (for all j and l). Then $u_{j,\,l}^{(m+1)}$ will be a reasonably good approximation to the exact solution $v_{j,\,l}$. (Recall that $v_{j,\,l}$ itself is only an approximate solution of Laplace's equation.)

The changes that occur at each updating may be emphasized by writing Jacobi iteration as

$$\boxed{u_{j,\,l}^{(m+1)} = u_{j,\,l}^{(m)} + \frac{1}{4}\left(u_{j+1,\,l}^{(m)} + u_{j-1,\,l}^{(m)} + u_{j,\,l+1}^{(m)} + u_{j,\,l-1}^{(m)} - 4u_{j,\,l}^{(m)}\right).} \tag{6.6.5}$$

In this way Jacobi iteration is seen to be the standard discretization (centered spatial and forward time differences) of the two-dimensional diffusion equation, $\partial u/\partial t = k(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)$,

with $s = k\Delta t/(\Delta x)^2 = \frac{1}{4}$ [see (6.4.4)]. Each iteration corresponds to a time step $\Delta t = (\Delta x)^2/4k$. The earlier example of computing the heat equation on an L-shaped region (see Section 6.4) is exactly Jacobi iteration. For large m, we see the solution approaching values independent of m. The resulting spatial distribution is an accurate approximate solution of the discrete version of Laplace's equation (satisfying the given boundary conditions).

Although Jacobi iteration converges, it will be shown to converge very slowly. To analyze *roughly* the rate of convergence, we investigate the decay of spatial oscillations for an $L \times L$ square ($\Delta x = \Delta y = L/N$). In (6.6.4), m is analogous to time ($t = m\Delta t$), and (6.6.4) is analogous to the type of partial difference equation analyzed earlier. Thus, we know that there are special solutions

$$u_{j,l}^{(m)} = Q^m e^{i(\alpha x + \beta y)}, \tag{6.6.6}$$

where $\alpha = n_1\pi/L$, $\beta = n_2\pi/L$, $n_j = 1, 2, \ldots, N-1$. We assume that in this calculation there are zero boundary conditions along the edge. The solution should converge to zero as $m \to \infty$; we will determine the speed of convergence in order to know how many iterations are expected to be necessary. Substituting (6.6.6) into (6.6.4) yields

$$Q = \frac{1}{4}\left(e^{i\alpha\Delta x} + e^{-i\alpha\Delta x} + e^{i\beta\Delta y} + e^{-i\beta\Delta y}\right) = \frac{1}{2}\left(\cos\alpha\Delta x + \cos\beta\Delta y\right).$$

Since $-1 < Q < 1$ for all α and β, it follows from (6.6.6) that $\lim_{m\to\infty} u_{j,l}^{(m)} = 0$, as desired. However, the convergence can be very slow. The slowest rate of convergence occurs for Q nearest to 1. This happens for the smallest and largest α and β, $\alpha = \beta = \pi/L$ and $\alpha = \beta = (N-1)\pi/L$, in which case

$$|Q| = \cos\frac{\pi\Delta x}{L} = \cos\frac{\pi}{N} \approx 1 - \frac{\pi^2}{2}\frac{1}{N^2}, \tag{6.6.7}$$

since $\Delta x = L/N$ and N is large. In this case $|Q|^m$ is approximately $\left[1 - \frac{1}{2}(\pi/N)^2\right]^m$. This represents the error associated with the worst spatial oscillation on a square. If N is large, this error converges slowly to zero. For example, for the error to be reduced by a factor of $\frac{1}{2}$,

$$\left[1 - \frac{1}{2}\left(\frac{\pi}{N}\right)^2\right]^m = \frac{1}{2}.$$

Solving for m using natural logarithms yields

$$m\log\left[1 - \frac{1}{2}\left(\frac{\pi}{N}\right)^2\right] = -\log 2.$$

A simpler formula is obtained since π/N is small. From the Taylor series for x small, $\log(1-x) \approx -x$, it follows that the number of iterations m necessary to reduce the error by $\frac{1}{2}$ is approximately

$$m = \frac{\log 2}{\frac{1}{2}(\pi/N)^2} = N^2\frac{2\log 2}{\pi^2},$$

using Jacobi iteration. The number of iterations required may be quite large, proportional to N^2, the number of lattice points squared (just to reduce the error in half).

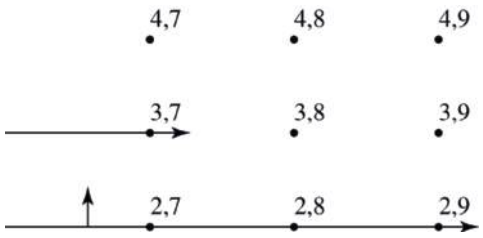
Gauss–Seidel iteration

Jacobi iteration is quite time-consuming to compute accurate solutions because of its slow convergence. Fortunately, there is a scheme that is easier to implement and that converges faster to the solution of the discretized version of Laplace's equation. It is usual in Jacobi iteration to obtain the updated temperature $u_{j,l}^{(m+1)}$ first in the lower left spatial region. Then we scan completely across a row of mesh points (from left to right) before updating the temperature on the next row above (again from left to right), as indicated in Fig. 6.6.1. For example,

$$u_{3,8}^{(m+1)} = \frac{1}{4}\left(u_{2,8}^{(m)} + u_{3,7}^{(m)} + u_{3,9}^{(m)} + u_{4,8}^{(m)}\right).$$

In Jacobi iteration, we use the old values $u_{2,8}^{(m)}$, $u_{3,7}^{(m)}$, $u_{3,9}^{(m)}$, and $u_{4,8}^{(m)}$ even though new values for two, $u_{3,7}^{(m+1)}$ and $u_{2,8}^{(m+1)}$, already have been calculated. In doing a computer implementation of Jacobi iteration, we cannot destroy immediately the old values (as we have shown some are needed even after new values have been calculated).

Figure 6.6.1 Gauss–Seidel iteration.



The calculation will be easier to program if old values are destroyed as soon as a new one is calculated. We thus propose to use the updated temperatures when they are known. For example,

$$u_{3,8}^{(m+1)} = \frac{1}{4}\left(u_{2,8}^{(m+1)} + u_{4,8}^{(m)} + u_{3,7}^{(m+1)} + u_{3,9}^{(m)}\right).$$

In general, we obtain

$$\boxed{u_{j,l}^{(m+1)} = \frac{1}{4}\left(u_{j-1,l}^{(m+1)} + u_{j+1,l}^{(m)} + u_{j,l-1}^{(m+1)} + u_{j,l+1}^{(m)}\right),} \tag{6.6.8}$$

known as **Gauss–Seidel iteration**. If this scheme converges, the solution will satisfy the discretized version of Laplace's equation.

There is no strong reason at this point to believe this scheme converges faster than Jacobi iteration. To investigate the speed at which Gauss–Seidel converges (for a square), we substitute again

$$u_{j,l}^{(m)} = Q^m e^{i(\alpha x+\beta y)}, \tag{6.6.9}$$

where

$$\alpha = \frac{n_1\pi}{L},\ \beta = \frac{n_2\pi}{L},\ n_i = 1, 2, \ldots, N-1.$$

The result is that

$$Q = \frac{1}{4}[e^{i\alpha\Delta x} + e^{i\beta\Delta y} + Q(e^{-i\alpha\Delta x} + e^{-i\beta\Delta y})]. \tag{6.6.10}$$

To simplify the algebra, we let

$$z = \frac{e^{i\alpha\Delta x} + e^{i\beta\Delta y}}{4} = \xi + i\eta, \tag{6.6.11}$$

and thus obtain $Q = z/(1-\bar{z})$. Q is complex, $Q = |Q|e^{i\theta}$, $u_{j,l}^{(m)} = |Q|^m e^{i\theta m} e^{i(\alpha x+\beta y)}$. The convergence rate is determined from $|Q|$,

$$|Q|^2 = \frac{z\bar{z}}{(1-\bar{z})(1-z)} = \frac{|z|^2}{1+|z|^2-2\mathrm{Re}(z)} = \frac{\xi^2+\eta^2}{1+\xi^2+\eta^2-2\xi} = \frac{\xi^2+\eta^2}{(1-\xi)^2+\eta^2}. \tag{6.6.12}$$

Since $|z| < \frac{1}{2}$ and thus $|\xi| < \frac{1}{2}$, it follows that $|Q| < 1$, yielding the convergence of Gauss–Seidel iteration. However, the rate of convergence is slow if $|Q|$ is near 1. Equation (6.6.12) shows that $|Q|$ is near 1 only if ξ is near $\frac{1}{2}$, which (6.6.11) requires α and β to be as small as possible. For a square, $\alpha = \pi/L$ and $\beta = \pi/L$, and thus

$$\xi = \frac{1}{2}\cos\frac{\pi\Delta x}{L} \text{ and } \eta = \frac{1}{2}\sin\frac{\pi\Delta x}{L}.$$

Therefore,

$$|Q|^2 = \frac{1}{5-4\cos\pi\Delta x/L} = \frac{1}{5-4\cos\pi/N} \approx \frac{1}{1+2(\pi/N)^2} \approx 1-2\left(\frac{\pi}{N}\right)^2,$$

since π/N is small. Thus

$$|Q| \approx 1-\left(\frac{\pi}{N}\right)^2.$$

This $|Q|$ is twice as far from 1 compared to Jacobi iteration [see (6.6.7)]. By doing the earlier analysis, half the number of iterations are required to reduce the error by any fixed fraction. Jacobi iteration should never be used. Gauss–Seidel iteration is a feasible and better alternative.

SOR

Both Jacobi and Gauss–Seidel iterative schemes require the number of iterations to be proportional to N^2, where N is the number of intervals (in one dimension). A remarkably faster scheme is successive overrelaxation, or simply SOR.

Gauss–Seidel can be rewritten to emphasize the change that occurs at each iteration,

$$u_{j,l}^{(m+1)} = u_{j,l}^{(m)} + \frac{1}{4}\left[u_{j-1,l}^{(m)} + u_{j+1,l}^{(m+1)} + u_{j,l-1}^{(m)} + u_{j,l+1}^{(m+1)} - 4u_{j,l}^{(m)}\right].$$

The bracketed term represents the change after each iteration as $u_{j,l}^{(m)}$ is updated to $u_{j,l}^{(m+1)}$. Historically, it was observed that one might converge faster if larger or smaller changes were introduced. Gauss–Seidel iteration with a **relaxation parameter** ω yields an **SOR iteration**:

$$\boxed{u_{j,l}^{(m+1)} = u_{j,l}^{(m)} + \omega\left[u_{j-1,l}^{(m)} + u_{j+1,l}^{(m+1)} + u_{j,l-1}^{(m)} + u_{j,l+1}^{(m+1)} - 4u_{j,l}^{(m)}\right].} \tag{6.6.13}$$

If $\omega = \frac{1}{4}$, this reduces to Gauss–Seidel. If SOR converges, it clearly converges to the discretized version of Laplace's equation. We will choose ω so that (6.6.13) converges as fast as possible.

We again introduce

$$u_{j,l}^{(m)} = Q^m e^{i(\alpha x + \beta y)}$$

in order to investigate the rate of convergence, where $\alpha = n_1\pi/L$, $\beta = n_2\pi/L$, $n_i = 1, 2, \ldots, N-1$. We obtain

$$Q = 1 + \omega\left[(e^{i\alpha\Delta x} + e^{i\beta\Delta y} + Q(e^{-i\alpha\Delta x} + e^{-i\beta\Delta y}) - 4\right].$$

The algebra here is somewhat complicated. We let $z = \omega\,(e^{i\alpha\Delta x} + e^{i\beta\Delta y}) = \xi + i\eta$ and obtain $Q = (1 - 4\omega + z)/(1 - \bar{z})$. Q is complex; $|Q|$ determines the convergence rate:

$$|Q|^2 = \frac{(1 - 4\omega + \xi)^2 + \eta^2}{(1 - \xi)^2 + \eta^2}.$$

Again $|z| < 2\omega$ and thus $|\xi| < 2\omega$. In Exercise 6.6.1 we show $|Q| < 1$ if $\omega < \frac{1}{2}$, guaranteeing the convergence of SOR. If $\omega < \frac{1}{2}$, $|Q|$ is near 1 only if ξ is near 2ω. This occurs only if α and β are as small as possible; (for a square) $\alpha = \pi/L$ and $\beta = \pi/L$ and thus $\xi = 2\omega\cos\pi\Delta x/L$ and $\eta = 2\omega\sin\pi\Delta x/L$. In this way

$$|Q|^2 = \frac{4\omega^2 + (1 - 4\omega)^2 + (1 - 4\omega)4\omega\cos(\pi/N)}{4\omega^2 + 1 - 4\omega\cos(\pi/N)}.$$

Exercise 6.6.1 shows that $|Q|^2$ is minimized if $\omega = \frac{1}{2} - \left(\sqrt{2}/2\right)\sqrt{1 - \cos\pi/N}$. Since π/N is large, we use (for a square) $\omega = \frac{1}{2}(1 - \pi/N)$, in which case

$$|Q| \approx 1 - \frac{\pi}{2N};$$

see Exercise 6.6.2. With the proper choice of ω, $|Q|$, although still near 1, is an order of magnitude further away from 1 than for either Jacobi or Gauss–Seidel iteration. In fact (see Exercise 6.6.2), errors are reduced by $\frac{1}{2}$ in SOR with the number of iterations being proportional to N (not N^2).

For nonsquare regions, the best ω for SOR is difficult to approximate. However, there exist computer library routines that approximate ω. Thus, often SOR is a practical, *relatively* quickly convergent iterative scheme for Laplace's equation.

Other improved schemes have been developed, including the alternating direction-implicit (ADI) method, which was devised in the mid-1950s by Peaceman, Douglas, and Rachford.

More modern techniques for solving sparse linear systems of equations include methods such as preconditioned conjugate-gradient methods, Krylov space methods, and multigrid methods. These are significantly more efficient than the classical Jacobi, Gauss–Seidel, and SOR techniques we have focused on, but the classical methods are useful for developing the fundamental idea of solving sparse linear systems by iteration, and analyzing their rates of convergence. A certain step

of the multigrid method uses a version of Jacobi called underrelaxed Jacobi, which is tuned to converge the fastest for the highest frequencies of the solution. A nice introduction to these modern techniques can be found in [41].

Exercises 6.6

6.6.1. **(a)** Show that $|Q| < 1$ if $\omega < \frac{1}{2}$ in SOR.

(b) Determine the optimal relaxation parameter ω in SOR for a square, by minimizing $|Q|^2$.

6.6.2. **(a)** If $\omega = \frac{1}{2}(1 - \pi/N)$, show that $|Q| \approx 1 - \pi/2N$ (for large N) in SOR.

(b) Show that with this choice the number of iterations necessary to reduce the error by $\frac{1}{2}$ is proportional to N (not N^2).

6.6.3. Describe a numerical scheme to solve Poisson's equation

$$\nabla^2 u = f(x, y),$$

(assuming that $\Delta x = \Delta y$) analogous to

(a) Jacobi iteration

(b) Gauss–Seidel iteration

(c) SOR

6.6.4. Describe a numerical scheme (based on Jacobi iteration) to solve Laplace's equation in three dimensions. Estimate the number of iterations necessary to reduce the error in half.

6.6.5. Modify Exercise 6.6.4 for Gauss–Seidel iteration.

6.6.6. Show that Jacobi iteration corresponds to the two-dimensional diffusion equation, by taking the limit as $\Delta x = \Delta y \to 0$ and $\Delta t \to 0$ in some appropriate way.

6.6.7. What partial differential equation does SOR correspond to? [*Hint*: Take the limit as $\Delta x = \Delta y \to 0$ and $\Delta t \to 0$ in various ways.] Specialize your result to Gauss–Seidel iteration by letting $\omega = \frac{1}{4}$.

6.6.8. Consider Laplace's equation on a square $0 \le x \le 1$, $0 \le y \le 1$ with $u = 0$ on three sides and $u = 1$ on the fourth.

(a) Solve using Jacobi iteration (let $\Delta x = \Delta y = \frac{1}{10}$).

(b) Solve using Gauss–Seidel iteration (let $\Delta x = \Delta y = \frac{1}{10}$).

(c) Solve using SOR iteration [let $\Delta x = \Delta y = \frac{1}{10}$ and $\omega = \frac{1}{2}(1 - \pi/10)$].

(d) Solve by separation of variables. Evaluate numerically the first 10 or 20 terms.

(e) Compare as many of the previous parts as you did.

6.7 Finite Element Method

In our modern world, many interesting physical problems formulated by partial differential equations are solved numerically. Finite difference methods may be used to approximate the continuous partial differential equation by a large number of difference equations. We describe another method known as the finite element method. For more in-depth explanations, we refer the reader to Strang (1986).

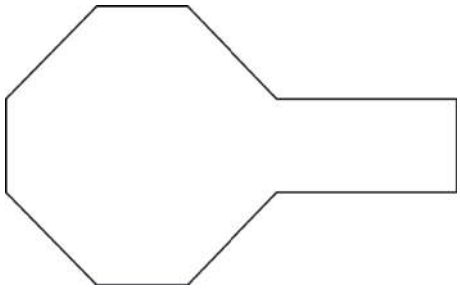
6.7.1 APPROXIMATION WITH NONORTHOGONAL FUNCTIONS (WEAK FORM OF THE PARTIAL DIFFERENTIAL EQUATION)

We solve a relatively simple partial differential equation, Poisson's equation in two dimensions with homogeneous boundary conditions,

$$\nabla^2 u = f(x, y) \text{ with } u = 0 \text{ on the closed boundary,} \tag{6.7.1}$$

where $f(x, y)$ is given. In problems whose geometry is not rectangular or circular, the method of separation of variables may be difficult or impossible. As an example, we consider the complicated 10-sided polygonal geometry, as illustrated in Fig. 6.7.1.

Figure 6.7.1 Region (polygonal example).



We approximate the solution with a large number of **test functions** $T_i(x, y)$, which have nothing to do with the partial differential equation nor their eigenfunctions. However, we assume the test functions satisfy the corresponding homogeneous boundary conditions:

$$T_i(x, y) = 0 \text{ on the closed boundary.} \tag{6.7.2}$$

We introduce $U(x, y)$, **an approximation to the solution** $u(x, y)$ using a sum (linear combination) of these test functions (instead of a series of eigenfunctions):

$$U(x, y) = \sum_{j=1}^{n} U_j T_j(x, y). \tag{6.7.3}$$

In practice, we use a large number n of test functions.

We do not insist that our approximation satisfies the partial differential equation (6.7.1). Instead, we determine the n constants so that the approximation satisfies the n conditions known as the weak form of the partial differential equation. We insist that the left- and right-hand sides of the partial differential equation (6.7.1) when multiplied by each trial function $T_i(x, y)$ and integrated over the entire region are the same:

$$\iint_R \nabla^2 u T_i \, dA = \iint_R f T_i \, dA, \tag{6.7.4}$$

where $dA = dxdy$. We can simplify this by integrating the left-hand side by parts. Using $\nabla \cdot (T_i \nabla u) = T_i \nabla^2 u + \nabla T_i \cdot \nabla u$, we obtain

$$\iint_R [\nabla \cdot (T_i \nabla u) - \nabla T_i \cdot \nabla u] \, dA = \iint_R f T_i \, dA. \tag{6.7.5}$$

Now we use the (two-dimensional) divergence theorem $\left(\iint_R \nabla \cdot \boldsymbol{B} \, dA = \oint \boldsymbol{B} \cdot \hat{\mathbf{n}} \, ds \right.$, where ds is differential arc length), and we obtain $\iint_R \nabla T_i \cdot \nabla u \, dA = \oint T_i \nabla u \cdot \hat{\mathbf{n}} \, ds - \iint_R f T_i \, dA$. Since the boundary contribution vanishes, $\oint T_i \nabla u \cdot \hat{\mathbf{n}} \, ds = 0$, because the trial functions satisfy the homogeneous boundary conditions (6.7.2), we obtain the **weak form (Galerkin) of the partial differential equation**:

$$\iint_R \nabla T_i \cdot \nabla u \, dA = -\iint_R f T_i \, dA. \tag{6.7.6}$$

All solutions $u(x, y)$ of the partial differential equation (6.7.1) satisfy the weak form (6.7.6). We insist that our approximation $U(x, y)$ (6.7.3) satisfies the weak form:

$$\sum_{j=1}^{n} U_j \iint_R \nabla T_i \cdot \nabla T_j dA = -\iint_R f T_i dA. \tag{6.7.7}$$

This is n equations for the n unknown coefficients U_i (one equation for each test function).

We introduce the symmetric **stiffness matrix** $\boldsymbol{K}$ with entries $K_{ij} = K_{ji}$:

$$K_{ij} = \iint_R \nabla T_i \cdot \nabla T_j \, dA \tag{6.7.8}$$

and the vector $\boldsymbol{F}$ with entries F_i:

$$F_i = -\iint_R f T_i \, dA. \tag{6.7.9}$$

The n equations, (6.7.7),

$$\sum_{j=1}^{n} K_{ij} U_j = F_i, \tag{6.7.10}$$

can be written in matrix form:

$$\boldsymbol{KU} = \boldsymbol{F}. \tag{6.7.11}$$

With given test functions, the right-hand side of (6.7.11) is known. Our solution is

$$\boldsymbol{U} = \boldsymbol{K}^{-1}\boldsymbol{F}. \tag{6.7.12}$$

However, a computational solution will usually be obtained without finding the general inverse of the usually very large $n \times n$ matrix $\boldsymbol{K}$.

Orthogonal two-dimensional eigenfunctions

If the boundary was exceptionally simple, then we could use trial functions that are two-dimensional eigenfunctions $T_i(x, y) = \phi_i(x, y)$ satisfying

$$\nabla^2 \phi_i = -\lambda_i \phi_i. \tag{6.7.13}$$

These eigenfunctions are orthogonal in a two-dimensional sense (see Section 7.5):

$$\iint_R \phi_i \phi_j \, dA = 0 \qquad \text{if } i \neq j. \tag{6.7.14}$$

In the Exercises we show that integration by parts is not necessary, so that there is an alternate expression for the stiffness matrix that can be derived directly from (6.7.4):

$$K_{ij} = -\iint_R T_i \nabla^2 T_j \, dA = -\iint_R \phi_i \nabla^2 \phi_j \, dA = \lambda_j \iint_R \phi_i \phi_j \, dA, \tag{6.7.15}$$

using (6.7.13). Using the two-dimensional orthogonality of eigenfunctions (6.7.14), we see that the stiffness matrix is diagonal:

$$K_{ij} = 0 \text{ if } i \neq j. \tag{6.7.16}$$

The diagonal elements from (6.7.15) are given by

$$K_{ii} = \lambda_i \iint_R \phi_i^2 \, dA. \tag{6.7.17}$$

The approximate solution is given by the finite series (6.7.3), where the coefficients are easily determined from (6.7.11) or (6.7.12) (if $\lambda = 0$ is not an eigenvalue):

$$U_i = \frac{F_i}{K_{ii}}, \tag{6.7.18}$$

where K_{ii} is given by (6.7.17). This approximate solution is a truncation of an infinite series of orthogonal functions. An example of this infinite series is described in a small portion of Section 8.6.

Galerkin numerical approximation for frequencies (eigenvalues)

Suppose we wish to obtain a numerical approximation to the frequencies of vibration of a membrane with shape perhaps as sketched in Fig. 6.7.1. In Section 7.2, it is shown that a vibrating membrane satisfies the two-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ with $u = 0$ on the boundary. After separation of time, $u = \phi(x, y)h(t)$, the frequencies of vibration $c\sqrt{\lambda}$ are determined from the eigenvalue problem

$$\nabla^2 \phi = -\lambda \phi \tag{6.7.19}$$

with $\phi = 0$ on the boundary. Since (6.7.19) corresponds to (6.7.1) with the right-hand side $f(x) = -\lambda\phi$, the Galerkin method may be used to approximate the eigenvalues and the eigenfunctions. The eigenfunction can be approximated by a series of test functions (6.7.3), $\phi(x, y) = \sum_{j=1}^{n} \phi_j T_j(x, y)$. Using the weak form (6.7.6) of the partial differential equation (6.7.19), the left-hand side of (6.7.7) can be again written in terms of the stiffness matrix $\boldsymbol{K}$. However, the right-hand side of (6.7.7) using $f(x) = -\lambda\phi$ involves a different matrix $\boldsymbol{M}$, so that the matrix form (6.7.11) becomes

$$\boldsymbol{K\phi} = \boldsymbol{\lambda M\phi}, \tag{6.7.20}$$

where we introduce the vector $\boldsymbol{\phi}$ with entries ϕ_j and the symmetric mass matrix $\boldsymbol{M}$ with entries given by

$$M_{ij} = \iint_R T_i T_j \, dA. \tag{6.7.21}$$

The n eigenvalues of (6.7.20) or $\boldsymbol{M}^{-1}\boldsymbol{K}$ approximate the eigenvalues of (6.7.19). The approximation improves as n increases. Strang (1986) shows that instead of computing the eigenvalues of $\boldsymbol{M}^{-1}\boldsymbol{K}$ directly, it is better to use the Cholesky decomposition of the mass matrix $\boldsymbol{M}$.

Finite elements

The trial functions we will use are not orthogonal so that the stiffness matrix will not be diagonal. However, finite elements will yield a stiffness matrix that has many zeros, known as a **sparse** matrix. In practice, the stiffness matrix is quite large and general methods to obtain its inverse as needed in (6.7.12) are not practical. However, there are practical methods to obtain the inverse since we will show that the stiffness matrix will be sparse.

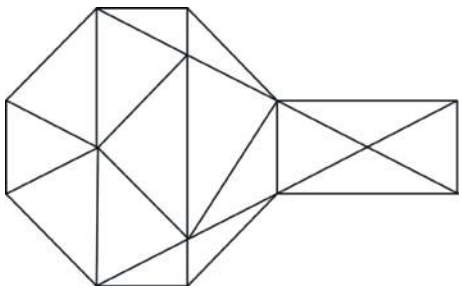
6.7.2 THE SIMPLEST TRIANGULAR FINITE ELEMENTS

One of the simplest ways to approximate any region is to break up the region into small triangles. The approximation improves as the number of triangles increases (and the size of the largest triangle decreases). We describe the method for a polygonal region. There are many ways to divide the region up into triangles (forming a triangular mesh). You can begin by dividing the region into triangles in a somewhat arbitrary way.

Let us first describe a very coarse mesh for our 10-sided figure (see Fig. 6.7.1). We somewhat arbitrarily pick 4 interior points forming 16 triangles (see Fig. 6.7.2). In general,

$$\text{number of triangles} = \text{number of boundary triangles} - 2 + 2\,(\text{number of interior points}). \tag{6.7.22}$$

Figure 6.7.2 Triangulated region (not unique).



This is easily seen by summing the interior angles of the triangles (180 times the number of triangles) in two different ways. Each interior point is completely surrounded (360 times the number of interior points) and the angles around the boundary points are the interior angles of a polygon (180 times number of sides minus 2).

The four interior points (x_i, y_i) give four unknowns $U(x_i, y_i)$. This gives us a four-dimensional vector space. We choose the test functions as the simplest basis functions of the four-dimensional vector space. We choose the test function $T_1(x, y)$ such that $T_1(x_1, y_1) = 1$, $T_1(x_2, y_2) = 0$, $T_1(x_3, y_3) = 0$, $T_1(x_4, y_4) = 0$. Similarly, the test function $T_2(x, y)$ is chosen to satisfy $T_2(x_1, y_1) = 0$, $T_2(x_2, y_2) = 1$, $T_2(x_3, y_3) = 0$, $T_2(x_4, y_4) = 0$, and so on. This is easily generalized to many more than three interior points (test functions).

Our approximation to the partial differential equation is (6.7.3), a linear combination of the four test functions,

$$U(x, y) = \sum_{j=1}^{4} U_j T_j(x, y).$$

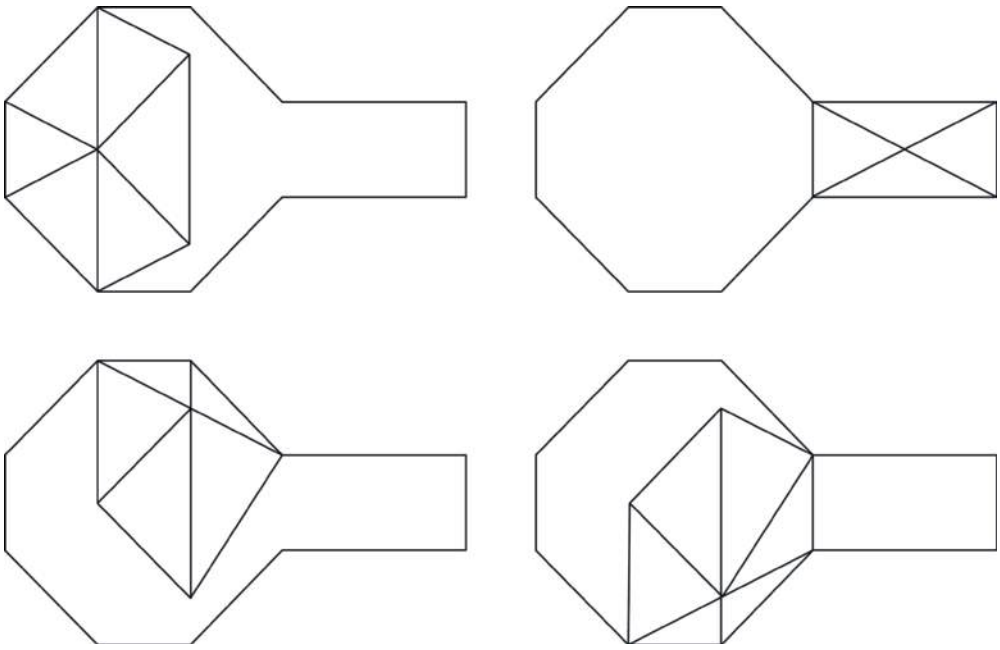
Note that we have the very nice property that

$$\begin{aligned} U(x_1, y_1) &= U_1 \\ U(x_2, y_2) &= U_2 \\ U(x_3, y_3) &= U_3 \\ U(x_4, y_4) &= U_4. \end{aligned} \tag{6.7.23}$$

Here, the coefficients are precisely the function evaluated at the nth point. That is why we use the notation U_j for the coefficients. In this way we have obtained a set of difference equations (6.7.10) or (6.7.11), where the stiffness matrix will be shown to be sparse. Because of (6.7.23), these represent difference equations for the value of u at each interior point. Thus, the difference equations derived by the finite element method are usually different from the difference equations that are derived by finite difference methods.

Our trial functions need to be defined everywhere, not just at the interior points. There are different finite elements, and we describe the simplest. We assume each trial function is linear (and looks like a simple plane) in each triangular region. Each trial function will be peaked at 1 and will linearly and continuously decay to 0 within the set of triangles that surrounds that vertex. Each trial function looks like a pyramid. **Each trial function will be identically zero for those regions that do not connect to the one peaked vertex**. The four trial functions for this problem are illustrated in Fig. 6.7.3. When there is a large number of degrees of freedom (vertices, trial functions), then each trial function is identically zero over most of the region.

Figure 6.7.3 Four trial functions (one corresponds to each interior point).



Calculation of stiffness matrix

The stiffness matrix,

$$K_{ij} = \iint_R \nabla T_i \bullet \nabla T_j \, dA, \tag{6.7.24}$$

is best calculated by summing up (called **assembling**) its contributions for each small triangular finite element. The entries of the stiffness matrix will be non-zero, only for its diagonal entries and for entries corresponding to adjacent interior vertices. The entries for the stiffness matrix will be zero, corresponding to non-adjacent interior vertices (since each trial function is mostly zero). The **stiffness matrix will be sparse** when there are many interior points.

The diagonal entries of the stiffness matrix corresponding to a specific interior vertex will be assembled only from the triangles that surround that interior vertex. The entries of the stiffness

matrix corresponding to adjacent interior vertices will be assembled from the two small triangles with the side in common from the line connecting the two adjacent interior vertices.

For a specific triangle with interior angles θ_i, it can be shown (in the Exercises) that the diagonal contributions satisfy

$$k_{11} = \text{portion of } K_{11} \text{ due to specific triangle} = \frac{1}{2\tan\theta_2} + \frac{1}{2\tan\theta_3}, \text{ etc.,}$$

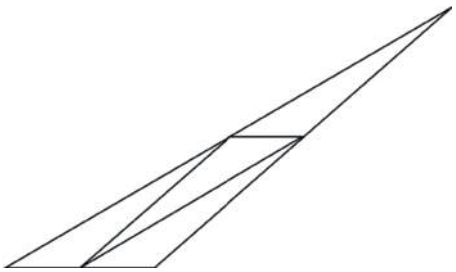
and the contributions to adjacent vertices are

$$k_{12} = \text{portion of } K_{12} \text{ due to specific triangle} = -\frac{1}{2\tan\theta_3}, \text{ etc.}$$

Mesh refinement

We begin with a specific mesh (for example, with four interior points as in Fig. 6.7.2). Then it is usual to obtain a finer mesh by subdividing each triangle by connecting the midpoints of each side. In this way each triangle becomes four similar triangles (see Fig. 6.7.4). Keeping similar triangles in the process can make numerical procedures somewhat easier since the above preceding formula shows that additional calculations are not needed for the stiffness matrix after the first mesh.

Figure 6.7.4 Refined mesh (each triangle).



Exercises 6.7

6.7.1. Consider a polygonal region of your choice. Sketch the trial functions:

(a) Assuming 5-sided figure with 2 interior points

(b) Assuming 5-sided figure with 3 interior points

(c) Assuming 5-sided figure with 5 interior points

(d) Assuming 5-sided figure with 7 interior points

6.7.2. For the finite elements of Exercise 6.7.1, which entries of the stiffness matrix are zero?

6.7.3. Derive a formula for K_{ij} from (6.7.4).

6.7.4. Show that $\iint_R \nabla U \cdot \nabla U \, dA = \boldsymbol{U}^T \boldsymbol{K} \boldsymbol{U}$. Thus, the entire matrix $\boldsymbol{K}$ can be identified by calculating this integral.

6.7.5. Show (by completing square of quadratics) that the minimum of

$$\iint_R \left[\frac{1}{2}\nabla U \cdot \nabla U - f(x, y)U\right] dA,$$

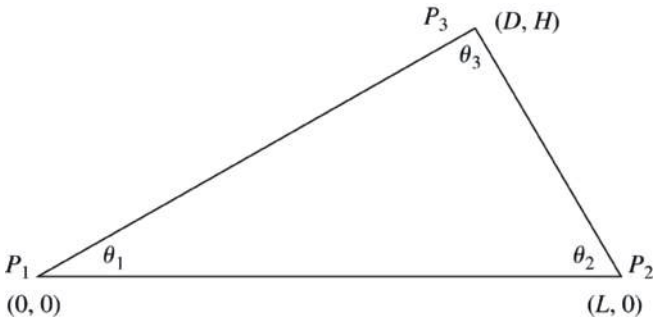
where U satisfies (6.7.3), occurs when $\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}$.

6.7.6. Consider a somewhat arbitrary triangle (as illustrated in Fig. 6.7.5) with $P_1 = (0, 0)$, $P_2 = (L, 0)$, $P_3 = (D, H)$ and interior angles θ_i. The solution on the triangle will be linear $U = a + bx + cy$.

(a) Show that $\iint_R \nabla U \cdot \nabla U \, dA = (b^2 + c^2)\frac{1}{2}LH$.

(b) The coefficients a, b, c are determined by the conditions at the three vertices $U(P_i) = U_i$. Demonstrate that $a = U_1$, $b = \frac{U_2 - U_1}{L}$, and $c = \frac{U_3 - U_1 - \frac{D}{L}(U_2 - U_1)}{H}$.

Figure 6.7.5 Triangular finite element.



(c) Show that $\frac{1}{\tan\theta_1} = \frac{D}{H}$, $\frac{1}{\tan\theta_2} = \frac{L-D}{H}$, and using $\tan\theta_3 = -\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{\tan\theta_1 \tan\theta_2 - 1}$, show that $\frac{1}{\tan\theta_3} = \frac{H}{L} - \frac{D}{H} + \frac{D^2}{HL}$.

(d) Using Exercise 6.7.4 and parts (a)–(c) of this exercise, show that for the contribution from this one triangle, $K_{12} = -\frac{1}{2\tan\theta_3}$. The other entries of the stiffness matrix follow in this way.

6.7.7. Continue with part (d) of Exercise 6.7.6 to obtain

(a) K_{11} **(b)** K_{22} **(c)** K_{33} **(d)** K_{23} **(e)** K_{13}

7

Higher-Dimensional Partial Differential Equations

7.1 Introduction

In our discussion of partial differential equations, we have solved many problems by the method of separation of variables, but all involved only two independent variables:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right), \rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2}.$$

In this chapter, we show how to extend the method of separation of variables to problems with more than two independent variables.

In particular, we discuss techniques to analyze the heat equation (with constant thermal properties) in two and three dimensions,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{(two dimensions)} \tag{7.1.1}$$

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \text{(three dimensions)} \tag{7.1.2}$$

for various physical regions with various boundary conditions. Also of interest will be the steady-state heat equation, Laplace's equation, in three dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

In all these problems, the partial differential equation has at least three independent variables. Other physical problems, not related to the flow of thermal energy, may also involve more than two independent variables. For example, the vertical displacement u of a vibrating membrane satisfies the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

It should also be mentioned that in acoustics, the perturbed pressure u satisfies the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

We will discuss and analyze some of these problems.

7.2 Separation of the Time Variable

We will show that similar methods can be applied to a variety of problems. We will begin by discussing the vibrations of a membrane of any shape, and follow that with some analysis for the conduction of heat in any two- or three-dimensional region.

7.2.1 VIBRATING MEMBRANE: ANY SHAPE

Let us consider the displacement u of a vibrating membrane of any shape. Later (Sections 7.3 and 7.7) we will specialize our result to rectangular and circular membranes. The displacement $u(x, y, t)$ satisfies the two-dimensional wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).} \tag{7.2.1}$$

The initial conditions will be

$$u(x, y, 0) = \alpha(x, y) \tag{7.2.2}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y), \tag{7.2.3}$$

but as usual they will be ignored at first when separating variables. A homogeneous boundary condition will be given along the entire boundary; $u = 0$ on the boundary is the most common condition. However, it is possible, for example, for the displacement to be zero on only part of the boundary and for the rest of the boundary to be "free." There are many other possible boundary conditions.

Let us now apply the method of separation of variables. We begin by showing that the time variable can be separated out from the problem for a membrane of any shape by seeking product solutions of the following form:

$$\boxed{u(x, y, t) = h(t)\phi(x, y).} \tag{7.2.4}$$

Here, $\phi(x, y)$ is an as yet unknown function of the two variables x and y. We do not (at this time) specify further $\phi(x, y)$ since we might expect different results in different geometries or with different boundary conditions. Later, we will show that for rectangular membranes $\phi(x, y) = F(x)G(y)$, while for circular membranes $\phi(x, y) = F(r)G(\theta)$; that is, the form of further separation depends on the geometry. It is for this reason that we begin by analyzing the general form (7.2.4). In fact, for most regions that are *not* geometrically as simple as rectangles and circles, $\phi(x, y)$ cannot be separated further. If (7.2.4) is substituted into the equation for a vibrating membrane, (7.2.1), then the result is

$$\phi(x, y)\frac{d^2 h}{dt^2} = c^2 h(t)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right). \tag{7.2.5}$$

We will attempt to proceed as we did when there were only two independent variables. Time can be separated from (7.2.5) by dividing by $h(t)\phi(x, y)$ (and an additional division by the constant c^2 is convenient):

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2 h}{dt^2} = \frac{1}{\phi}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = -\lambda. \tag{7.2.6}$$

The left-hand side of the first equation is a function only of time, while the right-hand side is a function only of space (x and y). Thus, the two (as before) must equal a separation constant. Again, we must decide what notation is convenient for the separation constant, λ or $-\lambda$. A quick glance at the resulting ordinary differential equation for $h(t)$ shows that $-\lambda$ is more convenient (as will be explained). We thus obtain two equations, but unlike the case of two independent variables, one of the equations is itself still a partial differential equation:

$$\boxed{\frac{d^2 h}{dt^2} = -\lambda c^2 h} \tag{7.2.7}$$

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda\phi.} \tag{7.2.8}$$

The notation $-\lambda$ for the separation constant was chosen because the time-dependent differential equation (7.2.7) has oscillatory solutions if $\lambda > 0$, although this is not guaranteed at the moment. If $\lambda > 0$, then h is a linear combination of $\sin c\sqrt{\lambda}t$ and $\cos c\sqrt{\lambda}t$; it oscillates with frequency $c\sqrt{\lambda}$. The values of λ determine the natural frequencies of oscillation of a vibrating membrane. To show that $\lambda > 0$, we must analyze the resulting eigenvalue problem, (7.2.8), where ϕ is subject to a

homogeneous boundary condition along the entire boundary (e.g., $\phi = 0$ on the boundary). Here, the eigenvalue problem itself involves a linear homogeneous partial differential equation. Shortly, we will show that $\lambda > 0$ by introducing a Rayleigh quotient applicable to (7.2.8). Before analyzing (7.2.8), we will show that it arises in other contexts.

7.2.2 HEAT CONDUCTION: ANY REGION

We will analyze the flow of thermal energy in any two-dimensional region. We begin by seeking product solutions of the form

$$\boxed{u(x, y, t) = h(t)\phi(x, y)} \tag{7.2.9}$$

for the two-dimensional heat equation, assuming constant thermal properties and no sources, (7.1.1). By substituting (7.2.9) into (7.1.1) and after dividing by $kh(t)\phi(x, y)$, we obtain

$$\frac{1}{k}\frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right). \tag{7.2.10}$$

A separation constant in the form $-\lambda$ is introduced so that the time-dependent part of the product solution exponentially decays (if $\lambda > 0$) as expected, rather than exponentially grows. Then, the two equations are

$$\boxed{\begin{aligned} \frac{dh}{dt} &= -\lambda k h \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= -\lambda\phi. \end{aligned}} \tag{7.2.11}$$

The eigenvalue λ relates to the decay rate of the time-dependent part. The eigenvalue λ is determined by the boundary value problem, again consisting of the partial differential equation (7.2.11) with a corresponding boundary condition on the entire boundary of the region.

For heat flow in any three-dimensional region, (7.1.2) is valid. A product solution,

$$\boxed{u(x, y, z, t) = h(t)\phi(x, y, z),} \tag{7.2.12}$$

may still be sought, and after separating variables, we obtain equations similar to (7.2.11),

$$\boxed{\begin{aligned} \frac{dh}{dt} &= -\lambda k h \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} &= -\lambda\phi. \end{aligned}} \tag{7.2.13}$$

The eigenvalue λ is determined by finding those values of λ for which nontrivial solutions of (7.2.13) exist, subject to a homogeneous boundary condition on the entire boundary.

7.2.3 SUMMARY

In situations described in this section, the spatial part $\phi(x, y)$ or $\phi(x, y, z)$ of the solution of the partial differential equation satisfies the eigenvalue problem consisting of the partial differential equation,

$$\boxed{\nabla^2\phi = -\lambda\phi,} \tag{7.2.14}$$

with ϕ satisfying appropriate homogeneous boundary conditions, which may be of the form [see (1.5.12) and (4.5.6)]

$$\alpha\phi + \beta\nabla\phi \cdot \hat{\mathbf{n}} = 0, \tag{7.2.15}$$

where α and β can depend on x, y, and z. If $\beta = 0$, (7.2.15) is the fixed boundary condition. If $\alpha = 0$, (7.2.15) is the insulated or free boundary condition. If both $\alpha \neq 0$ and $\beta \neq 0$, then (7.2.15) is the higher-dimensional version of Newton's law of cooling or the elastic boundary condition. In Section 7.4, we will describe general results for this two- or three-dimensional eigenvalue

problem, similar to our theorems concerning the general one-dimensional Sturm–Liouville eigenvalue problem. However, first we will describe the solution of a simple two-dimensional eigenvalue problem in a situation in which $\phi(x, y)$ may be further separated, producing two one-dimensional eigenvalue problems.

Exercises 7.2

7.2.1. For a vibrating membrane of any shape that satisfies (7.2.1), show that (7.2.14) results after separating time.

7.2.2. For heat conduction in any two-dimensional region that satisfies (7.1.1), show that (7.2.14) results after separating time.

7.2.3. **(a)** Obtain product solutions, $\phi = f(x)g(y)$, of (7.2.14) that satisfy $\phi = 0$ on the four sides of a rectangle. [*Hint*: If necessary, see Section 7.3.]

(b) Using part (a), solve the initial value problem for a vibrating rectangular membrane (fixed on all sides).

(c) Using part (a), solve the initial value problem for the two-dimensional heat equation with zero temperature on all sides.

7.3 Vibrating Rectangular Membrane

In this section, we analyze the vibrations of a rectangular membrane, with a domain as sketched in Figure 7.3.1. The vertical displacement $u(x, y, t)$ of the membrane satisfies the two-dimensional wave equation,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).} \tag{7.3.1}$$

We suppose that the boundary is given such that all four sides are fixed with zero displacement:

$$u(0, y, t) = 0, \qquad u(x, 0, t) = 0 \tag{7.3.2}$$

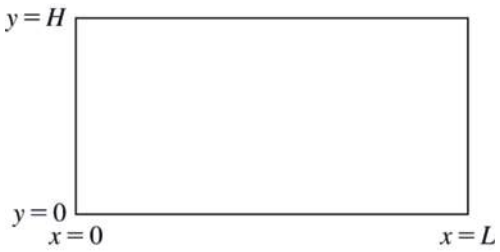
$$u(L, y, t) = 0, \qquad u(x, H, t) = 0. \tag{7.3.3}$$

We seek the displacement of the membrane at time t if the initial position and velocity are given:

$$u(x, y, 0) = \alpha(x, y) \tag{7.3.4}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y). \tag{7.3.5}$$

Figure 7.3.1 Rectangular membrane.



As we indicated in Section 7.2.1, since the partial differential equation and the boundary conditions are linear and homogeneous, we apply the method of separation of variables. First, we separate only the time variable by seeking product solutions in the form

$$u(x, y, t) = h(t)\phi(x, y). \tag{7.3.6}$$

According to our earlier calculation, we are able to introduce a separation constant $-\lambda$, and the following two equations result:

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \tag{7.3.7}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi. \tag{7.3.8}$$

We will show that $\lambda > 0$, in which case $h(t)$ is a linear combination of $\sin c\sqrt{\lambda}t$ and $\cos c\sqrt{\lambda}t$. The homogeneous boundary conditions imply that the eigenvalue problem is

$$\boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi} \tag{7.3.9}$$

$$\boxed{\begin{array}{l}\phi(0, y) = 0,\ \phi(x, 0) = 0\\ \phi(L, y) = 0,\ \phi(x, H) = 0;\end{array}} \tag{7.3.10}$$

that is, $\phi = 0$ along the entire boundary. We call (7.3.9)–(7.3.10) a two-dimensional eigenvalue problem.

The eigenvalue problem itself is a linear homogeneous PDE in two independent variables with homogeneous boundary conditions. As such (since the boundaries are simple), we can expect that (7.3.9)–(7.3.10) can be solved by separation of variables in Cartesian coordinates. In other words, we look for product solutions of (7.3.9)–(7.3.10) in the form

$$\phi(x, y) = f(x)g(y). \tag{7.3.11}$$

Before beginning our calculations, let us note that it follows from (7.3.6) that our assumption (7.3.11) is equivalent to

$$u(x, y, t) = f(x)g(y)h(t), \tag{7.3.12}$$

a product of functions of each independent variable. We claim, as we show in an appendix to this section, that we could obtain the same result by substituting (7.3.12) into the wave equation (7.3.1) as we now obtain by substituting (7.3.11) into the two-dimensional eigenvalue problem (7.3.9):

$$g(y)\frac{d^2f}{dx^2} + f(x)\frac{d^2g}{dy^2} = -\lambda f(x)g(y). \tag{7.3.13}$$

The x and y parts may be separated by dividing (7.3.13) by $f(x)g(y)$ and rearranging terms:

$$\frac{1}{f}\frac{d^2f}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2g}{dy^2} = -\mu. \tag{7.3.14}$$

Since the first expression is a function only of x, while the second is a function only of y, we introduce a *second* separation constant. We choose it to be $-\mu$ so that the easily solved equation, $d^2f/dx^2 = -\mu f$, has oscillatory solutions (as expected) if $\mu > 0$. Two ordinary differential equations result from the separation of variables of a partial differential equation with two independent variables:

$$\boxed{\frac{d^2f}{dx^2} = -\mu f} \tag{7.3.15}$$

$$\boxed{\frac{d^2g}{dy^2} = -(\lambda - \mu)g.} \tag{7.3.16}$$

Equations (7.3.15) and (7.3.16) contain *two* separation constants λ and μ, both of which must be determined. In addition, $h(t)$ solves an ordinary differential equation:

$$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h.} \tag{7.3.17}$$

When we separate variables for a partial differential equation in three variables, $u(x, y, t) = f(x)g(y)h(t)$, we obtain three ordinary differential equations, one a function of each independent coordinate. However, there will be only two separation constants.

To determine the separation constants, we need to use the homogeneous boundary conditions (7.3.10). The product form (7.3.11) then implies that

$$\begin{aligned} f(0) &= 0 \quad \text{and} \quad f(L) = 0 \\ g(0) &= 0 \quad \text{and} \quad g(H) = 0. \end{aligned} \tag{7.3.18}$$

Of our three ordinary differential equations, only two will be eigenvalue problems. There are homogeneous boundary conditions in x and y. Thus,

$$\frac{d^2f}{dx^2} = -\mu f \quad \text{with} \quad f(0) = 0 \quad \text{and} \quad f(L) = 0 \tag{7.3.19}$$

is a Sturm–Liouville eigenvalue problem in the x-variable, where μ is the eigenvalue and $f(x)$ is the eigenfunction. Similarly, the y-dependent problem is a regular Sturm–Liouville eigenvalue problem:

$$\frac{d^2g}{dy^2} = -(\lambda - \mu)g \quad \text{with} \quad g(0) = 0 \quad \text{and} \quad g(H) = 0. \tag{7.3.20}$$

Here, λ is the eigenvalue and $g(y)$ the corresponding eigenfunction.

Not only are both (7.3.19) and (7.3.20) Sturm–Liouville eigenvalue problems, but they are both ones we should be quite familiar with. Without going through the well-known details, the eigenvalues are

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots, \tag{7.3.21}$$

and the corresponding eigenfunctions are

$$f_n(x) = \sin\frac{n\pi x}{L}. \tag{7.3.22}$$

This determines the allowable values of the separation constant μ_n.

For each value of μ_n, (7.3.20) is still an eigenvalue problem. There are an infinite number of eigenvalues λ for each n. Thus, λ should be double subscripted, λ_{nm}. In fact, from (7.3.20) the eigenvalues are

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \qquad m = 1, 2, 3, \ldots, \tag{7.3.23}$$

where we *must* use a different index to represent the various y-eigenvalues (for each value of n). The corresponding y-eigenfunction is

$$g_{nm}(y) = \sin\frac{m\pi y}{H}. \tag{7.3.24}$$

The separation constant λ_{nm} can now be determined from (7.3.23):

$$\boxed{\lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2,} \tag{7.3.25}$$

where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. The two-dimensional eigenvalue problem (7.3.9) has eigenvalues λ_{nm} given by (7.3.25) and eigenfunctions given by the product of the two one-dimensional eigenfunctions. Using the notation $\phi_{nm}(x, y)$ for the two-dimensional eigenfunction corresponding to the eigenvalue λ_{nm}, we have

$$\boxed{\phi_{nm}(x, y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H},} \quad \begin{matrix} n = 1, 2, 3, \ldots \\ m = 1, 2, 3, \ldots. \end{matrix} \tag{7.3.26}$$

Note how easily the homogeneous boundary conditions are satisfied.

From (7.3.25), we have explicitly shown that all the eigenvalues are positive (for this problem). Thus, the time-dependent part of the product solutions are (as previously guessed) $\sin c\sqrt{\lambda_{nm}}t$ and $\cos c\sqrt{\lambda_{nm}}t$, oscillations with natural frequencies $c\sqrt{\lambda_{nm}} = c\sqrt{(n\pi/L)^2 + (m\pi/H)^2}$, $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. In considering the displacement u, we have obtained two doubly-infinite families of product solutions: $\sin n\pi x/L \sin m\pi y/H \sin c\sqrt{\lambda_{nm}}t$ and $\sin n\pi x/L \sin m\pi y/H \cos c\sqrt{\lambda_{nm}}t$. As with the vibrating string, each of these special product solutions is known as a *mode of vibration.* We sketch

in Fig. 7.3.2 a representation of some of these modes. In each, we sketch level contours of displacement in dotted lines at a fixed t. As time varies, the shape stays the same; only the amplitude varies periodically. Each mode is a standing wave. Curves along which the displacement is always zero in a mode are called **nodal curves** and are sketched in solid lines. Cells are apparent with neighboring cells always being out of phase; that is, when one cell has a positive displacement, the neighbor has negative displacement (as represented by the + and − signs).

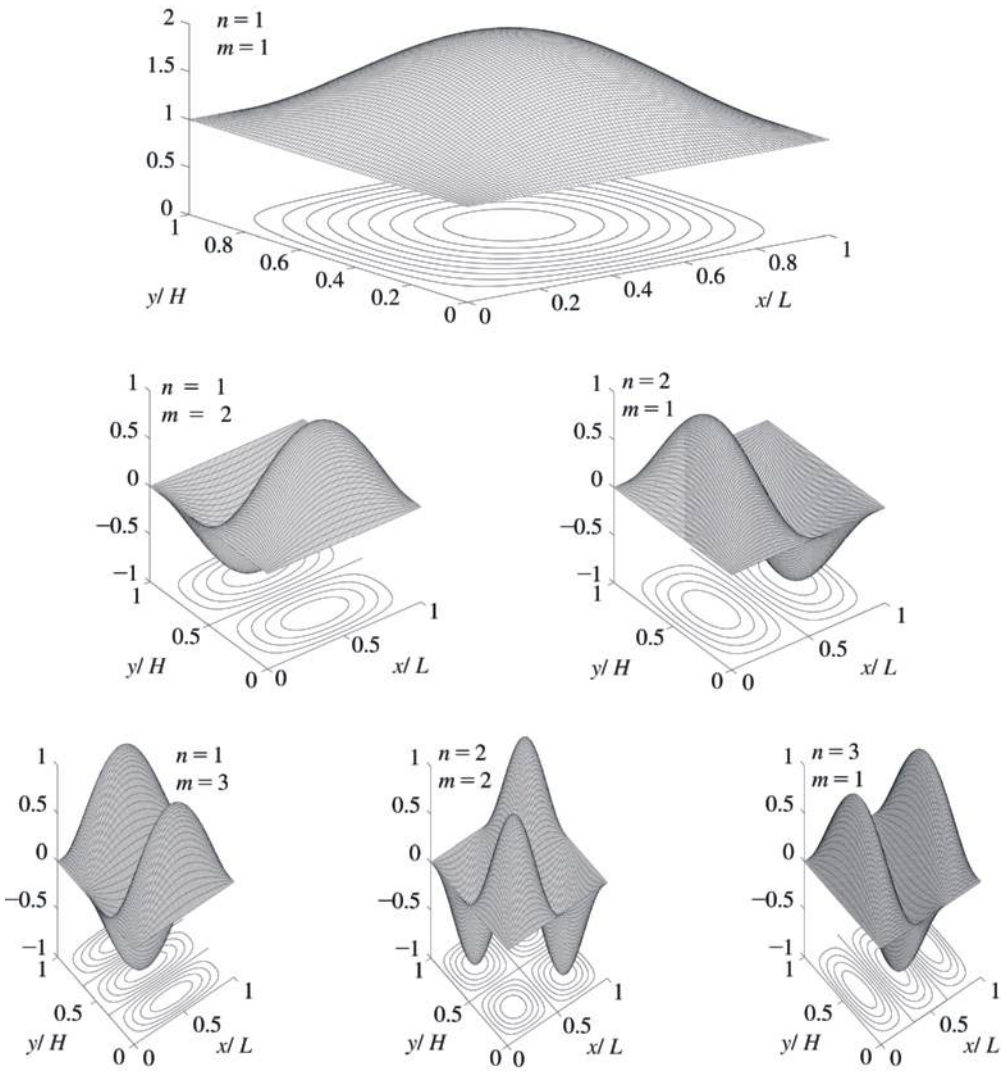
The principle of superposition implies that we should consider a linear combination of all possible product solutions. Thus, we must include both families, summing over both n and m,

$$\boxed{\begin{aligned} u(x, y, t) = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\cos c\sqrt{\lambda_{nm}}t \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\sin c\sqrt{\lambda_{nm}}t. \end{aligned}} \tag{7.3.27}$$

The two families of coefficients A_{nm} and B_{nm} hopefully will be determined from the two initial conditions. For example, $u(x, y, 0) = \alpha(x, y)$ implies that

$$\alpha(x, y) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L}\right)\sin\frac{m\pi y}{H}. \tag{7.3.28}$$

Figure 7.3.2 Nodal curves for modes of a vibrating rectangular membrane.



MATLAB Files

Download Animation of Figure 7.3.2 (.zip)

https://bit.ly/3xJGEc0

These files are provided as zipped .m files for use in MATLAB on desktop platform.

The series in (7.3.28) is an example of what is called a **double Fourier series**. Instead of discussing the theory, we show one method to calculate A_{nm} from (7.3.28). (In Section 7.4 we will discuss a simpler way.) For *fixed* x, we note that $\sum_{n=1}^{\infty} A_{nm} \sin n\pi x/L$ depends only on m. Furthermore, it must be the coefficients of the Fourier sine series in y of $\alpha(x, y)$ over $0 < y < H$. From our theory of Fourier sine series, we therefore know that we may easily determine the coefficients:

$$\sum_{n=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} = \frac{2}{H} \int_0^H \alpha(x, y) \sin \frac{m\pi y}{H}\, dy, \tag{7.3.29}$$

for each m. Equation (7.3.29) is valid for all x; the right-hand side is a function of x (not y, because y is integrated from 0 to H). For each m, the left-hand side is a Fourier sine series in x; in fact, it is the Fourier sine series of the right-hand side, $2/H \int_0^H \alpha(x, y) \sin m\pi y/H\, dy$. The coefficients of this Fourier sine series in x are easily determined:

$$\boxed{A_{nm} = \frac{2}{L} \int_0^L \left[\frac{2}{H} \int_0^H \alpha(x, y) \sin \frac{m\pi y}{H}\, dy \right] \sin \frac{n\pi x}{L}\, dx.} \tag{7.3.30}$$

This may be simplified to one double integral over the entire rectangular region, rather than two iterated one-dimensional integrals. In this manner, we have determined one set of coefficients from one of the initial conditions.

The other coefficients B_{nm} can be determined in a similar way. In particular, from (7.3.27), $\partial u/\partial t(x, y, 0) = \beta(x, y)$, which implies that

$$\beta(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}} B_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{7.3.31}$$

Thus, again using a Fourier sine series in y and a Fourier sine series in x, we obtain

$$\boxed{c\sqrt{\lambda_{nm}} B_{nm} = \frac{4}{LH} \int_0^L \int_0^H \beta(x, y) \sin \frac{m\pi y}{H} \sin \frac{n\pi x}{L}\, dy\, dx.} \tag{7.3.32}$$

The solution of our initial value problem is the doubly infinite series given by (7.3.27), where the coefficients are determined by (7.3.30) and (7.3.32).

We have shown that when all three independent variables separate for a partial differential equation, there results three ordinary differential equations, two of which are eigenvalue problems. In general, for a partial differential equation in N variables that completely separates, there will be N ordinary differential equations, $N - 1$ of which are one-dimensional eigenvalue problems (to determine the $N - 1$ separation constants). We have already shown this for $N = 3$ (this section) and $N = 2$.

Exercises 7.3

***7.3.1.** Solve the heat equation on the rectangle $0 \le x \le 3, 0 \le y \le 2$,

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

with initial condition

$$u(x, y, 0) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

and homogeneous boundary conditions,

$$u(0, y, t) = 0, \quad u(3, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, 2, t) = 0.$$

Explicitly calculate the coefficients in the double Fourier series expansion.

7.3.2. Consider the heat equation in a two-dimensional rectangular region $0 < x < L$, $0 < y < H$,

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial condition

$$u(x, y, 0) = \alpha(x, y).$$

[*Hint*: You may assume without derivation that product solutions $u(x, y, t) = \phi(x, y)h(t) = f(x)g(y)h(t)$ satisfy $\frac{dh}{dt} = -\lambda kh$, the two-dimensional eigenvalue problem $\nabla^2\phi + \lambda\phi = 0$ with further separation

$$\frac{d^2f}{dx^2} = -\mu f, \qquad \frac{d^2g}{dy^2} + (\lambda - \mu)g = 0,$$

or you may use results of the two-dimensional eigenvalue problem.]

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

***(a)** $u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, H, t) = 0$

(b) $\frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$

***(c)** $\frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, H, t) = 0$

(d) $u(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$

(e) $u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad u(x, 0, t) = 0,$

$\frac{\partial u}{\partial y}(x, H, t) + hu(x, H, t) = 0 \quad (h > 0)$

7.3.3. Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L, 0 < y < H, 0 < z < W$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

subject to the initial condition

$$u(x, y, z, 0) = \alpha(x, y, z).$$

[*Hint*: You may assume without derivation that the product solutions $u(x, y, z, t) = \phi(x, y, z)h(t)$ satisfy $\frac{dh}{dt} = -\lambda kh$ and satisfy the three-dimensional eigenvalue problem $\nabla^2\phi + \lambda\phi = 0$, or you may use results of the three-dimensional eigenvalue problem.]

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

(a) $u(0, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, z, t) = 0, \quad \frac{\partial u}{\partial z}(x, y, 0, t) = 0,$

$u(L, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, z, t) = 0, \quad u(x, y, W, t) = 0$

***(b)** $\frac{\partial u}{\partial x}(0, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, z, t) = 0, \quad \frac{\partial u}{\partial z}(x, y, 0, t) = 0,$

$\frac{\partial u}{\partial x}(L, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, z, t) = 0, \quad \frac{\partial u}{\partial z}(x, y, W, t) = 0$

7.3.4. Consider the heat equation in a three-dimensional box-shaped region, $0 < x < 3$, $0 < y < 5, 0 < z < 7$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

subject to the initial condition

$$u(x, y, z, 0) = \sin 2\pi x \sin \pi y \sin 3\pi z,$$

and the boundary conditions

$$u(0, y, z, t) = 0, \qquad u(x, 0, z, t) = 0, \qquad u(x, y, 0, t) = 0,$$
$$u(3, y, z, t) = 0, \qquad u(x, 5, z, t) = 0, \qquad u(x, y, 7, t) = 0.$$

Explicitly evaluate all the coefficients in the triple Fourier series. [*Hint*: All the coefficients but one evaluate to zero.]

7.3.5. Solve

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}$$

on a rectangle $(0 < x < L, 0 < y < H)$ subject to

$$u(0, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0,$$
$$u(x, y, 0) = \alpha(x, y),$$
$$u(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0.$$

7.3.6. Solve the wave equation on the rectangle $0 \le x \le 3$, $0 \le y \le 2$,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

with initial conditions

$$u(x, y, 0) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

and with homogeneous boundary conditions,

$$u(0, y, t) = 0, \qquad u(3, y, t) = 0, \qquad u(x, 0, t) = 0, \qquad u(x, 2, t) = 0.$$

Explicitly calculate the coefficients in the double Fourier series expansion.

7.3.7. Consider the wave equation for a vibrating rectangular membrane $(0 < x < L, 0 < y < H)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial conditions

$$u(x, y, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, 0) = \alpha(x, y).$$

[*Hint*: You may assume without derivation that the product solutions $u(x, y, t) = \phi(x, y)h(t)$ satisfy $\frac{d^2h}{dt^2} = -\lambda c^2 h$ and the two-dimensional eigenvalue problem $\nabla^2 \phi + \lambda \phi = 0$, and you may use results of the two-dimensional eigenvalue problem.]

Solve the initial value problem if

(a) $u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$

***(b)** $\frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$

(c) $u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, H, t) = 0.$

7.3.8. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - k \frac{\partial u}{\partial t} \quad \text{with } k > 0.$$

(a) Give a *brief* physical interpretation of this equation.

(b) Suppose that $u(x, y, t) = f(x)g(y)h(t)$. What ordinary differential equations are satisfied by f, g, and h?

7.3.9. Consider Laplace's equation

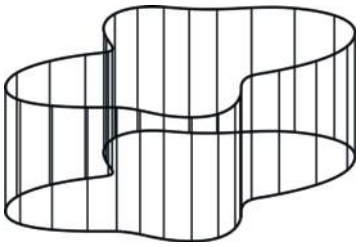
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is $z = H$, and the bottom is $z = 0$. Assume that

$$\frac{\partial}{\partial z} u(x, y, 0) = 0$$
$$u(x, y, H) = \alpha(x, y)$$

and $u = 0$ on the "lateral" sides.

Figure 7.3.3



(a) Separate the z-variable in general.

***(b)** Solve for $u(x, y, z)$ if the region is a rectangular box, $0 < x < L$, $0 < y < W$, $0 < z < H$.

7.3.10. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

in a box $0 < x < L, 0 < y < W, 0 < z < H$. [*Hint*: If there are homogeneous conditions in x and y, you may assume without derivation that the product solutions $u(x, y, z) = \phi(x, y)h(z)$ satisfy $\frac{d^2h}{dz^2} = \lambda h$ and the two-dimensional eigenvalue problem $\nabla^2\phi + \lambda\phi = 0$, and you may use results of the two-dimensional eigenvalue problem. If there are homogeneous conditions in two other directions, then make the corresponding changes.] Solve the equation subject to the boundary conditions

(a) $\frac{\partial u}{\partial x}(0, y, z) = 0$, $u(x, 0, z) = 0$, $u(x, y, 0) = \alpha(x, y)$
$\frac{\partial u}{\partial x}(L, y, z) = 0$, $u(x, W, z) = 0$, $u(x, y, H) = 0$

(b) $u(0, y, z) = 0$, $u(x, 0, z) = 0$, $u(x, y, 0) = 0$,
$u(L, y, z) = 0$, $u(x, W, z) = \alpha(x, z)$, $u(x, y, H) = 0$

***(c)** $\frac{\partial u}{\partial x}(0, y, z) = 0$, $\frac{\partial u}{\partial y}(x, 0, z) = 0$, $\frac{\partial u}{\partial z}(x, y, 0) = 0$
$\frac{\partial u}{\partial x}(L, y, z) = \alpha(y, z)$, $\frac{\partial u}{\partial y}(x, W, z) = 0$, $\frac{\partial u}{\partial z}(x, y, H) = 0$

***(d)** $\frac{\partial u}{\partial x}(0, y, z) = 0$, $\frac{\partial u}{\partial y}(x, 0, z) = 0$, $\frac{\partial u}{\partial z}(x, y, 0) = 0$
$u(L, y, z) = \alpha(y, z)$, $\frac{\partial u}{\partial y}(x, W, z) = 0$, $\frac{\partial u}{\partial z}(x, y, H) = 0$

Appendix to 7.3: Outline of Alternative Method to Separate Variables

An alternative (and equivalent) method to separate variables for

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{7.3.33}$$

is to assume product solutions of the form

$$u(x, y, t) = f(x)g(y)h(t). \tag{7.3.34}$$

By substituting (7.3.34) into (7.3.33) and dividing by $c^2 f(x)g(y)h(t)$, we obtain

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} = -\lambda, \tag{7.3.35}$$

after introducing a separation constant $-\lambda$. This shows that

$$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h.} \tag{7.3.36}$$

Equation (7.3.35) can be separated further,

$$\frac{1}{f}\frac{d^2f}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2g}{dy^2} = -\mu, \tag{7.3.37}$$

enabling a second separation constant $-\mu$ to be introduced:

$$\boxed{\frac{d^2f}{dx^2} = -\mu f} \tag{7.3.38}$$

$$\boxed{\frac{d^2g}{dy^2} = -(\lambda - \mu)g.} \tag{7.3.39}$$

In this way, we have derived the same three ordinary differential equations (with two separation constants).

7.4 Statements and Illustrations of Theorems for the Eigenvalue Problem $\nabla^2\phi + \lambda\phi = 0$

In solving the heat equation and the wave equation in any two- or three-dimensional region R (with constant physical properties, such as density), we have shown that the spatial part $\phi(x, y, z)$ of product form solutions $u(x, y, z, t) = \phi(x, y, z)h(t)$ satisfies the following multidimensional eigenvalue problem:

$$\boxed{\nabla^2\phi + \lambda\phi = 0,} \tag{7.4.1}$$

with

$$a\phi + b\nabla\phi \cdot \hat{\boldsymbol{n}} = 0 \tag{7.4.2}$$

on the entire boundary. Here a and b can depend on x, y, and z. Equation (7.4.1) is known as the **Helmholtz equation**.

Equation (7.4.1) can be generalized to

$$\nabla \cdot (p\nabla\phi) + q\phi + \lambda\sigma\phi = 0, \tag{7.4.3}$$

where p, q, and σ are functions of x, y, and z. This eigenvalue problem [with boundary condition (7.4.2)] is directly analogous to the one-dimensional regular Sturm–Liouville eigenvalue problem. We prefer to deal with a somewhat simpler case of (7.4.1), corresponding to $p = \sigma = 1$ and $q = 0$. We will state and prove results for (7.4.1). We leave the discussion of (7.4.3) to some exercises (in Section 7.5).

Only for very simple geometries [for example, rectangles (see Section 7.3), or circles (see Section 7.7)] can (7.4.1) be solved explicitly. In other situations, we may have to rely on numerical treatments. However, certain general properties of (7.4.1) are quite useful, all analogous to results we understand for the one-dimensional Sturm–Liouville problem. The reasons for the analogy will be discussed in the next section. We begin by simply stating the theorems for the two-dimensional case of (7.4.1) and (7.4.2):

Theorem

1. All the eigenvalues are real.
2. There exist an infinite number of eigenvalues. There is a smallest eigenvalue, but no largest one.
3. Corresponding to an eigenvalue, there *may* be many eigenfunctions (unlike regular Sturm–Liouville eigenvalue problems).
4. The eigenfunctions $\phi(x, y)$ form a "complete" set, meaning that any piecewise smooth function $f(x, y)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x, y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x, y). \tag{7.4.4}$$

   Here $\sum_\lambda a_\lambda \phi_\lambda$ means a linear combination of all the eigenfunctions. The series converges in the mean if the coefficients a_λ are chosen correctly.
5. Eigenfunctions belonging to different eigenvalues (λ_1 and λ_2) are orthogonal relative to the weight $\sigma(\sigma = 1)$ over the entire region R. Mathematically,

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \, dx \, dy = 0 \qquad \text{if } \lambda_1 \neq \lambda_2, \tag{7.4.5}$$

   where $\iint_R dx \, dy$ represents an integral over the entire region R. Furthermore, different eigenfunctions belonging to the same eigenvalue can be made orthogonal by the Gram–Schmidt process (see Section 7.5). Thus, we may assume that (7.4.5) is valid even if $\lambda_1 = \lambda_2$ as long as ϕ_{λ_1} is linearly independent of ϕ_{λ_2}.
6. An eigenvalue λ can be related to the eigenfunction by the Rayleigh quotient:

$$\lambda = \frac{-\oint \phi \nabla\phi \cdot \hat{\mathbf{n}} \, ds + \iint_R |\nabla\phi|^2 \, dx \, dy}{\iint_R \phi^2 \, dx \, dy}. \tag{7.4.6}$$

   The boundary conditions often simplify the boundary integral.

Here $\hat{\mathbf{n}}$ is a unit outward normal and $\oint ds$ is a closed line integral over the entire boundary of the plane two-dimensional region, where ds is the differential arc length. The three-dimensional result is nearly identical; $\iint$ must be replaced by $\iiint$ and the boundary line integral $\oint ds$ must be replaced by the boundary surface integral $\oiint dS$, where dS is the differential surface area.

EXAMPLE

We will prove some of these statements in Section 7.5. To understand their meaning, we will show how the example of Section 7.3 illustrates most of these theorems. For the vibrations of a rectangular ($0 < x < L$, $0 < y < H$) membrane with fixed zero boundary conditions, we have shown that the relevant eigenvalue problem is

$$\boxed{\begin{array}{c} \nabla^2\phi + \lambda\phi = 0 \\ \phi(0, y) = 0, \quad \phi(x, 0) = 0 \\ \phi(L, y) = 0, \quad \phi(x, H) = 0. \end{array}} \tag{7.4.7}$$

We have determined that the eigenvalues and corresponding eigenfunctions are

$$\boxed{\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2,} \quad \begin{array}{l} n = 1, 2, 3, \ldots \\ m = 1, 2, 3, \ldots \end{array} \quad \text{with} \quad \boxed{\phi_{nm}(x, y) = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}.} \tag{7.4.8}$$

1. **Real eigenvalues**. In our calculation of the eigenvalues for (7.4.7), we *assumed* that the eigenfunctions existed in a product form. Under that assumption, (7.4.8) showed the eigenvalues to be real. Our theorem guarantees that the eigenvalues will always be real.
2. **Ordering of eigenvalues**. There is a doubly infinite set of eigenvalues for (7.4.7), namely, $\lambda_{nm} = (n\pi/L)^2 + (m\pi/H)^2$ for $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. There is a smallest eigenvalue, $\lambda_{11} = (\pi/L)^2 + (\pi/H)^2$, but no largest eigenvalue.
3. **Multiple eigenvalues**. For $\nabla^2\phi + \lambda\phi = 0$, our theorem states that, in general, it is possible for there to be more than one eigenfunction corresponding to the same eigenvalue. To illustrate this, consider (7.4.7) in the case in which $L = 2H$. Then

$$\lambda_{nm} = \frac{\pi^2}{4H^2}(n^2 + 4m^2) \tag{7.4.9}$$

with

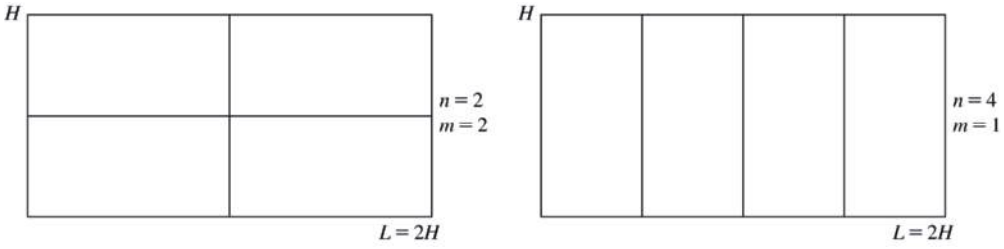
$$\phi_{nm} = \sin\frac{n\pi x}{2H}\sin\frac{m\pi y}{H}. \tag{7.4.10}$$

We note that it is possible to have different eigenfunctions corresponding to the same eigenvalue. For example, $n = 4, m = 1$ and $n = 2, m = 2$ yield the same eigenvalue:

$$\lambda_{41} = \lambda_{22} = \frac{\pi^2}{4H^2}20.$$

For $n = 4, m = 1$, the eigenfunction is $\phi_{41} = \sin 4\pi x/2H \sin \pi y/H$, while for $n = 2, m = 2$, $\phi_{22} = \sin 2\pi x/2H \sin 2\pi y/H$. The nodal curves for these eigenfunctions are sketched in Fig. 7.4.1. They are different eigenfunctions with the same eigenvalue, $\lambda = (\pi^2/4H^2)20$. It is not surprising that the eigenvalue is the same, since a membrane vibrating in these modes has cells of the same dimensions: one $H \times H/2$ and the other $H/2 \times H$. By symmetry they will have the same natural frequency (and hence the same eigenvalue since the natural frequency is $c\sqrt{\lambda}$). In fact, in general by symmetry [as well as by formula (7.4.9)], $\lambda_{(2n)m} = \lambda_{(2m)n}$.

Figure 7.4.1 Nodal curves for eigenfunctions with the same eigenvalue (symmetric).

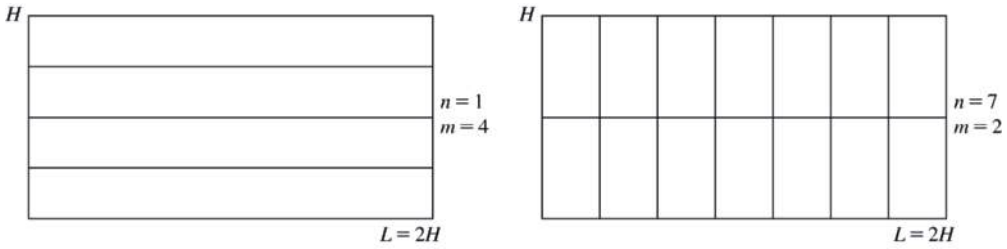


However, it is also possible for more than one eigenfunction to occur for reasons having nothing to do with symmetry. For example, $n = 1, m = 4$ and $n = 7, m = 2$ yield the same eigenvalue: $\lambda_{14} = \lambda_{72} = (\pi^2/4H^2)65$. The corresponding eigenfunctions are $\phi_{14} = \sin \pi x/2H \sin 4\pi y/H$ and $\phi_{72} = \sin 7\pi x/2H \sin 2\pi y/H$, which are sketched in Fig. 7.4.2. It is only coincidental that both of these shapes vibrate with the same frequency. In these situations, it is possible for two eigenfunctions to correspond to the same eigenvalue. We can find situations with even more multiplicities (or degeneracies). Since $\lambda_{14} = \lambda_{72} = (\pi^2/4H^2)65$, it is also true that $\lambda_{28} = \lambda_{(14)4} = (\pi^2/4H^2)260$. However, by symmetry, $\lambda_{28} = \lambda_{(16)1}$ and $\lambda_{(14)4} = \lambda_{87}$.
Thus,

$$\lambda_{28} = \lambda_{(16)1} = \lambda_{(14)4} = \lambda_{87} = \left(\frac{\pi^2}{4H^2}\right)260.$$

Here, there are four eigenfunctions corresponding to the same eigenvalue.

Figure 7.4.2 Nodal curves for eigenfunction with the same eigenvalue (asymmetric).



4a. Series of eigenfunctions. According to this theorem, (7.4.4), the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$ can always be used to represent any piecewise smooth function $f(x, y)$. In our illustrative example, (7.4.7), $\sum_\lambda$ becomes a double sum,

$$f(x, y) \sim \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}. \tag{7.4.11}$$

5. Orthogonality of eigenfunctions. We will show that the multidimensional orthogonality of the eigenfunctions, as expressed by (7.4.5) for any two different eigenfunctions, can be used to determine the generalized Fourier coefficients in (7.4.4).[1] We will do this in exactly the way we did for one-dimensional Sturm–Liouville eigenfunction expansions. We simply multiply (7.4.4) by ϕ_{λ_i} and integrate over the entire region R:

$$\iint_R f\phi_{\lambda_i}\, dx\, dy = \sum_\lambda a_\lambda \iint_R \phi_\lambda \phi_{\lambda_i}\, dx\, dy. \tag{7.4.12}$$

Since the eigenfunctions are all orthogonal to each other (with weight 1 because $\nabla^2\phi + \lambda\phi = 0$), it follows that

$$\iint_R f\phi_{\lambda_i}\, dx\, dy = a_{\lambda_i}\iint_R \phi_{\lambda_i}^2\, dx\, dy, \tag{7.4.13}$$

or, equivalently,

$$\boxed{a_{\lambda_i} = \frac{\iint_R f\phi\lambda_i\, dx\, dy}{\iint_R \phi_{\lambda_i}^2\, dx\, dy}.} \tag{7.4.14}$$

There is no difficulty in forming (7.4.14) from (7.4.13) since the denominator of (7.4.14) is necessarily positive.

For the special case that occurs for a rectangle with fixed zero boundary conditions, (7.4.7), the generalized Fourier coefficients a_{nm} are given by (7.4.14):

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\, dx\, dy}{\int_0^H \int_0^L \sin^2\frac{n\pi x}{L}\sin^2\frac{m\pi y}{H}\, dx\, dy}. \tag{7.4.15}$$

The integral in the denominator may be easily shown to equal $(L/2)(H/2)$ by calculating two one-dimensional integrals; in this way we rederive (7.3.30). In this case, (7.4.11), the generalized Fourier coefficient a_{nm} can be evaluated in two equivalent ways:

(a) Using one two-dimensional orthogonality formula for the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$

(b) Using two one-dimensional orthogonality formulas

4b. Convergence. As with any Sturm–Liouville eigenvalue problem (see Section 5.10), a *finite* series of the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$ may be used to approximate a function $f(x, y)$. In particular, we could show that if we measure error in the mean-square sense,

$$E \equiv \iint_R \left(f - \sum_\lambda a_\lambda\phi_\lambda\right)^2 dx\, dy, \tag{7.4.16}$$

[1]If there is more than one eigenfunction corresponding to the same eigenvalue, then we assume that the eigenfunctions have been made orthogonal (if necessary by the Gram–Schmidt process).

with weight function 1, then this mean-square error is minimized by the coefficients a_λ being chosen by (7.4.14), the generalized Fourier coefficients. It is known that the approximation improves as the number of terms increases. Furthermore, $E \to 0$ as all the eigenfunctions are included. We say that the series $\sum_\lambda a_\lambda \phi_\lambda$ **converges in the mean** to f. ■

Exercises 7.4

7.4.1. Consider the eigenvalue problem

$$\nabla^2\phi + \lambda\phi = 0$$
$$\frac{\partial \phi}{\partial x}(0, y) = 0, \quad \phi(x, 0) = 0$$
$$\frac{\partial \phi}{\partial x}(L, y) = 0, \quad \phi(x, H) = 0.$$

***(a)** Show that there is a doubly infinite set of eigenvalues.

(b) If $L = H$, show that most eigenvalues have more than one eigenfunction.

(c) Show that the eigenfunctions are orthogonal in a two-dimensional sense using two one-dimensional orthogonality relations.

7.4.2. Without using the explicit solution of (7.4.7), show that $\lambda \geq 0$ from the Rayleigh quotient, (7.4.6).

7.4.3. If necessary, see Section 7.5:

(a) Derive that $\iint (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \oint (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, ds$.

(b) From part (a), derive (7.4.5).

7.4.4. Derive (7.4.6). If necessary, see Section 7.6. [*Hint*: Multiply (7.4.1) by ϕ and integrate.]

7.5 Green's Formula, Self-Adjoint Operators, and Multidimensional Eigenvalue Problems

Introduction

In this section, we prove some of the properties of the multidimensional eigenvalue problem:

$$\boxed{\nabla^2\phi + \lambda\phi = 0} \tag{7.5.1}$$

with

$$\boxed{\beta_1\phi + \beta_2\nabla\phi \cdot \hat{\mathbf{n}} = 0} \tag{7.5.2}$$

on the entire boundary. Here, β_1 and β_2 are real functions of the location in space. As with Sturm–Liouville eigenvalue problems, we will simply assume that there is an infinite number of eigenvalues for (7.5.1) with (7.5.2) and that the resulting set of eigenfunctions is complete. Proofs of these statements are difficult and are beyond the intent of this text. The proofs for various other properties of the multidimensional eigenvalue problem are quite similar to corresponding proofs for the one-dimensional Sturm–Liouville eigenvalue problem. We introduce the Laplacian operator

$$L \equiv \nabla^2, \tag{7.5.3}$$

in which case the notation for the multidimensional eigenvalue problem becomes

$$L(\phi) + \lambda\phi = 0. \tag{7.5.4}$$

By comparing (7.5.4) to (7.4.3), we notice that the weight function for this multidimensional problem is expected to be 1.

Multidimensional Green's formula

The proofs for the one-dimensional Sturm–Liouville eigenvalue problem depended on $uL(v) - vL(u)$ being an exact differential (known as Lagrange's identity). The corresponding integrated form (known as Green's formula) was also needed. Similar identities will be derived for the Laplacian operator, $L = \nabla^2$, a multidimensional analog of the Sturm–Liouville differential operator. We will calculate $uL(v) - vL(u) = u\nabla^2 v - v\nabla^2 u$. We recall that $\nabla^2 u = \nabla \cdot (\nabla u)$ and $\nabla \cdot (a\boldsymbol{B}) = a\nabla \cdot \boldsymbol{B} + \nabla a \cdot \boldsymbol{B}$ (where a is a scalar and $\boldsymbol{B}$ a vector). Thus,

$$\begin{aligned} \nabla \cdot (u\nabla v) &= u\nabla^2 v + \nabla u \cdot \nabla v \\ \nabla \cdot (v\nabla u) &= v\nabla^2 u + \nabla v \cdot \nabla u. \end{aligned} \tag{7.5.5}$$

By subtracting these,

$$\boxed{u\nabla^2 v - v\nabla^2 u = \nabla \cdot (u\nabla v - v\nabla u).} \tag{7.5.6}$$

The differential form, (7.5.6), is the multidimensional version of Lagrange's identity, (5.5.7). Instead of integrating from a to b as we did for one-dimensional problems, we integrate over the entire two-dimensional region:

$$\iint_R (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \iint_R \nabla \cdot (u\nabla v - v\nabla u)\, dx\, dy.$$

The right-hand side is in the correct form to apply the divergence theorem (recall that $\iint_R \nabla \cdot \boldsymbol{A}\, dx\, dy = \oint \boldsymbol{A} \cdot \hat{\mathbf{n}}\, ds$). Thus,

$$\boxed{\iint_R (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \oint (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, ds.} \tag{7.5.7}$$

Equation (7.5.7) is analogous to Green's formula, (5.5.8). It is known as **Green's second identity**,[2] but we will just refer to it as **Green's formula**.

We have shown that $L = \nabla^2$ is a multidimensional **self-adjoint** operator in the following sense:

If u and v are any two functions such that

$$\oint (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, ds = 0, \tag{7.5.8}$$

then

$$\iint_R [u\nabla^2 v - v\nabla^2 u]\, dx\, dy = 0, \tag{7.5.9}$$

where $L = \nabla^2$.

In many problems, prescribed homogeneous boundary conditions will cause the boundary term to vanish. For example, (7.5.8) and thus (7.5.9) are valid if u and v both vanish on the boundary. Again for three-dimensional problems, $\iint$ must be replaced by $\iiint$ and $\oint$ must be replaced by $\oiint$.

Orthogonality of the eigenfunctions

As with the one-dimensional Sturm–Liouville eigenvalue problem, we can prove a number of theorems directly from Green's formula (7.5.7). To show eigenfunctions corresponding to different

[2] *Green's first identity* arises from integrating (7.5.5) [rather than (7.5.6)] with $v = u$ and applying the divergence theorem.

eigenvalues are orthogonal, we consider two eigenfunctions ϕ_1 and ϕ_2 corresponding to the eigenvalues λ_1 and λ_2:

$$\begin{aligned} \nabla^2\phi_1 + \lambda_1\phi_1 &= 0 \quad \text{or} \quad L(\phi_1) + \lambda_1\phi_1 = 0 \\ \nabla^2\phi_2 + \lambda_2\phi_2 &= 0 \quad \text{or} \quad L(\phi_2) + \lambda_2\phi_2 = 0. \end{aligned} \tag{7.5.10}$$

If both ϕ_1 and ϕ_2 satisfy the same homogeneous boundary conditions, then (7.5.8) is satisfied, in which case (7.5.9) follows. Thus

$$\iint_R (-\phi_1\lambda_2\phi_2 + \phi_2\lambda_1\phi_1)\,dx\,dy = (\lambda_1 - \lambda_2)\iint_R \phi_1\phi_2\,dx\,dy = 0.$$

If $\lambda_1 \neq \lambda_2$, then

$$\iint_R \phi_1\phi_2\,dx\,dy = 0, \tag{7.5.11}$$

which means that eigenfunctions corresponding to different eigenvalues are orthogonal (in a multidimensional sense with weight 1). If two or more eigenfunctions correspond to the same eigenvalue, they can be made orthogonal to each other (as well as all other eigenfunctions) by a procedure shown in the appendix of this section known as the Gram–Schmidt method.

We can now prove that the eigenvalues will be real. The proof is left for an exercise since the proof is identical to that used for the one-dimensional Sturm–Liouville problem (see Section 5.5).

Exercises 7.5

7.5.1. The vertical displacement of a nonuniform membrane satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

where c depends on x and y. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

(a) Show that the time variable can be separated by assuming that

$$u(x, y, t) = \phi(x, y)h(t).$$

Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \quad \text{with} \quad \phi = 0 \text{ on the boundary.} \tag{7.5.12}$$

What is $\sigma(x, y)$?

(b) If the eigenvalues are known (and $\lambda > 0$), determine the frequencies of vibration.

7.5.2. See Exercise 7.5.1. Consider the two-dimensional eigenvalue problem given in (7.5.12).

(a) Prove that the eigenfunctions belonging to different eigenvalues are orthogonal using Green's formula (with what weight?).

(b) Prove that all the eigenvalues are real.

(c) Do Exercise 7.6.1.

7.5.3. Redo Exercise 7.5.2 if the boundary condition is instead

(a) $\nabla\phi \cdot \hat{\mathbf{n}} = 0$ on the boundary

(b) $\nabla\phi \cdot \hat{\mathbf{n}} + h(x, y)\phi = 0$ on the boundary

(c) $\phi = 0$ on part of the boundary and $\nabla\phi \cdot \hat{\mathbf{n}} = 0$ on the rest of the boundary

7.5.4. Consider the heat equation in three dimensions with no sources but with nonconstant thermal properties

$$c\rho\frac{\partial u}{\partial t} = \nabla \cdot (K_0\nabla u),$$

where $c\rho$ and K_0 are functions of x, y, and z. Assume that $u = 0$ on the boundary. Show that the time variable can be separated by assuming that

$$u(x, y, z, t) = \phi(x, y, z)h(t).$$

Show that $\phi(x, y, z)$ satisfies the eigenvalue problem

$$\nabla \cdot (p\nabla\phi) + \lambda\sigma(x, y, z)\phi = 0 \qquad \text{with} \qquad \phi = 0 \quad \text{on the boundary.} \qquad (7.5.13)$$

What are $\sigma(x, y, z)$ and $p(x, y, z)$?

7.5.5. See Exercise 7.5.4. Consider the three-dimensional eigenvalue problem given in (7.5.13).

(a) Prove that the eigenfunctions belonging to different eigenvalues are orthogonal (with what weight?).

(b) Prove that all the eigenvalues are real.

(c) Do Exercise 7.6.3.

7.5.6. Derive an expression for

$$\iint [uL(v) - vL(u)]\, dx\, dy$$

over a two-dimensional region R, where

$$L = \nabla^2 + q(x, y) \quad [\text{i.e., } L(u) = \nabla^2 u + q(x, y)u].$$

7.5.7. Consider Laplace's equation $\nabla^2 u = 0$ in a three-dimensional region R (where u is the temperature). Suppose that the heat flux is given on the boundary (not necessarily a constant).

(a) Explain *physically* why $\oiint \nabla u \cdot \hat{\mathbf{n}}\, dS = 0$.

(b) Show this mathematically.

7.5.8. Suppose that in a three-dimensional region R

$$\nabla^2\phi = f(x, y, z)$$

with f given and $\nabla\phi \cdot \hat{\mathbf{n}} = 0$ on the boundary.

(a) Show mathematically that (if there is a solution)

$$\iiint_R f\, dx\, dy\, dz = 0.$$

(b) Briefly explain physically (using the heat flow model) why condition (a) must hold for a solution. What happens in a heat flow problem if

$$\iiint_R f\, dx\, dy\, dz > 0?$$

7.5.9. Show that the boundary term (7.5.8) vanishes if both u and v satisfy (7.5.2):

(a) Assume that $\beta_2 \neq 0$.

(b) Assume $\beta_2 = 0$ for part of the boundary.

7.5.10. Evaluate $\nabla \cdot (\phi\nabla\phi)$.

7.5.11. Derive Green's formula (for $L = \nabla^2$) in three dimensions.

7.5.12. For the two-dimensional eigenvalue problem $\nabla^2\phi = -\lambda\phi$ with $\phi = 0$ on the boundary, show that eigenfunctions corresponding to different eigenvalues are orthogonal using Green's formula.

Appendix to 7.5: Gram–Schmidt Method

We wish to show in general that eigenfunctions *corresponding to the same eigenvalue* can be made orthogonal. The process is known as **Gram–Schmidt orthogonalization**. Let us suppose that $\phi_1, \phi_2, \ldots, \phi_n$ are independent eigenfunctions corresponding to the *same eigenvalue.* We will form a set of n-independent eigenfunctions denoted $\psi_1, \psi_2, \ldots, \psi_n$ that are mutually orthogonal, even if $\phi_1, \ldots, \phi_n$ are not. Let $\psi_1 = \phi_1$ be any one eigenfunction. Any linear combination of the

eigenfunctions is also an eigenfunction (since they satisfy the *same* linear homogeneous differential equation and boundary conditions). Thus, $\psi_2 = \phi_2 + c\psi_1$ is also an eigenfunction (automatically independent of ψ_1), where c is an arbitrary constant. We choose c so that $\psi_2 = \phi_2 + c\psi_1$ is orthogonal to ψ_1: $\iint_R \psi_1\psi_2 \, dx \, dy = 0$ becomes

$$\iint_R (\phi_2 + c\psi_1)\psi_1 \, dx \, dy = 0.$$

c is uniquely determined:

$$c = \frac{-\iint_R \phi_2\psi_1 \, dx \, dy}{\iint_R \psi_1^2 \, dx \, dy}. \tag{7.5.14}$$

Since there may be more than two eigenfunctions corresponding to the same eigenvalue, we continue this process.

A third eigenfunction is $\psi_3 = \phi_3 + c_1\psi_1 + c_2\psi_2$, where we choose c_1 and c_2 so that ψ_3 is orthogonal to the previous two,

$$\iint_R \psi_3\psi_1 \, dx \, dy = 0 \quad \text{and} \quad \iint_R \psi_3\psi_2 \, dx \, dy = 0.$$

Thus,

$$\iint_R (\phi_3 + c_1\psi_1 + c_2\psi_2)\psi_1 \, dx \, dy = 0 \quad \text{and} \quad \iint_R (\phi_3 + c_1\psi_1 + c_2\psi_2)\psi_2 \, dx \, dy = 0$$

However, ψ_2 is already orthogonal to ψ_1, and hence

$$\iint_R \phi_3\psi_1 \, dx \, dy + c_1 \iint_R \psi_1^2 \, dx \, dy = 0$$
$$\iint_R \phi_3\psi_2 \, dx \, dy + c_2 \iint_R \psi_2^2 \, dx \, dy = 0,$$

easily determining the two constants. This process can be continued to determine n mutually orthogonal eigenfunctions. In general,

$$\psi_j = \phi_j - \sum_{i=1}^{j-1} \left(\frac{\iint_R \phi_j\psi_i \, dx \, dy}{\iint_R \psi_i^2 \, dx \, dy} \right) \psi_i.$$

This calculation is called the Gram–Schmidt orthogonalization process. It can be understood in terms of the *projection* of one function ϕ_j onto an orthogonal set of functions $S = \{\psi_1, \psi_2, \ldots, \psi_{j-1}\}$,

$$\text{proj}_S \phi_j = \sum_{i=1}^{j-1} \left(\frac{\iint_R \phi_j\psi_i \, dx \, dy}{\iint_R \psi_i^2 \, dx \, dy} \right) \psi_i.$$

The projection finds the linear combination of $\psi_1, \ldots, \psi_{j-1}$ that is closest (in a least-squares sense) to ϕ_j, and has the property that

$$\phi_j - \text{proj}_S \phi_j \quad \text{is orthogonal to any function in } S.$$

Thus, by construction, ψ_j is orthogonal to $\psi_1, \ldots, \psi_{j-1}$. That is why, even in the case of a multiple eigenvalue, we are always able to restrict our attention to orthogonal eigenfunctions, if necessary by this Gram–Schmidt process.

7.6 Rayleigh Quotient and Laplace's Equation

7.6.1 RAYLEIGH QUOTIENT

In Section 5.6, we obtained the Rayleigh quotient for the one-dimensional Sturm–Liouville eigenvalue problem. The result was obtained by multiplying the differential equation by ϕ, integrating over the entire region, solving for λ, and simplifying using integration by parts. We will derive a similar result for

$$\nabla^2\phi + \lambda\phi = 0. \tag{7.6.1}$$

We proceed as before by multiplying (7.6.1) by ϕ. Integrating over the entire two-dimensional region and solving for λ yields

$$\lambda = \frac{-\iint_R \phi \nabla^2 \phi \, dx \, dy}{\iint_R \phi^2 \, dx \, dy}. \tag{7.6.2}$$

Next, we want to generalize integration by parts to multi-dimensional functions. Integration by parts is based on the product rule for the derivative, $d/dx(fg) = f\,dg/dx + g\,df/dx$. Instead of using the derivative, we use a product rule for the divergence of a scalar function f multiplied by a vector function $\boldsymbol{g}$,

$$\nabla \cdot (f\boldsymbol{g}) = f\nabla \cdot \boldsymbol{g} + \boldsymbol{g} \cdot \nabla f.$$

Letting $f = \phi$ and $\boldsymbol{g} = \nabla\phi$, if follows that $\nabla \cdot (\phi\nabla\phi) = \phi\,\nabla \cdot (\nabla\phi) + \nabla\phi \cdot \nabla\phi$. Since $\nabla \cdot (\nabla\phi) = \nabla^2\phi$ and $\nabla\phi \cdot \nabla\phi = |\nabla\phi|^2$,

$$\phi\nabla^2\phi = \nabla \cdot (\phi\nabla\phi) - |\nabla\phi|^2. \tag{7.6.3}$$

Using (7.6.3) in (7.6.2) yields an alternative expression for the eigenvalue,

$$\lambda = \frac{-\iint_R \nabla \cdot (\phi\nabla\phi) \, dx \, dy + \iint_R |\nabla\phi|^2 \, dx \, dy}{\iint_R \phi^2 \, dx \, dy}. \tag{7.6.4}$$

Now we use (again) the divergence theorem to evaluate the first integral in the numerator of (7.6.4). Since $\iint_R \nabla \cdot \boldsymbol{A} \, dx \, dy = \oint \boldsymbol{A} \cdot \hat{\mathbf{n}} \, ds$, it follows that

$$\boxed{\lambda = \frac{-\oint \phi\nabla\phi \cdot \hat{\mathbf{n}} \, ds + \iint_R |\nabla\phi|^2 \, dx \, dy}{\iint_R \phi^2 \, dx \, dy},} \tag{7.6.5}$$

known as the **Rayleigh quotient**. This is quite similar to the Rayleigh quotient for Sturm–Liouville eigenvalue problems. Note that there is a boundary contribution for each: $-p\phi \; d\phi/dx|_a^b$ for (5.6.3) and $-\oint \phi\nabla\phi \cdot \hat{\mathbf{n}} \, ds$ for (7.6.5).

EXAMPLE

We consider any region in which the boundary condition is $\phi = 0$ on the entire boundary. Then $\oint \phi\nabla\phi \cdot \hat{\mathbf{n}} \, ds = 0$, and hence from (7.6.5), $\lambda \geq 0$. If $\lambda = 0$, then (7.6.5) implies that

$$0 = \iint_R |\nabla\phi|^2 \, dx \, dy. \tag{7.6.6}$$

Thus,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} = \mathbf{0} \tag{7.6.7}$$

everywhere. From (7.6.7), it follows that $\partial\phi/\partial x = 0$ and $\partial\phi/\partial y = 0$ everywhere. Thus, ϕ is a constant everywhere, but since $\phi = 0$ on the boundary, $\phi = 0$ everywhere. $\phi = 0$ everywhere is not an eigenfunction, and thus we have shown that $\lambda = 0$ is not an eigenvalue. In conclusion, $\lambda > 0$. ■

7.6.2 EQUILIBRIUM SOLUTIONS AND LAPLACE'S EQUATION

Equilibrium solutions of the time-dependent heat equation satisfy Laplace's equation. Solving Laplace's equation $\nabla^2\phi = 0$ subject to homogeneous boundary conditions corresponds to investigating whether $\lambda = 0$ is an eigenvalue for (7.6.1).

Zero temperature boundary condition

Consider $\nabla^2\phi = 0$ subject to $\phi = 0$ along the entire boundary. It can be concluded from (7.6.6) that $\phi = 0$ everywhere inside the region (since $\lambda = 0$ is not an eigenvalue). For an object of any

shape subject to the zero temperature boundary condition on the entire boundary, the steady-state (equilibrium) temperature distribution is zero temperature, which is somewhat obvious physically. For the time-dependent heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$, (7.6.1) arises by separation of variables, and since $\lambda > 0$ (from the Rayleigh quotient), we can prove that $u(x, y, t) \to 0$ as $t \to \infty$, the time-dependent temperature approaches the equilibrium temperature distribution for large time.

Insulated boundary condition

Consider $\nabla^2\phi = 0$ subject to $\nabla\phi \cdot \hat{\mathbf{n}} = 0$ along the entire boundary. It can be concluded from (7.6.6) that $\phi = c =$ arbitrary constant everywhere inside the region (since $\lambda = 0$ is an eigenvalue and $\phi = c$ is the eigenfunction). The constant equilibrium solution can be determined from the initial value problem for the time-dependent diffusion (heat) equation $\frac{\partial u}{\partial t} = k\nabla^2 u$. The fundamental integral conservation law (see Section 1.5), using the entire region is $\frac{d}{dt}\iint u\, dx\, dy = k\oint \nabla u \cdot \hat{\mathbf{n}}\, ds = 0$, where we have used the insulated boundary condition. Thus, the total thermal energy $\iint u\, dx\, dy$ is constant in time, and its equilibrium value $(t \to \infty)$, $\iint c\, dx\, dy = cA$, equals its initial value $(t = 0)$, $\iint f(x, y)\, dx\, dy$. In this way, $c = \frac{1}{A}\iint f(x, y)\, dx\, dy$, so that the constant equilibrium temperature with insulated boundaries must be the average of the initial temperature distribution. Here, A is the area of the two-dimensional region. For the time-dependent heat equation with insulated boundary conditions, it can be shown that $u(x, y, t) \to c = \frac{1}{A}\iint f(x, y)\, dx\, dy$ as $t \to \infty$ since $\lambda \geq 0$ (with $\phi = 1$ corresponding to $\lambda = 0$ from the Rayleigh quotient). That is, the time-dependent temperature approaches the equilibrium temperature distribution for large time.

Similar results hold in three dimensions.

Exercises 7.6

7.6.1. See Exercise 7.5.1. Consider the two-dimensional eigenvalue problem with $\sigma > 0$

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \text{ with } \phi = 0 \text{ on the boundary.}$$

(a) Prove that $\lambda \geq 0$.

(b) Is $\lambda = 0$ an eigenvalue, and if so, what is the eigenfunction?

7.6.2. Redo Exercise 7.6.1 if the boundary condition is instead

(a) $\nabla\phi \cdot \hat{\mathbf{n}} = 0$ on the boundary

(b) $\nabla\phi \cdot \hat{\mathbf{n}} + h(x, y)\phi = 0$ on the boundary

(c) $\phi = 0$ on part of the boundary and $\nabla\phi \cdot \hat{\mathbf{n}} = 0$ on the rest of the boundary

7.6.3. Redo Exercise 7.6.1 if the differential equation is

$$\nabla \cdot (p\nabla\phi) + \lambda\sigma(x, y, z)\phi = 0$$

with boundary condition

(a) $\phi = 0$ on the boundary

(b) $\nabla\phi \cdot \hat{\mathbf{n}} = 0$ on the boundary

7.6.4. **(a)** If $\nabla^2\phi = 0$ with $\phi = 0$ on the boundary, prove that $\phi = 0$ everywhere. [*Hint*: Use the fact that $\lambda = 0$ is not an eigenvalue for $\nabla^2\phi = -\lambda\phi$.]

(b) Prove that there cannot be two different solutions of the problem

$$\nabla^2 u = f(x, y, z)$$

subject to the given boundary condition $u = g(x, y, z)$ on the boundary. [*Hint*: Consider $u_1 - u_2$ and use part (a).]

7.6.5. Derive the Rayleigh quotient in three dimensions for $\nabla^2\phi = -\lambda\phi$.

7.7 Vibrating Circular Membrane and Bessel Functions

7.7.1 INTRODUCTION

An interesting application of both one-dimensional (Sturm–Liouville) and multi-dimensional eigenvalue problems occurs when considering the vibrations of a circular membrane. The vertical displacement u satisfies the two-dimensional wave equation,

$$\text{PDE: } \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.} \tag{7.7.1}$$

The geometry suggests that we use polar coordinates, in which case $u = u(r, \theta, t)$. We assume that the membrane has zero displacement at the circular boundary, $r = a$:

$$\text{BC: } \boxed{u(a, \theta, t) = 0.} \tag{7.7.2}$$

The initial position and velocity are given:

$$\text{IC: } \boxed{\begin{aligned} u(r, \theta, 0) &= \alpha(r, \theta) \\ \frac{\partial u}{\partial t}(r, \theta, 0) &= \beta(r, \theta). \end{aligned}} \tag{7.7.3}$$

7.7.2 SEPARATION OF VARIABLES

We first separate out the time variable by seeking product solutions,

$$u(r, \theta, t) = \phi(r, \theta) h(t). \tag{7.7.4}$$

Then, as shown earlier, $h(t)$ satisfies

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h, \tag{7.7.5}$$

where λ is a separation constant. From (7.7.5), the natural frequencies of vibration are $c\sqrt{\lambda}$ (if $\lambda > 0$). In addition, $\phi(r, \theta)$ satisfies the two-dimensional eigenvalue problem

$$\boxed{\nabla^2 \phi + \lambda \phi = 0,} \tag{7.7.6}$$

with $\phi = 0$ on the entire boundary, $r = a$:

$$\phi(a, \theta) = 0. \tag{7.7.7}$$

We will attempt to obtain solutions of (7.7.6) in the product form appropriate for polar coordinates,

$$\phi(r, \theta) = f(r) g(\theta), \tag{7.7.8}$$

since for the circular membrane $0 < r < a, -\pi < \theta < \pi$. This is equivalent to originally seeking solutions to the wave equation in the form of a product of functions of each independent variable, $u(r, \theta, t) = f(r) g(\theta) h(t)$. We substitute (7.7.8) into (7.7.6); in polar coordinates $\nabla^2 \phi = 1/r \ \partial/\partial r (r \ \partial \phi/\partial r) + 1/r^2 \ \partial^2 \phi/\partial \theta^2$, and thus $\nabla^2 \phi + \lambda \phi = 0$ becomes

$$\frac{g(\theta)}{r} \frac{d}{dr}\left(r \frac{df}{dr}\right) + \frac{f(r)}{r^2} \frac{d^2 g}{d\theta^2} + \lambda f(r) g(\theta) = 0. \tag{7.7.9}$$

r and θ may be separated by multiplying by r^2 and dividing by $f(r) g(\theta)$:

$$-\frac{1}{g} \frac{d^2 g}{d\theta^2} = \frac{r}{f} \frac{d}{dr}\left(r \frac{df}{dr}\right) + \lambda r^2 = \mu. \tag{7.7.10}$$

We introduce a second separation constant in the form μ because our experience with circular regions (see Sections 2.4.2 and 2.5.2) suggests that $g(\theta)$ must oscillate in order to satisfy the periodic conditions in θ. Our three differential equations, with two separation constants, are thus

$$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h} \tag{7.7.11}$$

$$\boxed{\frac{d^2g}{d\theta^2} = -\mu g} \tag{7.7.12}$$

$$\boxed{r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu)f = 0.} \tag{7.7.13}$$

Two of these equations must be eigenvalue problems. However, ignoring the initial conditions, the only given boundary condition is $f(a) = 0$, which follows from $u(a, \theta, t) = 0$ or $\phi(a, \theta) = 0$. We must remember that $-\pi < \theta < \pi$ and $0 < r < a$. Thus, both θ and r are defined over finite intervals. As such, there should be boundary conditions at both ends. The periodic nature of the solution in θ implies that

$$g(-\pi) = g(\pi) \tag{7.7.14}$$

$$\frac{dg}{d\theta}(-\pi) = \frac{dg}{d\theta}(\pi). \tag{7.7.15}$$

We already have a condition at $r = a$. Polar coordinates are singular at $r = 0$; a singularity condition must be introduced there. Since the displacement must be finite, we conclude that

$$|f(0)| < \infty.$$

7.7.3 EIGENVALUE PROBLEMS (ONE-DIMENSIONAL)

After separating variables, we have obtained two eigenvalue problems. We are quite familiar with the θ-eigenvalue problem, (7.7.12) with (7.7.14) and (7.7.15). Although it is not a regular Sturm–Liouville problem due to the periodic boundary conditions, we know that the eigenvalues are

$$\mu_m = m^2, \quad m = 0, 1, 2, \ldots. \tag{7.7.16}$$

The corresponding eigenfunctions are both

$$g(\theta) = \sin m\theta \quad \text{and} \quad g(\theta) = \cos m\theta, \tag{7.7.17}$$

although for $m = 0$, this reduces to one eigenfunction (not two as for $m \neq 0$). This eigenvalue problem generates a full Fourier series in θ, as we already know. m is the number of crests in the θ-direction.

For each integral value of m, (7.7.13) helps to define an eigenvalue problem for λ:

$$\boxed{r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - m^2)f = 0} \tag{7.7.18}$$

$$\boxed{f(a) = 0} \tag{7.7.19}$$

$$\boxed{|f(0)| < \infty.} \tag{7.7.20}$$

Since (7.7.18) has nonconstant coefficients, it is not surprising that (7.7.18) is somewhat difficult to analyze. Equation (7.7.18) can be put in the Sturm–Liouville form by dividing it by r:

$$\boxed{\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0,} \tag{7.7.21}$$

or $Lf + \lambda rf = 0$, where $L = d/dr(r\, d/dr) - m^2/r$. By comparison to the general Sturm–Liouville differential equation,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q\phi + \lambda\sigma\phi = 0,$$

with independent variable r, we have that $x = r$, $p(r) = r$, $\sigma(r) = r$, and $q(r) = -m^2/r$. Our problem is not a regular Sturm–Liouville problem due to the behavior at the origin ($r = 0$):

1. The boundary condition (7.7.20) at $r = 0$, is not of the correct form.
2. $p(r) = 0$ and $\sigma(r) = 0$ at $r = 0$ (and hence are not positive everywhere).
3. $q(r)$ approaches ∞ as $r \to 0$ [and hence $q(r)$ is not continuous] for $m \neq 0$.

However, we claim that all the statements concerning regular Sturm–Liouville problems are still valid for this important singular Sturm–Liouville problem. To begin with, there are an infinite number of eigenvalues (for each m). We designate the eigenvalues as λ_{nm}, where $m = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$, and the eigenfunctions $f_{nm}(r)$. *For each fixed* m, these eigenfunctions are orthogonal with weight r [see (7.7.21)], since it can be shown that the boundary terms vanish in Green's formula (see Exercise 5.5.1). Thus,

$$\int_0^a f_{mn_1} f_{mn_2} r \, dr = 0 \quad \text{for } n_1 \neq n_2. \tag{7.7.22}$$

Shortly, we will state more explicit facts about these eigenfunctions.

7.7.4 BESSEL'S DIFFERENTIAL EQUATION

The r-dependent separation of variables solution satisfies a "singular" Sturm–Liouville differential equation, (7.7.21). An alternative form is obtained by using the product rule of differentiation and by multiplying by r:

$$\boxed{r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + (\lambda r^2 - m^2) f = 0.} \tag{7.7.23}$$

There is some additional analysis of (7.7.23) that can be performed. Equation (7.7.23) contains two parameters, m and λ. We already know that m is an integer, but the allowable values of λ are as yet unknown. It would be quite tedious to solve (7.7.23) numerically for various values of λ (for different integral values of m). Instead, we might notice that the simple scaling transformation,

$$\boxed{z = \sqrt{\lambda} r,} \tag{7.7.24}$$

removes the dependence of the differential equation on λ:

$$\boxed{z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0.} \tag{7.7.25}$$

We note that the change of variables (7.7.24) may be performed[3] since we showed in Section 7.6 from the multidimensional Rayleigh quotient that $\lambda > 0$ (for $\nabla^2 \phi + \lambda \phi = 0$) anytime $\phi = 0$ on the entire boundary, as it is here. We can also show that $\lambda > 0$ for this problem using the one-dimensional Rayleigh quotient based on (7.7.18)–(7.7.20) (see Exercise 7.7.13). Equation (7.7.25) has the advantage of not depending on λ; less work is necessary to compute solutions of (7.7.25) than of (7.7.23). However, we have gained more than that because (7.7.25) has been investigated for over 150 years. It is now known as **Bessel's differential equation of order *m***.

7.7.5 SINGULAR POINTS AND BESSEL'S DIFFERENTIAL EQUATION

In this subsection, we *briefly* develop some of the properties of Bessel's differential equation. Equation (7.7.25) is a second-order linear differential equation with variable coefficients. We will not be able to obtain an exact closed-form solution of (7.7.25) *involving elementary functions.* To analyze a differential equation, one of the first things we should do is search for any special values of z that might cause some difficulties. $z = 0$ is a singular point of (7.7.25).

[3]In other problems, if $\lambda = 0$, then the transformation (7.7.24) is invalid. However, (7.7.24) is unnecessary for $\lambda = 0$ since in this case (7.7.23) becomes an equidimensional equation and can be solved as in Section 2.5.2.

Perhaps we should define a singular point of a differential equation. We refer to the standard form:

$$\frac{d^2f}{dz^2} + a(z)\frac{df}{dz} + b(z)f = 0.$$

If $a(z)$ and $b(z)$ and all their derivatives are finite at $z = z_0$, then $z = z_0$ is called an *ordinary point*. Otherwise, $z = z_0$ is a singular point. For Bessel's differential equation, $a(z) = 1/z$ and $b(z) = 1 - m^2/z^2$. All finite[4] z except $z = 0$ are ordinary points. $z = 0$ is a singular point [since, for example, $a(0)$ does not exist].

In the neighborhood of any ordinary point, it is known from the theory of differential equations that all solutions of the differential equation are well behaved [i.e., $f(z)$ and all its derivatives exist at any ordinary point]. We thus are guaranteed that all solutions of Bessel's differential equation are well behaved at every finite point except possibly at $z = 0$. The only difficulty can occur in the neighborhood of $z = 0$. We will investigate the expected behavior of solutions of Bessel's differential equation in the neighborhood of $z = 0$. We will describe a crude (but important) approximation. If z is very close to 0, then we should expect that $z^2 f$ in Bessel's differential equation can be ignored, since it is much smaller than $m^2 f$.[5] We do not ignore $z^2\, d^2f/dz^2$ or $z\, df/dz$ because although z is small, it is possible that derivatives of f are large enough so that $z\, df/dz$ is as large as $-m^2 f$. Dropping $z^2 f$ yields

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} - m^2 f \approx 0, \tag{7.7.26}$$

a valid approximation near $z = 0$. The advantage of this *approximation* is that (7.7.26) is exactly solvable, since it is an equidimensional (also known as a Cauchy or Euler) equation (see Section 2.5.2). Equation (7.7.26) can be solved by seeking solutions in the form

$$f \approx z^s. \tag{7.7.27}$$

By substituting (7.7.27) into (7.7.26), we obtain a quadratic equation for s,

$$s(s-1) + s - m^2 = 0, \tag{7.7.28}$$

known as the **indicial equation**. Thus, $s^2 = m^2$, and the two roots (indices) are $s = \pm m$. If $m \neq 0$ (in which case we assume $m > 0$), then we obtain two independent approximate solutions:

$$f \approx z^m \quad \text{and} \quad f \approx z^{-m} \quad (m > 0). \tag{7.7.29}$$

However, if $m = 0$, we obtain only one independent solution $f \approx z^0 = 1$. A second solution is easily derived from (7.7.26). If $m = 0$,

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} \approx 0 \quad \text{or} \quad z\frac{d}{dz}\left(z\frac{df}{dz}\right) \approx 0.$$

Thus, $z\, df/dz$ is constant, and, in addition to $f \approx 1$, it is also possible for $f \approx \ln z$. In summary, for $m = 0$, two independent solutions have the expected behavior near $z = 0$,

$$f \approx 1 \quad \text{and} \quad f \approx \ln z \quad (m = 0). \tag{7.7.30}$$

The general solution of Bessel's differential equation will be a linear combination of two independent solutions, satisfying (7.7.29) if $m \neq 0$ and (7.7.30) if $m = 0$. We have obtained the expected approximate behavior only near $z = 0$. More will be discussed in the next subsection. Because of the singular point at $z = 0$, it is possible for solutions not to be well behaved at $z = 0$. We see from (7.7.29) and (7.7.30) that independent solutions of Bessel's differential equation can be chosen such that one is well behaved at $z = 0$ and one solution is not well behaved at $z = 0$ [note that for one solution, $\lim_{z\to 0} f(z) = \pm\infty$].

---

[4]With an appropriate definition, it can be shown that $z = \infty$ is not an ordinary point for Bessel's differential equation.

[5]Even if $m = 0$, we still claim that $z^2 f$ can be neglected near $z = 0$, and the result will give a reasonable approximation.

7.7.6 BESSEL FUNCTIONS AND THEIR ASYMPTOTIC PROPERTIES (NEAR $z = 0$)

We continue to discuss Bessel's differential equation of order m,

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0. \tag{7.7.31}$$

As motivated by the previous discussion, we claim there are two types of solutions, solutions that are well behaved near $z = 0$ and solutions that are singular at $z = 0$. Different values of m yield a different differential equation. Its corresponding solution will depend on m. We introduce the standard notation for a *well-behaved solution* of (7.7.31), $J_m(z)$, called the **Bessel function of the first kind of order *m***. In a similar vein, we introduce the notation for a singular solution of Bessel's differential equation, $Y_m(z)$, called the **Bessel function of the second kind of order *m***. You can solve a lot of problems using Bessel's differential equation by just remembering that $Y_m(z)$ *approaches* $\pm\infty$ *as* $z \to 0$.

The general solution of any linear homogeneous second-order differential equation is a linear combination of two independent solutions. Thus, the general solution of Bessel's differential equation (7.7.31) is

$$\boxed{f = c_1 J_m(z) + c_2 Y_m(z).} \tag{7.7.32}$$

Precise definitions of $J_m(z)$ and $Y_m(z)$ are given in Section 7.8. However, for our immediate purposes, we simply note that they satisfy the following asymptotic properties *for small* z $(z \to 0)$:

$$\begin{aligned} J_m(z) &\sim \begin{cases} 1 & m = 0 \\ \frac{1}{2^m m!} z^m & m > 0 \end{cases} \\ Y_m(z) &\sim \begin{cases} \frac{2}{\pi} \ln z & m = 0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m > 0. \end{cases} \end{aligned} \tag{7.7.33}$$

It should be seen that (7.7.33) is consistent with our approximate behavior, (7.7.29) and (7.7.30). We see that $J_m(z)$ is bounded as $z \to 0$, whereas $Y_m(z)$ is not.

7.7.7 EIGENVALUE PROBLEM INVOLVING BESSEL FUNCTIONS

In this section, we determine the eigenvalues of the singular Sturm–Liouville problem (m fixed):

$$\boxed{\frac{d}{dr}\left(r \frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0} \tag{7.7.34}$$

$$\boxed{f(a) = 0} \tag{7.7.35}$$

$$\boxed{|f(0)| < \infty.} \tag{7.7.36}$$

By the change of variables $z = \sqrt{\lambda} r$, (7.7.34) becomes Bessel's differential equation,

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0.$$

The general solution is a linear combination of Bessel functions, $f = c_1 J_m(z) + c_2 Y_m(z)$. The scale change implies that in terms of the radial coordinate r,

$$\boxed{f = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r).} \tag{7.7.37}$$

Applying the homogeneous boundary conditions (7.7.35) and (7.7.36) will determine the eigenvalues. $f(0)$ must be finite. However, $Y_m(0)$ is infinite. Thus, $c_2 = 0$, implying that

$$\boxed{f = c_1 J_m(\sqrt{\lambda} r).} \tag{7.7.38}$$

Thus, the condition $f(a) = 0$ determines the eigenvalues:

$$\boxed{J_m(\sqrt{\lambda} a) = 0.} \tag{7.7.39}$$

We see that $\sqrt{\lambda}a$ must be a zero of the Bessel function $J_m(z)$. Later in Section 7.8.1, we show that a Bessel function is a decaying oscillation. There is an infinite number of zeros of each Bessel function $J_m(z)$. Let z_{mn} designate the nth zero of $J_m(z)$. Then

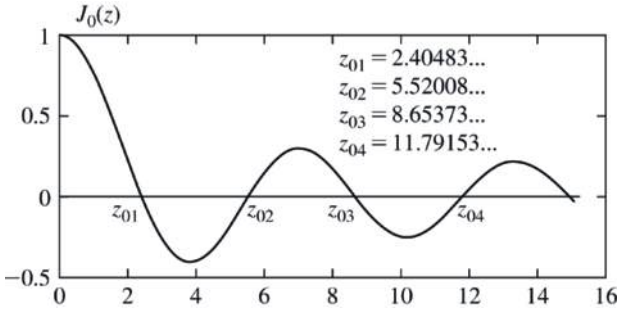
$$\sqrt{\lambda}a = z_{mn} \quad \text{or} \quad \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2. \tag{7.7.40}$$

For each m, there is an infinite number of eigenvalues, and (7.7.40) is analogous to $\lambda = (n\pi/L)^2$, where $n\pi$ are the zeros of $\sin x$.

EXAMPLE

Consider $J_0(z)$, sketched in detail in Fig. 7.7.1. From accurate tables, it is known that the first zero of $J_0(z)$ is $z = 2.4048255577\ldots$. Other zeros are recorded in Fig. 7.7.1. The eigenvalues are $\lambda_{0n} = (z_{0n}/a)^2$. Separate tables of the zeros are available. The *Handbook of Mathematical Functions* (Abramowitz and Stegun, 1974) is one source. Alternatively, over 700 pages are devoted to Bessel functions in *A Treatise on the Theory of Bessel Functions* by Watson (1995).

Figure 7.7.1 Sketch of $J_0(z)$ and its zeros.



Eigenfunctions

The eigenfunctions are thus

$$J_m\left(\sqrt{\lambda_{mn}}r\right) = J_m\left(z_{mn}\frac{r}{a}\right), \tag{7.7.41}$$

for $m = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$. For each m, these are an infinite set of eigenfunctions for the singular Sturm–Liouville problem, (7.7.34)–(7.7.36). *For fixed m*, they are orthogonal with weight r [as already discussed, see (7.7.22)]:

$$\int_0^a J_m\left(\sqrt{\lambda_{mp}}r\right)J_m\left(\sqrt{\lambda_{mq}}r\right)r\,dr = 0, \quad p \neq q. \tag{7.7.42}$$

It is known that this infinite set of eigenfunctions (m fixed) is complete. Thus, any piecewise smooth function of r can be represented by a generalized Fourier series of the eigenfunctions:

$$\alpha(r) = \sum_{n=1}^{\infty} a_n J_m\left(\sqrt{\lambda_{mn}}r\right), \tag{7.7.43}$$

where m is fixed. This is sometimes known as a **Fourier–Bessel series**. The coefficients can be determined by the orthogonality of the Bessel functions (with weight r):

$$a_n = \frac{\int_0^a \alpha(r)J_m\left(\sqrt{\lambda_{mn}}r\right)r\,dr}{\int_0^a J_m^2\left(\sqrt{\lambda_{mn}}r\right)r\,dr}. \tag{7.7.44}$$

This illustrates the one-dimensional orthogonality of the Bessel functions. We omit the evaluation of the normalization integrals $\int_0^a \left(\sqrt{\lambda_{mn}}r\right)\,r\,dr$ (e.g., see Churchill [1972] or Berg and McGregor [1966]).

7.7.8 INITIAL VALUE PROBLEM FOR A VIBRATING CIRCULAR MEMBRANE

The vibrations $u(r, \theta, t)$ of a circular membrane are described by the two-dimensional wave equation, (7.7.1), with u being fixed on the boundary, (7.7.2), subject to the initial conditions (7.7.3). When we apply the method of separation of variables, we obtain four families of product solutions, $u(r, \theta, t) = f(r)g(\theta)h(t)$:

$$J_m\left(\sqrt{\lambda_{mn}}r\right)\left\{\begin{array}{c}\cos m\theta\\ \sin m\theta\end{array}\right\}\left\{\begin{array}{c}\cos c\sqrt{\lambda_{mn}}t\\ \sin c\sqrt{\lambda_{mn}}t\end{array}\right\}. \tag{7.7.45}$$

To simplify the algebra, we will assume that the membrane is initially at rest,

$$\frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta) = 0.$$

Thus, the $\sin c\sqrt{\lambda_{mn}}t$-terms in (7.7.45) will not be necessary. Then according to the principle of superposition, we attempt to satisfy the initial value problem by considering the infinite linear combination of the remaining product solutions:

$$\begin{aligned} u(r, \theta, t) = &\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta\cos c\sqrt{\lambda_{mn}}t\\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta\cos c\sqrt{\lambda_{mn}}t. \end{aligned} \tag{7.7.46}$$

The initial position $u(r, \theta, 0) = \alpha(r, \theta)$ implies that

$$\begin{aligned} \alpha(r, \theta) = &\sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty}A_{mn}J_m\left(\sqrt{\lambda_{mn}}r\right)\right)\cos m\theta\\ &+\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}B_{mn}J_m\left(\sqrt{\lambda_{mn}}r\right)\right)\sin m\theta. \end{aligned} \tag{7.7.47}$$

By properly arranging the terms in (7.7.47), we see that this is an ordinary Fourier series in θ. Their Fourier coefficients are Fourier–Bessel series (note that m is fixed). Thus, the coefficients may be determined by the orthogonality of $J_m\left(\sqrt{\lambda_{mn}}r\right)$ with weight r [as in (7.7.44)]. As such, we can determine the coefficients by repeated application of one-dimensional orthogonality. Two families of coefficients A_{mn} and B_{mn} (including $m = 0$) can be determined from one initial condition since the periodicity in θ yielded two eigenfunctions corresponding to each eigenvalue.

However, it is somewhat easier to determine all the coefficients using two-dimensional orthogonality. Recall that for the two-dimensional eigenvalue problem,

$$\nabla^2\phi + \lambda\phi = 0$$

with $\phi = 0$ on the circle of radius a, the two-dimensional eigenfunctions are the doubly infinite families

$$\phi_\lambda(r, \theta) = J_m\left(\sqrt{\lambda_{mn}}r\right)\left\{\begin{array}{c}\cos m\theta\\ \sin m\theta\end{array}\right\}.$$

Thus,

$$\alpha(r, \theta) = \sum_\lambda A_\lambda\phi_\lambda(r, \theta), \tag{7.7.48}$$

where $\sum_\lambda$ stands for a summation over all eigenfunctions [actually two double sums, including both $\sin m\theta$ and $\cos m\theta$ as in (7.7.47)]. These eigenfunctions $\phi_\lambda(r, \theta)$ are orthogonal (in a two-dimensional sense) with weight 1. We then immediately calculate A_λ (representing both A_{mn} and B_{mn}),

$$A_\lambda = \frac{\iint \alpha(r, \theta)\phi_\lambda(r, \theta)\, dA}{\iint \phi_\lambda^2(r, \theta)\, dA}. \tag{7.7.49}$$

Here, $dA = r\,dr\,d\theta$. In two dimensions, the weighting function is constant. However, for geometric reasons, $dA = r\,dr\,d\theta$. Thus, **the weight r that appears in the one-dimensional orthogonality of Bessel functions is just a geometric factor**.

7.7.9 CIRCULARLY SYMMETRIC CASE

In this subsection, as an example, we consider the vibrations of a circular membrane, with $u = 0$ on the circular boundary, in the case in which the initial conditions are circularly symmetric (meaning independent of θ). We could consider this as a special case of the general problem, analyzed in Section 7.7.8. An alternative method, which yields the same result, is to reformulate the problem. The symmetry of the problem, including the initial conditions, suggests that the entire solution should be circularly symmetric; there should be no dependence on the angle θ. Thus,

$$u = u(r, t) \quad \text{and} \quad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \quad \text{since} \quad \frac{\partial^2 u}{\partial \theta^2} = 0.$$

The mathematical formulation is thus

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)} \tag{7.7.50}$$

$$\text{BC:} \quad \boxed{u(a, t) = 0} \tag{7.7.51}$$

$$\text{IC:} \quad \boxed{\begin{aligned} u(r, 0) &= \alpha(r) \\ \frac{\partial u}{\partial t}(r, 0) &= \beta(r). \end{aligned}} \tag{7.7.52}$$

We note that the partial differential equation has two independent variables. We need not study this problem in this chapter, which is reserved for problems with more than two independent variables. We could have analyzed this problem earlier. However, as we will see, Bessel functions are the radially dependent functions, and thus it is more natural to discuss this problem in the present part of this text.

We will apply the method of separation of variables to (7.7.50)–(7.7.52). Looking for product solutions,

$$u(r, t) = \phi(r)h(t), \tag{7.7.53}$$

yields

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda, \tag{7.7.54}$$

where $-\lambda$ is introduced because we suspect that the displacement oscillates in time. The time-dependent equation,

$$\frac{d^2h}{dt^2} = -\lambda c^2 h,$$

has solutions $\sin c\sqrt{\lambda}t$ and $\cos c\sqrt{\lambda}t$ if $\lambda > 0$. The eigenvalue problem for the separation constant is

$$\boxed{\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0} \tag{7.7.55}$$

$$\boxed{\phi(a) = 0} \tag{7.7.56}$$

$$\boxed{|\phi(0)| < \infty.} \tag{7.7.57}$$

Since (7.7.55) is in the form of a Sturm–Liouville problem, we immediately know that eigenfunctions corresponding to distinct eigenvalues are orthogonal with weight r.

From the Rayleigh quotient, we could show that $\lambda > 0$. Thus, we may use the transformation

$$z = \sqrt{\lambda}r, \tag{7.7.58}$$

in which case (7.7.55) becomes

$$\frac{d}{dz}\left(z\frac{d\phi}{dz}\right) + z\phi = 0 \quad \text{or} \quad z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + z^2\phi = 0. \tag{7.7.59}$$

We may recall that Bessel's differential equation of order m is

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\,\phi = 0, \tag{7.7.60}$$

with solutions being Bessel functions of order m, $J_m(z)$ and $Y_m(z)$. A comparison with (7.7.60) shows that (7.7.59) is Bessel's differential equation of order 0. The general solution of (7.7.59) is thus a linear combination of the zeroth-order Bessel functions:

$$\phi = c_1 J_0(z) + c_2 Y_0(z) = c_1 J_0\left(\sqrt{\lambda}r\right) + c_2 Y_0\left(\sqrt{\lambda}r\right), \tag{7.7.61}$$

in terms of the radial variable. The singularity condition at the origin (7.7.57) shows that $c_2 = 0$, since $Y_0\left(\sqrt{\lambda}r\right)$ has a logarithmic singularity at $r = 0$:

$$\phi = c_1 J_0\left(\sqrt{\lambda}r\right). \tag{7.7.62}$$

Finally, the eigenvalues are determined by the condition at $r = a$, (7.7.56), in which case

$$J_0\left(\sqrt{\lambda}a\right) = 0. \tag{7.7.63}$$

Thus, $\sqrt{\lambda}a$ must be a zero of the zeroth Bessel function. We thus obtain an infinite number of eigenvalues, which we label $\lambda_1, \lambda_2, \ldots$.

We have obtained two infinite families of product solutions

$$J_0\left(\sqrt{\lambda_n}r\right)\,\sin c\sqrt{\lambda_n}t \quad \text{and} \quad J_0\left(\sqrt{\lambda_n}r\right)\,\cos c\sqrt{\lambda_n}t.$$

According to the principle of superposition, we seek solutions to our original problem, (7.7.50)–(7.7.52), in the form

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} a_n J_0\left(\sqrt{\lambda_n}r\right)\,\cos c\sqrt{\lambda_n}t + \sum_{n=1}^{\infty} b_n J_0\left(\sqrt{\lambda_n}r\right)\,\sin c\sqrt{\lambda_n}t.} \tag{7.7.64}$$

As before, we determine the coefficients a_n and b_n from the initial conditions. $u(r, 0) = \alpha(r)$ implies that

$$\alpha(r) = \sum_{n=1}^{\infty} a_n J_0\left(\sqrt{\lambda_n}r\right). \tag{7.7.65}$$

The coefficients a_n are thus the Fourier–Bessel coefficients (of order 0) of $\alpha(r)$. Since $J_0\left(\sqrt{\lambda_n}r\right)$ forms an orthogonal set with weight r, we can easily determine a_n,

$$\boxed{a_n = \frac{\int_0^a \alpha(r) J_0\left(\sqrt{\lambda_n}r\right) r\,dr}{\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right) r\,dr}.} \tag{7.7.66}$$

In a similar manner, the initial condition $\partial/\partial t\ u(r, 0) = \beta(r)$ determines b_n.

Exercises 7.7

***7.7.1.** Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

with $u(a, \theta, t) = 0$, $u(r, \theta, 0) = 0$ and $\frac{\partial u}{\partial t}(r, \theta, 0) = \alpha(r)\,\sin 3\theta$.

7.7.2. Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \text{ subject to } \frac{\partial u}{\partial r}(a, \theta, t) = 0$$

with initial conditions

(a) $u(r, \theta, 0) = 0, \qquad \dfrac{\partial u}{\partial t}(r, \theta, 0) = \beta(r)\cos 5\theta$

(b) $u(r, \theta, 0) = 0, \qquad \dfrac{\partial u}{\partial t}(r, \theta, 0) = \beta(r)$

(c) $u(r, \theta, 0) = \alpha(r, \theta), \qquad \dfrac{\partial u}{\partial t}(r, \theta, 0) = 0$

***(d)** $u(r, \theta, 0) = 0, \qquad \dfrac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta)$

7.7.3. Consider a vibrating quarter-circular membrane, $0 < r < a, 0 < \theta < \pi/2$, with $u = 0$ on the entire boundary. [*Hint*: You may assume without derivation that $\lambda > 0$ and that product solutions

$$u(r, \theta, t) = \phi(r, \theta)h(t) = f(r)g(\theta)h(t)$$

satisfy

$$\nabla^2 \phi + \lambda \phi = 0$$
$$\frac{dh}{dt} = -\lambda k h$$
$$\frac{d^2 g}{d\theta^2} = -\mu g$$
$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu) f = 0.]$$

***(a)** Determine an expression for the frequencies of vibration.

(b) Solve the initial value problem if

$$u(r, \theta, 0) = g(r, \theta), \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

(c) Solve the wave equation inside a quarter-circle, subject to the conditions

$$\frac{\partial u}{\partial r}(a, \theta, t) = 0, \qquad u(r, 0, t) = 0$$
$$u\left(r, \frac{\pi}{2}, t\right) = 0, \qquad u(r, \theta, 0) = 0$$
$$\frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta)$$

7.7.4. Consider the displacement $u(r, \theta, t)$ of a "pie-shaped" membrane of radius a (and angle $\pi/3 = 60°$) that satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

Assume that $\lambda > 0$. Determine the natural frequencies of oscillation if the boundary conditions are

(a) $u(r, 0, t) = 0, \quad u(r, \pi/3, t) = 0, \quad \dfrac{\partial u}{\partial r}(a, \theta, t) = 0$

(b) $u(r, 0, t) = 0, \quad u(r, \pi/3, t) = 0, \quad u(a, \theta, t) = 0$

***7.7.5.** Consider the displacement $u(r, \theta, t)$ of a membrane whose shape is a 90° sector of an annulus, $a < r < b, 0 < \theta < \pi/2$, with the conditions that $u = 0$ on the entire boundary. Determine the natural frequencies of vibration.

7.7.6. Consider the displacement $u(r, \theta, t)$ of a circular membrane satisfying

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

subject to the boundary condition

$$u(a, \theta, t) = -\frac{\partial u}{\partial r}(a, \theta, t).$$

(a) Show that this membrane only oscillates.

(b) Obtain an expression that determines the natural frequencies.

(c) Solve the initial value problem if

$$u(r, \theta, 0) = 0, \quad \frac{\partial u}{\partial t}(r, \theta, 0) = \alpha(r)\sin 3\theta.$$

7.7.7. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a circle of radius a with zero temperature around the entire boundary, if initially

$$u(r, \theta, 0) = f(r, \theta)$$

Briefly analyze $\lim_{t\to\infty} u(r, \theta, t)$. Compare this to what you expect to occur using physical reasoning as $t \to \infty$.

***7.7.8.** Solve Exercise 7.7.7, but with the entire boundary insulated.

7.7.9. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a semicircle of radius a, and briefly analyze the $\lim_{t\to\infty}$ if the initial conditions are

$$u(r, \theta, 0) = f(r, \theta)$$

and the boundary conditions are

(a) $u(r, 0, t) = 0, \qquad u(r, \pi, t) = 0, \qquad \frac{\partial u}{\partial r}(a, \theta, t) = 0$

***(b)** $\frac{\partial u}{\partial \theta}(r, 0, t) = 0, \qquad \frac{\partial u}{\partial \theta}(r, \pi, t) = 0, \qquad \frac{\partial u}{\partial r}(a, \theta, t) = 0$

(c) $\frac{\partial u}{\partial \theta}(r, 0, t) = 0, \qquad \frac{\partial u}{\partial \theta}(r, \pi, t) = 0, \qquad u(a, \theta, t) = 0$

(d) $u(r, 0, t) = 0, \qquad u(r, \pi, t) = 0, \qquad u(a, \theta, t) = 0$

***7.7.10.** Solve for $u(r, t)$ if it satisfies the circularly symmetric heat equation

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

subject to the conditions

$$u(a, t) = 0$$
$$u(r, 0) = f(r).$$

Briefly analyze the $\lim_{t\to\infty}$.

7.7.11. Solve Exercise 7.7.10 with the boundary condition

$$\frac{\partial u}{\partial r}(a, t) = 0$$

but the same initial condition

7.7.12. For the following differential equations, what is the expected approximate behavior of all solutions *near* $x = 0$?

***(a)** $x^2\frac{d^2y}{dx^2} + (x - 6)y = 0$ **(b)** $x^2\frac{d^2y}{dx^2} + \left(x^2 + \frac{3}{16}\right)y = 0$

***(c)** $x^2\frac{d^2y}{dx^2} + (x + x^2)\frac{dy}{dx} + 4y = 0$ **(d)** $x^2\frac{d^2y}{dx^2} + (x + x^2)\frac{dy}{dx} - 4y = 0$

***(e)** $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (6 + x^3)y = 0$ **(f)** $x^2\frac{d^2y}{dx^2} + \left(x + \frac{1}{4}\right)y = 0$

7.7.13. Using the one-dimensional Rayleigh quotient, show that $\lambda > 0$ as defined by (7.7.18)–(7.7.20).

7.7.14. Solve the wave equation inside a $45°(\frac{\pi}{4})$ sector of a circle,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right],$$

subject to the conditions

$$\begin{aligned} &u(a, \theta, t) = 0, && u(r, 0, t) = 0 \\ &u\left(r, \frac{\pi}{4}, t\right) = 0, && u(r, \theta, 0) = \alpha(r, \theta) \\ &\frac{\partial u}{\partial t}(r, \theta, 0) = 0. \end{aligned}$$

[*Hint*: You may assume without derivation that $\lambda > 0$ and that product solutions

$$u(r, \theta, t) = \phi(r, \theta) h(t) = f(r) g(\theta) h(t)$$

satisfy

$$\begin{aligned} &(\nabla^2 \phi + \lambda \phi = 0) \\ &\frac{d^2 h}{dt^2} = -\lambda c^2 h \\ &\frac{d^2 g}{d\theta^2} = -\mu g \\ &r \frac{d}{dr} \left(r \frac{df}{dr} \right) + (\lambda r^2 - \mu) f = 0.] \end{aligned}$$

7.7.15. Solve the wave equation inside a $30°(\frac{\pi}{6})$ sector of a circle. [*Hint*: See hints for Exercise 7.7.14.]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right],$$

subject to the conditions

$$\begin{aligned} &u(a, \theta, t) = 0, && u(r, 0, t) = 0 \\ &u\left(r, \frac{\pi}{6}, t\right) = 0, && u(r, \theta, 0) = 0 \\ &\frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta). \end{aligned}$$

7.7.16. Solve the heat equation inside a quarter-circle,

$$\frac{\partial u}{\partial t} = k \nabla^2 u = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right],$$

subject to the conditions

$$\begin{aligned} &\frac{\partial u}{\partial t}(a, \theta, t) = 0, && u(r, 0, t) = 0 \\ &u\left(r, \frac{\pi}{2}, t\right) = 0, && u(r, \theta, 0) = \alpha(r, \theta). \end{aligned}$$

[*Hint*: You may assume without derivation that $\lambda > 0$ and that product solutions

$$u(r, \theta, t) = \phi(r, \theta) h(t) = f(r) g(\theta) h(t)$$

satisfy

$$\begin{aligned} &(\nabla^2 \phi + \lambda \phi = 0) \\ &\frac{dh}{dt} = -\lambda k h \\ &\frac{d^2 g}{d\theta^2} = -\mu g \\ &r \frac{d}{dr} \left(r \frac{df}{dr} \right) + (\lambda r^2 - \mu) f = 0.] \end{aligned}$$

7.8 More on Bessel Functions

7.8.1 QUALITATIVE PROPERTIES OF BESSEL FUNCTIONS

It is helpful to have some understanding of the sketch of Bessel functions. Let us rewrite Bessel's differential equation as

$$\frac{d^2f}{dz^2} = -\left(1 - \frac{m^2}{z^2}\right)f - \frac{1}{z}\frac{df}{dz}, \tag{7.8.1}$$

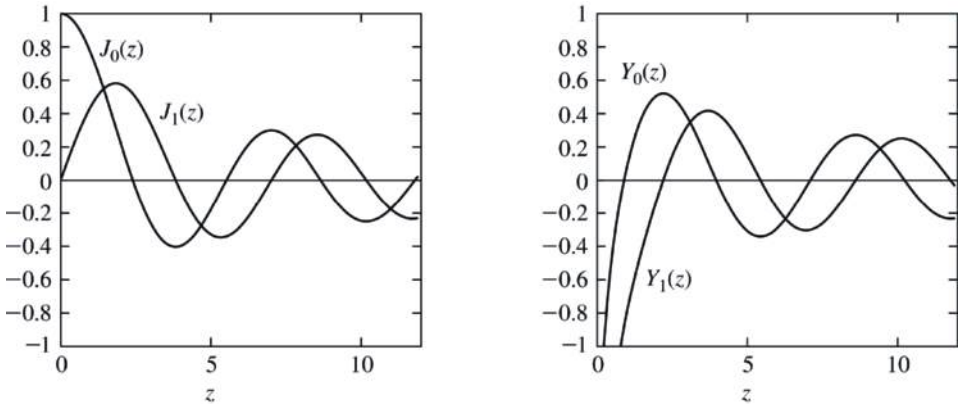
in order to compare it with the equation describing the motion of a spring-mass system (unit mass, spring "constant" k, and frictional coefficient c):

$$\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}.$$

The equilibrium is $y = 0$. Thus, we might think of Bessel's differential equation as representing a time-varying frictional force ($c = 1/t$) and a time-varying "restoring" force ($k = 1 - m^2/t^2$). The latter force is a variable restoring force only for $t > m$ ($z > m$). We might expect the solutions of Bessel's differential equation to be similar to a damped oscillator (at least for $z > m$). The larger z gets, the closer the variable spring constant k approaches 1, and the more the frictional force tends to vanish. The solution should oscillate with frequency approximately 1 but should slowly decay. This is similar to an underdamped spring-mass system, but the solutions to Bessel's differential equation should decay more slowly than any exponential since the frictional force is approaching zero. Detailed plots of Bessel functions are sketched in Fig. 7.8.1, verifying these points. Note that *for small* z,

$$\begin{aligned} J_0(z) &\approx 1, & Y_0(z) &\approx \frac{2}{\pi}\ln z \\ J_1(z) &\approx \frac{1}{2}z, & Y_1(z) &\approx -\frac{2}{\pi}z^{-1} \\ J_2(z) &\approx \frac{1}{8}z^2, & Y_2(z) &\approx -\frac{4}{\pi}z^{-2}. \end{aligned} \tag{7.8.2}$$

Figure 7.8.1 Sketch of various Bessel functions.



These sketches vividly show a property worth memorizing: **Bessel functions of the first and second kind look like decaying oscillations**. In fact, it is known that $J_m(z)$ and $Y_m(z)$ may be accurately approximated by simple algebraically decaying oscillations *for large* z:

$$\boxed{\begin{aligned} J_m(z) &\sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \text{ as } z \to \infty \\ Y_m(z) &\sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \text{ as } z \to \infty. \end{aligned}} \tag{7.8.3}$$

These are known as asymptotic formulas, meaning that the approximations improve as $z \to \infty$. In Section 5.9, we claimed that approximation formulas similar to (7.8.3) always exist for any Sturm–Liouville problem for the large eigenvalues $\lambda \gg 1$. Here, $\lambda \gg 1$ implies that $z \gg 1$ since $z = \sqrt{\lambda}r$ and $0 < r < a$ (as long as r is not too small).

A derivation of (7.8.3) requires facts beyond the scope of this text. However, information such as (7.8.3) is readily available from many sources.[6] We notice from (7.8.3) that the only difference in the approximate behavior for large z of all these Bessel functions is the precise phase shift. We also note that the frequency is approximately 1 (and period 2π) for large z, consistent with the comparison with a spring-mass system with vanishing friction and $k \to 1$. Furthermore, the amplitude of oscillation, $\sqrt{2/\pi z}$, decays more slowly as $z \to \infty$ than the exponential rate of decay associated with an underdamped oscillator, as previously discussed qualitatively.

7.8.2 ASYMPTOTIC FORMULAS FOR THE EIGENVALUES

Approximate values of the zeros of the eigenfunctions $J_m(z)$ may be obtained using these asymptotic formulas, (7.8.3). For example, for $m = 0$, for large z

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right).$$

The zeros approximately occur when $z - \pi/4 = -\pi/2 + n\pi$, but n must be large (in order for z to be large). Thus, the large zeros are given approximately by

$$z \sim \pi\left(n - \frac{1}{4}\right) \tag{7.8.4}$$

for large integer n. We claim that formula (7.8.4) becomes more and more accurate as n increases. In fact, since the formula is reasonably accurate already for $n = 2$ or 3 (see Table 7.8.1), it may be unnecessary to compute the zero to a greater accuracy than is given by (7.8.4). A further indication of the accuracy of the asymptotic formula is that we see that the differences of the first few eigenvalues are already nearly π (as predicted for the large eigenvalues).

TABLE 7.8.1 Zeros of $J_0(z)$

			Large z formula		Percentage	
n	z_{0n}	Exact	(7.8.4)	Error	error	$z_{0n} - z_{0(n-1)}$
1	z_{01}	2.40483...	2.35619	0.04864	2.0	—
2	z_{02}	5.52008...	5.49779	0.02229	0.4	3.11525
3	z_{03}	8.65373...	8.63938	0.01435	0.2	3.13365
4	z_{04}	11.79153...	11.78097	0.01156	0.1	3.13780

7.8.3 ZEROS OF BESSEL FUNCTIONS AND NODAL CURVES

We have shown that the eigenfunctions are $J_m\left(\sqrt{\lambda_{mn}}r\right)$, where $\lambda_{mn} = (z_{mn}/a)^2$, z_{mn} being the nth zero of $J_m(z)$. Thus, the eigenfunctions are

$$J_m\left(z_{mn}\frac{r}{a}\right).$$

For example, for $m = 0$, the eigenfunctions are $J_0(z_{0n}r/a)$, where the sketch of $J_0(z)$ is reproduced in Fig. 7.8.2 (and the zeros are marked). As r ranges from 0 to a, the argument of the eigenfunction $J_0(z_{0n}r/a)$ ranges from 0 to the nth zero, z_{0n}. At $r = a$, $z = z_{0n}$ (the nth zero). Thus, the nth eigenfunction has $n - 1$ zeros in the interior. Although originally stated for regular Sturm–Liouville problems, it is also valid for singular problems (if eigenfunctions exist).

The separation of variables solution of the wave equation is

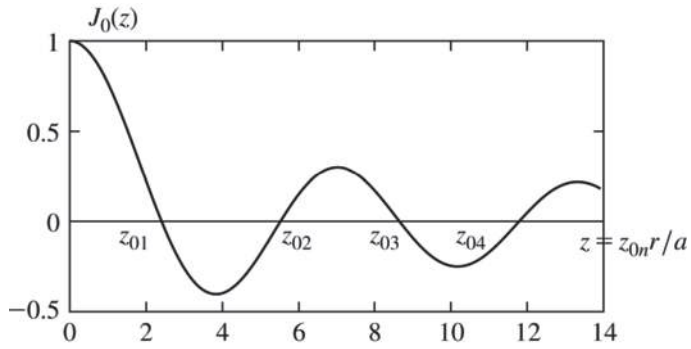
$$u(r, \theta, t) = f(r)g(\theta)h(t),$$

where

$$u(r, \theta, t) = J_m\left(z_{mn}\frac{r}{a}\right)\left\{\begin{array}{c}\sin m\theta\\ \cos m\theta\end{array}\right\}\left\{\begin{array}{c}\sin c\sqrt{\lambda_{mn}}t\\ \cos c\sqrt{\lambda_{mn}}t\end{array}\right\}, \tag{7.8.5}$$

[6] A personal favorite, highly recommended to students with a serious interest in the applications of mathematics to science and engineering, is *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, originally published *inexpensively* by the National Bureau of Standards in 1964 and in 1974 reprinted by Dover in paperback. This has been updated (2010) by NIST and edited by Olver, Lozier, Bolsvert, and Clark.

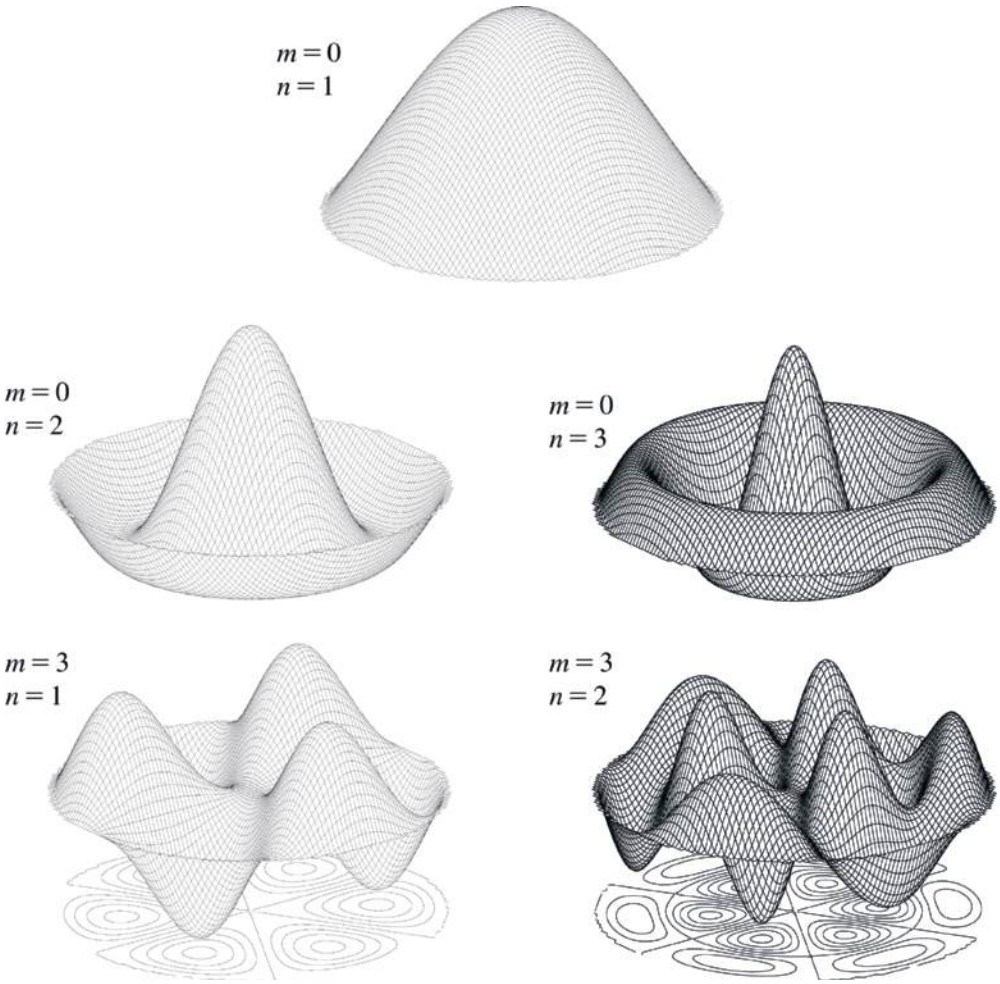
Figure 7.8.2 Sketch of $J_0(z)$ and its zeros.



known as a **normal mode of oscillation** and is graphed for fixed t in Fig. 7.8.3. For each $m \neq 0$, there are four families of solutions (for $m = 0$, there are two families). Each mode oscillates with a characteristic natural frequency, $c\sqrt{\lambda_{mn}}$. At certain positions along the membrane, known as **nodal curves**, the membrane will be unperturbed for all time (for vibrating strings, we called these positions nodes). The nodal curve for the $\sin m\theta$ mode is determined by

$$J_m\left(z_{mn}\frac{r}{a}\right)\sin m\theta = 0. \tag{7.8.6}$$

Figure 7.8.3 Normal nodes and nodal curves for a vibrating circular membrane.



MATLAB Files

Download Animation of Figure 7.8.3 (.zip)

https://bit.ly/3U7uHEp

These files are provided as zipped .m files for use in MATLAB on desktop platform.

The nodal curve consists of all points where $\sin m\theta = 0$ or $J_m(z_{mn}r/a) = 0$; $\sin m\theta$ is zero along $2m$ distinct rays, $\theta = s\pi/m$, $s = 1, 2, \ldots, 2m$. In order for there to be a zero of $J_m(z_{mn}r/a)$ for $0 < r < a$, $z_{mn}r/a$ must equal an earlier zero of $J_m(z)$, $z_{mn}r/a = z_{mp}, p = 1, 2, \ldots, n-1$. Thus, there are $n-1$ circles along which $J_m(z_{mn}r/a) = 0$ besides $r = a$. We illustrate this for $m = 3, n = 2$ in Fig. 7.8.3, where the nodal circles are determined from a table.

7.8.4 SERIES REPRESENTATION OF BESSEL FUNCTIONS

The usual method of discussing Bessel functions relies on series solution methods for differential equations. We will obtain little useful information by pursuing this topic. However, some may find it helpful to refer to the formulas that follow.

First we review some additional results concerning series solutions around $z = 0$ for second-order linear differential equations:

$$\frac{d^2f}{dz^2} + a(z)\frac{df}{dz} + b(z)f = 0. \tag{7.8.7}$$

Recall that $z = 0$ is an ordinary point if both $a(z)$ and $b(z)$ have Taylor series around $z = 0$. In this case, we are guaranteed that all solutions may be represented by a convergent Taylor series,

$$f = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots,$$

at least in some neighborhood of $z = 0$.

If $z = 0$ is not an ordinary point, then we call it a **singular point** (e.g., $z = 0$ is a singular point of Bessel's differential equation). If $z = 0$ is a singular point, we cannot state that all solutions have Taylor series around $z = 0$. However, if $a(z) = R(z)/z$ and $b(z) = S(z)/z^2$, with $R(z)$ and $S(z)$ having Taylor series, then we can say more about solutions of the differential equation near $z = 0$. For this case, which is known as a **regular singular point**, the coefficients $a(z)$ and $b(z)$ can have *at worst* a simple pole and double pole, respectively. It is possible for the coefficients $a(z)$ and $b(z)$ not to be that singular. For example, if $a(z) = 1 + z$ and $b(z) = (1 - z^3)/z^2$, then $z = 0$ is a regular singular point. Bessel's differential equation in the form (7.8.7) is

$$\frac{d^2f}{dz^2} + \frac{1}{z}\frac{df}{dz} + \frac{z^2 - m^2}{z^2}f = 0.$$

Here, $R(z) = 1$ and $S(z) = z^2 - m^2$; both have Taylor series around $z = 0$. Therefore, $z = 0$ is a regular singular point for Bessel's differential equation.

For a regular singular point at $z = 0$, it is known by the **method of Frobenius** that at least one solution of the differential equation is in the form

$$f = z^p \sum_{n=0}^{\infty} a_n z^n; \tag{7.8.8}$$

that is, z^p times a Taylor series, where p is one of the solutions of the quadratic indicial equation. One method to obtain the indicial equation is to substitute $f = z^p$ into the corresponding equidimensional equation that results by replacing $R(z)$ by $R(0)$ and $S(z)$ by $S(0)$. Thus

$$p(p-1) + R(0)p + S(0) = 0$$

is the indicial equation. If the two values of p (the roots of the indicial equation) differ by a noninteger, then two independent solutions exist in the form (7.8.8). If the two roots of the indicial equation are identical, then only one solution is in the form (7.8.8), and the other solution is more complicated but always involves logarithms. If the roots differ by an integer, then sometimes both solutions exist in the form (7.8.8), while other times form (7.8.8) exists only corresponding to the larger root p, and a series beginning with the smaller root p must be modified by the introduction of logarithms. Details of the method of Frobenius are presented in most elementary differential equations texts.

For Bessel's differential equation, we have shown that the indicial equation is

$$p(p-1) + p - m^2 = 0,$$

since $R(0) = 1$ and $S(0) = -m^2$. Its roots are $\pm m$. If $m = 0$, the roots are identical. Form (7.8.8) is valid for one solution, while logarithms must enter the second solution. For $m \neq 0$, the roots of the indicial equation differ by an integer. Detailed calculations also show that logarithms must enter. The following infinite series can be verified by substitution and are often considered as definitions of $J_m(z)$ and $Y_m(z)$:

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+m}}{k!(k+m)!} \tag{7.8.9}$$

$$\begin{aligned} Y_m(z) = \frac{2}{\pi}\Bigg[\left(\log\frac{z}{2} + \gamma\right)J_m(z) - \frac{1}{2}\sum_{k=0}^{m-1}\frac{(m-k-1)!(z/2)^{2k-m}}{k!} \\ + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}[\varphi(k)+\varphi(k+m)]\frac{(z/2)^{2k+m}}{k!(m+k)!}\Bigg], \end{aligned} \tag{7.8.10}$$

where

(i) $\varphi(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/k$, $\phi(0) = 0$,

(ii) $\gamma = \lim_{k\to\infty}[\varphi(k) - \ln k] = 0.5772157\ldots$, known as Euler's constant.

(iii) If $m = 0$, $\sum_{k=0}^{m-1} \equiv 0$.

We have obtained these from the previously mentioned handbook edited by Abramowitz and Stegun.

Exercises 7.8

7.8.1. The boundary value problem for a vibrating annular membrane $1 < r < 2$ (fixed at the inner and outer radii) is

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0$$

with $f(1) = 0$ and $f(2) = 0$, where $m = 0, 1, 2, \ldots$.

(a) Show that $\lambda > 0$.

***(b)** Obtain an expression that determines the eigenvalues.

(c) For what value of m does the smallest eigenvalue occur?

***(d)** Obtain an upper and lower bound for the smallest eigenvalue.

(e) Using a trial function, obtain an upper bound for the smallest eigenvalue.

(f) Compute approximately the smallest eigenvalue from part (b) using tables of Bessel functions. Compare with parts (d) and (e).

7.8.2. Consider the temperature $u(r, \theta, t)$ in a quarter-circle of radius a satisfying

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

subject to the conditions

$$\begin{aligned} u(r, 0, t) &= 0, \quad & u(a, \theta, t) &= 0 \\ u(r, \pi/2, t) &= 0, \quad & u(r, \theta, 0) &= G(r, \theta). \end{aligned}$$

(a) Show that the boundary value problem is

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{\mu}{r}\right)f = 0$$

with $f(a) = 0$ and $f(0)$ bounded.

(b) Show that $\lambda > 0$ if $\mu \geq 0$.

(c) Show that for each μ, the eigenfunction corresponding to the smallest eigenvalue has no zeros for $0 < r < a$.

***(d)** Solve the initial value problem.

7.8.3. Reconsider Exercise 7.8.2 with the boundary conditions

$$\frac{\partial u}{\partial \theta}(r, 0, t) = 0, \quad \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{2}, t\right) = 0, \quad u(a, \theta, t) = 0.$$

7.8.4. Consider the boundary value problem

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)f = 0$$

with $f(a) = 0$ and $f(0)$ bounded. For each integer m, show that the nth eigenfunction has $n - 1$ zeros for $0 < r < a$.

7.8.5. Using the known asymptotic behavior as $z \to 0$ and as $z \to \infty$, roughly sketch for all $z > 0$

(a) $J_4(z)$ **(b)** $Y_1(z)$ **(c)** $Y_0(z)$

(d) $J_0(z)$ **(e)** $Y_5(z)$ **(f)** $J_2(z)$

7.8.6. Determine approximately the *large* frequencies of vibration of a circular membrane.

7.8.7. Consider Bessel's differential equation

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2)f = 0.$$

Let $f = y/z^{1/2}$. Derive that

$$\frac{d^2y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2z^{-2}\right) = 0.$$

***7.8.8.** Using Exercise 7.8.7, determine exact expressions for $J_{1/2}(z)$ and $Y_{1/2}(z)$. Use and verify (7.8.3) and (7.7.33) in this case.

7.8.9. In this exercise use the result of Exercise 7.8.7. If z is large, verify as much as possible concerning (7.8.3).

7.8.10. In this exercise use the result of Exercise 7.8.7 in order to improve on (7.8.3):

(a) Substitute $y = e^{iz}w(z)$ and show that

$$\frac{d^2w}{dz^2} + 2i\frac{dw}{dz} + \frac{\gamma}{z^2}w = 0, \text{ where } \gamma = \frac{1}{4} - m^2.$$

(b) Substitute $w = \sum_{n=0}^{\infty}\beta_n z^{-n}$. Determine the first few terms β_n (assuming that $\beta_0 = 1$).

(c) Use part (b) to obtain an improved asymptotic solution of Bessel's differential equation. For real solutions, take real and imaginary parts.

(d) Find a recurrence formula for β_n. Show that the series diverges. (Nonetheless, a finite series is very useful.)

7.8.11. In order to "understand" the behavior of Bessel's differential equation as $z \to \infty$, let $x = 1/z$. Show that $x = 0$ is a singular point, but an irregular singular point. [The asymptotic solution of a differential equation in the neighborhood of an irregular singular point is analyzed in an unmotivated way in Exercise 7.8.10. For a more systematic presentation, see advanced texts on asymptotic or perturbation methods (such as Bender and Orszag, 1999).]

7.8.12. The smallest eigenvalue for (7.7.34)–(7.7.36) for $m = 0$ is $\lambda = (z_{01}/a)^2$. Determine a reasonably accurate upper bound by using the Rayleigh quotient with a trial function. Compare to the exact answer.

7.8.13. Explain why the nodal circles in Fig. 7.8.3 are nearly equally spaced.

7.9 Laplace's Equation in a Circular Cylinder

7.9.1 INTRODUCTION

Laplace's equation,

$$\nabla^2 u = 0, \tag{7.9.1}$$

represents the steady-state heat equation (without sources). We have solved Laplace's equation in a rectangle (Section 2.5.1) and Laplace's equation in a circle (Section 2.5.2). In both cases, when variables were separated, oscillations occur in one direction but not in the other. Laplace's equation in a rectangular box can also be solved by the method of separation of variables. As shown in some exercises in Chapter 7, the three independent variables yield two eigenvalue problems that have oscillatory solutions and solutions in the remaining direction that are not oscillatory.

A more interesting problem is to consider Laplace's equation in a circular cylinder of radius a and height H. Using circular cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z, \end{aligned}$$

Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{7.9.2}$$

We prescribe u (perhaps temperature) on the entire boundary of the cylinder:

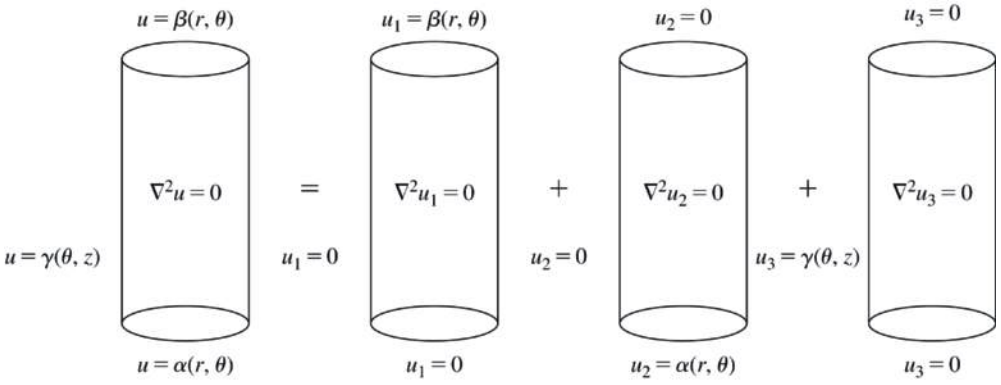
$$\begin{aligned} &\text{top:} && u(r, \theta, H) = \beta(r, \theta) \\ &\text{bottom:} && u(r, \theta, 0) = \alpha(r, \theta) \\ &\text{lateral side:} && u(a, \theta, z) = \gamma(\theta, z). \end{aligned}$$

There are three nonhomogeneous boundary conditions. One approach is to break the problem up into the sum of three simpler problems, each solving Laplace's equation,

$$\nabla^2 u_i = 0, \qquad i = 1, 2, 3,$$

where $u = u_1 + u_2 + u_3$. This is illustrated in Fig. 7.9.1. In this way, each problem satisfies two homogeneous boundary conditions, but the sum satisfies the desired nonhomogeneous conditions. We separate variables once, for all three cases, and then proceed to solve each problem individually.

Figure 7.9.1 Laplace's equation in a circular cylinder.



7.9.2 SEPARATION OF VARIABLES

We begin by looking for product solutions,

$$u(r, \theta, z) = f(r)g(\theta)h(z), \tag{7.9.3}$$

for Laplace's equation. Substituting (7.9.3) into (7.9.2) and dividing by $f(r)g(\theta)h(z)$ yields

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2}\frac{1}{g}\frac{d^2g}{d\theta^2} + \frac{1}{h}\frac{d^2h}{dz^2} = 0. \tag{7.9.4}$$

We immediately can separate the z-dependence, and hence

$$\frac{1}{h}\frac{d^2h}{dz^2} = \lambda. \tag{7.9.5}$$

Do we expect oscillations in z? From Fig. 7.9.1, we see that oscillations in z should be expected for the u_3-problem but not necessarily for the u_1- or u_2-problem. Perhaps $\lambda < 0$ for the u_3-problem but not for the u_1- and u_2-problems. Thus, we do not specify λ at this time. The r- and θ-parts also can be separated if (7.9.4) is multiplied by r^2 [and (7.9.5) is utilized]:

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \lambda r^2 = -\frac{1}{g}\frac{d^2g}{d\theta^2} = \mu. \tag{7.9.6}$$

A second separation constant μ is introduced, with the anticipation that $\mu > 0$ because of the expected oscillations in θ for all three problems. In fact, the implied periodic boundary conditions in θ dictate that

$$\mu = m^2 \tag{7.9.7}$$

and that $g(\theta)$ can be either $\sin m\theta$ or $\cos m\theta$, where m is a nonnegative integer, $m = 0, 1, 2, \ldots$. A Fourier series in θ will be appropriate for all these problems.

In summary, the θ-dependence is $\sin m\theta$ and $\cos m\theta$, and the remaining two differential equations are

$$\frac{d^2h}{dz^2} = \lambda h \tag{7.9.8}$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - m^2)f = 0. \tag{7.9.9}$$

These two differential equations contain only one unspecified parameter λ. Only one will become an eigenvalue problem. The eigenvalue problem needs two homogeneous boundary conditions. Different results occur for the various problems, u_1, u_2, and u_3. For the u_3-problem, there are two homogeneous boundary conditions in z, and thus (7.9.8) will become an eigenvalue problem [and (7.9.9) will have nonoscillatory solutions]. However, for the u_1- and u_2-problems, there do not exist two homogeneous boundary conditions in z. Instead, there should be two homogeneous conditions in r. One of these is at $r = a$. The other must be a boundedness condition at $r = 0$, which occurs due to the singular nature of polar (or circular cylindrical) coordinates at $r = 0$ and the singular nature of (7.9.9) at $r = 0$:

$$|f(0)| < \infty. \tag{7.9.10}$$

Thus, we will find that for the u_1- and u_2-problems, (7.9.9) will be the eigenvalue problem. The solution of (7.9.9) will oscillate, whereas the solution of (7.9.8) will not oscillate. We next describe the details of all three problems.

7.9.3 ZERO TEMPERATURE ON THE LATERAL SIDES AND ON THE BOTTOM OR TOP

The mathematical problem for u_1 is

$$\boxed{\nabla^2 u_1 = 0} \tag{7.9.11}$$

$$\boxed{u_1(r, \theta, 0) = 0} \tag{7.9.12}$$

$$\boxed{u_1(r, \theta, H) = \beta(r, \theta)} \tag{7.9.13}$$

$$\boxed{u_1(a, \theta, z) = 0.} \tag{7.9.14}$$

The temperature is zero on the bottom. By separation of variables in which the nonhomogeneous condition (7.9.13) is momentarily ignored, $u_1 = f(r)g(\theta)h(z)$. The θ-part is known to equal $\sin m\theta$ and $\cos m\theta$ (for integral $m \geq 0$). The z-dependent equation, (7.9.8), satisfies only one homogeneous condition, $h(0) = 0$. The r-dependent equation will become a boundary value problem determining the separation constant λ. The two homogeneous boundary conditions are

$$f(a) = 0 \tag{7.9.15}$$

$$|f(0)| < \infty. \tag{7.9.16}$$

The eigenvalue problem, (7.9.9) with (7.9.15) and (7.9.16), is one that was analyzed in Section 7.8. There we showed that $\lambda > 0$ (by directly using the Rayleigh quotient). Furthermore, we showed that the general solution of (7.9.9) is a linear combination of Bessel functions of order m with argument $\sqrt{\lambda}r$:

$$f(r) = c_1 J_m\left(\sqrt{\lambda}r\right) + c_2 Y_m\left(\sqrt{\lambda}r\right) = c_1 J_m\left(\sqrt{\lambda}r\right), \tag{7.9.17}$$

which has been simplified using the singularity condition, (7.9.16). Then the homogeneous condition, (7.9.15), determines λ:

$$\boxed{J_m\left(\sqrt{\lambda}a\right) = 0.} \tag{7.9.18}$$

Again, $\sqrt{\lambda}a$ must be a zero of the mth Bessel function, and the notation λ_{mn} is used to indicate the infinite number of eigenvalues for each m. The eigenfunction $J_m\left(\sqrt{\lambda_{mn}}r\right)$ oscillates in r.

Since $\lambda > 0$, the solution of (7.9.8) that satisfies $h(0) = 0$ is proportional to

$$h(z) = \sinh\sqrt{\lambda}z. \tag{7.9.19}$$

No oscillations occur in the z-direction. There are thus two doubly infinite families of product solutions:

$$\sinh\sqrt{\lambda_{mn}}z\; J_m\left(\sqrt{\lambda_{mn}}r\right)\begin{Bmatrix}\sin m\theta\\ \cos m\theta\end{Bmatrix}, \tag{7.9.20}$$

oscillatory in r and θ, but nonoscillatory in z. The principle of superposition implies that we should consider

$$\boxed{\begin{aligned} u(r, \theta, z) = &\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh\sqrt{\lambda_{mn}}z\; J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta \\ &+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sinh\sqrt{\lambda_{mn}}z\; J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta. \end{aligned}} \tag{7.9.21}$$

The nonhomogeneous boundary condition, (7.9.13), $u_1(r, \theta, H) = \beta(r, \theta)$, will determine the coefficients A_{mn} and B_{mn}. It will involve a Fourier series in θ and a Fourier–Bessel series in r. Thus, we can solve A_{mn} and B_{mn} using the two one-dimensional orthogonality formulas. Alternatively, the coefficients are more easily calculated using the two-dimensional orthogonality of $J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta$ and $J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta$ (see Section 7.8). We omit the details.

In a similar manner, one can obtain u_2. We leave as an exercise the solution of this problem.

7.9.4 ZERO TEMPERATURE ON THE TOP AND BOTTOM

A somewhat different mathematical problem arises if we consider the situation in which the top and bottom are held at zero temperature. The problem for u_3 is

$$\boxed{\nabla^2 u_3 = 0} \tag{7.9.22}$$

$$\boxed{u_3(r, \theta, 0) = 0} \tag{7.9.23}$$

$$\boxed{u_3(r, \theta, H) = 0} \tag{7.9.24}$$

$$\boxed{u_3(a, \theta, z) = \gamma(\theta, z).} \tag{7.9.25}$$

We may again use the results of the method of separation of variables. The periodicity again implies that the θ-part will relate to a Fourier series (i.e., $\sin m\theta$ and $\cos m\theta$). However, unlike what occurred in Section 7.9.3, the z-equation has two homogeneous boundary conditions:

$$\boxed{\frac{d^2h}{dz^2} = \lambda h} \tag{7.9.26}$$

$$\boxed{h(0) = 0} \tag{7.9.27}$$

$$\boxed{h(H) = 0.} \tag{7.9.28}$$

This is the simplest Sturm–Liouville eigenvalue problem (in a somewhat different form). In order for $h(z)$ to oscillate and satisfy (7.9.27) and (7.9.28), the separation constant λ must be negative. In fact, we should recognize that

$$\lambda = -\left(\frac{n\pi}{H}\right)^2, \quad n = 1, 2, \ldots \tag{7.9.29}$$

$$h(z) = \sin\frac{n\pi z}{H}. \tag{7.9.30}$$

The boundary conditions at the top and bottom imply that we will be using an ordinary Fourier sine series in z.

We have oscillations in z and θ. The r-dependent solution should not be oscillatory; they satisfy (7.9.9), which, using (7.9.29), becomes

$$\boxed{r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(-\left(\frac{n\pi}{H}\right)^2 r^2 - m^2\right)f = 0.} \tag{7.9.31}$$

A homogeneous condition, in the form of a singularity condition, exists at $r = 0$,

$$|f(0)| < \infty, \tag{7.9.32}$$

but there is no homogeneous condition at $r = a$.

Equation (7.9.31) looks similar to Bessel's differential equation but has the wrong sign in front of the r^2-term. It cannot be changed into Bessel's differential equation using a real transformation. If we let

$$s = i\left(\frac{n\pi}{H}\right)r \tag{7.9.33}$$

where $i = \sqrt{-1}$, then (7.9.31) becomes

$$s\frac{d}{ds}\left(s\frac{df}{ds}\right) + (s^2 - m^2)f = 0 \quad \text{or} \quad s^2\frac{d^2f}{ds^2} + s\frac{df}{ds} + (s^2 - m^2)f = 0.$$

We recognize this as exactly Bessel's differential equation, and thus

$$f = c_1 J_m(s) + c_2 Y(s) \quad \text{or} \quad f = c_1 J_m\left(i\frac{n\pi}{H}r\right) + c_2 Y_m\left(i\frac{n\pi}{H}r\right). \tag{7.9.34}$$

Therefore, the solution of (7.9.31) can be represented in terms of Bessel functions of an imaginary argument. This is not very useful since Bessel functions are not usually tabulated in this form.

Instead, we introduce a real transformation that eliminates the dependence on $n\pi/H$ of the differential equation:

$$w = \frac{n\pi}{H} r.$$

Then (7.9.31) becomes

$$\boxed{w^2 \frac{d^2 f}{dw^2} + w \frac{df}{dw} + (-w^2 - m^2) f = 0.} \tag{7.9.35}$$

Again the wrong sign appears for this to be Bessel's differential equation. Equation (7.9.35) is a modification of Bessel's differential equation, and its solutions, *which have been well tabulated*, are known as **modified Bessel functions**.

Equation (7.9.35) has the same kind of singularity at $w = 0$ as Bessel's differential equation. As such, the singular behavior could be determined by the method of Frobenius.[7] Thus, we can specify *one solution to be well defined at* $w = 0$, *called the* **modified Bessel function of order *m* of the first kind**, denoted $I_m(w)$. Another independent solution, which is *singular at the origin, is called the* **modified Bessel function of order *m* of the second kind**, denoted $K_m(w)$. Both $I_m(w)$ and $K_m(w)$ are well-tabulated functions. We will need very little information concerning $I_m(w)$ and $K_m(w)$. The general solution of (7.9.31) is thus

$$f = c_1 K_m\left(\frac{n\pi}{H} r\right) + c_2 I_m\left(\frac{n\pi}{H} r\right). \tag{7.9.36}$$

Since K_m is singular at $r = 0$ and I_m is not, it follows that $c_1 = 0$ and $f(r)$ is proportional to $I_m(n\pi r/H)$. We simply note that both $I_m(w)$ and $K_m(w)$ are nonoscillatory and are not zero for $w > 0$. A discussion of this and further properties are given in Section 7.9.5.

There are thus two doubly infinite families of product solutions:

$$I_m\left(\frac{n\pi}{H} r\right) \sin\frac{n\pi z}{H} \cos m\theta \quad \text{and} \quad I_m\left(\frac{n\pi}{H} r\right) \sin\frac{n\pi z}{H} \sin m\theta. \tag{7.9.37}$$

These solutions are oscillatory in z and θ, but nonoscillatory in r. The principle of superposition, equivalent to a Fourier sine series in z and a Fourier series in θ, implies that

$$\boxed{\begin{aligned} u_3(r, \theta, z) = &\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} E_{mn} I_m\left(\frac{n\pi}{H} r\right) \sin\frac{n\pi z}{H} \cos m\theta \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} I_m\left(\frac{n\pi}{H} r\right) \sin\frac{n\pi z}{H} \sin m\theta. \end{aligned}} \tag{7.9.38}$$

The coefficients E_{mn} and F_{mn} can be determined [if $I_m(n\pi a/H) \neq 0$] from the nonhomogeneous equation (7.9.25) either by two iterated one-dimensional orthogonality results or by one application of two-dimensional orthogonality. In the next section, we will discuss further properties of $I_m(n\pi a/H)$, including the fact that it has no positive zeros.

In this way, the solution for Laplace's equation inside a circular cylinder has been determined given any temperature distribution along the entire boundary.

7.9.5 MODIFIED BESSEL FUNCTIONS

The differential equation that defines the modified Bessel functions is

$$w^2 \frac{d^2 f}{dw^2} + w \frac{df}{dw} + (-w^2 - m^2) f = 0. \tag{7.9.39}$$

Two independent solutions are denoted $K_m(w)$ and $I_m(w)$. The behavior in the neighborhood of the singular point $w = 0$ is determined by the roots of the indicial equation, $\pm m$, corresponding to approximate solutions near $w = 0$ of the forms $w^{\pm m}$ (for $m \neq 0$), w^0 and $w^0 \ln w$ (for $m = 0$). We can choose the two independent solutions such that one is well behaved at $w = 0$ and the other is singular.

A good understanding of these functions comes from also analyzing their behavior as $w \to \infty$. Roughly speaking, for large w, (7.9.39) can be rewritten as

$$\frac{d^2 f}{dw^2} \approx -\frac{1}{w} \frac{df}{dw} + f. \tag{7.9.40}$$

[7]Here, it is easier to use the complex transformation (7.9.33). Then, the infinite series representation for Bessel functions is valid for complex arguments, avoiding additional calculations.

Thinking of this as Newton's law for a particle with certain forces, the $-1/w\, df/dw$-term is a weak damping force tending to vanish as $w \to \infty$. We might expect as $w \to \infty$ that

$$\frac{d^2 f}{dw^2} \approx f,$$

which suggests that the solution should be a linear combination of an exponentially growing e^w- and exponentially decaying e^{-w}-term. In fact, the weak damping has its effects (just as it did for ordinary Bessel functions). We state (but do not prove) a more advanced result, namely, that the asymptotic behavior for large w of solutions of (7.9.39) are approximately $e^{\pm w}/w^{1/2}$. Thus, both $I_m(w)$ and $K_m(w)$ are linear combinations of these two, one exponentially growing and the other decaying.

There is only one independent linear combination that decays as $w \to \infty$. There are many combinations that grow as $w \to \infty$. We *define* $K_m(w)$ to be a solution that decays as $w \to \infty$. It must be proportional to $e^{-w}/w^{1/2}$, and it is defined *uniquely* by

$$\boxed{K_m(w) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-w}}{w^{1/2}},} \tag{7.9.41}$$

as $w \to \infty$. As $w \to 0$, the behavior of $K_m(w)$ will be some linear combination of the two different behaviors (e.g., w^m and w^{-m} for $m \neq 0$). In general, it will be composed of both and hence will be singular at $w = 0$. In more advanced treatments, it is shown that

$$\boxed{K_m(w) \sim \begin{cases} \ln w & m = 0 \\ \frac{1}{2}(m-1)!\left(\frac{1}{2}w\right)^{-m} & m \neq 0, \end{cases}} \tag{7.9.42}$$

as $w \to 0$. The most important fact about this function is that $K_m(w)$ **exponentially decays as** $w \to \infty$ **but is singular at** $w = 0$.

Since $K_m(w)$ is singular at $w = 0$, we would like to define a second solution $I_m(w)$ that is not singular at $w = 0$. $I_m(w)$ is *defined* uniquely such that

$$\boxed{I_m(w) \sim \frac{1}{m!}\left(\frac{1}{2}w\right)^m,} \tag{7.9.43}$$

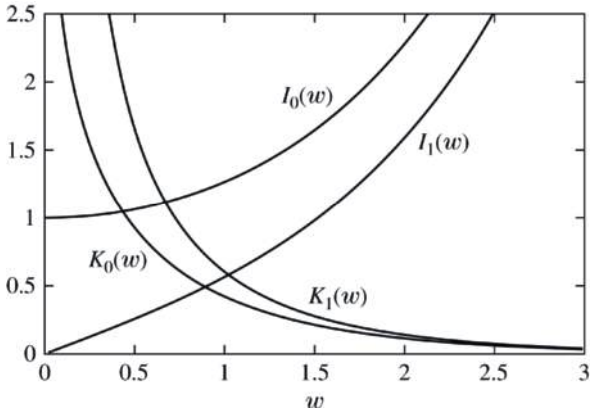
as $w \to 0$. As $w \to \infty$, the behavior of $I_m(w)$ will be some linear combination of the two different asymptotic behaviors ($e^{\pm w}/w^{1/2}$). In general, it will be composed of both and hence is expected to exponentially grow as $w \to \infty$. In more advanced works, it is shown that

$$\boxed{I_m(w) \sim \sqrt{\frac{1}{2\pi w}}\, e^w,} \tag{7.9.44}$$

as $w \to \infty$. The most important facts about this function is that $I_m(w)$ **is well behaved at** $w = 0$ **but grows exponentially as** $w \to \infty$.

Some modified Bessel functions are sketched in Fig. 7.9.2. Although we have not proved it, note that $I_m(w)$ and $K_m(w)$ are not both zero for $w > 0$.

Figure 7.9.2 Various modified Bessel functions (from Abramowitz and Stegun, 1974).



Exercises 7.9

7.9.1. Solve Laplace's equation inside a circular cylinder subject to the boundary conditions

(a) $u(r, \theta, 0) = \alpha(r, \theta)$, $u(r, \theta, H) = 0$, $u(a, \theta, z) = 0$

***(b)** $u(r, \theta, 0) = \alpha(r)\sin 7\theta$, $u(r, \theta, H) = 0$, $u(a, \theta, z) = 0$

(c) $u(r, \theta, 0) = 0$, $u(r, \theta, H) = \beta(r)\cos 3\theta$, $\dfrac{\partial u}{\partial r}(a, \theta, z) = 0$

(d) $\dfrac{\partial u}{\partial z}(r, \theta, 0) = \alpha(r)\sin 3\theta$, $\dfrac{\partial u}{\partial z}(r, \theta, H) = 0$, $\dfrac{\partial u}{\partial r}(a, \theta, z) = 0$

(e) $\dfrac{\partial u}{\partial z}(r, \theta, 0) = \alpha(r, \theta)$, $\dfrac{\partial u}{\partial z}(r, \theta, H) = 0$, $\dfrac{\partial u}{\partial r}(a, \theta, z) = 0$

For (e) only, under what condition does a solution exist?

7.9.2. Solve Laplace's equation inside a semicircular cylinder, subject to the boundary conditions

(a) $u(r, \theta, 0) = 0$, $u(r, \theta, H) = \alpha(r, \theta)$, $u(r, 0, z) = 0$, $u(r, \pi, z) = 0$, $u(a, \theta, z) = 0$

***(b)** $u(r, \theta, 0) = 0$, $\dfrac{\partial u}{\partial z}(r, \theta, H) = 0$, $u(r, 0, z) = 0$, $u(r, \pi, z) = 0$, $u(a, \theta, z) = \beta(\theta, z)$

(c) $\dfrac{\partial}{\partial z}u(r, \theta, 0) = 0$, $\dfrac{\partial}{\partial z}u(r, \theta, H) = 0$, $\dfrac{\partial u}{\partial \theta}(r, 0, z) = 0$, $\dfrac{\partial u}{\partial \theta}(r, \pi, z) = 0$, $\dfrac{\partial u}{\partial r}(a, \theta, z) = \beta(\theta, z)$

For (c) only, under what condition does a solution exist?

(d) $u(r, \theta, 0) = 0$, $u(r, 0, z) = 0$, $u(a, \theta, z) = 0$, $u(r, \theta, H) = 0$, $\dfrac{\partial u}{\partial \theta}(r, \pi, z) = \alpha(r, z)$

7.9.3. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a quarter-circular cylinder ($0 < \theta < \pi/2$ with radius a and height H) subject to the initial condition

$$u(r, \theta, z, 0) = f(r, \theta, z).$$

Briefly explain what temperature distribution you expect to be approached as $t \to \infty$. Consider the following boundary conditions:

(a) $u(r, \theta, 0) = 0$, $u(r, \theta, H) = 0$, $u(r, 0, z) = 0$, $u(r, \pi/2, z) = 0$, $u(a, \theta, z) = 0$

***(b)** $\dfrac{\partial u}{\partial z}(r, \theta, 0) = 0$, $\dfrac{\partial u}{\partial z}(r, \theta, H) = 0$, $\dfrac{\partial u}{\partial \theta}(r, 0, z) = 0$, $\dfrac{\partial u}{\partial \theta}(r, \pi/2, z) = 0$, $\dfrac{\partial u}{\partial r}(a, \theta, z) = 0$

(c) $u(r, \theta, 0) = 0$, $u(r, \theta, H) = 0$, $\dfrac{\partial u}{\partial \theta}(r, 0, z) = 0$, $u(r, \pi/2, z) = 0$, $\dfrac{\partial u}{\partial r}(a, \theta, z) = 0$

7.9.4. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a cylinder (of radius a and height H) subject to the initial condition

$$u(r, \theta, z, 0) = f(r, z),$$

independent of θ, if the boundary conditions are

*(a) $u(r, \theta, 0, t) = 0$, $u(r, \theta, H, t) = 0$, $u(a, \theta, z, t) = 0$

(b) $\frac{\partial u}{\partial z}(r, \theta, 0, t) = 0$, $\frac{\partial u}{\partial z}(r, \theta, H, t) = 0$, $\frac{\partial u}{\partial r}(a, \theta, z, t) = 0$

(c) $u(r, \theta, 0, t) = 0$, $u(r, \theta, H, t) = 0$, $\frac{\partial u}{\partial r}(a, \theta, z, t) = 0$

7.9.5. Determine the three ordinary differential equations obtained by separation of variables for Laplace's equation in spherical coordinates

$$\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

7.9.6. Solve Laplace's equation inside a cylinder,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

whose cross-section is pie-shaped with included angle $\frac{\pi}{3}$, subject to the temperature being zero everywhere on the boundary except the bottom:

$$u(a, \theta, z) = 0, \qquad u(r, 0, z) = 0$$
$$u(r, \theta, H) = 0, \qquad u\left(r, \frac{\pi}{3}, z\right) = 0$$
$$u(r, \theta, 0) = \alpha(r, \theta).$$

[*Hint*: You may assume without derivation that $\lambda > 0$ and that product solutions

$$u(r, \theta, z) = \phi(r, \theta)h(z) = f(r)g(\theta)h(z)$$

satisfy

$$(\nabla^2\phi + \lambda\phi = 0)$$
$$\frac{d^2h}{dz^2} = -\lambda h$$
$$\frac{d^2g}{d\theta^2} = -\mu g$$
$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu)f = 0.]$$

Note: $0 < r < a$, $0 < \theta < \frac{\pi}{3}$, and $0 < z < H$.

7.10 Spherical Problems and Legendre Polynomials

7.10.1 INTRODUCTION

Problems in a spherical geometry are of great interest in many applications. In the exercises, we consider the three-dimensional heat equation inside the spherical earth. Here, we consider the three-dimensional wave equation that describes the vibrations of the Earth:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u, \tag{7.10.1}$$

where u is a local displacement. In geophysics, the response of the real Earth to point sources is of particular interest due to earthquakes and nuclear testing. Solid vibrations of the real Earth are more complicated than (7.10.1). Compressional waves called P for primary are smaller than shear waves called S for secondary, arriving later because they propagate at a smaller velocity. There are also long (L) period surface waves, which are the most destructive in severe earthquakes because their energy is confined to a thin region near the surface. Real seismograms are more complicated because of scattering of waves due to the interior of the Earth not being uniform. Measuring the vibrations is frequently used to determine the interior structure of the Earth needed not only in seismology but also in mineral exploration, such as petroleum engineering. All displacements solve

wave equations. Simple mathematical models are most valid for the destructive long waves, since the variations in the Earth are averaged out for long waves. For more details, see Aki and Richards (1980), *Quantitative Seismology.* We use spherical coordinates (ρ, θ, ϕ), where ϕ is the angle from the pole and θ is the usual cylindrical angle. The boundary condition we assume is $u(a, \theta, \phi, t) = 0$, and the initial displacement and velocity distribution is given throughout the entire solid:

$$u(r, \theta, \phi, 0) = F(\rho, \theta, \phi) \tag{7.10.2}$$

$$\frac{\partial u}{\partial t}(r, \theta, \phi, 0) = G(\rho, \theta, \phi). \tag{7.10.3}$$

Problems with nonhomogeneous boundary conditions are treated in Chapter 8.

7.10.2 SEPARATION OF VARIABLES AND ONE-DIMENSIONAL EIGENVALUE PROBLEMS

We use the method of separation of variables. As before, we first introduce product solutions of space and time:

$$u(\rho, \theta, \phi, t) = w(\rho, \theta, \phi)h(t). \tag{7.10.4}$$

We have already separated space and time, so that we know

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \tag{7.10.5}$$

$$\nabla^2 w + \lambda w = 0, \tag{7.10.6}$$

where the first separation constant λ satisfies the multidimensional eigenvalue problem (7.10.6), subject to being zero on the boundary of the sphere. The frequencies of vibration of the solid sphere are given by $c\sqrt{\lambda}$.

Using the equation for the Laplacian in spherical coordinates (1.5.22), we have

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial w}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 w}{\partial \theta^2} + \lambda w = 0. \tag{7.10.7}$$

We seek product solutions of the form

$$w(\rho, \theta, \phi) = f(\rho)q(\theta)g(\phi). \tag{7.10.8}$$

To save some algebra, since the coefficients in (7.10.7) do not depend on θ and w is periodic in θ with period 2π, we note that it is clear that the eigenfunctions in θ are $\cos m\theta$ and $\sin m\theta$, corresponding to the periodic boundary conditions associated with the usual Fourier series in θ on the interval $-\pi \le \theta \le \pi$. In this case, the term $\frac{\partial^2 w}{\partial \theta^2}$ in (7.10.7) may be replaced by $-m^2 w$. We substitute (7.10.8) into (7.10.7), multiply by ρ^2, divide by $f(\rho)q(\theta)g(\phi)$, and introduce the third (counting $-m^2$ as number two) separation constant μ:

$$\frac{1}{f}\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \lambda\rho^2 = -\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \frac{m^2}{\sin^2\phi} = \mu. \tag{7.10.9}$$

The two ordinary differential equations that are the fundamental part of the eigenvalue problems in ϕ and ρ are

$$\boxed{\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu)f = 0} \tag{7.10.10}$$

$$\boxed{\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\mu\sin\phi - \frac{m^2}{\sin\phi}\right)g = 0.} \tag{7.10.11}$$

The homogeneous boundary conditions associated with (7.10.10) and (7.10.11) will be discussed shortly. We will solve (7.10.11) first because it does not depend on the eigenvalues λ of (7.10.10).

Equation (7.10.11) is a Sturm–Liouville differential equation (for each m) in the angular coordinate ϕ with eigenvalue μ and nonnegative weight $\sin\phi$ since ϕ is defined from $\phi = 0$ (North Pole) to $\phi = \pi$ (South Pole). However, (7.10.11) is not a regular Sturm–Liouville problem since $p = \sin\phi$ must be greater than 0 and $\sin\phi = 0$ at both ends. There is no physical boundary condition at the singular endpoints. Instead we will insist that the solution is bounded at each endpoint: $|g(0)| < \infty$ and $|g(\pi)| < \infty$. We claim that the usual properties of eigenvalues and eigenfunctions are valid. In particular, there is an infinite set of eigenfunctions (for each fixed m) corresponding to different eigenvalues μ_{nm}, and these eigenfunctions will be an orthogonal set with weight $\sin\phi$.

Equation (7.10.10) is a Sturm–Liouville differential equation (for each m and n) in the radial coordinate ρ with eigenvalue λ and weight ρ^2. One homogeneous boundary condition is $f(a) = 0$. Equation (7.10.10) is a singular Sturm–Liouville problem because of the zero at $\rho = 0$ in the coefficient in front of $df/d\rho$. Spherical coordinates are singular at $\rho = 0$, and solutions of the Sturm–Liouville differential equation must be bounded there: $|f(0)| < \infty$. We claim that this singular problem still has an infinite set of eigenfunctions (for each fixed m and n) corresponding to different eigenvalues λ_{knm}, and these eigenfunctions will form an orthogonal set with weight ρ^2.

7.10.3 ASSOCIATED LEGENDRE FUNCTIONS AND LEGENDRE POLYNOMIALS

A (not obvious) change of variables has turned out to simplify the analysis of the differential equation that defines the orthogonal eigenfunctions in the angle ϕ:

$$\boxed{x = \cos\phi.} \tag{7.10.12}$$

As ϕ goes from 0 to π, this is a one-to-one transformation in which x goes from 1 to -1. We will show that both endpoints remain singular points. Derivatives are transformed by the chain rule, $\frac{d}{d\phi} = \frac{dx}{d\phi}\frac{d}{dx} = -\sin\phi\frac{d}{dx}$. In this way (7.10.11) becomes, after dividing by $\sin\phi$ and recognizing that $\sin^2\phi = 1 - \cos^2\phi = 1 - x^2$:

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right] + \left(\mu - \frac{m^2}{1-x^2}\right)g = 0.} \tag{7.10.13}$$

This is also a Sturm–Liouville equation, and eigenfunctions will be orthogonal in x with weight 1. This corresponds to the weight $\sin\phi$ with respect to ϕ since $dx = -\sin\phi\, d\phi$. Equation (7.10.13) has singular points at $x = \pm 1$, which we will show are regular singular points (see Section 7.8.4). It is helpful to understand the local behavior near each singular point using a corresponding elementary equidimensional (Euler) equation. We analyze (7.10.13) near $x = 1$ (and claim due to symmetry that the local behavior near $x = -1$ will be the same). The troublesome coefficients $1 - x^2 = (1-x)(1+x)$ can be approximated by $-2(x-1)$ near $x = 1$, Thus, (7.10.13) may be approximated near $x = 1$ by

$$-2\frac{d}{dx}\left[(x-1)\frac{dg}{dx}\right] + \frac{m^2}{2(x-1)}g \approx 0 \tag{7.10.14}$$

since only the singular term that multiplies g is significant. Equation (7.10.14) is an equidimensional (Euler) differential equation whose exact solutions are easy to obtain by substituting $g = (x-1)^p$, from which we obtain $p^2 = m^2/4$ or $p = \pm m/2$. If $m \neq 0$, we conclude that one independent solution is bounded near $x = 1$ [and approximated by $(x-1)^{m/2}$], and the second independent solution is unbounded [and approximated by $(x-1)^{-m/2}$].

Since we want our solution to be bounded at $x = 1$, we can use only the one solution that is bounded at $x = 1$. When we compute this solution (perhaps numerically) at $x = -1$, its behavior must be a linear combination of the two local behaviors near $x = -1$. Usually, the solution that is bounded at $x = 1$ will be unbounded at $x = -1$. Only for certain very special values of μ_{mn} (which we have called the eigenvalues) will the solution of (7.10.13) be bounded at both $x = \pm 1$. To simplify significantly the presentation, we will not explain the mysterious but elegant result that the only values of μ for which the solution is bounded at $x = \pm 1$ are

$$\mu = n(n+1), \tag{7.10.15}$$

where n is an integer with some restrictions we will mention. It is quite remarkable that the eigenvalues do not depend on the important parameter m. Equation (7.10.13) is a linear differential equation whose two independent solutions are called **associated Legendre functions (spherical harmonics)** of the first $P_n^m(x)$ and second kind $Q_n^m(x)$. The first kind is bounded at both $x = \pm 1$ for integer n, so that the eigenfunctions are given by $g(x) = P_n^m(x)$.

If m = 0: Legendre polynomials

$m = 0$ corresponds to solutions of the partial differential equation with no dependence on the cylindrical angle θ. In this case, the differential equation (7.10.13) becomes

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right] + n(n+1)g = 0,} \tag{7.10.16}$$

given that it can be shown that the eigenvalues satisfy (7.10.15). By series methods, it can be shown that there are elementary Taylor series solutions around $x = 0$ that terminate (hence, a polynomial) only when $\mu = n(n+1)$, and hence are bounded at $x = \pm 1$ when $\mu = n(n+1)$. It can be shown (not easily) that if $\mu \neq n(n+1)$, then the solution to the differential equation is not bounded at either ± 1. These important bounded solutions are called **Legendre polynomials** and are not difficult to compute:

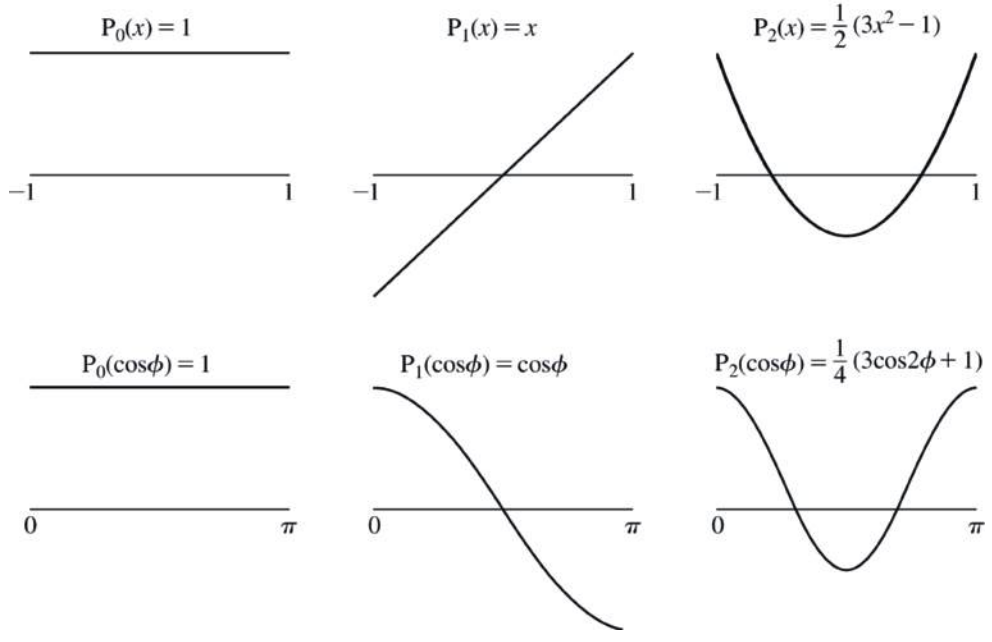
$$\boxed{\begin{aligned} &n = 0\text{: } P_0(x) = 1 \\ &n = 1\text{: } P_1(x) = x = \cos\phi \\ &n = 2\text{: } P_2(x) = \frac{1}{2}(3x^2-1) = \frac{1}{4}(3\cos 2\phi + 1). \end{aligned}} \tag{7.10.17}$$

These have been chosen such that they equal 1 at $x = 1$ ($\phi = 0$, North Pole). It can be shown that the Legendre polynomials satisfy **Rodrigues' formula**:

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1^n).} \tag{7.10.18}$$

Since Legendre polynomials are orthogonal with weight 1, they can be obtained using the Gram–Schmidt procedure (see Appendix of Section 7.5). We graph (see Fig. 7.10.1) in x and ϕ the first few eigenfunctions (Legendre polynomials). It can be shown that the Legendre polynomials are a complete set of polynomials, and therefore, there are no other eigenvalues besides $\mu = n(n+1)$.

Figure 7.10.1 Legendre polynomials.



If m > 0: The associated Legendre functions

Remarkably, the eigenvalues when $m > 0$ are basically the same as when $m = 0$ and are given by (7.10.15). Even more remarkable is that the eigenfunctions when $m > 0$ (which we have called **associated Legendre functions**) can be related to the eigenfunctions when $m = 0$ (Legendre polynomials):

$$\boxed{g(x) = P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x).} \tag{7.10.19}$$

We note that $P_n(x)$ is the nth-degree Legendre polynomial. The mth derivative will be zero if $n < m$. Thus, the eigenfunctions exist only for $n \geq m$, and the eigenvalues do depend weakly on m. The infinite number of eigenvalues is

$$\boxed{\mu = n(n+1),} \tag{7.10.20}$$

with the restriction that $n \geq m$. These formulas are also valid when $m = 0$; the associated Legendre functions when $m = 0$ are the Legendre polynomials, $P_n^0(x) = P_n(x)$.

7.10.4 RADIAL EIGENVALUE PROBLEMS

The radial Sturm–Liouville differential equation, (7.10.10), with $\mu = n(n+1)$,

$$\boxed{\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda\rho^2 - n(n+1))f = 0,} \tag{7.10.21}$$

has the restriction $n \geq m$ for fixed m. The boundary conditions are $f(a) = 0$, and the solution should be bounded at $\rho = 0$. Equation (7.10.21) is nearly Bessel's differential equation. The parameter λ can be eliminated by instead considering $\sqrt{\lambda}\rho$ as the independent variable. However, the result is not quite Bessel's differential equation. It is easy to show (see the Exercises) that if $Z_p(x)$ solves Bessel's differential equation (7.7.25) of order p, then $f(\rho) = \rho^{-1/2} Z_{n+\frac{1}{2}}(\sqrt{\lambda}\rho)$, called **spherical Bessel functions**, satisfy (7.10.21). Since the radial eigenfunctions must be bounded at $\rho = 0$, we have

$$\boxed{f(\rho) = \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho),} \tag{7.10.22}$$

for $n \geq m$. [If we recall the behavior of the Bessel functions at the origin (7.7.33), we can verify that these solutions are bounded at the origin. In fact, they are zero at the origin except for $n = 0$.] The eigenvalues λ are determined by applying the homogeneous boundary condition at $\rho = a$:

$$\boxed{J_{n+\frac{1}{2}}(\sqrt{\lambda}a) = 0.} \tag{7.10.23}$$

The eigenvalues are determined by the zeros of the Bessel functions of order $n + \frac{1}{2}$. There are an infinite number of eigenvalues for each n and m. Note that the frequencies of vibration are the same for all values of $m \leq n$.

The spherical Bessel functions can be related to trigonometric functions:

$$\boxed{x^{-1/2} J_{n+\frac{1}{2}}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right).} \tag{7.10.24}$$

7.10.5 PRODUCT SOLUTIONS, MODES OF VIBRATION, AND THE INITIAL VALUE PROBLEM

Product solutions for the wave equation in three dimensions are

$$u(\rho, \theta, \phi, t) = \begin{Bmatrix} \cos c\sqrt{\lambda}t \\ \sin c\sqrt{\lambda}t \end{Bmatrix} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} P_n^m(\cos\phi),$$

where the frequencies of vibration are determined from (7.10.23). There are four possible solutions, one from each pair. The angular parts $Y_n^m \equiv \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} P_n^m(\cos\phi)$ are called **surface harmonics** of

the first kind. Initial value problems are solved by using superposition of these infinite modes, summing over m, n, and the infinite radial eigenfunctions characterized by the zeros of the Bessel functions. The weights of the three one-dimensional orthogonality give rise to $d\theta$, $\sin\phi\, d\phi$, $\rho^2 d\rho$, which is equivalent to orthogonality in three dimensions with weight 1, since the differential volume in spherical coordinates is $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. This can be checked using the Jacobian J of the original transformation since $dx\, dy\, dz = J\, d\rho\, d\theta\, d\phi$ and

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2 \sin\phi.$$

Normalization integrals for associated Legendre functions can be found in reference books such as Abramowitz and Stegun:

$$\int_{-1}^{1} [P_n^m(x)]^2\, dx = \left(n + \frac{1}{2}\right)^{-1} \frac{(n+m)!}{(n-m)!} \tag{7.10.25}$$

EXAMPLE

For the purely radial mode $n = 0$ ($m = 0$ only), using (7.10.24) the frequencies of vibration satisfy $\sin\left(\sqrt{\lambda}a\right) = 0$, so that

$$\text{circular frequency} = c\sqrt{\lambda} = \frac{j\pi c}{a},$$

where a is the radius of the Earth, for example. The fundamental mode $j = 1$ has circular frequency $\frac{\pi c}{a}$ radians per second or a frequency of $\frac{c}{2a}$ cycles per second (hertz) or a period of $\frac{2a}{c}$ seconds. For the Earth we can take $a = 6000$ km and $c = 5$ km/s, giving a period of $\frac{12000}{5} = 2400$ seconds or 40 minutes. ■

7.10.6 LAPLACE'S EQUATION INSIDE A SPHERICAL CAVITY

In electrostatics, it is of interest to solve Laplace's equation inside a sphere with the potential u specified on the boundary $\rho = a$:

$$\boxed{\nabla^2 u = 0} \tag{7.10.26}$$

$$\boxed{u(a, \theta, \phi) = F(\theta, \phi).} \tag{7.10.27}$$

This corresponds to determining the electric potential given the distribution of the potential along the spherical conductor. We can use the previous computations, where we solved the problem by separation of variables. The θ and ϕ equations and their solutions will be the same, a Fourier series in θ involving $\cos m\theta$ and $\sin m\theta$ and a generalized Fourier series in ϕ involving the associated Legendre functions $P_n^m(\cos\phi)$. However, we need to insist that $\lambda = 0$, so that the radial equation (7.10.21) will be different and will not be an eigenvalue problem:

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) - n(n+1)f = 0. \tag{7.10.28}$$

Here, (7.10.28) is an equidimensional equation and can be solved exactly by substituting $f = \rho^r$. By substitution we have $r(r+1) - n(n+1) = 0$, which is a quadratic equation with two different roots $r = n$ and $r = -n - 1$ since n is an integer. Since the potential must be bounded at the center $\rho = 0$, we reject the unbounded solution ρ^{-n-1}. Product solutions for Laplace's equation are

$$\rho^n \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} P_n^m(\cos\phi), \tag{7.10.29}$$

so that the solution of Laplace's equation is in the form

$$\boxed{u(\rho, \theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \rho^n [A_{mn} \cos m\theta + B_{mn} \sin m\theta] P_n^m(\cos\phi).} \tag{7.10.30}$$

The nonhomogeneous boundary condition implies that

$$F(\theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a^n [A_{mn} \cos m\theta + B_{mn} \sin m\theta] P_n^m(\cos\phi). \tag{7.10.31}$$

By orthogonality, for example,

$$\boxed{a^n B_{mn} = \frac{\iint F(\theta, \phi) \sin m\theta P_n^m(\cos\phi) \sin\phi \, d\phi \, d\theta}{\iint \sin^2 m\theta [P_n^m(\cos\phi)]^2 \sin\phi \, d\phi \, d\theta}.} \tag{7.10.32}$$

A similar expression exists for A_{mn}.

EXAMPLE

In electrostatics, it is of interest to determine the electric potential inside a conducting sphere if the hemispheres are at different constant potentials. This can be done experimentally by separating two hemispheres by a negligibly thin insulated ring. For convenience, we assume the upper hemisphere is at potential $+V$ and the lower hemisphere at potential $-V$. The boundary condition at $\rho = a$ is cylindrically (azimuthally) symmetric; there is no dependence on the angle θ. We solve Laplace's equation under this simplifying circumstance, or we can use the general solution obtained previously. We follow the later procedure. Since there is no dependence on θ, all terms for the Fourier series in θ will be zero in (7.10.30) except for the $m = 0$ term. Thus, the solution of Laplace's equation with cylindrical symmetry can be written as a series involving the Legendre polynomials:

$$u(\rho, \phi) = \sum_{n=0}^{\infty} A_n \rho^n P_n(\cos\phi). \tag{7.10.33}$$

The boundary condition becomes

$$\left.\begin{array}{ll} V & \text{for} \quad 0 < \phi < \pi/2 \ (0 < x < 1) \\ -V & \text{for} \ \pi/2 < \phi < \pi \quad (-1 < x < 0) \end{array}\right\} = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\phi). \tag{7.10.34}$$

Thus, using orthogonality (in $x = \cos\phi$) with weight 1,

$$\begin{aligned} A_n a^n &= \frac{\int_{-1}^{0} -V P_n(x) \, dx + \int_0^1 V P_n(x) \, dx}{\int_{-1}^{1} [P_n(x)]^2 \, dx} \\ &= \begin{cases} 0 \text{ for } n \text{ even} \\ 2\int_0^1 V P_n(x) \, dx \Big/ \int_{-1}^{1} [P_n(x)]^2 \, dx \text{ for } n \text{ odd}, \end{cases} \end{aligned} \tag{7.10.35}$$

since $P_n(x)$ is even for n even, $P_n(x)$ is odd for n odd, and the potential on the surface of the sphere is an odd function of x. Using the normalization integral (7.10.25) for the denominator and Rodrigues' formula for Legendre polynomials, (7.10.18), for the numerator, it can be shown that

$$u(\rho, \phi) = V\left[\frac{3}{2}\frac{\rho}{a} P_1(\cos\phi) - \frac{7}{8}\left(\frac{\rho}{a}\right)^3 P_3(\cos\phi) + \frac{11}{16}\left(\frac{\rho}{a}\right)^5 P_5(\cos\phi) + \ldots\right]. \tag{7.10.36}$$

For a more detailed discussion of this, see Jackson (1998), *Classical Electrodynamics.* ■

Exercises 7.10

7.10.1. Solve the initial value problem for the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ inside a sphere of radius a subject to the boundary condition $u(a, \theta, \phi, t) = 0$ and the initial conditions

(a) $u(\rho, \theta, \phi, 0) = F(\rho, \theta, \phi)$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = 0$

(b) $u(\rho, \theta, \phi, 0) = 0$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = G(\rho, \theta, \phi)$

(c) $u(\rho, \theta, \phi, 0) = F(\rho, \phi)$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = 0$

(d) $u(\rho, \theta, \phi, 0) = 0$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = G(\rho, \phi)$

(e) $u(\rho, \theta, \phi, 0) = F(\rho, \phi)\cos 3\theta$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = 0$

(f) $u(\rho, \theta, \phi, 0) = F(\rho)\sin 2\theta$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = 0$

(g) $u(\rho, \theta, \phi, 0) = F(\rho)$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = 0$

(h) $u(\rho, \theta, \phi, 0) = 0$ and $\dfrac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = G(\rho)$

7.10.2. Solve the initial value problem for the heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $u(a, \theta, \phi, t) = 0$ and the initial conditions

(a) $u(\rho, \theta, \phi, 0) = F(\rho, \theta, \phi)$

(b) $u(\rho, \theta, \phi, 0) = F(\rho, \phi)$

(c) $u(\rho, \theta, \phi, 0) = F(\rho, \phi)\cos\theta$

(d) $u(\rho, \theta, \phi, 0) = F(\rho)$

7.10.3. Solve the initial value problem for the heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $\frac{\partial u}{\partial \rho}(a, \theta, \phi, t) = 0$ and the initial conditions

(a) $u(\rho, \theta, \phi, 0) = F(\rho, \theta, \phi)$

(b) $u(\rho, \theta, \phi, 0) = F(\rho, \phi)$

(c) $u(\rho, \theta, \phi, 0) = F(\rho, \phi)\sin 3\theta$

(d) $u(\rho, \theta, \phi, 0) = F(\rho)$

7.10.4. Using the one-dimensional Rayleigh quotient, show that $\mu \geq 0$ (if $m \geq 0$) as defined by (7.10.11). Under what conditions does $\mu = 0$?

7.10.5. Using the one-dimensional Rayleigh quotient, show that $\mu \geq 0$ (if $m \geq 0$) as defined by (7.10.13). Under what conditions does $\mu = 0$?

7.10.6. Using the one-dimensional Rayleigh quotient, show that $\lambda \geq 0$ (if $n \geq 0$) as defined by (7.10.6) with the boundary condition $f(a) = 0$. Can $\lambda = 0$?

7.10.7. Using the three-dimensional Rayleigh quotient, show that $\lambda \geq 0$ as defined by (7.10.11) with $u(a, \theta, \phi, t) = 0$. Can $\lambda = 0$?

7.10.8. Show that

$$x^2\frac{d^2 f}{dx^2} + x(1 - 2a - 2bx)\frac{df}{dx} + [a^2 - p^2 + (2a - 1)bx + (d^2 + b^2x^2)]f = 0 \tag{7.10.37}$$

has solutions $x^a e^{bx} Z_p(dx)$, where $Z_p(x)$ satisfies Bessel's differential equation (7.7.25). By comparing (7.10.21) and (7.10.37), we have $a = -\frac{1}{2}$, $b = 0$, $\frac{1}{4} - p^2 = -n(n + 1)$, and $d^2 = \lambda$. We find that $p = (n + \frac{1}{2})$.

7.10.9. Solve Laplace's equation inside a sphere $\rho < a$ subject to the following boundary conditions on the sphere:

(a) $u(a, \theta, \phi) = F(\phi)\cos 4\theta$

(b) $u(a, \theta, \phi) = F(\phi)$

(c) $\dfrac{\partial u}{\partial \rho}(a, \theta, \phi) = F(\phi)\cos 4\theta$

(d) $\dfrac{\partial u}{\partial \rho}(a, \theta, \phi) = F(\phi)$ with $\displaystyle\int_0^\pi F(\phi)\sin\phi\, d\phi = 0$

(e) $\dfrac{\partial u}{\partial \rho}(a, \theta, \phi) = F(\theta, \phi)$ with $\displaystyle\int_0^\pi \int_0^{2\pi} F(\theta, \phi)\sin\phi\, d\theta\, d\phi = 0$

7.10.10. Solve Laplace's equation outside a sphere $\rho > a$ subject to the potential given on the sphere:

(a) $u(a, \theta, \phi) = F(\theta, \phi)$

(b) $u(a, \theta, \phi) = F(\phi)$, with cylindrical (azimuthal) symmetry

(c) $u(a, \theta, \phi) = V$ in the upper hemisphere, $-V$ in the lower hemisphere (do not simplify; do not evaluate definite integrals)

7.10.11. Solve Laplace's equation inside a sector of a sphere $\rho < a$ with $0 < \theta < \frac{\pi}{2}$ subject to $u(\rho, 0, \phi) = 0$ and $u(\rho, \frac{\pi}{2}, \phi) = 0$ and the potential given on the sphere: $u(a, \theta, \phi) = F(\theta, \phi)$.

7.10.12. Solve Laplace's equation inside a hemisphere $\rho < a$ with $z > 0$ subject to $u = 0$ at $z = 0$ and the potential given on the hemisphere: $u(a, \theta, \phi) = F(\theta, \phi)$. [*Hint*: Use symmetry and solve a different problem, a sphere with the antisymmetric potential on the lower hemisphere.]

7.10.13. Show that Rodrigues' formula agrees with the given Legendre polynomials for $n = 0$, $n = 1$, and $n = 2$.

7.10.14. Show that Rodrigues' formula satisfies the differential equation for Legendre polynomials.

7.10.15. Derive (7.10.36) using (7.10.35), (7.10.18), and (7.10.25).

8

Nonhomogeneous Problems

8.1 Introduction

In the previous chapters we have developed only one method to solve partial differential equations: the method of separation of variables. In order to apply the method of separation of variables, the partial differential equation (with n independent variables) must be linear and homogeneous. In addition, we must be able to formulate a problem with linear and homogeneous boundary conditions for $n - 1$ variables. However, some of the most fundamental physical problems do not have homogeneous conditions.

8.2 Heat Flow with Sources and Nonhomogeneous Boundary Conditions

TIME-INDEPENDENT BOUNDARY CONDITIONS

As an elementary example of a nonhomogeneous problem, consider the heat flow (without sources) in a uniform rod of length L with the temperature fixed at the left end at $A°$ and the right at $B°$. If the initial condition is prescribed, the mathematical problem for the temperature $u(x, t)$ is

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \tag{8.2.1}$$

$$\text{BC1: } \boxed{u(0, t) = A} \tag{8.2.2}$$

$$\text{BC2: } \boxed{u(L, t) = B} \tag{8.2.3}$$

$$\text{IC: } \boxed{u(x, 0) = f(x).} \tag{8.2.4}$$

The method of separation of variables cannot be used directly since for even this simple example the boundary conditions are not homogeneous.

Equilibrium temperature

To analyze this problem, we first obtain an equilibrium temperature distribution, $u_E(x)$. If such a temperature distribution exists, it must satisfy the steady-state (time-independent) heat equation,

$$\boxed{\frac{d^2 u_E}{dx^2} = 0,} \tag{8.2.5}$$

as well as the given time-independent boundary conditions,

$$u_E(0) = A \tag{8.2.6}$$

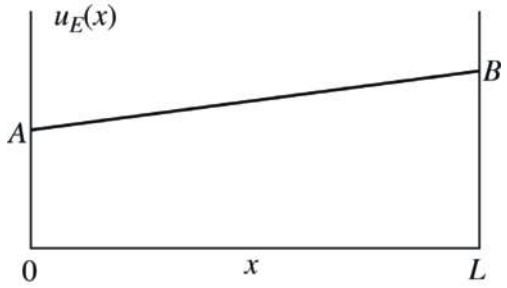
$$u_E(L) = B. \tag{8.2.7}$$

We ignore the initial conditions in defining an equilibrium temperature distribution. As shown in Section 1.4, (8.2.5) implies that the temperature distribution is linear, and the unique one that satisfies (8.2.2) and (8.2.3) can be determined geometrically or algebraically:

$$\boxed{u_E(x) = A + \frac{B - A}{L}x,} \tag{8.2.8}$$

which is sketched in Fig. 8.2.1. Usually, $u_E(x)$ will *not* be the desired time-dependent solution, since it satisfies the initial conditions (8.2.4) only if $f(x) = A + [(B - A)/L]x$.

Figure 8.2.1 Equilibrium temperature distribution.



Displacement from equilibrium

For more general initial conditions, we consider the temperature displacement from the equilibrium temperature,

$$\boxed{v(x, t) \equiv u(x, t) - u_E(x).} \tag{8.2.9}$$

Instead of solving for $u(x, t)$, we will determine $v(x, t)$. Since $\partial v/\partial t = \partial u/\partial t$ and $\partial^2 v/\partial x^2 = \partial^2 u/\partial x^2$ [note that $u_E(x)$ is linear in x], it follows that $v(x, t)$ also satisfies the heat equation

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}. \tag{8.2.10}$$

Furthermore, both $u(x, t)$ and $u_E(x)$ equal A at $x = 0$ and equal B at $x = L$, and hence their difference is zero at $x = 0$ and at $x = L$:

$$v(0, t) = 0 \tag{8.2.11}$$

$$v(L, t) = 0. \tag{8.2.12}$$

Initially, $v(x, t)$ equals the difference between the given initial temperature and the equilibrium temperature,

$$v(x, 0) = f(x) - u_E(x). \tag{8.2.13}$$

Fortunately, the mathematical problem for $v(x, t)$ is a linear homogeneous partial differential equation with linear homogeneous boundary conditions. Thus, $v(x, t)$ can be determined by the method of separation of variables. In fact, this problem is one we have encountered in (2.3.1)–(2.3.3). Hence, we note that from (2.3.30),

$$v(x, t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}, \tag{8.2.14}$$

where the initial conditions imply that

$$f(x) - u_E(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \tag{8.2.15}$$

Thus, a_n equals the Fourier sine coefficients of $f(x) - u_E(x)$:

$$a_n = \frac{2}{L}\int_0^L [f(x) - u_E(x)]\sin\frac{n\pi x}{L}\,dx. \tag{8.2.16}$$

From (8.2.9), we easily obtain the desired temperature, $u(x, t) = u_E(x) + v(x, t)$. Thus,

$$\boxed{u(x, t) = u_E(x) + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t},} \tag{8.2.17}$$

where a_n is given by (8.2.16) and $u_E(x)$ is given by (8.2.8). As $t \to \infty$, $u(x, t) \to u_E(x)$ irrespective of the initial conditions. **The temperature approaches its equilibrium distribution for all initial conditions**.

Steady nonhomogeneous terms

The previous method also works if there are steady sources of thermal energy:

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)} \tag{8.2.18}$$

$$\text{BC: } \boxed{\begin{array}{l} u(0, t) = A \\ u(L, t) = B \end{array}} \tag{8.2.19}$$

$$\text{IC: } \boxed{u(x, 0) = f(x).} \tag{8.2.20}$$

If an equilibrium solution exists (see Exercise 1.4.6 for a somewhat different example in which an equilibrium solution does *not* exist), then we determine it and again consider the displacement from equilibrium,

$$v(x, t) = u(x, t) - u_E(x).$$

We can show that $v(x, t)$ satisfies a linear homogeneous partial differential equation (8.2.10) with linear homogeneous boundary conditions (8.2.11) and (8.2.12). Thus, again $u(x, t) \to u_E(x)$ as $t \to \infty$.

Time-dependent nonhomogeneous terms

Unfortunately, nonhomogeneous problems are not always as easy to solve as the previous examples. In order to clarify the situation, we again consider the heat flow in a uniform rod of length L. However, we make two substantial changes. First, we introduce temperature-independent heat sources distributed in a prescribed way throughout the rod. Thus, the temperature will solve the following nonhomogeneous partial differential equation:

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t).} \tag{8.2.21}$$

Here, the sources of thermal energy $Q(x, t)$ vary in space and time. In addition, we allow the temperature at the ends to vary in time. This yields time-dependent and nonhomogeneous linear boundary conditions,

$$\text{BC: } \boxed{\begin{array}{l} u(0, t) = A(t) \\ u(L, t) = B(t), \end{array}} \tag{8.2.22}$$

instead of the time-independent ones, (8.2.2) and (8.2.3). Again the initial temperature distribution is prescribed:

$$\text{IC: } \boxed{u(x, 0) = f(x).} \tag{8.2.23}$$

The mathematical problem defined by (8.2.21)–(8.2.23) consists of a nonhomogeneous partial differential equation with nonhomogeneous boundary conditions.

Related homogeneous boundary conditions

We claim that we *cannot* always reduce this problem to a homogeneous partial differential equation with homogeneous boundary conditions, as we did for the first example of this section. Instead, we will find it quite useful to note that we can always transform our problem into one

with homogeneous boundary conditions, although in general the partial differential equation will remain nonhomogeneous. We consider *any* **reference temperature distribution** $r(x, t)$ (the simpler the better) with only the property that it satisfy the given nonhomogeneous boundary conditions. In our example, this means only that

$$r(0, t) = A(t)$$
$$r(L, t) = B(t).$$

It is usually not difficult to obtain many candidates for $r(x, t)$. Perhaps the simplest choice is

$$r(x, t) = A(t) + \frac{x}{L}[B(t) - A(t)], \tag{8.2.24}$$

although there are other possibilities.[1] Again the difference between the desired solution $u(x, t)$ and the chosen function $r(x, t)$ (now *not* necessarily an equilibrium solution) is employed:

$$v(x, t) \equiv u(x, t) - r(x, t). \tag{8.2.25}$$

Since both $u(x, t)$ and $r(x, t)$ satisfy the same linear (although nonhomogeneous) boundary condition at both $x = 0$ and $x = L$, it follows that $v(x, t)$ satisfies the *related* homogeneous boundary conditions:

$$v(0, t) = 0 \tag{8.2.26}$$
$$v(L, t) = 0. \tag{8.2.27}$$

The partial differential equation satisfied by $v(x, t)$ is derived by substituting

$$u(x, t) = v(x, t) + r(x, t)$$

into the heat equation with sources, (8.2.21). Thus,

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left[Q(x, t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}\right] \equiv k\frac{\partial^2 v}{\partial x^2} + \overline{Q}. \tag{8.2.28}$$

In general, the partial differential equation for $v(x, t)$ is of the same type as for $u(x, t)$, but with a different nonhomogeneous term, since $r(x, t)$ usually does not satisfy the homogeneous heat equation. The initial condition is also usually altered:

$$v(x, 0) = f(x) - r(x, 0) = f(x) - A(0) - \frac{x}{L}[B(0) - A(0)] \equiv g(x). \tag{8.2.29}$$

It can be seen that in general, only the boundary conditions have been made homogeneous. In Section 8.3, we will develop a method to analyze nonhomogeneous problems with homogeneous boundary conditions.

Exercises 8.2

8.2.1. Solve the heat equation with time-independent sources and boundary conditions

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$
$$u(x, 0) = f(x)$$

if an equilibrium solution exists. Analyze the limits as $t \to \infty$. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundary conditions (but do not solve). Assume

***(a)**	$Q(x) = 0,$	$u(0, t) = A,$	$\frac{\partial u}{\partial x}(L, t) = B$
(b)	$Q(x) = 0,$	$\frac{\partial u}{\partial x}(0, t) = 0,$	$\frac{\partial u}{\partial x}(L, t) = B \neq 0$
(c)	$Q(x) = 0,$	$\frac{\partial u}{\partial x}(0, t) = A \neq 0$	$\frac{\partial u}{\partial x}(L, t) = A$
***(d)**	$Q(x) = k,$	$u(0, t) = A,$	$u(L, t) = B$
(e)	$Q(x) = k,$	$\frac{\partial u}{\partial x}(0, t) = 0,$	$\frac{\partial u}{\partial x}(L, t) = 0$
(f)	$Q(x) = \sin\frac{2\pi x}{L},$	$\frac{\partial u}{\partial x}(0, t) = 0,$	$\frac{\partial u}{\partial x}(L, t) = 0$

[1] Other choices for $r(x, t)$ yield equivalent solutions to the original nonhomogeneous problem.

8.2.2. Consider the heat equation with time-dependent sources and boundary conditions

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$u(x,0) = f(x).$$

Reduce the problem to one with homogeneous boundary conditions if

***(a)** $\frac{\partial u}{\partial x}(0,t) = A(t)$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$

(b) $u(0,t) = A(t)$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$

***(c)** $\frac{\partial u}{\partial x}(0,t) = A(t)$ and $u(L,t) = B(t)$

(d) $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) + h(u(L,t) - B(t)) = 0$

(e) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) + h(u(L,t) - B(t)) = 0$

8.2.3. Solve the two-dimensional heat equation with circularly symmetric time-independent sources, boundary conditions, and initial conditions (inside a circle)

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with

$$u(r,0) = f(r) \text{ and } u(a,t) = T.$$

8.2.4. Solve the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the time-independent boundary conditions

$$u(0,y,t) = 0,\ \frac{\partial}{\partial y}u(x,0,t) = 0$$
$$u(L,y,t) = 0,\ u(x,H,t) = g(x)$$

and the initial condition

$$u(x,y,0) = f(x,y).$$

Analyze the limit as $t \to \infty$.

8.2.5. Solve the initial value problem for a two-dimensional heat equation inside a circle (of radius a) with time-independent boundary conditions

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$
$$u(a,\theta,t) = g(\theta)$$
$$u(r,\theta,0) = f(r,\theta).$$

8.2.6. Solve the wave equation with time-independent sources

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + Q(x)$$
$$u(x,0) = f(x)$$
$$\frac{\partial}{\partial t}u(x,0) = g(x)$$

if an "equilibrium" solution exists. Analyze the behavior for large t. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundary conditions. Assume that

***(a)** $Q(x) = 0$, $u(0,t) = A$, $u(L,t) = B$

(b) $Q(x) = 1$, $u(0,t) = 0$, $u(L,t) = 0$

(c) $Q(x) = 1$, $u(0,t) = A$, $u(L,t) = B$
[*Hint*: Add problems (a) and (b).]

***(d)** $Q(x) = \sin\frac{\pi x}{L}$, $u(0,t) = 0$, $u(L,t) = 0$

8.3 Method of Eigenfunction Expansion with Homogeneous Boundary Conditions (Differentiating Series of Eigenfunctions)

In Section 8.2, we showed how to introduce a problem with homogeneous boundary conditions, even if the original problem of interest has nonhomogeneous boundary conditions. For that reason, we will investigate nonhomogeneous linear partial differential equations with homogeneous boundary conditions. For example, consider

$$\text{PDE: } \boxed{\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t)} \tag{8.3.1}$$

$$\text{BC: } \boxed{\begin{aligned} v(0,t) &= 0 \\ v(L,t) &= 0 \end{aligned}} \tag{8.3.2}$$

$$\text{IC: } \boxed{v(x,0) = g(x).} \tag{8.3.3}$$

We will solve this problem by the **method of eigenfunction expansion**. Consider the eigenfunctions of the related homogeneous problem. The related homogeneous problem is

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \\ u(0,t) &= 0 \\ u(L,t) &= 0. \end{aligned} \tag{8.3.4}$$

The eigenfunctions of this related homogeneous problem satisfy

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned} \tag{8.3.5}$$

We know that the eigenvalues are $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \ldots$, and the corresponding eigenfunctions are $\phi_n(x) = \sin n\pi x/L$. However, the eigenfunctions will be different for other problems. We do not wish to emphasize the method of eigenfunction expansion solely for this one example. Thus, we will speak in some generality. We assume that the eigenfunctions (of the related homogeneous problem) are known, and we designate them $\phi_n(x)$. The eigenfunctions satisfy a Sturm–Liouville eigenvalue problem and as such they are complete (any piecewise smooth function may be expanded in a series of these eigenfunctions). **The method of eigenfunction expansion, employed to solve the nonhomogeneous problem (8.3.1) with homogeneous boundary conditions, (8.3.2), consists of expanding the unknown solution $v(x, t)$ in a series of the eigenfunction of the related homogeneous problem**:

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).} \tag{8.3.6}$$

For each fixed t, $v(x, t)$ is a function of x, and hence $v(x, t)$ will have a generalized Fourier series. In our example, $\phi_n(x) = \sin n\pi x/L$, and this series is an ordinary Fourier sine series. The generalized Fourier coefficients are a_n, but the coefficients will vary as t changes. Thus, the generalized Fourier coefficients are functions of time, $a_n(t)$. At first glance, expansion (8.3.6) may appear similar to what occurs in separating variables for homogeneous problems. However, (8.3.6) is substantially different. Here, $a_n(t)$ are *not* the time-dependent separated solutions $e^{-k(n\pi/L)^2 t}$. Instead, $a_n(t)$ are just the generalized Fourier coefficients for $v(x, t)$. We will determine $a_n(t)$ and show that usually $a_n(t)$ is not proportional to $e^{-k(n\pi/L)^2 t}$.

Equation (8.3.6) automatically satisfies the homogeneous boundary conditions. We emphasize this by stating that both $v(x, t)$ and $\phi_n(x)$ satisfy the same homogeneous boundary conditions. The initial condition is satisfied if

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x).$$

Due to the orthogonality of the eigenfunctions [with weight 1 in this problem because of the constant coefficient in (8.3.5)], we can determine the initial values of the generalized Fourier coefficients:

$$a_n(0) = \frac{\int_0^L g(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}. \tag{8.3.7}$$

"All" that remains is to determine $a_n(t)$ such that (8.3.6) solves the nonhomogeneous partial differential equation (8.3.1). We will show in two different ways that $a_n(t)$ satisfies a first-order differential equation in order for (8.3.6) to satisfy (8.3.1).

One method to determine $a_n(t)$ is by direct substitution. This is easy to do but requires calculation of $\partial v/\partial t$ and $\partial^2 v/\partial x^2$. Since $v(x, t)$ is an infinite series, the differentiation can be a delicate process. We simply state that with some degree of generality, if v and $\partial v/\partial x$ are continuous and **if $v(x, t)$ solves the same homogeneous boundary conditions as does $\phi_n(x)$, then the necessary term-by-term differentiations can be justified**. For the cases of Fourier sine and cosine series, a more detailed investigation of the properties of the term-by-term differentiation of these series was presented in Section 3.4, which proved this result. For the general case, we omit a proof. However, we obtain the same solution in Section 8.4 by an alternative method, which thus justifies the somewhat simpler technique of the present section. We thus proceed to differentiate $v(x, t)$ term-by-term:

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n(t)}{dt}\phi_n(x)$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t)\frac{d^2\phi_n(x)}{dx^2} = -\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x),$$

since $\phi_n(x)$ satisfies $d^2\phi_n/dx^2 + \lambda_n\phi_n = 0$. Substituting these results into (8.3.1) yields

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt} + \lambda_n k a_n\right]\phi_n(x) = \bar{Q}(x, t). \tag{8.3.8}$$

The left-hand side is the generalized Fourier series for $\bar{Q}(x, t)$. Due to the orthogonality of $\phi_n(x)$, we obtain a first-order differential equation for $a_n(t)$:

$$\frac{da_n}{dt} + \lambda_n k a_n = \frac{\int_0^L \bar{Q}(x, t)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} \equiv \bar{q}_n(t). \tag{8.3.9}$$

The right-hand side is a known function of time (and n), namely, the Fourier coefficient of $\bar{Q}(x, t)$:

$$\bar{Q}(x, t) = \sum_{n=1}^{\infty} \bar{q}_n(t)\phi_n(x).$$

Equation (8.3.9) requires an initial condition, and sure enough $a_n(0)$ equals the generalized Fourier coefficients of the initial condition [see (8.3.7)].

Equation (8.3.9) is a nonhomogeneous linear first-order equation. Perhaps the easiest method[2] to solve it [unless $\bar{q}_n(t)$ is particularly simple] is to multiply it by the integrating factor $e^{\lambda_n k t}$. Thus,

$$e^{\lambda_n k t}\left(\frac{da_n}{dt} + \lambda_n k a_n\right) = \frac{d}{dt}(a_n e^{\lambda_n k t}) = \bar{q}_n e^{\lambda_n k t}.$$

Integrating from 0 to t yields

$$a_n(t)e^{\lambda_n k t} - a_n(0) = \int_0^t \bar{q}_n(\tau)e^{\lambda_n k \tau}d\tau.$$

We solve for $a_n(t)$ and obtain

$$\boxed{a_n(t) = a_n(0)e^{-\lambda_n k t} + e^{-\lambda_n k t}\int_0^t \bar{q}_n(\tau)e^{\lambda_n k \tau}\,d\tau.} \tag{8.3.10}$$

[2]Another method is variation of parameters.

Note that $a_n(t)$ is in the form of a constant, $a_n(0)$, times the homogeneous solution $e^{-\lambda_n kt}$ plus a particular solution. This completes the method of eigenfunction expansions. The solution of our nonhomogeneous partial differential equation with homogeneous boundary conditions is

$$v(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

where $\phi_n(x) = \sin n\pi x/L$, $\lambda_n = (n\pi/L)^2$, $a_n(t)$ is given by (8.3.10), $\overline{q}_n(\tau)$ is given by (8.3.9), and $a_n(0)$ is given by (8.3.7). The solution is rather complicated.

As a check, if the problem was homogeneous, $Q(x, t) = 0$, then the solution simplifies to

$$v(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

where

$$a_n(t) = a_n(0)e^{-\lambda_n kt}$$

and $a_n(0)$ is given by (8.3.7), exactly the solution obtained by separation of variables.

EXAMPLE

As an elementary example, suppose that for $0 < x < \pi$ (i.e., $L = \pi$),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}\sin 3x, \text{ subject to } \begin{cases} u(0, t) = 0, \\ u(\pi, t) = 1, \\ u(x, 0) = f(x). \end{cases}$$

To make the boundary conditions homogeneous, we introduce the displacement from equilibrium $v(x, t) = u(x, t) - x/\pi$, in which case

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + e^{-t}\sin 3x, \text{ subject to } \begin{cases} v(0, t) = 0, \\ v(\pi, t) = 1, \\ v(x, 0) = f(x) - \dfrac{x}{\pi}. \end{cases}$$

The eigenfunctions are $\sin n\pi x/L = \sin nx$ (since $L = \pi$), and thus

$$v(x, t) = \sum_{n=1}^{\infty} a_n(t)\sin nx. \tag{8.3.11}$$

This eigenfunction expansion is substituted into the PDE, yielding

$$\sum_{n=1}^{\infty}\left(\frac{da_n}{dt} + n^2 a_n\right)\sin nx = e^{-t}\sin 3x.$$

Thus, the unknown Fourier sine coefficients satisfy

$$\frac{da_n}{dt} + n^2 a_n = \begin{cases} 0 & n \neq 3, \\ e^{-t} & n = 3. \end{cases}$$

The solution of this does not require (8.3.10):

$$a_n(t) = \begin{cases} a_n(0)e^{-n^2 t}, & n \neq 3, \\ \dfrac{1}{8}e^{-t} + \left[a_3(0) - \dfrac{1}{8}\right]e^{-9t}, & n = 3, \end{cases} \tag{8.3.12}$$

where

$$a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left[f(x) - \frac{x}{\pi}\right]\sin nx\, dx. \tag{8.3.13}$$

The solution to the original nonhomogeneous problem is given by $u(x, t) = v(x, t) + x/\pi$, where v is given by (8.3.11) with $a_n(t)$ determined from (8.3.12) and (8.3.13). ■

Exercises 8.3

8.3.1. Solve the initial value problem for the heat equation with time-dependent sources

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$u(x,0) = f(x)$$

subject to the following boundary conditions:

(a) $u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$

(b) $u(0,t) = 0, \qquad u(L,t) + 2\frac{\partial u}{\partial x}(L,t) = 0$

***(c)** $u(0,t) = A(t), \qquad \frac{\partial u}{\partial x}(L,t) = 0$

(d) $u(0,t) = A \neq 0, \qquad u(L,t) = 0$

(e) $\frac{\partial u}{\partial x}(0,t) = A(t), \qquad \frac{\partial u}{\partial x}(L,t) = B(t)$

***(f)** $\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$

(g) Specialize part (f) to the case $Q(x,t) = Q(x)$ (independent of t) such that $\int_0^L Q(x)\,dx \neq 0$. In this case, show that there are no time-independent solutions. What happens to the time-dependent solution as $t \to \infty$? Briefly explain.

8.3.2. Consider the heat equation with a steady source

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$

subject to the initial and boundary conditions:

$$u(0,t) = 0,\ u(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Obtain the solution by the method of eigenfunction expansion. Show that the solution approaches a steady-state solution.

***8.3.3.** Solve the initial value problem

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + qu + f(x,t),$$

where c, ρ, K_0, and q are functions of x only, subject to the conditions

$$u(0,t) = 0,\ u(L,t) = 0, \quad \text{and} \quad u(x,0) = g(x).$$

Assume that the eigenfunctions are known. [*Hint*: Let $L \equiv \frac{d}{dx}\left(K_0\frac{d}{dx}\right) + q$.]

8.3.4. Consider

$$\frac{\partial u}{\partial t} = \frac{1}{\sigma(x)}\frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] \qquad (K_0 > 0, \sigma > 0)$$

with the boundary conditions and initial conditions

$$u(x,0) = g(x),\ u(0,t) = A, \quad \text{and} \quad u(L,t) = B.$$

***(a)** Find a time-independent solution, $u_0(x)$.

(b) Show that $\lim_{t\to\infty} u(x,t) = f(x)$ independent of the initial conditions. [Show that $f(x) = u_0(x)$.]

***8.3.5.** Solve

$$\frac{\partial u}{\partial t} = k\nabla^2 u + f(r,t)$$

inside the circle ($r < a$) with $u = 0$ at $r = a$ and initially $u(r,0) = 0$.

8.3.6. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-2t}\sin 5x$$

subject to $u(0,t) = 1$, $u(\pi,t) = 0$, and $u(x,0) = 0$.

***8.3.7.** Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to $u(0, t) = 0$, $u(L, t) = t$, and $u(x, 0) = 0$.

8.4 Method of Eigenfunction Expansion Using Green's Formula (with or without Homogeneous Boundary Conditions)

In this section, we reinvestigate problems that may have nonhomogeneous boundary conditions. We still use the method of eigenfunction expansion. For example, consider

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t)} \tag{8.4.1}$$

$$\text{BC: } \boxed{\begin{aligned} u(0, t) &= A(t) \\ u(L, t) &= B(t) \end{aligned}} \tag{8.4.2}$$

$$\text{IC: } \boxed{u(x, 0) = f(x).} \tag{8.4.3}$$

The eigenfunctions of the related homogeneous problem,

$$\frac{d^2\phi_n}{dx^2} + \lambda_n\phi_n = 0 \tag{8.4.4}$$

$$\phi_n(0) = 0 \tag{8.4.5}$$

$$\phi_n(L) = 0, \tag{8.4.6}$$

are known to be $\phi_n(x) = \sin n\pi x/L$, corresponding to the eigenvalues $\lambda_n = (n\pi/L)^2$. Any piecewise smooth function can be expanded in terms of these eigenfunctions. Thus, even though $u(x, t)$ satisfies nonhomogeneous boundary conditions, it is still true that

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x).} \tag{8.4.7}$$

Note that the equality in (8.4.7) is not valid right at $x = 0$ or $x = L$ since $\phi_n(x)$ satisfies homogeneous boundary conditions, while $u(x, t)$ does not. However, the sum converges to the true solution $u(x, t)$ for all interior points $0 < x < L$ and thus the nonhomogeneous boundary conditions are satisfied in the limit as $x = 0$ and $x = L$ are approached from the interior of the domain. It is difficult to determine $b_n(t)$ by substituting (8.4.7) into (8.4.1); the required term-by-term differentiations with respect to x are not justified since $u(x, t)$ and $\phi_n(x)$ do not satisfy the same homogeneous boundary conditions $\left[\partial^2 u/\partial x^2 \neq \sum_{n=1}^{\infty} b_n(t)d^2\phi_n/dx^2\right]$. However, term-by-term differentiation in time were shown valid in Section 3.4:

$$\boxed{\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{db_n}{dt}\phi_n(x).} \tag{8.4.8}$$

We will determine a first-order differential equation for $b_n(t)$. Unlike in Section 8.3, it will be obtained *without* calculating spatial derivatives of an infinite series of eigenfunctions. From (8.4.8) it follows that

$$\sum_{n=1}^{\infty} \frac{db_n}{dt}\phi_n(x) = k\frac{\partial^2 u}{\partial x^2} + Q(x, t),$$

and thus

$$\frac{db_n}{dt} = \frac{\int_0^L \left[k\dfrac{\partial^2 u}{\partial x^2} + Q(x, t)\right]\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx}. \tag{8.4.9}$$

Equation (8.4.9) allows the partial differential equation to be satisfied. Already we note the importance of the generalized Fourier series of $Q(x, t)$:

$$Q(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x), \quad \text{where} \quad q_n(t) = \frac{\int_0^L Q(x, t)\phi_n(x)\, dx}{\int_0^L \phi_n^2\, dx}.$$

Thus, (8.4.9) simplifies to

$$\boxed{\frac{db_n}{dt} = q_n(t) + \frac{\int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_n(x)\, dx}{\int_0^L \phi_n^2\, dx}.} \tag{8.4.10}$$

We will show how to evaluate the integral in (8.4.10) in terms of $b_n(t)$, yielding a first-order differential equation.

If we integrate $\int_0^L \partial^2 u/\partial x^2 \phi_n(x)\, dx$ twice by parts, then we would obtain the desired result. However, this would be a considerable effort. There is a better way; we have already performed the required integration by parts in a more general context. Perhaps the operator notation, $L \equiv \partial^2/\partial x^2$, will help to remind you of the result we need. Using $L = \partial^2/\partial x^2$,

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\, dx = \int_0^L \phi_n L(u)\, dx.$$

Now this may be simplified by employing Green's formula (derived by repeated integrations in Section 5.5). Let us restate **Green's formula**:

$$\boxed{\int_0^L [uL(v) - vL(u)]\, dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_0^L,} \tag{8.4.11}$$

where L is any Sturm–Liouville operator $[L \equiv d/dx(p\, d/dx) + q]$. In our context, $L = \partial^2/\partial x^2$ (i.e., $p = 1, q = 0$). Partial derivatives may be used, since $\partial/\partial x = d/dx$ with t fixed. Thus,

$$\boxed{\int_0^L \left(u\frac{\partial^2 v}{\partial x^2} - v\frac{\partial^2 u}{\partial x^2}\right) dx = \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\Big|_0^L.} \tag{8.4.12}$$

Now we let $v = \phi_n(x)$. Often both u and ϕ_n satisfy the same homogeneous boundary conditions, and the right-hand side of (8.4.12) vanishes. In this case, $\phi_n(x) = \sin n\pi x/L$ satisfies homogeneous boundary conditions, but $u(x, t)$ satisfies the nonhomogeneous conditions $u(0, t) = A(t)$ and $u(L, t) = B(t)$. Using $d\phi_n/dx = (n\pi/L)\cos n\pi x/L$, (8.4.12) becomes

$$\int_0^L \left(u\frac{d^2\phi_n}{dx^2} - \phi_n\frac{\partial^2 u}{\partial x^2}\right) dx = \frac{n\pi}{L}\big(B(t)(-1)^n - A(t)\big).$$

Our goal is to isolate $\int_0^L \phi_n\, \partial^2 u/\partial x^2\, dx$ in order to express the right-hand side of (8.4.10) in terms of known functions and b_n. Note that $\int_0^L u\, \partial^2\phi_n/\partial x^2 dx = -\lambda_n \int_0^L u\phi_n$ since $d^2\phi_n/dx^2 + \lambda_n\phi_n = 0$. We further simplify $-\lambda_n\int_0^L u\phi_n$ since $b_n(t)$ are the generalized Fourier coefficients of $u(x, t)$, so that

$$b_n(t) = \frac{\int_0^L u\phi_n\, dx}{\int_0^L \phi_n^2\, dx}.$$

Finally, (8.4.10) reduces to a first-order differential equation for $b_n(t)$:

$$\boxed{\frac{db_n}{dt} + k\lambda_n b_n = q_n(t) + \frac{k(n\pi/L)\big[A(t) - (-1)^n B(t)\big]}{\int_0^L \phi_n^2(x)\, dx}.} \tag{8.4.13}$$

The nonhomogeneous terms arise in two ways: $q_n(t)$ is due to the source terms in the PDE, while the term involving $A(t)$ and $B(t)$ is a result of the nonhomogeneous boundary conditions

at $x = 0$ and $x = L$. Equation (8.4.13) is again solved by introducing the integrating factor $e^{k\lambda_n t}$. The required initial condition for $b_n(t)$ follows from the given initial condition, $u(x, 0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x)$$

$$b_n(0) = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx}.$$

It is interesting to note that the differential equation for the coefficients $b_n(t)$ for problems with nonhomogeneous boundary conditions is quite similar to the one that occurred in the preceding section for homogeneous boundary conditions; only the nonhomogeneous term is modified.

EXAMPLE

This example demonstrates the method of eigenfunction expansion with nonhomogeneous and time-dependent boundary conditions. Suppose that for $0 < x < 7$,

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} + xt,$$

subject to

$$u(0, t) = e^{-2t}, \quad u(7, t) = e^{-5t}, \quad \text{and} \quad u(x, 0) = 1 + x.$$

The eigenfunctions $\phi_n(x)$ are defined by (8.4.4), (8.4.5), and (8.4.6) with $L = 7$, so that $\phi_n(x) = \sin(n\pi x/7)$. Note that $\int_0^7 \phi_n^2\,dx = 7/2$. Let

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{7}.$$

It remains only to solve (8.4.13) for $b_n(t)$. The right-hand side terms include

$$q_n(t) = \frac{\int_0^7 xt\sin\frac{n\pi x}{7}\,dx}{\int_0^7 \phi_n^2\,dx} = \frac{2}{7}\int_0^7 xt\sin\frac{n\pi x}{7}\,dx = \frac{14t}{n\pi}(-1)^{n+1}$$

and

$$\frac{k(n\pi/L)\left[A(t) - (-1)^n B(t)\right]}{\int_0^L \phi_n^2\,dx} = \frac{3n\pi}{7}\frac{\left[e^{-2t} - (-1)^n e^{-5t}\right]}{7/2} = \frac{6n\pi}{49}\left[e^{-2t} - (-1)^n e^{-5t}\right].$$

The differential equation for $b_n(t)$ is therefore

$$\frac{db_n}{dt} + \frac{3n^2\pi^2}{49}b_n = R(t), \quad \text{where} \quad R(t) = \frac{14t}{n\pi}(-1)^{n+1} + \frac{6n\pi}{49}\left[e^{-2t} - (-1)^n e^{-5t}\right].$$

The solution is

$$b_n(t) = b_n(0)e^{-k\lambda_n t} + e^{-k\lambda_n t}\int_0^t R(\tau)e^{k\lambda_n \tau}\,d\tau$$

where $k = 3$, $\lambda_n = n^2\pi^2/49$, and

$$b_n(0) = \frac{\int_0^7 (1 + x)\sin\frac{n\pi x}{7}\,dx}{7/2} = \frac{2}{n\pi}\left(1 - 8(-1)^n\right).$$

The computation of the integral $\int_0^t R(\tau)e^{k\lambda_n \tau}\,d\tau$ is tedious but doable. Ultimately, we find

$$b_n(t) = \frac{2}{n\pi}\left(1 - 8(-1)^n\right)e^{-k\lambda_n t} + \frac{14t}{n\pi k\lambda_n}(-1)^{n+1} - \frac{14}{n\pi k^2\lambda_n^2}(-1)^{n+1}(1 - e^{-k\lambda_n t})$$

$$+ \frac{6n\pi}{49(k\lambda_n - 2)}(e^{-2t} - e^{-k\lambda_n t}) - \frac{6n\pi(-1)^n}{49(k\lambda_n - 5)}(e^{-5t} - e^{-k\lambda_n t}).$$

■

If the boundary conditions are homogeneous, $u(0, t) = 0$ and $u(L, t) = 0$, then (8.4.13) reduces to

$$\frac{db_n}{dt} + k\lambda_n b_n = q_n(t),$$

the differential equation derived in the preceding section. Using Green's formula is an alternative procedure to derive the eigenfunction expansion. It can be used even if the boundary conditions are homogeneous. In fact, it is this derivation that justifies the differentiation of infinite series of eigenfunctions used in Section 8.3.

We now have two procedures to solve nonhomogeneous partial differential equations with nonhomogeneous boundary conditions. By subtracting any function that just solves the nonhomogeneous boundary conditions, we can solve a related problem with homogeneous boundary conditions by the eigenfunction expansion method. Alternatively, we can solve directly the original problem with nonhomogeneous boundary conditions by the method of eigenfunction expansions. In both cases, we need the eigenfunction expansion of some function $w(x, t)$:

$$w(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

If $w(x, t)$ satisfies the same homogeneous boundary conditions as $\phi_n(x)$, then we claim that this series will converge reasonably fast. However, if $w(x, t)$ satisfies nonhomogeneous boundary conditions, then not only will the series not satisfy the boundary conditions (at $x = 0$ and $x = L$), but the series will converge more slowly everywhere. Thus, *the advantage of reducing a problem to homogeneous boundary conditions is that the corresponding series converges faster.*

Exercises 8.4

8.4.1. Solve using the method of eigenfunction expansion without making a reduction to homogeneous boundary conditions.

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} + x^2 + t^2,$$

subject to

$$u(0, t) = e^{-t}, \quad u(3, t) = 0, \quad \text{and} \quad u(x, 0) = \sin \pi x.$$

8.4.2. Consider again the example first presented at the end of Section 8.3. For $0 < x < \pi$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t} \sin 3x,$$

subject to

$$u(0, t) = 0, \quad u(\pi, t) = 1, \quad \text{and} \quad u(x, 0) = f(x).$$

This time, solve using the method of eigenfunction expansion without making a reduction to homogeneous boundary conditions. Show that $b_n(t)$ contains a term proportional to $1/n$, and therefore the series $\sum b_n(t) \sin nx$ converges slower than the series $\sum a_n(t) \sin nx$ found in that example, where the problem was first reduced to homogeneous boundary conditions.

8.4.3. In these exercises, do not make a reduction to homogeneous boundary conditions. Solve the initial value problem for the heat equation with time-dependent sources

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t)$$
$$u(x, 0) = f(x)$$

subject to the following boundary conditions:

(a) $u(0, t) = A(t), \qquad \dfrac{\partial u}{\partial x}(L, t) = B(t)$

***(b)** $\dfrac{\partial u}{\partial x}(0, t) = A(t), \qquad \dfrac{\partial u}{\partial x}(L, t) = B(t)$

8.4.4. Use the method of eigenfunction expansions to solve, without reducing to homogeneous boundary conditions,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x, 0) = f(x), \qquad \begin{matrix} u(0, t) = A \\ u(L, t) = B \end{matrix}$$

where A and B are constants.

8.4.5. Consider

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + q(x)u + f(x, t)$$
$$u(x, 0) = g(x), \qquad \begin{matrix} u(0, t) = \alpha(t) \\ u(L, t) = \beta(t). \end{matrix}$$

Assume that the eigenfunctions $\phi_n(x)$ of the related homogeneous problem are known.

(a) Solve without reducing to a problem with homogeneous boundary conditions.

(b) Solve by first reducing to a problem with homogeneous boundary conditions.

8.4.6. Reconsider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t)$$
$$u(x, 0) = f(x), \qquad \begin{matrix} u(0, t) = 0 \\ u(L, t) = 0. \end{matrix}$$

Assume that the solution $u(x, t)$ has the appropriate smoothness, so that it may be represented by a Fourier cosine series,

$$u(x, t) = \sum_{n=0}^{\infty} c_n(t)\cos\frac{n\pi x}{L}.$$

Solve for dc_n/dt. Show that c_n satisfies a first-order nonhomogeneous ordinary differential equation, but part of the nonhomogeneous term is not known. Make a brief philosophical conclusion.

8.5 Forced Vibrating Membranes and Resonance

The method of eigenfunction expansion may also be applied to nonhomogeneous partial differential equations with more than two independent variables. An interesting example is a vibrating membrane of arbitrary shape. In our previous analysis of membranes, vibrations were caused by the initial conditions. Another mechanism that will put a membrane into motion is an external force. The linear nonhomogeneous partial differential equation that describes a vibrating membrane is

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + Q(x, y, t),} \tag{8.5.1}$$

where $Q(x, y, t)$ represents a time- and spatially-dependent external force. To be completely general, there should be some boundary condition along the boundary of the membrane. However, it is more usual for a vibrating membrane to be fixed with zero vertical displacement. Thus, we will specify this homogeneous boundary condition,

$$\boxed{u = 0,} \tag{8.5.2}$$

on the entire boundary. Both the initial position and initial velocity are specified:

$$\boxed{u(x, y, 0) = \alpha(x, y)} \tag{8.5.3}$$

$$\boxed{\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y).} \tag{8.5.4}$$

To use the method of eigenfunction expansion, we must assume that we "know" the eigenfunctions of the related homogeneous problem. By applying the method of separation of variables

to (8.5.1) with $Q(x, y, t) = 0$, where the boundary condition is (8.5.2), we obtain the problem satisfied by the eigenfunctions:

$$\nabla^2\phi = -\lambda\phi, \tag{8.5.5}$$

with $\phi = 0$ on the entire boundary. We know that these eigenfunctions are complete, and that different eigenfunctions are orthogonal (in a two-dimensional sense) with weight 1. We have also shown that $\lambda > 0$. However, the specific eigenfunctions depend on the geometric shape of the region. Explicit formulas can be obtained only for certain relatively simple geometries. Recall that for a rectangle ($0 \le x \le L$, $0 \le y \le H$) the eigenvalues are $\lambda_{nm} = (n\pi/L)^2 + (m\pi/H)^2$, and the corresponding eigenfunctions are $\phi_{nm}(x, y) = \sin n\pi x/L \sin m\pi y/H$, where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. Also, for a circle of radius a, we have shown that the eigenvalues are $\lambda_{mn} = (z_{mn}/a)^2$, where z_{mn} are the nth zeros of the Bessel function of order m, $J_m(z_{mn}) = 0$, and the corresponding eigenfunctions are both $J_m(z_{mn}r/a)\sin m\theta$ and $J_m(z_{mn}r/a)\cos m\theta$, where $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, 3, \ldots$.

In general, we designate the related homogeneous eigenfunctions $\phi_i(x, y)$. Any (piecewise smooth) function, including the desired solution for our forced vibrating membrane, may be expressed in terms of an infinite series of these eigenfunctions. Thus,

$$\boxed{u(x, y, t) = \sum_i A_i(t)\phi_i(x, y).} \tag{8.5.6}$$

Here, the $\sum_i$ represents a summation over *all* eigenfunctions. For membranes it will include a double sum if we are able to separate variables for $\nabla^2\phi + \lambda\phi = 0$.

TERM-BY-TERM DIFFERENTIATION

We will obtain an ordinary differential equation for the time-dependent coefficients, $A_i(t)$. The differential equation will be derived in two ways: direct substitution (with the necessary differentiation of infinite series of eigenfunctions) and use of the multidimensional Green's formula. In either approach, we need to assume there are no difficulties with the term-by-term differentiation of (8.5.6) with respect to t. Thus,

$$\frac{\partial^2 u}{\partial t^2} = \sum_i \frac{d^2 A_i}{dt^2}\phi_i(x, y). \tag{8.5.7}$$

Term-by-term spatial differentiations are allowed since both u and ϕ_i solve the same homogeneous boundary conditions:

$$\nabla^2 u = \sum_i A_i(t)\nabla^2\phi_i(x, y). \tag{8.5.8}$$

This would not be valid if $u \neq 0$ on the entire boundary. Since $\nabla^2\phi_i = -\lambda_i\phi_i$, it follows that (8.5.1) becomes

$$\sum_i \left(\frac{d^2 A_i}{dt^2} + c^2\lambda_i A_i\right)\phi_i = Q(x, y, t). \tag{8.5.9}$$

If we expand $Q(x, y, t)$ in terms of these same eigenfunctions,

$$Q(x, y, t) = \sum_i q_i(t)\phi_i(x, y), \quad \text{where} \quad q_i(t) = \frac{\iint Q\phi_i \, dx \, dy}{\iint \phi_i^2 \, dx \, dy}, \tag{8.5.10}$$

then

$$\boxed{\frac{d^2 A_i}{dt^2} + c^2\lambda_i A_i = q_i(t).} \tag{8.5.11}$$

Thus, A_i solves a linear nonhomogeneous second-order differential equation.

Green's formula

An alternative way to derive (8.5.11) is to use Green's formula. We begin this derivation by determining $d^2 A_i/dt^2$ directly from (8.5.7), using only the two-dimensional orthogonality of $\phi_i(x, y)$ (with weight 1):

$$\frac{d^2 A_i}{dt^2} = \frac{\iint \frac{\partial^2 u}{\partial t^2}\phi_i \, dx \, dy}{\iint \phi_i^2 \, dx \, dy}. \tag{8.5.12}$$

We then eliminate $\partial^2 u/\partial t^2$ from the partial differential equation (8.5.1):

$$\frac{d^2A_i}{dt^2} = \frac{\iint (c^2\nabla^2 u + Q)\phi_i \, dx \, dy}{\iint \phi_i^2 \, dx \, dy}. \tag{8.5.13}$$

Recognizing the latter integral as the generalized Fourier coefficients of Q [see (8.5.10)], we have that

$$\frac{d^2A_i}{dt^2} = q_i(t) + \frac{\iint c^2\nabla^2 u \, \phi_i \, dx \, dy}{\iint \phi_i^2 \, dx \, dy}. \tag{8.5.14}$$

It is now appropriate to use the two-dimensional version of Green's formula:

$$\iint (\phi_i \nabla^2 u - u\nabla^2 \phi_i) \, dx \, dy = \oint (\phi_i \nabla u - u\nabla\phi_i) \cdot \hat{\mathbf{n}} \, ds, \tag{8.5.15}$$

where ds represents the differential arc length along the boundary and $\hat{\mathbf{n}}$ is a unit outward normal to the boundary. In our situation, u and ϕ_i satisfy homogeneous boundary conditions, and hence the boundary term in (8.5.15) vanishes:

$$\iint (\phi_i \nabla^2 u - u\nabla^2 \phi_i) \, dx \, dy = 0. \tag{8.5.16}$$

Equation (8.5.16) is perhaps best remembered as $\iint [uL(v) - vL(u)] \, dx \, dy = 0$, where $L = \nabla^2$. If the membrane did not have homogeneous boundary conditions, then (8.5.15) would be used instead of (8.5.16), as we did in Section 8.4. The advantage of the use of Green's formula is that we can also solve the problem when the boundary condition is nonhomogeneous. Through the use of (8.5.16),

$$\iint \phi_i \nabla^2 u \, dx \, dy = \iint u\nabla^2 \phi_i \, dx \, dy = -\lambda_i \iint u\phi_i \, dx \, dy = -\lambda_i A_i(t) \iint \phi_i^2 \, dx \, dy,$$

since $\nabla^2 \phi_i + \lambda_i \phi_i = 0$ and since $A_i(t)$ is the generalized Fourier coefficient of $u(x, y, t)$:

$$A_i(t) = \frac{\iint u\phi_i \, dx \, dy}{\iint \phi_i^2 \, dx \, dy}. \tag{8.5.17}$$

Consequently, we derive from (8.5.14) that

$$\boxed{\frac{d^2A_i}{dt^2} + c^2\lambda_i A_i = q_i,} \tag{8.5.18}$$

the same second-order differential equation as already derived [see (8.5.11)], justifying the simpler term-by-term differentiation performed there.

Variation of parameters

We will need some facts about ordinary differential equations in order to solve (8.5.11) or (8.5.18). Equation (8.5.18) is a second-order linear nonhomogeneous differential equation with constant coefficients (since $\lambda_i c^2$ is constant). The general solution is a particular solution plus a linear combination of homogeneous solutions. In this problem, the homogeneous solutions are $\sin c\sqrt{\lambda_i}t$ and $\cos c\sqrt{\lambda_i}t$ since $\lambda_i > 0$. A particular solution can always be obtained by variation of parameters. However, the method of undetermined coefficients is usually easier and should be used if $q_i(t)$ is a polynomial, exponential, sine, or cosine (or products and/or sums of these). Using the method of variation of parameters (see Section 9.3.2), it can be shown that the general solution of (8.5.18) is

$$\boxed{A_i(t) = c_1 \cos c\sqrt{\lambda_i}t + c_2 \sin c\sqrt{\lambda_i}t + \int_0^t q_i(\tau) \frac{\sin c\sqrt{\lambda_i}(t-\tau)}{c\sqrt{\lambda_i}} \, d\tau.} \tag{8.5.19}$$

Using this form, the initial conditions may be easily satisfied:

$$A_i(0) = c_1 \tag{8.5.20}$$

$$\frac{dA_i}{dt}(0) = c_2 c\sqrt{\lambda_i}. \tag{8.5.21}$$

From the initial conditions (8.5.3) and (8.5.4), it follows that

$$A_i(0) = \frac{\iint \alpha(x, y)\phi_i(x, y)\, dx\, dy}{\iint \phi_i^2\, dx\, dy} \tag{8.5.22}$$

$$\frac{dA_i}{dt}(0) = \frac{\iint \beta(x, y)\ \phi_i(x, y)\, dx\, dy}{\iint \phi_i^2\, dx\, dy}. \tag{8.5.23}$$

The solution, in general, for a forced vibrating membrane is

$$u\ (x, y, t) = \sum_i A_i(t)\phi_i(x, y),$$

where ϕ_i is given by (8.5.5) and $A_i(t)$ is determined by (8.5.19)–(8.5.23).

If there is no external force, $Q(x, y, t) = 0$ [i.e., $q_i(t) = 0$], then $A_i(t) = c_1 \cos c\sqrt{\lambda_i}t + c_2 \sin c\sqrt{\lambda_i}t$. In this situation, the solution is

$$u(x, y, t) = \sum_i \left(a_i \cos c\sqrt{\lambda_i}t + b_i \sin c\sqrt{\lambda_i}t\right)\phi_i(x, y),$$

exactly the solution obtained by separation of variables. The natural frequencies of oscillation of a membrane are $c\sqrt{\lambda_i}$.

Periodic forcing

We have just completed the analysis of the vibrations of an arbitrarily shaped membrane with arbitrary external forcing. We could easily specialize this result to rectangular and circular membranes. Instead of doing that, we continue to discuss an arbitrarily shaped membrane. However, let us suppose that the forcing function is purely oscillatory in time; specifically,

$$\boxed{Q(x, y, t) = \bar{Q}(x, y)\cos\omega t;} \tag{8.5.24}$$

that is, the **forcing frequency** is ω. We do not specify the spatial dependence, $\bar{Q}(x, y)$. The eigenfunction expansion of the forcing function is also needed. From (8.5.10), it follows that

$$q_i(t) = \gamma_i \cos\omega t, \tag{8.5.25}$$

where γ_i are constants,

$$\gamma_i = \frac{\iint \bar{Q}(x, y)\phi_i(x, y)\, dx\, dy}{\iint \phi_i^2\, dx\, dy}.$$

From (8.5.18), the generalized Fourier coefficients solve the second-order differential equation,

$$\frac{d^2A_i}{dt^2} + c^2\lambda_i A_i = \gamma_i \cos\omega t. \tag{8.5.26}$$

Since the r.h.s. of (8.5.26) is simple, a particular solution is more easily obtained by the method of undetermined coefficients [rather than by using the general form (8.5.19)]. Homogeneous solutions are again a linear combination of $\sin c\sqrt{\lambda_i}t$ and $\cos c\sqrt{\lambda_i}t$, representing the natural frequencies $c\sqrt{\lambda_i}$ of the membrane. The membrane is being forced at frequency ω.

The solution of (8.5.26) is not difficult. We might guess that a *particular solution* is in the form[3]

$$A_i(t) = B_i \cos\omega t. \tag{8.5.27}$$

Substituting (8.5.27) into (8.5.26) shows that

$$B_i(c^2\lambda_i - \omega^2) = \gamma_i \quad \text{or} \quad B_i = \frac{\gamma_i}{c^2\lambda_i - \omega^2},$$

but this division is valid only if $\omega^2 \neq C^2\lambda_i$. The physical meaning of this result is that *if the forcing frequency w is different from a natural frequency*, then a particular solution is

$$A_i(t) = \frac{\gamma_i \cos\omega t}{c^2\lambda_i - \omega^2}, \tag{8.5.28}$$

[3]If the first derivative term was present in (8.5.26), representing a frictional force, then a particular solution must include both $\cos\omega t$ and $\sin\omega t$.

and the general solution is

$$A_i(t) = \frac{\gamma_i \cos \omega t}{c^2\lambda_i - \omega^2} + c_1 \cos c\sqrt{\lambda_i}t + c_2 \sin c\sqrt{\lambda_i}t. \tag{8.5.29}$$

$A_i(t)$ represents the amplitude of the mode $\phi(x, y)$. Each mode is composed of a vibration at its natural frequency $c\sqrt{\lambda_i}$ and a vibration at the forcing frequency ω. The closer these two frequencies are (for a given mode), the larger the amplitude of that mode.

Resonance

However, **if the forcing frequency ω is the same as one of the natural frequencies** $c\sqrt{\lambda_i}$, then a phenomenon known as **resonance** occurs. Mathematically, if $\omega^2 = c^2\lambda_i$, then for those modes [i.e., *only* those $\phi_i(x, y)$ such that $\omega^2 = c^2\lambda_i$], (8.5.27) is not the appropriate solution since the r.h.s. of (8.5.26) is a homogeneous solution. Instead, the solution is not periodic in time. The amplitude of oscillations grows proportionally to t. Some algebra shows that a particular solution of (8.5.26) is

$$A_i(t) = \frac{\gamma_i}{2\omega} t \sin \omega t, \tag{8.5.30}$$

and hence the general solution is

$$A_i(t) = \frac{\gamma_i}{2\omega} t \sin \omega t + c_1 \cos \omega t + c_2 \sin \omega t, \tag{8.5.31}$$

where $\omega = c\sqrt{\lambda_i}$, for any mode that resonates. At resonance, natural modes corresponding to the forcing frequency grow unbounded in time. The other oscillating modes remain bounded. After a while, the resonating modes will dominate. Thus, the spatial structure of a solution will be primarily due to the eigenfunctions of the resonant modes. The other modes are not significantly excited. We present a brief derivation of (8.5.30), which avoids some tedious algebra. If $\omega^2 \neq c^2\lambda_i$, we obtain the general solution (8.5.28) relatively easily. Unfortunately, we cannot take the limit as $\omega \to c\sqrt{\lambda_i}$ since the amplitude then approaches infinity. However, from (8.5.29), we see that

$$A_i(t) = \frac{\gamma_i}{c^2\lambda_i - \omega^2}\left(\cos \omega t - \cos c\sqrt{\lambda_i}t\right) \tag{8.5.32}$$

is also an allowable solution[4] if $\omega^2 \neq c^2\lambda_i$. However, (8.5.32) may have a limit as $\omega^2 \to c^2\lambda_i$, since $A_i(t)$ is in the form of 0/0 as $\omega \to c\sqrt{\lambda_i}$. We calculate the limit of (8.5.32) as $\omega \to c\sqrt{\lambda_i}$ using l'Hôpital's rule:

$$A_i(t) = \lim_{\omega \to c\sqrt{\lambda_i}} \frac{\gamma_i\left(\cos \omega t - \cos c\sqrt{\lambda_i}t\right)}{c^2\lambda_i - \omega^2} = \lim_{\omega \to c\sqrt{\lambda_i}} \frac{-\gamma_i t \sin \omega t}{-2\omega},$$

verifying (8.5.30).

The displacement of the resonant mode cannot grow indefinitely, as (8.5.31) suggests. The mathematics is correct, but some physical assumptions should be modified. Perhaps it is appropriate to include a frictional force, which limits the growth as is shown in an exercise. Alternatively, perhaps the mode grows to such a large amplitude that the linearization assumption, needed in a physical derivation of the two-dimensional wave equation, is no longer valid; a different partial differential equation should be appropriate for sufficiently large displacements. Perhaps the amplitude growth due to resonance would result in the snapping of the membrane (but this is not likely to happen until after the linearization assumption has been violated).

Note that we have demonstrated the result for any geometry. The introduction of the details of rectangular or circular geometries might just cloud the basic mathematical and physical phenomena.

Resonance for a vibrating membrane is similar mathematically to resonance for spring-mass systems (also without friction). In fact, resonance occurs for any mechanical system when a forcing frequency equals one of the natural frequencies. Disasters such as the infamous Tacoma Bridge collapse and various jet airplane crashes have been blamed on resonance phenomena.

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[4]This solution corresponds to the initial conditions $A_i(0) = 0$, $dA_i/dt = 0$.

Exercises 8.5

8.5.1. By substitution show that

$$y(t) = \frac{1}{\omega_0}\int_0^t f(\bar{t})\sin\omega_0(t-\bar{t})\,d\bar{t}$$

is a particular solution of

$$\frac{d^2y}{dt^2} + \omega_0^2 y = f(t).$$

What is the general solution? What solution satisfies the initial conditions $y(0) = y_0$ and $\frac{dy}{dt}(0) = v_0$?

8.5.2. Consider a vibrating string with time-dependent forcing

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$u(0,t) = 0, \qquad u(x,0) = f(x),$$
$$u(L,t) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

(a) Solve the initial value problem.

***(b)** Solve the initial value problem if $Q(x,t) = g(x)\cos\omega t$. For what values of ω does resonance occur?

8.5.3. Consider a vibrating string with friction and time-periodic forcing

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial t} + g(x)\cos\omega t$$
$$u(0,t) = 0, \quad u(x,0) = f(x),$$
$$u(L,t) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

(a) Solve this initial value problem if β is moderately small $(0 < \beta < 2c\pi/L)$.

(b) Compare this solution to Exercise 8.5.2(b).

8.5.4. Solve the initial value problem for a vibrating string with time-dependent forcing

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + Q(x,t), \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0,$$

subject to the following boundary conditions. Do not reduce to homogeneous boundary conditions:

(a) $u(0,t) = A(t)$, $\qquad u(L,t) = B(t)$

(b) $u(0,t) = 0$, $\qquad \frac{\partial u}{\partial x}(L,t) = 0$

(c) $\frac{\partial u}{\partial x}(0,t) = A(t)$, $\qquad u(L,t) = 0$

8.5.5. Solve the initial value problem for a membrane with time-dependent forcing and fixed boundaries $(u = 0)$

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + Q(x,y,t),$$
$$u(x,y,0) = f(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = 0,$$

if the membrane is

(a) a rectangle $(0 < x < L, 0 < y < H)$

(b) a circle $(r < a)$

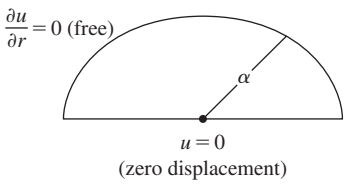
***(c)** a semicircle $(0 < \theta < \pi, r < a)$

(d) a circular annulus $(a < r < b)$

8.5.6. Consider the displacement $u(r,\theta,t)$ of a forced semicircular membrane of radius α (Fig. 8.5.1) that satisfies the partial differential equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + g(r,\theta,t),$$

Figure 8.5.1



with the homogeneous boundary conditions:

$$u(r, 0, t) = 0, \quad u(r, \pi, t) = 0, \quad \text{and} \quad \frac{\partial u}{\partial r}(\alpha, \theta, t) = 0$$

and the initial conditions

$$u(r, \theta, 0) = H(r, \theta) \quad \text{and} \quad \frac{\partial u}{\partial t}(r, \theta, 0) = 0.$$

***(a)** Assume that $u(r, \theta, t) = \sum\sum a(t)\phi(r, \theta)$, where $\phi(r, \theta)$ are the eigenfunctions of the related homogeneous problem. What initial conditions does $a(t)$ satisfy? What differential equation does $a(t)$ satisfy?

(b) What are the eigenfunctions?

(c) Solve for $u(r, \theta, t)$. [*Hint*: See Exercise 8.5.1.]

8.6 Poisson's Equation

We have applied the method of eigenfunction expansion to nonhomogeneous time-dependent boundary value problems for PDEs (with or without homogeneous boundary conditions). In each case, the method of eigenfunction expansion,

$$u = \sum_i a_i(t)\phi_i,$$

yielded an *initial value problem* for the coefficients $a_i(t)$, where ϕ_i are the related homogeneous eigenfunctions satisfying, for example,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{or} \quad \nabla^2\phi + \lambda\phi = 0.$$

Time-independent nonhomogeneous problems must be solved in a slightly different way. Consider the equilibrium temperature distribution with time-independent sources that satisfies **Poisson's equation**,

$$\boxed{\nabla^2 u = Q,} \tag{8.6.1}$$

where Q is related to the sources of thermal energy. For now we do not specify the geometric region. However, we assume the temperature is specified on the entire boundary,

$$u = \alpha,$$

where α is given and usually *not* constant. This problem is nonhomogeneous in two ways due to the forcing function Q and the boundary condition α. We can decompose the equilibrium temperature into two parts, $u = u_1 + u_2$, one u_1 due to the forcing and the other u_2 due to the boundary condition:

$$\begin{aligned} \nabla^2 u_1 &= Q, & \nabla^2 u_2 &= 0 \\ u_1 &= 0 \text{ on the boundary}, & u_2 &= \alpha \text{ on the boundary}. \end{aligned}$$

It is easily checked that $u = u_1 + u_2$ satisfies Poisson's equation and the nonhomogeneous BC. The problem for u_2 is the solution of Laplace's equation (with nonhomogeneous boundary conditions). For simple geometries this can be solved by the method of separation of variables (where in Sections 2.5.1 and 7.9.1 we showed how homogeneous boundary conditions could be introduced).

Thus, at first in this section, we focus our attention on Poisson's equation

$$\nabla^2 u_1 = Q,$$

with *homogeneous* boundary conditions ($u_1 = 0$ on the boundary). Since u_1 satisfies homogeneous BC, we should expect that the method of eigenfunction expansion is appropriate. The problem can be analyzed in two somewhat different ways: (1) We can expand the solutions in eigenfunctions of the related homogeneous problem, coming from separation of variables of $\nabla^2 u_1 = 0$ (as we did for the time-dependent problems) or (2) we can expand the solution in the eigenfunctions

$$\nabla^2\phi + \lambda\phi = 0.$$

The two methods are different (but are related).

ONE-DIMENSIONAL EIGENFUNCTIONS

To be specific, let us consider the two-dimensional Poisson's equation in a rectangle with zero boundary conditions:

$$\boxed{\nabla^2 u_1 = Q,} \tag{8.6.2}$$

as illustrated in Fig. 8.6.1. We first describe the use of one-dimensional eigenfunctions. The related homogeneous problem, $\nabla^2 u_1 = 0$, which is Laplace's equation, can be separated (in rectangular coordinates). We may recall that the solution oscillates in one direction, and it is a combination of exponentials in the other direction. Thus, eigenfunctions of the related homogeneous problem (needed for the method of eigenfunction expansion) might be x-eigenfunctions or y-eigenfunctions. Since we have two homogeneous boundary conditions in *both* directions, we can use *either* x-dependent or y-dependent eigenfunctions. Without loss of generality, we use x-dependent eigenfunctions, which are $\sin n\pi x/L$ since $u_1 = 0$ at $x = 0$ and $x = L$. The method of eigenfunction expansion consists of expanding $u_1(x, y)$ in a series of these eigenfunctions:

$$\boxed{u_1 = \sum_{n=1}^{\infty} b_n(y) \sin \frac{n\pi x}{L},} \tag{8.6.3}$$

where the sine coefficients $b_n(y)$ are functions of y. Differentiating (8.6.3) twice with respect to y and substituting this into Poisson's equation, (8.6.2), yields

$$\sum_{n=1}^{\infty} \frac{d^2 b_n}{dy^2} \sin \frac{n\pi x}{L} + \frac{\partial^2 u_1}{\partial x^2} = Q. \tag{8.6.4}$$

$\partial^2 u_1/\partial x^2$ can be determined in two related ways (as we also showed for nonhomogeneous time-dependent problems): use term-by-term differentiation with respect to x of the series (8.6.3), which is more direct, or by use of Green's formula. In either way, we obtain, from (8.6.4)

$$\sum_{n=1}^{\infty} \left[\frac{d^2 b_n}{dy^2} - \left(\frac{n\pi}{L} \right)^2 b_n \right] \sin \frac{n\pi x}{L} = Q, \tag{8.6.5}$$

since both u_1 and $\sin n\pi x/L$ satisfy the same homogeneous boundary conditions. Thus, the Fourier sine coefficients satisfy the following second-order ordinary differential equation:

$$\boxed{\frac{d^2 b_n}{dy^2} - \left(\frac{n\pi}{L} \right)^2 b_n = \frac{2}{L} \int_0^L Q(x, y) \sin \frac{n\pi x}{L}\, dx \equiv q_n(y),} \tag{8.6.6}$$

where the right-hand side is the sine coefficient of Q,

$$Q = \sum_{n=1}^{\infty} q_n \sin \frac{n\pi x}{L}. \tag{8.6.7}$$

We must solve (8.6.6). Two conditions are needed. We have satisfied Poisson's equation and the boundary condition at $x = 0$ and $x = L$. The boundary condition at $y = 0$ (for all x), $u_1 = 0$, and that at $y = H$ (for all x), $u_1 = 0$, imply that

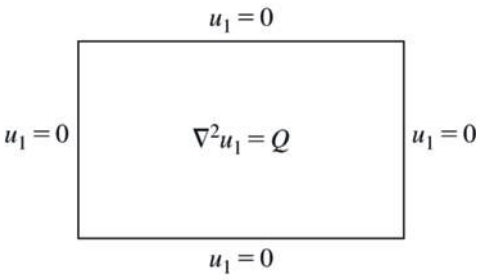
$$b_n(0) = 0 \quad \text{and} \quad b_n(H) = 0. \tag{8.6.8}$$

Thus, the unknown coefficients in the method of eigenfunction expansion [see (8.6.6)] themselves solve a one-dimensional *nonhomogeneous boundary value problem*. Compare this result to the time-dependent nonhomogeneous PDE problems, in which the coefficients satisfied one-dimensional initial value problems. One-dimensional boundary value problems are more difficult to satisfy than initial value problems. Later we will discuss boundary value problems for ordinary differential equations. We will find different ways to solve (8.6.6) subject to the BC (8.6.8). One form of the solution we can obtain using the method of variation of parameters (see Section 9.3.2) is

$$\boxed{\begin{aligned} b_n(y) = {} & \sinh \frac{n\pi(H - y)}{L} \int_0^y q_n(\xi) \sinh \frac{n\pi \xi}{L}\, d\xi \\ & + \sinh \frac{n\pi y}{L} \int_y^H q_n(\xi) \sinh \frac{n\pi(H - \xi)}{L}\, d\xi. \end{aligned}} \tag{8.6.9}$$

Thus, we can solve Poisson's equation (with homogeneous boundary conditions) using the x-dependent related one-dimensional homogeneous eigenfunctions. Problems with nonhomogeneous boundary conditions can be solved in the same way, introducing the appropriate modifications following from the use of Green's formula with nonhomogeneous conditions. In Exercise 8.6.1, the same problem is solved using the y-dependent related homogeneous eigenfunctions.

Figure 8.6.1 Poisson equation in a rectangle.



Two-dimensional eigenfunctions

A somewhat different way to solve Poisson's equation,

$$\boxed{\nabla^2 u_1 = Q,} \tag{8.6.10}$$

on a rectangle with zero BC is to consider the related two-dimensional eigenfunctions:

$$\nabla^2 \phi = -\lambda \phi$$

with $\phi = 0$ on the boundary. For a rectangle, we know that this implies a sine series in x *and* a sine series in y:

$$\phi_{nm} = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}$$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2.$$

The method of eigenfunction expansion consists of expanding the solution u_1 in terms of these two-dimensional eigenfunctions:

$$\boxed{u_1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}.} \tag{8.6.11}$$

Here b_{nm} are constants (not a function of another variable) since u_1 depends only on x and y. The substitution of (8.6.11) into Poisson's equation (8.6.10) yields

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} -b_{nm} \lambda_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} = Q,$$

since $\nabla^2 \phi_{nm} = -\lambda_{nm} \phi_{nm}$. The Laplacian can be evaluated by term-by-term differentiation since both u_1 and ϕ_{nm} satisfy the same homogeneous boundary conditions. The eigenfunctions ϕ_{nm} are orthogonal (in a two-dimensional sense) with weight 1. Thus,

$$-b_{nm} \lambda_{nm} = \frac{\int_0^H \int_0^L Q \sin n\pi x/L \sin m\pi y/H \, dx \, dy}{\int_0^H \int_0^L \sin^2 n\pi x/L \sin^2 m\pi y/H \, dx \, dy}, \tag{8.6.12}$$

determining the b_{nm}. The expression on the r.h.s. of (8.6.12) is recognized as the generalized Fourier coefficients of Q. Dividing by λ_{nm} to solve for b_{nm} poses no difficulty since $\lambda_{nm} > 0$ (explicitly or by use of the Rayleigh quotient). It is easier to obtain the solution using the expansion in terms of two-dimensional eigenfunctions than using one-dimensional ones. However, doubly infinite series such as (8.6.11) may converge quite slowly. Numerical methods may be preferable except in simple cases. In Exercise 8.6.2 we show that the Fourier sine coefficients in y of $b_n(y)$ [see (8.6.3)] equal b_{nm} [see (8.6.11)]. This shows the equivalence of the one- and two-dimensional eigenfunction expansion approaches.

Nonhomogeneous boundary conditions (any geometry)

The two-dimensional eigenfunctions can also be directly used for Poisson's equation subject to nonhomogeneous boundary conditions. It is no more difficult to indicate the solution for a rather general geometry. Suppose that

$$\boxed{\nabla^2 u = Q,} \tag{8.6.13}$$

with $u = \alpha$ on the boundary. Consider the eigenfunctions ϕ_i of $\nabla^2\phi = -\lambda\phi$ with $\phi = 0$ on the boundary. We represent u in terms of these eigenfunctions:

$$\boxed{u = \sum_i b_i\phi_i.} \tag{8.6.14}$$

Now, it is *no longer* true that

$$\nabla^2 u = \sum_i b_i \nabla^2\phi_i,$$

since u does not satisfy homogeneous boundary conditions. Instead, from (8.6.14), we know that

$$b_i = \frac{\iint u\phi_i \, dx \, dy}{\iint \phi_i^2 \, dx \, dy} = -\frac{1}{\lambda_i}\frac{\iint u\nabla^2\phi_i \, dx \, dy}{\iint \phi_i^2 \, dx \, dy}, \tag{8.6.15}$$

since $\nabla^2\phi_i = -\lambda_i\phi_i$. We can evaluate the numerator using Green's two-dimensional formula:

$$\iint (u\nabla^2 v - v\nabla^2 u) \, dx \, dy = \oint (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}} \, ds. \tag{8.6.16}$$

Letting $v = \phi_i$, we see

$$\iint u\nabla^2\phi_i \, dx \, dy = \iint \phi_i\nabla^2 u \;\; dx \;\; dy + \oint (u\nabla\phi_i - \phi_i\nabla u) \cdot \hat{\mathbf{n}} \, ds.$$

However, $\nabla^2 u = Q$, and on the boundary $\phi_i = 0$ and $u = \alpha$. Thus,

$$\boxed{b_i = -\frac{1}{\lambda_i}\frac{\iint \phi_i Q \, dx \, dy + \oint \alpha\nabla\phi_i \cdot \hat{\mathbf{n}} \, ds}{\iint \phi_i^2 \;\; dx \, dy}.} \tag{8.6.17}$$

This is the general expression for b_i, since λ_i, ϕ_i, α, and Q are considered to be known. Again, dividing by λ_i causes no difficulty, since $\lambda_i > 0$ from the Rayleigh quotient. For problems in which $\lambda_i = 0$, see Section 9.4.

If u also satisfies homogeneous boundary conditions, $\alpha = 0$, then (8.6.17) becomes

$$b_i = -\frac{1}{\lambda_i}\frac{\iint \phi_i Q \, dx \, dy}{\iint \phi_i^2 \, dx \, dy},$$

agreeing with (8.6.12) in the case of a rectangular region. This shows that (8.6.11) may be term-by-term differentiated if u and ϕ satisfy the same homogeneous boundary conditions.

Exercises 8.6

8.6.1. Solve

$$\nabla^2 u = Q(x, y)$$

on a rectangle $(0 < x < L, 0 < y < H)$ subject to

(a) $u(0, y) = 0, \quad u(x, 0) = 0,$
$u(L, y) = 0, \quad u(x, H) = 0.$

Use a Fourier sine series in y.

***(b)** $u(0, y) = 0, \quad u(x, 0) = 0,$
$u(L, y) = 1, \quad u(x, H) = 0.$

Do not reduce to homogeneous boundary conditions.

(c) Solve part (b) by first reducing to homogeneous boundary conditions.

***(d)** $\dfrac{\partial u}{\partial x}(0, y) = 0, \quad \dfrac{\partial u}{\partial y}(x, 0) = 0,$
$\dfrac{\partial u}{\partial x}(L, y) = 0, \quad \dfrac{\partial u}{\partial y}(x, H) = 0.$

In what situations are there solutions?

(e) $\dfrac{\partial u}{\partial x}(0, y) = 0, \quad u(x, 0) = 0,$
$\dfrac{\partial u}{\partial x}(L, y) = 0, \quad \dfrac{\partial u}{\partial y}(x, H) = 0.$

8.6.2. The solution of (8.6.6),

$$\frac{d^2b_n}{dy^2} - \left(\frac{n\pi}{L}\right)^2 b_n = q_n(y),$$

subject to $b_n(0) = 0$ and $b_n(H) = 0$ is given by (8.6.9).

(a) Solve this instead by letting $b_n(y)$ equal a Fourier sine series.

(b) Show that this series is equivalent to (8.6.9).

(c) Show that this series is equivalent to the answer obtained by an expansion in the two-dimensional eigenfunctions, (8.6.11).

8.6.3. Solve (using two-dimensional eigenfunctions) $\nabla^2 u = Q(r, \theta)$ inside a circle of radius a subject to the given boundary condition. In what situations are there solutions?

***(a)** $u(a, \theta) = 0$ **(b)** $\dfrac{\partial u}{\partial r}(a, \theta) = 0$

(c) $u(a, \theta) = f(\theta)$ **(d)** $\dfrac{\partial u}{\partial r}(a, \theta) = g(\theta)$

8.6.4. Solve Exercise 8.6.3 using one-dimensional eigenfunctions.

8.6.5. Consider

$$\nabla^2 u = Q(x, y)$$

inside an unspecified region with $u = 0$ on the boundary. Suppose that the eigenfunctions $\nabla^2\phi = -\lambda\phi$ subject to $\phi = 0$ on the boundary are known. Solve for $u(x, y)$.

***8.6.6.** Solve the following example of Poisson's equation:

$$\nabla^2 u = e^{2y} \sin x$$

subject to the following boundary conditions:

$$u(0, y) = 0, \qquad u(x, 0) = 0$$
$$u(\pi, y) = 0, \qquad u(x, L) = f(x).$$

8.6.7. Solve

$$\nabla^2 u = Q(x, y, z)$$

inside a rectangular box $(0 < x < L, 0 < y < H, 0 < z < W)$ subject to $u = 0$ on the six sides.

8.6.8. Solve

$$\nabla^2 u = Q(r, \theta, z)$$

inside a circular cylinder $(0 < r < a, \theta < 2\pi, 0 < z < H)$ subject to $u = 0$ on the sides.

8.6.9. On a rectangle $(0 < x < L, 0 < y < H)$ consider

$$\nabla^2 u = Q(x, y)$$

with $\nabla u \cdot \hat{\mathbf{n}} = 0$ on the boundary.

(a) Show that a solution exists only if $\iint Q(x, y)\, dx\, dy = 0$. Briefly explain, using physical reasoning.

(b) Solve using the method of eigenfunction expansion. Compare to part (a). [*Hint*: $\lambda = 0$ is an eigenvalue.]

(c) If $\iint Q\, dx\, dy = 0$, determine the arbitrary constant in the solution of part (b) by consideration of the time-dependent problem $\frac{\partial u}{\partial t} = k(\nabla^2 u - Q)$, subject to the initial condition $u(x, y, 0) = g(x, y)$.

8.6.10. Reconsider Exercise 8.6.9 for an arbitrary two-dimensional region.

9

Green's Functions for Time-Independent Problems

9.1 Introduction

Solutions to linear partial differential equations are nonzero due to initial conditions, nonhomogeneous boundary conditions, and forcing terms. If the partial differential equation is homogeneous and there is a set of homogeneous boundary conditions, then we usually attempt to solve the problem by the method of separation of variables. In Chapter 8 we developed the method of eigenfunction expansions to obtain solutions in cases in which there were forcing terms (and/or nonhomogeneous boundary conditions).

In this chapter, we will primarily consider problems without initial conditions (ordinary differential equations and Laplace's equation with sources). We will show that there is one function for each problem called the Green's function, which can be used to describe the influence of both nonhomogeneous boundary conditions and forcing terms. We will develop properties of these Green's functions and show direct methods to obtain them. Time-dependent problems with initial conditions, such as the heat and wave equations, are more difficult. They will be used as motivation, but a detailed study of their Green's functions will not be presented until Chapter 11.

9.2 Green's Functions for the One-Dimensional Heat Equation

We begin by reanalyzing the one-dimensional heat equation with no sources and homogeneous boundary conditions:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \tag{9.2.1}$$

$$u(0, t) = 0 \tag{9.2.2}$$

$$u(L, t) = 0 \tag{9.2.3}$$

$$u(x, 0) = g(x). \tag{9.2.4}$$

In Chapter 2, according to the method of separation of variables, we obtained

$$\boxed{u(x, t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t},} \tag{9.2.5}$$

where the initial condition implied that a_n are the coefficients of the Fourier sine series of $g(x)$,

$$g(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \tag{9.2.6}$$

$$\boxed{a_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.} \tag{9.2.7}$$

We examine this solution (9.2.5) more closely in order to investigate the effect of the initial condition $g(x)$. We eliminate the Fourier sine coefficients from (9.2.7) (introducing a dummy integration variable x_0):

$$u(x, t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L g(x_0)\sin\frac{n\pi x_0}{L}\,dx_0\right]\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}.$$

If we interchange the order of operations of the infinite summation and integration, we obtain

$$u(x, t) = \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}\right)dx_0. \tag{9.2.8}$$

We define the quantity in parentheses as the **influence function** for the initial condition. It quantitatively describes how the initial temperature at $x = x_0$ contributes to the temperature at some time t and position x. To obtain the temperature $u(x, t)$, we sum (integrate) the influences of all possible initial positions.

We have expressed the solution $u(x, t)$ into a very specific form (9.2.8) because in that form the influence function takes on a special role in defining the solution. The **Green's function** for (9.2.1)–(9.2.4) is defined to be the function $G(x, t; x_0)$ that relates the solution $u(x, t)$ to the initial conditions $g(x)$ according to

$$u(x, t) = \int_0^L g(x_0)G(x, t; x_0)\,dx_0.$$

Thus, equation (9.2.8) shows that the quantity in parentheses is the Green's function for (9.2.1)–(9.2.4).

In this example, we had the solution $u(x, t)$ by another technique, and we made an effort to express it in the form where we could identify the Green's function. In some future problems, we will solve for G directly, and that will be our only means of finding the solution u.

We will be defining Green's functions for a wide variety of problems—including ordinary and partial differential equations, time-dependent and time-independent problems, as well as one-, two-, and three-dimensional problems. In all cases, the fundamental idea that defines the Green's function is *the function u that solves the problem can be expressed as an integral of the Green's function multiplied by the nonhomogeneous parts of the problem generating the solution.*

For a more general heat equation including sources, but still having homogeneous boundary conditions,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t) \tag{9.2.9}$$

$$u(0, t) = 0 \tag{9.2.10}$$

$$u(L, t) = 0 \tag{9.2.11}$$

$$u(x, 0) = g(x), \tag{9.2.12}$$

the Green's function $G(x, t; x_0, t_0)$ is defined to be the function that allows us to express the solution in the form

$$u(x, t) = \int_0^L g(x_0)G(x, t; x_0, 0)\,dx_0 + \int_0^L\int_0^t Q(x_0, t_0)\,G(x, t; x_0, t_0)\,dt_0\,dx_0. \tag{9.2.13}$$

This nonhomogeneous problem is suited for the method of eigenfunction expansions,

$$u(x, t) = \sum_{n=1}^{\infty}a_n(t)\sin\frac{n\pi x}{L}. \tag{9.2.14}$$

This Fourier sine series can be differentiated term by term since both $\sin n\pi x/L$ and $u(x, t)$ solve the same homogeneous boundary conditions. Hence, $a_n(t)$ solves the following first-order differential equation:

$$\frac{da_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n = q_n(t) = \frac{2}{L}\int_0^L Q(x, t)\sin\frac{n\pi x}{L}\,dx, \tag{9.2.15}$$

where $q_n(t)$ are the coefficients of the Fourier sine series of $Q(x, t)$,

$$Q(x, t) = \sum_{n=1}^{\infty}q_n(t)\sin\frac{n\pi x}{L}. \tag{9.2.16}$$

The solution of (9.2.15) [using the integrating factor $e^{k(n\pi/L)^2 t}$] is

$$a_n(t) = a_n(0)e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t}\int_0^t q_n(t_0)e^{k(n\pi/L)^2 t_0}\, dt_0. \tag{9.2.17}$$

$a_n(0)$ are the coefficients of the Fourier sine series of the initial condition, $u(x, 0) = g(x)$:

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{L} \tag{9.2.18}$$

$$a_n(0) = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\, dx. \tag{9.2.19}$$

These Fourier coefficients may be eliminated, yielding

$$u(x, t) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{L}\int_0^L g(x_0)\sin\frac{n\pi x_0}{L}\, dx_0\right)e^{-k(n\pi/L)^2 t}\right.$$
$$\left. + e^{-k(n\pi/L)^2 t}\int_0^t\left(\frac{2}{L}\int_0^L Q(x_0, t_0)\sin\frac{n\pi x_0}{L}\, dx_0\right)e^{k(n\pi/L)^2 t_0} dt_0\right]\sin\frac{n\pi x}{L}.$$

After interchanging the order of performing the infinite summation and the integration (over both x_0 and t_0), we obtain

$$u(x, t) = \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}\right)\, dx_0$$
$$+\int_0^L\int_0^t Q(x_0, t_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 (t-t_0)}\right)\, dt_0\, dx_0.$$

By comparison with (9.2.13), we see that the **Green's function for the heat equation**, $G(x, t; x_0, t_0)$ is

$$\boxed{G(x, t; x_0, t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 (t-t_0)}.} \tag{9.2.20}$$

The Green's function at $t_0 = 0$, $G(x, t; x_0, 0)$, expresses the influence of the initial temperature at x_0 on the temperature at position x and time t. In addition, $G(x, t; x_0, t_0)$ shows the influence on the temperature at the position x and time t of the forcing term $Q(x_0, t_0)$ at position x_0 and time t_0. Instead of depending on the source time t_0 and the response time t, independently, we note that the Green's function depends only on the **elapsed time** $t - t_0$:

$$G(x, t; x_0, t_0) = G(x, t - t_0; x_0, 0).$$

This occurs because the heat equation has coefficients that do not change in time; the laws of thermal physics are not changing. The Green's function exponentially decays in elapsed time $(t - t_0)$ [see (9.2.20)]. For example, this means that the influence of the source at time t_0 diminishes rapidly. It is only the most recent sources of thermal energy that are important at time t.

Equation (9.2.20) is an extremely useful representation of the Green's function if time t is large. However, for small t the series converges more slowly. In Chapter 11 we will obtain an alternative representation of the Green's function useful for small t.

In (9.2.13), we integrate over all positions x_0. The solution is the result of adding together the influences of all sources and initial temperatures. We also integrate the sources over all *past* times $0 < t_0 < t$. This is part of a **causality principle**. The temperature at time t is due only to the thermal sources that acted *before* time t. Any future sources of heat energy cannot influence the temperature now.

Among the questions we will investigate later for this and other problems are the following:

1. Are there more direct methods to obtain the Green's function?
2. Are there any simpler expressions for the Green's function [can we simplify (9.2.20)]?
3. Can we explain the relationships between the influence of the initial condition and the influence of the forcing terms?
4. Can we easily account for nonhomogeneous boundary conditions?

Exercises 9.2

9.2.1. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$u(x,0) = g(x).$$

In all cases, obtain formulas similar to (9.2.20) by introducing a Green's function.

(a) Use Green's formula instead of term-by-term spatial differentiation if

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

(b) Modify part (a) if

$$u(0,t) = A(t) \quad \text{and} \quad u(L,t) = B(t).$$

Do not reduce to a problem with homogeneous boundary conditions.

(c) Solve using any method if

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0.$$

***(d)** Use Green's formula instead of term-by-term differentiation if

$$\frac{\partial u}{\partial x}(0,t) = A(t) \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = B(t).$$

9.2.2. Solve by the method of eigenfunction expansion

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q(x,t)$$

subject to $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,0) = g(x)$, if c, ρ and K_0 are functions of x. Assume that the eigenfunctions are known. Obtain a formula similar to (9.2.20) by introducing a Green's function.

***9.2.3.** Solve by the method of eigenfunction expansion

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$
$$u(0,t) = 0, \qquad u(x,0) = f(x)$$
$$u(L,t) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Define functions (in the simplest possible way) such that a relationship similar to (9.2.20) exists. It must be somewhat different due to the two initial conditions. [*Hint*: See Exercise 8.5.1.]

9.2.4. Modify Exercise 9.2.3 (using Green's formula if necessary) if instead

(a) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$

(b) $u(0,t) = A(t)$ and $u(L,t) = 0$

(c) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$

9.3 Green's Functions for Boundary Value Problems for Ordinary Differential Equations

9.3.1 ONE-DIMENSIONAL STEADY-STATE HEAT EQUATION

Introduction

Investigating the Green's functions for the time-dependent heat equation is not an easy task. Instead, we first investigate a simpler problem. Most of the techniques discussed will be valid for more difficult problems.

We will investigate the steady-state heat equation with homogeneous boundary conditions, arising in situations in which the source term $Q(x,t) = Q(x)$ is independent of time:

$$0 = k\frac{d^2u}{dx^2} + Q(x).$$

We prefer the form

$$\boxed{\frac{d^2u}{dx^2} = f(x),} \tag{9.3.1}$$

in which case $f(x) = -Q(x)/k$. The boundary conditions we consider are

$$\boxed{u(0) = 0 \quad \text{and} \quad u(L) = 0.} \tag{9.3.2}$$

We will solve this problem in many different ways in order to suggest methods for other, harder problems.

Summary

We will describe three fundamental methods to obtain Green's functions:

1. Variation of parameters (valid only for ordinary differential equations)
2. Method of eigenfunction expansion
3. Using the defining differential equation for the Green's function

In addition, steady-state Green's functions can be obtained as the limit as $t \to \infty$ of the solution with steady sources. To obtain Green's functions for partial differential equations, we will discuss one important additional method. It will be described in Section 9.5.

9.3.2 THE METHOD OF VARIATION OF PARAMETERS (OPTIONAL)

There are more direct ways to obtain the solution of (9.3.1) with (9.3.2) than the method we discuss in this section. We consider a nonhomogeneous problem

$$L(u) = \frac{d^2u}{dx^2} = f(x), \tag{9.3.3}$$

defined for $0 < x < L$, subject to two homogeneous boundary conditions,

$$u(0) = 0 \quad \text{and} \quad u(L) = 0, \tag{9.3.4}$$

where L is the Sturm–Liouville operator:

$$L \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q. \tag{9.3.5}$$

For this simple steady-state heat equation of the preceding subsection, $p = 1$ and $q = 0$, so that $L = d^2/dx^2$.

Nonhomogeneous ordinary differential equations can always be solved by the **method of variation of parameters** if two[1] linearly independent solutions of the homogeneous problem are known, $u_1(x)$ and $u_2(x)$. The homogeneous differential equation is

$$\frac{d^2y}{dx^2} = 0. \tag{9.3.6}$$

Two homogeneous solutions of (9.3.1) are 1 and x. However, the algebra is easier if we pick $u_1(x)$ to be a homogeneous solution satisfying one of the boundary conditions $u(0) = 0$ and $u_2(x)$ to be a homogeneous solution satisfying the other boundary condition:

$$u_1(x) = x \tag{9.3.7}$$

$$u_2(x) = L - x. \tag{9.3.8}$$

We briefly review this technique. In the method of variation of parameters, a particular solution of (9.3.3) is sought in the form

$$\boxed{u = v_1 \bullet u_1 + v_2 \bullet u_2 = v_1x + v_2(L - x),} \tag{9.3.9}$$

[1]Actually, only one homogeneous solution is necessary as the method of reduction of order is a procedure for obtaining a second homogeneous solution if one is known.

where v_1 and v_2 are functions of x to be determined. The original differential equation has one unknown function, so that the extra degree of freedom allows us to assume du/dx is the same as if v_1 and v_2 were constants:

$$\frac{du}{dx} = v_1\frac{du_1}{dx} + v_2\frac{du_2}{dx} = v_1 - v_2.$$

Since v_1 and v_2 are not constant, this is valid only if the other terms, arising from the variation of v_1 and v_2, vanish:

$$\frac{dv_1}{dx}x + \frac{dv_2}{dx}(L - x) = 0. \tag{9.3.10}$$

The differential equation $\frac{d^2u}{dx^2} = f(x)$ is then satisfied if

$$\frac{dv_1}{dx} \cdot 1 + \frac{dv_2}{dx} \cdot (-1) = f(x). \tag{9.3.11}$$

The method of variation of parameters at this stage yields two linear equations for the unknowns dv_1/dx and dv_2/dx. The solution is obtained by elimination. We multiply (9.3.11) by $L - x$ and add (9.3.10), and we obtain

$$\frac{dv_1}{dx} = f\frac{L - x}{L} \tag{9.3.12}$$

$$\frac{dv_2}{dx} = -f\frac{x}{L}. \tag{9.3.13}$$

By integrating (9.3.12) and (9.3.13), we obtain

$$v_1(x) = \frac{1}{L}\int_0^x f(x_0)(L - x_0)\,dx_0 + c_1$$

$$v_2(x) = -\frac{1}{L}\int_0^x f(x_0)x_0\,dx_0 + c_2,$$

which is needed in the method of variation of parameters ($u = u_1v_1 + u_2v_2$). The boundary condition $u(0) = 0$ yields $0 = c_2L$, whereas $u(L) = 0$ yields

$$0 = \int_0^L f(x_0)(L - x_0)\,dx_0 + c_1L,$$

so that $v_1(x) = -\frac{1}{L}\int_x^L f(x_0)(L - x_0)\,dx_0$. Thus, the solution of the nonhomogeneous boundary value problem is

$$u(x) = -\frac{x}{L}\int_x^L f(x_0)(L - x_0)\,dx_0 - \frac{L - x}{L}\int_0^x f(x_0)x_0\,dx_0. \tag{9.3.14}$$

We will transform (9.3.14) into the desired form,

$$\boxed{u(x) = \int_0^L f(x_0)G(x, x_0)\,dx_0.} \tag{9.3.15}$$

By comparing (9.3.14) with (9.3.15), we obtain

$$\boxed{G(x, x_0) = \begin{cases} \dfrac{-x(L - x_0)}{L} & x < x_0 \\ \dfrac{-x_0(L - x)}{L} & x > x_0. \end{cases}} \tag{9.3.16}$$

A sketch and interpretation of this solution will be given in Section 9.3.4. Although somewhat complicated, the symmetry can be seen:

$$G(x, x_0) = G(x_0, x).$$

The solution also can be derived by directly integrating (9.3.1) twice:

$$u = \int_0^x \left[\int_0^{x_0} f(\bar{x})\, d\bar{x} \right] dx_0 + c_1 x + c_2. \tag{9.3.17}$$

In Exercise 9.3.2, you are asked to show that (9.3.16) can be obtained from (9.3.17) by interchanging the order of integration in (9.3.17) or by integrating (9.3.17) by parts.

9.3.3 THE METHOD OF EIGENFUNCTION EXPANSION FOR GREEN'S FUNCTIONS

In Chapter 8, nonhomogeneous partial differential equations were solved by the eigenfunction expansion method. Here, we show how to apply the same ideas to the general Sturm–Liouville nonhomogeneous ordinary differential equation:

$$L(u) = f(x) \tag{9.3.18}$$

subject to two homogeneous boundary conditions. We introduce a related eigenvalue problem,

$$L(\phi) = -\lambda \sigma \phi, \tag{9.3.19}$$

subject to the *same* homogeneous boundary conditions. The weight σ here can be chosen arbitrarily. However, there is usually at most one choice of $\sigma(x)$ such that the differential equation (9.3.19) is in fact well known.[2] We solve (9.3.18) by seeking $u(x)$ as a generalized Fourier series of the eigenfunctions

$$\boxed{u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).} \tag{9.3.20}$$

We can differentiate this twice term by term[3] since both $\phi_n(x)$ and $u(x)$ solve the same homogeneous boundary conditions:

$$\sum_{n=1}^{\infty} a_n L(\phi_n) = -\sum_{n=1}^{\infty} a_n \lambda_n \sigma \phi_n = f(x),$$

where (9.3.19) has been used. The orthogonality of the eigenfunctions (with weight σ) implies that

$$-a_n \lambda_n = \frac{\int_a^b f(x)\phi_n\, dx}{\int_a^b \phi_n^2 \sigma\, dx}. \tag{9.3.21}$$

The solution of the boundary value problem for the nonhomogeneous ordinary differential equation is thus (after interchanging summation and integration)

$$u(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(x_0)}{-\lambda_n \int_a^b \phi_n^2 \sigma\, dx} dx_0 = \int_a^b f(x_0) G(x, x_0)\, dx_0. \tag{9.3.22}$$

For this problem, the Green's function has the representation in terms of the eigenfunctions:

$$\boxed{G(x, x_0) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(x_0)}{-\lambda_n \int_a^b \phi_n^2 \sigma\, dx}.}$$

Again the symmetry is explicitly shown. Note the appearance of the eigenvalues λ_n in the denominator. The Green's function does not exist if one of the eigenvalues is zero. This will be explained in Section 9.4. For now we assume that all $\lambda_n \neq 0$.

---

[2]For example, if $L = d^2/dx^2$, we pick $\sigma = 1$ giving trigonometric functions, but if $L = \frac{d}{dx}(x\frac{d}{dx}) - \frac{m^2}{x}$, we pick $\sigma = x$ so that Bessel functions occur.

[3]Green's formula can be used to justify this step (see Section 8.4).

EXAMPLE

For the boundary value problem

$$\frac{d^2u}{dx^2} = f(x)$$
$$u(0) = 0 \quad \text{and} \quad u(L) = 0,$$

the related eigenvalue problem,

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$
$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0,$$

is well known. The eigenvalues are $\lambda_n = (n\pi/L)^2$, $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $\sin n\pi x/L$. The Fourier sine series of $u(x)$ is given by (9.3.20). In particular,

$$u(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0,$$

where the Fourier sine series of the Green's function is

$$\boxed{G(x, x_0) = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin n\pi x/L \sin n\pi x_0/L}{(n\pi/L)^2}} \tag{9.3.23}$$

from equation after (9.3.22), agreeing with the answer (9.3.56) obtained by the limit as $t \to \infty$ of the time-dependent problem. ■

9.3.4 THE DIRAC DELTA FUNCTION AND ITS RELATIONSHIP TO GREEN'S FUNCTIONS

We have shown that

$$\boxed{u(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0,} \tag{9.3.24}$$

where we have obtained different representations of the Green's function. The Green's function shows the influence of each position x_0 of the source on the solution at x. In this section, we will find a more direct way to derive (9.3.24) and to determine the Green's function.

Dirac delta function

Our source $f(x)$ represents a forcing of our system at all points. $f(x)$ is sketched in Fig. 9.3.1. In order to isolate the effect of each individual point, we decompose $f(x)$ into a linear combination of unit pulses of duration Δx (see Fig. 9.3.2):

$$f(x) \approx \sum_i f(x_i)(\text{unit pulse starting at } x = x_i).$$

This is somewhat reminiscent of the definition of an integral. Only Δx is missing, which we introduce by multiplying and dividing by Δx:

$$f(x) = \lim_{\Delta x \to 0}\sum_i f(x_i)\frac{\text{unit pulse}}{\Delta x}\Delta x. \tag{9.3.25}$$

In this way, we have motivated a rectangular pulse of width Δx and height $1/\Delta x$, sketched in Fig. 9.3.3. It has unit area. In the limit as $\Delta x \to 0$, this approaches an infinitely concentrated pulse (not really a function) $\delta(x - x_i)$, which would be zero everywhere except ∞ at $x = x_i$, still with unit area:

$$\delta(x - x_i) = \begin{cases} 0 & x \neq x_i \\ \infty & x = x_i. \end{cases} \tag{9.3.26}$$

We can think of $\delta(x - x_i)$ as a **concentrated source** or **impulsive force** at $x = x_i$. According to (9.3.25), we have

$$f(x) = \int f(x_i)\delta(x - x_i)\, dx_i. \tag{9.3.27}$$

Figure 9.3.1 A typical function and its piecewise constant representation.

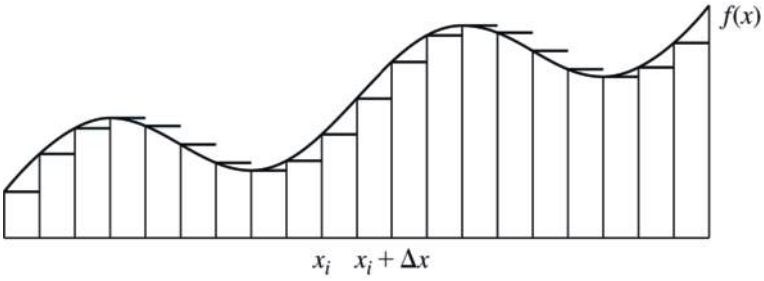


Figure 9.3.2 Pulse with unit height.

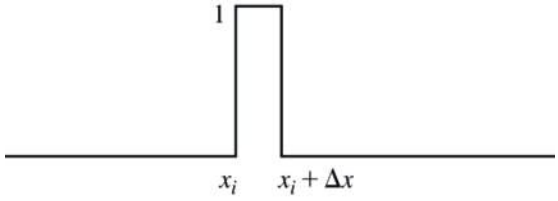
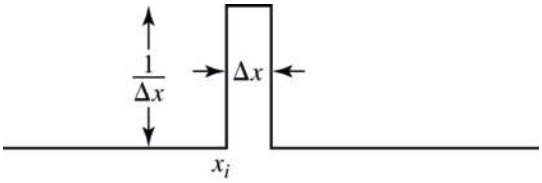


Figure 9.3.3 Rectangular pulse with unit area.



Since $\delta(x - x_i)$ is not a function, we define it as an operator with the property that for any continuous $f(x)$:

$$\boxed{f(x) = \int_{-\infty}^{\infty} f(x_i)\delta(x - x_i)\, dx_i,} \tag{9.3.28}$$

as is suggested by (9.3.27). We call $\delta(x - x_i)$ the **Dirac delta function**.[4] It is so concentrated that in integrating it with any continuous function $f(x_i)$, it "sifts" out the value at $x_i = x$. The Dirac delta function may be motivated by the "limiting function" of any sequence of concentrated pulses (the shape need not be rectangular).

Other important properties of the Dirac delta function are that it has unit area:

$$\boxed{1 = \int_{-\infty}^{\infty} \delta(x - x_i)\, dx_i;} \tag{9.3.29}$$

it operates like an even function

$$\delta(x - x_i) = \delta(x_i - x). \tag{9.3.30}$$

[4]Named after Paul Dirac, a 20th-century mathematical physicist (1902–1984).

This means that the definition (9.3.28) may be used without worrying about whether $\delta(x - x_i)$ or $\delta(x_i - x)$ appears. The Dirac delta function is also the derivative of the Heaviside unit step function $H(x - x_i)$:

$$H(x - x_i) \equiv \begin{cases} 0 & x < x_i \\ 1 & x > x_i; \end{cases} \tag{9.3.31}$$

$$\delta(x - x_i) = \frac{d}{dx} H(x - x_i); \tag{9.3.32}$$

$$H(x - x_i) = \int_{-\infty}^{x} \delta(x_0 - x_i)\, dx_0; \tag{9.3.33}$$

it has the following scaling property:

$$\delta[c(x - x_i)] = \frac{1}{|c|} \delta(x - x_i). \tag{9.3.34}$$

These properties are proved in the Exercises.

Green's function

The solution of the nonhomogeneous problem

$$\boxed{L(u) = f(x), a < x < b} \tag{9.3.35}$$

subject to two homogeneous boundary conditions is

$$\boxed{u(x) = \int_a^b f(x_0) G(x, x_0)\, dx_0.} \tag{9.3.36}$$

Here, the Green's function is the influence function for the source $f(x)$. As an example, suppose that $f(x)$ is a concentrated source at $x = x_s$, $f(x) = \delta(x - x_s)$. Then the response $u(x)$ at x, satisfies

$$u(x) = \int_a^b \delta(x_0 - x_s) G(x, x_0)\, dx_0 = G(x, x_s)$$

due to (9.3.28). This yields the fundamental interpretation of the **Green's function $G(x, x_s)$**; *it is the* **response at x due to a concentrated source at x_s**:

$$\boxed{L[G(x, x_s)] = \delta(x - x_s),} \tag{9.3.37}$$

where $G(x, x_s)$ will also satisfy the same homogeneous boundary conditions at $x = a$ and $x = b$.

As a check, let us verify that if (9.3.36) and (9.3.37) are true, then (9.3.35) is satisfied. To satisfy (9.3.35), we must use the operator L (in the simple case, $L = d^2/dx^2$);

$$L(u) = \int_a^b f(x_0) L[G(x, x_0)]\, dx_0 = \int_a^b f(x_0) \delta(x - x_0)\, dx_0 = f(x),$$

where the fundamental property of both the Green's function (9.3.37) and the Dirac delta function (9.3.28) has been used.

Often, (9.3.37) with two homogeneous boundary conditions is thought of as an *independent definition of the Green's function*. In this case, we might want to *derive* (9.3.36), the representation of the solution of the nonhomogeneous problem in terms of the Green's function satisfying (9.3.37). The usual method to derive (9.3.36) involves Green's formula:

$$\int_a^b [uL(v) - vL(u)]\, dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b. \tag{9.3.38}$$

If we let $v = G(x, x_0)$, then the right-hand side vanishes since both $u(x)$ and $G(x, x_0)$ satisfy the same homogeneous boundary conditions. Furthermore, from the respective differential equations (9.3.35) and (9.3.37), it follows that

$$\int_a^b [u(x)\delta(x - x_0) - G(x, x_0) f(x)]\, dx = 0.$$

Thus, from the definition of the Dirac delta function,

$$u(x_0) = \int_a^b f(x) G(x, x_0)\, dx.$$

If we interchange the variables x and x_0, we obtain (9.3.36):

$$u(x) = \int_a^b f(x_0)G(x_0, x)\,dx_0 = \int_a^b f(x_0)G(x, x_0)\,dx_0, \tag{9.3.39}$$

since the Green's function is known to be symmetric [see (9.3.16)], $G(x_0, x) = G(x, x_0)$.

Maxwell's reciprocity

The symmetry of the Green's function is very important. We will prove it without using the eigenfunction expansion. Instead, we will directly use the defining differential equation (9.3.37). We again use Green's formula (9.3.38). Here, we let $u = G(x, x_1)$ and $v = G(x, x_2)$. Since both satisfy the same homogeneous boundary conditions, it follows that the right-hand side is zero. In addition, $L(u) = \delta(x - x_1)$ while $L(v) = \delta(x - x_2)$, and thus

$$\int_a^b [G(x, x_1)\delta(x - x_2) - G(x, x_2)\delta(x - x_1)]\,dx = 0.$$

From the fundamental property of the Dirac delta function, it follows that

$$\boxed{G(x_1, x_2) = G(x_2, x_1),} \tag{9.3.40}$$

proving the symmetry from the differential equation defining the Green's function. This symmetry is remarkable; we call it **Maxwell's reciprocity**. **The response at x_1 due to a concentrated source at x_2 is the same as the response at x_2 due to a concentrated source at x_1.** This is *not* physically obvious.

Continuity and jump conditions

The Green's function $G(x, x_s)$ may be determined from (9.3.37). We can do this by solving the differential equation separately in the two regions $x < x_s$ and $x > x_s$, and matching the two parts with continuity and jump conditions across $x = x_s$. For $x < x_s$, $G(x, x_s)$ solves the homogeneous differential equation and the boundary condition at $x = a$. For $x > x_s$, $G(x, x_s)$ also solves the homogeneous differential equation and the boundary condition at $x = b$.

What are the conditions that the two parts of $G(x, x_s)$ must satisfy at $x = x_s$? If $G(x, x_s)$ were to have a jump discontinuity at $x = x_s$, then dG/dx would have a delta function singularity at $x = x_s$, and d^2G/dx^2 would be even more singular than the delta function on the right-hand side of (9.3.37). Thus, **the Green's function $G(x, x_s)$ is continuous at $x = x_s$.** However, dG/dx is not continuous at $x = x_s$; it has a jump discontinuity obtained by integrating (9.3.37) across $x = x_s$,

$$\int_{x_s^-}^{x_s^+} \frac{d^2G(x, x_s)}{dx^2}\,dx = \int_{x_s^-}^{x_s^+} \delta(x - x_s)\,dx.$$

Thus, the jump condition is

$$\left.\frac{dG}{dx}\right|_{x=x_s^+} - \left.\frac{dG}{dx}\right|_{x=x_s^-} = 1.$$

The jump in dG/dx is not equal to 1 in all problems; the value depends on the differential equation. For the Sturm–Liouville operator $L = (d/dx)\,(p(x)\,d/dx) + q(x)$, the jump is found by the calculation

$$\int_{x_s^-}^{x_s^+} \left[\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) + q(x)G\right] dx = \int_{x_s^-}^{x_s^+} \delta(x - x_s)\,dx,$$

which, since $q(x)$ and $G(x, x_s)$ are continuous, simplifies to

$$p(x)\left.\frac{dG}{dx}\right|_{x=x_s^+} - p(x)\left.\frac{dG}{dx}\right|_{x=x_s^-} = 1.$$

The function $p(x)$ is also continuous, so the jump condition in this case is

$$\left.\frac{dG}{dx}\right|_{x=x_s^+} - \left.\frac{dG}{dx}\right|_{x=x_s^-} = \frac{1}{p(x_s)}.$$

If the differential equation is not of second order, the jump will not be in the derivative dG/dx. For example, consider the differential equation

$$\frac{d^nG(x, x_s)}{dx^n} = \delta(x - x_s)$$

for $n > 2$. The continuity and jump conditions at $x = x_s$ are

$$G,\ \frac{dG}{dx},\ \frac{d^2G}{dx^2},\ \ldots,\ \frac{d^{n-2}G}{dx^{n-2}} \quad \text{are each continuous at } x = x_s$$

and

$$\left.\frac{d^{n-1}G}{dx^{n-1}}\right|_{x=x_s^+} - \left.\frac{d^{n-1}G}{dx^{n-1}}\right|_{x=x_s^-} = 1.$$

EXAMPLE

Consider the solution of the steady-state heat flow problem

$$\boxed{\begin{aligned} &\frac{d^2u}{dx^2} = f(x) \\ &u(0) = 0 \quad \text{and} \quad u(L) = 0. \end{aligned}} \tag{9.3.41}$$

We have shown that the solution can be represented in terms of the Green's function:

$$\boxed{u(x) = \int_0^L f(x_0)G(x, x_0)\,dx_0,} \tag{9.3.42}$$

where the Green's function satisfies the following problem:

$$\boxed{\begin{aligned} &\frac{d^2G(x, x_0)}{dx^2} = \delta(x - x_0) \\ &G(0, x_0) = 0 \quad \text{and} \quad G(L, x_0) = 0. \end{aligned}} \tag{9.3.43}$$

One reason for defining the Green's function by the differential equation is that it gives an alternative (and often easier) way to calculate the Green's function. Here, x_0 is a parameter, representing the position of a concentrated source. For $x \neq x_0$ there are no sources, and hence the steady-state heat distribution $G(x, x_0)$ must be linear ($d^2G/dx^2 = 0$):

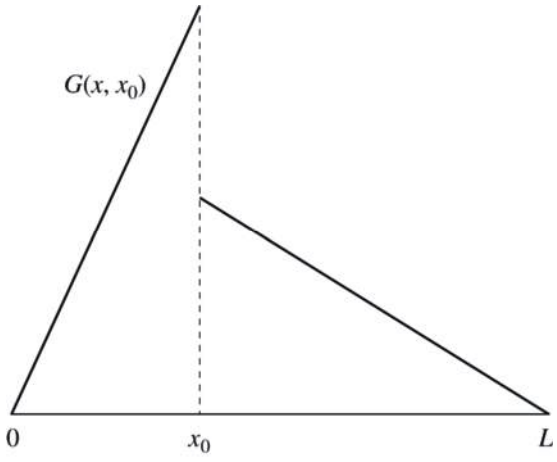
$$G(x, x_0) = \begin{cases} a + bx & x < x_0 \\ c + dx & x > x_0, \end{cases}$$

but the constants may be different. The boundary condition at $x = 0$ applies for $x < x_0$. $G(0, x_0) = 0$ implies that $a = 0$. Similarly, $G(L, x_0) = 0$ implies that $c + dL = 0$:

$$G(x, x_0) = \begin{cases} bx & x < x_0 \\ d(x - L) & x > x_0. \end{cases}$$

This preliminary result is sketched in Fig. 9.3.4.

Figure 9.3.4 Green's function before application of jump conditions at $x = x_0$.



The two remaining constants are determined by two conditions at $x = x_0$. The temperature $G(x, x_0)$ must be continuous at $x = x_0$,

$$\boxed{G(x_0^-, x_0) = G(x_0^+, x_0),} \tag{9.3.44}$$

and there is a jump in the derivative of $G(x, x_0)$, most easily derived by integrating the defining differential equation (9.3.43) from $x = x_0^-$ to $x = x_0^+$:

$$\boxed{\left.\frac{dG}{dx}\right|_{x=x_0^+} - \left.\frac{dG}{dx}\right|_{x=x_0^-} = 1.} \tag{9.3.45}$$

Equation (9.3.44) implies that

$$bx_0 = d(x_0 - L),$$

while (9.3.45) yields

$$d - b = 1.$$

By solving these simultaneously, we obtain

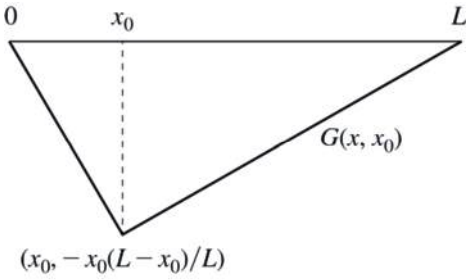
$$d = \frac{x_0}{L} \text{ and } b = \frac{x_0 - L}{L},$$

and thus

$$\boxed{G(x, x_0) = \begin{cases} -\dfrac{x}{L}(L - x_0) & x < x_0 \\ -\dfrac{x_0}{L}(L - x) & x > x_0, \end{cases}} \tag{9.3.46}$$

agreeing with (9.3.16). We sketch the Green's function in Fig. 9.3.5. The negative nature of this Green's function is due to the negative concentrated source of thermal energy, $-\delta(x - x_0)$, since $0 = d^2G/dx^2(x, 0) - \delta(x - x_0)$.

Figure 9.3.5 Green's function.



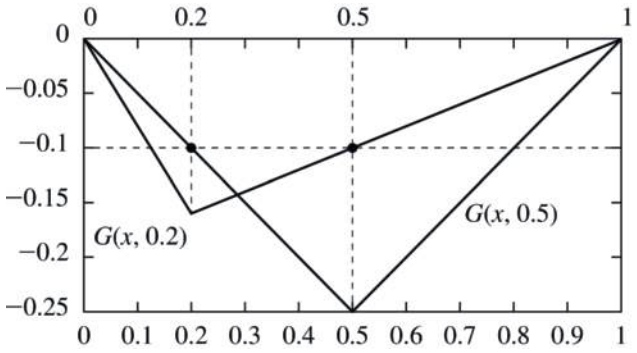
The symmetry of the Green's function (proved earlier) is apparent in all representations we have obtained. For example, letting $L = 1$,

$$G(x, x_0) = \begin{cases} -x(1 - x_0) & x < x_0 \\ -x_0(1 - x) & x > x_0 \end{cases} \quad \text{and} \quad G\left(\frac{1}{2}, \frac{1}{5}\right) = G\left(\frac{1}{5}, \frac{1}{2}\right) = -\frac{1}{10}.$$

We sketch $G(x, \frac{1}{5})$ and $G(x, \frac{1}{2})$ in Fig. 9.3.6. Their equality cannot be explained by simple physical symmetries.

For the steady-state heat equation, we have obtained two Green's functions, (9.3.46) and (9.3.23). They appear quite different. In Exercise 9.3.1, they are shown to be the same. In particular, (9.3.46) yields a piecewise smooth function (actually continuous), and its Fourier sine series can be shown to be given by (9.3.23).

Figure 9.3.6 Illustration of Maxwell's reciprocity.



9.3.5 NONHOMOGENEOUS BOUNDARY CONDITIONS

We have shown how to use Green's functions to solve nonhomogeneous differential equations with homogeneous boundary conditions. In this subsection, we extend these ideas to include problems with nonhomogeneous boundary conditions:

$$\boxed{\frac{d^2u}{dx^2} = f(x)} \tag{9.3.47}$$

$$\boxed{u(0) = \alpha \text{ and } u(L) = \beta.} \tag{9.3.48}$$

We will use the *same* Green's function as we did previously with problems with homogeneous boundary conditions:

$$\boxed{\frac{d^2G}{dx^2} = \delta(x - x_0)} \tag{9.3.49}$$

$$\boxed{G(0, x_0) = 0 \text{ and } G(L, x_0) = 0;} \tag{9.3.50}$$

the Green's function always satisfies the related homogeneous boundary conditions.

To obtain the representation of the solution of (9.3.47) with (9.3.48) involving the Green's function, we again utilize Green's formula, with $v = G(x, x_0)$:

$$\int_0^L \left[u(x)\frac{d^2G(x, x_0)}{dx^2} - G(x, x_0)\frac{d^2u}{dx^2}\right] dx = u\frac{dG(x, x_0)}{dx} - G(x, x_0)\frac{du}{dx}\Big|_0^L.$$

The right-hand side now does not vanish since $u(x)$ does not satisfy homogeneous boundary conditions. Instead, using only the definitions of our problem (9.3.47)–(9.3.48) and the Green's function (9.3.49)–(9.3.50), we obtain

$$\int_0^L [u(x)\delta(x - x_0) - G(x, x_0)f(x)]\, dx = u(L)\frac{dG(x, x_0)}{dx}\Big|_{x=L} - u(0)\frac{dG(x, x_0)}{dx}\Big|_{x=0}.$$

We analyze this as before. Using the property of the Dirac delta function (and reversing the roles of x and x_0) and using the symmetry of the Green's function, we obtain

$$\boxed{u(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0 + \beta\frac{dG(x, x_0)}{dx_0}\Big|_{x_0=L} - \alpha\frac{dG(x, x_0)}{dx_0}\Big|_{x_0=0}.} \tag{9.3.51}$$

This is a representation of the solution of our nonhomogeneous problem (including non-homogeneous boundary conditions) in terms of the standard Green's function. We must be careful in evaluating the boundary terms. In our problem, we have already shown that

$$G(x, x_0) = \begin{cases} -\dfrac{x}{L}(L - x_0) & x < x_0 \\ -\dfrac{x_0}{L}(L - x) & x > x_0. \end{cases}$$

The derivative with respect to the source position of the Green's function is thus

$$\frac{dG(x, x_0)}{dx_0} = \begin{cases} \dfrac{x}{L} & x < x_0 \\ -\left(1 - \dfrac{x}{L}\right) & x > x_0. \end{cases}$$

Evaluating this at the endpoints yields

$$\left.\frac{dG(x, x_0)}{dx_0}\right|_{x_0=L} = \frac{x}{L} \text{ and } \left.\frac{dG(x, x_0)}{dx_0}\right|_{x_0=0} = -\left(1 - \frac{x}{L}\right).$$

Consequently,

$$u(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0 + \beta\frac{x}{L} + \alpha\left(1 - \frac{x}{L}\right). \tag{9.3.52}$$

The solution is the sum of a particular solution of (9.3.42) satisfying homogeneous boundary conditions obtained earlier, $\int_0^L f(x_0)G(x, x_0)\, dx_0$, and a homogeneous solution satisfying the two required nonhomogeneous boundary conditions, $\beta(x/L) + \alpha(1 - x/L)$.

9.3.6 LIMIT OF TIME-DEPENDENT PROBLEM

One way (not the most obvious nor easiest) to solve (9.3.1) is to analyze our solution (9.2.13) of the time-dependent problem, obtained in the preceding section, in the special case of a steady source:

$$\begin{aligned} u(x, t) = &\int_0^L g(x_0)G(x, t; x_0, 0)\, dx_0 \\ &+ \int_0^L -kf(x_0)\left(\int_0^t G(x, t; x_0, t_0)\, dt_0\right)dx_0, \end{aligned} \tag{9.3.53}$$

where

$$G(x, t; x_0, t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2(t-t_0)}. \tag{9.3.54}$$

As $t \to \infty$, $G(x, t; x_0, 0) \to 0$ such that the effect of the initial condition $u(x, 0) = g(x)$ vanishes at $t \to \infty$. However, even though $G(x, t; x_0, t_0) \to 0$ as $t \to \infty$, the steady source is still important as $t \to \infty$ since

$$\int_0^t e^{-k(n\pi/L)^2(t-t_0)}\, dt_0 = \left.\frac{e^{-k(n\pi/L)^2(t-t_0)}}{k(n\pi/L)^2}\right|_{t_0=0}^{t} = \frac{1 - e^{-k(n\pi/L)^2 t}}{k(n\pi/L)^2}.$$

Thus, as $t \to \infty$,

$$u(x, t) \to \boxed{u(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0,} \tag{9.3.55}$$

where

$$\boxed{G(x, x_0) = -\sum_{n=1}^{\infty}\frac{2}{L}\frac{\sin n\pi x_0/L \sin n\pi x/L}{(n\pi/L)^2}.} \tag{9.3.56}$$

Here, we obtained the steady-state temperature distribution $u(x)$ by taking the limit as $t \to \infty$ of the time-dependent problem with a steady source $Q(x) = -kf(x)$. $G(x, x_0)$ is the **influence** or **Green's function** for the steady-state problem. The symmetry,

$$G(x, x_0) = G(x_0, x),$$

will be discussed later.

Exercises 9.3

9.3.1. The Green's function for (9.3.1) is given explicitly by (9.3.16). The method of eigenfunction expansion yields (9.3.23). Show that the Fourier sine series of (9.3.16) yields (9.3.23).

9.3.2. **(a)** Derive (9.3.17).

(b) Integrate (9.3.17) by parts to derive (9.3.16).

(c) Instead of part (b), simplify the double integral in (9.3.17) by interchanging the orders of integration. Derive (9.3.16) this way.

9.3.3. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t)$$

subject to $u(0, t) = 0$, $\frac{\partial u}{\partial x}(L, t) = 0$, and $u(x, 0) = g(x)$.

(a) Solve by the method of eigenfunction expansion.

(b) Determine the Green's function for this time-dependent problem.

(c) If $Q(x, t) = Q(x)$, take the limit as $t \to \infty$ of part (b) in order to determine the Green's function for

$$\frac{d^2u}{dx^2} = f(x) \text{ with } u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

9.3.4. **(a)** Derive (9.3.29) from (9.3.28). [*Hint*: Let $f(x) = 1$.]

(b) Show that (9.3.33) satisfies (9.3.31).

(c) Derive (9.3.30). [*Hint*: Show for any continuous $f(x)$ that

$$\int_{-\infty}^{\infty} f(x_0)\delta(x - x_0)\,dx_0 = \int_{-\infty}^{\infty} f(x_0)\delta(x_0 - x)\,dx_0$$

by letting $x_0 - x = s$ in the integral on the right.]

(d) Derive (9.3.34). [*Hint*: Evaluate $\int_{-\infty}^{\infty} f(x)\delta[c(x - x_0)]\,dx$ by making the change of variables $y = c(x - x_0)$.]

9.3.5. Consider

$$\frac{d^2u}{dx^2} = f(x) \text{ with } u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

***(a)** Solve by direct integration.

***(b)** Solve by the method of variation of parameters.

***(c)** Determine $G(x, x_0)$ so that (9.3.15) is valid.

(d) Solve by the method of eigenfunction expansion. Show that $G(x, x_0)$ is given by (9.3.23).

9.3.6. Consider

$$\frac{d^2G}{dx^2} = \delta(x - x_0) \text{ with } G(0, x_0) = 0 \quad \text{and} \quad \frac{dG}{dx}(L, x_0) = 0.$$

***(a)** Solve directly.

***(b)** Graphically illustrate $G(x, x_0) = G(x_0, x)$.

(c) Compare to Exercise 9.3.5.

9.3.7. Redo Exercise 9.3.5 with the following change: $\frac{du}{dx}(L) + hu(L) = 0, h > 0$.

9.3.8. Redo Exercise 9.3.6 with the following change: $\frac{dG}{dx}(L) + hG(L) = 0, h > 0$.

9.3.9. Consider

$$\frac{d^2u}{dx^2} + u = f(x) \text{ with } u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Assume that $(n\pi/L)^2 \neq 1$ (i.e., $L \neq n\pi$ for any n).

(a) Solve by the method of variation of parameters.

***(b)** Determine the Green's function so that $u(x)$ may be represented in terms of it [see (9.3.15)].

9.3.10. Redo Exercise 9.3.9 using the method of eigenfunction expansion.

9.3.11. Consider

$$\frac{d^2G}{dx^2} + G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0 \quad \text{and} \quad G(L, x_0) = 0.$$

***(a)** Solve for this Green's function directly. Why is it necessary to assume that $L \neq n\pi$?

(b) Show that $G(x, x_0) = G(x_0, x)$.

9.3.12. For the following problems, determine a representation of the solution in terms of the Green's function. Show that the nonhomogeneous boundary conditions can also be understood using homogeneous solutions of the differential equation:

(a) $\frac{d^2u}{dx^2} = f(x), u(0) = A, \frac{du}{dx}(L) = B$. (See Exercise 9.3.6.)

(b) $\frac{d^2u}{dx^2} + u = f(x), u(0) = A, u(L) = B$. Assume $L \neq n\pi$. (See Exercise 9.3.11.)

(c) $\frac{d^2u}{dx^2} = f(x), u(0) = A, \frac{du}{dx}(L) + hu(L) = 0$. (See Exercise 9.3.8.)

9.3.13. Consider the one-dimensional infinite space wave equation with a periodic source of frequency ω:

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} + g(x)e^{-i\omega t}. \tag{9.3.57}$$

(a) Show that a particular solution $\phi = u(x)e^{-i\omega t}$ of (9.3.57) is obtained if u satisfies a nonhomogeneous Helmholtz equation

$$\frac{d^2u}{dx^2} + k^2u = f(x).$$

***(b)** The Green's function $G(x, x_0)$ satisfies

$$\frac{d^2G}{dx^2} + k^2G = \delta(x - x_0).$$

Determine this infinite space Green's function so that the corresponding $\phi(x, t)$ is an outward-propagating wave.

(c) Determine a particular solution of (9.3.57).

9.3.14. Consider $L(u) = f(x)$ with $L = \frac{d}{dx}(p\frac{d}{dx}) + q$. Assume that the appropriate Green's function exists. Determine the representation of $u(x)$ in terms of the Green's function if the boundary conditions are nonhomogeneous:

(a) $u(0) = \alpha$ and $u(L) = \beta$

(b) $\frac{du}{dx}(0) = \alpha$ and $\frac{du}{dx}(L) = \beta$

(c) $u(0) = \alpha$ and $\frac{du}{dx}(L) = \beta$

***(d)** $u(0) = \alpha$ and $\frac{du}{dx}(L) + hu(L) = \beta$

9.3.15. Consider $L(G) = \delta(x - x_0)$ with $L = \frac{d}{dx}(p\frac{d}{dx}) + q$ subject to the boundary conditions $G(0, x_0) = 0$ and $G(L, x_0) = 0$. Introduce for all x two homogeneous solutions, y_1 and y_2, such that each solves one of the homogeneous boundary conditions:

$$\begin{aligned} L(y_1) &= 0, \quad & L(y_2) &= 0 \\ y_1(0) &= 0, \quad & y_2(L) &= 0 \\ \frac{dy_1}{dx}(0) &= 1, \quad & \frac{dy_2}{dx}(0) &= 1. \end{aligned}$$

Even if y_1 and y_2 cannot be explicitly obtained, they can be easily calculated numerically on a computer as two *initial value problems*. Any homogeneous solution must be a linear combination of the two.

***(a)** Solve for $G(x, x_0)$ in terms of $y_1(x)$ and $y_2(x)$. You may assume that $y_1(x) \neq cy_2(x)$.

(b) What goes wrong if $y_1(x) = cy_2(x)$ for all x and why?

9.3.16. Reconsider (9.3.41), whose solution we have obtained, (9.3.46). For (9.3.41), what are y_1 and y_2 in Exercise 9.3.15? Show that $G(x, x_0)$ obtained in Exercise 9.3.15 reduces to (9.3.46) for (9.3.41).

9.3.17. Consider

$$L(u) = f(x) \quad \text{with} \quad L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$
$$u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Introduce two homogeneous solutions y_1 and y_2, as in Exercise 9.3.15.

(a) Determine $u(x)$ using the method of variation of parameters.

(b) Determine the Green's function from part (a).

(c) Compare to Exercise 9.3.15.

9.3.18. Reconsider Exercise 9.3.17. Determine $u(x)$ by the method of eigenfunction expansion. Show that the Green's function satisfies (9.3.23).

9.3.19. **(a)** If a concentrated source is placed at a node of some mode (eigenfunction), show that the amplitude of the response of that mode is zero. [*Hint*: Use the result of the method of eigenfunction expansion and recall that a node x^* of an eigenfunction means anyplace where $\phi_n(x^*) = 0$.]

(b) If the eigenfunctions are $\sin n\pi x/L$ and the source is located in the middle, $x_0 = L/2$, show that the response will have no even harmonics.

9.3.20. Derive the eigenfunction expansion of the Green's function (9.3.23) directly from the defining differential equation (9.3.41) by letting

$$G(x, x_0) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Assume that term-by-term differentiation is justified.

***9.3.21.** Solve

$$\frac{dG}{dx} = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0.$$

Show that $G(x, x_0)$ is not symmetric even though $\delta(x - x_0)$ is.

9.3.22. Solve

$$\frac{dG}{dx} + G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0.$$

Show that $G(x, x_0)$ is not symmetric even though $\delta(x - x_0)$ is.

9.3.23. Solve

$$\frac{d^4G}{dx^4} = \delta(x - x_0)$$
$$G(0, x_0) = 0, \qquad G(L, x_0) = 0$$
$$\frac{dG}{dx}(0, x_0) = 0, \qquad \frac{d^2G}{dx^2}(L, x_0) = 0.$$

9.3.24. Use Exercise 9.3.23 to solve

$$\frac{d^4u}{dx^4} = f(x)$$
$$u(0) = 0, \qquad u(L) = 0$$
$$\frac{du}{dx}(0) = 0, \qquad \frac{d^2u}{dx^2}(L) = 0.$$

[*Hint*: Exercise 5.5.8 is helpful.]

9.3.25. Use the convolution theorem for Laplace transforms to obtain particular solutions of

(a) $\frac{d^2u}{dx^2} = f(x)$ (See Exercise 9.3.5.)

***(b)** $\frac{d^4u}{dx^4} = f(x)$ (See Exercise 9.3.24.)

9.3.26. Determine the Green's function satisfying $\frac{d^2G}{dx^2} - G = \delta(x - x_0)$:

(a) Directly on the interval $0 < x < L$ with $G(0, x_0) = 0$ and $G(L, x_0) = 0$

(b) Directly on the interval $0 < x < L$ with $G(0, x_0) = 0$ and $\frac{dG}{dx}(L, x_0) = 0$

(c) Directly on the interval $0 < x < L$ with $\frac{dG}{dx}(0, x_0) = 0$ and $\frac{dG}{dx}(L, x_0) = 0$

(d) Directly on the interval $0 < x < \infty$ with $G(0, x_0) = 0$

(e) Directly on the interval $0 < x < \infty$ with $\frac{dG}{dx}(0, x_0) = 0$

(f) Directly on the interval $-\infty < x < \infty$

9.3.27. Consider the Green's function problem

$$\frac{d}{dx}\left(\frac{1}{1+x}\frac{dG}{dx}\right) = \delta(x - x_0),\ G(0, x_0) = 0,\ G(3, x_0) = 0.$$

(a) What are the continuity and jump conditions at $x = x_0$? [*Hint*: The usual jump condition, $G'(x_0^+, x_0) - G'(x_0^-, x_0) = 1$, is *not* true in this problem.]

(b) Determine the Green's function $G(x, x_0)$. [*Hint*: Start by integrating both sides of $\frac{d}{dx}\left(\frac{1}{1+x}\frac{dG}{dx}\right) = 0$.]

Appendix to 9.3: Establishing Green's Formula with Dirac Delta Functions

Green's formula is very important when analyzing Green's functions. However, our derivation of Green's formula requires integration by parts. Here we will show that Green's formula,

$$\int_a^b [uL(v) - vL(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b, \tag{9.3.58}$$

where $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$, is valid even if v is a Green's function,

$$L(v) = \delta(x - x_0). \tag{9.3.59}$$

We will derive (9.3.58). We calculate the left-hand side of (9.3.58). Since there is a singularity at $x = x_0$, we are not guaranteed that (9.3.58) is valid. Instead, we divide the region into three parts:

$$\int_a^b = \int_a^{x_0^-} + \int_{x_0^-}^{x_0^+} + \int_{x_0^+}^b.$$

In the regions that exclude the singularity, $a \le x \le x_0^-$ and $x_0^+ \le x \le b$, Green's formula can be used. In addition, due to the property of the Dirac delta function,

$$\int_{x_0^-}^{x_0^+} [uL(v) - vL(u)]\,dx = \int_{x_0^-}^{x_0^+} [u\delta(x - x_0) - vL(u)]\,dx = u(x_0),$$

since $\int_{x_0^-}^{x_0^+} vL(u)\,dx = 0$. Thus, we obtain

$$\begin{aligned}\int_a^b [uL(v) - vL(u)]\,dx &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^{x_0^-} + p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0^+}^b + u(x_0)\\ &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b + p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0^+}^{x_0^-} + u(x_0).\end{aligned} \tag{9.3.60}$$

Since u, du/dx, and v are continuous at $x = x_0$, it follows that

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0^+}^{x_0^-} = p(x_0)u(x_0)\frac{dv}{dx}\Big|_{x_0^+}^{x_0^-}.$$

However, by integrating (9.3.59), we know that $pdv/dx|_{x_0^-}^{x_0^+} = 1$. Thus, (9.3.58) follows from (9.3.60). Green's formula may be utilized even if Green's functions are present.

9.4 Fredholm Alternative and Generalized Green's Functions

9.4.1 INTRODUCTION

In the nonhomogeneous problem

$$L(u) = f(x), \tag{9.4.1}$$

if $\lambda = 0$ is an eigenvalue of the differential operator L, then the Green's function does not exist. In order to understand the difficulty, we reexamine (9.4.1) *subject to homogeneous boundary conditions*. By the method of eigenfunction expansion, in the preceding section we obtained

$$u = \sum_{n=1}^{\infty} a_n \phi_n(x), \tag{9.4.2}$$

where by substitution

$$-a_n \lambda_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2 \sigma\,dx}. \tag{9.4.3}$$

If $\lambda_n = 0$ (for some n, often the lowest eigenvalue), there may not be any solutions to the nonhomogeneous boundary value problem. In particular, if $\int_a^b f(x)\phi_n(x)\,dx \neq 0$, for the eigenfunction corresponding to $\lambda_n = 0$, then (9.4.3) cannot be satisfied. This warrants further investigation.

EXAMPLE

Let us consider the following simple nonhomogeneous boundary value problem:

$$\frac{d^2u}{dx^2} = e^x \quad \text{with} \quad \frac{du}{dx}(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0. \tag{9.4.4}$$

After one integration, we get

$$\frac{du}{dx} = e^x + c.$$

The two boundary conditions cannot be satisfied as they are contradictory:

$$0 = 1 + c$$
$$0 = e^L + c,$$

unless $L = 0$. There is no guarantee that there are *any* solutions to a nonhomogeneous boundary value problem when $\lambda = 0$ is an eigenvalue for the related eigenvalue problem [$d^2\phi_n/dx^2 = -\lambda_n\phi_n$, with $d\phi_n/dx(0) = 0$ and $d\phi_n/dx(L) = 0$], which is the case here.

In this example, from one physical point of view, we are searching for an equilibrium temperature distribution. Since there are sources and the boundary conditions are of the insulated type, we know that an equilibrium temperature can exist only if there is no net input of thermal energy:

$$\int_0^L e^x\,dx = 0,$$

which is not valid. Since thermal energy is being constantly removed, there can be no equilibrium ($0 = d^2u/dx^2 - e^x$). ■

Zero eigenvalue

If $\lambda = 0$ is an eigenvalue, we have shown that there may be difficulty in solving `

$$L(u) = f(x), \tag{9.4.5}$$

subject to homogeneous boundary conditions. The eigenfunctions ϕ_n satisfy

$$L(\phi_n) = -\lambda_n \sigma \phi_n,$$

subject to the same homogeneous boundary conditions. Thus, if $\lambda = 0$ is an eigenvalue, the corresponding eigenfunction $\phi_h(x)$ satisfies

$$L(\phi_h) = 0 \tag{9.4.6}$$

with the same homogeneous boundary conditions. Thus, $\phi_h(x)$ is a nontrivial homogeneous solution of (9.4.5). This is important: **Nontrivial homogeneous solutions of (9.4.5) solving the same homogeneous boundary conditions are equivalent to eigenfunctions corresponding to the zero eigenvalue**. If there are no nontrivial homogeneous solutions (solving the same homogeneous boundary conditions), then $\lambda = 0$ is not an eigenvalue. If there are nontrivial homogeneous solutions, then $\lambda = 0$ is an eigenvalue.

The notion of a nontrivial homogeneous solution is less confusing than the existence of a zero eigenvalue. For example, consider

$$\frac{d^2u}{dx^2} + u = e^x \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(\pi) = 0. \tag{9.4.7}$$

Are there nontrivial homogeneous solutions? The answer is yes, $\phi = \sin x$. However, it may cause some confusion to say that $\lambda = 0$ is an eigenvalue (although it is true). The definition of the eigenvalues for (9.4.7) is

$$\frac{d^2\phi}{dx^2} + \phi = -\lambda\phi \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(\pi) = 0.$$

This is best written as $d^2\phi/dx^2 + (\lambda + 1)\phi = 0$. Therefore, $\lambda + 1 = (n\pi/L)^2 = n^2$, $n = 1, 2, 3, \ldots$, and it is now clear that $\lambda = 0$ is an eigenvalue (corresponding to $n = 1$).

9.4.2 FREDHOLM ALTERNATIVE

Important conclusions can be reached from (9.4.3), obtained by the method of eigenfunction expansion. The Fredholm alternative summarizes these results for nonhomogeneous problems

$$\boxed{L(u) = f(x),} \tag{9.4.8}$$

(where L is self-adjoint) *subject to homogeneous boundary conditions*.

1. Either $u = 0$ is the only homogeneous solution (i.e., $\lambda = 0$ is not an eigenvalue), in which case the nonhomogeneous problem has a unique solution, or
2. there are nontrivial homogeneous solutions $\phi_h(x)$ (i.e., $\lambda = 0$ is an eigenvalue), in which case the nonhomogeneous problem has no solutions or an infinite number of solutions.

Let us describe in more detail what occurs if $\phi_h(x)$ is a nontrivial homogeneous solution. By (9.4.3) **there are an infinite number of solutions of (9.4.8) if**

$$\boxed{\int_a^b f(x)\phi_h(x)\,dx = 0,} \tag{9.4.9}$$

because the corresponding a_n is arbitrary. These nonunique solutions correspond to an arbitrary additive multiple of a homogeneous solution $\phi_h(x)$. Equation (9.4.9) corresponds to the forcing function being orthogonal to the homogeneous solution (with weight 1). If

$$\int_a^b f(x)\phi_h(x)\,dx \neq 0, \tag{9.4.10}$$

then the nonhomogeneous problem (with homogeneous boundary conditions) has no solutions. These results are illustrated in Table 9.4.1.

TABLE 9.4.1 Number of Solutions of $L(u) = f(x)$ Subject to Homogeneous Boundary Conditions

	# of solutions	$\int_a^b f(x)\phi_h(x)\,dx$
$\phi_h = 0$ $(\lambda \neq 0)$	1	0
$\phi_h \neq 0$ $(\lambda = 0)$	∞	0
$\phi_h \neq 0$ $(\lambda = 0)$	0	$\neq 0$

A different phrasing of the Fredholm alternative states that **for the nonhomogeneous problem (9.4.8) with homogeneous boundary conditions, solutions exist only if the forcing function is orthogonal to all homogeneous solutions**.[5] Note that if $u = 0$ is the only homogeneous solution, then $f(x)$ is automatically orthogonal to it (in a somewhat trivial way), and there is a solution.

Part of the Fredholm alternative can be shown without using an eigenfunction expansion. If the nonhomogeneous problem has a solution, then

$$L(u) = f(x).$$

All homogeneous solutions, $\phi_h(x)$, satisfy

$$L(\phi_h) = 0.$$

We now use Green's formula with $v = \phi_h$ and obtain

$$\int_a^b [u \cdot 0 - \phi_h f(x)]\, dx = 0 \qquad \text{or} \qquad \boxed{\int_a^b f(x)\phi_h(x)\, dx = 0,}$$

since u and ϕ_h satisfy the same homogeneous boundary conditions.

EXAMPLES

We consider three examples. First, suppose that

$$\frac{d^2u}{dx^2} = e^x \quad \text{with} \quad \frac{du}{dx}(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0. \tag{9.4.11}$$

$u = 1$ is a homogeneous solution. According to the Fredholm alternative, there is a solution to (9.4.11) only if e^x is orthogonal to this homogeneous solution. Since $\int_0^L e^x \cdot 1\, dx \neq 0$, there are no solutions of (9.4.11).

For another example, suppose that

$$\frac{d^2u}{dx^2} + 2u = e^x \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(\pi) = 0.$$

Since there are no solutions of the corresponding homogeneous problem[6] (other than $u = 0$), the Fredholm alternative implies that there is a unique solution. However, to obtain that solution we must use standard techniques to solve nonhomogeneous differential equations, such as the methods of undetermined coefficients, variation of parameters, or eigenfunction expansion (using $\sin nx$).

As a more nontrivial example, we consider

$$\frac{d^2u}{dx^2} + \left(\frac{\pi}{L}\right)^2 u = \beta + x \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Since $\phi = \sin \pi x/L$ is a solution of the homogeneous problem, the nonhomogeneous problem has a solution only if the right-hand side is orthogonal to $\sin \pi x/L$:

$$0 = \int_0^L (\beta + x) \sin \frac{\pi x}{L}\, dx.$$

This can be used to determine the only value of β for which there is a solution:

$$\beta = \frac{-\int_0^L x \sin \pi x/L\, dx}{\int_0^L \sin \pi x/L\, dx} = -\frac{L}{2}.$$

However, again the Fredholm alternative cannot be used to actually obtain the solution, $u(x)$. ■

9.4.3 GENERALIZED GREEN'S FUNCTIONS

In this section, we will analyze

$$\boxed{L(u) = f} \tag{9.4.12}$$

[5]Here, the operator L is self-adjoint. For non-self-adjoint operators, the solutions exist if the forcing function is orthogonal to all solutions of the corresponding homogeneous *adjoint* problem (see Exercises 5.5.11–5.5.14).

[6]For $d^2\phi/dx^2 + \lambda\phi = 0$, with $\phi(0) = 0$ and $\phi(L) = 0$, the eigenvalues are $(n\pi/L)^2$. Here, $2 \neq n^2$.

subject to homogeneous boundary conditions when $\lambda = 0$ is an eigenvalue. If a solution to (9.4.12) exists, we will produce a particular solution of (9.4.12) by defining and constructing a modified or generalized Green's function.

If $\lambda = 0$ is not an eigenvalue, then there is a unique solution of the nonhomogeneous boundary value problem, (9.4.12), subject to homogeneous boundary conditions. In Section 9.3, we represented the solution using a Green's function $G(x, x_0)$ satisfying

$$L[G(x, x_0)] = \delta(x - x_0), \tag{9.4.13}$$

subject to the same homogeneous boundary conditions.

Here, we analyze the case in which $\lambda = 0$ is an eigenvalue: There are nontrivial homogeneous solutions $\phi_h(x)$ of (9.4.12), $L(\phi_h) = 0$. We will assume that there are solutions of (9.4.12), that is,

$$\int_a^b f(x)\phi_h(x)\,dx = 0. \tag{9.4.14}$$

However, the Green's function defined by (9.4.13) does not exist for all x_0 following from (9.4.10) since $\delta(x - x_0)$ is not orthogonal to solutions of the homogeneous problem for all x_0:

$$\int_a^b \delta(x - x_0)\phi_h(x)\,dx = \phi_h(x_0) \not\equiv 0.$$

Instead, we introduce a simple comparison problem that has a solution. $\delta(x - x_0)$ is not orthogonal to $\phi_h(x)$ because it has a "component in the direction" $\phi_h(x)$. However, there is a solution of (9.4.12) for all x_0 for the forcing function

$$\delta(x - x_0) + c\phi_h(x),$$

if c is properly chosen. In particular, we determine c easily such that this function is orthogonal to $\phi_h(x)$:

$$0 = \int_a^b \phi_h(x)[\delta(x - x_0) + c\phi_h(x)]\,dx = \phi_h(x_0) + c\int_a^b \phi_h^2(x)\,dx.$$

Thus, we introduce the **generalized Green's function** $G_m(x, x_0)$, which satisfies

$$\boxed{L[G_m(x, x_0)] = \delta(x - x_0) - \frac{\phi_h(x)\phi_h(x_0)}{\displaystyle\int_a^b \phi_h^2(x)\,dx},} \tag{9.4.15}$$

subject to the same homogeneous boundary conditions.

Since the right-hand side of (9.4.15) is orthogonal to $\phi_h(x)$, unfortunately there are an infinite number of solutions. In Exercise 9.4.9, it is shown that the generalized Green's function can be chosen to be symmetric:

$$\boxed{G_m(x, x_0) = G_m(x_0, x).} \tag{9.4.16}$$

If $g_m(x, x_0)$ is one symmetric generalized Green's function, then the following is also a symmetric generalized Green's function:

$$G_m(x, x_0) = g_m(x, x_0) + \beta\phi_h(x_0)\phi_h(x)$$

for any *constant* β (independent of x and x_0). Thus, there are an infinite number of symmetric generalized Green's functions. We can use any of these.

We use Green's formula to derive a representation formula for $u(x)$ using the generalized Green's function. Letting $u = u(x)$ and $v = G_m(x, x_0)$, Green's formula states that

$$\int_a^b \{u(x)L[G_m(x, x_0)] - G_m(x, x_0)L[u(x)]dx = 0,$$

since both $u(x)$ and $G_m(x, x_0)$ satisfy the same homogeneous boundary conditions. The defining differential equations (9.4.12) and (9.4.15) imply that

$$\int_a^b \left\{ u(x)\left[\delta(x - x_0) - \frac{\phi_h(x)\phi_h(x_0)}{\displaystyle\int_a^b \phi_h^2(\overline{x})\,d\overline{x}}\right] - G_m(x, x_0)f(x)\,dx = 0.\right.$$

Using the fundamental Dirac delta property (and reversing the roles of x and x_0) yields

$$u(x) = \int_a^b f(x_0)G_m(x, x_0)\,dx_0 + \frac{\phi_h(x)}{\displaystyle\int_a^b \phi_h^2(\overline{x})\,d\overline{x}}\int_a^b u(x_0)\phi_h(x_0)\,dx_0,$$

where the symmetry of $G_m(x, x_0)$ has also been utilized. The last expression is a multiple of the homogeneous solution, and thus a simple *particular* solution of (9.4.12) is

$$\boxed{u(x) = \int_a^b f(x_0) G_m(x, x_0)\, dx_0,} \tag{9.4.17}$$

the same form as occurs when $\lambda = 0$ is not an eigenvalue [see (9.3.36)].

EXAMPLE

The simplest example of a problem with a nontrivial homogeneous solution is

$$\frac{d^2u}{dx^2} = f(x) \tag{9.4.18}$$

$$\frac{du}{dx}(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0. \tag{9.4.19}$$

A constant is a homogeneous solution (eigenfunction corresponding to the zero eigenvalue). For a solution to exist, by the Fredholm alternative,[7] $\int_0^L f(x)\, dx = 0$. We assume $f(x)$ is of this type [e.g., $f(x) = x - L/2$]. The generalized Green's function $G_m(x, x_0)$ satisfies

$$\frac{d^2G_m}{dx^2} = \delta(x - x_0) + c \tag{9.4.20}$$

$$\frac{dG_m}{dx}(0) = 0 \quad \text{and} \quad \frac{dG_m}{dx}(L) = 0, \tag{9.4.21}$$

since a constant is the eigenfunction. For there to be such a generalized Green's function, the r.h.s. must be orthogonal to the homogeneous solutions:

$$\int_0^L [\delta(x - x_0) + c]\, dx = 0 \quad \text{or} \quad c = -\frac{1}{L}.$$

We use properties of the Dirac delta function (9.4.20) with (9.4.21). For $x \neq x_0$,

$$\frac{d^2G_m}{dx^2} = -\frac{1}{L}.$$

By integration

$$\frac{dG_m}{dx} = \begin{cases} -\dfrac{x}{L} & x < x_0 \\ -\dfrac{x}{L} + 1 & x > x_0, \end{cases} \tag{9.4.22}$$

where the constants of integration have been chosen to satisfy the boundary conditions at $x = 0$ and $x = L$. The jump condition for the derivative $\left(dG_m/dx|_{x_0-}^{x_0+} = 1\right)$, obtained by integrating (9.4.20), is already satisfied by (9.4.22). We integrate again to obtain $G_m(x, x_0)$. Assuming that $G_m(x, x_0)$ is continuous at $x = x_0$ yields

$$G_m(x, x_0) = \begin{cases} -\dfrac{1}{L}\dfrac{x^2}{2} + x_0 + c(x_0) & x < x_0 \\ -\dfrac{1}{L}\dfrac{x^2}{2} + x + c(x_0) & x > x_0. \end{cases}$$

$c(x_0)$ is an arbitrary additive constant that depends on x_0 and corresponds to an arbitrary multiple of the homogeneous solution. This is the representation of all possible generalized Green's functions. Often, we desire $G_m(x, x_0)$ to be symmetric. For example, $G_m(x, x_0) = G_m(x_0, x)$ for $x < x_0$ yields

$$-\frac{1}{L}\frac{x_0^2}{2} + x_0 + c(x) = -\frac{1}{L}\frac{x^2}{2} + x_0 + c(x_0)$$

or

$$c(x_0) = -\frac{1}{L}\frac{x_0^2}{2} + \beta,$$

[7] Physically, with insulated boundaries, there must be zero *net* thermal energy generated for equilibrium.

where β is an arbitrary *constant*. Thus, finally we obtain the generalized Green's function,

$$G_m(x, x_0) = \begin{cases} -\frac{1}{L}\frac{(x^2 + x_0^2)}{2} + x_0 + \beta & x < x_0 \\ -\frac{1}{L}\frac{(x^2 + x_0^2)}{2} + x + \beta & x > x_0. \end{cases} \tag{9.4.23}$$

A solution of (9.4.18)–(9.4.19) is given by (9.4.17) with $G_m(x, x_0)$ given previously. ■

An alternative generalized Green's function

In order to solve problems with homogeneous solutions, we could introduce instead a comparison problem satisfying nonhomogeneous boundary conditions. For example, the Neumann function G_a is defined by

$$\frac{d^2G_a}{dx^2} = \delta(x - x_0) \tag{9.4.24}$$

$$\frac{dG_a}{dx}(0) = -c \tag{9.4.25}$$

$$\frac{dG_a}{dx}(L) = c. \tag{9.4.26}$$

Physically, this represents a unit negative source $-\delta(x - x_0)$ of thermal energy with heat energy flowing into both ends at the rate of c per unit time. Thus, physically there will be a solution only if $2c = 1$. This can be verified by integrating (9.4.24) from $x = 0$ to $x = L$ or by using Green's formula. This alternate generalized Green's function can be obtained in a manner similar to the previous one. In terms of this Green's function, the representation of the solution of a nonhomogeneous problem can be obtained using Green's formula (see Exercise 9.4.12).

Exercises 9.4

9.4.1. Consider

$$L(u) = f(x) \quad \text{with} \quad L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

subject to two homogeneous boundary conditions. All homogeneous solutions ϕ_h (if they exist) satisfy $L(\phi_h) = 0$ and the same two homogeneous boundary conditions. Apply Green's formula to prove that there are no solutions u if $f(x)$ is not orthogonal (weight 1) to all $\phi_h(x)$.

9.4.2. Modify Exercise 9.4.1 if

$$L(u) = f(x)$$
$$u(0) = \alpha \quad \text{and} \quad u(L) = \beta.$$

***(a)** Determine the condition for a solution to exist.

(b) If this condition is satisfied, show that there are an infinite number of solutions using the method of eigenfunction expansion.

9.4.3. Without determining $u(x)$, how many solutions are there of

$$\frac{d^2u}{dx^2} + \gamma u = \sin x$$

(a) If $\gamma = 1$ and $u(0) = u(\pi) = 0$?

***(b)** If $\gamma = 1$ and $\frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0$?

(c) If $\gamma = -1$ and $u(0) = u(\pi) = 0$?

(d) If $\gamma = 2$ and $u(0) = u(\pi) = 0$?

9.4.4. For the following examples, obtain the general solution of the differential equation using the method of undetermined coefficients. Attempt to solve the boundary conditions, and show that the result is consistent with the Fredholm alternative:

(a) Equation (9.4.7)

(b) Equation (9.4.11)

(c) Example after (9.4.11)

(d) Second example after (9.4.11)

9.4.5. Are there any values of β for which there are solutions of

$$\frac{d^2u}{dx^2} + u = \beta + x$$

$$u(-\pi) = u(\pi) \quad \text{and} \quad \frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)?$$

***9.4.6.** Consider

$$\frac{d^2u}{dx^2} + u = 1.$$

(a) Find the general solution of this differential equation. Determine all solutions with $u(0) = u(\pi) = 0$. Is the Fredholm alternative consistent with your result?

(b) Redo part (a) if $\frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0$.

(c) Redo part (a) if $\frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)$ and $u(-\pi) = u(\pi)$.

9.4.7. Consider

$$\frac{d^2u}{dx^2} + 4u = \cos x$$

$$\frac{du}{dx}(0) = \frac{du}{dx}(\pi) = 0.$$

(a) Determine all solutions using the hint that a particular solution of the differential equation is in the form, $u_p = A\cos x$.

(b) Determine all solutions using the eigenfunction expansion method.

(c) Apply the Fredholm alternative. Is it consistent with parts (a) and (b)?

9.4.8. Consider

$$\frac{d^2u}{dx^2} + u = \cos x,$$

which has a particular solution of the form, $u_p = Ax\sin x$.

***(a)** Suppose that $u(0) = u(\pi) = 0$. Find all solutions. Is your result consistent with the Fredholm alternative?

(b) Answer the same questions as in part (a) if $u(-\pi) = u(\pi)$ and $\frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)$.

9.4.9. **(a)** Since (9.4.15) (with homogeneous boundary conditions) is solvable, there is an infinite number of solutions. Suppose that $g_m(x, x_0)$ is one such solution that is not orthogonal to $\phi_h(x)$. Show that there is a unique generalized Green's function $G_m(x, x_0)$ that is orthogonal to $\phi_h(x)$.

(b) Assume that $G_m(x, x_0)$ is the generalized Green's function that is orthogonal to $\phi_h(x)$. Prove that $G_m(x, x_0)$ is symmetric. [*Hint*: Apply Green's formula with $G_m(x, x_1)$ and $G_m(x, x_2)$.]

***9.4.10.** Determine the generalized Green's function that is needed to solve

$$\frac{d^2u}{dx^2} + u = f(x)$$

$$u(0) = \alpha \quad \text{and} \quad u(\pi) = \beta.$$

Assume that $f(x)$ satisfies the solvability condition (see Exercise 9.4.2). Obtain a representation of the solution $u(x)$ in terms of the generalized Green's function.

9.4.11. Consider

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad \frac{du}{dx}(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

A different generalized Greens function may be defined:

$$\frac{d^2G_a}{dx^2} = \delta(x - x_0)$$

$$\frac{dG_a}{dx}(0) = 0$$

$$\frac{dG_a}{dx}(L) = c.$$

***(a)** Determine c using mathematical reasoning.

***(b)** Determine c using physical reasoning.

(c) Explicitly determine all possible $G_a(x, x_0)$.

***(d)** Determine all symmetric $G_a(x, x_0)$.

***(e)** Obtain a representation of the solution $u(x)$ using $G_a(x, x_0)$.

9.4.12. The alternate generalized Green's function (Neumann function) satisfies

$$\frac{d^2G_a}{dx^2} = \delta(x - x_0)$$
$$\frac{dG_a}{dx}(0) = -c$$
$$\frac{dG_a}{dx}(L) = c, \quad \text{where we have shown } c = \tfrac{1}{2}.$$

(a) Determine all possible $G_a(x, x_0)$.

(b) Determine all symmetric $G_a(x, x_0)$.

(c) Determine all $G_a(x, x_0)$ that are orthogonal to $\phi_h(x)$.

(d) What relationship exists between β and γ for there to be a solution to

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad \frac{du}{dx}(0) = \beta \quad \text{and} \quad \frac{du}{dx}(L) = \gamma?$$

In this case, derive the solution $u(x)$ in terms of a Neumann function, defined above.

9.4.13. Consider $\frac{d^2u}{dx^2} + u = f(x)$ with $u(0) = 1$ and $u(\pi) = 4$. How many solutions are there, and how does this depend on $f(x)$? Do not determine $u(x)$.

9.4.14. Consider $\frac{d^2u}{dx^2} + u = f(x)$ with periodic boundary conditions $u(-L) = u(L)$ and $\frac{du}{dx}(-L) = \frac{du}{dx}(L)$. Note that $\phi_h = \sin\frac{\pi x}{L}$ and $\phi_h = \cos\frac{\pi x}{L}$ are homogeneous solutions.

(a) Under what condition is there a solution?

(b) Suppose a generalized Green's function satisfies

$$\frac{d^2G_m}{dx^2} + G_m = \delta(x - x_0) + c_1\cos\frac{\pi x}{L} + c_2\sin\frac{\pi x}{L}$$

with periodic boundary conditions. What are the constants c_1 and c_2? Do *not* solve for $G_m(x, x_0)$.

(c) Assume that the generalized Green's function is symmetric. Derive a representation for $u(x)$ in terms of $G_m(x, x_0)$.

9.5 Green's Functions for Poisson's Equation

9.5.1 INTRODUCTION

In Sections 9.3 and 9.4, we discussed Green's functions for Sturm–Liouville–type ordinary differential equations $L(u) = f$, where $L = d/dx(pd/dx) + q$. Before discussing Green's functions for time-dependent partial differential equations (such as the heat and wave equations), we will analyze Green's functions for Poisson's equation, a time-independent partial differential equation:

$$L(u) = f, \tag{9.5.1}$$

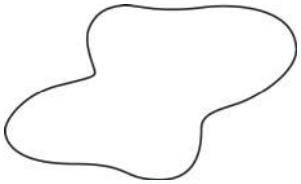
where $L = \nabla^2$, the Laplacian. At first, we will assume that u satisfies homogeneous boundary conditions. Later we will show how to use the same ideas to solve problems with nonhomogeneous boundary conditions. We will begin by assuming that the region is finite, as illustrated in Fig. 9.5.1. The extension to infinite domains will be discussed in some depth.

One-dimensional Green's functions were introduced to solve the nonhomogeneous Sturm–Liouville problem. Key relationships were provided by Green's formula. The analysis of Green's functions for Poisson's equation is quite similar. We will frequently use **Green's formula** for the Laplacian, either in its two- or three-dimensional forms:

$$\boxed{\begin{aligned}\iiint (u\nabla^2 v - v\nabla^2 u)\, dV &= \oiint (u\nabla v - v\nabla u)\cdot \hat{\mathbf{n}}\, dS\\ \iint (u\nabla^2 v - v\nabla^2 u)\, dA &= \oint (u\nabla v - v\nabla u)\cdot \hat{\mathbf{n}}\, ds.\end{aligned}}$$

We will assume that these formulas are valid even for the more exotic functions we will be discussing.

Figure 9.5.1 Finite two-dimensional region.



9.5.2 MULTIDIMENSIONAL DIRAC DELTA FUNCTION AND GREEN'S FUNCTIONS

The Green's function is defined as the solution to the nonhomogeneous problem with a concentrated source, subject to homogeneous boundary conditions. We define a two-dimensional Dirac delta function as an operator with a concentrated source with unit volume. It is the product of two one-dimensional Dirac delta functions. If the source is concentrated at $\boldsymbol{x} = \boldsymbol{x}_0 \left(\boldsymbol{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}, \boldsymbol{x}_0 = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}}\right)$, then

$$\boxed{\delta(\boldsymbol{x} - \boldsymbol{x}_0) = \delta(x - x_0)\delta(y - y_0).} \tag{9.5.2}$$

Similar ideas hold in three dimensions. The fundamental operator property of this multidimensional Dirac delta function is that

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}_0)\, dA = f(\boldsymbol{x}_0)} \tag{9.5.3}$$

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x - x_0)\delta(y - y_0)\, dA = f(x_0, y_0),} \tag{9.5.4}$$

in vector or two-dimensional component form, where $f(\boldsymbol{x}) = f(x, y)$. We will use the vector notation.

Green's function

In order to solve the nonhomogeneous partial differential equation

$$\boxed{\nabla^2 u = f(\boldsymbol{x}),} \tag{9.5.5}$$

subject to homogeneous conditions along the boundary, we introduce the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ for Poisson's equation[8]:

$$\boxed{\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0),} \tag{9.5.6}$$

subject to the *same homogeneous* boundary conditions. Here, $G(\boldsymbol{x}, \boldsymbol{x}_0)$ represents the response at $\boldsymbol{x}$ due to a source at $\boldsymbol{x}_0$.

Representation formula using Green's function

Green's formula (in its two-dimensional form) with $v = G(\boldsymbol{x}, \boldsymbol{x}_0)$ becomes

$$\iint (u\nabla^2 G - G\nabla^2 u)\, dA = 0,$$

since both $u(\boldsymbol{x})$ and $G(\boldsymbol{x}, \boldsymbol{x}_0)$ solve the same homogeneous boundary conditions such that $\oint (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, ds$ vanishes. From (9.5.5) and (9.5.6), it follows that

$$u(\boldsymbol{x}_0) = \iint f(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{x}_0)\, dA.$$

If we reverse the role of $\boldsymbol{x}$ and $\boldsymbol{x}_0$, we obtain

$$u(\boldsymbol{x}) = \iint f(\boldsymbol{x}_0)G(\boldsymbol{x}_0, \boldsymbol{x})\, dA_0.$$

As we will show, the Green's function is symmetric:

$$\boxed{G(\boldsymbol{x}, \boldsymbol{x}_0) = G(\boldsymbol{x}_0, \boldsymbol{x}),} \tag{9.5.7}$$

and hence

$$\boxed{u(\boldsymbol{x}) = \iint f(\boldsymbol{x}_0)G(\boldsymbol{x}, \boldsymbol{x}_0)\, dA_0.} \tag{9.5.8}$$

This shows how the solution of the partial differential equation may be computed if the Green's function is known.

[8]Sometimes this is called the Green's function for Laplace's equation.

Symmetry

As in one-dimensional problems, to show the symmetry of the Green's function we use Green's formula with $G(\boldsymbol{x}, \boldsymbol{x}_1)$ and $G(\boldsymbol{x}, \boldsymbol{x}_2)$. Since both satisfy the same homogeneous boundary conditions, we have

$$\iint [G(\boldsymbol{x}, \boldsymbol{x}_1)\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_2) - G(\boldsymbol{x}, \boldsymbol{x}_2)\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_1)]\, dA = 0.$$

Since $\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_1) = \delta(\boldsymbol{x} - \boldsymbol{x}_1)$ and $\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_2) = \delta(\boldsymbol{x} - \boldsymbol{x}_2)$, it follows using the fundamental property of the Dirac delta function that $G(\boldsymbol{x}_1, \boldsymbol{x}_2) = G(\boldsymbol{x}_2, \boldsymbol{x}_1)$; the Green's function is symmetric.

9.5.3 GREEN'S FUNCTIONS BY THE METHOD OF EIGENFUNCTION EXPANSION AND THE FREDHOLM ALTERNATIVE

One method to solve Poisson's equation in a finite region with homogeneous boundary conditions,

$$\boxed{\nabla^2 u = f(\boldsymbol{x}),} \tag{9.5.9}$$

is to use an eigenfunction expansion. We consider the related eigenfunctions, $\nabla^2\phi = -\lambda\phi$, subject to the same homogeneous boundary conditions. We assume that the eigenvalues λ and corresponding eigenfunctions $\phi_\lambda(\boldsymbol{x})$ are known. Simple examples occur in rectangular and circular regions. We solve for $u(\boldsymbol{x})$ using the method of eigenfunction expansion:

$$u(\boldsymbol{x}) = \sum_\lambda a_\lambda \phi_\lambda(\boldsymbol{x}). \tag{9.5.10}$$

Since $u(\boldsymbol{x})$ and $\phi_\lambda(\boldsymbol{x})$ solve the same homogeneous boundary condition, we expect to be able to differentiate term by term:

$$f = \nabla^2 u = \sum_\lambda a_\lambda \nabla^2 \phi_\lambda(\boldsymbol{x}) = -\sum_\lambda \lambda a_\lambda \phi_\lambda(\boldsymbol{x}).$$

This can be verified using Green's formula. Due to the multidimensional orthogonality of $\phi_\lambda(\boldsymbol{x})$, it follows that

$$-\lambda a_\lambda = \frac{\iint f(\boldsymbol{x}_0)\phi_\lambda(\boldsymbol{x}_0)\, dA_0}{\iint \phi_\lambda^2\, dA}. \tag{9.5.11}$$

If $\lambda = 0$ is not an eigenvalue, then we can determine a_λ. The representation of the solution $u(\boldsymbol{x})$ follows from 9.5.10 after interchanging $\sum_\lambda$ and $\iint$:

$$\boxed{u(\boldsymbol{x}) = \iint f(\boldsymbol{x}_0)G(\boldsymbol{x}, \boldsymbol{x}_0)\, dA_0,} \tag{9.5.12}$$

where the eigenfunction expansion of the Green's function is given by

$$\boxed{G(\boldsymbol{x}, \boldsymbol{x}_0) = \sum_\lambda \frac{\phi_\lambda(\boldsymbol{x})\phi_\lambda(\boldsymbol{x}_0)}{-\lambda \iint \phi_\lambda^2\, dA}.} \tag{9.5.13}$$

This is the natural generalization of the one-dimensional result (ordinary differential equation) corresponding to the Green's function for a nonhomogeneous Sturm–Liouville boundary value problem (see Section 9.3.3).

EXAMPLE

For a rectangle, $0 < x < L$, $0 < y < H$, with zero boundary conditions on all four sides, we have shown (see Chapter 7) that the eigenvalues are $\lambda_{nm} = (n\pi/L)^2 + (m\pi/H)^2$ ($n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$) and the corresponding eigenfunctions are $\phi_\lambda(\boldsymbol{x}) = \sin(n\pi x/L)\sin m\pi y/H$. In this case, the normalization constants are $\iint \phi_\lambda^2\, dx\, dy = \frac{L}{2} \cdot \frac{H}{2}$. The Green's function can be expanded in a series of these eigenfunctions, a Fourier sine series in x and y,

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{-4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sin n\pi x/L \sin m\pi y/H \sin n\pi x_0/L \sin m\pi y_0/H}{(n\pi/L)^2 + (m\pi/H)^2}.$$

Later in this section, as well as in Exercise 9.5.23(a), we will obtain alternative forms of this Green's function. ■

Fredholm alternative

We show that there is a Fredholm alternative as in Section 9.4. If $\lambda \neq 0$, we have obtained a unique solution of the nonhomogeneous problem (9.5.9) subject to homogenous boundary conditions. As before, difficulties occur if $\lambda = 0$ is an eigenvalue. In this case, there is at least one nontrivial homogeneous solution ϕ_h of Laplace's equation $\nabla^2\phi_h = 0$ (the homogeneous equation related to Poisson's equation) satisfying homogeneous boundary conditions. From (9.5.11), it follows that for the nonhomogeneous boundary value problem (9.5.9) subject to homogeneous boundary conditions, exactly one of the following statements holds:

There is an infinite number of solutions if the right-hand side is orthogonal to all homogeneous solutions $\iint f(\boldsymbol{x}_0)\phi_h(\boldsymbol{x}_0)\,dA_0 = 0$. From (9.5.11), the corresponding a_λ is arbitrary. (9.5.14)

There are no solutions if $\iint f(\boldsymbol{x}_0)\phi_h(\boldsymbol{x}_0)\,dA_0 = 0$. (9.5.15)

EXAMPLE

If the entire boundary is insulated, $\nabla\phi \cdot \hat{\mathbf{n}} = 0$, then ϕ_h equaling any constant is a nontrivial solution of $\nabla^2\phi = 0$ satisfying the boundary conditions. $\phi_h = 1$ is the eigenfunction corresponding to $\lambda = 0$. Solutions of $\nabla^2 u = f(\boldsymbol{x})$ then exist only if $\iint f(\boldsymbol{x})\,dA = 0$. Physically for a steady-state heat equation with insulated boundaries, the net heat energy generated must be zero. This is just the two-dimensional version of the problem discussed in Section 9.4. In particular, we could introduce in some way a generalized Green's function (which is also known as a **Neumann function**). We leave any discussion of this for the Exercises. For the remainder of Section 9.5, we will assume that $\lambda = 0$ is not an eigenvalue. ■

9.5.4 DIRECT SOLUTION OF GREEN'S FUNCTIONS (ONE-DIMENSIONAL EIGENFUNCTIONS) (OPTIONAL)

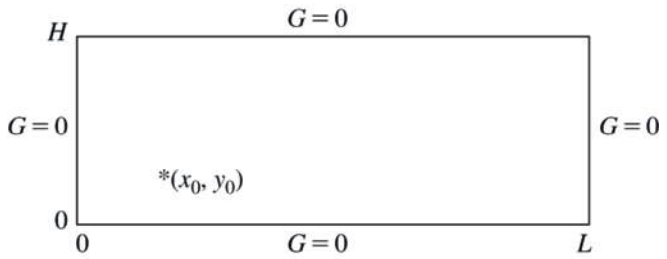
Green's functions can also be obtained by more direct methods. Consider the Green's function for Poisson's equation,

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{9.5.16}$$

inside a rectangle with zero boundary conditions, as illustrated in Fig. 9.5.2. Instead of solving for this Green's function using a series of two-dimensional eigenfunctions (see Section 9.5.3), we will use one-dimensional eigenfunctions, a sine series in either x or y due to the boundary conditions. Using a Fourier sine series in x,

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \sum_{n=1}^{\infty} a_n(y)\sin\frac{n\pi x}{L}. \tag{9.5.17}$$

Figure 9.5.2 Green's function for Poisson's equation on a rectangle.



By substituting (9.5.17) into (9.5.16), we obtain [since both $G(\boldsymbol{x}, \boldsymbol{x}_0)$ and $\sin(n\pi x/L)$ satisfy the same set of homogeneous boundary conditions]

$$\sum_{n=1}^{\infty}\left[\frac{d^2a_n}{dy^2} - \left(\frac{n\pi}{L}\right)^2 a_n\right]\sin\frac{n\pi x}{L} = \delta(x - x_0)\delta(y - y_0)$$

or

$$\begin{aligned}\frac{d^2a_n}{dy^2} - \left(\frac{n\pi}{L}\right)^2 a_n &= \frac{2}{L}\int_0^L \delta(x - x_0)\delta(y - y_0)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\sin\frac{n\pi x_0}{L}\delta(y - y_0).\end{aligned} \tag{9.5.18}$$

The boundary conditions at $y = 0$ and $y = H$ imply that the Fourier coefficients must satisfy the corresponding boundary conditions,

$$a_n(0) = 0 \quad \text{and} \quad a_n(H) = 0. \tag{9.5.19}$$

Equation (9.5.18) with boundary conditions (9.5.19) may be solved by a Fourier sine series in y, but this will yield the earlier double-sine series analysis. On the other hand, since the nonhomogeneous term for $a_n(y)$ is a one-dimensional Dirac delta function, we may solve (9.5.18) as we have done for Green's functions. The differential equation is homogeneous if $y \neq y_0$. In addition, if we utilize the boundary conditions, we obtain

$$a_n(y) = \begin{cases} c_n \sinh\dfrac{n\pi y}{L}\sinh\dfrac{n\pi(y_0 - H)}{L} & y < y_0 \\ c_n \sinh\dfrac{n\pi(y - H)}{L}\sinh\dfrac{n\pi y_0}{L} & y > y_0, \end{cases}$$

where in this form continuity at $y = y_0$ is automatically satisfied. In addition, we integrate (9.5.18) from y_{0-} to y_{0+} to obtain the jump in the derivative:

$$\left.\frac{da_n}{dy}\right|_{y_{0-}}^{y_{0+}} = \frac{2}{L}\sin\frac{n\pi x_0}{L}$$

or

$$c_n\frac{n\pi}{L}\left[\sinh\frac{n\pi y_0}{L}\cosh\frac{n\pi(y_0 - H)}{L} - \sinh\frac{n\pi(y_0 - H)}{L}\cosh\frac{n\pi y_0}{L}\right] = \frac{2}{L}\sin\frac{n\pi x_0}{L}. \tag{9.5.20}$$

Using an addition formula for hyperbolic functions, we obtain

$$c_n = \frac{2\sin n\pi x_0/L}{n\pi\sinh n\pi H/L}.$$

This yields the Fourier sine series (in x) representation of the Green's function

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \sum_{n=1}^{\infty}\frac{2\sin n\pi x_0/L\sin n\pi x/L}{n\pi\sinh n\pi H/L}\begin{cases}\sinh\dfrac{n\pi(y_0 - H)}{L}\sinh\dfrac{n\pi y}{L} & y < y_0 \\ \sinh\dfrac{n\pi(y - H)}{L}\sinh\dfrac{n\pi y_0}{L} & y > y_0.\end{cases} \tag{9.5.21}$$

The symmetry is exhibited explicitly. In the preceding subsection, this same Green's function was represented as a double Fourier sine series in both x and y.

A third representation of this Green's function is also possible. Instead of using a Fourier sine series in x, we could have used a Fourier sine series in y. We omit the nearly identical analysis.

9.5.5 USING GREEN'S FUNCTIONS FOR PROBLEMS WITH NONHOMOGENEOUS BOUNDARY CONDITIONS

As with one-dimensional problems, the *same* Green's function determined in Sections 9.5.2–9.5.4, $\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$ [with $G(\boldsymbol{x}, \boldsymbol{x}_0)$ satisfying homogeneous boundary conditions], may be used to solve Poisson's equation $\nabla^2 u = f(\boldsymbol{x})$ subject to *nonhomogeneous* boundary conditions.

For example, consider

$$\boxed{\nabla^2 u = f(\boldsymbol{x})} \tag{9.5.22}$$

with

$$\boxed{u = h(\boldsymbol{x})} \tag{9.5.23}$$

on the boundary. The Green's function is defined by

$$\boxed{\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0),} \tag{9.5.24}$$

with

$$\boxed{G(\boldsymbol{x}, \boldsymbol{x}_0) = 0} \tag{9.5.25}$$

for $\boldsymbol{x}$ on the boundary ($\boldsymbol{x}_0$ is often not on the boundary). The Green's function satisfies the related homogeneous boundary conditions. To obtain the Green's function representation of the solution of (9.5.22) and (9.5.23), we again employ Green's formula,

$$\iint (u\nabla^2 G - G\nabla^2 u)\, dA = \oint (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, ds.$$

Using the defining differential equations and the boundary conditions,

$$\iint [u(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}_0) - f(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{x}_0)]\, dA = \oint h(\boldsymbol{x})\nabla G \cdot \hat{\mathbf{n}}\, ds,$$

and thus,

$$u(\boldsymbol{x}_0) = \iint f(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{x}_0) dA + \oint h(\boldsymbol{x})\nabla G(\boldsymbol{x}, \boldsymbol{x}_0) \cdot \hat{\mathbf{n}}\, ds.$$

We interchange $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and we use the symmetry of $G(\boldsymbol{x}, \boldsymbol{x}_0)$ to obtain

$$\boxed{u(\boldsymbol{x}) = \iint f(\boldsymbol{x}_0)G(\boldsymbol{x}, \boldsymbol{x}_0) dA_0 + \oint h(\boldsymbol{x}_0)\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, \boldsymbol{x}_0) \cdot \hat{\mathbf{n}}\, ds_0.} \tag{9.5.26}$$

We must be especially careful with the closed line integral, representing the effect of the nonhomogeneous boundary condition. $\nabla_{\boldsymbol{x}_0}$ is a symbol for the gradient with respect to the position of the source,

$$\nabla_{\boldsymbol{x}_0} \equiv \frac{\partial}{\partial x_0}\hat{\mathbf{i}} + \frac{\partial}{\partial y_0}\hat{\mathbf{j}}$$

So, $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the influence function for the source term, while $\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, \boldsymbol{x}_0) \cdot \hat{\mathbf{n}}$ is the influence function for the nonhomogeneous boundary conditions. Let us attempt to give an interpretation to the influence function for the nonhomogeneous boundary conditions, $\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, \boldsymbol{x}_0) \cdot \hat{\mathbf{n}}$. This is an ordinary derivative with respect to the source position in the normal direction. Using the definition of a directional derivative,

$$\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, \boldsymbol{x}_0) \cdot \hat{\mathbf{n}} = \lim_{\Delta s \to 0} \frac{G(\boldsymbol{x}, \boldsymbol{x}_0 + \Delta s\hat{\mathbf{n}}) - G(\boldsymbol{x}, \boldsymbol{x}_0)}{\Delta s}.$$

This yields an interpretation of this normal derivative of the Green's function. $G(\boldsymbol{x}, \boldsymbol{x}_0 + \Delta s\hat{\mathbf{n}})/\Delta s$ is the response to a positive source of strength $1/\Delta s$ located at $\boldsymbol{x}_0 + \Delta s\hat{\mathbf{n}}$, while $-G(\boldsymbol{x}, \boldsymbol{x}_0)/\Delta s$ is the response to a negative source (strength $-1/\Delta s$) located at $\boldsymbol{x}_0$. The influence function for the nonhomogeneous boundary condition consists of two concentrated sources of opposite effects whose strength is $1/\Delta s$ and distance apart is Δs, in the limit as $\Delta s \to 0$. This is called a **dipole source**. Thus, *this nonhomogeneous boundary condition has an equivalent effect as a surface distribution of dipoles.*

9.5.6 INFINITE SPACE GREEN'S FUNCTIONS

In Sections 9.5.2–9.5.4, we obtained representations of the Green's function for Poisson's equation, $\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$. However, these representations were complicated. The resulting infinite series do not give a very good understanding of the effect at $\boldsymbol{x}$ of a concentrated source at $\boldsymbol{x}_0$. As we will show, the difficulty is caused by the presence of boundaries.

In order to obtain simpler representations, we begin by considering solving Poisson's equation

$$\nabla^2 u = f(\boldsymbol{x})$$

in infinite space with no boundaries. We introduce the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ defined by

$$\boxed{\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0),} \tag{9.5.27}$$

to be valid for all $\boldsymbol{x}$. Since this is a model of steady-state heat flow with a concentrated source located at $\boldsymbol{x} = \boldsymbol{x}_0$ *with no boundaries*, there should be a solution that is symmetric around the source point $\boldsymbol{x} = \boldsymbol{x}_0$. Our results are somewhat different in two and three dimensions. We simultaneously solve both. We let r represent radial distance (from $\boldsymbol{x} = \boldsymbol{x}_0$) in two dimensions and ρ represent radial distance (from $\boldsymbol{x} = \boldsymbol{x}_0$) in three dimensions:

two	three
$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$	$\rho = \boldsymbol{x} - \boldsymbol{x}_0$
$r = \lvert\boldsymbol{r}\rvert = \lvert\boldsymbol{x} - \boldsymbol{x}_0\rvert$	$\rho = \lvert\rho\rvert = \lvert\boldsymbol{x} - \boldsymbol{x}_0\rvert$
$= \sqrt{(x - x_0)^2 + (y - y_0)^2}$	$= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$

(9.5.28)

Our derivation continues with three-dimensional results in parentheses and on the right. We assume that $G(\boldsymbol{x}, \boldsymbol{x}_0)$ depends only on $r(\rho)$:

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = G(r) = G(\lvert\boldsymbol{x} - \boldsymbol{x}_0\rvert) \mid G(\boldsymbol{x}, \boldsymbol{x}_0) = G(\rho) = G(\lvert\boldsymbol{x} - \boldsymbol{x}_0\rvert).$$

Away from the source $r \neq 0 (\rho \neq 0)$, the forcing function is zero $[\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = 0]$. In two dimensions, we look for circularly symmetric solutions of Laplace's equation for $r \neq 0$ (in three dimensions the solutions should be spherically symmetric). From our earlier work,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0, \ (r \neq 0) \quad \Big| \quad \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dG}{d\rho}\right) = 0. \ (\rho \neq 0)$$

The general solution can be obtained by integration:

$$G(r) = c_1 \ln r + c_2 \quad \Big| \quad G(\rho) = \frac{c_3}{\rho} + c_4. \tag{9.5.29}$$

We will determine the constants that account for the singularity at the source. We can obtain the appropriate singularity by integrating (9.5.27) around a small circle (sphere) of radius $r(\rho)$:

$$\iint \nabla^2 G \, dA = 1 \qquad \iiint \nabla^2 G \, dV = 1$$

$$\iint \nabla \cdot (\nabla G) \, dA = \oint \nabla G \cdot \hat{\mathbf{n}} \, ds = 1 \quad \Big| \quad \iiint \nabla \cdot (\nabla G) dV = \oiint \nabla G \cdot \hat{\mathbf{n}} \, dS = 1,$$

where the divergence theorem has been used. In two dimensions, the derivative of the Green's function normal to a circle, $\nabla G \cdot \hat{\mathbf{n}}$, is $\partial G/\partial r$ (in three dimensions $\partial G/\partial \rho$). It depends only on the radial distance [see (9.5.29)]. On the circle (sphere), the radius is constant. Thus,

$$2\pi r \frac{\partial G}{\partial r} = 1 \quad \Big| \quad 4\pi\rho^2 \frac{\partial G}{\partial \rho} = 1,$$

since the circumference of a circle is $2\pi r$ (the surface area of a sphere is $4\pi\rho^2$). In other problems involving infinite space Green's functions, it may be necessary to consider the limit of an infinitesimally small circle (sphere). Thus, we will express the singularity condition as

$$\boxed{\lim_{r \to 0} r \frac{\partial G}{\partial r} = \frac{1}{2\pi}} \quad \Big| \quad \boxed{\lim_{\rho \to 0} \rho^2 \frac{\partial G}{\partial \rho} = \frac{1}{4\pi}.} \tag{9.5.30}$$

From (9.5.29) and (9.5.30),

$$c_1 = \frac{1}{2\pi} \quad \Big| \quad c_3 = -\frac{1}{4\pi}.$$

c_2 and c_4 are arbitrary, indicating that the infinite space Green's function for Poisson's equation is determined to within an arbitrary additive constant. For convenience, we let $c_2 = 0$ and $c_4 = 0$:

$\boxed{G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi} \ln r}$	$\boxed{G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi\rho}}$
$r = \lvert\boldsymbol{x} - \boldsymbol{x}_0\rvert$	$\rho = \lvert\boldsymbol{x} - \boldsymbol{x}_0\rvert$
$= \sqrt{(x - x_0)^2 + (y - y_0)^2}$	$= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$

(9.5.31)

Note that these are symmetric. These infinite space Green's functions are themselves singular at the concentrated source. (This does not occur in one dimension.)

In order to obtain the solution of Poisson's equation, $\nabla^2 u = f(\boldsymbol{x})$, *in infinite space,* using the infinite space Green's function, we need to utilize Green's formula:

$$\iint (u\nabla^2 G - G\nabla^2 u)\, dA = \oint (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, ds \quad \Bigg| \quad \iiint (u\nabla^2 G - G\nabla^2 u)\, dV = \oiint (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, dS. \tag{9.5.32}$$

The closed line integral $\oint$ (closed surface integral $\oiint$) represents integrating over the entire boundary. For infinite space problems with no boundaries, we must consider large circles (spheres) and take the limit as the radius approaches infinity. We would like the contribution to this closed integral "from infinity" to vanish:

$$\lim_{|x|\to\infty} \oint (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, ds = 0 \quad \Bigg| \quad \lim_{|x|\to\infty} \oiint (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, dS = 0. \tag{9.5.33}$$

In this case, using the defining differential equations, integrating (9.5.32) with the Dirac delta function, and reversing the roles of $\boldsymbol{x}$ and $\boldsymbol{x}_0$ [using the symmetry of $G(\boldsymbol{x}, \boldsymbol{x}_0)$] yields a representation formula for a solution of Poisson's equation in infinite space:

$$\boxed{u(\boldsymbol{x}) = \iint f(\boldsymbol{x}_0) G(\boldsymbol{x}, \boldsymbol{x}_0)\, dA_0} \quad \Bigg| \quad \boxed{u(\boldsymbol{x}) = \iiint f(\boldsymbol{x}_0) G(\boldsymbol{x}, \boldsymbol{x}_0)\, dV_0,} \tag{9.5.34}$$

where $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is given by (9.5.31).

The condition necessary for (9.5.33) to be valid is obtained by integrating in polar (spherical) coordinates centered at $\boldsymbol{x} = \boldsymbol{x}_0$:

$$\lim_{r\to\infty} r\left(u\frac{\partial G}{\partial r} - G\frac{\partial u}{\partial r}\right) = 0 \quad \Bigg| \quad \lim_{\rho\to\infty} \rho^2 \left(u\frac{\partial G}{\partial \rho} - G\frac{\partial u}{\partial \rho}\right) = 0,$$

since $ds = r d\theta$ [$dS = \rho^2 \sin\phi\, d\phi d\theta$]. By substituting the known Green's functions, we obtain conditions that must be satisfied at ∞ in order for the "boundary" terms to vanish there:

$$\boxed{\lim_{r\to\infty}\left(u - r\ln r \frac{\partial u}{\partial r}\right) = 0} \quad \Bigg| \quad \boxed{\lim_{\rho\to\infty}\left(u + \rho\frac{\partial u}{\partial \rho}\right) = 0.} \tag{9.5.35}$$

These are important conditions. For example, they are satisfied if $u \sim 1/r$ ($u \sim 1/\rho$) as $r \to \infty (\rho \to \infty)$. There are solutions of Poisson's equation in infinite space other than the solutions given by (9.5.34), but they do not satisfy these decay estimates (9.5.35).

9.5.7 GREEN'S FUNCTIONS FOR BOUNDED DOMAINS USING INFINITE SPACE GREEN'S FUNCTIONS

In this subsection we solve for the Green's function,

$$\boxed{\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0),} \tag{9.5.36}$$

on a bounded *two-dimensional* domain, subject to homogeneous boundary conditions. We have already discussed some methods of solution only for simple geometries, and even these involve a considerable amount of computation. However, we now know a particular solution of (9.5.36), namely, the infinite space Green's function

$$G_p(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi}\ln r = \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0| = \frac{1}{2\pi}\ln\sqrt{(x - x_0)^2 + (y - y_0)^2}. \tag{9.5.37}$$

Unfortunately, this infinite space Green's function will not solve the homogeneous boundary conditions. Instead, consider

$$\boxed{G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0| + v(\boldsymbol{x}, \boldsymbol{x}_0).} \tag{9.5.38}$$

$v(\boldsymbol{x}, \boldsymbol{x}_0)$ represents the effect of the boundary. It will be a homogeneous solution, solving Laplace's equation,

$$\nabla^2 v = 0,$$

subject to some nonhomogeneous boundary condition. For example, if $G = 0$ on the boundary, then $v = -(1/2\pi)\ln|\boldsymbol{x} - \boldsymbol{x}_0|$ on the boundary. $v(\boldsymbol{x}, \boldsymbol{x}_0)$ may be solved by standard methods for Laplace's equation based on separation of variables (if the geometry allows). It may be quite involved to calculate $v(\boldsymbol{x}, \boldsymbol{x}_0)$; nevertheless, this representation of the Green's function is quite important. In particular, since $v(\boldsymbol{x}, \boldsymbol{x}_0)$ will be well behaved everywhere, including $\boldsymbol{x} = \boldsymbol{x}_0$, (9.5.38) shows that **the Green's function on a finite domain will have the same singularity at source location $\boldsymbol{x} = \boldsymbol{x}_0$ as does the infinite space Green's function**. This can be explained in a somewhat physical manner. The response at a point due to a concentrated source nearby should not depend significantly on any boundaries. This technique represented by (9.5.38) removes the singularity.

9.5.8 GREEN'S FUNCTIONS FOR A SEMI-INFINITE PLANE ($Y > 0$) USING INFINITE SPACE GREEN'S FUNCTIONS: THE METHOD OF IMAGES

The infinite space Green's function can be used to obtain Green's functions for certain semi-infinite problems. Consider Poisson's equation in the two-dimensional semi-infinite region $y > 0$:

$$\boxed{\nabla^2 u = f(\boldsymbol{x}),} \tag{9.5.39}$$

subject to a nonhomogeneous condition (given temperature) on $y = 0$:

$$\boxed{u(x, 0) = h(x).} \tag{9.5.40}$$

The defining problem for the Green's function,

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{9.5.41}$$

satisfies the corresponding homogeneous boundary conditions,

$$G(x, 0; x_0, y_0) = 0, \tag{9.5.42}$$

as illustrated in Fig. 9.5.3. Here, we use the notation $G(\boldsymbol{x}, \boldsymbol{x}_0) = G(x, y; x_0, y_0)$. The semi-infinite space ($y > 0$) has no sources except a concentrated source at $\boldsymbol{x} = \boldsymbol{x}_0$. The infinite space Green's function, $(1/2\pi)\ln|\boldsymbol{x} - \boldsymbol{x}_0|$, is not satisfactory since it will not be zero at $y = 0$.

Figure 9.5.3 Image source for a semi-infinite plane.

Image source

There is a simple way to obtain a solution that is zero at $y = 0$. Consider an infinite space problem (i.e., no boundaries) with source $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ at $\boldsymbol{x} = \boldsymbol{x}_0$, and a negative image source $-\delta(\boldsymbol{x} - \boldsymbol{x}_0^*)$ at $\boldsymbol{x} = \boldsymbol{x}_0^*$ (where $\boldsymbol{x}_0 = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}}$ and $x_0^* = x_0\hat{\mathbf{i}} - y_0\hat{\mathbf{j}}$):

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0) - \delta(\boldsymbol{x} - \boldsymbol{x}_0^*). \tag{9.5.43}$$

According to the principle of superposition for nonhomogeneous problems, the response should be the sum of two individual responses:

$$G = \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0| - \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0^*|. \tag{9.5.44}$$

By symmetry, the response at $y = 0$ due to the source at $\boldsymbol{x} = \boldsymbol{x}_0^*$ should be minus the response at $y = 0$ due to the source at $\boldsymbol{x} = \boldsymbol{x}_0$. Thus, the sum should be zero at $y = 0$ (as we will verify shortly). We call this the **method of images**. In this way, we have obtained the Green's function for Poisson's equation on a semi-infinite space ($y > 0$):

$$\boxed{G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi} \ln \frac{|\boldsymbol{x} - \boldsymbol{x}_0|}{|\boldsymbol{x} - \boldsymbol{x}_0^*|} = \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}.} \tag{9.5.45}$$

Let us check that this is the desired solution. Equation (9.5.43), which is satisfied by (9.5.45), is not (9.5.41). However, in the upper half-plane ($y > 0$), $\delta(\boldsymbol{x} - \boldsymbol{x}_0^*) = 0$, since $\boldsymbol{x} = \boldsymbol{x}_0^*$ is in the lower half-plane. Thus, for $y > 0$, (9.5.41) is satisfied. Furthermore, we now show that at $y = 0$, $G(\boldsymbol{x}, \boldsymbol{x}_0) = 0$:

$$G(\boldsymbol{x}, \boldsymbol{x}_0)|_{y=0} = \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + y_0^2}{(x - x_0)^2 + y_0^2} = \frac{1}{4\pi} \ln 1 = 0.$$

Solution

To solve Poisson's equation with nonhomogeneous boundary conditions, we need the solution's representation in terms of this Green's function, (9.5.45). We again use Green's formula. We need to consider a large semicircle (in the limit as the radius tends to infinity):

$$\iint (u\nabla^2 G - G\nabla^2 u)\, dA = \oint (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, ds = \int_{-\infty}^{\infty} \left(G \frac{\partial u}{\partial y} - u \frac{\partial G}{\partial y} \right) \Big|_{y=0} dx,$$

since for the wall the outward unit normal is $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$ and since the contribution at ∞ tends to vanish if $u \to 0$ sufficiently fast [in particular, from (9.5.35) if $\lim_{r\to\infty}(u - r \ln r\, \partial u/\partial r) = 0$]. Substituting the defining differential equations and interchanging $\boldsymbol{x}$ and $\boldsymbol{x}_0$ [using the symmetry of $G(\boldsymbol{x}, \boldsymbol{x}_0)$] shows that

$$\boxed{u(\boldsymbol{x}) = \iint f(\boldsymbol{x}_0) G(\boldsymbol{x}, \boldsymbol{x}_0)\, dA_0 - \int_{-\infty}^{\infty} h(\boldsymbol{x}_0) \frac{\partial}{\partial y_0} G(\boldsymbol{x}, \boldsymbol{x}_0) \Big|_{y_0=0} dx_0,} \tag{9.5.46}$$

since $G = 0$ on $y = 0$. This can be obtained directly from (9.5.26). $G(\boldsymbol{x}, \boldsymbol{x}_0)$ from (9.5.45) is given by

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{4\pi} \left[\ln \left((x - x_0)^2 + (y - y_0)^2 \right) - \ln \left((x - x_0)^2 + (y + y_0)^2 \right) \right].$$

Thus,

$$\frac{\partial}{\partial y_0} G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{4\pi} \left[\frac{-2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} - \frac{2(y + y_0)}{(x - x_0)^2 + (y + y_0)^2} \right].$$

Evaluating this at $y_0 = 0$ (corresponding to the source point on the boundary) yields

$$\boxed{\frac{\partial}{\partial y_0} G(\boldsymbol{x}, \boldsymbol{x}_0) \Big|_{y_0=0} = -\frac{y/\pi}{(x - x_0)^2 + y^2}.} \tag{9.5.47}$$

This is an example of a dipole source (see Section 9.5.5).

EXAMPLE

Consider Laplace's equation for the two-dimensional semi-infinite space ($y > 0$):

$$\boxed{\begin{aligned} \nabla^2 u &= 0 \\ u(x, 0) &= h(x_0). \end{aligned}} \tag{9.5.48}$$

Equation (9.5.46) can be utilized with a zero source term. Here, the solution is due only to the nonhomogeneous boundary condition. Using the normal derivative of the Green's function, (9.5.47), we obtain

$$\boxed{u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} h(x_0) \frac{1}{(x - x_0)^2 + y^2}\, dx_0.} \tag{9.5.49}$$

The influence function for the boundary condition $h(x)$ is not the Green's function, but

$$-\frac{\partial}{\partial y_0}G(\boldsymbol{x}, \boldsymbol{x}_0)\Big|_{y_0=0} = \frac{y}{\pi[(x - x_0)^2 + y^2]}.$$

This is not only an influence function, but it is the dipole source described in Section 9.5.5, which is an elementary solution of Laplace's equation corresponding to the boundary condition itself being a delta function. In Chapter 10 we will obtain the same answer using Fourier transform techniques rather than using Green's functions. ■

Insulated boundaries

If the boundary condition at $y = 0$ for the Green's function is the insulated kind $\partial/\partial y G(\boldsymbol{x}, \boldsymbol{x}_0)|_{y=0} = 0$, then a positive image source must be used for $y < 0$. In this way, equal sources of thermal energy are located at $\boldsymbol{x} = \boldsymbol{x}_0$ and $\boldsymbol{x} = \boldsymbol{x}_0^*$. By symmetry, no heat will flow across $y = 0$, as desired. The resulting solution is obtained in an exercise.

9.5.9 GREEN'S FUNCTIONS FOR A CIRCLE: THE METHOD OF IMAGES

The Green's function for Poisson's equation for a circle of radius a (with zero boundary conditions),

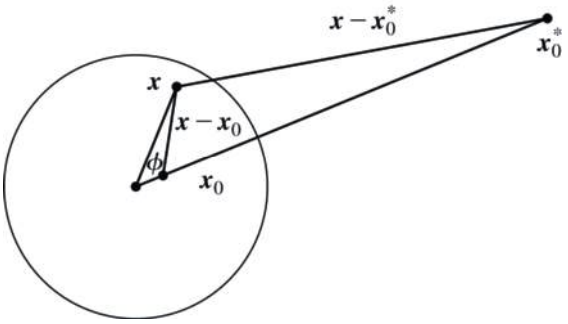
$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0) \tag{9.5.50}$$

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = 0 \quad \text{for} \quad |\boldsymbol{x}| = a, \tag{9.5.51}$$

rather remarkably is obtained using the method of images. The idea is that for geometric reasons there exists an image point $\boldsymbol{x} = \boldsymbol{x}_0^*$ (as sketched in Fig. 9.5.4) such that the response along the circumference of a circle is constant. Consider an infinite space Green's function corresponding to a source at $\boldsymbol{x} = \boldsymbol{x}_0$ and a negative image source at $\boldsymbol{x} = \boldsymbol{x}_0^*$, where we do not define $\boldsymbol{x}_0^*$ yet:

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0) - \delta(\boldsymbol{x} - \boldsymbol{x}_0^*). \tag{9.5.52}$$

Figure 9.5.4 Green's function for Poisson's equation for a circle (image source).



According to the principle of superposition, the solution will be the sum of the two infinite space Green's functions. We also introduce a constant homogeneous solution of Laplace's equation so that

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0| - \frac{1}{2\pi}\ln|\boldsymbol{x} - \boldsymbol{x}_0^*| + c = \frac{1}{4\pi}\ln\frac{|\boldsymbol{x} - \boldsymbol{x}_0|^2}{|\boldsymbol{x} - \boldsymbol{x}_0^*|^2} + c. \tag{9.5.53}$$

We will show that there exists a point $\boldsymbol{x}_0^*$ such that $G(\boldsymbol{x}, \boldsymbol{x}_0)$ given by (9.5.53) vanishes on the circle $|\boldsymbol{x}| = a$. In order for this to occur,

$$|\boldsymbol{x} - \boldsymbol{x}_0|^2 = k|\boldsymbol{x} - \boldsymbol{x}_0^*|^2 \tag{9.5.54}$$

when $|\boldsymbol{x}| = a$ (where $c = -1/4\pi \ln k$).

We show that there is an image point $\boldsymbol{x}_0^*$ along the same radial line as the source point $\boldsymbol{x}_0$ as illustrated in Fig. 9.5.4:

$$\boldsymbol{x}_0^* = \gamma \boldsymbol{x}_0. \tag{9.5.55}$$

We introduce the angle ϕ between $\boldsymbol{x}$ and $\boldsymbol{x}_0$ (the same as the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0^*$). Therefore,

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{x}_0|^2 &= (\boldsymbol{x}-\boldsymbol{x}_0)\cdot(\boldsymbol{x}-\boldsymbol{x}_0) = |\boldsymbol{x}|^2+|\boldsymbol{x}_0|^2-2\boldsymbol{x}\cdot\boldsymbol{x}_0 \\ &= |\boldsymbol{x}|^2+|\boldsymbol{x}_0|^2-2|\boldsymbol{x}||\boldsymbol{x}_0|\cos\phi \\ |\boldsymbol{x}-\boldsymbol{x}_0^*|^2 &= (\boldsymbol{x}-\boldsymbol{x}_0^*)\cdot(\boldsymbol{x}-\boldsymbol{x}_0^*) = |\boldsymbol{x}|^2+|\boldsymbol{x}_0^*|^2-2\boldsymbol{x}\cdot\boldsymbol{x}_0^* \\ &= |\boldsymbol{x}|^2+|\boldsymbol{x}_0^*|^2-2|\boldsymbol{x}||\boldsymbol{x}_0^*|\cos\phi, \end{aligned} \tag{9.5.56}$$

otherwise known as the *law of cosines*. Equation (9.5.54) will be valid on the circle $|\boldsymbol{x}| = a$ [using (9.5.55)] only if

$$a^2+r_0^2-2ar_0\cos\phi = k(a^2+\gamma^2r_0^2-2a\gamma r_0\cos\phi),$$

where $r_0 = |\boldsymbol{x}_0|$. This must hold for all angles ϕ, requiring γ and k to satisfy the following two equations:

$$\begin{aligned} a^2+r_0^2 &= k(a^2+\gamma^2r_0^2) \\ -2ar_0 &= k(-2a\gamma r_0). \end{aligned}$$

We obtain $k = 1/\gamma$, and thus

$$a^2+r_0^2 = \frac{1}{\gamma}a^2+\gamma r_0^2 \quad \text{or} \quad \gamma = \frac{a^2}{r_0^2}.$$

The image point is located at

$$\boldsymbol{x}_0^* = \frac{a^2}{r_0^2}\boldsymbol{x}_0.$$

Note that $|\boldsymbol{x}_0^*| = a^2/r_0$ (the product of the radii of the source and image points is the radius squared of the circle). The closer $\boldsymbol{x}_0$ is to the center of the circle, the farther the image point moves away.

Green's function

From $k = 1/\gamma = r_0^2/a^2$, we obtain the Green's function

$$G(\boldsymbol{x},\boldsymbol{x}_0) = \frac{1}{4\pi}\ln\left(\frac{|\boldsymbol{x}-\boldsymbol{x}_0|^2a^2}{|\boldsymbol{x}-\boldsymbol{x}_0^*|^2r_0^2}\right).$$

Using the law of cosines, (9.5.56),

$$G(\boldsymbol{x},\boldsymbol{x}_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{r^2+r_0^2-2rr_0\cos\phi}{r^2+r_0^{*2}-2rr_0^*\cos\phi}\right),$$

where $r = |\boldsymbol{x}|$, $r_0 = |\boldsymbol{x}_0|$, and $r_0^* = |\boldsymbol{x}_0^*|$. Since $r_0^* = a^2/r_0$,

$$G(\boldsymbol{x},\boldsymbol{x}_0) = \frac{1}{4\pi}\ln\left(\frac{a^2}{r_0^2}\frac{r^2+r_0^2-2rr_0\cos\phi}{r^2+a^4/r_0^2-2ra^2/r_0\cos\phi}\right),$$

or, equivalently,

$$\boxed{G(\boldsymbol{x},\boldsymbol{x}_0) = \frac{1}{4\pi}\ln\left(a^2\frac{r^2+r_0^2-2rr_0\cos\phi}{r^2r_0^2+a^4-2rr_0a^2\cos\phi}\right),} \tag{9.5.57}$$

where ϕ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$ and $r = |\boldsymbol{x}|$ and $r_0 = |\boldsymbol{x}_0|$. In these forms, it can be seen that on the circle $r = a, G(\boldsymbol{x},\boldsymbol{x}_0) = 0$.

Solution

The solution of Poisson's equation is directly represented in terms of the Green's function. In general, from (9.5.26),

$$u(\boldsymbol{x}) = \iint f(\boldsymbol{x}_0)G(\boldsymbol{x},\boldsymbol{x}_0)\,dA_0 + \oint h(\boldsymbol{x}_0)\nabla_{\boldsymbol{x}_0}G(\boldsymbol{x},\boldsymbol{x}_0)\cdot\hat{\mathbf{n}}\,ds_0. \tag{9.5.58}$$

This line integral on the circular boundary can be evaluated. It is best to use polar coordinates ($ds_0 = a\,d\theta_0$), in which case

$$\oint h(\boldsymbol{x}_0)\nabla_{\boldsymbol{x}_0}G(\boldsymbol{x},\boldsymbol{x}_0)\cdot\hat{\mathbf{n}}\,ds_0 = \int_0^{2\pi}h(\theta_0)\frac{\partial}{\partial r_0}G(\boldsymbol{x},\boldsymbol{x}_0)\Big|_{r_0=a}a\;d\theta_0, \tag{9.5.59}$$

where $r_0 = |\boldsymbol{x}_0|$. From (9.5.57),

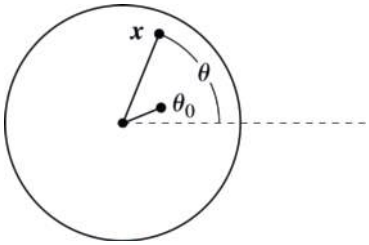
$$\frac{\partial G}{\partial r_0} = \frac{1}{4\pi}\left(\frac{2r_0 - 2r\cos\phi}{r^2 + r_0^2 - 2rr_0\cos\phi} - \frac{2r^2r_0 - 2ra^2\cos\phi}{r^2r_0^2 + a^4 - 2rr_0a^2\cos\phi}\right).$$

Evaluating this for source points on the circle $r_0 = a$ yields

$$\begin{aligned}\left.\frac{\partial G}{\partial r_0}\right|_{|r_0=a|} &= \frac{1}{4\pi}\frac{2a - 2r\cos\phi - (2r^2/a - 2r\cos\phi)}{r^2 + a^2 - 2ar\cos\phi} \\ &= \frac{a}{2\pi}\frac{1 - (r/a)^2}{r^2 + a^2 - 2ar\cos\phi},\end{aligned} \tag{9.5.60}$$

where ϕ is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}_0$. If polar coordinates are used for both $\boldsymbol{x}$ and $\boldsymbol{x}_0$, then $\phi = \theta - \theta_0$, as indicated in Fig. 9.5.5.

Figure 9.5.5 Polar coordinates.



EXAMPLE

For Laplace's equation, $\nabla^2 u = 0$ [i.e., $f(\boldsymbol{x}) = 0$ in Poisson's equation] inside a circle with $u(x, y) = h(\theta)$ at $r = a$, we obtain from (9.5.58)–(9.5.60) in polar coordinates

$$\boxed{u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta_0)\frac{a^2 - r^2}{r^2 + a^2 - 2ar\cos(\theta - \theta_0)}\,d\theta_0,} \tag{9.5.61}$$

known as **Poisson's formula**. Previously, we obtained a solution of Laplace's equation in this situation by the method of separation of variables (see Section 2.5.2). It can be shown that the infinite series solution so obtained can be summed to yield Poisson's formula (see Exercise 9.5.19). ■

Exercises 9.5

9.5.1. Consider (9.5.10), the eigenfunction expansion for $G(\boldsymbol{x}, \boldsymbol{x}_0)$. Assume that $\nabla^2 G$ has some eigenfunction expansion. Using Green's formula, verify that $\nabla^2 G$ may be obtained by term-by-term differentiation of (9.5.10).

9.5.2. **(a)** Solve

$$\nabla^2 u = f(x, y)$$

on a rectangle $(0 < x < L, 0 < y < H)$ with $u = 0$ on the boundary using the method of eigenfunction expansion.

(b) Write the solution in the form

$$u(\boldsymbol{x}) = \int_0^L \int_0^H f(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{x}_0)\,dx_0\,dy_0.$$

Show that this $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function obtained previously.

9.5.3. Using the method of (multidimensional) eigenfunction expansion, determine $G(\boldsymbol{x}, \boldsymbol{x}_0)$ if

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

(a) inside the rectangle $(0 < x < L, 0 < y < H)$ and

$$\text{at } x = 0,\ G = 0; \qquad \text{at } y = 0,\ \frac{\partial G}{\partial y} = 0$$

$$\text{at } x = L,\ \frac{\partial G}{\partial x} = 0; \qquad \text{at } y = H,\ \frac{\partial G}{\partial y} = 0$$

(b) inside the rectangular-shaped box ($0 < x < L, 0 < y < H, 0 < z < W$) with $G = 0$ on the six sides

***(c)** inside the semicircle ($0 < r < a, 0 < \theta < \pi$) with $G = 0$ on the entire boundary

(d) inside the quarter-circle ($0 < r < a, 0 < \theta < \pi/2$) with $G = 0$ on the straight sides and $\partial G/\partial r = 0$ at $r = a$

***9.5.4.** Consider in some three-dimensional region

$$\nabla^2 u = f$$

with $u = h(\boldsymbol{x})$ on the boundary. Represent $u(\boldsymbol{x})$ in terms of the Green's function (assumed to be known).

9.5.5. Consider inside a circle of radius a

$$\nabla^2 u = f$$

with

$$u(a, \theta) = h_1(\theta) \qquad \text{for} \qquad 0 < \theta < \pi$$

$$\frac{\partial u}{\partial r}(a, \theta) = h_2(\theta) \qquad \text{for} \qquad -\pi < \theta < 0.$$

Represent $u(r, \theta)$ in terms of the Green's function (assumed to be known).

9.5.6. Consider $\nabla^2 u = f(\boldsymbol{x})$ in two dimensions, satisfying homogeneous boundary conditions. Suppose that ϕ_h is a homogeneous solution,

$$\nabla^2 \phi_h = 0,$$

satisfying the same homogeneous boundary conditions. There may be more than one function ϕ_h.

(a) Show that there are no solutions $u(\boldsymbol{x})$ if $\iint f(\boldsymbol{x})\phi_h(\boldsymbol{x})\, dA \neq 0$ for any $\phi_h(\boldsymbol{x})$.

(b) Show that there are an infinite number of solutions if $\iint f(\boldsymbol{x})\phi_h(\boldsymbol{x})\, dA = 0$.

9.5.7. Consider in three dimensions

$$\text{with on the boundary } \left.\begin{array}{l} \nabla^2 u = f(\boldsymbol{x}) \\ \nabla u \cdot \hat{\mathbf{n}} = 0. \end{array}\right\} \tag{9.5.62}$$

(a) Show that $\phi_h = 1$ is a homogeneous solution satisfying homogeneous boundary conditions.

(b) Under what condition is there a solution of (9.5.62)?

(c) What problem defines a generalized Green's function for (9.5.62)? (Do *not* attempt to determine a generalized Green's function.)

(d) *Assume* that the generalized Green's function is symmetric. *Derive* a representation formula for $u(\boldsymbol{x})$ in terms of your generalized Green's function.

9.5.8. Redo Exercise 9.5.7 if on the boundary $\nabla u \cdot \hat{\mathbf{n}} = h(\boldsymbol{x})$.

9.5.9. Using the method of one-dimensional eigenfunction expansion, determine $G(\boldsymbol{x}, \boldsymbol{x}_0)$ if

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

(a) inside the rectangle ($0 < x < L, 0 < y < H$) and

$$\text{at } x = 0,\ G = 0; \qquad \text{at } y = 0,\ \frac{\partial G}{\partial y} = 0$$

$$\text{at } x = L,\ \frac{\partial G}{\partial x} = 0; \qquad \text{at } y = H,\ \frac{\partial G}{\partial y} = 0.$$

Use y-dependent eigenfunctions.

***(b)** inside the semicircle ($0 < r < a, 0 < \theta < \pi$) with $G = 0$ on the entire boundary. Use θ-dependent eigenfunctions.

9.5.10. Consider the problem solved in Section 9.5.4, where the Green's function for Poisson's equation

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

is solved inside a rectangle with homogeneous boundary conditions, $G = 0$ at $x = 0$, $x = L$, $y = 0$, and $y = H$. Perform the analysis starting with a Fourier sine series in y.

***9.5.11.** Consider the wave equation with a periodic source of frequency $\omega > 0$

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi + g(\boldsymbol{x}) e^{-i\omega t}.$$

Show that a particular solution at the same frequency, $\phi = u(\boldsymbol{x}) e^{-i\omega t}$, satisfies a nonhomogeneous Helmholtz equation

$$\nabla^2 u + k^2 u = f(\boldsymbol{x}). \qquad [\text{What are } k^2 \text{ and } f(\boldsymbol{x})?]$$

The Green's function satisfies

$$\nabla^2 G + k^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0).$$

(a) What is Green's formula for the operator $\nabla^2 + k^2$?

(b) In infinite three-dimensional space, show that

$$G = \frac{c_1 e^{ik\rho} + c_2 e^{-ik\rho}}{\rho}.$$

Choose c_1 and c_2 so that the corresponding $\phi(\boldsymbol{x}, t)$ is an *outward*-propagating wave. [*Hint*: Make the unmotivated change of variables $G = h/\rho$.]

(c) In infinite two-dimensional space, show that the Green's function is a linear combination of Bessel functions. Determine the constants so that the corresponding $\phi(\boldsymbol{x}, t)$ is an *outward*-propagating wave for r sufficiently large. [*Hint*: See (7.7.33) and (7.8.3).]

9.5.12. **(a)** Determine the Green's function for $y > 0$ (in two dimensions) for $\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$ subject to $\partial G/\partial y = 0$ on $y = 0$. [*Hint*: Consider a positive source at $\boldsymbol{x}_0$ and a positive image source at $\boldsymbol{x}_0^*$.]

(b) Use part (a) to solve $\nabla^2 u = f(\boldsymbol{x})$ with

$$\frac{\partial u}{\partial y} = h(x) \qquad \text{at } y = 0.$$

Ignore the contribution at ∞.

9.5.13. Modify Exercise 9.5.12 if the physical region is $y < 0$.

***9.5.14.** Modify Exercise 9.5.12 if the region is three-dimensional with $y > 0$. [Note that $h(x)$ in part (b) becomes $h(x, z)$.]

***9.5.15.** Using the method of images, solve

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

in the first quadrant ($x \geq 0$ and $y \geq 0$) with $G = 0$ on the boundaries.

9.5.16. **(a)** Reconsider Exercise 9.5.15 if $G = 0$ at $x = 0$ and $\partial G/\partial y = 0$ at $y = 0$.

(b) Use part (a) to solve ($x \geq 0$ and $y \geq 0$)

$$\begin{aligned} \nabla^2 u &= f(x, y) \\ u(0, y) &= g(y) \\ \frac{\partial u}{\partial y}(x, 0) &= h(x). \end{aligned}$$

9.5.17. **(a)** Using the method of images, solve

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

in the 60° wedge-shaped region ($0 < \theta < \pi/3$, $0 < r < \infty$) with $G = 0$ on the boundaries.

(b) For what other angled wedges can the method of images be used?

9.5.18. A generalized Green's function $G_m(\boldsymbol{x}, \boldsymbol{x}_0)$ satisfies

$$\nabla^2 G_m = \delta(\boldsymbol{x} - \boldsymbol{x}_0) + c$$

with

$$\nabla G_m \cdot \hat{\mathbf{n}} = 0$$

on the boundary of the rectangle ($0 < x < L$, $0 < y < H$).

(a) Show that the method of eigenfunction expansion (two-dimensional) works only for $c = -1/LH$. For this c, determine $G_m(\boldsymbol{x}, \boldsymbol{x}_0)$. If possible, make $G_m(\boldsymbol{x}, \boldsymbol{x}_0)$ symmetric.

(b) Show that the method of eigenfunction expansion (one-dimensional) works only for $c = -1/LH$. For this c, determine $G_m(\boldsymbol{x}, \boldsymbol{x}_0)$. If possible, make $G_m(\boldsymbol{x}, \boldsymbol{x}_0)$ symmetric.

9.5.19. Solve $\nabla^2 u = 0$ inside a circle of radius a with $u(x, y) = h(\theta)$ at $r = a$, using the method of separation of variables. Show that

$$u(r, \theta) = \int_0^{2\pi} h(\theta_0) I(r, \theta, \theta_0)\, d\theta_0.$$

Show that the infinite series for $I(r, \theta, \theta_0)$ can be summed yielding Poisson's formula (9.5.61).

***9.5.20.** Determine the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ inside the semicircle $(0 < r < a, 0 < \theta < \pi)$

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

with $G = 0$ on the boundary.

9.5.21. Modify Exercise 9.5.20 if $G = 0$ on $r = a$, but $\partial G/\partial\theta = 0$ on $\theta = 0$ and $\theta = \pi$.

9.5.22. Determine the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ inside the sphere of radius a

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

with $G = 0$ on the boundary.

9.5.23. Use the method of multiple images to obtain the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

(a) inside the rectangle $(0 < x < L, 0 < y < H)$ if $G = 0$ at $x = 0$ and $x = L$ and $G = 0$ at $y = 0$ and $y = H$

(b) inside the infinite strip $(0 < x < L, -\infty < y < \infty)$ if $G = 0$ at $x = 0$ and $\partial G/\partial x = 0$ at $x = L$

***(c)** inside the infinite strip $(0 < x < L, -\infty < y < \infty, -\infty < z < \infty)$ if $G = 0$ at $x = 0$ and $G = 0$ at $x = L$

(d) inside the semi-infinite strip $(0 < x < L, 0 < y < \infty)$ if $G = 0$ along the boundaries

(e) inside the semi-infinite strip $(0 < x < L, -\infty < y < 0)$ if $G = 0$ at $x = 0, G = 0$ at $x = L$, and $\partial G/\partial y = 0$ at $y = 0$

9.5.24. Determine a particular solution of

$$\nabla^2 u = f(\boldsymbol{x})$$

in infinite two-dimensional space if $f(\boldsymbol{x}) = g(r)$, where $r = |\boldsymbol{x}|$:

(a) Use the infinite space Green's function (9.5.31).

(b) Use a Green's function for the ordinary differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = g(r).$$

(c) Compare parts (a) and (b).

9.5.25. Consider in two dimensions $\nabla^2 u = f(\boldsymbol{x})$

(a) with $u = h(\boldsymbol{x})$ on the boundary. How many solutions are there, and how does this depend on $f(\boldsymbol{x})$ and $h(\boldsymbol{x})$? Do not determine $u(x, y)$.

(b) with $\nabla u \cdot \hat{\mathbf{n}} = h(\boldsymbol{x})$ on the boundary. How many solutions are there, and how does this depend on $f(\boldsymbol{x})$ and $h(\boldsymbol{x})$? Do not determine $u(x, y)$.

9.6 Perturbed Eigenvalue Problems

9.6.1 INTRODUCTION

When a small change (called a **perturbation**) is made to a problem that we know how to solve, then the resulting problem may not have a simple exact solution. Here, we will develop an approximate (asymptotic) procedure to analyze perturbed eigenvalue problems. Nonhomogeneous boundary value problems with nontrivial homogeneous solutions will occur, and hence our development in

Section 9.4 of the Fredholm alternative will be helpful. We begin with an elementary mathematical example before considering the more interesting case of a perturbed circular membrane.

9.6.2 MATHEMATICAL EXAMPLE

The simplest example of a perturbed eigenvalue problem is

$$\boxed{\frac{d^2\phi}{dx^2} + (\lambda + \epsilon f(x))\phi = 0} \tag{9.6.1}$$

with

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0. \tag{9.6.2}$$

If $\epsilon = 0$, this is the usual eigenvalue problem $\left[\lambda = (n\pi/L)^2, \phi = \sin n\pi x/L\right]$. If ϵ is a small nonzero parameter, then the coefficient deviates from a constant by a small given amount, $\epsilon f(x)$. It is known that the eigenvalues and eigenfunctions are well-behaved functions of ϵ:

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \cdots \quad \text{and} \quad \phi = \phi_0 + \epsilon\phi_1 + \cdots. \tag{9.6.3}$$

This is called a **perturbation expansion**. By substituting (9.6.3) into (9.6.1), we obtain

$$\frac{d^2}{dx^2}(\phi_0 + \epsilon\phi_1 + \cdots) + [\lambda_0 + \epsilon\lambda_1 + \cdots + \epsilon f(x)](\phi_0 + \epsilon\phi_1 + \cdots) = 0. \tag{9.6.4}$$

Equation (9.6.4) is valid for all ϵ. The terms without ϵ must equal zero (resulting from letting $\epsilon = 0$). Thus,

$$\frac{d^2\phi_0}{dx^2} + \lambda_0\phi_0 = 0. \tag{9.6.5}$$

The boundary conditions [obtained by substituting (9.6.3) into (9.6.2)] are $\phi_0(0) = 0$ and $\phi_0(L) = 0$. Consequently, as expected, the *leading-order* eigenvalues λ_0 and corresponding eigenfunctions ϕ_0 are the same as those of the unperturbed problem ($\epsilon = 0$):

$$\lambda_0 = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad \phi_0 = \sin\frac{n\pi x}{L}, \tag{9.6.6}$$

where $n = 1, 2, 3, \ldots$. A more precise notation would be $\lambda_n^{(0)}$.

The ϵ terms in (9.6.4) must also vanish:

$$\frac{d^2\phi_1}{dx^2} + \lambda_0\phi_1 = -f(x)\phi_0 - \lambda_1\phi_0, \tag{9.6.7}$$

where

$$\phi_1(0) = 0 \quad \text{and} \quad \phi_1(L) = 0 \tag{9.6.8}$$

follow from (9.6.2). This is a nonhomogeneous differential equation with homogeneous boundary conditions. We note that $\phi_0 = \sin(n\pi x/L)$ is a nontrivial homogeneous solution satisfying the homogeneous boundary conditions. Thus, by the Fredholm alternative, there is a solution to (9.6.7)–(9.6.8) only if the right-hand side of (9.6.7) is orthogonal to ϕ_0:

$$0 = \int_0^L f(x)\phi_0^2\,dx + \lambda_1\int_0^L \phi_0^2\,dx. \tag{9.6.9}$$

From (9.6.9) we determine the resulting perturbation of the eigenvalue:

$$\lambda_1 = -\frac{\int_0^L f(x)\phi_0^2\,dx}{\int_0^L \phi_0^2\,dx} = -\frac{2}{L}\int_0^L f(x)\phi_0^2\,dx. \tag{9.6.10}$$

Instead of using the Fredholm alternative, we could have applied the method of eigenfunction expansion to (9.6.7) with (9.6.8). In this latter way, we obtain ϕ_1 as well as λ_1 given by (9.6.10).

9.6.3 VIBRATING NEARLY CIRCULAR MEMBRANE

For a physical problem involving similar ideas, consider the vibrations of a nearly circular membrane with mass impurities. We have already determined (see Section 7.7) the natural frequencies for a circular membrane with constant mass density. We want to know how these frequencies are

changed due to small changes in both the density and geometry. In general, a vibrating membrane satisfies the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{9.6.11}$$

where $c^2 = T/\rho$ may be a function of r and θ. We assume that $u = 0$ on the boundary. By separating variables, $u(r,\theta,t) = \phi(r,\theta)h(t)$, we obtain

$$\frac{d^2 h}{dt^2} = -\lambda h \quad \text{and} \quad \nabla^2 \phi = -\frac{\lambda}{c^2}\phi. \tag{9.6.12}$$

Here, the separation constants λ are such that $\sqrt{\lambda}$ are the natural frequencies of oscillation. We know how to solve this problem (see Section 7.7) if c^2 is constant and the membrane is circular. However, we want to consider the case in which the constant mass density is slightly perturbed (perhaps due to a small imperfection), $\rho = \rho_0 + \epsilon\rho_1(r, \theta)$, where the perturbation of the density $\epsilon\rho_1(r, \theta)$ is given and ϵ is a very small parameter $(0 < |\epsilon| \ll 1)$. Thus

$$\frac{1}{c^2} = \frac{\rho}{T} = \frac{\rho_0 + \epsilon\rho_1(r, \theta)}{T} = \frac{1}{c_0^2} + \epsilon\frac{\rho_1(r, \theta)}{T},$$

where c_0 is the sound speed for a uniform membrane. The perturbed eigenvalue problem is to solve the partial differential equation

$$\boxed{\nabla^2 \phi = -\lambda\left(\frac{1}{c_0^2} + \epsilon\frac{\rho_1(r, \theta)}{T}\right)\phi,} \tag{9.6.13}$$

subject to $\phi = 0$ on the boundary:

$$\boxed{\phi(a + \epsilon g(\theta), \theta) = 0,} \tag{9.6.14}$$

since we express a perturbed circle as $r = a + \epsilon g(\theta)$ with $g(\theta)$ given.

Boundary condition

The boundary condition (9.6.14) is somewhat difficult; we may wish to consider the simple case in which the boundary is circular [$r = a$ or $g(\theta) = 0$]. In general, $\phi = 0$ along a complicated boundary, which is near to the simpler boundary $r = a$. This suggests that we utilize a Taylor series, in which case (9.6.14) may be replaced by

$$\phi + \epsilon g(\theta)\frac{\partial \phi}{\partial r} + \frac{\epsilon^2 g^2(\theta)}{2!}\frac{\partial^2 \phi}{\partial r^2} + \cdots = 0, \tag{9.6.15}$$

evaluated at $r = a$.

Perturbation expansion

To solve (9.6.13) with (9.6.15), we assume the eigenvalues and eigenfunctions depend on the small parameter such that

$$\phi = \phi_0 + \epsilon\phi_1 + \cdots \quad \text{and} \quad \lambda = \lambda_0 + \epsilon\lambda_1 + \cdots. \tag{9.6.16}$$

We substitute (9.6.16) into (9.6.13) and (9.6.15). The terms of order ϵ^0 are

$$\nabla^2 \phi_0 = -\frac{\lambda_0 \phi_0}{c_0^2} \tag{9.6.17}$$

with $\phi_0 = 0$ at $r = a$. Thus, λ_0 are the known unperturbed eigenvalues, and ϕ_0 are the corresponding known eigenfunctions for a circular membrane with uniform density ρ_0 (see Section 7.7). We are most interested in determining λ_1, the leading-order change of each eigenvalue due to the perturbed density and shape. We will determine λ_1 by considering the equations for ϕ_1 obtained by keeping only the ϵ-terms when the perturbation expansions (9.6.16) are substituted into (9.6.13) and (9.6.15):

$$\nabla^2 \phi_1 + \frac{\lambda_0}{c_0^2}\phi_1 = -\frac{\lambda_1}{c_0^2}\phi_0 - \frac{\lambda_0}{T}\rho_1(r,\theta)\phi_0, \tag{9.6.18}$$

subject to the boundary condition (at $r = a$)

$$\phi_1 = -g(\theta)\frac{\partial \phi_0}{\partial r}. \tag{9.6.19}$$

The right-hand side of (9.6.18) contains the known perturbation of the density ρ_1 and the unknown perturbation of each eigenvalue λ_1, whereas the right-hand side of (9.6.19) involves the known perturbation of the shape ($r = a + \epsilon g(\theta)$).

Compatibility condition

Boundary value problem (9.6.18) with (9.6.19) is a nonhomogeneous partial differential equation with nonhomogeneous boundary conditions. Most importantly, there is a nontrivial homogeneous solution, $\phi_{1_h} = \phi_0$; the leading-order eigenfunction satisfies the *corresponding homogeneous* partial differential equation and the *homogeneous* boundary conditions [see (9.6.17)]. Thus, there is a solution to (9.6.18)–(9.6.19) only if the compatibility equation is valid. This is most easily obtained using Green's formula (with $u = \phi_0$ and $v = \phi_1$):

$$\iint [\phi_0 L(\phi_1) - \phi_1 L(\phi_0)]\, dA = \oint (\phi_0 \nabla\phi_1 - \phi_1 \nabla\phi_0) \cdot \hat{\mathbf{n}}\, ds. \tag{9.6.20}$$

The right-hand side of (9.6.20) does not vanish since ϕ_1 does not satisfy homogeneous boundary conditions. Using (9.6.17) and (9.6.18)–(9.6.19) yields the solvability condition

$$-\iint \phi_0 \left(\frac{\lambda_1}{c_0^2}\phi_0 + \frac{\lambda_0}{T}\rho_1(r,\theta)\phi_0 \right) dA = \oint g(\theta) \frac{\partial \phi_0}{\partial r} \nabla\phi_0 \cdot \hat{\mathbf{n}}\, ds. \tag{9.6.21}$$

We easily determine the perturbed eigenvalue λ_1 from (9.6.21):

$$\lambda_1 = \frac{\dfrac{\lambda_0}{T}\displaystyle\iint \rho_1(r,\theta)\phi_0^2 r\, dr\, d\theta + \int_0^{2\pi} g(\theta)\left(\frac{\partial \phi_0}{\partial r}\right)^2 a\, d\theta}{-\dfrac{1}{c_0^2}\displaystyle\iint \phi_0^2 r\, dr\, d\theta}, \tag{9.6.22}$$

using $\nabla\phi_0 \cdot \hat{\mathbf{n}} = \partial\phi_0/\partial r$ (evaluated at $r = a$), $dA = r\, dr\, d\theta$, and $ds = a\, d\theta$. The eigenvalues decrease if the density is increased ($\rho_1 > 0$) or if the membrane is enlarged [$g(\theta) > 0$].

As is elaborated on in the Exercises, this result is valid only if there is one eigenfunction ϕ_0 corresponding to the eigenvalue λ_0. In fact, for a circular membrane, usually there are two eigenfunctions corresponding to each eigenvalue (from $\sin m\theta$ and $\cos m\theta$). Both must be considered.

If we were interested in ϕ_1, it could now be obtained from (9.6.18) using the method of eigenfunction expansion. However, in many applications, it is the perturbed eigenvalues (here frequencies) that are of greater importance.

Fredholm alternative

If $g(\theta) = 0$, then ϕ_0 and ϕ_1 satisfy the same set of homogeneous boundary conditions. Then (9.6.21) is equivalent to the Fredholm alternative; that is, solutions exist to (9.6.18) with (9.6.19) if and only if the right-hand side of (9.6.18) is orthogonal to the homogeneous solution ϕ_0. Equation (9.6.21) shows the appropriate modification for nonhomogeneous boundary conditions.

Exercises 9.6

9.6.1. Consider the perturbed eigenvalue problem (9.6.1). Determine the perturbations of the eigenvalue λ_1 if

(a) $\dfrac{d\phi}{dx}(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$

(b) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$

9.6.2. Reconsider Exercise 9.6.1. Determine the perturbations of the eigenvalues λ_1 and the eigenfunctions ϕ_1 using the method of eigenfunction expansion:

(a) $\dfrac{d\phi}{dx}(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$

(b) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$

(c) $\phi(0) = 0$ and $\phi(L) = 0$

9.6.3. Reconsider Exercise 9.6.1 subject to the periodic boundary conditions $\phi(-L) = \phi(L)$ and $d\phi/dx(-L) = d\phi/dx(L)$. For $n \neq 0$, the eigenvalue problem is **degenerate** if $\epsilon \neq 0$;

that is, there is more than one eigenfunction [$\sin(n\pi x/L)$ and $\cos(n\pi x/L)$] corresponding to the same eigenvalue. *Determine the perturbed eigenvalues* λ_1. Show that the eigenvalue **splits**. This means that if $\epsilon \neq 0$, there is one eigenfunction for each eigenvalue, but as $\epsilon \to 0$, two eigenvalues will approach each other (**coalesce**), yielding eigenvalues with two eigenfunctions. [*Hint*: It is necessary to consider a linear combination of both eigenfunctions ($\epsilon = 0$). For each eigenvalue, *determine the specific combination of these eigenfunctions that is the unique eigenfunction when* $\epsilon \neq 0$.]

9.6.4. Reconsider Exercise 9.6.1 subject to the boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$. Do additional calculations to obtain λ_2. Insist that the eigenfunctions are normalized, $\int_0^L \phi^2\, dx = 1$. This is solved for λ_1 in the text.

9.6.5. Consider the *nonlinearly* perturbed eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = \epsilon\phi^3$$

with $\phi(0) = 0$ and $\phi(L) = 0$. Determine the perturbation of the eigenvalue λ_1. Since the problem is nonlinear, the amplitude is important. Assume $\int_0^L \phi^2\, dx = a^2$. Sketch a as a function of λ.

9.6.6. Consider a vibrating string with approximately uniform tension T and mass density $\rho_0 + \epsilon\rho_1(x)$ subject to fixed boundary conditions. Determine the changes in the natural frequencies induced by the mass variation.

9.6.7. Consider a uniform membrane of fixed shape with known frequencies and known natural modes of vibration. Suppose the mass density is perturbed. Determine how the frequencies are perturbed. You may assume there is only one mode of vibration for each frequency.

9.6.8. For a circular membrane, determine the change in the natural frequencies of the circularly symmetric ($m = 0$) eigenfunctions due to small mass and shape variations.

9.6.9. Consider a circular membrane $r = a$. For noncircularly symmetric eigenfunctions ($m \neq 0$), (9.6.18) is valid with $\phi_0 = c_1\phi_0^{(1)} + c_2\phi_0^{(2)}$, where $\phi_0^{(1)}$ and $\phi_0^{(2)}$ are two mutually orthogonal eigenfunctions corresponding to the same eigenvalue λ_0. Here, c_1 and c_2 are arbitrary constants.

(a) Determine a homogeneous linear system of equations for c_1 and c_2 derived from the fact that ϕ_1 has two homogeneous solutions $\phi_0^{(1)}$ and $\phi_0^{(2)}$. This will be the compatibility condition for (9.6.18) with (9.6.19).

(b) Solve the linear system of part (a) to determine the perturbed frequencies and the corresponding natural modes of vibration.

9.7 Summary

We have calculated a few examples of time-independent Green's functions by some different techniques:

1. Limit of time-dependent problem
2. Variation of parameter (ordinary differential equation only)
3. Eigenfunction expansion of Green's function
4. Direct solution of differential equation defining Green's function
5. Use of infinite space Green's function:
   - **(a)** Removal of singularity
   - **(b)** Method of images

For time-independent problems, perhaps the best techniques are based on infinite space Green's functions. We will find the same to be true for time-dependent problems. In that case, we will need to discuss more techniques to solve partial differential equations in an infinite domain. For that reason in Chapter 10, we investigate solutions of homogeneous partial differential equations on infinite domains using Fourier transforms. Then in Chapter 11 we return to nonhomogeneous problems for time-dependent partial differential equations using Green's functions.

10

Infinite Domain Problems: Fourier Transform Solutions of Partial Differential Equations

10.1 Introduction

Most of the partial differential equations that we have analyzed previously were defined on finite regions (e.g., heat flow in a finite one-dimensional rod or in enclosed two- or three-dimensional regions). The solutions we obtained depended on conditions at these boundaries. In this chapter, we analyze problems that extend indefinitely in at least one direction. Physical problems never are infinite, but by introducing a mathematical model with infinite extent, we are able to determine the behavior of problems in situations in which the influence of actual boundaries is expected to be negligible. We will solve problems with infinite or semi-infinite extent by generalizing the method of separation of variables.

10.2 Heat Equation on an Infinite Domain

We begin by considering heat conduction in one dimension, unimpeded by any boundaries. For the simplest case with constant thermal properties and no sources, the temperature $u(x, t)$ satisfies the heat equation,

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},} \tag{10.2.1}$$

defined for all x, $-\infty < x < \infty$. We impose an initial condition,

$$u(x, 0) = f(x). \tag{10.2.2}$$

We would like to use (10.2.1) to predict the future temperature.

For problems in a finite region, boundary conditions are needed at both ends (usually $x = 0$ and $x = L$). Frequently, problems on an infinite domain ($-\infty < x < \infty$) seem to be posed without any boundary conditions. However, usually there are physical conditions at $\pm\infty$, even if they are not stated as such. In the simplest case, suppose that the initial temperature distribution $f(x)$ approaches 0 as $x \to \pm\infty$. This means that *initially* for all x sufficiently large, the temperature is approximately 0. Physically, *for all time* the temperature approaches 0 as $x \to \pm\infty$:

$$u(-\infty, t) = 0 \quad \text{and} \quad u(\infty, t) = 0.$$

In this way, our problem has homogeneous "boundary" conditions.

SEPARATION OF VARIABLES

Let us seek solutions using the familiar separation of variables technique. We will show that the solution can be expressed in a form that resembles the celebrated Fourier transform, which we will develop more carefully in Section 10.3. We separate variables as before:

$$u(x, t) = \phi(x)h(t), \tag{10.2.3}$$

so that

$$\frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda.$$

This yields the same pair of ordinary differential equations as for a finite geometry,

$$\frac{dh}{dt} = -\lambda k h \tag{10.2.4}$$

$$\frac{d^2\phi}{dx^2} = -\lambda\phi. \tag{10.2.5}$$

Determining the separation constant λ is not difficult, but it is very subtle in this case; doing the obvious would be *wrong*. We would expect that the boundary conditions at $\pm\infty$ for $\phi(x)$ are $\phi(-\infty) = 0$ and $\phi(\infty) = 0$. However, we would quickly see that there are no values of λ for which there are nontrivial solutions of $d^2\phi/dx^2 = -\lambda\phi$ that approach 0 at both $x = \pm\infty$. For example, for $\lambda > 0$, $\phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$, and these solutions do *not* approach 0 as $x \to \pm\infty$. As we will verify later, the *correct boundary condition for the separated spatial part* $\phi(x)$ *at* $x = \pm\infty$ *is different from the boundary condition for* $u(x, t)$ *at* $x = \pm\infty$. Instead, we specify that $\phi(x)$ is just bounded at $x = \pm\infty$, $|\phi(-\infty)| < \infty$, and $|\phi(\infty)| < \infty$. This is rather strange, but later we will show that although $|\phi(\pm\infty)| < \infty$, the eventual solution of the partial differential equation (after superposition) will in fact satisfy $u(\pm\infty, t) = 0$.

Eigenvalue problem

We thus claim that the boundary value problem of interest on an infinite domain is

$$\boxed{\frac{d^2\phi}{dx^2} + \lambda\phi = 0} \tag{10.2.6}$$

$$\boxed{|\phi(\pm\infty)| < \infty.} \tag{10.2.7}$$

Let us determine those values of λ for which both $|\phi(\pm\infty)| < \infty$. If $\lambda < 0$, the solution is a linear combination of exponentially growing and exponentially decaying solutions. It is impossible for both $|\phi(\pm\infty)| < \infty$; there are no negative eigenvalues. However, if $\lambda > 0$, then

$$\phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x.$$

No matter what λ is, the solution remains bounded for all x. Thus, all values of $\lambda > 0$ are eigenvalues. Furthermore, the eigenfunctions are both sines and cosines (since both c_1 and c_2 are arbitrary). We can also verify that $\lambda = 0$ is an eigenvalue whose eigenfunction is a constant and hence bounded. This is very similar to a Fourier series in that both sines and cosines (including a constant) are eigenfunctions. However, in a Fourier series, the eigenvalues were **discrete**, $\lambda = (n\pi/L)^2$, whereas here all non-negative values of λ are allowable. The set of eigenvalues for a problem is sometimes referred to as the **spectrum**. In this case, we have a **continuous spectrum**, $\lambda \geq 0$ (rather than discrete).

Superposition principle

The time-dependent ordinary differential equation is easily solved, $h = ce^{-\lambda kt}$, and thus we obtain the following product solutions:

$$\sin\sqrt{\lambda}x\ e^{-\lambda kt} \text{ and } \cos\sqrt{\lambda}x\ e^{-\lambda kt},$$

for all $\lambda \geq 0$. The principle of superposition suggests that we can form another solution by the most general linear combination of these. Instead of summing over all $\lambda \geq 0$, we integrate:

$$u(x, t) = \int_0^\infty \left[c_1(\lambda)\cos\sqrt{\lambda}x\ e^{-\lambda kt} + c_2(\lambda)\sin\sqrt{\lambda}x\ e^{-\lambda kt}\right] d\lambda,$$

where $c_1(\lambda)$ and $c_2(\lambda)$ are arbitrary functions of λ. This is a generalized principle of superposition. It may be verified by direct computation that the integral satisfies (10.2.1). It is usual to let $\lambda = \omega^2$, so that

$$\boxed{u(x, t) = \int_0^\infty \left[A(\omega)\cos\omega x\ e^{-k\omega^2 t} + B(\omega)\sin\omega x\ e^{-k\omega^2 t}\right] d\omega,} \tag{10.2.8}$$

where $A(\omega)$ and $B(\omega)$ are arbitrary functions.[1] This is analogous to the solution for finite regions (with periodic boundary conditions):

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} + b_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}\right].$$

In order to solve for the arbitrary functions $A(\omega)$ and $B(\omega)$, we must insist that (10.2.8) satisfies the initial condition $u(x,0) = f(x)$:

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega. \tag{10.2.9}$$

In later sections, we will explain that there exist $A(\omega)$ and $B(\omega)$ such that (10.2.9) is valid for most functions $f(x)$. More importantly, we will discover how to determine $A(\omega)$ and $B(\omega)$.

Complex exponentials

The solution (10.2.8) is traditionally written in a more compact form, using Euler's formula to express the solution in terms of complex exponentials,

$$\cos\omega x = \frac{1}{2}(e^{i\omega x} + e^{-i\omega x}), \quad \sin\omega x = \frac{1}{2i}(e^{i\omega x} - e^{-i\omega x}).$$

The solution becomes

$$\begin{aligned} u(x,t) &= \int_0^{\infty}\left[A(\omega)\left(\frac{1}{2}e^{i\omega x} + \frac{1}{2}e^{-i\omega x}\right) + B(\omega)\left(-\frac{i}{2}e^{i\omega x} + \frac{i}{2}e^{-i\omega x}\right)\right] e^{-k\omega^2 t}\, d\omega \\ &= \int_0^{\infty}\left(\frac{1}{2}A(\omega) + \frac{i}{2}B(\omega)\right) e^{-i\omega x}e^{-k\omega^2 t}\, d\omega + \int_0^{\infty}\left(\frac{1}{2}A(\omega) - \frac{i}{2}B(\omega)\right) e^{i\omega x}e^{-k\omega^2 t}\, d\omega. \end{aligned}$$

We introduce the change of variables $\tilde{\omega} = -\omega$ in the second integral,

$$\begin{aligned} u(x,t) = &\int_0^{\infty}\left(\frac{1}{2}A(\omega) + \frac{i}{2}B(\omega)\right) e^{-i\omega x}e^{-k\omega^2 t}\, d\omega \\ &+ \int_{-\infty}^{0}\left(\frac{1}{2}A(-\tilde{\omega}) - \frac{i}{2}B(-\tilde{\omega})\right) e^{-i\tilde{\omega}x}e^{-k\tilde{\omega}^2 t}\, d\tilde{\omega}. \end{aligned}$$

Finally, with an appropriate definition of $c(\omega)$ (see Exercise 10.2.1), this can be written in the form

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}e^{-k\omega^2 t}\, d\omega.} \tag{10.2.10}$$

We have chosen to express the solution in terms of $e^{-i\omega x}$. Note that we could have written the solution in terms of $e^{i\omega x}$ instead. $|\omega|$ is the *wave number*, the number of waves in 2π distance. It is a spatial frequency. Some authors use k for spatial frequency, and reserve ω for a temporal frequency. Since we have used k extensively as the thermal diffusivity, we use ω for the frequency variable, either temporal or spatial.

With the solution $u(x,t)$ in the form (10.2.10), the initial condition $u(x,0) = f(x)$ is satisfied if

$$f(x) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}\, d\omega. \tag{10.2.11}$$

$u(x,t)$ is real if $f(x)$ is real (see Exercises 10.2.1 and 10.2.2). Given an arbitrary real function $f(x)$, how can we determine $c(\omega)$ so that (10.2.11) is true? This question will lead us to the **Fourier transform** (see next section).

Exercises 10.2

***10.2.1.** Determine complex $c(\omega)$ so that (10.2.10) is equivalent to (10.2.9) with real $A(\omega)$ and $B(\omega)$. Show that $c(-\omega) = c^*(\omega)$, where the star denotes the complex conjugate.

10.2.2. If $c(-\omega) = c^*(\omega)$ (see Exercise 10.2.1), show that $u(x,t)$ given by (10.2.1) is real.

[1] To be precise, note that $c_1(\lambda)\, d\lambda = c_1(\omega^2)2\omega\, d\omega = A(\omega)\, d\omega$.

10.3 Fourier Transform Pair

10.3.1 MOTIVATION FROM FOURIER SERIES IDENTITY

In solving boundary value problems on a finite interval ($-L < x < L$, with periodic boundary conditions), we can use the complex form of a Fourier series (see Section 3.6):

$$\boxed{\frac{f(x+) + f(x-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}.} \tag{10.3.1}$$

Here $f(x)$ is represented by a linear combination of all possible sinusoidal functions that are periodic with period $2L$. The complex Fourier coefficients were determined in Section 3.6,

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx.} \tag{10.3.2}$$

The entire region of interest $-L < x < L$ is the domain of integration. We will extend these ideas to functions defined for $-\infty < x < \infty$ and apply it to the heat equation (in the next section).

The **Fourier series identity** follows by substituting (10.3.2) for c_n (and using a dummy integration variable $\bar{x}$ to distinguish it from the spatial position x):

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2L}\int_{-L}^{L} f(\bar{x})e^{in\pi\bar{x}/L}\,d\bar{x}\right]e^{-in\pi x/L}. \tag{10.3.3}$$

For periodic functions, $-L < x < L$, the allowable wave numbers ω (number of waves in 2π distance) are the infinite set of discrete values (see Fig. 10.3.1),

$$\omega = \frac{n\pi}{L} = 2\pi\frac{n}{2L}.$$

The wavelengths are $2L/n$, so that n complete waves fit exactly within the original region of length $2L$. The distance between successive values of the wave number is

$$\Delta\omega = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L};$$

they are equally spaced. From (10.3.3),

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=-\infty}^{\infty}\frac{\Delta\omega}{2\pi}\int_{L}^{L} f(\bar{x})e^{i\omega\bar{x}}\,d\bar{x}\;e^{-i\omega x}. \tag{10.3.4}$$

Figure 10.3.1 Discrete wave numbers.

10.3.2 FOURIER TRANSFORM

We will show that the fundamental Fourier integral identity may be roughly defined as the limit of (10.3.3) or (10.3.4) as $L \to \infty$. In other words, functions defined for $-\infty < x < \infty$ may be thought of in some sense as periodic functions with an infinite period.

The values of ω are the square root of the eigenvalues. As $L \to \infty$, and $\Delta\omega \to 0$ they become closer and closer. The eigenvalues approach a continuum; all possible wave numbers are allowable. The function $f(x)$ should be represented by a "sum" (which we will show becomes an integral) of waves of all possible wavelengths. Equation (10.3.4) represents a sum of rectangles (starting from $\omega = -\infty$ and going to $\omega = +\infty$) of base $\Delta\omega$ and height $(1/2\pi)\left[\int_{-L}^{L} f(\bar{x})e^{i\omega\bar{x}}\,d\bar{x}\right]e^{-i\omega x}$. As $L \to \infty$, this height is not significantly different from

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})e^{i\omega\bar{x}}\,d\bar{x}\;e^{-i\omega x}.$$

Thus, as $L \to \infty$ we expect that the areas of the rectangles approach the Riemann sum. Since $\Delta\omega \to 0$ as $L \to \infty$, (10.3.4) becomes the **Fourier integral identity**:

$$\boxed{\frac{f(x+) + f(x-)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\bar{x})e^{i\omega\bar{x}}\,d\bar{x}\right]e^{-i\omega x}\,d\omega.} \tag{10.3.5}$$

A careful proof of this fundamental identity (see Exercise 10.3.9) closely parallels the somewhat complicated proof for the convergence of a Fourier series.

Fourier transform

We now accept (10.3.5) as fact. We next introduce $F(\omega)$ and *define* it to be the **Fourier transform of $f(x)$**:

$$\boxed{F(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})e^{i\omega\bar{x}}\,d\bar{x}.} \tag{10.3.6}$$

From (10.3.5), it then follows that

$$\boxed{\frac{f(x+) + f(x-)}{2} = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega.} \tag{10.3.7}$$

The multiplicative constant $\frac{1}{2\pi}$ in the definition (10.3.6) of the Fourier transform is somewhat arbitrary. We can put any multiplicative factor in front of the integrals in (10.3.6) and (10.3.7) as long as their product is $\frac{1}{2\pi}$. Some other books choose $\frac{1}{\sqrt{2\pi}}$ in both for symmetry, so that you must be careful in using tables to make sure the definitions are the same.

If $f(x)$ is continuous, then $[f(x+) + f(x-)]/2 = f(x)$. Equation (10.3.7) shows that $f(x)$ is composed of waves $e^{-i\omega x}$ of all[2] wave numbers ω (and all wavelengths); it is known as the **Fourier integral representation of $f(x)$**, or simply the **Fourier integral**. $F(\omega)$, the Fourier transform of $f(x)$, represents the amplitude of the wave with wave number ω; it is analogous to the Fourier coefficients of a Fourier series. It is determined by integrating over the entire infinite domain. Compare this to (10.3.2), where for periodic functions defined for $-L < x < L$, integration occurred only over that finite interval. Similarly, $f(x)$ may be determined from (10.3.7) if the Fourier transform $F(\omega)$ is known. $f(x)$, as determined from (10.3.7), is called the **inverse Fourier transform of $F(\omega)$**.

These relationships, (10.3.6) and (10.3.7), are quite important. They are also known as the **Fourier transform pair**. In (10.3.7), when you integrate over ω (called the **transform variable**), a function of x occurs, whereas in (10.3.6) when you integrate over x, a function of ω results. One integrand contains $e^{-i\omega x}$; the other has $e^{i\omega x}$. It is difficult to remember which is which. It hardly matters, but we must be consistent throughout. We claim that (10.3.6) and (10.3.7) are valid if $f(x)$ satisfies $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$, in which case we say that $f(x)$ is absolutely integrable.[3]

An alternative notation $\mathcal{F}[f(x)]$ is sometimes used for $F(\omega)$, the Fourier transform of $f(x)$, given by (10.3.6). Similarly, the inverse Fourier transform of $F(\omega)$ is given the notation $\mathcal{F}^{-1}[F(\omega)]$.

10.3.3 INVERSE FOURIER TRANSFORM OF A GAUSSIAN

In Section 10.4, in order to complete our solution of the heat equation, we will need the inverse Fourier transform of the "bell-shaped" curve, known as a **Gaussian**, sketched in Fig. 10.3.2. The function $g(x)$, whose Fourier transform is $G(\omega)$,

$$\boxed{G(\omega) = e^{-\beta\omega^2},}$$

[2]Not just wave number $n\pi/L$ as for periodic problem for $-L < x < L$.

[3]If $f(x)$ is piecewise smooth and if $f(x) \to 0$ as $x \to \pm\infty$ *sufficiently* fast, then $f(x)$ is absolutely integrable. However, there are other kinds of functions that are absolutely integrable for which the Fourier transform pair may be used.

is given by

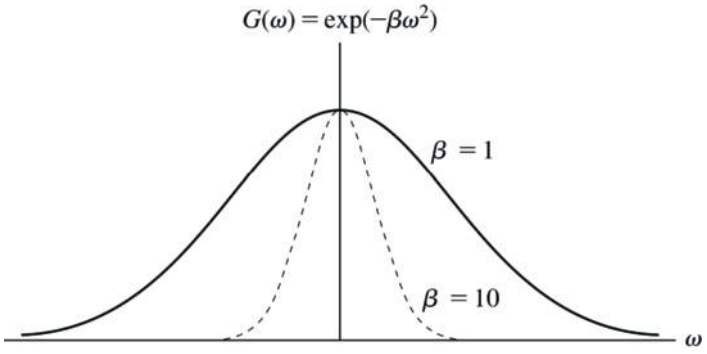
$$g(x) = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega x}\,d\omega = \int_{-\infty}^{\infty} e^{-\beta\omega^2}e^{-i\omega x}\,d\omega, \tag{10.3.8}$$

according to (10.3.7). By evaluating the integral in (10.3.8), we will derive shortly (in the appendix to this section) that

$$\boxed{g(x) = \sqrt{\frac{\pi}{\beta}}e^{-x^2/4\beta}} \tag{10.3.9}$$

if $G(\omega) = e^{-\beta\omega^2}$. As a function of x, $g(x)$ is also bell-shaped. We will have shown the unusual result that **the inverse Fourier transform of a Gaussian is itself a Gaussian**.

Figure 10.3.2 Bell-shaped Gaussian.



This result can be used to obtain the Fourier transform $F(\omega)$ of a Gaussian $e^{-\alpha x^2}$. Due to the linearity of the Fourier transform pair, the Fourier transform of $e^{-x^2/4\beta}$ is $\sqrt{\beta/\pi}e^{-\beta\omega^2}$. Letting $\alpha = 1/4\beta$, the Fourier transform of $e^{-\alpha x^2}$ is $1/\sqrt{4\alpha\pi}e^{-\omega^2/4\alpha}$. Thus, the Fourier transform of a Gaussian is also a Gaussian. We summarize these results in Table 10.3.1. If α is small, then $f(x)$ is a "broadly spread" Gaussian; its Fourier transform is "sharply peaked" near $\omega = 0$. On the other hand, if $f(x)$ is a narrowly peaked Gaussian function corresponding to α being large, its Fourier transform is broadly spread.

TABLE 10.3.1 Fourier Transform of a Gaussian

$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$	$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$
$\sqrt{\frac{\pi}{\beta}}e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$

Appendix to 10.3: Derivation of the Inverse Fourier Transform of a Gaussian

The inverse Fourier transform of a Gaussian $e^{-\beta\omega^2}$ is given by (10.3.8):

$$g(x) = \int_{-\infty}^{\infty} e^{-\beta\omega^2}e^{-i\omega x}\,d\omega.$$

It turns out that $g(x)$ solves an elementary ordinary differential equation, since

$$g'(x) = \int_{-\infty}^{\infty} -i\omega e^{-\beta\omega^2}e^{-i\omega x}\,d\omega$$

can be simplified using an integration by parts:

$$g'(x) = \frac{i}{2\beta}\int_{-\infty}^{\infty}\frac{d}{d\omega}(e^{-\beta\omega^2})e^{-i\omega x}\,d\omega = -\frac{x}{2\beta}\int_{-\infty}^{\infty} e^{-\beta\omega^2}e^{-i\omega x}\,d\omega = -\frac{x}{2\beta}g(x).$$

The solution of the initial value problem for this ordinary differential equation (by separation) is

$$g(x) = g(0)e^{-x^2/4\beta}.$$

Here

$$g(0) = \int_{-\infty}^{\infty} e^{-\beta\omega^2}\, d\omega.$$

The dependence on β in the preceding integral can be determined by the transformation $z = \sqrt{\beta}\omega\,(dz = \sqrt{\beta}\, d\omega)$, in which case

$$g(0) = \frac{1}{\sqrt{\beta}} \int_{-\infty}^{\infty} e^{-z^2}\, dz.$$

This yields the desired result, (10.3.9), since it is well known (but we will show) that

$$I = \int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}. \tag{10.3.10}$$

Perhaps you have not seen (10.3.10) derived. The $\int_{-\infty}^{\infty} e^{-z^2}\, dz$ can be evaluated by a clever procedure. We do not know yet how to evaluate I, but we will show that I^2 is easy. Introducing different dummy integration variables for each I, we obtain

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy.$$

We will evaluate this double integral, although each single integral is unknown. Polar coordinates are suggested:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned} \qquad x^2 + y^2 = r^2, \qquad dx\, dy = r\, dr\, d\theta.$$

The region of integration is the entire two-dimensional plane. Thus,

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} re^{-r^2}\, dr.$$

Both of these integrals are easily evaluated; $I^2 = 2\pi \cdot \frac{1}{2} = \pi$, completing the proof of (10.3.10).

DERIVATION USING COMPLEX VARIABLES

By completing the square, $g(x)$ becomes

$$\begin{aligned} g(x) = \int_{-\infty}^{\infty} e^{-\beta[\omega^2 + i(x/\beta)\omega]}\, d\omega &= \int_{-\infty}^{\infty} e^{-\beta[\omega + i(x/2\beta)]^2} e^{-x^2/4\beta}\, d\omega \\ &= e^{-x^2/4\beta} \int_{-\infty}^{\infty} e^{-\beta[\omega + i(x/2\beta)]^2}\, d\omega. \end{aligned}$$

The change of variables $s = \omega + i(x/2\beta)(ds = d\omega)$ appears to simplify the calculation,

$$\boxed{g(x) = e^{-x^2/4\beta} \int_{-\infty}^{\infty} e^{-\beta s^2}\, ds.} \tag{10.3.11}$$

However, although (10.3.11) is correct, we have not given the correct reasons. Actually, the change of variables $s = \omega + i(x/2\beta)$ introduces complex numbers into the calculation. Since ω is being integrated "along the real axis" from $\omega = -\infty$ to $\omega = +\infty$, the variable s has nonzero imaginary part and does not vary along the real axis as indicated by (10.3.11). Instead,

$$g(x) = e^{-x^2/4\beta} \int_{-\infty + i(x/2\beta)}^{\infty + i(x/2\beta)} e^{-\beta s^2}\, ds. \tag{10.3.12}$$

The full power of the theory of complex variables is necessary to show that (10.3.11) is equivalent to (10.3.12).

This is not the place to attempt to teach complex variables, but a little hint of what is involved may be of interest to many readers. We sketch a complex s-plane in Fig. 10.3.3. To compute integrals from $-\infty$ to $+\infty$, we integrate from a to b (and later consider the limits as $a \to -\infty$ and $b \to +\infty$). Equation (10.3.11) involves integrating along the real axis, while (10.3.12) involves

shifting off the real axis [with s having a constant imaginary part, $\text{Im}(s) = ix/2\beta$]. According to Cauchy's theorem (from complex variables), the closed line integral is zero, $\oint e^{-\beta s^2}\, ds = 0$, since the integrand $e^{-\beta s^2}$ has no "singularities" inside (or on) the contour. Here, we use a rectangular contour, as sketched in Fig. 10.3.3. The closed line integral is composed of four simpler integrals, and hence

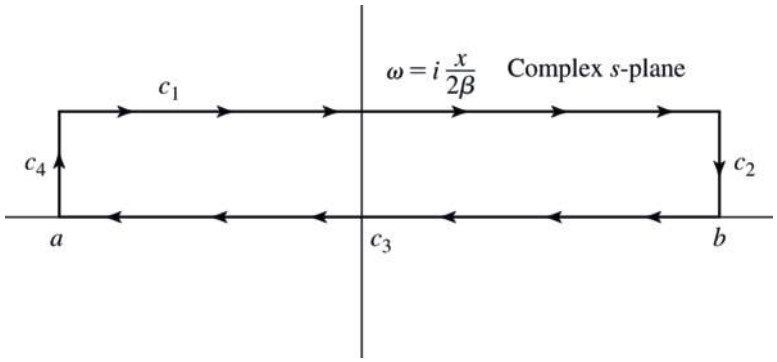
$$\int_{c_1} + \int_{c_2} + \int_{c3} + \int_{c_4} = 0.$$

It can be shown that in the limit as $a \to -\infty$ and $b \to +\infty$, both $\int_{c_2} = 0$ and $\int_{c_4} = 0$, since the integrand is exponentially vanishing on that path (and these paths are finite, of length $x/2\beta$). Thus,

$$\int_{-\infty+i(x/2\beta)}^{\infty+i(x/2\beta)} e^{-\beta s^2}\, ds + \int_{\infty}^{-\infty} e^{-\beta\omega^2}\, d\omega = 0.$$

This verifies that (10.3.11) is equivalent to (10.3.12) (where we use $\int_{\infty}^{-\infty} = -\int_{-\infty}^{\infty}$).

Figure 10.3.3 Closed contour integral in the complex plane.



Exercises 10.3

10.3.1. Show that the Fourier transform is a linear operator; that is, show that

(a) $\mathcal{F}[c_1 f(x) + c_2 g(x)] = c_1 F(\omega) + c_2 G(\omega)$

(b) $\mathcal{F}[f(x)g(x)] \neq F(\omega)G(\omega)$

10.3.2. Show that the inverse Fourier transform is a linear operator; that is, show that

(a) $\mathcal{F}^{-1}[c_1 F(\omega) + c_2 G(\omega)] = c_1 f(x) + c_2 g(x)$

(b) $\mathcal{F}^{-1}[F(\omega)G(\omega)] \neq f(x)g(x)$

10.3.3. Let $F(\omega)$ be the Fourier transform of $f(x)$. Show that if $f(x)$ is real, then $F^*(\omega) = F(-\omega)$, where $*$ denotes the complex conjugate.

10.3.4. Show that

$$\mathcal{F}\left[\int f(x;\alpha)\, d\alpha\right] = \int F(\omega;\alpha)\, d\alpha.$$

10.3.5. If $F(\omega)$ is the Fourier transform of $f(x)$, show that the inverse Fourier transform of $e^{i\omega\beta}F(\omega)$ is $f(x-\beta)$. This result is known as the **shift theorem** for Fourier transforms.

***10.3.6.** If

$$f(x) = \begin{cases} 0 & |x| > a \\ 1 & |x| < a, \end{cases}$$

determine the Fourier transform of $f(x)$. (The answer is given in the table of Fourier transforms in Section 10.4.4.)

***10.3.7.** If $F(\omega) = e^{-|\omega|\alpha}$ $(\alpha > 0)$, determine the inverse Fourier transform of $F(\omega)$. (The answer is given in the table of Fourier transforms in Section 10.4.4.)

10.3.8. If $F(\omega)$ is the Fourier transform of $f(x)$, show that $-idF/d\omega$ is the Fourier transform of $xf(x)$.

10.3.9. **(a)** Multiply (10.3.6) by $e^{-i\omega x}$, and integrate from $-L$ to L to show that

$$\int_{-L}^{L} F(\omega)e^{-i\omega x}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\frac{2\,\sin L(\bar{x}-x)}{\bar{x}-x}\,d\bar{x}. \tag{10.3.13}$$

(b) Derive (10.3.7). For simplicity, assume that $f(x)$ is continuous. [*Hints*: Rewrite $f(\bar{x}) = f(x) + f(\bar{x}) - f(x)$. Use the sine integral, $\int_0^\infty \frac{\sin s}{s}\,ds = \frac{\pi}{2}$. Integrate (10.3.13) by parts and then take the limit as $L \to \infty$.]

10.3.10. Consider the circularly symmetric heat equation on an infinite two-dimensional domain:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$
$$u(0, t) \text{ bounded}$$
$$u(r, 0) = f(r).$$

(a) Solve by separation. It is usual to let

$$u(r, t) = \int_0^\infty A(s)J_0(sr)e^{-s^2kt}s\,ds,$$

in which case the initial condition is satisfied if

$$f(r) = \int_0^\infty A(s)J_0(sr)s\,ds.$$

$A(s)$ is called the **Fourier–Bessel** or **Hankel transform** of $f(r)$.

***(b)** Use Green's formula to evaluate $\int_0^L J_0(sr)J_0(s_1r)r\,dr$. Determine an approximate expression for large L using (7.8.3).

***(c)** Apply the answer of part (b) to part (a) to derive $A(s)$ from $f(r)$. [*Hint*: See Exercise 10.3.9.]

10.3.11. **(a)** If $f(x)$ is a function with unit area, show that the scaled and stretched function $(1/\alpha)f(x/\alpha)$ also has unit area.

(b) If $F(\omega)$ is the Fourier transform of $f(x)$, show that $F(\alpha\omega)$ is the Fourier transform of $(1/\alpha)f(x/\alpha)$.

(c) Show that part (b) implies that broadly spread functions have sharply peaked Fourier transforms near $\omega = 0$, and vice versa.

10.3.12. Show that $\lim_{b\to\infty}\int_b^{b+ix/2\alpha} e^{-\alpha s^2}\,ds = 0$, where $s = b + iy$ $(0 < y < x/2\alpha)$.

10.3.13. Evaluate $I = \int_0^\infty e^{-k\omega^2 t}\cos\omega x\,d\omega$ in the following way: determine $\partial I/\partial x$, and then integrate by parts.

10.3.14. The gamma function $\Gamma(x)$ is defined for $x > 0$ as follows:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

Show that

(a) $\Gamma(1) = 1$

(b) $\Gamma(x + 1) = x\Gamma(x)$

(c) $\Gamma(n + 1) = n!$

(d) $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}$

(e) What is $\Gamma(\frac{3}{2})$?

10.3.15. **(a)** Using the definition of the gamma function in Exercise 10.3.14, show that

$$\Gamma(x) = 2\int_0^\infty u^{2x-1}e^{-u^2}\,du.$$

(b) Using double integrals in polar coordinates, show that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

[*Hint*: It is known from complex variables that

$$2\int_0^{\pi/2} (\tan\theta)^{2z-1}\,d\theta = \frac{\pi}{\sin \pi z}.]$$

***10.3.16.** Evaluate

$$\int_0^\infty y^p e^{-ky^n}\,dy$$

in terms of the gamma function (see Exercise 10.3.14).

10.3.17. From complex variables, it is known that

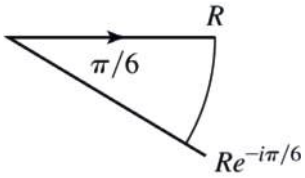
$$\oint e^{-i\omega^3/3}\,d\omega = 0$$

for any closed contour. By considering the limit as $R \to \infty$ of the 30° pie-shaped wedge (of radius R) sketched in Fig. 10.3.4, show that

$$\int_0^\infty \cos\left(\frac{\omega^3}{3}\right) d\omega = \frac{\sqrt{3}}{2}3^{-2/3}\Gamma\left(\frac{1}{3}\right)$$
$$\int_0^\infty \sin\left(\frac{\omega^3}{3}\right) d\omega = \frac{1}{2}3^{-2/3}\Gamma\left(\frac{1}{3}\right).$$

Exercise 10.3.16 may be helpful.

Figure 10.3.4



10.3.18. **(a)** For what α does $\alpha e^{-\beta(x-x_0)^2}$ have unit area for $-\infty < x < \infty$?

(b) Show that the limit as $\beta \to \infty$ of the resulting function in part (a) satisfies the properties of the Dirac delta function $\delta(x - x_0)$.

(c) Obtain the Fourier transform of $\delta(x - x_0)$ in two ways:

1. Take the transform of part (a) and take the limit as $\beta \to \infty$.

2. Use integration properties of the Dirac delta function.

(d) Show that the transform of $\delta(x - x_0)$ is consistent with the following idea: "Transforms of sharply peaked functions are spread out (contain a significant amount of many frequencies)."

(e) Show that the Fourier transform representation of the Dirac delta function is

$$\delta(x - x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(x-x_0)}\,d\omega. \tag{10.3.14}$$

Why is this not mathematically precise? However, what happens if $x = x_0$? Similarly,

$$\delta(\omega - \omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix(\omega-\omega_0)}\,dx. \tag{10.3.15}$$

(f) Equation (10.3.15) may be interpreted as an orthogonality relation for the eigenfunctions $e^{-i\omega x}$. If

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x}\, d\omega,$$

determine the "Fourier coefficient (transform)" $F(\omega)$ using the orthogonality condition (10.3.15).

10.4 Fourier Transform and the Heat Equation

10.4.1 HEAT EQUATION

In this subsection, we begin to illustrate how to use Fourier transforms to solve the heat equation on an infinite interval. Earlier we showed that $e^{-i\omega x}e^{-k\omega^2 t}$ for all ω solves the heat equation, $\partial u/\partial t = k\partial^2 u/\partial x^2$. A generalized principle of superposition showed that the heat equation is solved by

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t}\, d\omega.} \tag{10.4.1}$$

The initial condition $u(x, 0) = f(x)$ is satisfied if

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x}\, d\omega. \tag{10.4.2}$$

From the definition of the Fourier transform, we observe that (10.4.2) is a Fourier integral representation of $f(x)$. Thus, $c(\omega)$ is the Fourier transform of the initial temperature distribution $f(x)$:

$$\boxed{c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx.} \tag{10.4.3}$$

Equations (10.4.1) and (10.4.3) describe the solution of our initial value problem for the heat equation.[4]

In this form, this solution is too complicated to be of frequent use. We therefore describe a simplification. We substitute $c(\omega)$ into the solution, recalling that the x in (10.4.3) is a dummy variable (and hence we introduce $\bar{x}$):

$$u(x,t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega\bar{x}}\, d\bar{x}\right] e^{-i\omega x}\, e^{-k\omega^2 t}\, d\omega.$$

Instead of doing the $\bar{x}$ integration first, we interchange the orders:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) \left[\int_{-\infty}^{\infty} e^{-k\omega^2 t}\, e^{-i\omega(x-\bar{x})}\, d\omega\right] d\bar{x}. \tag{10.4.4}$$

Equation (10.4.4) shows the importance of $g(x)$, the inverse Fourier transform of $e^{-k\omega^2 t}$:

$$g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t}\, e^{-i\omega x}\, d\omega. \tag{10.4.5}$$

Thus, the integrand of (10.4.4) contains $g(x - \bar{x})$, not $g(x)$.

Influence function

We need to determine the function $g(x)$ whose Fourier transform is $e^{-k\omega^2 t}$ [and then make it a function of $x - \bar{x}$, $g(x - \bar{x})$]. $e^{-k\omega^2 t}$ is a Gaussian. From the previous section (or most tables of Fourier transforms; see Section 10.4.4), letting $\beta = kt$, we obtain the Gaussian $g(x) = \sqrt{\pi/kt}\, e^{-x^2/4kt}$, and thus the solution of the heat equation is

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt}\, d\bar{x}.} \tag{10.4.6}$$

[4] In particular, in Exercise 10.4.2, we show that $u \to 0$ as $x \to \infty$, even though $e^{-i\omega x} \not\to 0$ as $x \to \infty$.

This form clearly shows the solution's dependence on the entire initial temperature distribution, $u(x, 0) = f(x)$. Each initial temperature "influences" the temperature at time t. We define

$$\boxed{G(x, t; \bar{x}, 0) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt}} \tag{10.4.7}$$

and call it the **influence function**. Its relationship to an infinite space Green's function for the heat equation will be explained in Chapter 11. Equation (10.4.7) measures in some sense the effect of the initial ($\bar{t} = 0$) temperature at position $\bar{x}$ on the temperature at time t at location x. As $t \to 0$, the influence function becomes more and more concentrated. In fact, in Exercise 10.3.18, it is shown that

$$\lim_{t \to 0+} \frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt} = \delta(x - \bar{x})$$

(the Dirac delta function of Chapter 9), thus verifying that (10.4.6) satisfies the initial conditions.

The solution (10.4.6) of the heat equation on an infinite domain was derived in a complicated fashion using Fourier transforms. We required that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, a restriction on the initial temperature distribution, in order for the Fourier transform to exist. However, the final form of the solution does not even refer to Fourier transforms. Thus, we never need to calculate any Fourier transforms to utilize (10.4.6). In fact, we claim that the restriction $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ on (10.4.6) is not necessary. Equation (10.4.6) is valid (although the derivation we gave is not), roughly speaking, whenever the integral in (10.4.6) converges.

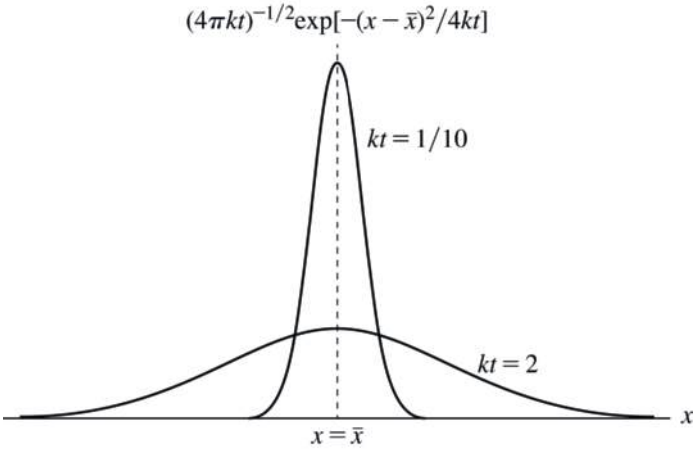
Fundamental solution of the heat equation

We consider solving the heat equation subject to an initial condition concentrated at $x = 0$, $u(x, 0) = f(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function with properties described in Section 9.3.4. According to (10.4.6), the solution of the heat equation with this initial condition is

$$\boxed{u(x, t) = \int_{-\infty}^{\infty} \delta(\bar{x}) \frac{1}{\sqrt{4\pi kt}} e^{-(x-\bar{x})^2/4kt}\, d\bar{x} = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt},} \tag{10.4.8}$$

using the basic property of the Dirac delta function. This is one of the most elementary solutions of the heat equation on an infinite domain, and it is called the **fundamental solution**. It is sketched in Fig. 10.4.1. The fundamental solution of the heat equation is the same as the infinite space Green's function for the heat equation, which will be explained in Chapter 11.

Figure 10.4.1 Fundamental solution for the heat equation.



EXAMPLE

To investigate how discontinuous initial conditions propagate, we consider the following interesting initial condition:

$$u(x, 0) = f(x) = \begin{cases} 0 & x < 0 \\ 100 & x > 0. \end{cases}$$

We thus ask how the thermal energy, initially uniformly concentrated in the right half of a rod, diffuses into the entire rod. According to (10.4.6),

$$u(x, t) = \frac{100}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x-\bar{x})^2/4kt}\, d\bar{x} = \frac{100}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^\infty e^{-z^2}\, dz,$$

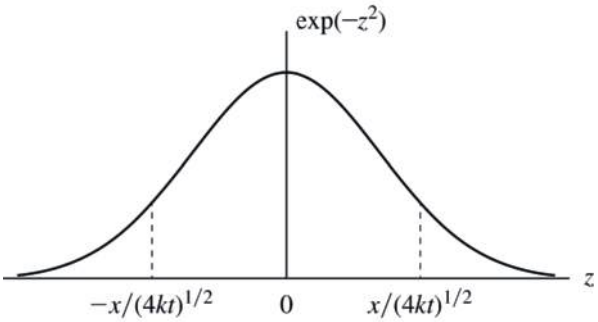
where the integral has been simplified by introducing the change of variables, $z = (\bar{x} - x)/\sqrt{4kt}$ $\left(dz = d\bar{x}/\sqrt{4kt}\right)$. The integrand no longer depends on any parameters. The integral represents the area under a Gaussian (or normal) curve, as illustrated in Fig. 10.4.2. Due to the evenness of e^{-z^2},

$$\int_{-x/\sqrt{4kt}}^\infty e^{-z^2}\, dz = \int_0^\infty e^{-z^2}\, dz + \int_0^{x/\sqrt{4kt}} e^{-z^2}\, dz.$$

Since, as shown earlier, $\int_0^\infty e^{-z^2}\, dz = \sqrt{\pi}/2$,

$$u(x, t) = 50 + \frac{100}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-z^2}\, dz. \tag{10.4.9}$$

Figure 10.4.2 Area under a Gaussian.



The temperature is constant along any curve $x/\sqrt{4kt} = \text{constant}$. These curves are parabolas in the x-t plane (see Fig. 10.4.3). $x/\sqrt{4kt}$ is called the **similarity variable**. For example, the distance between 60° and 75° increases proportionally to $\sqrt{t}$. The temperature distribution spreads out, a phenomenon known as **diffusion**. The temperature distribution given by (10.4.9) is sketched in Fig. 10.4.4 for various fixed values of t.

Figure 10.4.3 Curves of constant temperatures.

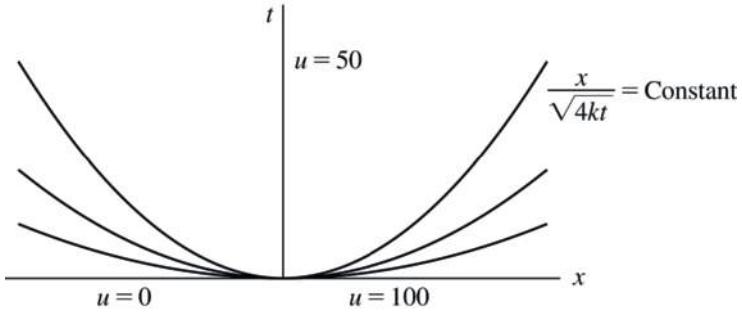
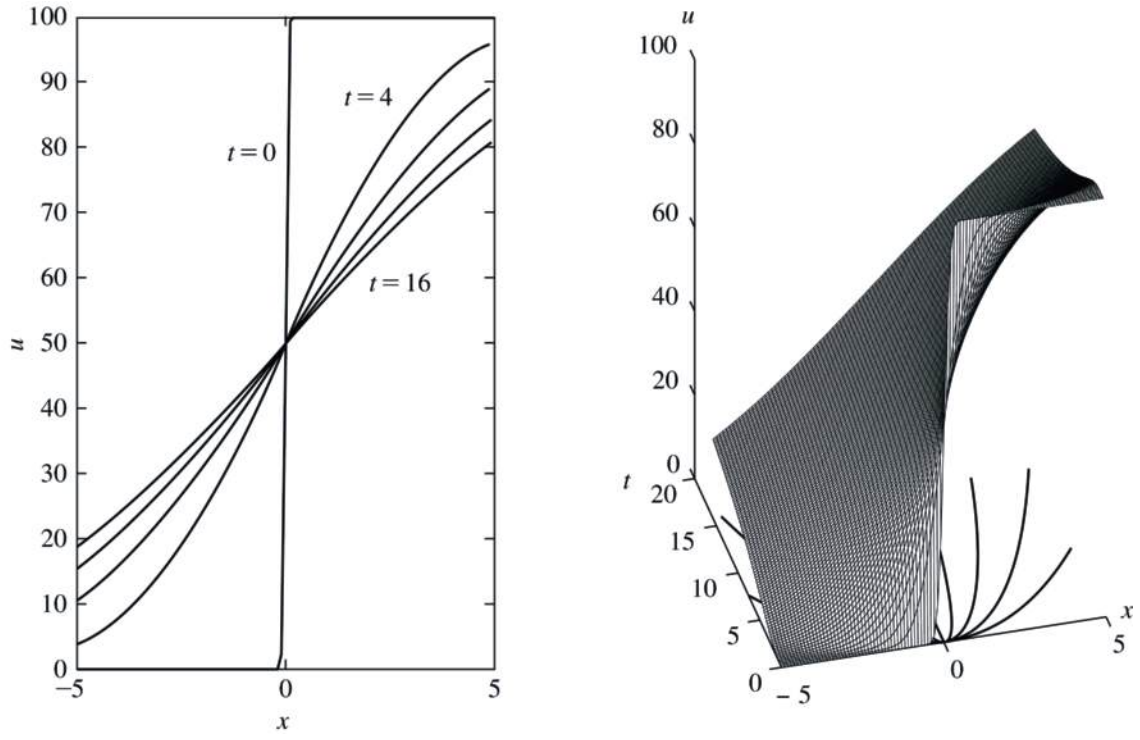


Figure 10.4.4 Temperature diffusion in an infinite rod.



MATLAB Files

Download Animation of Figure 10.4.4 (.zip)

https://bit.ly/3U8yilG

These files are provided as zipped .m files for use in MATLAB on desktop platform.

We note that the temperature is nonzero at all x for any positive t ($t > 0$) even though $u = 0$ for $x < 0$ at $t = 0$. The **thermal energy spreads at an infinite propagation speed**. This is a fundamental property of the diffusion equation. It contrasts with the finite propagation speed of the wave equation, described in Chapter 12 (see also Section 10.6.1).

The area under the normal curve is well tabulated. We can express our solution in terms of the **error function**, $\operatorname{erf} z = \left(2/\sqrt{\pi}\right)\int_0^z e^{-t^2}\,dt$, or the **complementary error function**, $\operatorname{erfc} z = \left(2/\sqrt{\pi}\right)\int_z^\infty e^{-t^2}\,dt = 1 - \operatorname{erf} z$. Using these functions, the solution satisfying

$$u(x, 0) = \begin{cases} 0 & x < 0 \\ 100 & x > 0 \end{cases}$$

is

$$\boxed{u(x, t) = 50\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right] = 50\left[2 - \operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right)\right].}$$

■

Similarity solution

We seek very special solutions of the diffusion equation, $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, with the property that the solutions remain the same under the elementary spatial scaling $x = Lx'$. The partial differential equation will remain the same only if time is scaled, $t = L^2 t'$. For the solution to be the same in both scalings,

$$u(x, t) = f\left(x/t^{1/2}\right),$$

and $\xi = x/t^{1/2}$ is called the **similarity variable**. Since $\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{x}{t^{3/2}}f'(\xi)$ and $\frac{\partial^2 u}{\partial x^2} = \frac{1}{t}f''(\xi)$, it follows that $f(\xi)$ solves the following linear ordinary differential equation:

$$-\frac{1}{2}\xi f' = kf''.$$

This is a first-order equation for f', whose general solution (following from separation) is

$$f' = c_1 e^{-\frac{\xi^2}{4k}}.$$

Integrating yields a similarity solution of the diffusion equation

$$u(x,t) = f(x/t^{1/2}) = c_2 + c_1 \int_0^{x/t^{1/2}} e^{-\frac{s^2}{4k}} ds = c_2 + c_3 \int_0^{x/\sqrt{4kt}} e^{-z^2} dz,$$

where the dimensionless form $\left(s = \sqrt{4k}z\right)$ is better. This can also be derived by dimensional analysis. These **self-similar** solutions must have very special self-similar initial conditions, which have a step at $x = 0$, so that these solutions correspond precisely to (10.4.9). The fundamental solution (10.4.7) could be obtained by instead assuming $u = t^{-\frac{1}{2}} g(\xi)$, which can be shown to be correct for the Dirac delta function initial condition. There are other solutions that can be obtained in related ways, but we restrict our attention to these elementary results.

10.4.2 FOURIER TRANSFORMING THE HEAT EQUATION: TRANSFORMS OF DERIVATIVES

We have solved the heat equation on an infinite interval:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty \\ u(x,0) &= f(x). \end{aligned} \tag{10.4.10}$$

Using separation of variables, we motivated the introduction of Fourier transforms. Here, we describe this simpler method; we Fourier transform the entire problem in terms of the spatial variable. From the heat equation, (10.4.10), the Fourier transform of $\partial u/\partial t$ equals k times the Fourier transform of $\partial^2 u/\partial x^2$:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]. \tag{10.4.11}$$

Thus, we need to calculate Fourier transforms of derivatives of $u(x,t)$. We begin by defining the spatial Fourier transform of $u(x,t)$,

$$\mathcal{F}[u] = \bar{U}(\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t) e^{i\omega x} dx. \tag{10.4.12}$$

Note that it is also a function of time; it is an ordinary Fourier transform with t fixed. To obtain a Fourier transform (in space), we multiply by $e^{i\omega x}$ and integrate. Spatial Fourier transforms of time derivatives are not difficult:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\omega x} dx = \frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t) e^{i\omega x} dx\right] = \frac{\partial}{\partial t}\bar{U}(\omega,t). \tag{10.4.13}$$

The spatial Fourier transform of a time derivative equals the time derivative of the spatial Fourier transform.

More interesting (and useful) results occur for the spatial Fourier transform of spatial derivatives:

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\omega x} dx = \frac{u e^{i\omega x}}{2\pi}\Big|_{-\infty}^{\infty} - \frac{i\omega}{2\pi}\int_{-\infty}^{\infty} u(x,t) e^{i\omega x} dx, \tag{10.4.14}$$

which has been simplified using integration by parts:

$$\begin{aligned} df &= \frac{\partial u}{\partial x} dx, \quad g = e^{i\omega x} \\ f &= u, \qquad\quad dg = i\omega e^{i\omega x} dx. \end{aligned}$$

If $u \to 0$ as $x \to \pm\infty$, then the endpoint contributions of integration by parts vanish. Thus, (10.4.14) becomes

$$\boxed{\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -i\omega\mathcal{F}[u] = -i\omega\bar{U}(\omega,t).} \tag{10.4.15}$$

In a similar manner, Fourier transforms of higher derivatives may be obtained:

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -i\omega\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = (-i\omega)^2 \bar{U}(\omega,t). \tag{10.4.16}$$

The Fourier transform of the nth derivative of a function with respect to x equals $(-i\omega)^n$ times the Fourier transform of the function, assuming that $u(x, t) \to 0$ sufficiently fast as $x \to \pm\infty$.[5]

Using (10.4.13) and (10.4.16), (10.4.11) becomes

$$\frac{\partial}{\partial t}\bar{U}(\omega, t) = k(-i\omega)^2\bar{U}(\omega, t) = -k\omega^2\bar{U}(\omega, t). \tag{10.4.17}$$

The Fourier transform operation converts a linear partial differential equation with constant coefficients into an ordinary differential equation, since spatial derivatives are transformed into algebraic multiples of the transform.

Equation (10.4.17) is a first-order constant-coefficient differential equation. Its general solution is

$$\bar{U}(\omega, t) = c\, e^{-k\omega^2 t}.$$

However, $\partial/\partial t$ is an ordinary derivative *keeping ω fixed,* and thus c is constant if ω is fixed. For other fixed values of ω, c may be a different constant; c depends on ω. c is actually *an arbitrary function of ω, $c(\omega)$.* Indeed, you may easily verify by substitution that

$$\boxed{\bar{U}(\omega, t) = c(\omega)e^{-k\omega^2 t}} \tag{10.4.18}$$

solves (10.4.17). To determine $c(\omega)$, we note from (10.4.18) that $c(\omega)$ equals the initial value of the transform [obtained by transforming the initial condition, $u(x, 0) = f(x)$] $c(\omega) = 1/2\pi \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$:

$$\bar{U}(\omega, t) = F(\omega)e^{-k\omega^2 t}.$$

This is the same result as obtained by separation of variables. We could reproduce the entire solution obtained earlier. Instead, we will show a simpler way to obtain those results.

10.4.3 CONVOLUTION THEOREM

We observe that $\bar{U}(\omega, t)$ is the product of two functions of ω, $c(\omega)$ and $e^{-k\omega^2 t}$, *both transforms of other functions*; $c(\omega)$ is the transform of the initial condition $f(x)$, and $e^{-k\omega^2 t}$ is the transform of some function (fortunately, since $e^{-k\omega^2 t}$ is a Gaussian, we know that it is the transform of another Gaussian, $\sqrt{\pi/kt}\; e^{-x^2/4kt}$). The mathematical problem of inverting a transform that is a product of transforms of known functions occurs very frequently (especially in using the Fourier transform to solve partial differential equations). Thus, we study this problem in some generality.

Suppose that $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively:

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx, \quad G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{i\omega x}\,dx \\ f(x) &= \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega, \qquad g(x) = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega x}\,d\omega. \end{aligned} \tag{10.4.19}$$

We will determine the function $h(x)$ whose Fourier transform $H(\omega)$ equals the product of the two transforms:

$$\boxed{H(\omega) = F(\omega)G(\omega)} \tag{10.4.20}$$

$$h(x) = \int_{-\infty}^{\infty} H(\omega)e^{-i\omega x}\,d\omega = \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{-i\omega x}\,d\omega. \tag{10.4.21}$$

[5]We also need higher derivatives to vanish as $x \to \pm\infty$. Furthermore, each integration by parts is not valid unless the appropriate function is continuous.

We eliminate either $F(\omega)$ or $G(\omega)$ using (10.4.19); it does not matter which:

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} g(\bar{x})e^{i\omega\bar{x}}\,d\bar{x}\right]e^{-i\omega x}\,d\omega.$$

Assuming that we can interchange orders of integration, we obtain

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})\left[\int_{-\infty}^{\infty} F(\omega)e^{-i\omega(x-\bar{x})}\,d\omega\right]d\bar{x}.$$

We now recognize the inner integral as $f(x-\bar{x})$ [see (10.4.19)], and thus

$$\boxed{h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})f(x-\bar{x})\,d\bar{x}.} \tag{10.4.22}$$

The integral in (10.4.22) is called the **convolution** of $g(x)$ and $f(x)$; it is sometimes denoted $g * f$. The **inverse Fourier transform of the product of two Fourier transforms is $1/2\pi$ times the convolution of the two functions**.

If we let $x-\bar{x} = w$ $\left(d\bar{x} = -dw \text{ but } \int_{\infty}^{-\infty} = -\int_{-\infty}^{\infty}\right)$, we obtain an alternative form,

$$\boxed{h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(w)g(x-w)\,dw,} \tag{10.4.23}$$

which would be denoted $f * g$. Thus, $g * f = f * g$.

Heat equation

We now apply the convolution theorem to our partial differential equation. The transform $\bar{U}(\omega, t)$ of the solution $u(x, t)$ is the product of $c(\omega)$ and $e^{-k\omega^2 t}$, where $c(\omega)$ is the transform of the initial temperature distribution and $e^{-k\omega^2 t}$ is the transform of $\sqrt{\pi/kt}\,e^{-x^2/4kt}$. Thus, according to the convolution theorem,

$$\boxed{u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\sqrt{\frac{\pi}{kt}}e^{-(x-\bar{x})^2/4kt}\,d\bar{x}.}$$

This is the same result as obtained (and discussed) earlier. In summary, the procedure is as follows:

1. Fourier transform the partial differential equation in one of the variables.
2. Solve the ordinary differential equation.
3. Apply the initial conditions, determining the initial Fourier transform.
4. Use the convolution theorem.

By using the convolution theorem, we avoid for each problem substituting the inverse Fourier transform and interchanging the order of integration.

Parseval's identity

Since $h(x)$ is the inverse Fourier transform of $F(\omega)G(\omega)$, the convolution theorem can be stated in the following form:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})f(x-\bar{x})\,d\bar{x} = \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{-i\omega x}\,d\omega. \tag{10.4.24}$$

Equation (10.4.24) is valid for all x. In particular, at $x = 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})f(-\bar{x})\,d\bar{x} = \int_{-\infty}^{\infty} F(\omega)G(\omega)\,d\omega. \tag{10.4.25}$$

An interesting result occurs if we pick $g(x)$ such that

$$\boxed{g^*(x) = f(-x)} \tag{10.4.26}$$

Here * is the complex conjugate. [For real functions, $g(x)$ is the reflection of $f(x)$ around $x = 0$.] In general, their Fourier transforms are related:

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(-s)e^{-i\omega s}\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g^*(x)e^{-i\omega x}\,dx = G^*(\omega), \end{aligned} \tag{10.4.27}$$

where we let $s = -x$. Thus, (10.4.25) becomes **Parseval's identity**:

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)g^*(x)\,dx = \int_{-\infty}^{\infty} G(\omega)G^*(\omega)\,d\omega,} \tag{10.4.28}$$

where $g(x)g^*(x) = |g(x)|^2$ and $G(\omega)G^*(\omega) = |G(\omega)|^2$. We showed a similar relationship for all generalized Fourier series (see Section 5.10). This result, (10.4.28), is given the following interpretation. Often energy per unit distance is proportional to $|g(x)|^2$, and thus, $1/2\pi\int_{-\infty}^{\infty}|g(x)|^2\,dx$ represents the total energy. From (10.4.28), $|G(\omega)|^2$ may be defined as the energy per unit wave number (the **spectral energy density**). All the energy is contained within all the wave numbers. **The Fourier transform $G(\omega)$ of a function $g(x)$ is a complex quantity whose magnitude squared is the spectral energy density (the amount of energy per unit wave number)**.

10.4.4 SUMMARY OF PROPERTIES OF THE FOURIER TRANSFORM

Tables of Fourier transforms exist and can be very helpful. The results we have obtained are summarized in Table 10.4.1. There are, of course, many more examples of functions with known Fourier transforms. A nice online tool to explore these is WolframAlpha (wolframalpha.com). We list below some classic textbooks that contain vast tables of Fourier transforms. Beware of various different notations.

F. Oberhettinger, *Tables of Fourier Transforms and Fourier Transforms of Distributions*, Springer-Verlag, New York, 1990.

R.V. Churchill, *Operational Mathematics,* 3rd ed., McGraw-Hill, New York, 1972.

G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications,* Van Nostrand, Princeton, NJ, 1948.

TABLE 10.4.1 Fourier Transform

$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$	$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$	Reference
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Gaussian (Section 10.3.3)
$\sqrt{\frac{\pi}{\beta}}e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$	
$\frac{\partial f}{\partial t}$	$\frac{\partial F}{\partial t}$	Derivatives (Section 10.4.2)
$\frac{\partial f}{\partial x}$	$-i\omega F(\omega)$	
$\frac{\partial^2 f}{\partial x^2}$	$(-i\omega)^2 F(\omega)$	
$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})g(x-\bar{x})d\bar{x}$	$F(\omega)G(\omega)$	Convolution (Section 10.4.3)
$\delta(x-x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	Dirac delta function (Exercise 10.3.18)
$f(x-\beta)$	$e^{i\omega\beta}F(\omega)$	Shifting theorem (Exercise 10.3.5)
$xf(x)$	$-i\frac{dF}{d\omega}$	Multiplication by x (Exercise 10.3.8)
$\frac{2\alpha}{x^2+\alpha^2}$	$e^{-\lvert\omega\rvert\alpha}$	Exercise 10.3.7
$f(x) = \begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$	$\frac{1}{\pi}\frac{\sin a\omega}{\omega}$	Exercise 10.3.6

Exercises 10.4

10.4.1. Using Green's formula, show that

$$\mathcal{F}\left[\frac{d^2f}{dx^2}\right] = -\omega^2 F(\omega) + \frac{e^{i\omega x}}{2\pi}\left(\frac{df}{dx} - i\omega f\right)\Big|_{-\infty}^{\infty}.$$

10.4.2. For the heat equation, $u(x, t)$ is given by (10.4.1). Show that $u \to 0$ as $x \to \infty$ even though $\phi(x) = e^{-i\omega x}$ does not decay as $x \to \infty$. [*Hint*: Integrate by parts.]

10.4.3. ***(a)** Solve the diffusion equation with **convection**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}, \qquad -\infty < x < \infty$$
$$u(x, 0) = f(x).$$

[*Hint*: Use the convolution theorem and the shift theorem (see Exercise 10.3.5).]

(b) Consider the initial condition to be $\delta(x)$. Sketch the corresponding $u(x, t)$ for various values of $t > 0$. Comment on the significance of the convection term $c\,\partial u/\partial x$.

10.4.4. **(a)** Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \gamma u, \qquad -\infty < x < \infty$$
$$u(x, 0) = f(x).$$

(b) Does your solution suggest a simplifying transformation?

10.4.5. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t), \qquad -\infty < x < \infty$$
$$u(x, 0) = f(x).$$

(a) Show that a particular solution for the Fourier transform $\bar{U}$ is

$$\bar{U} = e^{-k\omega^2 t}\int_0^t \bar{Q}(\omega, \tau)e^{k\omega^2\tau}\,d\tau.$$

(b) Determine $\bar{U}$.

***(c)** Solve for $u(x, t)$ (in the simplest form possible).

***10.4.6.** The **Airy function** $\mathrm{Ai}(x)$ is the unique solution of

$$\frac{d^2y}{dx^2} - xy = 0$$

that satisfies

(1) $\lim_{x\to\pm\infty} y = 0$

(2) $y(0) = 3^{-2/3}/\Gamma(\frac{2}{3}) = 3^{-2/3}\Gamma(\frac{1}{3})\sqrt{3}/2\pi = 1/\pi\int_0^\infty \cos(\omega^3/3)\,d\omega$ (It is not necessary to look at Exercises 10.3.15 and 10.3.17).

Determine a Fourier transform representation of the solution of this problem, $\mathrm{Ai}(x)$. [*Hint*: See Exercise 10.3.8].

10.4.7. **(a)** Solve the **linearized Korteweg–deVries equation**

$$\frac{\partial u}{\partial t} = k\frac{\partial^3 u}{\partial x^3}, \qquad -\infty < x < \infty$$
$$u(x, 0) = f(x).$$

(b) Use the convolution theorem to simplify.

***(c)** See Exercise 10.4.6 for a further simplification.

(d) Specialize your result to the case in which

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

10.4.8. Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \begin{matrix} 0 < x < L \\ -\infty < y < \infty \end{matrix}$$

subject to

$$u(0, y) = g_1(y)$$
$$u(L, y) = g_2(y).$$

10.4.9. Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \begin{matrix} y > 0 \\ -\infty < x < \infty \end{matrix}$$

subject to

$$u(x, 0) = f(x).$$

[*Hint*: If necessary, see Section 10.6.3.]

10.4.10. Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty$$
$$u(x, 0) = f(x) \qquad t > 0$$
$$\frac{\partial u}{\partial t}(x, 0) = 0$$

[*Hint*: If necessary, see Section 10.6.1.]

10.4.11. Derive an expression for the Fourier transform of the product $f(x)g(x)$.

10.4.12. Solve the heat equation, $-\infty < x < \infty$,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad t > 0$$

subject to the condition $u(x, 0) = f(x)$.

10.5 Fourier Sine and Cosine Transforms: The Heat Equation on Semi-Infinite Intervals

10.5.1 INTRODUCTION

The Fourier series has been introduced to solve partial differential equations on the finite interval $-L < x < L$ with periodic boundary conditions. For problems defined on the interval $0 < x < L$, special cases of Fourier series, the sine and cosine series, were analyzed in order to satisfy the appropriate boundary conditions.

On an infinite interval, $-\infty < x < \infty$, instead we use the Fourier transform. In this section, we show how to solve partial differential equations on a semi-infinite interval, $0 < x < \infty$. We will introduce special cases of the Fourier transform, known as the **sine and cosine transforms**. The modifications of the Fourier transform will be similar to the ideas we used for series on finite intervals.

10.5.2 HEAT EQUATION ON A SEMI-INFINITE INTERVAL I

We will motivate the introduction of Fourier sine and cosine transforms by considering a simple physical problem. If the temperature is fixed at 0° at $x = 0$, then the mathematical problem for heat diffusion on a semi-infinite interval $x > 0$ is

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad x > 0} \tag{10.5.1}$$

$$\text{BC: } \boxed{u(0,t) = 0} \tag{10.5.2}$$

$$\text{IC: } \boxed{u(x,0) = f(x).} \tag{10.5.3}$$

Here, we have one boundary condition, which is homogeneous.

If we separate variables,

$$u(x,t) = \phi(x)h(t),$$

for the heat equation, we obtain as before

$$\frac{dh}{dt} = -\lambda k h \tag{10.5.4}$$

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi.} \tag{10.5.5}$$

The boundary conditions to determine the allowable eigenvalues λ are

$$\boxed{\phi(0) = 0} \tag{10.5.6}$$

$$\boxed{\lim_{x\to\infty} |\phi(x)| < \infty.} \tag{10.5.7}$$

The latter condition corresponds to $\lim_{x\to\infty} u(x,t) = 0$, since we usually assume that $\lim_{x\to\infty} f(x) = 0$.

There are nontrivial solutions of (10.5.5) with (10.5.6) and (10.5.7) only for all positive $\lambda(\lambda > 0)$,

$$\phi(x) = c_1 \sin\sqrt{\lambda}x = c_1 \sin\omega x, \tag{10.5.8}$$

where, as with the Fourier transform, we prefer the variable $\omega = \sqrt{\lambda}$. Here $\omega > 0$ only. The corresponding time-dependent part is

$$h(t) = ce^{-\lambda k t} = ce^{-kt\omega^2}, \tag{10.5.9}$$

and thus product solutions are

$$u(x,t) = A\sin\omega x\, e^{-kt\omega^2}. \tag{10.5.10}$$

The generalized principle of superposition implies that we should seek a solution to the initial value problem in the form

$$\boxed{u(x,t) = \int_0^\infty A(\omega)\sin\omega x\, e^{-k\omega^2 t}\, d\omega.} \tag{10.5.11}$$

The initial condition $u(x,0) = f(x)$ is satisfied if

$$f(x) = \int_0^\infty A(\omega)\sin\omega x\, d\omega. \tag{10.5.12}$$

In the next subsection, we will show that $A(\omega)$ is the Fourier sine transform of $f(x)$; we will show that $A(\omega)$ can be determined from (10.5.12):

$$\boxed{A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx.} \tag{10.5.13}$$

10.5.3 FOURIER SINE AND COSINE TRANSFORMS

In the preceding subsection, we are asked to represent a function using only sine functions. We already know how to represent a function using complex exponentials, the Fourier transform:

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^\infty F(\omega)e^{-i\omega x}\, d\omega \tag{10.5.14}$$

$$F(\omega) = \frac{\gamma}{2\pi}\int_{-\infty}^\infty f(x)e^{i\omega x}\, dx. \tag{10.5.15}$$

Recall that (10.5.14)–(10.5.15) is valid for any γ.

Fourier sine transform

Since we want to use only $\sin \omega x$ (for all ω), we consider cases in which $f(x)$ is an odd function. If our physical region is $x \geq 0$, then our functions do not have physical meaning for $x < 0$. In this case, we can define these functions in any way we choose for $x < 0$; we introduce the odd extension of the given $f(x)$. If $f(x)$ is odd in this way, then its Fourier transform $F(\omega)$ can be simplified:

$$F(\omega) = \frac{\gamma}{2\pi}\int_{-\infty}^{\infty} f(x)(\cos \omega x + i \sin \omega x)\, dx = \frac{2i\pi}{2\pi}\int_{0}^{\infty} f(x) \sin \omega x\, dx, \tag{10.5.16}$$

since $f(x)\cos \omega x$ is odd in x and $f(x) \sin \omega x$ is even in x. Note that $F(\omega)$ is an odd function of ω [when $f(x)$ is an odd function of x]. Thus, in a similar manner,

$$f(x) = \frac{1}{\gamma}\int_{-\infty}^{\infty} F(\omega)(\cos \omega x - i \sin \omega x)\, d\omega = \frac{-2i}{\gamma}\int_{0}^{\infty} F(\omega) \sin \omega x\, d\omega. \tag{10.5.17}$$

We can choose γ in any way that we wish. Note that the product of the coefficients in front of both integrals must be $(2i\gamma/2\pi) \cdot (-2i/\gamma) = 2/\pi$ for Fourier transforms rather than $1/2\pi$. For convenience, we let $-2i/\gamma = 1$ (i.e., $\gamma = -2i$) so that *if $f(x)$ is odd*,

$$\boxed{f(x) = \int_{0}^{\infty} F(\omega) \sin \omega x\, d\omega} \tag{10.5.18}$$

$$\boxed{F(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(x) \sin \omega x\, dx.} \tag{10.5.19}$$

Others may prefer a symmetric definition, $-2i/\gamma = \sqrt{2/\pi}$. These are called the **Fourier sine transform pair**. $F(\omega)$ is called the **sine transform** of $f(x)$, sometimes denoted $S[f(x)]$, while $f(x)$ is the **inverse sine transform** of $F(\omega)$, $S^{-1}[F(\omega)]$. Equations (10.5.18) and (10.5.19) are related to the formulas for the Fourier transform of an odd function of x.

If the sine transform representation of $f(x)$ (10.5.18) is utilized, then zero is always obtained at $x = 0$. This occurs even if $\lim_{x \to 0} f(x) \neq 0$. Equation (10.5.18) as written is not always valid at $x = 0$. Instead, the odd extension of $f(x)$ has a jump discontinuity at $x = 0$ [if $\lim_{x \to 0} f(x) \neq 0$], from $-f(0)$ to $f(0)$. The Fourier sine transform representation of $f(x)$ converges to the average, which is zero (at $x = 0$).

Fourier cosine transform

Similarly, *if $f(x)$ is an even function,* we can derive the **Fourier cosine transform pair**:

$$\boxed{f(x) = \int_{0}^{\infty} F(\omega) \cos \omega x\, d\omega} \tag{10.5.20}$$

$$\boxed{F(\omega) = \frac{2}{\pi}\int_{0}^{\infty} f(x) \cos \omega x\, dx.} \tag{10.5.21}$$

Other forms are equivalent (as long as the product of the two numerical factors is again $2/\pi$). $F(\omega)$ is called the **cosine transform** of $f(x)$, sometimes denoted $C[f(x)]$, while $f(x)$ is the **inverse cosine transform** of $F(\omega)$, $C^{-1}[F(\omega)]$. Again if $f(x)$ is defined only for $x > 0$, then in order to use the Fourier cosine transform, we must introduce the even extension of $f(x)$.

Short tables of both the Fourier sine and cosine transforms appear at the end of this section (Tables 10.5.1 and 10.5.2).

10.5.4 TRANSFORMS OF DERIVATIVES

In Section 10.5.2, we derived important properties of the Fourier transform of derivatives. Here, similar results for the Fourier sine and cosine transform will be shown.

Our definitions of the Fourier cosine and sine transforms are

$$\boxed{C[f(x)] = \frac{2}{\pi}\int_{0}^{\infty} f(x) \cos \omega x\, dx} \tag{10.5.22}$$

$$\boxed{S[f(x)] = \frac{2}{\pi}\int_{0}^{\infty} f(x) \sin \omega x\, dx.} \tag{10.5.23}$$

Integration by parts can be used to obtain formulas for the transforms of first derivatives:

$$C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^\infty \frac{df}{dx}\cos\omega x\,dx = \frac{2}{\pi}f(x)\cos\omega x\Big|_0^\infty + \omega\frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx$$

$$S\left[\frac{df}{dx}\right] = \frac{2}{\pi}\int_0^\infty \frac{df}{dx}\sin\omega x\,dx = \frac{2}{\pi}f(x)\sin\omega x\Big|_0^\infty - \omega\frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx.$$

We have assumed that $f(x)$ is continuous. We obtain the following formulas:

$$\boxed{C\left[\frac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega S[f]} \tag{10.5.24}$$

$$\boxed{S\left[\frac{df}{dx}\right] = -\omega C[f],} \tag{10.5.25}$$

assuming that $f(x) \to 0$ as $x \to \infty$. Sine and cosine transforms of first derivatives always involve the other type of semi-infinite transform. Thus, if a partial differential equation contains first derivatives with respect to a potential variable to be transformed, the Fourier sine or Fourier cosine transform will never work. Do not use Fourier sine or cosine transforms in this situation. Note that for the heat equation, $\partial u/\partial t = k\,\partial^2 u/\partial x^2$, the variable to be transformed is x. No first derivatives in x appear.

Transforms of second derivatives have simpler formulas. According to (10.5.24) and (10.5.25),

$$\boxed{C\left[\frac{d^2f}{dx^2}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) + \omega S\left[\frac{df}{dx}\right] = -\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 C[f]} \tag{10.5.26}$$

$$\boxed{S\left[\frac{d^2f}{dx^2}\right] = -\omega C\left[\frac{df}{dx}\right] = \frac{2}{\pi}\omega f(0) - \omega^2 S[f].} \tag{10.5.27}$$

We learn some important principles from (10.5.26) and (10.5.27). **In order to use the Fourier cosine transform** to solve a partial differential equation (containing a second derivative) defined on a semi-infinite interval ($x \geq 0$), ***df/dx*(0) must be known**. Similarly, **the Fourier sine transform may be used for semi-infinite problems if *f*(0) is known**. Furthermore, problems are more readily solved if the boundary conditions are homogeneous. If $f(0) = 0$, then a Fourier sine transform will often yield a relatively simple solution. If $df/dx(0) = 0$, then a Fourier cosine transform will often be extremely convenient. These conditions are not surprising. If $f(0) = 0$, separation of variables motivates the use of sines only. Similarly, $df/dx(0) = 0$ implies the use of cosines.

10.5.5 HEAT EQUATION ON A SEMI-INFINITE INTERVAL II

Let us show how to utilize the formulas for the transforms of derivatives to solve partial differential equations. We consider a problem that is somewhat more general than the one presented earlier. Suppose that we are interested in the heat flow in a semi-infinite region with the temperature prescribed as a function of time at $x = 0$:

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \tag{10.5.28}$$

$$\text{BC: } \boxed{u(0, t) = g(t)} \tag{10.5.29}$$

$$\text{IC: } \boxed{u(x, 0) = f(x).} \tag{10.5.30}$$

The boundary condition $u(0, t) = g(t)$ is nonhomogeneous. We cannot use the method of separation of variables. Since $0 < x < \infty$, we may wish to use a transform. Since u is specified at $x = 0$, we should try to use Fourier sine transforms (and not the Fourier cosine transform). Thus, we introduce $\bar{U}(\omega, t)$, the Fourier sine transform of $u(x, t)$:

$$\bar{U}(\omega, t) = \frac{2}{\pi}\int_0^\infty u(x, t)\sin\omega x\,dx. \tag{10.5.31}$$

The partial differential equation (10.5.28) becomes an ordinary differential equation,

$$\frac{\partial \bar{U}}{\partial t} = k\left(\frac{2}{\pi}\omega g(t) - \omega^2 \bar{U}\right), \tag{10.5.32}$$

using (10.5.27). The initial condition yields the initial value of the Fourier sine transform:

$$\bar{U}(\omega, 0) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx. \tag{10.5.33}$$

Solving (10.5.32) is somewhat complicated in general (involving the integrating factor $e^{k\omega^2 t}$; see Section 8.3). We leave discussion of this to the Exercises.

EXAMPLE

In the special case with homogeneous boundary conditions, $g(t) = 0$, it follows from (10.5.32) that

$$\bar{U}(\omega, t) = c(\omega)e^{-k\omega^2 t}, \tag{10.5.34}$$

where, from the initial condition,

$$c(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx. \tag{10.5.35}$$

The solution is thus

$$u(x, t) = \int_0^\infty c(\omega)e^{-k\omega^2 t}\sin\omega x\, d\omega. \tag{10.5.36}$$

This is the solution obtained earlier by separation of variables. To simplify this solution, we note that $c(\omega)$ is an odd function of ω. Thus,

$$u(x, t) = \frac{1}{2}\int_{-\infty}^\infty c(\omega)e^{-k\omega^2 t}\sin\omega x\, d\omega = \int_{-\infty}^\infty \frac{c(\omega)}{2i}e^{-k\omega^2 t}e^{i\omega x}\, d\omega. \tag{10.5.37}$$

If we introduce the odd extension of $f(x)$, then

$$\frac{c(\omega)}{2i} = \frac{2}{\pi}\int_0^\infty f(x)\frac{\sin\omega x}{2i}\, dx = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\frac{\sin\omega x}{2i}\, dx = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\omega x}\, dx. \tag{10.5.38}$$

We note that (10.5.37) and (10.5.38) are exactly the results for the heat equation on an infinite interval. Thus,

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^\infty f(\bar{x})e^{-(x-\bar{x})^2/4kt}d\bar{x}.$$

Here, $f(x)$ has been extended to $-\infty < x < \infty$ as an odd function $[f(-x) = -f(x)]$. In order to utilize only $f(\bar{x})$ for $\bar{x} > 0$, we use the oddness property

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}}\left[\int_{-\infty}^0 -f(-\bar{x})\, e^{-(x-\bar{x})^2/4kt}d\bar{x} + \int_0^\infty f(\bar{x})e^{-(x-\bar{x})^2/4kt}d\bar{x}\right].$$

In the first integral, we let $\bar{x} = -\bar{\bar{x}}$ (and then we replace $\bar{\bar{x}}$ by $\bar{x}$). In this manner

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty f(\bar{x})\left[e^{-(x-\bar{x})^2/4kt} - e^{-(x+\bar{x})^2/4kt}\right] d\bar{x}. \tag{10.5.39}$$

The influence function for the initial condition is in brackets. We will discuss this solution further in Chapter 11. An equivalent (and simpler) method to obtain (10.5.39) from (10.5.34) and (10.5.35) is to use the convolution theorem for Fourier sine transforms (see Exercise 10.6.6). ■

10.5.6 TABLES OF FOURIER SINE AND COSINE TRANSFORMS

We present short tables of the Fourier sine transform (Table 10.5.1) and Fourier cosine transform (Table 10.5.2).

TABLE 10.5.1 Fourier Sine Transform

$f(x) = \int_0^\infty F(\omega)\sin\omega x\, d\omega$	$S[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx$	Reference
$\frac{df}{dx}$	$-\omega C[f(x)]$	Derivatives (Section 10.5.4)
$\frac{d^2f}{dx^2}$	$\frac{2}{\pi}\omega f(0) - \omega^2 F(\omega)$	
$\frac{x}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi}\cdot\frac{\omega}{\epsilon^2+\omega^2}$	Exercise 10.5.2
1	$\frac{2}{\pi}\cdot\frac{1}{\omega}$	Exercise 10.5.9
$\frac{1}{\pi}\int_0^\infty f(\bar{x})[g(x-\bar{x}) - g(x+\bar{x})]\,d\bar{x} = \frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x+\bar{x}) - f(\bar{x}-x)]\,d\bar{x}$	$S[f(x)]C[g(x)]$	Convolution (Exercise 10.5.6)

TABLE 10.5.2 Fourier Cosine Transform

$f(x) = \int_0^\infty F(\omega)\cos\omega x\, d\omega$	$C[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx$	Reference
$\frac{df}{dx}$	$-\frac{2}{\pi}f(0) + \omega S[f(x)]$	Derivatives (Section 10.5.4)
$\frac{d^2f}{dx^2}$	$-\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 F(\omega)$	
$\frac{\beta}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi}\cdot\frac{\epsilon}{\epsilon^2+\omega^2}$	Exercise 10.5.2
$e^{-\alpha x^2}$	$2\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Exercise 10.5.3
$\int_0^\infty g(\bar{x})[f(x-\bar{x}) + f(x+\bar{x})]d\bar{x}$	$F(\omega)G(\omega)$	Convolution (Exercise 10.5.7)

Exercises 10.5

10.5.1. Consider $F(\omega) = e^{-\beta\omega}$, $\beta > 0$ $(\omega \geq 0)$.

(a) Derive the inverse Fourier sine transform of $F(\omega)$.

(b) Derive the inverse Fourier cosine transform of $F(\omega)$.

10.5.2. Consider $f(x) = e^{-\alpha x}$, $\alpha > 0$ $(x \geq 0)$.

(a) Derive the Fourier sine transform of $f(x)$.

(b) Derive the Fourier cosine transform of $f(x)$.

***10.5.3.** Derive *either* the Fourier cosine transform of $e^{-\alpha x^2}$ or the Fourier sine transform of $e^{-\alpha x^2}$.

10.5.4. **(a)** Derive (10.5.26) using Green's formula.

(b) Do the same for (10.5.27).

10.5.5. **(a)** Show that the Fourier sine transform of $f(x)$ is an odd function of ω (if defined for all ω).

(b) Show that the Fourier cosine transform of $f(x)$ is an even function of ω (if defined for all ω).

10.5.6. There is an interesting convolution-type theorem for Fourier sine transforms. Suppose that we want $h(x)$ but know its sine transform $H(\omega)$ to be a product

$$H(\omega) = \overline{S}(\omega)\overline{C}(\omega),$$

where $\overline{S}(\omega)$ is the sine transform of $s(x)$ and $\overline{C}(\omega)$ is the cosine transform of $c(x)$. Assuming that $c(x)$ is even and $s(x)$ is odd, show that

$$h(x) = \frac{1}{\pi}\int_0^\infty s(\overline{x})[c(x-\overline{x}) - c(x+\overline{x})]d\overline{x} = \frac{1}{\pi}\int_0^\infty c(\overline{x})[s(x+\overline{x}) - s(\overline{x}-x)]\,d\overline{x}.$$

10.5.7. Derive the following: If a Fourier cosine transform in x, $H(\omega)$, is the product of two Fourier cosine transforms,

$$H(\omega) = F(\omega)G(\omega),$$

then

$$h(x) = \frac{1}{\pi}\int_0^\infty g(\overline{x})[f(x-\overline{x}) + f(x+\overline{x})]d\overline{x}.$$

In this result, f and g can be interchanged.

10.5.8. Solve (10.5.1)–(10.5.3) using the convolution theorem of Exercise 10.5.6. Exercise 10.5.3 may be of some help.

10.5.9. Let $S[f(x)]$ designate the Fourier sine transform.

(a) Show that

$$S[e^{-\epsilon x}] = \frac{2}{\pi}\frac{\omega}{\epsilon^2 + \omega^2} \quad \text{for} \quad \epsilon > 0.$$

Show that $\lim_{\epsilon\to 0+} S[e^{-\epsilon x}] = 2/\pi\omega$. We will let $S[1] = 2/\pi\omega$. Why isn't $S[1]$ technically defined?

(b) Show that

$$S^{-1}\left[\frac{2/\pi}{\omega}\right] = \frac{2}{\pi}\int_0^\infty \frac{\sin z}{z}\,dz,$$

which is known to equal 1.

***10.5.10.** Determine the inverse cosine transform of $\omega e^{-\alpha\omega}$. [*Hint*: Use differentiation with respect to a parameter.]

***10.5.11.** Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \qquad x > 0 \\ & \qquad\qquad t > 0 \\ u(0,t) &= 1 \\ u(x,0) &= f(x). \end{aligned}$$

(a) Solve directly using sine transforms. [*Hint*: Use Exercise 10.5.8 and the convolution theorem, Exercise 10.5.6.]

(b) If $f(x) \to 1$ as $x \to \infty$, let $v(x,t) = u(x,t) - 1$ and solve for $v(x,t)$.

(c) Compare part (b) to part (a).

10.5.12. Solve

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \quad (x > 0) \\ \frac{\partial u}{\partial x}(0,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

10.5.13. Solve (10.5.28)–(10.5.30) by solving (10.5.32).

10.5.14. Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x} \quad (x > 0) \\ u(0, t) &= 0 \\ u(x, 0) &= f(x). \end{aligned}$$

(a) Show that the Fourier sine transform does not yield an immediate solution.

(b) Instead, introduce

$$u = e^{[x-(v_0/2)t]v_0/2k}w,$$

and show that

$$\begin{aligned} \frac{\partial w}{\partial t} &= k\frac{\partial^2 w}{\partial x^2} \\ w(0, t) &= 0 \\ w(x, 0) &= f(x)e^{-v_0 x/2k}. \end{aligned}$$

(c) Use part (b) to solve for $u(x, t)$.

10.5.15. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \qquad \begin{matrix} 0 < x < L \\ 0 < y < \infty \end{matrix} \\ u(x, 0) &= 0 \\ u(0, y) &= g_1(y) \\ u(L, y) &= g_2(y). \end{aligned}$$

[*Hint*: If necessary, see Section 10.6.2.]

10.5.16. Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \qquad \begin{matrix} 0 < x < \infty \\ 0 < y < \infty \end{matrix} \\ u(0, y) &= g(y) \\ \frac{\partial}{\partial y}u(x, 0) &= 0. \end{aligned}$$

[*Hint*: If necessary, see Section 10.6.4.]

10.5.17. The effect of periodic surface heating (either daily or seasonal) on the interior of the earth may be modeled by

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty \\ u(x, 0) &= 0 \\ u(0, t) &= Ae^{i\sigma_0 t}, \end{aligned}$$

where the real part of $u(x, t)$ is the temperature (and x measures distance from the surface).

(a) Determine $\bar{U}(\omega, t)$, the Fourier sine transform of $u(x, t)$.

***(b)** Approximate $\bar{U}(\omega, t)$ for large t.

(c) Determine the inverse sine transform of part (b) in order to obtain an approximation for $u(x, t)$ valid for large t. [*Hint*: See Exercise 10.5.2(a) or Table 10.5.1.]

(d) Sketch the approximate temperature (for fixed large t).

(e) At what distance below the surface are temperature variations negligible?

10.5.18. Reconsider Exercise 10.5.17. Determine $u(x, t)$ exactly. [*Hint*: See Exercise 10.5.6.]

10.5.19. **(a)** Determine a particular solution of Exercise 10.5.17, satisfying the boundary condition but not the initial condition, of the form $u(x, t) = F(x)G(t)$.

(b) Compare part (a) with either Exercise 10.5.17 or 10.5.18.

10.5.20. Solve the heat equation, $0 < x < \infty$,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the conditions $u(0, t) = 0$ and $u(x, 0) = f(x)$.

10.6 Worked Examples Using Transforms

10.6.1 ONE-DIMENSIONAL WAVE EQUATION ON AN INFINITE INTERVAL

Previously, we have analyzed vibrating strings on a finite interval, usually $0 \le x \le L$. Here, we will study a vibrating string on an infinite interval. The best way to analyze vibrating strings on an infinite (or semi-infinite) interval is to use the method of characteristics, which we describe in Chapter 12. There, concepts of wave propagation are more completely discussed. Here, we will analyze only the following example, to show briefly how Fourier transforms can be used to solve the one-dimensional wave equation:

$$\text{PDE: } \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty} \tag{10.6.1}$$

$$\text{IC1: } \boxed{u(x, 0) = f(x)} \tag{10.6.2}$$

$$\text{IC2: } \boxed{\frac{\partial u}{\partial t}(x, 0) = 0.} \tag{10.6.3}$$

We give the initial position $f(x)$ of the string but insist that the string is at rest, $\partial u/\partial t(x, 0) = 0$, in order to simplify the mathematics.

Product solutions can be obtained by separation of variables. Instead, we will introduce the Fourier transform in x of $u(x, t)$:

$$\bar{U}(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, t)e^{i\omega x}\, dx \tag{10.6.4}$$

$$u(x, t) = \int_{-\infty}^{\infty} \bar{U}(\omega, t)e^{-i\omega x}\, d\omega. \tag{10.6.5}$$

Taking the Fourier transform of the one-dimensional wave equation yields

$$\frac{\partial^2 \bar{U}}{\partial t^2} = -c^2\omega^2\bar{U}, \tag{10.6.6}$$

with the initial conditions becoming

$$\bar{U}(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx \tag{10.6.7}$$

$$\frac{\partial}{\partial t}\bar{U}(\omega, 0) = 0. \tag{10.6.8}$$

The general solution of (10.6.6) is a linear combination of sines and cosines:

$$\bar{U}(\omega, t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t. \tag{10.6.9}$$

The initial conditions imply that

$$B(\omega) = 0 \tag{10.6.10}$$

$$A(\omega) = \bar{U}(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\, dx. \tag{10.6.11}$$

Using the inverse Fourier transform, the solution of the one-dimensional wave equation is

$$\boxed{u(x, t) = \int_{-\infty}^{\infty} \bar{U}(\omega, 0)\cos c\omega t\; e^{-i\omega x}\, d\omega,} \tag{10.6.12}$$

where $\bar{U}(\omega, 0)$ is the Fourier transform of the initial position.
The solution can be considerably simplified. Using Euler's formula,

$$u(x, t) = \frac{1}{2}\int_{-\infty}^{\infty} \bar{U}(\omega, 0)[e^{-i\omega(x-ct)} + e^{-i\omega(x+ct)}]\, d\omega. \tag{10.6.13}$$

However, $\bar{U}(\omega, 0)$ is the Fourier transform of $f(x)$, and hence

$$f(x) = \int_{-\infty}^{\infty} \bar{U}(\omega, 0) e^{-i\omega x}\, d\omega. \tag{10.6.14}$$

By comparing (10.6.13) and (10.6.14), we obtain D'Alembert's solution

$$\boxed{u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)].} \tag{10.6.15}$$

For an infinite string (*started at rest*), the solution is the sum of two terms, $\frac{1}{2}f(x - ct)$ and $\frac{1}{2}f(x + ct)$. $\frac{1}{2}f(x - ct)$ is a waveform of fixed shape. Its height stays fixed if $x - ct =$ constant, and thus $dx/dt = c$. For example, the origin corresponds to $x = ct$. Assuming that $c > 0$, this fixed shape moves to the right with velocity c. It is called a **traveling wave**. Similarly, $\frac{1}{2}f(x + ct)$ is a wave of fixed shape traveling to the left (at velocity $-c$). Our interpretation of this result is that if started at rest, the initial position of the string breaks in two, half moving to the left and half moving to the right at equal speeds c; the solution is the simple sum of these two traveling waves.

10.6.2 LAPLACE'S EQUATION IN A SEMI-INFINITE STRIP

The mathematical problem for steady-state heat conduction in a semi-infinite strip ($0 < x < L$, $y > 0$) is

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \tag{10.6.16}$$

$$\boxed{u(0, y) = g_1(y)} \tag{10.6.17}$$

$$\boxed{u(L, y) = g_2(y)} \tag{10.6.18}$$

$$\boxed{u(x, 0) = f(x).} \tag{10.6.19}$$

We assume that $g_1(y)$ and $g_2(y)$ approach zero as $y \to \infty$. In Fig. 10.6.1, we illustrate the three nonhomogeneous boundary conditions and the useful simplification

$$\boxed{u(x, y) = u_1(x, y) + u_2(x, y).} \tag{10.6.20}$$

Here, both u_1 and u_2 satisfy Laplace's equation. Separation of variables for Laplace's equation in the semi-infinite strip geometry provides motivation for our approach:

$$u(x, y) = \phi(x)\theta(y), \tag{10.6.21}$$

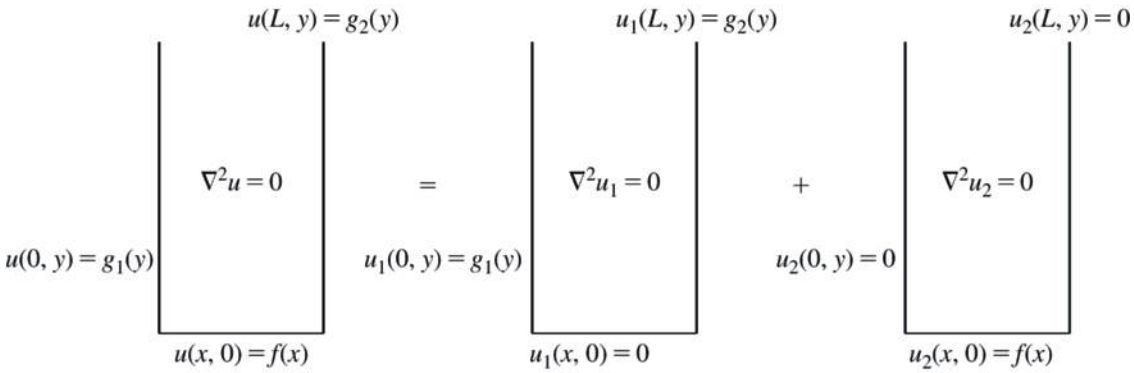
in which case

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{10.6.22}$$

$$\frac{d^2\theta}{dy^2} = \lambda\theta. \tag{10.6.23}$$

Two homogeneous boundary conditions are necessary for an eigenvalue problem. That is why we divided our problem into two parts.

Figure 10.6.1 Laplace's equation in a semi-infinite strip.



Zero temperature sides

For the u_2-problem, $u_2 = 0$ at both $x = 0$ and $x = L$. Thus, the differential equation in x is an eigenvalue problem, defined over a *finite* interval. The boundary conditions are exactly those of a Fourier sine series in x. The y-dependent solutions are exponentials. For u_2, separated solutions are

$$u_2(x, y) = \sin\frac{n\pi x}{L}e^{-n\pi y/L} \quad \text{and} \quad u_2(x, y) = \sin\frac{n\pi x}{L}e^{+n\pi y/L}.$$

The principle of superposition implies that

$$u_2(x, y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}e^{-n\pi y/L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}e^{n\pi y/L}. \tag{10.6.24}$$

There are two other conditions on u_2:

$$u_2(x, 0) = f(x) \tag{10.6.25}$$

$$\lim_{y\to\infty} u_2(x, y) = 0. \tag{10.6.26}$$

Since $u_2 \to 0$ as $y \to \infty$, $b_n = 0$. The nonhomogeneous condition is

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \tag{10.6.27}$$

and thus a_n are the Fourier sine coefficients of the nonhomogeneous boundary condition at $y = 0$:

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.} \tag{10.6.28}$$

Using these coefficients, the solution is

$$\boxed{u_2(x, y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}e^{-n\pi y/L}.} \tag{10.6.29}$$

There is no need to use Fourier transforms for the u_2-problem. The partial differential equation for u_2 could have been analyzed much earlier in this text (e.g., in Chapter 2).

Zero temperature bottom

For the u_1-problem, the second homogeneous boundary condition is less apparent:

$$u_1(x, 0) = 0 \tag{10.6.30}$$

$$\lim_{y\to\infty} u_1(x, y) = 0. \tag{10.6.31}$$

The y-dependent part is the boundary value problem. As $y \to \infty$, the "separated" solution must remain bounded (they do not necessarily vanish). The appropriate solutions of (10.6.23) are sines and cosines (corresponding to $\lambda < 0$). The homogeneous boundary condition at $y = 0$, (10.6.30), implies that only sines should be used. Instead of continuing to discuss the method of separation of variables, we now introduce the Fourier sine transform in y:

$$\boxed{u_1(x, y) = \int_0^{\infty} \bar{U}_1(x, \omega)\sin\omega y\,d\omega} \tag{10.6.32}$$

$$\bar{U}_1(x, \omega) = \frac{2}{\pi}\int_0^{\infty} u_1(x, y)\sin\omega y\,dy. \tag{10.6.33}$$

We directly take the Fourier sine transform with respect to y of Laplace's equation, (10.6.16). The properties of the transform of derivatives shows that Laplace's equation becomes an ordinary differential equation:

$$\frac{\partial^2}{\partial x^2}\bar{U}_1(x,\omega) - \omega^2\bar{U}_1(x,\omega) = 0. \tag{10.6.34}$$

The boundary condition at $y = 0$, $u_1(x, 0) = 0$, has been used to simplify our result. The solution of (10.6.34) is a linear combination of nonoscillatory (exponential) functions. It is most convenient (although not necessary) to utilize the following hyperbolic functions:

$$\boxed{\bar{U}_1(x,\omega) = a(\omega)\sinh\omega x + b(\omega)\sinh\omega(L-x).} \tag{10.6.35}$$

The two nonhomogeneous conditions at $x = 0$ and $x = L$ yield

$$\boxed{\bar{U}_1(0,\omega) = b(\omega)\sinh\omega L = \frac{2}{\pi}\int_0^\infty g_1(y)\sin\omega y\, dy} \tag{10.6.36}$$

$$\boxed{\bar{U}_1(L,\omega) = a(\omega)\sinh\omega L = \frac{2}{\pi}\int_0^\infty g_2(y)\sin\omega y\, dy.} \tag{10.6.37}$$

$\bar{U}_1(x,\omega)$, the Fourier sine transform of $u_1(x, y)$, is given by (10.6.35), where $a(\omega)$ and $b(\omega)$ are obtained from (10.6.36) and (10.6.37).[6] This completes the somewhat complicated solution of Laplace's equation in a semi-infinite channel. It is the sum of a solution obtained using a Fourier sine series and one using a Fourier sine transform.

Nonhomogeneous boundary conditions

If desired, Laplace's equation (10.6.16) with three nonhomogeneous boundary conditions, (10.6.17)–(10.6.19), can be solved by directly applying a Fourier sine transform in y without decomposing the problem into two:

$$\boxed{u(x,y) = \int_0^\infty \bar{U}(x,\omega)\sin\omega y\, d\omega.} \tag{10.6.38}$$

Since the boundary condition at $y = 0$, $u(x, 0) = f(x)$, is nonhomogeneous, an extra term is introduced into the Fourier sine transform of Laplace's equation (10.6.16):

$$\frac{\partial^2\bar{U}}{\partial y^2} - \omega^2\bar{U} = -\frac{2}{\pi}\omega f(x). \tag{10.6.39}$$

In this case, the Fourier sine transform satisfies a second-order linear constant-coefficient *nonhomogeneous* ordinary differential equation. This equation must be solved with two nonhomogeneous boundary conditions at $x = 0$ and $x = L$. Equation (10.6.39) can be solved by variation of parameters. This solution is probably more complicated than the one consisting of a sum of a series and a transform. Furthermore, this solution has a jump discontinuity at $y = 0$; the integral in (10.6.38) equals zero at $y = 0$ but converges to $f(x)$ as $y \to 0$. Usually, breaking the problem into two problems is preferable.

10.6.3 LAPLACE'S EQUATION IN A HALF-PLANE

If the temperature is specified to equal $f(x)$ on an infinite wall, $y = 0$, then the steady-state temperature distribution for $y > 0$ satisfies Laplace's equation,

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,} \tag{10.6.40}$$

[6]Unfortunately, $\bar{U}_1(x,\omega)$ is not the product of the transforms of two simple functions. Hence, we do not use the convolution theorem.

subject to the boundary condition,

$$\boxed{u(x, 0) = f(x).} \tag{10.6.41}$$

If $f(x) \to 0$ as $x \to \pm\infty$, then there are three other implied boundary conditions:

$$\lim_{x\to+\infty} u(x, y) = 0 \tag{10.6.42}$$

$$\lim_{x\to-\infty} u(x, y) = 0 \tag{10.6.43}$$

$$\lim_{y\to+\infty} u(x, y) = 0; \tag{10.6.44}$$

the temperature approaches zero at large distances from the wall.

The method of separation of variables suggests the use of a Fourier transform in x, since there are two homogeneous boundary conditions as $x \to \pm\infty$:

$$u(x, y) = \int_{-\infty}^{\infty} \bar{U}(\omega, y) e^{-i\omega x}\, d\omega \tag{10.6.45}$$

$$\bar{U}(\omega, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, y) e^{i\omega x}\, dx. \tag{10.6.46}$$

By taking the Fourier transform in x of (10.6.40), we obtain the ordinary differential equation satisfied by the Fourier transform,

$$\frac{\partial^2 \bar{U}}{\partial y^2} - \omega^2 \bar{U} = 0. \tag{10.6.47}$$

Since $u(x, y) \to 0$ as $y \to +\infty$, its Fourier transform in x also vanishes as $y \to +\infty$,

$$\bar{U}(\omega, y) \to 0, \quad \text{as } y \to +\infty. \tag{10.6.48}$$

In addition, at $y = 0$, $\bar{U}(\omega, 0)$ is the Fourier transform of the boundary condition,

$$\bar{U}(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx. \tag{10.6.49}$$

We must be careful solving (10.6.47). The general solution is

$$\bar{U}(\omega, y) = a(\omega) e^{\omega y} + b(\omega) e^{-\omega y}, \tag{10.6.50}$$

which is of interest *for all* ω [see (10.6.45)]. There are two boundary conditions to determine the two arbitrary functions $a(\omega)$ and $b(\omega)$. Equation (10.6.48) states that $\bar{U}(\omega, y) \to 0$ as $y \to +\infty$. At first you might think that this implies that $a(\omega) = 0$, but that is not correct. Instead, for $\bar{U}(\omega, y)$ to vanish as $y \to +\infty$, $a(\omega) = 0$ only for $\omega > 0$. If $\omega < 0$, then $b(\omega)e^{-\omega y}$ grows exponentially as $y \to +\infty$. Thus, $b(\omega) = 0$ for $\omega < 0$. We have shown that

$$\bar{U}(\omega, y) = \begin{cases} a(\omega) e^{\omega y} & \text{for } \omega < 0 \\ b(\omega) e^{-\omega y} & \text{for } \omega > 0, \end{cases}$$

where $a(\omega)$ is arbitrary for $\omega < 0$ and $b(\omega)$ is arbitrary for $\omega > 0$. It is more convenient to note that this is equivalent to

$$\boxed{\bar{U}(\omega, y) = c(\omega) e^{-|\omega| y},} \tag{10.6.51}$$

for all ω. The nonhomogeneous boundary condition (10.6.41) now shows that $c(\omega)$ is the Fourier transform of the temperature at the wall, $f(x)$. This completes our solution. However, we will determine a simpler representation of the solution.

Application of the convolution theorem

The easiest way to simplify the solution is to note that $\bar{U}(\omega, y)$ is the product of two Fourier transforms. $f(x)$ has the Fourier transform $c(\omega)$, and some function $g(x, y)$, as yet unknown, has the Fourier transform $e^{-|\omega| y}$. Using the convolution theorem, the solution of our problem is

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) g(x - \bar{x}, y)\, d\bar{x}. \tag{10.6.52}$$

We now need to determine what function $g(x, y)$ has the Fourier transform $e^{-|w|y}$. According to the inversion integral, (10.3.7)

$$g(x, y) = \int_{-\infty}^{\infty} e^{-|\omega|y} e^{-i\omega x}\, d\omega,$$

which may be integrated directly:

$$\begin{aligned} g(x, y) &= \int_{-\infty}^{0} e^{\omega y} e^{-i\omega x}\, d\omega + \int_{0}^{\infty} e^{-\omega y} e^{-i\omega x}\, d\omega \\ &= \frac{e^{\omega(y-ix)}}{y - ix}\bigg|_{-\infty}^{0} + \frac{e^{-\omega(y+ix)}}{-(y + ix)}\bigg|_{0}^{\infty} = \frac{1}{y - ix} + \frac{1}{y + ix} = \frac{2y}{x^2 + y^2}. \end{aligned}$$

Thus, the solution to Laplace's equation in semi-infinite space ($y > 0$) subject to $u(x, 0) = f(x)$ is

$$\boxed{u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) \frac{2y}{(x - \bar{x})^2 + y^2}\, d\bar{x}.} \tag{10.6.53}$$

This solution was derived assuming $f(x) \to 0$ as $x \to \pm\infty$. In fact, it is valid in other cases, roughly speaking, as long as the integral is convergent. In Chapter 9 we obtained (10.6.53) through use of the Green's function. It was shown [see (9.5.47)] that the influence function for the nonhomogeneous boundary condition [see (10.6.53)] is the outward normal derivative (with respect to source points) of the Green's function:

$$-\frac{\partial}{\partial \bar{y}} G(x, y; \bar{x}, \bar{y})\bigg|_{\bar{y}=0} = \frac{y}{\pi} \frac{1}{(x - \bar{x}) + y^2},$$

where $G(x, x_0)$ is the Green's function, the influence function for sources in the half-plane ($y > 0$).

EXAMPLE

A simple, but interesting, solution arises if

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \tag{10.6.54}$$

This boundary condition corresponds to the wall uniformly heated to two different temperatures. We will determine the equilibrium temperature distribution for $y > 0$. From (10.6.53),

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi}\int_{0}^{\infty} \frac{2y}{y^2 + (x - \bar{x})^2}\, d\bar{x} = \frac{1}{\pi} \tan^{-1}\left(\frac{\bar{x} - x}{y}\right)\bigg|_{0}^{\infty} \\ &= \frac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{-x}{y}\right)\right] = \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right)\right]. \end{aligned} \tag{10.6.55}$$

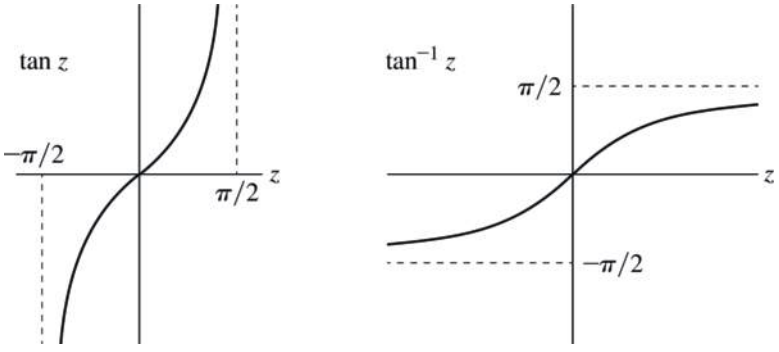
Some care must be used in evaluating the inverse tangent function along a continuous branch. Figure 10.6.2, in which the tangent and inverse tangent functions are sketched, is helpful. If we introduce the usual angle θ from the x-axis,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{y}\right),$$

then the temperature distribution becomes

$$\boxed{u(x, y) = 1 - \frac{\theta}{\pi}.} \tag{10.6.56}$$

Figure 10.6.2 Tangent and inverse tangent functions.



We can check this answer independently. Reconsider this problem, but pose it in the usual polar coordinates. Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0,$$

and the boundary conditions are $u(r, 0)=1$ and $u(r, \pi)=0$. The solution depends on only the angle, $u(r, \theta)=u(\theta)$, in which case

$$\frac{d^2u}{d\theta^2}=0,$$

subject to $u(0)=1$ and $u(\pi)=0$, confirming (10.6.56). This result and more complicated ones for Laplace's equation can be obtained using conformal mappings in the complex plane. ■

10.6.4 LAPLACE'S EQUATION IN A QUARTER-PLANE

In this subsection we consider the steady-state temperature distribution within a quarter-plane ($x>0, y>0$) with the temperature given on one of the semi-infinite walls and the heat flow (gradient of temperature) given on the other:

$$\boxed{\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0} \tag{10.6.57}$$

$$\boxed{u(0, y)=g(y), \quad y>0} \tag{10.6.58}$$

$$\boxed{\frac{\partial u}{\partial y}(x, 0)=f(x), \quad x>0.} \tag{10.6.59}$$

We assume that $g(y)\to 0$ as $y\to\infty$ and $f(x)\to 0$ as $x\to\infty$, such that $u(x, y)\to 0$ as both $x\to\infty$ and $y\to\infty$. There are two nonhomogeneous boundary conditions. Thus, it is convenient to decompose the problem into two, as illustrated in Fig. 10.6.3:

$$\boxed{u=u_1(x, y)+u_2(x, y),} \tag{10.6.60}$$

where

$$\boxed{\nabla^2 u_1=0} \tag{10.6.61}$$

$$\boxed{u_1(0, y)=g(y)} \tag{10.6.62}$$

$$\boxed{\frac{\partial}{\partial y}u_1(x, 0)=0} \tag{10.6.63}$$

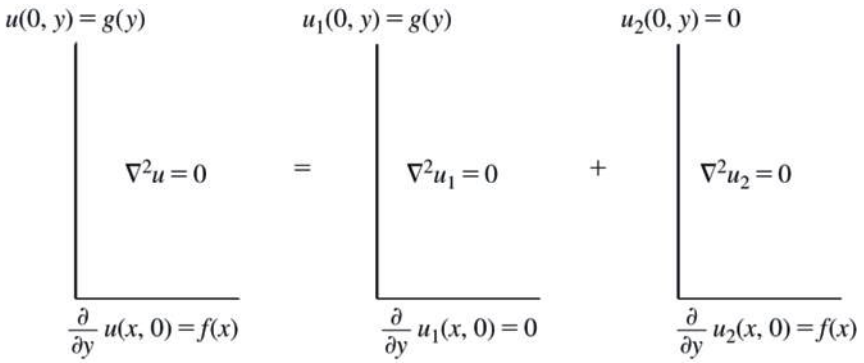
$$\boxed{\nabla^2 u_2=0} \tag{10.6.64}$$

$$\boxed{u_2(0, y)=0} \tag{10.6.65}$$

$$\boxed{\frac{\partial}{\partial y}u_2(x, 0)=f(x).} \tag{10.6.66}$$

Here, we will analyze only the problem for u_1, leaving the u_2-problem for an exercise.

Figure 10.6.3 Laplace's equation in a quarter-plane.



Cosine transform in y

The u_1-problem can be analyzed in two different ways. The problem is semi-infinite in both x and y. Since u_1 is given at $x = 0$, a Fourier sine transform in x can be used. However, $\partial u_1/\partial y$ is given at $y = 0$, and hence a Fourier cosine transform in y can also be used. In fact, $\partial u_1/\partial y = 0$ at $y = 0$. Thus, we prefer to use a Fourier cosine transform in y since we expect the resulting ordinary differential equation to be *homogeneous*:

$$\boxed{u_1(x, y) = \int_0^\infty \bar{U}_1(x, \omega) \cos \omega y \, d\omega} \tag{10.6.67}$$

$$\boxed{\bar{U}_1(x, \omega) = \frac{2}{\pi}\int_0^\infty u_1(x, y) \cos \omega y \, dy.} \tag{10.6.68}$$

If the u_1-problem is Fourier cosine transformed in y, then we obtain

$$\frac{\partial^2 \bar{U}_1}{\partial x^2} - \omega^2 \bar{U}_1 = 0. \tag{10.6.69}$$

The variable x ranges from 0 to ∞. The two boundary conditions for this ordinary differential equation are

$$\bar{U}_1(0, \omega) = \frac{2}{\pi}\int_0^\infty g(y) \cos \omega y \, dy \quad \text{and} \quad \lim_{x \to \infty} \bar{U}_1(x, \omega) = 0. \tag{10.6.70}$$

The general solution of (10.6.69) is

$$\boxed{\bar{U}_1(x, \omega) = a(\omega)e^{-\omega x} + b(\omega)e^{\omega x},} \tag{10.6.71}$$

for $x > 0$ and $\omega > 0$ only. From the boundary conditions (10.6.70), it follows that

$$\boxed{b(\omega) = 0 \quad \text{and} \quad a(\omega) = \frac{2}{\pi}\int_0^\infty g(y) \cos \omega y \, dy.} \tag{10.6.72}$$

Convolution theorem

A simpler form of the solution can be obtained using a convolution theorem for Fourier cosine transforms. We derived the following in Exercise 10.5.7, where we assume $f(x)$ is even:

If a Fourier cosine transform in x, $H(\omega)$, is the product of two Fourier cosine transforms,

$$H(\omega) = F(\omega)G(\omega),$$

then

$$h(x) = \frac{1}{\pi}\int_0^\infty g(\bar{x}) \left[f(x - \bar{x}) + f(x + \bar{x}) \right] d\bar{x}. \tag{10.6.73}$$

In our problem, $\bar{U}_1(x, \omega)$, the Fourier cosine transform of $u_1(x, y)$, is the product of $a(\omega)$, the Fourier cosine transform of $g(y)$, and $e^{-\omega x}$:

$$\bar{U}_1(x, \omega) = a(\omega)e^{-\omega x}.$$

We use the cosine transform pair [see (10.6.67–68)] to obtain the function $Q(y)$, which has the Fourier cosine transform $e^{-\omega x}$:

$$Q(y) = \int_0^\infty e^{-\omega x} \cos \omega y \, d\omega = \int_0^\infty e^{-\omega x} \frac{e^{i\omega y} + e^{-i\omega y}}{2} \, d\omega$$
$$= \frac{1}{2}\left(\frac{1}{x - iy} + \frac{1}{x + iy}\right) = \frac{x}{x^2 + y^2}.$$

Thus, according to the convolution theorem,

$$\boxed{u_1(x, y) = \frac{x}{\pi}\int_0^\infty g(\bar{y}) \left[\frac{1}{x^2 + (y - \bar{y})^2} + \frac{1}{x^2 + (y + \bar{y})^2}\right] d\bar{y}.} \tag{10.6.74}$$

This result also could have been obtained using the Green's function method (see Chapter 9). In this case, appropriate image sources may be introduced so as to utilize the infinite space Green's function for Laplace's equation.

Sine transform in x

An alternative method to solve for $u_1(x, y)$ is to use the Fourier sine transform in x:

$$u_1(x, y) = \int_0^\infty \bar{U}_1(\omega, y) \sin \omega x \, d\omega$$
$$\bar{U}_1(\omega, y) = \frac{2}{\pi}\int_0^\infty u_1(x, y) \sin \omega x \, dx.$$

The ordinary differential equation for $\bar{U}_1(\omega, y)$ is nonhomogeneous:

$$\frac{\partial^2 \bar{U}_1}{\partial y^2} - \omega^2 \bar{U}_1 = -\frac{2}{\pi}\omega g(y).$$

This equation must be solved with the following boundary conditions at $y = 0$ and $y = \infty$:

$$\frac{\partial \bar{U}_1}{\partial y}(\omega, 0) = 0 \quad \text{and} \quad \lim_{y\to\infty} \bar{U}_1(\omega, y) = 0.$$

This approach is further discussed in the Exercises.

10.6.5 HEAT EQUATION IN A PLANE (TWO-DIMENSIONAL FOURIER TRANSFORMS)

Transforms can be used to solve problems that are infinite in both x and y. Consider the heat equation in the x-y plane, $-\infty < x < \infty, -\infty < y < \infty$:

$$\boxed{\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),} \tag{10.6.75}$$

subject to the initial condition

$$\boxed{u(x, y, 0) = f(x, y).} \tag{10.6.76}$$

If we separate variables, we obtain product solutions of the form

$$u(x, y, t) = e^{-i\omega_1 x}e^{-i\omega_2 y}e^{-k(\omega_1^2 + \omega_2^2)t}$$

for all ω_1 and ω_2. Corresponding to $u \to 0$ as $x \to \pm\infty$ and $y \to \pm\infty$ are the boundary conditions that the separated solutions remain bounded, $x \to \pm\infty$ and $y \to \pm\infty$. Thus, these types of solutions are valid for all real ω_1 and ω_2. A generalized principle of superposition implies that the form of the integral representation of the solution is

$$\boxed{u(x, y, t) = \int_{-\infty}^\infty \int_{-\infty}^\infty A(\omega_1, \omega_2)e^{-i\omega_1 x}e^{-i\omega_2 y}e^{-k(\omega_1^2 + \omega_2^2)t} \, d\omega_1 \, d\omega_2.} \tag{10.6.77}$$

The initial condition is satisfied if

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega_1, \omega_2)e^{-i\omega_1 x}e^{-i\omega_2 y}\, d\omega_1\, d\omega_2.$$

We will show that we can determine $A(\omega_1, \omega_2)$,

$$\boxed{A(\omega_1, \omega_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{i\omega_1 x}e^{i\omega_2 y}\, dx\, dy,} \tag{10.6.78}$$

completing the solution. $\mathrm{A}(\omega_1, \omega_2)$ is called the **double Fourier transform** of $f(x, y)$.

Double Fourier transforms

We have used separation of variables to motivate **two-dimensional Fourier transforms**. Suppose that we have a function of two variables $f(x, y)$ that decays sufficiently fast as x and $y \to \pm\infty$. The Fourier transform in x (keeping y fixed, with transform variable ω_1) is

$$F(\omega_1, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x, y)e^{i\omega_1 x}\, dx;$$

its inverse is

$$f(x, y) = \int_{-\infty}^{\infty} F(\omega_1, y)e^{-i\omega_1 x}\, d\omega_1.$$

$F(\omega_1, y)$ is a function of y that also can be Fourier transformed (here with transform variable ω_2):

$$\tilde{F}(\omega_1, \omega_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega_1, y)e^{i\omega_2 y}\, dy$$
$$F(\omega_1, y) = \int_{-\infty}^{\infty} \tilde{F}(\omega_1, \omega_2)e^{-i\omega_2 y}\, d\omega_2.$$

Combining these, we obtain the **two-dimensional (or double) Fourier transform pair**:

$$\tilde{F}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{i\omega_1 x}e^{i\omega_2 y}\, dx\, dy \tag{10.6.79}$$

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(\omega_1, \omega_2)e^{-i\omega_1 x}e^{-i\omega_2 y}\, d\omega_1\, d\omega_2. \tag{10.6.80}$$

Wave number vector

Let us motivate a more convenient notation. When using $e^{i\omega x}$ we refer to ω as the wave number (the number of waves in 2π distance). Here, we note that

$$e^{i\omega_1 x}e^{i\omega_2 y} = e^{i(\omega_1 x + \omega_2 y)} = e^{i\boldsymbol{\omega}\cdot\boldsymbol{r}},$$

where $\boldsymbol{r}$ is a **position vector**[7] and $\boldsymbol{\omega}$ is a **wave number vector**:

$$\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \tag{10.6.81}$$

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}}. \tag{10.6.82}$$

To interpret $e^{i\boldsymbol{\omega}\cdot\boldsymbol{r}}$, we discuss, for example, its real part, $\cos(\boldsymbol{\omega}\cdot\boldsymbol{r}) = \cos(\omega_1 x + \omega_2 y)$. The crests are located at $\omega_1 x + \omega_2 y = n(2\pi)$, sketched in Fig. 10.6.4. The direction perpendicular to the crests is $\nabla(\omega_1 x + \omega_2 y) = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}} = \boldsymbol{\omega}$. This is called the direction of the wave. Thus, *the wave number vector is in the direction of the wave.* We introduce the magnitude ω of the wave number vector,

$$\omega^2 = \boldsymbol{\omega}\cdot\boldsymbol{\omega} = |\boldsymbol{\omega}|^2 = \omega_1^2 + \omega_2^2.$$

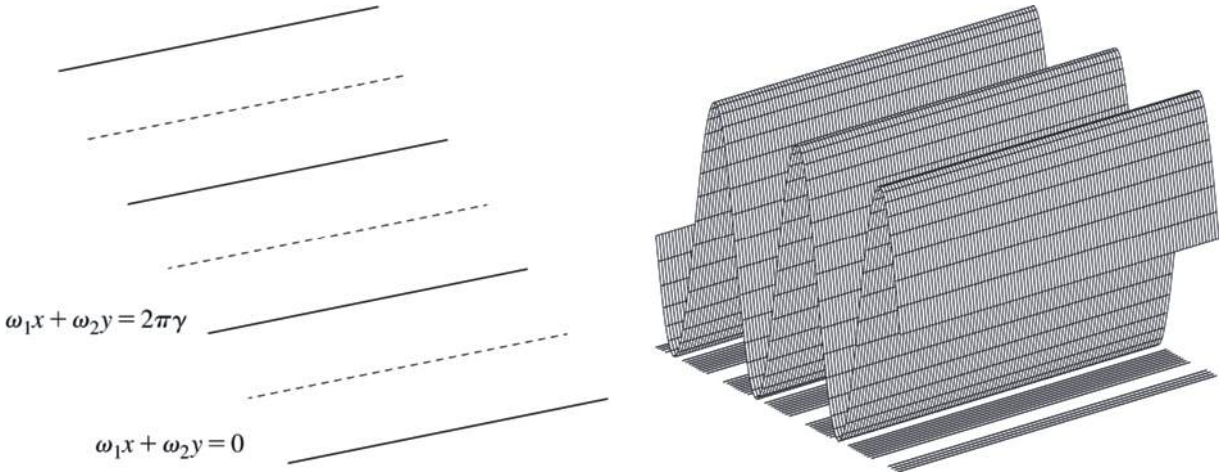
The unit vector in the wave direction is $\boldsymbol{\omega}/\omega$. If we move a distance s in the wave direction (from the origin), then $\boldsymbol{r} = s\boldsymbol{\omega}/\omega$. Thus,

$$\boldsymbol{\omega}\cdot\boldsymbol{r} = s\omega \quad \text{and} \quad \cos(\boldsymbol{\omega}\cdot\boldsymbol{r}) = \cos(\omega s).$$

[7] In other contexts, we use the notation $\boldsymbol{x}$ for the position vector. Thus $\boldsymbol{r} = \boldsymbol{x}$.

Thus, ω is the number of waves in 2π distance (in the direction of the wave). We have justified the name wave number vector for $\boldsymbol{\omega}$; that is, the **wave number vector is in the direction of the wave and its magnitude is the number of waves in 2π distance (in the direction of the wave)**.

Figure 10.6.4 Two-dimensional wave and its crests.



Using the position vector $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and the wave number vector $\boldsymbol{\omega} = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}}$, the **double Fourier transform pair**, (10.6.79) and (10.6.80), becomes

$$\boxed{F(\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{r})e^{i\boldsymbol{\omega}\cdot\boldsymbol{r}}\, d^2r} \tag{10.6.83}$$

$$\boxed{f(\boldsymbol{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\boldsymbol{\omega})e^{-i\boldsymbol{\omega}\cdot\boldsymbol{r}}\, d^2\omega,} \tag{10.6.84}$$

where $f(\boldsymbol{r}) = f(x, y)$, $d^2r = dx\, dy$, $d^2\omega = d\omega_1\, d\omega_2$, and $F(\boldsymbol{\omega})$ is the double Fourier transform of $f(\boldsymbol{r})$.

Using the notation $\mathcal{F}[u(x, y, t)]$ for the double spatial Fourier transform of $u(x, y, t)$, we have the following easily verified fundamental properties:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t}\mathcal{F}[u] \tag{10.6.85}$$

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -i\omega_1\mathcal{F}[u] \tag{10.6.86}$$

$$\mathcal{F}\left[\frac{\partial u}{\partial y}\right] = -i\omega_2\mathcal{F}[u] \tag{10.6.87}$$

$$\mathcal{F}[\nabla^2 u] = -\omega^2\mathcal{F}[u], \tag{10.6.88}$$

where $\omega^2 = \boldsymbol{\omega}\cdot\boldsymbol{\omega} = \omega_1^2 + \omega_2^2$, as long as u decays sufficiently rapidly as x and $y \to \pm\infty$. A short table of the double Fourier transform appears at the end of this subsection.

Heat equation

Instead of using the method of separation of variables, the two-dimensional heat equation (10.6.75) can be directly solved by applying the double Fourier transform to it:

$$\frac{\partial \bar{U}}{\partial t} = -k\omega^2\bar{U}, \tag{10.6.89}$$

where $\bar{U}$ is the double spatial Fourier transform of $u(x, y, t)$:

$$\mathcal{F}[u] = \bar{U}(\boldsymbol{\omega}, t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x, y, t)e^{i\boldsymbol{\omega}\cdot\boldsymbol{r}}\, dx\, dy. \tag{10.6.90}$$

The elementary solution of (10.6.89) is

$$\bar{U}(\boldsymbol{\omega}, t) = A(\boldsymbol{\omega}) e^{-k\omega^2 t}. \tag{10.6.91}$$

Applying (10.6.76), $A(\boldsymbol{\omega})$ is the Fourier transform of the initial condition:

$$\boxed{A(\boldsymbol{\omega}) = \bar{U}(\boldsymbol{\omega}, 0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i\boldsymbol{\omega} \cdot \mathbf{r}} \, dx \, dy.} \tag{10.6.92}$$

Thus, the solution of the two-dimensional heat equation is

$$\boxed{\begin{aligned} u(x, y, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{U}(\boldsymbol{\omega}, t) e^{-i\boldsymbol{\omega} \cdot \mathbf{r}} \, d\omega_1 \, d\omega_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\omega) e^{-k\omega^2 t} e^{-i\boldsymbol{\omega} \cdot \mathbf{r}} \, d\omega_1 \, d\omega_2. \end{aligned}} \tag{10.6.93}$$

This verifies what was suggested earlier by separation of variables.

Application of convolution theorem

In an exercise, we show that a **convolution theorem** holds directly *for double Fourier transforms*:

If $H(\boldsymbol{\omega}) = F(\boldsymbol{\omega}) G(\boldsymbol{\omega})$, then

$$\boxed{h(\mathbf{r}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{r}_0) g(\mathbf{r} - \mathbf{r}_0) \, dx_0 \, dy_0.} \tag{10.6.94}$$

For the two-dimensional heat equation, we have shown that $\bar{U}(\boldsymbol{\omega}, t)$ is the product of $e^{-k\omega^2 t}$ and $A(\boldsymbol{\omega})$, the double Fourier transform of the initial condition. Thus, we need to determine the function whose double Fourier transform is $e^{-k\omega^2 t}$:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\boldsymbol{\omega} \cdot \mathbf{r}} \, d\omega_1 \, d\omega_2 &= \int_{-\infty}^{\infty} e^{-k\omega_1^2 t} e^{-i\omega_1 x} \, d\omega_1 \int_{-\infty}^{\infty} e^{-k\omega_2^2 t} e^{-i\omega_2 y} \, d\omega_2 \\ &= \sqrt{\frac{\pi}{kt}} e^{-x^2/4kt} \sqrt{\frac{\pi}{kt}} e^{-y^2/4kt} = \frac{\pi}{kt} e^{-r^2/4kt}. \end{aligned}$$

The inverse transform of $e^{-k\omega^2 t}$ is the product of the two one-dimensional inverse transforms; it is a two-dimensional Gaussian, $(\pi/kt) e^{-r^2/4kt}$ where $r^2 = x^2 + y^2$. In this manner, the solution of the initial value problem for the two-dimensional heat equation on an infinite plane is

$$\boxed{u(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_0, y_0) \frac{1}{4\pi kt} \exp\left[-\frac{(x - x_0)^2 + (y - y_0)^2}{4kt} \right] dx_0 \, dy_0.} \tag{10.6.95}$$

The influence function for the initial condition is

$$g(x, y, t; x_0, y_0, 0) = \frac{1}{4\pi kt} \exp\left[-\frac{(x - x_0)^2 + (y - y_0)^2}{4kt} \right] = \frac{1}{4\pi kt} e^{-|\mathbf{r} - \mathbf{r}_0|^2/4kt}.$$

It expresses the effect at x, y (at time t) due to the initial heat energy at x_0, y_0. The influence function is the **fundamental solution** of the two-dimensional heat equation, obtained by letting the initial condition be a two-dimensional Dirac delta function, $f(x, y) = \delta(x)\delta(y)$, concentrated at the origin. **The fundamental solution for the two-dimensional heat equation is the product of the fundamental solutions of two one-dimensional heat equations**.

10.6.6 TABLE OF DOUBLE FOURIER TRANSFORMS

We present a short table of double Fourier transforms (Table 10.6.1).

TABLE 10.6.1 Double Fourier Transform

$f(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega\cdot\mathbf{r}}\, d\omega_1\, d\omega_2$	$F(\omega) = \frac{2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r})e^{i\omega\cdot\mathbf{r}}\, dx\, dy$	Reference
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$	$-i\omega_1 F(\omega), -i\omega_2 F(\omega)$	Derivatives (Section 10.6.5)
$\nabla^2 f$	$-\omega^2 F(\omega)$	
$\frac{\pi}{\beta}e^{-r^2/4\beta}$	$e^{-\beta\omega^2}$	Gaussian (Section 10.6.5)
$f(\mathbf{r}-\beta)$	$e^{i\omega\cdot\beta}F(\omega)$	Exercise 10.6.8
$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r}_0)g(\mathbf{r}-\mathbf{r}_0)dx_0\, dy_0$	$F(\omega)G(\omega)$	Convolution (Exercise 10.6.7)

Exercises 10.6

10.6.1. Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad 0 < y < H, -\infty < x < \infty,$$

subject to

***(a)** $u(x, 0) = f_1(x)$ and $u(x, H) = f_2(x)$

(b) $\frac{\partial}{\partial y}u(x, 0) = f_1(x)$ and $u(x, H) = f_2(x)$

(c) $u(x, 0) = 0$ and $\frac{\partial u}{\partial y}(x, H) + hu(x, H) = f(x)$

10.6.2. Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < L, y > 0$$

subject to the following boundary conditions. If there is a solvability condition, state it and explain it physically:

(a) $\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = g(y), \quad \frac{\partial u}{\partial y}(x, 0) = 0$

***(b)** $u(0, y) = g_1(y), \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0$

(c) $\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = f(x)$

(d) $\frac{\partial u}{\partial x}(0, y) = g_1(y), \quad \frac{\partial u}{\partial x}(L, y) = g_2(y), \quad u(x, 0) = f(x)$

10.6.3. **(a)** Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad x < 0, -\infty < y < \infty,$$

subject to $u(0, y) = g(y)$.

(b) Determine the simplest form of the solution if

$$g(y) = \begin{cases} 0 & |y| > 1 \\ 1 & |y| < 1. \end{cases}$$

10.6.4. Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad x > 0, y > 0$$

subject to

***(a)** $u(0, y) = 0$ and $\frac{\partial u}{\partial y}(x, 0) = f(x)$ [*Hint*: Invert $\frac{\partial \bar{U}}{\partial y}$.]

(b) $u(0, y) = 0$ and $u(x, 0) = f(x)$

10.6.5. Reconsider Exercise 10.6.4(a). Let $w = \partial u/\partial y$. Show that w satisfies Exercise 10.6.4(b). In this manner solve both Exercises 10.6.4(a) and (b).

10.6.6. Consider (10.6.61) with (10.6.62)–(10.6.63). In the text, we introduce the Fourier cosine transform in y. Instead, here we introduce the Fourier sine transform in x.

(a) Solve for $\bar{U}_1(\omega, y)$ if

$$\frac{\partial^2 \bar{U}_1}{\partial y^2} - \omega^2 \bar{U}_1 = -\frac{2}{\pi}\omega g(y)$$

$$\frac{\partial \bar{U}_1}{\partial y}(\omega, 0) = 0$$

$$\lim_{y\to\infty} \bar{U}_1(\omega, y) = 0.$$

[*Hint*: See (9.3.9)–(9.3.14) or (13.3.10), using exponentials.]

(b) Derive $u_1(x, y)$. Show that (10.6.74) is valid.

10.6.7. Derive the two-dimensional convolution theorem, (10.6.94).

10.6.8. Derive the following shift theorem for two-dimensional Fourier transforms: The inverse transform of $e^{i\boldsymbol{\omega}\cdot\boldsymbol{\beta}}F(\boldsymbol{\omega})$ is $f(\boldsymbol{r} - \boldsymbol{\beta})$.

10.6.9. Solve

$$\frac{\partial u}{\partial t} + \boldsymbol{v}_0 \cdot \nabla u = k\nabla^2 u \qquad \begin{array}{c} t > 0 \\ -\infty < x < \infty, -\infty < y < \infty \end{array}$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

[*Hint*: See Exercise 10.6.7.] Show how the influence function is altered by the convection term $\boldsymbol{v}_0 \cdot \nabla u$.

10.6.10. Solve

$$\frac{\partial u}{\partial t} = k_1\frac{\partial^2 u}{\partial x^2} + k_2\frac{\partial^2 u}{\partial y^2} \qquad \begin{array}{c} t > 0 \\ -\infty < x < \infty, -\infty < y < \infty \end{array}$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

10.6.11. Consider

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad \begin{array}{l} x > 0 \\ y > 0 \end{array}$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve with the following boundary conditions:

***(a)** $u(0, y, t) = 0$ and $u(x, 0, t) = 0$

(b) $\dfrac{\partial u}{\partial x}(0, y, t) = 0$ and $\dfrac{\partial u}{\partial y}(x, 0, t) = 0$

(c) $u(0, y, t) = 0$ and $\dfrac{\partial u}{\partial y}(x, 0, t) = 0$

10.6.12. Consider

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad \begin{array}{l} 0 < x < L \\ y > 0 \end{array}$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve with the following boundary conditions:

***(a)** $u(0, y, t) = 0$, $u(L, y, t) = 0$, $u(x, 0, t) = 0$

(b) $u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad \dfrac{\partial u}{\partial y}(x, 0, t) = 0$

(c) $\dfrac{\partial u}{\partial x}(0, y, t) = 0, \quad \dfrac{\partial u}{\partial x}(L, y, t) = 0, \quad \dfrac{\partial u}{\partial y}(x, 0, t) = 0$

10.6.13. Solve

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad \begin{matrix} t > 0 \\ 0 < y < H \\ -\infty < x < \infty \end{matrix}$$

subject to the initial condition

$$u(x, y, 0) = f(x, y)$$

and the boundary conditions

$$u(x, 0, t) = 0$$
$$u(x, H, t) = 0.$$

10.6.14. **(a)** Without deriving or applying a convolution theorem, solve for $u(x, y, z, t)$:

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ \text{such that} \\ u(x, y, z, 0) = f(x, y, z) \\ u(x, y, 0, t) = 0 \end{array}\right\} \quad \text{with} \quad \begin{matrix} t > 0 \\ -\infty < x < \infty \\ -\infty < y < \infty \\ 0 < z < \infty. \end{matrix}$$

(b) Simplify your answer (developing convolution ideas, as needed).

10.6.15. Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad z > 0$$
$$u(x, y, 0) = f(x, y). \quad -\infty < x < \infty, \quad -\infty < y < \infty$$

***(a)** Determine the double Fourier transform of u.

***(b)** Solve for $u(x, y, z)$ by calculating the inversion integral in polar coordinates. [*Hints*: It is easier first to solve for w, where $u = \partial w/\partial z$. Then the following integral may be useful:

$$\int_0^{2\pi} \frac{d\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta} = \frac{2\pi}{ab}.$$

(This integral may be derived using the change of variables $z = e^{i\theta}$ and the theory of complex variables.)]

(c) Compare your result with the Green's function result.

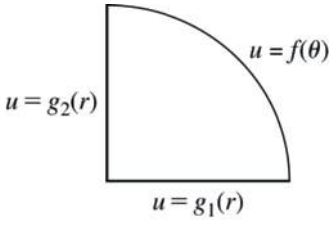
10.6.16. Consider Laplace's equation

$$\nabla^2 u = 0$$

inside a quarter-circle (Fig. 10.6.5) (a finite region) subject to the boundary conditions

$$u(a, \theta) = f(\theta), \quad u(r, 0) = g_1(r), \quad u\left(r, \frac{\pi}{2}\right) = g_2(r).$$

Figure 10.6.5



(a) Divide $u = u_1 + u_2 + u_3$ into three problems such that

$$\begin{aligned} &u_1(a, \theta) = 0, && u_2(a, \theta) = 0, && u_3(a, \theta) = f(\theta) \\ &u_1(r, 0) = g_1(r), && u_2(r, 0) = 0, && u_3(r, 0) = 0 \\ &u_1\left(r, \frac{\pi}{2}\right) = 0, && u_2\left(r, \frac{\pi}{2}\right) = g_2(r), && u_3\left(r, \frac{\pi}{2}\right) = 0. \end{aligned}$$

Solve for $u_3(r, \theta)$.

***(b)** Solve for $u_2(r, \theta)$. [*Hints*: Try to use the method of separation of variables, $u_2(r, \theta) = \phi(r)h(\theta)$. Show that $\phi(r) = \sin\left[\sqrt{\lambda}\ln(r/a)\right]$ for all $\lambda > 0$. It will be necessary to use a Fourier sine transform in the variable $\rho = -\ln(r/a)$.] [*Comments*: Here, a singular Sturm–Liouville problem on a *finite* interval occurs that has a continuous spectrum. For the wave equation on a quarter-circle, the corresponding singular Sturm–Liouville problem (involving Bessel functions) has a discrete spectrum.]

10.6.17. Reconsider the problem for $u_2(r, \theta)$ described in Exercise 10.6.16(b). Introduce the independent variable $\rho = -\ln(r/a)$ instead of r. [*Comment:* In complex variables this is the conformal transformation

$$w = -\ln\left(\frac{z}{a}\right), \quad \text{where} \quad z = x + iy$$

(i.e., $\ln(z/a) = \ln|z/a| + i\theta$).]

(a) Determine the partial differential equation for u_2 in the variables ρ and θ. Sketch the boundary on a set of (ρ, θ) axes.

(b) Solve this problem. [*Hint*: Section 10.6.2 may be helpful.]

(c) Compare to Exercise 10.6.16.

***10.6.18.** Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x, 0) &= 0, \\ \frac{\partial u}{\partial t}(x, 0) &= g(x). \end{aligned}$$

[*Hint*: Use the convolution theorem and see Exercise 10.3.6.]

10.6.19. For the problem in Section 10.6.1, we showed that

$$\bar{U}(\omega, t) = F(\omega)\cos c\omega t.$$

Obtain $u(x, t)$ using the convolution theorem. [*Hint*: $\cos c\omega t$ does not have an ordinary inverse Fourier transform. However, obtain the inverse Fourier transform of $\cos c\omega t$ using Dirac delta functions.]

10.6.20. Consider Exercise 10.6.17 for $u_3(r, \theta)$ rather than $u_2(r, \theta)$.

10.6.21. Solve the wave equation, $-\infty < x < \infty$,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(a) subject to the conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t}(x, 0) = 0$

(b) subject to the conditions $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = g(x)$

10.6.22. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with $x > 0$ and $y > 0$, subject to $u(x, 0) = 0$, $u(0, y) = g(y)$. Simplify as much as possible.

10.7 Scattering and Inverse Scattering

In our previous work, we have studied eigenvalue problems with a discrete spectrum and eigenvalue problems with a continuous spectrum. In this section, we briefly study an eigenvalue problem with both a continuous spectrum and a discrete spectrum. One motivation of this is from separation of variables $[\psi(x, t) = e^{-i\lambda t}\phi(x)]$ for the normalized version of the **Schrödinger**

equation from quantum mechanics, $i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + u(x)\psi$. If the potential $u(x)$ is time independent, then we obtain the boundary value problem

$$\boxed{\frac{d^2\phi}{dx^2} + (\lambda - u(x))\phi = 0.} \tag{10.7.1}$$

We will assume $u(x)$ decays sufficiently fast as $x \to \pm\infty$ so that we will be interested in solutions to the partial differential equation that decay to zero as $x \to \pm\infty$. If $u(x) = 0$, then $\lambda = k^2 > 0$ is the continuous spectrum so that $\phi(x) = e^{\pm ikx}$ corresponds to the Fourier transform. If $u(x) \neq 0$, then the differential equation (10.7.1) usually cannot be solved exactly, so that we will discuss theoretical aspects of (10.7.1). We claim that there is always a continuous spectrum analogous to the Fourier transform and that there may also be a discrete spectrum.

CONTINUOUS SPECTRUM

If $u(x) \to 0$ sufficiently fast as $x \to \pm\infty$, then intuitively (which can be derived mathematically) $\phi(x)$ should be approximated by $e^{\pm ikx}$ as $x \to \pm\infty$. It is helpful to imagine that e^{-ikx} corresponds to a left-going wave [since the corresponding time-dependent solution $e^{-ikx}e^{-ik^2t} = e^{-ik(x+kt)}$ is left-going for $k > 0$] and that e^{+ikx} corresponds to a right-going wave. We analyze a special solution of (10.7.1) that corresponds to an incoming wave (left-going from the right as $x \to +\infty$) of unit amplitude.

The variable coefficient (potential) $u(x)$ causes some reflection, and the amplitude transmitted is also changed so that

$$\boxed{\phi(x) \sim e^{-ikx} + R(k)e^{+ikx} \text{ as } x \to +\infty} \tag{10.7.2}$$

$$\boxed{\phi(x) \sim T(k)e^{-ikx} \text{ as } x \to -\infty,} \tag{10.7.3}$$

where $R(k)$ is called the **reflection coefficient** and $T(k)$ the **transmission coefficient**. The reflection and transmission coefficients are complex and can be shown to satisfy a conservation of energy equation $|R|^2 + |T|^2 = 1$. Other solutions to (10.7.1) can be obtained.

Discrete spectrum

There can be a discrete set of negative **eigenvalues** λ that satisfy (10.7.1) subject to the boundary condition that $\phi(x) \to 0$ as $x \to \pm\infty$:

$$\boxed{\lambda = -\kappa_n^2 \text{ (with } \kappa_n > 0), n = 1, 2, \ldots, N.} \tag{10.7.4}$$

The corresponding eigenfunctions $\phi_n(x)$ are known as **bound states** and may be chosen to satisfy

$$\boxed{\phi_n(x) \sim c_n e^{-\kappa_n x} \text{ as } x \to +\infty} \tag{10.7.5}$$

$$\boxed{\int_{-\infty}^{\infty} \phi_n^2(x)\,dx = 1.} \tag{10.7.6}$$

In order for the integral to be finite, the eigenfunction must decay exponentially as both $x \to \pm\infty$. Other normalizations (10.7.6) of the bound states are possible. In general, for bound states ($\lambda < 0$), it can be shown from differential equation (10.7.1) that if the coefficient $\lambda - u(x) < 0$ for all x, then the solution is like an exponential of one kind for all x and cannot be a bound state. Thus we conclude that a necessary (but not sufficient) condition for a bound state is that $u(x)$ must be less than zero somewhere.

EXAMPLE OF CONTINUOUS SPECTRUM: DELTA FUNCTION POTENTIAL

As an elementary example, we assume the potential $u(x)$ is a Dirac delta function

$$u(x) = U\delta(x), \tag{10.7.7}$$

and we assume ϕ is continuous and $\frac{d\phi}{dx}$ satisfies the jump condition found by integrating (10.7.1), $\frac{d\phi}{dx}\Big|_{0-}^{0+} = U\phi(0)$. The differential equation is easy for $x \neq 0$ so that (10.7.2) and (10.7.3) are valid not only asymptotically but also for $x > 0$ and $x < 0$, respectively. The continuity of $\phi(x)$ at $x = 0$ implies

$$1 + R(k) = T(k). \tag{10.7.8}$$

The jump condition $\frac{d\phi}{dx}\Big|_{0-}^{0+} = U\phi(0)$ becomes

$$-ik + ikR(k) + ikT(k) = UT(k). \tag{10.7.9}$$

Some elementary algebraic steps are necessary to show that $R(k) = \frac{U}{2ik-U}$ and $T(k) = \frac{2ik}{2ik-U}$. In this example, we can show that $|R|^2 + |T|^2 = 1$, which is a general result. ■

EXAMPLE OF DISCRETE EIGENVALUES AND EIGENFUNCTIONS: DELTA FUNCTION POTENTIAL

For the delta function potential (10.7.7), (10.7.5) is valid for $x > 0$, while it is simpler to note that $\phi_n(x) = b_n e^{\kappa_n x}$ for $x < 0$. The continuity at $x = 0$ implies that $b_n = c_n$. The jump condition at $x = 0$, $\frac{d\phi_n}{dx}\Big|_{0-}^{0+} = U\phi_n(0)$, becomes $-c_n\kappa_n - c_n\kappa_n = Uc_n$. From this we obtain

$$\kappa_n = -\frac{1}{2}U. \tag{10.7.10}$$

Since $\kappa_n > 0$, there are no discrete eigenvalues for the delta function potential if $U > 0$, but there is exactly one discrete negative eigenvalue if $U < 0$, namely,

$$\lambda = -\kappa_n^2 = -\frac{1}{4}U^2. \tag{10.7.11}$$

A bound state exists only for the negative delta function, which is consistent with the necessary condition mentioned previously: that $u(x)$ must be less than zero somewhere for a bound state to exist. ■

Inverse scattering

For a long time, theorists wondered whether a unique potential could be determined from the reflection and transmission coefficients. Gelfand and Levitan (1955) proved the remarkable result that the potential could be uniquely determined if the reflection and transmission coefficients were supplemented by the knowledge of the discrete spectrum. In particular, the potential could be reconstructed from

$$\boxed{u(x) = -2\frac{d}{dx}K(x, x),} \tag{10.7.12}$$

where $K(x, y)$ is the unique solution of the Gelfand–Levitan–Marchenko nonhomogeneous linear integral equation

$$\boxed{K(x, y) + F(x + y) + \int_x^\infty K(x, z)F(y + z)\,dz = 0, \text{ for } y > x.} \tag{10.7.13}$$

The kernel and nonhomogeneous term $F(s)$ is the generalized inverse Fourier transform of the reflection coefficient $R(k)$ (10.7.2) and hence includes contributions from the discrete spectrum (10.7.5) as well:

$$\boxed{F(s) = \sum_{n=1}^N c_n^2 e^{-\kappa_n s} + \frac{1}{2\pi}\int_{-\infty}^\infty R(k)e^{iks}\,dk.} \tag{10.7.14}$$

Here $\lambda = -\kappa_n^2$ are the discrete eigenvalues, and c_n are related to the bound states.

Exercises 10.7

10.7.1. Consider the step potential $u(x) = U$ for $-1 < x < 1$ and $u(x) = 0$ otherwise.

(a) Find reflection and transmission coefficients if $U < 0$.

(b) By solving (10.7.1), determine the discrete spectrum ($\lambda = -\kappa^2$ with $\kappa > 0$). [*Hints*: (i) Since we claim $\lambda - u(x)$ must be greater than zero somewhere, you may assume $U < \lambda = -\kappa^2$. (ii) The algebra is easier if even and odd bound states are analyzed separately.]

(c) Find reflection and transmission coefficients (for all positive λ) if $U > 0$.

10.7.2. Show that $|R|^2 + |T|^2 = 1$ for the delta function potential in the text.

10.7.3. **(a)** Show that the Wronskian $W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_2\phi_1'$ of two independent solutions of (10.7.1) is a constant.

(b) $\phi(x)$ satisfying (10.7.2) and (10.7.3) and its complex conjugate $\phi^*(x)$ are two linearly independent solutions of (10.7.1). By computing the Wronskian of these two solutions using the asymptotic conditions as $x \to \pm\infty$, show that $|R|^2 + |T|^2 = 1$.

10.7.4. **Orthogonality conditions**

(a) Show that the discrete eigenfunctions are orthogonal to eigenfunctions of the continuous spectrum.

(b) Show that one discrete eigenfunction is orthogonal to another discrete eigenfunction.

10.7.5. **(a)** Show that a pole of the transmission coefficient in the upper-half complex k-plane corresponds to a value of the discrete spectrum.

(b) Verify that this occurs for the delta function example in the text.

10.7.6. A **reflectionless potential** is one in which the reflection coefficient is zero for all k. Find an example of a reflectionless potential by solving the Gelfand–Levitan–Marchenko integral equation (10.7.13). Assume there is one discrete eigenvalue $\lambda = -\kappa^2$, and assume the corresponding coefficient c^2 is given. [*Hint*: The integral equation is separable.]

11
Green's Functions for Wave and Heat Equations

11.1 Introduction

In Chapter 9, we had some success in obtaining Green's functions for time-independent problems. One particularly important idea was the use of infinite space Green's functions. Here we will analyze Green's functions for the heat and wave equations. Problems with one, two, and three spatial dimensions will be considered. We will derive Green's formulas for the heat and wave equation and use them to represent the solution of nonhomogeneous problems (nonhomogeneous sources and nonhomogeneous boundary conditions) in terms of the Green's function. We will obtain elementary formulas for these infinite space Green's functions. For the wave equation, we will derive the one-dimensional infinite space Green's function by utilizing the general solution of the one-dimensional wave equation. We will derive the infinite space Green's function for the three-dimensional wave equation by making a well-known transformation to a one-dimensional wave equation. For the heat equation, we will derive the infinite space Green's function by comparing the Green's function with the appropriate solution of the initial value problem for the infinite space heat equation solved in Chapter 10.

11.2 Green's Functions for the Wave Equation

11.2.1 INTRODUCTION

In this section, we solve the wave equation with possibly time-dependent sources,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + Q(\boldsymbol{x}, t),} \tag{11.2.1}$$

subject to the two initial conditions,

$$\boxed{u(\boldsymbol{x}, 0) = f(\boldsymbol{x})} \tag{11.2.2}$$

$$\boxed{\frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = g(\boldsymbol{x}).} \tag{11.2.3}$$

If the problem is on a finite or semi-infinite region, then in general, $u(\boldsymbol{x}, t)$ will satisfy nonhomogeneous conditions on the boundary. We will determine simultaneously how to solve this problem in one, two, and three dimensions. (In one dimension $\nabla^2 = \partial^2/\partial x^2$.)

We introduce the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ as a solution due to a concentrated source at $\boldsymbol{x} = \boldsymbol{x}_0$ acting instantaneously only at $t = t_0$:

$$\boxed{\frac{\partial^2 G}{\partial t^2} = c^2\nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0),} \tag{11.2.4}$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ is the Dirac delta function of the appropriate dimension. For finite or semi-infinite problems, G will satisfy the related homogeneous boundary conditions corresponding to the nonhomogeneous ones satisfied by $u(\boldsymbol{x}, t)$.

The Green's function is the response at $\boldsymbol{x}$ at time t due to a source located at $\boldsymbol{x}_0$ at time t_0. Since we desire the Green's function G to be the response due only to this source acting at $t = t_0$ (not due to some nonzero earlier conditions), we insist that the response G will be zero before the source acts ($t < t_0$):

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \qquad \text{for } t < t_0,} \tag{11.2.5}$$

known as the **causality principle** (see Section 9.2).

The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ depends on only the time after the occurrence of the concentrated source. If we introduce the **elapsed** time, $T = t - t_0$,

$$\frac{\partial^2 G}{\partial T^2} = c^2 \nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(T)$$
$$G = 0 \qquad \text{for} \qquad T < 0,$$

then G is also seen to be the response due to a concentrated source at $\boldsymbol{x} = \boldsymbol{x}_0$ at $T = 0$. We call this the **translation** property,

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}, t - t_0; \boldsymbol{x}_0, 0).} \tag{11.2.6}$$

11.2.2 GREEN'S FORMULA FOR THE WAVE EQUATION

Before solving for the Green's function (in various dimensions), we will show how the solution of the nonhomogeneous wave equation (11.2.1) (with nonhomogeneous initial and boundary conditions) is obtained using the Green's function. For time-independent problems (nonhomogeneous Sturm–Liouville type or the Poisson equation), the relationship between the nonhomogeneous solution and the Green's function was obtained using Green's formula:

Sturm–Liouville operator $[L = d/dx(pd/dx) + q]$:

$$\int_a^b [uL(v) - vL(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b. \tag{11.2.7}$$

Three-dimensional Laplacian $(L = \nabla^2)$:

$$\iiint [uL(v) - vL(u)]d^3x = \oiint (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS, \tag{11.2.8}$$

where $d^3x = dV = dx\, dy\, dz$. There is a corresponding result for the two-dimensional Laplacian.

To extend these ideas to the nonhomogeneous wave equation, we introduce the appropriate linear differential operator:

$$\boxed{L = \frac{\partial^2}{\partial t^2} - c^2\nabla^2.} \tag{11.2.9}$$

Using this notation, the nonhomogeneous wave equation (11.2.1) satisfies

$$L(u) = Q(\boldsymbol{x}, t), \tag{11.2.10}$$

while the Green's function (11.2.4) satisfies

$$L(G) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0). \tag{11.2.11}$$

For the wave operator L [see (11.2.9)], we will derive a Green's formula analogous to (11.2.7) and (11.2.8). We will use a *notation* corresponding to three dimensions but will make clear modifications (when necessary) for one and two dimensions. For time-dependent problems, L has both space and time variables. Formulas analogous to (11.2.7) and (11.2.8) are expected to exist, but integration will occur over both space $\boldsymbol{x}$ and time t. Since for the wave operator

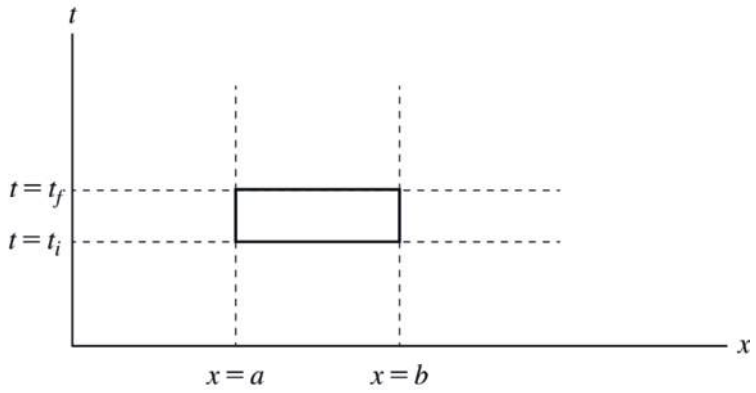
$$uL(v) - vL(u) = u\frac{\partial^2 v}{\partial t^2} - v\frac{\partial^2 u}{\partial t^2} - c^2(u\nabla^2 v - v\nabla^2 u),$$

the previous Green's formulas will yield the new **"Green's formula for the wave equation"**:

$$\boxed{\begin{aligned}&\int_{t_i}^{t_f}\iiint [uL(v) - vL(u)]d^3x\,dt \\ &\quad = \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\Big|_{t_i}^{t_f} d^3x - c^2\int_{t_i}^{t_f}\left(\oiint (u\nabla v - v\nabla u)\cdot\hat{\mathbf{n}}\,dS\right)dt,\end{aligned}} \tag{11.2.12}$$

where $\iiint$ indicates integration over the three-dimensional space ($\int_a^b$ for one-dimensional problems) and $\oiint$ indicates integration over its boundary ($|_a^b$ for one-dimensional problems). The terms on the right-hand side represent contributions from the boundaries: the spatial boundaries for all time and the temporal boundaries ($t = t_i$ and $t = t_f$) for all space. These space-time boundaries are illustrated (for a one-dimensional problem) in Fig. 11.2.1.

Figure 11.2.1 Space-time boundaries for a one-dimensional wave equation.



For example, *if* both u and v satisfy the usual type of homogeneous boundary conditions (in space, for all time), then $\oiint (u\nabla v - v\nabla u)\cdot\hat{\mathbf{n}}\,dS = 0$, but

$$\iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\Big|_{t_i}^{t_f} d^3x$$

may not equal zero due to contributions from the "initial" time t_i and "final" time t_f.

11.2.3 RECIPROCITY

For time-independent problems, we have shown that the Green's function is symmetric, $G(\boldsymbol{x}, \boldsymbol{x}_0) = G(\boldsymbol{x}_0, \boldsymbol{x})$. We proved this result using Green's formula for two different Green's functions [$G(\boldsymbol{x}, \boldsymbol{x}_1)$ and $[G(\boldsymbol{x}, \boldsymbol{x}_2)]$. The result followed because the boundary terms in Green's formula vanished.

For the wave equation there is a somewhat analogous property. The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ satisfies

$$\frac{\partial^2 G}{\partial t^2} - c^2\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0), \tag{11.2.13}$$

subject to the causality principle,

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \quad \text{for } t < t_0. \tag{11.2.14}$$

G will be nonzero for $t > t_0$. To utilize Green's formula (to prove reciprocity), we need a second Green's function. If we choose it to be $G(\boldsymbol{x}, t; \boldsymbol{x}_A, t_A)$, then the contribution $\int_{t_i}^{t_f}\iiint (u\nabla v - v\nabla u)\cdot\hat{\mathbf{n}}\,dS\,dt$ on the spatial boundary (or infinity) vanishes, but the contribution

$$\iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\Big|_{t_i}^{t_f} d^3x$$

on the time boundary will not vanish at both $t = t_i$ and $t = t_f$. In time, our problem is an initial value problem, not a boundary value problem. If we let $t_i \leq t_0$ in Green's formula, the "initial" contribution will vanish.

For a second Green's function, we are interested in varying the source time t, $G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)$, what we call the **source-varying Green's function**. From the translation property,

$$\boxed{G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = G(\boldsymbol{x}, -t; \boldsymbol{x}_1, -t_1),} \tag{11.2.15}$$

since the elapsed times are the same $[-t - (-t_1) = t_1 - t]$. By causality, these are zero if $t_1 < t$ (or, equivalently, $-t < -t_1$):

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = 0, \qquad t > t_1. \tag{11.2.16}$$

We call this the **source-varying causality principle**. By introducing this Green's function, we will show that the "final" contribution from Green's formula may vanish.

To determine the differential equation satisfied by the source-varying Green's function, we let $t = -\tau$, in which case, from (11.2.15),

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = G(\boldsymbol{x}, \tau; \boldsymbol{x}_1, -t_1).$$

This is the ordinary (variable-response position) Green's function, with τ being the time variable. It has a concentrated source located at $\boldsymbol{x} = \boldsymbol{x}_1$ when $\tau = -t_1 (t = t_1)$:

$$\left(\frac{\partial^2}{\partial \tau^2} - c^2 \nabla^2\right) G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1).$$

Since $\tau = -t$, from the chain rule $\partial/\partial\tau = -\partial/\partial t$, but $\partial^2/\partial\tau^2 = \partial^2/\partial t^2$. Thus, the wave operator is symmetric in time, and therefore

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right) G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = L[G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)] = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1).} \tag{11.2.17}$$

A reciprocity formula results from Green's formula (11.2.12) using two Green's functions, one with varying response time,

$$u = G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0), \tag{11.2.18}$$

and one with varying source time,

$$v = G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t). \tag{11.2.19}$$

Both satisfy partial differential equations involving the *same* wave operator, $L = \frac{\partial^2}{\partial t^2} - c^2 \nabla^2$. We integrate from $t = -\infty$ to $t = +\infty$ in Green's formula (11.2.12) (i.e., $t_i = -\infty$ and $t_f = +\infty$). Since both Green's functions satisfy the same homogeneous boundary conditions, Green's formula (11.2.12) yields

$$\begin{aligned} \int_{-\infty}^{\infty} \iiint [u\delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1) - v\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)]\, d^3x\, dt \\ = \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\Big|_{-\infty}^{+\infty} d^3x. \end{aligned} \tag{11.2.20}$$

From the causality principles, u and $\partial u/\partial t$ vanish for $t < t_0$, and v and $\partial v/\partial t$ vanish for $t > t_1$. Thus, the r.h.s. of (11.2.20) vanishes. Consequently, using the properties of the Dirac delta function, u at $\boldsymbol{x} = \boldsymbol{x}_1$, $t = t_1$ equals v at $\boldsymbol{x} = \boldsymbol{x}_0$, $t = t_0$:

$$\boxed{G(\boldsymbol{x}_1, t_1; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}_0, t_1; \boldsymbol{x}_1, t_0),} \tag{11.2.21}$$

the **reciprocity formula** for the Green's function for the wave equation. Assuming that $t_1 > t_0$, the response at $\boldsymbol{x}_1$ (at time t_1) due to a source at $\boldsymbol{x}_0$ (at time t_0) is the same as the response at $\boldsymbol{x}_0$ (at time t_1) due to a source at $\boldsymbol{x}_1$, *as long as the elapsed times from the sources are the same*. In this case, it is seen that interchanging the source and location points has no effect, what we called Maxwell reciprocity for time-independent Green's functions.

11.2.4 USING THE GREEN'S FUNCTION

As with our earlier work, the relationship between the Green's function and the solution of the nonhomogeneous problem is established using the appropriate Green's formula, (11.2.12). We let

$$u = u(\boldsymbol{x}, t) \tag{11.2.22}$$

$$v = G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t) = G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t), \tag{11.2.23}$$

where $u(\boldsymbol{x}, t)$ is the solution of the nonhomogeneous wave equation satisfying

$$L(u) = Q(\boldsymbol{x}, t)$$

subject to the given initial conditions for $u(\boldsymbol{x}, 0)$ and $\partial u/\partial t(\boldsymbol{x}, 0)$, and where $G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)$ is the source-varying Green's function satisfying (11.2.17):

$$L[G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)] = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)$$

subject to the source-varying causality principle

$$G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t) = 0 \qquad \text{for} \quad t > t_0.$$

G satisfies homogeneous boundary conditions, but u may not. We use Green's formula (11.2.12) with $t_i = 0$ and $t_f = t_{0+}$; we integrate just beyond the appearance of a concentrated source at $t = t_0$:

$$\int_0^{t_{0+}} \iiint [u(\boldsymbol{x}, t)\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0) - G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)Q(\boldsymbol{x}, t)]\, d^3x\, dt$$
$$= \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\Big|_0^{t_{0+}} d^3x - c^2 \int_0^{t_{0+}} \left[\oiint (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS\right] dt.$$

At $t = t_{0+}$, $v = 0$ and $\partial v/\partial t = 0$, since we are using the source-varying Green's function. We obtain, using the reciprocity formula (11.2.21),

$$u(\boldsymbol{x}_0, t_0) = \int_0^{t_{0+}} \iiint G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t)Q(\boldsymbol{x}, t) d^3x\, dt$$
$$+ \iiint \left[\frac{\partial u}{\partial t}(\boldsymbol{x}, 0)G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, 0) - u(\boldsymbol{x}, 0)\frac{\partial}{\partial t}G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, 0)\right] d^3x$$
$$- c^2 \int_0^{t_{0+}} \left[\oiint (u(\boldsymbol{x}, t)\nabla G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t) - G(\boldsymbol{x}_0, t_0; \boldsymbol{x}, t)\nabla u(\boldsymbol{x}, t)) \cdot \hat{\mathbf{n}}\, dS\right] dt.$$

It can be shown that t_{0+} may be replaced by t_0 in these limits. If the roles of $\boldsymbol{x}$ and $\boldsymbol{x}_0$ are interchanged (as well as t and t_0), we obtain a representation formula for $u(\boldsymbol{x}, t)$ in terms of the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$:

$$u(\boldsymbol{x}, t) = \int_0^t \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)Q(\boldsymbol{x}_0, t_0) d^3\boldsymbol{x}_0\, dt_0$$
$$+ \iiint \left[\frac{\partial u}{\partial t_0}(\boldsymbol{x}_0, 0)G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0) - u(\boldsymbol{x}_0, 0)\frac{\partial}{\partial t_0}G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)\right] d^3x_0$$
$$- c^2 \int_0^t \left[\oiint (u(\boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) - G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} u(\boldsymbol{x}_0, t_0)) \cdot \hat{\mathbf{n}}\, dS_0\right] dt_0. \tag{11.2.24}$$

Note that $\nabla_{\boldsymbol{x}_0}$ means a derivative with respect to the source position. Equation (11.2.24) expresses the response due to the three kinds of nonhomogeneous terms: source terms, initial conditions, and nonhomogeneous boundary conditions. In particular, the initial position $u(\boldsymbol{x}_0, 0)$ has an influence function

$$-\frac{\partial}{\partial t_0}G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)$$

(meaning the source time derivative evaluated initially), while the influence function for the initial velocity is $G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)$.

Furthermore, for example, if u is given on the boundary, then G satisfies the related homogeneous boundary condition; that is, $G = 0$ on the boundary. In this case, the boundary term in (11.2.24) simplifies to

$$-c^2 \int_0^t \left[\iint u(\boldsymbol{x}_0, t_0) \nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\mathbf{n}} \, dS_0 \right] dt_0.$$

The influence function for this nonhomogeneous boundary condition is

$$-c^2 \nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\mathbf{n}}.$$

This is $-c^2$ times the source outward normal derivative of the Green's function.

11.2.5 GREEN'S FUNCTION FOR THE WAVE EQUATION

We recall that the Green's function for the wave equation satisfies (11.2.4) and (11.2.5):

$$\boxed{\frac{\partial^2 G}{\partial t^2} - c^2 \nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)} \tag{11.2.25}$$

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \quad \text{for} \quad t < t_0, \tag{11.2.26}$$

subject to homogeneous boundary conditions. We will describe the Green's function in a different way.

11.2.6 ALTERNATE DIFFERENTIAL EQUATION FOR THE GREEN'S FUNCTION

Using Green's formula, the solution of the wave equation with homogeneous boundary conditions and with no sources, $Q(\boldsymbol{x}, t) = 0$, is represented in terms of the Green's function by (11.2.24),

$$u(\boldsymbol{x}, t) = \iiint \left[\frac{\partial u}{\partial t_0}(\boldsymbol{x}_0, 0) G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0) - u(\boldsymbol{x}_0, 0) \frac{\partial}{\partial t_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0) \right] d^3 \boldsymbol{x}_0.$$

From this we see that G is also the influence function for the initial condition for the derivative $\frac{\partial u}{\partial t}$, while $-\frac{\partial G}{\partial t_0}$ is the influence function for the initial condition for u. If we solve the wave equation with the initial conditions $u = 0$ and $\frac{\partial u}{\partial t} = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$, the solution is the Green's function itself. Thus, the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ satisfies the ordinary wave equation with no sources,

$$\boxed{\frac{\partial^2 G}{\partial t^2} - c^2 \nabla^2 G = 0,} \tag{11.2.27}$$

subject to homogeneous boundary conditions and the specific concentrated initial conditions at $t = t_0$:

$$G = 0 \tag{11.2.28}$$

$$\frac{\partial G}{\partial t} = \delta(\boldsymbol{x} - \boldsymbol{x}_0). \tag{11.2.29}$$

The Green's function for the wave equation can be determined directly from the initial value problem (11.2.27)–(11.2.29) rather than from its defining differential equation (11.2.4) or (11.2.26). Exercise 11.2.9 outlines another derivation of (11.2.27)–(11.2.29) in which the defining equation (11.2.4) is integrated from t_{0-} until t_{0+}.

11.2.7 INFINITE SPACE GREEN'S FUNCTION FOR THE ONE-DIMENSIONAL WAVE EQUATION AND D'ALEMBERT'S SOLUTION

We will determine the infinite space Green's function by solving the one-dimensional wave equation, $\frac{\partial^2 G}{\partial t^2} - c^2 \frac{\partial^2 G}{\partial x^2} = 0$, subject to initial conditions (11.2.28) and (11.2.29). In Chapter 12

(briefly mentioned in Chapter 4) it is shown that there is a remarkable general solution of the one-dimensional wave equation,

$$G = f(x - ct) + g(x + ct), \tag{11.2.30}$$

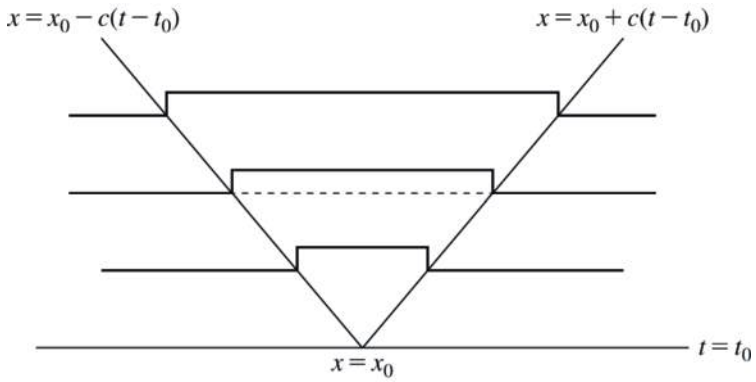
where $f(x - ct)$ is an arbitrary function moving to the right with velocity c and $g(x + ct)$ is an arbitrary function moving to the left with velocity $-c$. It can be verified by direct substitution that (11.2.30) solves the wave equation. For ease, we assume $t_0 = 0$ and $x_0 = 0$. Since from (11.2.28), $G = 0$ at $t = 0$, it follows that in this case $g(x) = -f(x)$, so that $G = f(x - ct) - f(x + ct)$. We calculate $\frac{\partial G}{\partial t} = -cf'(x - ct) - cf'(x + ct)$. In order to satisfy the initial condition (11.2.29), $\delta(x) = \frac{\partial G}{\partial t}\big|_{t=0} = -2c\frac{df(x)}{dx}$. By integration, $f(x) = -\frac{1}{2c}H(x) + k$, where $H(x)$ is the Heaviside step function (and k is an unimportant constant of integration):

$$\boxed{G(x, t; 0, 0) = \frac{1}{2c}[H(x + ct) - H(x - ct)] = \begin{cases} 0 & |x| > ct \\ \frac{1}{2c} & |x| < ct. \end{cases}} \tag{11.2.31}$$

Thus, **the infinite space Green's function for the one-dimensional wave equation is an expanding rectangular pulse moving at the wave speed c**, as is sketched in Fig. 11.2.2. Initially (in general, at $t = t_0$), it is located at one point $x = x_0$. Each end spreads out at velocity c. In general,

$$\boxed{G(x, t; x_0, t_0) = \frac{1}{2c}\{H[(x - x_0) + c(t - t_0)] - H[(x - x_0) - c(t - t_0)]\}.} \tag{11.2.32}$$

Figure 11.2.2 Green's function for the one-dimensional wave equation.



D'Alembert's solution

To illustrate the use of this Green's function, consider the initial value problem for the wave equation without sources on an infinite domain $-\infty < x < \infty$:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{11.2.33}$$

$$u(x, 0) = f(x) \tag{11.2.34}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x). \tag{11.2.35}$$

In the formula (11.2.24), the boundary contribution[1] vanishes since $G = 0$ for x sufficiently large (positive or negative); see Fig. 11.2.2. Since there are no sources, $u(x, t)$ is caused only by the initial conditions:

$$u(x, t) = \int_{-\infty}^{\infty}\left[g(x_0)G(x, t; x_0, 0) - f(x_0)\frac{\partial}{\partial t_0}G(x, t; x_0, 0)\right]dx_0.$$

[1]The boundary contribution for an infinite problem is the limit as $L \to \infty$ of the boundaries of a finite region, $-L < x < L$.

We need to calculate $\frac{\partial}{\partial t_0}G(x, t; x_0, 0)$ from (11.2.32). Using properties of the derivative of a step function [see (9.3.32)], it follows that

$$\frac{\partial}{\partial t_0}G(x, t; x_0, t_0) = \frac{1}{2}[-\delta(x - x_0 + c(t - t_0)) - \delta(x - x_0 - c(t - t_0))]$$

and thus,

$$\frac{\partial}{\partial t_0}G(x, t; x_0, 0) = \frac{1}{2}[-\delta(x - x_0 + ct) - \delta(x - x_0 - ct)].$$

Finally, we obtain the solution of the initial value problem:

$$\boxed{u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x_0)\, dx_0.} \tag{11.2.36}$$

This is known as **d'Alembert's solution** of the wave equation. It can be obtained more simply by the method of characteristics (see Chapter 12). There we will discuss the physical interpretation of the one-dimensional wave equation.

Related problems

Semi-infinite or finite problems for the one-dimensional wave equation can be solved by obtaining the Green's function by the method of images. In some cases, transform or series techniques may be used. Of greatest usefulness is the method of characteristics.

11.2.8 INFINITE SPACE GREEN'S FUNCTION FOR THE THREE-DIMENSIONAL WAVE EQUATION (HUYGENS' PRINCIPLE)

We solve the infinite space Green's function using (11.2.27)–(11.2.29). The solution should be spherically symmetric and depend only on the distance $\rho = |\boldsymbol{x} - \boldsymbol{x}_0|$. Thus, the Green's function satisfies the spherically symmetric wave equation, $\frac{\partial^2 G}{\partial t^2} - c^2 \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial G}{\partial \rho}\right) = 0$. Through an unmotivated but very well-known transformation, $G = \frac{h}{\rho}$, the spherically symmetric wave equation simplifies:

$$0 = \frac{1}{\rho}\frac{\partial^2 h}{\partial t^2} - \frac{c^2}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial h}{\partial \rho} - h\right) = \frac{1}{\rho}\left(\frac{\partial^2 h}{\partial t^2} - c^2 \frac{\partial^2 h}{\partial \rho^2}\right).$$

Thus, h satisfies the one-dimensional wave equation. In Chapter 12 it is shown that the general solution of the one-dimensional wave equation can be represented by the sum of left- and right-going waves moving at velocity c. Consequently, we obtain the exceptionally significant result that **the general solution of the spherically symmetric wave equation** is

$$\boxed{G = \frac{f(\rho - ct) + g(\rho + ct)}{\rho}.} \tag{11.2.37}$$

$f(\rho - ct)$ is spherically expanding at velocity c, while $g(\rho + ct)$ is spherically contracting at velocity c. To satisfy the initial condition (11.2.28), $G = 0$ at $t = 0$, $g(\rho) = -f(\rho)$, and hence $G = \frac{f(\rho - ct) - f(\rho + ct)}{\rho}$. We calculate $\frac{\partial G}{\partial t} = -\frac{c}{\rho}[f'(\rho - ct) + f'(\rho + ct)]$. Thus, applying the initial condition (11.2.29) yields

$$\delta(\boldsymbol{x} - \boldsymbol{x}_0) = \left.\frac{\partial G}{\partial t}\right|_{t=0} = -\frac{2c}{\rho}\frac{df(\rho)}{d\rho}.$$

Here, $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ is a three-dimensional delta function. The function f will be constant, which we set to zero away from $\rho = 0$. We integrate in three-dimensional space over a sphere of radius R and obtain

$$1 = -2c\int_0^R \frac{1}{\rho}\frac{df}{d\rho} 4\pi\rho^2\, d\rho = 8\pi c\int_0^R f\, d\rho,$$

after an integration by parts using $f = 0$ for $\rho > 0$ (One way to justify integrating by parts is to introduce the even extensions of f for $\rho < 0$ so that $\int_0^R f\, d\rho = \frac{1}{2}\int_{-R}^R f\, d\rho$). Thus, $f = \frac{1}{4\pi c}\delta(\rho)$,

where $\delta(\rho)$ is the one-dimensional delta function that is even and hence satisfies $\int_0^R \delta(\rho)\, d\rho = \frac{1}{2}$. Consequently,

$$\boxed{G = \frac{1}{4\pi c\rho}[\delta(\rho - ct) - \delta(\rho + ct)].} \tag{11.2.38}$$

However, since $\rho > 0$ and $t > 0$, the latter Dirac delta function is always zero. To be more general, t should be replaced by $t - t_0$. In this way we obtain the **infinite space Green's function for the three-dimensional wave equation**:

$$\boxed{G(x, t; x_0, t_0) = \frac{1}{4\pi c\rho}[\delta(\rho - c(t - t_0))],} \tag{11.2.39}$$

where $\rho = |\boldsymbol{x} - \boldsymbol{x}_0|$. The Green's function for the three-dimensional wave equation is a spherical shell impulse spreading out from the source ($x = x_0$ and $t = t_0$) at radial velocity c with an intensity decaying proportional to $\frac{1}{\rho}$.

Huygens' principle

We have shown that a concentrated source at $\boldsymbol{x}_0$ (at time t_0) influences the position $\boldsymbol{x}$ (at time t) only if $|\boldsymbol{x} - \boldsymbol{x}_0| = c(t - t_0)$. The distance from source to location equals c times the time. The point source emits a wave moving in all directions at velocity c. At time $t - t_0$ later, the source's effect is located on a spherical shell a distance $c(t - t_0)$ away. This is part of what is known as **Huygens' principle**.

EXAMPLE

To be more specific, let us analyze the effect of sources, $Q(\boldsymbol{x}, t)$. Consider the wave equation with sources in infinite three-dimensional space with zero initial conditions. According to Green's formula (11.2.24),

$$u(\boldsymbol{x}, t) = \int_0^t \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) Q(\boldsymbol{x}_0, t_0) d^3x_0\, dt_0 \tag{11.2.40}$$

since the "boundary" contribution vanishes. Using the infinite three-dimensional space Green's function,

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi c}\int_0^t \iiint \frac{1}{\rho}\delta[\rho - c(t - t_0)] Q(\boldsymbol{x}_0, t_0)\, d^3x_0\, dt_0, \tag{11.2.41}$$

where $\rho = |\boldsymbol{x} - \boldsymbol{x}_0|$. The only sources that contribute satisfy $|\boldsymbol{x} - \boldsymbol{x}_0| = c(t - t_0)$. The effect at $\boldsymbol{x}$ at time t is caused by all received sources; the velocity of propagation of each source is c. ■

EXAMPLE

Here we discuss an even more specific example about the propagation of sound. Suppose a car is driving along the x-axis at a speed of 25 m/s, producing a sound of a pure tone at a frequency of 440 Hz. We model the source of the sound with a delta function,

$$Q(\boldsymbol{x}_0, t_0) = \cos(440 \bullet 2\pi t_0)\delta(x_0 - 25t_0).$$

If you are standing at the origin, $(x, y, z) = (0, 0, 0)$, what is the sound you hear as a function of time? According to (11.2.41) with $\rho = x_0$,

$$u(0, 0, 0, t) = \frac{1}{4\pi c}\int_0^t \iiint \frac{1}{x_0}\delta(x_0 - c(t - t_0))\cos(440 \bullet 2\pi t_0)\delta(x_0 - 25t_0) d^3x_0\, dt_0,$$

where $c = 343$ m/s is the speed of sound. Note we are using the three-dimensional space solution, so we are neglecting effects such as the reflections of sound off the ground. The only contributions to the integral are for $x_0 = 25t_0$ (so the second delta function is nonzero, corresponding to the

location of the source) and $x_0 = c(t - t_0)$ (the first delta function is nonzero, corresponding to the time that the sound reaches the origin). Eliminating x_0 between these two equations,

$$25\,t_0 = c(t - t_0) \Rightarrow t_0 = \frac{ct}{c + 25}.$$

Using the sifting property of the delta function, we evaluate the integral,

$$u(t) = \frac{1}{4\pi c}\frac{25 + c}{25ct}\cos\left(440 \cdot 2\pi\frac{ct}{c + 25}\right).$$

The frequency you hear is $440c/(c + 25)$. The movement of the car away from you has lowered the pitch. This change in frequency due to the movement of the source is called the **Doppler effect**. ■

11.2.9 TWO-DIMENSIONAL INFINITE SPACE GREEN'S FUNCTION

The two-dimensional Green's function for the wave equation is not as simple as the one- and three-dimensional cases. In Exercise 11.2.12 the two-dimensional Green's function is derived by the method of descent by using the three-dimensional solution with a two-dimensional source. The signal again propagates with velocity c, so that the solution is zero before the signal is received, that is for the elapsed time $t - t_0 < \frac{r}{c}$, where $r = |\boldsymbol{x} - \boldsymbol{x}_0|$ in two dimensions. However, once the signal is received, it is largest (infinite) at the moment the signal is first received, and then the signal gradually decreases:

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = \begin{cases} 0 & \text{if } r > c(t - t_0) \\ \frac{1}{2\pi c}\frac{1}{\sqrt{c^2(t-t_0)^2 - r^2}} & \text{if } r < c(t - t_0) \end{cases}. \tag{11.2.42}$$

11.2.10 SUMMARY

For the wave equation in any dimension, information propagates at velocity c. The Green's functions for the wave equation in one and three dimensions are different. Huygens' principle is valid only in three dimensions in which the influence of a concentrated source is felt only on the surface of the expanding sphere propagating at velocity c. In one dimension, the influence is felt uniformly inside the expanding pulse. In two dimensions, the largest effect occurs on the circumference corresponding to the propagation velocity c, but the effect diminishes behind the pulse.

Exercises 11.2

11.2.1. **(a)** Show that for $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$, $\partial G/\partial t = -\partial G/\partial t_0$.

(b) Use part (a) to show that the response due to $u(\boldsymbol{x}, 0) = f(\boldsymbol{x})$ is the time derivative of the response due to $\frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = f(\boldsymbol{x})$.

11.2.2. Express (11.2.24) for a one-dimensional problem.

11.2.3. If $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0$ for $\boldsymbol{x}$ on the boundary, explain why the corresponding term in (11.2.24) vanishes (for any $\boldsymbol{x}$).

11.2.4. For the one-dimensional wave equation, sketch $G(x, t; x_0, t_0)$ as a function of

(a) x with t fixed (x_0, t_0 fixed)

(b) t with x fixed (x_0, t_0 fixed)

11.2.5. **(a)** For the one-dimensional wave equation, for what values of x_0 (x, t, t_0 fixed) is $G(x, t; x_0, t_0) \neq 0$?

(b) Determine the answer to part (a) using the reciprocity property.

11.2.6. **(a)** Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + Q(x, t), \quad -\infty < x < \infty$$

with $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = 0$.

***(b)** What space-time locations of the source $Q(x, t)$ influence u at position x_1 and time t_1?

11.2.7. Reconsider Exercise 11.2.6 if $Q(x, t) = g(x)e^{-i\omega t}$.

***(a)** Solve for $u(x, t)$. Show that the influence function for $g(x)$ is an outward-propagating wave.

(b) Instead, determine a particular solution of the form $u(x, t) = \psi(x)e^{-i\omega t}$. (See Exercise 9.3.13.)

(c) Compare parts (a) and (b).

11.2.8. ***(a)** In three-dimensional infinite space, solve

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + g(\boldsymbol{x})e^{-i\omega t}$$

with zero initial conditions, $u(\boldsymbol{x}, 0) = 0$ and $\frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = 0$. From your solution, show that the influence function for $g(\boldsymbol{x})$ is an outward-propagating wave.

(b) Compare with Exercise 9.5.11.

11.2.9. Consider the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ for the wave equation. From (11.2.24) we easily obtain the influence functions for $Q(\boldsymbol{x}_0, t_0)$, $u(\boldsymbol{x}_0, 0)$, and $\partial u/\partial t_0(\boldsymbol{x}_0, 0)$. These results may be obtained in the following alternative way:

(a) For $t > t_{0+}$ show that

$$\frac{\partial^2 G}{\partial t^2} = c^2\nabla^2 G, \tag{11.2.43}$$

where (by integrating from t_{0-} to t_{0+})

$$G(\boldsymbol{x}, t_{0+}; \boldsymbol{x}_0, t_0) = 0 \tag{11.2.44}$$

$$\frac{\partial G}{\partial t}(\boldsymbol{x}, t_{0+}; \boldsymbol{x}_0, t_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0). \tag{11.2.45}$$

From (11.2.32), briefly explain why $G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)$ is the influence function for $\frac{\partial u}{\partial t_0}(\boldsymbol{x}_0, 0)$.

(b) Let $\phi = \partial G/\partial t$. Show that for $t > t_{0+}$,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\nabla^2\phi \tag{11.2.46}$$

$$\phi(\boldsymbol{x}, t_{0+}; \boldsymbol{x}_0, t_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0) \tag{11.2.47}$$

$$\frac{\partial \phi}{\partial t}(\boldsymbol{x}, t_{0+}; \boldsymbol{x}_0, t_0) = 0. \tag{11.2.48}$$

From (11.2.46)–(11.2.48), briefly explain why $\frac{-\partial G}{\partial t_0}(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)$ is the influence function for $u(\boldsymbol{x}_0, 0)$.

11.2.10. Consider

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} + Q(x, t), \qquad x > 0\\
u(x, 0) &= f(x)\\
\frac{\partial u}{\partial t}(x, 0) &= g(x)\\
u(0, t) &= h(t).
\end{aligned}$$

(a) Determine the appropriate Green's function using the method of images.

***(b)** Solve for $u(x, t)$ if $Q(x, t) = 0$, $f(x) = 0$, and $g(x) = 0$.

(c) For what values of t does $h(t)$ influence $u(x_1, t_1)$? Briefly interpret physically.

11.2.11. Reconsider Exercise 11.2.10:

(a) if $Q(x, t) \neq 0$, but $f(x) = 0$, $g(x) = 0$, and $h(t) = 0$

(b) if $f(x) \neq 0$, but $Q(x, t) = 0$, $g(x) = 0$, and $h(t) = 0$

(c) if $g(x) \neq 0$, but $Q(x, t) = 0$, $f(x) = 0$, and $h(t) = 0$

11.2.12. Consider the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_1, t_1)$ for the *two*-dimensional wave equation as the solution of the following *three*-dimensional wave equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \nabla^2 u + Q(\boldsymbol{x}, t) \\ u(\boldsymbol{x}, 0) &= 0 \\ \frac{\partial u}{\partial t}(\boldsymbol{x}, 0) &= 0 \\ Q(\boldsymbol{x}, t) &= \delta(x - x_1)\delta(y - y_1)\delta(t - t_1). \end{aligned}$$

We will solve for the two-dimensional Green's function by this **method of descent** (descending from three dimensions to two dimensions).

***(a)** Solve for $G(\boldsymbol{x}, t; \boldsymbol{x}_1, t_1)$ using the general solution of the three-dimensional wave equation. Here, the source $Q(\boldsymbol{x}, t)$ may be interpreted either as a point source in two dimensions or as a line source in three dimensions. [*Hint*: $\int_{-\infty}^{\infty} \cdots dz_0$ may be evaluated by introducing the three-dimensional distance ρ from the point source,

$$\rho^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_0)^2.]$$

(b) Show that G is a function only of the elapsed time $t - t_1$ and the two-dimensional distance r from the line source,

$$r^2 = (x - x_1)^2 + (y - y_1)^2.$$

(c) Where is the effect of an impulse felt after a time τ has elapsed? Compare to the one- and three-dimensional problems.

(d) Sketch G for $t - t_1$ fixed.

(e) Sketch G for r fixed.

11.2.13. Modify the example considering the propagation of sound to show that if you are chasing the car with a velocity of $v < 25$ m/s, then the frequency you hear is shifted by a factor of $(c + v)/(c + 25)$.

11.2.14. Consider the three-dimensional wave equation. Determine the response to a unit point source moving at the constant velocity $\boldsymbol{v}$:

$$Q(\boldsymbol{x}, t) = \delta(\boldsymbol{x} - \boldsymbol{v}t).$$

11.2.15. Solve the wave equation in infinite three-dimensional space without sources, subject to the initial conditions

(a) $u(\boldsymbol{x}, 0) = 0$ and $\frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = g(\boldsymbol{x})$. The answer is called **Kirchhoff's formula**, although it is due to Poisson (according to Weinberger [1995]).

(b) $u(\boldsymbol{x}, 0) = f(\boldsymbol{x})$ and $\frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = 0$. [*Hint*: Use (11.2.24).]

(c) Solve part (b) in the following manner. Let $v(\boldsymbol{x}, t) = \frac{\partial}{\partial t} u(\boldsymbol{x}, t)$, where $u(\boldsymbol{x}, t)$ satisfies part (a). [*Hint*: Show that $v(\boldsymbol{x}, t)$ satisfies the wave equation with $v(\boldsymbol{x}, 0) = g(\boldsymbol{x})$ and $\frac{\partial v}{\partial t}(\boldsymbol{x}, 0) = 0$.]

11.2.16. Derive the one-dimensional Green's function for the wave equation by considering a three-dimensional problem with $Q(\boldsymbol{x}, t) = \delta(x - x_1)\delta(t - t_1)$. [*Hint*: Use polar coordinates for the y_0, z_0 integration centered at $y_0 = y, z_0 = z$.]

11.3 Green's Functions for the Heat Equation

11.3.1 INTRODUCTION

We are interested in solving the heat equation with possibly time-dependent sources,

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u + Q(\boldsymbol{x}, t),} \tag{11.3.1}$$

subject to the initial condition $u(\boldsymbol{x}, 0) = g(\boldsymbol{x})$. We will analyze this problem in one, two, and three spatial dimensions. In this subsection, we do not specify the geometric region or the possibly

nonhomogeneous boundary conditions. There can be three nonhomogeneous terms: the source $Q(\boldsymbol{x}, t)$, the initial condition, and the boundary conditions.

We define the **Green's function** $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ as the solution of

$$\boxed{\frac{\partial G}{\partial t} = k\nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)} \tag{11.3.2}$$

on the same region with the related homogeneous boundary conditions. Since the Green's function represents the temperature response at $\boldsymbol{x}$ (at time t) due to a concentrated thermal source at $\boldsymbol{x}_0$ (at time t_0), we will insist that this Green's function is zero before the source acts:

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \quad \text{for } t < t_0,} \tag{11.3.3}$$

the **causality principle**.

Furthermore, we show that only the elapsed time $t - t_0$ (from the initiation time $t = t_0$) is needed:

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}, t - t_0; \boldsymbol{x}_0, 0),} \tag{11.3.4}$$

the **translation property**. Equation (11.3.4) is shown by letting $T = t - t_0$, in which case the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ satisfies

$$\frac{\partial G}{\partial T} = k\nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(T) \quad \text{with} \quad G = 0 \quad \text{for} \quad T < 0.$$

This is precisely the response due to a concentrated source at $\boldsymbol{x} = \boldsymbol{x}_0$ at $T = 0$, implying (11.3.4).

We postpone until later subsections the actual calculation of the Green's function. For now, we will assume that the Green's function is known and ask how to represent the temperature $u(\boldsymbol{x}, t)$ in terms of the Green's function.

11.3.2 NON-SELF-ADJOINT NATURE OF THE HEAT EQUATION

To show how this problem relates to others discussed in this book, we introduce the linear operator notation,

$$L = \frac{\partial}{\partial t} - k\nabla^2, \tag{11.3.5}$$

called the **heat** or **diffusion operator**. In previous problems the relation between the solution of the nonhomogeneous problem and its Green's function was based on Green's formulas. We have solved problems in which L is the Sturm–Liouville operator, the Laplacian, and most recently the wave operator.

The heat operator L is composed of two parts. ∇^2 is easily analyzed by Green's formula for the Laplacian [see (11.2.8)]. However, as innocuous as $\partial/\partial t$ appears, it is much harder to analyze than any of the other previous operators. To illustrate the difficulty presented by first derivatives, consider

$$L = \frac{\partial}{\partial t}.$$

For second-order Sturm–Liouville operators, elementary integrations yielded Green's formula. The same idea for $L = \partial/\partial t$ will not work. In particular,

$$\int [uL(v) - vL(u)]\, dt = \int \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) dt$$

cannot be simplified. There is no formula to evaluate $\int [uL(v) - vL(u)]dt$. The operator $L = \partial/\partial t$ is not self-adjoint. Instead, by standard integration by parts,

$$\int_a^b uL(v)\, dt = \int_a^b u\frac{\partial v}{\partial t}\, dt = uv\Big|_a^b - \int_a^b v\frac{\partial u}{\partial t}\, dt,$$

and thus

$$\int_a^b \left(u\frac{\partial v}{\partial t} + v\frac{\partial u}{\partial t}\right)dt = uv\Big|_a^b. \tag{11.3.6}$$

For the operator $L = \partial/\partial t$, we introduce the **adjoint operator**,

$$\boxed{L^* = -\frac{\partial}{\partial t}.} \tag{11.3.7}$$

From (11.3.6),

$$\boxed{\int_a^b [uL^*(v) - vL(u)]\,dt = -uv\Big|_a^b.} \tag{11.3.8}$$

This is analogous to Green's formula.[2]

11.3.3 GREEN'S FORMULA FOR THE HEAT EQUATION

We now return to the nonhomogeneous heat problem:

$$\boxed{L(u) = Q(\boldsymbol{x}, t)} \tag{11.3.9}$$

$$\boxed{L(G) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0),} \tag{11.3.10}$$

where

$$\boxed{L = \frac{\partial}{\partial t} - k\nabla^2.} \tag{11.3.11}$$

For the nonhomogeneous heat equation, our results are more complicated since we must introduce the **adjoint heat operator**,

$$\boxed{L^* = -\frac{\partial}{\partial t} - k\nabla^2.} \tag{11.3.12}$$

By a direct calculation,

$$uL^*(v) - vL(u) = -u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} + k(v\nabla^2 u - u\nabla^2 v),$$

and thus Green's formula for the heat equation is

$$\boxed{\begin{aligned}&\int_{t_i}^{t_f}\iiint [uL^*(v) - vL(u)]d^3x\,dt\\ &= -\iiint uv\Big|_{t_i}^{t_f} d^3x + k\int_{t_i}^{t_f}\oiint (v\nabla u - u\nabla v)\cdot\hat{\mathbf{n}}\,dS\,dt.\end{aligned}} \tag{11.3.13}$$

We have integrated over all space and from some time $t = t_i$ to another time $t = t_f$. We have used (11.3.6) for the $\partial/\partial t$-terms and Green's formula (11.2.8) for the ∇^2 operator. The "boundary contributions" are of two types, the spatial part (over $\oiint$[3]) and a temporal part (at the initial t_i and final t_f times). If both u and v satisfy the same homogeneous boundary condition (of the usual types), then the spatial contributions vanish:

$$\oiint (v\nabla u - u\nabla v)\cdot\hat{\mathbf{n}}\;dS = 0.$$

Equation (11.3.13) will involve initial contributions (at $t = t_i$) and final contributions $(t = t_f)$.

11.3.4 ADJOINT GREEN'S FUNCTION

In order to eventually derive a representation formula for $u(\boldsymbol{x}, t)$ in terms of the Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$, we must consider summing up various source times. Thus, we consider the **source-varying Green's function**,

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = G(\boldsymbol{x}, -t; \boldsymbol{x}_1, -t_1),$$

[2]For a first-order operator, typically there is only one "boundary condition," $u(a) = 0$. For the integrated-by-parts term to vanish, we must introduce an *adjoint boundary condition*, $v(b) = 0$.

[3]For infinite or semi-infinite geometries, we consider finite regions in some appropriate limit. The boundary terms at infinity will vanish if u and v decay sufficiently fast.

where the translation property has been used. This is precisely the procedure we employed when analyzing the wave equation [see (11.2.15)]. By causality, these are zero if $t > t_1$:

$$G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = 0 \quad \text{for } t > t_1. \tag{11.3.14}$$

Letting $\tau = -t$, we see that the source-varying Green's function $G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)$ satisfies

$$\left(-\frac{\partial}{\partial t} - k\nabla^2\right)G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t) = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1), \tag{11.3.15}$$

as well as the source-varying causality principle (11.3.14). The heat operator L does not occur. Instead, the adjoint heat operator L^* appears:

$$\boxed{L^*[G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)] = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1).} \tag{11.3.16}$$

We see that $G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)$ is the Green's function for the adjoint heat operator (with the source-varying causality principle). Sometimes it is called the **adjoint Green's function**, $G^*(\boldsymbol{x}, t; \boldsymbol{x}_1, t_1)$. However, it is unnecessary to ever calculate or use it since

$$G^*(\boldsymbol{x}, t; \boldsymbol{x}_1, t_1) = G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t), \tag{11.3.17}$$

and both are zero for $t > t_1$.

11.3.5 RECIPROCITY

As with the wave equation, we derive a reciprocity formula. Here, there are some small differences because of the occurrence of the adjoint operator in Green's formula, (11.3.13). In (11.3.13) we introduce

$$u = G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \tag{11.3.18}$$

$$v = G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t), \tag{11.3.19}$$

the latter having been shown to be the source-varying or adjoint Green's function. Thus, the defining properties for u and v are

$$L(u) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0), \; L^*(v) = \delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1)$$
$$u = 0 \quad \text{for} \quad t < t_0, \qquad\qquad v = 0 \quad \text{for} \quad t > t_1.$$

We integrate from $t = -\infty$ to $t = +\infty$ [i.e., $t_i = -\infty$ and $t_f = +\infty$ in (11.3.13)], obtaining

$$\int_{-\infty}^{\infty}\iiint [G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\delta(\boldsymbol{x} - \boldsymbol{x}_1)\delta(t - t_1) - G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0)]d^3x\, dt$$
$$= -\iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)G(\boldsymbol{x}, t_1; \boldsymbol{x}_1, t)\Big|_{t=-\infty}^{t=\infty} d^3x,$$

since u and v both satisfy the same homogeneous boundary conditions, so that

$$\oiint (v\nabla u - u\nabla v) \cdot \hat{\mathbf{n}}\, dS$$

vanishes. The contributions also vanish at $t = \pm\infty$ due to causality. Using the properties of the Dirac delta function, we obtain **reciprocity**:

$$\boxed{G(\boldsymbol{x}_1, t_1; \boldsymbol{x}_0, t_0) = G(\boldsymbol{x}_0, t_1; \boldsymbol{x}_1, t_0).} \tag{11.3.20}$$

As we have shown for the wave equation [see (11.2.21)], interchanging the source and location positions does not alter the responses if the elapsed times from the sources are the same. In this sense, the Green's function for the heat (diffusion) equation is symmetric because of Green's formula.

11.3.6 REPRESENTATION OF THE SOLUTION USING GREEN'S FUNCTIONS

To obtain the relationship between the solution of the nonhomogeneous problem and the Green's function, we apply Green's formula (11.3.13) with u satisfying (11.3.1) subject to nonhomogeneous boundary and initial conditions. We let v equal the source-varying or adjoint Green's function,

$v = G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)$. Using the defining differential equations (11.3.9) and (11.3.10), Green's formula (11.3.13) becomes

$$\begin{aligned}\int_0^{t_{0+}} \iiint [u\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0) - G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)Q(\boldsymbol{x}, t)]\, d^3x\, dt \\ = \iiint u(\boldsymbol{x}, 0)G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, 0)\, d^3x \\ + k\int_0^{t_{0+}} \oiint [G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)\nabla u - u\nabla G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)] \cdot \hat{\mathbf{n}}\, dS\, dt,\end{aligned}$$

since $G = 0$ for $t > t_0$. Solving for u, we obtain

$$\begin{aligned}u(\boldsymbol{x}_0, t_0) = \int_0^{t_0} \iiint G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)Q(\boldsymbol{x}, t)\, d^3x\, dt \\ + \iiint u(\boldsymbol{x}, 0)G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, 0)\, d^3x \\ + k\int_0^{t_0} \oiint [G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)\nabla u - u\nabla G(\boldsymbol{x}, t_0; \boldsymbol{x}_0, t)] \cdot \hat{\mathbf{n}}\, dS\, dt.\end{aligned}$$

It can be shown that the limits t_{0+} may be replaced by t_0. We now (as before) interchange $\boldsymbol{x}$ with $\boldsymbol{x}_0$ and t with t_0. In addition, we use reciprocity and derive

$$\boxed{\begin{aligned}u(\boldsymbol{x}, t) = \int_0^t \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)Q(\boldsymbol{x}_0, t_0)\, d^3x_0\, dt_0 \\ + \iiint G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)u(\boldsymbol{x}_0, 0)\, d^3x_0 \\ + k\int_0^t \oiint [G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} u - u(\boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)] \cdot \hat{\mathbf{n}}\, dS_0\, dt_0.\end{aligned}}$$

(11.3.21)

Equation (11.3.21) illustrates how the temperature $u(\boldsymbol{x}, t)$ is affected by the three nonhomogeneous terms. The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ is the influence function for the source term $Q(\boldsymbol{x}_0, t_0)$ as well as for the initial temperature distribution $u(\boldsymbol{x}_0, 0)$ (if we evaluate the Green's function at $t_0 = 0$, as is quite reasonable). Furthermore, nonhomogeneous boundary conditions are accounted for by the term $k\int_0^t \oiint (G\nabla_{\boldsymbol{x}_0} u - u\nabla_{\boldsymbol{x}_0} G) \cdot \hat{\mathbf{n}}\, dS_0\, dt_0$. Equation (11.3.21) illustrates the causality principle; at time t, the sources and boundary conditions have an effect only for $t_0 < t$. Equation (11.3.21) generalizes the results obtained by the method of eigenfunction expansion in Section 8.2 for the one-dimensional heat equation on a finite interval with zero boundary conditions.

EXAMPLE

Both u and its normal derivative seem to be needed on the boundary. To clarify the effect of the nonhomogeneous boundary conditions, we consider an example in which the temperature is specified along the entire boundary:

$$u(\boldsymbol{x}, t) = u_B(\boldsymbol{x}, t) \quad \text{along the boundary.}$$

The Green's function satisfies the related homogeneous boundary conditions, in this case

$$G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = 0 \quad \text{for all } \boldsymbol{x} \text{ along the boundary.}$$

Thus, the effect of this imposed temperature distribution is

$$-k\int_0^t \oiint u_B(\boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\mathbf{n}}\, dS_0\, dt_0.$$

The influence function for the nonhomogeneous boundary conditions is minus k times the normal derivative of the Green's function (a dipole distribution). ■

One-dimensional case

It may be helpful to illustrate the modifications necessary for one-dimensional problems. Volume integrals $\iiint d^3x_0$ become one-dimensional integrals $\int_a^b dx_0$. Boundary contributions on the closed surface $\oiint dS_0$ become contributions at the two ends $x = a$ and $x = b$. For example, if the temperature is prescribed at both ends, $u(a, t) = A(t)$ and $u(b, t) = B(t)$, then these nonhomogeneous boundary conditions influence the temperature $u(\boldsymbol{x}, t)$:

$$-k\int_0^t \oiint u_B(\boldsymbol{x}_0, t_0)\nabla_{\boldsymbol{x}_0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) \cdot \hat{\mathbf{n}}\ dS_0\ dt_0 \text{ becomes}$$

$$-k\int_0^t \left[B(t_0)\frac{\partial G}{\partial x_0}(x, t; b, t_0) - A(t_0)\frac{\partial G}{\partial x_0}(x, t; a, t_0)\right] dt_0.$$

This agrees with results that could be obtained by the method of eigenfunction expansions (Chapter 9) for nonhomogeneous boundary conditions.

11.3.7 ALTERNATE DIFFERENTIAL EQUATION FOR THE GREEN'S FUNCTION

Using Green's formula, we derived (11.3.21), which shows the influence of sources, nonhomogeneous boundary conditions, and as is initial condition for the heat equation. The Green's function for the heat equation is not only the influence function for the sources, but also the influence function for the initial condition. If there are no sources, if the boundary conditions are homogeneous, and if the initial condition is a delta function, then the response is the Green's function itself. The Green's function $G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0)$ may be determined directly from the diffusion equation with no sources:

$$\boxed{\frac{\partial G}{\partial t} = k\nabla^2 G,} \tag{11.3.22}$$

subject to homogeneous boundary conditions and the concentrated initial conditions at $t = t_0$:

$$G = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{11.3.23}$$

rather than its defining differential equation (11.3.2).

11.3.8 INFINITE SPACE GREEN'S FUNCTION FOR THE DIFFUSION EQUATION

If there are no boundaries and no sources, $\frac{\partial u}{\partial t} = k\nabla^2 u$ with initial conditions $u(\boldsymbol{x}, 0) = f(\boldsymbol{x})$, then (11.3.21) represents the solution of the diffusion equation in terms of its Green's function:

$$u(x, t) = \iiint u(x_0, 0)G(x, t; x_0, 0)\, d^3x_0 = \iiint f(x_0)G(x, t; x_0, 0)\, d^3x_0. \tag{11.3.24}$$

Instead of solving (11.3.22), we note that this initial value problem was analyzed using the Fourier transform in Chapter 10, and we obtained the one-dimensional solution (10.4.6):

$$u(x, t) = \int_{-\infty}^{\infty} f(x_0)\frac{1}{\sqrt{4\pi kt}} e^{-(x-x_0)^2/4kt}\ dx_0. \tag{11.3.25}$$

By comparing (11.3.24) and (11.3.25), we are able to determine the one-dimensional infinite space Green's function for the diffusion equation:

$$G(x, t; x_0, 0) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-x_0)^2/4kt}. \tag{11.3.26}$$

Due to translational invariance, the more general Green's function involves the elapsed time:

$$\boxed{G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}} e^{-(x-x_0)^2/4k(t-t_0)}.} \tag{11.3.27}$$

For n dimensions ($n = 1, 2, 3$), the solution was also obtained in Chapter 10 (10.6.95), and hence the ***n*-dimensional infinite space Green's function for the diffusion equation** is

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = \left[\frac{1}{4\pi k(t - t_0)}\right]^{\frac{n}{2}} e^{-|\boldsymbol{x}-\boldsymbol{x}_0|^2/4k(t-t_0)}.} \tag{11.3.28}$$

This Green's function shows the symmetry of the response and source positions as long as the elapsed time is the same. As with one-dimensional problems discussed in Section 10.4, the influence of a concentrated heat source diminishes exponentially as one moves away from the source. For small times (t near t_0) the decay is especially strong.

EXAMPLE

In this manner, we can obtain the solution of the heat equation with sources on an infinite domain:

$$\boxed{\frac{\partial u}{\partial t} = k\nabla^2 u + Q(\boldsymbol{x}, t)} \tag{11.3.29}$$

$$u(x, 0) = f(x).$$

According to (11.3.21) and (11.3.28), the solution is

$$\boxed{\begin{aligned} u(\boldsymbol{x}, t) = &\int_0^t \int_{-\infty}^{\infty} \left[\frac{1}{4\pi k(t - t_0)}\right]^{n/2} e^{-(\boldsymbol{x}-\boldsymbol{x}_0)^2/4k(t-t_0)} Q(\boldsymbol{x}_0, t_0)\, d^n x_0\, dt_0 \\ &+ \int_{-\infty}^{\infty} \left(\frac{1}{4\pi kt}\right)^{n/2} e^{-(\boldsymbol{x}-\boldsymbol{x}_0)^2/4kt} f(\boldsymbol{x}_0) d^n x_0. \end{aligned}} \tag{11.3.30}$$

If $Q(\boldsymbol{x}, t) = 0$, this simplifies to the solution obtained in Chapter 10 using Fourier transforms directly without using the Green's function. ■

11.3.9 GREEN'S FUNCTION FOR THE HEAT EQUATION (SEMI-INFINITE DOMAIN)

In this subsection, we obtain the Green's function needed to solve the nonhomogeneous heat equation on the semi-infinite interval in one dimension ($x > 0$), subject to a nonhomogeneous boundary condition at $x = 0$:

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x, t),} \quad x > 0, \tag{11.3.31}$$

$$\text{BC: } \boxed{u(0, t) = A(t)} \tag{11.3.32}$$

$$\text{IC: } \boxed{u(x, 0) = f(x).} \tag{11.3.33}$$

Equation (11.3.21) can be used to determine $u(x, t)$ if we can obtain the Green's function. The Green's function $G(x, t; x_0, t_0)$ is the response due to a concentrated source:

$$\frac{\partial G}{\partial t} = k\frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\delta(t - t_0).$$

The Green's function satisfies the corresponding *homogeneous* boundary condition,

$$G(0, t; x_0, t_0) = 0,$$

and the causality principle,

$$G(x, t; x_0, t_0) = 0 \qquad \text{for } t < t_0.$$

The Green's function is determined by the method of images (see Section 9.5.8). Instead of a semi-infinite interval with one concentrated positive source at $x = x_0$, we consider an infinite interval with an additional negative source (the image source) located at $x = -x_0$. By symmetry, the temperature G will be zero at $x = 0$ for all t. The Green' function is thus the sum of two infinite space Green's functions:

$$G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}}\left\{\exp\left[-\frac{(x - x_0)^2}{4k(t - t_0)}\right] - \exp\left[-\frac{(x + x_0)^2}{4k(t - t_0)}\right]\right\}. \tag{11.3.34}$$

We note that the boundary condition at $x = 0$ is automatically satisfied.

11.3.10 GREEN'S FUNCTION FOR THE HEAT EQUATION (ON A FINITE REGION)

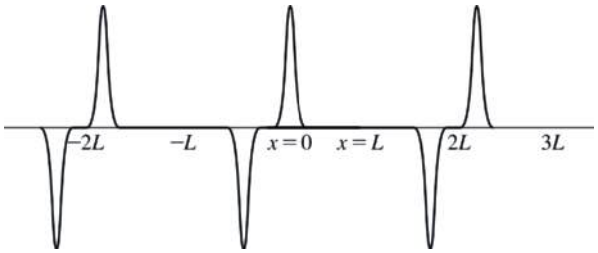
For a one-dimensional rod, $0 < x < L$, we have already determined in Chapter 9 the Green's function for the heat equation by the method of eigenfunction expansions. With zero boundary conditions at both ends,

$$G(x, t; x_0, t_0) = \sum_{n=1}^{\infty} \frac{2}{L} \sin\frac{n\pi x}{L} \sin\frac{n\pi x_0}{L} e^{-k(n\pi/L)^2(t-t_0)}. \tag{11.3.35}$$

We can obtain an alternative representation for this Green's function by utilizing the method of images. By symmetry (see Fig. 11.3.1), the boundary conditions at $x = 0$ and at $x = L$ are satisfied if positive concentrated sources are located at $x = x_0 + 2Ln$ and negative concentrated sources are located at $x = -x_0 + 2Ln$ (for all integers n, $-\infty < n < \infty$). Using the infinite space Green's function, we have an alternative representation of the Green's function for a one-dimensional rod:

$$G(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}} \sum_{n=-\infty}^{\infty} \left\{\exp\left[-\frac{(x - x_0 - 2Ln)^2}{4k(t - t_0)}\right] - \exp\left[-\frac{(x + x_0 - 2Ln)^2}{4k(t - t_0)}\right]\right\}. \tag{11.3.36}$$

Figure 11.3.1 Multiple image sources for the Green's function for the heat equation for a finite one-dimensional rod.



Each form has its own advantage. The eigenfunction expansion, (11.3.35), is an infinite series that converges rapidly if $(t - t_0)k/L^2$ is large. It is thus most useful for $t \gg t_0$. In fact, if $t \gg t_0$,

$$G(x, t; x_0, t_0) \approx \frac{2}{L} \sin\frac{\pi x_0}{L} \sin\frac{\pi x}{L} e^{-k(\pi/L)^2(t-t_0)}.$$

However, if the elapsed time $t - t_0$ is small, then many terms of the infinite series are needed.

Using the method of images, the Green's function is also represented by an infinite series, (11.3.36). The infinite space Green's function (at fixed t) exponentially decays away from the source position,

$$\frac{1}{\sqrt{4\pi k(t-t_0)}}e^{-(x-x_0)^2/4k(t-t_0)}.$$

It decays in space very sharply if t is near t_0. If t is near t_0, then only sources near the response location x are important; sources far away will not be important (if t is near t_0); see Fig. 11.3.1. Thus, the image sources can be neglected if t is near t_0 (and if x or x_0 is neither near the boundaries 0 or L, as is explained in Exercise 11.3.8). As an approximation,

$$G(x,t;x_0,t_0) \approx \frac{1}{\sqrt{4\pi k(t-t_0)}}e^{-(x-x_0)^2/4k(t-t_0)};$$

if t is near t_0, the Green's function with boundaries can be approximated (in regions away from the boundaries) **by the infinite space Green's function**. This means that for small times the boundary can be neglected (away from the boundary).

To be more precise, the effect of every image source is much smaller than the actual source if $L^2/k(t-t_0)$ is large. This yields a better understanding of a "small time" approximation. The Green's function may be approximated by the infinite space Green's function if $t-t_0$ is small (i.e., if $t-t_0 \ll L^2/k$, where L^2/k is a ratio of physically measurable quantities). Alternatively, this approximation is valid for a "long rod" in the sense that $L \gg \sqrt{k(t-t_0)}$.

In summary, the image method yields a rapidly convergent infinite series for the Green's function if $L^2/k(t-t_0) \gg 1$, while the eigenfunction expansion yields a rapidly convergent infinite series representation of the Green's function if $L^2/k(t-t_0) \ll 1$. If $L^2/k(t-t_0)$ is neither small nor large, then the two expansions are competitive, but both require at least a moderate number of terms.

Exercises 11.3

11.3.1. Show that for the Green's functions defined by (11.3.2) with (11.3.3),

$$G^*(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = G(\boldsymbol{x}_0,t_0;\boldsymbol{x},t).$$

11.3.2. Consider

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad x>0\\ u(0,t) &= A(t)\\ u(x,0) &= f(x).\end{aligned}$$

(a) Solve if $A(t)=0$ and $f(x)=0$. Simplify this result if $Q(x,t)=1$.

(b) Solve if $Q(x,t)=0$ and $A(t)=0$. Simplify this result if $f(x)=1$.

***(c)** Solve if $Q(x,t)=0$ and $f(x)=0$. Simplify this result if $A(t)=1$.

***11.3.3.** Determine the Green's function for

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t), x>0\\ \frac{\partial u}{\partial x}(0,t) &= A(t)\\ u(x,0) &= f(x).\end{aligned}$$

11.3.4. Consider (11.3.34), the Green's function for (11.3.31). Show that the Green's function for this semi-infinite problem may be approximated by the Green's function for the infinite problem if

$$\frac{xx_0}{k(t-t_0)} \gg 1 \qquad \text{(i.e., } t-t_0 \text{ small)}.$$

Explain physically why this approximation fails if x or x_0 is near the boundary.

11.3.5. Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x, t) \\ u(x, 0) &= f(x) \\ \frac{\partial u}{\partial x}(0, t) &= A(t) \\ \frac{\partial u}{\partial x}(L, t) &= B(t). \end{aligned}$$

(a) Solve for the appropriate Green's function using the method of eigenfunction expansion.

(b) Approximate the Green's function of part (a). Under what conditions is your approximation valid?

(c) Solve for the appropriate Green's function using the infinite space Green's function.

(d) Approximate the Green's function of part (c). Under what conditions is your approximation valid?

(e) Solve for $u(x, t)$ in terms of the Green's function.

11.3.6. Determine the Green's function for the heat equation subject to zero boundary conditions at $x = 0$ and $x = L$ by applying the method of eigenfunction expansions directly to the defining differential equation. [*Hint*: The answer is given by (11.3.35).]

11.3.7. Derive the two-dimensional infinite space Green's function by taking two-dimensional transforms.

11.3.8. Derive the three-dimensional infinite space Green's function by taking three-dimensional transforms.

12

The Method of Characteristics for Linear and Quasilinear Wave Equations

12.1 Introduction

In previous chapters, we obtained certain results concerning the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{12.1.1}$$

subject to the initial conditions

$$u(x, 0) = f(x) \tag{12.1.2}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x). \tag{12.1.3}$$

For a vibrating string with zero displacement at $x = 0$ and $x = L$, we obtained a somewhat complicated Fourier sine series solution by the method of separation of variables in Chapter 4:

$$u(x, t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}\right). \tag{12.1.4}$$

Further analysis of this solution [see (4.4.14) and Exercises 4.4.9 and 4.4.10] shows that the solution can be represented as the sum of a forward- and backward-moving wave. In particular,

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x_0)\ dx_0, \tag{12.1.5}$$

where $f(x)$ and $g(x)$ are the odd-periodic extensions of the functions given in (12.1.2) and (12.1.3). We also obtained (12.1.5) in Chapter 11 for the one-dimensional wave equation without boundaries, using the infinite space Green's function.

In this chapter, we introduce the more powerful method of characteristics to solve the one-dimensional wave equation. We will show that, in general, $u(x, t) = F(x - ct) + G(x + ct)$, where F and G are arbitrary functions. We will show that (12.1.5) follows for infinite space problems. Then we will discuss modifications needed to solve semi-infinite and finite domain problems. In Section 12.6, the method of characteristics will be applied to quasilinear partial differential equations. Traffic flow models will be introduced in Section 12.6.2, and expansion waves will be discussed (Section 12.6.3). When characteristics intersect, we will show that a shock wave must occur, and we will derive an expression for the shock velocity. The dynamics of shock waves will be discussed in considerable depth (Section 12.6.4). In Section 12.7, the method of characteristics will be used to solve the eikonal equation, which we will derive from the wave equation.

12.2 Characteristics for First-Order Wave Equations

12.2.1 INTRODUCTION

The one-dimensional wave equation can be rewritten as

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0.} \tag{12.2.1}$$

A short calculation shows that it can be "factored" in two ways:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$
$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0,$$

since the mixed second-derivative terms vanish in both. If we let

$$\boxed{w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}} \tag{12.2.2}$$

$$\boxed{v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x},} \tag{12.2.3}$$

we see that the one-dimensional wave equation (involving second derivatives) yields two **first-order wave equations**:

$$\boxed{\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0} \tag{12.2.4}$$

$$\boxed{\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0.} \tag{12.2.5}$$

12.2.2 METHOD OF CHARACTERISTICS FOR FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

We begin by discussing either one of these simple first-order partial differential equations:

$$\boxed{\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0.} \tag{12.2.6}$$

The methods we will develop will be helpful in analyzing the one-dimensional wave equation (12.2.1). We consider the rate of change of $w(x(t), t)$ as measured by a moving observer, $x = x(t)$. The chain rule[1] implies that

$$\boxed{\frac{d}{dt}w(x(t), t) = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x}.} \tag{12.2.7}$$

The first term, $\partial w/\partial t$, represents the change in w at the fixed position, while the term $(dx/dt)(\partial w/\partial x)$ represents the change due to the fact that the observer moves into a region of possibly different w. Compare (12.2.7) with the partial differential equation for w, Equation (12.2.6). If $x(t)$ is a function that solves

$$\boxed{\frac{dx}{dt} = c,} \tag{12.2.8}$$

then the first-order wave equation (12.2.6) becomes

$$\boxed{\frac{d}{dt}w(x(t), t) = 0.} \tag{12.2.9}$$

Thus, $w(x(t), t)$ is constant. An observer moving with this special speed c would measure no changes in w.

[1]Here d/dt as measured by a moving observer is sometimes called the *substantial derivative*.

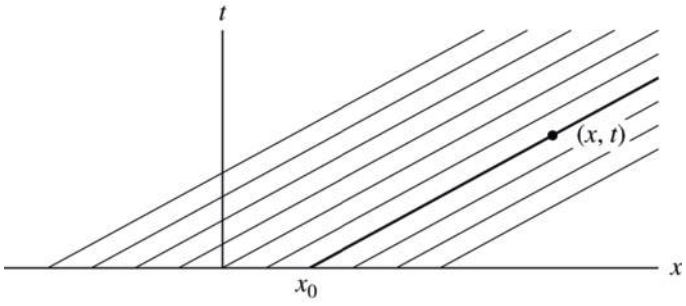
Characteristics

In this way, the partial differential equation (12.2.6) has been replaced by two ordinary differential equations, (12.2.8) and (12.2.9). Integrating (12.2.8) yields

$$\boxed{x = ct + x_0,} \tag{12.2.10}$$

the equation for the family of **characteristics**[2] of (12.2.6), sketched in Fig. 12.2.1. Each curve in the family of characteristics is distinguished by a different value of x_0. We carefully note that x_0 is the value of x at $t = 0$ on that particular curve. On each curve, $w(x, t)$ takes on a constant value (possibly different values on different curves in the family).

Figure 12.2.1 Characteristics for the first-order wave equation.



General solution

We exploit the property that $w(x, t)$ is constant along each characteristic curve to find a general solution to the wave equation (12.2.6). If $w(x, t)$ is given initially at $t = 0$,

$$w(x, 0) = P(x), \tag{12.2.11}$$

then let us determine w at the point (x, t) in terms of the given function $P(x)$.

Consider the characteristic that begins at the point $(0, 0)$. This characteristic carries the value of w along it. Hence, at every point along that curve, $w(x, t) = w(0, 0) = P(0)$. Similarly, on a characteristic beginning at a general point $(x_0, 0)$, we have $w(x, t) = w(x_0, 0) = P(x_0)$. All that is left to do is to determine the value of x_0 on the characteristic through a given point (x, t). Equation (12.2.10) tells us $x_0 = x - ct$, and thus

$$\boxed{w(x, t) = P(x - ct),} \tag{12.2.12}$$

which we call the general solution of (12.2.6).

We introduced $P(x)$ as some given initial condition, but more generally (12.2.12) solves the first-order wave equation (12.2.6) for an arbitrary function $P(x)$. To verify this, we substitute (12.2.12) back into the partial differential equation (12.2.6). Using the chain rule,

$$\frac{\partial w}{\partial x} = P'(x - ct)\frac{\partial}{\partial x}(x - ct) = P'(x - ct)$$

and

$$\frac{\partial w}{\partial t} = P'(x - ct)\frac{\partial}{\partial t}(x - ct) = -cP'(x - ct).$$

Thus, it is verified that (12.2.6) is satisfied by (12.2.12). The general solution of a first-order partial differential equation contains an arbitrary *function*, just as the general solution of a first-order ordinary differential equation contains an arbitrary *constant*. The interpretation of the solution (12.2.12) is that w **propagates as a wave** with wave speed c. This is why we call the partial differential equation (12.2.6) a wave equation.

[2]A characteristic is a curve along which a PDE reduces to an ODE.

EXAMPLE

Consider

$$\frac{\partial w}{\partial t} + 2\frac{\partial w}{\partial x} = 0,$$

subject to the initial condition

$$w(x, 0) = \begin{cases} 0 & x < 0 \\ 4x & 0 < x < 1 \\ 0 & x > 1. \end{cases}$$

We have shown that w is constant along the characteristics $x - 2t =$ constant, keeping its same shape moving at velocity 2 (to the right). The important characteristics, $x = 2t + 0$ and $x = 2t + 1$, as well as a sketch of the solution at various times, appear in Fig. 12.2.2. $w(x, t) = 0$ if $x > 2t + 1$ or if $x < 2t$. Otherwise, by shifting,

$$w(x, t) = 4(x - 2t) \qquad \text{if } 2t < x < 2t + 1.$$

To derive this analytic solution, we use the characteristic that starts at $x = x_0$:

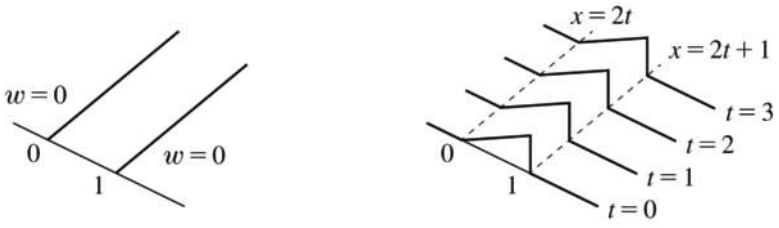
$$x = 2t + x_0.$$

Along this characteristic, $w(x, t)$ is constant. If $0 < x_0 < 1$, then

$$w(x, t) = w(x_0, 0) = 4x_0 = 4(x - 2t),$$

as before. This is valid if $0 < x_0 < 1$ or, equivalently, $0 < x - 2t < 1$.

Figure 12.2.2 Propagation for the first-order wave equation.

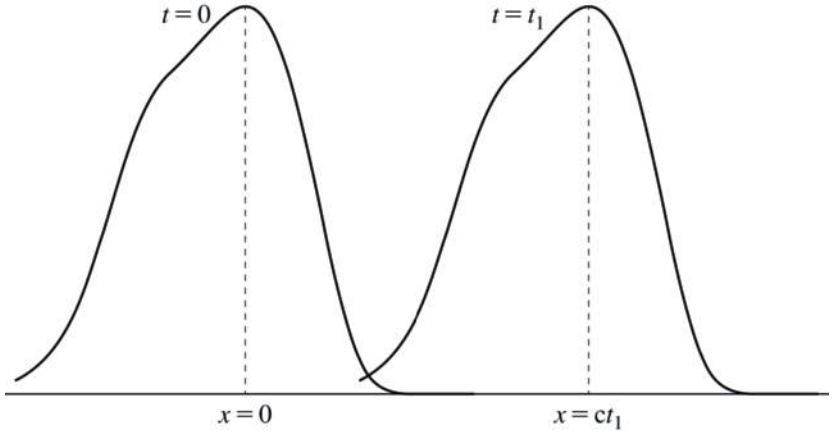


■

Same shape

In general, $w(x, t) = P(x - ct)$. *At fixed t, the solution of the first-order wave equation is the same shape shifted a distance ct* (distance = velocity times time). We illustrate this in Fig. 12.2.3.

Figure 12.2.3 Shape invariance for the first-order wave equation.



MATLAB Files

Download Animation of Figure 12.2.3 (.zip)

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These files are provided as zipped .m files for use in MATLAB on desktop platform.

EXAMPLE TO SOLVE INITIAL VALUE PROBLEM AND FIND GENERAL SOLUTION

Consider

$$\frac{\partial w}{\partial t} + 3t^2 \frac{\partial w}{\partial x} = 2tw, \tag{12.2.13}$$

subject to the initial conditions $w(x, 0) = P(x)$. By the method of characteristics,

$$\text{if } \frac{dx}{dt} = 3t^2, \tag{12.2.14}$$

$$\text{then } \frac{dw}{dt} = 2tw. \tag{12.2.15}$$

The characteristics are not straight lines but satisfy

$$x = t^3 + x_0, \tag{12.2.16}$$

where the characteristics start ($t = 0$) at $x = x_0$. Along the characteristics, by integrating the ODE (12.2.15), we obtain

$$w = ke^{t^2}. \tag{12.2.17}$$

To satisfy the initial condition at x_0, $w(x_0, 0) = P(x_0)$, we have $P(x_0) = k$, so that the solution of the initial value problem by the method of characteristics is

$$w(x, t) = P(x_0)e^{t^2} = P(x - t^3)e^{t^2}. \tag{12.2.18}$$

Since $P(x - t^3)$ is an arbitrary function of $(x - t^3)$, (12.2.18) is the general solution of the partial differential equation (12.2.13). The method of characteristics can be used to determine the general solution in a slightly different way. The arbitrary constants that solve the ordinary differential equations are arbitrary functions of each other. In this way, k in (12.2.17) is an arbitrary function of $x_0 = x - t^3$, and we obtain directly from (12.2.17) the general solution of this partial differential equation:

$$w(x, t) = f(x - t^3)e^{t^2}, \tag{12.2.19}$$

where f is an arbitrary function of $x - t^3$. The initial value problem could now be solved from the general solution (12.2.19). ■

Summary

The method of characteristics solves the first-order wave equation (12.2.6). In Sections 12.3–12.5, this method is applied to solve the wave equation (12.1.1). The reader may proceed directly to Section 12.6, where the method of characteristics is described for quasilinear partial differential equations.

Exercises 12.2

12.2.1. Show that the wave equation can be considered as the following system of two coupled first-order partial differential equations:

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w$$

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0.$$

***12.2.2.** Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0 \quad \text{with } w(x, 0) = \cos x.$$

12.2.3. Solve

$$\frac{\partial w}{\partial t} + 4\frac{\partial w}{\partial x} = 0 \quad \text{with } w(0, t) = \sin 3t.$$

12.2.4. Solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad (c > 0)$$

for $x > 0$ and $t > 0$ if

$$w(x, 0) = f(x), \quad x > 0$$
$$w(0, t) = h(t), \quad t > 0.$$

12.2.5. Solve using the method of characteristics.

(a) $\dfrac{\partial w}{\partial t} + t^2 \dfrac{\partial w}{\partial x} = t + x$ with $w(x, 0) = \sin x$

(b) $\dfrac{\partial w}{\partial t} + t^4 \dfrac{\partial w}{\partial x} = x$ with $w(x, 0) = \dfrac{x}{1 + x^2}$

12.2.6. Solve using the method of characteristics (if necessary, see Section 12.6):

(a) $\dfrac{\partial w}{\partial t} + c\dfrac{\partial w}{\partial x} = e^{2x}$ with $w(x, 0) = f(x)$

***(b)** $\dfrac{\partial w}{\partial t} + x\dfrac{\partial w}{\partial x} = 1$ with $w(x, 0) = f(x)$

(c) $\dfrac{\partial w}{\partial t} + t\dfrac{\partial w}{\partial x} = 1$ with $w(x, 0) = f(x)$

***(d)** $\dfrac{\partial w}{\partial t} + 3t\dfrac{\partial w}{\partial x} = w$ with $w(x, 0) = f(x)$

***12.2.7.** Consider (if necessary, see Section 12.6):

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u(x, 0) = f(x).$$

Show that the characteristics are straight lines.

12.2.8. Consider Exercise 12.2.7 with

$$u(x, 0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + x/L & 0 < x < L \\ 2 & x > L. \end{cases}$$

(a) Determine equations for the characteristics. Sketch the characteristics.

(b) Determine the solution $u(x, t)$. Sketch $u(x, t)$ for t fixed.

***12.2.9.** Consider Exercise 12.2.7 with

$$u(x, 0) = f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0. \end{cases}$$

Obtain the solution $u(x, t)$ by considering the limit as $L \to 0$ of the characteristics obtained in Exercise 12.2.8. Sketch characteristics and $u(x, t)$ for t fixed.

12.2.10. As motivated by the analysis of a moving observer, make a change of independent variables from (x, t) to a coordinate system moving with velocity c, (ξ, t'), where $\xi = x - ct$ and $t' = t$, in order to solve (12.2.7).

12.2.11. For the first-order "quasilinear" partial differential equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c,$$

where a, b, and c are functions of x, y, and u, show that the method of characteristics (if necessary, see Section 12.6) yields

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

12.3 Method of Characteristics for the One-Dimensional Wave Equation

12.3.1 GENERAL SOLUTION

From the one-dimensional wave equation,

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,} \tag{12.3.1}$$

we derived two first-order partial differential equations, $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ and $\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$, where $w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$ and $v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$. We have shown that w remains the same shape moving at velocity c:

$$w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = P(x - ct). \tag{12.3.2}$$

The problem for v is identical (replace c by $-c$). Thus, we could have shown that v is translated unchanged at velocity $-c$:

$$v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = Q(x + ct). \tag{12.3.3}$$

By combining (12.3.2) and (12.3.3), we obtain, for example,

$$2\frac{\partial u}{\partial t} = P(x - ct) + Q(x + ct) \quad \text{and} \quad 2c\frac{\partial u}{\partial x} = Q(x + ct) - P(x - ct),$$

and thus

$$\boxed{u(x, t) = F(x - ct) + G(x + ct),} \tag{12.3.4}$$

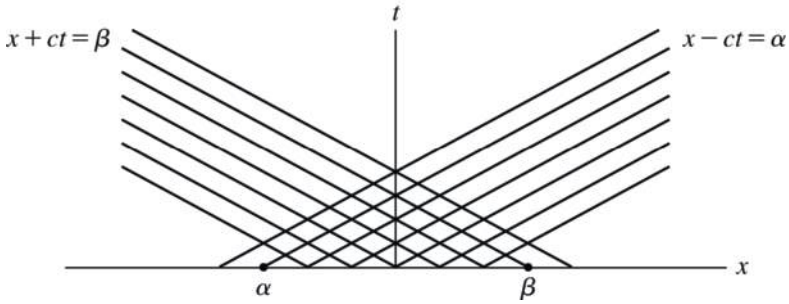
where F and G are arbitrary functions ($-cF' = \frac{1}{2}P$ and $cG' = \frac{1}{2}Q$). This result was obtained by d'Alembert in 1747. **Equation (12.3.4) is a remarkable result, as it is a general solution of the one-dimensional wave equation, (12.3.1)**, a very nontrivial partial differential equation.

The general solution is the sum of $F(x - ct)$, a wave of fixed shape moving to the right with velocity c, and $G(x + ct)$, a wave of fixed shape moving to the left with velocity $-c$. The solution may be sketched if $F(x)$ and $G(x)$ are known. We shift $F(x)$ to the right a distance ct, shift $G(x)$ to the left a distance ct, and add the two. Although each shape is unchanged, the sum will in general be a shape that is changing in time. In Section 12.3.2, we will show how to determine $F(x)$ and $G(x)$ from initial conditions.

Characteristics

Part of the solution is constant along the family of characteristics $x - ct =$ constant, while a different part of the solution is constant along $x + ct =$ constant. For the one-dimensional wave equation, (12.3.1), there are two families of characteristic curves, as sketched in Fig. 12.3.1.

Figure 12.3.1 Characteristics for the one-dimensional wave equation.



Alternate derivation of the general solution

We may derive the general solution of the wave equation, by making a somewhat unmotivated change of variables to "characteristic coordinates" $\xi = x - ct$ and $\eta = x + ct$ moving with the velocities $\pm c$. Using the chain rule, first derivatives are given by $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$. Substituting the corresponding second derivatives into the wave equation (12.3.1) yields

$$c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right) = c^2\left(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right).$$

After canceling the terms $c^2\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right)$, we obtain

$$4c^2 \frac{\partial^2 u}{\partial \xi \, \partial \eta} = 0.$$

By integrating with respect to $\xi (fixed \eta)$, we obtain $\frac{\partial u}{\partial \eta} = g(\eta)$, where $g(\eta)$ is an arbitrary function of η. Now integrating with respect to $\eta (fixed \xi)$ yields the general solution (12.3.4), $u = F(\xi) + G(\eta) = F(x - ct) + G(x + ct)$, where F and G are arbitrary functions.

12.3.2 INITIAL VALUE PROBLEM (INFINITE DOMAIN)

In Section 12.3.1, we showed that the general solution of the one-dimensional wave equation is

$$\boxed{u(x, t) = F(x - ct) + G(x + ct).} \tag{12.3.5}$$

Here we will determine the arbitrary functions in order to satisfy the initial conditions:

$$\boxed{u(x, 0) = f(x),} \quad -\infty < x < \infty \tag{12.3.6}$$

$$\boxed{\frac{\partial u}{\partial t}(x, 0) = g(x),} \quad -\infty < x < \infty. \tag{12.3.7}$$

These initial conditions imply that

$$f(x) = F(x) + G(x) \tag{12.3.8}$$

$$\frac{g(x)}{c} = -\frac{dF}{dx} + \frac{dG}{dx}. \tag{12.3.9}$$

We solve for $G(x)$ by eliminating $F(x)$; for example, adding the derivative of (12.3.8) to (12.3.9) yields

$$\frac{dG}{dx} = \frac{1}{2}\left(\frac{df}{dx} + \frac{g(x)}{c}\right).$$

By integrating this, we obtain

$$\boxed{G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\, d\bar{x}} + k \tag{12.3.10}$$

$$\boxed{F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\, d\bar{x}} - k, \tag{12.3.11}$$

where the latter equation was obtained from (12.3.8). k can be neglected since $u(x, t)$ is obtained from (12.3.5) by adding (12.3.10) and (12.3.11) (with appropriate shifts).

Sketching technique

The solution $u(x, t)$ can be graphed based on (12.3.5) in the following straightforward manner:

1. Given $f(x)$ and $g(x)$, obtain the graphs of

$$\frac{1}{2} f(x) \quad \text{and} \quad \frac{1}{2c}\int_0^x g(\bar{x})\, d\bar{x},$$

   the latter by integrating first.
2. By addition and subtraction, form $F(x)$ and $G(x)$; see (12.3.10) and (12.3.11).
3. Translate (shift) $F(x)$ to the right a distance ct and $G(x)$ to the left ct.
4. Add the two shifted functions, thus satisfying (12.3.5).

Initially at rest

If a vibrating string is initially at rest $[\frac{\partial u}{\partial t}(x, 0) = g(x) = 0]$, then from (12.3.10) and (12.3.11) $F(x) = G(x) = \frac{1}{2}f(x)$. Thus,

$$u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]. \tag{12.3.12}$$

The initial condition $u(x, 0) = f(x)$ splits into two parts; half moves to the left and half to the right.

EXAMPLE

Suppose that an infinite vibrating string is initially stretched into the shape of a single rectangular pulse and is let go from rest. The corresponding initial conditions are

$$u(x, 0) = f(x) = \begin{cases} 1 & |x| < h \\ 0 & |x| > h \end{cases}$$

and

$$\frac{\partial u}{\partial t}(x, 0) = g(x) = 0.$$

The solution is given by (12.3.12). By adding together these two rectangular pulses, we obtain Fig. 12.3.2. The pulses overlap until the left end of the right-moving one passes the right end of the other. Since each is traveling at speed c, they are moving apart at velocity $2c$. The ends are initially a distance $2h$ apart, and hence the time at which the two pulses separate is

$$t = \frac{\text{distance}}{\text{velocity}} = \frac{2h}{2c} = \frac{h}{c}.$$

Important characteristics are sketched in Fig. 12.3.3. F stays constant moving to the right at velocity c, while G stays constant moving to the left. From (12.3.10) and (12.3.11),

$$F(x) = G(x) = \begin{cases} \frac{1}{2} & |x| < h \\ 0 & |x| > h. \end{cases}$$

This information also appears in Fig. 12.3.3.

Figure 12.3.2
Solution for the one-dimensional wave equation with rectangular pulse initial condition. (Top) The solution is shown for three different times. The pulse of width $2h$ and height 1 separates into a left-traveling pulse and a right-traveling pulse each of height $\frac{1}{2}$. (Middle) The solution at $t = 0$, 3, and 8 are shown for $h = 1$, $c = \frac{1}{5}$, superimposed on xt axes. (Bottom) The solution at many times on the same xt axes, with $t = 0$, 3, and 8 highlighted for cross-reference.

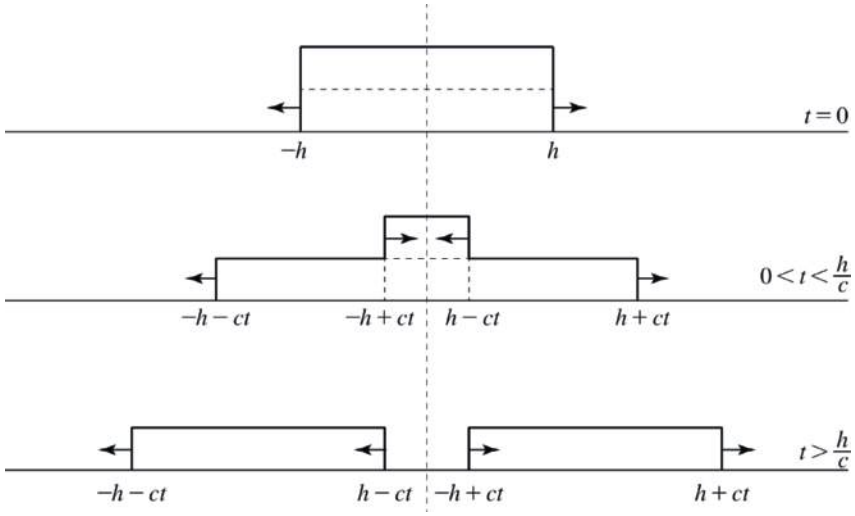


Figure 12.3.2 (*continued*)

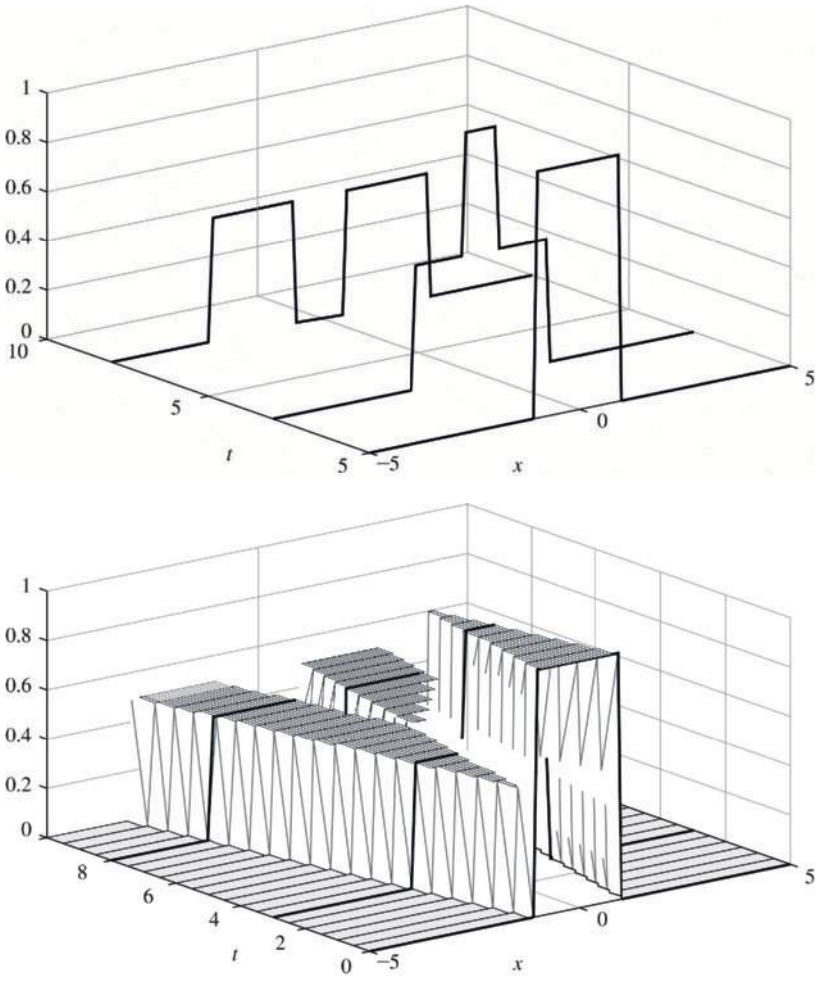
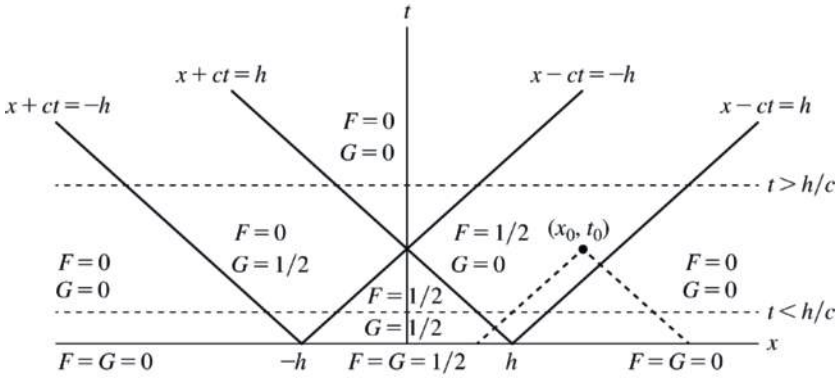


Figure 12.3.3 Method of characteristics for the one-dimensional wave equation.



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EXAMPLE NOT AT REST

Suppose that an infinite string is initially horizontally stretched with prescribed initial velocity as follows:

$$u(x, 0) = f(x) = 0$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) = \begin{cases} 1 & |x| < h \\ 0 & |x| > h. \end{cases}$$

In Exercise 12.3.3, it is shown that this corresponds to instantaneously applying a constant impulsive force to the entire region $|x| < h$, as though the string is being struck by a broad ($|x| < h$)

MATLAB Files

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https://bit.ly/3Ul2lm5

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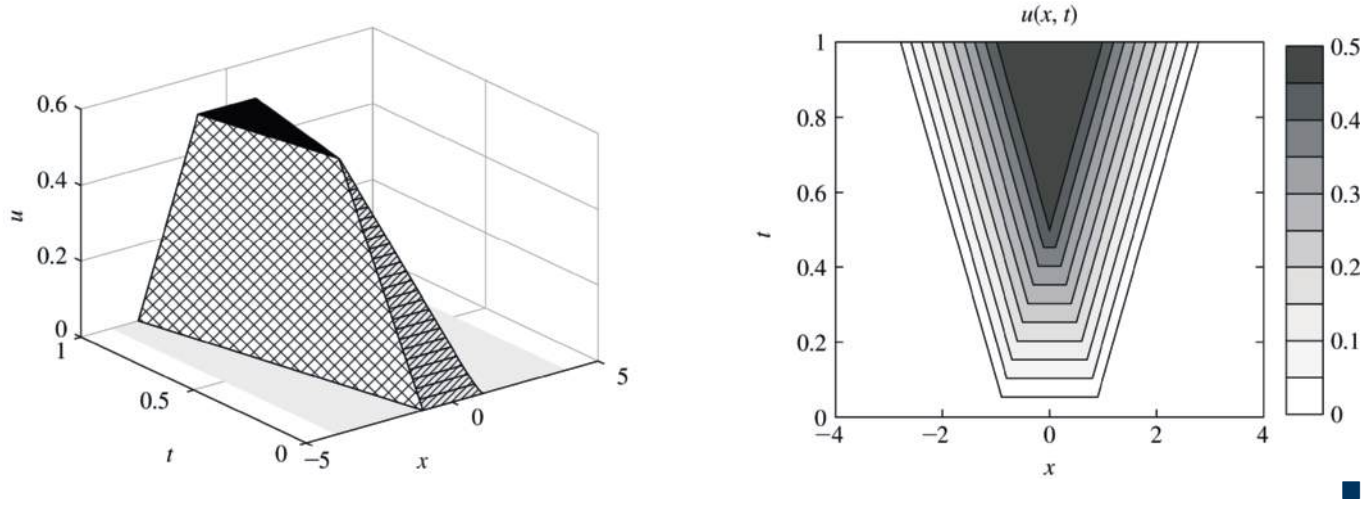
hammer. The calculation of the solution of the wave equation with these initial conditions is more involved than in the preceding example. From (12.3.10) and (12.3.11), we need $\int_0^x g(\bar{x})\,d\bar{x}$, representing the area under $g(x)$ from 0 to x:

$$2cG(x) = -2cF(x) = \int_0^x g(\bar{x})\,d\bar{x} = \begin{cases} -h & x < -h \\ x & -h < x < h \\ h & x > h. \end{cases}$$

The solution $u(x, t)$ is the sum of $F(x)$ shifted to the right (at velocity c) and $G(x)$ shifted to the left (at velocity c). $F(x)$ and $G(x)$ are sketched in Fig. 12.3.4, as is their shifted sum. The striking of the broad hammer causes the displacement of the string to gradually increase near where the hammer hit and to have this disturbance spread out to the left and right as time increases. Eventually, the string reaches an elevated rest position. Alternatively, the solution can be obtained in an algebraic way (see Exercise 12.3.6). The characteristics sketched in Fig. 12.3.3 are helpful.

Figure 12.3.4 Time evolution for a struck string ($h = 1$, $c = 2$).



12.3.3 D'ALEMBERT'S SOLUTION

The general solution of the one-dimensional wave equation can be simplified somewhat. Substituting (12.3.10) and (12.3.11) into the general solution (12.3.5) yields

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c}\left[\int_0^{x+ct} g(\bar{x})\,d\bar{x} - \int_0^{x-ct} g(\bar{x})\,d\bar{x}\right]$$

or

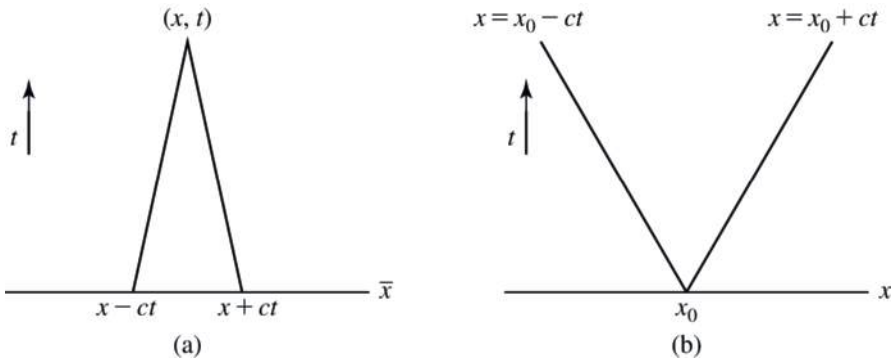
$$\boxed{u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x},} \tag{12.3.13}$$

known as **d'Alembert's solution** (obtained in Chapter 11 by using Green's formula and the infinite space Green's function for the one-dimensional wave equation). It is a very elegant result. However, for sketching solutions often it is easier to work directly with (12.3.10) and (12.3.11), where these are shifted according to (12.3.5).

Domain of dependence and range of influence

The importance of the characteristics $x - ct =$ constant and $x + ct =$ constant is clear. At position x at time t the initial position data are needed at $x \pm ct$, while all the initial velocity data between $x - ct$ and $x + ct$ is needed. The region between $x - ct$ and $x + ct$ is called the **domain of dependence** of the solution at (x, t), as sketched in Fig. 12.3.5. In addition, we sketch the **range of influence**, the region affected by the initial data at one point.

Figure 12.3.5 (a) Domain of dependence; (b) range of influence.



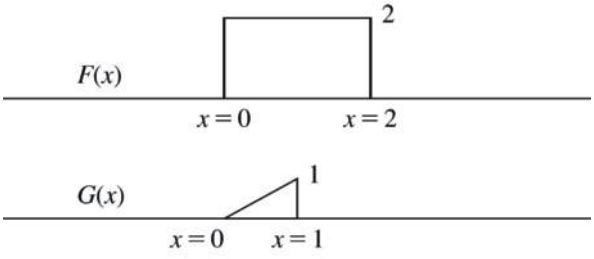
Space- and timelike boundaries

Two initial conditions (12.3.6) and (12.3.7) are specified along $t = 0$, which is the x-axis, called a **spacelike boundary**. In the next section, it is shown that one boundary condition is specified at a fixed boundary—for example, $x = 0$—along which time varies, so that $x = 0$ is called a **timelike boundary**. For a subsonic moving boundary, moving at a speed less than the characteristic velocity c, $|\frac{dx}{dt}| < c$, then one condition is specified on that moving boundary, and the boundary is said to be **timelike**. For a supersonic moving boundary, moving at a speed greater than the characteristic velocity c, $|\frac{dx}{dt}| > c$, two conditions are specified (but sometimes information propagates out of the region), and the boundary is said to be **spacelike**. The boundaries do not have to move at constant velocities.

Exercises 12.3

12.3.1. Suppose that $u(x, t) = F(x - ct) + G(x + ct)$, where F and G are sketched in Fig. 12.3.6. Sketch the solution for $c = \frac{1}{2}$ and the following times: $t = 1$, 2, 4, and 5.

Figure 12.3.6



12.3.2. Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2} \quad u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x),$$

where

$$f(x) = \begin{cases} x & \text{if} \quad x \le -3, \\ x^2 & \text{if} \quad -3 < x \le 1, \\ x^3 & \text{if} \quad 1 < x \le 5, \\ x^4 & \text{if} \quad 5 < x, \end{cases} \qquad \text{and} \quad g(x) = 0.$$

(a) What are the two families of characteristic curves for this PDE?

(b) What are the two specific characteristic curves passing through the point $x = 2, t = 3$?

(c) If $u(x, t)$ solves the above problem, what is the value of $u(-4, 1)$? What is the value of $u(5, 3/4)$?

12.3.3. Suppose that a stretched string is unperturbed (horizontal, $u = 0$) and at rest ($\partial u/\partial t = 0$). If an impulsive force is applied at $t = 0$, the initial value problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \alpha(x)\delta(t)$$

$$u(x, t) = 0, \quad t < 0.$$

(a) Without using explicit solutions, show that this is equivalent to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \; t > 0,$$

subject to $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = \alpha(x)$. Thus, the initial velocity $\alpha(x)$ is equivalent to an impulsive force.

(b) Do part (a) using the explicit solution of both problems.

12.3.4. An alternative way to solve the one-dimensional wave equation (12.3.1) is based on (12.3.2) and (12.3.3). Solve the wave equation by introducing a change of variables from (x, t) to two moving coordinates (ξ, η), one moving to the left (with velocity $-c$) and one moving to the right (with velocity c):

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct.$$

***12.3.5.** Suppose that $u(x, t) = F(x - ct)$. Evaluate:

(a) $\dfrac{\partial u}{\partial t}(x, 0)$ **(b)** $\dfrac{\partial u}{\partial x}(0, t)$

12.3.6. Determine analytic formulas for $u(x, t)$ if

$$u(x, 0) = f(x) = 0$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) = \begin{cases} 1 & |x| < h \\ 0 & |x| > h. \end{cases}$$

[*Hint*: Using characteristics as sketched in Fig. 12.3.3, show that there are two distinct regions, $t < h/c$ and $t > h/c$. In each, show that the solution has five different forms, depending on x.]

12.3.7. Consider the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

Assume that the solution is **spherically symmetric**, so that

$$\nabla^2 u = (1/\rho^2)(\partial/\partial\rho)(\rho^2 \, \partial u/\partial\rho).$$

(a) Make the transformation $u = (1/\rho)w(\rho, t)$ and verify that

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial \rho^2}.$$

(b) Show that the most general spherically symmetric solution of the wave equation consists of the sum of two spherically symmetric waves, one moving outward at speed c and the other inward at speed c. Note the decay of the amplitude.

12.4 Semi-Infinite Strings and Reflections

We will solve the one-dimensional wave equation on a semi-infinite interval, $x > 0$:

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \tag{12.4.1}$$

$$\text{IC1:} \quad \boxed{u(x, 0) = f(x)} \tag{12.4.2}$$

$$\text{IC2:} \quad \boxed{\frac{\partial u}{\partial t}(x, 0) = g(x).} \tag{12.4.3}$$

A condition is necessary at the boundary $x = 0$. We suppose that the string is fixed at $x = 0$:

$$\text{BC: } \boxed{u(0, t) = 0.} \tag{12.4.4}$$

Although a Fourier sine transform can be used, we prefer to indicate how to use the general solution and the method of characteristics:

$$\boxed{u(x, t) = F(x - ct) + G(x + ct).} \tag{12.4.5}$$

As in Section 12.3, the initial conditions are satisfied if

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\, d\bar{x}, \quad x > 0 \tag{12.4.6}$$

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\, d\bar{x}, \quad x > 0. \tag{12.4.7}$$

However, it is very important to note that (unlike the case of the infinite string) (12.4.6) and (12.4.7) are valid only for $x > 0$; the arbitrary functions are determined from the initial conditions only for positive arguments. In the general solution, $G(x + ct)$ requires only positive arguments of G (since $x > 0$ and $t > 0$). On the other hand, $F(x - ct)$ requires positive arguments if $x > ct$, but requires negative arguments if $x < ct$. As indicated by a space-time diagram, Fig. 12.4.1, the information that there is a fixed end at $x = 0$ travels at a finite velocity c. Thus, if $x > ct$, the string does not know that there is any boundary. In this case ($x > ct$), the solution is obtained as before [using (12.4.6) and (12.4.7)],

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\, d\bar{x}, \quad x > ct, \tag{12.4.8}$$

d'Alembert's solution. However, here this is not valid if $x < ct$. Since $x + ct > 0$,

$$G(x + ct) = \frac{1}{2}f(x + ct) + \frac{1}{2c}\int_0^{x+ct} g(\bar{x})\, d\bar{x},$$

as determined earlier. To obtain F for negative arguments, we cannot use the initial conditions. Instead, the boundary condition must be utilized. $u(0, t) = 0$ implies that [from (12.4.5)]

$$0 = F(-ct) + G(ct) \quad \text{for } t > 0. \tag{12.4.9}$$

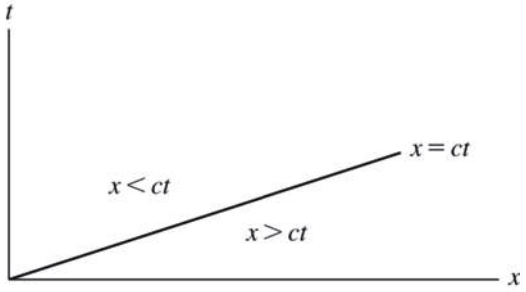
Thus, F for negative arguments is $-G$ of the corresponding positive argument:

$$F(z) = -G(-z) \quad \text{for } z < 0. \tag{12.4.10}$$

Thus, the solution for $x - ct < 0$ is

$$\begin{aligned} u(x, t) &= F(x - ct) + G(x + ct) = G(x + ct) - G(ct - x) \\ &= \frac{1}{2}[f(x + ct) - f(ct - x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(\bar{x})\, d\bar{x} - \int_0^{ct-x} g(\bar{x})\, d\bar{x}\right] \\ &= \frac{1}{2}[f(x + ct) - f(ct - x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\, d\bar{x}. \end{aligned}$$

Figure 12.4.1 Characteristic emanating from the boundary.



To interpret this solution, the method of characteristics is helpful. Recall that for infinite problems, $u(x, t)$ is the sum of F (moving to the right) and G (moving to the left). For semi-infinite problems with $x > ct$, the boundary does not affect the characteristics (see Fig. 12.4.2). If $x < ct$,

then Fig. 12.4.3 shows the left-moving characteristic (G constant) not affected by the boundary, but the right-moving characteristic emanates from the boundary. F is constant moving to the right. Due to the boundary condition, $F + G = 0$ at $x = 0$, the right-moving wave is minus the left-moving wave. The wave inverts as it "bounces off" the boundary. The resulting right-moving wave $-G(ct - x)$ is called the **reflected wave**. For $x < ct$, the total solution is the reflected wave plus the as-yet unreflected left-moving wave:

$$u(x, t) = G(x + ct) - G(-(x - ct)).$$

Figure 12.4.2 Characteristics.

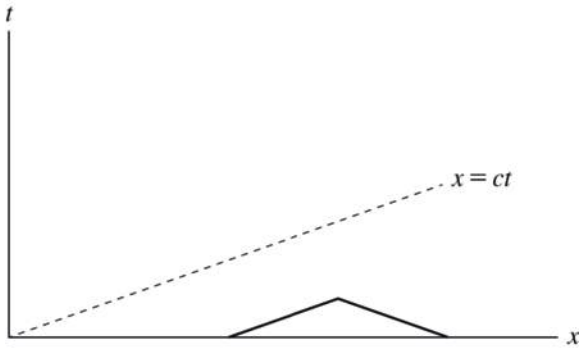
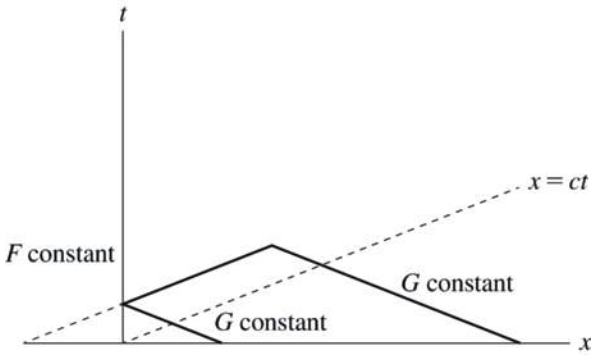


Figure 12.4.3 Reflected characteristics.



The negatively reflected wave $-G(-(x - ct))$ moves to the right. It behaves *as if* initially at $t = 0$ it were $-G(-x)$. If there were no boundary, the right-moving wave $F(x - ct)$ would be initially $F(x)$. Thus, the reflected wave is exactly the wave that would have occurred if

$$F(x) = -G(-x) \quad \text{for} \quad x < 0,$$

or, equivalently,

$$\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x} = -\frac{1}{2}f(-x) - \frac{1}{2c}\int_0^{-x} g(\overline{x})\, d\overline{x}.$$

One way to obtain this is to extend the initial position $f(x)$ for $x > 0$ as an odd function [such that $f(-x) = -f(x)$] and also extend the initial velocity $g(x)$ for $x > 0$ as an odd function [then its integral, $\int_0^x g(\overline{x})\, d\overline{x}$, will be an even function]. *In summary,* **the solution of the semi-infinite problem with $u = 0$ at $x = 0$ is the same as an infinite problem with the initial positions and velocities extended as odd functions**.

As further explanation, suppose that $u(x, t)$ is any solution of the wave equation. Since the wave equation is unchanged when x is replaced by $-x$, $u(-x, t)$ (and any multiple of it) is also a solution of the wave equation. If the initial conditions satisfied by $u(x, t)$ are odd functions of x, then both $u(x, t)$ and $-u(-x, t)$ solve these initial conditions and the wave equation. Since the initial value problem has a unique solution, $u(x, t) = -u(-x, t)$; that is, $u(x, t)$, which is odd initially, will remain odd for all time. Thus, odd initial conditions yield a solution that will satisfy a zero boundary condition at $x = 0$.

EXAMPLE

Consider a semi-infinite string $x > 0$ with a fixed end $u(0, t) = 0$, which is initially at rest, $\frac{\partial u}{\partial t}(x, 0) = 0$, with an initial unit rectangular pulse,

$$f(x) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

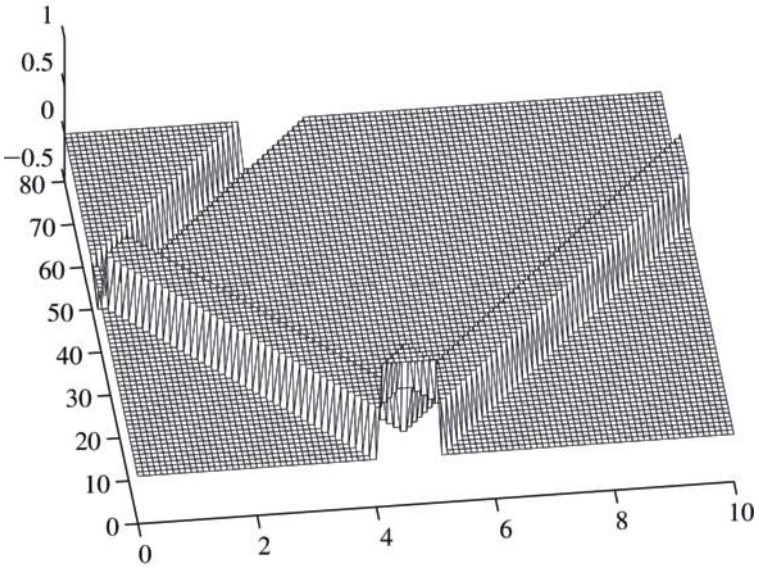
Since $g(x) = 0$, it follows that

$$F(x) = G(x) = \frac{1}{2}f(x) = \begin{cases} \frac{1}{2} & 4 < x < 5 \\ 0 & \text{otherwise (with } x > 0). \end{cases}$$

F moves to the right; G moves to the left, negatively reflecting off $x = 0$. This can also be interpreted as an initial condition (on an infinite domain) with $f(x)$ and $g(x)$ extended as odd functions. The solution is sketched in Fig. 12.4.4. Note the *negative* reflection.

Problems with nonhomogeneous boundary conditions at $x = 0$ can be analyzed in a similar way.

Figure 12.4.4 Reflected pulse.



MATLAB Files

Download Animation of Figure 12.4.4 (.zip)

https://bit.ly/3U9qr7B

These files are provided as zipped .m files for use in MATLAB on desktop platform.

Exercises 12.4

***12.4.1.** Solve by the method of characteristics:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \; x > 0,$$

subject to $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = 0$, and $u(0, t) = h(t)$.

***12.4.2.** Determine $u(x, t)$ if

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } x < 0 \text{ only,}$$

where

$$\begin{aligned} u(x, 0) &= \cos x, && x < 0 \\ \frac{\partial u}{\partial t}(x, 0) &= 0, && x < 0 \\ u(0, t) &= e^{-t}, && t > 0. \end{aligned}$$

Do not sketch the solution. However, draw a space-time diagram, including all important characteristics.

12.4.3. Consider the wave equation on a semi-infinite interval

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{for } 0 < x < \infty$$

with the free boundary condition

$$\frac{\partial u}{\partial x}(0, t) = 0$$

and the initial conditions

$$u(x, 0) = \begin{cases} 0 & 0 < x < 2 \\ 1 & 2 < x < 3, \\ 0 & x > 3 \end{cases} \quad \frac{\partial u}{\partial t}(x, 0) = 0.$$

Determine the solution. Sketch the solution for various times. (Assume that u is continuous at $x = 0, t = 0$.)

12.4.4. **(a)** Solve for $x > 0, t > 0$ (using the method of characteristics):

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) \end{aligned}\right\} \quad x > 0$$
$$\frac{\partial u}{\partial x}(0, t) = 0, \quad t > 0.$$

(Assume that u is continuous at $x = 0, t = 0$.)

(b) Show that the solution of part (a) may be obtained by extending the initial position and velocity as even functions (around $x = 0$).

(c) Sketch the solution if $g(x) = 0$ and

$$f(x) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

12.4.5. **(a)** Show that if $u(x, t)$ and $\partial u/\partial t$ are initially even around $x = x_0$, $u(x, t)$ will remain even for all time.

(b) Show that this type of even initial condition yields a solution that will satisfy a zero derivative boundary condition at $x = x_0$.

***12.4.6.** Solve ($x > 0, t > 0$)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = 0$, and $\frac{\partial u}{\partial x}(0, t) = h(t)$.

***12.4.7.** Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \begin{matrix} x > 0 \\ t > 0 \end{matrix}$$

subject to $u(x, 0) = f(x)$, $\frac{\partial u}{\partial t}(x, 0) = 0$, and $\frac{\partial u}{\partial x}(0, t) = h(t)$. (Assume that u is continuous at $x = 0, t = 0$.)

12.4.8. Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ with } u(x, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = 0,$$

subject to $u(x, t) = g(t)$ along $x = \frac{c}{2}t (c > 0)$.

12.5 Method of Characteristics for a Vibrating String of Fixed Length

In Chapter 4, we solved for the vibration of a finite string satisfying

$$\text{PDE: } \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \tag{12.5.1}$$

$$\text{BC: } \boxed{\begin{aligned} u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned}} \tag{12.5.2}$$

$$\text{IC: } \boxed{\begin{aligned} u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x), \end{aligned}} \tag{12.5.3}$$

using Fourier series methods. We can obtain an equivalent, but in some ways more useful, result by using the general solution of the one-dimensional wave equation:

$$u(x, t) = F(x - ct) + G(x + ct). \tag{12.5.4}$$

The initial conditions are prescribed only for $0 < x < L$, and hence the formulas for $F(x)$ and $G(x)$ previously obtained are valid only for $0 < x < L$:

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} \tag{12.5.5}$$

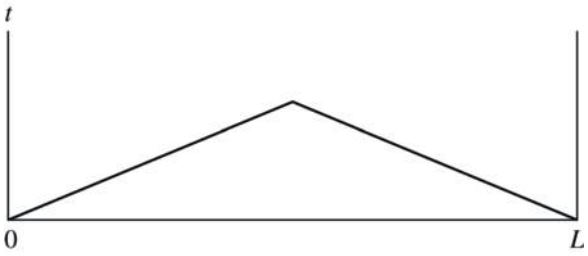
$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}. \tag{12.5.6}$$

If $0 < x - ct < L$ and $0 < x + ct < L$, which is the triangular region in Fig. 12.5.1, then d'Alembert's solution is valid:

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}. \tag{12.5.7}$$

In this region, the string does not know that either boundary exists; the information that there is a boundary propagates at velocity c from $x = 0$ and $x = L$.

Figure 12.5.1 Characteristics.



If one's position and time are such that signals from the boundary have already arrived, then modifications in (12.5.7) must be made. The boundary condition at $x = 0$ implies that

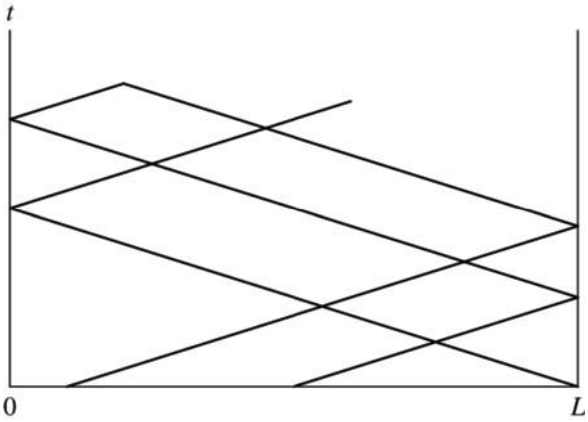
$$0 = F(-ct) + G(ct) \qquad \text{for } t > 0, \tag{12.5.8}$$

while at $x = L$ we have

$$0 = F(L - ct) + G(L + ct) \qquad \text{for } t > 0. \tag{12.5.9}$$

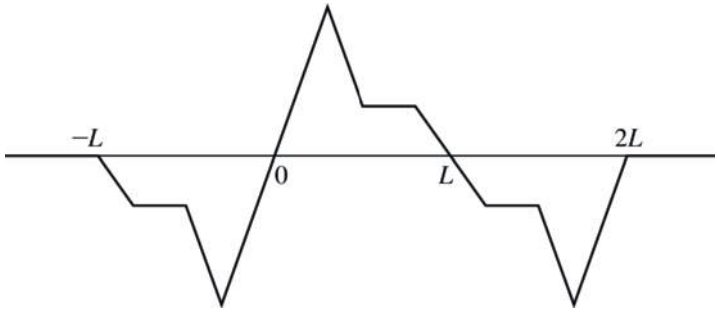
These, in turn, imply reflections and multiple reflections, as illustrated in Fig. 12.5.2.

Figure 12.5.2 Multiply reflected characteristics.



Alternatively, a solution on an infinite domain without boundaries can be considered that is odd around $x = 0$ and odd around $x = L$, as sketched in Fig. 12.5.3. In this way, the zero condition at both $x = 0$ and $x = L$ will be satisfied. We note that $u(x, t)$ is periodic with period $2L$. In fact, we ignore the oddness around $x = L$, since periodic functions that are odd around $x = 0$ are automatically odd around $x = L$. Thus, the simplest way to obtain the solution is to **extend the initial conditions as odd functions (around $x = 0$) that are periodic (with period $2L$)**. With these odd-periodic initial conditions, the method of characteristics can be utilized as well as d'Alembert's solution (12.5.7).

Figure 12.5.3 Odd-periodic extension.



EXAMPLE

Suppose that a string is initially at rest with prescribed initial conditions $u(x, 0) = f(x)$. The string is fixed at $x = 0$ and $x = L$. Instead of using Fourier series methods, we extend the initial conditions as odd functions around $x = 0$ and $x = L$. Equivalently, we introduce the *odd-periodic extension*. (The odd-periodic extension is also used in the Fourier series solution.) Since the string is initially at rest, $g(x) = 0$; the odd-periodic extension is $g(x) = 0$ for all x. Thus, the solution of the one-dimensional wave equation is the sum of two simple waves:

$$u(x, t) = \frac{1}{2}[f_{\text{ext}}(x - ct) + f_{\text{ext}}(x + ct)],$$

where $f_{\text{ext}}(x)$ is the odd-periodic extension of the given initial position. This solution is much simpler than the summation of the first 100 terms of its Fourier sine series. ■

SEPARATION OF VARIABLES

By separation of variables, the solution of the wave equation with fixed boundary conditions $[u(0, t) = 0$ and $u(L, t) = 0]$ satisfying the initial conditions $u(x, 0) = f(x)$ and $\frac{\partial u}{\partial t} = 0$ is

$$u(x, t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L},$$

where the given initial conditions are specified only for $[0, L]$. To be precise, the infinite Fourier series $f_{\text{ext}}(x)$ equals the odd-periodic extension (with period $2L$) of $f(x)$: $f_{\text{ext}}(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$. Using $\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} = \frac{1}{2}\sin\frac{n\pi(x+ct)}{L} + \frac{1}{2}\sin\frac{n\pi(x-ct)}{L}$, we obtain

$$u(x, t) = \frac{1}{2}f_{\text{ext}}(x + ct) + \frac{1}{2}f_{\text{ext}}(x - ct),$$

which is the same result obtained by the method of characteristics.

Exercises 12.5

12.5.1. Consider

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} \\ u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) \\ u(0, t) &= 0 \\ u(L, t) &= 0. \end{aligned}\right\} \quad 0 < x < L$$

(a) Obtain the solution by Fourier series techniques.

***(b)** Show that, in the case where $g(x) = 0$, the Fourier series solution found in part (a) is equivalent to the solution found by the method of characteristics.

(c) Show that, in the case where $f(x) = 0$, the Fourier series solution found in part (a) is equivalent to the solution found by the method of characteristics.

12.5.2. Solve using the method of characteristics:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x, 0) = 0, \qquad u(0, t) = h(t)$$
$$\frac{\partial u}{\partial t}(x, 0) = 0, \qquad u(L, t) = 0.$$

12.5.3. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 10$$
$$u(x, 0) = f(x) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise,} \end{cases} \qquad u(0, t) = 0$$
$$\frac{\partial u}{\partial t}(x, 0) = g(x) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0.$$

(a) Sketch the solution using the method of characteristics.

(b) Obtain the solution using Fourier series–type techniques.

(c) Obtain the solution by converting to an equivalent problem on an infinite domain.

12.5.4. How should initial conditions be extended if $\frac{\partial u}{\partial x}(0, t) = 0$ and $u(L, t) = 0$?

12.6 The Method of Characteristics for Quasilinear Partial Differential Equations

12.6.1 METHOD OF CHARACTERISTICS

Most of this text describes methods for solving linear partial differential equations (separation of variables, eigenfunction expansions, Fourier and Laplace transforms, Green's functions) that cannot be extended to nonlinear problems. However, the method of characteristics, used to solve the wave equation, can be applied to partial differential equations of the form

$$\boxed{\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = Q,} \tag{12.6.1}$$

where c and Q may be functions of x, t, and ρ. When Q is not a linear function of ρ or, more important to us, when the coefficient c depends on the unknown solution ρ, then (12.6.1) is not linear. Superposition is not valid. Nonetheless (12.6.1) is called a **quasilinear** partial differential equation, since it is linear in the first partial derivatives, $\partial\rho/\partial t$ and $\partial\rho/\partial x$. To solve (12.6.1), we again consider an observer moving in some prescribed way $x(t)$. By comparing (12.2.7) and (12.6.1), we obtain

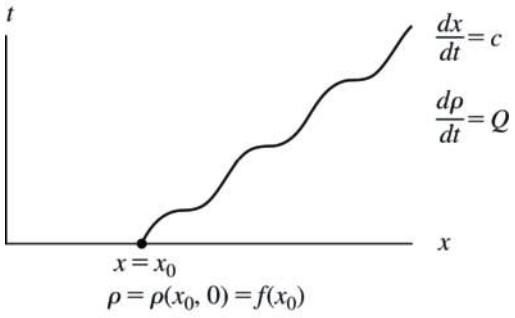
$$\boxed{\frac{d\rho}{dt} = Q(\rho, x, t)} \tag{12.6.2}$$

if

$$\boxed{\frac{dx}{dt} = c(\rho, x, t).} \tag{12.6.3}$$

The partial differential equation (12.6.1) reduces to two coupled ordinary differential equations along the special trajectory or direction defined by (12.6.3), known as a **characteristic curve**, or simply a **characteristic** for short. The velocity defined by (12.6.3) is called the **characteristic velocity**, or **local wave velocity**. A characteristic starting from $x = x_0$, as illustrated in Fig. 12.6.1, is determined from the coupled differential equations (12.6.2) and (12.6.3) using the initial conditions $\rho(x, 0) = f(x)$. Along the characteristic, the solution ρ changes according to (12.6.2). Other initial positions yield other characteristics, generating a family of characteristics.

Figure 12.6.1 Characteristic starting from $x = x_0$ at time $t = 0$.



EXAMPLE

If the local wave velocity c is a constant c_0 and $Q = 0$, then the quasilinear partial differential equation (12.6.1) becomes the linear one, (12.2.6), which arises in the analysis of the wave equation. In this example, the characteristics may be obtained by directly integrating (12.6.3) without using (12.6.2). Each characteristic has the same constant velocity, c_0. The family of characteristics are parallel straight lines, as sketched in Fig. 12.2.1. ■

Quasilinear in two-dimensional space

If the independent variables are x and y instead of x and t, then a quasilinear first-order partial differential equation is usually written in the form

$$a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = c, \tag{12.6.4}$$

where a, b, and c may be functions of x, y, and ρ. The method of characteristics is

$$\frac{d\rho}{dx} = \frac{c}{a} \tag{12.6.5}$$

if

$$\frac{dy}{dx} = \frac{b}{a}. \tag{12.6.6}$$

This is written in the following (easy to memorize) equivalent form:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{d\rho}{c}. \tag{12.6.7}$$

12.6.2 TRAFFIC FLOW

Traffic density and flow

As an approximation it is possible to model a congested one-directional highway by a quasilinear partial differential equation. We introduce the **traffic density** $\rho(x, t)$, the number of cars per mile at time t located at position x. An easily observed and measured quantity is the **traffic flow** $q(x, t)$, the number of cars per hour passing a fixed place x (at time t).

Conservation of cars

We consider an arbitrary section of roadway, between $x = a$ and $x = b$. If there are neither entrances nor exits on this segment of the road, then the number of cars between $x = a$ and $x = b$ [$N = \int_a^b \rho(x, t)\, dx$, the definite integral of the density] might still change in time. The rate of change of the number of cars, dN/dt, equals the number per unit time entering at $x = a$ [the traffic flow $q(a, t)$ there] minus the number of cars per unit time leaving at $x = b$ [the traffic flow $q(b, t)$ there]:

$$\boxed{\frac{d}{dt}\int_a^b \rho(x, t)\ \ dx = q(a, t) - q(b, t).} \tag{12.6.8}$$

Equation (12.6.8) is called the integral form of **conservation of cars**. As with heat flow, a partial differential equation may be derived from (12.6.8) in several equivalent ways. One way is to note that the boundary contribution may be expressed as an integral over the region:

$$q(a, t) - q(b, t) = -\int_a^b \frac{\partial}{\partial x} q(x, t)\, dx. \tag{12.6.9}$$

Thus, by taking the time derivative inside the integral (making it a partial derivative) and using (12.6.9), it follows that

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0,} \tag{12.6.10}$$

since a and b are arbitrary (see Section 1.2). We call (12.6.10) **conservation of cars**.

Car velocity

The number of cars per hour passing a place equals the density of cars times the velocity of cars. By introducing $u(x, t)$ as the **car velocity**, we have

$$\boxed{q = \rho u.} \tag{12.6.11}$$

To derive (12.6.11), it may be helpful to consider the number of cars passing a fixed observer in a short time Δt, the flow times time: $q\Delta t$. On the other hand, cars are moving at velocity u, will have traveled a distance $u\Delta t$, with car density, cars per mile ρ, so that the number of cars passing is $\rho u \Delta t$.

In the mid-1950s, Lighthill and Whitham and, independently, Richards made a simplifying assumption, namely, that the car velocity depends only on the density, $u = u(\rho)$, with cars slowing down as the traffic density increases (i.e., $du/d\rho \leq 0$). For further discussion, the interested reader is referred to Whitham [1999] and Haberman [1998]. Under this assumption, the traffic flow is a function only of the traffic density, $q = q(\rho)$. In this case, conservation of cars (12.6.10) becomes

$$\boxed{\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0,} \tag{12.6.12}$$

where $c(\rho) = q'(\rho)$, a quasilinear partial differential equation with $Q = 0$ [see (12.6.1)]. Here $c(\rho)$ is considered to be a known function of the unknown solution ρ. In any physical problem in which a density ρ is conserved and the flow q is a function of density, ρ satisfies (12.6.12).

Elementary traffic model

In general, the car velocity u should be a decreasing function of ρ. At zero density, cars move fastest, which we denote $u_{\max}$. At some maximum density $\rho_{\max}$, the car velocity will be zero. The simplest relationship which satisfies these properties is $u(\rho) = u_{\max}(1 - (\rho/\rho_{\max}))$, in which case from (12.6.11) the flow is given by $q(\rho) = u_{\max}\rho(1 - (\rho/\rho_{\max})) = u_{\max}(\rho - (\rho^2/\rho_{\max}))$, and the density wave velocity satisfies $c(\rho) = q'(\rho) = u_{\max}(1 - (2\rho/\rho_{\max}))$.

12.6.3 METHOD OF CHARACTERISTICS ($Q = 0$)

The equations for the characteristics for (12.6.12) are

$$\frac{d\rho}{dt} = 0 \tag{12.6.13}$$

along

$$\frac{dx}{dt} = c(\rho). \tag{12.6.14}$$

The **characteristic velocity** c is not constant but depends on the density ρ. It is known as the **density wave velocity**. From (12.6.13), it follows that the density ρ remains constant along each as-yet undetermined characteristic. The velocity of each characteristic, $c(\rho)$, will be constant, since ρ is constant. Each characteristic is thus a straight line [as in the case in which $c(\rho)$ is a constant c_0]. However, different characteristics will move at different constant velocities because they may start with different densities. The characteristics, though each is straight, are not parallel to one another. Consider the characteristic that is initially at the position $x = x_0$, as shown in Fig. 12.6.2. Along

the curve $dx/dt = c(\rho)$, $d\rho/dt = 0$ or ρ is constant. Initially ρ equals the value at $x = x_0$ (i.e., at $t = 0$). Thus, along this one characteristic,

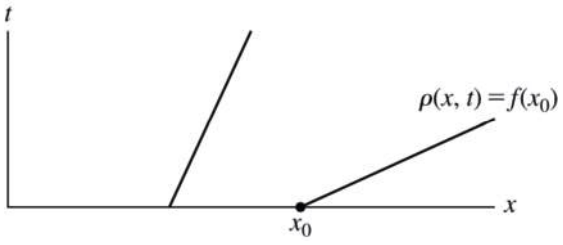
$$\rho(x, t) = \rho(x_0, 0) = f(x_0), \tag{12.6.15}$$

which is a known constant. The local wave velocity that determines the characteristic is a constant, $dx/dt = c(f(x_0))$. Consequently, this characteristic is a straight line,

$$x = c(f(x_0))t + x_0, \tag{12.6.16}$$

since $x = x_0$ at $t = 0$. Different values of x_0 yield different straight-line characteristics, perhaps as illustrated in Fig. 12.6.2. Along each characteristic, the traffic density ρ is a constant; see (12.6.15). To determine the density at some later time, the characteristic with parameter x_0 that goes through that space-time point must be obtained from (12.6.16).

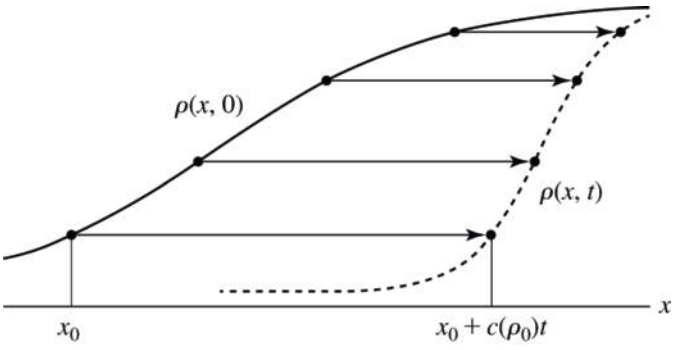
Figure 12.6.2 Possibly nonparallel straight-line characteristics.



Graphical solution

In practice, it is often difficult and not particularly interesting to determine x_0 from (12.6.16) as an explicit function of x and t. Instead, a graphical procedure may be used to determine $\rho(x, t)$. Suppose the initial density is as sketched in Fig. 12.6.3. We know that each density ρ_0 stays the same, moving at its own constant-density wave velocity $c(\rho_0)$. At time t, the density ρ_0 will have moved a distance $c(\rho_0)t$, as illustrated by the arrow in Fig. 12.6.3. This process must be carried out for a large number of points (as is elementary to do on any computer). In this way, we could obtain the density at time t.

Figure 12.6.3 Graphical solution.



Fanlike characteristics

As an example of the method of characteristics, we consider the following initial value problem:

$$\frac{\partial \rho}{\partial t} + 2\rho\frac{\partial \rho}{\partial x} = 0$$

$$\rho(x, 0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0. \end{cases}$$

The density $\rho(x, t)$ is constant moving with the characteristic velocity 2ρ:

$$\frac{dx}{dt} = 2\rho.$$

Thus, the characteristics are given by

$$x = 2\rho(x_0, 0)t + x_0. \tag{12.6.17}$$

If $x_0 > 0$, then $\rho(x_0, 0) = 4$, while if $x_0 < 0$, then $\rho(x_0, 0) = 3$. The characteristics, sketched in Fig. 12.6.4, show that

$$\rho(x, t) = \begin{cases} 4 & x > 8t \\ 3 & x < 6t, \end{cases}$$

as illustrated in Fig. 12.6.5. The distance between $\rho = 3$ and $\rho = 4$ is increasing; we refer to the solution as an **expansion wave**. But what happens for $6t < x < 8t$? The difficulty is caused by the initial density having a discontinuity at $x = 0$. We imagine that all values of ρ between 3 and 4 are present initially at $x = 0$. There will be a straight-line characteristic along which ρ equals each value between 3 and 4. Since these characteristics start from $x = 0$ at $t = 0$, it follows from (12.6.17) that the equation for these characteristics is

$$x = 2\rho t, \quad \text{for } 3 < \rho < 4,$$

also sketched in Fig. 12.6.4. In this way, we obtain the density in the wedge-shaped region,

$$\rho = \frac{x}{2t} \quad \text{for } 6t < x < 8t,$$

which is linear in x (for fixed t). We note that the characteristics fan out from $x = 6t$ to $x = 8t$ and hence are called **fanlike characteristics**. The resulting density is sketched in Fig. 12.6.5. It could also be obtained by the graphical procedure.

Figure 12.6.4 Characteristics (including the fanlike ones).

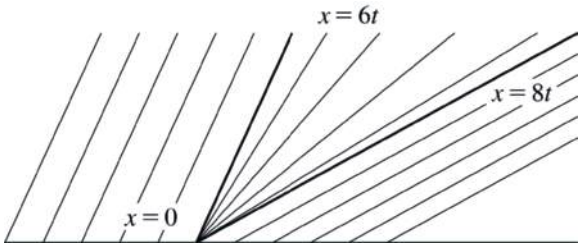
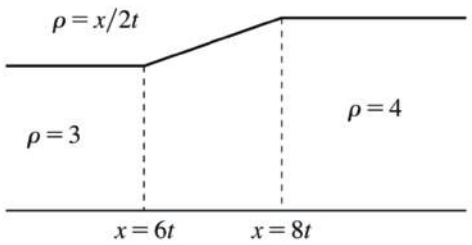
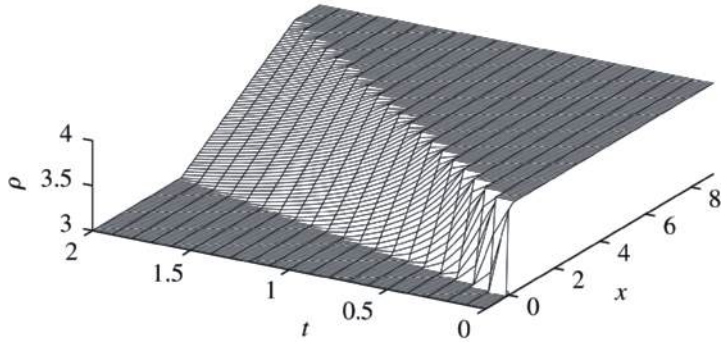


Figure 12.6.5 Expansion wave.





Red light turning green

We assume the elementary model of traffic flow, so that the traffic density satisfies

$$\frac{\partial \rho}{\partial t} + u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)\frac{\partial \rho}{\partial x} = 0.$$

Behind a red light ($x = 0$), the traffic density is maximum, while ahead of the light the traffic density is zero, so that at the moment ($t = 0$) the light turns green, the initial conditions are

$$\rho(x, 0) = \begin{cases} \rho_{\max} & x < 0 \\ 0 & x > 0. \end{cases}$$

The characteristic velocity is $\frac{dx}{dt} = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$. The density is constant along the characteristics so that the characteristics satisfy

$$x = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)t + x_0.$$

The characteristic velocity is $u_{\max}$ for $\rho = 0$, while the characteristic velocity is $-u_{\max}$ for $\rho = \rho_{\max}$. Thus,

$$\rho(x, t) = \begin{cases} \rho_{\max} & x < -u_{\max}t \\ 0 & x > u_{\max}t. \end{cases}$$

The information that the traffic light has turned green propagates backward at density velocity $-u_{\max}$. That is why you have to wait after a light turns green before you can move. The characteristics are similar to Fig. 12.6.4. To obtain the density elsewhere, we note that there is a family of fanlike characteristics that all start at $x_0 = 0$. Thus, in this region, $x = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)t$. Given x and t in this region, we can solve for the density

$$\rho(x, t) = \frac{\rho_{\max}}{2}\left(1 - \frac{x}{u_{\max}t}\right) \quad \text{for } -u_{\max}t < x < u_{\max}t.$$

The solution is similar to the expansion wave shown in Fig. 12.6.5, but for an expansion wave for traffic flow, the higher density (slowly moving traffic) is behind the lower density (faster moving traffic), as a result of the traffic light turning green.

12.6.4 SHOCK WAVES

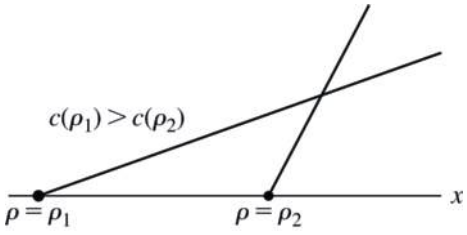
Intersecting characteristics

The method of characteristics will not always work as we have previously described. For quasilinear partial differential equations, it is quite usual for characteristics to intersect. The resolution will require the introduction of moving discontinuities called *shock waves*. In order to make the mathematical presentation relatively simple, we restrict our attention to quasilinear partial differential equations with $Q = 0$, in which case

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{12.6.18}$$

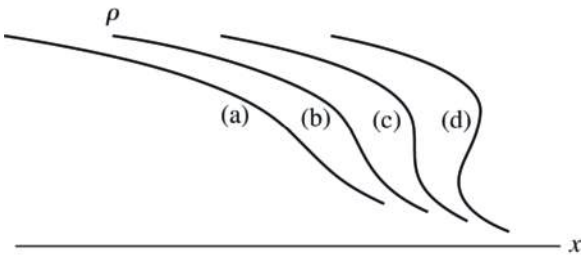
In Fig. 12.6.6, two characteristics are sketched, one starting at $x = x_1$, with $\rho = f(x_1, 0) \equiv \rho_1$, and the other starting at $x = x_2$ with $\rho = f(x_2, 0) \equiv \rho_2$. These characteristics intersect if $c(\rho_1) > c(\rho_2)$, the faster catching up to the slower. The density is constant along characteristics. As time increases, the distance between the densities ρ_1 and ρ_2 decreases. Thus, this is called a **compression wave**. We sketch the initial condition in Fig. 12.6.7a. The density distribution becomes

Figure 12.6.6 Intersecting characteristics.



steeper as time increases (Figs. 12.6.7b and c). Eventually characteristics intersect; the theory predicts that the density is simultaneously ρ_1 and ρ_2. If we continue to apply the method of characteristics, the faster-moving characteristic passes the slower. Then we obtain Fig. 12.6.7d. The method of characteristics predicts that the density becomes a "multivalued" function of position; that is, at some later time, our mathematics predicts there will be three densities at some positions (as illustrated in Fig. 12.6.7d). We say the density wave *breaks*. However, in many physical problems (such as traffic flow) it makes no sense to have three values of density at one place.[3] The density must be a single-valued function of position.

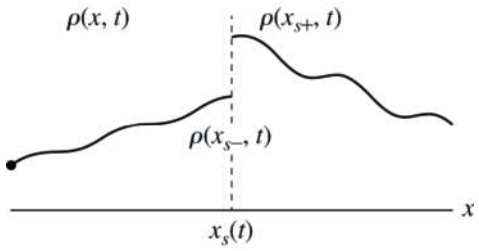
Figure 12.6.7 Density wave steepens (density becomes triple valued).



Discontinuous solutions

On the basis of the quasilinear partial differential equation (12.6.18), we predicted the physically impossible phenomenon that the density becomes multivalued. Since the method of characteristics is mathematically justified, it is the partial differential equation itself that must not be entirely valid. Some approximation or assumption that we used must at times be invalid. We will assume that the density (as illustrated in Fig. 12.6.8) and velocity have a jump discontinuity, which we call a **shock wave**, or simply a **shock**.[4] The shock occurs at some unknown position x_s and propagates in time, so that $x_s(t)$. We introduce the notation x_{s-} and x_{s+} for the position of the shock on the two sides of the discontinuity. The shock velocity, dx_s/dt, is as yet unknown.

Figure 12.6.8 Density discontinuity at $x = x_s(t)$.



Shock velocity

On either side of the shock, the quasilinear partial differential equation applies, $\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$, where $c(\rho) = \frac{dq(\rho)}{d\rho}$. We need to determine how the discontinuity propagates. If ρ is conserved even at a discontinuity, then the flow relative to the moving shock on one side of the shock must equal

[3]The partial differential equations describing the height of water waves near the shore (i.e., in shallow water) are similar to the equations for traffic density waves. In this situation, the prediction of breaking is then significant!

[4]The terminology *shock wave* is introduced because of the analogous behavior that occurs in gas dynamics. There, changes in pressure and density of air, for example, propagate and are heard (due to the sensitivity of the human ear). They are called *sound waves*. When fluctuations of pressure and density are small, the equations describing sound waves can be linearized. Then sound is propagated at a constant speed known as the *sound speed*. However, if the amplitudes of the fluctuations of pressure and density are not small, then the partial differential equations are quasilinear. Characteristics may intersect. In this case, the pressure and density can be modeled mathematically as being discontinuous, the result being called a *shock wave*. Examples are the sound emitted from an explosion or the thunder resulting from lightning. If a shock wave results from exceeding the sound barrier, it is known as a *sonic boom*.

the flow relative to the moving shock on the other side. This statement of relative inflow equaling relative outflow becomes

$$\rho(x_{s-}, t)\left[u(x_{s-}, t) - \frac{dx_s}{dt}\right] = \rho(x_{s+}, t)\left[u(x_{s+}, t) - \frac{dx_s}{dt}\right], \tag{12.6.19}$$

since flow equals density times velocity (here relative velocity). Solving for the **shock velocity** from (12.6.19) yields

$$\boxed{\frac{dx_s}{dt} = \frac{q(x_{s+}, t) - q(x_{s-}, t)}{\rho(x_{s+}, t) - \rho(x_{s-}, t)} = \frac{[q]}{[\rho]},} \tag{12.6.20}$$

where we recall that $q = \rho u$ and where we introduce the notation $[q]$ and $[\rho]$ for the jumps in q and ρ, respectively. In gas dynamics, (12.6.20) is called the Rankine–Hugoniot condition. **In summary, for the conservation law $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$ (if the quantity $\int \rho\, dx$ is actually conserved), the shock velocity equals the jump in the flow divided by the jump in the density of the conserved quantity**. At points of discontinuity, this shock condition replaces the use of the partial differential equation, which is valid elsewhere. However, we have not yet explained where shocks occur and how to determine $\rho(x_{s+}, t)$ and $\rho(x_{s-}, t)$.

Alternate derivation of the shock velocity

Let us consider the conservation law, conservation of cars, (12.6.8), for a finite region $(a < x < b)$ that includes the moving shock $(a < x_s(t) < b)$ (see Figure 12.6.8):

$$\frac{d}{dt}\int_a^b \rho\, dx = q(a, t) - q(b, t). \tag{12.6.21}$$

Since there is a discontinuity in the density, it is best to break the integral up into two pieces:

$$\frac{d}{dt}\left[\int_a^{x_s(t)} \rho(x, t)\ dx + \int_{x_s(t)}^b \rho(x, t)\ dx\right] = q(a, t) - q(b, t).$$

Using Leibniz's rule for the derivative of integrals with variable limits, we obtain

$$\int_a^{x_s(t)} \frac{\partial \rho}{\partial t}\, dx + \frac{dx_s}{dt}\rho(x_{s-}, t) + \int_{x_s(t)}^b \frac{\partial \rho}{\partial t} dx - \frac{dx_s}{dt}\rho(x_{s+}, t) = q(a, t) - q(b, t).$$

However, away from the shock, the solution is smooth, and the partial differential equation $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$ is valid in each region $a < x < x_s$ and $x_s < x < b$. The integrals $\int \frac{\partial q}{\partial x}\, dx$ are then easily computed so that

$$q(a, t) - q(x_{s-}, t) + \frac{dx_s}{dt}[\rho(x_{s-}, t) - \rho(x_{s+}, t)] + q(x_{s+}, t) - q(b, t) = q(a, t) - q(b, t).$$

Canceling $q(a, t)$ and $q(b, t)$ then yields the fundamental equation for the shock velocity (12.6.20):

$$\boxed{\frac{dx_s}{dt} = \frac{q(x_{s-}, t) - q(x_{s+}, t)}{\rho(x_{s-}, t) - \rho(x_{s+}, t)} = \frac{[q]}{[\rho]},} \tag{12.6.22}$$

where $[\rho]$ and $[q]$ are notations for the jump in density across the discontinuous shock and the jump in the flow.

EXAMPLE

We consider the initial value problem

$$\frac{\partial \rho}{\partial t} + 2\rho\frac{\partial \rho}{\partial x} = 0$$

$$\rho(x, 0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0. \end{cases}$$

We assume that ρ is a conserved density. Putting the partial differential equation in conservation form $\left(\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0\right)$ shows that the flow $q = \rho^2$. Thus, if there is a discontinuity, the shock

velocity satisfies $\frac{dx}{dt} = [q]/[\rho] = [\rho^2]/[\rho]$. The density $\rho(x, t)$ is constant moving at the characteristic velocity 2ρ:

$$\frac{dx}{dt} = 2\rho.$$

Therefore, the equation for the characteristics is

$$x = 2\rho(x_0, 0)t + x_0.$$

If $x_0 < 0$, then $\rho(x_0, 0) = 4$. This parallel group of characteristics intersects those starting from $x_0 > 0$ [with $\rho(x_0, 0) = 3$] in the cross-hatched region in Fig. 12.6.9a. The method of characteristics yields a multivalued solution of the partial differential equation. This difficulty is remedied by introducing a shock wave (Fig. 12.6.9b), a propagating wave indicating the path at which densities and velocities abruptly change (i.e., are discontinuous). On one side of the shock, the method of characteristics suggests the density is constant $\rho = 4$, and on the other side, $\rho = 3$. We do not know as yet the path of the shock. The theory for such a discontinuous solution implies that the path for any shock must satisfy the shock condition, (12.6.20). Substituting the jumps in flow and density yields the following equation for the shock velocity:

$$\frac{dx_s}{dt} = \frac{q(4) - q(3)}{4 - 3} = \frac{4^2 - 3^2}{4 - 3} = 7,$$

since in this case $q = \rho^2$. Thus, the shock moves at a constant velocity. The initial position of the shock is known, giving a condition for this first-order ordinary differential equation. In this case, the shock must initiate at $x_s = 0$ at $t = 0$. Consequently, applying the initial condition results in the position of the shock,

$$x_s = 7t.$$

The resulting space-time diagram is sketched in Fig. 12.6.9c. For any time $t > 0$, the traffic density is discontinuous, as shown in Fig. 12.6.10.

Figure 12.6.9 Shock caused by intersecting characteristics.

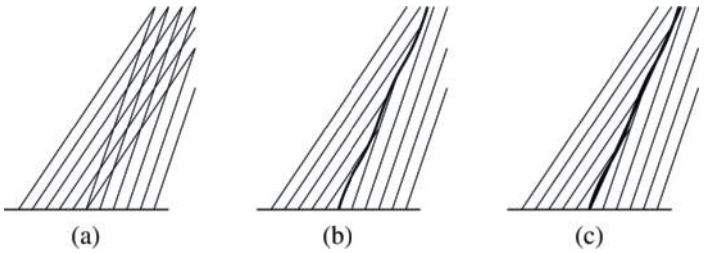
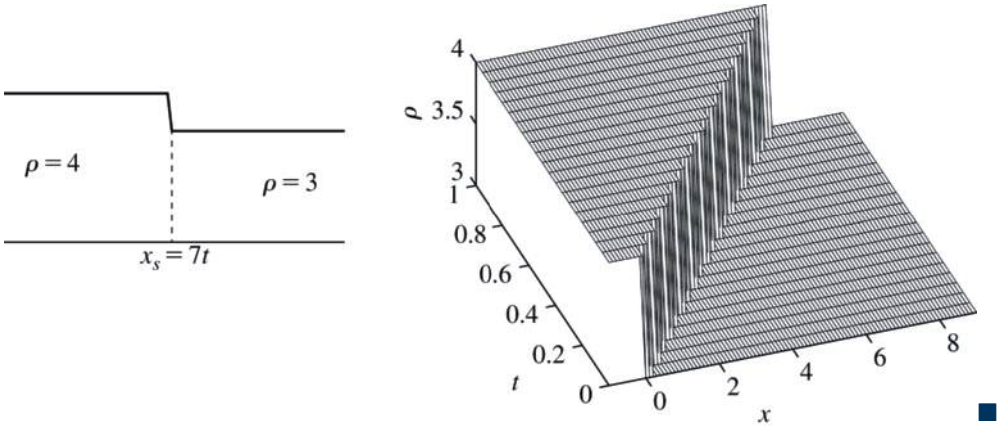


Figure 12.6.10 Density shock wave.



Entropy condition

We note that as shown in Fig. 12.6.9c, the characteristics must flow into the shock on both sides. The characteristic velocity on the left $(2\rho = 8)$ must be greater than the shock velocity $\left(\frac{dx_s}{dt} = 7\right)$, and the characteristic velocity on the right $(2\rho = 6)$ must be less than the shock velocity. This is a general principle, called the **entropy condition**,

$$c(\rho(x_{s-})) > \frac{dx_s}{dt} > c(\rho(x_{s+})). \tag{12.6.23}$$

Nonuniqueness of shock velocity and its resolution

The shock velocity for $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$, where $q = q(\rho)$, is $\frac{dx_s}{dt} = \frac{[q]}{[\rho]}$. It is sometimes claimed that the shock velocity is not unique because mathematically the partial differential equation can be multiplied by any function of ρ and the resulting shock velocity would be different. For the previous example of a shock with $\rho = 4$ and $\rho = 3$, $\frac{dx_s}{dt} = \frac{[q]}{[\rho]} = \frac{[\rho^2]}{[\rho]} = 7$. However, if we multiply the PDE by ρ, we obtain $\frac{\partial(\frac{1}{3}\rho^2)}{\partial t} + \frac{\partial(\frac{2}{3}\rho^3)}{\partial x} = 0$, so that the shock velocity assuming ρ^2 is conserved would be different: $\frac{dx_s}{dt} = \frac{[q]}{[\rho^2]} = \frac{\left[\frac{4}{3}\rho^3\right]}{[\rho^2]} = \frac{4}{3}\frac{4^3-3^3}{7} = \frac{148}{21}$. However, the point of view we wish to take is that only one of the conservation laws is the correct conservation law from physical principles. For example, for traffic flow problems, ρ, not ρ^2, is the conserved quantity, and the shock velocity formula can be used only for the density that is physically conserved.

Green light turns red

We assume that the traffic density satisfies the partial differential equation corresponding to the elementary model of traffic flow:

$$\frac{\partial \rho}{\partial t} + u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)\frac{\partial \rho}{\partial x} = 0.$$

Before the light turns red, we assume a very simple traffic situation in which the traffic density is a constant $\rho = \rho_0$ so that all cars are moving at the same velocity, $u = u_{\max}\left(1 - \frac{\rho_0}{\rho_{\max}}\right)$. We assume the green light $(x = 0)$ turns red at $t = 0$. Behind the light $(x < 0)$, the traffic density is initially uniform $\rho = \rho_0$, but the density is maximum $\rho = \rho_{\max}$ at the light itself since the cars are stopped there:

$$\rho(x, 0) = \rho_0 \text{ for } x < 0$$
$$\rho(0, t) = \rho_{\max} \text{ for } t > 0.$$

The characteristic velocity is $\frac{dx}{dt} = u_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$. The characteristics from $x = 0$ move backward since the characteristic velocity is $-u_{\max}$ for $\rho = \rho_{\max}$. The characteristic velocity is $u_{\max}\left(1 - \frac{2\rho_0}{\rho_{\max}}\right)$ for $\rho = \rho_0$. The two parallel families of characteristics will intersect because $u_{\max}\left(1 - \frac{2\rho_0}{\rho_{\max}}\right) > -u_{\max}$. A shock will form separating $\rho = \rho_0$ from $\rho = \rho_{\max}$. The shock velocity is determined from (12.6.20):

$$\frac{dx_s}{dt} = \frac{[q]}{[\rho]} = \frac{q(\rho_{\max}) - q(\rho_0)}{\rho_{\max} - \rho_0} = \frac{-q(\rho_0)}{\rho_{\max} - \rho_0},$$

where we have noted that $q(\rho_{\max}) = 0$. Since the shock starts $(t = 0)$ at $x = 0$, the shock path is

$$x_s = \frac{-q(\rho_0)}{\rho_{\max} - \rho_0}t.$$

The shock velocity is negative since $q(\rho_0) > 0$, and the shock propagates backward. In this case there is a formula, $q(\rho_0) = u_{\max}\left(1 - \frac{\rho_0}{\rho_{\max}}\right)\rho_0$, but we do not need it. The traffic density is ρ_0 before the shock and increases to $\rho_{\max}$ at the shock (unlike Fig. 12.6.10). Characteristics and the shock are qualitatively similar to Fig. 12.6.9c, but traffic shocks occur when faster-moving traffic (lower density) is behind slower-moving traffic (higher density). Here, the shock velocity represents the line of stopped cars due to the red light moving backward behind the light. Many accidents are caused by the suddenness of traffic shock waves.

Conditions for shock formation

A shock forms if initially the characteristic velocity $c(\rho) = q'(\rho)$ is a decreasing function of x (so that faster waves are behind slower waves). Thus, a shock forms if initially $\frac{\partial}{\partial x} q'(\rho) = q''(\rho)\frac{\partial \rho}{\partial x} < 0$. It follows that for traffic problems, where $q''(\rho) < 0$, shocks form when the density is an increasing function of x, and the density must increase with x at a shock. However, if $q''(\rho) > 0$, then shocks form only if $\frac{\partial \rho}{\partial x} < 0$ so that the density must decrease at a shock. Otherwise, characteristics do not intersect, and discontinuous initial conditions correspond to expansion waves. If $q''(\rho)$ does not change signs, then discontinuous initial conditions result in either a shock or an expansion wave.

Example with a shock and an expansion wave

If $q''(\rho)$ does change sign at least once, then a discontinuous initial condition may result in both a shock and an expansion wave. A simple mathematical example with this property is $q = \frac{1}{3}\rho^3$, so that $q' = \rho^2$ and $q'' = 2\rho$. If $\rho < 0$, this is similar to a traffic problem, but we will assume initially that ρ is both positive and negative. As an example we assume the initial condition is increasing from negative to positive:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0$$

$$\rho(x, 0) = \begin{cases} -1 & x < 0 \\ 2 & x > 0. \end{cases}$$

Since ρ will be an increasing function of x, shocks can occur where $q' = 2\rho < 0$ (meaning $-1 < \rho < 0$), while an expansion wave occurs for $\rho > 0$. In general, according to the method of characteristics, ρ is constant moving at velocity ρ^2, so that characteristics satisfy

$$x = \rho^2(x_0, 0)t + x_0.$$

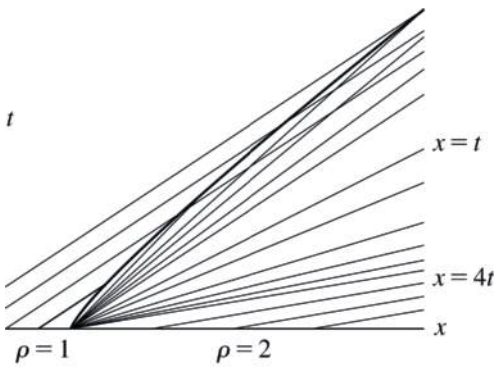
Important characteristics that correspond to $\rho = -1$, $\rho = 2$, and (only at $t = 0$) $\rho = 0$ are graphed in Fig. 12.6.11. The density $\rho = 2$ for $x > 4t$. The expansion wave satisfies $x = \rho^2(x_0, 0)t$, so that there

$$\rho_{\text{fan}} = +\sqrt{\frac{x}{t}},$$

where we have carefully noted that in this problem the expansion wave corresponds to $\rho > 0$. (The characteristic $x = 0t$ corresponding to $\rho = 0$ in some subtle sense bounds the expansion wave ranging from $\rho = 0$ to $\rho = 2$.) However, the characteristics from $x_0 < 0$ in which $\rho = -1$ moving at velocity $+1$ intersect characteristics in the region of the expansion wave. A **nonuniform shock** (nonconstant velocity) will form, separating the region with constant density $\rho = -1$ from the expansion wave $\rho_{\text{fan}} = +\sqrt{\frac{x}{t}}$. The path of the shock wave is determined from the shock condition

$$\frac{dx}{dt} = \frac{[q]}{[\rho]} = \frac{1}{3}\frac{1 + (\frac{x}{t})^{3/2}}{1 + (\frac{x}{t})^{1/2}}.$$

Figure 12.6.11 Characteristics for shock and expansion wave.



It is somewhat complicated to solve exactly this ordinary differential equation. In any event, the ordinary differential equation could be solved (with care) numerically (let $y = \frac{x}{t}$). It can be shown that this nonuniform shock persists for these initial conditions. In other problems (but not this one), this shock might no longer persist beyond some time, in which case it would be replaced by a simpler **uniform shock** (constant velocity) separating $\rho = -1$ and $\rho = 2$.

Diffusive conservation law

Here we will introduce a different way to understand the relationship between the unique shock velocity and the unique conservation law. Suppose the partial differential equation was diffusive, but only slightly different from the one previously studied:

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = \epsilon \frac{\partial^2 \rho}{\partial x^2}, \tag{12.6.24}$$

where ϵ is a very small positive parameter, $0 < \epsilon << 1$. Note that ρ is conserved in the sense that $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(q - \epsilon \frac{\partial \rho}{\partial x}\right) = 0$. Under most circumstances the partial differential equation can be accurately approximated by the equation with $\epsilon = 0$, which can be solved by the method of characteristics. The term $\epsilon \frac{\partial^2 \rho}{\partial x^2}$ would be important only where $\frac{\partial^2 \rho}{\partial x^2}$ was large, near a region in which the density changes rapidly in a short distance, what we have called a shock. This (12.6.24) could be derived by assuming the car velocity satisfies $u = \frac{q}{\rho} - \epsilon \frac{\partial \rho / \partial x}{\rho}$, which corresponds to drivers slowing down if they see increased density ahead $\left(\frac{\partial \rho}{\partial x} > 0\right)$. We are interested in solving (12.6.24) with initial conditions that approach two different constants ρ_1 and ρ_2 as $x \to \pm\infty$. This would correspond to a transition wave between the two constant states. There are two very different cases, one corresponding to shock-like initial conditions and the other corresponding to expansion-type initial conditions. We will show that a traveling transition wave exists only for the shock-like case.

Velocity of traveling shock wave

There are no methods to find the general solution of such nonlinear partial differential equations, (12.6.24), so we look for very special traveling wave solutions with unknown traveling wave velocity c:

$$\rho = f(\xi) = f(x - ct), \tag{12.6.25}$$

where ξ is the spatial coordinate moving with velocity c. Substituting (12.6.25) into (12.6.24) yields a second-order nonlinear ordinary differential equation:

$$(-c + q'(f))\frac{df}{d\xi} = \epsilon \frac{d^2 f}{d\xi^2}, \tag{12.6.26}$$

which can be integrated to yield a first-order nonlinear differential equation:

$$-A - cf + q(f) = \epsilon \frac{df}{d\xi}, \tag{12.6.27}$$

where $-A$ a constant. We will determine the two constants c and A, though the velocity is more important. To satisfy the condition that $f \to \rho_1$ and ρ_2 as $x \to \pm\infty$, it follows that

$$-A - c\rho_1 + q(\rho_1) = 0 \tag{12.6.28}$$

$$-A - c\rho_2 + q(\rho_2) = 0. \tag{12.6.29}$$

By subtracting (12.6.28) from (12.6.29), we determine the traveling wave speed:

$$c = \frac{q(\rho_2) - q(\rho_1)}{\rho_2 - \rho_1} = \frac{[q]}{[\rho]}. \tag{12.6.30}$$

Thus, **the traveling wave velocity equals the shock velocity**, but we will show that this is valid for the shock-like conditions and not valid for the expansion wave-like conditions.

Spatial structure of traveling shock wave

The first-order autonomous differential equation (12.6.27) can be analyzed using a one-dimensional phase portrait. We need to know properties of $q(\rho)$. For the traffic flow problem, we assume, as graphed in Fig. 12.6.12, that $q''(\rho) < 0$, since that corresponds to experiments for traffic flow as well as our simple example. We are concerned with the difference between $q(f)$ and the straight line $A + cf$. We adjust the constant A so that there are two intersections, which we call ρ_1

and ρ_2. These are equilibrium solutions and must satisfy (12.6.28) and (12.6.29). It is very important that we label the solutions such that $\rho_2 > \rho_1$. For (12.6.27), $\frac{df}{d\xi}$ is a known function of f, and it is graphed in the top portion of Fig. 12.6.12. The **one-dimensional phase line** is graphed in the bottom portion of Fig. 12.6.12 for the autonomous first-order ordinary differential equation (12.6.27). We note that in the upper half-plane $\frac{df}{d\xi} > 0$ and hence f is an increasing function of ξ, and right arrows are introduced, indicating that f increases as ξ increases. Similarly, in the lower half-plane f is a decreasing function of ξ. Solutions with $f > \rho_2$ or $f < \rho_1$ explode either forward or backward in the traveling wave coordinate ξ and are of no interest to us. The only bounded traveling wave solutions correspond to $\rho_1 < f < \rho_2$. Most important, f is an increasing function of ξ for $\rho_1 < f < \rho_2$, and thus it is graphed in Fig. 12.6.13 as a traveling wave with the property that $f \to \rho_1$ as $x - ct \to -\infty$ and $f \to \rho_2$ as $x - ct \to \infty$. This is important as it corresponds to the case corresponding to the shock wave ($\rho_2 > \rho_1$) for traffic flow equations. It can be shown that if ϵ is small, then the transition from ρ_1 to ρ_2 occurs in a thin region. The moving discontinuity is a good approximation to the more precise continuous traveling wave. We have shown that their velocities are the same. Thus, the traveling wave corresponds to what we still call a shock wave. In fact, as $\epsilon \to 0$, the spatial structure of the continuous traveling wave approaches the discontinuous shock wave.

Figure 12.6.12 Phase line for traveling shock wave.

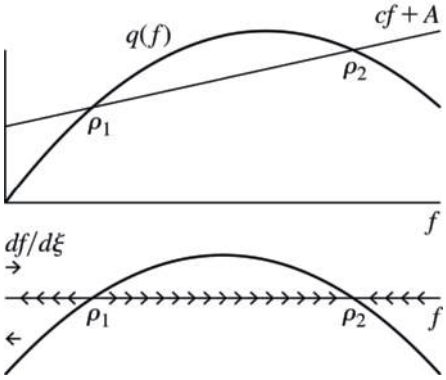
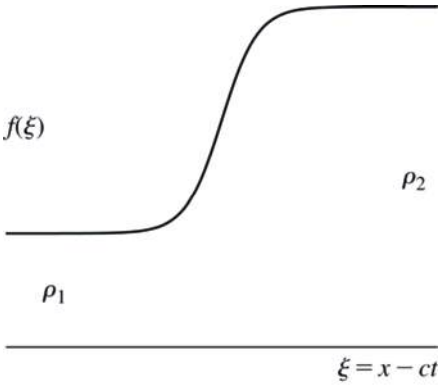


Figure 12.6.13 Spatial structure for traveling shock wave.



Formation of a shock

We have described the propagation of shock waves. In the examples previously considered, the density was initially discontinuous, and the shock wave formed immediately. However, we will now show that shock waves take a finite time to form if the initial density is continuous, and we will compute that finite time. Suppose the initial condition is such that characteristics intersect as shown in Fig. 12.6.6 because faster-moving characteristics are behind more slowly moving characteristics. For the partial differential equation $\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$, where the density wave velocity is $c(\rho) = q'(\rho)$, the family of straight-line characteristics [along which the density $\rho(x, t) = \rho(x_0, 0) = f(x_0)$ is constant determined from the given initial conditions] satisfies

$$x = c[f(x_0)t + x_0 = F(x_0)t + x_0, \tag{12.6.31}$$

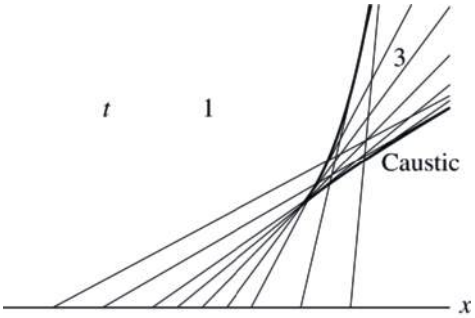
where we have introduced a simplifying notation for the constant-density wave velocity $F(x_0) = c[f(x_0)]$. Any characteristic starting between the two intersecting characteristics shown in Fig. 12.6.6 will almost certainly intersect one of the other two characteristics at an earlier time. Thus, it is usual for neighboring characteristics to intersect, and we first discuss that.

Caustic (envelope of a family of curves, rays, or characteristics)

The family of characteristics (12.6.31) for the quasilinear partial differential equation may converge as shown in Fig. 12.6.6. We wish to explain the envelope of the characteristics shown in Fig. 12.6.14. To be more general, we consider instead of (12.6.31) any family of curves parameterized by x_0:

$$G(x, t, x_0) = 0. \tag{12.6.32}$$

Figure 12.6.14 Formation of caustic, envelope of characteristics.



A **caustic** is an envelope of the family of curves, and we claim that it simultaneously satisfies (12.6.32) and the partial derivative of (12.6.32) with respect to the parameter x_0:

$$\frac{\partial}{\partial x_0} G(x, t, x_0) = 0. \tag{12.6.33}$$

To prove that (12.6.33) is valid, we consider the intersection of neighboring curves, namely, (12.6.32) and $G(x, t, x_0 + \Delta x_0) = 0$. Using the Taylor series, this becomes $G(x, t, x_0) + \Delta x_0 \frac{\partial}{\partial x_0} G(x, t, x_0) + \cdots = 0$. By using (12.6.32), dividing by Δx_0, and taking the limit $\Delta x_0 \to 0$, (12.6.33) is proved.

Caustic for the characteristics for the partial differential equation

In our example, the family of characteristics is straight lines, (12.6.31), $G(x, t, x_0) = 0 = F(x_0)t + x_0 - x$, where $F(x_0) = c[f(x_0)]$. Thus, the caustic (envelope of the characteristics or rays) is formed by simultaneously solving $0 = F'(x_0)t + 1$. This gives the parametric representation of the caustic (easily graphed numerically from given initial conditions):

$$x = -\frac{F(x_0)}{F'(x_0)} + x_0 \tag{12.6.34}$$

$$t = -\frac{1}{F'(x_0)}. \tag{12.6.35}$$

Characteristics will intersect ($t > 0$) only if $F'(x_0) < 0$. Since $F(x_0) = c[f(x_0)]$, we conclude that neighboring characteristics will intersect if they emanate from regions where the characteristic velocity is locally decreasing (faster-moving characteristics behind slower characteristics). The caustic is shown in Fig. 12.6.14. It is shown in Section 14.6.2 that the caustic has a cusp.

Initiation of a shock

The solution stays continuous until the finite time at which the caustic appears. A shock wave begins at this time. The times for the caustic are given by (12.6.35). To determine the first time that characteristics intersect, we must minimize the intersection times. The absolute minimum of t given by (12.6.35) corresponds to an absolute minimum of $F'(x_0)$ since $F'(x_0) < 0$. Hence, the shock starts at t given by (12.6.35), where x_0 corresponds to the minimum of $F'(x_0)$ and satisfies $F''(x_0) = 0$.

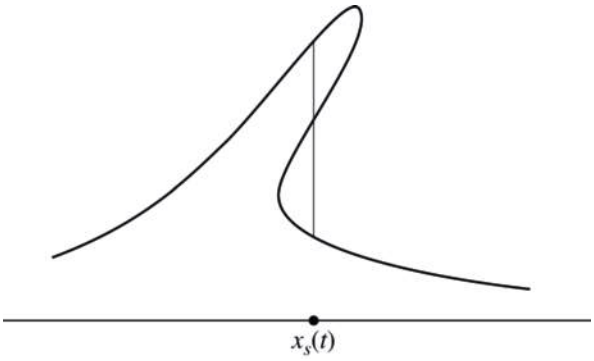
Triple valuedness

Inside the caustic (after the caustic has formed), look carefully at Fig. 12.6.14 to see that three different characteristics (marked 3) go through each space-time point, corresponding to the solution of the partial differential equation being triple valued as shown in Fig. 12.6.15 or 12.6.7d. Outside the caustic, a unique characteristic (marked 1) goes through each point, and the solution is single valued. The caustic is the boundary between these two regions. We will show that the slope of the solution is infinite (see Fig. 12.6.15) along the caustic. Since $\rho(x, t) = f(x_0)$, we have

$$\frac{\partial \rho}{\partial x} = f'(x_0)\frac{\partial x_0}{\partial x} = \frac{f'(x_0)}{F'(x_0)t + 1},$$

using the partial derivative of (12.6.31) with respect to x, $1 = [F'(x_0)t + 1]\frac{\partial x_0}{\partial x}$. The slope is infinite at the caustic, satisfying (12.6.35).

Figure 12.6.15 Whitham's equal-area principle.



Shock dynamics

The triple-valued folded-over solution (within the caustic) makes no sense. Instead, as discussed earlier, a shock wave exists satisfying the shock condition $\frac{dx_s}{dt} = \frac{[q]}{[\rho]}$, but here the shock wave begins at the cusp of the caustic at a time determined by minimizing the time in (12.6.35). This gives a differential equation for the position of the shock, but the shock does not necessarily have constant velocity. The shock is located somewhere within the caustic. Whitham [1999] has shown that the correct location of the shock may be determined by cutting off the lobes to form equal areas (Fig. 12.6.15). The reason for this is that the method of characteristics conserves cars and that, when a shock is introduced, the number of cars (represented by the area $\int \rho\, dx$) must also be the same as it is initially.

12.6.5 QUASILINEAR EXAMPLE

Consider the quasilinear example

$$\frac{\partial \rho}{\partial t} - \rho \frac{\partial \rho}{\partial x} = -2\rho, \tag{12.6.36}$$

subject to the initial conditions

$$\rho(x, 0) = f(x). \tag{12.6.37}$$

This could model a traffic flow problem (with ρ a scaled version of the density), where cars are not conserved but instead leave the roadway (at exits) at a rate proportional to the density (as though cars exit the highway to avoid congestion).

The method of characteristics yields

$$\frac{d\rho}{dt} = -2\rho, \tag{12.6.38}$$

along the characteristic

$$\frac{dx}{dt} = -\rho. \tag{12.6.39}$$

These equations are sometimes written in the equivalent form

$$\frac{d\rho}{-2\rho} = \frac{dx}{-\rho} = dt. \tag{12.6.40}$$

Sometimes the coupled system of ordinary differential equations can be directly solved. In this case the ordinary differential equation for ρ may be solved first and its solution used to determine the characteristic x. Because of the initial condition (12.6.37), we introduce the parameter x_0 representing the characteristic that emanates from $x = x_0$ (at $t = 0$). From (12.6.38), along the characteristic, we obtain

$$\rho(x, t) = \rho(x_0, 0)e^{-2t} = f(x_0)e^{-2t}. \tag{12.6.41}$$

The parameter x_0 is constant along each characteristic. The solution (density) exponentially decays along the characteristic as time increases. Thus, the characteristic velocity becomes

$$\frac{dx}{dt} = -f(x_0)e^{-2t}.$$

By integrating the velocity, we obtain the position of the characteristic:

$$x = \frac{1}{2}f(x_0)e^{-2t} - \frac{1}{2}f(x_0) + x_0, \tag{12.6.42}$$

since x must equal x_0 at $t = 0$. Here, the characteristics are not straight lines. The parametric representation of the solution is obtained from (12.6.41), where x_0 should be considered as a function of x and t from (12.6.42). Usually, an explicit solution is impractical.

Explicit solution of an initial value problem

For the quasilinear partial differential equation (12.6.36), suppose the initial conditions are

$$\rho(x, 0) = f(x) = x.$$

In this case, from (12.6.42), the characteristics satisfy

$$x = \frac{1}{2}x_0e^{-2t} + \frac{1}{2}x_0.$$

Thus, an explicit solution can be obtained:

$$x_0 = \frac{2x}{1 + e^{-2t}}.$$

Note that for each x and t there is only one characteristic x_0 (because the initial condition was chosen such that the family of characteristics does not intersect itself). From (12.6.41), the solution of the initial value problem for the partial differential equation is

$$\rho(x, t) = \frac{2xe^{-2t}}{1 + e^{-2t}} = \frac{2x}{1 + e^{2t}}.$$

General solution

Quasilinear partial differential equations have general solutions (as with the linear wave equation). Without specifying initial conditions, integrating (12.6.38) and (12.6.39) yields

$$\rho(x, t) = c_1e^{-2t}$$
$$x = \frac{1}{2}c_1e^{-2t} + c_2.$$

In general, one constant can be an arbitrary function of the other constant. Thus, we obtain the **general solution** of (12.6.36),

$$x = \frac{1}{2}\rho + f(\rho e^{2t}).$$

Exercises 12.6

12.6.1. Determine the solution $\rho(x, t)$ satisfying the initial condition $\rho(x, 0) = f(x)$ if

***(a)** $\dfrac{\partial \rho}{\partial t} = 0$ **(b)** $\dfrac{\partial \rho}{\partial t} = -3\rho + 4e^{7t}$

***(c)** $\dfrac{\partial \rho}{\partial t} = -3x\rho$ **(d)** $\dfrac{\partial \rho}{\partial t} = x^2t\rho$

***12.6.2.** Determine the solution of $\partial\rho/\partial t = \rho$ that satisfies $\rho(x, t) = 1 + \sin x$ along $x = -2t$.

12.6.3. Suppose $\frac{\partial \rho}{\partial t} + c_0\frac{\partial \rho}{\partial x} = 0$ with c_0 constant.

***(a)** Determine $\rho(x, t)$ if $\rho(x, 0) = \sin x$.

***(b)** If $c_0 > 0$, determine $\rho(x, t)$ for $x > 0$ and $t > 0$, where $\rho(x, 0) = f(x)$ for $x > 0$ and $\rho(0, t) = g(t)$ for $t > 0$.

(c) Show that part (b) cannot be solved if $c_0 < 0$.

***12.6.4.** If $u(\rho) = \alpha + \beta\rho$, determine α and β such that $u(0) = u_{\max}$ and $u(\rho_{\max}) = 0$.

(a) What is the flow as a function of density? Graph the flow as a function of the density.

(b) At what density is the flow maximum? What is the corresponding velocity? What is the maximum flow (called the *capacity*)?

12.6.5. Redo Exercise 12.6.4 if $u(\rho) = u_{\max}\left(1 - \frac{\rho^3}{\rho_{\max}}\right)$.

12.6.6. Consider the traffic flow problem

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0.$$

Assume $u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$. Solve for $\rho(x, t)$ if the initial conditions are

(a) $\rho(x, 0) = \rho_{\max}$ for $x < 0$ and $\rho(x, 0) = 0$ for $x > 0$. This corresponds to the traffic density that results after an infinite line of stopped traffic is started by a red light turning green.

(b) $$\rho(x, 0) = \begin{cases} \rho_{\max} & x < 0 \\ \dfrac{\rho_{\max}}{2} & 0 < x < a \\ 0 & x > a \end{cases}$$

(c) $$\rho(x, 0) = \begin{cases} \dfrac{3\rho_{\max}}{5} & x < 0 \\ \dfrac{\rho_{\max}}{5} & x > 0 \end{cases}$$

12.6.7. Solve the following problems:

(a) $\dfrac{\partial \rho}{\partial t} + \rho^2\dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x, 0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0 \end{cases}$

(b) $\dfrac{\partial \rho}{\partial t} + 4\rho\dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x, 0) = \begin{cases} 2 & x < 1 \\ 3 & x > 1 \end{cases}$

(c) $\dfrac{\partial \rho}{\partial t} + 3\rho\dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x, 0) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x < 1 \\ 4 & x > 1 \end{cases}$

(d) $\dfrac{\partial \rho}{\partial t} + 6\rho\dfrac{\partial \rho}{\partial x} = 0$ for $x > 0$ only, $\quad \rho(x, 0) = 5 \quad x > 0$, $\quad \rho(0, t) = 2 \quad t > 0$

12.6.8. Solve subject to the initial condition $\rho(x, 0) = f(x)$:

***(a)** $\dfrac{\partial \rho}{\partial t} + c\dfrac{\partial \rho}{\partial x} = e^{-3x}$

(b) $\dfrac{\partial \rho}{\partial t} + 3x\dfrac{\partial \rho}{\partial x} = 4$

***(c)** $\dfrac{\partial \rho}{\partial t} + t\dfrac{\partial \rho}{\partial x} = 5$

(d) $\dfrac{\partial \rho}{\partial t} + 5t\dfrac{\partial \rho}{\partial x} = 3\rho$

***(e)** $\dfrac{\partial \rho}{\partial t} - t^2\dfrac{\partial \rho}{\partial x} = -\rho$

(f) $\dfrac{\partial \rho}{\partial t} + t^2\dfrac{\partial \rho}{\partial x} = 0$

***(g)** $\dfrac{\partial \rho}{\partial t} + x\dfrac{\partial \rho}{\partial x} = t$

12.6.9. Determine a parametric representation of the solution satisfying $\rho(x, 0) = f(x)$:

***(a)** $\dfrac{\partial \rho}{\partial t} - \rho^2\dfrac{\partial \rho}{\partial x} = 3\rho$

(b) $\dfrac{\partial \rho}{\partial t} + \rho\dfrac{\partial \rho}{\partial x} = t$

***(c)** $\dfrac{\partial \rho}{\partial t} + t^2\rho\dfrac{\partial \rho}{\partial x} = -\rho$

(d) $\dfrac{\partial \rho}{\partial t} + \rho\dfrac{\partial \rho}{\partial x} = -x\rho$

12.6.10. Solve $\frac{\partial \rho}{\partial t} + t^2\frac{\partial \rho}{\partial x} = 4\rho$ for $x > 0$ and $t > 0$ with $\rho(0, t) = h(t)$ and $\rho(x, 0) = 0$.

***12.6.11.** Solve $\frac{\partial \rho}{\partial t} + (1 + t)\frac{\partial \rho}{\partial x} = 3\rho$ for $t > 0$ and $x > -t/2$ with $\rho(x, 0) = f(t)$ for $x > 0$ and $\rho(x, t) = g(t)$ along $x = -t/2$.

12.6.12. Consider (12.6.8) if there is a moving shock x, such that $a < x_s(t) < b$. By differentiating the integral [with a discontinuous integrand at $x_s(t)$], *derive* (12.6.20).

12.6.13. Suppose that, instead of $u = U(\rho)$, a car's velocity u is

$$u = U(\rho) - \frac{\nu}{\rho}\frac{\partial \rho}{\partial x},$$

where ν is a constant.

(a) What sign should ν have for this expression to be physically reasonable?

(b) What equation now describes conservation of cars?

(c) Assume that $U(\rho) = u_{\max}(1 - \rho/\rho_{\max})$. Derive **Burgers' equation**:

$$\frac{\partial \rho}{\partial t} + u_{\max}\left[1 - \frac{2\rho}{\rho_{\max}}\right]\frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2}.$$

12.6.14. Consider Burgers' equation as derived in Exercise 12.6.13. Suppose that a solution exists as a density wave moving without change of shape at velocity V, $\rho(x, t) = f(x - Vt)$.

***(a)** What ordinary differential equation is satisfied by f?

(b) Integrate this differential equation once. By graphical techniques show that a solution exists such that $f \to \rho_2$ as $x \to +\infty$ and $f \to \rho_1$ as $x \to -\infty$ only if $\rho_2 > \rho_1$. Roughly sketch this solution. Give a physical interpretation of this result.

***(c)** Show that the velocity of wave propagation, V, is the same as the shock velocity separating $\rho = \rho_1$ from $\rho = \rho_2$ (occurring if $\nu = 0$).

12.6.15. Consider Burgers' equation as derived in Exercise 12.6.13. Show that the change of dependent variables

$$\rho = \frac{\nu \rho_{\max}}{u_{\max}} \frac{\phi_x}{\phi},$$

introduced independently by E. Hopf in 1950 and J. D. Cole in 1951, transforms Burgers' equation into a diffusion equation, $\frac{\partial \phi}{\partial t} + u_{\max}\frac{\partial \phi}{\partial x} = \nu \frac{\partial^2 \phi}{\partial x^2}$. Use this to solve the initial value problem $\rho(x, 0) = f(x)$ for $-\infty < x < \infty$. [In Whitham [1999], it is shown that this exact solution can be asymptotically analyzed as $\nu \to 0$ using Laplace's method for exponential integrals to show that $\rho(x, t)$ approaches the solution obtained for $\nu = 0$ using the method of characteristics with shock dynamics.]

12.6.16. Suppose that the initial traffic density is $\rho(x, 0) = \rho_0$ for $x < 0$ and $\rho(x, 0) = \rho_1$ for $x > 0$. Consider the two cases, $\rho_0 < \rho_1$ and $\rho_1 < \rho_0$. For which of the preceding cases is a density shock necessary? Briefly explain.

12.6.17. Consider a traffic problem, with $u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$. Determine $\rho(x, t)$ if

***(a)** $\rho(x, 0) = \begin{cases} \dfrac{\rho_{\max}}{5} & x < 0 \\ \dfrac{3\rho_{\max}}{5} & x > 0 \end{cases}$ **(b)** $\rho(x, 0) = \begin{cases} \dfrac{\rho_{\max}}{3} & x < 0 \\ \dfrac{2\rho_{\max}}{3} & x > 0 \end{cases}$

12.6.18. Assume that $u(\rho) = u_{\max}\left(1 - \frac{\rho^2}{\rho_{\max}^2}\right)$. Determine the traffic density ρ (for $t > 0$) if $\rho(x, 0) = \rho_1$ for $x < 0$ and $\rho(x, 0) = \rho_2$ for $x > 0$.

(a) Assume that $\rho_2 > \rho_1$. ***(b)** Assume that $\rho_2 < \rho_1$.

12.6.19. Solve the following problems (assuming ρ is conserved):

(a) $\dfrac{\partial \rho}{\partial t} + \rho^2 \dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x, 0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$

(b) $\dfrac{\partial \rho}{\partial t} + 4\rho \dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x, 0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$

(c) $\dfrac{\partial \rho}{\partial t} + 3\rho \dfrac{\partial \rho}{\partial x} = 0, \qquad \rho(x, 0) = \begin{cases} 4 & x < 0 \\ 2 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$

(d) $\dfrac{\partial \rho}{\partial t} + 6\rho \dfrac{\partial \rho}{\partial x} = 0$ for $x > 0$ only, $\quad \rho(x, 0) = 2 \quad x > 0$, $\quad \rho(0, t) = 5 \quad t > 0$

12.6.20. Redo Exercise 12.6.19, assuming that ρ^2 is conserved.

12.6.21. Compare Exercise 12.6.19(a) with 12.6.20(a). Show that the shock velocities are different.

12.6.22. Solve $\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0$. If a nonuniform shock occurs, give only its differential equation. Do you believe the nonuniform shock persists or is eventually replaced by a uniform shock?

(a) $\rho(x, 0) = \begin{cases} -1 & x < 0 \\ 3 & x > 0 \end{cases}$ **(b)** $\rho(x, 0) = \begin{cases} -2 & x < 0 \\ 1 & x > 0 \end{cases}$

(c) $\rho(x,0)=\begin{cases}1 & x<0\\ -3 & x>0\end{cases}$ **(d)** $\rho(x,0)=\begin{cases}2 & x<0\\ -1 & x>0\end{cases}$

12.6.23. Solve $\frac{\partial\rho}{\partial t}-\rho^2\frac{\partial\rho}{\partial x}=0$. If a nonuniform shock occurs, give only its differential equation. Do you believe the nonuniform shock persists or is eventually replaced by a uniform shock?

(a) $\rho(x,0)=\begin{cases}-2 & x<0\\ 1 & x>0\end{cases}$ **(b)** $\rho(x,0)=\begin{cases}-1 & x<0\\ 4 & x>0\end{cases}$

(c) $\rho(x,0)=\begin{cases}1 & x<0\\ -3 & x>0\end{cases}$ **(d)** $\rho(x,0)=\begin{cases}5 & x<0\\ -2 & x>0\end{cases}$

12.7 First-Order Nonlinear Partial Differential Equations

12.7.1 EIKONAL EQUATION DERIVED FROM THE WAVE EQUATION

For simplicity, we consider the two-dimensional wave equation

$$\frac{\partial^2 E}{\partial t^2}=c^2\left(\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial y^2}\right). \tag{12.7.1}$$

Plane waves and their reflections were analyzed in Section 4.6. Nearly plane waves exist under many circumstances. If the coefficient c is not constant but varies slowly, then over a few wavelengths the wave sees nearly constant c. However, over long distances (relative to short wavelengths) we may be interested in the effects of variable c. Another situation in which nearly plane waves arise is the reflection of a plane wave by a curved boundary (or reflection and refraction by a curved interface between two media with different indices of refraction). We assume the radius of curvature of the boundary is much longer than typical wavelengths. In many of these situations the temporal frequency ω is fixed (by an incoming plane wave). Thus,

$$E=A(x,y)e^{-i\omega t}, \tag{12.7.2}$$

where that $A(x,y)$ satisfies the **Helmholtz** or **reduced wave equation**:

$$-\omega^2 A=c^2\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}\right). \tag{12.7.3}$$

Again the temporal frequency ω is fixed (and given), but $c=c(x,y)$ for inhomogeneous media or $c=$ constant for uniform media. In uniform media ($c=$ constant), plane waves of the form $E=A_0e^{i(k_1x+k_2y-\omega t)}$ or

$$A=A_0e^{i(k_1x+k_2y)} \tag{12.7.4}$$

exist if

$$\omega^2=c^2(k_1^2+k_2^2). \tag{12.7.5}$$

For nearly plane waves, we introduce the phase $u(x,y)$ of the reduced wave equation:

$$A(x,y)=R(x,y)e^{iu(x,y)}. \tag{12.7.6}$$

The wave numbers k_1 and k_2 for uniform media are usually called p and q, respectively, and are defined by

$$p=\frac{\partial u}{\partial x} \tag{12.7.7}$$

$$q=\frac{\partial u}{\partial y}. \tag{12.7.8}$$

As an approximation (which can be derived using perturbation methods), it can be shown that the (slowly varying) wave numbers satisfy (12.7.5), corresponding to the given temporal frequency associated with plane waves,

$$\boxed{\omega^2=c^2(p^2+q^2).} \tag{12.7.9}$$

This is a first-order nonlinear partial differential equation (not quasilinear) for the phase $u(x,y)$, known as the **eikonal equation**:

$$\boxed{\frac{w^2}{c^2}=\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2,} \tag{12.7.10}$$

where ω is a fixed reference temporal frequency and $c = c(x, y)$ for inhomogeneous media or $c =$ constant for uniform media. Sometimes the **index of refraction** $n(x, y)$ is introduced proportional to $\frac{1}{c}$. The amplitude $R(x, y)$ solves equations (which we do not discuss) known as the transport equations, which describe the propagation of energy of these nearly plane waves.

12.7.2 SOLVING THE EIKONAL EQUATION IN UNIFORM MEDIA AND REFLECTED WAVES

The simplest example of the eikonal equation (12.7.10) occurs in uniform media ($c =$ constant):

$$\boxed{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\omega^2}{c^2},} \tag{12.7.11}$$

where ω and c are constants. Rather than solve for $u(x, y)$ directly, we will show that it is easier to solve first for $p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial y}$. Thus, we consider

$$p^2 + q^2 = \frac{\omega^2}{c^2}. \tag{12.7.12}$$

Differentiating (12.7.11) or (12.7.12) with respect to x yields

$$p\frac{\partial p}{\partial x} + q\frac{\partial q}{\partial x} = 0.$$

Since $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$, p satisfies a first-order quasilinear partial differential equation:

$$p\frac{\partial p}{\partial x} + q\frac{\partial p}{\partial y} = 0. \tag{12.7.13}$$

Equation (12.7.13) may be solved by the method of characteristics [see specifically (12.6.7)]:

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dp}{0}. \tag{12.7.14}$$

If there is a boundary condition for p, then (12.7.14) can be solved for p since $q = \pm\sqrt{\frac{\omega^2}{c^2} - p^2}$ [from (12.7.12)]. Since (12.7.14) shows that p is constant along each characteristic, it also follows from (12.7.14) that each characteristic is a straight line. In this way p can be determined. However, given $p = \frac{\partial u}{\partial x}$, integrating for u is not completely straightforward.

We have differentiated the eikonal equation with respect to x. If instead we differentiate with respect to y, we obtain

$$p\frac{\partial p}{\partial y} + q\frac{\partial q}{\partial y} = 0.$$

A first-order quasilinear partial differential equation for q can be obtained by again using $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$:

$$p\frac{\partial q}{\partial x} + q\frac{\partial q}{\partial y} = 0. \tag{12.7.15}$$

Thus, $\frac{dx}{p} = \frac{dy}{q} = \frac{dq}{0}$, which, when combined with (12.7.14), yields the more general result

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dp}{0} = \frac{dq}{0}. \tag{12.7.16}$$

However, usually we want to determine u so that we wish to determine how u varies along this characteristic: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = p\,dx + q\,dy = p^2\frac{dx}{p} + q^2\frac{dy}{p} = (p^2 + q^2)\frac{dx}{p} = \frac{\omega^2}{c^2}\frac{dx}{p}$, where we have used (12.7.16) and (12.7.12). Thus, for the eikonal equation,

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dp}{0} = \frac{dq}{0} = \frac{du}{\omega^2/c^2}. \tag{12.7.17}$$

The characteristics are straight lines since p and q are constants along the characteristics.

Reflected waves

We consider $e^{i(\boldsymbol{k}_I \cdot \boldsymbol{x} - \omega t)}$, an elementary incoming plane wave where $\boldsymbol{k}_I$ represents the given constant incoming wave number vector and where $\omega = c|\boldsymbol{k}_I|$. We assume the plane wave reflects off a curved boundary (as illustrated in Fig. 12.7.1), which we represent with a parameter τ as

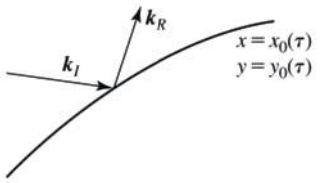
$x = x_0(\tau)$ and $y = y_0(\tau)$. We introduce the unknown reflected wave, $R(x, y)e^{iu(x,y)}e^{-i\omega t}$, and we wish to determine the phase $u(x, y)$ of the reflected wave. The eikonal equation

$$p^2 + q^2 = \frac{\omega^2}{c^2} = |\boldsymbol{k}_I|^2$$

can be interpreted as saying the slowly varying reflected wave number vector (p, q) has the same length as the constant incoming wave number vector (physically, the slowly varying reflected wave will always have the same wave length as the incident wave). We assume the boundary condition on the curved boundary is that the total field is zero (other boundary conditions yield the same equations for the phase): $0 = e^{i(\boldsymbol{k}_I \cdot \boldsymbol{x} - \omega t)} + R(x, y)e^{iu(x,y)}e^{-i\omega t}$. Thus, on the boundary, the phase of the incoming wave and the phase of the reflected wave must be the same:

$$u(x_0, y_0) = \boldsymbol{k}_I \cdot \boldsymbol{x}_0. \tag{12.7.18}$$

Figure 12.7.1 Reflected wave from curved boundary.



Taking the derivative of (12.7.18) with respect to the parameter τ shows that

$$\frac{\partial u}{\partial x}\frac{dx_0}{d\tau} + \frac{\partial u}{\partial y}\frac{dy_0}{d\tau} = p\frac{dx_0}{d\tau} + q\frac{dy_0}{d\tau} = \boldsymbol{k}_R \cdot \frac{d\boldsymbol{x}_0}{d\tau} = \boldsymbol{k}_I \cdot \frac{d\boldsymbol{x}_0}{d\tau}. \tag{12.7.19}$$

where we have noted that the vector (p, q) is the unknown reflected wave number vector $\boldsymbol{k}_R$ (because p and q are constant along the characteristic). Since $\frac{d\boldsymbol{x}_0}{d\tau}$ is a vector tangent to the boundary, (12.7.19) shows that the tangential component of the incoming and reflecting wave numbers must be the same. Since the magnitude of the incident and reflecting wave number vectors are the same, it follows that the normal component of the reflected wave must be minus the normal component of the incident wave. Thus, the angle of reflection off a curved boundary is the same as the angle of incidence. Thus, at any point along the boundary the constant value of p and q is known for the reflected wave. Because $q = \pm\sqrt{\frac{\omega^2}{c^2} - p^2}$, there are two solutions of the eikonal equation; one represents the incoming wave and the other (of interest to us) the reflected wave. To obtain the phase of the reflected wave, we must solve the characteristic equations (12.7.17) for the eikonal equation with the boundary condition specified by (12.7.18). Since for uniform media $\frac{\omega^2}{c^2} = |\boldsymbol{k}_I|^2$ is a constant, the differential equation for u along the characteristic, $\frac{du}{dx} = \frac{\omega^2}{c^2}\frac{1}{p} = \frac{|\boldsymbol{k}_I|^2}{p}$, can be integrated (since p is constant) using the boundary condition to give

$$u(x, y) = \frac{|\boldsymbol{k}_I|^2}{p}(x - x_0) + \boldsymbol{k}_I \cdot \boldsymbol{x}_0,$$

along a specific characteristic. The equation for the characteristics $p(y - y_0) = q(x - x_0)$ corresponds to the angle of reflection equaling the angle of incidence. Since $p^2 + q^2 = |\boldsymbol{k}_I|^2$, the more pleasing representation of the phase (solution of the eikonal equation) follows along a specific characteristic:

$$u(x, y) = p(x - x_0) + q(y - y_0) + \boldsymbol{k}_I \cdot \boldsymbol{x}_0, \tag{12.7.20}$$

where $u(x_0, y_0) = \boldsymbol{k}_I \cdot \boldsymbol{x}_0$ is the phase of the incident wave on the boundary.

Wave front

The characteristic direction is the direction in which the light propagates. For the eikonal equation, according to (12.7.14), $\frac{dx}{p} = \frac{dy}{q}$. This is also valid for nonuniform media. Light propagates (characteristic direction) in the direction of the gradient of the phase, ∇u, since $\nabla u = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} = p\hat{\mathbf{i}} + q\hat{\mathbf{j}} = \frac{p}{dx}\left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}\right)$. Thus, **light rays propagate normal to the wave fronts**.

12.7.3 FIRST-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Any first-order nonlinear partial differential equation can be put in the form

$$\boxed{F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.} \tag{12.7.21}$$

As with the eikonal equation example of the previous subsection, we show that $p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial y}$ solve quasilinear partial differential equations, and hence (12.7.21) can be solved by the method of characteristics. Using p and q gives

$$F(x, y, u, p, q) = 0. \tag{12.7.22}$$

Taking the partial derivative of (12.7.22) with respect to x, we obtain

$$F_x + F_u p + F_p \frac{\partial p}{\partial x} + F_q \frac{\partial q}{\partial x} = 0,$$

where we use the subscript notation for partial derivatives. For example, $F_u \equiv \frac{\partial F}{\partial u}$, keeping x, y, p, and q constant. Since $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$, we obtain a quasilinear partial differential equation for p:

$$F_p \frac{\partial p}{\partial x} + F_q \frac{\partial p}{\partial y} = -F_x - F_u p.$$

Thus, the method of characteristics for p yields

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dp}{-F_x - F_u p}. \tag{12.7.23}$$

Similarly, taking the partial derivative of (12.7.22) with respect to y yields

$$F_y + F_u q + F_p \frac{\partial p}{\partial y} + F_q \frac{\partial q}{\partial y} = 0.$$

Here $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ yields a quasilinear partial differential equation for q:

$$F_p \frac{\partial q}{\partial x} + F_q \frac{\partial q}{\partial y} = -F_y - F_u q.$$

The characteristic direction is the same as in (12.7.23), so that (12.7.23) is amended to become

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dp}{-F_x - F_u p} = \frac{dq}{-F_y - F_u q}. \tag{12.7.24}$$

In order to solve for u, we want to derive a differential equation for $u(x, y)$ along the characteristics:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = p dx + q dy = p F_p \frac{dx}{F_p} + q F_q \frac{dy}{F_q} = (p F_p + q F_q) \frac{dx}{F_p}.$$

The complete system to solve for p, q, and u is

$$\boxed{\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dp}{-F_x - F_u p} = \frac{dq}{-F_y - F_u q} = \frac{du}{p F_p + q F_q}.} \tag{12.7.25}$$

Exercises 12.7

12.7.1. Show that light rays propagate normal to the wave fronts for nonuniform media.

12.7.2. For the normalized eikonal equation for uniform media, $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$,

(a) Derive a partial differential equation for q without using (12.7.25).

(b) Use the method of characteristics for q, assuming q is given at $y = 0$.

(c) Show that the result is the same as using (12.7.25).

13

Laplace Transform Solution of Partial Differential Equations

13.1 Introduction

We have introduced some techniques to solve linear partial differential equations. For problems with a simple geometry, the method of separation of variables motivates using Fourier series, its various generalizations, or variants of the Fourier transform. Of most importance is the type of boundary condition, including whether the domain is finite, infinite, or semi-infinite. In some problems a Green's function can be utilized, while for the one-dimensional wave equation, the method of characteristics exists. Whether or not any of these methods may be appropriate, numerical methods (introduced in Chapter 6) are often most efficient.

Another technique, to be elaborated on in this chapter, relies on the use of Laplace transforms. We will see that Laplace transforms are particularly well suited to solve linear initial value problems. *Most problems in partial differential equations that can be analyzed by Laplace transforms also can be analyzed by one of our earlier techniques, and substantially equivalent answers can be obtained.* We will present the elementary aspects of Laplace transforms in order to enable the reader to become somewhat familiar with them. However, whole books[1] have been written concerning their use in partial differential equations. Consequently, in this chapter, we only briefly discuss Laplace transforms and describe their application to partial differential equations with only a few examples.

13.2 Properties of the Laplace Transform

13.2.1 INTRODUCTION

Definition

One technique for solving ordinary differential equations (mostly with constant coefficients) is to introduce the **Laplace transform of** $f(t)$ as follows:

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt.} \tag{13.2.1}$$

For the Laplace transform to be defined, the integral in (13.2.1) must converge. For many functions, $f(t)$, s is restricted. For example, if $f(t)$ approaches a nonzero constant as $t \to \infty$, then the integral converges only if $s > 0$. If s is complex, $s = \mathrm{Re}(s) + i\,\mathrm{Im}(s)$, and $e^{-st} = e^{-\mathrm{Re}(s)t}[\cos(\mathrm{Im}(s)t) - i\sin(\mathrm{Im}(s)t)]$, then it follows in this case that $\mathrm{Re}(s) > 0$ for convergence.

We will assume that the reader has studied (at least briefly) Laplace transforms. We will quickly review the important properties of Laplace transforms. Tables exist,[2] and we include a short one here. The Laplace transform of some elementary functions can be obtained by direct integration. Some fundamental properties can be derived from the definition; these and others are summarized in Table 13.2.1.

[1]For example, Churchill (1972).

[2]Some better ones are in Churchill (1972); CRC Standard Mathematical Tables (2002) (from Churchill's table); Abramowitz and Stegun (1974) (also has Churchill's table); and Roberts and Kaufman (1966).

From the definition of the Laplace transform, $f(t)$ is needed only for $t > 0$. So that there is no confusion, we usually define $f(t)$ to be zero for $t < 0$. One formula (13.2.2l) requires the Heaviside unit step function:

$$H(t-b) = \begin{cases} 0 & t < b \\ 1 & t > b. \end{cases} \tag{13.2.2}$$

Inverse Laplace transforms

If instead we are given $F(s)$ and want to calculate $f(t)$, then we can also use the same tables. $f(t)$ is called the **inverse Laplace transform of** $F(s)$. The notation $f(t) = \mathcal{L}^{-1}[F(s)]$ is also used. For example, from Table 13.2.1, the inverse Laplace transform of $1/(s-3)$ is e^{3t}, $\mathcal{L}^{-1}[1/(s-3)] = e^{3t}$.

Not all functions of s have inverse Laplace transforms. From (13.2.1), we notice that if $f(t)$ is any type of ordinary function, then $F(s) \to 0$ as $s \to \infty$. All functions in our table have this property.

TABLE 13.2.1 Laplace Transforms (short table of formulas and properties)

	$f(t)$	$F(s) \equiv \mathcal{L}[f(t)] \equiv \int_0^\infty f(t)e^{-st}\,dt$	
Elementary functions (Exercises 13.2.1 and 13.2.2)	1	$\dfrac{1}{s}$	(13.2.2a)
	$t^n\,(n > -1)$	$n!s^{-(n+1)}$	(13.2.2b)
	e^{at}	$\dfrac{1}{s-a}$	(13.2.2c)
	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$	(13.2.2d)
	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$	(13.2.2e)
	$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$	$\frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$	(13.2.2f)
	$\cosh at = \frac{1}{2}(e^{at} + e^{-at})$	$\frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$	(13.2.2g)
Fundamental properties (Section 13.2.3 and Exercise 13.2.3)	$\dfrac{df}{dt}$	$sF(s) - f(0)$	(13.2.2h)
	$\dfrac{d^2f}{dt^2}$	$s^2F(s) - sf(0) - \dfrac{df}{dt}(0)$	(13.2.2i)
	$-tf(t)$	$\dfrac{dF}{ds}$	(13.2.2j)
	$e^{at}f(t)$	$F(s-a)$	(13.2.2k)
	$H(t-b)f(t-b)$	$e^{-bs}F(s) \quad (b > 0)$	(13.2.2l)
Convolution (Section 13.2.4)	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	(13.2.2m)
Dirac delta function (Section 13.2.4)	$\delta(t-b)$	$e^{-bs}\,(b \geq 0)$	(13.2.2n)
Inverse transform (Section 13.7)	$\dfrac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds$	$F(s)$	(13.2.2o)
Miscellaneous (Exercise 13.2.9)	$t^{-1/2}e^{-a^2/4t}$	$\sqrt{\frac{\pi}{s}}e^{-a\sqrt{s}}\,(a \geq 0)$	(13.2.2p)
	$t^{-3/2}e^{-a^2/4t}$	$\frac{2\sqrt{\pi}}{a}e^{-a\sqrt{s}}\,(a > 0)$	(13.2.2q)

13.2.2 SINGULARITIES OF THE LAPLACE TRANSFORM

We note that when $f(t)$ is a simple exponential, $f(t) = e^{at}$, the growth rate a is also the point at which its Laplace transform $F(s) = 1/(s-a)$ has a singularity. As $s \to a$, the Laplace transform approaches ∞. We claim *in general* that as a check in *any* calculation, **the singularities of a Laplace transform** $F(s)$ (*the zeros of its denominator*) **correspond** (*in some way*) **to the exponential growth rates of** $f(t)$. We refer to this as the **singularity property** of Laplace transforms. Later we will show this using complex variables. Throughout this chapter we illustrate this correspondence.

EXAMPLES

For now we briefly discuss some examples. Both the Laplace transforms $\omega/(s^2+\omega^2)$ and $s/(s^2+\omega^2)$ have singularities at $s^2+\omega^2 = 0$ or $s = \pm i\omega$. Thus, their inverse Laplace transforms will involve exponentials e^{st}, where $s = \pm i\omega$. According to Table 13.2.1, their inverse Laplace transforms are, respectively, $\sin\omega t$ and $\cos\omega t$, which we know from Euler's formulas can be represented as linear combinations of $e^{\pm i\omega t}$.

As another example, consider the Laplace transform $F(s) = 3/[s(s^2+4)]$. One method to determine $f(t)$ is to use partial fractions (with real factors):

$$\frac{3}{s(s^2+4)} = \frac{a}{s} + \frac{bs+c}{s^2+4} = \frac{3/4}{s} + \frac{-(3/4)s}{s^2+4}.$$

Now the inverse transform is easy to obtain using tables:

$$f(t) = \frac{3}{4} - \frac{3}{4}\cos 2t.$$

As a check we note that $3/[s(s^2+4)]$ has singularities at $s = 0$ and $s = \pm 2i$. The singularity property of Laplace transforms then implies that its inverse Laplace transform must be a linear combination of e^{0t} and $e^{\pm 2it}$, as we have already seen. ■

Partial fractions

In doing inverse Laplace transforms, we are frequently faced with the ratio of two polynomials $q(s)/p(s)$. To be a Laplace transform, it must approach 0 as $s \to \infty$. Thus, we can assume that the degree of p is greater than the degree of q. A partial fraction expansion will yield immediately the desired inverse Laplace transform. We describe this technique only in the case in which the roots of the denominator are simple; there are no repeated or multiple roots. First, we factor the denominator:

$$p(s) = \alpha(s-s_1)(s-s_2)\cdots(s-s_n),$$

where $s_1, \ldots, s_n$ are the n *distinct* roots of $p(s)$, also called the **simple poles** of $q(s)/p(s)$. The partial fraction expansion of $q(s)/p(s)$ is

$$\frac{q(s)}{p(s)} = \frac{c_1}{s-s_1} + \frac{c_2}{s-s_2} + \cdots + \frac{c_n}{s-s_n}. \tag{13.2.3}$$

The coefficients c_i of the partial fraction expansion can be obtained by cumbersome algebraic manipulations using a common denominator. A more elegant and sometimes quicker method utilizes the singularities s_i of $q(s)/p(s)$. To determine c_i, we multiply (13.2.3) by $s-s_i$ and then take the limit as $s \to s_i$. All the terms except c_i vanish on the right:

$$\boxed{c_i = \lim_{s\to s_i} \frac{(s-s_i)q(s)}{p(s)}.} \tag{13.2.4}$$

Often, this limit is easy to evaluate. Since $s-s_i$ is a factor of $p(s)$, we cancel it in (13.2.4) and then evaluate the limit.

EXAMPLE

Using complex roots,

$$\frac{3}{s(s^2+4)} = \frac{c_1}{s} + \frac{c_2}{s+2i} + \frac{c_3}{s-2i},$$

where

$$c_1 = \lim_{s\to 0} s\frac{3}{s(s^2+4)} = \frac{3}{4}$$

$$c_2 = \lim_{s\to -2i}(s+2i)\frac{3}{s(s^2+4)} = \lim_{s\to -2i}(s+2i)\frac{3}{s(s+2i)(s-2i)} = -\frac{3}{8}$$

$$c_3 = \lim_{s\to 2i}(s-2i)\frac{3}{s(s^2+4)} = \lim_{s\to 2i}\ s-2i\frac{3}{s(s+2i)(s-2i)} = -\frac{3}{8}.$$

■

Simple poles

In some problems, we can make the algebra even easier. The limit in (13.2.4) is 0/0 since $p(s_i) = 0$ [$s = s_i$ is a root of $p(s)$]. L'Hôpital's rule for evaluating 0/0 yields

$$\boxed{c_i = \lim_{s\to s_i}\frac{d/ds[(s-s_i)q(s)]}{d/ds\ p(s)} = \frac{q(s_i)}{p'(s_i)},} \tag{13.2.5}$$

Equation (13.2.5) is valid only for simple poles.

Once we have a partial fraction expansion of a Laplace transform, its inverse transform may be easily obtained. In summary, if

$$\boxed{F(s) = \frac{q(s)}{p(s)},} \tag{13.2.6}$$

then by inverting (13.2.3),

$$\boxed{f(t) = \sum_{i=1}^{n}\frac{q(s_i)}{p'(s_i)}e^{s_i t},} \tag{13.2.7}$$

where we assumed that $p(s)$ has only simple poles at $s = s_i$.

EXAMPLE

To apply this formula for $q(s)/p(s) = 3/[s(s^2+4)]$, we let $q(s) = 3$ and $p(s) = s(s^2+4) = s^3+4$. We need $p'(s) = 3s^2+4$. Thus, if

$$F(s) = \frac{3}{s(s^2+4)} = \frac{c_1}{s} + \frac{c_2}{s+2i} + \frac{c_3}{s-2i},$$

then

$$c_1 = \frac{q(0)}{p'(0)} = \frac{3}{4},\ c_2 = \frac{q(-2i)}{p'(-2i)} = \frac{3}{-8}, \qquad \text{and} \qquad c_3 = \frac{q(2i)}{p'(2i)} = -\frac{3}{8},$$

as before. For this example,

$$f(t) = \frac{3}{4} - \frac{3}{8}e^{-2it} - \frac{3}{8}e^{2it} = \frac{3}{4} - \frac{3}{4}\cos 2t.$$

■

Quadratic expressions (completing the square)

Inverse Laplace transforms for quadratic expressions

$$F(s) = \frac{\alpha s+\beta}{as^2+bs+c}$$

can be obtained by partial fractions if the roots are real or complex. However, if the roots are complex, it is often easier to complete the square. For example, consider

$$F(s) = \frac{1}{s^2+2s+8} = \frac{1}{(s+1)^2+7},$$

whose roots are $s = -1 \pm i\sqrt{7}$. Since a function of $s+1$ appears, we use (13.2.2k) in Table 13.2.1 which is known as the shift theorem,

$$F(s) = G(s+1), \quad \text{where} \quad G(s) = \frac{1}{s^2+7}.$$

According to the shift theorem, the inverse transform of $G(s+1)$ is (using $a=-1$) $f(t)=e^{-t}g(t)$, where $g(t)$ is the inverse transform of $1/(s^2+7)$. From Table 13.2.1, $g(t)=\left(1/\sqrt{7}\right)\sin\sqrt{7}t$ and thus,

$$f(t)=\frac{1}{\sqrt{7}}e^{-t}\sin\sqrt{7}t.$$

This result is consistent with the singularity property; the solution is a linear combination of e^{st}, where $s=-1\pm i\sqrt{7}$.

13.2.3 TRANSFORMS OF DERIVATIVES

One of the most useful properties of the Laplace transform is the way in which it operates on derivatives. For example, by elementary integration by parts,

$$\boxed{\mathcal{L}\left[\frac{df}{dt}\right]}=\int_0^\infty \frac{df}{dt}e^{-st}\,dt=fe^{-st}\Big|_0^\infty+s\int_0^\infty fe^{-st}\,dt\ \boxed{=sF(s)-f(0).} \tag{13.2.8}$$

Similarly,

$$\boxed{\mathcal{L}\left[\frac{d^2f}{dt^2}\right]}=s\mathcal{L}\left[\frac{df}{dt}\right]-\frac{df}{dt}(0)=s(sF(s)-f(0))-\frac{df}{dt}(0)\ \boxed{=s^2F(s)-sf(0)-\frac{df}{dt}(0).} \tag{13.2.9}$$

This property shows that the transform of derivatives can be evaluated in terms of the transform of the function. Certain "initial" conditions are needed. For the transform of the first derivative df/dt, $f(0)$ is needed. For the transform of the second derivative d^2f/dt^2, $f(0)$ and $df/dt(0)$ are needed. These are just the types of information that are known if the variable t is time. Usually, *if a Laplace transform is used, the independent variable t is time*. Furthermore, the Laplace transform method will often simplify if the initial conditions are all zero.

Application to ordinary differential equations

For ordinary differential equations, the use of the Laplace transform reduces the problem to an algebraic equation. For example, consider

$$\boxed{\begin{aligned}&\frac{d^2y}{dt^2}+4y=3\\ \text{with}\quad &y(0)=1\\ &\frac{\partial y}{\partial t}(0)=5.\end{aligned}}$$

Taking the Laplace transform of the differential equation yields

$$s^2Y(s)-s-5+4Y(s)=\frac{3}{s},$$

where $Y(s)$ is the Laplace transform of $y(t)$. Thus,

$$Y(s)=\frac{1}{s^2+4}\left(\frac{3}{s}+s+5\right)=\frac{3}{s(s^2+4)}+\frac{s}{s^2+4}+\frac{5}{s^2+4}.$$

The inverse transforms of $s/(s^2+4)$ and $5/(s^2+4)$ are easily found in tables. The function whose transform is $3/[s(s^2+4)]$ has been obtained in different ways. Thus,

$$y(t)=\frac{3}{4}-\frac{3}{4}\cos 2t+\cos 2t+\frac{5}{2}\sin 2t.$$

13.2.4 CONVOLUTION THEOREM

Another method to obtain the inverse Laplace transform of $3/[s(s^2+4)]$ is to use the convolution theorem. We begin by stating and deriving the convolution theorem. Often, as in this example, we need to obtain the function whose Laplace transform is the product of two transforms, $F(s)G(s)$. The **convolution theorem** states that

$$\boxed{\mathcal{L}^{-1}[F(s)G(s)] = g * f = \int_0^t g(\bar{t})f(t-\bar{t})d\bar{t},} \tag{13.2.10}$$

where $g * f$ is called the **convolution** of g and f. Here $f(t) = \mathcal{L}^{-1}[F(s)]$ and $g(t) = \mathcal{L}^{-1}[G(s)]$. Equivalently, the convolution theorem states

$$\boxed{\mathcal{L}\left[\int_0^t g(\bar{t})f(t-\bar{t})d\bar{t}\right] = F(s)G(s).} \tag{13.2.11}$$

By letting $t - \bar{t} = w$, we also derive that $g * f = f * g$; the order is of no importance. Earlier, when studying Fourier transforms (see Section 10.4), we also introduced the convolution of g and f in a slightly different way,

$$g * f \equiv \int_{-\infty}^{\infty} f(t-\bar{t})g(\bar{t})d\bar{t}.$$

However, in the context of Laplace transforms, both $f(t)$ and $g(t)$ are zero for $t < 0$, and thus, (13.2.10) follows since $f(t-\bar{t}) = 0$ for $\bar{t} > t$ and $g(\bar{t}) = 0$ for $\bar{t} < 0$.

Laplace transform of Dirac delta functions

The Laplace transform of a Dirac delta function is often useful together with the convolution, especially to help find Green's functions for certain problems. From the definition of the transform,

$$\boxed{\mathcal{L}[\delta(t-b)] = \int_0^{\infty} \delta(t-b)e^{-st}dt = e^{-sb},} \tag{13.2.12}$$

if $b > 0$. Thus, the inverse Laplace transform of an exponential is a Dirac delta function:

$$\boxed{\mathcal{L}^{-1}[e^{-sb}] = \delta(t-b).} \tag{13.2.13}$$

In the limit as $b \to 0$, we also obtain

$$\mathcal{L}[\delta(t-0+)] = 1 \quad \text{and} \quad \mathcal{L}^{-1}[1] = \delta(t-0+). \tag{13.2.14}$$

Derivation of convolution theorem

To derive the convolution theorem, we introduce two transforms $F(s)$ and $G(s)$ and their product $F(s)G(s)$:

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \tag{13.2.15}$$

$$G(s) = \int_0^{\infty} g(t)e^{-st}\,dt \tag{13.2.16}$$

$$F(s)G(s) = \int_0^{\infty}\int_0^{\infty} f(\bar{t})g(T)e^{-s(\bar{t}+T)}dT\,d\bar{t}. \tag{13.2.17}$$

We will show

$$\mathcal{L}[(f*g)(t)] = F(s)G(s).$$

The derivation of this theorem is a nice application of switching the order of integration in a double integral. The transform of the convolution is

$$\begin{aligned}\mathcal{L}[f*g] &= \int_0^{\infty} e^{-st}\left(\int_0^t f(\bar{t})g(t-\bar{t})\,d\bar{t}\right)dt \\ &= \int_0^{\infty}\int_0^t e^{-st}f(\bar{t})g(t-\bar{t})d\bar{t}\,dt.\end{aligned}$$

We reverse the order of integration (noting that $0 \le \bar{t} \le t$ for $0 \le t < \infty$ implies $\bar{t} \le t < \infty$ for $0 \le \bar{t} < \infty$),

$$\mathcal{L}[f*g] = \int_0^{\infty}\int_{\bar{t}}^{\infty} e^{-st}f(\bar{t})g(t-\bar{t})dt\,d\bar{t}.$$

Then, we make the substitution $T = t - \bar{t}$ for the variable t, using $dt = dT$ and $T = 0$ when $t = \bar{t}$, so that

$$\begin{aligned}\mathcal{L}[f*g] &= \int_0^{\infty}\int_0^{\infty} e^{-s(\bar{t}+T)}f(\bar{t})g(T)dT\,d\bar{t} \\ &= F(s)G(s),\end{aligned}$$

the convolution theorem for Laplace transforms.

EXAMPLE

Determine the function whose Laplace transform is $3/[s(s^2 + 4)]$, using the convolution theorem. We introduce

$$F(s) = \frac{3}{s} \quad [\text{so that } f(t) = 3] \quad \text{and} \quad G(s) = \frac{1}{s^2 + 4} \quad \left[\text{so that } g(t) = \frac{1}{2}\sin 2t\right].$$

It follows from the convolution theorem that the inverse transform of $(3/s)[1/(s^2 + 4)]$ is $\int_0^t f(t - \bar{t})g(\bar{t})\,d\bar{t}$:

$$\int_0^t 3 \cdot \frac{1}{2}\sin 2\bar{t}\; d\bar{t} = -\frac{3}{4}\cos 2\bar{t}\Big|_0^t = \frac{3}{4}(1 - \cos 2t),$$

as we obtained earlier using the partial fraction expansion. ■

Exercises 13.2

13.2.1. From the definition of the Laplace transform (i.e., using explicit integration), determine the Laplace transform of $f(t) =$:

(a) 1 **(b)** e^{at}

(c) $\sin\omega t$ [*Hint*: $\sin\omega t = \text{Im}(e^{i\omega t})$.] **(d)** $\cos\omega t$ [*Hint*: $\cos\omega t = \text{Re}(e^{i\omega t})$.]

(e) $\sinh at$ **(f)** $\cosh at$

(g) $H(t - t_0)$, $t_0 > 0$

13.2.2. The gamma function $\Gamma(x)$ was defined in Exercise 10.3.14. Derive that $\mathcal{L}[t^n] = \Gamma(n + 1)/s^{n+1}$ for $n > -1$. Why is this not valid for $n \leq -1$?

13.2.3. Derive the following fundamental properties of Laplace transforms:

(a) $\mathcal{L}[-tf(t)] = dF/ds$

(b) $\mathcal{L}[e^{at}f(t)] = F(s - a)$

(c) $\mathcal{L}[H(t - b)f(t - b)] = e^{-bs}F(s)$ $(b > 0)$

***13.2.4.** Using Table 13.2.1, determine the Laplace transform of $\int_0^t f(\bar{t})\,d\bar{t}$ in terms of $F(s)$.

13.2.5. Using Table 13.2.1, determine the Laplace transform of $f(t) =$:

(a) t^3e^{-2t} ***(b)** $t\sin 4t$

(c) $H(t - 3)$ ***(d)** $e^{3t}\sin 4t$

(e) $te^{-4t}\cos 6t$ ***(f)** $f(t) = \begin{cases} 0 & t < 5 \\ t^2 & 5 < t < 8 \\ 0 & 8 < t \end{cases}$

(g) $t^2H(t - 1)$ ***(h)** $(t - 1)^4H(t - 1)$

13.2.6. Using Table 13.2.1, determine the inverse Laplace transform of $F(s) =$:

(a) $\dfrac{1}{s^2 + 4}$ **(b)** $\dfrac{e^{-3s}}{s^2 - 4}$

(c) s^{-3} **(d)** $(s - 4)^{-7}$

***(e)** $\dfrac{s}{s^2 + 8s + 7}$ **(f)** $\dfrac{2s + 1}{s^2 - 4s + 9}$

(g) $\dfrac{s}{(s^2 + 1)(s^2 + 4)}$ **(h)** $\dfrac{s}{s^2 - 4s - 5}$

(i) $\dfrac{s}{s^2 - 4s - 5}(1 - 4e^{-7s})$ ***(j)** $\dfrac{s + 2}{s(s^2 + 9)}(1 - 5e^{-4s})$

(k) $\dfrac{1}{(s + 1)^2}$ **(l)** $\dfrac{1}{(s^2 + 1)^2}$

13.2.7. Solve the following ordinary differential equations using Laplace transforms:

(a) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + y = t^3$ with $y(0) = 7$ and $\dfrac{dy}{dt}(0) = 5$

***(b)** $\dfrac{dy}{dt} + y = 1$ with $y(0) = 2$

(c) $\dfrac{dy}{dt} + 3y = \begin{cases} 4e^{-t} & t < 8 \\ 2 & t > 8 \end{cases}$ with $y(0) = 1$

***(d)** $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} - 6y = \begin{cases} 0 & 0 < t < 3 \\ e^{-t} & t > 3 \end{cases}$ with $y(0) = 3$ and $\dfrac{dy}{dt}(0) = 7$

(e) $\dfrac{d^2y}{dt^2} + y = \cos t$ with $y(0) = 0$ and $\dfrac{dy}{dt}(0) = 0$

***(f)** $\dfrac{d^2y}{dt^2} + 4y = \sin t$ with $y(0) = 0$ and $\dfrac{dy}{dt}(0) = 0$

13.2.8. Derive the convolution theorem for Laplace transforms in the form $\mathcal{L}^{-1}[F(s)G(s)] = f * g$ using a Dirac delta function. [*Hint*: Apply the inverse Laplace transform to (13.2.17). Linearity of the transform implies

$$\mathcal{L}^{-1}[F(s)G(s)] = \int_0^\infty \int_0^\infty f(\bar{t})\, g(T)\mathcal{L}^{-1}\left[e^{-s(\bar{t}+T)}\right] dT\, d\bar{t}.]$$

13.2.9. In this exercise, we will determine

$$I = \mathcal{L}\left\{t^{-3/2}e^{-a^2/4t}\right\} \quad \text{and} \quad J = \mathcal{L}\left\{t^{-1/2}e^{-a^2/4t}\right\}.$$

(a) Determine a relationship between I and J by substituting the expression $u = s^{1/2}t^{1/2} - (a/2)t^{-1/2}$ into $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$.

(b) Determine a relationship between I and J by introducing the change of variables $sw = (a^2/4)t^{-1}$ into the definition of I.

(c) Derive that $I = \left(2\sqrt{\pi}/a\right)e^{-a\sqrt{s}}$ and $J = \left(\sqrt{\pi/s}\right)e^{-a\sqrt{s}}$ using parts (a) and (b).

13.3 Green's Functions for Initial Value Problems for Ordinary Differential Equations

The convolution theorem is very useful in solving nonhomogeneous ordinary differential equations. For example, consider

$$\alpha\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + \gamma y = f(t), \tag{13.3.1}$$

subject to zero[3] initial conditions

$$y(0) = 0 \tag{13.3.2}$$

$$\frac{dy}{dt}(0) = 0. \tag{13.3.3}$$

Taking the Laplace transform of the differential equation (13.3.1) yields

$$(\alpha s^2 + \beta s + \gamma)Y(s) = F(s) \quad \text{or} \quad Y(s) = F(s) \cdot \frac{1}{\alpha s^2 + \beta s + \gamma}, \tag{13.3.4}$$

where $Y(s)$ and $F(s)$ are the Laplace transforms of $y(t)$ and $f(t)$, respectively. The solution $y(t)$ can be obtained using the convolution theorem:

$$y(t) = \int_0^t f(t_0)q(t - t_0)\, dt_0, \tag{13.3.5}$$

where $q(t)$ is the inverse Laplace transform of $1/(\alpha s^2 + \beta s + \gamma)$. We can determine $q(t)$ using tables and/or partial fractions. This result, (13.3.5), will be equivalent to the solution of nonhomogeneous problems as is usually obtained by the method of variation of parameters in most elementary texts on ordinary differential equations.

There is an important alternative interpretation of this result. $q(t)$ is the solution of (13.3.1) if $F(s) = 1$. The inverse Laplace transform of $F(s) = 1$ is $f(t) = \delta(t - 0+)$ [see (13.2.14)]. Thus, $q(t)$ is the response due to an impulse at $t = 0+$:

$$\begin{aligned} \alpha\frac{d^2q}{dt^2} + \beta\frac{dq}{dt} + \gamma q &= \delta(t - 0+) \\ q(0) &= 0 \\ \frac{dq}{dt}(0) &= 0. \end{aligned} \tag{13.3.6}$$

[3]Nonzero initial conditions can be analyzed by adding appropriate homogeneous solutions of the differential equation.

We can introduce the terminology of Chapters 9 and 11. We call $q(t)$ the **Green's function for the initial value problem**, the response at t due to a concentrated source at $t = 0$:

$$q(t) = G(t, 0).$$

The convolution theorem shows that we are interested in $q(t - t_0)$:

$$q(t - t_0) = G(t - t_0, 0).$$

However, due to the constant coefficients present in (13.3.6), **the response at t due to an impulse at t_0**, $G(t, t_0)$, is the same as the response due to an impulse at 0 if the elapsed time is the same:

$$G(t, t_0) = G(t - t_0, 0), \tag{13.3.7}$$

the **translation property** of the Green's function. Thus,

$$q(t - t_0) = G(t, t_0).$$

Therefore, from (13.3.5), through the use of Laplace transforms, we have obtained a representation of the solution of the nonhomogeneous initial value problem (13.3.1)–(13.3.3) involving the Green's function

$$\boxed{y(t) = \int_0^t f(t_0)G(t, t_0)dt_0.} \tag{13.3.8}$$

The solution is the generalized superposition of all sources acting before the time t, an example of the **causality principle** for initial value problems for ordinary differential equations. In this form, the result appears quite similar to our results concerning Green's functions for boundary value problems for ordinary and partial differential equations. Here the Green's function $h(t) = G(t, 0)$ is simply the inverse Laplace transform of $1/(\alpha s^2 + \beta s + \gamma)$.

EXAMPLE

Consider the differential equation

$$\alpha^2 \frac{d^2y}{dt^2} - \gamma^2 y = f(t). \tag{13.3.9}$$

The solution that satisfies zero initial conditions is

$$y(t) = \int_0^t f(t_0)G(t, t_0)dt_0,$$

where the Green's function $G(t, t_0)$ satisfies $q(t - t_0) = G(t, t_0)$. We introduce the Laplace transform of $q(t)$:

$$\mathcal{L}[q(t)] = \frac{1}{\alpha^2 s^2 - \gamma^2} = \frac{1}{\alpha^2\left(s^2 - \frac{\gamma^2}{\alpha^2}\right)}.$$

Directly from tables, we may obtain the Green's function, $G(t, 0)$,

$$G(t, 0) = q(t) = \frac{1}{\alpha^2}\frac{\alpha}{\gamma}\sinh\frac{\gamma}{\alpha}t.$$

Thus, the solution of (13.3.9) is

$$y(t) = \frac{1}{\alpha\gamma}\int_0^t f(t_0)\sinh\frac{\gamma}{\alpha}(t - t_0)dt_0, \tag{13.3.10}$$

where $y(0) = 0$ and $\frac{dy}{dt}(0) = 0$. ■

Exercises 13.3

13.3.1. By using Laplace transforms, determine the effect of the initial conditions in terms of the Green's function for the initial value problem

$$\alpha\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + \gamma y = 0,$$

subject to $y(0) = y_0$ and $\frac{dy}{dt}(0) = v_0$.

***13.3.2.** What is the Green's function for

$$\frac{d^2y}{dt^2} + y = f(t)$$

with $y(0) = 0$ and $\frac{dy}{dt}(0) = 0$? Solve for $y(t)$.

13.3.3. **(a)** Do Exercise 9.3.25(a).

(b) Do Exercise 9.3.25(b).

13.3.4. Show that for $t > t_0$, $G(t, t_0)$ for (13.3.1)–(13.3.3) satisfies

$$\alpha\frac{d^2G}{dt^2} + \beta\frac{dG}{dt} + \gamma G = 0$$

with $G(t_0, t_0) = 0$ and $\frac{dG}{dt}(t_0, t_0) = \frac{1}{\alpha}$.

13.3.5. Solve Exercise 13.3.2 using Exercise 13.3.4.

13.4 A Signal Problem for the Wave Equation

Using Laplace transforms to solve partial differential equations often requires great skill in the use of Laplace transforms. We pursue only some relatively simple examples.[4] Consider a semi-infinite string ($x > 0$), whose motion is caused only by a time-dependent boundary condition at $x = 0$:

$$\text{PDE: } \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}} \tag{13.4.1}$$

$$\text{BC: } \boxed{u(0, t) = f(t)} \tag{13.4.2}$$

$$\text{IC: } \boxed{\begin{aligned} u(x, 0) &= 0 \\ \frac{\partial u}{\partial t}(x, 0) &= 0. \end{aligned}} \tag{13.4.3}$$

The string is initially horizontal and at rest. The left end is being moved vertically (while maintaining the large tension). Since the problem is defined for all $x > 0$, we need a boundary condition as $x \to \infty$:

$$\lim_{x\to\infty} u(x, t) = 0. \tag{13.4.4}$$

All the initial conditions of this problem are zero. Consequently, the use of Laplace transforms in the time variable is expected to yield a simple solution:

$$\mathcal{L}[u(x, t)] = \overline{U}(x, s) = \int_0^\infty u(x, t)e^{-st}\,dt.$$

As with ordinary differential equations, we take the Laplace transform in the time variable of (13.4.1),

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2\mathcal{L}[u] - su(x, 0) - \frac{\partial u}{\partial t}(x, 0) = s^2\mathcal{L}[u]. \tag{13.4.5}$$

Here, we also need the Laplace transforms in the time variable of partial derivatives with respect to x. We obtain

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^\infty \frac{\partial^2 u}{\partial x^2}e^{-st}dt = \frac{\partial^2}{\partial x^2}\int_0^\infty u(x, t)e^{-st}dt = \frac{\partial^2}{\partial x^2}\mathcal{L}[u]. \tag{13.4.6}$$

In this manner, the Laplace transform of a partial differential equation yields an "ordinary" differential equation

$$s^2\overline{U}(x, s) = c^2\frac{\partial^2\overline{U}}{\partial x^2}, \tag{13.4.7}$$

defined for $0 < x$. At $x = 0$, $u(x, t)$ is given for all t, and thus, its Laplace transform is known:

$$\overline{U}(0, s) = \int_0^\infty u(0, t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt = F(s), \tag{13.4.8}$$

where $F(s)$ is the Laplace transform of the boundary condition. Also, since $u(x, t) \to 0$ as $x \to \infty$ (for all fixed t), we have the same result for its Laplace transform,

$$\lim_{x\to\infty} \overline{U}(x, s) = 0. \tag{13.4.9}$$

[4]For more difficult examples, see Churchill (1972) and Weinberger (1995).

The general solution of (13.4.7) is

$$\overline{U}(x, s) = A(s)e^{-(s/c)x} + B(s)e^{(s/c)x},$$

where $A(s)$ and $B(s)$ are arbitrary functions of the transform variable s. For $s > 0$ [more precisely $\mathrm{Re}(s) > 0$], $B(s) = 0$ to satisfy the decay as $x \to \infty$, (13.4.9). In addition, the boundary condition at $x = 0$, (13.4.8), implies that $A(s) = F(s)$, and thus,

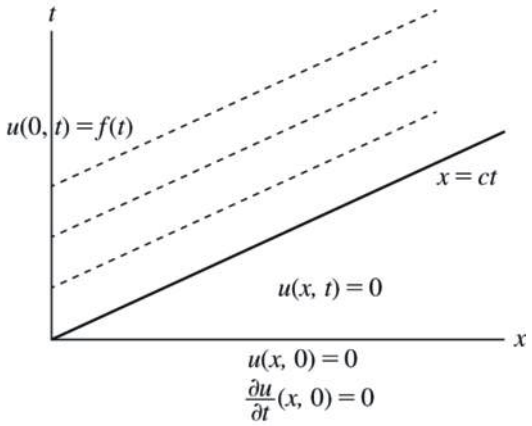
$$\overline{U}(x, s) = F(s)e^{-(s/c)x}. \tag{13.4.10}$$

To invert this transform, we could use the convolution theorem. Instead, a quick glance at Table 13.2.1 shows that an exponential multiple in the transform yields a time shift in the solution. Consequently,

$$\boxed{u(x, t) = H\left(t - \frac{x}{c}\right) f\left(t - \frac{x}{c}\right),} \tag{13.4.11}$$

where H is the Heaviside unit step function. The solution is zero for $x > ct$. In fact, the solution is constant whenever $x - ct$ is constant. The solution travels as a wave of fixed shape at velocity c. We obtained similar results in Chapter 12 using the method of characteristics. We illustrate this in a space-time diagram in Fig. 13.4.1. The signal propagates with velocity c, and thus, at time t it has traveled only a distance ct. If $x > ct$, the "wiggling" of the string at $x = 0$ has not been noticed.

Figure 13.4.1 Signal problem for the one-dimensional wave equation.



However, it is educational to obtain the same result using the convolution theorem. Since $\overline{U}(x, s) = F(s)e^{-(s/c)x}$,

$$u(x, t) = \int_0^t f(t_0)g(t - t_0)dt_0,$$

where $e^{-(s/c)x}$ is the Laplace transform of $g(t)$. From a table of Laplace transforms (as can be easily verified), $g(t) = \delta(t - x/c)$. Thus,

$$u(x, t) = \int_0^t f(t_0)\delta\left(t - t_0 - \frac{x}{c}\right)dt_0 = \begin{cases} 0 & t < x/c \\ f(t - x/c) & t > x/c, \end{cases}$$

which is equivalent to (13.4.11).

Exercises 13.4

13.4.1. Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

subject to $\frac{\partial u}{\partial x}(0, t) = f(t)$, $u(x, 0) = 0$, and $\frac{\partial u}{\partial t}(x, 0) = 0$.

13.4.2. Solve

$$\begin{aligned} \frac{\partial w}{\partial t} &= c\frac{\partial w}{\partial x}, \qquad c < 0, x > 0, t > 0 \\ w(0, t) &= f(t) \\ w(x, 0) &= 0. \end{aligned}$$

***13.4.3.** Solve

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty \\ u(x, 0) &= \sin x \\ \frac{\partial u}{\partial t}(x, 0) &= 0. \end{aligned}$$

***13.4.4.** Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \qquad x > 0 \\ u(x, 0) &= 0 \\ u(0, t) &= f(t). \end{aligned}$$

Determine the Laplace transform of $u(x, t)$. Invert to obtain $u(x, t)$. [*Hint*: See Table 13.2.1 of Laplace transforms.]

13.4.5. Reconsider Exercise 13.4.4 if instead the boundary and initial conditions are

$$u(x, 0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(0, t) = f(t).$$

13.4.6. Reconsider Exercise 13.4.4 if $f(t) = Ae^{i\sigma_0 t}$ (see Exercise 10.5.17).

(a) Determine an expression for $u(x, t)$ using Laplace transforms.

(b) Simplify part (a) with the change of variables $w = x/2\sqrt{t - \overline{t}}$, where $\overline{t}$ is the variable of integration in part (a).

(c) Approximate $u(x, t)$ if t is large.

13.5 A Signal Problem for a Vibrating String of Finite Length

In partial differential equations, the Laplace inversion integrals that are needed are often not as simple as in Section 13.4. We illustrate this for a vibrating string of length L initially at rest in the horizontal equilibrium position subject to the following time-dependent boundary condition at one end $x = L$:

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}} \tag{13.5.1}$$

$$\text{BC:} \quad \boxed{\begin{aligned} u(0, t) &= 0 \\ u(L, t) &= b(t) \end{aligned}} \tag{13.5.2}$$

$$\text{IC:} \quad \boxed{\begin{aligned} u(x, 0) &= 0 \\ \frac{\partial u}{\partial t}(x, 0) &= 0. \end{aligned}} \tag{13.5.3}$$

The zero initial conditions facilitate the use of the Laplace transform in t of $u(x, t)$:

$$\overline{U}(x, s) = \int_0^\infty e^{-st}u(x, t)dt. \tag{13.5.4}$$

By transforming (13.5.1), $\overline{U}(x, s)$ satisfies the ordinary differential equation

$$s^2\overline{U} = c^2\frac{\partial^2 \overline{U}}{\partial x^2}, \tag{13.5.5}$$

subject to the boundary conditions

$$\overline{U}(0, s) = 0 \tag{13.5.6}$$

$$\overline{U}(L, s) = B(s), \tag{13.5.7}$$

where $B(s)$ is the Laplace transform of $b(t)$. We can easily determine $\overline{U}(x, s)$:

$$\overline{U}(x, s) = B(s)\frac{\sinh(s/c)x}{\sinh(s/c)L}. \tag{13.5.8}$$

The convolution theorem implies that

$$u(x, t) = \int_0^t b(t_0) f(t - t_0) dt_0, \tag{13.5.9}$$

where $f(t)$ is the inverse Laplace transform of

$$F(s) = \frac{\sinh(s/c)x}{\sinh(s/c)L}. \tag{13.5.10}$$

To obtain this inverse Laplace transform is not straightforward. One method to obtain the inverse transform of (13.5.10) is to attempt to use our elementary tables, in which primarily exponentials appear. We note that

$$\frac{\sinh(s/c)x}{\sinh(s/c)L} = \frac{e^{(s/c)x} - e^{-(s/c)x}}{e^{(s/c)L} - e^{-(s/c)L}} = \frac{e^{(s/c)x} - e^{-(s/c)x}}{e^{(s/c)L}(1 - e^{-(2s/c)L})}.$$

However, due to the denominator, this cannot be analyzed in a simple way. Instead, we can introduce an infinite series of exponentials based on the geometric series $[1/(1 - x) = 1 + x + x^2 + \cdots, \text{ if } |x| < 1]$

$$\begin{aligned} F(s) &= \frac{e^{(s/c)x} - e^{-(s/c)x}}{e^{(s/c)L}(1 - e^{-(2s/c)L})} \\ &= e^{-(s/c)L}(e^{(s/c)x} - e^{-(s/c)x})(1 + e^{-(2L/c)s} + e^{-(4L/c)s} + \cdots) \\ &= \sum_{n=0}^{\infty} \left\{ \exp\left[-s\left(\frac{2nL - x + L}{c}\right)\right] - \exp\left[-s\left(\frac{2nL + x + L}{c}\right)\right] \right\}. \end{aligned} \tag{13.5.11}$$

These are all decaying exponentials since $x < L$, and hence each can be inverted using formula (13.2.2n) in Table 13.2.1. The Laplace transform is a linear combination of

$$\exp\left[-s\left(\frac{2nL \pm x + L}{c}\right)\right] (n \geq 0).$$

The inverse Laplace transform of $F(s)$ is thus a linear combination of Dirac delta functions, $\delta[t - (2nL \pm x + L)/c]$:

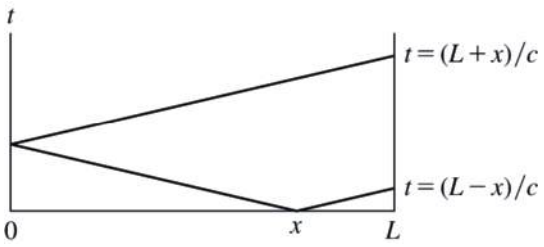
$$f(t) = \sum_{n=0}^{\infty} \left[\delta\left(t - \frac{2nL - x + L}{c}\right) - \delta\left(t - \frac{2nL + x + L}{c}\right) \right].$$

Since $f(t - t_0)$ in (13.5.9) is the influence function for the boundary condition, these Dirac delta functions represent signals whose travel times are $(2nL \pm x + L)/c$ [elapsed from the signal time t_0]. These can be interpreted as direct signals and their reflections off the boundaries $x_0 = 0$ and $x_0 = L$. Since the nonhomogeneous boundary condition is at $x_0 = L$, we imagine these signals are initiated there (at $t = t_0$). The signal can travel to x in different ways, as illustrated in Fig. 13.5.1. The direct signal must travel a distance $L - x$ at velocity c, yielding the retarded time $(L - x)/c$ (corresponding to $n = 0$). A signal can also arrive at x by additionally making an integral number of complete circuits, an added travel distance of $2Ln$. The other terms correspond to waves first reflecting off the wall $x_0 = 0$ before impinging on x. In this case, the total travel distance is $L + x + 2nL (n \geq 0)$. Further details of the solution are left as exercises.

Using Laplace transforms (and inverting in the way described in this section) yields a representation of the solution as an infinite sequence of reflecting waves. Similar results can be obtained by the method of characteristics (see Chapter 12) or (in some cases) by using the method of images (sequences of infinite space Green's functions).

Alternatively, in subsequent sections, we will describe the use of contour integrals in the complex plane to invert Laplace transforms. This technique will yield a significantly different representation of the same solution.

Figure 13.5.1 Space-time signal paths.



Exercises 13.5

13.5.1. Consider

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty \\ u(x, 0) &= f(x) \\ \frac{\partial u}{\partial x}(x, 0) &= g(x). \end{aligned}$$

Solve using Laplace transforms

(a) if $g(x) = 0$

(b) if $f(x) = 0$

13.5.2. **(a)** Using the results of this section, invert (13.5.8) based on the convolution theorem. Solve for $u(x, t)$.

(b) Without using the convolution theorem, from (13.5.8) and (13.5.11), determine $u(x, t)$.

***13.5.3.** Solve for $u(x, t)$ using Laplace transforms:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ u(0, t) &= 0, \quad u(x, 0) = 0 \\ \frac{\partial u}{\partial x}(L, t) &= b(t), \quad \frac{\partial u}{\partial t}(x, 0) = 0. \end{aligned}$$

13.5.4. Reconsider Exercise 13.5.3 if instead

(a) $\frac{\partial u}{\partial x}(0, t) = 0$ and $u(L, t) = b(t)$

(b) $\frac{\partial u}{\partial x}(0, t) = 0$ and $\frac{\partial u}{\partial x}(L, t) = b(t)$

13.5.5. Solve for $u(x, t)$ using Laplace transforms:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty \\ u(x, 0) &= f(x). \end{aligned}$$

[*Hint*: See Table 13.2.1 of Laplace transforms.]

13.5.6. Solve for $u(x, t)$ using Laplace transforms:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

subject to $u(x, 0) = f(x)$, $u(0, t) = 0$, and $u(L, t) = 0$. By what other method(s) can this representation of the solution be obtained?

13.6 The Wave Equation and Its Green's Function

The Laplace transform can be used to determine the relationship between solutions of nonhomogeneous partial differential equations and its corresponding Green's function. Consider the wave equation on a finite interval ($0 < x < L$) with sources and time-dependent boundary conditions

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + q(x, t)} \tag{13.6.1}$$

$$\text{BC:} \quad \boxed{\begin{aligned} u(0, t) &= a(t) \\ u(L, t) &= b(t) \end{aligned}} \tag{13.6.2}$$

$$\text{IC:} \quad \boxed{\begin{aligned} u(x, 0) &= f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= g(x). \end{aligned}} \tag{13.6.3}$$

GREEN'S FUNCTION

The Green's function $G(x, t; x_0, t_0)$ satisfies

$$\frac{\partial^2 G}{\partial t^2} = c^2 \frac{\partial^2 G}{\partial x^2} + \delta(x - x_0)\delta(t - t_0) \tag{13.6.4}$$

$$G(0, t; x_0, t_0) = 0 \tag{13.6.5}$$

$$G(L, t; x_0, t_0) = 0 \tag{13.6.6}$$

subject to the causality principle

$$G(x, t; x_0, t_0) = 0 \quad \text{for} \quad t < t_0. \tag{13.6.7}$$

Laplace transform of Green's function

In this section, we determine the Green's function using the Laplace transform. The transform of (13.6.4)–(13.6.6) yields

$$s^2 \overline{G} = c^2 \frac{\partial^2 \overline{G}}{\partial x^2} + \delta(x - x_0)e^{-st_0} \tag{13.6.8}$$

$$\overline{G}(0, s; x_0, t_0) = 0 \tag{13.6.9}$$

$$\overline{G}(L, s; x_0, t_0) = 0, \tag{13.6.10}$$

where $\overline{G}(x, s; x_0, t_0)$ is the Laplace transform in time of $G(x, t; x_0, t_0)$. The transform of (13.6.4) simplifies because the causality principle implies that G satisfies zero initial conditions (if $t_0 > 0$).

The Laplace transform of the Green's function satisfies (13.6.8), an ordinary differential equation of the Green's function type. To satisfy the boundary conditions, $\overline{G}(x, s; x_0, t_0)$ must be proportional to $\sinh(s/c)x$ for $x < x_0$ and it must be proportional to $\sinh(s/c)(L - x)$ for $x > x_0$. Since it will be symmetric, we know that

$$\overline{G}(x, s; x_0, t_0) = \begin{cases} \gamma \sinh\frac{s}{c}(L - x_0)\sinh\frac{s}{c}x & x < x_0 \\ \gamma \sinh\frac{s}{c}x_0 \sinh\frac{s}{c}(L - x) & x > x_0, \end{cases} \tag{13.6.11}$$

where γ is a constant (independent of x and x_0). In this manner, the continuity of $\overline{G}$ at $x = x_0$ is automatically satisfied. The additional jump condition, which is found by integrating (13.6.8) from $x_0 -$ to $x_0 +$,

$$0 = c^2 \frac{d\overline{G}}{dx}\bigg|_{x_0-}^{x_0+} + e^{-st_0},$$

determines γ:

$$0 = -c^2\gamma\frac{s}{c}\left[\sinh\frac{s}{c}x_0 \cosh\frac{s}{c}(L - x_0) + \sinh\frac{s}{c}(L - x_0)\cosh\frac{s}{c}x_0\right] + e^{-st_0}.$$

By using an addition formula for hyperbolic functions [$\sinh(a + b) = \sinh a \cosh b + \cosh a \sinh b$], we obtain

$$\gamma = \frac{e^{-st_0}}{cs\sinh(s/c)L}. \tag{13.6.12}$$

In Exercise 13.6.2 the Green's function itself is obtained by determining the inverse Laplace transform of (13.6.11) with (13.6.12).

Representation of solution in terms of the Green's function

Using Laplace transforms, we further investigate the relationships between $u(x, t)$ and its Green's function. To simplify some of our work, we consider the special case of Section 13.5, $q(x, t) = 0$, $a(t) = 0$, $f(x) = 0$, and $g(x) = 0$; the only nonhomogeneous term is the boundary condition at $x = L$. In the preceding section we showed (13.5.8) that

$$\overline{U}(x, s) = B(s)\frac{\sinh(s/c)x}{\sinh(s/c)L}. \tag{13.6.13}$$

Here we will relate this to the Laplace transform of the Green's function (13.6.11) with (13.6.12). Since the source is located at $x_0 = L$, we need $\overline{G}$ for $x < x_0$:

$$\overline{G}(x, s; x_0, t_0) = \frac{e^{-st_0}\sinh(s/c)(L - x_0)\sinh(s/c)x}{cs\sinh(s/c)L}. \tag{13.6.14}$$

To compare this with (13.6.13), we take the derivative with respect to x_0:

$$\frac{\partial \overline{G}}{\partial x_0}(x, s; x_0, t_0) = -\frac{e^{-st_0}\cosh(s/c)(L - x_0)\sinh(s/c)x}{c^2\sinh(s/c)L}.$$

We note that at $x_0 = L$ and $t_0 = 0$,

$$\frac{\partial \overline{G}}{\partial x_0}(x, s; L, 0) = -\frac{\sinh(s/c)x}{c^2\sinh(s/c)L}, \tag{13.6.15}$$

similar to the term appearing in (13.6.13). Thus,

$$\overline{U}(x, s) = -c^2 B(s)\frac{\partial \overline{G}}{\partial x_0}(x, s; L, 0).$$

Using the convolution theorem, we obtain

$$u(x, t) = -c^2\int_0^t b(t_0)\frac{\partial G}{\partial x_0}(x, t - t_0; L, 0)dt_0,$$

which may be replaced by the more usual expression,

$$\boxed{u(x, t) = -c^2\int_0^t b(t_0)\frac{\partial G}{\partial x_0}(x, t; L, t_0)dt_0,} \tag{13.6.16}$$

due to the time-translation invariance of the Green's function. Equation (13.6.16), obtained using Laplace transforms, is equivalent to the representation formula (11.2.24) obtained using Green's formula for the special case $q(x, t) = 0$, $a(t) = 0$, $f(x) = 0$, and $g(x) = 0$. The general case (11.2.24) may be derived in the same way.

Exercises 13.6

13.6.1. **(a)** Determine the Laplace transform of $u(x, t)$ satisfying (13.6.1)–(13.6.3).

(b) Represent $u(x, t)$ in terms of its Green's function using Laplace transforms [i.e., use (13.6.11) with (13.6.12)].

13.6.2. Determine the Green's function by inverting (13.6.11) with (13.6.12). Show that signals are appropriately reflected.

13.6.3. Determine the Laplace transform of the Green's function for the wave equation if the boundary conditions are

(a) $u(0, t) = a(t)$ and $\dfrac{\partial u}{\partial x}(L, t) = b(t)$

(b) $\dfrac{\partial u}{dx}(0, t) = a(t)$ and $\dfrac{\partial u}{\partial x}(L, t) = b(t)$

13.6.4. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x, t), \qquad \begin{matrix} x > 0 \\ t > 0 \end{matrix}$$
$$u(0, t) = h(t)$$
$$u(x, 0) = f(x).$$

***(a)** Determine the Laplace transform of the Green's function for this exercise.

(b) Determine $\overline{U}(x, s)$ if $f(x) = 0$ and $q(x, t) = 0$ (see Exercise 13.4.4).

(c) By comparing parts (a) and (b), derive a representation of $u(x, t)$ in terms of the Green's function [if $f(x) = 0$ and $q(x, t) = 0$]. Compare with (11.3.21).

(d) From part (a), determine the Green's function. [*Hint*: See Table 13.2.1 of Laplace transforms.]

13.6.5. Reconsider Exercise 13.6.4 if

(a) $h(t) = 0$ and $q(x, t) = 0$

(b) $h(t) = 0$ and $f(x) = 0$

13.6.6. Reconsider Exercise 13.6.4 if, instead, the boundary condition were

$$\frac{\partial u}{\partial x}(0, t) = h(t).$$

[Restrict attention to $f(x) = 0$ and $q(x, t) = 0$.]

13.7 Inversion of Laplace Transforms Using Contour Integrals in the Complex Plane

Laplace transforms sometimes can be inverted by using tables. However, one of the most important properties of Laplace transforms is that they can be inverted by a contour integral in the complex plane. Furthermore, we will show how to evaluate this integral using results from the theory of functions of a complex variable.

FOURIER AND LAPLACE TRANSFORMS

First, we show that the Laplace transform can be considered a special case of the Fourier transform. As a review, we introduce $g(x)$ and its Fourier transform $G(\omega)$:

$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{i\omega x}dx \tag{13.7.1}$$

$$g(x) = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega x}d\omega. \tag{13.7.2}$$

Suppose, as is usual for functions that will be Laplace transformed, we discuss functions $g(x)$ that are zero for $x < 0$:

$$g(x) = \begin{cases} 0 & x < 0 \\ 2\pi f(x)e^{-\gamma x} & x > 0. \end{cases} \tag{13.7.3}$$

The $e^{-\gamma x}$ is introduced (and γ chosen) so that $g(x)$ automatically decays sufficiently rapidly as $x \to \infty$ for certain $f(x)$. For this function, the Fourier transform pair (13.7.1)–(13.7.2) becomes

$$G(\omega) = \int_0^{\infty} f(x)e^{-(-i\omega+\gamma)x}\,dx$$

$$2\pi f(x)e^{-\gamma x} = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega x}d\omega \quad (x > 0).$$

If we introduce

$$s = \gamma - i\omega \quad (ds = -id\omega)$$

and

$$F(s) \equiv G(\omega) = G\left(\frac{s-\gamma}{-i}\right), \tag{13.7.4}$$

then using t instead of $x (x = t)$ yields

$$\boxed{F(s) = \int_0^{\infty} f(t)e^{-st}dt} \tag{13.7.5}$$

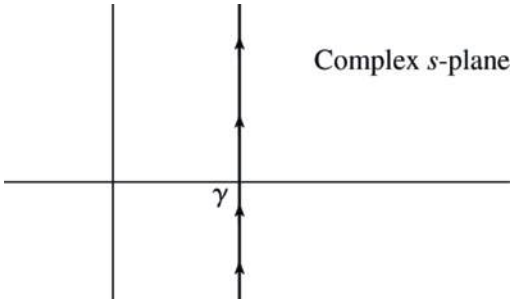
$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds} \quad (t > 0). \tag{13.7.6}$$

Equation (13.7.5) shows that $F(s)$ *is the Laplace transform of* $f(t)$. (We usually use t instead of x when discussing Laplace transforms.) $F(s)$ *is also the Fourier transform of*

$$g(t) = \begin{cases} 0 & t < 0 \\ 2\pi f(t)e^{-\gamma t} & t > 0. \end{cases}$$

More importantly, **given the Laplace transform $F(s)$, (13.7.6) shows how to compute its inverse Laplace transform**. It involves a line integral in the complex s-plane, as illustrated in Fig. 13.7.1. From the theory of complex variables, it can be shown that the line integral is to the right of all singularities of $F(s)$. Other than that, the evaluation of the integral is independent of the value of γ. All singularities are in the "left half-plane."

Figure 13.7.1 Line integral in complex s-plane for inverse Laplace transforms.



Cauchy's theorem and residues

We give only an extremely brief discussion of evaluating integrals using the theory of complex variables. The fundamental tool is **Cauchy's theorem**, which states that if $g(s)$ is analytic (no singularities) at all points inside and on a closed contour C, then the closed line integral is zero:

$$\oint_C g(s)ds = 0. \tag{13.7.7}$$

Closed line integrals are nonzero only due to singularities of $g(s)$. The **residue theorem** states that the closed line integral (counterclockwise) can be evaluated in terms of contributions (called **residues**) of the singularities s_n *inside the contour* (if there are no branch points, which usually are square-root or logarithmic-type singularities):

$$\oint_C g(s)ds = 2\pi i \sum_n \text{res}(s_n). \tag{13.7.8}$$

The evaluation of residues is often straightforward. If $g(s) = R(s)/Q(s)$ *has simple poles at simple zeros* s_n of $Q(s)$ inside the contour, then in complex variables it is shown that

$$\text{res}\,(s_n) = \frac{R(s_n)}{Q'(s_n)}, \tag{13.7.9}$$

and thus

$$\oint_C g(s)ds = 2\pi i \sum_n \frac{R(s_n)}{Q'(s_n)}. \tag{13.7.10}$$

Inversion integral

The inversion integral for Laplace transforms is not a closed line integral but instead an infinite straight line (with constant real part γ) to the right of all singularities. In order to utilize a finite closed line integral, we can consider either of the two large semicircles illustrated in Fig. 13.7.2. We will allow the radius to approach infinity so that the straight part of the closed contour approaches the desired infinite straight line. We want the line integral along the arc of the circle to vanish as the radius approaches infinity. The integrand $F(s)e^{st}$ in the inversion integral (13.7.6) must be sufficiently small. Since $F(s) \to 0$ as $s \to \infty$ [see Section 13.2 or (13.7.5)], we will need e^{st} to vanish as the radius approaches infinity. If $t < 0$, e^{st} exponentially decays as the radius increases only on the right-facing semicircle (real part of $s > 0$). Thus, if $t < 0$, we "close the contour" to the right. Since there are no singularities to the right and the contribution of the large semicircle vanishes, we conclude that

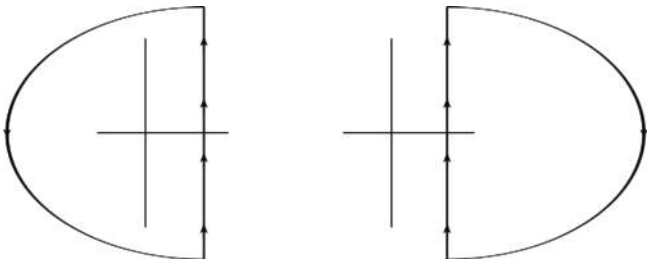
$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds = 0$$

if $t < 0$; when we use Laplace transforms, we insist that $f(t) = 0$ for $t < 0$. Of more immediate importance is our analysis of the inversion integral (13.7.6) for $t > 0$. If $t > 0$, e^{st} exponentially decays in the left half-plane (real part of $s < 0$). Thus, if $t > 0$, we close the contour to the left. There is a contribution to the integral from all the singularities. For $t > 0$, the inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds = \frac{1}{2\pi i}\oint F(s)e^{st}\,ds = \sum_n \text{res}\,(s_n). \tag{13.7.11}$$

This is valid if $F(s)$ has no branch points. The summation includes all singularities (since the path is to the right of all singularities).

Figure 13.7.2 Closing the line integrals.



Simple poles

If $F(s) = p(s)/q(s)$ [so that $g(s) = p(s)e^{st}/q(s)$] and if all the singularities of the Laplace transform $F(s)$ are simple poles [at simple zeros of $q(s)$], then $\text{res}(s_n) = p(s_n)e^{s_n t}/q'(s_n)$, and thus,

$$f(t) = \sum_n \frac{p(s_n)}{q'(s_n)}e^{s_n t}. \tag{13.7.12}$$

Equation (13.7.12) is the same result we derived earlier by partial fractions if $F(s)$ is a rational function [i.e., if $p(s)$ and $q(s)$ are polynomials].

EXAMPLE OF THE CALCULATION OF THE INVERSE LAPLACE TRANSFORM

Consider $F(s) = (s^2 + 2s + 4)/[s(s^2 + 1)]$. The inverse transform yields

$$f(t) = \sum_n \text{res}(s_n).$$

The poles of $F(s)$ are the zeros of $s(s^2 + 1)$, namely, $s = 0, \pm i$. The residues at these simple poles are

$$\text{res}(0) = 4e^{0t} = 4$$
$$\text{res}(s_n = \pm i) = \frac{s_n^2 + 2s_n + 4}{3s_n^2 + 1}e^{s_n t} = \frac{3 + 2s_n}{-2}e^{s_n t}.$$

Thus,

$$f(t) = 4 + \left(-\frac{3}{2} - i\right)e^{it} + \left(-\frac{3}{2} + i\right)e^{-it} = 4 - \frac{3}{2} \cdot 2\cos t - i \cdot 2i\sin t$$
$$= 4 - 3\cos t + 2\sin t,$$

using Euler's formulas. In the next section, we apply these ideas to solve for the inverse Laplace transform that arises in a partial differential equation's problem. ■

Exercises 13.7

13.7.1. Use the inverse theory for Laplace transforms to determine $f(t)$ if $F(s) =$:

(a) $1/(s - a)$

***(b)** $1/(s^2 + 9)$

(c) $(s + 3)/(s^2 + 16)$

13.7.2. The residue b_{-1} at a singularity s_0 of $f(s)$ is the coefficient of $1/(s - s_0)$ in an expansion of $f(s)$ valid near $s = s_0$. In general,

$$f(s) = \sum_{m=-\infty}^{\infty} b_m (s - s_0)^m,$$

called a **Laurent series** or **expansion**.

(a) For a simple pole, the most negative power is $m = -1$ ($b_m = 0$ for $m < -1$). In this case, show that

$$\text{res}(s_0) = \lim_{s \to s_0} (s - s_0) f(s).$$

(b) If s_0 is a simple pole and $f(s) = R(s)/Q(s)$ [with $Q(s_0) = 0$, $R(s_0) \neq 0$, and $dQ/ds(s_0) \neq 0$], show that

$$\text{res}(s_0) = \frac{R(s_0)}{Q'(s_0)},$$

assuming that both $R(s)$ and $Q(s)$ have Taylor series around s_0.

(c) For an Mth-order pole, the most negative power is $m = -M$ ($b_m = 0$ for $m < -M$). In this case, show that

$$\text{res}(s_0) = \frac{1}{(M-1)!} \frac{d^{M-1}}{ds^{M-1}} \left[(s - s_0)^M f(s) \right] \Big|_{s=s_0}. \tag{13.7.13}$$

(d) In part (c), show that the M appearing in (13.7.13) may be replaced by any integer greater than M.

13.7.3. Using Exercise 13.7.2, determine $f(t)$ if $F(s) =$

(a) $1/s^3$

(b) $1/(s^2 + 4)^2$

13.7.4. If $|F(s)| < \alpha/r^2$ for large $r \equiv |s|$, prove that

$$\int_{C_R} F(s) e^{st} ds \to 0 \quad \text{as} \quad r \to \infty \text{ for } t > 0,$$

where C_R is any arc of a circle in the left half-plane ($\text{Re } s \leq 0$). [If $|F(s)| < \alpha/r$ instead, it is more difficult to prove the same conclusion. The latter case is equivalent to **Jordan's lemma** in complex variables.]

13.8 Solving the Wave Equation Using Laplace Transforms (with Complex Variables)

In Section 13.6, we showed that

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \tag{13.8.1}$$

$$\text{BC:} \quad \boxed{\begin{aligned} u(0, t) &= 0 \\ u(L, t) &= b(t) \end{aligned}} \tag{13.8.2}$$

$$\text{IC:} \quad \boxed{\begin{aligned} u(x, 0) &= 0 \\ \frac{\partial u}{\partial t}(x, 0) &= 0 \end{aligned}} \tag{13.8.3}$$

could be analyzed by introducing $\overline{U}(x, s)$, the Laplace transform in t of $u(x, t)$. We obtained $u(x, t)$,

$$u(x, t) = -c^2 \int_0^t b(t_0) \frac{\partial G}{\partial x_0}(x, t; L, t_0) dt_0, \tag{13.8.4}$$

in terms of the Green's function. The Laplace transform of this influence function was known [see (13.6.15)]:

$$\frac{\partial \overline{G}}{\partial x_0}(x, s; L, 0) = F(s) = -\frac{\sinh(x/c)s}{c^2 \sinh(L/c)s}. \tag{13.8.5}$$

In this section, we use the complex inversion integral for the Laplace transform:

$$\frac{\partial G}{\partial x_0}(x, t; L, 0) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} -\frac{\sinh(x/c)s}{c^2 \sinh(L/c)s} e^{st}\, ds. \tag{13.8.6}$$

The singularities of $F(s)$ only are simple poles s_n, located at the zeros of the denominator:

$$\sinh \frac{L}{c} s_n = 0. \tag{13.8.7}$$

However, $s = 0$ is not a pole since, near $s = 0$, $F(s) \approx -(x/c)s/[c^2(L/c)s] \not\approx \infty$. There is an infinite number of these poles located on the imaginary axis:

$$\frac{L}{c} s_n = in\pi, \quad n = \pm 1, \pm 2, \pm 3, \ldots. \tag{13.8.8}$$

The location of the poles, $s_n = ic(n\pi/L)$, corresponds to the eigenvalues, $\lambda = (n\pi/L)^2$. This follows from the singularity property of Laplace transforms (see Section 13.2). The singularity of the Laplace transform $[s = ic(n\pi/L)]$ corresponds to a complex exponential solution $[e^{ic(n\pi/L)t}]$.

The residue at each pole may be evaluated:

$$\text{res}(s_n) = \frac{R(s_n)}{Q'(s_n)} = \frac{-\sinh(x/c)s_n e^{s_n t}}{cL\cosh(L/c)s_n} = \frac{-i\sin(n\pi x/L)e^{i(n\pi ct/L)}}{cL\cos n\pi},$$

since $\sinh ix = i\sin x$ and $\cosh ix = \cos x$. Thus, the influence function for this problem is

$$\begin{aligned}\frac{\partial}{\partial x_0} G(x, t; L, 0) &= \sum_{n=-\infty}^{\infty} \frac{-i\sin(n\pi x/L)e^{i(n\pi ct/L)}}{cL\cos n\pi} \\ &\quad (n \neq 0) \\ &= \frac{2}{cL}\sum_{n=1}^{\infty}(-1)^n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L},\end{aligned} \tag{13.8.9}$$

where the positive and negative n contributions have been combined into one term. Finally, using (13.8.4), we obtain

$$u(x, t) = \sum_{n=1}^{\infty} A_n(t)\sin\frac{n\pi x}{L}, \text{ where } A_n = -(-1)^n \frac{2c}{L}\int_0^t b(t_0)\sin\frac{n\pi c}{L}(t - t_0)\, dt_0,$$

the same result as would be obtained by the method of eigenfunction expansion.

The influence function is an infinite series of the eigenfunctions. For homogeneous problems with homogeneous boundary conditions, inverting the Laplace transform using an infinite sequence of poles also will yield a series of eigenfunctions, the same result as obtained by separation of variables. In fact, it is the Laplace transform method that is often used to *prove* the validity of the method of separation of variables.

Exercises 13.8

***13.8.1.** Solve for $u(x, t)$ using Laplace transforms:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = 0, \qquad u(0, t) = 0, \qquad u(L, t) = 0.$$

Invert the Laplace transform of $u(x, t)$ using the residue theorem for contour integrals in the s-plane. Show that this yields the same result as derivable by separation of variables.

13.8.2. Modify Exercise 13.8.1 if instead

(a) $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = g(x)$

(b) $u(0, t) = 0$ and $\frac{\partial u}{\partial x}(L, t) = 0$

(c) $\frac{\partial u}{\partial x}(0, t) = 0$ and $\frac{\partial u}{\partial x}(L, t) = 0$

13.8.3. Solve for $u(x, t)$ using Laplace transforms:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to $u(x, 0) = f(x)$, $u(0, t) = 0$, and $u(L, t) = 0$.

Invert the Laplace transform of $u(x, t)$ using the residue theorem for contour integrals in the complex s-plane. By what other method can this representation of the solution be obtained? (Compare to Exercise 13.5.6.)

13.8.4. Consider

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \sin\sigma_0 t \\
&u(x, 0) = 0, \quad u(0, t) = 0 \\
&\frac{\partial u}{\partial t}(x, 0) = 0, \quad u(L, t) = 0.
\end{aligned}$$

(a) Solve using Laplace transforms (with contour inversion) if $\sigma_0 \neq c(m\pi/L)$.

(b) Solve if $\sigma_0 = c(3\pi/L)$. Show that resonance occurs (see Section 8.5).

14

Dispersive Waves: Slow Variations, Stability, Nonlinearity, and Perturbation Methods

14.1 Introduction

This chapter is intentionally concise, giving the instructor or reader just an overview of many more advanced and important topics in partial differential equations. In Section 14.2, we introduce the dispersion relation, a relationship between the wave number and frequency, and show the important distinction between phase velocity and group velocity. Examples of dispersive waves discussed are water waves (14.2) and, more extensively, wave guides (14.3) and fiber optics (14.4). In Section 14.5, the importance of group velocity in the propagation of linear dispersive waves is developed using the method of stationary phase. In Section 14.6, we show that dispersive waves may be more easily analyzed by allowing the amplitude, wave number, and frequency to be slowly varying functions of space and time. Slowly varying waves are shown (14.6) to be solved by the method of characteristics, which may form caustics, as we describe. By assuming that the wave number and frequency are nearly constant, in Section 14.7, we derive a wave envelope equation known as the Schrödinger equation. We also derive the linearized Korteweg–de Vries equation and show its application to the caustic of a rainbow. We analyze solitary waves for the corresponding nonlinear dispersive wave equations and briefly discuss solitons and the inverse scattering transform. Section 14.8 describes bifurcation phenomena and stability analysis for partial differential equations, including an introduction to dynamical systems for ordinary differential equations (14.8.1). We discuss a typical unstable equilibrium for a partial differential equation in Section 14.8.3, giving rise to a neutral stability curve and showing how patterns arise in partial differential equations. Such slightly unstable dispersive wave situations are analyzed in Section 14.8.5, deriving in a simple way the linearized complex Ginzburg–Landau equation. In Section 14.8.6, we describe bifurcation phenomena for partial differential equations. Nonlinear terms give rise to the complex Ginzburg–Landau equation, and we analyze the modulational instability. Section 14.8.7 introduces the long-wave instability. Pattern formation for reaction–diffusion equations and the Turing instability are presented in considerable detail in Section 14.8.8. The last two sections describe singular perturbation methods, independent from most of the rest of the text. To facilitate learning these perturbation methods, we begin by presenting a few simpler examples from ordinary differential equations. We introduce in Section 14.9 singular perturbation problems that are solved by the method of multiple scales, doing the wave equation in a slowly varying media (14.9.4) and slowly varying dispersive waves (14.9.5). Boundary layer problems are solved in Section 14.10 by the method of matched asymptotic expansions. Section 14.10.2 presents a partial differential equation example that corresponds to diffusion of a pollutant dominated by convection.

14.2 Dispersive Waves and Group Velocity

14.2.1 TRAVELING WAVES AND THE DISPERSION RELATION

Solutions for linear partial differential equations in uniform media (constant coefficients), when the boundary conditions are particularly simple, often may be obtained in a way that is equivalent to but perhaps simpler than the method of separation of variables. Since the form of the solution

is known from our experience in separating variables, we may directly substitute into the partial differential equation

	wave propagation	nonwave propagation
one dim	$e^{i(kx-\omega t)}$	$e^{ikx}e^{\sigma t}$
two dim	$e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$	$e^{i\boldsymbol{k}\cdot\boldsymbol{x}}e^{\sigma t}$.

In finite rectangular geometries with homogeneous boundary conditions such as zero-ends, zero-slope, or periodic, $k(\boldsymbol{k})$ represents one of the discrete wave numbers (vectors) permitted by the appropriate Fourier series. The wavelength is $\frac{2\pi}{k}$, and hence the wave number k is the number of periods in 2π distance. In infinite geometries (or semi-infinite geometries with appropriate homogeneous boundary condition), $k(\boldsymbol{k})$ represents a continuous wave number (vector) permitted by the appropriate Fourier transform. In these problems, ω is the temporal frequency, and σ represents the exponential time dependence of the solution of the partial differential equation. (In this chapter, we use k for wave number and ω for temporal frequency, while in the rest of the book k is the diffusivity for the heat equation and ω is the spatial wave number. We hope this does note cause confusion.)

In general, to solve partial differential equations that correspond to **wave-propagation** problems, we will find that the temporal frequency is a real function of the wave number

$$\boxed{\omega = \omega(k),} \tag{14.2.1}$$

which we call the **dispersion relation**. In multidimensional problems $\omega = \omega(\boldsymbol{k})$, meaning $\omega = \omega(k_1, k_2, k_3)$. We will do a few examples shortly. For one-dimensional wave propagation, $e^{i(kx-\omega t)} = e^{ik\left(x-\frac{\omega}{k}t\right)}$, so that this substitution represents a **traveling wave** with **wave or phase velocity** $= \frac{\omega(k)}{k}$. If the velocity really depends on the wave number, then waves of different lengths will move at different velocities. Since initial conditions are composed of many different wavelengths (via Fourier analysis), initial conditions will generally spread out or disperse. This will be shown more clearly later. Meanwhile, we say a wave-propagation problem is **dispersive** if the wave velocity depends on the wave number, and specifically if the wave velocity is not constant. To be **nondispersive**, $\frac{\omega(k)}{k} =$ constant, so that $\omega(k) =$ constant $\times\, k$. More technically, we say a one-dimensional problem is dispersive if $\frac{d\omega}{dk} \neq$ constant. In higher dimensions, a partial differential equation is dispersive if $\nabla_k \omega = \left(\frac{\partial\omega}{\partial k_1}, \frac{\partial\omega}{\partial k_2}, \frac{\partial\omega}{\partial k_3}\right) \neq$ **constant vector**.

EXAMPLE REAL SOLUTIONS

Consider the partial differential equation (considered in Chapter 12)

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}, \tag{14.2.2}$$

to be solved as an initial value problem for all x. By substituting $u = e^{i(kx-\omega t)}$, we obtain first $-i\omega = cik$, and then the dispersion relation

$$\omega = -ck.$$

The velocity $= \frac{\omega}{k} = -c$ is constant for all k, so we say that (14.2.2) is nondispersive. Each wave number moves at the same velocity $-c$. Thus, we expect an arbitrary initial condition to move as a **permanent wave** without change of shape, moving at velocity $-c$. This can be shown using the Fourier transform:

$$u(x, t) = \int_{-\infty}^{+\infty} A(k)e^{i(kx-\omega t)}\,dk = \int_{-\infty}^{+\infty} A(k)e^{ik(x+ct)}\,dk, \tag{14.2.3}$$

using (14.2.2). To relate this to the initial condition $u(x, 0) = f(x)$:

$$u(x, 0) = f(x) = \int_{-\infty}^{+\infty} A(k)e^{ikx}\,dk. \tag{14.2.4}$$

We can solve for $A(k)$, but it is not necessary. From (14.2.3) and (14.2.4), we see that

$$u(x, t) = f(x + ct). \tag{14.2.5}$$

This is shown by the method of characteristics in Chapter 12.

One solution is $Ae^{i(kx-\omega(k)t)}$, with A a constant. You may not like this solution because it is not real. To show how real solutions of this kind can be derived, we simply note that both k and $-k$ are allowable wave numbers. Thus, by superposition, an allowable solution of the differential equation is $Ae^{i(kx-\omega(k)t)} + Ae^{i(-kx-\omega(-k)t)} = A(e^{i(kx-\omega(k)t)} + e^{-i(kx-\omega(k)t)}) = 2A\cos(kx - \omega(k)t)$, since $\omega(k)$ is an odd function of k. In many (but not all) problems it is possible to create real solutions in this way. Thus, most physical scientists use the notation $Ae^{i(kx-\omega t)}$ to represent a real wave. Be careful if the partial differential equation itself is nonlinear or not real. ■

EXAMPLE LINEARIZED KORTEWEG–DE VRIES EQUATION

This linear partial differential equation arises in many physical situations, especially as an approximation for long waves (waves whose wavelengths are long compared with other length scales in a physical problem) when more general dispersion exists:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \beta\frac{\partial^3 u}{\partial x^3}.$$

We will discuss its derivation later in this chapter. The traveling wave assumption, $u = e^{i(kx-\omega t)}$, yields $-i\omega + c(ik) = \beta(ik)^3$, from which we obtain the dispersion relation:

$$\omega = ck + \beta k^3. \tag{14.2.6}$$

The phase velocity is not a constant but depends on the wave number: $\frac{\omega}{k} = c + \beta k^2$. Here, $\frac{d\omega}{dk} \neq$ constant, so we have our first example of a dispersive wave. Using a Fourier transform,

$$u(x, t) = \int_{-\infty}^{+\infty} A(k)e^{i(kx-\omega t)}\,dk = \int_{-\infty}^{+\infty} A(k)e^{i(k(x-ct)-\beta k^3 t)}\,dk.$$

We cannot learn much directly from this exact solution, which is typical of dispersive wave problems. Later in this chapter, we show a method for analyzing dispersive wave problems based on approximating the exact solution for long distances and large times. ■

Operator result for dispersion relations

For an elementary traveling wave solution of a linear dispersive wave, $u(x, t) = Ae^{i(kx-\omega t)}$, where A represents a constant amplitude. Note that

$$\frac{\partial u}{\partial t} = -i\omega u$$
$$\frac{\partial u}{\partial x} = iku.$$

Thus, it is often useful to note that wave number and frequency are equivalent to the appropriate multiplication of the spatial and temporal differential operators:

$$\boxed{\frac{\partial}{\partial t} \Rightarrow -i\omega(k) \text{ or } \omega(k) \Rightarrow i\frac{\partial}{\partial t}} \tag{14.2.7}$$

$$\boxed{\frac{\partial}{\partial x} \Rightarrow ik \text{ or } k \Rightarrow -i\frac{\partial}{\partial x}.} \tag{14.2.8}$$

We will find these observations useful when we discuss wave envelope equations.

14.2.2 GROUP VELOCITY I

Linear dispersive waves $e^{i(kx-\omega t)}$ are characterized by the frequency depending on the wave number so that the phase velocity $\frac{\omega(k)}{k}$ depends on the wave number. Waves of different lengths have different phase velocities. However, it turns out that for dispersive waves, there is a more important velocity, the group velocity, which is important because energy moves at this velocity (not the phase velocity). This is not particularly easy to show, so we begin by discussing a simple example. Instead of considering a continuum of waves of all wave numbers (using Fourier transforms), we consider a solution that consists of just two waves with nearby wave numbers, k and $k + \Delta k$,

where Δk is very small. For convenience we choose waves of the form $\cos(kx - \omega(k)t)$, and for simplicity we assume that both waves have the same constant amplitude A:

$$u(x, t) = A\cos(kx - \omega(k)t) + A\cos[(k + \Delta k)x - \omega(k + \Delta k)t].$$

Using the trigonometric addition formula for cosines [$\cos\theta + \cos\phi = 2\cos\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta - \phi)$], we obtain

$$u(x, t) = 2A\cos\left[\left(k + \frac{\Delta k}{2}\right)x - \frac{\omega(k) + \omega(k + \Delta k)}{2}t\right]\cos\left[\frac{\Delta k}{2}x - \frac{\omega(k + \Delta k) - \omega(k)}{2}t\right].$$

The sum is a product of two trigonometric functions whose spatial behavior differs. One wave number is $k + \frac{\Delta k}{2}$, nearly the same as the original two wave numbers, while the other has a very small wave number, $\frac{\Delta k}{2}$, corresponding to a very large wavelength. The solution has two vastly different length scales. To graph the solution (at fixed t), we first graph the very long sinusoidal wave. Periodically (with a relatively short wavelength) the solution lies on this long wave and on its negative, as graphed in Fig. 14.2.1. The rapidly oscillating waves have their amplitude slowly changing in space. The two waves alternatively interfere constructively or destructively. The long wave acts as a **wave envelope** of the short waves. Between each zero of the wave envelope, there appears a **group** of short waves. The short waves are nearly the individual dispersive waves with one wave number $k + \frac{\Delta k}{2}$ and a phase velocity nearly the phase velocity of an individual dispersive wave:

$$\frac{\omega(k + \frac{\Delta k}{2})}{k + \frac{\Delta k}{2}} \approx \frac{\omega(k) + \omega(k + \Delta k)}{2(k + \frac{\Delta k}{2})} \approx \frac{\omega(k)}{k}.$$

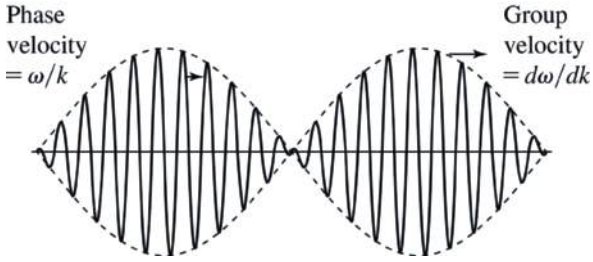
MATLAB Files

Download Animation of Figure 14.2.1 (.zip)

https://bit.ly/3UBndtV

These files are provided as zipped .m files for use in MATLAB on desktop platform.

Figure 14.2.1 Group and phase velocity.



However, the wave envelope or the group of waves moves with an entirely different velocity, $\frac{\omega(k+\Delta k)-\omega(k)}{\Delta k}$. In the limit as $\Delta k \to 0$, this is called the **group velocity**:

$$\boxed{\text{group velocity} \equiv \frac{d\omega}{dk}.} \tag{14.2.9}$$

In some sense, to be discussed later in more detail, the amplitude of a group of dispersive waves moves with the group velocity. Energy moves with the group velocity. The individual waves appear to move through the wave envelope since the phase velocity is usually different from the group velocity.

EXAMPLE

For the linearized KdV equation, the dispersion relation is (14.2.6), $\omega = ck + \beta k^3$. In this case

$$\text{phase velocity} = \frac{\omega}{k} = c + \beta k^2$$

$$\text{group velocity} = \frac{d\omega}{dk} = c + 3\beta k^2.$$

It is even possible that the group velocity is in the opposite direction from the phase velocity. This could occur, for example, if $c + 3\beta k^2 > 0$, while $c + \beta k^2 < 0$. ■

Water waves

For surface water waves, it is known that the dispersion relation is $\omega^2 = gk \tanh kh$, where $g = 9.8\ \text{m/s}^2$ is the gravitational constant and h is the constant depth of the water. For deep water waves, $kh \gg 1$, so that $\tanh kh \simeq 1$, the dispersion relation can be approximated by $\omega^2 = gk$ or $\omega = \pm\sqrt{gk}$. For deep water waves, $\frac{\omega}{k} = \pm\sqrt{\frac{g}{k}}$, while the group velocity is $\frac{d\omega}{dk} = \pm\frac{1}{2}\sqrt{\frac{g}{k}}$. For deep water waves, the group velocity is half the phase velocity. Most of the waves at a beach are generated from storms far away. Most of these waves are deep water waves, since the typical depth of the ocean is 5 km. The group velocity $= \frac{1}{2}\sqrt{\frac{gL}{2\pi}}$, since the wave number can be related to the wave length $L = \frac{2\pi}{k}$. The longer waves have larger group velocities. If a localized storm generates waves of all wavelengths, the longer waves move faster and will arrive at a distant shoreline sooner. Thus, the changing frequencies at the shoreline are due to these different arrival times. (The frequency observed at a beach is meaningful since the frequency of a wave stays the same as the depth changes slowly.) Thus, at the beach the longer waves with smaller frequencies (longer periods) will be observed first and will be a precursor to the other waves (and perhaps the storm system itself coming ashore). The interested reader is referred to Kinsman (1984).

Exercises 14.2

14.2.1. For the following one-dimensional partial differential equations, find the dispersion relation:

(a) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

(b) $\dfrac{\partial^2 u}{\partial t^2} = -\gamma^2 \dfrac{\partial^4 u}{\partial x^4}$

***(c)** $i\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

(d) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} - u$

14.2.2. For the following two-dimensional partial differential equations, find the dispersion relation:

(a) $\dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

(b) $\dfrac{\partial^2 u}{\partial t^2} = c_1^2 \dfrac{\partial^2 u}{\partial x^2} + c_2^2 \dfrac{\partial^2 u}{\partial y^2}$

14.2.3. Show that any linear partial differential equation with a one-mode dispersion relation $\omega = \omega(k)$ will have real solutions if the dispersion relation is odd, $\omega(-k) = -\omega(k)$.

14.2.4. Show that any linear partial differential equation (with higher spatial dimensions) with a one-mode dispersion relation $\omega = \omega(k)$ will have real solutions if the dispersion relation is odd, $\omega(-k) = -\omega(k)$.

***14.2.5.** Water waves satisfy $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ for $y \le 0$, where ϕ is a velocity potential such that the fluid velocity can be found from $u = \nabla\phi$. The boundary condition at a flat bottom $y = -h$ is $\frac{\partial \phi}{\partial y} = 0$ there. For water waves of very small amplitude (still of great physical interest), the boundary condition at the unknown free surface $y = s(x, t)$ can be approximated by the two conditions $\frac{\partial \phi}{\partial t} + gs = 0$ and $\frac{\partial s}{\partial t} = \frac{\partial \phi}{\partial y}$, applied at $y = 0$, where g is the gravitational constant. Find the dispersion relation by assuming $\phi = A(y)e^{i(kx-\omega t)}$ and $s = Be^{i(kx-\omega t)}$.

14.2.6. Determine the dispersion relation for water waves with surface tension by replacing $\frac{\partial \phi}{\partial t} + gs = 0$ by $\frac{\partial \phi}{\partial t} + gs - \frac{\gamma}{\rho}\frac{\partial^2 s}{\partial x^2} = 0$, where γ is the coefficient of surface tension and ρ is the constant mass density of water [Whitham, 1999].

***14.2.7.** Determine the dispersion relation for deep water waves by solving Exercise 14.2.5 with the condition $\frac{\partial \phi}{\partial y} \to 0$ as $y \to -\infty$ instead of $\frac{\partial \phi}{\partial y} = 0$ at $y = -h$.

14.2.8. Derive the dispersion relation for an internal wave assuming a two-fluid model with density ρ_1 for $y \ge 0$ and density ρ_2 for $y \le 0$. In Exercise 14.2.5 one of the boundary conditions corresponding to zero pressure is now replaced by the pressure being continuous: $\rho_1\left(\frac{\partial \phi}{\partial t} + gs\right) = \rho_2\left(\frac{\partial \phi}{\partial t} + gs\right)$ at $y = 0$. The other condition at $y = 0$ holds for both fluids. Assume $y \to \pm\infty$.

14.2.9. Compare phase and group velocities for

(a) $\dfrac{\partial u}{\partial t} = \beta \dfrac{\partial^3 u}{\partial x^3}$

(b) $i\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

(c) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - u$. Show that the phase velocity is greater than the sound speed c, but the group velocity is less than the sound speed c.

14.2.10. Determine the group velocity for water waves satisfying $\omega^2 = gk \tanh kh$.

14.2.11. Tsunamis (water waves generated by earthquakes) are long waves with $kh \ll 1$. Show that long waves satisfy $\omega = \pm k\sqrt{gh}$. Approximate the phase (and group velocity) for tsunamis assuming the ocean is 5 km deep.

14.3 Wave Guides

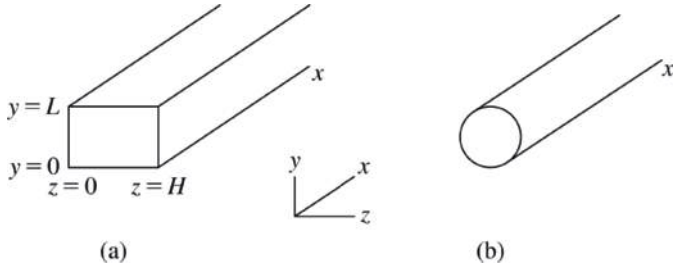
In wave propagation problems (where physical phenomena are described by the wave equation or other partial differential equations), local disturbances decay rapidly in three-dimensional space. In order to communicate efficiently in a three-dimensional world, energy must be confined to two or one dimension. Wave guides are introduced for this purpose for electromagnetic (light) or acoustic (sound) waves. Typical wave guides are long, hollow tubes with a circular cross section that (as we will show) can be designed to permit the propagation of electromagnetic or acoustic waves in one dimension. Acoustic waves solve the three-dimensional wave equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).} \tag{14.3.1}$$

Electromagnetic waves satisfy a system of wave equations, more complicated because the different components of the electric and magnetic fields are coupled through various boundary conditions. However, the mathematical procedures to analyze these electromagnetic waves are the same as for acoustic waves so that to simplify the presentation we will restrict our attention to the three-dimensional wave equation (14.3.1).

There are a few new ideas in analyzing wave guides, so we wish to make our mathematical analysis as easy to follow as possible. Therefore, we will analyze a wave guide with a rectangular cross section rather than the more realistic circular cross section (see Fig. 14.3.1). The analysis of rectangular wave guides $(0 < y < L,\ 0 < z < H)$ uses the trigonometric functions of a Fourier series, while circular wave guides require Bessel functions. We leave circular wave guides to the Exercises, but we assure the reader that the circular problem presents no new problems once the rectangular wave guide is understood.

Figure 14.3.1 (a) Rectangular and (b) circular wave guides.



For electromagnetic waves, different boundary conditions apply for different phenomena. We choose to study the boundary condition that $u = 0$ on the boundary of the wave guide. Solutions of the partial differential equation (14.3.1) will be in the form

$$\boxed{u = e^{i(kx-\omega t)} \sin\frac{n\pi y}{L} \sin\frac{m\pi z}{H},} \quad n = 1, 2, 3, \ldots \quad \text{and} \quad m = 1, 2, 3, \ldots, \tag{14.3.2}$$

corresponding to Fourier sine series in y and z due to the boundary conditions. The traveling wave $e^{i(kx-\omega t)}$ in the x-direction corresponds to a Fourier transform in x. Solutions to the initial value problem for (14.3.1) are obtained by summing over n and m and integrating over k. Substituting (14.3.2) into (14.3.1) yields the all-natural frequencies of vibration ω for the wave guide, the dispersion relation

$$\omega^2 = c^2\left[k^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right], \; n = 1, 2, 3, \ldots \quad \text{and} \quad m = 1, 2, 3, \ldots \tag{14.3.3}$$

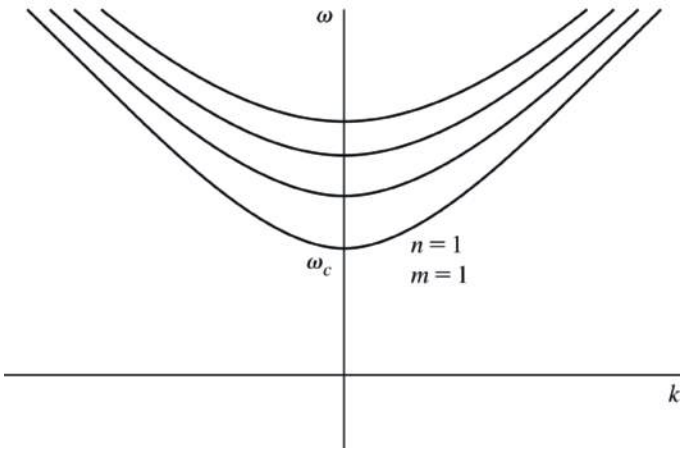
This also follows by separation of variables. It is important to distinguish between the continuous nature of k and the discrete nature of n and m. Each solution (14.3.2) is called a **mode** of the wave guide. There is a doubly infinite set of modes, $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$. Each mode (fixed m and n) is dispersive. The group velocity can be obtained by differentiating (14.3.3) with respect to the wave number k, $2\omega\frac{d\omega}{dk} = c^2 2k$, so that

$$\frac{d\omega}{dk} = c^2\frac{k}{\omega}.$$

For wave guides, the group velocity is in the same direction as the phase velocity $\frac{\omega}{k}$. For wave guides, waves whose phase travels to the left have energy that travels to the left. (For other partial differential equations, the phase and the energy do *not* have to travel in the same direction.) In Fig. 14.3.2 we graph the frequency as a function of the wave number for the first few modes. There is a lowest natural frequency, which is called a *cut-off frequency*, for reasons to be described. It occurs for $n = 1$, $m = 1$ with $k = 0$ (infinitely long waves):

$$\omega_c = c\sqrt{\left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{H}\right)^2}. \tag{14.3.4}$$

Figure 14.3.2 Dispersion relation for various modes of a wave guide.



The propagation of waves in wave guides occurs by forcing the wave guide in some way with a forcing frequency ω_f. Roughly speaking, resonance is involved. We will show that if the forcing frequency is greater than the cut-off frequency $(\omega_f > \omega_c)$, then some kind of resonance is possible in which a wave propagates along the wave guide with constant amplitude. Given a specific forcing frequency $\omega = \omega_f$, the dispersion relation (14.3.3) determines the wave number k_f (for each n and m) that will propagate:

$$k_f \equiv \pm\sqrt{\frac{\omega_f^2}{c^2} - \left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]}.$$

The number of waves that can propagate depends on ω_f, and from Fig. 14.3.2 the larger ω_f, the more modes that can propagate. We will show that if the forcing frequency is less than the cut-off frequency $(\omega_f < \omega_c)$, then the response of the system is much smaller.

14.3.1 RESPONSE TO CONCENTRATED PERIODIC SOURCES WITH FREQUENCY ω_f

To understand how a wave guide responds to periodic forcing, we consider a periodic source concentrated inside the wave guide at $x = x_0$, $y = y_0$, $z = z_0$ with forcing frequency ω_f:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + e^{-i\omega_f t}\delta(x - x_0)\delta(y - y_0)\delta(z - z_0).} \tag{14.3.5}$$

We assume the same boundary conditions $u = 0$ on the boundary. Because of the boundary conditions, we seek a solution as a double Fourier sine series in y and z:

$$u(x, y, z, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}(x, t)\sin\frac{n\pi y}{L}\sin\frac{m\pi z}{H}. \tag{14.3.6}$$

The equation that the amplitudes A_{nm} satisfy (dropping subscripts for convenience) is

$$\begin{aligned}\frac{\partial^2 A}{\partial t^2} = c^2\frac{\partial^2 A}{\partial x^2} - c^2\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]A \\ + \frac{4}{LH}e^{-i\omega_f t}\delta(x - x_0)\sin\frac{n\pi y_0}{L}\sin\frac{m\pi z_0}{H}.\end{aligned} \tag{14.3.7}$$

The amplitude of each mode satisfies (14.3.7), a one-dimensional wave equation (in the propagation direction x of the wave guide) with an extra restoring force and a concentrated periodic source. Since (14.3.7) has simple periodic forcing, we can find a particular solution of (14.3.7) with the same periodic forcing:

$$A(x, t) = G(x, x_0)e^{-i\omega_f t}, \tag{14.3.8}$$

where the one-dimensional Green's function $G(x, x_0)$ for each mode (fixed n and m) along the wave guide satisfies

$$c^2\frac{d^2G}{dx^2} + \left\{\omega_f^2 - c^2\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]\right\}G = \frac{4}{LH}\delta(x - x_0)\sin\frac{n\pi y_0}{L}\sin\frac{m\pi z_0}{H}. \tag{14.3.9}$$

14.3.2 GREEN'S FUNCTION IF MODE PROPAGATES

Since the right-hand side of (14.3.9) is a Dirac delta function in space, the differential equation is solved by using homogeneous solutions for $x < x_0$ and for $x > x_0$ and applying the jump condition as described in Chapter 8. There are two cases for the homogeneous solutions, depending on whether the forcing frequency (ω_f) is greater than or less than the natural frequency of the (m, n)th mode.

If the forcing frequency is greater than the natural frequency of the particular mode (n and m fixed), then homogeneous solutions are $\sin k_f x$ and $\cos k_f x$, where k_f is precisely the wave number that would occur using the dispersion relation (14.3.3) if the frequency was prescribed to be ω_f: $k_f \equiv \sqrt{\frac{\omega_f^2}{c^2} - \left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]}$. However, the differential equation in x (14.3.9) does not have any boundary conditions in x. Even if we insist the solution is bounded in x, all of these solutions are bounded in x. There is no easy, mathematically precise way to derive a unique Green's function. Instead we apply a **radiation condition** (which follows from more advanced mathematics). The analysis is clearer if we use the equivalent homogeneous solutions $e^{\pm ik_f(x-x_0)}$. According to (14.3.8), to obtain the amplitude of the(m, n)th mode, we must multiply by $e^{-i\omega_f t}$. Thus, the homogeneous solutions for the (m, n)th mode correspond to $e^{\pm ik_f(x-x_0)-i\omega_f t}$. These are seen to be one-dimensional waves whose energy is propagating to the left and right. Since we have a source at $x = x_0$, the **radiation condition** asserts that we have waves with energy moving to the right for $x > x_0$ and waves with energy moving to the left for $x < x_0$. In this way [if ω_f is greater than the natural frequency of the (m, n)th mode], we derive that the Green's function

$$G(x, x_0) = \frac{4}{2ik_f LHc^2}\sin\frac{n\pi y_0}{L}\sin\frac{m\pi z_0}{H}\begin{cases} e^{ik_f(x-x_0)} & \text{for } x > x_0 \\ e^{-ik_f(x-x_0)} & \text{for } x < x_0, \end{cases}$$

where we have also applied the jump condition following from (14.3.9) that

$$\left.\frac{dG}{dx}\right|_{x=x_0-}^{x=x_0+} = \frac{4}{LHc^2}\sin\frac{n\pi y_0}{L}\sin\frac{m\pi z_0}{H}.$$

Thus, the amplitude of the (m, n)th mode is quite simple and corresponds to a wave with energy propagating outward in the wave guide (due to the concentrated source) if the forcing frequency is greater than the natural frequency of that mode. Since ω_f is given, the wave number is discrete. From (14.3.6), the solution in the wave guide consists of a *sum* of all electromagnetic or acoustic waves with energy traveling outward with wave numbers k_f corresponding to a given forcing frequency ω_f. These solutions are traveling waves; they do not decay in space or time as they travel away from the source.

14.3.3 GREEN'S FUNCTION IF MODE DOES NOT PROPAGATE

We continue to consider part of the response that generates the (m, n)th mode with structure $\sin\frac{n\pi y}{L}\sin\frac{m\pi z}{H}$ transverse to the wave guide. The amplitude of this mode due to the concentrated periodic source satisfies (14.3.9). If the forcing frequency is less than the natural frequency of the (m, n)th mode, then the homogeneous solutions of (14.3.9) are not sinusoidal but are growing and decaying exponentials $e^{\pm\beta_f x}$, where $\beta_f = \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 - \frac{\omega_f^2}{c^2}} > 0$. We insist our solution is bounded for all x. Thus, the solution must exponentially decay both for $x > x_0$ and for $x < x_0$. Thus, the Green's function (14.3.9) is given by

$$G(x, x_0) = \frac{4}{-2\beta_f LHc^2}\sin\frac{n\pi y_0}{L}\sin\frac{m\pi z_0}{H}\begin{cases} e^{-\beta_f(x-x_0)} \text{ for } x > x_0 \\ e^{\beta_f(x-x_0)} \text{ for } x < x_0. \end{cases}$$

For example, for $x > x_0$, we have a simple exact elementary solution of the three-dimensional wave equation, $u = Be^{-\beta_f x}\sin\frac{n\pi y}{L}\sin\frac{m\pi z}{H}e^{-i\omega_f t}$, called an **evanescent** wave.

14.3.4 DESIGN CONSIDERATIONS

Often it is desirable to design a wave guide such that only one wave propagates at a given frequency. Otherwise the signal is more complicated, composed of two or more waves with different wavelengths but the same frequency. This can be accomplished easily only for the mode with the lowest frequency (here $n = 1, m = 1$). We design the wave guide such that the desired frequency is slightly greater than the cut-off frequency. For a square $(L = H)$ wave guide, satisfying the three-dimensional wave equation, $\omega_c = \frac{c\pi}{L}\sqrt{2} < \omega_f$. The next-lowest frequency corresponds to $n = 1$ and $m = 2$ (or $m = 1$ and $n = 2$): $\omega_2 = \frac{c\pi}{L}\sqrt{5}$. To guarantee that only one wave of frequency ω_f propagates, we insist that

$$\frac{c\pi}{L}\sqrt{2} < \omega_f < \frac{c\pi}{L}\sqrt{5}.$$

If we assume that the material through which the electromagnetic or acoustic wave travels is fixed, then we would know c. The lengths of the sides would satisfy

$$\frac{c\pi}{\omega_f}\sqrt{2} < L < \frac{c\pi}{\omega_f}\sqrt{5}.$$

Exercises 14.3

14.3.1. Consider a two-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ with $u = 0$ at $y = 0$ and $y = L$.

(a) Determine the dispersion relation.

(b) Determine a cut-off frequency.

(c) For what forcing frequencies will only one mode propagate?

14.3.2. Redo Exercise 14.3.1 with the boundary condition $\frac{\partial u}{\partial y} = 0$ at $y = 0$ and $y = L$. (Do not call $n = 0$ a propagating wave.)

14.3.3. Consider the three-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ inside a circular conductor $(y^2 + z^2 = a^2)$ with $u = 0$ at $r = a$. Only consider circularly symmetric solutions where $u = u(r, x, t)$ propagating in the x-direction.

(a) Determine the dispersion relation.

(b) Determine a cut-off frequency.

(c) For what forcing frequencies will only one mode propagate?

14.3.4. Redo Exercise 14.3.3, but do not make the assumption that the solution is circularly symmetric.

14.3.5. Consider the three-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ inside a rectangular conductor with $u = 0$ at $y = 0$ and $y = L$ and $\frac{\partial u}{\partial z} = 0$ at $z = 0$ and $z = H$. Answer the same questions as for Exercise 14.3.1.

14.3.6. Determine the Green's function corresponding to a propagating mode for

(a) Exercise 14.3.1

(b) Exercise 14.3.2

(c) Exercise 14.3.4

(d) Exercise 14.3.5

14.3.7. Determine the Green's function when a mode is not propagating for

(a) Exercise 14.3.1

(b) Exercise 14.3.2

(c) Exercise 14.3.4

(d) Exercise 14.3.5

14.3.8. Show that allowable traveling wave modes that can propagate in a rectangular wave guide of this section have zero amplitude if the concentrated source is located at a node of that mode.

14.3.9. Using the radiation condition (involving group velocity), determine the Green's function for the following partial differential equations:

***(a)** $\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \delta(x)e^{-i\omega_f t}$

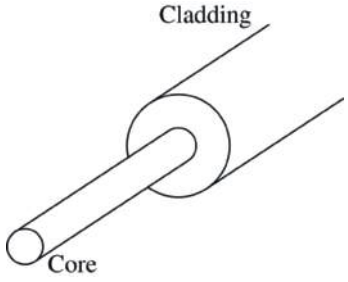
(b) $\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + \delta(x)e^{-i\omega_f t}$

(c) $i\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \delta(x)e^{-i\omega_f t}$ (analyze both cases $\omega_f > 0$ and $\omega_f < 0$)

14.4 Fiber Optics

In wave guides of the previous section, energy is confined to one dimension by a hollow metallic conductor. In this section, we will show that energy can be confined to one dimension in a fundamentally different way. Information in the form of electromagnetic waves can also be propagated in a wave guide made up of a non-conducting (dielectric) material such as glass. Typically, a fiber consists of a narrow glass core surrounded by a thicker glass clad with different optical properties with a protective jacket, as shown in Fig. 14.4.1. To make the problem a little simpler, we assume the cladding is infinitely thick. We will show that a small glass core can be designed so that most of the light energy propagates in the thin core. To enable us to obtain interesting results quickly, we will assume incorrectly that the physics of fiber optics is described by the three-dimensional wave equation.

Figure 14.4.1 Fiber with circular cross section.



A realistic problem with polar coordinates is analyzed in the Exercises. Here, we simplify the problem further. We imagine a two-dimensional region with an inner core (from $y = -L$ to $y = L$) and an outer core that extends to infinity in both directions $y \to \pm\infty$. This geometric region is symmetric around $y = 0$. We claim that all solutions can be made either symmetric or

antisymmetric around $y = 0$. We analyze here only one of these problems, namely, the antisymmetric modes. Thus, we solve a two-dimensional wave guide for $y > 0$, subject to the antisymmetry boundary condition:

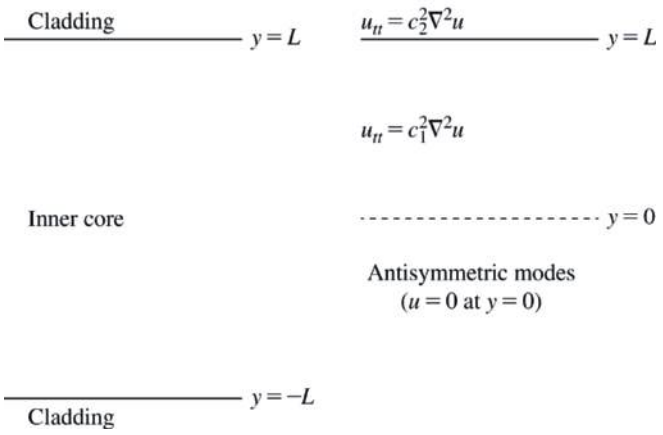
$$u = 0 \text{ at } y = 0. \tag{14.4.1}$$

We assume the two-dimensional wave equation is valid in each region (as shown in Fig. 14.4.2) with different propagation constants, so that there is velocity for each region c_1 and c_2:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c_2^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\ y > L} \tag{14.4.2}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c_1^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\ L > y > 0.} \tag{14.4.3}$$

Figure 14.4.2 Fiber with plane cross section.



Conditions depending on the physics occur on the boundary between the two different media. We assume that

$$u \quad \text{and} \quad \frac{\partial u}{\partial y} \text{ are both continuous at } y = L \tag{14.4.4}$$

(often in electromagnetic wave propagation $\partial u/\partial y$ is not continuous, but some physical constant in each region times $\partial u/\partial y$ is continuous).

We want waves to propagate in the x-direction, $e^{i(kx-\omega t)}$. One of our goals is to determine the dispersion relation $\omega = \omega(k)$. The boundary condition forces the horizontal and time structure of the solution to be the same in both materials:

$$u = B(y)e^{i(kx-\omega t)},\ y > L \tag{14.4.5}$$

$$u = A(y)e^{i(kx-\omega t)},\ L > y > 0. \tag{14.4.6}$$

Substituting the traveling wave assumption (14.4.5) and (14.4.6) into the wave equations (14.4.2) and (14.4.3) yields ordinary differential equations for the transverse structure in each material of the wave guide:

$$-\omega^2 B = c_2^2\left(\frac{d^2B}{dy^2} - k^2 B\right),\ y > L,\ \frac{d^2B}{dy^2} = \left(k^2 - \frac{\omega^2}{c_2^2}\right)B \tag{14.4.7}$$

$$-\omega^2 A = c_1^2\left(\frac{d^2A}{dy^2} - k^2 A\right),\ L > y > 0,\ \frac{d^2A}{dy^2} = \left(k^2 - \frac{\omega^2}{c_1^2}\right)A.$$

We want to investigate those frequencies for which most of the energy propagates in the core $0 < y < L$. Thus, we assume that ω is such that the transverse behavior is oscillatory in the core and exponential in the "cladding":

$$k^2 - \frac{\omega^2}{c_2^2} > 0, \text{ but } k^2 - \frac{\omega^2}{c_1^2} < 0. \tag{14.4.8}$$

From (14.4.8), the frequency must satisfy

$$c_1|k| < |\omega| < c_2|k|.$$

To have this kind of wave guide, it is necessary (but not sufficient) that $c_1 < c_2$. The wave speed in the core must be smaller than the wave speed in the cladding. Under conditions (14.4.8), the solutions of (14.4.7) show that in the core and the cladding

$$u = B_0 e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}y} e^{i(kx-\omega t)},\ y > L$$
$$u = A_0 \sin\sqrt{\frac{\omega^2}{c_1^2} - k^2}\, y e^{i(kx-\omega t)},\ L > y > 0,$$

where A_0 and B_0 are constants. We have assumed the solution exponentially decays in the cladding, and we have used the antisymmetric boundary condition (14.4.1) at $y = 0$.

The dispersion relation is determined by satisfying the two continuity conditions (14.4.4) at the interface between the two media:

$$B_0 e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}L} = A_0 \sin\sqrt{\frac{\omega^2}{c_1^2} - k^2}\, L$$
$$-B_0\sqrt{k^2 - \frac{\omega^2}{c_2^2}}\, e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}L} = A_0\sqrt{\frac{\omega^2}{c_1^2} - k^2}\cos\sqrt{\frac{\omega^2}{c_1^2} - k^2}\, L.$$

These are two linear homogeneous equations in two unknowns (A_0 and B_0). By elimination or the determinant condition, the solution will usually be zero except when

$$\det\begin{bmatrix} \sin\sqrt{\frac{\omega^2}{c_1^2} - k^2}\, L & -e^{-\sqrt{k^2 - \frac{\omega^2}{c_2^2}}L} \\ \sqrt{\frac{\omega^2}{c_1^2} - k^2}\cos\sqrt{\frac{\omega^2}{c_1^2} - k^2} & L\sqrt{k^2 - \frac{\omega^2}{c_2^2}}\, e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}L} \end{bmatrix} = 0.$$

This condition will be the dispersion relation

$$\sqrt{k^2 - \frac{\omega^2}{c_2^2}}\sin\sqrt{\frac{\omega^2}{c_1^2} - k^2}\, L + \sqrt{\frac{\omega^2}{c_1^2} - k^2}\cos\sqrt{\frac{\omega^2}{c_1^2} - k^2}\, L = 0,$$

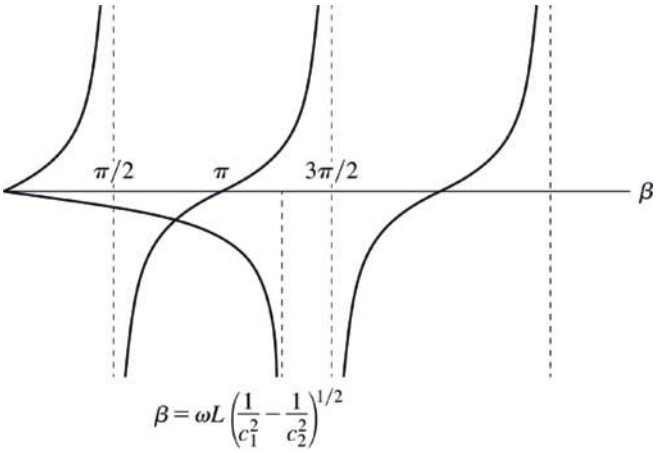
or, equivalently,

$$\boxed{\tan\sqrt{\frac{\omega^2}{c_1^2} - k^2}\, L = -\frac{\sqrt{\frac{\omega^2}{c_1^2} - k^2}}{\sqrt{k^2 - \frac{\omega^2}{c_2^2}}}.} \tag{14.4.9}$$

Solutions ω of (14.4.9) for a given k are the dispersion relation.

In designing wave guides, we specify ω and determine k from (14.4.9). To determine solutions of (14.4.9), we graph in Fig. 14.4.3 the ordinary tangent function as a function of $\beta \equiv \sqrt{\frac{\omega^2}{c_1^2} - k^2}\, L$ and the right-hand side of (14.4.9) also as a function of β. Intersections are solutions. The qualitative features of the right-hand side are not difficult since the right-hand side is always negative. In addition, the right-hand side is 0 at $\beta = 0$, and the right-hand side is infinite at $k = \frac{\omega}{c_2}$ (which means $\beta = \omega L\sqrt{\frac{1}{c_1^2} - \frac{1}{c_2^2}}$), as graphed in Fig. 14.4.3. From the figure, we conclude that if $\omega L\sqrt{\frac{1}{c_1^2} - \frac{1}{c_2^2}} < \frac{\pi}{2}$, there are no intersections, so that there are no modes that propagate in the inner core. This gives a cut-off frequency $\omega_c = \frac{\frac{\pi}{2}}{L\sqrt{\frac{1}{c_1^2} - \frac{1}{c_2^2}}} = \frac{\pi}{2L}\frac{c_1c_2}{\sqrt{c_2^2 - c_1^2}}$ below which there are no wave guide–type modes. However, the desirable situation of only one mode propagating in the core of the fiber occurs if $\frac{\pi}{2} < \omega L\sqrt{\frac{1}{c_1^2} - \frac{1}{c_2^2}} < \frac{3\pi}{2}$ or $\frac{\pi}{2L}\frac{c_1c_2}{\sqrt{c_2^2 - c_1^2}} < \omega < \frac{3\pi}{2L}\frac{c_1c_2}{\sqrt{c_2^2 - c_1^2}}$. It is interesting to note that if c_2 is only slightly greater than c_1, a large range of large frequencies will support one wave that travels with its energy focused in the core of the fiber. All the other solutions will not be traveling waves along the fiber.

Figure 14.4.3 Graphical solution of traveling wave modes in fiber.



Exercises 14.4

14.4.1. Determine the dispersion relation for materials described by (14.4.2) and (14.4.3) with the energy primarily in the core:

(a) For symmetric modes

(b) For antisymmetric modes with the boundary conditions at the interface of the two materials that u and $c\frac{\partial u}{\partial y}$ are continuous

(c) For antisymmetric modes where the cladding goes to $y = H > L$ with $u = 0$ there

(d) For symmetric modes where the cladding goes to $y = H > L$ with $u = 0$ there

14.4.2. For antisymmetric modes, determine the dispersion relation for materials described by (14.4.2) and (14.4.3) with the energy distributed throughout the core and the cladding?

14.4.3. Consider antisymmetric modes described by (14.4.2) and (14.4.3). For what frequencies will waves not propagate in the core and not propagate in the cladding?

14.4.4. Determine the dispersion relation for a fiber with a circular cross section solving the three-dimensional wave equation with coefficient c_1 for $r < L$ and c_2 for $r > L$ (with u and $\frac{\partial u}{\partial r}$ continuous at $r = L$) with energy primarily in the core:

(a) Assuming the solutions are circularly symmetric

(b) Assuming the solutions are not circularly symmetric

(c) Assuming the solutions are circularly symmetric but the cladding stops at $r = H$, where $u = 0$

14.5 Group Velocity II and the Method of Stationary Phase

The solution of linear dispersive partial differential equations with a dispersion relation $\omega = \omega(k)$ is of the form

$$u(x, t) = \int_{-\infty}^{\infty} G(k) e^{i(kx - \omega(k)t)}\, dk = \int_{-\infty}^{\infty} G(k) e^{it\left[k\frac{x}{t} - \omega(k)\right]}\, dk. \tag{14.5.1}$$

The function $G(k)$ is related to the Fourier transform of the initial condition $u(x, 0)$. Although this integral can be numerically evaluated for fixed x and t, this is a tedious process and little would be learned about the partial differential equation. There are some well-known analytic approximations based on t being large (with $\frac{x}{t}$ fixed), which we now discuss.

14.5.1 METHOD OF STATIONARY PHASE

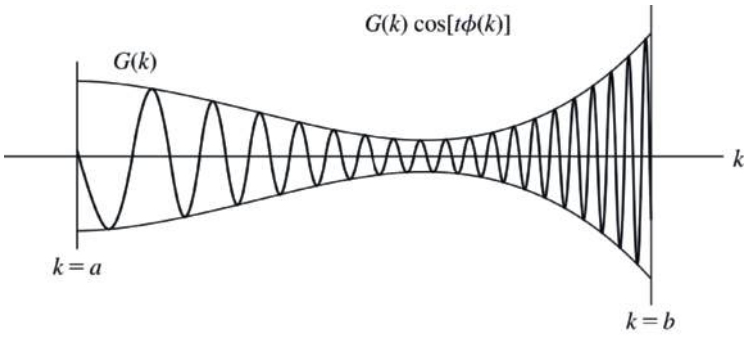
We begin by analyzing integrals depending on a parameter t:

$$\boxed{I(t) = \int_a^b G(k) e^{it\phi(k)}\, dk.} \tag{14.5.2}$$

Later we will specialize our results to linear dispersive waves where the integral is specifically described by (14.5.1). For large values of t, the integrand in (14.5.2) oscillates quickly, as shown in Fig. 14.5.1. We expect significant cancellation, so that we expect $I(t)$ to be small for large t. In the Exercises (by integration by parts) it is shown that

$$\boxed{I(t) = O\left(\frac{1}{t}\right) \quad \text{if} \quad \phi'(k) \neq 0.}$$

Figure 14.5.1 Oscillatory integral cancellation (t large).



The formula by integration by parts includes a boundary contribution and an integral. For infinite domains of integration, often the integral is much smaller since the boundary contribution vanishes and the integral can be integrated by parts again, making it smaller. If $\phi(k)$ is flat somewhere $[\phi'(k_0) = 0]$, then there is less cancellation. That part of the integrand contributes the most. We claim that for large t, the largest contribution to the integral (14.5.2) comes from points near where the phase is **stationary**, places k_0 where

$$\boxed{\phi'(k_0) = 0.} \tag{14.5.3}$$

We begin by assuming that $\phi(k)$ has one stationary point between a and b. We claim that the largest contribution to the integral comes from a small neighborhood of k_0, so that for large t

$$I(t) \sim \int_{k_0-\varepsilon}^{k_0+\varepsilon} G(k) e^{it\phi(k)}\, dk,$$

where ε is small (and to justify this approximation we must let ε vanish in some way as $t \to \infty$). Furthermore, $G(k)$ can be approximated by $G(k_0)$ [if $G(k_0) \neq 0$], and $\phi(k)$ can be approximated by its Taylor series around $k = k_0$:

$$I(t) \sim G(k_0) \int_{k_0-\varepsilon}^{k_0+\varepsilon} e^{it\left[\phi(k_0)+\frac{(k-k_0)^2}{2!}\phi''(k_0)+\frac{(k-k_0)^3}{3!}\phi'''(k_0)+\cdots\right]}\, dk,$$

using (14.5.3). If we assume that ε is small enough for large t such that $\varepsilon^3 t$ is small, then the cubic multiplicative term $e^{it\frac{(k-k_0)^3}{3!}\phi'''(k_0)}$ can be approximated by 1. Thus,

$$I(t) \sim G(k_0) e^{it\phi(k_0)} \int_{k_0-\varepsilon}^{k_0+\varepsilon} e^{it\frac{(k-k_0)^2}{2!}\phi''(k_0)} dk.$$

The following linear change of variables simplifies the exponent in the integrand:

$$y = (k - k_0)\left(\frac{t|\phi''(k_0)|}{2}\right)^{\frac{1}{2}},$$

in which case

$$I(t) \sim \frac{\sqrt{2}G(k_0)e^{it\phi(k_0)}}{(t|\phi''(k_0)|)^{\frac{1}{2}}} \int_{-\varepsilon\left(\frac{t|\phi''(k_0)|}{2}\right)^{\frac{1}{2}}}^{+\varepsilon\left(\frac{t|\phi''(k_0)|}{2}\right)^{\frac{1}{2}}} e^{i(\operatorname{sign}\phi''(k_0))y^2}\, dy.$$

We can choose ε so that $\varepsilon t^{\frac{1}{2}}$ is large as $t \to \infty$. However, recall that $\varepsilon^3 t$ must be small. (For those of you who are skeptical, we can satisfy both by choosing $\varepsilon = t^{-p}$ for any p between $\frac{1}{3}$ and $\frac{1}{2}$.) In this way, the limits approach $\pm\infty$ as $t \to \infty$. Thus,

$$I(t) \sim \frac{2\sqrt{2}G(k_0)e^{it\phi(k_0)}}{(t|\phi''(k_0)|)^{\frac{1}{2}}}\int_0^\infty e^{i(\text{sign}\phi''(k_0))y^2}\,dy.$$

The integral is just a number, albeit an interesting one. It is known that

$$\int_0^\infty \cos(y^2)\,dy = \int_0^\infty \sin(y^2)\,dy = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Thus, with a little gamesmanship (algebra based on $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$), we can obtain what is called the

Method of Stationary Phase

The asymptotic expansion of the integral $I(t) = \int_a^b G(k)e^{it\phi(k)}\,dk$ as $t \to \infty$ is given by

$$I(t) \sim \left\{\frac{2\pi}{t|\phi''(k_0)|}\right\}^{\frac{1}{2}} G(k_0)e^{it\phi(k_0)}\,e^{i(\text{sign}\phi''(k_0))\frac{\pi}{4}} \tag{14.5.4}$$

[assuming there is a simple stationary point k_0 satisfying $\phi'(k_0) = 0$ with $\phi''(k_0) \neq 0$)].

The symbol $\sim$ (read "is asymptotic to") means that the right-hand side is a good approximation to the left and that approximation improves as t increases. The most important part of this result is that $I(t)$ is small for large t but much larger than the contribution for regions without stationary points:

$$I(t) = O\left(\frac{1}{t^{\frac{1}{2}}}\right),$$

for a simple stationary point [assuming $\phi'(k_0) = 0$ with $\phi''(k_0) \neq 0$]. The asymptotic approximation to the integral is just the integrand of the integral evaluated at the stationary point multiplied by an amplitude factor and a phase factor. In many applications the phase shift is not particularly important.

If there is more than one stationary point, the approximation to the integral is the sum of the contribution from each stationary point since the integral can be broken up into pieces with one stationary point each.

14.5.2 APPLICATION TO LINEAR DISPERSIVE WAVES

Here, we consider a linear dispersive partial differential equation that has elementary solutions of the form $e^{i(kx-\omega t)}$, where ω satisfies the dispersion relation $\omega = \omega(k)$. Each mode of the solution to the initial value problem has the form

$$\boxed{u(x,\,t) = \int_{-\infty}^{\infty} A(k)e^{i[kx-\omega(k)t]}\,dk = \int_{-\infty}^{\infty} A(k)e^{it\left[k\frac{x}{t}-\omega(k)\right]}\,dk,} \tag{14.5.5}$$

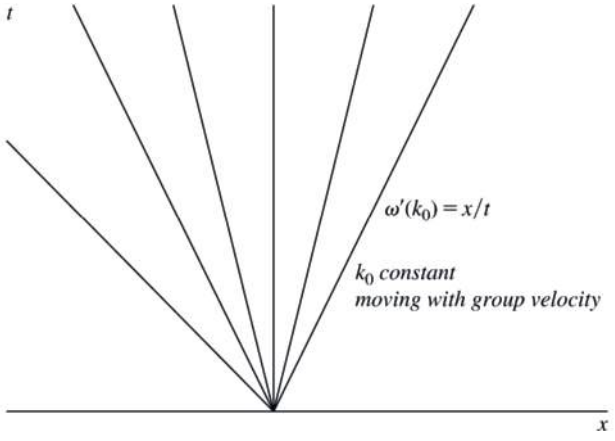
where $A(k)$ is related to the Fourier transform of the initial condition. Usually this integral cannot be evaluated explicitly. Numerical methods could be used, but little can be learned this way concerning the underlying physical processes. However, important approximate behavior for large x and t can be derived using the method of stationary phase. To use the method of stationary phase, we assume t is large (and $\frac{x}{t}$ is fixed). For large t, destructive interference occurs and the integral is small. The largest contribution to the integral (14.5.5) occurs from any point k_0 where the phase is stationary:

$$\boxed{\frac{x}{t} = \omega'(k_0).} \tag{14.5.6}$$

Given x and t, (14.5.6) determines the stationary point k_0, as shown in Fig. 14.5.2. According to the method of stationary phase (14.5.4), the following is a good approximation for large x and t:

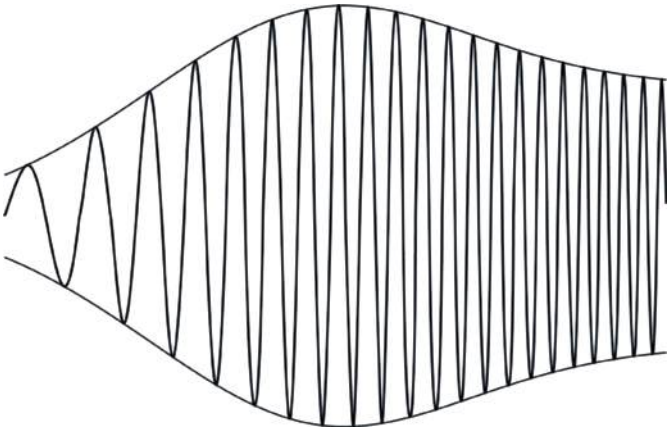
$$u(x, t) \sim A(k_0)\left|\frac{2\pi}{t\omega''(k_0)}\right|^{\frac{1}{2}} e^{i[k_0 x - \omega(k_0)t]} e^{-i(\operatorname{sign}\omega''(k_0))\frac{\pi}{4}} \tag{14.5.7}$$

Figure 14.5.2 Wave number moves with group velocity.



since $\phi'' = -\omega''$, where k_0 satisfies (14.5.6). (We ignore in our discussion the usually constant phase factor.) The solution (14.5.7) looks like an elementary plane traveling wave with wave number k_0 and frequency $\omega(k_0)$, whose amplitude decays. The solution is somewhat more complicated as the wave number and frequency are not constant. The wave number and frequency depend on x and t, satisfying (14.5.6). However, the wave number and frequency are nearly constant since they change slowly in space and time. Thus, the solution is said to be a **slowly varying dispersive wave**. The solution is a relatively elementary sinusoidal wave with a specific wavelength at any particular place, but the wavelength changes appreciably only over many wavelengths (see Fig. 14.5.3).

Figure 14.5.3 Slowly varying dispersive wave.



Equation (14.5.6) shows the importance of the group velocity. The wave number is a function of x and t. To understand how the wave number travels, imagine an initial condition in which all the wave numbers are simultaneously concentrated near $x = 0$ at $t = 0$. (This is a reasonable assumption since we are approximating the solution for large values of x, and from that point of view the initial condition is localized near $x = 0$.) Equation (14.5.6) shows that at later times each

wave number is located at a position that is understood if **the wave number moves at its group velocity** $\omega'(k_0)$. In more advanced discussions, it can be shown that the energy propagates with the group velocity (not the phase velocity). Energy is propagated with all wave numbers. If there is a largest group velocity, then energy cannot travel faster than it. If one is located at a position such that $\frac{x}{t}$ is greater than the largest group velocity, then there are no stationary points. If there are no stationary points, the solution (14.5.5) is much smaller than the results (14.5.7) obtained by the method of stationary phase. It is possible for more than one point to be stationary (in which case the solution is the sum of terms of the form to be presented).

The amplitude decays (as $t \to \infty$) because the partial differential equation is dispersive. The solution is composed of waves of different wavelengths, and their corresponding phase velocities are different. The wave then spreads apart (disperses).

Exercises 14.5

14.5.1. Consider $u_t = u_{xxx}$.

(a) Solve the initial value problem.

(b) Approximate the solution for large x and t using the method of stationary phase.

(c) From (b), solve for the wave number as a function of x and t.

(d) From (b), graph lines in space-time along which the wave number is constant.

(e) From (b), graph lines in space-time along which the phase is constant (called phase lines).

14.5.2. Consider $u_t = iu_{xx}$.

(a) Solve the initial value problem.

(b) Approximate the solution for large x and t using the method of stationary phase.

(c) From (b), solve for the wave number as a function of x and t.

(d) From (b), graph lines in space-time along which the wave number is constant.

(e) From (b), graph lines in space-time along which the phase is constant (called phase lines).

14.5.3. Consider the dispersion relation $\omega = \frac{1}{3}k^3 - \frac{1}{2}k^2$.

(a) Find a partial differential equation with this dispersion relation.

***(b)** Where in space-time are there 0, 1, 2, 3, . . . waves?

14.5.4. Assume (14.5.5) is valid for a dispersive wave. Suppose the maximum of the group velocity $\omega'(k)$ occurs at $k = k_1$. This also corresponds to a **rainbow caustic** (see Fig. 14.6.8).

(a) Show that there are two stationary points if $x < \omega'(k_1)t$ and no stationary points if $x > \omega'(k_1)t$.

(b) State (but do not prove) the order of magnitude of the wave envelope in each region.

(c) Show that $\omega''(k_1) = 0$ with usually $\omega'''(k_1) < 0$.

(d) Approximate the integral in (14.5.5) by the region near k_1 to derive

$$u(x, t) \sim A(k_1)e^{i(k_1x-\omega(k_1)t)}\int_{k_1-\epsilon}^{k_1+\epsilon} e^{i[(k-k_1)(x-\omega'(k_1)t)-(\omega'''(k_1)/3!)(k-k_1)^3t]}\,dk \qquad (14.5.8)$$

(e) Introduce instead the integration variable s, defined by

$$k - k_1 = \frac{s}{(t|\omega'''(k_1)|/2)^{1/3}},$$

and show that the following integral is important:

$$\int_{-\infty}^{\infty} e^{i(ps+s^3/3)}\,ds, \quad \text{where } p = \frac{x - \omega'(k_1)t}{(t|\omega'''(k_1)|/2)^{1/3}}.$$

(f) The **Airy function** is defined as $\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^{\infty} \cos(zs + \frac{1}{3}s^3)ds$. Do not show that it satisfies the ordinary differential equation $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$. Express the wave solution in terms of this Airy function.

(g) As $z \to \pm\infty$, the Airy function has the following well-known properties:

$$\mathrm{Ai}(z) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}}|z|^{-\frac{1}{4}} \sin\left(\dfrac{2}{3}|z|^{\frac{3}{2}} + \dfrac{\pi}{4}\right) & z \to -\infty \\ \dfrac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\left(-\dfrac{2}{3}z^{\frac{3}{2}}\right) & z \to +\infty. \end{cases} \tag{14.5.9}$$

Using this, show that the solution decays at the appropriate rates in each region.

14.5.5. By integrating by parts, show that $I(t)$ given by (14.5.2) is $O(\frac{1}{t})$ if $\phi'(k) \neq 0$.

14.5.6. Approximate (for large t) $I(t)$ given by (14.5.2) if there is one stationary point k_0 with $\phi'(k_0) = 0$ and $\phi''(k_0) = 0$ but $\phi'''(k_0) \neq 0$. Do not prove your result. What is the order of magnitude for large t?

14.5.7. The coefficients of Fourier sine and cosine series can be observed to decay for large n. If $f(x)$ is continuous [and $f'(x)$ is piecewise smooth], show by integrating by parts (twice) that *for large n*

(a) $\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = O\left(\frac{1}{n^2}\right)$

(b) $\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = O\left(\frac{1}{n^2}\right)$ if $f(0) = f(L) = 0$

14.5.8. Consider $u_{tt} = u_{xx} - u$.

(a) Solve the initial value problem.

(b) Approximate the solution for large x and large t using the method of stationary phase.

(c) Assume the group velocity is zero at $k = 0$, the group velocity steadily increases, and the group velocity approaches 1 as $k \to \infty$. How many wave numbers are there as a function of x and t?

14.6 Slowly Varying Dispersive Waves (Group Velocity and Caustics)

14.6.1 APPROXIMATE SOLUTIONS OF DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

In this section, we will show how to obtain approximate solutions to dispersive partial differential equations. Some of these results follow from the method of stationary phase (and generalizations). However, we attempt here to develop material independent of the section on the method of stationary phase.

Linear dispersive waves have solutions of the form $u(x, t) = Ae^{i(kx-\omega t)}$, where the dispersion relation $\omega = \omega(k)$ is known from the partial differential equation. Arbitrary initial conditions can be satisfied by expressions such as

$$u(x, t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}\,dk,$$

which are often too complicated to be directly useful. We wish to consider classes of solutions that are slightly more complicated than an elementary traveling wave $u(x, t) = Ae^{i(kx-\omega t)}$. For these elementary traveling waves, the wave is purely periodic with a constant amplitude A and a constant $O(1)$ wave number (wavelength). We want to imagine that over very long distances and very large times, there are solutions in which the amplitude and the wave number might change. For example, this might be due to initial conditions in which the wave number and/or amplitude are not constant but slowly change over long distances. We introduce a **slowly varying wave train** with a slowly varying amplitude $A(x, t)$ and a phase $\theta(x, t)$:

$$\boxed{u(x, t) = A(x, t)e^{i\theta(x, t)}.} \tag{14.6.1}$$

The wave number and frequency will be slowly varying, and they are defined in the manner that would be used if the wave were a simple traveling wave:

$$\boxed{\text{slowly varying wave number } k \equiv \frac{\partial \theta}{\partial x}} \tag{14.6.2}$$

$$\boxed{\text{slowly varying frequency } \omega \equiv -\frac{\partial \theta}{\partial t}.} \tag{14.6.3}$$

From (14.6.2) and (14.6.3), we derive a conservation law,

$$\boxed{\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0,} \tag{14.6.4}$$

which is called **conservation of waves** (see Exercises 14.6.3 and 14.6.4 for further discussion of this). We claim that with uniform media, the frequency ω will satisfy the usual dispersion relation

$$\boxed{\omega = \omega(k),} \tag{14.6.5}$$

even though the solution (14.6.1) is not an elementary traveling wave upon which (14.6.5) was derived. This can be derived using perturbation methods, but the derivation involves more technical details than we have time to discuss here.

If the dispersion relation (14.6.5) is substituted into conservation of waves (14.6.4), we determine a first-order quasi-linear (really nonlinear) partial differential equation that the wave number $k(x, t)$ must satisfy:

$$\frac{\partial k}{\partial t} + \frac{d\omega}{dk}\frac{\partial k}{\partial x} = 0. \tag{14.6.6}$$

The initial value problem can be solved using the method of characteristics (Section 12.6):

$$\text{if } \frac{dx}{dt} = \frac{d\omega}{dk}, \text{ then } \frac{dk}{dt} = 0, \tag{14.6.7}$$

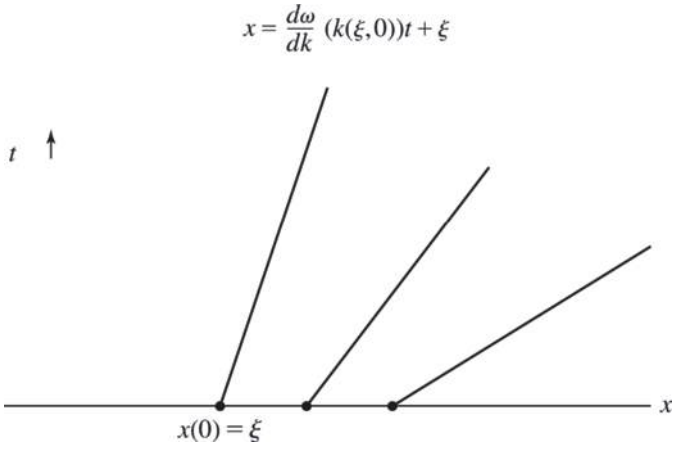
showing that the wave number stays constant moving with the group velocity. The characteristics are straight lines but not parallel in general (see Fig. 14.6.1), since the characteristic velocity, the group velocity, $\frac{d\omega}{dk}$, depends on k. The coupled system of ordinary differential equations is easy to solve. If the characteristic (moving observer) is parameterized by its initial position ξ, $x(0) = \xi$, then the solution of (14.6.7) is

$$k(x, t) = k(\xi, 0), \tag{14.6.8}$$

where the equation for the straight-line characteristics follows from (14.6.7):

$$x = \frac{d\omega}{dk}(k(\xi, 0))t + \xi. \tag{14.6.9}$$

Figure 14.6.1 Characteristics for propagation of dispersive waves.



Given x and t, we can try to solve for ξ from (14.6.9), so that ξ is considered a function of x and t. Once $k(x, t)$ is obtained, the phase can be determined by integrating (14.6.2) and (14.6.3).

The dispersion relation $\omega = \omega(k)$ can be interpreted using (14.6.2) and (14.6.3) as a nonlinear partial differential equation for the unknown phase:

$$-\frac{\partial \theta}{\partial t} = \omega\left(\frac{\partial \theta}{\partial x}\right), \tag{14.6.10}$$

called the **Hamilton–Jacobi** equation. In the specific case in which the original partial differential equation is the two-dimensional wave equation, then (14.6.10) is called the **eikonal** equation. However, the simplest way to solve (14.6.10) for θ is to use the method of characteristics as preceding (see also Section 12.7):

$$\frac{d\theta}{dt} = \theta_t + \theta_x \frac{dx}{dt} = -\omega + k\frac{d\omega}{dk}.$$

Since ω and $\frac{d\omega}{dk}$ depend only on k, and k is a constant moving with the group velocity,

$$\theta(x, t) = \left(-\omega + k\frac{d\omega}{dk}\right)t + \theta(\xi, 0),$$

where $\theta(\xi, 0)$ is the initial phase, which should be given. The phase can be expressed in a more physically intuitive way using (14.6.9):

$$\theta(x, t) = k(x - \xi) - \omega t + \theta(\xi, 0).$$

14.6.2 FORMATION OF A CAUSTIC

We consider a linear partial differential equation with a dispersion relation $\omega = \omega(k)$. We assume a slowly varying wave exists of the form $u(x, t) = A(x, t)e^{i\theta(x,t)}$ with $k \equiv \frac{\partial\theta}{\partial x}$ and $\omega \equiv -\frac{\partial\theta}{\partial t}$. The wave number propagates according to the nonlinear partial differential equation

$$\boxed{\frac{\partial k}{\partial t} + \omega'(k)\frac{\partial k}{\partial x} = 0,} \tag{14.6.11}$$

which approximates the original linear partial differential equation. Equation (14.6.11) can be solved by the method of characteristics, but here we consider problems in which the characteristics intersect (see Fig. 14.6.2). We assume the initial condition is such that k is a smooth function of x. We will show that initial conditions can be chosen so that the wave number evolves from being single valued to being triple valued. Using the method of characteristics [$\frac{dk}{dt} = 0$ if $\frac{dx}{dt} = \omega'(k)$] yields

$$k(x, t) = k(\xi, 0), \tag{14.6.12}$$

where ξ is the location of the characteristic at $t = 0$ and where we assume that the initial distribution for k is given. The equation for the characteristics is

$$x = \omega'(k(\xi,0))t + \xi = F(\xi)t + \xi, \tag{14.6.13}$$

where we introduce $F(\xi)$ as the velocity of the characteristics. The characteristics are straight lines, and some of them are graphed in Fig. 14.6.2. Characteristics intersect if characteristics to the right move more slowly, $F'(\xi) < 0$, as shown in Fig. 14.6.2.

Neighboring characteristics intersect (and are visible as the boundary between lighter and darker regions) at a curve called a **caustic**. In Fig. 14.6.3 we show the caustic generated by a computer-drawn plot of a family of straight-line characteristics satisfying (14.6.13). [It can be shown that the amplitude $A(x, t)$ predicted by the appropriate slow variation or geometrical optics or ray theory becomes infinite on a caustic.] This is the same focusing process for light waves, which is why it is called a caustic (**caustic** means "capable of burning," as the location where light can be focused

Figure 14.6.2 Intersection of characteristics.

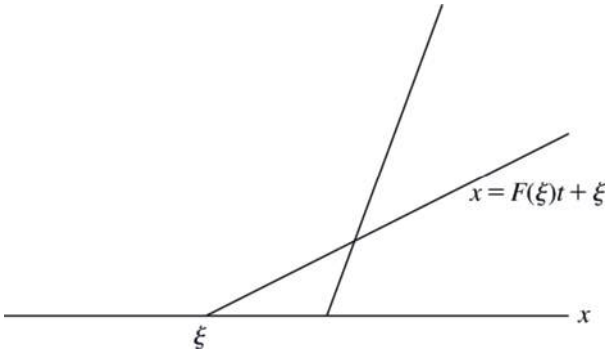


Figure 14.6.3 Caustic formed from characteristics.

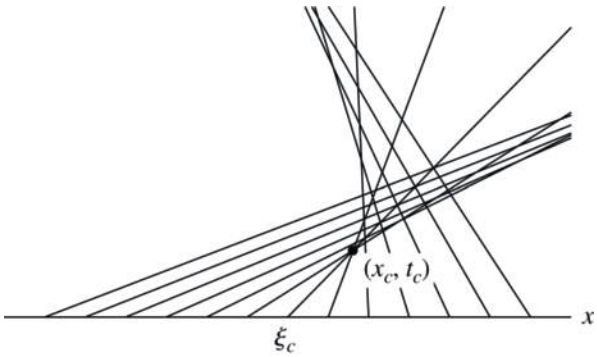
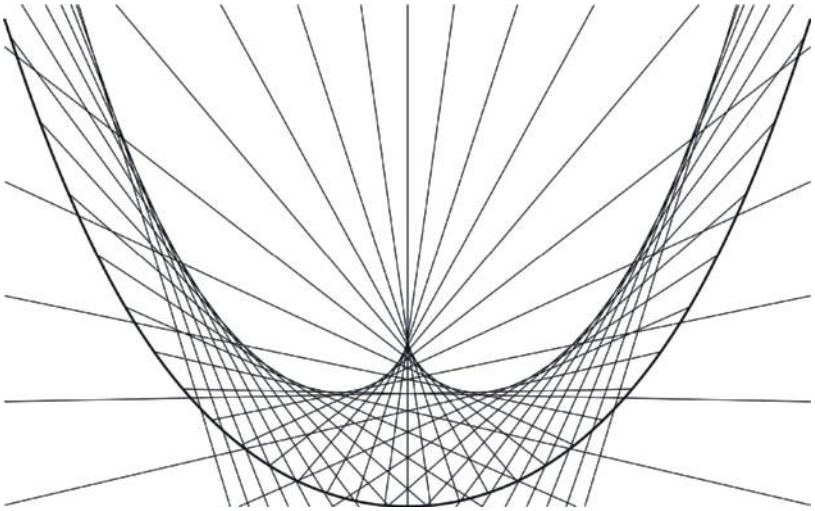


Figure 14.6.4 Caustic formed by reflected rays of nonparabolic reflector.



to burn material). Light waves can focus and form a caustic as they bounce off a nonparabolic reflector (using the angle of incidence equaling the angle of reflection as shown in Fig. 14.6.4). If you look carefully, you will see three reflected rays reaching each point within the caustic, while only one reflected ray reaches points outside the caustic.

This caustic [envelope of a family of curves (14.6.13)] can be obtained analytically by simultaneously solving (14.6.13) with the derivative of (14.6.13) with respect to ξ (as described in Section 12.6):

$$0 = F'(\xi)t + 1 \quad \text{or, equivalently,} \quad t = \frac{-1}{F'(\xi)}, \tag{14.6.14}$$

which determines the time at which the caustic occurs (for a given characteristic). We must assume the initial conditions are such that $F'(\xi) < 0$ in order for the time to be positive. (The spatial position of the caustic is $x = -\frac{F(\xi)}{F'(\xi)} + \xi$, giving the parametric representation of the caustic.) It can be shown that $\frac{\partial k}{\partial x}$ is infinite at the caustic, so that the caustic is the location of the turning points of the triple-valued wave-number curve. Two solutions coalesce on the caustic.

The caustic first forms at

$$t_c = \frac{-1}{F'(\xi_c)} \quad \text{and} \quad x_c = F(\xi_c)t_c + \xi_c, \tag{14.6.15}$$

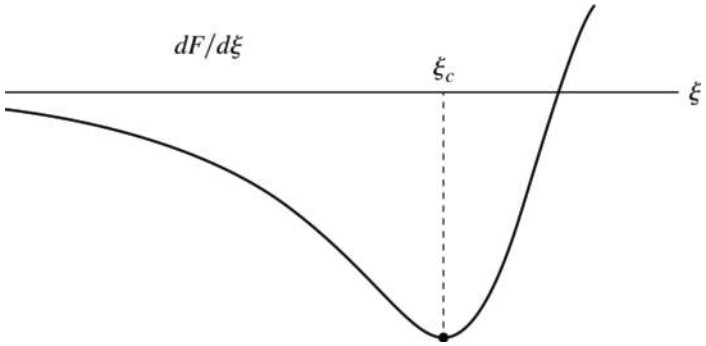
where ξ_c is the position where $F'(\xi)$ has a negative minimum (see Fig. 14.6.5), so that

$$F''(\xi_c) = 0 \tag{14.6.16}$$

with

$$F'''(\xi_c) > 0. \tag{14.6.17}$$

Figure 14.6.5 Minimum (first intersection).



We will show that the caustic is cusp shaped in the neighborhood of its formation. We will assume that x is near x_c and t is near t_c so that the parameter ξ is near ξ_c. The solution can be determined in terms of ξ:

$$k(x,t) = k(\xi,0) = k(\xi_c,0) + k_\xi(\xi_c,0)(\xi-\xi_c) + \dots . \tag{14.6.18}$$

Thus, the wave number is approximately a constant, and the spatial and temporal dependence of the wave number are approximately proportional to $\xi - \xi_c$. To determine how $\xi - \xi_c$ depends on x and t (near the first focusing time of the caustic), we approximate the characteristics (14.6.13) for ξ near ξ_c. Using a Taylor series for $F(\xi)$, we obtain

$$x = F(\xi_c)t + \xi_c + (\xi-\xi_c)[F'(\xi_c)t+1] + \frac{(\xi-\xi_c)^2}{2!}F''(\xi_c)t + \frac{(\xi-\xi_c)^3}{3!}F'''(\xi_c)t + \dots . \tag{14.6.19}$$

If we note that $t = t_c + t - t_c$, then using (14.6.15) and (14.6.16), (14.6.19) becomes

$$\boxed{x - x_c - F(\xi_c)(t-t_c) = (\xi-\xi_c)F'(\xi_c)(t-t_c) + \frac{(\xi-\xi_c)^3}{3!}F'''(\xi_c)t_c,} \tag{14.6.20}$$

where in the last expression we have approximated t by t_c since $t - t_c$ is small. Equation (14.6.20) shows the importance of $F(\xi_c)$, the (group) velocity of the critical characteristic.

Equation (14.6.20) is a **cubic** equation. Each root corresponds to a characteristic for a given x and t. To understand the cubic, we introduce $X = x - x_c - F(\xi_c)(t-t_c)$, $T = t - t_c$, and $s = \xi - \xi_c$. We recall $F'(\xi_c) < 0$ and $F'''(\xi_c) > 0$, so that for convenience we choose $F'(\xi_c) = -1$ and $F'''(\xi_c)t_c = 2$, so that X is an elementary cubic function of s,

$$X = -sT + \frac{1}{3}s^3. \tag{14.6.21}$$

For elementary graphing for fixed T, we need $\frac{dX}{ds} = -T + s^2$. We see that there are no critical points if $T < 0 (t < t_c)$, so that $X(s)$ is graphed on its side in Fig. 14.6.6 and $s(X)$ is single valued corresponding to one root for $t < t_c$. However, if $T > 0 (t > t_c)$, two elementary critical points, as seen in Fig. 14.6.6, are located at $s = \pm T^{1/2}$, which corresponds to the caustic. For $t > t_c$, there are three real roots within the caustic region shown in Figs. 14.6.6 and 14.6.7. In this scaling the caustic satisfies $0 = -T + s^2$, so that $s = \pm T^{1/2}$. Using (14.6.21), the caustic is located at $X = s(-T + \frac{1}{3}s^2) = -\frac{2}{3}sT = \mp\frac{2}{3}T^{3/2}$ and is **cusp shaped** (see Fig. 14.6.7) because $\frac{dx}{dt} = F(\xi_c)$ at $t = 0$.

The characteristics form a caustic. Reflected waves from a nonparabolic reflector form this kind of caustic, as shown in Fig. 14.6.3. Near the caustic, the approximations that yielded the nonlinear partial differential equations (14.6.11) are no longer valid. Instead, near the caustic we must return to the original linear partial differential equation and obtain a different solution. The triple-valued solution predicted by the method of characteristics is meaningful and corresponds to the linear superposition of three slowly varying waves. We explain this in the next sections. (When characteristics intersect and form a caustic, the energy focuses, and the amplitude increases dramatically, though not to infinity, as predicted by slow variation theory, ray theory, or geometrical optics.)

Figure 14.6.6 Smooth solution becoming triple valued.

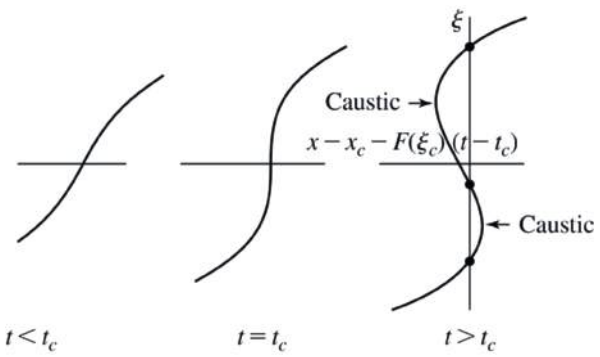
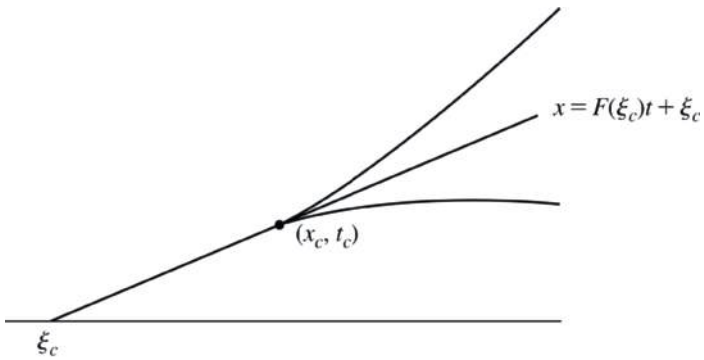
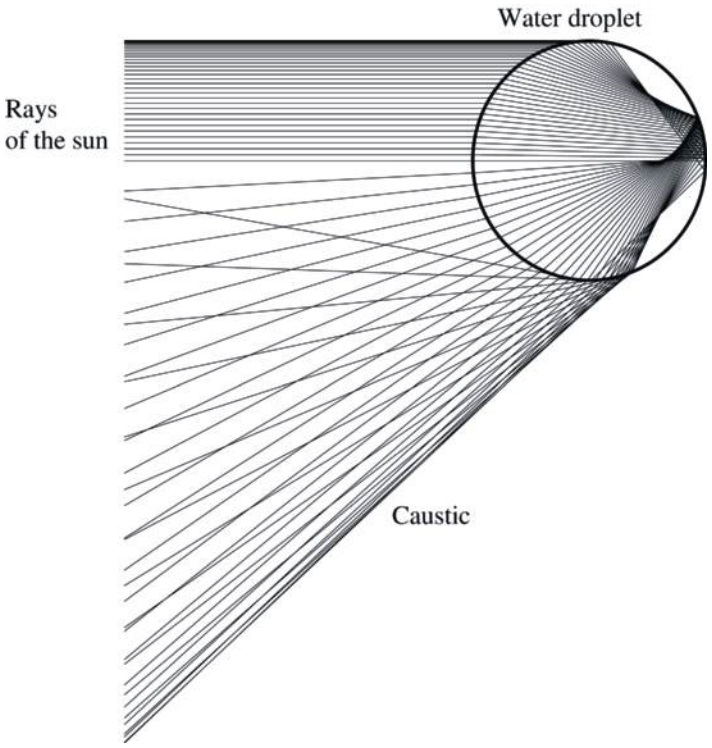


Figure 14.6.7 Cusped caustic.



The curved portions of caustics (away from the cusp) are described in Exercises 14.6.5 and 14.7.1. In Exercise 14.5.4 and in Section 14.7.2, we describe a straight-line caustic that separates a region with two rays from a region with no rays. This is the explanation of the **rainbow**. Parallel light rays from the sun (see Fig. 14.6.8) pass through water droplets (using Snell's law of refraction) before reflecting internally. There is a minimum angle at which the intensity is large. The wave speed of light in the water depends a little bit on the wavelength (color) so that the minimum angle is slightly different for the different colors. This gives rise to the common rainbow.

Figure 14.6.8 Caustic formed by refraction and reflection through water droplet.



Exercises 14.6

***14.6.1.** The solution (14.5.7) obtained by the method of stationary phase has the phase $\theta(x, t) = k_0 x - \omega(k_0)t$, where k_0 is a function of x and t given by the formula (14.5.6) for a stationary point. Determine the k and ω defined by (14.6.2) and (14.6.3).

14.6.2. In a uniform rectangular wave guide, we have learned that for a particular mode, $\omega^2 = c^2\left[k^2 + \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right]$. Let us consider a slowly varying rectangular wave guide in a uniform media, where the width $L = L(x)$ varies slowly in the propagation direction. We claim that

$$\omega^2 = c^2\left[k^2 + \left(\frac{n\pi}{L(x)}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right].$$

What partial differential equation (do not solve it) does the wave number $k(x, t)$ satisfy?

14.6.3. Assume the number of waves is conserved. $\frac{k}{2\pi}$ is the number of waves per unit spatial distance, and $\frac{\omega}{2\pi}$ is the number of waves per unit time. Consider the number of waves between two fixed points $x = a$ and $x = b$.

(a) Explain why the number of waves between $x = a$ and $x = b$ is $\frac{1}{2\pi}\int_a^b k(x, t)\,dx$.

(b) If the number of waves is conserved, show that $\frac{d}{dt}\int_a^b k(x, t)\,dx = \omega(a, t) - \omega(b, t)$.

(c) From part (b), derive that $\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$.

14.6.4. Reconsider Exercise 14.6.3:

(a) Why does $\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} k(x,t)\,dx = 0$ if the endpoints are not constant but move with the phase velocity?

(b) By differentiating the integral in part (a), derive $\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$.

14.6.5. **Curved caustic**. We wish to analyze any one small portion of the curved caustic (see Fig. 14.6.3) away from the cusp. Characteristics of a dispersive wave problem (14.6.11) satisfy (14.6.12) and (14.6.13). At each point (x_c, t_c) in space and time along the curved caustic, there is a specific characteristic ξ_c. We analyze the region near this point, and we will determine the region in which there are two and zero characteristics. The caustic will satisfy (14.6.14), so that (14.6.15) is also valid. However, we will assume $F''(\xi_c) \neq 0$. Assuming that ξ is near ξ_c, it follows that (14.6.18) is still valid.

(a) If x is near x_c and t is near t_c, derive that the following **quadratic** is valid instead of the cubic (14.6.20):

$$x - x_c - F(\xi_c)(t - t_c) = (\xi - \xi_c)F'(\xi_c)(t - t_c) + \frac{(\xi - \xi_c)^2}{2!}F''(\xi_c)t_c. \quad (14.6.22)$$

(b) Using (14.6.22), in what region are there two and zero characteristics? Show that your answer depends on the sign of $F''(\xi_c)$.

14.6.6. Consider $\frac{\partial u}{\partial t} = \beta(x,t)\frac{\partial^3 u}{\partial x^3}$, where $\beta(x,t)$ is a slowly varying coefficient. We assume the dispersion relation is $\omega = \beta(x,t)k^3$.

(a) If $\beta(x,t)$ is constant, determine k and the characteristics.

(b) If $\beta(x,t)$ is constant, determine the phase θ along characteristics.

(c) If $\beta(x,t)$ is not constant, what differential equations determine k and the characteristics?

(d) If $\beta(x,t)$ is not constant, what differential equations determine θ along characteristics?

(e) If $\beta(t)$ only, determine the characteristics and θ.

14.7 Wave Envelope Equations (Concentrated Wave Number)

For linear dispersive partial differential equations, plane traveling waves of the form $u(x,t) = Ae^{i(kx-\omega(k)t)}$ exist with constant wave number k. The most general situations are somewhat difficult to analyze since they involve the superposition of all wave numbers using a Fourier transform. A greater understanding can be achieved by considering some important special situations.

In Section 14.6, we assume that the wave number is slowly varying. Here, instead we assume most of the energy is concentrated in one wave number k_0. We assume the solution of the original partial differential equation is in the form

$$\boxed{u(x,t) = A(x,t)e^{i[k_0x-\omega(k_0)t]}.} \quad (14.7.1)$$

We assume the amplitude $A(x,t)$ is not constant but varies slowly in space and time. The amplitude $A(x,t)$ acts as an **wave envelope** of the traveling wave, and our goal is to determine a partial differential equation that describes the propagation of that wave envelope $A(x,t)$. Some ways in which energy can be concentrated into one wave number are as follows:

1. The initial conditions can be chosen with one wave number but with the amplitude slowly varying as in (14.7.1).
2. It is known that arbitrary initial conditions with all wave numbers disperse (spread out). The wave number is known to move with the group velocity. If one is investigating the solution in some special region of space and time, then in that region most of the energy may be concentrated in one wave number.
3. Rays along which the wave number is constant may focus and form a caustic. In a caustic energy is focused in one wave number.

We will determine partial differential equations that the wave envelope $A(x, t)$ will always satisfy for any dispersive wave equation. We first note that since $u(x, t)$ has the exact solution $u(x, t) = e^{i[kx-\omega(k)t]}$ for all k, it follows that the partial differential equation for $A(x, t)$ must have the very special but simple exact solution

$$A(x, t) = e^{i[(k-k_0)x-(\omega(k)-\omega(k_0))t]}.$$

We note that $\frac{\partial A}{\partial x} = i(k - k_0)A$ and $\frac{\partial A}{\partial t} = -i(\omega(k) - \omega(k_0))A$. In this way we have shown that first- and higher-derivative operators acting on the amplitude correspond to elementary multiplications:

$$-i\frac{\partial}{\partial x} \Longleftrightarrow k - k_0 \tag{14.7.2}$$

$$i\frac{\partial}{\partial t} \Longleftrightarrow \omega(k) - \omega(k_0). \tag{14.7.3}$$

The partial differential equation for the wave amplitude follows from the dispersion relation $\omega = \omega(k)$. Since we assume energy is focused in the wave number k_0, we can use a Taylor series for the dispersion relation around the special wave number k_0:

$$\omega(k) = \omega(k_0) + \omega'(k_0)(k - k_0) + \frac{\omega''(k_0)}{2!}(k - k_0)^2 + \frac{\omega'''(k_0)}{3!}(k - k_0)^3 + \ldots. \tag{14.7.4}$$

Moving $\omega(k_0)$ to the left-hand side, using the operator relations, and dividing by i yields the **wave envelope equation** in all cases:

$$\boxed{\frac{\partial A}{\partial t} + \omega'(k_0)\frac{\partial A}{\partial x} = i\frac{\omega''(k_0)}{2!}\frac{\partial^2 A}{\partial x^2} + \frac{\omega'''(k_0)}{3!}\frac{\partial^3 A}{\partial x^3} + \ldots.} \tag{14.7.5}$$

This shows the importance of the group velocity $c_g = \omega'(k_0)$. These results can also be obtained by perturbation methods.

14.7.1 SCHRÖDINGER EQUATION

To truncate the Taylor expansion (14.7.4) in a useful and accurate way, we must assume that $k - k_0$ is small. From (14.7.2) it follows that the spatial derivatives of the wave envelope must be small. This corresponds to the assumptions of a slowly varying wave amplitude alluded to earlier. The wave amplitude must not change much over one wavelength $\frac{2\pi}{k_0}$ for the wave envelope equation (14.7.5) to be valid. Each spatial derivative of the amplitude in (14.7.5) is smaller. Thus, if $\omega''(k_0) \neq 0$, we are justified in using the **Schrödinger equation**,

$$\boxed{\frac{\partial A}{\partial t} + \omega'(k_0)\frac{\partial A}{\partial x} = i\frac{\omega''(k_0)}{2!}\frac{\partial^2 A}{\partial x^2},} \tag{14.7.6}$$

the approximation that results from ignoring the third and higher derivatives. Anytime energy is focused in one wave number (called the **nearly monochromatic** approximation), $u(x, t) \approx A(x, t)e^{i(k_0x-\omega(k_0)t)}$, the wave amplitude or wave envelope satisfies the Schrödinger equation (14.7.6). The Schrödinger equation is a linear partial differential equation with plane wave solutions $A = e^{i(\alpha x-\Omega(\alpha)t)}$ so that its dispersion relation is quadratic: $\Omega(\alpha) = \omega'(k_0)\alpha + \frac{\omega''(k_0)}{2!}\alpha^2$. The solution of the Schrödinger equation corresponding to an infinite domain can be obtained by Fourier transforms:

$$A(x, t) = \int_{-\infty}^{\infty} G(\alpha)e^{i\left[\alpha(x-\omega'(k_0)t)-\frac{\omega''(k_0)}{2!}\alpha^2 t\right]}\, d\alpha. \tag{14.7.7}$$

In this nearly monochromatic approximation, the dispersive term is small. However, the dispersion cannot be ignored if we wish to understand the behavior for relatively long times. Perhaps the relations between space and time are better understood, making a change of variables to a coordinate system moving with the group velocity:

$$X = x - \omega'(k_0)t \tag{14.7.8}$$

$$T = t. \tag{14.7.9}$$

In this moving coordinate system the Schrödinger equation has the following simpler form:

$$\frac{\partial A}{\partial T} - \omega'(k_0)\frac{\partial A}{\partial X} + \omega'(k_0)\frac{\partial A}{\partial X} = \frac{\partial A}{\partial T} = i\frac{\omega''(k_0)}{2!}\frac{\partial^2 A}{\partial X^2}.$$

In this way, small spatial derivatives are balanced by small time derivatives (in the moving coordinate system).

Caustics

Away from caustics, slowly varying linear dispersive waves can be analyzed approximately by the method of characteristics. However, this approximation fails near the caustic, where characteristics focus the energy. Near a caustic the solution is more complicated. In the region near this caustic (x near x_c and t near t_c), the wave energy is focused in one wave number [the critical value $k_c = k(\xi_c, 0)$] so that $u(x,t) \approx A(x,t)e^{i(k_c(x-x_c)-\omega(k_c)(t-t_c))}$, and the wave amplitude $A(x,t)$ approximately solves the linear Schrödinger equation whose solutions are given by (14.7.7). We may replace x by $x - x_c$ and t by $t - t_c$ in (14.7.7), though this corresponds to a different arbitrary function $G(\alpha)$. We wish to determine the complex function $G(\alpha) = R(\alpha)e^{i\Phi(\alpha)}$, which agrees with the known caustic behavior:

$$A(x,t) = \int_{-\infty}^{\infty} R(\alpha)e^{i\Phi(\alpha)}e^{i\left[\alpha(x-x_c-\omega'(k_c)(t-t_c))-\frac{\omega''(k_c)}{2!}\alpha^2(t-t_c)\right]}\,d\alpha. \tag{14.7.10}$$

This exact solution can be approximated by evaluating the phase at the value of α at which the phase is stationary:

$$x - x_c - \omega'(k_c)(t - t_c) - \omega''(k_c)\alpha(t - t_c) + \Phi'(\alpha) = 0. \tag{14.7.11}$$

By comparing (14.7.11) with the fundamental cubic equation (14.6.20), first we see that $\alpha = k_\xi(\xi - \xi_c)$, since from (14.6.13), $F' = \omega'' k_\xi$. It follows that $\Phi'(\alpha) = -\frac{\alpha^3}{3!}\frac{F'''(\xi_c)}{k_\xi^3}t_c$, so that $\Phi(\alpha) = -\frac{\alpha^4}{4!}\frac{F'''(\xi_c)}{k_\xi^3}t_c$. In this way, we derive an integral representation of the solution in the neighborhood of a cusped caustic:

$$\boxed{A(x,t) = \int_{-\infty}^{\infty} e^{i\left[\alpha(x-x_c-\omega'(k_c)(t-t_c))-\frac{\omega''(k_c)}{2!}\alpha^2(t-t_c)-\frac{\alpha^4}{4!}\frac{F'''(\xi_c)}{k_\xi^3}t_c\right]}\,d\alpha,} \tag{14.7.12}$$

where for simplicity we have taken $R(\alpha) = 1$. Equation (14.7.12) is known as the **Pearcey integral**, though Brillouin seems to have been the first to study it. Stationary points for (14.7.12) satisfy the cubic (14.7.11), so that asymptotically the number of oscillatory phases varies from one outside the cusped caustic to three inside.

14.7.2 LINEARIZED KORTEWEG–DE VRIES EQUATION

Usually, the wave envelope satisfies the Schrödinger equation (14.7.6). However, if wave energy is focused in one wave number and that wave number corresponds to a maximum or minimum of the group velocity $\omega'(k)$, then $\omega''(k_0) = 0$. Usually, when the group velocity is at an extrema, then the wave envelope is approximated by the **linearized Korteweg–de Vries equation**:

$$\boxed{\frac{\partial A}{\partial t} + \omega'(k_0)\frac{\partial A}{\partial x} = \frac{\omega'''(k_0)}{3!}\frac{\partial^3 A}{\partial x^3},} \tag{14.7.13}$$

which follows directly from (14.7.5). The dispersive term is small, but over large times its effects must be kept. [The transformation (14.7.8) and (14.7.9) corresponding to moving with the group velocity could be used.]

Long waves

Partial differential equations arising from physical problems usually have odd dispersion relations $\omega(-k) = -\omega(k)$ so that the phase velocities corresponding to k and $-k$ are the same. For that reason, here we assume the dispersion relation is odd. **Long waves** are waves with wavelengths much longer than any other length scale in the problem. For long waves, the wave number k will be small. The approximate dispersion relation for long waves can be obtained from the Taylor series of the dispersion relation:

$$\omega(k) = \omega(0) + \omega'(0)k + \frac{\omega''(0)}{2!}k^2 + \frac{\omega'''(0)}{3!}k^3 + \cdots = \omega'(0)k + \frac{\omega'''(0)}{3!}k^3 + \cdots, \tag{14.7.14}$$

since for odd dispersion relations $\omega(0) = 0$ and $\omega''(0) = 0$. Thus, because of the usual operator assumptions (14.2.7) and (14.2.8) ($k = -i\frac{\partial}{\partial x}$ and $\omega = i\frac{\partial}{\partial t}$), long waves should satisfy the linearized Korteweg–de Vries (linearized KdV) equation:

$$\frac{\partial u}{\partial t} + \omega'(0)\frac{\partial u}{\partial x} = \frac{\omega'''(0)}{3!}\frac{\partial^3 u}{\partial x^3}.$$

This can be understood in another way. If energy is focused in one wave (long wave) $k_0 = 0$, then the wave amplitude equation follows from (14.7.5):

$$\frac{\partial A}{\partial t} + \omega'(0)\frac{\partial A}{\partial x} = \frac{\omega'''(0)}{3!}\frac{\partial^3 A}{\partial x^3} + \dots.$$

Here the solution and the wave envelope are the same, satisfying the same partial differential equation because for nearly monochromatic waves

$$u(x, t) = A(x, t)e^{i(k_0 x - \omega(k_0)t)} = A(x, t)$$

since $k_0 = 0$ and $\omega(0) = 0$. The group velocity for long waves (with an odd dispersion relation) is obtained by differentiating (14.7.14), $\omega'(k) = \omega'(0) + \frac{\omega'''(0)}{2}k^2 + \dots$. Thus, the group velocity has a minimum or maximum for long waves ($k = 0$). Thus, the first or last waves often observed will be long waves. To understand how long waves propagate, we just study the linearized Korteweg–de Vries equation. Since it is dispersive, the amplitudes observed should be very small (as shown by the method of stationary phase). Largeamplitude long dispersive waves must have an alternate explanation (see the next section).

Maximum group velocity and rainbow caustic

We briefly investigate the solution that occurs (from the method of stationary phase) when the group velocity $\omega'(k)$ has a maximum. Thus, $\omega''(k_1) = 0$, in which case the linearized KdV (14.7.13) governs. Specifically, following from (14.5.8) in Exercise 14.5.4, the wave envelope satisfies

$$A(x, t) = \int_{-\infty}^{\infty} e^{i\left[(k-k_1)(x-\omega'(k_1)t) - \frac{(k-k_1)^3}{3!}\omega'''(k_1)t\right]}\, dk.$$

From this it can be seen that $A(x, t)$ satisfies the linearized KdV (14.7.13), as should follow theoretically from (14.7.14). This is perhaps easier to see using a coordinate system moving with the group velocity, in which case very

$$A_T = -A_{XXX}$$

[since $\omega'''(k_1) < 0$]. Further analysis in Exercise 14.5.4 shows that

$$A(x, t) = \frac{1}{t^{1/3}}\mathrm{Ai}\left(\frac{x - \omega'(k_1)t}{t^{1/3}}\right),$$

where Ai is an Airy function. Thus, $A(x, t)$ should be a **similarity solution** of the linearized KdV. It will be instructive to show the form taken by similarity solutions of the linearized KdV:

$$A(X, t) = \frac{1}{t^{1/3}}f\left(\frac{X}{t^{1/3}}\right) = \frac{1}{t^{1/3}}f(\xi),$$

where the similarity variable ξ is given by

$$\xi = \frac{X}{t^{1/3}}.$$

Derivatives with respect to X are straightforward $\left(\frac{\partial}{\partial X} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial X} = \frac{1}{t^{1/3}}\frac{\partial}{\partial \xi}\right)$, but we must be more careful with t-derivatives. The linearized KdV ($A_T = -A_{XXX}$) becomes

$$-\frac{1}{3}\frac{1}{t^{4/3}}f + \frac{1}{t^{1/3}}f'\left(-\frac{1}{3}\frac{\xi}{t}\right) = -\frac{1}{t^{1/3}}\frac{1}{t}f''',$$

which, after multiplying by $t^{4/3}$, becomes a third-order ordinary differential equation ($-\frac{1}{3}f - \frac{1}{3}f'\xi = -f'''$) that can be integrated to $-\frac{1}{3}f\xi = -f'' + c$. The constant $c = 0$ (since we want $f \to 0$ as $\xi \to +\infty$), and hence the similarity solution of the linearized KdV is related to **Airy's equation**:

$$f'' - \frac{1}{3}f\xi = 0.$$

Here, regions with two and zero characteristics are caused by a maximum group velocity. Regions with two and zero characteristics are separated by a straight-line characteristic (caustic) $x = \omega'(k_1)t$ with $\omega''(k_1) = 0$. This is the same situation that occurs for the characteristics for a rainbow (see Fig. 14.6.8), where there is a maximum group velocity.

14.7.3 NONLINEAR DISPERSIVE WAVES: KORTEWEG–DE VRIES EQUATION

These amplitude equations, the Schrödinger equation (14.7.6) or the linearized Korteweg–de Vries equation (14.7.13), balance small spatial and temporal changes (especially when viewed from moving coordinate systems). Often in physical problems small nonlinear terms have been neglected, and they are often just as important as the small dispersive terms. The specific nonlinear terms can be derived for each specific application using multiple-scale singular perturbation methods (which are beyond the scope of this text). In different physical problems, the nonlinear terms frequently have similar forms (since they are derived as small but finite amplitude expansions much like Taylor series approximations for the amplitude).

For long waves, the usual nonlinearity that occurs yields the **Korteweg–de Vries** (KdV) equation:

$$\boxed{\frac{\partial u}{\partial t} + [\omega'(0) + \beta u]\frac{\partial u}{\partial x} = \frac{\omega'''(0)}{3!}\frac{\partial^3 u}{\partial x^3}.} \tag{14.7.15}$$

If for the moment we ignore the dispersive term $\frac{\partial^3 u}{\partial x^3}$, then (14.7.15) is a quasi-linear partial differential equation solvable by the method of characteristics. The characteristic velocity, $\omega'(0) + \beta u$, can be thought of as the linearization around $u = 0$ (small-amplitude approximation) of some unknown characteristic velocity $f(u)$. Taller waves move faster or slower (depending on β) and smooth initial conditions steepen (and eventually break). Some significant effort (usually using perturbation methods corresponding to long waves) is required to derive the coefficient β from the equations of motion for a specific physical problem. Korteweg and de Vries first derived (14.7.15) in 1895 when trying to understand unusually persistent surface water waves observed in canals.

The KdV equation is an interesting model nonlinear partial differential equation because two different physical effects are present. There is an expectation that solutions of the KdV equation decay due to the dispersive term. However, the nonlinear term causes waves to steepen. By moving with the linearized group velocity and scaling x and u, we obtain the standard form of the KdV equation:

$$\boxed{\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.} \tag{14.7.16}$$

We limit our discussion here to elementary traveling wave solutions of the KdV equation:

$$\boxed{u(x, t) = f(\xi), \quad \text{where } \xi = x - ct.} \tag{14.7.17}$$

When (14.7.17) is substituted into (14.7.16), a third-order ordinary differential equation arises:

$$f''' - cf' + 6ff' = 0.$$

This can be integrated to yield a nonlinear second-order ordinary differential equation (of the type corresponding to $F = ma$ in mechanics, where $a = f''$):

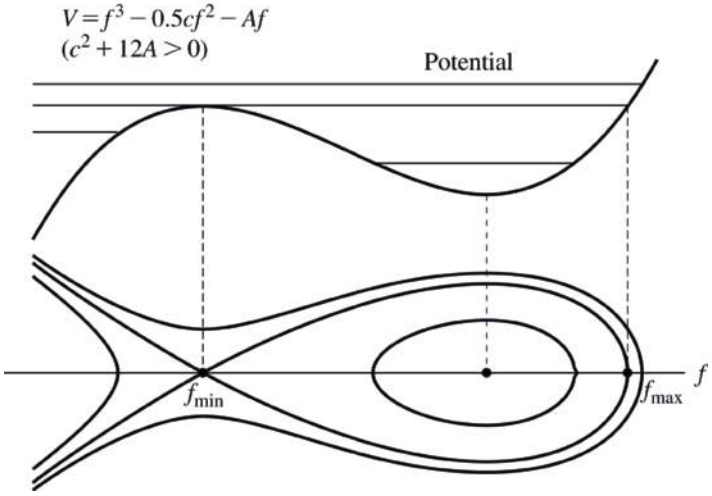
$$f'' + 3f^2 - cf - A = 0, \tag{14.7.18}$$

where A is a constant. Multiplying by f' and integrating with respect to ξ yields an equation corresponding to **conservation of energy** [if (14.7.18) were Newton's law]:

$$\frac{1}{2}(f')^2 + f^3 - \frac{1}{2}cf^2 - Af = E, \tag{14.7.19}$$

where E is the constant total energy [and $\frac{1}{2}(f')^2$ represents **kinetic energy** and $f^3 - \frac{1}{2}cf^2 - Af$ **potential energy**]. In Fig. 14.7.1 we graph the potential energy as a function of f. Critical points for the potential occur if $3f^2 - cf - A = 0$, corresponding to equilibrium solutions of (14.7.18). The discriminant of this quadratic ($b^2 - 4ac$) is $c^2 + 12A$. If $c^2 + 12A \le 0$, then the potential energy is monotonically increasing, and it can be shown that the traveling waves are not bounded. Thus, we assume $c^2 + 12A > 0$, in which case two equilibria exist. Constant-energy lines (in the potential energy sketch) enable us to draw the phase portrait in Fig. 14.7.1. We note that one equilibria is a saddle point ($f_{\min}$) and the other is a center.

Figure 14.7.1 Potential and phase portrait for traveling wave for the KdV equation.



Periodic traveling waves (cnoidal waves)

Most of the bounded traveling waves are periodic. Some analysis is performed in the Exercises.

Solitary traveling waves

If the constant energy E is just right, then the traveling wave has an infinite period. The cubic potential energy has two coincident roots at f_{min} and a larger single root at $f_{max} > f_{min}$, so that

$$\frac{1}{2}(f')^2 = -(f - f_{max})(f - f_{min})^2. \tag{14.7.20}$$

The phase portrait shows that solution has a single maximum at $f = f_{max}$ and tails off exponentially to $f = f_{min}$. It is graphed in Fig. 14.7.2 and is called a **solitary wave**. This permanent traveling wave exists when the steepening effects of the nonlinearity balance the dispersive term. An expression for the wave speed can be obtained by comparing the quadratic terms in (14.7.19) and (14.7.20): $\frac{1}{2}c = f_{max} + 2f_{min} = 3f_{min} + (f_{max} - f_{min})$. The simplest example is when $f_{min} = 0$, requiring $f_{max} > 0$, in which case

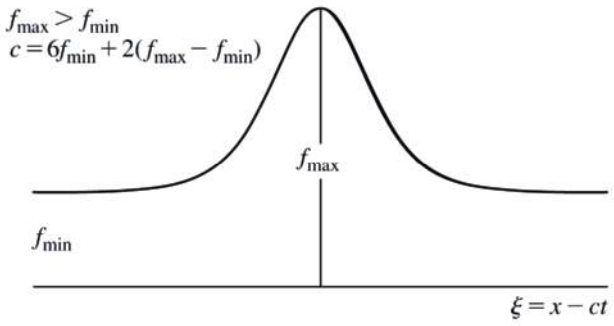
$$\frac{1}{2}c = f_{max}. \tag{14.7.21}$$

These solitary waves occur only for $f_{max} > 0$, as sketched in Fig. 14.7.2. Thus, taller waves move faster (to the right). There is an analytic formula for these solitary waves. If $f_{min} = 0$, it can be shown that

$$\boxed{u(x, t) = \frac{1}{2}c\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{c}(x - ct)\right],} \tag{14.7.22}$$

where $c > 0$ is given by (14.7.21). This shows that the taller waves (which move faster) are more sharply peaked.

Figure 14.7.2 Solitary wave for the KdV equation.



14.7.4 SOLITONS AND INVERSE SCATTERING

For many other nonlinear partial differential equations, solitary waves exist. For most nonlinear dispersive wave equations, no additional analytic results are known since the equations are nonlinear. Modern numerical experiments usually show that solitary waves of different velocities interact in a somewhat complex way. However, for the KdV equation (14.7.16) Zabusky and Kruskal [1965] showed that different solitary waves interact like particles (preserving their amplitude exactly after interaction) and hence are called **solitons**. These solitons have become quite important because it has been shown that solutions of this form develop even if the initial conditions are not in this shape and that this property also holds for many other nonlinear partial differential equations that describe other physically interesting nonlinear dispersive waves. In attempting to understand these numerical experiments, Gardner, Greene, Kruskal, and Miura [1967] showed that the nonlinear KdV equation could be related to a scattering problem associated with the Schrödinger eigenvalue problem (see Section 10.7) and the time evolution of the scattering problem. Lax [1968] generalized this to two linear nonconstant differential operators L and M that depend on an unknown function $u(x, t)$:

$$L\phi = \lambda\phi \tag{14.7.23}$$

$$\frac{\partial \phi}{\partial t} = M\phi. \tag{14.7.24}$$

The operator L describes the spectral (scattering) problem with ϕ the usual eigenfunction, and M describes how the eigenfunctions evolve in time. The consistency of these equations [solving both for $L\frac{\partial \phi}{\partial t}$ by taking the time derivative of (14.7.23)] yields $L\frac{\partial \phi}{\partial t} = LM\phi = -\frac{\partial L}{\partial t}\phi + \lambda\frac{\partial \phi}{\partial t} + \frac{d\lambda}{dt}\phi = -\frac{\partial L}{\partial t}\phi + ML\phi + \frac{d\lambda}{dt}\phi$, where (14.7.23) and (14.7.24) have been used. The **spectral parameter is constant** ($\frac{d\lambda}{dt} = 0$) if and only if an equation known as **Lax's equation** holds:

$$\frac{\partial L}{\partial t} + LM - ML = 0, \tag{14.7.25}$$

which in practice will be a nonlinear partial differential equation for $u(x, t)$ since the commutator $LM - ML$ of two nonconstant operators is usually nonzero.

In an exercise, it is shown that for the specific operators

$$\boxed{L = -\frac{\partial^2}{\partial x^2} + u} \tag{14.7.26}$$

$$\boxed{M = \gamma - \frac{\partial u}{\partial x} + 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3},} \tag{14.7.27}$$

where γ is a constant, Lax's equation is a version of the Korteweg–de Vries equation:

$$\boxed{\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.} \tag{14.7.28}$$

Inverse scattering transform

The initial value problem for the KdV equation on the infinite interval $-\infty < x < \infty$ is solved by utilizing the difficult relationships between the nonlinear KdV equation and the linear scattering problem for $-\infty < x < \infty$. The eigenfunction ϕ satisfies the Schrödinger eigenvalue problem

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + (\lambda - u(x, t))\phi = 0.} \tag{14.7.29}$$

Here time is an unusual parameter. In the brief Section 10.7 on inverse scattering, we claimed that the potential $u(x, t)$ for fixed t can be reconstructed from the scattering data at that fixed t:

$$\boxed{u(x, t) = -2\frac{\partial}{\partial x}K(x, x, t),} \tag{14.7.30}$$

using the unique solution of the Gelfand–Levitan–Marchenko integral equation:

$$\boxed{K(x, y, t) + F(x + y, t) + \int_x^\infty K(x, z, t)F(y + z, t)\,dz = 0, \text{ for } y > x.} \tag{14.7.31}$$

Here the nonhomogeneous term and the kernel are related to the inverse Fourier transform of the reflection coefficient $R(k, t)$ (defined in Section 10.7), including a contribution from the bound states (discrete eigenvalues $\lambda = -\kappa_n^2$):

$$\boxed{F(s, t) = \sum_{n=1}^{N} c_n^2(t) e^{-\kappa_n s} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k, t) e^{iks}\, dk.} \tag{14.7.32}$$

Here the scattering data depend on a parameter, time. Unfortunately, we do not know the time-dependent scattering data since only $u(x, 0)$ is given as the initial condition for the KdV equation. Thus, at least the initial scattering data can be determined, and we assume those data are known. If the initial condition has discrete eigenvalues, then these discrete eigenvalues for the time evolution $u(x, t)$ of the KdV equation miraculously do not change in time because we have shown that $\frac{d\lambda}{dt} = 0$ for the KdV equation. However, for the KdV equation it has also been shown that the time-dependent scattering data can be determined easily from the initial scattering data using only (14.7.29) with (14.7.26) and (14.7.27):

$$R(k, t) = R(k, 0) e^{8ik^3 t} \tag{14.7.33}$$

$$c_n(t) = c_n(0) e^{4\kappa_n^3 t}. \tag{14.7.34}$$

This method is called the **inverse scattering transform**. The initial condition is transformed to the scattering data, and the scattering data satisfy a simple time-dependent linear ordinary differential equation whose solution appears in (14.7.33) and (14.7.34). The time-dependent solution is then obtained by an inverse scattering procedure.

It can be shown that the solution of the inverse scattering transform corresponding to a initial condition that is a reflectionless potential with one discrete eigenvalue yields the solitary wave solution discussed earlier. However, solutions can be obtained corresponding to initial conditions that are reflectionless potentials with two or more discrete eigenvalues. The corresponding solutions to the KdV equation are interacting, strongly nonlinear solitary waves with exact interaction properties first observed numerically by Zabusky and Kruskal [1965]. We have been very brief. Ablowitz, Kaup, Newell, and Segur developed a somewhat simpler procedure, equivalent to (14.7.23) and (14.7.24), which is described (along with many other things) in the books by Ablowitz and Segur [1981] and Ablowitz and Clarkson [1991].

14.7.5 NONLINEAR SCHRÖDINGER EQUATION

When wave energy is focused in one wave number, $u(x, t) = A(x, t) e^{i(k_0 x - \omega(k_0)t)}$, the wave amplitude of a linear dispersive wave can be approximated by (14.7.6). Small temporal changes are balanced by small spatial changes. If the underlying physical equations are nonlinear, small but finite amplitude effects can be developed using perturbation methods. In many situations, the nonlinearity and spatial dispersion balance in the following way. The amplitude is said to solve the (cubic) **nonlinear Schrödinger equation** (NLS):

$$\boxed{\frac{\partial A}{\partial t} + \omega'(k_0)\frac{\partial A}{\partial x} = i\frac{\omega''(k_0)}{2!}\frac{\partial^2 A}{\partial x^2} + i\beta|A|^2 A.} \tag{14.7.35}$$

To understand the nonlinear aspects of this equation, first note that there is a solution with the wave amplitude constant in space: $u(x, t) = A(t) e^{i(k_0 x - \omega(k_0)t)}$ if $\frac{\partial A}{\partial t} = i\beta|A|^2 A$. To solve this differential equation, we let $A = re^{i\theta}$, in which case by equating the real and imaginary parts, we obtain $\frac{d\theta}{dt} = \beta r^2$ and $\frac{dr}{dt} = 0$. Thus, $A(t) = r_0 e^{i\beta r_0^2 t}$, which corresponds to $u(x, t) = r_0 e^{i\beta r_0^2 t} e^{i(k_0 x - \omega(k_0)t)}$. Here the frequency $\omega(k_0, |A|) = \omega(k_0) - \beta|A|^2$ depends on the amplitude $r_0 = |A|$. It is fairly typical that the frequency depends on the amplitude of the wave in this way as an approximation for small wave amplitudes. When spatial dependence is included, the nonlinear dispersive wave equation, (14.7.35), results.

We will show that the NLS has solutions that correspond to an oscillatory traveling wave with a wave envelope shaped like a solitary wave. We let

$$\boxed{A(x, t) = r(x, t) e^{i(\theta(x, t))} = r(x, t) e^{i(\alpha x - \Omega t)},}$$

where $r(x, t)$ is real and represents the amplitude of an elementary traveling wave with wave number α and frequency Ω. The wave number α is arbitrary, but we will determine the frequency Ω

corresponding to this solitary wave envelope. Since $A_x = (r_x + i\alpha r)e^{i(\alpha x - \Omega t)}$, it follows that $A_{xx} = (r_{xx} + 2i\alpha r_x - \alpha^2 r)e^{i(\alpha x - \Omega t)}$. The real part of the NLS (14.7.35) yields

$$r_t + [\omega'(k_0) + \alpha\omega''(k_0)]r_x = 0. \tag{14.7.36}$$

The method of characteristics can be applied to (14.7.36), and it shows that

$$r(x, t) = r(x - ct),$$

where the wave speed of the solitary wave envelope satisfies

$$c = \omega'(k_0) + \alpha\omega''(k_0). \tag{14.7.37}$$

This shows the magnitude of the complex amplitude stays constant, moving with the group velocity. Since α represents a small perturbed wave number, this is just an approximation of the group velocity at the wave number $k_0 + \alpha$. The imaginary part of the NLS (14.7.35) yields

$$-\Omega r + \omega'(k_0)\alpha r = \frac{\omega''(k_0)}{2!}(r_{xx} - \alpha^2 r) + \beta r^3.$$

We can rewrite this as the nonlinear ordinary differential equation

$$0 = r_{xx} + \delta r + \gamma r^3, \tag{14.7.38}$$

where $\gamma = \frac{2\beta}{\omega''(k_0)}$ and $\delta = -\alpha^2 + 2\frac{\Omega - \omega'(k_0)\alpha}{\omega''(k_0)}$. Multiplying (14.7.38) by r_x and integrating yields the energy equation:

$$\frac{1}{2}(r_x)^2 + \frac{\delta}{2}r^2 + \frac{\gamma}{4}r^4 = E = 0.$$

We have chosen $E = 0$ in order to look for a wave envelope with the property that $r \to 0$ as $x \to \infty$. The potential $\frac{\delta}{2}r^2 + \frac{\gamma}{4}r^4$ is graphed in Fig. 14.7.3. From the potential, the phase portrait (r_x as a function of r) is obtained (Fig. 14.7.3), which shows that a solitary wave (Fig. 14.7.4) exists only if $\gamma > 0$ [corresponding to β having the same sign as $\omega''(k_0)$] and $\delta < 0$. Here the nonlinearity prevents the wave packet from dispersing.

Figure 14.7.3 Potential and phase portrait for NLS.

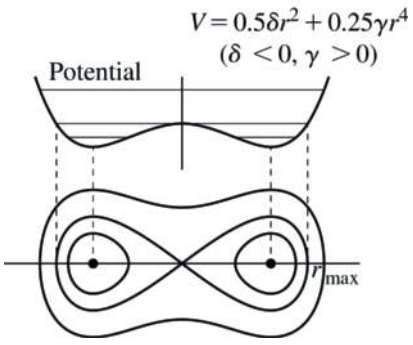
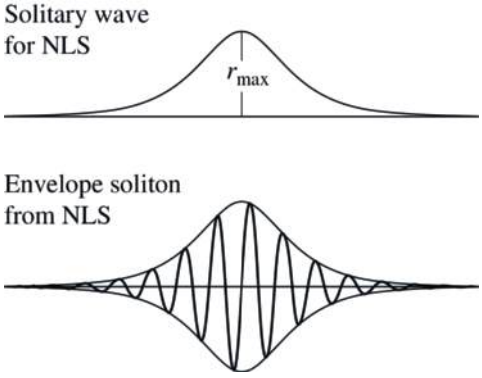


Figure 14.7.4 Solitary wave for the amplitude is used to obtain wave envelope soliton for the NLS equation.



The maximum value of r, the amplitude of the solitary wave envelope, is given by $r_{max}^2 = -2\frac{\delta}{\gamma} = -\frac{\delta}{\beta}\omega''(k_0)$. This equation can be used to determine the frequency Ω if r_{max} is known:

$$\Omega = \omega'(k_0)\alpha + \frac{\omega''(k_0)}{2}\alpha^2 - \frac{\beta}{2}r_{max}^2. \tag{14.7.39}$$

In addition to the frequency caused by the perturbed wave number, there is an amplitude dependence of the frequency. It can be shown that this wave envelope soliton with $r \to 0$ as $x \to \infty$ for the NLS (14.7.35) is given by

$$\boxed{A(x, t) = r_{max}\text{sech}\left[\sqrt{\frac{\beta}{\omega''(k_0)}}r_{max}(x - ct)\right]e^{i(\alpha x - \Omega t)},}$$

where Ω is given by (14.7.39) and c given by (14.7.37). (Note that α and $r_{\max}$ are arbitrary.) The real part of $A(x, t)$ is sketched in Fig. 14.7.4. Note that the phase velocity of the individual waves is different from the velocity of the wave envelope. These wave envelope solitary waves are known as **wave envelope solitons** because of surprising exact nonlinear interaction properties.

Exercises 14.7

14.7.1. **Curved caustic**. Near a curved caustic the wave number is approximately a constant $k_0 = k_c = k(\xi_c, 0)$ so that the Schrödinger equation (14.7.6) applies.

(a) From (14.7.7) [assuming $R(\alpha) = 1$], using the fundamental quadratic (14.6.22), derive that

$$A(x, t) = \int_{-\infty}^{\infty} e^{i\left[\alpha(x - x_c - \omega'(k_c)(t-t_c)) - \frac{\omega''(k_c)}{2!}\alpha^2(t-t_c) - \frac{\alpha^3 F''(\xi_c)}{3!\, k_\xi^2} t_c\right]} d\alpha.$$

To make the algebra easier for (b)–(d), consider

$$B(z,\tau) = \int_{-\infty}^{\infty} e^{i[\beta z + \beta^2 \tau + \beta^3/3]}\, d\beta.$$

(b) Show that B satisfies a dimensionless form of the Schrödinger equation $B\tau = -iB_{zz}$.

(c) Show that the quadratic term in the integrand can be transformed away by letting $\beta = \gamma - \tau$, in which case

$$B(z, \tau) = e^{i\left(-\tau z + \frac{2}{3}\tau^3\right)} \int_{-\infty}^{\infty} e^{i[\gamma(z - \tau^2) + \gamma^3/3]}\, d\gamma.$$

(d) This describes the intensity of light inside the caustic. The remaining integral is an Airy function, usually defined as

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i[\gamma x + \gamma^3/3]}\, d\gamma.$$

Express $B(z, \tau)$ in terms of an Airy function. [It can be shown that this Airy function satisfies $\omega'' - xw = 0$. The asymptotic expansion for large arguments of the Airy function can be used to show that the curved caustic (related to the Airy function) separates a region with two rays from a region with zero rays.]

(e) Determine $A(x, t)$ in terms of the Airy function.

14.7.2. The dispersion relation for water waves is $\omega^2 = gk \tanh kh$, where g is the usual gravitational acceleration and h is the constant depth. Determine the coefficients of the linearized KdV equation that is valid for long waves.

14.7.3. Sketch a phase portrait that shows that periodic and solitary nonlinear waves exist:

(a) **Modified KdV equation**: $\frac{\partial u}{\partial t} + 6u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$

(b) **Klein–Gordon equation**: $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u - u^3 = 0$

(c) **Sine-Gordon equation**: $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0$

14.7.4. Determine an integral formula for the period of periodic solutions of the KdV equation. Determine the wave speed in terms of the three roots of the cubic equation. Periodic solutions cannot be represented in terms of sinusoidal functions. Instead it can be shown that the solution is related to the Jacobian elliptic function *cn* and hence are called *cnoidal waves*. If you wish a project, study Jacobian elliptic functions in Abramowitz and Stegun [1974] or elsewhere.

14.7.5. Derive (using integral tables) the formula in the text for the solitary wave for

(a) the KdV equation

(b) the nonlinear Schrödinger equation

(c) derive solution for the modified KdV (see Exercise 14.7.3a)

14.7.6. Using differentiation formulas and identities for hyperbolic functions, verify the formula in the text for the solitary wave for

(a) the KdV equation

(b) the nonlinear Schrödinger equation

14.7.7. If the eigenfunction satisfies the Schrödinger equation but the time evolution of the eigenfunction satisfies $\frac{\partial \phi}{\partial t} = P\frac{\partial \phi}{\partial x} + Q\phi$, show that the equations are consistent only if $Q = -\frac{1}{2}\frac{\partial P}{\partial x}$ and $u(x, t)$ satisfies the partial differential equation $u_t = -\frac{1}{2}P_{xxx} + 2P_x(u - \lambda) + Pu_x$.

***14.7.8.** Refer to Exercise 14.7.7. If $P = A + B\lambda + C\lambda^2$ with C constant, determine A and B and a nonlinear partial differential equation for $u(x, t)$.

14.7.9. Show that Lax's equation is the Korteweg–de Vries equation for operators L and M given by (14.7.26) and (14.7.27). [*Hint*: Compute the compatibility of (14.7.23) and (14.7.24) directly using (14.7.26) and (14.7.27).]

14.7.10. Using the definitions of the reflection and transmission coefficients in Section 10.7, derive (14.7.33). In doing so, you should also derive that $\gamma = 4ik^3$ in (14.7.27). The bound states are more complicated.

14.7.11. Assume the initial condition for the KdV equation is a reflectionless potential $R(k, 0) = 0$ with one discrete eigenvalue. Solve the Gelfand–Levitan–Marchenko integral equation (it is separable) and show that $u(x, t)$ is the solitary (soliton) wave described earlier.

14.7.12. Generalize Exercise 14.7.11 to the case of a reflectionless potential with two discrete eigenvalues. The integral equation is still separable. The solution represents the interaction of two solitons.

14.8 Stability and Instability

14.8.1 BRIEF ORDINARY DIFFERENTIAL EQUATIONS AND BIFURCATION THEORY

Equilibrium solutions of partial differential equations may be stable or unstable. We will briefly develop these ideas first for ordinary differential equations, which are more fully discussed in many recent books on **dynamical systems**, such as the ones by Glendinning [1994], Strogatz [1994], and Verhulst [1997].

First-order ordinary differential equations

The concepts of equilibrium and stability are perhaps simplest in the case of autonomous first-order ordinary differential equations:

$$\boxed{\frac{dx}{dt} = f(x).} \tag{14.8.1}$$

An **equilibrium** solution x_0 is a solution of (14.8.1) independent of time:

$$\boxed{0 = f(x_0).} \tag{14.8.2}$$

An equilibrium solution is **stable** if all nearby initial conditions stay near the equilibrium. If there exists a nearby initial condition for which the solution goes away from the equilibrium, we say the equilibrium is **unstable**.

To analyze whether x_0 is stable or unstable requires considering only the differential equation (14.8.1) near x_0. We approximate the differential equation using a Taylor series of $f(x)$ around x_0 [the linearization or tangent line approximation for $f(x)$ near x_0]:

$$\frac{dx}{dt} = f(x) = f(x_0) + (x - x_0)f'(x_0) + \dots.$$

We usually can ignore the nonlinear terms since we assume x is near x_0. Since x_0 is an equilibrium, $f(x_0) = 0$, and the differential equation (14.8.1) can be approximated by a linear differential equation (with constant coefficients):

$$\boxed{\frac{dx}{dt} = (x - x_0)f'(x_0).} \tag{14.8.3}$$

The solution of (14.8.3) is straightforward: $x - x_0 = ce^{f'(x_0)t}$. We conclude that

the equilibrium x_0 is stable if $f'(x_0) < 0$
the equilibrium x_0 is unstable if $f'(x_0) > 0$.

If $f'(x_0) = 0$, the neglected nonlinear terms are needed to determine the stability of x_0.

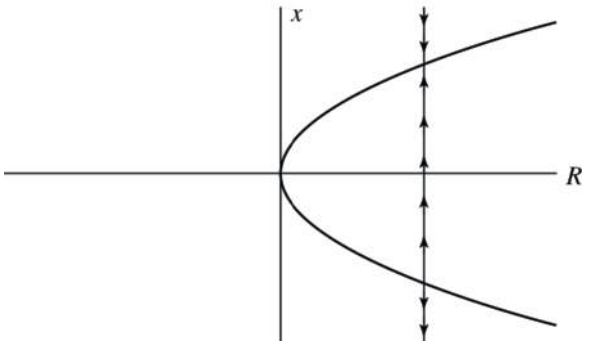
EXAMPLE OF BIFURCATION POINT

We wish to study how solutions of differential equations depend on a parameter R. As a specific elementary example, we begin by considering

$$\frac{dx}{dt} = R - x^2. \tag{14.8.4}$$

For an equilibrium, $x^2 = R$. If $R > 0$, there are two equilibria, $x = \pm\sqrt{R}$. These two equilibria coalesce to $x = 0$ when $R = 0$. If $R < 0$, there are no equilibria. $R = 0$ is called a **bifurcation point** because the number of equilibria changes there. In other examples, different kinds of bifurcation occur. Sometimes a **bifurcation diagram** (as in Fig. 14.8.1) is drawn in which the equilibria are graphed as a function of the parameter R. The stability of the equilibria can be determined using the linearization (see the Exercises). However, in the figure we illustrate the determination of stability using a **one-dimensional phase portrait**. We fix the parameter R (drawing a vertical line). If $\frac{dx}{dt} > 0$, we introduce upward arrows (and downward arrows if $\frac{dx}{dt} < 0$). In this example (14.8.4), if x is large and positive, then $\frac{dx}{dt} < 0$ (and the sign of $\frac{dx}{dt}$ changes each time an equilibria is reached since in this example the roots are simple roots). Thus we see in this example that the upper branch $(x > 0)$ is stable and the lower branch $(x < 0)$ is unstable.

Figure 14.8.1 One-dimensional phase portrait and bifurcation diagram for an example of a saddle-node bifurcation.



Definition of a bifurcation point

First-order differential equations that depend on a parameter R may be written:

$$\boxed{\frac{dx}{dt} = f(x, R).} \tag{14.8.5}$$

First, we just study equilibrium solutions x_0:

$$0 = f(x_0, R).$$

Generally, the equilibrium x_0 will depend on the parameter R. From our previous discussion, x_0 will be stable if $f_x(x_0, R) < 0$ and unstable if $f_x(x_0, R) > 0$. (In specific examples the stability can be determined using one-dimensional phase portraits, as in the previous example.)

We wish to investigate how one specific equilibrium changes as the parameter changes a small amount. We assume there is a special value of the parameter of interest R_c and we know an equilibrium corresponding to that value x_c, so that

$$0 = f(x_c, R_c).$$

If R is near R_c, we assume that x_0 will be near x_c. Thus, we use a Taylor series for a function of two variables:

$$0 = f(x_0, R) = f(x_c, R_c) + (x_0 - x_c) f_x(x_c, R_c) + (R - R_c) f_R(x_c, R_c) + \dots .$$

As an approximation, for R near R_c we conclude that usually the equilibrium is changed by a small amount:

$$x_0 - x_c = -\frac{f_R}{f_x}(R - R_c) + \dots . \tag{14.8.6}$$

This is guaranteed to occur if $f_x \neq 0$. Thus, the number of equilibria cannot change if $f_x \neq 0$. Equation (14.8.6) is the tangent line approximation to the bifurcation diagram.

The only *interesting* things can happen when $f_x = 0$. A point (x, R) is called a **bifurcation point** if

$$\boxed{f(x, R) = 0 \text{ at the same time } f_x(x, R) = 0.} \tag{14.8.7}$$

The number of equilibria can change only at a bifurcation point. Also, the stability of an equilibria may change at a bifurcation point since an equilibrium is stable if $f_x < 0$ and unstable if $f_x > 0$.

We reconsider the previous example (14.8.4), in which $0 = f(x, R) = R - x^2$. The bifurcation point can be determined by insisting that simultaneously $f_x = -2x = 0$. We see the bifurcation point is $x = 0$, in which case $R = 0$.

Saddle-node bifurcation

In this subsection, we will show that the type of bifurcation illustrated by (14.8.4) is typical. We consider the first-order problem depending on the parameter R:

$$\boxed{\frac{dx}{dt} = f(x, R).} \tag{14.8.8}$$

We assume that we have found a bifurcation point (x_c, R_c) satisfying

$$\boxed{f(x_c, R_c) = 0 \quad \text{and} \quad f_x(x_c, R_c) = 0.} \tag{14.8.9}$$

Since we are interested in solving the differential equation if R is near R_c, we will, in addition, assume that x is near x_c. Thus, we approximate the differential equation using the Taylor series of $f(x, R)$ around (x_c, R_c):

$$\begin{aligned}\frac{dx}{dt} &= f(x, R) \\ &= f(x_c, R_c) + (x - x_c) f_x(x_c, R_c) + (R - R_c) f_R(x_c, R_c) \\ &\quad + \frac{1}{2}(x - x_c)^2 f_{xx}(x_c, R_c) + \dots .\end{aligned}$$

This simplifies since (x_c, R_c) is a bifurcation point satisfying (14.8.9):

$$\boxed{\frac{dx}{dt} = (R - R_c) f_R(x_c, R_c) + \frac{1}{2}(x - x_c)^2 f_{xx}(x_c, R_c) + \dots .} \tag{14.8.10}$$

It can be shown that the other terms in the Taylor series are much smaller, such as terms containing $(R - Rc)^2$, $(x - x_c)(R - R_c)$, and $(x - x_c)^3$ because we will be assuming R is near R_c and x is near x_c with $(x - x_c)^2 = O(R - R_c)$. Thus, we will be justified (as an approximation valid near the bifurcation point) in ignoring these other higher-order terms. The bifurcation diagram near (x_c, R_c) will be approximately parabolic (similar to Fig. 14.8.1), which is derived by considering the equilibrium associated with the simple approximate differential equation (14.8.10). Here, the bifurcation point is (x_c, R_c). The parabola can open to the left or right depending on the signs of f_R and f_{xx}. For this to be a good approximation, we assume $f_R(x_c, R_c) \neq 0$ and $f_{xx}(x_c, R_c) \neq 0$. Whenever $f_R(x_c, R_c) \neq 0$ and $f_{xx}(x_c, R_c) \neq 0$, the bifurcation point is a

turning point, which is also known as a **saddle-node bifurcation** (for complicated reasons discussed briefly later). The stability of the equilibria can be determined in each case by a one-dimensional phase portrait. One branch (depending again on the signs of the Taylor coefficients) of equilibrium will be stable and the other branch unstable.

Other bifurcations

Other kinds of bifurcations (see the Exercises) such as transcritical (exchange of stabilities) and pitchfork will occur if $f_R(x_c, R_c) = 0$ or if $f_{xx}(x_c, R_c) = 0$.

Systems of first-order differential equations

Consider a system of first-order autonomous differential equations:

$$\boxed{\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f(\boldsymbol{x}) \\ g(\boldsymbol{x}) \end{bmatrix}.} \tag{14.8.11}$$

Equilibria $\boldsymbol{x_0}$ satisfy $\boldsymbol{f}(\boldsymbol{x_0}) = 0$. To analyze the stability of an equilibrium, we consider $\boldsymbol{x}$ near $\boldsymbol{x_0}$, and thus use a Taylor series. The displacement from equilibrium is introduced, $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x_0}$, and we obtain the linear system instead of (14.8.3) involving the **Jacobian matrix** $\boldsymbol{J}$ evaluated at the equilibrium

$$\boxed{\frac{d\boldsymbol{y}}{dt} = \boldsymbol{J}\boldsymbol{y},} \tag{14.8.12}$$

where

$$\boxed{\boldsymbol{J} = \begin{bmatrix} J_{11}J_{12} \\ J_{21}J_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}_{\boldsymbol{x}=\boldsymbol{x}_0}.} \tag{14.8.13}$$

To solve the linear systems of differential equation, we substitute $\boldsymbol{y}(t) = e^{\lambda t}\boldsymbol{v}$, in which case

$$\boxed{\boldsymbol{J}\boldsymbol{v} = \lambda\boldsymbol{v} \text{ or } (\boldsymbol{J} - \lambda\boldsymbol{I})\boldsymbol{v} = 0,} \tag{14.8.14}$$

so that $\boldsymbol{\lambda}$ **are the eigenvalues of the Jacobian matrix** $\boldsymbol{J}$ and $\boldsymbol{v}$ the corresponding eigenvectors. For nontrivial solutions, the eigenvalues are obtained from the determinant condition

$$\boxed{0 = \det(\boldsymbol{J} - \lambda\boldsymbol{I}) = \det\begin{bmatrix} J_{11} - \lambda & J_{12} \\ J_{21} & J_{22} - \lambda \end{bmatrix},} \tag{14.8.15}$$

so that

$$\boxed{\lambda^2 - T\lambda + D = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0,} \tag{14.8.16}$$

where the **trace** $J = T = J_{11} + J_{22}$ and **determinant** $J = D = J_{11}J_{22} - J_{12}J_{21}$ have been introduced and where the quadratic has been factored with two roots λ_1 and λ_2, which are the two eigenvalues. By comparing the two forms of the quadratic, the product of the eigenvalues equals the determinant and the sum of the eigenvalues equals the trace:

$$\boxed{\prod\lambda_i = \det \boldsymbol{J}} \tag{14.8.17}$$

$$\boxed{\sum\lambda_i = \operatorname{tr} \boldsymbol{J}.} \tag{14.8.18}$$

This is also valid for n by n matrices.

Stability for systems

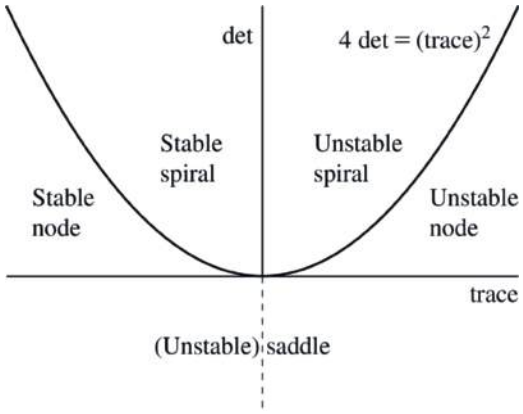
To be unstable, the displacement from the equilibrium must grow for *some* initial condition. To be stable, the displacement should stay near the equilibrium for *all* initial conditions. Since solutions are proportional to $e^{\lambda t}$ and λ may be real or complex:

The equilibrium is unstable if one eigenvalue has $\mathbf{Re}(\boldsymbol{\lambda}) > \mathbf{0}$.

The equilibrium is stable if all eigenvalues satisfy $\mathbf{Re}(\boldsymbol{\lambda}) < \mathbf{0}$.

Furthermore, the trace and determinant are particularly useful to analyze the stability of the equilibrium for 2 by 2 matrices, as summarized in Fig. 14.8.2. The well-known result (as we will explain) is that the **equilibrium is stable if and only if both tr $\boldsymbol{J} < 0$ and det $\boldsymbol{J} > 0$**. This is particularly useful when we study the Turing bifurcation. For real eigenvalues, the change from stable to unstable can occur only with one eigenvalue negative and other eigenvalue zero (changing from negative to positive), so that $\prod\lambda_i = \det \boldsymbol{J} = 0$ and $\sum\lambda i = \operatorname{tr} \boldsymbol{J} < 0$, which corresponds to a saddle-node (described later), transcritical, or pitchfork bifurcation. For complex eigenvalues, the change from stable to unstable can occur only with imaginary eigenvalues so that $\prod\lambda_i = \det \boldsymbol{J} > 0$ and $\sum\lambda_i = \operatorname{tr} \boldsymbol{J} = 0$, which corresponds to the Hopf bifurcation described later. In detail, $0 = \det(\boldsymbol{J} - \lambda \boldsymbol{I}) = \det\begin{bmatrix} J_{11} - \lambda & J_{12} \\ J_{21} & J_{22} - \lambda \end{bmatrix} = \lambda^2 - T\lambda + D = 0$, and thus $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$. If the eigenvalues are real ($4D < T^2$), the stable case (both λ negative whose phase portrait is a **stable node**) has $\prod\lambda_i = \det \boldsymbol{J} > 0$ and $\sum\lambda_i = \operatorname{tr} \boldsymbol{J} < 0$, the unstable node (both λ positive) has $\prod\lambda_i = \det \boldsymbol{J} > 0$ and $\sum\lambda_i = \operatorname{tr} \boldsymbol{J} > 0$, and the **unstable saddle point** (one λ positive and the other negative) has $\prod\lambda_i = \det \boldsymbol{J} < 0$ with positive or negative trace. If the eigenvalues are complex ($4D > T^2$), $\prod\lambda_i = |\lambda|^2 = \det \boldsymbol{J} > 0$, and **unstable spirals** have $\sum\lambda_i = \operatorname{tr} \boldsymbol{J} > 0$ while **stable spirals** have $\sum\lambda_i = \operatorname{tr} \boldsymbol{J} < 0$.

Figure 14.8.2 Stability (and phase portrait) of 2 by 2 matrices in terms of the determinant and trace.



Bifurcation theory for systems

Suppose there is a first-order system that depends on a parameter R:

$$\boxed{\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, R).} \tag{14.8.19}$$

An equilibrium solution satisfies $\boldsymbol{f}(\boldsymbol{x}_0, R) = 0$ so that, in general, the equilibrium $\boldsymbol{x}_0$ depends on the parameter R. Suppose, at some value of the parameter $R = R_c = 0$, the equilibrium is known to be at $\boldsymbol{x}_c = 0$, for convenience, so that $\boldsymbol{f}(0, 0) = 0$. We first ask how the equilibrium changes as we vary R near 0. Using a Taylor series, we have $0 = R\boldsymbol{f}_R + \boldsymbol{J}\boldsymbol{x}_0$, where $\boldsymbol{J}$ is again the Jacobian matrix (evaluated at the known equilibrium). We can solve for the equilibrium $\boldsymbol{x}_0$ uniquely, $\boldsymbol{x}_0 = -R\boldsymbol{J}^{-1}\boldsymbol{f}_R$ (known as the *implicit function theorem*), except if $\boldsymbol{J}^{-1}$ doesn't exist. The case in

which $\boldsymbol{J}^{-1}$ doesn't exist corresponds to $\det \boldsymbol{J} = 0$, so that one eigenvalue ($\lambda = 0$) of the Jacobian matrix is zero. We analyze 2 by 2 systems, and we assume the second eigenvalue is negative. It can be shown that the two-dimensional system is reduced to a one-dimensional system because the solutions exponentially decay in one direction. The solutions decay to the other direction (called the *center manifold*), on which a first-order equation such as (14.8.8) is valid. We assume a **turning point** occurs, but **transcritical** and **pitchfork bifurcations** are also possible. Each branch of the turning point bifurcation for our first-order differential equation represents an equilibrium, one stable (a negative growth rate) and one unstable (with a positive growth rate). For the two-dimensional problem, the stable branch of equilibria has two negative growth rates or eigenvalues (one hidden) and is called a stable **node**. An unstable equilibrium has one visible positive growth rate (and one hidden negative growth rate). The unstable equilibrium is called a **saddle** point (growing in one direction, decaying in the other direction). Thus, the turning point bifurcation is called a **saddle-node bifurcation** because the stable branch is a branch of (stable) nodes and the unstable branch is a branch of saddle points. The two branches coalesce at the bifurcation point. The importance of (14.8.8) is that it describes bifurcation phenomena even for higher-order systems, provided one eigenvalue is zero and the others have negative real parts.

Hopf bifurcation

For systems, at some value of the parameter R_h, called a **Hopf bifurcation**, it is possible that an equilibrium exists with the eigenvalues of the Jacobian matrix purely imaginary $\lambda = \pm i\omega$, corresponding to a frequency ω. According to the implicit function theorem, the equilibrium will exist for R near R_h since $\lambda \neq 0$. In general the equilibrium changes from being stable to unstable since the growth rate ($e^{\lambda t}$) is complex ($\lambda = s \pm i\omega$), with the real part changing from negative ($s < 0$) to positive ($s > 0$). The careful nonlinear analysis of Hopf bifurcation is too involved and would divert us from our main purposes. Hopf bifurcation is similar to the stability problem for the partial differential equation, which we will discuss shortly. We give only a crude analysis of the nonlinearity in Section 14.8.6, to which we refer the reader. There are two cases of Hopf bifurcation (see Fig. 14.8.5). In one case, called supercritical Hopf bifurcation, a stable periodic solution exists ($R > R_h$) only when the equilibrium is unstable, and for subcritical Hopf bifurcation, an unstable periodic solution exists ($R < R_h$) only when the equilibrium is stable.

14.8.2 ELEMENTARY EXAMPLE OF A STABLE EQUILIBRIUM FOR A PARTIAL DIFFERENTIAL EQUATION

In this book, all the equilibrium solutions of partial differential equations that have been considered so far are stable. For example, consider the heat equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$$

with prescribed nonzero temperature at two ends, $u(0, t) = A$ and $u(L, t) = B$. The equilibrium solution is $u_e(x) = A + (B - A)\frac{x}{L}$. To determine whether the equilibrium is stable, we consider initial conditions that are near to this: $u(x, 0) = u_e(x) + g(x)$, where $g(x)$ is small. We let

$$u(x, t) = u_e(x) + v(x, t),$$

where $v(x, t)$ is the **displacement from the equilibrium**. We determine that $v(x, t)$ satisfies

$$\frac{\partial v}{\partial t} = K\frac{\partial^2 v}{\partial x^2}, \quad \text{with} \quad v(0, t) = 0, v(L, t) = 0, \quad \text{and} \quad v(x, 0) = g(x).$$

Using our earlier results, we have

$$v(x, t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-K\left(\frac{n\pi}{L}\right)^2 t},$$

where a_n can be determined from the initial conditions (but we do not need it here). Since $v(x, t) \to 0$ as $t \to \infty$, it follows that $u(x, t) \to u_e(x)$ as $t \to \infty$. We say that the equilibrium solution $u_e(x)$ is (asymptotically) **stable**.

If $u(x, t) - u_e(x)$ is bounded as $t \to \infty$ [and we assume initially u is near $u_e(x)$], then we say the equilibrium solution is **stable**. If for some initial condition [near $u_e(x)$] $u(x, t) - u_e(x)$ is large as $t \to \infty$, then we say $u_e(x)$ is **unstable**.

In general, the displacement from an equilibrium satisfies a linear partial differential equation that will have an infinite number of degrees of freedom (modes). If one or more of the modes exponentially increases in time, the equilibrium will be unstable. To be stable, all the modes must have time dependence that exponentially decays or oscillates.

14.8.3 TYPICAL UNSTABLE EQUILIBRIUM FOR A PARTIAL DIFFERENTIAL EQUATION AND PATTERN FORMATION

One physical example of an instability is the heating from below of a fluid between two parallel plates. The interest for this type of problem is generated from meteorology in which much of the interesting weather phenomena are due to the sun's heating at the surface of the earth. For the simpler situation of heating the bottom plate, it can be observed in simple experiments that if the bottom plate is gently heated, then a simple state of conductive heat flow arises [solving the usual heat equation, so that the conductive state satisfies $u = u(0) + \frac{y}{L}(u(L) - u(0))$ if the bottom is $y = 0$ and the top $y = L$]. Heating the bottom of the fluid causes the bottom portions of the fluid to be less dense than the top. Due to buoyancy, there is a tendency for the fluid to move (the less dense hot fluid rising and the more dense cold falling). The partial differential equations that describe this must include Newton's laws for the velocity of the fluid in addition to the heat equation for the temperature. The gravitational force tends to stabilize the situation, and the buoyant force tends to destabilize the situation. Experimentally it is observed that if the bottom is heated sufficiently, then the conductive state becomes unstable. The fluid tends to move more dramatically and rotating cells of fluid are formed between the plates (reminiscent of large-scale atmospheric motions). A preferred horizontal length scale of motion is observed when no horizontal length scales are present to begin with. This process of pattern formation is fundamental in physical and natural sciences. We wish to explain these types of features. However, a good mathematical model of this buoyancy instability is perhaps a little too difficult for a first example.

We will analyze the following partial differential equation (which will have many desirable features), which is related to the linearized **Kuramoto–Sivashinsky** equation:

$$\boxed{\frac{\partial u}{\partial t} = -u - R\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}.} \tag{14.8.20}$$

We note that $u = 0$ is considered to be an equilibrium solution of (14.8.20). We assume $R > 0$ is a parameter of interest. To understand this equation, we substitute $u = e^{i(kx-\omega t)}$ or $u = e^{\sigma t}e^{ikx}$. In either way, we find

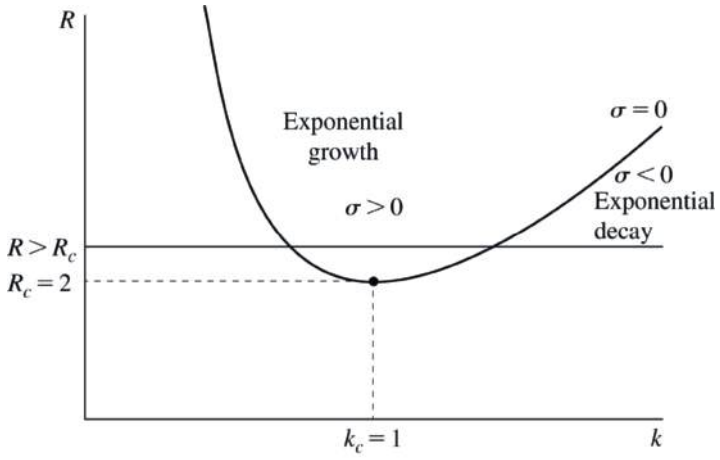
$$\sigma = -1 + Rk^2 - k^4. \tag{14.8.21}$$

The growth rate is a function of R and k. In this example, the exponential growth rate σ is real for all values of the wave number k (for all wavelengths). It is very important to distinguish between $\sigma > 0$ (exponential growth) and $\sigma < 0$ (exponential decay). In Fig. 14.8.3, we graph regions of exponential growth and exponential decay by graphing $\sigma = 0$:

$$\sigma = 0 \quad \text{if} \quad R = \frac{1}{k^2} + k^2.$$

This **neutral stability curve**, which separates stable from unstable (see Fig. 14.8.3), has a vertical asymptote at $k = 0$ and for large k is approximately the parabola $R = k^2$. There is only one critical point, an absolute minimum, where $\frac{dR}{dk} = 0$. To determine the minimum, we note that $0 = -2\frac{1}{k^3} + 2k$ yields $k_c = 1$, in which case $R_c = 1 + 1 = 2$. If $R < R_c$, then we say $u = 0$ is stable because the time dependence of all modes (all values of k) exponentially decays. If $R > R_c$, then there is a band of wave numbers that grow exponentially. If any of these wave numbers occurs, then we say $u = 0$ is unstable. For example, if the partial differential equation (14.8.20) is to be solved on the infinite interval, then all values of k are relevant (from the Fourier transform), and we say that $u = 0$ is unstable if $R > R_c$. Typically experiments are performed in which R is gradually increased from a value in which $u = 0$ is stable to a value in which $u = 0$ is unstable. If R is slightly greater than the critical value R_c, then we expect waves to grow for wave numbers in a small band surrounding $k_c = 1$. Thus, we expect the solution to involve the wavelength $\frac{2\pi}{k_c}$, a preferred wavelength. This is the way that patterns form in nature from rather arbitrary initial conditions.

Figure 14.8.3 Neutral stability curve.



However (this is a little subtle), if the boundary conditions (after separation) are $u(0) = u''(0) = 0$ and $u(L) = u''(L) = 0$, then it can be shown that the eigenfunctions are $\sin\frac{n\pi x}{L}$, so that k takes on the discrete values $k = \frac{n\pi}{L}$. If R is not much greater than R_c, then there is only a thin band of unstable wave numbers, and it is possible that $u = 0$ is stable. The first instability could occur when R is some specific value greater than R_c and would correspond to specific value of $n = n_c$. Patterns would be expected to be formed with this different wavelength $\frac{2L}{n_c}$. In the rest of this chapter, we will assume we are solving partial differential equations on the infinite interval.

In Fig. 14.8.4, an alternate method is used to illustrate the growth rate. We graph the growth rate $\sigma = \sigma(k, R)$ given by (14.8.21) for fixed R. We note that $\sigma < 0$ for all k if $R < R_c = 2$. At $R = R_c = 2$, the growth rate first becomes zero at $k = k_c = 1$:

$$\sigma(k_c, R_c) = 0 \tag{14.8.22}$$

$$\sigma_k(k_c, R_c) = 0 \tag{14.8.23}$$

$$\sigma_{kk}(k_c, R_c) < 0. \tag{14.8.24}$$

If R is slightly greater than R_c, then there is a band of wave numbers near k_c in which $\sigma > 0$. This is the same band of unstable wave numbers shown in Fig. 14.8.3. This band of unstable wave numbers can be estimated (if R is slightly greater than R_c) using the Taylor series of a function of two variables for $\sigma(k, R)$ around $k = k_c$ and $R = R_c$:

$$\sigma(k, R) = \sigma(k_c, R_c) + \sigma_k(k_c, R_c)(k - k_c) + \frac{1}{2}\sigma_{kk}(k_c, R_c)(k - k_c)^2 + \\ + \ldots + \sigma_R(k_c, R_c)(R - R_c) + \ldots.$$

This simplifies due to (14.8.22) and (14.8.23):

$$\sigma(k, R) = \frac{1}{2}\sigma_{kk}(k_c, R_c)(k - k_c)^2 + \sigma_R(k_c, R_c)(R - R_c) + \ldots. \tag{14.8.25}$$

It can be shown that other terms in the Taylor series can be neglected. Since the band of unstable wave numbers terminates at $\sigma(k, R) = 0$, we have as an approximation that unstable wave numbers satisfy for $R > R_c$

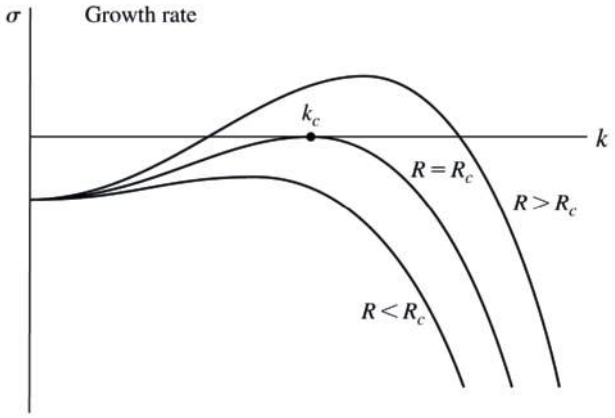
$$|k - k_c| < \sqrt{\frac{-2\sigma_R}{\sigma_{kk}}(R - R_c)}.$$

We also need to assume that

$$\sigma_R(k_c, R_c) > 0, \tag{14.8.26}$$

since (at fixed $k = k_c$) we want σ to be increasing at $R = R_c$ (from $\sigma < 0$ to $\sigma > 0$).

Figure 14.8.4 Growth rate as function of wave number.



In other linear partial differential equations, the exponential time dependence is complex, $e^{st} = e^{(\sigma - i\omega)t}$. Thus, σ is the real part of s and $-\omega$ is the imaginary part of s, where $\omega(k)$ is a frequency.

14.8.4 ILL-POSED PROBLEMS

A linear time-dependent partial differential equation is said to be **ill posed** if the largest exponential growth rate is positive but **unbounded** for allowable wave numbers. When a problem is ill posed, it suggests that the partial differential equation does not correctly model some physical phenomena. Instead some important physics has been ignored or incorrectly modeled.

We show that the backward (in time) heat equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} \tag{14.8.27}$$

is ill posed because the exponential growth rate ($u = e^{\sigma t} e^{ikx}$) is

$$\sigma = k^2.$$

$u = 0$ is unstable because $\sigma > 0$ for values of k of interest. However, the growth rate is positive and unbounded as $k \to \infty$, and thus (14.8.27) is ill posed. This difficulty occurs when $k \to \infty$, which corresponds to indefinitely short waves (the wavelengths approaching zero). The backward heat equation is even ill posed with zero boundary conditions at $x = 0$ and $x = L$, since then $k = \frac{n\pi}{L}$ and the growth rate is positive and unbounded as $n \to \infty$. (If indefinitely short wavelengths could be excluded, then the backward heat equation would not be ill posed.)

Without the fourth derivative, (14.8.20) would be ill posed like the backward heat equation. However, the fourth derivative term in (14.8.20) prevents short waves from growing. Although $u = 0$ is unstable for (14.8.20), for fixed $R > R_c = 2$ the largest growth rate is bounded. From (14.8.21), $\sigma_{\max} = -1 + \frac{1}{4}R^2$ (which is finite).

14.8.5 SLIGHTLY UNSTABLE DISPERSIVE WAVES AND THE LINEARIZED COMPLEX GINZBURG–LANDAU EQUATION

For linear **purely** dispersive waves, solutions of the partial differential equation are in the form $e^{i(kx - \omega t)}$, where the frequency ω is real and said to satisfy the dispersion relation $\omega = \omega(k)$. For other partial differential equations, the frequency is not real. We prefer to analyze these more general problems using the complex growth rate $s = \sigma - i\omega$:$e^{ikx}e^{st} = e^{ikx}e^{(\sigma - i\omega)t}$. If $\sigma > 0$ for some allowable k, we say that $u = 0$ is unstable. Often the partial differential equation depends on a parameter R in which the solution $u = 0$ is stable for $R < R_c$ and becomes unstable at $R = R_c$. As in Section 14.7, we assume there is a preferred wave number k_c such that the real exponential growth rate $\sigma(k, R)$ satisfies (14.8.22), (14.8.23), (14.8.24), and (14.8.26) near $k = k_c$ and $R = R_c$.

In this section, we will assume R is slightly greater than R_c so that $u = 0$ is unstable, but the largest positive value of σ will be small, so we call $u = 0$ **slightly unstable**. In this case, there is a small band of unstable wave numbers near k_c. We expect that energy is focused in one wave number k_c. We have discussed (in Section 14.6) the nearly monochromatic assumption $u(x, t) = A(x, t)e^{i(k_0 x - \omega(k_0)t)}$ for purely dispersive waves and have shown that the wave envelope

approximately satisfies the linear Schrödinger equation, $\frac{\partial A}{\partial t} + \omega'(k_0)\frac{\partial A}{\partial x} = i\frac{\omega''(k_0)}{2!}\frac{\partial^2 A}{\partial x^2}$. Here, we wish to generalize that result to the case of slightly unstable waves, where the time dependence is oscillatory and exponential, $e^{st} = e^{(\sigma - i\omega)t}$. At $R = R_c$ and $k = k_c$, the exponential growth is zero $\sigma(k_c, R_c) = 0$, but often there is a nonzero critical frequency $\omega(k_c, R_c)$. Thus, as before, we expect $A(x, t)$ to be the wave envelope of an elementary oscillatory traveling wave:

$$\boxed{u(x, t) = e^{ikx}e^{s(k,R)t} = A(x, t)e^{i[k_c x - \omega(k_c, R_c)t]},} \tag{14.8.28}$$

so that

$$A(x, t) = e^{i(k-k_c)x}e^{[s(k,\, r) - s(k_c,\, R_c)]t}.$$

There are elementary solutions for u of the form $u = e^{ikx}e^{[\sigma(k,\, R) - i\omega(k,\, R)]t}$. If we apply our earlier ideas concerning spatial and temporal derivatives, (14.7.2) and (14.7.3), we have here

$$-i\frac{\partial}{\partial x} \Leftrightarrow (k - k_c) \tag{14.8.29}$$

$$\frac{\partial}{\partial t} \Leftrightarrow s(k, R) - s(k_c, R_c) = s(k, R) + i\omega(k_c, R_c). \tag{14.8.30}$$

We can derive a partial differential equation that the amplitude $A(x, t)$ satisfies by considering the Taylor expansion of $s(k, R)$ around $k = k_c$ and $R = R_c$:

$$s(k, R) = s(k_c, R_c) + s_k(k - k_c) + \frac{s_{kk}}{2!}(k - k_c)^2 + \ldots + s_R(R - R_c) + \ldots. \tag{14.8.31}$$

All partial derivatives are to be evaluated at $k = k_c$ and $R = R_c$. We will assume $(k - k_c)^2 = O(R - R_c)$, so that it can be shown that other terms in the Taylor series are smaller than the ones kept. Recall that $s(k, R)$ is complex, $s(k, R) = \sigma(k, R) - i\omega(k, R)$, so that s_R and s_{kk} are complex. However, $s_k = \sigma_k - i\omega_k = -i\omega_k$ since $\sigma_k = 0$ at $k = k_c$ and $R = R_c$ from (14.8.23). From (14.8.31), using (14.8.29) and (14.8.30), we obtain

$$\frac{\partial A}{\partial t} = -is_k\frac{\partial A}{\partial x} - \frac{s_{kk}}{2!}\frac{\partial^2 A}{\partial x^2} + s_R(R - R_c)A.$$

Since $is_k = \omega_k$ is the usual real group velocity, the complex wave amplitude solves a partial differential equation known as the **linearized complex Ginzburg–Landau** (LCGL) equation:

$$\boxed{\frac{\partial A}{\partial t} + \omega_k\frac{\partial A}{\partial x} = -\frac{s_{kk}}{2}\frac{\partial^2 A}{\partial x^2} + s_R(R - R_c)A.} \tag{14.8.32}$$

The diffusion coefficient (the coefficient in front of the second derivative) is complex since $s_{kk} = \sigma_{kk} - i\omega_{kk}$. In order for LCGL (14.8.32) to be well posed, the real part of the diffusion coefficient $-\sigma_{kk}/2$ must be positive. This occurs because the real growth rate σ has a local maximum there (14.8.24). Equation (14.8.32) is the wave envelope equation that generalizes the linear Schrödinger equation to virtually any physical situation (in any discipline) where $u = 0$ is slightly unstable. Wavelike solutions of (14.8.32) grow exponentially in time.

Two spatial dimensions

It can be shown that in two-dimensional problems, $s(k_1, k_2, R)$, so that (14.8.29) is generalized to $k_1 - k_{10} = -i\frac{\partial}{\partial x}$ and $k_2 - k_{20} = -i\frac{\partial}{\partial y}$. We can orient our coordinate system so that the basic wave is in the x-direction, in which case $k_{20} = 0$ and thus $k_{10} = k_c$:

$$k_1 - k_c = -i\frac{\partial}{\partial x} \tag{14.8.33}$$

$$k_2 = -i\frac{\partial}{\partial y}. \tag{14.8.34}$$

It can be shown that very little change is needed in (14.8.32). The linear term $s_k(k - k_c)$ in (14.8.31) yields the group velocity term in (14.8.32) so that $\omega_k\frac{\partial A}{\partial x}$ in one dimension becomes $\omega_{k_1}\frac{\partial A}{\partial x} + \omega_{k_2}\frac{\partial A}{\partial y}$ in two dimensions. We must be more careful with the quadratic term. We assume the original physical partial differential equation has no preferential direction, so that $s(k)$ where $k = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2}$. The Taylor series in (14.8.31) remains valid. We must evaluate only the

quadratic term $(k - k_c)^2$, where from (14.8.33) and (14.8.34) k_1 is near k_c but k_2 is near zero. Using the Taylor series of two variables (k_1, k_2) around $(k_c, 0)$ as an approximation,

$$k = \sqrt{k_1^2 + k_2^2} = k_c + 1(k_1 - k_c) + \frac{1}{2k_c}k_2^2 + \dots .$$

It is perhaps easier to first derive an expression for k^2. It is very important to note that we are assuming that $(k_1 - k_c)$ has the same small size as k_2^2. In particular, we are assuming $(k_1 - k_c)^2 = O(k_2^4) = O(R - R_c)$. In this way, we derive the equation known as the **linearized Newell–Whitehead–Segel** equation:

$$\frac{\partial A}{\partial t} + \omega_{k_1}\frac{\partial A}{\partial x} + \omega_{k_2}\frac{\partial A}{\partial y} = -\frac{s_{kk}}{2}\left(\frac{\partial}{\partial x} - \frac{i}{2k_c}\frac{\partial^2}{\partial y^2}\right)^2 A + S_R(R - R_c)A. \tag{14.8.35}$$

When nonlinear terms are included using perturbation methods, the complex cubic nonlinearity $\gamma A|A|^2$ is added to the right-hand side.

14.8.6 NONLINEAR COMPLEX GINZBURG–LANDAU EQUATION

The LCGL (14.8.32) describes the wave amplitude of a solution to a linear partial differential equation when the solution $u = 0$ is slightly unstable. Linear partial differential equations are often small-amplitude approximations of the real nonlinear partial differential equations of nature. Since solutions of (14.8.32) grow exponentially in time, eventually the solution is sufficiently large so that the linearization approximation is no longer valid. Using perturbation methods, it can be shown that for most physical problems with a slightly unstable solution, (14.8.32) must be modified to account for small nonlinear terms. In this way $u(x, t)$ can be approximated by $A(x, t)e^{i[k_c x - \omega(k_c, R_c)t]}$ as before, but $A(x, t)$ satisfies the **complex Ginzburg–Landau** (CGL) equation:

$$\boxed{\frac{\partial A}{\partial t} + \omega_k\frac{\partial A}{\partial x} = -\frac{s_{kk}}{2}\frac{\partial^2 A}{\partial x^2} + s_R(R - R_c)A + \gamma A|A|^2,} \tag{14.8.36}$$

where $\gamma = \alpha + i\beta$ is an important complex coefficient that can be derived with considerable effort for specific physical problems using perturbation methods. Note that the CGL equation (14.8.36) generalizes the NLS equation (14.7.35) by having a complex coefficient ($s_{kk} = \sigma_{kk} - i\omega_{kk}$ with $\sigma_{kk} < 0$) in front of the second derivative instead of an imaginary coefficient, by having the instability term $s_R(R - R_c)A$, and by the nonlinear coefficient being complex. As with nearly all nonlinear partial differential equations, a complete analysis of the initial value problem for CGL (14.8.36) is impossible. We will discuss only some simpler results.

Bifurcation diagrams and the Landau equation

The amplitude equation (14.8.36) has no spatial dependence if the original unstable problem is an ordinary differential equation or if the wave number is fixed at k_c (as would occur if the partial differential equation was solved in a finite geometry). With no spatial dependence, the CGL amplitude equation becomes an ordinary differential equation, the **Landau** equation,

$$\boxed{\frac{dA}{dt} = s_R(R - R_c)A + \gamma A|A|^2.} \tag{14.8.37}$$

The solution $u = 0$ becomes unstable $[u(x, t) = A(t)e^{i[k_c x - \omega(k_c, R_c)t]}]$ with the frequency $\omega(k_c, R_c)$ at $R = R_c$. This is characteristic of a phenomenon in ordinary differential equations known as **Hopf bifurcation**. We should recall that the coefficients $s_R = \sigma_R - i\omega_R$ and $\gamma = \alpha + i\beta$ are complex with $\sigma_R > 0$ [see (14.8.26)]. The Landau equation can be solved by introducing polar coordinates $A(t) = r(t)e^{i\theta(t)}$. Since $\frac{dA}{dt} = e^{i\theta(t)}(\frac{dr}{dt} + ir\frac{d\theta}{dt})$, it follows by appropriately taking real and imaginary parts that (14.8.37) becomes two equations for the magnitude r and phase θ:

$$\boxed{\frac{dr}{dt} = \sigma_R(R - R_c)r + \alpha r^3} \tag{14.8.38}$$

$$\frac{d\theta}{dt} = -\omega_R(R - R_c) + \beta r^2. \tag{14.8.39}$$

We assume $\sigma_R > 0$, corresponding to the solution $u = 0$ being stable for $R < R_c$ and unstable for $R > R_c$. The radial equation (14.8.38) is most important and can be solved independently of the phase equation. First we look for equilibrium solutions r_e of the radial equation (14.8.38):

$$0 = \sigma_R(R - R_c)r_e + \alpha r_e^3.$$

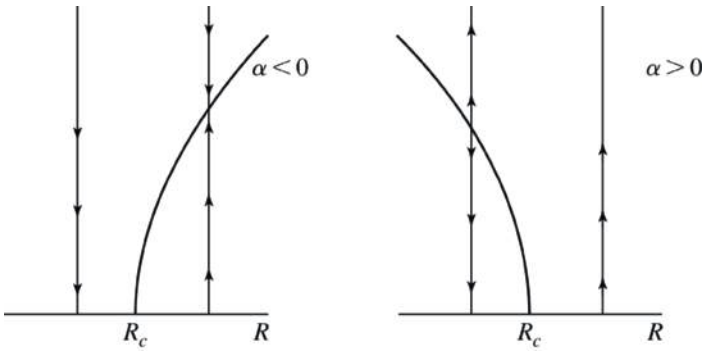
The equilibrium solution $r_e = 0$ corresponds to $A = 0$ or $u = 0$. We will shortly verify what we should already know: that $u = 0 (r_e = 0)$ is stable for $R < R_c$ and unstable for $R > R_c$. There is another important (nearby) equilibrium satisfying

$$0 = \sigma_R(R - R_c) + \alpha r_e^2.$$

For this nonzero equilibrium to exist, $\frac{\sigma_R(R-R_c)}{\alpha} < 0$. If $\alpha < 0$, the nonzero equilibrium exists if $R > R_c$ (and vice versa). We graph two cases (depending on whether $\alpha > 0$ or $\alpha < 0$) in Fig. 14.8.5; the equilibrium r_e as a function of the parameter R is known as a **bifurcation diagram**. $R = R_c$ is called a **bifurcation point** since the number of equilibrium solutions changes (bifurcates) there. Since $r = |A|$, we restrict our attention to $r \geq 0$. However, these are only equilibrium solutions of the first-order nonlinear differential equation (14.8.38).

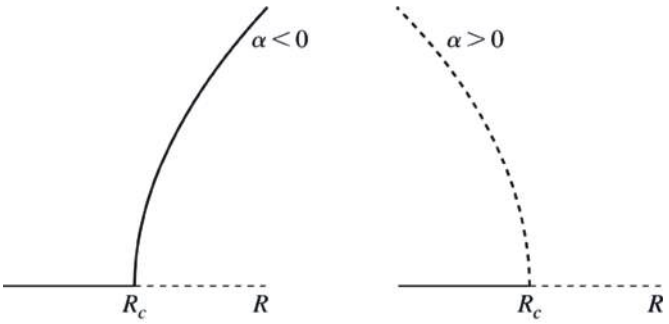
To determine all solutions (not just equilibrium solutions) and determine whether these equilibrium solutions are stable or unstable, we graph **one-dimensional phase portraits** in Fig. 14.8.5. It is best to draw vertical lines (corresponding to fixing R) and introduce arrows upward if r is increasing $(\frac{dr}{dt} > 0)$ and arrows downward if r is decreasing $(\frac{dr}{dt} < 0)$. The sign of $\frac{dr}{dt}$ is determined from the differential equation (14.8.38).

Figure 14.8.5 Bifurcation diagram and one-dimensional phase portrait for Hopf bifurcation.



There are two cases depending on the sign of α (the real part of γ). In words, we describe only the case in which $\alpha < 0$, but both figures are presented. We note from (14.8.38) that $\frac{dr}{dt} < 0$ (and introduce downward arrows) if r is sufficiently large in the case $\alpha < 0$. The direction of the arrows changes at the equilibria (if the equilibria are a simple root). We now see that $r = 0$ (corresponding to $u = 0$) is stable if $R < R_c$since all nearby solutions approach the equilibria as time increases, and $r = 0$ is unstable if $R > R_c$. The bifurcated nonzero equilibria (which coalesce on $r = 0$ as $R \to R_c$) exist only for $R > R_c$ and are stable (if $\alpha < 0$). If $\alpha < 0$ $(\alpha > 0)$, this is called **supercritical** (subcritical) Hopf bifurcation because the nonzero equilibria exist for R greater (less) than R_c. It is usual in bifurcation diagrams to mark unstable equilibria with dashed lines (and stable with solid lines), and we have done so in Fig. 14.8.6. The nonlinearity has two possible effects: If the nonlinearity is stabilizing $(\alpha < 0)$, then a stable solution is created for $R > R_c$. This stable solution has small amplitude since r_e is proportional to $(R - R_c)^{1/2}$, consistent with our analysis based on Taylor series assuming $R - R_c$ is small and r is near zero. If the nonlinearity is destabilizing $(\alpha > 0)$, then an unstable solution is created for $R < R_c$. If the nonlinearity is destabilizing $(\alpha > 0)$, a better question is what occurs if $R > R_c$? In this case the linear dynamics are unstable, but the nonlinear terms do not stabilize the solution. Instead it can be shown that the solution explodes (goes to infinity in a finite time). If $R > R_c$ (and $\alpha > 0$), there are no stable solutions with small amplitudes amenable to a linear or weakly nonlinear analysis; we must understand the original fully nonlinear problem.

Figure 14.8.6 Bifurcation diagram (including stability) for Hopf bifurcation.



The bifurcated equilibrium solution (stable or unstable) has r equaling a specific constant (depending on $R - R_c$). From (14.8.39) this bifurcated equilibrium solution ($A = re^{i\theta} = r_e e^{i\theta}$) is actually a periodic solution with frequency $\omega_R(R - R_c) - \beta r_e^2 = (R - R_c)(\alpha\omega_R - \beta\sigma_R)/\alpha$. For the original partial differential equation [$u = A(x, t)e^{i[k_c x - \omega(k_c, R_c)t]}$], the frequency is $\omega(k_c, R_c) + (R - R_c)(\alpha\omega_R - \beta\sigma_R)/\alpha$, corresponding to dependence of the frequency on the parameter R and the equilibrium amplitude $|A|^2 = r_e^2$. The bifurcated solution is periodic in time (with an unchanged wave number). For this relatively simple Landau equation (14.8.37), the bifurcated solution is stable when the zero solution is unstable (and vice versa).

Bifurcated solutions for complex Ginzburg–Landau equation

We will show that the CGL equation (14.8.36) has elementary nonzero plane traveling wave solutions

$$A(x, t) = A_0 e^{i[Kx - \Omega t + \phi_0]}, \tag{14.8.40}$$

where A_0 and ϕ_0 are real constants. Since $u = A(x, t)e^{i[k_c x - \omega(k_c, R_c)t]}$, it follows that the wave number for $u(x, t)$ is $k_c + K$. Previously we have called k the wave number for $u(x, t)$, and thus

$$K = k - k_c$$

and the frequency is $\omega(k_c, R_c) + \Omega$. For CGL (14.8.36) to be valid, K must be small (corresponding to the wave number being near k_c). We show (14.8.40) are solutions by substituting (14.8.40) into (14.8.36):

$$-i\Omega + i\omega_k K = \frac{s_{kk}}{2}K^2 + s_R(R - R_c) + \gamma A_0^2.$$

Since $s_{kk} = \sigma_{kk} - i\omega_{kk}$ (with $\sigma_{kk} < 0$), $s_R = \sigma_R - i\omega_R$ (with $\sigma_R > 0$), and $\gamma = \alpha + i\beta$ are complex, we must take real and imaginary parts:

$$\alpha A_0^2 = -\left[\frac{\sigma_{kk}}{2}K^2 + \sigma_R(R - R_c)\right] \tag{14.8.41}$$

$$\Omega = \omega_k K + \frac{\omega_{kk}}{2}K^2 + \omega_R(R - R_c) - \beta A_0^2. \tag{14.8.42}$$

Solutions of the form (14.8.40) exist only if the right-hand side of (14.8.41) has the same sign of α. In (14.8.41), we recognize $\frac{\sigma_{kk}}{2}K^2 + \sigma_R(R - R_c)$ as the Taylor expansion (14.8.25) of the growth rate $\sigma(k, r)$. As before, there are two completely different cases depending on the sign of α, the real part of the nonlinear coefficient γ. If $\alpha < 0$ (the case in which the nonlinearity is stabilizing for the simpler Landau equation), solutions exist only if $\frac{\sigma_{kk}}{2}K^2 + \sigma_R(R - R_c) > 0$, which corresponds to the growth rate being positive (the small band of exponentially growing wave numbers near k_c that occurs only for $R > R_c$). If $\alpha > 0$ (the case in which the bifurcated solution is unstable for the simpler Landau equation), solutions exist if $\frac{\sigma_{kk}}{2}K^2 + \sigma_R(R - R_c) < 0$, which corresponds to the growth rate being negative (excluding the small band of exponentially growing wave numbers near k_c). However, it is not easy to determine whether these solutions are stable. We will briefly comment on this in the next subsection.

We should comment on (14.8.42), though it is probably less important than (14.8.41). Equation (14.8.42) shows that the frequency for solutions (14.8.40) changes because the frequency depends on the wave number, the parameter R, and the amplitude.

Stability of bifurcated solutions for complex Ginzburg–Landau equation

It is difficult to determine the stability of these elementary traveling wave solutions of the CGL equation (14.8.36). First we derive the modulational instability for the nonlinear Schrödinger equation (the CGL when the coefficients are purely imaginary) and then outline some results for the case in which the coefficients are real. The general case (which we omit) is closer to the real case.

Modulational (Benjamin–Feir) instability

When the coefficients of the CGL equation are purely imaginary ($s_{kk} = -i\omega_{kk}$ with $\sigma_{kk} = 0$, $s_R = \sigma_R - i\omega_R$ with $\sigma_R = 0$, and $\gamma = i\beta$ with $\alpha = 0$), the CGL equation reduces to the nonlinear Schrödinger equation (NLS):

$$\frac{\partial A}{\partial t} + \omega_k \frac{\partial A}{\partial x} = i\frac{\omega_{kk}}{2}\frac{\partial^2 A}{\partial x^2} + i\beta A|A|^2. \tag{14.8.43}$$

Actually, there is an additional term $-i\omega_R(R - R_c)A$ on the right-hand side of (14.8.43), but we ignore it since that term can be shown to contribute only a frequency shift corresponding to changing the parameter R. This does not correspond to the instability of $u = 0$ but instead describes a situation in which the energy is focused in one wave number k_0. The elementary traveling wave solution $A(x, t) = A_0 e^{i[Kx - \Omega t + \phi_0]}$ always exists for arbitrary A_0. Equation (14.8.41) becomes $0 = 0$, so that the amplitude A_0 is arbitrary, while (14.8.42) determines the frequency Ω. (A special example, frequently discussed in optics, occurs with $K = 0$, in which case $\Omega = -\beta A_0^2$.) We will show (not easily) that these solutions are stable if β has a different sign from ω_{kk}, and thus in some cases these nonlinear traveling waves will be observed. However, if β has the same sign as ω_{kk}, then these traveling waves are unstable, as first shown by Benjamin and Feir in 1967. We follow Newell [1985]. We move with the group velocity so that the ω_k-term can be neglected in (14.8.43). We substitute $A = re^{i\phi}$ into (14.8.43), assuming nonconstant wave number $k = \phi_x$ and frequency $\Omega = -\phi_t$. We note $A_x = (r_x + ikr)e^{i\phi}$, $A_t = (r_t - i\Omega r)e^{i\phi}$, and $A_{xx} = [r_{xx} - k^2 r + i(2kr_x + k_x r)]e^{i\phi}$. Thus, the imaginary part gives $-\Omega r = \frac{\omega_{kk}}{2}(r_{xx} - k^2 r) + \beta r^3$, the exact dispersion relation for the nonlinear Schrödinger equation, and the real part yields an exact equation for the modulation of the amplitude:

$$r_t + \frac{\omega_{kk}}{2}(2kr_x + k_x r) = 0. \tag{14.8.44}$$

Using the dispersion relation, conservation of waves $k_t + \Omega_x = 0$ becomes

$$k_t + \left[\frac{\omega_{kk}}{2}\left(-\frac{r_{xx}}{r} + k^2\right) - \beta r^2\right]_x = 0. \tag{14.8.45}$$

Equations (14.8.44) and (14.8.45) give an exact representation of solutions of the nonlinear Schrödinger equation. We note the simple exact solution $k = 0$, $r = r_0$. From the dispersion relation, $-\Omega = \beta r_0^2$, which corresponds to the exact solution $A = r_0 e^{-i\beta r_0^2 t}$ of the nonlinear Schrödinger equation. To investigate the stability of this solution, we linearize the nonlinear system by letting $k = \mu k_1$ and $r = r_0 + \mu r_1$, where μ is small, and obtain (after dropping the subscripts 1)

$$r_t + \frac{\omega_{kk}}{2}k_x r_0 = 0 \tag{14.8.46}$$

$$k_t - \frac{\omega_{kk}}{2}\frac{r_{xxx}}{r_0} - 2\beta r_0 r_x = 0. \tag{14.8.47}$$

Eliminating k_{xt} yields the linear constant-coefficient dispersive partial differential equation

$$r_{tt} = \frac{\omega_{kk}}{2}\left(-\frac{\omega_{kk}}{2}r_{xxxx} - 2\beta r_0^2 r_{xx}\right). \tag{14.8.48}$$

We analyze (14.8.48) in the usual way by letting $r = e^{i(\alpha x - \Omega_1 t)}$, where α corresponds to a sideband wave number. The dispersion relation for (14.8.48) is $\Omega_1^2 = \left(\frac{\omega_{kk}}{2}\right)^2 \alpha^4 - \omega_{kk}\beta r_0^2 \alpha^2$. If ω_{kk} and β have opposite signs, the frequency Ω_1 is real for all α, and this solution is stable (to

sidebands). The **Benjamin–Feir sideband instability** occurs when ω_{kk} and β have the same sign (called the focusing nonlinear Schrödinger) in which waves are unstable for wave numbers satisfying $0 < \alpha < 2\sqrt{\frac{\beta}{\omega_{kk}}}|r_0|$. Sufficiently long-wave sideband perturbations are unstable.

Recurrence for nonlinear Schrödinger equation

Analysis of the initial value problem for (14.8.43) with periodic boundary conditions shows that (see Ablowitz and Clarkson [1991]) more than one wavelength corresponding to an elementary Fourier series may be unstable. At first, these unstable waves will grow exponentially in the manner associated with a linear partial differential equation. However, eventually, the nonlinearity prevents the growth from continuing. For the case in which the elementary solution is unstable (β has the same sign as ω_{kk}), experiments (Yuen and Lake [1975]), numerical solutions (Yuen and Ferguson [1978]), and advanced theory remarkably show that the solution nearly returns to its initial condition after a long time, and then the instability nearly repeats over and over again. This phenomenon is called **recurrence**.

Stable and unstable finite amplitude waves

If $u = 0$ becomes unstable at $R = R_c$ with a preferred wave number $k = k_c$ but the growth rate is real ($s_{kk} = \sigma_{kk}$ with $\sigma_{kk} < 0$ and $\omega_{kk} = 0$, $s_R = \sigma_R$ with $\sigma_R > 0$ and $\omega_R = 0$) and the nonlinear coefficient is real ($\gamma = \alpha$), then the CGL (14.8.36) becomes

$$\frac{\partial A}{\partial t} = -\frac{\sigma_{kk}}{2}\frac{\partial^2 A}{\partial x^2} + \sigma_R(R - R_c)A + \alpha A|A|^2,$$

where we have also assumed $\omega_k = 0$. The elementary traveling wave solution $A(x, t) = A_0 e^{i[Kx - \Omega t + \phi_0]}$ corresponds to $K = k - k_c$. Using our earlier result, (14.8.41) and (14.8.42), solutions exist if

$$\alpha A_0^2 = -\left[\frac{\sigma_{kk}}{2}K^2 + \sigma_R(R - R_c)\right] \tag{14.8.49}$$

$$\Omega = 0. \tag{14.8.50}$$

This solution is periodic in space and constant in time. As in the general case, if $\alpha < 0$, solutions exist only in the small band of exponentially growing wave numbers $\left(|k - k_c| < \sqrt{\frac{2\sigma_R(R - R_c)}{-\sigma_{kk}}}\right)$ that occur for $R > R_c$.However, Eckhaus in 1965 showed that these waves are stable only in the smaller bandwidth $|k - k_c| < \frac{1}{\sqrt{3}}\sqrt{\frac{2\sigma_R(R - R_c)}{-\sigma_{kk}}}$.

14.8.7 LONG-WAVE INSTABILITIES

We again consider linear partial differential equations with solutions of the form $u(x, t) = e^{ikx}e^{st} = e^{ikx}e^{(\sigma - i\omega)t}$, allowing for unstable growth ($\sigma > 0$) or decay ($\sigma < 0$). If waves first become unstable with wave number $k \neq 0$, then usually the complex Ginzburg–Landau equation (14.8.32) or (14.8.36) is valid. However, long waves $k = 0$ usually become unstable in a different manner. Usually, the growth rate $\sigma(0) = 0$ for $k = 0$. Otherwise, a spatially uniform solution would grow or decay exponentially. The growth rate will be an even function of k since waves with positive and negative wave numbers should grow at the same rate. We use the Taylor series for small k and obtain $\sigma(k) \sim \frac{\sigma_{kk}}{2}k^2$ (we will be more accurate shortly). If long waves are stable, then $\sigma_{kk} < 0$. The simplest way that long waves can become unstable, as we vary a parameter R, is for σ_{kk} to change signs at R_c, so that for long waves near R_c, $\sigma(k, R) \sim \frac{\sigma_{Rkk}}{2}k^2(R - R_c)$, with $\sigma_{Rkk} > 0$. Since we wish to investigate long waves for R near R_c, it is best to include the next term in the Taylor series,

$$\sigma(k, R) \sim \frac{\sigma_{Rkk}}{2}k^2(R - R_c) + \frac{\sigma_{kkkk}}{4!}k^4, \tag{14.8.51}$$

with $\sigma_{Rkk} > 0$. If the fourth-order term is stabilizing, $\sigma_{kkkk} < 0$, then we see that for $R > R_c$ there is a band of unstable ($\sigma < 0$) long waves $k^2 < -\frac{12\sigma_{Rkk}}{\sigma_{kkkk}}(R - R_c)$. Since ω is an odd function of k (for the phase velocity to be even),

$$\omega(k, R) \sim [\omega_k + \omega_{kR}(R - R_c) + \ldots]k + \frac{\omega_{kkk}}{3!}k^3. \tag{14.8.52}$$

The term in brackets is an improved approximation for the group velocity. To find the corresponding partial differential equation for long waves in the most general situation, we allow purely dispersive terms. Since $s = \sigma - i\omega = \frac{\partial}{\partial x}$ and $k = -i\frac{\partial}{\partial x}$, we obtain (after moving with the group velocity)

$$u_t = -\frac{\sigma_{Rkk}}{2}(R - R_c)u_{xx} + \frac{\sigma_{kkkk}}{4!}u_{xxxx} + \frac{\omega_{kkk}}{3!}u_{xxx}. \tag{14.8.53}$$

Using perturbations methods, the appropriate nonlinear term may be as simple as in the Korteweg–de Vries equation (14.7.15) or more complicated, as in the long-wave instability of thin-liquid films, as shown by Benney [1966].

14.8.8 PATTERN FORMATION FOR REACTION–DIFFUSION EQUATIONS AND THE TURING INSTABILITY

In 1952 Turing (also known for breaking the German code for England in World War II and the Turing machine in computer science) suggested that patterns in biological organisms (morphogenesis) might emerge from an instability of spatially dependent chemical concentrations described by partial differential equations. Particularly extensive and well-written presentations are given by Murray [1993] and Nicolis [1995]. General theory exists, but we prefer to add spatial diffusion to a well-known example with two reacting chemicals, known as the Brusselator due to Prigogine and Lefever [1968]:

$$\frac{\partial u}{\partial t} = 1 - (b+1)u + au^2v + D_1\nabla^2 u \tag{14.8.54}$$

$$\frac{\partial v}{\partial t} = bu - au^2v + D_2\nabla^2 v. \tag{14.8.55}$$

There are four parameters, $a > 0, b > 0, D_1 > 0, D_2 > 0$, but we wish to vary only b. We first determine all spatially uniform steady-state equilibria, which must satisfy both $1 - u[(b+1) - auv] = 0$ and $u(b - auv) = 0$. The second equation suggests $u = 0$, but that cannot satisfy the first equation. Thus, the second yields $auv = b$, and the first now yields $u = 1$. For this example, there is a unique uniform steady state $u = 1$ and $v = \frac{b}{a}$, corresponding to a chemical balance. In this example, there cannot be a bifurcation to a different uniform steady state.

Linearization

To analyze the stability of the uniform steady state, we approximate the system of partial differential equations near the steady state. We introduce small displacements from the uniform steady state: $u - 1 = u_1$ and $v - \frac{b}{a} = v_1$. As in the stability analysis of equilibria for systems of ordinary differential equations (see Section 14.8.1), we **linearize** each chemical reaction, including the linear terms of the Taylor series for functions of two variables. We must include the diffusive term, but that is elementary since the equilibria are spatially constant. In this way we obtain a linear system of partial differential equations that the displacements $\boldsymbol{u_1} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$ must satisfy:

$$\frac{\partial \boldsymbol{u}_1}{\partial t} = \boldsymbol{J}\boldsymbol{u}_1 + \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}\nabla^2 \boldsymbol{u}_1, \tag{14.8.56}$$

where $\boldsymbol{J}$ is the usual **Jacobian matrix** evaluated at the uniform steady state,

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} = \begin{bmatrix} -(b+1) + 2auv & au^2 \\ b - 2auv & -au^2 \end{bmatrix} = \begin{bmatrix} b-1 & a \\ -b & -a \end{bmatrix}. \tag{14.8.57}$$

Spatial and time dependence

The linear partial differential equation (14.8.56) has constant coefficients, so elementary Fourier analysis based on separation of variables is valid:

$$\boldsymbol{u_1} = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{w}(t). \tag{14.8.58}$$

If there are no boundaries, $\boldsymbol{k}$ is arbitrary, corresponding to using a Fourier transform. Note that $\nabla^2 \boldsymbol{u}_1 = -k^2 e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{w}(t)$, where $k = |\boldsymbol{k}|$. In this way the time-dependent part satisfies a linear system of ordinary differential equations with constant coefficients:

$$\frac{d\boldsymbol{w}}{dt} = \boldsymbol{A}\boldsymbol{w}. \tag{14.8.59}$$

where the matrix $\boldsymbol{A}$ is related to the Jacobian,

$$\boldsymbol{A} = \boldsymbol{J} - \begin{bmatrix} D_1 k^2 & 0 \\ 0 & D_2 k^2 \end{bmatrix} = \begin{bmatrix} b - 1 - D_1 k^2 & a \\ -b & -a - D_2 k^2 \end{bmatrix}. \tag{14.8.60}$$

Linear system

The linear system (14.8.59) is solved by letting $\boldsymbol{w}(t) = e^{\lambda t}\boldsymbol{v}$, in which case $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$, so that λ are the eigenvalues of the matrix $\boldsymbol{A}$ and $\boldsymbol{v}$ the corresponding eigenvectors. The eigenvalues are determined from

$$0 = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{bmatrix} b - 1 - D_1 k^2 - \lambda & a \\ -b & -a - D_2 k^2 - \lambda \end{bmatrix}. \tag{14.8.61}$$

The two eigenvalues satisfy

$$\lambda^2 + \lambda(a - b + 1 + (D_1 + D_2)k^2) + a + aD_1 k^2 - (b-1)D_2 k^2 + D_1 D_2 k^4 = 0. \tag{14.8.62}$$

The goal is to determine when the uniform state is stable or unstable as a function of the four parameters and a function of the wave number k. The stability of the uniform state is determined by the eigenvalues λ since solutions are proportional to $e^{\lambda t}$. The uniform state is unstable if one eigenvalue is positive or has positive real part $\mathrm{Re}(\lambda) > 0$. The uniform state is stable if all eigenvalues have negative real parts $\mathrm{Re}(\lambda) < 0$. For 2×2 matrices $\boldsymbol{A}$, it is best to note (see Section 14.8.1) that the solution is stable if and only if both its trace < 0 and its determinant > 0. However, we first do a somewhat easier procedure (which can also be used in higher-order problems but which has serious limitations). There are two ways in which the solutions can change from stable to unstable.

One eigenvalue with $\lambda = 0$ [and all other eigenvalues with $\mathrm{Re}(\lambda) < 0$]

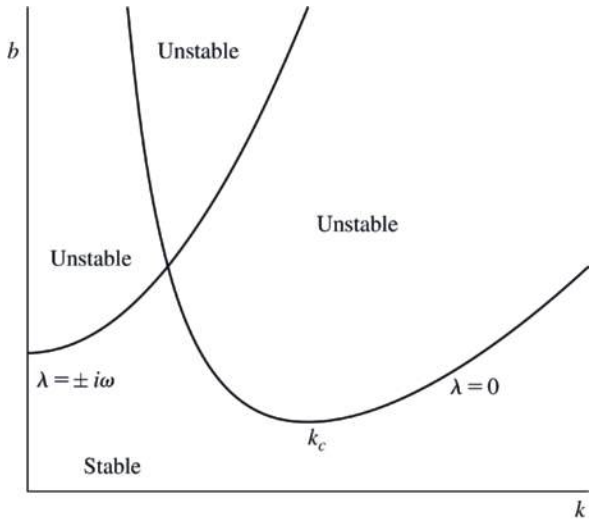
If $\lambda = 0$, then from (14.8.62) it follows that we think of b as being a function of the wave number k:

$$b = D_1 k^2 + 1 + \frac{aD_1}{D_2} + \frac{a}{D_2 k^2}. \tag{14.8.63}$$

The parameter b is graphed as a function of the wave number k in Fig. 14.8.7 (note the asymptotic behavior as $k \to 0$ and $k \to \infty$). The function has a minimum $b_{\min} = D_1 k_c^2 + 1 + \frac{aD_1}{D_2} + \frac{a}{D_2 k_c^2}$ at a critical wave number k_c,

$$k_c^4 = \frac{a}{D_1 D_2}. \tag{14.8.64}$$

Figure 14.8.7 Turing instability.



Along this curve $\lambda = 0$, so that this curve may not be on the boundary of stability if the other eigenvalue is positive. Another important question is which side of this curve is stable and which side unstable? It is not easy to answer these questions. First determine (perhaps numerically) the other eigenvalue at one point on this curve. Stability will remain the same until this curve intersects any other curve for which stability changes. For the portion of the curve in which stability changes, determine stability everywhere by computing (perhaps numerically) all the eigenvalues at one point on each side of the curve.

One set of complex conjugate eigenvalues satisfying $\lambda = \pm i\omega$ since $\mathrm{Re}(\lambda) = 0$ [and any other eigenvalues with $\mathrm{Re}(\lambda) < 0$]

Complex conjugate eigenvalues can be determined by substituting $\lambda = \pm i\omega$ into (14.8.62). The imaginary part of (14.8.62) must vanish, so that b is a different function of k:

$$b = a + 1 + (D_1 + D_2)k^2, \tag{14.8.65}$$

and the frequency satisfies $\omega^2 = a + aD_1k^2 - (b-1)D_2k^2 + D_1D_2k^4$. Thus, (14.8.65) is not valid for all k, but for only those values where $\omega^2 > 0$. For the curve (14.8.65), b has a minimum $b = a + 1$ at $k = 0$. The portion of the curve with real frequency will separate stable from unstable, but again in this way we do not know which side is stable and which side unstable.

Necessary and sufficient (trace and determinant) condition for stability

We will be careful in determining where the uniform state is stable. As shown in Fig. 14.8.2, an equilibrium is stable if and only if for the matrix both its trace < 0 and its determinant > 0. To be stable, both inequalities must be satisfied:

$$\mathrm{tr}\, \boldsymbol{A} = b - a - 1 - (D_1 + D_2)k^2 < 0 \tag{14.8.66}$$

$$\det \boldsymbol{A} = a + aD_1k^2 - (b-1)D_2k^2 + D_1D_2k^4 > 0. \tag{14.8.67}$$

This is more accurate than (14.8.63) and (14.8.65). (The determinant condition relates to $\lambda = 0$, while the trace condition relates to complex eigenvalues.) By solving each inequality in (14.8.66) and (14.8.67) for b, we see that the stable region is below the intersection of the two curves (14.8.63) and (14.8.65).

Pattern formation

There is a competition between these two minima $a + 1$ and $b_{\min}$. If the parameter b is less than both, then the uniform state is stable. We assume that we gradually increase b from the stable region. If the first instability occurs at $k = 0$ (infinite wavelength), this corresponds to an instability in the equilibrium solution of the partial differential equation identical to the ordinary differential equation in which the uniform state becomes unstable without developing any spatial structure. If the first instability occurs at $k = k_c$ given by (14.8.64), there is a preferred wave number (preferred wavelength $\frac{2\pi}{k_c}$). From a uniform state, **a pattern emerges with predictable wave-length from the theory of the instability of the uniform state**, and this would be called a **Turing instability**. The case of a Turing instability is philosophically significant since the spatial wavelength of patterns is predicted from the mathematical equations for arbitrary initial conditions without spatial structure, explaining spatial structures observed in nature. The pattern is a one-dimensional approximately linear wave with a predictable wavelength $\frac{2\pi}{k_c}$.

Complex Ginzburg–Landau

If the Turing instability occurs and the parameter b is slightly greater than its critical value, then there is a small band of unstable wave numbers around k_c. Energy will be focused in these wave numbers, and as discussed in Sections 14.8.5 and 14.8.6, the amplitude can be approximated by the complex Ginzburg–Landau (CGL) equation (14.8.36) with the typical cubic nonlinearity $\gamma A|A|^2$, which can be derived using perturbation methods. The complex coefficient $\gamma = \alpha + i\beta$. To get a crude understanding of the effect of the nonlinearity, we discuss the more elementary Landau equation (14.8.37) associated with Hopf bifurcation for ordinary differential equations. In this way it is seen that if $\alpha > 0$, the nonlinearity is destabilizing, while if $\alpha < 0$, the nonlinearity is stabilizing. It is not particularly easy to use perturbation methods to determine whether $\alpha > 0$ or $\alpha < 0$, Is the pitchfork bifurcation subcritical or supercritical? This can sometimes be determined

by numerically computing the original nonlinear partial differential equation for the parameter slightly greater than the value of the instability. If there is an equilibrated solution with a simple spatial structure, it can often be seen from the numerics, and the pattern that is formed is due to this type of pitchfork bifurcation. On the other hand, if numerics do not show a solution with a simple spatial structure as predicted by the nonlinear analysis, it is possible that the nonlinearity is destabilizing, and the simple solution is unstable.

Two-dimensional patterns

In two-dimensional instability problems, there is a preferred wave number $|\boldsymbol{k}| = k_c$. Since $\boldsymbol{k}$ is a vector, waves with any direction are possible, though they are all characterized by the same wavelength. This is a difficult subject, and we make only a few brief remarks. The linear theory predicts that the solution can be a complicated superposition of many of these waves. **Hexagonal patterns** are observed in nature in a variety of different disciplines, and it is believed that they result from the superposition of six $\boldsymbol{k}$ differing each by 60°. The wavelength of the hexagonal patterns is predicted from the linear instability theory. Many systems of reaction–diffusion equations seem to be characterized by having **spiral wave patterns**, as are now frequently observed in the laboratory. Three-dimensional generalizations (see Scott [1999]) are called **scroll waves** and **scroll rings**. According to Winfree [1987], human cardiac arrhythmias preceding heart attacks are characterized by waves with electrical activity like scroll rings. The study of spiral and scroll wave solutions of partial differential equations and their stability is an area of contemporary research.

Exercises 14.8

14.8.1. The (nonlinear) pendulum satisfies the ordinary differential equation $\frac{d^2x}{dt^2} + \sin x = 0$, where x is the angle. Equilibrium solutions satisfy $\sin x_0 = 0$. The natural position is $x_0 = 0$, and the inverted position is $x_0 = \pi$. Determine whether an equilibrium solution is stable or unstable by considering initial conditions near the equilibrium and approximating the differential equation there. [*Hint*: Since x is near x_0, we use the Taylor series of $\sin x$ around $x = x_0$.]

14.8.2. **(a)** Graph four bifurcation diagrams for saddle-node bifurcation at $(x_c, R_c) = (0,0)$ corresponding to different choices of signs of f_R and f_{xx}.

(b) Determine stability using one-dimensional phase diagrams.

14.8.3. Assume that at a bifurcation point $(x_c, R_c) = (0,0)$, in addition to the usual criteria for a bifurcation point, $f_R = 0$ and $f_{xx} \neq 0$. Using a Taylor series analysis, show that as an approximation,

$$\frac{dx}{dt} = \frac{f_{xx}}{2}x^2 + f_{xR}Rx + \frac{f_{RR}}{2}R^2.$$

(a) If $f_{xR}^2 > f_{xx}f_{RR}$, then the bifurcation is called **transcritical**. Analyze stability using one-dimensional phase diagrams (assuming $f_{xx} > 0$). Explain why the transcritical bifurcation is also called **exchange of stabilities**.

(b) If $f_{xR}^2 < f_{xx}f_{RR}$, then show that the equilibrium is isolated to $R = R_c$ only.

14.8.4. Assume that at a bifurcation point $(x_c, R_c) = (0,0)$, in addition to the usual criteria for a bifurcation point, $f_R = 0$ and $f_{xx} = 0$. Using a Taylor series analysis, show that as an approximation,

$$\frac{dx}{dt} = f_{xR}Rx + \frac{f_{RRR}}{6}R^3 + \cdots.$$

Assume $f_{xR} > 0$. Show there are two cases depending on the sign of f_{RRR}. Analyze stability using one-dimensional phase diagrams (assuming $f_{xx} > 0$). Explain why this bifurcation is called **pitchfork bifurcation**.

14.8.5. For the following examples, draw bifurcation diagrams examples and determine stability using one-dimensional phase portraits:

(a) $\frac{dx}{dt} = 2x + 5R$

(b) $\frac{dx}{dt} = -5x + 2R$

(c) $\frac{dx}{dt} = 2x^2 + 5R$

(d) $\frac{dx}{dt} = -2x^2 - 5R$

(e) $\frac{dx}{dt} = xR + x^2$

(f) $\frac{dx}{dt} = -(x - 3R)(x - 5R)$

(g) $\frac{dx}{dt} = Rx - x^3$

(h) $\frac{dx}{dt} = Rx + x^3$

(i) $\frac{dx}{dt} = -(x - 4)(x - e^R)$

(j) $\frac{dx}{dt} = 1 + (R - 1)x + x^2$ (graph R as a function of x first)

(k) $\frac{dx}{dt} = R^2 + Rx + x^2$

(l) $\frac{dx}{dt} = R^2 + 4Rx + x^2$

14.8.6. Find ($e^{st} = e^{(\sigma - i\omega)t}$) the exponential decay rate σ and the frequency ω for the following partial differential equations:

(a) $\frac{\partial u}{\partial t} = -u - R\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} - c\frac{\partial u}{\partial x}$

***(b)** $\frac{\partial u}{\partial t} = -u - R\frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} + \frac{\partial^3 u}{\partial x^3}$

(c) $\frac{\partial u}{\partial t} = u + \frac{1}{R}\frac{\partial^2 u}{\partial x^2}$

14.8.7. Find ($e^{st} = e^{(\sigma - i\omega)t}$) the exponential decay rate σ and the frequency ω. Briefly explain why the following partial differential equations are ill posed or not:

(a) $\frac{\partial u}{\partial t} = 4\frac{\partial u}{\partial x}$

(b) $\frac{\partial u}{\partial t} = i\frac{\partial u}{\partial x}$

***(c)** $\frac{\partial u}{\partial t} = -i\frac{\partial u}{\partial x}$

(d) $\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial x^2}$

(e) $\frac{\partial u}{\partial t} + i\frac{\partial u}{\partial x} = i\frac{\partial^2 u}{\partial x^2}$

(f) $\frac{\partial u}{\partial t} - i\frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3}$

(g) $\frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4}$

(h) $\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4}$

14.8.8. Derive (14.8.20) and (14.8.21).

14.8.9. Determine the dispersion relation for the linearized complex Ginzburg–Landau equation.

14.8.10. Draw bifurcation diagrams and determine stability of the solutions using a one-dimensional phase portrait for $\frac{dr}{dt} = \sigma_R(R - R_c)r + \alpha r^3$:

(a) Assume $\sigma_R < 0$ and $\alpha > 0$.

(b) Assume $\sigma_R < 0$ and $\alpha < 0$.

14.8.11. Under what circumstances (parameters) does the Turing bifurcation occur for lower b than the bifurcation of the uniform state?

14.8.12. Consider the dynamical system that arises by ignoring diffusion in the model that exhibits the Turing bifurcation.

(a) Show that a Hopf bifurcation occurs at $b = 1 + a$.

(b) Investigate numerically the dynamical system.

14.9 Singular Perturbation Methods: Multiple Scales

Often problems of physical interest can be expressed as difficult mathematical problems that are near (in some sense) to a problem that is easy to solve. For example, as in Section 9.6, the geometric region may be nearly circular, and we wish to determine the effect of the small perturbation. There we determined how the solution differs from the solution corresponding to a circular geometry. We usually have (or can introduce) a small parameter ε, and the solution (for example) $u(x, y, t, \varepsilon)$ depends on ε. If

$$u(x, y, t, \varepsilon) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \dots, \tag{14.9.1}$$

the solution is said to be a **regular perturbation problem**. The first term (called the leading-order term) $u_0(x, y, t)$ is often a well-known solution corresponding to the unperturbed problem $\varepsilon = 0$. Usually, only the first few additional terms are needed, and the higher-order terms such as $u_1(x, y, t)$ can be successively determined by substituting the regular expansion into the original partial differential equation.

More difficult (and interesting) situations arise when a regular perturbation expansion is not valid, in which case we call the problem a **singular perturbation problem**. Whole books (for example, see the one by Kevorkian and Cole [1996]) exist on the subject. Sometimes simple expansions exist like (14.9.1), but expansions of different forms are valid in different regions. In this case, **boundary layer** methods can be developed (see Section 14.10). Sometimes different scaled variables (to be defined shortly) are simultaneously valid, in which case we use the **method of multiple scales**.

14.9.1 ORDINARY DIFFERENTIAL EQUATION: WEAKLY NONLINEARLY DAMPED OSCILLATOR

As a motivating example to learn the method of multiple scales in its simplest context (ordinary differential equations), we consider a linear oscillator with small nonlinear damping (proportional to the velocity cubed). We introduce dimensionless variables so that the model ordinary differential equation is

$$\boxed{\frac{d^2u}{dt^2} + u = -\varepsilon\left(\frac{du}{dt}\right)^3}, \tag{14.9.2}$$

where ε is a small positive parameter, $0 < \varepsilon \ll 1$. The perturbation corresponds to some form of nonlinear damping because it is an odd function of the velocity $\frac{du}{dt}$.

Regular perturbation expansion (naive)

We begin by assuming the solution has a regular perturbation expansion:

$$u(t, \varepsilon) = u_0(t) + \varepsilon u_1(t) + \dots. \tag{14.9.3}$$

By substituting (14.9.3) into (14.9.2) and comparing coefficients of ε to different powers, we obtain

$$O(\varepsilon^0): \frac{d^2u_0}{dt^2} + u_0 = 0 \tag{14.9.4}$$

$$O(\varepsilon^1): \frac{d^2u_1}{dt^2} + u_1 = -\left(\frac{du_0}{dt}\right)^3. \tag{14.9.5}$$

The leading-order terms yield the unperturbed equation (14.9.4). The solution of the unperturbed equation is periodic, a linear combination of $\sin t$ and $\cos t$. We will use instead complex exponentials (in order to simplify some of the later nonlinear calculations):

$$u_0(t) = Ae^{it} + A^*e^{-it}. \tag{14.9.6}$$

We use the complex amplitude A, and in order for the solution to be real we let the other coefficient be A^*, the complex conjugate of A.

Using this leading-order solution (14.9.6), we obtain the differential equation for the most important perturbed term u_1:

$$\frac{d^2u_1}{dt^2} + u_1 = i(Ae^{it} - A^*e^{-it})^3 = i(A^3e^{3it} - 3A^2A^*e^{it}) + (*), \tag{14.9.7}$$

where $(*)$ stands for the complex conjugate of the other terms presented on the right-hand side of (14.9.7). The right-hand side contains third harmonic terms $e^{\pm 3it}$ and the fundamental $e^{\pm it}$. Using the method of undetermined coefficients, a particular solution is easy to obtain corresponding to the third harmonic term. However, here in the regular perturbation method, the forcing frequency associated with e^{it} equals the natural frequency, and thus resonance occurs. The corresponding solution grows linearly in time and is said to be **secular**. In this way, a particular solution of (14.9.7) is

$$u_1(t) = i\frac{A^3}{1-9}e^{3it} + i\#A^2A^*te^{it} + (*),$$

where # is a number that could be computed (with some effort using the method of undetermined coefficients), but we do not need to know its value. Homogeneous solutions should also be included.

The regular expansion of the solution $u_0(t) + \varepsilon u_1(t) + \ldots$ is a valid approximation if εt is small. However, if t is large enough so that εt is an order 1 quantity or larger, then the regular expansion is no longer valid because of the secular term.

The method of multiple scales

The regular perturbation expansion is not valid for large times. However, the regular perturbation expansion suggests that interesting dynamics occur when $\varepsilon t = O(1)$. If we are interested in the solution on that long time scale, then we assume the solution $u(t, T, \varepsilon)$ depends on two variables, a **fast variable** t and a **slow variable**

$$\boxed{T = \varepsilon t,} \tag{14.9.8}$$

known as the **method of multiple scales**. (Sometimes this is called the **method of slow variation**.) We may use the chain rule

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T},} \tag{14.9.9}$$

where derivatives are now treated as partial derivatives (and a subscript notation for partial derivatives will be used). In this case the equation for the nonlinearly damped oscillator (14.9.2) becomes

$$\boxed{u_{tt} + 2\varepsilon u_{tT} + \varepsilon^2 u_{TT} + u = -\varepsilon(u_t + \varepsilon u_T)^3.} \tag{14.9.10}$$

Now the regular expansion, $u(t, T, \varepsilon) = u_0(t, T) + \varepsilon u_1(t, T) + \ldots$, yields

$$O(\varepsilon^0): \frac{\partial^2 u_0}{\partial t^2} + u_0 = 0 \tag{14.9.11}$$

$$O(\varepsilon^1): \frac{\partial^2 u_1}{\partial t^2} + u_1 = -\left(\frac{\partial u_0}{\partial t}\right)^3 - 2\frac{\partial}{\partial T}\left(\frac{\partial u_0}{\partial t}\right). \tag{14.9.12}$$

Only the order ε term $-2\frac{\partial}{\partial T}\left(\frac{\partial u_0}{\partial t}\right)$ differs from the naive perturbation expansion (14.9.4) and (14.9.5).

The leading-order equation is the unperturbed linear oscillator, but we must remember that $\frac{\partial}{\partial t}$ means keeping T fixed. Thus, the general solution of (14.9.11) is

$$\boxed{u_0(t, T) = A(T)e^{it} + A^*(T)e^{-it},} \tag{14.9.13}$$

where $A(T)$ is an arbitrary complex function of the slow time variable.

Using (14.9.13), the first perturbed term satisfies

$$\boxed{\begin{aligned}\frac{d^2u_1}{dt^2} + u_1 &= i(Ae^{it} - A^*e^{-it})^3 - 2i\left(\frac{dA}{dT}e^{it} - \frac{dA^*}{dT}e^{-it}\right)\\ &= i(A^3e^{3it} - 3A^2A^*e^{it}) - 2i\frac{dA}{dT}e^{it} + (*).\end{aligned}} \tag{14.9.14}$$

We will determine $A(T)$ by eliminating secular terms in the perturbed problem. The multiply scaled variables are introduced in order for the solution to be valid for long times $t = O(\frac{1}{\varepsilon})$. The secular terms (resonant terms on the right-hand side proportional to e^{it}) are not allowable, in which case we derive

$$\boxed{2\frac{dA}{dT} = -3A^2A^*.} \tag{14.9.15}$$

Equation (14.9.15) can be derived in another way. Equation (14.9.14) is a nonhomogeneous linear differential equation with two homogeneous boundary conditions (here the periodicity constraint). According to the Fredholm alternative, periodic solutions to (14.9.14) exist only if the right-hand side is orthogonal to the solutions of the homogeneous differential equation satisfying homogeneous boundary conditions, in this case e^{it} and e^{-it}. Equation (14.9.15) follows from this Fredholm alternative since from the theory of Fourier series e^{3it} is orthogonal to $e^{\pm it}$ on the interval 0 to 2π.

The differential equation for the slow variation of the complex amplitude can be solved using the amplitude and phase form of a complex quantity,

$$A(T) = r(T)e^{i\phi(T)}, \tag{14.9.16}$$

in which case

$$u_0(t, T) = r(T)e^{i(t+\phi(T))} + r(T)e^{-i(t+\phi(T))} = 2r(T)\cos(t + \phi(T)).$$

Thus, $2r(T)$ is the real amplitude of oscillation, which slowly varies, and the phase of the oscillation is $t + \phi(T)$. Physically, the derivative of the phase with respect to time is the frequency, and thus the frequency is $1 + \varepsilon\frac{d\phi}{dT}$.

Since $\frac{dA}{dT} = \left(\frac{dr}{dT} + i\frac{d\phi}{dT}r\right)e^{i\phi(T)}$ and $A^2A^* = r^3e^{i\phi(T)}$, it follows from (14.9.15), using the real and imaginary parts, that

$$2\frac{dr}{dT} = -3r^3 \tag{14.9.17}$$

$$r\frac{d\phi}{dT} = 0. \tag{14.9.18}$$

The one complex equation (14.9.15) is equivalent to two real equations, (14.9.17) and (14.9.18). In this specific example, the phase shift $\phi(T)$ is a constant. However, the amplitude of oscillation decays due to the nonlinear damping satisfying (14.9.17). By separation, $\frac{dr}{r^3} = -\frac{3}{2}dT$. It follows after some algebra that the amplitude of oscillation algebraically decays due to the small nonlinear damping:

$$\boxed{r(T) = \frac{r(0)}{\sqrt{1 + 3r^2(0)T}}.}$$

Typically, the phase $(t + \phi(T))$ varies quickly while the amplitude $[r(T)]$ varies slowly. This is typical of problems with multiple scales. A graph of the solution is illuminating since it shows the simultaneous appearance of two scaled variables. We call the oscillator a **slowly varying oscillator** as the amplitude decays slowly.

14.9.2 ORDINARY DIFFERENTIAL EQUATION: SLOWLY VARYING OSCILLATOR

Solutions of oscillatory ordinary differential equations also may require the method of multiple scales if the coefficients are slowly varying. Consider the spring-mass system with slowly varying spring constant:

$$\boxed{\frac{d^2u}{dt^2} + \omega^2(\varepsilon t)u = 0.} \tag{14.9.19}$$

This can also represent the propagation of light with a slowly varying index of refraction $\omega(\varepsilon t)$. The one-dimensional wave equation $\left[\frac{\partial^2 E}{\partial t^2} = c^2(x)\frac{\partial^2 E}{\partial x^2}\right]$ in a variable spatial medium with fixed temporal frequency $\omega_f\left[E = u(x)e^{-i\omega_f t}\right]$ becomes $-\omega_f^2 u = c^2(x)\frac{\partial^2 E}{\partial x^2}$. This is an ordinary

differential equation in x which can be shown equivalent to the time-dependent (14.9.19). If the typical length over which $c(x)$ varies is much longer than a typical wavelength $\frac{2\pi c}{\omega_f}$ of solutions, then the medium is slowly varying.

Solutions of (14.9.19) depend on two variables: a fast phase θ and the slow time $T = \varepsilon t$. If the coefficient $\omega(\varepsilon t)$ were a constant, then the frequency of vibration would be ω. Intuitively, if ω varies very slowly, then the frequency of oscillations should change slowly but be precisely $\omega(\varepsilon t)$. Our notions of frequency are such that if we have a phase θ, then $\frac{d\theta}{dt}$ is the frequency. Thus, we postulate that the fast phase satisfies exactly

$$\boxed{\frac{d\theta}{dt} = \omega(\varepsilon t).} \tag{14.9.20}$$

[It can be shown that perturbation methods based on phases that do not satisfy (14.9.20) will not work.] The following expressions for the fast phase are equivalent:

$$\boxed{\theta = \int \omega(\varepsilon t)dt = \frac{\int \omega(T)dT}{\varepsilon}.} \tag{14.9.21}$$

We assume the solution $u(\theta, T, \varepsilon)$ depends on two variables: a fast phase θ [satisfying (14.9.20)] and a slow time $T = \varepsilon t$. According to the chain rule (the method of multiply scaled variables),

$$\boxed{\frac{d}{dt} = \omega(T)\frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial T}.} \tag{14.9.22}$$

Since the operator in (14.9.22) does not have constant coefficients, we must calculate the second derivative with care:

$$\begin{aligned}\frac{d^2}{dt^2} &= \left(\omega(T)\frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial T}\right)\left(\omega(T)\frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial T}\right)\\ &= \omega^2\frac{\partial^2}{\partial\theta^2} + \varepsilon\left(2\omega(T)\frac{\partial}{\partial T}\frac{\partial}{\partial\theta} + \omega'(T)\frac{\partial}{\partial\theta}\right) + \varepsilon^2\frac{\partial^2}{\partial T^2}.\end{aligned}$$

The differential equation (14.9.19) becomes

$$\boxed{\omega^2\frac{\partial^2 u}{\partial\theta^2} + \varepsilon\left(2\omega(T)\frac{\partial}{\partial T}\frac{\partial u}{\partial\theta} + \omega'(T)\frac{\partial u}{\partial\theta}\right) + \varepsilon^2\frac{\partial^2 u}{\partial T^2} + \omega^2 u = 0.} \tag{14.9.23}$$

We now assume a perturbation expansion in the multiply scaled variables:

$$u(\theta, T, \varepsilon) = u_0(\theta, T) + \varepsilon u_1(\theta, T) + \ldots. \tag{14.9.24}$$

By substituting (14.9.24) into (14.9.23), we obtain a sequence of equations of which we need only the first two:

$$O(\varepsilon^0)\colon \omega^2\left(\frac{\partial^2 u_0}{\partial\theta^2} + u_0\right) = 0 \tag{14.9.25}$$

$$O(\varepsilon^1)\colon \omega^2\left(\frac{\partial^2 u_1}{\partial\theta^2} + u_1\right) = -2\omega(T)\frac{\partial}{\partial T}\frac{\partial u_0}{\partial\theta} - \omega'(T)\frac{\partial u_0}{\partial\theta}. \tag{14.9.26}$$

To leading order, the solution is an elementary slowly varying oscillator in the phase variable θ:

$$\boxed{u_0(\theta, T) = A(T)e^{i\theta} + A^*(T)e^{-i\theta},} \tag{14.9.27}$$

where $A(T)$ is an arbitrary function of the slow time variable $T = \varepsilon t$, which we will determine shortly. Equation (14.9.27), with (14.9.20), shows that the slowly varying frequency is ω, as we have insisted.

We substitute our form of solution (14.9.27) into (14.9.26) and obtain

$$\omega^2\left(\frac{\partial^2 u_1}{\partial\theta^2} + u_1\right) = -[2\omega(T)A'(T) + \omega'(T)A(T)]ie^{i\theta} + (*).$$

Coefficients on the right-hand side proportional to $e^{\pm i\theta}$ are secular (resonant since the forcing frequency equals the natural frequency), and they must be eliminated:

$$\boxed{2\omega(T)A'(T) + \omega'(T)A(T) = 0.} \tag{14.9.28}$$

Equation (14.9.28) is a differential equation for the slowly varying complex amplitude. Since $A(T)$ may be complex,

$$A(T) = r(T)e^{i\psi}. \tag{14.9.29}$$

However, it may be shown that ψ is a constant (in this problem), and the amplitude satisfies

$$2\omega(T)r'(T) + \omega'(T)r(T) = 0. \tag{14.9.30}$$

This equation can be solved by separation or usual linear techniques or by just noticing that

$$\frac{d}{dT}\left(\omega^{\frac{1}{2}}r\right) = 0 \tag{14.9.31}$$

is equivalent to (14.9.30). Thus, by integration,

$$\boxed{r(T) = cw^{-\frac{1}{2}},} \tag{14.9.32}$$

where c is an arbitrary constant.

This example illustrates the **conservation of action** first investigated by Einstein. We will show that, with slowly varying coefficients, the energy is not conserved. However, the action is conserved:

$$\frac{d}{dT}(\text{action}) = 0, \text{ where action} \equiv \frac{\text{energy}}{\text{frequency}}.$$

The energy is defined to be the energy that would occur if the coefficients were constant. If ω in our differential equation were constant, $\frac{d^2u}{dt^2} + \omega^2 u = 0$, then we would define the energy as follows: $E = \frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{\omega^2}{2}u^2$ (kinetic energy plus potential energy). The solution of the differential equation (in our complex notation) would be $u = Ae^{i\omega t} + A^*e^{-i\omega t} = 2r\cos(\omega t + \phi_0)$. In this case the energy $E = \frac{\omega^2}{2}(2r)^2(\cos^2 + \sin^2) = 2\omega^2 r^2$. Thus, according to the conservation of action,

$$\frac{d}{dT}\left(\frac{2\omega^2 r^2}{\omega}\right) = 0,$$

equivalent to (14.9.32), which we have derived using the method of multiply-scaled variables. In more difficult problems, physicists use conservation of action to determine the slow variation of the amplitude. Note that energy is not conserved:

$$\frac{dE}{dT} = \frac{d}{dT}(\omega^2 r^2) \neq 0,$$

according to (14.9.32). It is not easy to figure out how a slowly varying restoring force puts energy into the system or takes energy out of the system. That is why conservation of action is quite useful.

In summary, we have analyzed the linear differential equation with slowly varying coefficients:

$$\boxed{\frac{d^2u}{dt^2} + \omega^2(\varepsilon t)u = 0.} \tag{14.9.33}$$

We have obtained the following approximate solution:

$$\boxed{u(t) \approx u_0(\theta, T) = c\omega^{-\frac{1}{2}}e^{i\theta+\psi} + (*) = 2c\omega^{-\frac{1}{2}}\cos\left(\int \omega(\varepsilon t)dt + \psi\right),} \tag{14.9.34}$$

where c and ψ are arbitrary constants since θ satisfies

$$\boxed{\frac{d\theta}{dt} = \omega(\varepsilon t).} \tag{14.9.35}$$

This is not difficult to memorize as the derivative of the phase can be obtained from the physically intuitive notions of frequency. We just need to memorize the magical amplitude factor of $\omega^{-\frac{1}{2}}$ (or use conservation of action). This is a well-known formula for wave propagation problems with variable media. It can also be used to approximate the eigenfunctions of any Sturm–Liouville problem for large eigenvalues (see Section 6.9). This method is sometimes incorrectly called the WKB method, while perhaps Liouville–Green (working independently in 1837) would be more appropriate. WKB (Wentzel, Kramers, Brillouin) in the 1920s solved problems in which the coefficient ω^2 evolves from positive to negative (in the context of tunneling in quantum mechanics), obtaining connection formulas between the oscillatory solutions as in (14.9.34) and the corresponding exponential solutions.

14.9.3 SLIGHTLY UNSTABLE PARTIAL DIFFERENTIAL EQUATION ON FIXED SPATIAL DOMAIN

We consider the following model weakly nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Ru - \varepsilon u^3, \tag{14.9.36}$$

subject to the boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$. The linearized problem ($\varepsilon = 0$) has solutions $\sin\frac{n\pi x}{L}e^{\sigma t}$, where $\sigma = R - k(\frac{n\pi}{L})^2$. Some modes (fixed n) grow exponentially depending on the parameter R. For n sufficiently large, the modes decay exponentially (similar to the heat equation). The first instability (of $u = 0$) occurs when R is slightly greater than $k(\frac{\pi}{L})^2$, in which case only the mode $n = 1$ exponentially grows; all the other modes decay exponentially. Thus, we assume $R = k(\frac{\pi}{L})^2 + \varepsilon R_1$ with $R_1 > 0$ so that

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \left[k\left(\frac{\pi}{L}\right)^2 + \varepsilon R_1\right]u - \varepsilon u^3.} \tag{14.9.37}$$

It can be shown that a naive perturbation expansion fails on the long time scale:

$$\boxed{T = \varepsilon t.} \tag{14.9.38}$$

Thus, we use the method of multiple scales:

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}.} \tag{14.9.39}$$

Using (14.9.39), the partial differential equation (14.9.37) becomes

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + k\left(\frac{\pi}{L}\right)^2 u + \varepsilon\left[-\frac{\partial u}{\partial T} + R_1 u - u^3\right].} \tag{14.9.40}$$

Substituting the perturbation expansion $u = u_0 + \varepsilon u_1 + \ldots$ into (14.9.40) yields

$$O(\varepsilon^0): \frac{\partial u_0}{\partial t} = k\frac{\partial^2 u_0}{\partial x^2} + k\left(\frac{\pi}{L}\right)^2 u_0 \tag{14.9.41}$$

$$O(\varepsilon): \frac{\partial u_1}{\partial t} = k\frac{\partial^2 u_1}{\partial x^2} + k\left(\frac{\pi}{L}\right)^2 u_1 - \frac{\partial u_0}{\partial T} + R_1 u_0 - u_0^3. \tag{14.9.42}$$

We use an elementary solution of the leading-order equation (14.9.41):

$$\boxed{u_0 = A(T)\sin\frac{\pi x}{L}.} \tag{14.9.43}$$

The other modes can be included, but the other modes quickly decay exponentially in time. The amplitude $A(T)$ of the slightly unstable mode $\sin\frac{\pi x}{L}$ is an arbitrary function of the slow time. We will determine $A(T)$ by eliminating secular terms from the $O(\varepsilon)$ equation.

Using (14.9.43), the perturbed equation (14.9.42) becomes

$$\frac{\partial u_1}{\partial t} = k\frac{\partial^2 u_1}{\partial x^2} + k\left(\frac{\pi}{L}\right)^2 u_1 - \frac{dA}{dT}\sin\frac{\pi x}{L} + R_1 A\sin\frac{\pi x}{L} - A^3\sin^3\frac{\pi x}{L}, \tag{14.9.44}$$

to be solved with homogeneous boundary conditions $u_1(0, t) = 0$ and $u_1(L, t) = 0$. Since (from trigonometric tables) $\sin^3 \frac{\pi x}{L} = \frac{3}{4}\sin\frac{\pi x}{L} - \frac{1}{4}\sin\frac{3\pi x}{L}$, the nonlinearity generates the third harmonic in x. The right-hand side of (14.9.44) involves only the first and third harmonics, and thus (the method of eigenfunction expansion)

$$u_1 = B_1 \sin\frac{\pi x}{L} + B_3 \sin\frac{3\pi x}{L}. \tag{14.9.45}$$

Substituting (14.9.45) into (14.9.44) yields ordinary differential equations for the coefficients:

$$\frac{\partial B_1}{\partial t} = -\frac{dA}{dT} + R_1 A - \frac{3}{4}A^3 \tag{14.9.46}$$

$$\frac{\partial B_3}{\partial t} + 8k\left(\frac{\pi}{L}\right)^2 B_3 = \frac{1}{4}A^3. \tag{14.9.47}$$

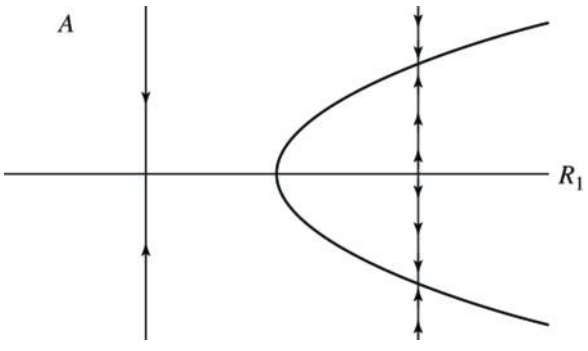
A particular solution $\left[B_3 = \frac{1}{32k}\left(\frac{L}{\pi}\right)^2 A^3\right]$ for the third harmonic is not needed since it is not secular. Homogeneous solutions for the third harmonic in x decay exponentially.

We consider (14.9.46) the higher-order term for the first harmonic. The terms on the right-hand side of (14.9.46) are functions of the slow time $T = \varepsilon t$ and hence constant with respect to the fast time variable t on the left-hand side. If a constant c were on the right-hand side, the solution corresponding to it would be $B_1 = ct$, algebraic growth in time. This algebraic growth is not acceptable in the asymptotic expansion $u = u_0 + \varepsilon u_1 + \ldots$. All the first harmonic in space terms on the right-hand side of (14.9.46) are secular, giving rise to algebraic growth in time (proportional to t). The secular terms must be zero (eliminating them), which implies that the amplitude $A(T)$ varies slowly and satisfies (what we have called the **Landau equation** in Section 14.8.6)

$$\boxed{\frac{dA}{dT} = R_1 A - \frac{3}{4}A^3.} \tag{14.9.48}$$

According to the linear theory (A small), the amplitude grows exponentially (if $R_1 > 0$). However, as shown in Fig. 14.9.1 using a one-dimensional phase portrait, the **nonlinearity prevents this growth**. The amplitude $A(T)$ **equilibrates** (the limit as $t \to \infty$) to $A = \pm\sqrt{\frac{4R_1}{3}}$ (depending on the initial condition). This is referred to as **pitchfork bifurcation**, as can be seen from the bifurcation diagram in Fig. 14.9.1 (where the equilibrated amplitude is graphed as a function of the parameter R_1). If $R_1 > 0$, $A = 0$ is unstable, but $A = \pm\sqrt{\frac{4R_1}{3}}$ are stable. The parameter R_1 measures the distance from the critical value of the parameter R, $\varepsilon R_1 = R - k\left(\frac{\pi}{L}\right)^2$.

Figure 14.9.1 One-dimensional phase portrait and bifurcation diagram for pitchfork bifurcation (Landau equation).



14.9.4 SLOWLY VARYING MEDIUM FOR THE WAVE EQUATION

The wave equation in a spatially variable two- or three-dimensional medium is

$$\frac{\partial^2 E}{\partial t^2} = c^2(x, y, z)\nabla^2 E.$$

Plane waves in a uniform medium (c constant) satisfy $E = E_0 e^{i(k_1 x + k_2 y + k_3 z - i\omega t)}$, where $k = |\boldsymbol{k}| = \frac{\omega}{c}$. For uniform medium, the spatial part of the phase can be defined as follows: $\theta \equiv k_1 x + k_2 y + k_3 z$. Thus, the wave number vector satisfies $k_1 \equiv \frac{\partial\theta}{\partial x}$, $k_2 \equiv \frac{\partial\theta}{\partial y}$, $k_3 \equiv \frac{\partial\theta}{\partial z}$, which can be summarized as $\boldsymbol{k} = \nabla\theta$.

For **nonuniform medium** (spatially varying c), often the temporal frequency ω is fixed (perhaps due to some incoming wave at infinity), $E = ue^{-i\omega t}$, so that u satisfies the **reduced wave equation** (also called the **Helmholtz equation**)

$$\boxed{\nabla^2 u + n^2 u = 0,} \tag{14.9.49}$$

where $n = \frac{\omega}{c}$ is the **index of refraction**. Here, it is assumed that we have a **slowly varying medium**, which means that the medium varies over a much longer distance than typical wavelengths, so that $n^2 = n^2(\varepsilon x, \varepsilon y, \varepsilon z)$. For example, when x varies by a very long distance $O(\frac{1}{\varepsilon})$ (many wavelengths), the index of refraction can change. The analogous one-dimensional problem was discussed in Section 14.9.2.

For a **slowly varying wave train**, we introduce a fast unknown phase θ and define the wave number as we did for uniform media $k_1 \equiv \frac{\partial\theta}{\partial x}$, $k_2 \equiv \frac{\partial\theta}{\partial y}$, $k_3 \equiv \frac{\partial\theta}{\partial z}$. Thus, the wave number vector is defined as follows:

$$\boxed{\boldsymbol{k} = \nabla\theta.} \tag{14.9.50}$$

We assume the solution depends on this fast phase θ and the slow spatial scales $X = \varepsilon x$, $Y = \varepsilon y$, $Z = \varepsilon z$. In the method of multiply scaled variables (by the chain rule),

$$\boxed{\frac{\partial}{\partial x} = k_1 \frac{\partial}{\partial\theta} + \varepsilon \frac{\partial}{\partial X}}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(k_1 \frac{\partial}{\partial\theta} + \varepsilon \frac{\partial}{\partial X}\right)\left(k_1 \frac{\partial}{\partial\theta} + \varepsilon \frac{\partial}{\partial X}\right) \\ &= k_1^2 \frac{\partial^2}{\partial\theta^2} + \varepsilon\left(2k_1 \frac{\partial}{\partial X}\frac{\partial}{\partial\theta} + k_{1_X} \frac{\partial}{\partial\theta}\right) + \varepsilon^2 \frac{\partial^2}{\partial X^2}.\end{aligned}$$

In general, using the vector notation (subscripts on the gradient operator mean spatial derivatives with respect to the slow spatial variables),

$$\nabla^2 = k^2 \frac{\partial^2}{\partial\theta^2} + \varepsilon\left(2\boldsymbol{k} \cdot \nabla_X \frac{\partial}{\partial\theta} + \nabla \cdot \boldsymbol{k} \frac{\partial}{\partial\theta}\right) + \varepsilon^2 \nabla_X^2. \tag{14.9.51}$$

Thus, the reduced wave equation (14.9.49) becomes

$$\boxed{k^2 \frac{\partial^2 u}{\partial\theta^2} + \varepsilon\left(2\boldsymbol{k} \cdot \nabla_X \frac{\partial u}{\partial\theta} + \nabla \cdot \boldsymbol{k} \frac{\partial u}{\partial\theta}\right) + \varepsilon^2 \nabla_X^2 u + n^2 u = 0.} \tag{14.9.52}$$

It can be shown (not easily) that the perturbation methods to follow will work only if the **eikonal equation** is satisfied:

$$\boxed{k^2 = \nabla\theta \cdot \nabla\theta = n^2,} \tag{14.9.53}$$

where $k = |\boldsymbol{k}|$. As described in Section 12.7, the eikonal equation says that the wave number of a slowly varying wave is always the wave number associated with the infinite plane wave ($k = \frac{\omega}{c} = n$). The eikonal equation describes **refraction** for slowly varying media (and is the basis of **geometrical optics**). The eikonal equation can be solved for the phase θ by the method of characteristics (see Section 12.6).

Assuming the eikonal equation (14.9.53) is valid, (14.9.52) becomes

$$n^2 \frac{\partial^2 u}{\partial\theta^2} + \varepsilon\left(2\boldsymbol{k} \cdot \nabla_X \frac{\partial u}{\partial\theta} + \nabla \cdot \boldsymbol{k} \frac{\partial u}{\partial\theta}\right) + \varepsilon^2 \nabla_X^2 u + n^2 u = 0. \tag{14.9.54}$$

We now introduce the perturbation expansion $u = u_0 + \varepsilon u_1 + \ldots$ and obtain

$$O(\varepsilon^0)\colon n^2\left(\frac{\partial^2 u_0}{\partial\theta^2} + u_0\right) = 0 \tag{14.9.55}$$

$$O(\varepsilon)\colon n^2\left(\frac{\partial^2 u_1}{\partial\theta^2} + u_1\right) = -2\boldsymbol{k} \cdot \nabla_X \frac{\partial u_0}{\partial\theta} - \nabla \cdot \boldsymbol{k} \frac{\partial u_0}{\partial\theta}. \tag{14.9.56}$$

The leading-order solution of (14.9.55) is a slowly varying plane wave

$$\boxed{u_0 = A(X, Y, Z)e^{i\theta}.} \tag{14.9.57}$$

We will determine the slow dependence of the amplitude A by eliminating secular terms from the next term in the perturbation expansion.

By substituting (14.9.57) into (14.9.56), the first perturbation satisfies

$$n^2\left(\frac{\partial^2 u_1}{\partial\theta^2} + u_1\right) = -(2\boldsymbol{k} \cdot \nabla_X A + \nabla \cdot \boldsymbol{k}A)ie^{i\theta}. \tag{14.9.58}$$

All the terms on the right-hand side of (14.9.58) are secular terms (resonant with forcing frequency equaling the natural frequency). Thus, to obtain a valid asymptotic expansion over the long distances associated with the slowly varying coefficient, the amplitude A must satisfy the **transport equation**:

$$\boxed{2\boldsymbol{k} \cdot \nabla_X A + \nabla \cdot \boldsymbol{k}A = 0.} \tag{14.9.59}$$

Since $\boldsymbol{k} = \nabla\theta$, an equivalent expression for the transport equation is

$$\boxed{2\nabla\theta \cdot \nabla_X A + \nabla^2\theta A = 0.} \tag{14.9.60}$$

Elegant expressions (see, for example, Bleistein [1984]) for the amplitude A and phase θ can be obtained using the method of characteristics. These equations are the basis of Keller's geometric theory of diffraction of plane waves by blunt objects (such as cylinders or airplanes or submarines).

In summary, the leading-order solution of the reduced wave equation $(\nabla^2 u + n^2 u = 0)$ for slowly varying media is

$$u_0 = Ae^{i\theta}, \tag{14.9.61}$$

where θ satisfies the eikonal equation (14.9.53) and A solves the transport equation (14.9.59) or (14.9.60). Higher-order terms may be obtained.

14.9.5 SLOWLY VARYING LINEAR DISPERSIVE WAVES (INCLUDING WEAK NONLINEAR EFFECTS)

As a simple example of the method of multiply scaled variables for partial differential equations, we consider the following model linear dispersive wave (with dimensionless variables) with a weakly nonlinear perturbation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = \varepsilon\beta u^3.} \tag{14.9.62}$$

The unperturbed partial differential equation ($\varepsilon = 0$) is a linear wave equation with an additional restoring force $-u$. Plane wave solutions of the unperturbed problem $u = e^{i(kx-\omega t)}$ satisfy the dispersion relation $\omega^2 = k^2 + 1$, and hence the unperturbed problem is a linear dispersive wave. For plane waves, we introduce the phase $\theta = kx - \omega t$ such that the wave number $k = \frac{\partial\theta}{\partial x}$ and frequency $\omega = -\frac{\partial\theta}{\partial t}$.

We seek slowly varying plane wave solutions due either to initial conditions that are slowly varying or to the perturbation. We introduce the unknown phase θ and define the wave number and frequency as for plane waves:

$$\boxed{k = \frac{\partial\theta}{\partial x}} \tag{14.9.63}$$

$$\boxed{\omega = -\frac{\partial\theta}{\partial t}.} \tag{14.9.64}$$

We assume initially that the wave number k may not be constant but may vary slowly, changing appreciably only over many wavelengths. We assume that there are slow spatial and temporal scales,

$$\boxed{X = \varepsilon x} \tag{14.9.65}$$

$$\boxed{T = \varepsilon t,} \tag{14.9.66}$$

of the same order of magnitude as induced by the perturbation to the partial differential equation. We use the **method of multiply scaled variables** with the fast phase θ and the slow spatial and temporal scales X, T. According to the chain rule,

$$\boxed{\frac{\partial}{\partial x} = k\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial X}} \tag{14.9.67}$$

$$\boxed{\frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial T}.} \tag{14.9.68}$$

For the partial differential equation, we need the second derivatives

$$\frac{\partial^2}{\partial x^2} = \left(k\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial X}\right)\left(k\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial X}\right) = k^2\frac{\partial^2}{\partial \theta^2} + \varepsilon\left(2k\frac{\partial}{\partial X}\frac{\partial}{\partial \theta} + k_X\frac{\partial}{\partial \theta}\right) + \varepsilon^2\frac{\partial^2}{\partial X^2} \tag{14.9.69}$$

$$\frac{\partial^2}{\partial t^2} = \left(-\omega\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial T}\right)\left(-\omega\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial T}\right) = \omega^2\frac{\partial^2}{\partial \theta^2} + \varepsilon\left(-2\omega\frac{\partial}{\partial T}\frac{\partial}{\partial \theta} - \omega_T\frac{\partial}{\partial \theta}\right) + \varepsilon^2\frac{\partial^2}{\partial T^2}. \tag{14.9.70}$$

Substituting these expressions (14.9.69) and (14.9.70) into the partial differential equation (14.9.62) yields

$$(\omega^2 - k^2)\frac{\partial^2 u}{\partial \theta^2} + u + \varepsilon\left(-2\omega\frac{\partial}{\partial T}\frac{\partial u}{\partial \theta} - \omega_T\frac{\partial u}{\partial \theta} - 2k\frac{\partial}{\partial X}\frac{\partial u}{\partial \theta} - k_X\frac{\partial u}{\partial \theta}\right) + \varepsilon^2\left(\frac{\partial^2 u}{\partial T^2} - \frac{\partial^2 u}{\partial X^2}\right) = \varepsilon\beta u^3. \tag{14.9.71}$$

We claim that the perturbation method that follows will not work unless the frequency of the slowly varying wave satisfies the dispersion relation for elementary plane waves:

$$\boxed{\omega^2 = k^2 + 1.} \tag{14.9.72}$$

The slowly varying wave number k and frequency ω (and phase θ) can be solved by the method of characteristics from given initial conditions, as we outline later (and have shown in Section 14.6). In particular, conservation of waves follows from the definitions of k and ω [(14.9.63) and (14.9.64)]:

$$k_T + \omega_X = 0. \tag{14.9.73}$$

A quasilinear partial differential equation for the wave number k follows from (14.9.73) using the dispersion relation (14.9.72):

$$k_T + \omega_k k_X = 0. \tag{14.9.74}$$

Thus, the wave number stays constant moving with the group velocity (where in this problem $\omega\omega_k = k$).

Using the dispersion relation (14.9.72), Equation (14.9.71) simplifies to

$$\frac{\partial^2 u}{\partial \theta^2} + u + \varepsilon\left(-2\omega\frac{\partial}{\partial T}\frac{\partial u}{\partial \theta} - \omega_T\frac{\partial u}{\partial \theta} - 2k\frac{\partial}{\partial X}\frac{\partial u}{\partial \theta} - k_X\frac{\partial u}{\partial \theta}\right) + \varepsilon^2\left(\frac{\partial^2 u}{\partial T^2} - \frac{\partial^2 u}{\partial X^2}\right) = \varepsilon\beta u^3. \tag{14.9.75}$$

Using a perturbation expansion $u = u_0 + \varepsilon u_1 + \ldots$, we obtain from (14.9.75)

$$O(\varepsilon^0): \frac{\partial^2 u_0}{\partial \theta^2} + u_0 = 0 \tag{14.9.76}$$

$$O(\varepsilon): \frac{\partial^2 u_1}{\partial \theta^2} + u_1 = \left(\omega_T + 2\omega\frac{\partial}{\partial T}\right)\frac{\partial u_0}{\partial \theta} + \left(k_X + 2k\frac{\partial}{\partial X}\right)\frac{\partial u_0}{\partial \theta} + \beta u_0^3. \tag{14.9.77}$$

The solution of the leading-order equation (14.9.76) is a **slowly varying (modulating) plane wave**

$$\boxed{u_0 = A(X, T)e^{i\theta} + (*),} \tag{14.9.78}$$

where $(*)$ represents the complex conjugate. In general, $A(X, T)$ is the complex wave amplitude and will be determined by eliminating secular terms from the next order equation.

We now consider the $O(\varepsilon)$ equation (14.9.77), whose right-hand side includes the effect of the nonlinear perturbation and the slow variation assumptions. If the nonlinearity is present ($\beta \neq 0$), then the nonlinearity generates first and third harmonics since $u_0^3 = A^3e^{3i\theta} + 3A^2A^*e^{i\theta} + (*)$. Substituting (14.9.78) into the right-hand side of (14.9.77) yields

$$\frac{\partial^2 u_1}{\partial \theta^2} + u_1 = i\Big(\omega_T A + 2\omega\frac{\partial A}{\partial T}\Big)e^{i\theta} + i\Big(k_X A + 2k\frac{\partial A}{\partial X}\Big)e^{i\theta} + \beta(A^3e^{3i\theta} + 3A^2A^*e^{i\theta}) + (*). \tag{14.9.79}$$

The first harmonic terms $e^{i\theta}$ on the right-hand side of (14.9.79) are secular (resonant with forcing frequency equaling the natural frequency). Eliminating the secular terms yields a partial differential equation the wave amplitude $A(X, T)$ must satisfy:

$$\boxed{2\omega\frac{\partial A}{\partial T} + 2k\frac{\partial A}{\partial X} + (\omega_T + k_X)A - i\beta 3A^2A^* = 0.} \tag{14.9.80}$$

From this equation we see that the amplitude moves with the group velocity (since, for this dispersion relation $\omega^2 = k^2 + 1$, the group velocity is given by $\omega_k = \frac{k}{\omega}$), but the amplitude is not constant, moving with the group velocity.

Multiplying (14.9.80) by A^* and adding the complex conjugate yields an equation that represents an important nontrivial physical concept:

$$\boxed{\frac{\partial}{\partial T}(k|A|^2) + \frac{\partial}{\partial X}(k|A|^2) = 0,} \tag{14.9.81}$$

where we have used $|A|^2 = AA^*$. Equation (14.9.81) is called **conservation of wave action** and is a general principle. Conservation laws can be put in the form $\frac{\partial p}{\partial t} + \frac{\partial q}{\partial x} = 0$, where p is the conserved quantity and q is its flux. Differential conservation laws follow from integral conservation laws (as is briefly discussed in Section 1.2). Sometimes on infinite domains it can be shown that $\frac{d}{dt}\int_{-\infty}^{\infty}\rho\, dx = 0$, and hence for all time $\int_{-\infty}^{\infty}\rho\, dx$ equals its initial value (and is thus constant in time or conserved). The general statement of **conservation of wave action** for **linear dispersive partial differential equations even when the coefficients in the partial differential equation are slowly varying** is

$$\boxed{\frac{\partial}{\partial T}\Big(\frac{E}{\omega}\Big) + \frac{\partial}{\partial X}\Big(c_g\frac{E}{\omega}\Big) = 0.} \tag{14.9.82}$$

The wave action $\frac{E}{\omega}$ is conserved, where E is defined to be the average energy density and c_g is the usual group velocity. It can be shown for our example that $E = \omega^2|A^2$. Consequently, the flux of wave action satisfies $c_g\frac{E}{\omega} = k|A^2$ since in our example $\omega_k = \frac{k}{\omega}$. [If $\beta = 0$ and thus A could be real, this result would follow by just multiplying (14.9.80) by A.] Wave action provides a generalization to linear partial differential equations of the idea of action for linear ordinary differential equations. Action has been further generalized to nonlinear dispersive partial differential equations by Whitham [1999].

In this example, the nonlinearity affects only the phase since the nonlinear force here is just a restoring force. If the nonlinearity had been a damping force, the nonlinearity would have affected the amplitude (see the Exercises). To see how the nonlinear perturbation effects the phase, we let

$$A = re^{i\phi}, \tag{14.9.83}$$

where $r = |A|$ has already been analyzed. Substituting (14.9.83) into (14.9.80) and keeping only the imaginary part yields the equation for the phase (of the complex amplitude A):

$$2\omega\frac{\partial \phi}{\partial T} + 2k\frac{\partial \phi}{\partial X} = 3\beta r^2, \tag{14.9.84}$$

which shows that the phase moves with the group velocity but evolves depending on $r^2 = |A|^2$. Since $\frac{\partial \phi}{\partial T}$ is a frequency and $\frac{\partial \phi}{\partial X}$ a spatial wave number, (14.9.84) represents the dependence of the spatial and temporal frequencies on the amplitude $r = |A|$ due to the nonlinear perturbation. (If $\beta = 0$, the phase ϕ would be constant if it were initially constant.)

Typically, small perturbations of oscillatory phenomena cause small modulations (slow variations) of the amplitude (as in Section 14.9.1) and the phase (as in this subsection).

Exercises 14.9

In all of the following exercises, assume $0 < \varepsilon \ll 1$. In Exercises 14.9.1–14.9.7, use the method of multiple scales to obtain equations for the amplitude and phase.

14.9.1. $\frac{d^2u}{dt^2} + u = -\varepsilon \frac{du}{dt}$ (compare to approximations of exact solution)

14.9.2. $\frac{d^2u}{dt^2} + \omega^2(\varepsilon t)u = -\varepsilon \frac{du}{dt}$

***14.9.3.** $\frac{d^2u}{dt^2} + \omega^2(\varepsilon t)u = -\varepsilon\left(\frac{du}{dt}\right)^3$

14.9.4. $\frac{d^2u}{dt^2} + u = \varepsilon\left(u^3 - \frac{du}{dt}\right)$

***14.9.5.** A linearized pendulum with varying length L can be shown to satisfy the following equation:

$$\frac{d^2u}{dt^2} + \frac{g}{L}u = -2\frac{1}{L}\frac{dL}{dt}\frac{du}{dt}.$$

Note that the right-hand side is not zero. Suppose L is a slowly varying length $L = L(\varepsilon t)$. Use the method of multiple scales to obtain a solution valid when $\varepsilon t = O(1)$. [*Hint*: $\frac{1}{L}\frac{dL}{dt} = \frac{\varepsilon}{L}\frac{dL}{dT}$, where $T = \varepsilon t$.]

14.9.6. Consider the system

$$\frac{d^2x}{dt^2} + 9x = \varepsilon y^3$$
$$\frac{d^2y}{dt^2} + y = 4\varepsilon x y^2.$$

(a) By discussing the frequencies, show why the slow time is $T = \varepsilon t$.

(b) Obtain the leading-order equations (but do *not* solve them) that are valid for $T = O(1)$, describing the long-time behavior of this system.

14.9.7. Approximate the solution of Airy's differential equation $\frac{d^2y}{dx^2} + xy = 0$ in the region of oscillation (assume large x can be interpreted as a slowly varying coefficient) using the Liouville–Green approximation.

In Exercises 14.9.8–14.9.10, use the method of multiply scaled variables to obtain the equation for the slowly varying amplitude and phase.

***14.9.8.** $\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$

14.9.9. $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = \varepsilon\beta\left(\frac{\partial u}{\partial t}\right)^3$

14.9.10. $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \beta u = 0$, assuming c and β depend slowly on X and T. You may assume the dispersion relation is still valid for slowly varying media.

14.9.11. Because the reduced wave equation (14.9.49) is linear, we can simple let $u = A(X, Y, Z)e^{i\theta}$, where θ solves the eikonal equation (14.9.53). Find the exact equation for A and the leading-order equation.

14.10 Singular Perturbation Methods: Boundary Layers Method of Matched Asymptotic Expansions

A large variety of significant physical problems [see Kevorkian and Cole (1996)] is described by ordinary or partial differential equations, where different approximations are valid in different regions (of space and/or time). Often one of the regions is much smaller than another and is located at a boundary, and thus this thin region is referred to as a **boundary layer**. Perturbation expansions can be determined in each region, but they must be related by a procedure called the

method of matched asymptotic expansions. We begin with an elementary example of an ordinary differential equation to explain the method of matched asymptotic expansions in its simplest context. Our second (and last) example is a partial differential equation representing the convection of a pollutant where diffusion is only important near a boundary.

14.10.1 BOUNDARY LAYER IN AN ORDINARY DIFFERENTIAL EQUATION

As an example to motivate the method of matched asymptotic expansions, we consider the boundary value problem for the following *second-order* ordinary differential equation:

$$\boxed{\varepsilon \frac{d^2u}{dx^2} - \frac{du}{dx} + 2xu = 0,} \tag{14.10.1}$$

subject to the *two* boundary conditions $u(0) = 3$ and $u(1) = 2$. We assume ε is a small positive parameter, $0 < \varepsilon << 1$. Even this elementary problem cannot be solved analytically because of the variable coefficient. Numerical methods are difficult to apply if ε is small (and impossible if ε is sufficiently small). In addition, numerical methods often do not add any insight into the boundary layer behavior we will derive.

Reduced problem

Intuitively, we expect that if ε is very small (10^{-8}, for example), then for all practical purposes the **reduced** *first-order* differential equation should be a very good approximation:

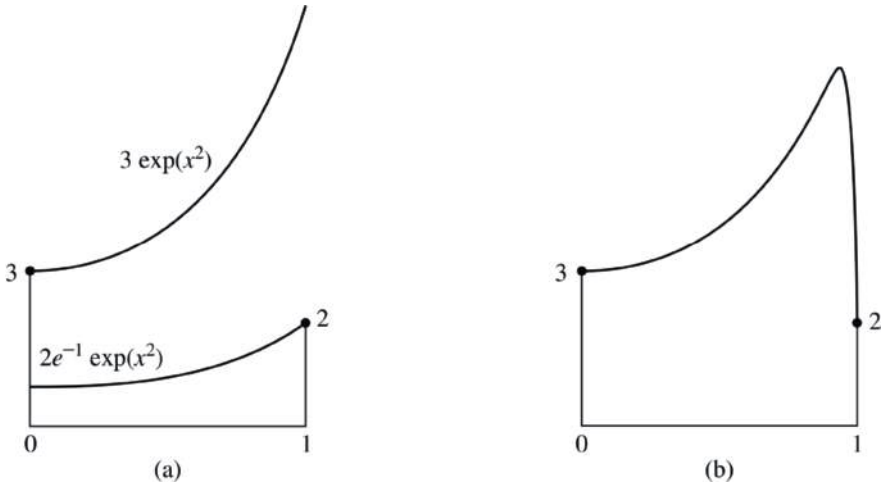
$$\boxed{-\frac{du}{dx} + 2xu = 0.} \tag{14.10.2}$$

The general solution of the reduced equation (14.10.2) (by separation or by an integrating factor) has one arbitrary constant:

$$\boxed{u = ce^{x^2}.} \tag{14.10.3}$$

Unfortunately, this solution has one arbitrary constant, which cannot be used to solve these two boundary conditions (see Fig. 14.10.1). We will show that this solution is approximately valid almost everywhere. The constant c can be determined from one of the boundary conditions, and we will determine which of the two boundary conditions should be used to determine c. There is a thin region near one of the boundaries (called a **boundary layer**) where (14.10.3) is not a good approximation.

Figure 14.10.1 (a) Solution of reduced equation can satisfy only one boundary condition; (b) boundary layer at $x = 1$.



Boundary layer location and thickness

We are in trouble if ε is small since neglecting the $\varepsilon \frac{d^2u}{dx^2}$-term reduces the differential equation to first order. Somewhere $\varepsilon \frac{d^2u}{dx^2}$ cannot be neglected even though ε is small. We conclude that our simplifying approximation (14.10.2) or (14.10.3) may fail if somewhere $\frac{d^2u}{dx^2}$ is large. There are many possibilities, some of which are that $\frac{d^2u}{dx^2}$ is large only near the left boundary $x = 0$, large only near the right boundary $x = 1$, large only near some interior point, large only near both boundaries, or large everywhere. The main idea of a boundary layer will be a region where the derivatives are large. Derivatives can be large if the solution u changes by an order 1 amount in a short distance $x - x_0$ near some (unknown) point x_0. The small distance near x_0 is called the **boundary layer thickness**, which we assume is $O(\varepsilon^p)$, where $p > 0$ in order for the distance to be small:

$$\boxed{x - x_0 = \varepsilon^p X,} \tag{14.10.4}$$

or, equivalently,

$$\boxed{X = \frac{x - x_0}{\varepsilon^p}.} \tag{14.10.5}$$

Here, we are rescaling the independent variable x. The new variable X is called the **boundary layer variable**. Derivatives will be large because of the chain rule $\frac{du}{dx} = \varepsilon^{-p} \frac{du}{dX}$, where we assume $\frac{du}{dX}$ is $O(1)$ in the boundary layer. Using the boundary layer variable (14.10.5), the differential equation (14.10.1) becomes

$$\varepsilon^{1-2p} \frac{d^2u}{dX^2} - \varepsilon^{-p} \frac{du}{dX} + 2(x_0 + \varepsilon^p X)u = 0. \tag{14.10.6}$$

Since ε is small, the middle term is large $O(\varepsilon^{-p})$, much larger than the term $2(x_0 + \varepsilon^p X)u$. In order for the differential equation to imply some nontrivial balance, in this example, the first and second terms must have the same order of magnitude:

$$1 - 2p = -p. \tag{14.10.7}$$

We therefore conclude *in this example* that

$$p = 1, \tag{14.10.8}$$

so that in this example the boundary layer thickness is $O(\varepsilon^1)$, $x - x_0 = \varepsilon X$.

However, we need to determine the location of the boundary layer. In the boundary layer the first and second terms are both $O(\varepsilon^{-1})$ larger than the remaining term. Roughly (we will improve this later), the leading-order boundary layer equation is

$$\boxed{\frac{d^2u}{dX^2} - \frac{du}{dX} = 0.} \tag{14.10.9}$$

The general solution of (14.10.9) is

$$\boxed{u = A + Be^X = A + Be^{\frac{x - x_0}{\varepsilon}},} \tag{14.10.10}$$

which is valid in the boundary layer. One term exponentially grows as x increases with very large growth rate $O(\frac{1}{\varepsilon})$ on the x-scale. If x is more than $O(\varepsilon)$ greater than x_0, the exponential term will be so large that it cannot connect to the solution of the reduced equation. This kind of nearly unlimited exponential growth of the solution in a boundary layer must be prevented. One way to prevent this fast exponential growth is to let $B = 0$. If $B = 0$, the solution in the boundary layer is a constant, which is not acceptable since derivatives of the solution are usually large in the boundary layer. Thus, we must include $Be^{\frac{x - x_0}{\varepsilon}}$ in the boundary layer solution. In this problem, x_0 cannot be some interior point (or the left endpoint $x_0 = 0$) since the solution in the boundary layer exponentially grows too large if $x > x_0$. Only if the boundary layer is located at the right

endpoint $x_0 = 1$ will the undesirable large exponential growth not occur. If $x_0 = 1$, then x cannot be greater than x_0:

$$x - 1 = \varepsilon X. \tag{14.10.11}$$

In this case, the leading-order boundary layer solution (14.10.10) (near $x = 1$) is

$$u = A + Be^{\frac{x-1}{\varepsilon}} \tag{14.10.12}$$

and decays exponentially (not grows) as the solution leaves the boundary layer. If $x - 1$ is negative, then $e^{\frac{x-1}{\varepsilon}}$ is transcendentally small. The solution varies quickly (in a small region), desirable for a boundary layer. Now that we know that the boundary layer in this problem is located at the right end, we can proceed with the mathematical solution.

Outer expansion

We begin by determining an asymptotic expansion of the solution of the differential equation away from any boundary layer. We introduce a perturbation expansion (which we assume is in the following elementary form):

$$u(x, \varepsilon) = u_0(x) + \varepsilon u_1(x) + \dots. \tag{14.10.13}$$

We call this the **outer expansion**, since it is valid away from a boundary layer. Since ε is very small, we may need only a few terms in this expansion [sometimes only the leading-order term $u_0(x)$]. Substituting the perturbation expansion (14.10.13) into the differential equation (14.10.1) yields

$$O(\varepsilon^0): \frac{du_0}{dx} - 2xu_0 = 0 \tag{14.10.14}$$

$$O(\varepsilon^1): \frac{du_1}{dx} - 2xu_1 = \frac{d^2u_0}{dx^2}. \tag{14.10.15}$$

We will restrict our attention to the leading-order outer equation (14.10.14), whose solution (as before) is

$$u_0(x) = ce^{x^2}. \tag{14.10.16}$$

However, now we know there is a boundary layer at $x = 1$, so that the outer solution must solve the boundary condition at $x = 0$:

$$u_0(0) = c = 3. \tag{14.10.17}$$

The leading-order outer solution

$$u_0(x) = 3e^{x^2} \tag{14.10.18}$$

satisfies the boundary condition at $x = 0$ but does not satisfy the boundary condition at $x = 1$.

Inner (boundary layer) expansion

In the boundary layer near $x = 1$, we introduce the inner variable $x = 1 + \varepsilon X$ (14.10.11). The exact inner (boundary layer) equation follows by multiplying (14.10.6) by ε:

$$\frac{d^2u}{dX^2} - \frac{du}{dX} + 2\varepsilon(1 + \varepsilon X)u = 0. \tag{14.10.19}$$

We can introduce a perturbation expansion of the solution valid in the boundary layer. This is called the **inner expansion** or **boundary layer expansion**:

$$u = U_0(X) + \varepsilon U_1(X) + \dots. \tag{14.10.20}$$

By substituting (14.10.20) into (14.10.19), we obtain

$$\frac{d^2U_0}{dX^2} - \frac{dU_0}{dX} = 0 \tag{14.10.21}$$

$$\frac{d^2U_1}{dX^2} - \frac{dU_1}{dX} = -2U_0. \tag{14.10.22}$$

As we obtained earlier, the solution of the leading-order equation (14.10.21) is

$$U_0(X) = A + Be^X = A + Be^{\frac{x-1}{\varepsilon}}, \tag{14.10.23}$$

when expressed in terms of the usual inner variable x. The boundary layer equation should satisfy the boundary condition at $x = 1$. $u(x = 1) = 2$ becomes $U_0(X = 0) = 2$. From this we conclude that

$$2 = A + B. \tag{14.10.24}$$

In this example the boundary layer equation is a second-order equation with two arbitrary constants. In this example, one condition is a boundary condition, while the second condition will be that the outer solution must match the inner (boundary layer) solution.

Matching

We have obtained an outer expansion valid away from the boundary layer,

$$u = 3e^{x^2} + \varepsilon u_1(x) + \ldots, \tag{14.10.25}$$

and an inner expansion valid in the boundary layer,

$$u = A + Be^X + \varepsilon U_1(X) + \ldots, \tag{14.10.26}$$

where from the boundary condition $2 = A + B$. These two asymptotic expansions must be expansions of the same solution of the differential equation. We assume there is an overlap region where both expansions are simultaneously valid. Thus, we equate the two expansions in a manner that is called the **method of matched asymptotic expansions**. The overlap region is near the boundary layer (from the point of view of the outer solution) and far from the boundary layer (from the point of view of the scales in the boundary layer). In simple examples (such as this one), the **matching principle** states that

the inner limit of the outer solution = the outer limit of the inner solution. (14.10.27)

From the point of view of the outer solution, the inner limit is the limit as $x \to 1$. In this example, the outer limit means $X \to -\infty$. In this example,

$$\lim_{x\to 1} u_0(x) + \varepsilon u_1(x) = \lim_{X\to -\infty} U_0(X) + \varepsilon U_1(X). \tag{14.10.28}$$

Using only the leading-order terms, we have

$$\lim_{x\to 1} 3e^{x^2} = \lim_{X\to -\infty} A + Be^X. \tag{14.10.29}$$

Thus $3e = A$. Since $A + B = 2$, we also have $B = 2 - 3e$. Consequently, in summary,

Leading-order outer solution (away from $x = 1$): $u = 3e^{x^2}$

Leading-order inner solution (near $x = 1$): $u = 3e + (2 - 3e)e^X$,

which is graphed in Fig. 14.10.1.

Often these inner and outer limits do not exist and must be replaced by the corresponding asymptotic expansions:

the inner asymptotic expansion of the outer solution = the outer asymptotic expansion of the inner solution. (14.10.30)

These are two very different calculations that must be the same. In the case where $x - 1 = \varepsilon X$, the two expansions are as follows:

1. Fixing x and letting $\varepsilon \to 0$. (This means $X \to -\infty$, the outer expansion of the inner solution.)
2. Fixing X and letting $\varepsilon \to 0$. (This means $x \to 1$, the outer expansion of the inner solution.)

There are practical procedures that take into account the fact that often only the first few terms are worth the effort of computation. Remember that ε is very small, and higher-order terms are often negligible.

The form of the inner and outer expansions may have to be adjusted in other problems. Higher-order terms can be included and matched, but that is an important (and often necessary) process that we do not have the time to develop. The asymptotic expansions of the inner and outer solutions are often doable, but more effort is required than in the elementary example here. Nonlinear problems can be more difficult and interesting.

14.10.2 DIFFUSION OF A POLLUTANT DOMINATED BY CONVECTION

We will analyze one elementary example of a singular perturbation problem for a partial differential equation by the boundary layer method (the method of matched asymptotic expansions). We consider a pollutant with unknown chemical concentration $u(x, y, t)$. Without convection (the motion of a fluid) the concentration satisfies the diffusion equation $\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$. If the fluid moves at a known velocity $\boldsymbol{v}$ (where in fluid dynamics it is shown that usually $\nabla \cdot \boldsymbol{v} = 0$), then the time derivative in the diffusion equation should be replaced by the **convective derivation** (the time derivative moving with the fluid) $\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u$:

$$\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{14.10.31}$$

We consider an idealized situation in which the pollution is caused by a series of smoke stacks (industrial parks), each of which we assume produces a known (but perhaps different) level of pollution that does not change in time. Thus, we know the level of pollution on the boundary of some region, and we wish to determine the steady-state level of pollution inside the large region:

$$\boldsymbol{v} \cdot \nabla u = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{14.10.32}$$

We first assume the convective atmospheric velocity is constant and in a somewhat general direction $\boldsymbol{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. In this case the steady equation for diffusion becomes

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{14.10.33}$$

where (for the convenience of singular perturbation methods) we have introduced ε as a small (in some sense) diffusion coefficient. This is not an easy mathematical problem, especially if the geometric region is not a rectangle.

Outer Expansion

If ε is a very small parameter, we expect (as a good approximation) that the following reduced first-order partial differential equation should be important:

$$a\frac{\partial u_0}{\partial x} + b\frac{\partial u_0}{\partial y} = 0. \tag{14.10.34}$$

We justify this by introducing into (14.10.33) a regular perturbation expansion:

$$u = u_0 + \varepsilon u_1 + \ldots, \tag{14.10.35}$$

whose leading-order term satisfies (14.10.34). Equation (14.10.34) can be solved by the method of characteristics. If $\frac{dy}{dx} = \frac{b}{a}$, then $\frac{du_0}{dx} = 0$ or $u_0 =$ constant. The characteristics of the reduced equation (14.10.34) are $ay - bx =$ constant. Thus,

$$u_0 = f(ay - bx), \tag{14.10.36}$$

where the arbitrary function f should be determined from one boundary condition. However, for the diffusion equation we have two boundary conditions (determined by the manner in which the characteristics divide the boundary of the closed two-dimensional region, as will be clearer by the specific example to follow). We will show that (14.10.36) is valid away from a boundary layer (region where derivatives are large). In addition, we will show that the boundary layer is located only at a specific portion of the boundary. We call (14.10.35) the **outer expansion** and (14.10.36) the leading-order **outer solution** since it is valid away from the boundary layer.

Simple example

For the rest of this subsection, we assume the fluid velocity is in the positive y-direction $\boldsymbol{v} = (0, 1)$ so that the partial differential equation (14.10.33) for steady diffusion becomes

$$\boxed{\frac{\partial u}{\partial y} = \varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).} \tag{14.10.37}$$

The solution of the leading-order outer equation $\frac{\partial u_0}{\partial y} = 0$ is

$$u_0 = f(x), \tag{14.10.38}$$

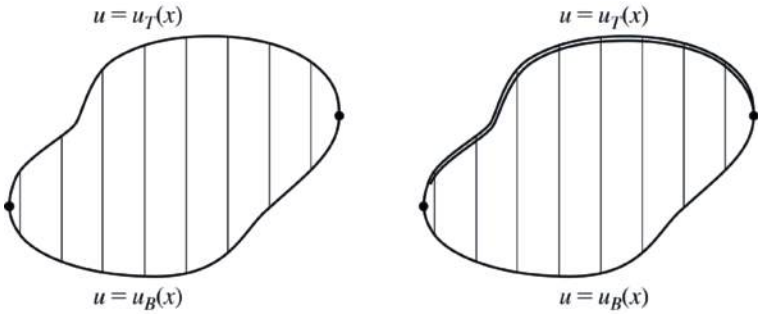
an arbitrary function of x. We assume the characteristics (of the reduced equation) $x =$ constant divide the boundary of the geometric region into two pieces, which we call the top $y = y_T(x)$ and bottom $y = y_B(x)$ (see Fig. 14.10.2). The boundary condition for the diffusive partial differential equation is that the level of pollution is specified on the entire boundary. We introduce two functions, the level of pollution on the top:

$$u = u_T(x) \text{ on } y = y_T(x), \tag{14.10.39}$$

and the level of pollution on the bottom:

$$u = u_B(x) \text{ on } y = y_B(x). \tag{14.10.40}$$

Figure 14.10.2 (a) Characteristics; (b) convection dominates (boundary layer at the top).



The one arbitrary function can satisfy one but not both of these conditions. On the basis of physical intuition, we expect that since the atmosphere is moving in the positive vertical direction, the wind caries the pollutant upward. We expect that the solution of the reduced problem satisfies the boundary condition on the bottom:

$$u_0 = f(x) = u_B(x). \tag{14.10.41}$$

Since this does not solve the boundary condition on the top, we expect that there is a thin region near the top where derivatives are large and cannot be neglected. In the next subsection, we show mathematically that the boundary layer is located on the top and determine its thickness.

Boundary layer (inner) expansion

We want to develop a procedure to determine where a boundary layer is allowed (and determine its thickness). We first analyze the possibility that the boundary layer is located near the top. However, our mathematical analysis will be general enough to show that the boundary layer in this example cannot be located on the bottom. We introduce as a scaled set of independent spatial variables (ξ, η) with unknown thickness $O(\varepsilon^p)$ with $p > 0$:

$$\boxed{\begin{aligned} \xi &= x \\ \eta &= \frac{y - y_T(x)}{\varepsilon^p}. \end{aligned}} \tag{14.10.42}$$

Since the boundary is not necessarily straight, both x- and y-derivatives are large:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} - \frac{y_T'(x)}{\varepsilon^p}\frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial y} &= \frac{1}{\varepsilon^p}\frac{\partial u}{\partial \eta}. \end{aligned}}$$

Second derivatives are needed:

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial \xi} - \frac{y_T'(x)}{\varepsilon^p}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} - \frac{y_T'(x)}{\varepsilon^p}\frac{\partial u}{\partial \eta}\right) = \frac{[y_T'(x)]^2}{\varepsilon^{2p}}\frac{\partial^2 u}{\partial \eta^2} + \dots \tag{14.10.43}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\varepsilon^{2p}}\frac{\partial^2 u}{\partial \eta^2}. \tag{14.10.44}$$

If only the leading-order terms are needed (in the boundary layer), we can avoid the more complicated expressions for the second derivative. Substituting these expressions (14.10.43) and (14.10.44) into our example (14.10.37), we obtain the leading-order equation $u = U_0(\xi, \eta) + \dots$ in a boundary layer:

$$\frac{1}{\varepsilon^p}\frac{\partial U_0}{\partial \eta} = \varepsilon^{1-2p} k_T(\xi)\frac{\partial^2 U_0}{\partial \eta^2}, \tag{14.10.45}$$

where the important coefficient is positive:

$$k_T(\xi) = 1 + [y_T'(x)]^2. \tag{14.10.46}$$

By balancing order of magnitudes of the largest terms in (14.10.45), we obtain

$$p = 1, \tag{14.10.47}$$

so that the **thickness of the boundary layer** is $O(\varepsilon^1)$. Because derivatives are largest in one direction, the partial differential equation to leading order reduces to an ordinary differential equation:

$$\boxed{\frac{\partial U_0}{\partial \eta} = k_T(\xi)\frac{\partial^2 U_0}{\partial \eta^2}.} \tag{14.10.48}$$

The general solution of (14.10.48) is

$$U_0(\xi, \eta) = A(\xi) + B(\xi)e^{\frac{\eta}{k_T(\xi)}}, \tag{14.10.49}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions of ξ since $\frac{\partial}{\partial \eta}$ involves ξ fixed.

Now it is possible to determine whether or not a boundary layer exists at the top (or the bottom). The method of matched asymptotic expansions states that the boundary layer solution must match the interior (outer) solution. Since

$$\eta = \frac{y - y_T(x)}{\varepsilon}, \tag{14.10.50}$$

the interior is approached by the limit process:

$\eta \to -\infty$ if the boundary layer is at the top

$\eta \to +\infty$ if the boundary layer is at the bottom.

Fast exponential growth (14.10.49) in a boundary layer is not allowed as the solution becomes transcendentally large. However, exponential decay is typical. Since $k_T(\xi) > 0$ and the corresponding $k_B(\xi) > 0$, in this example we must prevent $\eta \to +\infty$ as one leaves a boundary layer. This can be prevented only if the boundary layer for this problem is located at the top. This is the same conclusion that we reached based on the physical intuition that the pollution levels are convected into the interior of the region by the given upward atmospheric velocity.

In this way we have obtained the leading-order solution in the boundary layer located at the top:

$$U_0(\xi, \eta) = A(\xi) + B(\xi)e^{\frac{\eta}{k_T(\xi)}}. \tag{14.10.51}$$

If the boundary layer is at the top, then the solution in the boundary layer should satisfy the boundary condition at the top, which is $u = u_T(x)$ at $y = y_T(x)$. Since $y = y_T(x)$ corresponds to $\eta = 0$, we satisfy the top boundary condition with

$$A(\xi) + B(\xi) = u_T(\xi). \tag{14.10.52}$$

The top boundary condition prescribes one condition for the two arbitrary functions $A(\xi)$ and $B(\xi)$. The second condition (needed to determine the boundary layer solution) is that the inner and outer solutions must match.

Matching of the inner and outer expansions

In summary, the leading-order (outer) solution is valid away from a thin boundary layer near the top portion of the boundary:

$$u_0 = u_B(x). \tag{14.10.53}$$

The leading-order (inner) solution in the boundary layer near the top is

$$U_0 = A(x) + B(x)e^{\frac{\eta}{k_T(x)}}, \tag{14.10.54}$$

where the boundary layer variable is $\eta = \frac{y - y_T(x)}{\varepsilon}$ and where $A(x) + B(x) = u_T(x)$ to satisfy the boundary condition at the top.

These two asymptotic expansions represent a good approximation to the solution of the original partial differential equation in different regions. Since there is an overlap region where both expansions are valid, the **matching principle** states that

the inner limit of the outer solution = the outer limit of the inner solution.

In this example, the inner limit is $y \to y_T(x)$, and the outer limit is $\eta \to -\infty$. Thus, matching the leading-order inner and outer solutions yields

$$\lim_{y \to y_T(x)} u_B(x) = \lim_{\eta \to -\infty} A(x) + B(x)e^{\frac{\eta}{k_T(x)}}, \tag{14.10.55}$$

or, equivalently,

$$u_B(x) = A(x). \tag{14.10.56}$$

Since $A(x) + B(x) = u_T(x)$, we have $B(x) = u_T(x) - u_B(x)$. The leading-order solution in the boundary layer is

$$U_0 = u_B(x) + [u_T(x) - u_B(x)]e^{\frac{y - y_T(x)}{\varepsilon k_T(x)}}. \tag{14.10.57}$$

The concentration of the pollutant is convected from the bottom into the interior, as described by the leading-order outer solution (14.10.53). The leading-order inner solution (14.10.57) shows that a thin boundary layer at the top exists (illustrated in Fig. 14.10.2) in which diffusion dominates and the level of pollution suddenly changes from level in the interior (convected from the bottom) to the level specified by the top boundary condition.

Exercises 14.10

In Exercises 14.10.1–14.10.7, find one term of the inner and outer expansions and match them:

14.10.1. $\varepsilon \frac{d^2u}{dx^2} - 2\frac{du}{dx} + 3u = 0$ with $u(0) = 1$ and $u(1) = 2$

14.10.2. $\varepsilon \frac{d^2u}{dx^2} + 2\frac{du}{dx} - 3u = 0$ with $u(0) = 1$ and $u(1) = 2$

***14.10.3.** $\varepsilon \frac{d^2u}{dx^2} - 4u = x$ with $u(0) = 1$ and $u(1) = 2$

14.10.4. $\varepsilon \frac{d^2u}{dx^2} + \varepsilon \frac{du}{dx} - 9u = 0$ with $u(0) = 1$ and $u(1) = 2$

14.10.5. $\varepsilon \frac{d^2u}{dx^2} + 2e^{-x}u = 8$ with $u(0) = 1$ and $u(1) = 2$

***14.10.6.** $\varepsilon \frac{d^2u}{dx^2} + (2x + 1)\frac{du}{dx} + 2u = 0$ with $u(0) = 1$ and $u(1) = 2$

14.10.7. $\varepsilon \frac{d^2u}{dx^2} + 2e^{-x}u = 8$ with $u(0) = 1$ and $u(1) = 2$

14.10.8. Find an exact solution of Exercise 14.10.1 (and graph using software).

14.10.9. Sometimes there can be a boundary layer within the boundary layer. Consider for $0 < x < 1$ with $0 < \varepsilon \ll 1$:

$$\varepsilon^4 \frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} - \varepsilon y = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y(1) = 2.$$

You may assume (without verifying) that there is no boundary layer at $x = 1$. Solve this problem, including two terms of the outer expansion. Show that two different scalings near $x = 0$ are valid, a thin and thick one. Determine only one term of these two other

expansions. Match the outer solution to the thicker inner solution, and match the thicker inner solution to the thinner inner solution.

14.10.10. Consider $\varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$, with u specified on the boundaries of the square $0 < x < 1$, $0 < y < 1$. Using physical reasoning, where will there be a boundary layer?

14.10.11. $\varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = u - f(x, y)$ with boundary conditions on the unit square: $u(x, 0) = g(x)$, $u(x, 1) = h(x)$, $u(0, y) = r(y)$, and $u(1, y) = s(y)$. Determine the leading-order outer solution. To save time, just determine the leading-order inner solution in one boundary layer (since the others are quite similar). Match these.

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ANSWERS TO STARRED EXERCISES

1.2.9. $\int_0^L c\rho u\, A\, dx$

1.2.10. **(e)** $u(t) = u_0 \exp\left[-\left(\frac{2h}{c\rho r}\right)t\right]$

1.3.2. $K_0(x_{0-})\frac{\partial u}{\partial x}(x_{0-}, t) = K_0(x_{0+})\frac{\partial u}{\partial x}(x_{0+}, t)$

1.3.3. $Vc_f\rho_f\frac{\partial u}{\partial t}(L, t) = -K_0(L)\frac{\partial u}{\partial x}(L, t)A$, where V is the volume of the bath

1.4.2. **(a)** $u(x) = \frac{Tx}{L}$

1.4.2. **(d)** $u = T + \alpha x$

1.4.2. **(f)** $u(x) = -\frac{x^4}{12} + \frac{L^3x}{3} + T$

1.4.2. **(h)** $u = T + \alpha(x + 1)$

1.4.3. **(a)** $\frac{K_0L^2}{2}$

1.4.8. **(a)** $\beta = 1 - L$

1.5.2. $\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u = k\nabla^2 u - u\nabla \cdot \boldsymbol{v}$(often, physically, $\nabla \cdot \boldsymbol{v} = 0$)

1.5.9. **(a)** $u = \frac{T_1 \ln\frac{r_2}{r} + T_2 \ln\frac{r}{r_1}}{\ln\frac{r_2}{r_1}}$

1.5.11. $\beta = \frac{b}{a}$

1.5.13. $u(r) = \frac{320}{3}(1 - \frac{1}{r})$

2.3.1. **(a)** $\frac{dh}{dt} = -\lambda kh$ and $\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda\phi$

2.3.1. **(c)** $\frac{d^2\phi}{dx^2} = -\lambda\phi$ and $\frac{d^2h}{dy^2} = \lambda h$

2.3.1. **(e)** $\frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^4\phi}{dx^4} = \lambda$

2.3.1. **(f)** $\frac{1}{c^2h}\frac{d^2h}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$

2.3.4. **(b)** $\lambda = (n\pi/L)^2$ with $L = 1$ so that $\lambda = n^2\pi^2$, $n = 1, 2, \ldots$

2.3.4. **(c)** $\lambda = \left(\frac{n\pi}{b-a}\right)^2$, $n = 1, 2, 3, \ldots$

2.3.5. **(c)** $u(x, t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}$, where

$$A_n = \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\, dx$$

2.3.6. **(a)** $c\rho A\sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(\frac{1-\cos n\pi}{\frac{n\pi}{L}}\right)$

2.3.6. **(c)** Heat energy equals initial heat energy plus the time integral of the flow *in* of the heat energy at the boundaries.

2.3.8. $0(n \neq m)$, $\frac{L}{2}(n = m \neq 0)$, $L(n = m = 0)$

2.3.10. **(a)** $u = 0$

2.3.10. **(b)** $u = e^{-\alpha t}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}$

2.3.11. **(a)** If $\sqrt{\frac{-\alpha}{k}}L = n\pi$, then $u(x) = A\sin\frac{n\pi x}{L}$.

2.3.11. **(b)** If $-\frac{\alpha}{k} = \left(\frac{\pi}{L}\right)^2$, $u(x, t) \to B\sin\frac{\pi x}{L}$, as $t \to \infty$. If $-\frac{\alpha}{k} < \left(\frac{\pi}{L}\right)^2$, then $u \to 0$ as $t \to \infty$. However, if $-\frac{\alpha}{k} > \left(\frac{\pi}{L}\right)^2$, $u \to \infty$ as $t \to \infty$.

2.3.13. **(c)** $\left[\int_0^L A(x)B(x)\, dx\right]^2 \leq \left(\int_0^L A^2\, dx\right)\left(\int_0^L B^2\, dx\right)$

2.4.7. For all parts, $u = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}$, where A_0 and A_n are as follows:

2.4.7. (a) $A_0 = \frac{1}{2}$, $A_n = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$ for $n > 0$

2.4.7. (b) $A_0 = 6$, $A_3 = 4$, and $A_n = 0$ for all $n \neq 0$, $n \neq 3$

2.4.7. (c) $A_0 = -\frac{4}{\pi}$, $A_n = -\frac{8}{\pi(1-n^2)}$ for n even, $A_n = 0$ for n odd

2.4.7. (d) $A_8 = -3$, $A_n = 0$ for $n \neq 8$

2.4.7. (e) $A_0 = 2L$, $A_n = \frac{8L}{n^2\pi^2}\big((-1)^n - 1\big)$ for $n > 0$

2.4.7. (f) $A_0 = 8L^2/3$, $A_n = \frac{32L^2}{n^2\pi^2}(-1)^n$ for $n > 0$

2.4.10. $\lambda = n^2$, $\phi = \sin nx$ and $\cos nx$, $n = 0, 1, 2, 3, \ldots$

2.5.3. (a) $u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$

2.5.3. (c) $u(x, y) = \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi x}{H}\sin\frac{n\pi y}{H}$, where $A_n \cosh\frac{n\pi L}{H} = \frac{2}{H}\int_0^H g(y)\sin\frac{n\pi y}{H}\,dy$

2.5.3. (e) $u(x, y) = \sum_{n=1}^{\infty} A_n h_n(y)\sin\frac{n\pi x}{L}$, $A_n h_n(H) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$, where

$h_n(y) = \cosh\frac{n\pi y}{L} + \frac{L}{n\pi}\sinh\frac{n\pi y}{L}$

2.5.4. (a) $\int_0^L f(x)\,dx = 0$

2.5.5. $u(r, \theta) = \sum_{n=0}^{\infty} A_n r^{-n}\cos n\theta + \sum_{n=1}^{\infty} B_n r^{-n}\sin n\theta$

2.5.5. (a) $A_0 = \ln 2$, $A_3 a^{-3} = 4$, other $A_n = 0$, $B_n = 0$

2.5.5. (b) See (2.5.47) with a^n replaced by a^{-n}.

2.5.6. $u(r, \theta) = \frac{a^2 - r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\bar{\theta})\,d\bar{\theta}}{a^2 + r^2 - 2ar\cos(\theta - \bar{\theta})}$

2.5.7. (a) $u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{2n-1}\cos(2n-1)\theta$

2.5.7. (c) $u(r, \theta) = \sum_{n=1}^{\infty} A_n r^{2n}\sin 2n\theta$, $A_n = \frac{2}{\pi n}\int_0^{\pi/2} f(\theta)\sin 2n\theta\,d\theta$

2.5.8. (a) $u(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta$

2.5.9. (b) $u(r, \theta) = \sum_{n=0}^{\infty} A_n r^{3n}\cos 3n\theta$

2.5.10. (a) $\phi_1(r) = \begin{cases} \ln(r/a) & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n & n \neq 0 \end{cases}$, $\phi_2(r) = \begin{cases} \ln(r/b) & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n & n \neq 0 \end{cases}$

$u(r, \theta) = \sum_{n=0}^{\infty}\cos n\theta[A_n\phi_1(r) + B_n\phi_2(r)] + \sum_{n=1}^{\infty}\sin n\theta[C_n\phi_1(r) + D_n\phi_2(r)]$

$D_n\phi_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$, etc.

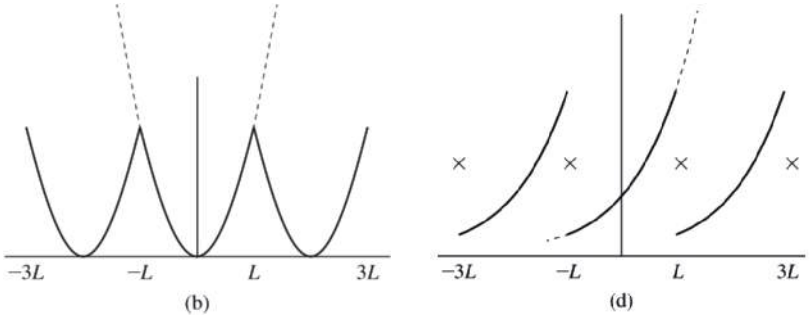
2.5.11. (a) $u(r, \theta) = \sum_{n=1}^{\infty} A_n\left[\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right]\sin 2n\theta$, where $A_n\left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right]$

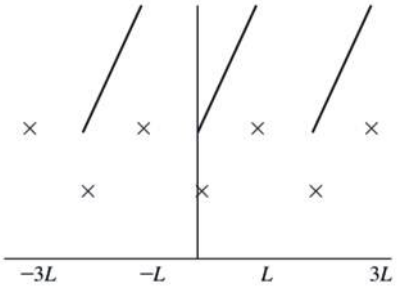
$= \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta\,d\theta$

2.5.11. (b) $u(r, \theta) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi\theta}{\ln(b/a)}\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]$, where $A_n \sinh\frac{n\pi^2}{2\ln(b/a)}$

$= \frac{2}{\ln(b/a)}\int_a^b f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]dr/r$

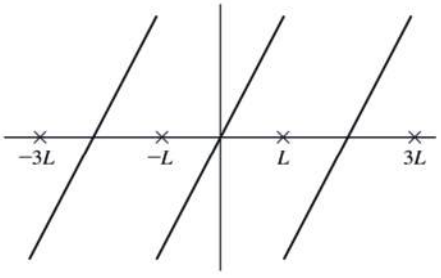
3.2.3.





(f)

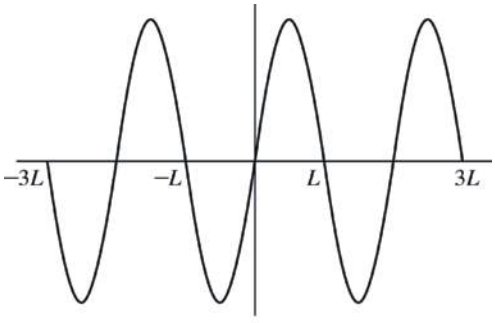
3.2.4. (a)



(a)

$a_n = 0,\ b_n = \frac{2L}{n\pi}(-1)^{n+1}$

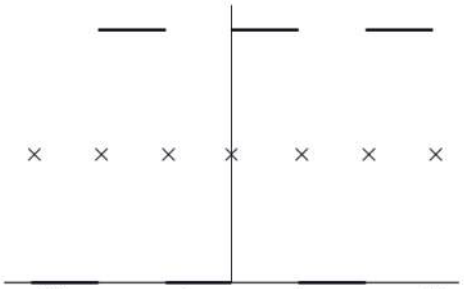
3.2.4. (c)



(c)

$b_1 = 1,$ all others $= 0$

3.2.4. (f)

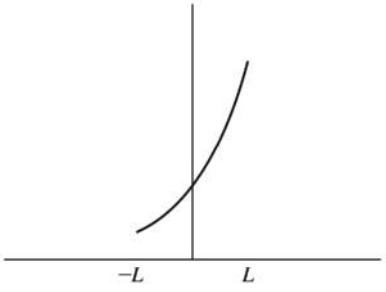


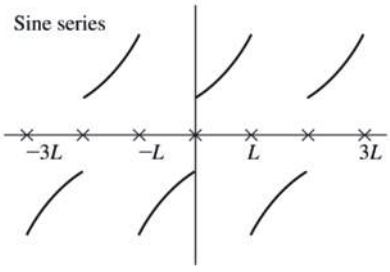
(f)

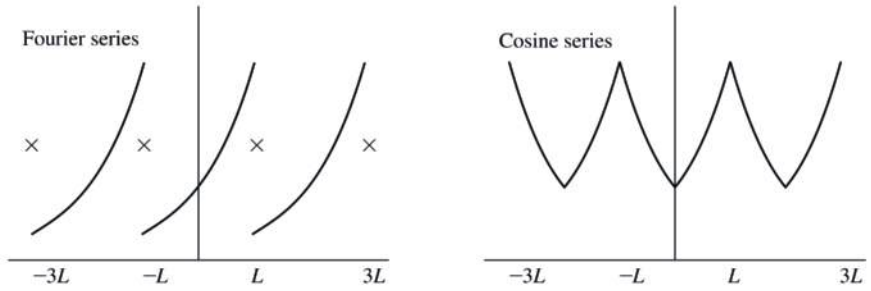
$a_0 = \frac{1}{2},$ other $a_n = 0$

$b_n = \frac{2}{n\pi}(n \text{ odd}),$ other $b_n = 0$

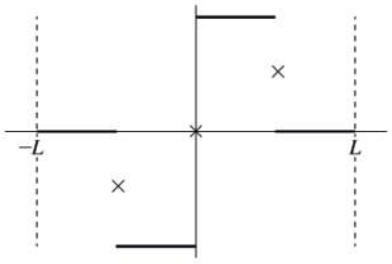
3.3.2. (d)







3.3.3. (d)



repeat periodically,
$b_n = \frac{2}{n\pi}(1 - \cos\frac{n\pi}{2})$

3.3.11. $f_e(x) = \frac{1}{2}[f(x) + f(-x)] = \frac{1}{2}\begin{cases} x^2 + e^x & x < 0 \\ x^2 + e^{-x} & x > 0 \end{cases}$

$f_o(x) = \frac{1}{2}[f(x) - f(-x)] = \frac{1}{2}\begin{cases} x^2 - e^x & x < 0 \\ e^{-x} - x^2 & x > 0 \end{cases}$

3.3.14. $b_n = \frac{2}{L}\int_0^L f(x)\sin n\pi x/L\, dx$, $b_n = 0$ for n even, since $\sin n\pi x/L$ is odd around $L/2$ for n even

3.4.1. (a) $\int_a^b u\frac{dv}{dx}\, dx = uv\Big|_a^b + uv\Big|_{c+}^{c-} - \int_a^b v\frac{du}{dx}\, dx$

3.4.3. (a) $b_n = \frac{2}{L}\sin\frac{n\pi x_0}{L}(\alpha - \beta) - \frac{n\pi}{L}a_n$

3.4.9. $\frac{db_n}{dt} + k(\frac{n\pi}{L})^2 b_n = \frac{2}{L}\int_0^L q(x, t)\sin\frac{n\pi x}{L}\, dx$

3.4.12. $u = \sum_{n=0}^{\infty} A_n(t)\cos\frac{n\pi x}{L}$,

$n \neq 0, 3 \quad A_n(t) = A_n(0)e^{-k(n\pi/L)^2 t}$

$A_0(t) = A_0(0) + 1 - e^{-t}$

$A_3(t) = A_3(0)e^{-k(3\pi/L)^2 t} + \frac{e^{2t} - e^{-k(3\pi/L)^2 t}}{k(\frac{3\pi}{L})^2 - 2}$,

with $A_0(0) = \frac{1}{L}\int_0^L f(x)\, dx$ and $(n \neq 0)A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$

3.5.1. (c) $(\frac{x}{L})^3 = \frac{1}{4} + \sum_{n=1}^{\infty}\Big\{6\frac{(-1)^n}{(n\pi)^2} + 12\frac{[1-(-1)^n]}{(n\pi)^4}\cos\frac{n\pi x}{L}$, $0 \le x \le L$

3.5.4. $b_n = \frac{2n\pi}{L^2 + n^2\pi^2}[1 - (-1)^n \cosh L]$

3.5.7. $\pi^3/32$

3.6.1. $C_m = \frac{1}{m\pi\Delta}e^{im\pi/L(x_0 + \Delta/2)}\sin\frac{m\pi\Delta}{2L}$

4.4.4. (a) $\frac{n\pi c}{L}$, $n = 1, 2, 3, \ldots$

4.4.4. (b) $(m - \frac{1}{2})\pi c/H$, $m = 1, 2, 3, \ldots$

4.4.5. (c) Frequencies of vibration are $\sqrt{\lambda_n} = \sqrt{\frac{(n\pi/L)^2 T_0 - \alpha}{\rho_0}}$

4.4.6. (b) $u = e^{-\beta t/2\rho_0}\sum_{n=1}^{\infty}(a_n\cos\omega_n t + b_n\sin\omega_n t)\sin\frac{n\pi x}{L}$, where $\omega_n = \sqrt{\frac{T_0}{\rho_0}(\frac{n\pi}{L})^2 - \frac{\beta^2}{4\rho_0^2}}$

5.3.1. $T_0\frac{d^2\phi}{dx^2} + \alpha\phi + \lambda\rho_0\phi = 0$

5.3.3. $H = c_1\exp\Big[\int^x \alpha(t)dt\Big]$, let $c_1 = 1$. Then $p(x) = H$, $q(x) = \gamma H$, and $\sigma(x) = \beta H$.

5.3.4. **(b)** $u = e^{\frac{v_0}{2k}x}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-\lambda_n t}$, where $A_n = \frac{2}{L}\int_0^L f(x) e^{-\frac{v_0}{2k}x} \sin\frac{n\pi x}{L}\, dx$.
Note that $\lambda = \frac{v_0^2}{4k} + k\left(\frac{n\pi}{L}\right)^2$.

5.3.9. **(c)** $n = 1, 2, \ldots$. $\lambda = (n\pi/\ln b)^2$

5.4.2. **(b)** $u = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$ (with $\lambda_1 = 0$, $\phi_1 = 1$)

5.4.3. $u = \sum_{n=1}^{\infty} a_n \phi_n(r) e^{-\lambda k t}$, where $a_n = \dfrac{\int_0^a f(r)\phi_n(r) r\, dr}{\int_0^a \phi_n^2(r) r\, dr}$

5.4.6. $u = \sum_{n=1}^{\infty} A_n \phi_n(x) \cos\sqrt{\lambda_n} t$

5.5.1. **(g)** $\alpha\delta - \beta\gamma = 1$

5.5.9. $\lambda = \dfrac{-\int_0^1 \left(\frac{d^2\phi}{dx^2}\right)^2 dx}{\int_0^1 e^x \phi^2\, dx}$

5.5.11. **(a)** $H(x) = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) + uv\left(\frac{dp}{dx} - r\right)$

5.5A.4. **(a)** $\boldsymbol{v}(t) = \frac{4}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t} - \frac{3}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t}$

5.5A.5. **(b)** $\lambda = 2 \pm \sqrt{3}$

5.6.1. **(c)** $u_T = ax + bx^2$ with $a = 3$, $b = -2$, $\lambda_1 \le 4\frac{1}{6}$

5.7.3. The circular frequency (cycles per 2π units of time) is $\sqrt{\lambda}$, but the actual frequency is $\sqrt{\lambda}/2\pi$ (cycles per 1 unit of time):

$$\frac{1}{2} \le \frac{\sqrt{\lambda}}{2\pi} \le \frac{1}{2}\sqrt{1+\alpha^2}.$$

5.8.2. **(c)** By extension of Fig. 5.8.2a, $(n-l)\pi < \sqrt{\lambda_n} L < (n - \frac{1}{2})\pi$.

5.8.3. **(b)** $(n-1)\pi < \sqrt{\lambda_n} L < (n - \frac{1}{2})\pi$, $n = 1, 2, 3, \ldots$

5.8.7. **(b)** $\lambda = 1/4$ **(c)** none **(d)** no **(e)** yes

5.8.8. **(c)** $\sqrt{\lambda_n} \sim (n-1)\pi$

5.8.10. **(a)** $\lambda_1 \approx 4.12$ **(b)** $\lambda_1 = 4.11586\ldots$

5.8.13. $\cos\sqrt{\lambda} L = \frac{1}{\sqrt{1+(\lambda/h^2)}}$. Thus, $\int_0^L \sin^2\sqrt{\lambda} x\, dx = \frac{L}{2} + \frac{1}{2h[1+(\lambda/h^2)]}$.

5.9.1. **(b)** $\lambda^{1/2}\int_0^L (\sigma/\rho)^{1/2}\, dx_0 \approx (n + \frac{1}{2})\pi$

5.9.3. **(a)** $A'' + i\lambda^{1/2}(2A'\sigma^{1/2} + \frac{1}{2}\sigma^{-1/2}\sigma' A) + qA = 0$

5.9.3. **(e)** $A_{n+1} = \frac{i}{2}\sigma^{-1/4}\int_0^x \sigma^{-1/4}(A_n'' + qA_n)\, dx_0$

5.10.2. **(b)** $\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots$

6.2.5. $a = -3/(2\Delta x)$, $b = 2/(\Delta x)$, and $c = -1/(2\Delta x)$.
Truncation error $= -\frac{1}{3}(\Delta x)^2 f'''(x_0)$.

6.2.7. $\frac{\partial^2 u}{\partial x\, \partial y} \approx \frac{1}{4(\Delta x)^2}[u(x+\Delta x, y+\Delta y) - u(x-\Delta x, y+\Delta y) - u(x+\Delta x, y-\Delta y)$
$+ u(x-\Delta x, y-\Delta y)]$ assuming that $\Delta x = \Delta y$

6.3.4. **(a)** $\beta_n = \dfrac{\sum_{j=1}^{N-1} f_j \sin\frac{n\pi j}{N}}{\sum_{j=1}^{N-1} \sin^2\frac{n\pi j}{N}}$

6.3.4. **(b)** $\frac{N-1}{2}$

6.3.6. **(d)** Stable if $s < 0.5125$

6.3.9. **(b)** $\boldsymbol{A} = \frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

6.3.10. **(b)** $\frac{\partial u}{\partial t} = \frac{ka}{s}\frac{\partial^2 u}{\partial x^2}$

6.3.14. **(c)** $\lambda = 0,\ 3,\ 4;\ |\lambda - 1| \le 5,\ |\lambda - 4| \le 8,\ |\lambda - 2| \le \frac{1}{3}$

6.4.1. $k\Delta t\left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right] \le \frac{1}{2}$

6.5.5. **(b)** Unstable

6.5.6. **(b)** Stable if $\frac{c}{\Delta x/\Delta t} \le 1$

7.3.1. $u(x,\ y,\ t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin\left(\frac{n\pi x}{3}\right)\sin\left(\frac{m\pi y}{2}\right)e^{-\lambda_{mn}t}$,

where $\lambda_{mn} = \frac{n^2\pi^2}{9} + \frac{m^2\pi^2}{4}$

and $c_{mn} = \frac{4}{\pi^2 mn}(\cos\frac{m\pi}{2} - 1)(\cos\frac{n\pi}{3} - 1)$

7.3.2. **(a)** $u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}e^{-\lambda_{nm}kt}$, where $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

7.3.2. **(c)** $u = \sum_{n=0}^{\infty}\sum_{n=1}^{\infty} a_{nm} \cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}e^{-\lambda_{nm}kt}$, where $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

7.3.3. **(b)** $u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} a_{nm\ell} \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\cos\frac{\ell\pi z}{W}e^{-\lambda_{nm\ell}kt}$,

where $\lambda_{nm\ell} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{\ell\pi}{W}\right)^2$

$u(x,\ y,\ z,\ t) \to a_{000} = \frac{1}{LHW}\int_0^W\int_0^H\int_0^L f(x,\ y,\ z)\,dx\,dy\,dz$ as $t \to \infty$

7.3.7. **(b)** $u(x,\ y,\ t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{nm} \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}h_{nm}(t)$

$$h_{nm}(t) = \begin{cases} t & n = 0,\ m = 0 \\ \sin\omega_{nm}t & \text{otherwise,} \end{cases} \quad \text{where } \omega_{nm}^2 = c^2\pi^2\left[(n/L)^2 + (m/H)^2\right]$$

$$A_{nm}h'_{nm}(0) = \frac{\int_0^H\int_0^L f\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\,dx\,dy}{\int_0^H\int_0^L \cos^2\frac{n\pi x}{L}\cos^2\frac{m\pi y}{H}\,dx\,dy}$$

7.3.9. **(b)** $u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{W}\cosh\sqrt{\lambda_{nm}}z$, where $\lambda_{nm} = (n\pi/L)^2 + (m\pi/W)^2$

7.3.10. **(c)(d)** $u(x,\ y,\ z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \cos\frac{n\pi y}{W}\cos\frac{m\pi z}{H}\cosh\beta_{nm}x$,

where $\beta_{nm}^2 = (n\pi/W)^2 + (m\pi/H)^2$

For (c) a solution only exists if $\int_0^H\int_0^W f(y,\ z)\,dy\,dz = 0$, in which case A_{00} is arbitrary, and otherwise

$$A_{nm}\beta_{nm}\sinh\beta_{nm}L = \frac{\int_0^H\int_0^W f\cos\frac{n\pi y}{W}\cos\frac{m\pi z}{H}\,dy\,dz}{\int_0^H\int_0^W \cos^2\frac{n\pi y}{W}\cos^2\frac{m\pi z}{H}\,dy\,dz}.$$

For (d) $A_{nm}\cosh\beta_{nm}L = \frac{\int_0^H\int_0^W g\cos\frac{n\pi y}{W}\cos\frac{m\pi z}{H}\,dy\,dz}{\int_0^H\int_0^W \cos^2\frac{n\pi y}{W}\cos^2\frac{m\pi z}{H}\,dy\,dz}$.

7.4.1. **(a)** $\lambda_{nm} = (n\pi/H)^2 + (m\pi/H)^2$, where $n = 1,\ 2,\ 3,\ \ldots$ and $m = 0,\ 1,\ 2,\ \ldots$

7.7.1. $u(r,\ \theta,\ t) = \sum_{n=1}^{\infty} A_n J_3\left(\sqrt{\lambda_{3n}}r\right)\sin 3\theta \sin c\sqrt{\lambda_{3n}}t$

7.7.2. **(d)** $u(r,\ \theta,\ t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos m\theta + B_{mn}\sin m\theta)\,H_{mn}(r,\ t)$, where

$$H_{mn}(r,\ t) = \begin{cases} t & m = 0,\ n = 1 \\ J_m\left(\sqrt{\lambda_{mn}}r\right)\sin c\sqrt{\lambda_{mn}}t & \text{otherwise with } J'_m\left(\sqrt{\lambda_{mn}}a\right) = 0 \end{cases}$$

$$A_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \beta(r,\ \theta)\phi_{mn}(r)\cos m\theta\ r\ dr\ d\theta}{\int_{-\pi}^{\pi}\int_0^a \phi_{mn}^2(r)\cos^2 m\theta\ r\ dr\ d\theta}$$

where

$$\phi_{mn}(r) = \begin{cases} 1 & m = 0,\ n = 1 \\ c\sqrt{\lambda_{mn} J_m\left(\sqrt{\lambda_{mn}} r\right)} & \text{otherwise} \end{cases}$$

and B_{mn} is the same as A_{mn} with $\cos m\theta$ above replaced by $\sin m\theta$.

7.7.3. **(a)** $c\sqrt{\lambda_{mn}}$, where $J_{2m}\left(\sqrt{\lambda_{mn}} a\right) = 0$

7.7.5. The frequencies are $c\sqrt{\lambda}$, where $J_{2m}\left(\sqrt{\lambda} a\right) Y_{2m}\left(\sqrt{\lambda} b\right) - J_{2m}\left(\sqrt{\lambda} b\right) Y_{2m}\left(\sqrt{\lambda} a\right) = 0$.

7.7.8. For the heat equation, Section 7.7 is valid with (7.7.5) replaced by $dh/dt = -\lambda k h$. The boundary condition introduces more substantial changes. The Rayleigh quotient shows $\lambda \geq 0$ with $\lambda = 0$ only when $\phi(r, \theta)$ is constant, which means $m = 0$. The other eigenfunctions still satisfy (7.7.38) with the boundary condition $f'(a) = 0$ yielding $J'_m\left(\sqrt{\lambda} a\right) = 0$. Thus

$$\phi_{mn}(r, \theta) = \begin{cases} 1 & m = 0,\ n = 1 \\ J_m\left(\sqrt{\lambda} r\right) \cos m\theta & \text{otherwise.} \end{cases}$$

As $t \to \infty$, $e^{-\lambda_{mn} k t} \to 0$ except for $m = 0$, $n = 1$ since then $\lambda_{mn} = 0$. Thus $u(r, \theta, t) \to A_{01}$, where $A_{01} = \left(\int_{-\pi}^{\pi} \int_0^a f(r, \theta) r \, dr \, d\theta\right) \Big/ (\pi a^2)$.

7.7.9. **(b)** $u(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{mn} \phi_{mn}(r) \cos m\theta e^{-\lambda_{mn} k t}$, where

$$\phi_{mn}(r) = \begin{cases} 1 & m = 0,\ n = 1 \\ J_m\left(\sqrt{\lambda_{mn}} r\right) & \text{otherwise, where } J'_m\left(\sqrt{\lambda} a\right) = 0 \end{cases}$$

7.7.10. $u(r, t) = \sum_{n=1}^{\infty} a_n J_0\left(\sqrt{\lambda_n} r\right) e^{-\lambda_n k t}$, where $J_0\left(\sqrt{\lambda_n} a\right) = 0$

7.7.12. **(a)** $y = c_1(x^{-2} + \ldots) + c_2(x^3 + \ldots)$

7.7.12. **(c)** $y = c_1[\cos(2 \ln x) + \ldots] + c_2[\sin(2 \ln x) + \ldots]$

7.7.12. **(e)** $y = c_1(x^2 + \ldots) + c_2(x^3 + \ldots)$

7.8.1. **(b)** $J_m\left(\sqrt{\lambda}\right) Y_m\left(2\sqrt{\lambda}\right) - J_m\left(2\sqrt{\lambda}\right) Y_m\left(\sqrt{\lambda}\right) = 0$

7.8.1. **(d)** $\frac{1}{2}\pi^2 \leq \lambda_1 \leq 2\pi^2$

7.8.2. **(d)** $u(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{mn} J_{2m}\left(\sqrt{\lambda_{mn}} r\right) \sin 2m\theta e^{-\lambda_{mn} k t}$, where $J_{2m}\left(\sqrt{\lambda_{mn}} a\right) = 0$

7.8.8. $J_{1/2}(z) = \sqrt{2/\pi z} \sin z$

7.9.1. **(b)** $u(r, \theta, z) = \sum_{n=1}^{\infty} A_n \sinh \sqrt{\lambda_{7n}} (H - z) J_7\left(\sqrt{\lambda_{7n}} r\right) \sin 7\theta$, where $J_7\left(\sqrt{\lambda_{7n}} a\right) = 0$

7.9.2. **(b)** $u(r, \theta, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} I_m\left[\left(n - \frac{1}{2}\right)\frac{\pi}{H} r\right] \sin\left(n - \frac{1}{2}\right)\frac{\pi z}{H} \sin m\theta$

7.9.3. **(b)** $u(r, \theta, z, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} A_{\ell mn} \phi_{\ell mn}(r, \theta, z) e^{-\lambda_{\ell mn} k t}$,
where $\lambda_{\ell mn} = (\ell\pi/H)^2 + \lambda'_{mn}$ and $J'_{2m}\left(\sqrt{\lambda'_{mn}} a\right) = 0$

$$\phi_{\ell mn}(r, \theta, z) = \begin{cases} 1 & m = 0,\ \ell = 0,\ n = 1 \\ \cos\frac{\ell\pi z}{H} \cos 2m\theta J_{2m}\left(\sqrt{\lambda'_{mn}} r\right) & \text{otherwise} \end{cases}$$

7.9.4. **(a)** $u(r, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} J_0\left(\sqrt{\lambda'_n} r\right) \sin\frac{m\pi z}{H} e^{-\lambda k t}$,
where $\lambda = \lambda' + (m\pi/H)^2$ and $J_0\left(\sqrt{\lambda'_n} a\right) = 0$. Here,

$$A_{nm} = \frac{\iint f(r, z) J_0\left(\sqrt{\lambda'_n} r\right) \sin\frac{m\pi z}{H} \, r \, dr \, dz}{\iint J_0^2\left(\sqrt{\lambda'_n} r\right) \sin^2\frac{m\pi z}{H} \, r \, dr \, dz}.$$

8.2.1. **(a)** $u(x, t) = A + Bx + \sum_{n=1}^{\infty} a_n \sin\left(n - \frac{1}{2}\right)\frac{\pi x}{L} e^{-[(n-1/2)\pi/L]^2 k t}$,
where $a_n = \frac{2}{L} \int_0^L g(x) \sin\left(n - \frac{1}{2}\right)\frac{\pi x}{L} \, dx$

8.2.1. **(d)** $u_E(x) = -\frac{x^2}{2} + \left(\frac{B-A}{L} + \frac{L}{2}\right)x + A.$

8.2.2. **(a)** $r(x, t) = A(t)x + \frac{[B(t)-A(t)]x^2}{2L}$

8.2.2. **(c)** $r(x, t) = A(t)x + B(t) - LA(t)$

8.2.6. **(a)** $u(x, t) = A + (B - A)\frac{x}{L} + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)$,

where $A_n = \frac{2}{L}\int_0^L \{f(x) - [A + (B - A)\frac{x}{L}]\sin\frac{n\pi x}{L}\, dx$ and

$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\, dx$

8.2.6. **(d)** $u_E(x) = \frac{L^2}{c^2\pi^2}\sin\frac{\pi x}{L}$

8.3.1. **(c)** $u(x, t) = A(t) - \sum_{n=1}^{\infty} B_n(t)\sin\frac{\left(n-\frac{1}{2}\right)\pi x}{L}$

8.3.1. **(f)** $u(x, t) = \sum_{n=0}^{\infty} A_n(t)\cos\frac{n\pi x}{L}$,

where $\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n = \frac{\int_0^L Q(x, t)\cos\frac{n\pi x}{L}\, dx}{\int_0^L \cos^2\frac{n\pi x}{L}\, dx}$

8.3.3. $u(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$, where $\frac{da_n}{dt} = -\lambda_n a_n + \frac{\int_0^L \phi_n f\, dx}{\int_0^L \phi_n^2 c\rho\, dx}$

8.3.4. **(a)** $u_E(x) = A + (B - A)\frac{\int_0^x \frac{d\bar{x}}{K_0(\bar{x})}}{\int_0^L \frac{d\bar{x}}{K_0}(\bar{x})}$

8.3.5. $u(r, t) = \sum_{n=1}^{\infty} A_n(t)J_0\left(\lambda_n^{1/2} r\right)$, where $J_0\left(\lambda_n^{1/2} a\right) = 0$

and where $\frac{dA_n}{dt} + k\lambda_n A_n = \frac{\int_0^a f(r, t)J_0\left(\lambda_n^{1/2} r\right) r\, dr}{\int_0^a J_0^2\left(\lambda_n^{1/2} r\right) r\, dr}$

8.3.7. $r(x, t) = \frac{x^3}{6L} + \left(\frac{t}{L} - \frac{L}{6}\right)x$

8.4.3. **(b)** $u(x, t) = \sum_{n=0}^{\infty} A_n(t)\cos\frac{n\pi x}{L}$, where

$$A_n(t) = e^{-k(n\pi/L)^2 t}\left(A_n(0) + \int_0^t e^{k(n\pi/L)^2 \bar{t}}\left\{q_n(\bar{t}) + \frac{I_n k}{L}\left[(-1)^n B(\bar{t}) - A(\bar{t})\right]\right\} d\bar{t}\right)$$

with $A_n(0) = \frac{\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx}{\int_0^L \cos^2\frac{n\pi x}{L}\, dx}$ and $q_n(t) = \frac{\int_0^L Q(x, t)\cos\frac{n\pi x}{L}\, dx}{\int_0^L \cos^2\frac{n\pi x}{L}\, dx}$ and $I_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0 \end{cases}$

8.5.2. **(b)** $\omega^2 = \left(\frac{n\pi c}{L}\right)^2$

8.5.5. **(c)** $u(r, \theta, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm}(t)J_m\left(\sqrt{\lambda_{nm}} r\right)\sin m\theta$ with $J_m\left(\sqrt{\lambda_{nm}} a\right) = 0$, where

$A_{nm} = \frac{1}{c\sqrt{\lambda}}\int_0^t Q_{nm}(\bar{t})\sin c\sqrt{\lambda_{nm}}(t - \bar{t})\, d\bar{t} + c_{nm}\cos c\sqrt{\lambda_{nm}} t,$

$$Q_{nm} = \frac{\iint Q(x, y, t)J_m\left(\sqrt{\lambda_{nm}} r\right)\sin m\theta r\, dr\, d\theta}{\iint J_m^2\left(\sqrt{\lambda_{nm}} r\right)\sin^2 m\theta r\, dr\, d\theta},\ c_{nm} = \frac{\iint f(x, y)J_m\left(\sqrt{\lambda_{nm}} r\right)\sin m\theta r\, dr\, d\theta}{\iint J_m^2\left(\sqrt{\lambda_{nm}} r\right)\sin^2 m\theta r\, dr\, d\theta}$$

8.5.6. **(a)** $\frac{1}{c^2}\frac{d^2 a}{dt^2} + \lambda a = \frac{\int_0^{\pi}\int_0^{\alpha} g\phi r\, dr\, d\theta}{\int_0^{\pi}\int_0^{\alpha} \phi^2 r\, dr\, d\theta}$, where $\frac{d}{dr}J_m(\lambda^{1/2}\alpha) = 0$

8.6.1. **(b)** $u(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$, where

$A_{nm} = \frac{-Q_{nm} - \frac{m}{mL^2}(-1)^n\left[1-(-1)^m\right]}{(n\pi/L)^2 + (m\pi/H)^2}$

8.6.1. **(d)** If $\iint Q\ dx\ dy = 0$, then $u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}$, where A_{00} is arbitrary and the others are given by

$$-A_{nm}\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right] = \frac{\iint Q\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\, dx\ dy}{\iint \cos^2\frac{n\pi x}{L}\cos^2\frac{m\pi y}{H}\, dx\ dy}.$$

8.6.3. **(a)** $u(r,\theta)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos m\theta J_m\left(\sqrt{\lambda_{mn}}r\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin m\theta J_m\left(\sqrt{\lambda_{mn}}r\right)$,

where $J_m\left(\sqrt{\lambda_{mn}}a\right)=0$ and $\begin{pmatrix}A_{mn}\\B_{mn}\end{pmatrix}=\dfrac{-\frac{1}{\lambda_{mn}}\iint Q\begin{pmatrix}\cos m\theta\\ \sin m\theta\end{pmatrix}J_m\left(\sqrt{\lambda_{mn}}r\right)r\,dr\,d\theta}{\iint\begin{pmatrix}\cos^2 m\theta\\ \sin^2 m\theta\end{pmatrix}J_m^2\left(\sqrt{\lambda_{mn}}r\right)r\,dr\,d\theta}$

8.6.6. $u(x,y)=\sum_{n=1}^{\infty}a_n(y)\sin nx$, where $a_n(y)=\frac{1}{3}e^{2y}\delta_{n_1}+\alpha_n\sinh ny+\beta_n\cosh ny$

and $\delta_{n_1}=\begin{cases}1 & n=1\\ 0 & n\neq 1\end{cases}$

9.2.1. **(d)** $G(x,t;x_0,t_0)=\sum_{n=0}^{\infty}\frac{1}{I_n}\cos\frac{n\pi x}{L}\cos\frac{n\pi x_0}{L}e^{-k(n\pi/L)^2(t-t_0)}$,

where $I_n=\begin{cases}L & n=0\\ L/2 & n\neq 0\end{cases}$

$$u(x,t)=\int_0^L g(x_0)G(x,t;x_0,0)\,dx_0+\int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\,dt_0\,dx_0$$
$$+\int_0^t kB(t_0)G(x,t;L,t_0)\,dt_0-\int_0^t kA(t)G(x,t;0,t_0)\,dt_0$$

9.2.3. $G(x,t;x_0,t_0)=\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\frac{\sin n\pi c(t-t_0)/L}{n\pi c/L}$

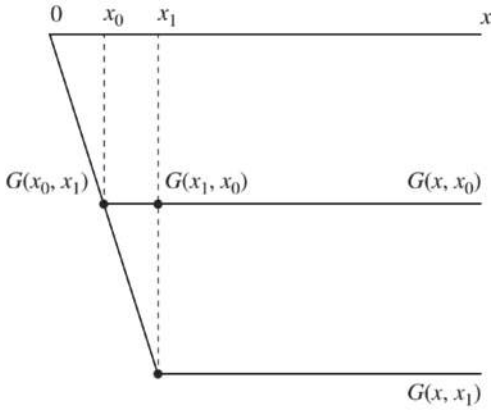
$$u(x,t)=\int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\,dt_0\,dx_0+\int_0^L g(x_0)G(x,t;x_0,0)\,dx_0$$
$$+\int_0^L f(x_0)\tfrac{\partial G}{\partial t}(x,t;x_0,0)\,dx_0$$

9.3.5. **(a), (b)** $u(x)=\int_0^x(x-x_0)f(x_0)\,dx_0-x\int_0^L f(x_0)\,dx_0$

9.3.5. **(c)** $G(x,x_0)=\begin{cases}-x & x<x_0\\ -x_0 & x>x_0\end{cases}$

9.3.6. **(a)** See answer to 9.3.5(c).

9.3.6. **(b)**



9.3.9. **(b)** See answer to 9.3.11.

9.3.11. **(a)** $G(x,x_0)=\begin{cases}\frac{\sin(x_0-L)\sin x}{\sin L} & x<x_0\\ \frac{\sin(x-L)\sin x_0}{\sin L} & x>x_0\end{cases}$

9.3.13. **(b)** $G(x,x_0)=\frac{1}{2ik}e^{ik|x-x_0|}$

9.3.14. **(d)** $u(x)=\int_0^L G(x,x_0)f(x_0)\,dx_0-\alpha p(0)\frac{dG}{dx_0}(x,x_0)\Big|_{x_0=0}-\beta p(L)G(x,L)$

9.3.15. **(a)** $G(x,x_0)=\begin{cases}\frac{1}{k}y_1(x)y_2(x_0)x<x_0\\ \frac{1}{k}y_1(x_0)y_2(x)x>x_0,\end{cases}$ where k is a constant

9.3.21. $G(x, x_0) = \begin{cases} 0 & x < x_0 \\ 1 & x > x_0 \end{cases}$

9.3.25. **(b)** $u(x) = \frac{1}{6}\int_0^x f(t_0)(x - t_0)^3\, dt_0$

9.4.2. **(a)** $0 = \int_0^L \phi_h(x) f(x)\, dx - \alpha p(0)\frac{d\phi_h}{dx}\Big|_{x=0} + \beta p(L)\frac{d\phi_h}{dx}\Big|_{x=L}$

9.4.3. **(b)** Infinite number of solutions

9.4.6. **(a)** $u = 1 + c_1 \cos x + c_2 \sin x$; no solutions

9.4.6. **(b)** $c_2 = 0$, c_1 arbitrary

9.4.6. **(c)** c_1 and c_2 arbitrary

9.4.8. **(a)** $u = \frac{1}{2}x \sin x + c_2 \sin x$

9.4.10. $G_m(x, x_0) = a \sin x \sin x_0 + \begin{cases} \frac{1}{\pi}(x \cos x \sin x_0 + x_0 \cos x_0 \sin x) - \cos x_0 \sin x, & x < x_0 \\ \frac{1}{\pi}(x_0 \cos x_0 \sin x + x \cos x \sin x_0) - \cos x \sin x_0, & x > x_0 \end{cases}$

$u(x) = \int_0^\pi f(x_0) G_m(x, x_0)\, dx_0 - \frac{\beta}{\pi}(x \cos x + \sin x) - \alpha[\frac{1}{\pi}(\sin x + x \cos x) - \cos x]$
$+ k \sin x$, where k is an arbitrary constant

9.4.11. **(a), (b)** $c = 1$

9.4.11. **(d)** $G_a(x, x_0) = \alpha + \begin{cases} x_0 & x < x_0 \\ x & x > x_0, \end{cases}$ where α is an arbitrary constant

9.4.11. **(e)** $u(x) = \int_0^L f(x_0) G_a(x, x_0)\, dx_0 + k_1$, where k_1 is arbitrary

9.5.3. **(c)** $G(r, \theta; r_0, \theta_0) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\sin m\theta \sin m\theta_0 J_m(\sqrt{\lambda_{mn}} r) J_m(\sqrt{\lambda_{mn}} r_0)}{-\lambda \iint J_m^2(\sqrt{\lambda_{mn}} r) \sin^2 m\theta r\, dr\, d\theta}$,

where $J_m\left(\sqrt{\lambda_{mn}} a\right) = 0$

9.5.4. **(a)** See (9.5.26) with extra integral signs

9.5.9. **(b)** $G(\boldsymbol{x}, \boldsymbol{x}_0) = \sum_{m=1}^\infty \frac{\sin m\theta \sin m\theta_0}{m\pi} \begin{cases} \left(\frac{r}{a}\right)^m \left[\left(\frac{r_0}{a}\right)^m - \left(\frac{a}{r_0}\right)^m\right] & r < r_0 \\ \left(\frac{r_0}{a}\right)^m \left[\left(\frac{r}{a}\right)^m - \left(\frac{a}{r}\right)^m\right] & r > r_0 \end{cases}$

9.5.10. **(c)** $G = \frac{1}{4}[Y_0(kr) - iJ_0(kr)]$

9.5.11. **(a)** $L = \nabla^2 + k^2$, $\iiint [uL(v) - vL(u)]\, dV = \oiint (u\nabla v - v\nabla u) \bullet \hat{\mathbf{n}}\, dS$

9.5.11. **(b)** $c_2 = 0$, $c_1 = \frac{-1}{(4\pi)}$

9.5.11. **(c)** $G = \frac{1}{4}[Y_0(kr) - iJ_0(kr)]$

9.5.14. **(a)** $G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2+(z-z_0)^2}}\right]$

9.5.15. $G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{4\pi} \ln \frac{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}{[(x-x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2]}$

9.5.20. $G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{4\pi} \ln\left[a^2 \frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta-\theta_0)}\right] - \frac{1}{4\pi} \ln\left[a^2 \frac{r^2+r_0^2-2rr_0\cos(\theta+\theta_0)}{r^2r_0^2+a^4-2rr_0a^2\cos(\theta+\theta_0)}\right]$

9.5.23. **(c)** $G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi}\sum_{n=-\infty}^\infty \left(\frac{1}{|\boldsymbol{x}-\boldsymbol{\alpha}_n|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{\beta}_n|}\right)$,

where $\boldsymbol{\alpha}_n = (x_0 + 2Ln, y_0, z_0)$ and $\boldsymbol{\beta}_n = (-x_0 + 2Ln, y_0, z_0)$

10.2.1. $c(\omega) = \begin{cases} \frac{1}{2}[A(-\omega) - iB(-\omega)] & \omega < 0 \\ \frac{1}{2}[A(\omega) + iB(\omega)] & \omega > 0 \end{cases}$

10.3.6. See Table 10.4.1.

10.3.7. See Table 10.4.1.

10.3.10. **(b)** $\int_0^L J_0(sr) J_0(s_1 r) r\, dr \approx \frac{2}{\pi} \frac{-\sqrt{s_1/s}\cos\left(sL-\frac{\pi}{4}\right)\sin\left(s_1L-\frac{\pi}{4}\right)+\sqrt{s/s_1}\cos\left(s_1L-\frac{\pi}{4}\right)\sin\left(sL-\frac{\pi}{4}\right)}{s^2-s_1^2}$

10.3.10. **(c)** $A(s_1) = \int_0^\infty f(r)J_0(s_1 r)r\,dr$

10.3.16. $\int_0^\infty y^p e^{-ky^n}\,dy = \frac{1}{n}k^{-(1+p)/n}\Gamma\left(\frac{1+p}{n}\right)$

10.4.3. **(a)** $u(x,\,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\bar{x})e^{-(x+ct-\bar{x})^2/4kt}\,d\bar{x}$

10.4.5. **(c)** $u(x,\,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\sqrt{\frac{\pi}{kt}}e^{-(x-\bar{x})^2/4kt}\,d\bar{x}$
$+\ \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} Q(\bar{x},\,\tau)\sqrt{\frac{\pi}{k(t-\tau)}}e^{-(x-\bar{x})^2/4k(t-\tau)}\,d\bar{x}\,d\tau$

10.4.6. $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{\omega^3}{3} + \omega x\right)\,d\omega$

10.4.7. **(c)** $u(x,\,t) = 1/(3kt)^{1/3}\int_{-\infty}^{\infty} f(\bar{x})\mathrm{Ai}\left[\frac{\bar{x}-x}{(3kt)^{1/3}}\right]d\bar{x}$

10.5.3. $C[e^{-\alpha x^2}] = 2\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$

10.5.10. $C^{-1}(\omega e^{-\omega\alpha}) = \frac{(\alpha^2-x^2)}{(\alpha^2+x^2)^2}$

10.5.11. $u(x,\,t) = 1 + \frac{1}{\sqrt{4\pi kt}}\int_0^\infty (f(\bar{x}) - 1)\left(e^{-(x-\bar{x})^2/4kt} - e^{-(x+\bar{x})^2/4kt}\right)d\bar{x}$

10.5.17. **(b)** $\bar{U}(\omega,\,t) \approx \frac{(2/\pi)\omega A e^{i\sigma_0 t}}{\omega^2 + i\sigma_0/k}$

10.6.1. **(a)** $\bar{u}(\omega,\,y) = F_2(\omega)\frac{\sinh\omega y}{\sinh\omega H} + F_1(\omega)\frac{\sinh\omega(H-y)}{\sinh\omega H}$, where $\bar{u}(\omega,\,y)$ is the Fourier transform of $u(x,\,y)$

10.6.2. **(b)** $\bar{u}(x,\,\omega) = \frac{G_1(\omega)\cosh\omega(L-x)}{\cosh\omega L}$, where $\bar{u}(x,\,\omega)$ is the cosine transform of $u(x,\,y)$

10.6.4. **(a)** $u(x,\,t) = \frac{1}{2\pi}\int_0^\infty f(\bar{x})\ln\frac{(x-\bar{x})^2+y^2}{(x+\bar{x})^2+y^2}\,d\bar{x}$

10.6.11. **(a)** $u(x,\,y,\,t) = \int_0^\infty\int_0^\infty f(x_0,\,y_0)\frac{1}{4\pi kt}\left\{\exp\left[\frac{-(x-x_0)^2-(y-y_0)^2}{4kt}\right]\right.$
$+\exp\left[\frac{-(x+x_0)^2-(y+y_0)^2}{4kt}\right] - \exp\left[\frac{-(x+x_0)^2-(y-y_0)^2}{4kt}\right]$
$\left.-\exp\left[\frac{-(x-x_0)^2-(y+y_0)^2}{4kt}\right]\right\}dx_0 dy_0$

10.6.12. **(a)** $u(x,\,y,\,t) = \int_0^\infty\sum_{n=1}^\infty \bar{A}_n(\omega,\,t)\sin\frac{n\pi x}{L}\sin\omega y\,d\omega$, where
$\bar{A}_n(\omega,\,t) = c(\omega)e^{-k[\omega^2+(n\pi/L)^2]t}$

and

$$c(\omega) = \frac{4}{L\pi}\int_0^\infty\int_0^L f(x,\,y)\sin\frac{n\pi x}{L}\sin\omega y\,dx\,dy$$

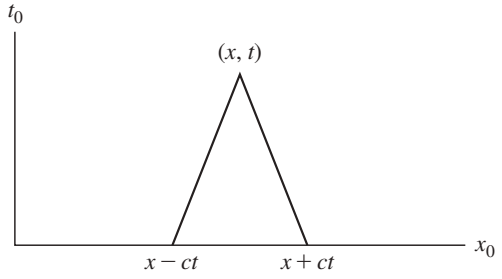
10.6.15. **(a)** $\bar{U} = F(\omega)e^{-\omega z}$

10.6.15. **(b)** $u(x,\,y,\,z) = \frac{z}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{f(x_0,\,y_0)\,dx_0 dy_0}{\left[(x-x_0)^2+(y-y_0)^2+z^2\right]^{3/2}}$

10.6.16. **(b)** $u(r,\,\theta) = \int_0^\infty A(\omega)\sinh\omega\theta\sin\left(\omega\ln\frac{r}{a}\right)d\omega$

10.6.18. $u(x,\,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$

11.2.6. **(b)**



11.2.7. **(a)** $u(x,\,t) = \int_{x-ct}^{x+ct} g(x_0)\frac{1-e^{-i\omega(t-|x-x_0|/c)}}{2i\omega c}\,dx_0$

11.2.8. **(a)** Influence function $= \begin{cases} 0 & \text{if} |\boldsymbol{x}-\boldsymbol{x}_0| > ct \\ \frac{e^{-i\omega(t-r/c)}}{4\pi c^2 r} & \text{if} |\boldsymbol{x}-\boldsymbol{x}_0| < ct \end{cases}$, where $r = |\boldsymbol{x}-\boldsymbol{x}_0|$

11.2.10. **(b)** $u(x,\,t) = \begin{cases} 0 & \text{if } x > ct \\ h\big(t-\frac{x}{c}\big) & \text{if } x < ct \end{cases}$

11.2.12. **(a)** $G(\boldsymbol{x},\,t;\,\boldsymbol{x}_1,\,t_1) = \begin{cases} 0 & \text{if } r > c(t-t_1) \\ \frac{1}{2\pi c\sqrt{c^2(t-t_1)^2-r^2}} & \text{if } r < c(t-t_1) \end{cases}$

11.3.2. **(c)** If $A = 1$, $u(x,\,t) = \frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4kt}}^{\infty} e^{-\eta^2}\,d\eta$.

11.3.3. $G(x,\,t;\,x_0,\,t_0) = \frac{1}{\sqrt{4\pi k(t-t_0)}}\Big\{\exp\Big[\frac{-(x-x_0)^2}{4k(t-t_0)}\Big] + \exp\Big[\frac{-(x+x_0)^2}{4k(t-t_0)}\Big]$

12.2.2. $w(x,\,t) = \cos(x+3t)$

12.2.6. **(b)** $w(x,\,t) = t + f(xe^{-t})$

12.2.6. **(d)** $w(x,\,t) = e^t f\big(x-\frac{3}{2}t^2\big)$

12.2.7. $x = 2f(x_0)t + x_0$

12.2.9. $u(x,\,t) = \begin{cases} 1 & x \le 2t \\ \frac{x}{2t} & 2t < x < 4t \\ 2 & x \ge 4t \end{cases}$

12.3.5. **(a)** $\frac{\partial u}{\partial t}(x,\,0) = -c\frac{dF(x)}{dx}$

12.3.5. **(b)** $\frac{\partial u}{\partial x}(0,\,t) = -\frac{1}{c}\frac{dF(-ct)}{dt}$

12.4.1. $u(x,\,t) = \begin{cases} 0 & x > ct \\ h\big(t-\frac{x}{c}\big) & x < ct \end{cases}$

12.4.2. $u(x,\,t) = \begin{cases} \cos x\cos ct & x < -ct \\ e^{-(t+x/c)} + \sin x\sin ct & 0 > x > -ct \end{cases}$

12.4.6. $u(x,\,t) = \begin{cases} 0 & x > ct \\ -c\int_0^{t-x/c} h(\overline{t})\,d\overline{t} & x < ct \end{cases}$

12.4.7. $u(x,\,t) = \begin{cases} \frac{1}{2}[f(x-ct)+f(x+ct)] & x > ct \\ \frac{1}{2}[f(x+ct)+f(ct-x)] - c\int_0^{t-x/c} h(\overline{t})d\overline{t} & x < ct \end{cases}$

12.5.1. **(b)** $u(x,\,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$

12.6.1. **(a)** $\rho(x,\,t) = f(x)$

12.6.1. **(c)** $\rho(x,\,t) = f(x)e^{-3xt}$

12.6.2. $\rho(x,\,t) = (1+\sin x)e^{t+x/2}$

12.6.3. **(a)** $\rho(x,\,t) = \sin(x-c_0t)$

12.6.3. **(b)** $\rho(x,\,t) = \begin{cases} g\big(t-\frac{x}{c_0}\big) & x < c_0t \\ f(x-c_0t) & x > c_0t \end{cases}$

12.6.4. **(a)** $q = u_{\max}\rho\big(1-\frac{\rho}{\rho_{\max}}\big)$

12.6.4. **(b)** $\rho = \rho_{\max}/2$, $u = u_{\max}/2$, $q = \frac{\rho_{\max}u_{\max}}{4}$

12.6.8. **(a)** $\rho(x,\,t) = \frac{e^{-3(x-ct)}}{3c}(1-e^{-3ct}) + f(x-ct)$

12.6.8. **(c)** $\rho(x,\,t) = 5t + f\big(x-\frac{1}{2}t^2\big)$

12.6.8. **(e)** $\rho(x,\,t) = e^{-t}f\big(x+\frac{1}{3}t^3\big)$

12.6.8. **(g)** $\rho(x,\,t) = \frac{1}{2}t^2 + f(xe^{-t})$

12.6.9. **(a)** $\rho(x,\,t) = e^{3t} f(x_0)$, where $x = x_0 - \frac{1}{6}(e^{6t} - 1) f^2(x_0)$

12.6.9. **(c)** $\rho(x,\,t) = e^{-t} f(x_0)$, where $x = x_0 + f(x_0)\int_0^t \tau^2 e^{-\tau}\, d\tau$

12.6.11. $\rho(x,\,t) = \begin{cases} e^{3t} f\left(x - t - \frac{t^2}{2}\right) & x > t + \frac{t^2}{2} \\ e^{3(t-\tau)} g(\tau), & \text{where } x = t + \frac{t^2}{2} - \frac{3}{2}\tau - \frac{\tau^2}{2},\ x < t + \frac{t^2}{2} \end{cases}$

12.6.14. **(a)** $-Vf' + u_{\max}\left(1 - \frac{2f}{\rho_{\max}}\right) f' = \nu f''$

12.6.14. **(c)** $V = \frac{[q]}{[\rho]} = u_{\max}\left(1 - \frac{\rho_1 + \rho_2}{\rho_{\max}}\right)$

12.6.17. **(a)** $\rho(x,\,t) = \begin{cases} \frac{\rho_{\max}}{5} & x < u_{\max} t/5 \\ \frac{3\rho_{\max}}{5} & x > u_{\max} t/5 \end{cases}$

12.6.18. **(b)** $\rho(x,\,t) = \begin{cases} \rho_1 & x < u_{\max}\left(1 - 3\frac{\rho_1^2}{\rho_{\max}^2}\right) t \\ \frac{\rho_{\max}}{\sqrt{3}}\sqrt{1 - \frac{x}{u_{\max} t}} & \text{otherwise} \\ \rho_2 & x > u_{\max}\left(1 - \frac{3\rho_2^2}{\rho_{\max}^2}\right) t \end{cases}$

13.2.4. $\mathcal{L}\left[\int_0^t f(\bar{t})\, d\bar{t}\right] = \frac{F(s)}{s}$

13.2.5. **(b)** $\frac{8s}{(s^2+16)^2}$

13.2.5. **(d)** $\frac{4}{s^2 - 6s + 25}$

13.2.5. **(f)** $e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}\right) - e^{-8s}\left(\frac{2}{s^3} + \frac{16}{s^2} + \frac{64}{s}\right)$

13.2.5. **(h)** $\frac{24e^{-s}}{s^5}$

13.2.6. **(e)** $\frac{7}{6}e^{-7t} - \frac{1}{6}e^{-t}$

13.2.6. **(j)** $\frac{2}{9} + \frac{1}{3}\sin 3t - \frac{2}{9}\cos 3t - 5H(t-4)\left[\frac{2}{9} + \frac{1}{3}\sin 3(t-4) - \frac{2}{9}\cos 3(t-4)\right]$

13.2.7. **(b)** $y = 1 + e^{-t}$

13.2.7. **(d)** $y = \begin{cases} \frac{25}{7}e^t - \frac{4}{7}e^{-6t} & 0 < t < 3 \\ -\frac{1}{10}e^{-t} + e^t\left(\frac{25}{7} + \frac{1}{14}e^{-6}\right) + e^{-6t}\left(\frac{1}{35}e^{15} - \frac{4}{7}\right) & t > 3 \end{cases}$

13.2.7. **(f)** $y = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$

13.3.2. $G(t,\,t_0) = \sin(t - t_0)$

13.4.3. $u(x,\,t) = \sin x\, \cos ct$

13.4.4. $\bar{U}(x,\,s) = F(s)e^{-\sqrt{s/k}\,x}$

13.5.3. $\bar{U}(x,\,s) = \frac{cB(s)\sinh\frac{sx}{c}}{s\cosh\frac{sL}{c}}$

13.6.4. **(a)** $\bar{G}(x,\,s;\,x_0,\,t_0) = \frac{e^{-st_0}}{\sqrt{sk}}\begin{cases} e^{-\sqrt{s/k}\,x_0}\sinh\sqrt{\frac{s}{k}}x & x < x_0 \\ e^{-\sqrt{s/k}\,x}\sinh\sqrt{\frac{s}{k}}x_0 & x > x_0 \end{cases}$

13.7.1. **(b)** $f(t) = \frac{1}{3}\sin 3t$

13.8.1. $u(x,\,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$, where $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx$

14.2.1. **(c)** $\omega = -k^2$

14.2.5. $\omega^2 = gk\,\tanh kh$

14.2.7. $\omega^2 = g|k|$

14.3.9. **(a)** Let $k = \omega_f^{1/3}$ and $s = \text{sign}(\omega_f)$. $G(x,\,0) = \frac{1}{3}\left(is\sqrt{3} + 1\right)e^{-ikx/2 + \sqrt{3}|k|x/2}$, for $x < 0$. $G(x,\,0) = \frac{2}{3}e^{ikx} + \frac{1}{3}\left(is\sqrt{3} - 1\right)e^{-ikx/2 - \sqrt{3}|k|x/2}$, for $x > 0$.

14.5.3. **(b)** two waves if $x > -\frac{1}{4}t$ and zero waves if $x < -\frac{1}{4}t$

14.6.1. $k = k_0$ and $\omega = \omega(k_0)$

14.7.8. $2B = Cu + 2B_0$, $8A = -Cu_{xx} + 3Cu^2 + 4B_0u + 8A_0$, $2u_t = -A_{xxx} + 4uA_x + 2Au_x$

14.8.6. **(b)** $\sigma = -1 + Rk^2 - k^4$, $\omega = k^3$

14.8.7. **(c)** $\sigma = k$, $\omega = 0$ ill posed since $\sigma \to \infty$ as $k \to +\infty$

14.9.3. $\psi =$ constant, and $2\omega\frac{dr}{dT} + \frac{d\omega}{dT}r + 3\omega^3 r^3 = 0$

14.9.5. $u = cL^{-3/4}\cos\Big(\int\sqrt{\frac{g}{L(\varepsilon t)}}\,dt + \psi\Big)$

14.9.8. $k = \theta_x$, $\omega = -\theta_t$, $k_T + 3k^2k_X = 0$, $u_0 = Ae^{i\theta} + (*)$, $A_T + 3k^2A_X + 3kk_XA = 0$

14.10.3. Outer $u_0 = -\frac{x}{4}$. Inner(right)$U_0 = -\frac{1}{4} + \frac{9}{4}e^{2X}$, where $x - 1 = \varepsilon^{1/2}X$.
Inner (left)$U_0 = e^{-2X}$, where $x = \varepsilon^{1/2}X$.

14.10.6. Outer $u_0 = 6(2x + 1)^{-1}$. Inner $U_0 = 6 - 5e^{-X}$, where $x = \varepsilon X$.

List of Downloadable MATLAB Animations

1. **heat_fig234.m** Animation of heat equation solution for a rod with fixed temperature boundary conditions. Compare to Figure 2.3.4.
2. **heat_fig241.m** Animation of heat equation solution for a rod with insulated boundary conditions. Compare to Figure 2.4.1.
3. **fourier_fig338.m** Partial sums of the Fourier series of $f(x) = 100$ demonstrating the Gibbs phenomenon. Compare to Figure 3.3.8.
4. **fourier_fig339.m** Fourier sine series of $f(x) = x$, $0 < x < L$, showing the odd periodic extension. Compare to Figure 3.3.9.
5. **fourier_fig3314.m** Fourier cosine series of $f(x) = x$, $0 < x < L$, showing the even periodic extension. Compare to Figure 3.3.13 and Figure 3.3.14.
6. **wave_modes_fig441.m** Animation of normal modes of vibration for a string. Compare to Figure 4.4.1.
7. **wave_example_ch4.m** Animation of wave equation solution with initial condition $u(x, t = x)$, $0 < x < 5$, and ends held at 0, demonstrating wave reflection. This is an example in Chapter 4.
8. **numerical_fig635.m** Numerical solution of the heat equation. Compare to Figure 6.3.5.
9. **analytic_fig635.m** Analytic solution of heat equation, for the same problem solved numerically in Figure 6.3.5.
10. **heat_fig643.m** Numerical solution of the heat equation on an L-shaped domain. Compare to Figures 6.4.2 and 6.4.3.
11. **wave_fig732.m** Modes of vibration of a rectangular membrane. Compare to Figure 7.3.2.
12. **wave_fig783.m** Modes of vibration of a circular membrane. Compare to Figure 7.8.3.
13. **heat_fig1044.m** Animation of temperature diffusion in an infinite rod. Compare to Figure 10.4.4.
14. **wave_fig1223.m** Animation of wave equation solution, showing shape invariance of a traveling wave. Compare to Figure 12.2.3.
15. **wave_fig1232.m** Animation of initial value problem solution for the one-dimensional wave equation, showing left- and right-traveling waves. Compare to Figure 12.3.2.
16. **wave_fig1234.m** Animation of initial value problem solution for the one-dimensional wave equation, with no initial displacement but with an initial velocity. Compare to Figure 12.3.4.
17. **wave_fig1244.m** Animation of solution for wave equation with a reflected pulse. Compare to Figure 12.4.4.
18. **phase_group_fig1421.m** Demonstration of phase and group velocity. Compare to Figure 14.2.1.

Formulas and Equations

	Sturm–Liouville	Helmholtz (2-dim)
Eigenvalue problem	$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + (\lambda\sigma + q)\phi = 0$	$\nabla^2\phi + \lambda\phi = 0$
Operator	$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$	$L = \nabla^2$
Green's formula	$\int_a^b [uL(v) - vL(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big\|_a^b$	$\iint_R (u\nabla^2 v - v\nabla^2 u)\,dx\,dy = \oint (u\nabla v - v\nabla u)\cdot\hat{n}\,ds$
Rayleigh quotient	$\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big\|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]dx}{\int_a^b \phi^2\sigma\,dx}$	$\lambda = \frac{-\oint \phi\nabla\phi\cdot\hat{n}\,ds + \iint_R \lvert\nabla\phi\rvert^2\,dx\,dy}{\iint_R \phi^2\,dx\,dy}$

EIGENVALUE OR BOUNDARY VALUE PROBLEMS for $\frac{d^2\phi}{dx^2} = -\lambda\phi$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\frac{d\phi}{dx}(0) = 0$ $\frac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

FOURIER TRANSFORM

$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$	$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx$	Reference
$e^{-\alpha x^2}$ $\sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$ $e^{-\beta\omega^2}$	Gaussian (Section 10.3.3)
$\frac{\partial f}{\partial t}$ $\frac{\partial f}{\partial x}$ $\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial F}{\partial t}$ $-i\omega F(\omega)$ $(-i\omega)^2 F(\omega)$	Derivatives (Section 10.4.2)
$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) g(x - \bar{x}) d\bar{x}$	$F(\omega) G(\omega)$	Convolution (Section 10.4.3)
$\delta(x - x_0)$	$\frac{1}{2\pi} e^{i\omega x_0}$	Dirac delta function (Exercise 10.3.18)
$f(x - \beta)$	$e^{i\omega\beta} F(\omega)$	Shifting theorem (Exercise 10.3.5)
$xf(x)$	$-i\frac{dF}{d\omega}$	Multiplication by x (Exercise 10.3.8)
$\frac{2\alpha}{x^2 + \alpha^2}$	$e^{-\lvert\omega\rvert\alpha}$	Exercise 10.3.7
$f(x) = \begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$	$\frac{1}{\pi} \frac{\sin a\omega}{\omega}$	Exercise 10.3.6

DOUBLE-FOURIER TRANSFORM

$f(\boldsymbol{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega \cdot r} \, d\omega_1 \, d\omega_2$	$F(\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\boldsymbol{r}) e^{i\omega \cdot r} \, dx \, dy$	Reference
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ $\nabla^2 f$	$-i\omega_1 F(\omega), -i\omega_2 F(\omega)$ $-\omega^2 F(\omega)$	Derivatives (Section 10.6.5)
$\frac{\pi}{\beta} e^{-\boldsymbol{r}^2/4\beta}$	$e^{-\beta\omega^2}$	Gaussian (Section 10.6.5)
$f(\boldsymbol{r} - \beta)$	$e^{i\omega \cdot \beta} F(\omega)$	Exercise 10.6.8
$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\boldsymbol{r}_0) g(\boldsymbol{r} - \boldsymbol{r}_0) dx_0 \, dy_0$	$F(\omega) G(\omega)$	Convolution (Exercise 10.6.7)

FOURIER SINE TRANSFORM

$f(x) = \int_0^\infty F(\omega)\sin\omega x\, d\omega$	$S[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx$	Reference
$\frac{df}{dx}$ $\frac{d^2f}{dx^2}$	$-\omega C[f(x)]$ $\frac{2}{\pi}\omega f(0) - \omega^2 F(\omega)$	Derivatives (Section 10.5.4)
$\frac{x}{x^2 + \beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi}\cdot\frac{\omega}{\epsilon^2 + \omega^2}$	Exercise 10.5.2
1	$\frac{2}{\pi}\cdot\frac{1}{\omega}$	Exercise 10.5.9
$\frac{1}{\pi}\int_0^\infty f(\bar{x})[g(x - \bar{x}) - g(x + \bar{x})]\, d\bar{x}$ $= \frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x + \bar{x}) - f(\bar{x} - x)]\, d\bar{x}$	$S[f(x)]C[g(x)]$	Convolution (Exercise 10.5.6)

FOURIER COSINE TRANSFORM

$f(x) = \int_0^\infty F(\omega)\cos\omega x\, d\omega$	$C[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\, dx$	Reference
$\frac{df}{dx}$ $\frac{d^2f}{dx^2}$	$-\frac{2}{\pi}f(0) + \omega S[f(x)]$ $-\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 F(\omega)$	Derivatives (Section 10.5.4)
$\frac{\beta}{x^2 + \beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi}\cdot\frac{\epsilon}{\epsilon^2 + \omega^2}$	Exercise 10.5.2
$e^{-\alpha x^2}$	$2\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Exercise 10.5.3
$\frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x - \bar{x}) + f(x + \bar{x})]d\bar{x}$	$F(\omega)G(\omega)$	Convolution (Exercise 10.5.7)

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