

Dyscalculia and Numeracy



New Strategies for Helping
Dyscalculic Students Master Math

MAHESH C. SHARMA,
STEVE CHINN, AND
SANJAY RAGHAV



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Dyscalculia and Numeracy: New Strategies for Helping Dyscalculic Students Master Math presents a groundbreaking and detailed framework for addressing dyscalculia and numeracy challenges.

Drawing on Professor Mahesh C. Sharma's extensive research, this book introduces the CPVA/S model—an evolution of the CPA model—that integrates cognitive development, effective questioning, and language strategies to foster deep mathematical understanding, helping to ensure that all students, regardless of learning difficulties, can master numeracy. The book provides diagnostic tools for early intervention and practical strategies to build confidence, or a procedure to prevent math anxiety, and ensure long-term success. This text is more than a guide for dyscalculia; it offers transformative solutions for all students.

By addressing the root causes of learning difficulties, it equips educators, parents, and researchers with actionable strategies to help children thrive in mathematics and develop a love for learning.

Mahesh C. Sharma is the Founder/President of the Center for Teaching/Learning of Mathematics, Inc., of Framingham, Massachusetts. He is the former President and Professor of Mathematics Education at Cambridge College, where he trained teachers from many countries. He has consulted with schools and has been an invited speaker to organizations in the Americas, Europe, Asia, and Africa. Professor Sharma was Chief Editor and Publisher of *Focus on Learning Problems in Mathematics*, an

international interdisciplinary research journal, and the Editor of *The Math Notebook*. Among the many awards, he has received British Dyslexia Associations' the Marion Welchman International Award for lifetime contributions to the service of Dyslexia. Currently, he writes a blog (www.mathlanguage.wordpress) on math learning and teaching and is engaged in services to children and adults with learning problems in mathematics, including dyscalculia and teacher training through his webinars.

Steve Chinn is an independent lecturer, writer, researcher, and a retired visiting professor at the University of Derby, UK. He spent 24 years as headteacher of three specialist schools, one in the USA, two in the UK, and was a mainstream teacher for 14 years. He was Founder and Principal of Mark College in Somerset, UK, a multi-award winning specialist residential school for young people with specific learning difficulties. In 2011 he set up and chaired the BDA's first subcommittee on dyscalculia. He has received the Marion Welchman International Award for Services to Dyslexia, as well as the Lady Radnor Award for Lifetime Services to Dyslexia.

Sanjay Raghav is a mathematics educator, teacher trainer, and curriculum developer with over 15 years of experience in teaching and mentoring students and teachers across the K–12 spectrum. His work centers on developing deep conceptual understanding in mathematics, particularly in numbers, algebraic thinking, and geometric reasoning. He is based in New Delhi, India, where he runs a dedicated mathematics and teacher development center at Bluebells School International. He leads teacher training programs, curriculum development initiatives, and research-based interventions in mathematics education.

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Sight Facts of Ten

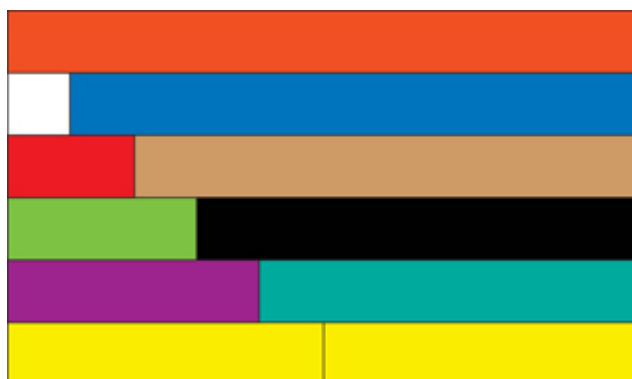


Figure 0.1 Sight Facts

$$10 = 9 + 1 = 1 + 9$$

$$10 = 8 + 2 = 2 + 8$$

$$10 = 7 + 3 = 3 + 7$$

$$10 = 4 + 6 = 6 + 4$$

$$10 = 5 + 5$$

*As we expect every child to read fluently, with comprehension, by age nine, so that they can use reading to learn anything and everything, similarly, we should expect and work diligently for every child to have mastery of numeracy, with understanding, by age ten, so that they can learn mathematics, science, and technology easily, effectively, and efficiently. We should aspire to see our children **literate, numerate, and eager to learn**. We believe it can be done!*

***Numeracy** is a child's ability and facility in executing and applying **four whole number operations** (addition, subtraction, multiplication, and division) correctly, consistently, and fluently with understanding in **diverse situations in multiple forms**, including the **standard procedures**.¹*

It requires effective, efficient, and elegant methods and tools of instruction. Instructional tools are the life of a lesson. And a master teacher makes the tools live and effective for her students. Beautiful² tools make work a joyful and fulfilling experience. Mathematics is a language and science of patterns.

Notes

1. See *Dyslexia and Other Language Related Mathematics Difficulties* by Sharma (2008) for interplay of language and numeracy, and also see the blog: "Dysgraphia, dyslexia, and dyscalculia" by Sharma on www.mathematicsforall.org. ↵
2. Tools are 'good' when they help us to be effective, efficient, and elegant. Tools are *efficient* if they can be used for modeling a variety of problems with ease and transparency. Tools are *elegant* when they help

us in transiting from concrete to pictorial and pictorial to visualization and finally to abstractions. The choice of ‘good tools’ is the mark of an *effective* teacher; where all students thrive. [!\[\]\(125d701e9425b54c764340b5671b38cd_img.jpg\)](#)

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Introduction

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Whether you think you can, or you think you can't, you're right.

—Henry Ford

Mathematics is a language, science, and study of patterns. It is also the language of science—natural and social, technology and engineering. Even aspects of medicine and humanities are studied, understood, communicated, and made comprehensible by the help of this language.

It is easier to identify the feel, the movement, the production of the sound than it is to identify the perception of the sound. The experience with quantity and space is just like that. We see and perceive the quantity and space, but it is difficult to describe it without number, its language, and its properties.

From calibration, reading, and use of carpenters' tools and using artificial intelligence to a chemist's lab and developing the strategic plans for marketing and healthcare campaigns, and discovering new tools and methods to deal with the current and future problems of the world, every student of today needs a high level of mathematics proficiency. To be well-prepared for the careers and opportunities that await the students following graduation, they need a strong background in mathematics ([Tiles, 1991](#)). While the amount of new mathematics and its methods being generated keeps on increasing, students' math proficiency in countries around the globe, from the Americas to Europe and even in many Asian countries, is declining. Of course, there are bright spots here and there, but the

mathematics achievement data are mostly sobering. Strengthening mathematics achievement is on every government's agenda for the present and for the foreseeable future. One of the causes of poor achievement in mathematics is the poor preparation of students in the foundation years. How can we strengthen students' first three foundation years of school mathematics?

There is a solution to the problem of improving mathematics instruction from primary grades to college and university classrooms. The foundations of such instruction are set in the primary grades and in homes where the preparation for such a future begins.

In this book, we explore how to prepare children effectively in primary school so that they do not face difficulties in the college classroom. As the saying goes: 'If you want to build a hut, there is no need for a basement, but if you want to build a sixteen-story building, you need a strong and deep foundation.' In this book, that is our goal: providing suggestions and activities for building the foundations of solid mathematics learning and preventing the manifestation of learning problems in mathematics, such as dyscalculia, acquired dyscalculia, and debilitating mathematics anxiety. To achieve this effectively, we discuss the roles of:

- What should children know, when to know, and how to know that?
- The instructional hierarchy: what is it, and how does it guide instruction at the primary level?
- Mathematical fluency: what is it, and why is it important?
- What are the most effective ways of teaching and mastering arithmetic facts?
- What is the role of fact fluency in mastering numeracy and then mathematics?
- What does research say about teaching multiple strategies for mathematical operations?
- Math facts: simple techniques that develop automaticity.
- How and when to support all children, with and without learning disabilities and/or dyscalculia.

There is a substantial body of evidence about how to effectively teach mathematics. We make a serious effort to provide suggestions for building the foundations for that.

Primary schooling is mandatory in almost all countries; however, there are different facilities and opportunities for children to take advantage of it. Mathematics, after language and reading development, is a central subject in primary education, and the delivery of the mathematics curriculum is important in all countries. Throughout history, the level of mathematics in a country has driven innovations, inventions in science and technology, and that, in turn, affects the level of standards of living. Thus, the aim of higher achievement in mathematics, individually and collectively, in any country, is the same: make life better everywhere. It is seen as a vehicle for realizing the potential of their children, higher aspirations for their society, more productive and responsible citizens, and high levels of competence whatever they decide to do. This process, however, is unique to a country. The importance of mathematics has increasingly been acknowledged by all nations, which in turn has led to more intense scrutiny of our mathematics education system (Hourigan and O'Donoghue, 2015).

Numeracy in primary school lays the foundation for secondary school and beyond. It is one of the goals of education for all and is a part of the *UN Global Education First Initiative* ([UNESCO, 2012](#)). Several national and international professional groups, the National Mathematics Advisory Panel and the Institute for Educational Sciences have concluded that all students can learn mathematics, and most can succeed through algebraic concepts, procedures, and their applications. However, the abstractness and complexity of algebraic concepts and missing precursor skills and understandings of—number conceptualization, place value, arithmetic facts and procedures, and concepts such as fractions/decimals and integers—make learning in algebra and other-higher order concepts overwhelming for many students, not just dyscalculics. At the same time, it is difficult for so many teachers to teach all children effectively.

Mathematics does not come naturally to all; it is a learned human trait. The brain functions developed for other purposes, such as survival, have

been recycled for reading and mathematics. The cultivation of numeracy skills, just like reading, is a delicate and complex process. Some problems and difficulties of learning mathematics are unique to the demands of middle and high school mathematics and to the adolescent years. However, *many of the difficulties in learning middle and high school mathematics originate in the lack of students' mastery and understanding of numeracy*. Teaching of numeracy, with mastery, in the primary grades conveys the important message of laying a solid foundation as early as possible for further mathematics learning smoothly. Solid numeracy skills predict success at high school level, not just in mathematics, but in all academic areas ([Wilson et al., 2007](#)).

Acquiring the *number concept* or *numberness*—understanding number, its representation, and its applications—is a fundamental skill. It is acquiring the alphabet (the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) of the mathematics language with arithmetic facts as its words. Playing with numbers—seeing the patterns and structures and observing and understanding their interrelationships—begins the fundamentals of numeracy and mathematics. It is ways of making sense of the quantitative and spatial properties and characteristics of our physical and intellectual environment. It is the beginning of a way of thinking in a new language and new and different skills and methods for solving problems.

Much of the research ([Geary, 1993](#)) has focused on developing a theoretical understanding of mathematics learning difficulties and their impact on learners and their aspirations. In this book, we look at the role of *number concept*, *numbersense*, and *additive reasoning* in understanding the nature of mathematics learning difficulties, such as dyscalculia, and search for ways of preventing any difficulties in learning mathematics and designing strategies and methods for effective initial instruction, meaningful interventions, and lasting remediations. From our experience, we find that children's understanding and level of mastery of number concept and numbersense provide a window into their arithmetic difficulties, particularly dyscalculia ([Dehaene et al., 1999](#); [Gersten and Chard, 1999](#); [Chinn, 2021c](#); [Sharma, 2019](#)).

Students with difficulties with numeracy and/or mathematics are neither a small nor a homogeneous group. There are well-identified large clusters and groups in this population with specific and defining characterizing problems and difficulties in learning mathematics—features of performance, symptoms, needs, and attitudes toward learning mathematics.

Some students have difficulty in numeracy and/or mathematics because of their *learning characteristics and differences*—neuropsychological and cognitive profiles; poor linguistic skills; lack of prerequisite skills for learning mathematics; difficulties with operating on a sequence of actions and thoughts, then understanding how they can be reversed; and/or poor executive functions. These deficits may result in mathematics-related learning difficulties/disabilities, such as *dyscalculia* and/or *dyslexia*¹ and *math anxiety* (state).

On the other hand, there are others who, *due to socio-cultural* and other *environmental factors* such as: poor instruction, poor instructional materials, lower standards of performance, lack of adequate practice, lack of clear standards of achievement, inappropriate expectations, experience difficulties in learning numeracy and/or mathematics, thereby resulting in *acquired dyscalculia* or *math anxiety* (trait).²

Some of the mathematics learning problems are also due to *ineffectual instructional resources* (ineffective and inefficient tools) *and strategies* (poor presentations in textbooks and lessons, misconceptions about what numeracy is all about, not enough time devoted to proper numeracy tasks),³ lack of or poor use of mathematics language, and lack of or inappropriate practice.

Children and adults coming out of these different situations may show similar symptoms or difficulties in learning and achieving in mathematics. However, for different reasons, these symptoms and conditions may not have any relationship to the child's learning ability, potential, and attitude toward mathematics. Because of these situations, negative attitudes toward mathematics are prevalent in all sectors of the population.

Poor performance and poor attitudes toward mathematics have prompted educators, policymakers, and administrators in many nations to seek and

introduce reforms to mathematics education from time to time.

The last 50 years have been a time of great action to bring reform in mathematics education for different reasons and from different perspectives, thereby influencing school mathematics—from designing new curricula overhauling mathematics content to special textbooks focusing on specific content; from teacher-centered whole classroom instruction to child-centered explorations as a mode of instruction; from influx of a variety of instructional materials to specially designed tools reflecting a specific philosophy of education and instruction; from competency-based assessment to whole classroom mandatory testing; from small group instruction to individually designed lesson plans; from ability grouping to heterogeneous group instruction; from separate classroom instruction for special education students to inclusion and intervention; from learning stations to team teaching, etc. At present, it is the technology or individualized special tutoring with or without technology that is seen as the savior. However, despite these reforms and infusions, the achievement in mathematics has minimally improved or declined, particularly for specific groups of students—minorities (e.g., social and linguistic), learning disabled, cognitively challenged, and the underserved.

There are many reasons for this disparity in achievement or lack of change. The most important factor is the nature of instruction. We believe that it is in the improvement in the quality of daily lessons in the classroom by individual teachers that truly impacts learning and achievement. Transformation of instruction that has *fidelity to content* and that *reaches all* children is the vehicle of change in children's achievement and their attitude toward learning and using mathematics.

The pedagogy that is responsible for the prevalent current problem, particularly at the numeracy level (pre-K through fourth grade), involves a variety of counting exercises for the development of all content: number conceptualization, concept formation, and deriving procedures and then executing those procedures. Similarly, at the middle and high school level, the emphasis continues introducing and practicing a variety of procedures without any linguistic and conceptual understanding.

The objective of most current instruction related to numeracy is to get the correct answer. And many students achieve it by paying a heavy price by using ineffective strategies. They appear to be proficient.⁴ However, being overtly proficient at arithmetic/numeracy skills (getting an answer to a problem somehow) is certainly an asset. It is a necessary condition for success, but not a sufficient condition. Children with mastery that has been achieved—using inefficient strategies and without understanding (i.e., how the answer has been achieved or why we do this or that)—get overwhelmed when the concepts get complex. Students with such strategies reach success plateaus quickly, and if they experience the slightest difficulty and challenge, they give up.

Unfortunately, many of these existing methods of *initial classroom instruction*, even *interventions* of various kinds, and ineffective, inefficient, and inappropriate *remediation* strategies and approaches have not made much headway in improving mathematics achievement. Clearly, there exists a need for instruction and interventions that go beyond ‘historical, typical’ instruction⁵ based just on counting—counting forward for addition, counting backwards for subtraction, skip counting forward for multiplication, and skip counting backward for division.⁶ These strategies simply do not work for all children and impart very little understanding of the four operations and their inter-relationships.

The real cause of learning problems in mathematics is poor numeracy skills, particularly poor preparation and lack of mastery of number concept, number relationships, and outcome of numerical operations. Having strong sense of number (number sense—integration of number concept, arithmetic facts, place value, estimation) is an antidote for learning problems in mathematics ([Geary, 1990](#), [2004](#), [2005](#)).

For the interventions to work, the strategies and methods should be *effective* (children should be able to learn and understand them with ease), *efficient* (they should be able to get answers quickly and correctly with less but noble effort), and *elegant* (strategies and methods should be such that they can be *generalized*, *extrapolated*, and *abstracted* for other concepts, procedures, and ideas, in the present and in the future). Such methods

develop efficiency, understanding, and confidence that translate into proficient performance and a healthy attitude toward mathematics and disciplines based on mathematics.

Achieving true proficiency is one of the important goals of effective instruction. The other equally important goal is to achieve it with understanding. In achieving proficiency and applying it, there are subjective and objective aspects. Subjective deals with the students—what progress they have made individually and how they feel about it. Objective deals with curricular and teacher objectives for the class and students. Instruction must reflect these needs and objectives. How we achieve this proficiency—the *language, strategies, models*, and the *level of knowing* a concept or procedure—also matters in learning new and higher-order concepts. Some of the strategies that have been used in achieving mastery and proficiency may neither be *effective, efficient*, nor *elegant*,⁷ nor *developmental* ([Carpenter et al., 1999](#)), although curricular and instructional aspects might be met. For example, the student ‘knows’ addition facts by counting or with the help of concrete materials only ([Butterworth et al., 2001](#)). When the methods and strategies are not efficient, the standards of mastery and proficiency are lowered or are absent. It is evident when you see that most students are leaving kindergarten without mastering number concepts; first graders without mastering addition facts; second graders without mastering subtraction facts; third graders leaving without mastering multiplication facts; and fourth graders without division facts, etc. Almost 45% of middle and high school students have not mastered the multiplication tables; as a result, they have difficulty operating on fractions, decimals, integers, and algebraic manipulations.

The criteria for *mastery* that curricula such as *Common Core State Standards in Mathematics (CCSS-M)*⁸—set for arithmetic for early elementary grades—are very specific, yet universal. In essence, they advocate: for a particular concept or procedure, students should have (a) its *understanding* (e.g., have appropriate language and possess efficient and effective strategies, based on authentic conceptual schemas, etc.), (b) consistent *accuracy* independent of instructional approach, materials used,

nature or form of instruction, or the context of learning, (c) *fluency* (at an acceptable level and at specified standards and in usable forms), and (d) *applicability* (can apply it in intra-mathematical, inter-disciplinary, and extra-curricular situations—to other concepts, procedures, and problem-solving).⁹

Our emphasis on this definition of mastery for numeracy has profound implications for not only classroom instruction but also for interventions and remediations. The implications of this kind of mastery for *number concept means—learning the fundamentals*: subitizing, numberness— instant recognition of and interrelationship between ‘the cluster of quantity,’ ‘orthographic symbol of the number,’ and ‘the number name’; decomposition/recomposition of cluster and number; 45 sight facts, making ten, and fluency with teens’ numbers. Mastery of *arithmetic facts* means understanding of the appropriate language, accuracy, fluency, and applicability in multiple contexts and language forms.

Mastering any concept or procedure means that the child is progressively making progress in the concept along its trajectory of the development of that concept. For example, mastering means understanding the organization of our Base Ten system, knowing the difference between digits and numbers, the difference between digit, place, and value of a digit in a multi-digit number (with canonical and non-canonical decomposition), and the place value contextually:

kindergarten $65 = 60 + 5$;

first grade $563 = 500 + 60 + 3$;

second grade: $7,589 = 7000 + 500 + 80 + 9$;

third grade: $62,869 = 60,000 + 2,000 + 800 + 60 + 9$

$= 6 \times 10,000 + 2 \times 1,000 + 8 \times 100 + 6 \times 10 + 3$; and

fourth grade = standard form: 75,623.458,

expanded

form:

$70,000 + 5,000 + 600 + 20 + 3 + 4/10 + 5/100 + 8/1000$

(fraction form)

expanded form: $70,000 + 5,000 + 600 + 20 + 3 + .4 + .05 + .008$
(decimal form)

place-value form: $7 \times 10,000 + 5 \times 1000 + 6 \times 100 + 2 \times 10 + 3 \times 1 + 4 \times 1/10 + 5 \times 1/100 + 8 \times 1/1000$), and exponential form: $7 \times 10^4 + 5 \times 10^3 + 6 \times 10^2 + 2 \times 10^1 + 3 \times 10^0 + 4 \times 10^{-1} + 5 \times 10^{-2} + 8 \times 10^{-3}$, and finally, it turns into Scientific notation: $75,623.458 = 7.623458 \times 10^4$ when they reach the seventh grade and latest by the eighth grade. As one can see, the place value concept becomes richer with the other concepts and operations being added to the curriculum through its trajectory. The trajectory of this concept begins in kindergarten and continues and ultimately transforms into scientific notation (Young-Loveridge and Bicknell, 2015).

For success in numeracy, they should also know *properties of numbers and operations; comparison of numbers*—bigger/larger and smaller, more and less, fewer and many, even and odd, tens' numbers; structure of the number system; prime/composite, divisibility tests, etc.

Finally, for mastery of numeracy, students should be proficient in *arithmetic operations* with clear definitions, diverse conceptual models for each operation; effective and efficient procedures, including the standard procedures; clear understanding of the properties of numbers and operations—commutative, associative, etc.

Adhering to the aforementioned criteria for mastery and students with such mastery will have a strong sense of number and numeracy skills and appreciation for them.

Feeling of confidence and competence in dealing with quantity, in different forms, displays, and usage of numeracy. It is the ease and familiarity with dealing with numbers. *Numeracy* is the integration of number concepts, arithmetic facts, place value, and choosing how to decompose/recompose to operate on numbers without losing track of the constituent numbers themselves. Students with mastery of numeracy look at the numbers involved in a problem and get a sense of how to relate with them and how to use them according to the context rather than the rigidity

of procedures and operations. They transform them, break them, and combine them to make other numbers that are of better use in the context.

Students struggle with numeracy because they have not learned numbers, place value, and facts with efficient and effective strategies and do not get the opportunities for using them properly in tasks such as: *Empty Number Line*, *alternate models*, and *procedures for arithmetic operations*. Most dyscalculics have poor numeracy. It is a key indicator of dyscalculia and acquired dyscalculia. Proper number concept in kindergarten and then mastery of additive reasoning can prevent the incidence of acquired dyscalculia, mitigate, or minimize the impact of dyscalculia and is the antidote to mathematics anxiety.

To get some guidance, let us take a chapter from the Science of Reading. As we expect every child to read fluently with comprehension by the end of third grade (approximately age 9), we should expect and work for every child to have mastery of numeracy by the end of fourth grade (approximately age 10) so that they can learn future mathematics easily, effectively, and efficiently.

There are decades of research supporting the inclusion of important language elements as part of a multi-linguistic (multicomponent) approach to teaching reading and writing—an approach that integrates *phonology*, *orthography*, *morphology*, *semantics*, and *prosody* at both the lexical (word) and sub-lexical levels, and further extends to include syntax, pragmatics, verbal reasoning, and more—beyond the word level. This functional integration of the many components of language is required for skilled reading and writing.

Similarly, to have strong number sense and mastery of its components: subitizing, counting, decomposition/recomposition, understanding of our decimal place value system (ten digits, making groups of ten, successively from one place to the other, and its versatility), mastery of appropriate grade level arithmetic facts:

- 45 sight facts at the end of kindergarten;
- 100 addition facts—sums up to 20 at the end of first grade,

- 100 subtraction facts—differences of numbers up to 20 at the end of second grade,
- 100 multiplication facts by the end of third grade, and
- 100 division facts by the end of fourth grade

ensures that students form strong, secure, and developmentally appropriate foundations for numeracy by the end of fourth grade. Then they are truly prepared for concepts such as: fractions, decimals, ratio and proportion, percents, and pre-algebraic concepts. The development of the foundations of numeracy is assured of the principles espoused in Standards of Mathematics Practices (SMP, NCTM, 2004) along with the CCSS-M content standards ([CCSS, 2010](#)) with their intent and fidelity. Our methods in this book are designed to develop, to strengthen, and optimize these processes with flexibility in their usage.

As a fluent reader with comprehension,¹⁰ a child can apply his/her reading skills to reading content in any discipline and context. Similarly, a student with fluent numeracy skills (with proper language, clear understanding, efficient strategies, meaningful fluency, and facility with diverse applications) can use those skills for learning any mathematics concepts, including algebra and geometry. Then they can get entry to higher mathematics and Science, Technology, Engineering, Economics, and Mathematics (STEEM) disciplines. Such a level of proficiency and experience prepares children to aspire to be part of vocations and professions of any type and choice.

Of course, in the process of acquiring reading skills—fluency and comprehension—many children face unique hurdles. Similarly, the aim of acquiring numeracy skills with fluency and understanding poses its unique challenges in learning its *language*, *concepts*, and *procedures*. In this matter, we can again learn a great deal from the journey of arriving at the principles of the science of reading.

The principles of pedagogy in mathematics education and special education related to acquiring numeracy skills have not reached a level of maturity as in the case of teaching strategies for acquiring reading skills.

Many governments at different levels, from local to federal governments and professional organizations, are trying to come up with new standards and guidelines for instruction to compete with the achievements of countries where children are excelling in mathematics. Every nation is seeking to adopt improved mathematics curricula and teaching methods to respond to the employment needs in the changing economies, technologies, and their developmental needs. All seek reforms in mathematics education practices in their countries. Discussions about mathematics education, going on now among mathematics educators, are quite animated, matching the political fights in almost every country as different nations realize the importance of mathematics in their future development and improvement in their standards of living.

Research in mathematics education, neurosciences, and special education shows that numeracy skills achieved with strategies that arise from fundamental concepts of numeracy: proper, precise, and concise language (e.g., multiple linguistic expressions and representations for key concepts such as addition, subtraction, multiplication, and division); robust and comprehensive conceptual schemas (e.g., subtraction has multiple expressions, multiplication is seen as ‘repeated addition,’ ‘groups of,’ ‘an array,’ and ‘area of a rectangle’; similarly, division also has the same four models, etc.), effective, efficient strategies and procedures. For example, the addition and subtraction facts and procedures are derived using 45 sight facts, making 10, teens’ numbers, and decomposition/recomposition rather than counting strategies of various kinds. Similarly, multiplication and division facts and procedures should be based on the mastery of tables of 1, 10, 2, and 5 first and then extending the tables to include tables of 9, 4, 3, 6, 7, and 8 (in that order) by decomposition/recomposition (distributive property of multiplication over addition and subtraction and commutative property of multiplication) of facts and using the area model for multiplication.

With such a strong base of conceptual understanding and language about fundamentals, students will have a better chance of achieving proficiency and mastery, so they can appreciate the role, power, and beauty of

mathematics in daily life, vocations, and professions. Our experience in multiple settings shows that this approach not only provides help for dyscalculics but also acts as an antidote to acquired dyscalculia and the resultant mathematics anxiety. Acquired dyscalculia is highly responsible for most mathematics anxiety.

The discussions about the pros and cons of the Common Core State Standards in Math (CCSS-M) in America are a clear example of that. New guidelines for curricular changes in the United Kingdom, India, and South Africa have generated serious discussions that involved every school in the country. Even those who have not adopted them are giving considerable thought to creating high standards with effective pedagogies.

As we see it, international comparisons of different kinds of curricular approaches and standards, such as CCSS-M, have considerably upped the ante in mastery, rigor, coherence, and understanding the developmental trajectories of mathematics ideas—skills, strategies, and procedures. But they have also opened the option of introducing ‘modeling,’ which means the mathematics we teach should relate to the real world and be responsive to new demands and the impact of new technologies not only in daily living but also on teaching and learning. For such a curriculum to be available for all children, it demands that children be prepared with strong numeracy skills.

What brought the authors together are our experiences working with children with a wide range of abilities, disabilities, and differences. We could relate many success stories because of the work we do. These include the many common factors that inform each other’s work and the issues that bring our focus on changing the mindset of learners to help them achieve not only their full potential in mathematics from early childhood to their entrance into higher education or their careers, but also help them modify and improve their learnability skills. We also want teachers and parents to be able to help make that possible.

We have arrived at very similar destinations, despite doing so via different aspects of mathematics teaching with different populations of learners with or without learning disabilities in mathematics, and from very

different parts of the globe. This brings interesting elements of commonalities and differences. The most noteworthy aspect of everyone's success rate is that great results have been achieved because of that very specific goal to change what goes on inside the heads of our students, whether they are learning numbers, encountering a new concept, or performing complex procedures in algebra, geometry, calculus, or handling large data.

The aim of this book is to take a closer look at the progression of the development of the key concepts that are directly or indirectly related to the incidence of mathematics learning problems, difficulties, and disabilities, such as dyscalculia. We have found that key concepts that become the cause of learning difficulties in mathematics are: (a) Mastering *number concept*, (b) Development of *numbersense*, and (c) Proper development of *operations*, particularly Additive Reasoning. This means we should know effective, efficient, and elegant methods of teaching and helping children learn number concept and additive reasoning well. This should include development of the proper language, flexible conceptual schemas, and strong communication of oral and written calculations for the diverse operations, including standard algorithms. Their mastery should be so strong and flexible that they can apply them in learning mathematics and communicate it effectively. They should master not only the operations (standard and alternate) and answers to questions such as: Why this operation here? How does it work? Why does it work? How is it derived? Will it work always? What are the limitations of this procedure? Is this procedure efficient in this situation only? Can this procedure be extended, generalized, and extrapolated? How does this operation relate to other operations?

In this book, we clearly identify how the mastery of these key concepts affects the mastery of multiplicative reasoning to proportional reasoning (fractions, decimals, ratio and proportion, percents, to rate, and other related concepts). In the process, we want to share ideas, activities, and strategies that help build the relationship between concepts, fluency in the execution, and application of numerical procedures. In other words, how to build

foundations for strong numerical skills. We also want to share methods and strategies that can help them overcome the impact of their deficits—poor learning skills and the condition of mathematics anxiety.

The aim is to take a closer look at the progression of the development of additive reasoning from number concept formation to place value to mental addition and subtraction and to written calculation for the standard operations of addition and subtraction to understanding their inverse relationship. In the process, we want to suggest ideas, activities, and strategies that help build the relationship between addition and subtraction concepts, fluency in the execution, and application of numerical procedures reflective of additive reasoning. In the process, it equips our children with ways of learning future concepts with ease and mastery.

We also want to make sure that the traditional dichotomy of *conceptual understanding* and *procedural fluency* as separate objectives is broken and suggest effective, efficient, and productive activities in classroom instruction. The aim is to suggest ideas, strategies, activities, and experiences that help the development of students' abilities to carry out operations with understanding and integrate proficiently the concepts and skills that in many mathematics classrooms and textbooks have been featured as separate entities.

The interaction of the various components of the mathematics ideas and learning—language, conceptual schema, strategies, facts, procedure, and practice is the key for effective teaching and learning. For example, repetition for practice and understanding—in multiple forms and meaningful conceptual learning—are not mutually exclusive. As one Chinese saying goes:

书读百遍其意自现

– meaning reveals itself when you read something many times.

In the case of mathematics, practice means: First, discussion and reflection on what we have just learned. Second, practice it with someone—your partner, with your parent—immediately after learning. Third, test yourself

or have your teacher test you, practice intermittently, retest, and apply it to other mathematics ideas, other disciplines, and practical problems. Practice also means to arrive at the answer using multiple strategies and pathways. Meaningful practice with definite incremental goals and objectives one achieves, keeps, and applies to further learning. The practice problems should be accessible to the child, but ultimately, they should reach a moderately difficult level. An important point that children should realize is that when one understands the concept and the meaning behind the strategy or procedure, one remembers it better and is able to use it effectively and productively, even in novel situations.

In our approach, we also integrate the generation of language, development and construction of concepts and their schemas, and then arriving at procedures using effective tools. In our approach, the notions of ‘conceptual knowledge’ and ‘procedural knowledge’ have been given equal and appropriate importance. For example, in subtracting a multi-digit number from a multi-digit number (with regrouping/renaming/borrowing), several procedures are derived that reflect aspects of additive reasoning and showcase the importance of understanding certain key features of additive reasoning. More on these in the chapters on Addition and Subtraction Procedures.

Furthermore, we show the need for constructing structures and schemas associated with understanding that are more general, rather than aspects of a mathematical idea. Accordingly, these fundamental cognitive schemas and strategies reflect fundamental epistemic elements of inquiry and problem structures that are developed in the context of the foundations of numeracy. They are equally important to the learning of mathematics in any context and level. For example, we cannot add or subtract any two quantities (later fractions, decimals, numbers, terms, and other mathematics elements such as determinants, matrices, functions, etc.) unless they share some common property, attribute, membership, or characteristic. Our children always ask: ‘Can I add or subtract them? When can I subtract them? What tools do I need to solve this problem?’ Etc.

We believe the discussions in this book have international and long-lasting implications for mathematics education for all children, including learning disabled. As a result, all school mathematics can be enlivened and made relevant, even exciting, to students by dipping into the vast array of applications (e.g., *intra-mathematical*, *interdisciplinary*, and *extracurricular*) that mathematics offers to problem solving in real life.

In this context, our message is: math, properly taught, can engage all students, including learning disabled, particularly dyscalculic students interested in mathematics, to realize their potential and acquire skills useful for life and their professions ([Geary 1990](#), [2004](#), [2005](#)). Here, we cover content and strategies related to key developmental milestones in mathematics—topics that form the backbone of school mathematics curriculum, particularly the *number concept*, *numbersense*, and *additive reasoning*.

The objective is to help teachers, parents, and researchers know, learn about, and implement efficient instructional practices for teaching mathematics to all students, including those who struggle. The approach emphasizes the principles and practices to teach and learn the critical concepts and skills necessary for mastering numeracy and preparing for success in algebra. It also ensures that they do not become anxious about learning mathematics and are not turned away due to their learning difficulties, such as dyslexia, dyscalculia, dysgraphia, and/or other specific learning disabilities in mathematics.

The book provides specific content, appropriate and efficient instructional and intervention recommendations, and draws upon currently available research-based evidence for teaching mathematics and our combined experience and skills from more than a century in teacher training and in providing direct services to children of all levels of mathematics, competence, and learner differences.

The chapters deal with understanding the issues of learning problems in mathematics, including dyscalculia and mathematics difficulties due to dyslexia and other language-related difficulties.

We feel that in the field of dyscalculia, we have arrived at a place where we need to move on from just focusing on the characteristics and symptoms of dyscalculia to paying attention and responding to:

- a. What are the possible reasons for the incidence and conditions of dyscalculia and acquired dyscalculia (assessment and diagnosis)?
- b. What to do to help students learn numeracy, particularly those who show such characteristics (planning and organizing—inclusion and/or individualization)?
- c. What should be the nature of mathematics instruction: during the regular classroom (inclusion), interventions (pre- and post-classroom settings), and remedial sessions (individual and small-group settings) to help these students learn mathematics easily and effectively?

So far, the suggestions and directions that most research on dyscalculia provides for instruction are too fragmented, compartmentalized, and, as the results show, have little or no consequence, as the number of mathematics-anxious and avoiders of mathematics learning is increasing rather than decreasing.

We present a comprehensive conceptual framework for mathematics instruction for school mathematics, with a focus on earlier grades, that can help education professionals—classroom teachers, special educators, and interventionists—to better understand what their students are facing as they acquire numeracy skills and learn mathematics: *language, concepts, procedures, and skills*. Why do they have difficulty in learning the key concepts of numeracy? We open avenues for effective instruction and intervention and provide future directions for research to be pursued to improve mathematics instruction for all students, including dyscalculics.

For example, several national and international panels and research studies have recommended certain instructional components for improving mathematics outcomes but presented these instructional components as a list of actions without explicitly addressing their interrelations, either in terms of instructional activities (lesson plans) that improve students'

mathematics learnability, content mastery, and/or their cognitive development. We systematically explore the key cognitive capacities underlying learning and conceptualizing mathematics ideas that specify the relationships between them.

The central objective of this book is: to help teachers and intervention specialists achieve better outcomes for students through *prevention* (how to effectively prepare for learning), better *initial instruction* (instruction that is accessible to all children in the classroom and that leaves residue on one's memory), *intervention* (identifying natural places for intervention and strategies for providing it), and *remediation* (focused and surgical approach to remediation—helping children to remove misconceptions and transplanting good learning habits and strategies). In other words, what cognitive capacities undergird learning specific prerequisite and mathematics skills, particularly those that provide the basis for developing numeracy skills in all children effectively, efficiently, and easily.

We present specific activities, strategies, teaching methods, lesson plans, and assessment tools—formal and informal—to teach key skills, concepts, and procedures in numeracy:

- a. Acquiring solid number concepts,
- b. Simple and sure methods for acquiring the mastery of arithmetic facts—addition and subtraction,
- c. Solid mental mathematics strategies based on properties of number and operations, and
- d. Introducing them to task analysis of the concept or procedure at hand and then developing a *script*¹¹ for that problem or procedure.¹²

These methods will assure that children are better prepared for learning concepts and procedures related to fractions, decimals, ratios, proportions, and percents with a clear understanding and mastery.

For example, in using task analysis of what cognitive, memory, linguistic, and numeracy-related skills are involved for success in mastering arithmetic facts, we provide specific activities and strategies for each area.

For example, the specific primary concepts and skills necessary for mastering arithmetic facts are *subitizing*, *decomposition/recomposition* of *quantity/number*, *45 sight facts*, *how to make ten*, *teens' numbers*, and the ability to derive other arithmetic facts. Therefore, assessment, evaluation, and diagnosis should focus on these elements.

We have developed specific *scripts* (linguistic conceptual schemas) based on the understanding of the nature of number concepts (*subitizing* and *numberness*), *decomposition/recomposition* of clusters and numbers, and the use of efficient and effective tools: *Visual Cluster cards*, *Invicta Balance*, *Cuisenaire rods*, and the *Empty Number Line*. The activities developed from these principles help children learn the number relationships easily and support the effective and efficient mastery of facts and their usage.

A. Background of Learning Problems in Mathematics

Far too many children find mathematics difficult to learn, especially as they reach the higher grades. Some show anxiety about mathematics as early as first grade. Most stay in mathematics courses only as long as they must, and most students never succeed in understanding the real meanings of mathematical concepts and procedures. Many become deft technicians in manipulating symbols—even complicated equations—and many others learn to avoid anything dealing with numbers. It is almost a badge of honor for people to declare and accept their incompetence in mathematics—real or imagined. Current research on mathematics anxiety shows that it is an international phenomenon.¹³ Although the number of people who perceive and believe they have difficulty in learning mathematics is much larger than the actual number of people who have either neuropsychological or cognitive conditions, or co-morbid reasons for mathematics difficulties.

Mathematics is considered by many as the worst curricular villain in driving students to failure in school. The mere mention of the word ‘math’ or ‘maths’ strikes fear into the hearts of many. More than any other subject, mathematics filters students out of programs leading to scientific and

professional careers. Each succeeding year of schooling, fewer students take the mathematics necessary for important professional, scientific, technical, and even ordinary careers. Few students display a positive attitude towards mathematics as they go from lower grades to upper grades. Unless required, from tenth grade on, the percentage of students taking ‘real’ mathematics courses decreases by half for each year of schooling. Only about 15 percent of high school seniors take four years of real mathematics, unless it is mandated. Many schools offer courses in the name of mathematics, but they are far from mathematics. Less than ten percent of American high school students elect to study higher mathematics—that is, calculus and advanced calculus and/or manipulation and understanding of data analysis, statistics, and probability. Millions of adolescents and adults screen themselves out of meaningful professional and personal opportunities because they fear performing poorly in mathematics.

Math phobia, math anxiety, or math trauma affect many children and adults, including even those with a ‘good’ education. They avoid mathematics as they have developed a negative attitude toward mathematics very early during their school experiences—sometimes as early as first and second grade. As a result, compared to non-anxious students, those who identify themselves as math-anxious use more fear words and negative comments such as ‘confusing,’ ‘difficult,’ ‘I am not a good memorizer,’ ‘I guess I do not have a head for mathematics,’ or ‘can’t do.’ Their words are highly charged with emotion and show that failure in mathematics in many cases has affected their self-esteem and attitude towards mathematics and mathematics-related activities and persons. Despite these self-declared difficulties, the problem in learning mathematics is much more pronounced in those learners who have been identified as learning disabled. Students classified as learning disabled or with emotional issues have specific constraints imposed on their facility to learn mathematics and show unique variances in their deficiencies in mathematical skills and concepts.

This dislike has a long-lasting impact on the lives of many individuals. For example, many adults make career decisions that involve minimum encounters with quantitative and spatial reasoning, closing many exciting

and challenging career opportunities. They may also be afraid of entering further education programs, as they know they will encounter mathematics.

However, most students, both with and without math anxiety, express that mathematics is important. Even math-anxious individuals realize that mathematics is required in virtually all academic programs and professions and that proficiency in mathematics can open doors to many desirable and lucrative career opportunities. Despite its importance, many students, such as budding scientists and engineers who are proficient in mathematics, see learning it as a chore to be accomplished in the pursuit of their real interest. The proportion of people going on to study for a mathematics degree is stagnant or declining, and there is a growing feeling among academics that many of these students no longer come with much enthusiasm for mathematics and mathematics-related subjects.

Most students with mathematics learning difficulties, disabilities, or anxiety can recall the exact moment they began to dislike mathematics. Generally, it is during the transitions from one mathematics developmental milestone to another. This is so because each developmental milestone in mathematics reaches a higher level of abstraction and complexity than the previous one. For example, the first hurdle takes place when children move from counting objects to representing them by a symbol or word; the next abstraction takes place when they realize that the smaller collection is part of a larger collection and one number is smaller or bigger than the other; the next is the number concept (integration of subitizing, numberness, each number is made of smaller numbers—sight facts) to number relationships (additive reasoning).

Those who have not mastered the component concepts and skills of number: subitizing, decomposition/recomposition of number, 45 sight facts, making ten, and teens numbers, have great difficulty in mastering addition and subtraction facts. It is almost certain that they will have difficulty in transiting from additive reasoning to multiplicative reasoning. They will have difficulty in mastering multiplication and division facts, and therefore, multi-digit operations of multiplication and long division will be difficult to learn.

We have observed that children with a lack of mastery of multiplication facts never feel comfortable with proportional reasoning (fractions, decimals, ratio, proportion, etc.). Thinking of algebraic reasoning without proportional reasoning is very difficult to envision. The level of distaste for mathematics increases with each transition of mathematics developmental milestone to the next.

Junctions of these milestones are pronounced hurdles for many of these students:

... ... *Subitizing* → *Numberness* → *Number concept* → *Number Facts* → *Place-Value* → *Additive Reasoning* → *Multiplicative Reasoning* → *Proportional Reasoning* → *Algebraic Thinking* →
... ..

With a poor understanding of numeracy (number concept, additive reasoning, place value, and multiplicative reasoning), one faces difficulty in most of the concepts after that.

Each junction poses new difficulties, and we lose a few more students. At each milestone, there is new and more language, more and higher-level conceptual schemas, more procedures, and more strategies, particularly when they are taught as disconnected concepts. If the previous level was mastered by counting, the next concept becomes more difficult. The symptoms and conditions of dyscalculia and acquired dyscalculia begin to manifest at each one of these junctions.

Each successive failure and hurdle reinforces their belief that they cannot learn mathematics. The transitions are poor if they have poor development of mathematics language (vocabulary, syntax, and semantics) and its usage for previous concepts, limited and poor conceptual models, ineffective and inefficient tools and strategies, and poor procedural development that is not generalizable (abundance of tricks, short-cuts, mnemonic aids, non-generalizable scripts for specific problems or specific types of problems), and lack of integration of primary concepts. Then a cycle of poor interventions and remediations begin that use the same strategies that were

the cause of the problems. The unsuccessful experiences and defeats bring frustrations and anxiety and distaste for mathematics begin and when it is continuous, the cognitive defeats turn into emotional defeats and mathematics anxiety begins to interfere with learning related to anything that appears like mathematics.

To compound the problems and difficulties due to the nature of the numeracy content other neuropsychological, neurolinguistic, and cognitive functions, such as executive functions, play a big role in numeracy and mathematics learning. For example, every component of executive functions—working memory, attention and focus, organization, and flexibility of thought is one way or the other involved in the simplest of arithmetic and mathematics learning, mastering, functioning, and applying. Interventions and remediations that do not integrate these in lessons, practice, and applying numerical and mathematics skills fail.

For these various reasons, students are often not able to improve their mathematics skills without ‘*proper*’ remediation. The remediation that emphasizes just practicing the same methods, same strategies, and same language that did not work the first time, so more of it is not enough. It might even be counterproductive.

Unfortunately, many numeracy teachers and mathematics instructors are not always trained in the diagnosis and remediation of learning problems in mathematics. At the same time, most special education teachers and interventionists are also not trained in mathematics learning problems—their diagnosis and remediation. As a result, learners with problems in mathematics compared to learners with reading and spelling difficulties receive little or inappropriate attention from professionals in special education. Because of their own training in study skills, behavior management, reading, and language difficulties, most special educators tend to emphasize the remediation of reading and spelling, and they do a good job of it. However, when it comes to mathematics, many of them transfer their own fears of mathematics and anxieties to their students or use inefficient methods of remediation that do not result in higher student

achievement and change in attitudes toward mathematics. Of course, there are exceptions.

A person feels anxious when one or more of the following happens when they encounter a mathematics problem that they are unable to solve:

- a. do not understand the problem—do not know the meaning of terms, symbols, phrases, or questions—poor vocabulary, lack of understanding of syntax, semantics, or organization of mathematics language and symbols ([Pimm, 1995](#)),
- b. cannot recognize, recall, or execute a procedure,
- c. lack of conceptual understanding,
- d. lack of mastery of pre-skills/primary concepts involved in the problem, and/or
- e. lack of resources—cognitive, linguistic, physical tools, strategies, knowledge, definition, formula, or experience in executing a procedure.

The treatment involves changes in systems, pedagogy, assessment, and a supportive environment. The treatment is a combination of improvements in initial teaching, intervention, remediation, and mediation of thinking and cognition.

In some school systems, although the real incidence of dyscalculia is about 4 to 6% of the student population, more than 20% of students are identified as having a learning difficulty or disability in mathematics. The population of those declaring that they have difficulty in learning mathematics is much higher. And those who self-declare that they are afraid of mathematics is even higher. Policy and decision-makers, mathematics and special educators, psychologists, academic therapists, social and community workers, classroom teachers, and tutors are increasingly concerned with why so many students have difficulty in and an aversion to learning mathematics. They want to understand the causes of the problem and strategies for its prevention and remediation.

Apart from the questions of diagnosis and remediation, of learning problems in mathematics—dyscalculia, acquired dyscalculia, mathematics anxiety, etc.—other important questions also arise:

- How do we focus the attention of special education professionals on mathematics learning problems?
- What kind of training and tools can we provide them for informal and formal diagnosis?
- How and what kind of training do we provide them in key mathematics concepts and remedial strategies?
- Should literacy tutors also be numeracy tutors? Can they be?
- What should be the skill set and role of these tutors—*therapeutic, instructional, remedial, or skill training*—such as cognitive and executive functions?
- What should be the *content* and the *level* of their training?
- How intensive should remedial programs be?
- Where should they occur? And when?
- What should be the focus of remediation in numeracy—*skill building, concept development, content mastery, improvement of self-esteem, cognitive preparation* for higher education, or *job-related*?
- What role should technology play in skill building?
- Who should train numeracy teachers/tutors?
- What should be the nature and content of tutoring?
- What does the structure of a remedial lesson look like?
- Etc., etc.

The gap in students' preparation and expected performance and functioning is enormous. For example, to make progress equivalent to one year of schooling (according to the curricular demands) in a subject area, one needs more than 150 to 200 hours of effective instruction per year, whereas it has been found that these students and programs serving and working with them are able to commit less than 100 hours of instruction per year. In some cases, it is as low as 30 hours a year. Time is limited in which

to bring these students to a level where they can take advantage of higher education or develop productive job-related skills. As a result, many reach adulthood without adequate numeracy/mathematics skills for life and professions. Less than 10% of these individuals as adults needing literacy and numeracy skills return to education and skill building to community colleges and tertiary education, and of these, more than 60% drop out again for the same reasons that were responsible for their difficulty the first time. Our community colleges and tertiary education programs have many remedial programs for them, yet the success rate in these programs is very low. One of the reasons, once again, is the nature of instruction.

Students in large numbers encounter unsuccessful experiences in mathematics, some of which can be traced to practices unique to the teaching of numeracy in the early schooling and teaching of mathematics during middle and high school. And a great deal of this is again repeated when they join remedial courses in community colleges. Teaching mathematics is a complex task. It is also a difficult subject to teach and learn. Individuals who are otherwise talented and capable of abstract reasoning in other subjects show frustration and anxiety when faced with mathematics tasks, particularly when improper language, inefficient teaching methods, and narrowly conceived models are used.

Math anxiety is often developed because of repeated unsuccessful experiences, most times not necessarily due to lack of ability. In such cases, anxiety, rather than competence, controls students' academic performance. Furthermore, there are students who perform well in one branch of mathematics but do poorly in another branch. For example, there are students who do well in arithmetic and algebra and do poorly in geometrical concepts, and some students who do well in geometry and then do poorly in arithmetic and algebra. Similarly, the failures, which some children and adults experience in arithmetic, often are at variance with their basic intellectual ability, and despite their difficulty in learning arithmetic, they can learn higher mathematics. On the other hand, some students can learn basic 'mechanical' arithmetic, but they may have difficulty in conceptualizing even numeracy concepts, let alone higher mathematics such

as algebra and trigonometry. Thus, arithmetic and mathematics learning problems are manifested differently by children and are carried over into adulthood with different attitudes toward mathematics.

Many people have problems in learning mathematics. The nature and causes of their problems vary. Therefore, there is no one explanation for a person's learning problems in mathematics or difficulty in conceptualizing mathematics concepts or executing procedures. Consequently, there is no one instructional or corrective technique that is most effective for every individual or for every mathematics concept. One must look at the problem, establish the causes, and then determine the course of remediation. To be more effective, educators need to learn special techniques to help students, as each individual learner comes with a unique combination of deficits and specific characteristics of their learning profile. They need to know which instructional materials and strategies are most successful for which topics and for which students under what settings. They also need to be able to gauge the most useful timing and sequencing of content, tasks, and strategies of their interventions. For example, the student who cannot recall arithmetic facts at age nine and the student who has to devote an enormous amount of time to figure out even the simplest of arithmetic facts at age 15 have similarities in the manifestation of the problem from the performance point of view, but they may have developed quite different attitudes toward mathematics and the reasons for their deficits might be quite different from innate developmental skills to the habits and skills developed in certain environments that are limiting. The nature and the course of remediation to be followed in these two different types of cases may be similar in content but quite different in the approach to instruction—different sequence of concepts, emphasis, practice, and language and questioning process, the selection of instructional materials, and in the sequence of tasks.

As far as remediation of numeracy problems is concerned, the earlier the onset of remediation, the easier it is to make up. When remediation is not provided early, the student has negative attitudes about learning simple facts and therefore, they have resistance to many simplistic methods. The

remediation should tackle current problems and weave into the remediation of earlier problems.

For example, one of the key problems students with dyscalculia and acquired dyscalculia demonstrate is mastering arithmetic facts for a variety of reasons. For example, on analyzing the trajectory of the development of arithmetic facts in general, and the role of the primary and secondary concepts in the development of the mastery of addition and subtraction facts, one realizes that it is dependent on a few simple primary concepts: (a) *Number Concept* (subitizing, visual clustering and counting, and numberness), (b) *Decomposition/recomposition of numbers (and clusters)*, (c) *45 sight facts*, (d) *Making 10*, and (e) *Teens numbers*. The remediation should focus on developing these primary concepts easily, effectively, and efficiently using proper tools and strategic scripts. The following chapters focus on these aspects in initial teaching, interventions, and remedial strategies.

For the mastery of these primary concepts and skills to be achieved, the most effective materials are: (a) Visual Cluster Cards, (b) Cuisenaire rods, (c) Invicta Balance, (d) Empty Number Line (ENL), and (e) Graphic Organizers for decomposition/recomposition of numbers in context. These tools will become essential in developing numeracy—its components: language, conceptual schemas, strategies, and procedures—properly and effectively. In the use of these tools, the process also helps develop cognition, executive functions, appropriate learning skills, particularly related to numeracy. The sequencing of these tasks and their mastery is very important for mastering arithmetic facts.

Proper application of these strategies will help the student to understand, learn, master, automatize, and apply arithmetic facts in diverse settings and see their importance in life, and therefore appreciate their learning. Just more counting and more rote memorization is not going to help the students; it is not an effective intervention nor remediation. We believe very strongly, based on our work with diverse students, that the brain's plasticity and its flexibility are the key to remediation. We need to mediate to change the student's thinking—help him grow in his ability and cognition. The

same principles apply to every and all arithmetic skills in the acquisition of numeracy. In the preventive instruction, intervention, and remediation of learning difficulties of all students, including learning disabled and/or dyscalculic students, we need to focus on the area of difficulty and its primary concepts and other supporting component skills and what change in the student—improvement in cognition, linguistic, neuropsychological—and pre-requisite skills can be brought about to make them a better learner.

B. Causes of Learning Problems in Mathematics

The types of arithmetical and mathematical difficulties that children with or without learning disabilities experience, as well as their diagnosis and remediation, are just beginning to receive serious attention from psychologists, special needs and mathematics educators, mathematics education and other researchers, such as psychologists, neurologists, and policymakers.

The number of research reports on identification, diagnosis, and remediation of children's mathematics learning problems now appearing in the professional literature is increasing every day. Compared to reading and language development, the interest in the subject of learning problems in numeracy and mathematics and their prevention and remediation is very recent; we still have fewer options for diagnosis and remediation. Teachers and parents of children with learning problems in mathematics find that there are still fewer assessment devices and remedial materials available than for problems in reading, writing, and spelling. Some of the reasons for this are: the widespread concern with literacy; emphasis on language arts skills in the primary grades; the lack of emphasis on arithmetic; the feeling on the part of some teachers and parents that arithmetic is not as vital to future success and preparation for careers as other academic subjects; and most importantly, the belief of many that some people just cannot learn mathematics ([Aunio & Niemivirta 2010](#), 18).

We need to emphasize that a person's literacy skills have been the 'door opener' and for centuries a key factor for advancement in society and the

cause of change in social, economic, and causes of higher standards of living. Today numeracy is the new literacy and the ‘gate opener’ to not only to STEEM professions, incentive for innovation, and higher standards of living. Today’s leaders are steeped in mathematics skills and the skills that it makes available to an individual. It is an instrument for achieving social and economic justice and equality. Every child has a right to be literate and numerate and the option to learn higher mathematics.

The sources of children’s learning difficulties and disabilities are far from clear. It is presumed that difficulties in learning mathematics may stem from dysfunctions of the central nervous system, but this does not provide sufficient explanation for the specific problems that children and adults face in learning mathematics in general and arithmetic, in particular. As learning disabilities can exist across the intellectual spectrum, the group of children having difficulty in mathematics is also heterogeneous; their underlying problems, while different, may result in similar as well as different patterns of underachievement. The pattern of difficulties may look similar because they cannot perform the right algorithms and cannot recall the right facts and appropriate operational procedures. Their pattern may look different because of their wrong answers, the wrong procedures that they have applied in apparently similar situations, the wrong facts that they have given at incorrect times. The difficulties they show at different age levels also indicate both different and similar patterns of errors. As a result, the effect of their failures on their other academic work, attitudes about mathematics, self-esteem, and mental health are also strikingly different.

For example, learning disabilities in mathematics may manifest because of deficits in the development of *prerequisite skills* necessary for numeracy and mathematics learning. These include:

- a. the ability to follow *sequential directions*,
- b. being aware of *spatial orientation/space organization*,
- c. recognizing, identifying, extending, creating, and/or using *patterns*,
- d. the ability to *visualize* and use *visual perception*,

- e. *estimating and approximating the outcomes of numerical operations*, and,
- f. *deductive and inductive* thinking.

These deficits may affect learning ability in different aspects of numeracy and mathematics. For example, all procedures are dependent on the ability to follow sequential directions. The number relationships remain elusive without the ability to discern patterns and relationships. Spatial sense is a prerequisite to place-value, fractions, decimals, geometry, etc. Children have difficulty when they do not have a ‘good’ sense of estimation and approximation. Because of the cumulative nature of mathematics, a deficit in a few isolated skills in one concept/procedure or several areas of arithmetic/mathematics may affect learning many other concepts and skills in numeracy and mathematics.

Some learning problems in numeracy and mathematics fall in the intersection of the skills related to *quantity, language, and spatial thinking*. For example, many children show difficulty in subtraction and division, fractions, and word problems due to their language-related difficulties ([Sharma, 1981](#)).

1. Diverse Perspectives

Because of the range of mathematics disabilities, we cannot clearly identify a cause-or-effect relationship with every kind of mathematics problem; no one explanation adequately addresses the nature of the diversity of learning problems in mathematics. Most mathematics problems and difficulties, such as *carryover problems*¹⁴ in numeracy or mathematics, *dyscalculia*, *acquired dyscalculia*, or *mathematics anxiety*, are manifested as a lack of quantitative thinking. Here in this book, we are only interested in mathematics disabilities and difficulties in the acquisition of fundamental skills of numeracy due to dyscalculia or acquired dyscalculia, but the suggestions extend beyond those conditions.

The causes of problems in arithmetic and mathematics are many. The causative factors considered for the explanations of the incidence and

nature of mathematics learning problems reflect the different perspectives that observers and researchers from various disciplines use in studying these problems.

For example, most *neurologists* explain mathematics learning problems (particularly dyscalculia and acalculia) in terms of deficits of the higher cortical functioning of the brain (ranging from very subtle to gross) and lack of development of parts of the brain that deal with quantity and spatial organization. This approach is like explaining the incidences of dyslexia and other language-related learning disabilities. According to neurologists, any dysfunction—a lesion in some specific part, an insult or injury to the brain potentially involving learning and, therefore, may become a cause of learning problems in mathematics ([Ashkenazi et al., 2013](#)). Luckily, neuroscience has identified many relationships between legions and deficits in particular areas and the learning they support. For example, different parts of the memory areas and the learning difficulties of different kinds are related ([Steves, 1983](#)). And opens doors for approaches, for actions we can take to assure that the proper methods are used to maximize the information entering the short-term memory and use working memory space properly so we can put information in long-term memory in specific locations of the brain effectively.

Many researchers attribute certain reading disabilities to *neuropsychological factors*. Similarly, the problems children experience in learning and the high incidence of failure to achieve in arithmetic may have a neuropsychological basis. Neuropsychological studies of patients with brain damage reveal a complex network in the brain that supports arithmetical processes. Here, we need to differentiate between the children with or without neuropsychological issues—deficits vs abnormalities. For example, in many cases, a neurological basis of their problems may be true; however, it is an inadequate explanation in other cases as many children exhibit difficulty in learning mathematics for a variety of other reasons and many times quite independently of any discernible neuropsychological deficit or anomaly. Some children demonstrate generalized deficits in

computational skills and particular difficulty in the rapid recall of basic facts in early school years.

Some *neuropsychologists* attribute the difficulties to a student's deficits in *executive functions* (deficits in *working memory*, *lack of inhibition/attention and focus*, *organization*: physical—tools, language, thought, order, pattern, and finally, *flexibility of thought*). Some others attribute these problems due to information processing of different types: *reception* or *expression* of information (difficulties due to poor processing of mathematical, linguistic, and symbolic information)—receptive and/or expressive ([De Smelt et al., 2011](#), 20).

Several findings link numerosity processing and arithmetical calculation related to diverse areas in the brain. See [Butterworth and Walsh \(2011\)](#) and [Dehaene \(2020\)](#) for a review of the neural basis of mathematics. Several specific areas of the brain are responsible for arithmetic functioning. For example, damage to the frontal lobes affects the ability to solve novel problems, while damage to the parietal lobes, usually the left parietal lobe, affects the ability to do routine tasks or to recall previously learned facts (Cipolotti and van Harskamp, 2001; [Badian, 1983](#); see [Butterworth, 1999, 2015, 2025](#) and 2026).

Neuroimaging shows that the parietal lobes are activated by very simple tasks, such as selecting the larger of two numbers or the display with more dots ([Dehaene et al., 2003](#); [Pinel et al., 2001](#)). In fact, small regions in the left and right parietal lobes (the intraparietal sulci) are specific for processing the numerosity of displays ([Castelli et al., 2006](#)). These regions are part of a brain network involving both the parietal and frontal lobes that are activated almost every time we carry out a numerical calculation, routine or novel ([Andres et al., 2011](#)).

Cognitivists suggest the lack of cognitive development and deficits in information processing capability of individuals as possible reasons for their difficulties in learning mathematics ([Price et al., 2007](#)). These include problems in:

- a. developing effective memorization skills,

- b. acquiring cognitive strategies to represent concrete experiences into abstract forms,
- c. generalizing and extrapolating specific experiences to form conjectures about quantitative/numerical information and observe emerging patterns in mathematics concepts,
- d. interpreting the given information in the context of the problem, and
- e. articulating strategies necessary to solve the problem, etc.

From this perspective, a lag in a particular cognitive skill may contribute to difficulty in learning a particular aspect of mathematics. For example, a student may give the answer to the problem $5496 \div 24$, if we let him perform the actual calculations procedurally (applying the standard division algorithm). However, when asked for an estimate for the answer to the problem, he has difficulty in coming up with an estimate. Or a student can give the answer for: ‘what is one half of eight?’ as ‘4,’ but cannot explain the process of arriving at the answer. He may answer that he ‘has divided eight by one half.’ Many students can find the correct answers, but they lack conceptual understanding. On the other hand, some may know the concept but may not have the quantitative skills: ‘There are 5,496 apples packed in 24 boxes. How many are in each box?’ The student answers: ‘I know I am supposed to divide, but I hate long division. Sorry! I cannot do it.’ In such cases, not only do they lack conceptual understanding, but they also have difficulty in generalizing the results from specific problems because of their difficulty in seeing patterns and underlying principles in information and relationships between different concepts. A child with this difficulty is generally good in basic procedural numerical computation but poor in mathematical and logical reasoning, and they plateau in their achievement in mathematics early.

Behaviorists attribute pupils’ mathematical difficulties to such factors as failure in mastering prerequisite skills, lack of instruction, inadequate understanding of the stimulus and lack of control in problem variables, inadequate practice, insufficient and ineffective procedures in learning skills, including improper sequencing of skill hierarchies and conceptual

links. From this perspective, therefore, contributing factors to the incidence of the learning problems in mathematics lie outside the child. The causes of the problem are mostly socio-cultural and environmental.

Special needs educators generally posit that the type of learning disability imposes specific limitations on a child's learning ability, such as limited working memory, diminished attention, poor recall, etc., therefore the resulting poor performance in mathematics. As a result, the nature of the child's learning disability provides some of the explanations for their learning problems and difficulties in mathematics. Unfortunately, many special educators, due to this information in their approach to remediation, have lower expectations because of these factors, placing limitations on the content to be exposed, thereby further limiting their ability to learn. They place lower expectations, pay attention to learning management kinds of skills, and lack focus on the development of mathematics concepts: the language, conceptual schemas, and effective and innovative procedures.

Mathematics educators look at the problem from the perspective of mathematics content. The lack of preparation for mathematics experiences, poor teaching, lack of appropriate conceptual models, lack of motivation and concentration, lack of adequate practice, poor memory, and the difficult nature of mathematics concepts are seen as the reasons for a student's difficulty in mathematics learning. Their approach is just to emphasize practice, ability grouping, and lower expectations.

The complexity of mathematical learning problems shows that their etiology is as complex and diverse as learning disabilities in general. No single perspective is adequate to fully explain the nature of mathematics learning problems, since there are several reasons a child has difficulty in learning mathematics. Therefore, there is no one approach that can fully explain a child's learning problems in mathematics. In the case of a particular child, they may emanate from several perspectives. Therefore, the causes of his/her their problems may be due to a combination of:

1. *intellectual attributes and cognitive factors* (mental deficiency, lack of performance capabilities, and/or cognitive lag in thinking and

- strategies),
2. *reading and other related language disabilities* (carryover problems, comorbidity with dyslexia, etc.),
 3. *lack of adequate mathematical experiences and/or the pre-requisite skills necessary for the development of concepts and procedures,*
 4. *emotional and other psychological and neuropsychological factors* (personality factors, ADD, ADHD, cognitive style, anxiety, and other learner attributes),
 5. *educational effects of instructional practices and curriculum patterns* (lack of matching in instructional approaches and the child's learning starts, preferences, and needs (carryover problems)),
 6. *individual neuropsychological pattern of developmental style* (development-related problems),
 7. *dyscalculia*: Phonological deficits have been identified as a major factor in the emergence of dyslexia (along with attention deficits—the child mixes up the letters in neighboring words) and visual perceptual deficits that cause mirror confusions. Virtually the same story could be told about dyscalculia—*decomposition and re-composition is to mathematics as phonemic awareness is to language and reading*, and *visual and attentional deficits* also create similar difficulties with numeracy. We have found that the processes such as *deficits in subitizing, difficulty with decomposition/re-composition, and disturbances in visual perceptual integration might be the reasons associated with dyscalculia* and other related learning problems in mathematics. Early neurological disorganization may cause dyscalculia because on core knowledge of *subitizing, one-to-one correspondence, decomposition/re-composition*. In other words, *dyscalculia is the outcome of disconnections in initial quantitative intuitions: Number Concept* (subitizing and visual clustering—numberness, decomposition/recomposition, making ten, and teens numbers), *Number relationships* (arithmetic facts, place value of numbers, and number sense), and appropriate initial teaching is a factor.

8. Any one or combination can result in difficulty in learning mathematics and failures in mathematics performance. Frequent failures in mathematics over a period result in *mathematics anxiety*. Mathematics anxiety produces a feeling of defeat, inadequacy, and helplessness and results in low self-esteem and apathy toward anything numerical. That attitude leads to a person avoiding mathematics-related work even in the simplest of situations. Many adults attribute their mathematics anxieties to poor and inadequate early mathematics experiences, particularly difficulty learning arithmetic facts.

In the earlier grades, the condition of mathematics anxiety is only a symptom of something not being right, but as the pattern of failures keeps repeating into later grades, the mathematics anxiety begins to *affect a student's self-esteem, and their attitude toward learning mathematics becomes negative, and avoidance of mathematics becomes a common habit*. It is much easier to remedy mathematics anxiety during the early years than later years. If not treated early, mathematics anxiety becomes a causative factor of future learning problems in mathematics. Because of anxiety, the student avoids mathematics and mathematics-related activity, which in turn creates a more serious lag in his mathematical thinking, and the same pattern is repeated.^{[15](#)}

In this book, we look at the nature of mathematics learning, mathematics learning problems, their causes, and prevention, intervention, and remediation. We place special emphasis on understanding the nature of dyscalculia and math anxiety and their implications to instruction and remediation. However, we believe, with others ([Dehaene, 2020](#)),

that none of the genes that are involved in dyslexia, dyscalculia, or, for that matter, any other developmental syndrome, including autism and schizophrenia, have 100 percent determinism... . the environment also has a huge share in the developmental trajectory that a child will ultimately take... . *no dyslexia or dyscalculia is so strong as to be beyond the reach of rehabilitation.*^{[16](#)}

(p. 82, italics added)

We also believe that recent neurological research shows that the brain's plasticity can be exploited both for improving its functioning and using it for learning effectively, efficiently, and with elegance. There is a definite interplay of genes, self-organization, environmental factors, such as instruction. Effective, efficient, appropriate early instruction, proper intervention, and well-thought-out and deftly delivered remediation can do wonders. 'The brain's early organization does not remain forever unchanged: experience refines and enriches it' ([Dehaene, 2020](#), p. 83).

Our neural synapses are constantly changing throughout our lives, and these changes reflect what we can learn and can apply. Brain plasticity is both vast and limited, and every change—the resulting learning—results from small adjustments in instruction and focused practice of that instruction. Brain plasticity is maximal only during a limited time interval, which is called the 'sensitive period.' The quality of initial instructions and early interventions, therefore, is important to the keys to the incidence of dyscalculia. Plasticity is an adjustment variable, fundamental for learning—changeable with limitations and constraints. It is the conjunction of a *fixed genome* and *unique quality experiences*.

We provide effective and efficient pedagogical suggestions for unique and quality experiences for initial classroom instruction, intervention, and remediation in the foundation concepts, skills, strategies, and procedures. The development and mastery of foundation concepts is the most effective way of minimizing the possibilities of developing effects of dyscalculia, acquired dyscalculia, and mathematics anxiety. Our goal is improving math instruction in the earlier grades—and making it work for all our students, with and without learning difficulties and disabilities in mathematics. Authors have worked on the one hand with children and adults on a continuum of abilities from children facing difficulty in learning in early years and across the grades to higher education on the other and children who are talented and gifted in mathematics. And we have seen children with disabilities who excel and are gifted and talented who fall behind

because of their needs were not met and they show the same symptoms as the learning disabled.

These methods work for slow learners as well as for gifted and talented individuals in mathematics. It is a much-needed change that schools should embrace. It does not call for a lot. We are not asking for increasing allotted instructional time, increasing fluency practices by mindless strategies, or setting up new or more math centers with tons of manipulatives. Our approach and methods require rethinking of the nature of the development of early and core mathematics skills and concepts and their trajectories—understanding the developmental milestones, examining current instructional practices, refocusing on the structure and balance of the components of a mathematical idea—language, concepts, strategies, skills, and procedures. Our experience tells us about this change in the early math classrooms can be managed and it doesn't have to mean reconfiguring everything, throwing out all your existing resources and your understanding of children and their needs.

Various environmental factors can all be associated with lower mathematics attainment, including socioeconomic and minority ethnic status, as well as gender, which should perhaps be considered a social rather than genetic factor in this context ([Royer and Walles, 2007](#)). All of these have an impact on children's preparation for mathematics learning. Therefore, we strongly solicit parental participation in their children's numeracy learning by suggesting what kinds of activities during early childhood and later can enhance the preparation and learning of mathematics.

Parents are a child's first teachers. Their knowledge about what daily activities contribute positively to a child's preparation for mathematics is important. Most of the prerequisite skills for mathematics learning are acquired through games and play activities. We have prepared a booklet that identifies games and toys and the related skills development in children. Parents can play these games with them and help develop the prerequisite skills necessary for learning.^{[17](#)}

Our suggestions are guided by research and what we have personally observed over a long span of time in mathematics education. Although it is difficult to assess the role of poor or inappropriate teaching, the fact that the introduction of a detailed new national primary school strategy for numeracy in the UK had only a minor and possibly nonsignificant effect on numeracy for the groups studied is indicative ([Gross et al., 2009](#)). The same can be said about mathematics performance of children in the US; the National Assessment of Educational Practice shows some changes in mathematics achievement, but many students are underperforming across the spectrum with some pockets of excellence, many of them attributable to environment/socio-cultural factors. It should also be noted that there are wide individual differences on even very simple tasks that depend relatively little on the quality of educational experience, such as comparison of the magnitude of two single-digit numbers or enumerating a small array of objects ([Reigosa-Crespo et al., 2012](#); [Wilson and Dehaene, 2007](#)).

Taken together, the evidence presented here suggests that factors specific to the domain of numbers and arithmetic make a major independent contribution to low arithmetic attainment throughout middle and high school in general, and one of the components in the condition of dyscalculia. This is supported by findings from studies that have found low attainment in learners matched for IQ and working memory. In a longitudinal study by Geary and colleagues, tests on understanding the numerosity of sets and on estimating the position of a number on a number line were two important predictors of low achievement in mathematics, affecting some 50% of the sample, and of mathematics learning disability, affecting approximately 7% of the sample ([Geary et al., 2009](#)). For example, we have observed in working with many children that knowing the location of a number in kindergarten and first grade, knowing that a number lies between what two 10s numbers, and how to use these 10s numbers is very important for mastery of arithmetic facts. If not properly dealt with, it appears to cause difficulty in locating fractions, decimals, and integers.

In a sample of 1,500 pairs of monozygotic (MZ) and 1,375 pairs of dizygotic (DZ) 7-year-old twins, Kovas and colleagues found that approximately 30% of the genetic variance was specific to mathematics ([Kovas et al., 2007](#)). In another genetic study, this time of first-degree relatives of dyslexic probands, it was found that numerical abilities constituted a separate factor ([Schulte-Körne et al., 2007](#)). In fact, recent reviews have proposed that developmental dyscalculia follows from a core deficit in this domain-specific capacity that is defined by number concept, place value, and number relationships ([Butterworth, 2005b](#); [Rubinsten and Henik, 2009](#); [Shalev et al. 2005](#); [Wilson and Dehaene, 2007](#)).

One obvious question arises: how do our numerical innate capacities relate to the learner's ability to acquire arithmetic? This question has become the focus of our work in general, as well as this work. A child is endowed with some evolutionary skills: perceptually recognizing a small collection of objects (*subitizing*), *comparing two collections of objects and numbers*, *observing the spatial display of objects being perceived* (spatial orientation and space organization), and *decomposition/re-composition of small collections of objects visually* (the ability to relate whole-to-part and part-to-whole). Children's numbersense development is the integration of evolutionary essential skills and culturally developed and learned skills. This interplay is manifested, strengthened, and made more complex by the cultural skill demands made on them by society. New cultural tools, rituals, and activities add to these skills and expectations.

Every culture transmits its essentials—values, history, language, accomplishments, expectations, and its formal membership—through its rituals, customs, and records. For example, exposure to its literature and artifacts is a strong means of this acculturation. The means through which a child accesses them is the language, including writing, reading, and unique symbols, including numerical symbols and their manipulations. In turn, for example, reading also prepares the child to receive the advantages of and contributes to this acculturation process. This skill allows the child to become part of the culture. Thus, reading is a formally organized process, and the type of reading and level of its mastery is a culturally determined

skill and activity. Modern societies expect children to read not only for pragmatic, social, and intellectual purposes but also for personal pleasure, creativity, and satisfaction. Children first learn to read and then they read to learn.

Numeracy skills and the type and level of fluency are also expected cultural activities, and its parameters are largely culturally determined. Mastery of numeracy skills cuts across cultural bounds and is truly a universal activity. In modern societies, a child is expected to be numerate with the expectations that include, but are not limited to, understanding the world quantitatively, spatially, and solving problems using the tools/knowledge unique to the child's culture (such as language, rituals, and traditions) and their level of achievement in mathematics.

Mathematics is a collection of concepts, procedures, and skills, but mathematics is also a language of quantity and a language for dealing with quantity and space. As a language, it uses numbers, symbols and objects—real, symbolic, and imaginary—following the rules of linguistic, quantitative, and spatial reasoning. In some respects, mathematics language is acquired in the same way as any language—through consistent formal and informal exposure to experiences and interactions with others in language-rich environments. Proper shaping of these experiences through language, rules, codes, symbols, and structures like learning other languages and reading helps children become fluent in this unique and powerful language.

At the same time, we observe that every culture supports *its* language more than mathematics. It is natural. For example, there are more opportunities for learning and using it. Most parents read to their young children to familiarize them with the patterns of sounds and the structure of the native language, that includes, among other functions—the exposure to its intonation, pronunciation, affect, symbols, shapes, orthographic representations of letters and sounds, names of objects and their purposes, words for action and emotion, and the general usage of language. For the child, language becomes the means of relating with others. Exposure to his native language—formal and informal—shows the child not only the

patterns in sound and linguistic symbols but it also helps create lexical entries (language containers) in the child's mind for concepts of knowledge in the cognitive, affective, and psycho-motoric domains. These experiences also help shape the physical apparatus of sound perception, reception, formation, production, and the intellectual systems of meaning-making from these sounds. These early language experiences are immensely useful for the child's cognitive development, not just linguistically but also neuro-psychologically, intellectually, socially, and emotionally.

Similarly, these experiences also shape the developments that affect and facilitate the perceptions, language, conceptual structures, and routines of numerical development and its usage. However, there is still need for indirect experiences that relate to quantity and space, but also direct experiences with and about quantity and space so they also perceive these elements as they perceive color, sound and emotions.

In technologically and data-rich societies of today, parents should also introduce children early to the language and symbols of quantity and space. This will prepare them to understand and negotiate the quantitative and spatial attributes and aspects of actions, space and objects—creating *lexical entries* (number words—*language containers* for *quantitative* and *spatial experiences*) and *symbols* for numbers and actions, representations, measurement, and shapes in the world around them. This includes the early development of *quantification* and *spatial sense*:

- What attributes of this object to quantify and how to quantify?
- Is this attribute or experience quantifiable?

And later, when the idea of exchange of quantity occurs naturally and is planned in a culture, the child begins to perceive:

- What and how to measure, and where and how to use quantities relating to measurements of multitudes of attributes.

Some quantitative experiences are unique for a child and adult.^{[18](#)} The related experiences with quantity and space expose the child to the

awareness, recognition, and use of spatial orientation and space organization and its importance to quantity and its usage. As children grow, they begin to explore the relationships between quantity and space—the discrete and continuous nature of number and their integration, in creating, relating, and forming numbers/numerals from the exploration of the space around them gives opportunity to generate number, data, and their manipulation. Use of number and space representation and measurements are examples of such integrations.

Family members and other adults often provide and choreograph environmental conditions that promote language development and interest in reading. When children are read to early and reared in homes with reading materials and formal/informal and rich use of language takes place, they begin to read early and with better comprehension. Activities that foster the development of decoding/encoding, phonemic awareness, chunking, and blending are common in some families, though less frequent in others. Some families tend to offer direct formal instruction in language quite early; they offer more reading materials—books, magazines, better television program viewing, and day-to-day discussion of topics with richer and varied vocabulary and accompanying questioning, reasoning, and discussions. Children in such homes hear their parents and siblings using rich language both formally and informally. Such literacy-rich environments have a profound impact on their preparation for formal language learning.

Similarly, children from homes where there is early and frequent use of quantity and quantitative reasoning in daily discourse develop an early and better sense of number, space, measurement, and quantitative problem-solving. Quantitatively (number)-rich homes and classrooms promote numberness and numbersense and their expression—orally and in writing.¹⁹ Therefore, teachers teaching mathematics (from K through to 8th grade, at least) should have mathematics-rich classrooms, e.g., display of a *Number Line* from 0 to beyond 125 at the eye level of children, a *Vertical Line* from -15 to 30, a *Hundreds Chart* in the usual format, a *Modified Hundreds Chart*, a *Monthly Calendar*, multiple types of *clocks* and *thermometers*

(analog, digital), *number formation chart*—with directional arrows, digit forms from different languages, a Place Value chart with several cycles, *Visual Cluster cards*, *dice*, *die*, *diagrams* and pictures of commonly found objects and geometrical objects in the environment, *Cuisenaire rods*, *Base-Ten blocks*, *Invicta Balance*, a variety of counting objects, *abaca abaci*, *Ten- and Five-Frames*, a rich collection of board games and toys, and displays of student work: their conjectures, definitions, exemplary papers, and work samples but do not overwhelm or confuse with too much information.

The goals of literacy programs in schools are ‘learning to read’ by third grade and then ‘reading to learn’—social studies, fiction, sciences, comic books, etc. Similarly, children should master numeracy skills by the end of fourth grade and then use the numeracy skills to learn models and methods of mathematics—quantitative, numerical, algebraic, spatial/geometric, probabilistic, statistical, functional, trigonometric, etc., for learning future mathematics and other subjects efficiently and effectively.

Reading is a complex process, and to achieve mastery in it (reading fluently with comprehension) calls for the involvement of a variety of brain components, centers, and systems. Both localized and global systems perform and integrate tasks such as recognizing language symbols—visual and aural, discerning and analyzing visual and sound patterns, perceiving and organizing sequences of symbols and sounds, spatial arrangements—source of speech or location of the symbol, and many other subtle and not-so-subtle activities and clues—both verbal and nonverbal. Fluent performance in these tasks results in understanding and producing language and reading with meaning, affect, and impact.

Some of the same processes and skills needed in learning to read and in the reading process are also called upon in aspects of conceptualizing number, acquiring number and spatial sense, mastering arithmetic facts, performing arithmetic operations, and thinking mathematically—quantitatively, spatially, and symbolically. A breakdown and/or deficits in any of these skills may affect mathematics learning also. For example, when a child is assigning a number (orally and orthographically) to a

quantity (collection), similar areas of action related to language, visuospatial, sequencing, and quantity light up on MRI images. It is not surprising, therefore, that almost 40% of dyslexics also exhibit symptoms of dyscalculia or acquired dyscalculia.

About 6–8% of children of the general population exhibit symptoms of dyslexia. However, research in the science of reading has resulted in a better understanding of this complex skill and has resulted in effective methods of teaching reading. This has equipped educators to help children learn to read. Early diagnosis and proper interventions of reading instruction are keys to overcoming these situations. The very small number of children who cannot read are those who are not afforded the effective methods and opportunities or have extreme cases of neurological, neuropsychological, and/or cognitive deficits.

Some children have low achievement in mathematics because of dyscalculia (about 6–8%). However, many more have trouble in learning due to poor/inappropriate instruction. For example, in many schools, the lowest-performing children are assigned to the least-trained personnel in mathematics. Screening and diagnosis are performed late. Ways and means must be found to train all elementary teachers and paraprofessionals to understand this problem and train in effective instruction for numeracy so that almost all children, by the end of fourth grade, are able to reach a satisfactory level of mastery in numeracy.

Problems in acquiring literacy and numeracy skills have common as well as different origins. There are enough parallels and common patterns of development in literacy and numeracy. English is an alphanumeric language, and so is mathematics. The acts of creating, representing, using, and applying mathematical symbols and sounds emerge and grow along with the child's native language structures, even in non-alphabetic languages, such as ideographic languages (e.g., Chinese, Japanese, Burmese, etc.) where characters, or ideographs (more precisely, logograms), represent entire ideas, concepts, words, or morphemes (meaningful units of language). However, there are important differences, and some unique abilities and systems are also needed to learn number and space and their

applications. There are people who can read well and have poor numeracy skills, but there are very few who are numerate but cannot learn to read.

In teaching numeracy skills, therefore, we need to explore the similarities and differences in the conceptualization of and skill development in number *and* language. Largely, instruction to develop numeracy and literacy, particularly reading, in young children can go together because the acquisition of one enhances the other.

We should examine and understand the commonalities and differences in these processes to design instructional/remedial activities for learning number and numeracy so that they are accessible to all children. We need to understand the:

- *Primary concepts and component skills: natural evolutionary* (subitizing and spatial sense) and *culture-specific* (language) and *formal* (symbols) involved in the development of number concept, number sense, and numeracy.
- Role of *prerequisite skills* and cognitive, neuropsychological, and psychological preparation in learning first numeracy and then mathematics, and
- Optimal methods/models, activities, and tools for connecting the content skill or concept with the child who has a particular learning profile and potential.

For example, the National Research Council's Committee on Early Childhood Mathematics released a summary of research and recommendations to address the crucial national need to prepare all children to do well in kindergarten. This report, *Mathematics learning in early childhood: Paths toward excellence and equity* ([National Research Council, 2009](#)), provides the major research basis for *Math Expressions Early Learning Resources*. Some of its major recommendations are summarized and exemplified in *Focus in Prekindergarten* (2010), published by the National Council of Teachers of Mathematics and the National Association for the Education of Young Children. The information is a good starting

point for numeracy instruction. According to this report, early childhood programs have short-changed children by underestimating their abilities and potential for numeracy by not providing quality and challenging programs. This is particularly so in the current situation in the USA.

Numeracy is the way to participate fully in today's world. For example, for success in any profession and career, in a technological society, one needs to be both *literate* and *numerate*. Numerates have a distinct advantage in the modern world—numeracy is the new literacy. Being numerate means having a keen sense of quantity and space, flexibility with the use of numbers and number relationships, and having the ability to use number skills to cope with the quantitative demands of everyday life. It involves, for instance, observing numerical trends in information—in graphs, charts, tables, or in reference to percentage increase or decrease. This requires mastery of numeracy and critical thinking—mastery of number concept, numbersense, and computational fluency (ability to execute four basic operations with whole numbers, in the standard and alternative forms, correctly, consistently, efficiently, with understanding). This facility in applying quantitative reasoning and the resultant confidence in numbersense strongly predicts later success in STEM fields.

In most cultures, most children, as part of the acculturation/socialization process, develop a basic sense of number and arithmetic (rudimentary number relations and procedures) without explicit instruction. A fully fledged member of the modern world requires more than that. Modern societies emphasize the mastery of arithmetic, reading, and writing as the focus of early school experience. The goal of teaching mathematics to young children is to develop a sound foundation in numbersense and in the mathematical way of thinking to prepare not only for arithmetic but also for higher-order mathematics. This book is a focused attempt to define and provide an important direction to this endeavor for all children, *with and without* learning disabilities. The chapters are organized so that teachers, researchers, interested parents, school administrators, and policy makers can get a comprehensive understanding of the mathematics content, teaching strategies, and learning methods that can help teachers teach and

children master foundational skills in numeracy to prepare them for higher mathematics.

The chapters emphasize its importance not only for numeracy, but they define the contours and content, at least during the early primary years. Each chapter deals with the fundamental concepts about numeracy and how they affect the nature and incidence of dyscalculia and other learning problems in mathematics. We examine the important issues:

- a. What constitutes numeracy and its importance not only to school mathematics, but also to higher mathematics?
- b. Why do individuals experience difficulty in mastering numeracy: Number Concept, number relationships (arithmetic fact), and the outcome of numerical operations?
- c. Why do many children have difficulty mastering numeracy?
- d. What role does instruction play in children's difficulties in numeracy?
And
- e. How do deficits in numeracy skills affect individuals' lives in multiple ways?

In each chapter, we discuss the multiple perspectives that guide our understanding of learning problems in mathematics, particularly dyscalculia and acquired dyscalculia.

In [Chapter One](#): *Curricular Goals for Numeracy for Primary Years*, we focus on the non-negotiable skills during the early primary years, particularly related to *Additive Reasoning*. Drawing upon more than a century of collective teaching experience, several key principles regarding the foundational skills in school mathematics have emerged. These principles inform our understanding of what constitutes essential mathematical skills, their significance, and the ways in which children can acquire them effectively. In this chapter, the emphasis is on:

The Importance of Foundational Skills

It is our conviction that if children master these fundamental skills early on, they will build a robust base that enables every child to pursue and achieve any level of mathematics required for their aspirations—be it for vocational, professional, or personal fulfillment. Without these foundational skills, many children pass through school without truly appreciating the power, beauty, reach, and breadth of mathematics.

Non-Negotiable Skills for Mathematical Success

There are certain indispensable skills that must be identified, taught, and mastered by students. When these are in place, the learning of advanced mathematical concepts becomes significantly more accessible and less daunting.

Teacher Preparation and Professional Development

To ensure children achieve these goals, it is equally important to consider the training, attitudes, and competencies required of from teachers. This chapter highlights the need to redesign teacher training programs and curriculum content so that educators are well-equipped to nurture these foundational mathematical skills in their students.

Research-Informed Content Selection

The structure and organization of this chapter are guided by interdisciplinary research across the fields of education, linguistics, neuropsychology, information theory, and human development. These diverse perspectives help shape a curriculum that is both comprehensive and effective in fostering early numeracy and its mastery.

In [*Chapter Two: Mathematics Difficulties and Mathematics Anxiety*](#), we explore why many learners develop anxiety around mathematics and how this affects their progress in professional and personal lives. It explains ideas such as *number anxiety*, fear of being judged, and the difference between *state anxiety* in specific math situations and the broader, personality-linked *trait anxiety* ([Chinn, 2011](#)). Drawing on research from

psychology, neurology, and education, it shows how early experiences, developmental gaps, and classroom expectations—especially emphasis on speed and poor assessment methods—shape confidence. The chapter also discusses how memory load, teaching practices, and social beliefs contribute to difficulties and result in *math anxiety*. It concludes by emphasizing early support in number concept and number operations through effective, elegant, and efficient instruction so that students experience success early and steadily, building competence, motivation, and reducing anxiety.

In [*Chapter Three: The Role of Language in Mathematics Learning and Language Related Mathematics Difficulties*](#), we explain that language is central to mathematical understanding, as children must translate between their native language and the symbolic language of mathematics. It highlights mathematics as a second language and contends that language and words are the containers that hold the conceptual and procedural ideas in the brain centers. Proper teaching transforms concrete ideas into abstract forms through these *language scripts*—stepwise instructional and conceptual actions. The chapter draws distinction between different types of words used in mathematics: words that share meanings with the native language, some that partially match, while some have entirely abstract meanings, requiring careful bridging. Effective instruction involving rich dialogue, and precise vocabulary strengthen conceptualization and support confident mathematical learning. It also carefully examines the structure of mathematics language, its syntax, semantics, and other nuances that are particular to mathematics and other linguistic rules and integrates them into the teaching of mathematics. It gives a structure to understand mathematical language and its role in pedagogy of mathematics.

[*Chapter Four: The CPVA/S Model of Mathematics Instruction*](#) is the focus of the book. This chapter is a fundamental breakthrough in mathematics pedagogy, and it delves deeply into the cognitive, behavioral, mathematical, developmental, and linguistic aspects of teaching numeracy. It is a fundamental shift and advancement from the traditional *Concrete Representational/Pictorial Abstract* (CRA or CPA) strategy for teaching

numeracy and for school mathematics in general and inserts the crucial and yet missing link of *visualization*: the ability to manipulate the idea in the mind's eye to ease the teaching and learning of the concept. Every concept starts with an intuitive idea about it, and concrete modeling makes it real. Such a schema then must transition to its pictorial representations, which are both iconic and non-iconic, and the child must develop visualization of that and take it further to its abstract forms finally. These transitions are helped by an active and supportive language in the form of '*scripts*' which moves these ideas through six levels of knowing. In the traditional approach, it is believed that the natural progression from pictorial to abstract happens automatically and easily. Many children may not move to abstract or symbolic representation easily. This approach makes children heavily dependent on the concrete or pictorial representation, and many children do not transition to the abstract naturally and easily. It has implications for all of school mathematics and makes a strong appeal to classroom and special educators to actively facilitate these transitions otherwise children depend heavily on only physical or visual materials for teaching and learning and face challenges in progressing to abstraction because of missing visualization and poor development of the scripts derived at concrete, pictorial, and visualization levels.

Chapter Five: *Nature of Tools in School Mathematics* provides the rationale and importance of effective, efficient, and elegant tools for instruction and learning. The discussion is not just why and what tools to use, but on how to use these tools to make mathematics accessible, meaningful, and efficient for all learners. Tools and their usage define teaching and the teacher. Effective tools can be categorized into content-specific, cognitive, linguistic, and physical instructional tools. Together and through proper usage they help students move beyond inefficient counting toward flexible, conceptually driven strategies that are generalizable and can be applied effectively. Grounded in cognitive psychology, neurosciences, and educational theory, effective tools support number relationships, decomposition and re-composition, and properties of operations and numbers, such as commutativity and associativity.

A key function of these tools is to generate mathematical language, help learners build robust and comprehensive conceptual schemas, and promote effective, efficient, and elegant strategies and procedures. The chapter stresses the importance of using tools in a hierarchical, developmental sequence, progressing from discrete and unorganized materials to more structured, continuous tools that build coherent understanding.

Within this progression, the chapter highlights two organized and conceptually powerful tools: the Special Number Rods (Mahesh Sharma and Sanjay Raghav) and the Visual Cluster Cards and Sight Fact cards (Sharma). These tools strengthen number sense, support efficient strategies, and deepen conceptual understanding. When used purposefully and in sequence, they help all learners—including gifted students and those with dyscalculia—by providing concrete, visual, linguistic, and cognitive scaffolds. However, we should be aware that over-reliance on poor tools or misuse of effective tools can weaken understanding. Thoughtful integration of appropriate instructional tools is therefore essential for developing lasting mathematical competence.

We strongly believe most numeracy difficulties, including dyscalculia and the resultant mathematics anxiety, emanate from poor introduction, mastery, and use of number concept. Number Concept is the first mathematics developmental milestone, it must be taught properly—effectively, efficiently, and easily. Therefore, [*Chapter Six: Mastering the Concept of Number*](#) provides rationale and strategies for teaching number concepts effectively and easily.

There are two fundamental goals in most countries in their foundational years, and these are Mastery of Numeracy and Literacy. This chapter comprehensively explores number-concept and its parallels and differences to reading. It goes on to explain the trajectory of learning number-concept from both historical and pedagogical perspectives. It describes in detail teaching number-concept using two important tools, which are *Visual Cluster Number cards* and *Number rods*. This chapter also compares traditional teaching of numbers through counting-based strategies and points out the limitations of such an approach. It suggests a more efficient,

effective, and elegant strategy of using decomposition and re-composition and extends the idea of *sight facts* which is a fundamental breakthrough in the teachings of numbers. This approach provides a sincere/authentic solution to numeracy challenges for all types of learners including children with special needs and both *acquired* and *innate/developmental dyscalculia*.

Many children begin to have trouble in mathematics classes when they are asked to memorize (rote learn) addition facts using counting of collections. Additive Reasoning is not only the second developmental milestone in mathematics, but the foundation of all future numeracy operations. In [*Chapter Seven: Understanding Additive Reasoning: Its Language, Concept, Procedures: Addition \(Part One and Two\)*](#), our focus is on teaching children how to derive and use addition facts easily, master them efficiently, and apply them appropriately.

Number concept is a higher level of abstraction than counting. What begins as counting abstracts into an abstract entity called a number, which is free from material, physical properties, and specific situations. The addition concept extends the number concept further and reaches an even higher level of abstraction than numbers. In this chapter, we discuss the trajectory of this development and the fundamental skills involved in understanding and performing addition operations. Addition is the first step in the mastery of the next milestone after the number concept, called Additive Reasoning. The suggestions and practices discussed in the chapter divert from traditional practices of teaching addition directly through standard procedures based on counting (counting up, counting all, etc.) to more conceptually aligned strategies that build on *subitizing*, *decomposition and re-composition*, *sight facts*, and use fundamental understanding of place value, such as making ten, next tens, and so on.

[*Chapter Eight: Understanding Additive Reasoning: Its Language, Concept, Procedures: Subtraction*](#) builds on the earlier chapter and examines the concept of subtraction from multiple perspectives. It also explains why subtraction is conceptually complex for many children, examining linguistic, cognitive, procedural, and pedagogical factors. It highlights common misconceptions and traces their roots through anecdotal

records using Sharma's CPVA/S framework ([Sharma, 2019](#)). The chapter distinguishes between continuous and discrete models of subtraction and reviews classroom tools such as Visual Cluster Number cards, Sight Fact cards, Number rods, empty number lines, and standard algorithms, emphasizing part-whole relationships and the influence of language and executive function. A Piagetian perspective and the CPVA/S model underscore the need for integrated learning across conceptual, linguistic, procedural, and social dimensions. The chapter outlines the developmental trajectory of subtraction, its connection to addition, and why students struggle with word problems related to subtraction. Teachers and parents will receive practical strategies for addressing gaps, while noting that effective implementation requires deep professional understanding and commitment.

[Chapter Nine](#): *Understanding Procedures of Additive Reasoning* focuses on extending the language, concepts, and strategies of Additive Reasoning to operations of addition and subtraction on multi-digit numbers effectively and easily. In our schools, mathematics instruction has been reduced in most cases into procedural routines, without much understanding of the language and conceptual schemas behind them. It is taught as a collection of shortcuts, mnemonics and gimmicks, without the language, concepts, strategies, and procedures. The purpose of mathematics is much more than just acquiring a collection of facts, mnemonics and procedures. Fundamentally, it is to develop a way of thinking, robust numbersense, and skilled numeracy skills.

Mathematics is learned in four ways: *concretely*, *discerning patterns*, by *analogies*, and through *reason*—deductive and inductive. In deriving the addition and subtraction procedures, we emphasize all the components of addition and subtraction: the language, concepts, and strategies. When we use a mathematics way of thinking, children acquire skills that are transferable to learning other disciplines. If CPVA/S strategies are used, such teaching develops and uses all the fundamental *executive functions* of (a) working memory, (b) organization, (c) focus and attention, and (d) flexibility of thought. In this chapter, we integrate the *four conceptual*

components of a mathematical idea: Language, conceptual schemas, strategies, and procedures. We derive, in both in addition and subtraction, multiple procedures—from alternative to standard, so that children are well-prepared for applications, problem solving, including word problems with robust numbersense, mental math, and the ability to use strategies according to the needs of the problem.

Principles, practices, and strategies described in these chapters have the potential of helping develop skills and competence in children that will assure deep, strong, and effective mastery of foundation skills that will prepare them well for future mathematics and mathematics related disciplines.

Notes

1. About 40% of dyslexics may show symptoms of dyscalculia, as some of the same brain areas that are called upon for learning numeracy/mathematics are used for acquiring reading skills. [!\[\]\(31b03e46ee8a80a1f1467b8c03bd76e8_img.jpg\)](#)
2. Anxiety is a mental state characterized by an intense sense of tension, worry, or apprehension, relative to something adverse that might happen in the future. Researchers differentiate aspects of anxiety into *state* and *trait*, respectively defined as a more transient reaction to an adverse situation, and as a more stable personality attribute in experiencing events. Trait and state anxiety are psychological responses akin to nervousness or worry. For more, see the blog on mathematics anxiety by Sharma and Chinn (www.wordpress.mathlanguage.org). [!\[\]\(7d9665ff04f9d2270c38081c6215a724_img.jpg\)](#)
3. In a typical elementary school (grades K through second), the amount of time devoted on reading and language arts instruction is about 90 minutes per day, whereas on numeracy it is less than an hour on average per day. Most parents read to their children on almost a daily basis. There is hardly any meaningful numeracy work at home. [!\[\]\(7cea648fec4dfc1e99934873e9173b69_img.jpg\)](#)
4. In a classroom session in a fourth-grade class, I asked the class: ‘What is 8×7 ?’ I saw four hands going up immediately. Others were trying

to get the answer by counting. Finally, a few more hands went up. I asked them for their strategies for arriving at the answer. Most of them who knew it fluently answered: ‘I just know it.’ Others said: ‘I just counted by 8 or 7.’ I asked one who correctly answered: ‘How many groups of 8 or 7 are in 56?’ The answer was: ‘No clue!’ [!\[\]\(a3ea015cc5581cad732d1eb81613fe7b_img.jpg\)](#)

5. The methods that emphasize only the *counting* of objects (these include: random objects, organized as number lines, fingers, and other such instructional tools). These are introductory tools, but they are neither effective nor efficient. [!\[\]\(1207edb9a08751d3d55970560645ed23_img.jpg\)](#)
6. Unfortunately, the same strategies of counting forward and backward are applied in teaching the concepts such as integers and their operations. These methods limit students’ preparation for and performance on proportional reasoning, pre-algebraic concepts, and then algebra. Counting backwards is usually more of a challenge. [!\[\]\(d7a34a706cfa4ef37c62a369101e1b36_img.jpg\)](#)
7. For example, instructional and remedial methods and strategies that are based on the child’s *innate propensities* of numbersense—subitizing, visual clustering, and approximate number sense (ANS) and spatial sense—organization, sequencing, patterns, visualization, and symmetry, and *tools of the culture of the child*—language (alphabet of mathematics—digits and counting), decomposition/recomposition (akin to phonemic awareness is essential for learning to read), properties of numbers (e.g., sight facts of first ten numbers) and operations (e.g., commutative and associative properties of addition and multiplication, distributive property of multiplication over addition and subtraction), place value (that the place determines the value of a number as the placement of phonemes, syllables to reading), and the instructional tools and approaches are effective, efficient, and elegant methods; counting-based strategies are not. [!\[\]\(7325769475e8f4bf67f57a0cbebc8ab9_img.jpg\)](#)
8. Unfortunately, curricula such as Common Core State Standards (CCSS-M) have become a political football due to poor instruction not because of their content. (CCSS-M are content standards and emphasize the mastery of important and non-negotiable content with a focus at each age and grade level with an emphasis on the language,

concepts, and procedures of mathematics appropriate to that age and grade level. They do not propose a particular pedagogy or instructional strategies. National Council of Teachers of Mathematics, on the other hand, have proposed and promoted a set of teaching and learning principles in conjunction with the CCSS-M Standards.) [!\[\]\(50ba758255c5d7cec2761495a31c7c80_img.jpg\)](#)

9. *Intra-mathematical* application means a child can apply mastery in one concept to learning a new concept in mathematics; for example, mastery of addition and multiplication facts by decomposition/recomposition (e.g., $8 + 7 = 8 + 2 + 5 = 10 + 5 = 15$ to $8 \times 7 = 8 \times 5 + 8 \times 2 = 40 + 16 = 56$). *Interdisciplinary* application means a child's mastery of arithmetic facts, concepts, and procedures to other subject areas as science, word problems, and projects. *Extracurricular applications* mean projects, real life problem solving in day-to-day living. [!\[\]\(467d80e979964f7f8c752fb22248b5b7_img.jpg\)](#)
10. Comprehension in reading and understanding in the context of numeracy are comparable skills. They require same kind of instruction—questioning, feedback (i.e., empathetic conversation), and practice. They happen in the same areas of the brain and require the same level of use executive functions. For details, see: Literacy and Numeracy: Comprehension and Understanding by [Sharma, 2008](#). [!\[\]\(b71552d33dbf62adf5e5199a70ee02bf_img.jpg\)](#)
11. *Script* is the S in the *CPVA/S* model. It is a powerful means to help the child Visualize the conceptual schema Abstracted from the Concrete and Pictorial levels' of learning and translated into Abstract level. The script is a *storage-rich* and *retrieval-rich* process of transferring information from short-term and working memories to long-term memory and internalize in a richer way and then retrieve and reproduce as and when needed. [!\[\]\(03134b765d1473836ff001925b1b0550_img.jpg\)](#)
12. More on this in the future chapters. [!\[\]\(aed6947356668967079310026052edc0_img.jpg\)](#)
13. The two senior authors have had extensive interaction with mathematics and special educators, students, classroom teachers, and the general public internationally, and this phenomenon of disliking mathematics is a pandemic. [!\[\]\(e61aeb0d9066d5d9e54d9b655f50da3d_img.jpg\)](#)

14. Difficulties in other areas, such as reading, language, handwriting, general anxiety, and pre-requisite skills bring their limitations to learning numeracy and mathematics. ↵
15. More on this in the chapter on Mathematics Anxiety. ↵
16. Emphasis is ours. ↵
17. See [Sharma \(2008\)](#): *Games and Their Uses and Prerequisite Skills and Mathematics Learning*. ↵
18. The senior author was born in a village in India. His early childhood was spent in an extended family setup of aunts and uncles and many nephews, where the patriarch of the family, his grandfather, was his *locus of learning*, a person who introduced him to all formal learning. At the age of three, there was a ceremony called *Vidya-armbh* (commencement of formal learning). It is a big occasion. The whole family gathers—there were several dozen members in this big clan. His grandfather had prepared a vedi (a platform) and a brand-new silver plate's bottom was covered with a layer of about $\frac{1}{10}$ of an inch of honey. After the usual part of the ceremony invoking the God of wisdom, Lord Ganesh and others from the pantheon of gods and goddesses, he asked the author to give him his forefinger to his hand. He asked if the author would like to write his name on the surface of the plate. First, the author hesitated, but then he let his grandfather use his finger to guide him to write the first letter of his name on the surface of honey-covered plate. By the time he finished the first letter of his name, the finger was covered in honey. The author asked his grandfather whether he could lick his finger? His grandfather said: 'Of course!' The author licked it. He still remembers the taste of that honey. Then, the author asked his grandfather if that was his full name. He said: 'No! Your *name is much longer*. It has several letters in it. Do you want to write *the whole name*?' Of course, the author wrote his name on the plate several times till the plate was completely clean of honey. Now, he wanted to write his name every day and then everybody else's name. That was the beginning of his education, and he has a very sweet memory of that commencement. There were

several other ceremonies where he thought learning was a wonderful thing. They included not just language but quantity and space, such as helping prepare a vedi organizing elements of alpana (a drawing with flour, colors, and rice around the vedi). His friend, a carpenter's son, introduced his toys that his father made with intricate designs, and both requested his father to design toys that they had described at the ages of 3 and 4 years. These experiences are unique in themselves as an introduction to the world of number, space, and measurement. They measured by their hands and their body parts. ↵

19. More suggestions in later chapters. ↵

1 The Curricular Goals for Numeracy in Primary Years

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What rules the world of science?

Not atoms. Not algorithms.

Mathematics does.

Carl Frederich Gauss, often called the “Prince of Mathematicians,” once said: “Mathematics is the queen of the sciences, and number theory is the queen of mathematics.”

This quote captures a powerful truth—mathematics is not just a subject; it is the language through which we understand the universe, let alone our individual worlds. Number is the foundation of mathematics. And mathematics, though most times abstract, forms the core of the

- cryptography that secures our data
- algorithms that drive artificial intelligence
- patterns that structure digital communication and the study of any phenomenon that happens repeatedly
- calculations that govern all transactions—real and imaginary.

Mathematics does not support science—it leads it. The foundations of this great language, a collection of great thinking and practical tools, and the basis of all scientific methods and skills are formed in the first three to ten years of education in schools—in learning number, number relationships, and numerical operations.

Numeracy takes an important role in the historical context of mathematics education in schools all over the world; many of the problems that afflict students in learning mathematics, at upper elementary, middle, and even high school, and also at the higher education level, are caused by deficiencies in mastering the basic language, concepts, and operations of arithmetic—patterns and operations on numbers.

Early numeracy skills directly impact student achievement across all subjects. The research is clear: students who develop strong mathematical foundations in K – 2 – 4 show consistently better academic outcomes throughout their education. Early numeracy, as a cornerstone of long-term academic success, is perfect for parents, teachers, and administrators to focus upon if they are truly committed to academic progress.

A. Early Numeracy Matters: What Children, Teachers, Education Leaders, and Parents Need to Know About Foundational Numeracy

Developing strong numeracy skills early on creates lasting benefits for students in:

- Improved problem-solving skills: Early math education effectively fosters critical thinking and logic.
- Academic growth: A strong foundation in numeracy leads to higher achievement in core academic areas.
- Closing learning gaps: Identifying and addressing numeracy gaps from the start ensures all students stay on track.

Strong number concept and additive reasoning play the most important role in the effort to help children master other arithmetic facts, concepts, and procedures effectively. Later success in higher-order concepts and their foundation is established on the mastery of number concept and additive reasoning. For example, proportional reasoning is based on the language, conceptual schemas, and procedures of multiplicative reasoning. And

multiplicative reasoning is based on additive reasoning. Mathematics is developmental.

In the following chapters, we show why strengthening early mathematics instruction for K – 2 students is essential for higher mathematics achievement, for lowering the incidence of learning problems in mathematics, and their outcomes—symptoms and characteristics of mathematics anxiety and its impact. We also focus on how teachers can better support young learners in building these critical foundations from day one effectively, efficiently, and elegantly.

The key objective of the mathematics curriculum—instruction and achievement, from a quantitative perspective from K through second grade should be to acquire, master, and retain *additive reasoning*: number concept, number relationships, place value, addition and subtraction procedures, and the inverse relationship between addition and subtraction. Achieving competence in additive reasoning should be the focus of the curriculum for K through second grade to avoid the development of acquired dyscalculia and in many cases, avoiding the possible impact of dyscalculia. And, in other cases, to mitigate the effects of dyscalculia.

Then the key objective of the mathematics curriculum—instruction and achievement, for third and fourth grade, should be acquiring and mastering *multiplicative reasoning*: extending number concept and additive reasoning, and place value to multiplication and division concepts, their various conceptual models and schemas, and related procedures. Mastery of multiplicative reasoning means, not only that they understand the language, concepts, and procedures of multiplication and division, but that they also understand the inverse relationship between multiplication and division. When students can successfully convert a multiplication relationship into a division relationship and vice versa, they have acquired multiplicative reasoning. *By the end of fourth grade, they should have mastered the language, concepts, and procedures of numeracy.*

B. Numeracy and Literacy: Numeracy is the New Literacy

It is certain that mankind made some sense of quantity in their minds, at least some *estimations*, considered the role of *shapes and figures* in the environment, and *discerned the patterns* in objects, events, and actions in their lives, first for survival and then for the benefit of their clan, long before the invention of reading and writing. These skills have been important foundations for early numeracy and, later, also for mathematics.

Every culture transmits its values, history, accomplishments, expectations, and the rules of membership in it—through unique rituals, customs, and records—literature, architecture, and artifacts. The means of accessing them, in the recent past, have been language, writing, and reading. In modern times, reading prepares its members to receive the advantages of and contributes to this acculturation process. This allows the child to become an integral part of their culture. Thus, the development of reading for every child is an important part of the acculturation process. The type of reading and the level of its mastery is culturally determined. Modern societies expect children to read not only for pragmatic and intellectual purposes but also for personal pleasure and satisfaction.

Children First Learn to Read and Then They Read to Learn

Every culture has supported the acquisition and development of language more than mathematics. Parents begin to communicate to their child from birth—through sounds, gestures, songs, mime, and dance. The observant young mind of the child returns the favor. In literate societies, for example, most parents read to their young children to familiarize them in the patterns of sounds and the structure of their language—its intonation, pronunciation, affect, symbols, shapes, orthographic representations of letters and their sounds, names of objects and their purposes, words for action and emotion, and the general usage of language in expressing and receiving ideas.

Formal mathematics learning and teaching, however, evolved much later. Now, numeracy skills and the nature of its content and the level of its mastery are expected cultural activities in numerate societies. Mastery of numeracy skills cuts across cultural bounds and is truly a universal activity.

Today, a person is expected to be numerate—the ability to understand the world quantitatively, spatially, and solve problems involving quantity and space using the tools unique to one's culture and mathematics.

Modern societies expect their children to be *literate* and *numerate*, and maybe in the near future use computing devices—as coders and posers of problems to AI devices.

To be *numerate* means having the ability to respond to and operate flexibly on problems involving quantity and efficiently manipulate numbers mentally, on paper, and with a calculating device (when the use of a device is appropriate). It means the ability to comprehensively understand and manipulate number, number relationships, and operations with proficiency, confidence, and competence. For example, one should understand the use of number in measurement—which integrates quantitative and spatial reasoning. It is also the ability to process, communicate, and interpret numerical information in a variety of ways and in diverse contexts. It is the understanding of quantitative concepts and the related procedure from multiple perspectives and forms and using multiple strategies.¹ To achieve these objectives, students should have the mastery of (a) Number Concept and (b) Numbersense. Then and only then is the development and competent display of numeracy possible.

Since the Industrial Age, literacy has been the key skill for gaining social status, advancement in careers, and in fulfilling one's potential and a requirement for responsible stations in a society. In most cultures, almost everyone knows what literacy is—the ability to read fluently with comprehension and to write with intent and the need of the function. Literacy is still important and is valued, but in the information- and knowledge-rich technological society *numeracy is the new literacy*. Therefore, the purpose of a comprehensive elementary education is for children to become *literate* and *numerate*.

In many ways, their development is parallel in nature. Many of the common prerequisite skills are required and involve similar focal areas of the brain. Many of the problems encountered and successes achieved by children becoming literate and numerate are also similar. In every culture,

there are many who are not able to achieve the status of a literate and numerate person. As a result, they suffer in multiple ways. As late as the 1980s and 1990s, they were called remedial, illiterate, intellectually deficient—these are the stigmas that used to define the educationally underprepared. Now, a few more qualifiers have been added. They are identified as illiterate, innumerate, dyslexic, dyscalculic, dyspraxic, and dysgraphic. The feelings of being stigmatized have always been profound. Apart from these individuals, there are those who live on the boundary of the two worlds of proficient and unprepared individuals who are afraid of both numeracy and literacy.

Concepts and strategies of numeracy and literacy have a great deal of commonality in their development, their manifestations, and their growth patterns. In most cultures, most people recognize the letters of the native language alphabet instantaneously and even recognize some words. We are able to do so by comparing the written letter with the idealized image of that letter in our brain. This happens even when the letter presented does not conform exactly to the form of the letter we have in our mind and we are able to decompose and re-compose sounds by the knowledge of phonemic awareness and phonological sensitivity.

The same phenomenon is true when we learn a new language. Until we form the idealized images of the letters, shapes, graphical forms of the symbols of the new language, and the rules of decomposing and re-composing the sounds in that language, we struggle in the process of learning it.

Mathematics, in many ways, is like learning a second language. Therefore, to develop number and number relationships, one needs to become familiar with the existence and awareness of quantity around them. Children begin to ‘*perceive*’ quantity quite early and are able to subtly differentiate between different quantities—represented by collections of objects. They also begin to approximate them. They begin to identify them by comparing the collection and the familiar idealized forms of the clusters in their mind’s eye.

As in learning a language and reading, the adults (culture) repeat the sound and provide the name of a sound or a collection of sounds (how to put the sounds together or how to break a complex sound into smaller sound elements and associating the sound to its representation), and the child begins to mimic the sounds and in due course learns the language of the culture. Similarly, a child perceives a collection, and the culture helps the child to name these perceived collections and break them into their components and units. The culture introduces the language for these collections—the process called *counting*. With this, they begin to associate the names to these collections with their cultural names—number words. This is called *subitizing*. They begin to form idealized images of these collections in their minds, their names, and their representations. The integration of these three component processes is called *numberness*. They form the idealized images of first ten numbers (as needed by our culture)—the alphabet of mathematics, called *digits*, in their minds. This is fundamental to number concept. In our experience, most children, who are presented for evaluation of dyscalculia and other conditions like acquired dyscalculia lack the mastery of *subitizing* ([Butterworth, 2025](#)).

In teaching numeracy skills, therefore, we need to explore the similarities and differences in the conceptualization of and the skill development in numeracy, language, and reading. Largely, instruction to develop numeracy and literacy, particularly reading, in young children can go together because acquisition of one enhances the other—some of the same brain areas and functions are involved. This process can give us directions to design preventive/instructional/interventional/remedial activities for learning number, numbersense, and numeracy that are accessible to all children. For example, we need to understand the:

- a. role of *prerequisite skills* and specific cognitive and neuropsychological preparation in learning numeracy and mathematics (these begin to be acquired at home through games, toys, rituals, and cultivation),

- b. *trajectory of skills* involved in the development of number concept, numbersense, and numeracy (this is many times initiated at home and continues during pre-kindergarten and kindergarten), and,
- c. *identifying optimal methods/models/tools and activities* for developing and connecting the content—language, concepts, and skills for all children, including for the child who has a particular learning profile and potential.

To facilitate the transformation of collections into related digits and into ‘*numberness*’ is the key to learning number concept. When children do not have these images—the numberness—they struggle in mastering arithmetic facts and procedures, particularly subtraction and division. Having the ability to reverse sequences is an essential component of mathematical giftedness ([Krutetskii, 1976](#)).

Numeracy is learning the integration of a complex of concepts, procedures, and skills, with the understanding that numeracy is also a language of quantity, patterns, and space.

Similarly, mathematics is the language of science and means of studying and expressing nature and its phenomena that exists and we experience it around us. As a language, it uses numbers and objects—real and imaginary, following the rules of quantitative and spatial reasoning and principles of algebra. In some respects, mathematics language is acquired in the same way as any other language—through consistent formal and informal exposures to experiences and interactions with others in language-rich environments. Proper shaping of these experiences through rules, codes, and structures like learning native languages help children become fluent in this unique language.

For the child, language, from the very beginning, becomes the means of relating to others. Exposure to the language—formal and informal, shows and motivates the child not only to learn the patterns in sound and linguistic symbols but also helps create *lexical entries* (language containers) for concepts of knowledge in the cognitive, affective, and psycho-motoric domains in their mind. These early language experiences also shape the

physical apparatus of sound reception, formation, and perception, and the intellectual systems of meaning making from these sounds. The brain is also a repository of innate systems, such as the structures of ‘universal grammar.’ These early language experiences are immensely useful for the child’s development, not just linguistically, but also neuro-psychologically, intellectually, socially, and emotionally, and shape and utilize this universal grammar into an active language with a systematic unique grammar of its own. The integration of natural development (osmosis, immersion, and mimicry) and cultural richness of language bring important skills and resources from a child’s development of language into the articulation of innate quantitative and spatial skills into fruition.

Parents and other caregivers, therefore, should also introduce children early to the language, symbols, and representations of quantity, space, and their interrelationships. This will prepare them to understand and negotiate the attributes and aspects of quantity, space, and objects—*creating lexical entries (number words) and symbols for numbers and actions on them*—representations, measurement, and shapes in the world around them. This includes the early development of quantification (numbersense) and spatialsense:

- What is the same and different about these objects or collections?
- What attributes do these objects or collections of objects quantify?
- Is the attribute or object quantifiable?
- How to quantify a particular attribute of an object or objects?
- What and how to measure?

And where, when, for what, and how to use quantities relating to measurement, such as age, days, and time left to an important event, being aware of even (we get even amounts), money (same and different about them), time (when, before, after, late, early), and data: number of siblings, number of toys, number of people in the family, etc.

These experiences expose the child to the awareness, recognition, and use of spatial orientation and space organization along with quantity. As

children grow, they begin to explore the relationships between quantity and space, and their integration with language, forming numbers/numerals and measurements, are examples.

Family members and other adults often provide environmental conditions that promote language development and interest in reading. When children are read to early, and are reared in homes with reading materials and formal/informal and rich usage of language, they begin to read early and acquire a larger and richer vocabulary, and with better comprehension.

Activities that foster the development of decoding/encoding, phonemic awareness, chunking, and blending are common in some families, though less frequent in others. Some families offer formal instruction in language; they offer more reading materials—books, magazines, better television program viewing, and day-to-day discussion of topics with richer and varied vocabulary and with accompanied questioning, reasoning, and discussions. Such literacy-rich environments have a profound impact on their preparation for formal language learning and development of reading skills.

Similarly, children from homes where there is early and frequent use of quantity, frequently employing quantitative vocabulary, and providing quantitative reasoning in daily discourse develop an early and better sense of number and quantitative problem solving. Quantitatively (number) rich homes and classrooms promote numberness and numbersense and their expression—orally and in writing. Therefore, teachers of numeracy and mathematics (from K through 8th grade, at least) should have mathematics-rich classrooms, e.g., a display of a number line from 0 to beyond 125 at the eye level of children with multiples of 5s and 10s highlighted in font and color to show their importance, a vertical line from -15 to $+30$, a hundreds chart in the usual format (numbers increasing from left to right and the tens displayed vertically), a modified hundreds chart (numbers increasing top to bottom and tens displayed horizontally), a monthly calendar, clocks with displays in several forms (analog, digital), thermometers (with Fahrenheit and Centigrade indicators), number formation chart—with directional arrows, a Place Value chart with several cycles of three digits, Visual

Cluster cards, dice, die, diagrams and pictures of commonly found geometrical objects in the environment, Cuisenaire rods, Base-Ten blocks, Invicta Balance, a variety of counting objects, Ten Frames, and a rich collection of board games and toys.²

The goal of literacy programs in any school is learning to read, with fluency and comprehension, by the third grade. Once children become fluent readers, they use reading to learn anything and everything—social studies, instructions of different kinds, sciences, etc., and for pleasure.

Similarly, children should master numeracy skills, with fluency and understanding, by the end of fourth grade. When children become numerate, they are prepared and competent to learn more mathematics easily and flexibly. They can use the numeracy skills to learn language, models, and the methods of mathematics—algebraic, geometric, probabilistic, statistical, functional, trigonometric, etc. They are prepared for learning future mathematics efficiently and effectively, and for understanding information in other disciplines where mathematics is used. They should learn to apply their number, geometric, and mathematics language to a variety of situations and forms. Mathematics also opens doors for many more disciplines, areas of study, professions, careers, and vocations.

Reading is a complex process, and to achieve mastery in it calls for the involvement of a variety of brain components, centers, functions, and systems. Both localized and global brain systems perform and integrate tasks such as recognizing language symbols—visual and aural, discerning and analyzing visual and sound patterns, perceiving and organizing sequences of symbols and sounds, spatial arrangements—source of speech or location of the symbol, and many other subtle and not so subtle activities and clues—both verbal and non-verbal. Fluent performance in these tasks results in understanding and producing language and reading with meaning, affect, and impact.

Some of the same processes and skills needed for acquiring reading skills are, more or less, the same skill sets and brain centers that are called upon in conceptualizing number, acquiring number sense, mastering arithmetic

facts, performing arithmetic algorithms and thinking mathematically—quantitatively, spatially, and symbolically.

A breakdown and/or deficit in any of these skills may affect numeracy and mathematics learning. For example, when a child is assigning a number (orally and orthographically) to a quantity (collection), similar areas of perception and action related to language, visuospatial, sequencing, and quantity light up on MRI pictures. It is not surprising, therefore, that almost 40% of dyslexics also exhibit symptoms of dyscalculia.

About 6–8% of children exhibit symptoms of dyslexia. However, research in the science of reading has resulted in a better understanding of the complex skill set of reading and has resulted in some very effective methods and approaches of teaching and learning reading. This has equipped educators to help children learn to read effectively and efficiently. Early diagnosis and proper interventions of reading instruction are keys to overcoming these situations. The very small number of children who are not able to read are those who are not afforded the effective methods and opportunities or have extreme cases of neurological, neuropsychological, and/or cognitive deficits.

Some children (with or without dyslexia) have low achievement in mathematics because of dyscalculia (about 6–8% of the population). However, many more have trouble in learning numeracy and mathematics due to environmental factors, such as poor instruction. For example, in many schools, the lowest-performing children are assigned to the least trained personnel in mathematics. Screening and diagnosis are performed late. There is no universal agreement on what the items for early screening of children for numeracy should be. What should be mastered at what level?

Ways and means must be found to train all elementary teachers—regular classroom and special education and paraprofessionals—to understand this problem. They should be well-trained in spotting the signs and nature of numeracy difficulties and disabilities, beyond that the child ‘cannot do mathematics.’ Training in effective instruction and remediation of numeracy difficulties, disabilities, and dyscalculia should be made available

for teachers so that almost all children are able to reach a satisfactory level of mastery in numeracy by the end of fourth grade.

Problems in acquiring literacy and numeracy skills have common as well as different origins and progression. There are enough parallels and patterns of development in literacy and numeracy to learn from the field of Science of Reading. For example, English is an alphanumeric language; so is mathematics. The acts of creating, representing, using, and applying mathematical symbols and sounds emerge and grow along with the child's native language structures and comprehension. However, there are important differences, and some unique abilities and systems are also needed to learn number and its applications.

The aim of early numeracy education is development of flexible numbersense and an ability and facility to perform the four arithmetic operations on whole numbers correctly, consistently, fluently, precisely, contextually, with understanding, and apply them in diverse situations.

As mentioned earlier, being numerate means the ability to have a comprehensive understanding of *number*, *number relationships*, *outcome of numerical operations* and *proficiency, confidence, and competence* in their *applications*. It is demonstrated, for example, in its use in measurement—which integrates the quantitative and spatial reasoning. To achieve these objectives, students should have the mastery of (a) *Number Concept*, (b) *Numbersense*, and (c) *Numeracy*.

Such mastery, in the case of *additive reasoning*, means³ that the child knows the addition and subtraction facts, concepts, and procedure with competence in:

- Language of addition/subtraction—vocabulary, syntax, semantics, and translation of quantitative problems expressed in native language and different contexts to numerical form (mathematics language and symbols) and vice-versa.
- Understanding the role of addition/subtraction in diverse situations.
- Accuracy and fluency of addition and subtraction facts.

- Applying strategies and procedures of addition and subtraction in solving contextual quantitative problems; and
- Understanding and using the inverse relationship between addition and subtraction in procedures and applications.

For example, mastery of addition facts at the end of first grade means: When a child is asked: find the sum of $8 + 6$, the child gives the answer 14 instantly (in two seconds or less orally and 3 seconds or less in writing—this shows accuracy and fluency—both in oral and written forms.). It needs to be pointed out that it is not enough to give this answer; a student must also demonstrate an efficient strategy or method used for arriving at the correct answer.

Therefore, when the student is asked how would you show, derive, prove, or explain it, they can give more than one strategy (and one of the strategies is other than counting) that is effective, efficient, and generalizable.

Effective strategies for learning addition and subtraction concepts, facts, and procedures are based on certain key number related concepts, such as *subitizing*, *sight facts*, *properties of numbers*, *decomposition/re-composition* of a number, *making ten*, and understanding *teens numbers*.⁴ For example,

$$8 + 6 = 8 + 2 + 4 = 10 + 4 = 14; \text{ or}$$

$$8 + 6 = 4 + 4 + 6 = 4 + 10 = 14; \text{ or}$$

$$8 + 6 = 2 + 6 + 6 = 2 + 12 = 14; \text{ or}$$

$$8 + 6 = 8 + 8 - 2 = 16 - 2 = 14; \text{ or}$$

$$8 + 6 = 7 + 1 + 6 = 7 + 7 = 14.$$

Every one of these responses shows that the child is using a thoughtful strategy and is demonstrating mastery of number concept—knows decomposition/re-composition, knows sight facts, knows making ten, and knows teens numbers. This is an example of understanding akin to comprehension in reading. Similarly, when the student is asked to solve the word problem:

‘Your team scored 8 points on Monday and 6 points on Tuesday, how many points did they score in all (what was the combined score of both days, or what was the total score in two days?’

she can answer: 14 without making any pictorial or concrete representations and can write the equation: $8 + 6 = 14$. When asked to supply a diagram, model, and reasoning, the child can do so (without counting):⁵ Through an Empty Number Line, Number Line, showing the application of pre-skills—Sight Facts, decomposition/re-composition, Making Ten, and Teens’ numbers.

Similarly, the four arithmetical operations (addition, subtraction, multiplication, and division) are dependent on the mastery of the three key fundamental concepts: *number concepts*, *arithmetic facts*, and *place value*. When children have a narrow and rigid conception of any of these topics and lack mastery, it can hinder their later mathematical progress in becoming fluent when executing arithmetic procedures and generalizing and applying them to proportional, pre-algebraic, algebraic reasoning, and geometrical applications.

I. Social and Academic Expectations About Numeracy

1. WHAT IS NUMBERSENSE?

To be fluent in numeracy, one of the components is having good numbersense. One’s open, positive, and flexible attitude toward number and the ability to display that flexibility and proficiency in handling quantity is called *numbersense*. As a skill set, numbersense refers to a person’s ability to look at the world quantitatively and make quantitative and spatial judgments and decisions with mental calculations using the properties of numbers and operations. It is the integration of (a) *number concept*, (b) *number relationships* (arithmetic facts), and (c) *place value*.

Numbersense consists of a cluster of ideas such as the meaning and ways of representing numbers, developing and seeing relationships among numbers, observing the relative magnitude of numbers, and proficiency in

working with them (ultimately leading to mastering arithmetic facts and their usage). Numbersense is not a set of discrete skills but a set of integrative skills. Students with good numbersense can move effortlessly between the real world of numbers and formal numerical expressions. They can represent the same number in multiple ways depending on the context and purpose. In operations with numbers, children with a good sense of number can decompose and re-compose numbers flexibly, with ease and fluency. When asked to give the sum $7 + 8$, a person produces 15. But they would also be able to work out the sum using efficient strategies as: $7 + 7$ is 14 and then add 1 to make 15, or, two 8s is 16 and subtract 1 and get 15, or, I take the 3 from 8 and add to 7 and $7 + 3 = 10$ and 5 more is 15, $8 + 2$ is 10 and 5 more is 15, or, $7 + 8 = 5 + 2 + 3 + 5 = 5 + 5 + 5 = 15$. When they know $7 + 8 = 15$, she can extend it easily to: $70 + 80 = 150$, $700 + 800 = 1500$, $7x + 8x = 15x$, etc.

Exploring number relationships in this manner, they become so familiar with this fact that it is committed to memory automatically. There is a need to automatize facts, but there is no need to memorize with rote. The learner should be developing a sense of security and accuracy for their answers. Automatized math facts, in themselves, are a small part of mathematics and they are best learned through the strategic use of numbers in different and flexible ways and situations. Unfortunately, many teachers and parents focus only on the rote memorization of math facts and use of unproductive strategies. This gives students the wrong impression that math facts are the essence of mathematics and that fast recall means being a strong mathematics student. This is a misunderstanding and misrepresentation of mathematics. We should minimize the dependency on rote memory in mathematics learning, as it may produce math-anxious and disaffected students. We should emphasize accuracy, fluency with understanding, and using efficient strategies for deriving them, and then automatization with practice.

‘Numeracy is a child’s ability and facility in executing and applying four whole number operations (addition, subtraction, multiplication, and division) correctly, consistently, and fluently with understanding in diverse situations in multiple forms, including the standard procedures.’

Numbersense is age- and grade-specific. It becomes more complex and sophisticated as a person encounters new number systems and operations. For example, at the *end of kindergarten the expected mastery of numbersense* is: (a) Number concept, (b) 45 sight facts, and (c) place value of 2-digits. Similarly, mastery of numbersense expected by the end of first grade is: (a) Number concept, (b) 100 Addition facts (sums up to 20), and, (c) Place value of 3-digits. And numbersense at the end of fourth grade is (a) Number concept, (b) All arithmetic facts (addition, subtraction, multiplication, and division), and (c) Place value of any number of digits in whole numbers and up to the hundredth place. At the end of sixth grade, it is: (a) Mastery of numeracy, (b) Flexible and effective use of whole numbers, fractions, decimals, and percents, and (c) Extended place value of whole numbers and decimals.

Through organized practice and experiences in various forms such as algorithms/procedures, this proficiency and fluency in numbersense is translated into numeracy.

Numeracy is a child’s ability and facility in arithmetic operations (addition, subtraction, multiplication, and division) on whole numbers correctly, consistently, fluently, in multiple forms of procedures, including the standard algorithm, with understanding.

By the end of fourth grade, a child should have mastered numeracy. If a student has deficits in it at the end of fourth grade, he needs intervention from specialists. That intervention should focus on developing number concept, arithmetic facts, place value, and then numeracy using efficient, effective, and elegant strategies.

C. Numeracy Goals for Kindergarten through 2nd Grade

The mathematics instruction goals and the objectives of the mathematics curriculum, in the quantitative strand of a mathematics curriculum, for K through 2nd grade are: acquiring and mastering additive reasoning: number concept, number relationships and operations, and the inverse relationship between addition and subtraction—can successfully convert an addition relationship into a subtraction relationship and vice-versa. The acquisition and mastery of number concept and additive reasoning are the first two developmental rungs in the ladder of mastering numeracy—this is truly a preparation for higher numeracy concepts such as multiplication, division, and fractions.⁶ This is parallel to ‘decoding’ and ‘encoding’ skills in reading. [NCTM \(1979\)](#), [Mulligan et al \(2013\)](#), [Mulligan et al \(2015\)](#), [Mulligan & Woolcut \(2015\)](#), [Mulligan \(2018\)](#)

Additive reasoning is the understanding that every number is a sum and/or difference of other numbers, that it can be broken into components, and that the components can give back that number. Practice with this inverse relationship leads children to the most important mathematical skills—decomposition and re-composition and numbersense and numeracy. When children have acquired these processes (e.g., decomposition and re-composition, etc.), quantitative problems make fewer demands on their memory system, particularly on the working memory.⁷

Quantitative Strand: Developmental Mile-Stones

1. *Mastering Number Concept:*

- a. *Concept of numberness*—from Subitizing to Visual Clusters,
- b. Integration of *subitizing, one-to-one correspondence, and counting,*
- c. *Digits*—the alphabet of numeracy,
- d. Mastering 45 *sight facts,*
- e. Concept of *decomposition* and *re-composition* of quantities and number,

- f. *Number attack skills*,
- g. Integrating these four skills into learning *strategies for addition and subtraction*,
- h. *Making tens and teens numbers*, and
- i. Extending these skills to learn *place value up to 100*.

2. *Language, Concepts, and Procedures of Additive Reasoning:*

- a. Concepts of *addition and subtraction*,
- b. Developing addition and subtraction *strategies*,
- c. Mastering addition and subtraction *facts*,
- d. Mastering the addition and subtraction *procedures*,
- e. Realizing that addition and subtraction are *inverse operations*, and
- f. Mastering the *applications* of additive reasoning to problem solving, including money (simple money transactions), measurement (linear), and time.

3. *Place Value: Language and Concept*

- a. The concept of *digit* (the alphabet of mathematics), *number*, and *place value* (knowing the difference between *digits*, *numbers*, and *quantity*),
- b. In a multi-digit number, the *value of each digit* is determined by its place⁸ in the number,
- c. Making and forming, reading, representing, *writing large numbers* (copying and writing from dictation), and *reading numbers*,
- d. *Decomposing/re-composing* multi-digit numbers: *standard form*, *expanded form* (Canonical and non-canonical forms of expansion), *place-value form*,⁹ and
- e. *Effective, efficient use of Instruction Materials*: Cuisenaire rods, Base-Ten/Place value/Dienes blocks, Unifix cubes, and Empty Number Line (ENL).

D. Continuum of the Use of Instructional Materials for Number Concept and Additive Reasoning

Counting (random materials, hash marks, other discrete marks and symbols) —> Fingers —> Five-Frame —> Ten-Frame —> Number Line —> Reken-Rek —> Dice and Dominoes —> Ordinary Playing Cards and Dot Patterns —> Visual Cluster Cards —> Sight Fact Cards —> Cuisenaire Rods —> Invicta (Number) Balance —> Empty Number Line (ENL).

E. Development of Strategies for Mastering Addition/Subtraction Facts (Application of CPVA/S)

- a. *Strategies*: Decomposition/re-composition; Properties of Operations: Commutative Property, Associative Property;
- b. *Representation*: Iconic and pictorial: Empty Number Line, Number Bonds, Numbers have families, Addition/Subtraction Ladders, Decomposition/Re-composition Graphic organizer;
- c. *Executive Functions*: Role of (a) Working Memory, (b) Attention and Focus, (c) Organization, and (d) Flexibility of thought;
- d. *Visualization*: Manipulating information derived from concrete and pictorial representation about strategies to derive facts and procedures and converting into symbolic and abstract forms
- e. *Automatization/Mastery*: (a) Mastering language of concepts, strategies, and procedures, (b) Understanding the integration of language and concepts, (c) Accuracy, (d) Fluency, and (e) Applicability.

F. Mastering Additive Reasoning

1. Understanding:

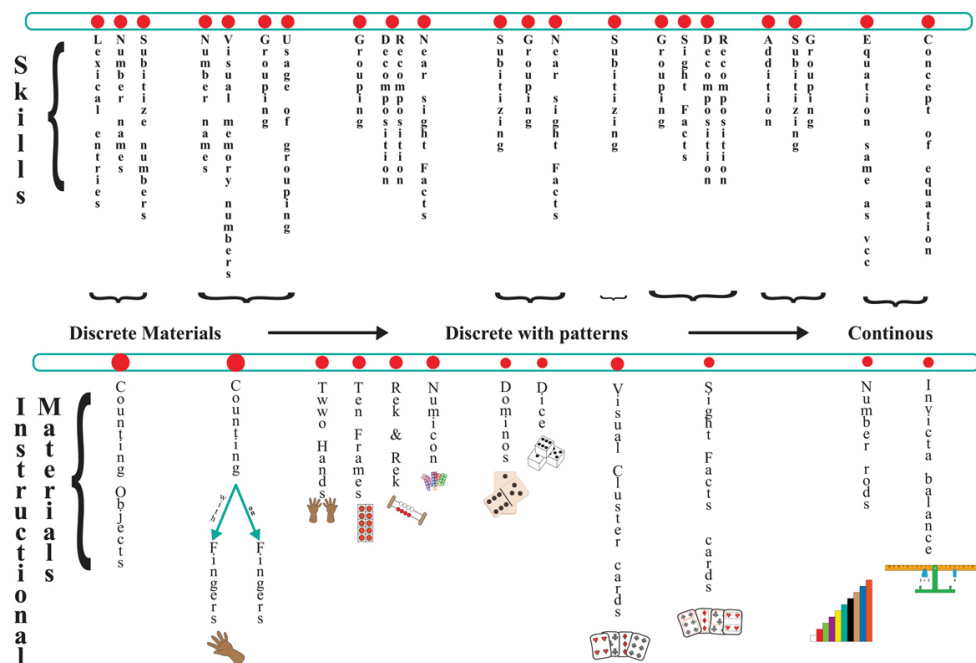
- a. Number Concept,

- b. Concept and language of addition and subtraction: definition, facts, number relationships, procedures, applications, and communication,
- c. Addition and subtraction are inverse relations—can convert an addition problem (fact, procedure, or equation) into a subtraction problem and vice-versa; extending addition strategies to subtraction strategies; can arrive at a fact using a combination of addition and subtraction strategies, and
- d. Applying additive reasoning to: (i) Learning new concepts in numeracy and mathematics, (ii) Solving problems in mathematics, (iii) Solving problems in other related disciplines, and (iv) Solving problems in extra-curricular situations—games, toys, and other academic and non-academic settings.

2. Operation of Addition

Flow of the Lesson: Concrete to Abstract: (Intuitive¹⁰ —> Concrete —> Pictorial —> Visualization —> Abstract/symbolic—> Applications —> Communication)

- a. *Concrete*: Using Cuisenaire rods to model the addition concept: Decomposition/re-composition of numbers and arriving at strategies (Concrete—continuous)



[Extended Description](#)

Figure 1.1 Instructional materials continuum for number concept

- b. *Pictorial*: Use of Visual Cluster Number Cards and Sight Fact Cards: Modeling decomposition/re-composition of number concept and arriving at strategies (Pictorial—semi concrete)
- c. *Use of Empty Number Line*: Modeling/representing the concept pictorially and arriving at strategies (Pictorial—continuous)
- d. *Mental Arithmetic/Addition*: Visualization Practicing strategies by visualization
- e. *Estimation*
- f. *Procedures* using Decomposition/recomposition and Place value Strategies: Partial Sum development and recording
- g. *Alternative Algorithms (Using decomposition/re-composition) Empty Number Line*
- h. *Standard Algorithm*: Deriving standard algorithm and practicing Alternative Procedures

G. Operation of Subtraction (Same Processes and Sequence as in Addition)

Flow of the lessons: Concrete to Abstract (Intuitive → Concrete → Pictorial → Visualization → Visualization → Abstract/symbolic → Applications → Communication)

- (i) *Use of Empty Number Line* (Cuisenaire rods → Number Line → Empty Number Line (ENL))
- (j) *Mental Arithmetic/Subtraction* (Application of strategies)
- (k) *Estimation* (Cuisenaire rods → Next Tens → Rounding → Estimation)
- (l) *Partial Sum Recording* (Cuisenaire rods → ENL → Abstract/Symbolic)
- (m) *Standard Algorithm* (Cuisenaire rods → ENL → Abstract/Symbolic)
- (n) *Alternative Procedures* (Cuisenaire rods → ENL → Abstract/Symbolic)

By design, almost 70% of instruction time, during these years (K-2nd grade), should be devoted to development, practice, and mastery of additive reasoning (K—Number Concept; 1st grade—Addition; 2nd grade—Subtraction and integration of addition and subtraction).

About 20% of instruction time to spatial reasoning (spatial sense and geometrical skills),¹¹ and the rest should be on applications of these two domains—quantitative and spatial reasoning.

However, all these three areas should be taught as an integrated whole, not as compartments. Every day, part of the lesson should be devoted to quantitative reasoning, part to spatial reasoning, and part to applications of these components. For example, calculating the perimeter of a rectangle or triangle is a good example of integration of quantitative and spatial reasoning in first grade. Making change is a good example of this integration.

Similarly, everyday, a part of the lesson (about 25%) is for '*Tool building*,'¹² 50% of the lesson is on the *main topic/concept/procedure/strategy* (its language, its conceptual schemas, and the related strategies and procedures): about 50% of that day, and the rest of the time on *review, practice, consolidation*, providing *remediation on demand*, and *feedback* to individual and small groups of children. This part of the lesson is also for *extra support* and to *focus on children with special needs*.

The goal of tool building is to assure the access to content and success of the lesson for each child. It consists of (a) facts, (b) earlier definitions, formulas, strategies, and procedures needed to be successful in the current lesson. It is to be done orally with the whole class. In oral settings more repetition takes place and children are exposed to repetition. However, the questions must be asked according to the child's level and to be scaffolded if needed by the teacher or by the help of another child.

The result of such a division of the instructional time will be that the number of students who may develop *acquired dyscalculia*¹³ is reduced. This part of the lesson is also for extra support and focus on children with special needs and is an example of the prevention of failure and lack of engagement.

Prevention, intervention, and remediation activities should take places in all three places: (a) *Tool building time*, (b) *Main lesson*—referring to old concepts and connecting the current concept or procedure to previous topics and relationships, and (c) *Individual and small group intervention* during the practice time. In addition to these places, students with serious difficulties should have specially arranged time outside the class as additional sessions specifically focused on the needs of the specific student or a group of students by *mathematics specialists*, who are trained in numeracy teaching with special training in diagnosis and remediation of learning problems in mathematics, including dyscalculia and other learning problems in mathematics, including a deep understanding of mathematics anxiety. It is to be noted that this kind of service is in addition to the classroom instruction (not a substitute for classroom instruction). A

mathematics specialist position is just like a *reading specialist*. A reading specialist is a professional with special training in the areas of teaching reading, diagnosis, and remediation of learning problems in reading, including dyslexia. Every school should have a reading specialist as well as a mathematics specialist.

Notes

1. See the comparison of the development of numeracy and literacy in Appendix A. [!\[\]\(2824aab9645d9fab95bae27ff6828dab_img.jpg\)](#)
2. See [Sharma, 2015](#) for List of Games. [!\[\]\(0fbf3ad74a6c8dc44ba9ea17fc2aca5e_img.jpg\)](#)
3. The example of teaching Additive Reasoning here is illustrative; all other concepts and procedures will follow this format. [!\[\]\(c42d0234b47eca423823087b9f2f5716_img.jpg\)](#)
4. These key skills, key states, and sequence of pre-skills are derived by careful task analysis of the concept under discussion; here the concept being discussed is additive reasoning. Similar collection of strategies derived by task analysis are available for all key concepts of numeracy. [!\[\]\(0a4e27b70a7e12aa1778c291185d94b0_img.jpg\)](#)
5. Arriving at the answer by hash mark, counting objects, on a number line, or even counting up or counting down are not the signs of true mastery. [!\[\]\(5fd3f9e293352907910bc12719d886a8_img.jpg\)](#)
6. The metaphor of a staircase is selected with great care. In our experience, research supports it. If these two rungs are not strong, future climbing is precarious and difficult. [!\[\]\(5624885e6ed09b96433c36df0b4bcc44_img.jpg\)](#)
7. One of the important components of executive function: (a) Working memory, (b) Focus and attention, (c) Organization, and (d) Flexibility of thought. [!\[\]\(d963f205c828afa6df04ce78ed1ad9fd_img.jpg\)](#)
8. Just as the sound of the letter or groups of letters is determined by their places in the word. [!\[\]\(6f04a4e5847f873f8b68ad9f105c3e74_img.jpg\)](#)
9. This should be introduced after multiplication has been introduced and can be expanded to decimal places when the concepts of fractions and decimals have been introduced. [!\[\]\(31c704f6a713ce068a0e82b13a7bca9b_img.jpg\)](#)

10. Before introducing the main concept, the teacher should, through informal questioning or activities, ascertain if the children have any intuitive idea about the main concept, strategy, procedure, or application. It is the preparation stage of instruction. It could also be the exploration of the primary concepts or the sharpening of the tools needed for the main concept. ↵
11. See the Development of Spatial Reasoning in Children by [Sharma \(2019\)](#). ↵
12. Tool building for additive reasoning includes: (a) Subitizing, (b) Sight facts, (c) Counting activities, (d) Decomposition/re-composition, (e) Making ten, and (f) Teens' numbers. ↵
13. To be defined later. ↵

2 Mathematics Difficulties and Mathematics Anxiety

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Numbers evoke joy and wonder; why doesn't math class?
Why does mathematics cause anxiety in so many people across the whole age range?

There are many perspectives and conditions we need to consider to be able to answer this important question in mathematics education. Mathematics anxiety is neither a recent nor a simple phenomenon. Despite a long history of negative influences on people learning and using mathematics, we are far from an acceptable level of understanding that enables us to address the problem successfully. People describe mathematics either in awe or with disdain; sometimes with both.

Since mathematics learning is one of the components of public and private school education in modern societies, mathematics learning difficulties and its impact as mathematics anxiety has become an international problem. There are some serious implications and important deductions we can draw from that statement.

In this chapter, we consider some of the issues, possible approaches, and suggestions for interventions and even amelioration of this condition. We shall select some key points and some suggestions to minimize its prevalence and help provide some directions for solutions.

A. Nature and Origins of Math Anxiety

The essentials are obvious. They are the subject itself:

- a. the nature of mathematics itself has great deal of complexities starting from acquisition of number concept to mastering higher order concepts,
- b. the way it is taught in schools, and
- c. the culture and beliefs in society and education that underpin attitudes toward mathematics and mathematics learning.

Many people complain of suffering from mathematics anxiety, and most times they express it even if uninvited. *Math anxiety*¹ is a negative, threatening, and potentially impairing emotional reaction to mathematics. It is the result of unsuccessful and unhappy experiences in learning it. It reminds the person of the memories of seemingly insurmountable failures in mathematics tasks in classrooms, examinations, and unsuccessful encounters with it in day-to-day usage. Those with the condition avoid encounters with anything mathematical in day-to-day, professional, and academic situations. Math anxiety is an affective, cognitive, and visceral reaction, sometimes appearing to be an irrational response to numeracy and/or mathematics. Individuals for a variety of reasons begin to experience and report anxiety related to mathematics learning and its usage. The people who perceive mathematics as a difficult subject are much higher than the number of people who experience some kind of learning disability and/or mathematics anxiety.

It is not just an issue for a 10-year-old who is failing to learn multiplication facts. We have people who are highly successful in many areas of endeavor, renowned for their confidence, who admit to having anxiety about anything quantitative. What students and even famous people do is to avoid failure by not attempting any quantitative tasks. However, the presence of mathematics anxiety is not necessarily a sign of inferior intelligence. [Gardner \(1999\)](#) suggested that there were eight intelligences, including Linguistic and Logical-mathematical.

David [Geary \(1994\)](#) defined mathematics anxiety as a:

state of fear or apprehension that is associated with many mathematical endeavors, including test taking, course taking, homework and so on ... many individuals show acute anxiety about mathematical activities, above and beyond any more general tendency to be anxious.

He also notes that ‘mathematics anxiety does not appear to be strongly related to general intelligence.’ There is an apprehension of harm, affliction, danger arising from repressed emotions or desire. Anxiety tension comes from stress. [Burton \(1979\)](#) quoting CG [Costello \(1976\)](#):

Anxiety occurs when an anticipated event is expected to make demands for which the person is unprepared ... not only is the anxious person not able to simply dismiss the anxiety-producing possibility, but they can become obsessed with it.

These negative emotional reactions to mathematics lead to varying degrees of helplessness, panic, and mental disorganization that arise among some people when faced with a mathematics problem. It is uneasiness or distress of mind regarding mathematics as one does with some uncertain event as if it may involve danger or misfortune. In the case of mathematics anxiety that event is uncertainty about the ability of dealing with a quantitative situation—solving a mathematics problem. This anxiety can cause students to worry and stress about, ‘what if I fail in it?’ ‘What if its impact is negative to my situation—status, impression, result?’ The person fears being wrong, possible failure, even if that failure would be on a small scale in others’ eyes. Such anxiety has a negative effect on a person’s normal behavior and mathematics learning and achievement. This, in turn, can lead to long-term detriment to career opportunities and earning potential and possibly decreased job satisfaction. It exacts huge costs to both individuals and societies.

In a nationally representative survey conducted in the spring of 2024, nearly all K-12 educators told the Ed Week Research Center that math

anxiety is a problem for their students. More than 1 in 5 elementary educators and more than 1 in 3 middle and high school educators said their students' math anxiety is severe—which could potentially be enough to interfere with learning, or sometimes even attempting to solve problems. It is pervasive.

The first academic paper to be published on *math anxiety* (on 'number anxiety') was by [Dreger and Aiken in 1957](#). Over the next three-quarters of a century, there have been many studies, papers ([Sunkarapalli and Agarwal, 2017](#)), and books about mathematics anxiety and about the 'fear of negative evaluation' (FNE). One of the most recent is the notable book from [Kirkland and Hunt \(2004\)](#) *Maths Anxiety: Solving the Equation*. Three key findings from this work worth noticing are:

- 'Number Anxiety' seems to be a separate factor from 'general anxiety';
- 'Number Anxiety' does not appear to be related to general intelligence, as measured by Wechsler-Bellevue or ACE scales; and
- Persons with high 'Number Anxiety' tend to make lower grades in mathematics.

The first book that popularized the term *math anxiety* and brought mathematics anxiety as a topic of general discussion and research was *Overcoming Math Anxiety* by Sheila [Tobias \(1978, 2007\)](#). However, Tobias saw 'math anxiety' as a socio-political issue. So long as people *see* themselves as disabled in mathematics and do not rise and confront the social and pedagogical origins of their disabilities, they will be denied 'math mental health.' Tobias defined math mental health as an important construct for personal development and growth for an individual as 'the willingness to learn the math you need when you need it.' To her, for the nature of mathematics learning in an ever-more technical society, having that willingness can make the difference between high and low self-esteem, failure and success. The problem to her is more motivation; thus counseling

could solve the problem, and she thought that teachers could address this situation quite well.

Obviously, it has not happened for many in our society. Ironically, the problem is still here with us in its full form. Plummeting math performance and rising rates of student mental health problems have created a perfect storm for math anxiety, but emerging research shows there are clear ways teachers can build students' confidence in the subject. However, there are some built-in difficulties in the technological era. Students are greatly distracted from learning due to technology, misguided early reliance on calculational devices, poor teaching and achievements at foundation levels, and lowered interest and expectations in learning.

There is a proliferation of articles, books, education programs on television and on the internet. For example, in the year 2025, there seems to have been more than 50 papers published internationally on math anxiety. In the UK, a Maths Anxiety Trust founded by the late Dame Shirley Conran has done much to raise the profile of and interest in the condition. In the United States and the UK, there are math anxiety help centers in many colleges and universities. Although math anxiety has been recognized in higher academic circles for over half a century and although it affects a significant proportion of the population worldwide, there is little known about it by the wider public or [Mamarella et al. \(2019\)](#) even the teaching profession. Not every learning difficulty in mathematics is dyscalculia.

Mathematics is a social justice issue in that around one in five adults in the UK and about three in ten in the USA lack even basic numeracy skills without which they find it difficult to budget, they feel stressed and insecure about money regardless of their income, and they struggle to overcome barriers to progression in work or even obtaining the work they desire. From Dame Shirley Coran: 'Also, a lack of math qualification in young people can significantly restrict their career choices. Thus, futures can be restricted by the consequences of math anxiety.'

Research indicates that math anxiety is the strongest predictor of both applied and basic math performance. Math anxiety is not simply an emotional response. It is also a physiological one that affects heart rate and

neural activity. Students who feel anxious about math will not pay attention voluntarily to any instruction, no matter how interesting it is.

A major article was published in the (UK's) Sunday Times of August 31, 2015, on their poll of 16–17-year-old students' reports on anxiety. They found that 14% of males and 34% of females often avoided going to school or college because of anxiety, that 9% of males and 11% of females were reported as having dyslexia while 1% of males were reported as having dyscalculia, and the figure for females was 2%. 48% of boys and 69% of girls reported having stayed home from school because of math anxiety. In the early grades (kindergarten to 3rd grade), parents have reported children having 'tummy aches' on the days when the teacher has planned 'mad minutes', 'facts in three minutes,' 'round-robin fact mastery', 'fact drills,' etc.

The instrument 'The development of the Numeracy Apprehension rating Scale for Children aged 4–7 years' ([Petronzi et al., 2017](#)) anxiety may stem from the early development of numeracy apprehension.'

We therefore believe that mathematics curriculum used in schools is significantly developmental ([Chinn, 2021](#); [Sharma, 2008](#)), so setting firm foundations will be hugely influential for future success. We need to address mathematics anxiety in our youngest pupils by securing foundational mathematics knowledge.

Why is mathematics learning and trying to master it such a problem in comparison to other disciplines of study?

[Skemp \(1986\)](#) suggested some possible causes of math anxiety, including an authoritarian teacher. He also comments on a situation that we have grave concern about:

'The increasing efforts the student makes will inevitably use the only approach that they know, chanting the facts as a way of memorizing.' (Many teachers and parents encourage this approach.) 'Problems with short-term, long-term, and working memory as mathematics becomes more advanced, more complex and more demanding on memory, impose an impossible burden on memory.' This and many others like this situation have compelled us to develop, identify, and invent methods and strategies

that make better use of children's resources—physical, intellectual, and learned to tackle these demands. That is why we encourage mathematics to be taught developmentally ([Chinn, 2021](#); [Sharma, 1980](#)). It is also a theme throughout this book. It is also a key finding from many other authors (such as [Holt, 1969](#)). Key and added benefits of such an approach is the improvement in a child's learnability and potential that is the hallmark of plasticity achieved by the brain through meaningful—effective, efficient, and elegant teaching methods.

B. Numeracy Anxiety

Numeracy anxiety (or *Arithmophobia*) is the extreme fear of numbers. Some people fear certain numbers, such as 'unlucky' number 13. Others fear all numbers. Numeracy anxiety can significantly interfere with daily life. People with this phobia may have a hard time holding onto certain jobs, paying bills, keeping time, or managing a budget.

There seems to be a culture around the way mathematics, particularly numeracy, is taught, whether in school or at home. Part of that culture is that math and numeracy is about remembering facts and procedures, working and thinking quickly, resulting in timed tasks and tests.

Observations and research indicate that early number concept anxiety (*numeracy anxiety*) can act as both a cause and consequence of poor numeracy achievement, and that this bi-directional relation is particularly apparent for many boys due to their lack of dexterity/fine motor coordination needed for counting, which is a prevalent method of teaching number and other number concepts. The early onset of number anxiety is likely to set the foundations for future anxiety in mathematics and related disciplines and a debilitating anxiety, and this should be concerning to us all.

There are two reasons for such an anxiety to happen: One is a lack of development of *subitizing* and the other is over-emphasis on counting. The absence of subitizing precludes decomposition/re-composition, therefore, on the lack of development of efficient and effective strategies. It is

teachers' lack of use of efficient and appropriate tools of teaching based on decomposition/re-composition that becomes the cause of numeracy anxiety. Moreover, early arithmetic is often made into a matter of being 'right' or 'wrong.' How teachers react to that and manage it can help the learners gain enough confidence to try new ideas and learn/realize that being wrong can be part of the learning process.

Early introduction to difficult abstract concepts, inefficient instruction, poorly selected or designed instruction materials, and a low level of mastery expected and achieved so that the information is not sufficiently mastered and internalized, can become causes of numeracy anxiety.

Another problem is that most teachers are not aware of the role of executive functions in learning, therefore they emphasize only the acquisition of arithmetic content and do not include strategies to improve children's learnability of that content. Therefore, many teachers emphasize only one way to 'get the answer' to children, parents, and special educators, then practice that 'getting the answer' routine. Because of this, children do not develop a 'flexibility of thought' of deriving and 'getting the answer.' Activities to improve executive functions, such as working memory and flexibility of thought are necessary for improving the learnability of the students. They become non-innovative too early in the learning process. When they do not get the answer by one method, they think they are incapable of doing the problem. At this point, proper feedback, exposure to diverse strategies, and an introduction to estimation are important. Discussion is preferable to judgment. Being judged as 'wrong' (especially publicly) can be very debilitating at that age and at any age, till a person has a strong self-image and has developed critical thinking themselves. In such a situation, helping the child (guiding using Socratic questioning) arrive at the answer is very important. Teachers should never leave a child with the judgment of being wrong. They should leave the child successful in one problem and possibly another problem. Let us consider the scenario:

Teacher: 'I have written an addition problem ($7 + 8$) on the board/screen. Please come and write your answer. And be quick.' (Thus, exposing

the pupil to judgment by the whole class.)

Student: (Somewhat hesitantly writes) 14.

There is some reaction from the class. Some of this conveys the message that this is not the correct answer.

The teacher writes a big red cross on the screen/board and says to the pupil, 'Sit down. Let's ask someone who will know the answer.' This kind of interaction can be the beginning of a child's fear of, anxiety about, and disinterest in mathematics at an early age. The teacher does not ask the student to appraise their answer by asking: 'Could you share how you arrived at this answer?' This will open an opportunity to show what he knows and that may become the stepping stone for deriving the answer to this problem correctly. 'Do you think the correct answer is bigger or smaller than your answer?' And possibly, 'Why do you think that 14 was the correct answer? How did you work it out?' If done in a non-threatening and encouraging way (the teacher's body language could well be a significant factor here), this may enable the pupil to provide a new answer and may even help with a reason for that new answer and a boost to their confidence.

If done properly, this could be a very rewarding experience for this student despite giving the wrong answer. This could be made into a diagnostic and effective teaching episode. Here is the alternate scenario.

Teacher: 'We have been adding numbers this week. You also have had some practice in it. I have written an addition problem ($7 + 8 = ?$) on the board/screen. Let me see, if someone could come forward and answer this question.'

He chooses a student, who may not have mastered addition yet, with confidence and accuracy. The teacher uses this opportunity to engage in some intervention, remediation, and confidence building.

Teacher: 'David, could you please come and help me write the answer?'

David comes forward and stands next to the teacher, facing the class.

Teacher: David! Is the answer going to be more than ten or less than 10?

David: It will be more than 10.

Teacher: Great! How do you know?

David: Because I know $8 + 2$ and $7 + 3$ is 10.

This shows to the teacher that David has one of the prerequisite skills for mastering arithmetic facts.²

Teacher: Could you try to give me the answer?

David: I think it is 14.

Teacher: Actually, $7 + 7$ is 14. Our problem is $7 + 8 = ?$

David: Oh! It is 15.

Teacher: Great! How did you figure it out?

David: Well, I now know that $7 + 7 = 14$, and 8 is 1 more than 7, so $7 + 8$ must be 15.

Teacher: Then, what is $8 + 7 = ?$

David: That will also be 15.

What if the child answered that he does not know the sum of $7 + 8$? The teacher asks: Do you know what two numbers make ten? Then, the teacher should help the child to arrive at the answer by making use of the prerequisite skills that the child already knows or builds those up.

Low-stress, carefully thought-out and non-judgmental discussions and dialogues can be very successful for situations like this, and for setting a positive and constructive ethos in the classroom.

Numeracy anxiety is the outcome of poor number concept and lack of mastery of the development of arithmetic facts, place value, and inability to perform and apply arithmetic procedures fluently, consistently with understanding. These are also the characteristic deficit markers of dyscalculia and of acquired dyscalculia. We suggest that mathematics anxiety and mathematics achievement are related in a recursive negative feedback loop: low achievements in numeracy (arithmetic facts, place-value, and properties of numbers and operations) and mathematics can lead to anxiety about numeracy and mathematics. For example, anxious thoughts about numeracy can create intrusive worries around ability that decrease numeracy performance above and beyond one's skill level. It is not unique

to numeracy or mathematics. A similar phenomenon is observed in reading anxiety and reading achievement ([Ramirez, et al., 2013](#), 15–34).

However, the opposite is also true. We have found that success in one small area of numeracy or mathematics—one concept, one procedure, using efficient language, conceptual schema and strategy, reduces their anxiety and encourages them to try new problems. We have found that small successes have a profound effect, particularly on *specific mathematics anxiety*.

Each one of the developmental milestones in mathematics are progressively more complex and more abstract than the previous ones, and this abstraction is cumulative in nature: *Number concept* is abstraction of *subitizing* and *counting* of discrete objects; *addition* and *subtractions* are abstractions of *number concept*; *multiplication* and *division* are further abstractions of *addition and subtraction*. And later, *exponents* are abstractions of multiplication. *Place value*—a very complex concept in scope and sequence—is an integration of all these concepts and spatial orientation/space organization with spatial impact of language.

Because of this complexity, the learning of these concepts is difficult for many children as each concept demands *attention*, *focus*, and an adequate *memory system*—short-term, working, and long-term. At the same time, the worry of learning these begins to generate moderate and then clinical levels of anxiety for many children. Lack of understanding of the trajectory of the development of the mathematics conceptual milestones, poor instruction, poor instructional materials and strategies, inappropriate expectations, and social factors undermine the timely development of the innate numerical concepts (i.e., subitizing, spatial orientation/space organization) and, as a result, these deficits may sometimes manifest as dyscalculia or acquired dyscalculia. Further, if proper instruction, intervention, and remediation are not provided, they may give rise first to numerical anxiety and then full-fledged mathematics anxiety.

It is fully true that students who have had difficulties in arithmetic are the only ones who may have difficulty in learning algebra. Students facing difficulty in algebra are not just those who have done poorly in elementary

school or those who have intellectual or cognitive deficits. Some students who are intellectually quite superior also face difficulty in algebra if it is presented, either earlier than they are ready for it, and/or it is presented only abstractly or procedurally. For example, in many schools, superior mathematics students from sixth or seventh grades are given the opportunity to study their first course of algebra as the first course in an accelerated mathematics program sequence that culminates with advanced placement calculus (AP Calculus) toward the end of high school. Unfortunately, this course in algebra proves to be a stumbling block for some students whether the course is accelerated or regular in nature. Unsuccessful students in the accelerated course either drop the algebra course and revert to regular grade-level mathematics or complete the course only to have to repeat it in the next grade. In either case, intelligent, previously high-achieving students experience their first trauma of underachievement in mathematics. These students experience their first bout of anxiety, and they begin to doubt their competence.

Although the difficulties, fear, and anxieties of these students are quite different from the students we mentioned earlier, a small number of them do begin to experience math anxiety and begin to avoid demanding courses. We have met many underachieving gifted and talented students.

From the perspective of their impact on a student's mental and mathematics health, the effects are similar—an aversion to and a fear of mathematics. Sometimes, it appears quite irrational. In terms of their diagnosis and remediation and the impact of remediation, numerical and mathematics anxiety present different situations. In many other ways, math anxiety functions differently than other types of anxiety (i.e., general anxiety). In observations, we have found that specific mathematics anxiety can be handled with efficient and effective methods of teaching of specific topics that gave rise to the initial numerical and/or mathematics anxiety. We have observed that when the arousal or interest in the task is due to efficient strategies, they taste success. Then, there is an immediate change in their level of engagement and attitude toward mathematics. Some recent research

suggests that developing a growth mindset as an arousal factor has positive results.

Although effective instruction has a positive effect on both kinds of math anxieties, people with specific math anxiety respond to targeted instruction in the specific area of difficulty; whilst the same instruction for global mathematics anxiety subjects has lower positive effects. In their case, there is a need for counseling for their trait anxiety, or a more personal level of instruction is more effective as their trait anxiety heightens their state of anxiety (mathematics anxiety) in demanding and trying situations.

The discrepancy between high- and low-math anxious people's performance appears to vary as a function of the size and nature of the problem being asked to be solved. It leads to a suspicion that there is a close relationship between the observed deficits and the memory system, particularly the working memory system as working memory skills are positively associated with academic performance, especially in mathematics. In contrast, high levels of trait anxiety are linked with educational underachievement in general. Research shows that the association between trait anxiety and academic performance is significantly mediated by verbal working memory for three of the six academic performance measures. This is particularly so for math, quantitative, and non-verbal reasoning. Spatial working memory does not significantly mediate the relationship between trait anxiety and academic performance.

C. What is it About Mathematics that Generates Anxiety?

The meta-analysis research in education often focuses on two prominent forms of anxiety and mathematics anxiety. In particular, the subject of mathematics promotes expressions of anxiety that take such forms as tension and dislike (*attitudinal features*), worry, helplessness and mental disorganization (*cognitive features*), and fear and stress (*emotional features*). [Spielberger \(1972\)](#) described anxiety as the outcome of a 'chain reaction consisting of a stressor, a perception of threat, a state reaction, cognitive reappraisal, and coping.'

Most research and our experience of teaching learners with learning difficulties (including dyslexics and dyscalculics) suggest that anxiety is predominantly an undesirable and unhelpful state. We believe most teachers recognize it as an unhelpful outcome of lack of success in mathematics. Another term used for mathematics anxiety is *Mathaphobia* (or *Mathphobia*). Mathaphobia is explained as a silent, numerical illiteracy that cripples the pre-frontal and frontal-brain lobes from understanding basic mathematics concepts and making executive decisions about quantitative relations, therefore, problems with *numbersense*. Mathphobia can be as crippling as a learning disability (such as dyscalculia). Unfortunately, simply ignoring it or wishing that mathematics anxiety and fear of mathematics will go away with time is very unlikely to be efficacious. Other conditions that relate to test and mathematics anxiety are phobia, fear, unrealistic expectations, and the fear of negative evaluation of one's performance setting the conditions of mathematics anxiety.

The concept of a 'fear of negative evaluation' ([Bath et al., 1986](#)) might suggest that anxiety is more prevalent amongst low achievers, but research ([Jamaludin et al., 2024](#)) from Singapore, a country that features in the top ranks of International Tables of mathematics achievements (for example, PISA and TIMMS) reveals that it can impact higher achievers too. For them, it is not lack of success, but the fear of 'negative evaluation' or 'achieving lower than their real or unreal, expectations of performance.' Fear of negative evaluation is defined as 'apprehension about others' evaluations, distress over their negative evaluations, avoidance of evaluative situations, and others that would evaluate oneself negatively.'

Another source of anxiety is seeking perfection in examinations. It is a false sense of a desire to achieve perfection, which is not possible to achieve every time. This puts unusual demands on one's performance, and as a result they become victims of the paralysis of inactivity. Perfectionism is a personality disposition characterized by striving for flawlessness and setting exceedingly high standards of performance accompanied by overly critical evaluations of one's own behavior.

[Witzel and Little \(2016\)](#) have listed eight factors that can influence mathematics learning and achievement:

- Learned helplessness
- Passivity
- Memory difficulties
- Attention difficulties
- Cognitive and metacognitive difficulties
- Processing deficits
- Low level of achievement
- Mathematics anxiety

These factors are interactive; they negatively influence each other. The external manifestations of math anxiety—classic symptoms of general anxiety, avoidance of mathematics, and increased number of mathematics errors begin to appear very early in children’s schooling and, if not addressed may persist for a long time, even into adulthood. The causes of these errors and reasons for a person’s failure in mathematics must be investigated and dealt with, to make success a possibility. The first learning experience of a new topic is a dominant entry to the brain. It is hard to replace this with new, correct learning ([Buswell and Judd, 1925](#)).

Much of the content of this book looks at these factors and how to minimize their effect and suggests strategies for reducing the chances of failure in learning mathematics concepts.

The consequences of anxiety include avoidance of courses and an inability to achieve in the subject ([Ma, 1999](#)). Avoidance can be at many levels, from avoiding just a few questions on a mathematics test/examination, to avoiding any mathematics at all, to avoiding careers that require mathematics proficiency. A big influence on these scenarios is the ambience of how these challenges are presented. However, wherever and whenever attention to these factors has been paid to the effects of math anxiety they have been reversed, indicating that gain in achievement can occur if the anxiety is reduced.

On the other hand, far from making people vigilant, math anxiety seems to result in people rushing to get through an unpleasant task as fast as possible, even at the cost of accuracy. This is to relieve the condition of anxiety. [Ashcroft and Faust \(1994\)](#) found that people with high levels of anxiety solved mathematics problems more quickly and less accurately than those with less anxiety. There may also be an element of feeling under pressure to come up with an answer—any answer—to a problem that one has been set. In their estimation, wrong answers, if one can get them, may be better than no answer at all ([Holt, 1966](#)).

To us, this suggests that anxiety is also about not achieving one's own goals and aspirations in mathematics. That is a question of not meeting expectations—one's own, imposed by others, or inferred from others' expectations. The presence of any of these expectations and not meeting them for some may trigger the onset of mathematics anxiety.

The fear of negative evaluation (in its many forms, including from teachers, peers, and parents) is called *atychiphobia*. It is a fear or phobia, responding to unrealistic expectations without skills or motivations; the resultant worries cripple the frontal lobes from understanding basic mathematics concepts and making executive learning decisions difficult. As the neuro-technology develops and we know more about the brain and its functions, it provides more concrete evidence that math anxiety is not just a nervous and imagined response to stressful situations related to quantity and quality of negative evaluations and fear of not meeting expectations.

For example, the work done in the first fMRI study to examine mathematics anxiety in children, [Young et al. \(2012\)](#), found brain activations in 7- to 9-year-old children that tell a similar story to that found in adult samples; that is, highly mathematics-anxious children exhibited stronger activation in the basolateral nucleus of the left amygdala, a site previously tied to the learned fear response in adults.

Early onset of number anxiety is likely to set the foundations for future anxiety of more debilitating anxiety, and this is disconcerting. The phenomenon of math anxiety—understanding it, seeking directions for

diagnosing it, and to some extent finding solutions to it—can be explained from three perspectives: psychological, mathematical, and pedagogical.

D. Test Anxiety

There is a third kind of anxiety that shows up more often in the mathematics context: mathematics anxiety due to test-taking anxiety. Testing anxiety manifests itself in different ways for different students. Test anxiety can stem from many reasons, including fear of failure and thus negative evaluation, negative testing experiences, the pressure of having to work quickly, or feeling unprepared. Its effect and response can range from minor protest or displeasure to refusing to do the work, crying, hiding in the bathroom, and verbal aggression to physical behavior like flipping tables and desks or hitting school staff. Some students avoid school on test days, and many suffer from symptoms such as stomachaches or headaches on pop-quizz days. In special education programs, many of our students' disabilities are closely related to anxiety, and testing can be a trigger that heightens those negative thoughts and feelings. For many students, the biggest obstacle to passing a test is not what they know, but the anxiety they feel. Stress and anxiety can wreak havoc on a student's ability to concentrate on tests, leading to poor performance and, ultimately, fewer future opportunities to succeed in school.

Therefore, most assessments can trigger negative thoughts and feelings, especially when students' disabilities are related to anxiety, particularly mathematics anxiety. Here's how to help students address testing anxiety in a healthy way.

The major source of this anxiety is timed tests. Many of the mathematics tests and assessments that students have to face are timed; even the assessment test designed by Chinn³ (2025) is short but timed—at 15 minutes (to reduce the time taken on the test during an assessment). Assessments are available from psychologists to support requests for many parents to let their children have extended time during examinations. What Chinn suggests for his test is that, if a student wishes to continue after 15

minutes, that they are given a different colored pen/pencil to use until such time as they feel they have completed as much as they can. That can be useful supporting evidence for an extra time allowance.

Tests compound the problems of anxiety; some individuals have an added anxiety known as *test-taking anxiety*—fear of tests, examinations, or any other kind of evaluation. It is a common belief that testing anxiety affects only older students, such as those taking high school or college placement examinations. However, testing anxiety affects students of all ages. In fact, studies have shown that test anxiety is actually at its worst in the middle grades. Even children in the first grade can begin to have test anxiety. If activities are such as ‘mad minutes’ or ‘rote memory activities to memorize facts and procedures’ are done too often in the classroom, then students begin to develop test and mathematics anxiety. The attributes of tests that can invoke higher levels of anxiety include the use of ‘none of the above’ as a multiple-choice option, distorted pictures (picture is not drawn to scale) with word problems, and the presence of extraneous information in word problems ([Hembree, 1990](#); [Ma, 1999](#)). Being asked to do something you are not comfortable with is very likely to create anxiety and depressed performance. So, within reason, we should try not to impose arbitrary time restraints on our students while they work on mathematics problems. As mentioned earlier, we appreciate the need for pragmatic management of the components of mathematics lessons, but it is definitely useful and in students’ interests to find alternatives which are not counter-productive or controversial.

E. Psychological Perspective

Any anxiety is a mental state characterized by mild to an intense sense of tension, worry, or apprehension relative to something adverse that might happen in the future. Fear is an intense biological response to immediate danger, while anxiety is an emotion regarding things we think may happen. Thus, fear may be experienced when a mathematics exam paper is placed in front of the learner, whereas anxiety will be the feelings about an impending

event, that the result of this examination will be an anticipated judgment about their performance on this paper. Therefore, the recurrence of the anxiety and the concomitant behaviors. However, the symptoms of mathematics anxiety are the same as symptoms for general anxiety—feelings of fear, worry, frustration, task avoidance, fidgetiness, dilation of pupils, restlessness, sweating of palms, etc.

Researchers differentiate aspects of any anxiety into *state anxiety* and *trait anxiety*, respectively defined as a more transient reaction to an adverse situation, and as a more stable personality attribute in experiencing such events. *Trait* and *state* anxiety are psychological responses akin to nervousness, worry, and fear. Anxiety generally brings with it a sense of unease about something with an uncertain or negative outcome. It causes both physical, emotional, and cognitive symptoms. Physically, it causes an increase in heart rate and blood pressure, as well as a display of symptoms of increased nervousness, poor learning and performance. Anxiety also stimulates our ‘fight or flight’ response releasing *adrenaline*—a hormone—into the bloodstream and usually has a negative effect on one’s learning capacity and performance. Recent neurological research shows that brain maps distinguish structural and functional features and aspects of these two types of anxieties.

Trait anxiety refers to anxiety as a characteristic of a person, so that the person is generally anxious about unknown outcomes; it is a part of who the person is. It is a *personality trait*. People have various levels of trait anxiety, and an athlete or a stage performer with high levels of trait anxiety will need to develop mechanisms to control the levels of anxiety that arise with any performance. In the context of mathematics anxiety, we will call this ‘*global mathematics anxiety*.’

State anxiety refers to anxiety that arises in a particular situation. Everyone experiences state anxiety, but the stimulus can vary. In sport, state anxiety may rise when an athlete is in a high-pressure situation and is called upon to perform. Examples of this include serving for the match in tennis, taking a penalty shot in soccer, converting a try in rugby, or shooting a free throw in basketball. In mathematics, it is a feeling of difficulty or fear when

one is called to solve a mathematics problem for which either one has a negative history or lack of preparation. We call it *specific math anxiety*.

One of the authors used to offer a course at the university that was open to anyone who was afraid of some aspect of mathematics. On the first day of the course, students were asked to write their mathematics autobiography. Almost all students would mention they were never successful in mathematics, at least in some aspects of mathematics. They would emphasize that they had a fear of mathematics. Many of them would identify a specific moment or the year or topic when they felt they became afraid of mathematics. They would begin the course with number concept and progress through all the developmental milestones of arithmetic. Every time a developmental milestone that gave difficulty to a student in the class was understood, the author would get the statement from that group: 'Now this makes sense. I did not realize that it was not difficult.' Generally, the person would hardly complain about the later work. Over time, the professor started sorting them as persons of specific math topic anxiety.

The nature of the impact of math anxiety—the feelings, the nature of expression, and the impact on learning and self-concept—varies with the individual. It begins with the person when they begin to feel uncertain and inadequate to deal with the quantitative situation/a challenge—an uncertainty. The person feels inadequate as they do not know and think, 'How should I deal with something that I do not understand that I have to do and that, despite my efforts, will most likely be unsuccessful?' As a result, they avoid even attempting the task.

For example, mathematics anxiety, up to about age ten, is only seen as a host of symptoms of arithmetic difficulties or disabilities. Up to this point, in many cases, it may not have been internalized, and as a result, it may not have yet affected the child's self-esteem. However, once the feelings of mathematics anxiety and the related attitudes about mathematics have been internalized, it becomes *a causative factor for more anxiety and lower mathematics achievement*. Then, because of mathematics anxiety, the learner does not do well and avoids mathematics.

In many cases the mathematics anxiety is triggered by personal characteristics or an consequence of early failures in mathematics which could have been due to a variety of factors: early introduction to difficult abstract concepts, inefficient instruction or poor instructional materials, low level of mastery expected and achieved so that the information is not internalized.

Most cases of math anxiety are examples of state anxiety, as the person may not display any anxiety in other aspects of his life, including other academic areas. We call it *specific math anxiety*. In this context, the anxiety becomes evident when one faces timed tests, oral recital of facts, an examination, or situations such as being called upon to demonstrate the solution on the board in public. It happens only in particular concepts, procedures, or tasks, for example, multiplication tables, long division, fractions, operations on signed numbers, geometrical proofs, solving algebraic equations, word problems, etc. when the learner is exposed to judgement of their proficiency and successes. People have reported specific instances and places in their mathematics careers. For example, most people express specific topics, and these topics generally are transitions from one developmental milestone to another. For example, number concept to additive reasoning, from additive reasoning to multiplicative reasoning, from multiplicative reasoning to proportional reasoning, and the beginning of formal algebra and geometry. One might say: 'Subtraction created special problems for me, particularly, when there were zeros in the top number.' 'I could never master all the multiplication tables properly. I tried all kinds of tricks, but to no avail.' 'Long division is still a problem for me. Why do they teach it?' 'I never understood the operations on negative and positive numbers.' 'And "x" is still a mystery for me. Algebra is an enigma to me.' All these statements are indications that every one of these students found some aspects of mathematics, particularly some developmental milestone of numeracy learning, anxiety-producing. If not resolved and the difficulty persists, it turns into math anxiety.

In some individuals, mathematics anxiety is a problem of the 'secondary type.' One may have learning difficulty or disability in mathematics or

dyscalculia, and the fear of mathematics may be the result of these conditions. Some dyscalculics and dyslexics can suffer from it. Sometimes a student has no disability but for socio-cultural reasons, poor teaching, etc. may do poorly in arithmetic and mathematics and may develop math anxiety.

On the other hand, some students would say, ‘I understand this and I have even mastered long-division, but I know I will not be able to handle the division of decimal numbers.’ Their mathematics anxiety only shifted from this topic to the next topic. The author started categorizing them in the ‘global mathematics anxiety group.’ We compare this behavior with the condition that when a person with trait anxiety experiences mathematics anxiety, it may not be just due to mathematics factors; they may display anxiety in other areas also. Thus, their anxiety is much more of a global nature to the person, and we want to differentiate it as *global mathematics anxiety*.

The nature of math anxiety—*global* or *specific*—depends on whether it has an emotional or cognitive base. For example, when a person is anxious, to start with anything can trigger anxiety (e.g., a demanding curriculum, a fact, a task, or a question), they may become anxious about the outcome. A person may be anxious about many situations, but mathematics tasks, because they demand accuracy and a certain expected speed of response, create especially anxiety-provoking learning situations. Similarly, a person with developmental cognitive delay or lack of logical reasoning may find mathematics tasks difficult. They may experience consistent failures and develop anxiety. This kind of anxiety is termed *global math anxiety*. On the other hand, if the cause of failure and resultant anxiety is due to unsuccessful experiences in one particular area of mathematics, the person may develop specific math anxiety. But the person is not globally prone to anxiety. Generally, their anxiety is a very focused one with difficulty with a particular concept, procedure, or an idea. Thus, trait anxiety may convert into global mathematics anxiety and state anxiety into specific mathematics anxiety. For example, people with general (state) anxiety respond to mathematics learning differently, as mathematics, in particular situations,

demands an exact answer, and these individuals are fearful of being wrong (a fear of negative evaluation) and they become anxious.

There are similarities and differences in the impact on mathematics learning due to the two types of anxieties: (a) specific math anxiety (state anxiety) developed due to problems and difficulties with specific mathematics developmental and conceptual milestones, and (b) global math anxiety developed due to trait anxiety created by personal and socio-environmental factors. This type of mathematics anxiety could also be termed a social-cultural type of mathematics anxiety.

If the basis of mathematics anxiety is emotional, we will have to significantly improve that student's self-esteem by providing successful experiences so that they may begin to participate in quantitative experiences and do better in arithmetic. On the other hand, if the cause of math anxiety is due to failure in a specific area of mathematics (i.e., difficulty in one or more of the developmental milestones in mathematics), we need to focus on and work on those mathematical ideas.

F. Mathematical Perspective

There seems to be a culture around the way numeracy and mathematics are taught, whether in school or at home. Some of the issues that act as a barrier are embedded in the current culture of mathematics, such as the necessity of having instant recall of basic arithmetic facts, such as multiplication tables, remembering steps of procedures, working and thinking quickly, resulting in timed tasks and tests. Being asked to do something you are not comfortable with is very likely to create anxiety and depressed performance. Where there are alternatives, as will be discussed in this book, these beliefs should be carefully evaluated.

Mathematics learning and the related difficulties do create their own stresses and anxieties in most people. For example, people with trait anxiety show mathematics anxiety earlier in demanding mathematics situations. And anxiety often shows up during the performance of any 'difficult' task (perceived or assumed) especially under testing, demanding contexts, and

social conditions where poor performance may have consequences—personal and/or social)—but there are, of course, exceptions. Difficulty in a task is defined by the means of: (a) the presence of many components—sub-tasks, prerequisite skills, primary concepts vs secondary concepts, etc., (b) the complexity of the sub-tasks, multi-concept and multi-step procedures, i.e., long division, systems of equations, word problems, etc., (c) lack of familiarity with the task and its components, (d) the cognitive complexity of the tasks and sub-tasks, i.e., subtraction problem: for example: $10,043 - 9.87 = ?$ (Involvement of re-grouping, zeros in several places, etc.), and (e) the seemingly irrationality of procedures, not being able to understand a procedure or an algorithm.

Some researchers on anxiety and performance argue that anxiety causes worry, and worry beyond a threshold always impairs performance on tasks with high attentional or short-term memory demands. As mathematics tasks demand close observation of tasks, intentional attention and focus, and sustained engagement with the task, many ADD and ADHD children become unsuccessful in mathematics tasks. As a result, they may develop mathematics anxiety.

It is also not true that students who have had difficulties in arithmetic are the only ones who have difficulty in learning algebra. Students facing difficulty in algebra are not just those who have done poorly in elementary school or those who have intellectual or cognitive deficits. Some students who are intellectually quite superior also face difficulty in algebra if it is presented either earlier than they are ready for it or presented purely abstractly as collection of unrelated procedures or tricks, and not related back to the numeracy that they have mastered. For example, in many schools, superior mathematics students from seventh grade are given the opportunity to study algebra in the middle-school as the first course in an accelerated mathematics program that culminates with advanced placement calculus (AP Calculus) toward the end of high school. Unfortunately, the first course in algebra proves to be a stumbling block for some students whether the course is accelerated or regular in nature. Unsuccessful students in the accelerated course either drop the algebra course and revert to regular

eighth-grade mathematics or complete the course only to have to repeat it in the ninth grade. In either case, intelligent, previously high achieving students experience their first trauma of failure or underachievement in mathematics. These students experience their first bout of debilitating math anxiety.

Developmental interventions that teach more than just the recall/access to a fact or procedure will build more enduring success with mathematics. Once again, it is often a matter of ‘what else am I teaching?’ So, within the pragmatic management principles, effective teachers do not impose arbitrary time constraints on their students while they work on mathematics problems. The strategy of changing the color of the pen, or pencil, after the time limit on a timed test, so that they can see the full extent of their efforts. They appreciate the need for pragmatic management of the components of mathematics lessons, but these teachers find alternatives which are not counter-productive or controversial.

A more sophisticated metaphor construct, metacognition, can be a factor in curtailing the impact of math anxiety. The seminal book *How People Learn*, ([Blandford et al., 2000](#)) has, as its third Key Finding, is that a metacognitive approach to instruction can help students to take control of their own learning by defining learning goals and monitoring their progress in achieving them.

The way we approach problem solving can be very diagnostic in the way that we focus on the perceived details, or whether we focus on getting the ‘big picture.’ Teachers should be aware of this in terms of students who (usually quickly) form an overview of the task/problems and are less drawn to details as opposed to those whose focus is on the details, often at the expense of not seeing the big picture (cognitive style/mathematics learning personality) ([Sharma, 1989](#)). Both thinking styles have their advantages and disadvantages and can be very influential for successful problem solving ([Bath et al., 1986](#); [Sharma, 2008](#)). The nature of a person’s mathematics anxiety is reflective of a person’s cognitive and affective profile. We consider metacognition to be an important component in problem solving and in the way that teachers explain mathematical concepts and processes to

their students. Metacognition thus acts as an antidote and a possible help for reducing math anxiety ([Roberts and Erdos, 1993](#)).

All of us make mistakes, and often our students also make mistakes. When they make mistakes and they do not learn how to correct them, they do not just get discouraged. Often, they will give up before starting because they shut down emotionally. Such emotions about making mistakes, feeling defeated, and experiencing failures result in students experiencing math anxiety—the feelings of fear and tension when doing math.

To reduce students' math anxiety, we need to learn different strategies for reacting to student mistakes, errors, and failures. Here are three common types of student errors and what we do in response.

Most of the errors in numeracy occur whenever the student shows deficits in the integration of the principles of subitizing, decomposing/recomposing, and recognizing whole-to-parts and parts-to-whole as related to number concept, numbersense, and numeracy because these elements are developmental and conceptual.⁴

Precision Errors

Precision errors are minor mistakes that occur when students don't read, write, speak, listen, or type carefully. These errors include arithmetic mistakes (such as saying that $3 + 4$ equals 6 instead of 7), transposition of numbers (dividing by 42 instead of 24), incorrect symbol use (using the variable $+$ instead of $\times$), incorrect rounding (rounding 36.5 as 36 instead of 37), or calculator mistakes (typing '233' instead of '223').

Precision errors also include misreading a word (thinking that a measurement is in feet instead of miles) or ignoring a word (not seeing the word 'not' in the question 'Which statement is not true?').

When students make precision errors, we ask them to look at their work and the original question to see if their work makes sense. Modeling a self-dialogue helps students to see how they should review their work and identify potential mistakes. We try to limit future precision errors by periodically giving assignments in which students only get credit if they get

the correct answer with no mistakes in their work. Online tools or paper assignments like vocabulary quizzes where the exact definitions are required help our students pay closer attention to their work. Reading their work out loud may help recognition of mistakes. Sometimes, these errors may also be due to visual perceptual integration.

Conceptual Errors

Conceptual errors happen when students do correct procedural work but apply the wrong concept (such as finding the area of a rectangle by adding its length and width instead of multiplying them). Students also make conceptual errors when they misinterpret words (such as thinking that an isosceles triangle has three sides of equal length instead of two).

Since conceptual errors usually occur when students misunderstand an idea, we should spend more time and effort to address them than precision errors. Diagnosing errors can be very helpful in targeting intervention and remediation. When students make conceptual errors, we encourage them to talk through their reasoning using a variety of methods. For example, we ask students, ‘Which method works best here?’ or ‘How would you grade this work?’ and implement the ‘think-pair-share’ strategy, in which students think silently on the question before sharing their thoughts with a partner and finally sharing with the rest of the class.

Conceptual errors also occur when there is an emphasis on procedures without the development of the linguistic and conceptual components of a mathematical idea.

Processing Errors

Processing errors take place when the language or notation of a problem is so dense that it requires a great deal of specialized knowledge or reading-comprehension skills to understand it, such as locating an improper fraction or a decimal number on a number line.

Unlike conceptual errors, in which students generally understand a problem but miss one important idea, students who make processing errors

often struggle just to get started. To help students process complex mathematical text, we teach students to use techniques that are familiar to many English language learners.

For example, we teach students to read the instructions several times. We want students to read the text of a word problem at least three times, each time focusing on a different aspect of the problem. The *first time* is reading for meaning, comprehension, and understanding the text and the context of the word problem. The *second time* is to know the ‘knowns’ and ‘unknowns’ and determining the ‘objective’ of the problem. During the *third reading*, one identifies and defines the unknowns and then translates the English language into mathematics language, relating the unknowns and the knowns into equations, expressions, or formulas, and then solves the equations and identifies the answer. In the *final reading*, the student checks the answer. We encourage students to annotate text, interacting with it so they can understand and remember concepts. We model different methods for students to rewrite text into simpler English or a figure or a diagram representing the knowns and unknowns before translating it into mathematical symbols.

All of the strategies we use to address mistakes focus on strengthening conceptualization and mathematical communication skills for all learners. When students become more comfortable reflecting on their work, they express themselves more precisely and accurately. As a result, students become more confident and make fewer errors, and the math anxiety begins to have a reduced impact.

As mentioned, anxiety characteristically impairs efficiency more than effectiveness. Impaired proficiency with a task consumes more mental resources and as a result fewer tasks are performed in each time resource, thereby decreasing task satisfaction and increasing fatigue. Inefficient, ineffective strategies create less success and more fatigue. Fatigue—both physical and cognitive, causes cessation of activity and then procrastination and avoidance of the task. Students have used terms such as: ‘My brain hurts.’ ‘It takes so much out of me.’ ‘It tires me.’ ‘It is so boring. I do not

want to do this.’ And the student wants to curtail the activity and withdraws from it. They may not return to the task soon or ever.

G. Pedagogical Perspective

Ben has just entered third grade and is having a great deal of difficulty in numeracy. Although his mother is a middle school math teacher, she cannot understand what she should do. Ben and his mother are both overwhelmed by his problems. He refused to do any mathematics work with her and in the classroom. Every second sentence was: ‘I hate math.’

Ben came to the Center with a diagnosis of ADHD, dyslexia, dyscalculia, poor executive functions development, and dysgraphia. After evaluation, I asked him what he would like to learn. He said: ‘Place value.’ I asked him to write the number I am going to say. I said: write: ‘One hundred forty-five.’ He wrote 10045.

Obviously, he has not mastered place value even at first grade level. I asked him: ‘Do you know the alphabet of mathematics?’ He said: ‘Numbers.’ ‘Give me an example,’ I asked him. He said: ‘All the numbers.’ I asked him: ‘What do you use to write words?’ He said: ‘Letters.’ ‘What are the letters of Mathematics?’ I asked him. He said: ‘I do not know.’ ‘Would you like to know?’ I asked him. He said, ‘Yes!’ I said the alphabet of mathematics are digits and they are used to write all the numbers. ‘How many are there?’ He sincerely and inquisitively asked. I said: ‘There are only ten digits: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} to write any number.’ He said: ‘Write a big number. At least four digits.’ I wrote the number: 1,435. I asked him: ‘What digits make this number?’ He said: ‘1, 4, 3, 5.’ I said: ‘Let us make the numbers using these blocks.’ I pointed to the Cuisenaire rods and Base Ten blocks. Within about 15 minutes of playing with them under my direction and a great deal of questioning, he derived the values of these blocks and made the number 1,435. He placed the blocks on the table, and I asked him to write the digits in front of those blocks. Then the following script developed.

Author: What digits make this number?

Ben: 1, 4, 3, and 5.

A: What does each digit represent? Just like the sounds represented by letters in the word.

B: Digit 1 is 1000, 4 is 400, 3 is 30, and 5 is 5. And every digit before the comma is in the thousand family.

A: Please write: 'Fourteen thousand four hundred thirty.'

B: I know 14 will be before the comma and then 432. The number will be 14,432. It will be made up of digits: 1, 4, 4, 3, and 2. Value of 1 is 10,000, 4 is four thousand, the next 4 is 400, and 3 is 30 and 2 is 2. It is like spelling. You have to break the number into its parts and sound out.

Ben is representative of a multitude of children who experience difficulty in learning, failures, errors; the resultant frustrations, helplessness, and resignation about further learning ends up with hating mathematics and bad math anxiety from an early age. We need to look at the phenomenon of math anxiety from the pedagogical perspective.

It seems undeniable that we need to make learning interesting, successful, secure, and to achieve this with minimal stress. This depends significantly on how teachers communicate with their pupils and what kinds of strategies they use in their instruction. Tasks should be interesting, engaging and successful. The new learning should be 'storage rich' and 'retrieval rich.' We need to minimize any fear of negative evaluations, in whatever forms it is transmitted.

The objective of this book is to show teachers and parents the most effective ways of teaching the foundational skills of numeracy effectively, efficiently, and in ways that do not produce stress, fear, and worry of and about mathematics.

Negative evaluations need to be handled sensitively and in ways that do not reduce or, worse still, destroy motivation. Two negatives in the affective domain rarely make a positive.

The emotion and cognitive areas of the brain are highly interlinked, so emotional factors like stress, anxiety, happiness, and belonging need to be considered when thinking about ways to improve teaching. During our visits to kindergartens, we observe and teachers share that many children crave more assurance in their relationships with teachers. They want to sit by us, sit next to us, they want more encouragement from us. It seems that they are better engaged in learning when their belonging needs are met. Emotion uses many of the same areas of the brain that learning uses. Learning is improved when teachers consider students' emotional needs during their teaching and in developing the classroom environment. We have found success is the greatest ingredient in improving the classroom environment.

Anxiety and stress of any type of place heavy demands on visual and verbal memories and may also block their optimal functioning. For example, both visual and verbal working memory are significant factors in accounting for the variance in math performance measured broadly. Mathematics learning, including numeracy is highly correlated to visual working memory, as a result, math anxiety primarily impacts visual working memory, although some research indicates that anxiety is also processed in verbal working memory; therefore, it may also affect mathematics learning.

While mathematics is often considered a hard subject, not all difficulties with the subject result from a person's cognitive difficulties or deficits. Many children and adults experience feelings of anxiety, apprehension, tension, or discomfort when confronted by a mathematics problem related to very specific reasons, such as developmental transitions of mathematics ideas and the interaction of language and abstract concepts as seen in applications and word problems.

H. Testing for Anxiety

When designing any test, it is good practice to be clear about the objectives and characteristics of the test; for example, are the responses it generates

reliably quantifiable? Is the test to be an assessment (focusing on a score), a diagnosis (focusing on why the answer was correct or incorrect), or both? In what way will it inform and help a teacher, a parent, a psychologist, and through them the learner? We have included an ‘Mathematics Anxiety Test’ ([Chinn, 2017](#)) here. It is not a test that you pass or fail; it is, we hope, used for a diagnostic purpose. A good test ([Suinn, 1972](#)) can initiate diagnostic discussions and lead to appropriate interventions, remedial, or reframed thinking about the problems and its causes; therefore, it can provide a new approach to instruction. For example, The Modified Abbreviated Mathematics Anxiety Scale: A Valid Reliable Instrument for Use with Children by [Carey et al. \(2017\)](#) is one such test.

The learner is asked to grade each item on a 1–4 scale, so in that sense it is quantitative (of course, based on qualitative decisions), but the information it can provide will be expanded considerably by empathetic dialog between the learner and the tester. That can change an evaluation, an assessment, into a diagnosis.

The favorite example of one of the authors was a 14-year-old male who quickly rated every item on the test as a ‘4’ except the last one (20), ‘Taking an end of term mathematics exam,’ which he rated as a 1, that is, ‘never anxious.’ The author asked him in a manner that he hoped was calm and neutral why he gave this item such a low anxiety score. He looked at the author and smiled, saying, ‘The key words are end of term.’ A smart young man!

We believe that for every assessment/evaluation to be diagnostic, it should have a face-to-face component or as close to a ‘face-to-face’ setting as possible. Setting the ‘right’ atmosphere for a diagnosis is important and will have a large impact on the knowledge one gleans from the learner. For example, even in a testing situation (without face-to-face), the assessment tool should be designed in such a way that it does not become threatening from the start. That is one of the main reasons that the authors start every protocol ([Chinn, 2017](#); [Sharma, 2008](#)) with low-stress items which are not harshly judgmental and where an answer can be acceptable and open to

genuine discussion and garnering of information. The following anecdote makes the point.

One of the authors had a presentation on strategies for basic facts to a large group of teachers, using the ‘death by Power Point strategy.’ The computer technician who watched over the author with care and with a positive attitude said to the author, ‘When I was at school, I used to work out 3×6 by halving 6×6 and all the facts of fives table by halving the related 10s table facts.’

Although it is not the most efficient and elegant strategy, the person had found a way of mastering something; it should be encouraged and if the opportunity and need presents itself or during the initial teaching, one should give the most effective, efficient, and elegant strategies. The author said to the technician, ‘That is good. Did your teacher know this?’ The technician said, ‘She never asked me or anyone else for our strategies.’ We think that is very sad if we do not have an ethos where pertinent and constructive questions can be asked and positive discussions can be held, we not only lose the first of three Key Findings in the major research study in the United States as ‘How People Learn’ ([Blandford et al., 2000](#)); in such classrooms, many children develop an apathy toward and a fear of mathematics. There are certain characteristics of teaching where these situations happen.

I. Interventions

1. Instructional Suggestions

In the case of many anxious students, it is difficult to distinguish between numerical anxiety, content-specific mathematics anxiety, mathematics anxiety due to one’s psychological profile (anxious personality), and mathematics test-taking anxiety. All of them require specific approaches to deal with the problem. For example, male and female reactions to general anxiety and mathematics anxiety are different and so is their impact, and the ensuing behaviors are quite different. Female students also face an extra

factor of ‘social stereotypes’ of female performance on mathematics. Therefore, the professional interventions should consider these factors in their planning and executions.

In dealing with people with numerical and mathematics anxiety, we believe that we should take a tripartite approach: *preventive*, *interventional*, and *remedial*. The preventive approach is to have strong and meaningful approach to initial teaching, from the very beginning—developing prerequisite skills for mathematics learning (more on this in the later chapters) and appropriate introduction of new topics—language, conceptual schemas, procedures, and adequate and proper practice of the learned materials in non-threatening and number-friendly classrooms. This will reassure students not to fear mathematics.

Mathematics as a subject has a certain ‘mathematics culture,’ and that influences learning. The established culture of numeracy and mathematics, especially where counting (fingers, hash-marks, objects, number line, number charts, etc.) is the only method of finding the answers, inefficient or no strategies, speed of response, emphasis on answer only, teaching the ‘terminal student,’⁵ and the impact of ‘quick answers’ have a negative impact on learning for many students. Several of the components of this culture can be counterproductive and have a negative influence on learning numeracy and mathematics and are conducive to creating numeracy anxiety and then math anxiety.

Finger counting⁶ is frowned upon, criticized by many, and discouraged in many classrooms, for good reasons, but the alternatives suggested are either poor or not superior to counting. Most of the book will be devoted to suggesting more productive strategies. In the interim, to be pragmatic, as in the paper by Jay and Betenson (2017), we also suggest that there are some positives in using finger counting, but all is not there for teaching numeracy. Use of counting should be to teach decomposition/recomposition of numbers. As shown earlier ($7 + 7 = ?$), 7 can be visualized as $5 + 2$ and that, in turn, as all the fingers on one hand, plus 2 fingers on the other hand, makes $7 + 7 = 5 + 5 + 2 + 2 = 10 + 4$ easily, even mentally, thereby improving ‘mental math.’ So, using ‘appropriate

chunking’ (another name for decomposition/re-composition) can help rather than counting in ones, which is rarely developmental.

Teachers should be aware of WEAYT: *What Else Are You Teaching?* Because mathematics is so developmental, teachers should keep in mind the trajectory of the concept, procedure, and ideas at hand and consider: where is this topic leading? Which future topics is this a key foundation for? What connections should we be making with this concept?

Authors, in their work with teachers and students, have always made them aware of cognitive styles ([Bath et al., 1986](#)) and mathematics learning personalities ([Sharma, 2008](#)) which helps them become more aware of and have an appreciation of the use of asking, ‘can you think of another way of doing that?’ Knowing one way is enough initially for many learners, especially those who are anxious. The prospect of having to find ‘another way’ can be daunting and unnecessary when one’s perception is that ‘one way is enough.’ But seeking and striving for another method—more efficient, more elegant, more effective—is what makes brain plasticity happen.

For effective teaching and success for all students, a sequence of activities by the learner and the teacher can play an important role in improving learning. The sequence is:

*Planning → Mindful Intention → Attention → Focus → Proper Practice
→ Testing (self and administered) → Error Analysis → Feedback
Review → Challenging tasks → Make it a Habit → Create a Script
→ Retrieval*

The application of this teaching and learning loop describes the structure of an effective lesson and learning from initial teaching to remediation.

Apart from the content-related interventions and remedial strategies, there is a need for changes in affective aspects of learning to create more and frequent success for these students. Many research reports set out a series of recommendations for teachers and parents. They are:

- be aware that math anxiety can affect students' math performance.
- be aware that their own math anxiety might influence their students' math anxiety.
- be aware that gendered stereotypes about math ability might contribute to the gender gap in math performance. Students—particularly girls and students of color, who experience stereotypes of poor math performance—have also been found to be more likely to develop math anxiety over time if they are taught by math-anxious teachers.

Teachers should be aware of the role of emotions on learning, and they should be aware of methods that reduce classroom pressure. These methods include free writing about emotions before a test to help alleviate test and math anxiety. They should:

- watch the expressions and body language of their students to spot any signs of anxiety.
- not be afraid to return to the development of basic skills and show their relevance and role in more advanced topics such as algebra.
- encourage students to share their thinking and methods while solving problems. Create that classroom culture.
- use constructive questions, expressed positively and empathetically.
- create a low-stress, open-discussion atmosphere in your classroom.

Researchers and educators from different disciplines have suggested a variety of interventions.

What generates the maximum anxiety in mathematics learning? We need to acknowledge that it is our mathematics behavior. No one likes negative feedback or negative evaluations. Evaluation without meaningful feedback, learning experiences without some success is a waste of time for the teacher and the student. Marks and feedback should not be demotivating or overly critical (positive criticism can be constructive and productive). The authors have performed thousands of evaluations/assessments and diagnostic workshops. It is one of our cardinal principles that the student should not

leave the session without learning something new that they feel enriched them in some way. Every teaching session should be an evaluation (a diagnosis), and every evaluation session should be a learning experience for the student and a teaching experience for the teacher.

We know from decades of teaching students with learning difficulties, at all levels of mathematics education, that teaching in ways that are as non-judgmental as possible, achieving ‘small successes’ on the way, helps to maintain motivation, intention, and engagement. It is often the perception that ‘the answer is the answer’ and that alternatives are not valid and that a close, but incorrect answer is still wrong. When a basketball circles around the rim of the hoop, it gives feedback to the player that he has come close and with a little more practice he may score a basket. The same attitude should be adopted in mathematics learning. Every step should provide feedback that tells the student you are getting close to ‘making the basket.’ We have realized that the greatest motivation to learning for anyone is success. The more genuine successes we can generate, the more willing our students are to persevere with mathematics tasks. Mathematics and mathematics teachers usually are very judgmental and prone to generating negative evaluations. For example, answering 15 to ‘what is $7 + 7$?’ And being told, ‘No. That is wrong.’ That response is closing the chapter on effort. It is much less motivating than a conversation that says, ‘You are close, but not quite correct. Do you think the correct answer is bigger or smaller than 15? And then, what do you think?’ And if the child said: ‘ $7 + 8$ is 15,’ then the comeback should be: ‘what do you think $7 + 7$ might be? More than 15 or less than 15?’ Invariably, a child would answer 14. Then you ask: ‘Why do you think so?’ After the child answers it correctly, you ask: ‘Could you find the answer by another strategy?’ If the answer is no, we need to make sure that the child knows a strategy that is based on ‘decomposition and re-composition’ of numbers, as it will become evident later on how important decomposition/re-composition is for arithmetic and many other mathematics operations.

2. Helping Students Beat Test Anxiety

The dominance of evaluation in mathematics is intimidating and demotivating for many learners. It can begin in very young children. If not remediated, it may last throughout life. Hearing ‘wrong’ too often is an inflexible and harsh judgment, leaving little room to alleviate its impact. As a result, students develop a fear of negative evaluation and then any evaluation, therefore ultimately facing failure. Of course, we need to know when we are wrong, but we also need to know why we are wrong and where the mistake is and how to correct it. It can be a constructive judgment, but it is rarely perceived as such. The consequence is avoidance of the task that can lead to the judgment in much the same strategy of avoiding situations that may lead to physical harm. The avoidance of specific situations may well permeate to avoiding any involvement in mathematics, which may include future math learning.

Another factor which is ingrained in the culture of mathematics is the speed of processing and answering questions. There is a general impression in the minds of many people that mathematics learning means acquiring speed in giving answers—facts, applying procedures, solving problems quickly, memorizing enormous amounts of information quickly. They also believe that every mathematics problem has just one correct answer. And there is one ‘optimal’ procedure for solving a problem. Numeracy and mathematics require fluency and understanding, not speed and rote memorization. The normal distribution applies to many factors and that includes the speed of processing mathematical data. A classic example is the demand to answer mental arithmetic questions quickly. A natural reaction to the demand to do something more quickly than you feel you can do is to withdraw from any involvement in the task. However, it is a misconception? on the part of most everyone that doing better in mathematics means speed and memorization. As mentioned earlier, many teachers teach numeracy either by counting or by rote memorization (whether cards, songs, apps, or computer-generated feedback), and in mathematics they teach with a focus on procedures. Most students, from such instructional situations, get misinformed about the nature of numeracy and mathematics. There is more on mental arithmetic in later chapters.

We should also realize that short-term working memory capacities are varied. The amount of data people can retain for a short time is not the same for all individuals. Teachers, or any person interacting with another, need to be aware when they are exceeding the short-term and working memory load of the other person. And they need to know that when short-term memory loses the data, it is irretrievable. Working memory is now recognized as a key component in mathematics ability. Again, the working memory capacity and its functioning varies among individuals. What is additionally critical is that relatively recent research has shown that this key component in mathematics ability is negatively impacted by anxiety. However, current research and our extensive experience shows that memory—all its components, from receiving information into the short-term memory to improving the input into short-term memory, enhancing functioning of the working memory to sending more information to long-term memory systems can be improved with efficient strategies. It is the coordinated effort that can maximize the functioning and the productivity of the memory systems ([Ranganath, 2024](#)). More on this on later chapters.

We think the worst thing one can do is to minimize students' concerns and feelings around testing and math anxiety. Students want to feel understood and that their input has value. Instead, we can help them put the assessment in perspective. Empathize with your students and tell them you understand their concerns. Talk about what they believe would be the absolute worst-case scenario if they fail the test. Often just by saying it aloud students will realize their fears are not as bad as they envisioned.

We believe that teachers should always tell their students that while they may not have power over everything about an assessment, there are actually a number of things that are within their control. An easy way to explain this is to create a visual concept map or web with your students, with ideas of things they can control that will help them before and on the test day. The day they are told about a test or examination, they should plan how to prepare for the test and follow through on it. On the day of the test or exam, they should have a routine: they can eat a good breakfast, get enough sleep the night before, study in advance, practice self-regulation strategies such as

breathing exercises, and organize notes and materials. Then, one can also work on teaching concrete test-taking strategies, such as eliminating multiple choice answers you know are not correct, systematically throughout the school year as mini lessons. These practices reinforce the idea that they are not helpless when faced with a test but can instead help themselves in many ways.

Sometimes teachers can add to student assessment stress by sharing their negative past academic experiences or current anxiety about their students' progress. It happens quite often in math classes for special needs students, where many special needs teachers would sometimes project their failures and negative thoughts about mathematics onto their children. One author's teacher used to say: 'No wonder so many students failed this test. It is a hard topic. When I was a student, my class used to do poorly.'

The objective is to illustrate to students that testing is not something to fear, but rather an opportunity to show what you know and discover where you might require some additional review to master the subject matter.

Finally, if you are giving out rewards, such as preferred activities or even items like stickers, make sure they are for *finishing the task at the level of standards that apply to all*, not based on artificial scores. One of the worst mishandling of rewards I have seen was when students completed their districtwide progress monitoring assessments and staff decided to give out a fun bracelet only to the students who had improved scores. So many factors go into student assessment performance that this is really not an equitable practice. Some of my students in special education were mortified to not earn a bracelet to wear around school the rest of the day, which led to more anxiety around testing. Instead, we want assessment experiences to conjure positive memories, so that students can have less worries in the future.

Anxiety, of any kind, does not 'look' the same for all students, so the teachers and parents need to know when to push students and when to empathize, when to listen, and when to set limits. It is important to reiterate that the entire point of assessment is not to measure who you are by a single score on a single day for a particular topic in math, but rather it is to be used

as an instrument to gauge progress, select interventions, and direct instruction for more optimal learning. Students are a heterogeneous group.

Although each student's needs and conditions are unique, there are some strategies to help students manage testing and math anxiety and math anxiety in general.

There are many adjustments that can be made in a classroom that will reduce anxiety. One of the goals of a teacher is to create a classroom culture that includes a variety of low-stress formative, constructive, and short assessments/diagnoses throughout the semester. A test should be a set of questions where the first three or four are straightforward and then the difficulty increases slowly. This is to reduce the behavior that does not support effort. When there is a sudden quantum leap in difficulty many learners give up, even though there may be subsequent questions that they could tackle.

Adjustments can be made to those classroom behaviors of teachers that can limit the escalation of anxiety. Largely these are rooted in understanding learner profiles: How do children at different stages learn? How does this child learn? Why do learning problems occur? What are the specific reasons for this child's learning problems and difficulties in learning in general, and what are the reasons for difficulty in this particular topic, problem, or situation? What are efficient strategies to make difficult concepts accessible and even easier, and adjust presentations accordingly.

Ensuring that accommodations are customized—for example, level of task, type of questions being asked, the amount of practice and praise are based on actual student needs. Not all students with IEPs need extended time or directions read aloud, for example. However, good teaching is not necessarily individualized teaching; it is teaching to meet the needs of individuals. There is a misunderstanding that individualization means one-to-one teaching. Children with and without learning disabilities, including dyscalculia and mathematics anxiety, should interact with other children as they can learn a great deal from them, if that interaction is well supervised and directed. The presence of those who are well-versed in the material provides examples of expert behaviors to emulate.

There is a beautiful śloka (a couplet) in Sanskrit about learning. It has always helped me (Sharma) understand the nature of teaching and learning.

Sanskrit (Devanagari):

आचार्यात् पादमादत्ते पादं शिष्यः स्वमेधया ।
पादं सब्रह्मचारिभ्यः पादं कालक्रमेण च ॥

Transliteration (IAST):

ācāryāt pādamādatte pādaṃ śiṣyaḥ svamedhayā |
pādaṃ sabrahmacāribhyaḥ pādaṃ kālakrameṇa ca ||

Meaning:

A *quarter* (of knowledge) is gained from the teacher (ācāryāt), a *quarter* by the student's own intelligence (svamedhayā), a *quarter* from fellow students (sabrahmacāribhyaḥ), and a *quarter* in the course of time (kālakrameṇa ca).

What are the implications of this to our situation? With instruction by a well-trained teacher, observing and interacting and discussing with other children functioning at different levels (from inferior to superior), self-practice of effective, efficient, and elegant strategies provided by an effective teacher, and with multiple applications, one learns effectively.

Help students to deal with difficulty, fear, stress, and anxiety effectively. A new study ([Roze et al., 2019](#)) highlights an effective solution: Guide students to view stress differently—as a boost instead of a burden. Simple 10-minute writing exercises given just before a test helped students see stress as ‘a beneficial and energizing force’ that could help them.

The writing exercises were most effective for disadvantaged students, who may be sensitive to situations that emphasize rank and status—a by-product of high-stakes testing. Their wealthier peers may have access to more resources like tutors who can boost their chances of success in school, and in comparison, students from low-income families may feel more pressure to do well because they have ‘less margin for error,’ which raises the stakes of failure.

This study uses a robust design to show that it is possible to reduce the anxiety associated with high-stakes tests in secondary school students. One key factor in reducing testing anxiety is increasing a student's sense of being in control.

Psychologists followed high school biology students for a year to study how stress affected their ability to pass major examinations. They noticed that low-income students were disproportionately harmed by difficulty regulating test anxiety. These students experienced 'worried thoughts about the possibility of failure' that became a self-fulfilling prophecy: Being stressed about failing increased the likelihood of failure. But 10-minute writing exercises that encouraged students to let go of negative thoughts, regulate their emotions, and reinterpret stress as a positive force helped them perform better.

In the study, two types of writing were assigned immediately before end-of-semester exams were taken:

Expressive writing. Students were asked to write about their thoughts and feelings about the exam they were about to take. They were also asked to write about other times in their lives when they had experienced worried thoughts.

Stress reappraisal. Students were asked to think about their symptoms of stress as helpful for test-taking. For example: 'If you find yourself feeling nervous or anxious while taking a test, think about how your body's responses can actually energize and help you.' Students also read a passage that explained how psychological responses to stress—a faster heartbeat and heavy breathing, for example—help improve performance by increasing oxygen flow into the brain, boosting alertness.

Both types of exercises proved to be effective at boosting student achievement, especially that of low-income students. (Both writing prompts are available online.) The achievement gap between low- and high-income students decreased by 29 percent, and the course failure rate for low-income

students was cut in half, making this intervention a potentially valuable tool for increasing equity.

3. Role of Stress and Its Remediation and Mathematics Anxiety

It is commonly believed that stress can be a cause of mathematics anxiety or other debilitating conditions for learning. Yes, frequent undue stress and distress can be the cause of anxiety and mathematics anxiety. However, stress is not always bad. Stress and its impact falls on a continuum, called a *stress continuum*. The stress continuum can be divided into four regions: *No stress*, *eustress*, *stress*, and *distress*.

Psychologists find that the *first region* on the stress continuum with *no stress* to the individual is not conducive to learning. When there are no expectations, no responsibilities, no demands of any kind, no decision making, then there is no motivation, no engagement, no innovations, no taste of success, and no creativity. Then there is no stress. Therefore, there is no desire to take any initiative and no engagement. There is no engagement of the brain, no decision making, no exercise for different components of the brain. You are not being challenged and have no incentive or motivation to go above and beyond. There is no cognitive growth. This means little or no learning. Too little or no arousal (no responsibilities, no accountability, no stress, no rewards, etc.) doesn't provide much in the way of motivation. Having no stress at all is not necessarily a good thing in terms of mathematics performance or any performance. With no surprises, only routines, then there is no arousal of curiosity, no interest. There is no stress. Then there is no learning or growth, either. For example, when classroom teaching is all about routines and nothing ever changes, there is less focus and then boredom sets in. Activities feel meaningless, and have no purpose so the student does the bare minimum or nothing. Having no stress at all is not necessarily a good thing in terms of mathematics performance, or any performance.

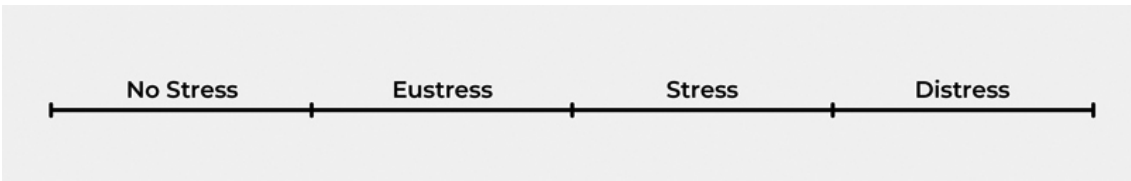


Figure 2.1 Stress continuum

Students frequently discern unspoken teacher and parental expectations and can influence both teacher and student behavior in class. Of course, they can come from many sources and may create conflict. Therefore, communication skills are not just about giving students information, but about the way it is given and the conduct of any subsequent discussions or questions.

Expectations can be expressed in multiple ways, with resources—support, effective and efficient strategies, moderately challenging, but accessible problems to solve, entice engagement. However, to expect a student to perform where the task is not accessible and difficult can create unrealistic expectations and unacceptable stress, therefore, anxiety. When the task is too easy, meaningless, and places low demands on intellect/cognition, it is not conducive to engagement, to cognitive growth, to innovation, or resilience. Therefore, there is no progress in learning.

However, we must remember that nothing works for everyone. We like Chinn's use of the made-up phrase 'unanxious expectations.' On the other hand, for some children the strategy of weakness, of incompetence, of impotence, may be a useful strategy to avoid the evaluation of performance. For after all, if they (meaning we adults and teachers) know that you can't do anything they won't expect you to do anything and they will not blame you or punish you for not being able to do what you have been told to do. The same strategy makes some children what we call 'answer grabbers.' The teacher asks a question and the students answers it by giving a 'bizarre' answer. The teacher moves on to another student; the student feels that he has avoided the situation and next time the answer is even more bizarre. Over a period, the student has not learned much in the class and the work becomes overwhelming and the student truly gives up for ever.

Meaningless expectations range from statements such as, ‘Your sister was good at math, so why aren’t you?’ to those star charts on the wall that are meant to motivate all learners. When there are expectations and there are meaningful resources, such as effective and efficient strategies, meaningful questions, and feedback and support, then they can entertain moderately challenging problems, if they are accessible. Students, both young and old, who depend heavily on adult approval may decide that, if they cannot have total success, their next best thing is to have total failure. That is a worrying consequence of a need for predictability and consistency. And inconsistency can be a powerful negative influence.

The *second region* to the right is named by German psychologist and psychiatrist Hans Selyé as *eustress*—a region of ‘good’ stress. Eustress refers to the state where there is necessary stress to be attentive and responsive to the task and people. Eustress is positive, desirable stress and this level of stress acts as an arousal mechanism. Many accomplished orators, athletes, public speakers, teachers, and politicians have expressed that every time they are in front of an audience to perform, there is an element of ‘stress’ that keeps them alert and attentive. This is good stress. Too little or too much arousal results in poorer performances.

Effective teachers, with their understanding of the students—their weaknesses and strengths, the content—its language, conceptual schemas, effective strategies, and trajectory, with effective teaching skills, and with proper tools keep on extending the range of eustress in both directions—to bring some from the left of the no-stress zone to eustress and help students’ tolerance for a little more stress to the region to the right, but do not cross the boundary between eustress and stress.

Being or remaining in the stress zone, the *third region*, begins to create affective problems for the students. They begin to be disengaged and disinterested in the subject matter, the process and the experience. Boredom, anxiety, fear, and withdrawal from the activities begin to take place.

When a teacher creates these situations, the student crosses into the *distress* area, the *fourth region*, the student begins to be angry, oppositional,

belligerent, and even threatening and possibly violent—verbally or even in actions.

Why would stress be beneficial at all? Researchers hypothesize that it played a key role in early humans' survival. It helped us not only to identify threats but also to take advantage of opportunities. We tend to think of stress as a reaction to negative events, and while this is often the case, *some stress does help sharpen our focus*. Feeling nervous about meeting someone famous? The butterflies in our stomach before giving a presentation? Those feelings are telling you, 'This is important, so pay attention.' That is difficult to pinpoint and will differ from person to person. A mild to moderate amount of short-term stress can result in an acute stress response that provides the motivation and energy one needs just when one needs it. It lasts only long enough to help one perform at one's best.

While chronic stress can cause serious health problems, short-term and limited stress can be beneficial, boosting attention and memory performance. In 1908, psychologists Robert Yerkes and John Dodson conducted a simple experiment on mice that laid the foundation for understanding the role between stress and learning. They observed that for challenging tasks, a moderate amount of stress is ideal: Too little leads to apathy, but too much impairs performance. In the beginning of every classroom session, lesson, or presentation at a conference, the authors, with almost 70 years of teaching experience, have always felt a bit of stress that they feel brings their total attention to the task and makes the presentation better.

The Yerkes-Dodson law in psychology models the functional relationship between the variables: *arousal* and *stress (anxiety)* as independent variables and *performance* as the dependent variable and task performance. Yerkes and Dodson's research into the relationship between stress and learning has shown that low levels of the stress hormone cortisol enhance memory—high levels impair it, including in humans. More recent research ([Corbett, 2015](#)) under more rigorous experimental conditions also found that low levels of the stress hormone cortisol enhance memory—and high levels impair it. It states that there is an ideal level of arousal that results in

optimal performance. Long-term exposure to cortisol damages cells in the hippocampus; this damage results in impaired learning and memory. Furthermore, high levels of cortisol inhibit memory retrieval of already stored information.

For productivity and learning, low to moderate stress is the right amount of stress. It is manageable, motivational, and performance-enhancing. One's heart beats a bit faster. One feels a sense of clarity and alertness. The brain and body are all fired up. It is that little extra push you need when a hard deadline looms or quiz is coming up. There is something the person wants. One has engagement with the task, and the person has definitely got skin in the game. (A US saying!) A moderate surge of stress boosts one's performance. One needs just enough stress to provide motivation, but not so much that one is overwhelmed.

Students should feel a sense of responsibility, expectation, and attentiveness in the classroom that creates a sense of *eustress* that is productive. The teacher should not cross this threshold; otherwise, she may enter the region of *stress*, then students begin to feel anxious about the content and the lesson. Students begin to lose engagement and interest wanes and disinterest in learning sets in, which leads to poor performance. The repetition of this poor performance creates the conditions of mathematics anxiety. When the classroom environment crosses from the 'region of *stress* to the region of *distress*,' some students may become very anxious, feel threatened, and their behavior may become appositional, belligerent, or even threatening. On the other hand, some may become withdrawn, closed down to further learning; they might even forget the earlier part of learning.

Therefore, only a mild to moderate amount of short-term stress can result in an acute stress response that provides the motivation and energy you need just when you need it. It lasts only long enough to help you perform at your best.

Psychologists also suggest the reframing of stress to help people be more productive. For example, receiving messages from counselors: 'That pounding heart is preparing you for action. If you're breathing faster, it's no

problem. It's getting more oxygen to your brain.' They have developed activities to maximize the impact of this reframing of stress conditions.

4. Transforming Stress

In the classroom, *math test anxiety* can be pervasive, affecting students' ability to perform at their full potential. But teachers can take a few steps to help students beat test anxiety:

Step 1: Recognize that tests measure more than just academic ability.

They also measure how much test anxiety a student suffers from.

Step 2: Reframe stress. It's not a burden but a way of energizing the body. Consider how top performers, from athletes to musicians, deal with stress. NBA champion Kobe Bryant said: 'Everything negative—pressure, challenges—is all an opportunity for me to rise.'

Step 3: Before a major test, give students a short break to flush out negative thoughts. Writing exercises are effective, as the recent study shows, but so are short brain breaks, mindfulness exercises, and movement breaks, things like teachers taking 10 minutes before a test to help students view stress positively. While it is easy to have faith in the accuracy of tests, students' anxiety can prevent them from showing what they know.

All too often, students think to themselves, 'This test is too hard, and I'll probably fail.' But teachers can step in and encourage a positive mindset toward stress, instead leading students to think, 'This test is really challenging, so I need to do my best.'

Intense stress can lead to a fight, flight, or freeze response. It is the test that could keep one from graduating or the poor evaluation that shatters the fragile self-esteem, and the resulting disappointment and humiliation are high-level stressors. In these types of situations, stress and anxiety ramp up to an unmanageable level. The heart is beating faster, but it is unsettling, distracting, even nerve-wracking. One might lose focus because of the resulting pressure in such situations, and then one may not reach one's full

potential. One is all too aware of the fact that it is working against you. It is too much. That state is difficult to pinpoint in advance and will differ from person to person.

Chronic stress is not likely to benefit students at all. In fact, it will negatively impact students' physical and mental health. Some classroom and school environments are chronic stressors. Daily inefficiencies and poorly-thought-out arithmetic fact drills, unnecessary and unproductive daily homework, unreasonable tasks, lack of and poor instructions for tasks, laborious and inefficient strategies, lack of recognition of student contributions to their learning, etc. are chronic stressors that create debilitating math anxiety. Once the math anxiety is present, it begins to have neuropsychological and neurophysiological impacts on the person's mathematics learning.

Many children, but not all, love competition, but it must have a purpose, and it should be healthy and have the possibility of encouraging growth. The incidence of failure, mistakes, errors, homework avoidance, and fear of assessment most times stem from poor preparation of students, poor and inappropriate methods of teaching, poor task analysis, poor sequencing of tasks, and poor reinforcement. By paying attention to these elements, teachers can reduce failure, errors, and negative attitudes. Encouraging guided group work and problem solving, asking a lot of questions ([Sharma, 2019](#)), and providing answers to 'how,' 'why,' and 'when' questions, discussing students' approaches to concepts, tasks, problems, and their answers, teaching students to support classmates, asking students to make observations and then making conjectures about their observations, and letting students do error analyses to observe patterns in errors, and then fixing their mistakes not only reduces stress and anxiety but increases and creates a positive attitude toward mathematics.

It is also helpful to have students get used to seeing their own data and tracking their own progress, whether on their IEP goals or classroom tests and quizzes. By creating an environment where it is not only acceptable to make mistakes but to realize that understanding the cause and nature of these mistakes is the beginning of their being mathematicians, or mastering

mathematics they need for their future, decreases not only test and assessment anxiety but also math anxiety.

For meaningful and productive engagement of students it is important to have their input into their learning. Students want to feel understood and that their input has value. With discussion and with appropriate examples, we can help them to see the role and utility of assessments and help put the assessment in perspective. A test may also be an assessment of the efficacy of the teaching of the topics(s), we should empathize with our students, and they should realize that we understand their concerns about learning and assessment. Talking about what they believe would be the absolute worst-case scenario if they fail a particular test and how we will support them in that situation can reassure them. Often just by talking about these concerns aloud, students will realize their fears are not as bad as they envisioned.

Sometimes teachers can add to student assessment stress by sharing their negative past academic experiences or current anxiety about their students' progress. It happens quite often in math classes for special needs students, where many special needs teachers would sometimes project their failures and negative thoughts onto their children. Children do identify with us. We should share how we overcame our limitations and ask if those strategies are applicable to them.

The objective is to illustrate to students that testing is not something to fear, but rather an opportunity to show what you know and discover where you might require some additional review to master the subject matter.

Anxiety does not 'look' the same for all students, so you must know when to push students and when to empathize, when to listen, and when to set limits. It is important to reiterate that the entire point of assessment is not to measure who you are by a single score on a single day for a particular area in school, but rather it is to be used as an instrument to gauge progress, potential, strengths, and direct instruction for more optimal learning. Students are a heterogeneous group.

J. Ways Teachers Can Support Students with Math Anxiety

1. Physical Environment and Equipment

We as mathematics teachers need to provide an atmosphere, an ambience, that is reassuring, that allows our learners to be confident that nothing will happen that creates unacceptable levels of failure. This could be the way the work is presented on a screen or a board, avoiding any hint of time pressure, and keeping the ‘score’ (evaluation) private for learners.

2. Instruction

With the use of concrete models,⁷ visual representations—be it drawings, computer visuals, or virtual manipulatives, visualization, alongside the digits and symbols, with proper use of language and questioning process, we can help all students to develop a broader and deeper picture of the tasks—language, concepts, strategies, and procedures. Students need as deep an understanding as we can give them as mathematics is developmental, and they are likely to meet the same concept or chunk of knowledge in several different applications as the syllabus progresses. Teachers should keep in mind: ‘What else am I teaching? How do I connect these concepts? What foundations am I putting in place here for future and related concepts?’⁸

The ambience and the mood of a classroom is significant for teaching and learning. It should be non-threatening—physically, emotionally, and intellectually; relaxing in mood and emotion; cooperative with healthy competition; active, purposeful, and productive; with good comfortable communications between teachers and students and as nonjudgmental as possible. The concept of Harris’s book on transactional analysis, *I’m OK—You’re OK*, sums that up admirably. It is about both sides of teaching and learning environments.

Seligman’s work about ‘attributional style’ is also useful in classroom management, particularly for addressing issues in the affective domain. Attributional style provides a way of channeling a person’s thinking so that they view situations and events from a positive, optimistic perspective. A positive attributional style is strongly associated with high performance and success. Seligman specifies three dimensions of attributional style:

Permanent, Pervasive, and Personal. Each style brings different type of attitude and engagement with learning, in general, and mathematics in particular.

Seligman explains *pervasive* as: When a bad thing happens to person, they can believe that

‘Nothing ever goes right for me.’ (*Universal*) Or they can believe that this was a one-off event and that it will not influence future events. (*Specific*).

For example,

Universal (pessimistic). ‘I can’t do long division; I can’t do mathematics.’

Specific (optimistic). ‘I am not good at long division, but I am very good at drawing and constructing graphs.’

Examples of *permanent* are:

Permanent (and pessimistic). ‘I am never able to do fractions.’

Temporary (and optimistic). ‘I am having difficulty with fractions, our topic for this week.’

And *personal* is:

When bad things happen to us, we can blame ourselves (internalize).

Or we can blame other people or circumstances (externalize).

These situations are controlled by the influences of teachers and maybe also parents.

In our teaching, we have found this construct helped us to recognize negative behaviors and gave us strategies that might address the problem. It is not a matter of having a set response script, but it is a matter of perception, attitude, and teaching style and the skills to respond appropriately and positively.

There are several factors and small actions that can facilitate the goal of creating an encouraging learning environment. For a starter, consistency in classroom environment can be very reassuring.

Paper: For some students, organizing their work (writing, graphing, illustrating, etc.) on a page is a big challenge. An ‘old tech’ intervention might be to provide squared paper (graph paper), with appropriately squared ($1\text{cm} \times 1\text{cm}$) for kindergarten and first grade children and $\frac{1}{4}\text{ in} \times \frac{1}{4}\text{ in}$ for higher grades. Mathematics work should always be done on graph paper.

Visual Stress: There was a time when students would be allowed to choose a colored piece of paper, rather than white paper. This intervention worked for some students, not for all, as is often the case. A low-cost variation was to supply colored overlays of transparent film to address visual stress.

Short-Term Memory

Some students suffering from mathematics anxiety or other mathematics difficulties have a very poor short-term memory as well as working memory issues, so they cannot remember oral instructions or instructions that are available only for a short while. Teachers and tutors should make sure that instructions are available to all students always, as they can be a great motivator and a strategy that obviously maintains open access to the problem that was set for students (Ashcraft & Krause, 2007; Ashcraft, 2002). Moreover, some students do not understand the meaning and purpose of many of the instructions associated with mathematics tasks. Teachers should take time to explain the meaning, purpose, and usage of specific instructions and mathematics terms involved in the instructions. We advise reading instructions several times: In the beginning to understand what is needed to be done; at every transition in the solution process of the problem; and at the end to check and confirm whether the instruction has been followed. For more, see the chapter *On the Role of Language in Mathematics*.

Communication

Communication can be very influential on anxiety in general, and math anxiety in particular. The vocabulary used, sentence structure, the tone of voice, and body language can all enhance or diminish the impact and the purpose of communication. A look at the ideas suggested by Transactional Analysis may be useful ([Harris, 1995](#)). The teacher should ask herself, ‘Is the work on the board, screen, or on the sheet clear and uncluttered? Is the vocabulary used explained and accessible to all students, and is there a preview of pertinent words and terms at the start of the lesson (particularly with class participation)?’

Collaboration: For some students, working in a group can be helpful in reducing anxiety. We have found that the initial work of discussion, practice, and understanding is better done when students work in pairs. However, if it was not monitored and was not held accountable, it could be counterproductive. We have found that two people discussing an idea or practicing together is a powerful method.

Adults’—teachers’ and parents’—own math discomfort and anxiety can also affect their students’ math anxiety. It turns out that teachers who have conquered their own math anxieties are very good instructors of math-anxious students. Studies find highly math-anxious parents are more likely to have math-anxious children—and that interventions encouraging families to explore math can build confidence in both parents and children. As a policy, we always encourage a parent to sit on the tutoring sessions. Parents have reported that their understanding of mathematics gets better and the quality of their support for their children improves.

Many teachers are also math-anxious. More than a third of educators surveyed told the Ed Week Research Center that they felt uneasy in the math classes they took as students, and the same proportion said they feel anxious when they had to teach or do math now.

Because math anxiety is not just about a student’s objective math performance, effective interventions must address a student’s previous academic gaps—language, concepts, facts, and procedures, learning habits,

emotional and cognitive issues associated with their math anxiety. First and foremost, anxious students must continue to be exposed to and appropriately challenged by math at the grade level. In the classroom, this means encouraging whole-class discussions rather than calling on just a few students—every student should be called upon. However, the question should be appropriate to the student and the topic being discussed. The grouping of students to do collaborative work should be planned carefully and assigning group work to ensure all students are actively solving problems.

In our method (CPVA/S), initial teaching instruction, interventions, and remediation strategies are congruent and supportive of each other. They benefit all children. Their applications improve math performance effectively at intervention and remediation levels to reduce anxiety. Well-planned and executed intensive tutoring based on effective diagnosis and methods involving individualized or small-group instruction by a trained tutor can be quite effective. A trained tutor means a person who is well-versed in mathematics, incorporating executive functions and special education strategies in mathematics. For severely math-anxious students, studies suggest it can be helpful to treat tutoring like exposure therapy, with tutors directly correcting not only conceptual misunderstandings but also false perceptions (e.g., ‘I’m not a math person.’).

Because of the impact of anxiety, anxious students have poor executive functions and they need help with executive functions and specific study skills. However, remedial strategies for math and executive functions must be integrated into the lesson itself to have an impact on their math anxiety. Actively solving math problems with support from a classmate or the teacher, with immediate feedback on their errors and difficulties, can help a student to understand where gaps may be; they may not notice those things if they just keep reviewing practice problems. Without challenging oneself in those ways, one is not exposing oneself to the information that one may then need for that final exam.

K. Teacher Personal Behaviors that Can Cause Anxiety

A teacher's behavior, personality traits, attitudes toward mathematics, teaching, and students can have a direct impact on student learning. We all remember effective and productive teachers and the impact they had or did not have.

1. You Are Boring

Partly it may be the subject matter. A teacher's posture and tone of voice can convey a critical atmosphere in the classroom. However, we all have had mathematics teachers who have made mathematics, even with the difficult content, so interesting that students became interested in the subject. One of the authors used to teach a course at the university called 'Mathematics Appreciation for mathematics and non-mathematics majors.' The first semester it was offered, only five students signed up (two nursing majors, one psychology major, and two real estate management majors). The author encouraged students to bring data from their other courses, from the workplace, from newspapers, and from their personal interests. They were encouraged to bring their children's homework sheets and newspaper articles with data and quantitative information. The course was rich with use of mathematics in medicine, management, sports, puzzles, and children's games and toys. There was a heavy emphasis on their own mathematics histories. They were encouraged to make conjectures, generalizations, and connections between ideas, concepts, strategies, and procedures. The final semester, the author had more than 300 students from almost all majors at the university, including mathematics majors.

2. You Are Inconsistent

As mentioned elsewhere, according to the triune theory of the brain,⁹ the reptilian brain (basal ganglia) seeks consistency and safety, the paleomammalian complex (limbic system) thrives in a positive and inviting environment, and the neo-mammalian complex (neo-cortex) seeks

intellectual and challenging situations (MacLean). Inconsistency can confuse an uncertain mind. This could be inconsistency inherent in the topic, e.g., time ([Chinn, 2020](#)), inconsistency in the ‘routines’ of the development of language, concept, strategies, or procedures, inconsistency in behavior, ways of communicating and teaching (Cialdini, 2007), or inconsistent use of language, symbols, tools, and strategies.

3. You Are Disorganized

What impression does your classroom make on students when they enter? Is it a bewildering onslaught of mathematics posters? Posters which display information they have been unable to commit to memory and act as a reminder of its failure, or do they aid memory and help make connections? Is the classroom welcoming to the student, or are they entering a room where they will experience more failure?

A disorganized classroom is the beginning of a disorganized lesson. In an elementary classroom, the role of concrete materials is very important. They should be available to children easily and properly. Disorganization is not related to only the use of materials. The disorganization may be present in the usage of the language for instruction, disorganization in thinking, use of the questioning process, calling on children randomly rather than thinking about who needs what kind of question or participation. The physical classroom environment—clutter, disorganization in activities, and clutter in displays—writing, handouts, diagrams, non-verbal and verbal communication (you talk too much, unclear, jumbled thoughts and expressions) is obvious in these areas. Keep displays uncluttered and accessible. What good are a number line, calendar, geometrical figures and objects in a kindergarten or first grade classroom, if they are not accessible to the child? What good is a number line or place value chart if it is 6 feet high on the wall? It should be accessible so that the child can trace the digits. Do not overload the short-term and working memory capacities—chunk the information. Teach them decomposition/re-composition in all situations. The most important skill in learning is seeing the relationships

from part-to-whole and whole-to parts. Teach them how to do task analysis by taking simple everyday tasks from real life and sports and then apply those skills to mathematics problems. Use ‘toolboxes’ for housing instructional manipulatives.

A teacher should carefully consider the design of the worksheets, selection of textbooks, instruction tools, and supporting/practice materials. One should avoid poor-quality worksheets, books, and tools. For a new learner, it is important that the quality of instructional materials where pages are cluttered with too much information poorly organized and crammed onto the page. Children deserve aesthetically prepared materials: the design of worksheets, the organization and progression of problems on the worksheet, the nature and quality of instruction, the work area for the student on the worksheet, the formation of questions and tasks on the worksheet, the nature and form of the answer being solicited. A worksheet should have areas for reflection for the student about that work.

The prevalence of dysgraphia in children is partly the result of emphasis on the practice of fine motor skills and lack of structure and practice of a proper pencil-grip, proper formation of letters and digits, artwork, and playing with real toys and games.

The need for a good role-model behavior for teachers, whether in a regular classroom or when working with special needs students, is essential. Role model behavior is the integration of the three roles of a teacher—didactic, Socratic, and coaching. When you help too little or help too much, both are unhelpful. All of these behaviors are not conducive to learning by any level of students.

Notes

1. The popular term for mathematics anxiety in the United States is *math anxiety* and in the United Kingdom and Commonwealth countries it is *maths anxiety*. In this book, we will use the term *math anxiety*. ↩
2. As you will see in the later chapters, the pre-requisite skills and primary concepts necessary for mastering arithmetic facts are: (a)

- Subitizing, (b) Knowledge of the alphabets of the mathematics language, (c) Decomposition/re-composition of clusters and numbers, (d) 45 sight facts, (e) Making Ten, and (f) Teens' Numbers. [↵](#)
3. [Chinn, S. \(2021\)](#). *The Trouble with Maths* (4th edn). [↵](#)
 4. More on these fundamental principles in the instructional sections. [↵](#)
 5. Treating the student as a terminal student means: I teach only third grade. I worry whether he can give the answer now—somehow. This method works now; I do not know if it will work later or not; that is not my worry. What happens to him in the future is someone else's job. [↵](#)
 6. Finger counting, its limitations, and its impact on numeracy learning are discussed in detail in later chapters. [↵](#)
 7. We developed a comprehensive model of instruction that is applicable to initial instruction, intervention, and remediation of learning problems created by dyscalculia, acquired dyscalculia, or mathematics anxiety. See the chapters on Tools of Instruction and Learning and the CPVA/S model. [↵](#)
 8. [Chinn \(2021\)](#). [↵](#)
 9. [MacLean \(1990\)](#). [↵](#)

3 The Role of Language in Mathematics Learning and Language-Related Mathematics Difficulties

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Without language, there is no understanding and for robust understanding one needs robust language.

Bhartahari—A Sanskrit scholar, 250 BCE

Communication in every domain of knowledge requires a representation system such as language, symbols, and images. To facilitate this communication, each discipline uses its own symbol system. These symbols, images, words, expressions, phrases, and/or collections of words have special meanings unique to the discipline. Mathematics has its unique symbols and communication system. In mathematics, symbols and graphical representations, words and phrases, and many everyday words and cultural artifacts and symbols are used with special meanings.

Language has four major functions: to receive, to comprehend, to think, and communicate ideas and concepts. Similarly, in mathematics we receive, comprehend, and communicate quantitative and spatial information. We use native language and the mathematics language to think, understand, and develop ideas—concepts, strategies, procedures, facts, and tools about quantity, space, and their integration.

Mathematics is a means to express ideas, thoughts, and methods dealing with quantity, space, and their intra- and interrelations. Initially, the number, concepts, and their relationships have their basis in quantity and space. Later, earlier concepts and problems and real data from outside of mathematics—other disciplines and real-life situations become the source of mathematics ideas. People in most vocations, science, mathematics, and technology think and express their ideas in the language of mathematics. Mathematics, thus, is the language of mathematics, sciences, and many other disciplines, and its development follows a similar trajectory as the native language of a child, but at a much faster rate.

Mathematics has often been referred to as a second language for most children ([Bunton, 1994](#); [Cockcroft, 1986](#); Ginsberg, 1977; [Sharma, 1985a, 1985b, 2008](#)). [Pimm \(1981\)](#) wrote: ‘Mathematics? I speak it fluently’. In his study *Children and Number: Difficulties in Learning Mathematics*, [Hughes \(1986\)](#) highlighted the gap between children’s language at home and the abstract symbolism and language of school mathematics. He pointed out that for many learners mathematics is ‘more like an unfamiliar foreign language’ (p. 42) and proposed that the difficulty is that children must learn to translate between mathematical language and their everyday language. Similarly, [Ginsburg \(1977\)](#), [Allardice \(1997\)](#), and [Sharma \(1980\)](#) identified this type of difficulty as one of the causes of some of children’s learning difficulties in numeracy and mathematics.

The National Council of Teachers of Mathematics (NCTM, 2000) has, through its numerous publications, pointed out the need for students to read, write, and discuss mathematics and to think in mathematics language. Then they would read the problems more easily, comprehend them better, engage in discourse about their understanding of mathematics behind these problems, and carry out the solution processes. Even after solving the problem, they should reflect on the whole process, using language that becomes a *script*¹ for a class of problems. Effective teachers have always recognized the importance of language as a tool for conceptual understanding of mathematics in children. For them, developing this language is to teach mathematics.

A. Role of Language in Mathematics Learning

For many students, achieving proficiency in their language is a difficult task; the use of language in content-specific disciplines such as mathematics makes that task even harder. Current mathematics textbooks and the new mathematics curricula call for a high degree of familiarity with words, grammar patterns, styles of presentation, and arguments that are not part of informal talk that is common in their native language. Communicating mathematics ideas requires a very formal and specific language—a second language. Children therefore need to translate between mathematical and everyday language constantly.

The language used to convey mathematical information in teaching and learning is a topic of interest from several perspectives:

- An increasing number of students are not well-prepared in the language components of mathematics because of limited language experiences.
- Pages of specialized, rigorous reading, and language in textbooks challenge students.
- Emphasis on applications of mathematics requires reading and interpretation of information presented in linguistic form.
- Conceptual components in curricula make demands on the linguistic abilities of students and place a heavy emphasis on an enlarged and appropriate vocabulary.

Some of the linguistic difficulties children have in mathematics are because of poorly written textbooks. Difficult new technical terms crowd textbook pages. Explanations are sometimes unintelligible. New words and terms are introduced without adequate explanations and examples and only as recipes. Some mathematics programs use ‘program-specific language and words’ that are not common in mathematics. For example, ‘Bar model,’ ‘Tape Diagram,’ ‘Peanut method of finding Greatest Common Factor,’ ‘Bow Tie Method of Factoring.’ Because many people perceive

mathematics learning as difficult in the first place, increased linguistic demands heighten this perception.

Most people see mathematics language only in the context of word problems, but it is much more than that. Mathematics language, just like any language, is learning a communication and representational code. Mathematics language acquisition is learning this code and its systems. However, communication occurs when parties collaborate in activity of joint concern, have a common code, and rules of engagement. Mathematics communication uses words and phrases and many everyday words in particular ways and with special meanings that describe mathematical phenomena and related concepts. In this exchange, errors are bound to happen. Some errors are due to the sender of the code, some errors are due to the incompetence or ignorance of the receiver, and some others are due to the medium of communication and comprehension of the code.

At the root of all higher mental functioning is language, symbols, and their combinations² and the root of all language development is the ensuing dialogue. Therefore, the richness of teaching is dependent on the quality of models, nature of language usage, and the quality of dialogue between the teacher and students. A dialogue is an exchange of statements and questions. Each exchange is productive and beneficial to both, if attention is paid to the message. Language always interacts with the message; the quality of the language modifies it. The syntactic and logical structures of language qualify and enhance the dialogue and therefore both play important roles in learning mathematics—any learning. Language is like a river: we can row a boat on it and carry things, or we can transport logs in it. One is planned, organized, and efficient. Other is disorganized, dangerous, and other anticipated or unanticipated activities can take place in the process of transporting logs.

The vital role of language in mathematics learning and performance is widely recognized. In general, positive correlations exist between mathematics achievement and mathematics vocabulary and other related language skills (receptive and verbal abilities), although it is uncertain

whether these result from a dependence of mathematics on language or because both involve general intellectual and cognitive skills.

Language is an artifact of the culture, and it is a key medium to communicate and to think. Even when we reflect, we use some aspect of language. Language plays a key role in conveying mathematics concepts for learning and during teaching. And it is an aid in the development of mathematical ideas and thinking about them. Numeracy is not free from the language and the culture it stems from, but rather is deeply rooted in the native language of the child and the cultural tools that he is surrounded by. These cultural tools: language, rituals, social interactions, and toys and games bring out a child's innate, evolutionary skills related to quantity, space, perception, and memory and help transform them into useful numeracy and mathematics concepts and tools to be successful in the modern world.

In a sense, the native language and quantitative and spatial content give rise to a new means of communication: the *mathematics language*. However, sometimes this interplay of native language and the mathematics content presents inherent difficulties. The native language and cultural perspectives may enhance or limit the mathematics language and the content it develops in the child's mind.

Ordinarily, the specific features of the native language and usage can help make numerical concepts transparent and support the understanding that occurs during a learning discourse. Researchers and mathematics educators through cross-cultural examinations of languages have shown us how to understand the linguistic support and limitations that may foster or hinder students' learning and teachers' teaching of numeracy and mathematics.

Language is not just an apparatus whereby mathematical facts, concepts, and procedures are communicated, but language is also a means of understanding, knowing, and constructing ideas. Its grammar, structures, and conventions help us to generate, organize, and sustain thought. Language operates by forming pictures of reality that faithfully reproduces its logical relationships that we translate into mathematical symbols. In teaching, for it to be successful, there should be a logical isomorphism

between concrete, pictorial, and abstract modes of mathematical representation ([Dienes, 1973](#); [Bruner, 1966, 1977](#); Vygotsky, 1992; Bartolini, 2004), and this happens by making direct associations between linguistic utterances and their concrete referents. The language may not always picture mathematical concepts, but it helps in consolidating them. In the teaching process the language and, more precisely, the words and structures are its tools. Tools have different functions; practice makes us flexible and experts in their usage—when to use them, how to use them, and how to use them efficiently and effectively.

For example, language, particularly vocabulary, plays several roles in mathematics learning:

- Gaining attention and focus
- Receiving and giving instructions
- Forming and reporting thoughts—conveying the content and desired outcomes
- Forming conceptual schemas—efficiently constructing and conveying a concept
- Naming, describing objects, concepts, and actions—forming an association between a word and a mathematical object
- Efficient ways of describing an action
- Modifying concepts and actions

Ultimately, by repetition, language also becomes a shorthand to describe a set of instructions, actions, and thoughts that are fully developed and determined by using a few words—codes. For example: place value, a long division algorithm, prime factorization, the area of a triangle is half the base times the height, division is inverse of multiplication, etc. convey a set of concepts and actions in a very compact form. This brevity and specialized terminology, although essential and important, also create difficulty for children. For many children, gaps exist

- between their language at home and the formal language of school and

- between their native language and the language of mathematics.

For those children, these gaps may create enormous difficulties in learning and using mathematics as they are not able to understand and to make effective use of instructions.

Children with the medium of mathematics instruction being a second language face some extra hurdles in acquiring proficiency in mathematics and its applications. They can be divided into two identifiable groups:

(a) The first group consists of those with strong native language and the related literacy skills and with ample opportunities for formal and informal communication of ideas in their native language. They may have done academic work in their native language. For them, their language and thinking are closely interwoven. They use language for communication and for concept formation. We will call them *cognitively academic language proficient in the native language* (CALP-N) after the term introduced by [Cummins \(1981\)](#) for cognitive academic language proficiency (CALP). They have an ‘adequate to rich’ formal vocabulary in the native language, some concept knowledge of mathematics, meta-linguistic insights, and, with little effort and instruction, they can apply their second language in the context of mathematics also.

(b) The second group consists of those who may lack literacy skills in their native language and who have had very little opportunity for conceptual communication of ideas at home even in their native language. They use their native language for everyday social communication only. For them, the native language is a transaction of instructions, orders, and identifiers. They have *surface fluency* in the native language. They have limited vocabulary and rely on contextual cues to process the native language. We will identify them as having *basic interpersonal communication skills in the native language* (BICS-N), after [Cummins’ \(1981\)](#) term for *basic interpersonal communicative skills* (BICS).

In the case of CALP-N students, the fluency of the language and the knowledge of language skills and structure learned in their native language translate well into their second language. Therefore, research-based

instructional design supports initial reading instruction in the first language to develop a skill platform, followed by a quick transition to the second language. In other words, if the student has a strong internalization of the first/native language—literacy, related thinking skills, and ability to communicate (and the child is living in a literate family—the parents read, write, discuss ideas, and contribute to the enlargement of the vocabulary and syntactic structure of the language), then initial mathematics instruction should be given in the native language to create a platform of skills, which are transferable to learning mathematics language.

In the case of BICS-N students, it may be better to begin mathematics instruction in the second language—the language of the child's new environment, where he is going to have most of his interactions with quantity and space and opportunities to use the new skills. Most children who are not developing or have not developed the academic part of their native language at home and are not acquiring a larger vocabulary and syntactic structure do not benefit from skill building in school in their native language. They live in environments where the parents speak only in the native language, using it for basic social communications and following instructions. The use of native language is purely transactional; it is not used for thinking—receiving and creating ideas. Children and parents may not read or discuss conceptual ideas in their native language. In such a situation, children need to have a wider, more cognitively demanding exposure to the language of instruction than the native language. For these children, the platform of mathematics skills should be developed in the medium of the instruction language itself.

Children with a facility in their native language, even if it is different from the language of instruction, have a better preparation for numeracy, whether it is in the first or second language. If they possess the vocabulary and terminology (language containers) for mathematics ideas and concepts in their native language, then they can develop language and conceptual schemas in the second language/medium of instruction also.

B. Components of a Mathematics Idea

To understand the role and the importance of mathematics language, we need to consider the language demands of mathematics learning. Every mathematical idea has three components: *linguistic*, *conceptual*, and *procedural*. To learn and use a mathematics idea means to create linguistic and conceptual models for it. Mathematics educators use the term ‘mathematics register’ ([Halliday, 1975](#)) or ‘language container’ ([Sharma, 2005](#)) to denote the meanings that ordinary language acquires in the context of mathematics and words and collection of words that represent the mathematics idea or concept. The terms—mathematics register or *language container*—mean a word, a collection of words, or a phrase to express a mathematics concept, procedure, or an idea. However, it is more than a collection of words, it carries with it the related schema for the mathematics concept. For example, ‘sum of 3 and 4 is 7 represents $3 + 4 = 7$,’ ‘The difference of 17 and 9 is written as: $17 - 9$,’ ‘3 repeatedly added 4 times ($3 + 3 + 3 + 3$) can be represented as 3×4 .’ The expression ‘3 repeatedly added 4 times’ is one of the several linguistic models and conceptual schemas for multiplication. We will refer to these linguistic models (the collection of words or phrases denoting a conceptual scheme) as ‘language containers.’ We need to develop and have a language container to receive, comprehend, retain, explain, and think about a concept. Language containers should be internalized. We cannot retain an abstract concept unless we have a language container for it. Just as in one’s native language, when a word has been internalized, it invokes an image, action, or concept, in mathematics too, when a word or a collection of words or phrases has been internalized, it invokes an image, concept or action. Without an internalized language container, we need to relearn the concept every time we encounter it. This is the reason that many children quickly forget the concept, procedure, or strategy, as they did not develop the language container for the mathematical entity. When the teaching is rote memorization of mathematics facts, procedures, or ‘tricks,’ it is easier to forget them.

[Halliday \(1975\)](#) and [Sharma \(2005\)](#) have observed that a mathematics register/language container has the following components:

- a. Natural language words reinterpreted in the contexts of mathematics such as set, point, field, column, sum, even (number), random, differentiation, summation, etc.
- b. Locutions such as square on the hypotenuse and least common multiple
- c. Terms created from combinations of natural language words such as feedback, function, line of best fit, smooth or discontinuous curve, natural number, and output
- d. Terms formed from combining elements of Sanskrit, Greek, Latin, or Old English words such as perimeter, decagon, parabola, denominator, coefficient, and asymptotic, etc.
- e. New words coined for use in mathematics such as conic section, cubic, ellipsoid, quadratic, etc.

In addition to vocabulary, these terms also include style meaning and ways of presenting arguments within the context of mathematics.

When children create and encounter a language container for a mathematics concept, they also create and/or invoke the related conceptual model or schema for the concept in their minds. For example, each word and expression such as sum, difference, rational number, least common multiple, denominator, rectangular solid, conic section, and asymptotic expresses a mathematics concept with the related conceptual model or schema. Any encounter with these words and expressions has the potential to invoke the related mathematical conceptual models. For example, if a person understands or is given a definition of multiplication as ‘repeated addition’ or ‘groups of,’ then either one of these expressions ‘repeated addition’ or ‘groups of’ invokes a conceptual schema for multiplication. When students encounter the problem 7×3 , they will automatically think of this as 7 repeated 3 times ($7 + 7 + 7$) or 7 groups of 3 ($3 + 3 + 3 + 3 + 3 + 3 + 3$). On the other hand, if the definition of

multiplication is given as the ‘area of a rectangle,’ then 7×3 will invoke a conceptual model that will be an image or picture of a rectangle with dimensions of 7 and 3 (7 is the vertical side and 3 is the horizontal side). Learning mathematics, then, is using, developing/creating, extending, and modifying language containers. *Students’ proficiency in mathematics is directly related to the size of the set of their language containers.* However, rote memorization of a collection of words is not enough to master the language of mathematics. One must acquire the related schema with understanding.

C. Language and Number

In this section, we examine the nature of mathematics as a language in the context of number concepts. Here, we examine how number naming, the language of the number-related conceptual schemas, and operations with the help of native language structures support or hinder learning mathematics and mathematics language—concepts, operations, and applications. We also discuss language issues related to our whole number structure, the arithmetic operations of addition and subtraction, and key concepts that have important educational implications for numeracy and later mathematics and applications to sciences.

Although the sense of quantity is an evolutionary and developmental concept, the ‘number language’—names, symbols, and expressions—is the first set of cultural impact children encounter in mathematics. They appear more salient than other teaching and learning factors in relation to number, numbersense, and numeracy. The notions of numbers are first learned as a linguistic routine of counting through which the number names are perceived as sign systems or cultural semiotic systems that enable the symbolic representation of quantitative knowledge ([Goswami, 2008](#)).

There are few studies on conflicts among number naming, how cultural identities and universal decimal features and how local language gets linked to universal Hindu-Arabic numerals. In the following, these two questions are discussed.

Children's mastery of most mathematical concepts is dependent on the inter- and intra-play of three basic structures:

- a. child's innate knowledge of quantity and space;
- b. cultural tools, such as language, rituals, games, and toys; and
- c. mathematics terms, concepts, and models.

Cognitive psychologists contend that mathematical conceptualization is independent of language in early childhood. However, in the light of recent neurological and anthropological information, we believe, once a child has acquired language fluency, mathematics and language interact for the benefit of each other and innate quantitative and spatial skills are made fully manifest and a mathematics edifice begins to take shape. Progress in one enhances development in the other, except, perhaps, in a situation where there is a specific learning disability where interventions and remediations are called for.

Children's initial number conceptualization is the integration of innate skills, social and concrete experiences, and language. Number conceptualization is neither totally dependent on concrete models, nor is it just an artifact of language only. Initially, the naming of numbers is purely a cultural language contribution; however, at that stage, language plays a limited role in conceptualization. Language provides the number words used to express the number orally. But conceptual understanding of number, even in the initial stage, is independent of language. One cannot learn the concept of number with the help of language alone. The conceptual schemas (a cognitive activity) for number come from the manipulation of objects with some language and with diverse concrete experiences. With experience and exposure, children can take it far. For example, young children acquire the concept of number earlier than they do pluralization rules, showing that they can learn the number one and generate other numbers from one.

Conceptualization of number begins with concrete experiences and observation of others using quantity, but number concept acquisition and

communication is facilitated and accelerated by appropriate language using efficient, elegant, and effective tools that support the innate perceptual skills. Thus, concrete experiences are essential for the development of the conceptual part of number. However, concrete experiences alone are also not enough. For every concrete experience to reach a conceptual level, to exist as a concept and for it to be communicated, it needs an appropriate and stable label (language container—the name of the number or concept) and native language to express the understanding. Of course, children without language (hearing, vocal, or speech impairment) use other means of labels and expressions. Language, in some form, is essential for creating names (language containers), bringing out the innate and learning new conceptual schemas of ideas, and the script and sequence of procedures.

Effective use of appropriate language—questions being asked, commentary being made, script being followed, and descriptions being furnished—converts innate and concrete experiences into abstractions with labels. These interactions between concrete experience and language provide such labelling (language containers) and abstractions inherent in them. People in the child's environment (parent, teacher, sibling) mediate this interaction in the form of posing questions, affirming responses, scaffolding thinking, and creating and posing practical problem-solving situations.

Number is never devoid of concreteness, but once number is conceptualized, its applications and related concepts are much more dependent on two languages: the native language, and the mathematics language. Even the emergence of number concept is aided by native language and then number concepts are the beginning of the development of the language of mathematics—its terms, words, order of usage, and translations from mathematical expressions to native language and vice versa.

Appropriate scaffolding of questions³ accelerates the symbolizing and abstracting process from concrete experiences. In other words, concrete experience is the starting point for number conceptualization, conceptual schemas, and procedures, and language furthers the development of the

abstraction process and transition from one level of abstraction to another. The facilitating questioning process mediates between concrete experiences and native language. This is an iterative process.

Effective teaching facilitates and accelerates the interaction of rich language, quantification, and spatial sense with efficient and appropriate concrete materials, resulting in the development of number concept. It is to be noted that number concept is the integration of perception, language, quantification, spatial sense, and memory. Efficient use of appropriate concrete instructional materials facilitates all these skills and their integration. Once the number is conceptualized and there are adequate language containers present, in most cases, further number-related concepts can be developed with language alone. However, it is not advisable, as the access to ideas will be limited to a few learners. Effective, efficient, and elegant tools—concrete models—make mathematics learning a democratizing process. The purpose of using concrete materials is to help:

- a. generate the language (language containers—vocabulary, terminology, context, etc.),
- b. create conceptual schemas, and then, finally,
- c. arrive at symbolic/procedural representation of the concept.

The role of concrete materials is not to solve every problem by using them. A child must transcend the concrete materials and language containers and model practical problems through mathematical ideas and ultimately express their understanding through mathematical symbols and their manipulations. Then they operate in the mathematics language to learn new and higher-order concepts. Many regular education and special education teachers remain too long at the concrete level, therefore delaying the abstraction and symbolizing process and making children overly dependent on concrete models.

D. Mathematics Language and Native Language

Mathematics as a language involves the communication of concepts through symbols, words, and expressions. In addition to typical language components—vocabulary, syntax, semantics, and usage, there are elements that make mathematics language unique and complex. There are other elements, such as numeric and other operational symbols that are like letters; sums and arithmetic facts are the words. Equations and other mathematical expressions are the sentences of this language. Thus, mathematics language, just like any live language, is a dynamic language.

Since mathematics is the language of science and technology, proficiency in this language—its acquisition, its understanding, and its use—is critical to meet the demands and advancement of a technological society.

There are common elements in learning a language and mathematics. As a result, certain components of mathematics learning, particularly word problems and mathematics conceptualization, are specifically influenced by any language disability of the child. Of course, they are further complicated by the conditions of dyscalculia and other numeric or mathematics learning difficulties and their by-products: numerical anxiety, test taking anxiety, and math anxiety.

Among the concepts embodied in this unique language are, at the primary level, classification and ordering of objects, quantifying and representing actions and collections, size, order, space, form, distance, and time, and establishing quantitative relationships between them. At a higher level, the language of mathematics is used in conceptualizing and conveying abstractions and arriving at and expressing generalizations, patterns, logical thinking, and reasoning. It is used in the collection, classification, and organization of data and ideas and then in the analysis, interpretation, and communication of information and concepts. At this level, mathematics language is a means of acquiring mathematics ideas and expressing them meaningfully.

Every mathematics idea, except the initial number conceptualization and simple arithmetic facts (sight facts), consists of three components: *linguistic*, *conceptual*, and *procedural*. Specific interactive approaches and strategies for teaching and learning are needed for children to acquire these

three components effectively. While initially concrete experiences and conceptual models help the acquisition of native and mathematics language, later the native and mathematics languages play an important role in the conceptualization of mathematics. With appropriate language and strong conceptual bases, procedures are easier to acquire and use. When the student has acquired the linguistic and conceptual schemas behind a procedure, he resorts less to rote memorization and becomes competent and more flexible in the use of a procedure. In the absence of conceptual understanding, the ‘robustness,’ ‘the reach,’ and ‘the power’ of algorithms are difficult to exploit; then, abstractions and generalizations from them are limited and extrapolations and connections are rare.

In mathematics conceptualization, language is the key in helping children conceptualize mathematics ideas and how different elements of mathematics (linguistic, conceptual, and procedural) are learned and mastered. Because of the interaction of these two languages, we encounter key issues in learning mathematics concepts, skills, and procedures. Our understanding of the linguistic and conceptual mechanisms and processes that underlie the formation and development of mathematics skills, concepts, and procedures is important.

Language is intricately involved in the mechanisms involved in conceptualizing numbers, procedures, and the key milestones in school mathematics. We explore processes in the transitions from one developmental milestone in mathematics to another. We illustrate how children progress along the journey of these key milestones: *number concept, arithmetic facts, place value, arithmetic operations*, proportional reasoning and algebraic thinking on the one hand, and development in the different stages of spatial thinking on the other. Linguistic fluency and the presence of conceptual schemas facilitate the transitions from one mathematical milestone to the other better, more smoothly, and more interwoven.

E. Mathematics: A Second Language

Mathematics is a unique language. For children, mathematics is essentially a language, and a *second language* or a foreign language at that. It may be a third or a fourth language for many children. Mathematics is a true second language with its own alphabet, symbols, vocabulary, syntax, and grammar. *Digits* {1, 2, 3, 4, 5, 6, 7, 8, 9, and 0}, *numeric*, and *operational symbols* are its alphabet and smallest thinking units; number and symbol combinations are its words. Equations and mathematical expressions are the sentences of this language. Mathematical symbols, concepts, and language modify the natural/native language, as a formal language, in turn, shapes mathematical thinking.

Acquiring proficiency in mathematics and solving mathematics word problems means learning this language thoroughly. To learn this new language, one must acquire its vocabulary, syntax, its structure, and its rules of translation from native language to mathematics language and vice-versa. Proficiency and a high level of competence in this language is demanded by a technological society for its functioning and its innovations. Along with mathematical processes, the child needs to acquire the related language to exchange his learning with others.

A teacher not only uses the language to mediate the learning but also to modify a child's thinking, thereby his learnability and capacity for learning. Children from literate environments learn to read early and with fluency. The same is true for numeracy. Children who grow up in numerate and literate environments (where children are exposed to language, quantitative, and spatial representations) will have prior knowledge about quantity and space to bring to school. Many children enter school having done much mathematics at home by helping with everyday activities. If children count apples in the fruit basket, set the table, or count the number of steps to go upstairs, they are interacting with numeracy. Children at every age continue to learn mathematics at home through simple questions or statements: Is this the twelfth month of the year? Is the jug nearly empty? Do you have enough money to buy the book you want? How many days before my birthday? Throw the ball a little higher. There were thousands of people at the fair last night. There are more animals than pets in the world. They

bring the skills of communicating this information and such thinking to school.

This prior knowledge of numeracy may be in a language other than the language of instruction. If the first language is internalized, the numeracy concepts will be easily transferable to the new language. Every child translates ideas into mathematics language by going through a transition from the native/natural language to the language of mathematics. That experience is transferable. These transitions—from the native language to the language of instruction and then to mathematics language—need careful guidance and mediation from sympathetic and knowledgeable adults who expand children's language and conceptual base. These expansions are both qualitative and quantitative.

One of these transitions is in understanding the role of numbers in the native language and in mathematics language. For example, in the child's native language and in the context of concrete experiences, numbers and quantification function as predicators and qualifiers—modifying the nouns in their scope and meaning quantitatively, such as *five* dishes, *two* toys, *many* books, *fewer* children, *some* days, *a* pen, *more* homework, a *bigger* cookie, a *longer* ride, etc. They function, in some respect, like adjectives in a sentence. Children are quite fluent in this usage of number before they come to kindergarten.

Later, in the language of mathematics, numbers are real, even concrete, and stable entities that continue to function as predicators and number characteristics that are described by other predicators. They become nouns themselves. For example, *six hundred* is a big number, an *even* number, *bigger* than *six* or *six* tens, or *much smaller* than the number *six hundred thousand*. It is far away on the right on the number line. Thus, in the formal language of mathematics, numbers are abstract, singular objects, but at the same time, in mathematical operations, numbers are 'real,' concrete entities—for example, when a child says: 'I will show you number four' and then shows four fingers or writes the number '4.' 'My house number is 4.'

Number exists at two levels: Every member of the class makes the class, and the class is a group—a whole. Similarly, there are 5 members in the

team, and one team is made of 5 people. Holding the two ideas of 5 as five discrete units ($1 + 1 + 1 + 1 + 1 = 5$) and 5 as a whole (as a team of $5 = 1 + 1 + 1 + 1 + 1$) is an abstract idea. The idea here is the ability to see two things simultaneously: part-to-whole and whole-to-parts. This is much more than just 5 objects being counted. It is the idea of ‘fiveness’ that is an abstraction—a new entity has emerged. It is the first order of abstraction that the child encounters. Holding the idea of part-to-whole and whole-to parts is behind the idea of number, and it is behind the idea of the place-value: 500 is whole and is parts ($100 + 100 + 100 + 100 + 100$). This idea will be repeated many times later in multiplication, division, fractions, decimals, and exponents. The question ‘seventeen what?’ is generally irrelevant and meaningless in the context of mathematical language. It has its own identity and address. And sometimes a number plays many roles and multiple personalities: 5,000 is 5 thousand; 50 hundreds; 500 tens, and 5,000 ones—what a chameleon!

Thus, in mathematics, numbers, albeit abstract, have their own entity, properties, relationships, and roles. Thus, numbers are abstract objects, more specifically, abstract singular objects—one thousand is a single number although it could also be considered as made up of many numbers in different combinations and forms. One thousand is also an abstraction ([Frege, 1950](#); [Nesher, 1982](#)). Consequently, the learning of mathematics requires that the child be able to perform a particular type of abstraction, which is nothing but the transition from the conception of number as a predicate in natural language to its conception as an abstract object in the language of mathematics ([Nesher and Katrid, 1986](#)). The processes of decomposition and re-composition are pertinent here, too.

This transition from one language system to another is conceptualized as an ontological process. It evolves through a prolonged constructive process which begins with the native language and ends in the child’s operational conception of number as part of the formal mathematical language system. [Nesher and Katrid \(1986\)](#) suggest that the interrelationship between the predicative function of numbers in native language and their status as objects in the language of mathematics is a constant accompaniment to the

process of arithmetic learning and instruction in the primary grades. Accordingly, the mediation between the natural language and the formal language of mathematics is important and dependent on a variety of concrete and linguistic experiences. It can be seen as a three-stage process.

- a. At the beginning of the process, children's familiarity with numbers is based on their encounters with numbers on a natural language level. They seek out objects to which numerical expressions apply. For example, 'It is three,' 'I have 5 books,' 'my birthday was 2 days ago,' 'I am 5 feet tall,' '3 books plus 2 books are 5 books,' etc. Here the flow of the recognition of a cluster as perceptual activity to giving it a name. That is facilitated by the integration of innate perceptual activity of subitizing to a culturally learned activity of counting a collection and calling it 5 or writing it as '5' integration of these three cyclic processes: *subitizing*—perceiving a collection, *saying f-i-v-e*, and *writing 5* is called '*fiveness*.'
- b. At the second stage, mathematical learning consists of the ability to make a transition from the natural language to formal language. For example: 'When will I be twelve?' 'His score is 12,' etc. However, lapses are possible where the child may use both forms; for example, when a young child says: 'I want five cookies. I have only three. Give me two more.'
- c. The third stage appears after children have acquired the ability to operate with the formal language of mathematics. This stage presupposes that the child can handle numbers as abstract objects and, given this ability, must make the transition in the opposite direction, that is, must apply the mathematics models to a concrete or real-life situation. For example, the child knows $3 + 4 = 7$; therefore, he is prepared to solve word problems involving addition and numbers such as 3 and 4.

At the beginning of a child's schooling, the foundations of all future interactions with quantitative and spatial models (along with speaking,

reading, writing, and problem solving) are being laid. When children emerge from these years in control of their natural language and mathematics language, they are on their way to becoming *literate* and *numerate*. They should be able to use and improve their literacy and numeracy skills as they experience more language involving quantity and space and engage in novel interactions of quantity and space. These skills are the tools needed by children for understanding mathematics, particularly in an information age.

Mathematics teachers have a dual task as far as supporting children's acquisition of mathematics language: development of language in the context of mathematics and the use of language in acquiring mathematical concepts. Because it is difficult to separate these two functions, we do not advise they be developed in isolation of each other. However, it is possible to see them as separate activities for the purpose of understanding these functions. For example, we are developing the language of mathematics when the child is learning to answer questions such as what a rectangle is. How do you describe a rectangle? What is the difference between a rectangle and a square? Is a square a rectangle? What should we change in a rectangle to make it into a square? These questions create a new language and modify the existing one. On the other hand, when the student tries to learn how to find the least common multiple of 4 and 6, they are using language in learning a mathematics concept or procedure. The second results in conceptual models and then in procedures. Therefore, the two most common activities in the process of arithmetic instruction that facilitate these two functions are basic *operations with numbers* (*mathematics exercises*) and appropriate *word problems*. The mathematics exercise is formal mathematics text in which numbers clearly have the status of objects. The word problem is made of sentences in natural language that include numbers having a predicative function. The integration of the two facilitates learning mathematics problem solving.

Word problems in arithmetic are designed to apply the mathematical language the child has acquired to some actual situation while operating within the formal language system. To solve the problem, the child must fit

a mathematical model, equation, or expression (mathematics exercise) to a situation described by a series of sentences in natural/native language. This process entails a change in the status of numbers from predicative to object status. Then, after the mathematics exercise has been solved, the numbers must again be attributed to the appropriate noun phrase so that the answer will again be given in natural/native language and the number again functions as a predicate. Two examples at different levels of complexity with common elements of translation,

Word Problem 1:

Danny has four stamps and Ron has three stamps. How many stamps do they have altogether?

Mathematics Exercise: $3 + 4 = 7$ (Three and four are seven or three and four is seven.)

Answer: Seven stamps.

Word Problem 2:

Find the area of a rectangle whose length is 3 more than twice the width. The perimeter is 96 cm.

Mathematics Exercise: perimeter = 96 cm; length = 2 width + 3; perimeter = 2 length + 2 width; Area = length $\times$ width

This translates, using mathematics symbols, into mathematics equations as: $p = 96$; $l = 2w + 3$; $p = 2l + 2w$; $A = l \times w$.

$$96 = 2l + 2w = 2(2w + 3) + 2w = 4w + 6 + 2w = 6w + 6;$$

$$96 = 6w + 6; \quad 90 = 6w; \quad w = 15. \quad \text{Thus} \quad l = 2(15) + 3 = 33.$$

$$A = l \times w = 33 \times 15 = 495 \text{ square units.}$$

The translation process as a flow of activities can be summarized as:

Native language $\rightarrow$ Mathematics language $\rightarrow$ Mathematics symbolization $\rightarrow$ Mathematization (mathematics expressions and equations) $\rightarrow$ Application of mathematics $\rightarrow$ Communication of the results

The mastery of a mathematical concept by children is the result of a long and continuous interactive process between native language, mathematics symbols, and their quantitative and spatial experiences. This process begins with children's intuitive play and concrete experiences in their environment. The concrete experiences are then represented through native language, pictorial, or other visual forms and means, which then may result in abstract mathematics formulations—problems that students need to solve. This *mathematics formulation*—devising abstract symbols, formulas, and equations, is then applied to more problems, and the result of this process is communicated back to the real world. Through these successful communications, we know that the child has mastered the particular concept. At each stage, a child must deal with the intersection of native language and mathematics language.

On the other hand, thinking of numbers as concrete and abstract at the same time is at the core of true number conceptualization. Holding in mind the concurrent idea that a number is concrete and abstract is difficult and creates hurdles for many children to become fluent in other mathematics concepts and in applications outside of mathematics. Their concreteness and their abstract representation are perhaps the most paradoxical thing about numbers, and this is exactly the feature that makes them so useful.

F. Transitions: Concrete to Abstract and Native to Formal Language

Acquisition of number concept and other subsequent mathematics concepts require that the child performs two types of abstractions: *The first abstraction* takes place when a child translates sensory, concrete representations of quantity into symbolic entities (five fingers, five books, five flowers, five kisses, etc. all are translated into the number 'five' and '5').⁴ *The second abstraction* takes place when the child transforms the conception of number as a predicate in his native language to its conception as an object in the language of mathematics. For example, when the child says: 'You know 5 is more than 3. Here, this is 5 and this is 3.' They show

five fingers in one hand and 3 fingers in the other. ‘Look at these. Isn’t that true that 5 is more than 3?’

For some children, particularly children with learning disabilities, these transitions are not easy and need to be facilitated and mediated carefully by a knowledgeable and sympathetic person using appropriate language, an enabling questioning process, and efficient and transparent concrete instructional models and tools. This process is a dual process where the child is being transformed both in the gathering of ‘new content’ and as a ‘learner.’ This modification in the child is possible when the mediating adult is cognizant of the nature of the content and the modifiability and the resilience of the human cognitive development. ([Feuerstein, 2006a](#)) Just like the ‘Reading Brain’ is a different brain (a reader child) than the brain that started reading ([Dehaene, 2010](#)), similarly, the ‘Numeracy Doing Brain’ is a different brain than the brain that started learning numeracy. The successful interaction and resulting integration of this dual process is the modified brain (a better container with more content and a better filler of that container—the *numerate child*). Every successful experience should result in the learner being a better learner.

These two abstract transitions are significant and involve complex processes. Just like language, they evolve through a prolonged constructive process (from about age 2 to ages 5-6-9). They begin with the native language and concrete experiences such as counting on fingers and ends in the child’s conception of number as part of the formal mathematics language and number relationships. In the elementary grades, the child is constantly struggling with these roles of number and the interrelationships between the predicative function of numbers in the native language and their status as objects in the language of mathematics.

A survey of recent research ([Clements and Sarama, 2014](#)) shows that the learning experiences should include two important learning activities: learning games with an emphasis on quantity—both from manipulating discrete objects and continuous models ([Ramani and Siegler, 2008](#); [Sharma, 2008](#)) and higher levels of experience with patterns (pattern recognition → extending patterns → creating patterns → applying patterns → making

conjectures Learning (including numeracy)). For example, repeated patterns of the types: ab, aab, aaab, aaaab, ...; ab, aabbb, aaabbbb, aaaabbbbb, ... elicit higher-order-thinking than the patterns of the type: ‘ab, ab, ab, ab, ...’ ([Rittle-Johnson et al., 2015](#)). We should not forget that all standard procedures of arithmetic and algebra are exercises in recognizing, extending, forming patterns, and applying patterns. Practising them with understanding is a great exercise in learning.

The mediation between the native language and the language of mathematics depends on the quality of concrete models—tools, the nature of language used, type and richness of questions asked, and modifying ability of the adult. Models, language, and questions should satisfy certain requirements. For example, about the models, we need to ask: Are they exact? Are they efficient? Are they effective? Are they elegant?⁵ *Elegant* here means: Are the models generalizable from specific cases to general situations? Ultimately, can the experience be generalized and the learning be extrapolated and concepts can be abstracted from it. Most learning difficulties begin to emerge in poor transitions, particularly, at the developmental milestones in mathematics development. *The most important and difficult transition is from number concept to additive reasoning.*

Effective teachers provide a variety of concrete learning experiences, and their expert questioning and deft use of language facilitate smooth transitions. Just like introducing children to the alphabet, we need to distinguish between the name of the letter, its orthographic image, and the sound associated with the letter, and then integrating the three. Similarly, it is important to help children distinguish the cluster (the quantity) associated with the number, the name of the number, the orthographic image of the number. The integration of these three is the end of the multi-stage abstraction process called ‘numberness.’

In the later chapters, we examine and demonstrate the role of ‘language and instruction tools’ in mathematics learning. There we demonstrate the ways in which language proficiency facilitates mathematics conceptualization and language difficulties interfere in learning mathematics. In this respect, we explore which aspects of mathematics are

particularly affected by dyslexia and other language-related disabilities. Similarly, we ask why learning the mathematics language is so important. The question then arises: Is mathematics language learned differently than other languages? Most importantly, how do we help children acquire the language of mathematics?

In the previous chapters, we outlined the nature of mathematics learning problems and their relationship to dyscalculia. Here we relate some of those problems. In particular, we look at specific problems dyslexic children and some dyscalculic children face in mathematics. As mentioned, some learning difficulties and problems exist uniquely in arithmetic and mathematics due to the nature of numeracy and mathematics. Of course, some of the other mathematical difficulties children face are a result of sociocultural factors and the unique nature of skills required for learning mathematics. However, we find that some mathematics learning problems are due to the interactive nature of language and mathematics. In some cases, the same difficulties that the child experiences in reading and other language concepts begin to interfere in learning arithmetic and mathematical concepts. We observe that some very specific arithmetic and mathematics problems are language and reading related, particularly those skills, which involve the same or similar kinds of underlying learning mechanisms and cognitive processes.

Further, the same intellectual, physical, social, and emotional factors that contribute to difficulties in other academic areas may be responsible for difficulties in arithmetic and mathematics.

Cognitive skills such as *visual perceptual integration*, *spatial*, and *temporal order* are important for reading and language development, and they are equally important for the conceptualization of arithmetic and mathematics concepts and for many of the procedures. Children with language-related difficulties may manifest difficulties in learning even the simplest of arithmetic concepts, such as number relationships, basic number facts, and algorithms in arithmetic. Because of these and many other underlying similarities in skills in both arithmetic and language, dyslexic students may begin to manifest problems in number conceptualization. And

as number is the basis of all future operations and processes, they may develop profound anxiety about anything mathematical. As we discussed, in the earlier chapter, initially this anxiety is only a symptom, but later it begins to be a causative factor in their difficulties in learning mathematics.

Children's problems in arithmetic and mathematics, coupled with their reading and language problems, form a substantial subset of children's mathematics learning difficulties and disabilities—almost 40% of dyslexics also have trouble in learning mathematics. This subset is the best understood of all the mathematics disabilities and difficulties because reading and other language-related problems have received attention from researchers for a longer time.

To understand some of children's arithmetic problems, we have tried to show why it is important to understand, on the one hand, the nature of language- and reading-related problems and, on the other hand, the linguistic demands made in learning arithmetic and mathematics.

Thus, children's native and mathematics languages play an important role in the conceptualization of mathematical ideas and in the application of mathematical information. When children say that they have difficulty solving mathematical word problems, the problem may not be just mathematical. To be proficient in mathematical problem solving, particularly word problem solving, children need competence in mathematical vocabulary, the mathematics concept, and the related conceptual schemas involved in the problem. Further, they need to understand the syntax and voice used in the mathematical language as well as the ability to translate from mathematical language to a child's native language and vice versa. Let us consider the following word problem:

David is 17 years old, and his sister is 9 years old. How much older is David than his sister?

Here, the operation is not obvious to the child as no commonly used terms like *subtraction*, *take away*, *minus*, *less*, etc. are used. Therefore, the child must infer the operation from context. The problem is asked during the

lesson when subtraction is the focus of the lesson, or the child has experience in using the term ‘how much more’ or other related terms. In the absence of all of these, the child comes to the teacher and asks: ‘Is it addition or subtraction?’ If the teacher says, ‘It is subtraction,’ that is not an adequate answer. The teacher should realize that the problem is language-related and deal with this situation from that perspective and provide some intervention and develop the concept right then that it is an example of finding the answer to a subtraction problem by considering addition to find the difference: ‘9 plus what is 17?’.

Children may have a good grasp of arithmetical concepts, but if they have difficulty translating the English expression of the word problem into mathematical expressions and equations, they will have difficulty solving word problems. Similarly, if children have difficulty translating mathematics equations into the English language, they will have difficulty seeing the meaning of mathematics formulations of real-life problems. Both types of translations, from English to mathematics and from mathematics to English, are essential for proficiency in mathematics problem solving.

Finally, children need help in understanding instructions associated with mathematics conceptual problems and word problems. We consider specific strategies for improving these areas. Because children should acquire the ability to think and communicate mathematics, exploring the role of asking questions in the classroom is vital. Questions help develop mathematics concepts and mathematics language.

G. Interaction of Mathematics and Language: Mathematization

The following represents a comprehensive model, outlining the roles of language in mathematics conceptualization, application, and problem solving.

Native Language and Mathematics Language Interaction

- a. Translation from the context (native language to pictorial and formal linguistic forms),
- b. Representation (in the native language),
- c. Discussion (in the native language),
- d. Mathematical formulation of the problem (in the mathematical language),
- e. Manipulation of mathematical language and concepts,
- f. Translation from mathematics language to native language, and
- g. Expressing in native language, if necessary.

The child's mastery of a mathematical concept begins with the quality and quantity of concrete experiences with quantity and space. They are communicated through visual representations and artifacts. Finally, the child is taught to represent them symbolically/abstractly. These abstract symbols, formulas, and equations are then applied to more complex problems. This iterative and cyclic process is called *mathematization*.

Mathematization

Concrete and social experience → native language → mathematics experience → mathematics language → mathematics symbolization → mathematics expressions and equations

The various linguistic activities in this process serve different purposes in the acquisition of mathematical concepts and skills. Students need considerable proficiency in the native and mathematical language to cope with the level of proficiency required for learning and applying mathematics.

1. Components of Mathematics Language

To be proficient in mathematics and in mathematics language, students need to develop mathematics vocabulary and recognizing, understanding, and forming symbols, awareness of the syntax and structure of mathematics sentences and expressions, and the two-way translation from English language to mathematics language and vice versa.

Vocabulary and Symbols: words, terms, and symbols representing a thought, concept, or complex information. Some children's mathematics difficulties are due to their limited vocabularies/language containers. By giving a name to something, you acquire power over it. We cannot receive, comprehend, hold in the memory, and communicate a concept unless we have a representation (name, picture, artifact—a symbol) for it. On the Massachusetts Comprehensive Assessment System examination,⁶ many students at the fifth-grade level did not solve the problems: *Find the sum of 8.7 and 5.2. Find the product of 1.2 and 1.3.* When one of the authors presented the same problems in the classroom and asked the students who did not answer the questions or answered incorrectly, the author found that the reason they did not answer these questions was that they did not know the meanings of 'sum' and 'product.'

Symbols: Cultures and civilizations, organizations, and events are identified by symbols. Mathematics is a collection of symbols and ideas. A symbol:

- has an orthographic shape/representation
- has a name
- has a function
- it acts with and on other symbols
- it has a behavior
- it may transform itself according to context and need
- it can be one of the forms: operator, operand, operation.

It is therefore important to introduce them properly, with all due respect and fanfare. Sometimes, they are an introduction to something—language, action, and outcome and sometimes they are the culmination, but they

introduce, explain, and represent ideas, and they are always an aid to understanding.

Symbols, initially, are a bewilderment for many children. These symbols have multiple meanings contextually; they seem so transitory to them. If we subject this symbolism to close scrutiny, we find that, although every attempt is made to refine it to make it explicit and unambiguous, its power lies in a very specific kind of ambiguity. For example, 23, 32, 3^2 , 2^3 , 2_3 , $\frac{2}{3}$, $\frac{3}{2}$, .23, 2.3, and 3.2 all involve the digital symbols 2 and 3, but all these symbols/abstractions (combinations of 2 and 3) have different values, meanings, and representations. It is in many ways no different than reading where the same letter or a word may have different sound and meaning according to the context. With appropriate explanations and practice, we resolve it in reading and with practice we resolve these dilemmas in mathematics also. Same processes and practice work here, in the case of mathematics symbol manipulation.

However, there is one more complexity, a symbol, word, or expression may often refer both to ‘something to do—an action,’ and to ‘the result of the doing—the result.’ So $\frac{3}{4}$ refers to the intention to divide 3 by 4, it indicates ‘a process to be applied,’ and to ‘the result of the action of applying the process, the fraction, the number, $\frac{3}{4}$.’ The symbol ‘–2’ refers both to the intention to subtract two, again indicating a process to be applied, and to the result of the action of subtracting 2, the negative number, –2. First encounters with such symbolism can lead to bewilderment and consternation for the learner in establishing precisely what the symbolism means. It may be easy to use the term ‘take away 2,’ but how can one have a negative number which is ‘less than nothing’? Most can see this problem for what it is: an amalgam of both process and concept, a process which in most cases is manifested through a procedure to get an answer, and a thing produced through the process or the manifestation of the process which can itself be manipulated as a mental object. This amalgam of process and concept must be understood clearly.

Consider the question: ‘What is $2 + 2$?’

The obvious answer is '4'. But how is it calculated? It requires the process of addition. When the child attempts this for the first time, the addition process is usually manifested through a procedure which involves counting two sets each with two objects. At its simplest, it requires either taking two objects and two more, then counting them all saying 'one, two, three, four,' or carrying the first two in the head and counting on from two, saying 'three, four.' Thus $2 + 2$ signals both the counting process and the product of that process: 'the number 4.' The symbol $2 + 2$ evokes both the process of addition and the concept of sum.

The use of a symbolism can mean 'both process and product occurring' throughout mathematics. A number such as 'three' involves both the process of counting and the concept of number. The symbolism $2 + 2$ stands for both the process of addition and the concept of sum, it also represents 2×2 or '2 repeatedly added 2 times' or '2 groups of 2,' etc. This phenomenon occurs repeatedly:

- The symbol 4×3 stands for the process of multiplication of four multiplied by three, which in procedural terms may involve 'repeated addition,' '4 groups of 3,' 'a 4 by 3 array,' or 'the area of a 4 by 3 rectangle' to produce the product of four and three which is the number 12.
- The symbol $\frac{3}{4}$ stands for both the process of division and the concept of fraction.
- The symbol $+4$ stands for both the process of 'add four' or 'shift four units along the number line,' and the concept of 'positive number +4.'
- The symbol -7 stands for both the process of 'subtract seven' or 'shift seven one's in the opposite direction along the number line,' and the concept of the 'negative number -7.'

As a child learns mathematics, the introduced symbol takes on a life of its own. It can be written (say, $3 + 2$), it can be read, it can be spoken ('three plus two'), it can be heard. It is an external object that different people can share, so it has, or seems to have, its own external reality. It is the

construction of meaning for such symbols, the processes required to compute them, and the higher mental procedures required to manipulate them, that constitute the abstraction of mathematics. Indeed, the ambiguity of notation to describe either procedure or output, whichever is more convenient or required at the time, proves to be a valuable thinking device for the child, a mathematician, and a layperson.

Syntax in mathematics is an organization of words, symbols, and structures of mathematical expressions and sentences (numerical and algebraic expressions). Some of children's mathematics difficulties are also due to not understanding the order of words in a mathematical sentence. For example, compare 'subtract 5 from 3' vs. 'subtract 3 from 5,' '.75 divided by .89' vs. '.89 divided by .75,' '2 more than 10' vs. '10 more than 2,' and '2 less than 10' vs. '10 less than 2.' Another important syntax is the order of operations, summarized as GEMDAS—Grouping, Exponent, Multiplication, Division, Addition, Subtraction—where multiplication/division and addition/subtraction are performed in the order they appear. These two types of syntax in mathematics do create difficulty.

Translation: translating from a mathematics sentence into English and from English to a mathematical expression. When students encounter a word problem, many ask the teacher to supply the operation involved in the problem. Once the teacher provides the operation, students perform the appropriate operation and provide the answer. The problem here is one of translation not of computation. Similarly, students solve a problem but do not know what the answer means. This is also a problem of translation.

2. Developing Mathematics Vocabulary and Language Containers

An underlying assumption of the recent mathematics education reform movement is that all children will learn mathematics, including children from groups who are traditionally underrepresented in mathematics, science, and technology careers. One of the recommendations of the mathematics education reform movement is that students are to

communicate mathematically using a variety of tools, including written and spoken language (National Council of Teachers of Mathematics [[NCTM](#)], [1989](#), 2000; CCSS, 2010).

That means there should be emphasis on teaching the development of the components of the language of mathematics and its usage.

(a) Building the Vocabulary of Mathematics

To acquire fluency and flexibility of thought in mathematical language, just like any other language, one should have a large usable vocabulary. Research shows that the size of a student's mathematics vocabulary predicts achievement in mathematics. However, the vocabulary in the mathematics language has some unique characteristics. There are five kinds of words and terms used in mathematics:

- The words that have the same meaning in the native and mathematics languages, e.g., *circle*, *connect*, *area*, etc. In this case, they already have conceptual framework (even though it may be rough or incomplete) meaning, schema, and rules of usage associated with the word or term.
- The words that have different meaning and conceptual schema in the native and mathematics languages, for example, 'distribute and distributive property,' 'irrational behavior and irrational number,' 'difference between seasons of the year,' 'find the difference between the numbers 17 and 9,' 'prime and prime number,' 'factor and factor of a number,' 'odd and odd number,' 'product of numbers 6 and 8,' etc. In such cases, the conceptual schema represented by the word used in native language is different from the conceptual schema needed for its usage in mathematics. The teacher needs to use either concrete or representational models or analogies to create the mathematical schema and examples for its usage.
- The words that are unique to mathematics and may not exist in the native language, parabola, exponent, differential coefficient, prime

number, scalene triangle, etc. In this case, a conceptual schema and meaning must also be created.

- Compound words formed by combining words (sometimes even with opposite meanings), e.g., *scalene right triangle*, *simultaneous linear equations*, *least common multiple*, *greatest common factor*, etc. Meanings and conceptual schemas of these words are best developed by examples, analogies, and focusing on the syntax (order of words).
- Compound words formed by combining words, terms, and actions from different areas, e.g., *miles per hour*, *velocity*, *acceleration*, *hypotenuse*, *trigonometric ratios*, etc. To develop these words and terms, the component primary concepts must be developed first.

Just like the development of vocabulary in any language is a key function of teaching that language, the development of mathematics vocabulary is an important activity for a mathematics teacher and their students alike.

Direct study of quantitative vocabulary (e.g., rote counting of numbers from 1 to 100 in kindergarten and first grade; skip counting by 1, 2, 5, 10, and a unit or proper fraction starting from any given number) within a mathematics context contributes significantly to improving mathematical conceptualization—building lexical entries, learning new concepts, creating deeper and robust conceptual schemas and more effective communication.

The size, level, and quality of a child's mathematics vocabulary depend on the intersection of three language sets:

- The level of the language and prior knowledge the child brings to the task. (C = Child's native language set)
- The language that the teacher uses during the mathematics lessons and interactions with the children. (T = Teacher's Language Set)
- The language set of the mathematics textbook or curriculum. (Textbook Vocabulary set = V)

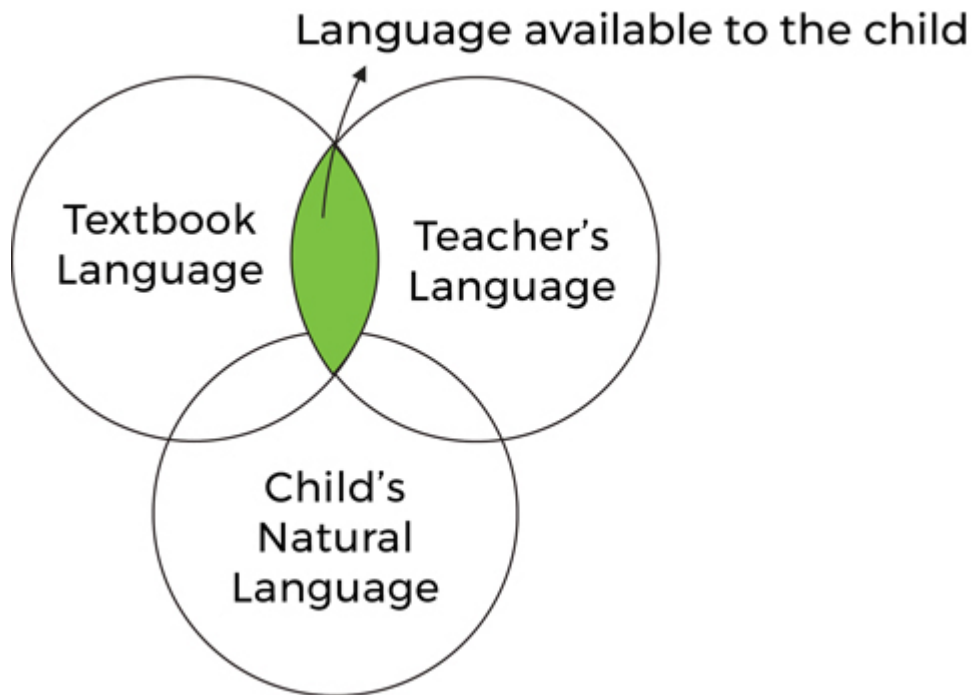


Figure 3.1 Intersection of languages

The intersection between these three language sets is the ‘available language set’ which the child may use to learn mathematics. If the size of this intersection set is small, the child has a limited vocabulary range. That has repercussions for learning mathematics ideas, creating mathematics conceptual schemas, applying mathematics strategies and procedures (e.g., word problems and others), and their communication (reception and production). The reasons for the small size of a child’s mathematics vocabulary are varied and many. A child may have a limited mathematics vocabulary because

- The initial medium of instruction in the classroom is not the child’s native language, and the child is not bilingual. The mathematics problems of the child with English as a second language in a classroom where the medium of instruction is other than the child’s native language are well documented.
- The child’s and teacher’s economic, sociocultural, and geographical backgrounds differ. The linguistic problems that many urban black

children and immigrant children generally face are an example of linguistic/cultural mismatch and assumptions teachers make in instructing these children. The size of a child's general and their mathematics vocabulary is a function of child's socioeconomic background.

- Textbook language sets differ from the language sets of the children and the teacher, and, sometimes, even the curricular guidelines.

Whatever the reasons, we need to help children acquire a large mathematics vocabulary. Careful attention to acquiring and using the vocabulary of mathematics with its technical language is a must if we want children to think mathematically. Used in context, mathematics vocabulary has a profound effect on children's mathematics achievement. Planned activities for developing, expanding, and using quantitative and spatial vocabulary contribute significantly to better mathematical word problem-solving ability and in supporting learning new concepts, deeper conceptual understanding, and more effective mathematical communication.

The objective, then, is to increase the size of the intersection set. School systems, state departments of education, teachers training colleges, and universities can take an active role in this matter. For example, to this end, the Japanese Ministry of Education regularly publishes commonly used vocabulary and new words that are coined to facilitate pedagogy and mathematics. Similarly, the Numeracy Project motivated by the National Curriculum in the United Kingdom has published a vocabulary list for use by classroom teachers. The Common Core State Standards/Mathematics have included a well-defined glossary of terms and their definitions. The National Council of Research and Training in India publishes textbooks and ancillary instructional materials to be used by teachers throughout the country.

Although more textbooks are responding to the ideas propagated in NCTM's Curriculum and CCSS-Mathematics Standards by emphasizing the language of mathematics, there is still little attempt to develop a coherent and comprehensive mathematics vocabulary in K-12 mathematics teaching.

For example, we examined six major textbook series in use in the United States for their language content and found that in one series the expression ‘find the sum’ was introduced quite early. In another series, the expression was introduced around page 50, and about 20 pages later the words ‘find the sum’ and ‘add’ were used interchangeably. In the fourth series, the word ‘sum’ was used sparingly, and so on. Consequently, a child may face different language sets from grade to grade and from school to school. Although the textbooks had many common language terms and vocabulary, they had many words that were not in common. Their individual language sets were quite different in distribution over the school years ([Sharma, 1979](#)). Unfortunately, some textbooks now use so much language without properly introducing the language set and the related conceptual schemas that many children and parents are finding the textbooks frustrating. At the same time, the mathematics exercises do not provide enough practice in basic skills, so children are not able to automatize the language or the conceptual skills associated with them.

To enhance children’s vocabulary, we recommend that school systems have a minimal mathematics vocabulary list at each grade level⁷ and that a copy of the list is available for parents and teachers. Mastery of words from such lists prepares children to develop conceptual schemas and strategies to communicate. This communication furthers children’s language development. The mathematics vocabulary list can also be used to estimate students’ grade-level language of mathematics as it suggests the target mastery grade for a particular word. The English language teachers should also get this list so that they can integrate such vocabulary into their classroom work.

A teacher needs to introduce new mathematics words in the same way as one would introduce them in a language arts class using the same kinds of techniques. The teacher should select a few words from the mathematics vocabulary list that are related to the concept or procedure being taught in the class and help develop them during the lesson, not by introducing them as a list and then asking children to memorize the list by rote as an isolated list of vocabulary words. All the words must be developed or introduced

and then used in the mathematics conceptual and procedural context. For example, the teacher can give a word and then ask children to use it in a mathematics context. The following exchange illustrates this process.

The teacher selects a word from the list related to the content being taught at that time, uses them contextually, and helps children master them. For example,

‘I have put these two rods (3-rod and 5-rod), end to end, as a train. Which one rod will be as long as the two rods together?’

I have joined the rods together. It is also called adding the two numbers. What two numbers are being added?

Can someone read this as an addition problem?’

After several words have been used with rods in the context of adding the rods, the teacher extends the exercises to language activity.

‘Give me a sentence that uses the word **‘add.’**’

‘You have \$5 and I have \$14. Let us add both amounts.’

‘That is great! Now use the word **‘sum’** in a sentence.’

‘That is easy. If we add our monies, what is the sum of our monies?’

‘That is also great! Now I am going to write some words on the board. I want you to write a sentence or two using each word. If you want, you can use more than one word in a sentence.’

For students with a history of word recognition difficulties, the mathematics vocabulary list may provide valuable information regarding a student’s mathematics difficulties with language and vocabulary.

The introduction of words, terms, and definitions can take place at any level, but the mathematics vocabulary list indicates where we want the specific words to be introduced, used, and mastered by the child. It is developmentally appropriate to point out these linguistic elements to children even younger than those in kindergarten.

Once the key root words have been introduced to children, the teacher can begin to extend the mathematics vocabulary words. Among the easiest

sets are the words formed with prefixes, suffixes, and the derivative words. The process is to introduce the math prefixes and roots casually and more formally in 1st-6th grades. In a casual manner, parents and teachers can remark, 'You know, a tricycle has 3 wheels. Tri- means 3 and cycle means wheels.'

Teacher: What will be the name of the object that has three angles?

Student: A triangle.

Teacher: Why?

Student: A triangle has 3 angles and tri- means 3.

Teacher: Can you make a triangle using the coffee-stirrers on your desk?

Take different length pieces and make three triangles in each in each team and write word 'triangle' next to it on paper.

Teacher: Now draw a triangle on your paper.

Children draw triangles on their papers in different orientations and different side lengths and angle measures.

Teacher: Now you should write the definition of a triangle in your team and then raise your hand, if you can share the definition with the class.

After some time, the teacher asks: 'Could someone define a triangle?'

Teacher: The word 'lateral' means a side. What will you call an object that has three sides?

Student: Trilateral.

Teacher: Now draw a trilateral on a paper.

Children draw a trilateral on their papers.

Teacher: If the word 'gon' means a corner, what will you call an object that has three corners?

Student: A trigon.

Teacher: So, a three straight-sided closed figure can be called: triangle, trilateral, and trigon.

Teacher: If 'octa' means eight, what does 'octagon' mean?

Children: A figure with eight corners.

With the introduction of each linguistic element, students are exposed to several words and concepts that contain it. The students use it in their own words, with as many examples as they can formulate. The concepts are then reviewed in the share-circular fashion, built upon, and tied into the new ideas that are introduced. This helps children to form a working vocabulary—a vocabulary that is constantly augmented with new words and terms. But more importantly, they are constructing their own vocabulary and learning skills to build it. The teacher is there to guide it, shape it, refine, and augment it.

As with all language development, there is a sequence in moving from speech ability to writing ability: the input to this process is auditory in its foundation (the child is immersed in oral linguistic experiences), followed by speech ability (the child produces language), and later by reading and writing ability. When young children have this kind of foundation, they will escape the anxiety of making sense of key foreign words later in a formal setting. They will be able to generalize and relate math concepts to their repertoire of daily experiences.

To help children develop mathematics vocabulary, the teacher must identify, introduce, develop, and display the words and phrases that children need to understand and use if they are to make good progress in mathematics. There are a few main ways in which children's failure to understand mathematical vocabulary may manifest:

- a. children have difficulty conceptualizing a mathematics idea because of language component.
- b. they do not respond to questions in lessons; they do not ask questions.
- c. they cannot perform a task because they do not understand the instructions and terms.
- d. they do poorly on tests, particularly on word problems.
- e. Their lack of conceptualization of a mathematical idea may be because they do not have the language container for the concept to receive it, comprehend it or express it, such as 'find the sum of,' 'union of two rays ...,' 'evaluate ...'

- f. Their lack of response may be because they do not understand spoken or written instructions such as ‘draw a line between ...,’ ‘touch the base of the triangle, place a positive sign next to the numeral,’ or ‘find two different ways to ...’
- g. They are not familiar with mathematics vocabulary words such as ‘difference,’ ‘subtract,’ ‘quotient,’ or ‘product.’
- h. They may be confused about mathematical terms such as ‘odd’ or ‘table,’ which have different meanings in everyday English and have more precise meanings in mathematics.
- i. They may be confused about words and symbols like ‘area’ and ‘perimeter,’ ‘factor and multiply,’ ‘and’.

3. Instructional Suggestions for Mathematics Language Proficiency

There are, then, practical reasons children need to acquire an appropriate vocabulary so that they can participate in the activities, lessons, and tests that are part of a classroom life. There is, however, an even more important reason: *vocabulary*, as part of mathematical language, is crucial to children’s development of thinking not only in mathematics problem-solving but in general problem-solving. Once children have control over their language usage, they have control over their meta-cognitive skills that produce insights into their learning and their interactions with learning tasks. Language proficiency refers to the degree to which learners exhibit control over their language. Language and thinking are interwoven in reasoning, problem solving, and applications of mathematics in multiple forms—*intra-mathematical*, *interdisciplinary*, and *extracurricular*. If children do not have the vocabulary to talk and think about a concept, they cannot make progress in its understanding and its applications.

Teachers often use informal, everyday language in mathematics lessons before or alongside technical mathematical vocabulary. Although they can help children to grasp the meaning of words and phrases, a structural approach to the teaching and learning of vocabulary is essential if children

are to move on and begin using the correct mathematical terminology. This also applies to proficiency. Because some children start school with a good understanding of mathematical words when used informally either in English or their native language, the teacher needs to determine the extent of children's mathematical vocabulary and the depth of their understanding and then build on this.

The teacher needs to plan the introduction of new words through a suitable context, for example, with relevant, real objects, mathematical apparatus, pictures, and/or diagrams. The teacher or a student can act the word out. The teacher should explain the meanings of words carefully and rehearse the pronunciation of the word several times. The teacher should ask the children to say the word clearly and slowly. The teacher should refer to a similar word and give the history and derivation of the word. Referring to new words only once will do little to promote the learning of mathematics vocabulary. The teacher should encourage the use of the word in context and help sort out any ambiguities or misconceptions students may have through a range of divergent (open) and convergent (closed) questions. Asking good questions is the mark of a good teacher.

The teacher should use every opportunity to draw attention to new words or symbols with the whole class, in a group, or when talking to individual students. Finally, the teacher should create opportunities for children to read and write new mathematics vocabulary in diverse circumstances and to use the word in sentences.

It is not just younger children who need regular, planned opportunities to develop their mathematical vocabulary. All children from K through 12 and adults returning to education need to experience a cycle of mathematization: concrete work, oral work, representation—pictorial and abstract, application, and communication—reading, writing, and:

- *Concrete work:* Students are encouraged to explore and solve problems using manipulative materials (both discrete and continuous models) and are asked to record the activity concurrently using words, pictures, and symbols by the teacher and students.

- *Oral work:* We sometimes think that students need reams and reams of practice worksheets to practice arithmetic facts and other mathematics ideas to be able to learn and master and to learn independently before they can access demanding arithmetic procedures. Yet, from the evolutionary perspective, we know that our hearing comprehension is more developed than our visual decoding and encoding abilities, particularly for novice learners. When we practice arithmetic facts aloud, after concrete work and the appropriate scripts have been developed, doing oral work our students are bathed in rich language—scripts and strategies—that they might not otherwise have access to.

Students describe the work done at the concrete level using mathematics words and expressions based on the visual and tactile experience of the meaning of mathematical words in a variety of contexts. This oral work may be facilitated in different contexts by listening to the teacher or other students using words correctly.

- Acquiring confidence and fluency in speaking, using complete sentences that include the new words and phrases, sometimes in chorus with others and sometimes individually,
 - discussing ways of solving a problem, collecting data, organizing data and discussing the properties of the data for a variety of reasons: to generate hypotheses, develop conjectures or make predictions about possible results or relationships between different elements and variables involved in the problem,
 - presenting, explaining, communicating, and justifying methods, results, solutions, or reasoning, to the whole class, to a group or a partner,
 - generalizing or describing examples that match a general statement.
- *Reading:* For example, during the reading of materials (books, stories, or videos) students must also be asked questions involving mathematics concepts because they cannot learn the meanings of

words in isolation. Activities such as reading aloud and silently, sometimes as a whole class and sometimes individually, particularly during elementary grades, can develop strong mathematics language.

Shared reading can be a valuable context for learning and teaching not only mathematics language but also mathematics. For example, the National Council of Teachers of Mathematics (NCTM) Curriculum and Evaluation Standards for School Mathematics (1989, 2000) advocates strategies such as the use of children's books as a vehicle for communicating mathematical ideas and therefore mathematical language. Students can be asked to read and explain:

- numbers, signs, and symbols, expressions, and equations in blackboard presentations
- instructions and explanations in workbooks, textbooks, and other multimedia presentations
- texts with mathematical references in fiction and non-fiction books, books of rhymes, children's books during the literacy hour, as well as mathematics lessons
- labels and captions on classroom displays, in diagrams, graphs, charts, and tables
- definitions in illustrated dictionaries, including dictionaries that the children have made themselves, to discover synonyms, origins of words, words that start with the same group of letters (e.g. triangle, tricycle, triplet, trisect ...), words made by coding pre-fixes or suffixes, words derived from other words.

H. Structure, Syntax and Voice in Mathematics Language

Most children entering kindergarten and first grade have developed a sophisticated syntax for their spoken language, but they have not acquired the same level of syntactical facility in the understanding and use of the written and oral language of mathematics. The syntax of mathematical language is quite rigid. In a mathematical expression such as:

$$7 + 4 = 11 \text{ (seven plus four equals eleven)}$$

the order of words as written is quite rigid. We can interchange the order of words (numbers and mathematical operational symbols) somewhat according to the rules of mathematics. We cannot substantially change the order. Unlike other languages, including the English language, where we can play with the order of words in the sentence, without losing the meaning of the expression, we do not have that freedom in mathematics expressions. For example, we cannot say: $7 + 4 = 11$ and other variations like this, but we can say $4 + 7 = 11$ which introduces an important concept: commutative property of addition. This shows that the order and arrangement of words in mathematical expressions is quite rigid, but multi-faceted.

Children who have difficulty in their native language, particularly with syntax, find the syntax of mathematics difficult to grasp. For example, the problem: $1.4 \div 56.43$ is misread and written incorrectly in operational sign and fraction form by more than 50% of middle school and high school students. They read it as 56.43 divided by 1.4 rather than 1.4 divided by 56.43, or that 1.4 divides into 56.43. Because many students find it difficult to realize the difference the syntactical structure makes in the meaning of a mathematical expression or sentence, it is important for a mathematics teacher to make children aware of the differences when they change the order of words or operations in mathematical expressions. For example, if it says '2 more than a number' and the student writes $2 + n$ and the teacher accepts it because ' $n + 2$ is the same as $2 + n$ without comment and does not correct it as $n + 2$, he is inviting a student to make a mistake in '2 less than a number.' Invariably, most students translate that as $2 - n$, which is obviously incorrect.

As a matter of course, the teacher must provide linguistic equivalents of a mathematical idea so that when the child sees the same idea in a different linguistic format, in different textbooks, or used by different teachers he is able to understand what that expression means.

Similarly, children find it easier to understand and express ideas in an active voice. They express operations of addition, subtraction, multiplication in the form ‘2 plus 4,’ ‘4 minus 1,’ ‘2 times 6,’ respectively. However, they have difficulty with division (e.g., 3 divided by 4), as most times, it is stated in the passive voice. Therefore, many children will write: $4 \div 3$ rather than $3 \div 4$.

1. Instructional Suggestions for Building Syntactical Structures

To make sure students understand the role of syntax in mathematics and in the language of instruction and acquire a variety of syntactical structures, teachers need to help students develop multiple linguistic expressions for the same arithmetical operation. For example, in the previous problem: 3 divided by 4, children should be encouraged to read this as: 4 divides into 3; how many groups of 4 are in 3? Similarly, the arithmetic operation of subtraction is not limited just to terms such as ‘take away’ or ‘minus.’ It is also an absolute comparison of quantities in several forms. It should be introduced to students in different forms. We need to create versatile models for each mathematics operation. For example, it is important that the students see $17 - 9 = ?$, as

- 17 minus 9 equals what?
- 17 take away 9 equals what?
- What do you get when you subtract 9 from 17?
- What is the difference between 17 and 9?
- 17 is how much more than 9?
- 9 is how much less than 17?
- What is 17 reduced by 9?
- What is 9 take away from 17?
- What is left when you take away 9 from 17?
- What is 17 subtract 9?
- What is 9 fewer than 17?
- What do you get when you subtract 9 from 17?
- Subtract 9 from 17?

- Deduct 9 from 17? Etc.

As one can see, these sentences are in different forms and use the words that signify the operation of subtraction in different places in the sentences. In these different forms of the same subtraction problem, their language of subtraction is broadened and the conceptual schema for subtraction is much more robust. If children are only taught that subtraction involves ‘taking away,’ they can experience real difficulties when confronted with ‘the difference between’ or ‘subtract -4.5 from -3.4 ’ as subtraction. The ‘take away’ expression does not create an adequate cognitive schema and model to accommodate the subtraction of integers, fractions, and decimals.

Subtraction occurs in real life in multiple linguistic forms. When students see an operation in a multiplicity of linguistic expressions, they do not become dependent on specific words, a specific voice, nor a specific order. They develop versatile language containers for the concept, operation, or procedure. Many teachers teach mathematics concepts as if children are ‘terminal students.’ As if they are going to see subtraction only in this form and as an operation on whole numbers only. We should teach them to acquire flexibility of thought in the use of language in mathematical operations. This flexibility of thought helps them to acquire new concepts more easily, and they can generalize ideas better.

Language is a means of forming conceptual models for mathematics ideas; limited language and its models limit the conceptualization of mathematics ideas. For example, we just read $17-9=?$ in several ways. When it is read as ‘17 minus 9 equals what?’ ‘17 subtract 9 equals what?’ or ‘17 take away 9 equals what?’ all these linguistic expressions create a particular conceptual model and suggest a certain kind of ‘action’ and ‘thought process’ in the mind of the learner. The model invokes a particular type of activity—physical, iconic, and abstract action. For example, in this problem and the model suggested by the language, you begin with 17 objects and then you ‘take away/subtract/minus 9 objects.’ We begin with 17 objects and perform a direct action (taking away, subtracting, etc. on those 17 objects).

The conceptual and linguistic schema formed by this language is adequate only for whole numbers when we are subtracting a smaller number from a larger number, but it is inadequate for later instances of subtraction problems dealing with fractions, decimals, integers, real numbers, and algebraic expressions. For example, $-32 - (-7)$ or $-1.04 - 3.9$ do not lend themselves to the ‘take away’ model. As a result, many students never understand and cannot master the subtraction of integers or how to manipulate algebraic expressions. Our experience and many teachers report that more than 70% of errors students make in algebra deal with operations with integers and fractions.

Acquiring the concept of numberness (numerosity), learning natural numbers (counting numbers), and performing simple arithmetic operations on them can be learned and mastered with very little language involvement, but their preparation is not solid for future number systems. However, when we introduce higher order concepts—place value, operations on numbers—whole numbers, fractions, integers, word problems, etc. the language, therefore, the vocabulary, the syntax, and the quality and quantity of language usage have a major impact on children’s thinking. Then we begin to see problems in conceptualization, applications, and communication of mastery of mathematics ideas.

Let us consider another example. When we asked students in sixth grade to define a square, most of them gave the standard definition with imprecision: ‘A square has all sides equal.’ When we have probed further, they begin to see the holes in their definition: it is not enough to have all sides equal. That gives us a rhombus, or it may not even be a closed figure, or the sides might be arcs of a circle. To avoid this, we need to take the linguistic approach. We drew a square on the board.

Teacher: What do you observe in this figure and about this figure?
Anything and everything about it. What are the key characteristics of this figure?

After some probing and coaching, students came up with the following:

Student 1: It has four sides.

Student 2: It is made up of straight sides.

Student 3: It is a closed figure.

Another shouted: 'It is a shape.'

Student 4: All the sides are equal.

Student 5: All the angles are equal.

Student 6: It is a figure.

Student 7: It is a two-dimensional figure.

Student 8: It is a quadrilateral.

Student 9: It is a parallelogram.

Student 10: It is a rhombus.

Student 11: It is a polygon.

Student 12: It is a four-sided polygon.

Teacher: In groups of three, I want you to come up with a 'consensus' definition of a square in your group. I will choose the group members to present your consensus definition.

Students came up with several definitions. The author collected them. We discussed each one of them for accuracy from the point of view of language and mathematics and posted those definitions on a wall in the class with the names of the group. Here are the definitions generated by students.

'A square is a four-sided polygon with equal sides and equal angles.'

'A square is a rectangle with four equal sides.'

'A square is a rhombus with four equal angles.'

'A square is a parallelogram with four equal angles and four equal sides.'

'A square is a quadrilateral with four equal sides and four equal angles.'

'A square is a four straight-sided closed figure with equal sides and equal angles.'

'A square is a two-dimensional closed shape, with four equal sides and four equal angles.'

To test their understanding of the concept, the author asked many questions based on those definitions. Then the author changed this into an exercise to be completed by each student independently.

- A rectangle with _____ is called a square.
- A parallelogram with _____ sides and four _____ is called a _____.
- All squares are _____.
- A rectangle is a square when _____.
- Etc.

I. Translations: Native Language to Mathematics and Vice Versa

One of the difficulties children face is to answer a problem posed in words even when they can do the relevant computations. This is evident on local, state, national, and international assessments. Word problems are a request for computations in linguistic form. They are examples of *integrating cognitive, linguistic, mathematical content, and thinking*. Most children and adults dread word problems as they view them as difficult and unnecessary. Many teachers devote the least amount of time to word problems. When the procedures of multiplication in the unit are done, then we say: ‘Let us apply it. Let us do word problems involving multiplication.’ Children practice a little on the word problems.

The problem exists at four levels:

First, when students have been introduced only to the procedural aspects of mathematics, and the conceptual and linguistic aspects of mathematics are not emphasized, they find themselves not able to respond to mathematics word problems. They see a word problem and give up. They have difficulty even in determining the content of the problem. The first problem is having difficulty translating words in the mathematics word problem into mathematics expressions. This is a language issue, and language skills such as reading,

comprehension, and translation are some of the causes of the problem.

Second, the students have been introduced to language, but they have been exposed to only one form of conceptual schema for a mathematics idea, therefore they cannot determine the concept, operation, or the formula needed to solve the problem. Therefore, they make false starts and give up.

Third, the student has a strong language knowledge and conceptual schemas but does not have enough practice of or has limited exposure to efficient procedures, they can translate the word problem but may have difficulty solving it.

Fourth, students have not been exposed to systematic problems solving strategies and heuristics and therefore cannot put together a scheme for solving the problem.

This requires helping students acquire, develop a practical systematic way of solving problems.

Because of the correlational nature, there has always been considerable interest in understanding the nature of the relationship between reading (particularly, reading comprehension) and mathematics word problem solving. This has prompted educators to focus on developing vocabulary and strengthening reading comprehension to improve problem solving. This usually means learning and identifying key phrases and the corresponding arithmetical operations. The teacher will ask students to underline the common key phrases or words such as ‘left,’ ‘all together,’ ‘of,’ or ‘part of’ in the word problem so that they can connect the mathematical operation with the problem. However, this does not help the child to make lasting connections between language, mathematical concepts, and problem-solving. This approach produces only moderate success because the focus is only on the limited vocabulary that child possesses.

Students have the most difficulty translating words in mathematics word problems into mathematical symbols and operations. For example, the sentence ‘There are twice as many bananas as apples’ is often expressed as

‘ $A = 2B$ ’ rather than ‘ $B = 2A$.’ Where A represents the number of apples and B represents the number of bananas in the collection. This conceptual error occurs whether the problem is given in the student’s primary language or in the second language. The sentence signals an extra level of difficulty for language minority students because of the additional translation required. Additional translation is a larger problem than just not being able to solve word problems. The suggestion that the remedy for this is to repeat solving many more problems of this type is not the answer. What is needed is increased exposure to mathematics content and thinking and translation activities from mathematics to English and then English to mathematics. Experience with mathematics content and language is the most important factor in predicting children’s achievement in mathematics word problem-solving.

To have an impact on children’s word problem-solving abilities, the size and nature of the mathematics vocabulary and practice in its usage must be done. For example, we have found that when we ask children *to create and write number stories* for a given mathematics expression. ‘Write a number story for the problem: $17 - 9 = ?$ ’ Children write their stories and then share them with the class. The teacher points out the different terms—words, expressions, and the order of words used in their stories. The teacher shares her story and asks them to write another story for the same problem, but use another word or expression for the operation involved in the given problem.

Solving word problems involves several language and mathematics content skills:

- Reading a word problem is a different skill than reading any other prose. A word problem should be *read slowly, in phrases, and with understanding*. In ordinary reading, if we do not know the meaning of a word, in the text, the context may help us in deriving the meaning or an approximate meaning. In word problems the vocabulary falls into one of at least five categories:

- Contextual words that describe the setting of the problem,
- Action words from ordinary vocabulary that may describe the operation directly or indirectly,
- Specific content words that might describe the operations involved,
- The nature of the question asked to indicate the operation, and
- Hidden words that describe the operation.

Therefore, the problem must be read more than once. We urge our students to read the word problem at least four times. The *first time* is to understand the context of the problem and the general idea of the problem. The *second time*, to determine and classify the quantitative and spatial information into knowns (*what is given*) and unknowns (*what needs to be found?*) and define the variables. What are they asking to find, calculate, determine? The *third time* is to translate the information using the knowns and unknowns (the variables) into mathematics expressions, equations, equalities/inequalities, and determining formulas. The *fourth time* is to determine whether we have answered the problem or not and whether it is in proper form? Is it in proper units or format?

- Recognizing and understanding explicit and embedded mathematical vocabulary. It involves encoding words in the problem to understand it and then to reformulate the problem in one's own words. (What is given? What is happening in the problem? What is the scenario?)
- Understanding the mathematical concepts and operations behind those words and expressions. (What is being asked? What needs to be done? What mathematics content or procedure is involved in the problem?)
- Translating English expressions into mathematical expressions. (Forming equations, strategies, formula seeking, and selecting procedures.)
- Solving the equation(s) or executing procedure(s). (What mathematics content is needed? What do I do? How do I solve it? Is there any information that I know about the problem or the items that I can use?)

For example, the sum of the angles of a triangle is always 180 degrees.)

- Translating the calculated values in terms of what was asked in the problem. (Is this what I needed to do? Did I get the right answer?)

Most times, when students know the operations involved in the word problem, they can handle the execution of the operation. The ability to solve word problems, therefore, lies in proficiency in translation at several levels. The complexity of these translations suggests why children find word problems difficult. Facility in translation, in turn, is dependent on the development of mathematical language—its vocabulary, syntax, grammar, and special phraseology and familiarity with language expressions used in word problems.

As mentioned earlier, a mathematical language expression suggests the related conceptual schema. Each language expression about a mathematics idea invokes a conceptual model. Limited language expressions invoke limited conceptual schemas. Schemas are helpful to us in these two-way translations. Let us consider the concept of simple subtraction:

- David has 14 books and John takes away 6 books from him. How many books are left with him?
- An elm tree is 16 feet tall, and a maple tree is 9 feet tall. How much taller is the elm than the maple?
- School is 16 blocks from David's house. David has already walked 9 blocks. How many-more blocks must he walk?

Each one of these three linguistic models suggests a conceptual schema for subtraction, appropriate instructional material, and a related teaching strategy:

- 'Take away' expression suggests use of discrete (counting) materials such as counters and 'counting on' or 'counting down' strategies to solve the subtraction problems.

- ‘How much more’ suggests the use of semi-discrete or continuous models: the Number Line (semi-discrete), the Empty Number Line (continuous), or Number Rods (continuous), and the solution is found by comparison of two locations on the number line; the jumps from 9 to 10 and then to 16 on Empty Number Line; with number-rods comparison of the 9-rod with 10-rod plus the 6-rod.
- ‘How many more’ suggests the missing addend as a teaching strategy and using the comparison of two Cuisenaire rods or the Empty Number Line as effective, efficient, and generalizable instructional materials.

If children have been exposed to only limited language models, they will have limited conceptual frameworks and strategies and will not have the flexibility of thought necessary for the translation skills necessary for many word problems. Thus, limited language and limited conceptual frameworks do not prepare children well for problem solving in general, and word problems in particular.

The linguistic variables also play an important role in word problem-solving. For example, children presented with word problems requiring identical operations on identical numbers exhibit differences in outcomes when the wording and problem contexts change. The language variables that contribute to problem difficulty are:

- familiarity of the situation described in the problem,
- the arrangement of the problem; the number of unfamiliar objects and nonessential elements,
- whether the story problem is intrinsically interesting,
- how difficult the vocabulary is,
- whether the problems are presented as declarative sentences, as questions, or as problem statements,
- placement of the question; and so forth.

To develop proficiency in solving word problems requires teaching the language of mathematics and developing word problem exercises that pay attention to these variables in an organized manner. Word problem exercises should be developed incrementally by introducing each one of these linguistic variables individually and then in combination (Looi et al., 2009).

J. Instructional Suggestions

1. Reading Comprehension in Mathematics Word Problems

Even if children have strong mathematics vocabulary and understanding of the syntactic structure, for efficient translation, they must have good reading comprehension as well. Teachers should devote some time to improving reading comprehension exercises when children are about to encounter reading material on a new topic or are getting ready to solve mathematics word problems. The following are some suggested activities. Reading comprehension in a content area such as mathematics is dependent on both the content knowledge and mathematics language.

Example 1: Here is an example of a reading exercise in a geometry lesson. The teacher is about to start a lesson on intersecting lines. She assigns the reading of a passage from the textbook before the start of the lesson.

Teacher: Before you read the passage, I want you to read these statements, and in the space to the left of each statement, place a check mark (✓) if you agree or think the statement is true.

Students read the material.

Teacher: When you're done reading, add new check marks or cross through those about which you have changed your mind. Use the space under each statement to note the page, column, and paragraph(s) where you found information to support your answers.

____ 1. All lines intersect.

- ___ 2. When they intersect, they intersect in at one point.
- ___ 3. Two intersecting lines make four angles.
- ___ 4. All intersecting lines are perpendicular.
- ___ 5. Supplementary angles have a sum of 180° .
- ___ 6. Some pairs of angles are not supplementary.
- ___ 7. Adjacent angles share a common ray.
- ___ 8. Vertical angles are complementary.
- ___ 9. Vertical angles can be right angles.
- ___ 10. Vertical angles form four linear pairs.
- ___ 11. Supplementary angles must be adjacent.
- ___ 12. Opposite rays form a straight line.
- ___ 13. Every point on a line segment is 180 degrees.
- ___ 14. When naming a vertical angle, the middle letter must be the intersection point of the two lines.
- ___ 15. When naming adjacent angles, one of the three letters will be the same for each angle.
- ___ 16. There are three ways of writing the name of an angle.
- ___ 17. With any two intersecting lines, angles adjacent to two vertical angles are vertical and congruent.
- ___ 18. A line perpendicular to a plane is perpendicular to exactly 2 lines on that plane.

After the students have answered these questions, she discusses the answers and students arrive at pertinent definitions, conjectures, results, and theorems. They provide examples and counterexamples to prove, disprove, or make conjectures. It can be made into a game.

2. Writing Mathematics

Students should be asked to write and record in a variety of ways, progressing from words, phrases and short sentences to paragraphs and longer pieces of writing.

- Writing prose to describe, compare, predict, interpret, explain, justify ...

- Writing mathematics expressions and formulas, first using words, then symbols [e.g. one less than 10 is $10 - 1$, one less than d is $d - 1$.]

Amy likes candy better than ice cream. She likes ice cream better than soda. She likes soda better than popcorn. Translate these sentences into mathematics expressions. [$C > I > S > P$, where C = candy, I = ice cream, S = soda, and P = popcorn.]

- Sketching and labeling diagrams to clarify their meanings and describing relationships between different elements
- Drawing and labeling graphs, charts, or tables and interpreting and making predictions from the data in mathematics and other subjects

Another method is to give children language prompts. Of course, the response to these prompt questions will be different depending on children's ages and experiences. Engaging them in writing mathematics-related topics is also a useful exercise. For example:

- Describe a square in your own words.
- How do you feel about mathematics? What is different about your mathematics experiences compared to other subjects?
- Define two mathematical terms from today's homework.
- Describe the step-by-step procedure for completing today's homework.
- Which operations are the inverse of each other (give three examples) and why?
- What is mathematics? What did you learn in mathematics today?
- What do you do really well in math?
- What is your favorite math topic and why?
- I work best in math when ...
- Write a letter to a classmate who has missed class explaining what was covered in math class today.
- Write a letter to an incoming ... grader telling him/her what to expect next year and how to prepare for this class.

- Write a word problem for one of the homework problems.
- Rewrite one of your homework problems in English, without using mathematical symbols.

3. Writing Number Stories as a Precursor to Word Problems

Before children are asked to solve word problems, we have found it to be helpful if they write number stories based on mathematics expressions and equations. Here is an example of creating such a story. This vignette is taken from the second week in a second-grade class. In the first week, the focus was on learning the vocabulary of addition and subtraction and on addition and subtraction facts. Here is a portion of a session during the second week:

Teacher: Look at the equation: $17 - 9 = \square$. Tell me what should go in the box.

Student: 8.

Teacher: How did you arrive at the answer?

Student 2: I know 17 take away 9 is 8.

Teacher: That is perfect. Now that you know the answer to this problem, I want you to make a story using all the numbers involved in this equation and perform the operation indicated in the equation.

Student 3: Do you want me to tell you a story using subtraction?

Teacher: Yes! Or any other word that goes for subtraction.

Student 4: Just like the stories we made for addition and then subtraction.

Teacher: Yes!

Student 5: I had 17 stickers in my collection. 9 of them were doubles, so I gave them to my friend. Is that what you want?

Teacher: Yes! You are on the right track. What question should you ask so I give you the answer? What should you do about the box?

Student 6: That is easy. I will have 8 in the box.

Teacher: That is right! What question would you have asked so that I would give you the answer as 8?

Student 7: I would have asked: How many stickers do I have now?

Teacher: That is right. Now give me the whole story together with the question.

Student 8: I had 17 stickers in my collection. 9 of them were doubles, so I gave them to my friend. How many stickers do I have now? Is that what you want?

Teacher: That is right. Let me give you another equation. Look at the equation: $17 - 9 + 9 - 7 =$. Tell me what should go in the box.

Student: 9.

Teacher: How did you arrive at the answer?

Student 9: I know 17 take away 9 is 8. Then I added 8 to 9 and that is 17 and then I subtracted 7 from that. That gives me 10. Right?

Teacher: That is perfect. Now that you know the answer to this problem, I want you to make a story using all the numbers involved in this equation and perform all the operations.

Student 10: Do you want me to tell you a story using addition and subtraction?

Teacher: Yes!

Student 11: Just like the stories we made for addition and then subtraction. Now you want me to combine those two.

Teacher: Yes!

Student 12: I had 17 stickers in my collection. 9 of them were doubles, so I gave them to my friend. He gave me 9 different stickers in return. I also sold 7 of them to my friend. Is that what you want?

Teacher: Yes! You are on the right track. What should you do about the box?

Student 13: That is easy. I have 10 of them.

Teacher: That is right! What question would you have asked so that I would give you the answer as 10?

Student 14: I would have asked: How many stickers do I have now?

Teacher: That is right. Now give me the whole story together with the question.

Student 15: I had 17 stickers in my collection. 9 of them were doubles, so I gave them to my friend. He gave me 9 different stickers in return. I

also sold 7 of them to my friend. How many stickers do I have now?
Is that what you want?

Teacher: Wonderful! Now I will give you a story and you give me the equation.

I had 27 stickers in my collection. 12 of them were doubles, so I gave them to my friend. He gave me 10 stickers in return. I also sold 11 of them to another of my friends. What does the equation look like?

Student 16: That is easy! It is the same thing. You have just changed the numbers. It is: I have 27, give away 12, I get 10, and I sell 11. The equation is: $27 - 12 + 10 - 11 = \square$

Teacher: That is correct. How many stickers do I have now?

Student 17: I have 14 stickers.

Teacher: I will give you another one:

I have to hike 15 miles of trails. We hiked 9 miles the first day. We had to go back to a spot 2 miles back that we passed the previous day. On the second day we hiked 7 miles. How many miles do we still have to go? Everyone should write the equation and share it with their partner and then a volunteer will share their equation.

The author moved around the class, participated in discussions, gave some suggestions, asked some questions, goaded some teams to action, praised some children's efforts, gave some examples, invited some teams to observe other children's efforts. After some time, some hands started being raised. He invited one team to the board and asked one member of the team to write the equation and the other to explain how they arrived at the equation.

Student 1: Our equation is: $15 - 9 + 2 - 7 = ?$

The teacher asked if any other team agreed with it. The teacher acknowledged them. He asked others to share their equations. Asked for their reasoning and asked other teams' opinions. When the class

decided the correct equation. The teacher asked volunteers to answer his questions. The teacher points to the correct equation.

Teacher: This equation as all of us have agreed is right. That is very good. Can someone answer: ‘How did you decide to add 2 to the equation?’

Student 2: Oh! That is because that added 2 extra miles of hiking.

Several other examples were used to give the students facility to think this way. This kind of activity should be done using different numbers and different contexts. In the beginning, the context should be familiar to the students and later, one can use other situations that may not be familiar.

The three components of mathematics learning: *linguistic*, *conceptual*, and *procedural* need specific pedagogical approaches.⁸ As the previous examples show, to improve children’s mathematics conceptualization and achievement in mathematics problem-solving requires a focus on the development of appropriate mathematics language.

Since formal and informal language have varying roles in learning concepts and applying mathematics, some difficulties emanate from the interaction of the systems responsible for number, calculations, procedures, and language. In the case of language, it may be vocabulary, syntax, or the ability to translate from mathematics to language and from language to mathematics and reading.

Many children with language-related problems do not have problems in mathematics. However, since some of the same prerequisite skills are involved in both language acquisition and mathematics learning, at least in the early years, the coincidence of dyslexia and dyscalculia is not uncommon. Many dyslexics can solve computational and spatial problems easily. They only have difficulty with language related problems in mathematics as they may not have the facility to receive, comprehend, and produce quantitative and spatial language (words, symbols and expressions) properly. They cannot solve problems that depend on language. Research studies have shown that many elementary-age students who perform poorly in mathematics also have basic language deficits.

Dyslexics' language-related problems in mathematics are of two kinds: *primary* and *secondary*. Language difficulties in reading, spelling, etc. cause primary problems in mathematics. Dyslexic children sometimes also manifest problems in arithmetic that are of a secondary nature. That is, since the dyslexic child is provided with extra instructional experiences in reading and language areas, their experiences in arithmetic during the critical developmental period may be limited or overlooked. As a result, the child may begin to do poorly in arithmetic. In many cases, once the child's problem in reading and other language areas is remedied, he may begin to do well in arithmetic unless the failure in arithmetic has converted into numerical anxiety and has affected the child's self-esteem. Then the problem translates into mathematics anxiety.

Given the complexities of mathematics learning, it is important to understand the nature of children's language and reading problems in order to understand their arithmetic problems as they shed light on mathematics learning and problem-solving abilities. These challenges are substantial. On the one hand, teachers are faced with the ever-increasing demands of accountability for children's performance on standardized tests. These tests are narrow in scope because they examine only a segment of mathematics learning that relates highly to procedural competence. On the other hand, the objective of a mathematics curriculum should not just be for children to learn a collection of procedures but also to think mathematically and to be able to communicate mathematics ideas. In order for children to have an interest in learning mathematics, teachers also need to provide them with an understanding of the language and concepts of mathematics. Proficiency in developing a mathematical way of thinking—competence in the language and concepts of mathematics—is important for intra-mathematical applications as well as for interdisciplinary and extracurricular applications. By developing children's proficiency in mathematics concepts and mathematics language and by understanding the nature of their language-related learning problems, we can improve their mathematics word problem-solving ability and their competence in general problem-solving, which are so essential in a highly technological society.

I. Language Difficulties, Dyslexia, and Mathematics Learning

The nature of reading and language disability sheds light on a child's mathematics learning. Some children are late in learning to read and have persistent difficulty in remembering spellings in contexts without any issues of intelligence or lack of opportunity to practice. These issues spill over into the area of mathematics. Some of them experience other difficulties also—long struggles over learning to tell time (on non-digital clocks), uncertainty over left and right, confusion over times and dates, an inability to recite without stumbling, any but the easiest arithmetic tables. These problems lie in the intersection of language, reading, and numeracy.

A. DYSLEXIA

When a child of average ability is late at learning to read and has special difficulty with spelling and shows confusion over getting things into spatial and temporal order, whether in language or arithmetic, he has a disability called dyslexia. A person with this disability (or group of disabilities) has difficulty in receiving, comprehending, and/or producing language orally and/or writing.

Dyslexia has been associated with slow speech development, speech and language difficulties, delay in motor development, sequencing problems (usually the inability to remember the days of the week or the months of year), impairments in temporal or spatial awareness, and visual perceptual deficits.

Most dyslexics are late in learning to read and have, especially, difficulty in learning to spell. Many dyslexics may remain slow readers, and although speeding up is possible, any task, which calls for the processing of symbolic material at speed, is likely to cause them trouble. Dyslexics have difficulty with phonology—with the remembering and ordering of speech sounds. Problems over left and right also persist in their work.

In general, if usual counting-based methods are used, dyslexics tend to be slow at certain basic aspects of mathematics—learning and recalling arithmetic facts, particularly multiplication tables, adding up columns of figures, etc.—but once they have understood the symbols, they may be

quite creative. At the same time, based on their profile, some dyslexics can be strong in certain aspects of mathematics and weak in other areas. It is important to distinguish dyslexic children from slow-learning children.

For individuals with dyslexia, learning mathematical concepts and vocabulary and the ability to process and use mathematical symbols can be impeded by problems like those which interfered with their acquisition of the written language. Too frequently and too readily, individuals with dyslexia who have difficulty with mathematics are misdiagnosed as having the neurologically based disability dyscalculia—literally trouble with numberness, knowing number relationships, and calculating, as against, the acquired dyscalculia where the difficulties are the by-product of environmental factors. Some dyslexics are only numerically dyslexic—having difficulty only in the numerical aspects of mathematics but may not have difficulty with spatially related concepts.

Difficulties with math for dyslexics can be identified by the following symptoms:

- The dyslexic may have a problem with number concept, number relationships, and outcomes of numerical operations, particularly, mental calculations involving adding, subtracting, and times tables.
- They may be confused by similar-looking mathematical signs: + and $\times$; and the comparing of quantities symbols: =; < (less than) and > (greater than).
- They may have limited vocabulary, as a result, they may not grasp that the words ‘difference,’ ‘reduce,’ ‘take away,’ and ‘minus’ all suggest the same operation: that is, ‘subtraction.’
- They may understand the term ‘adding’ yet be confused if asked to ‘find the total or sum.’
- The dyslexic may reverse numbers or expressions and read or write 71 for 17 or 3 divided by 4 for 4 divided by 3. They may transpose numbers i.e., 752 to 572 or some other arrangement of digits. Therefore, place-value, fractions, decimal numbers, and integers may pose difficulties for them.

- They may have problems with applications of numeracy, even simple ones, such as telling the time, reading a timetable, making change and word problems.

Individuals with dyslexia may have problems with the language of mathematics and the concepts associated with it. These include spatial and quantitative references such as ‘before, after, between, one more than, and one less than.’ Mathematical terms such as numerator and denominator, prime numbers and prime factors, and carrying and borrowing may also be challenging.

In many ways, some of the symptoms of dyscalculia closely parallel the behaviors exhibited by students with language dysfunctions; therefore, many neurologists believe that dyscalculia does not exist as a separate dysfunction but is a manifestation of a brain lesion which causes language and mathematics dysfunctions simultaneously. This view has some relevance, as mathematics is also a language—the language of quantity and space. Some mathematics problems are extensions of language difficulties such as *alexia*—an inability to read, and/or *agraphia*—an inability to write. For example, almost half of the students with difficulties in mathematics also have difficulties in spelling and writing and many show signs of dysgraphia.

Not all dyslexics have problems with mathematics, and not all dyscalculics have difficulty with reading and other language skills. Of course, there are many differences based on strengths and weaknesses of dyslexics and dyscalculics. Pure dyscalculia (the difficulty in conceptualizing number, number relationships, and the outcome of numerical operations) and dyslexia are independent. However, in many cases, dyscalculia (and acquired dyscalculia) and dyslexia influence each other, particularly when dyscalculia is defined more generally (as many psychologists and special educators do) as any poor performance in mathematics (including the failure of number and calculation mechanisms). For many dyslexics, the difficulties that affect their reading and spelling also cause problems with mathematics.

Some correlational evidence has emerged between the coincidence of dyslexia and dyscalculia. The British Dyslexia Association (On Dyscalculia) has suggested that from 20 to 60% of children have some difficulty with numbers or number relationships. According to studies, almost 51% of dyscalculics also show signs of dyslexia. Of those dyslexics who do not have mathematics difficulties, about 11% excelled in mathematics. The remaining dyslexics have the same mathematical abilities as those who do not have learning difficulties. These numbers have not significantly changed.

Arithmetic and mathematics learning deficits can be caused by a variety of factors, sometimes because of reading difficulties and at other times with no connection to reading difficulties. Children with arithmetic difficulties and children with combined arithmetic-and-reading difficulties represent two different underachievement subtypes whose problems may be due to different cognitive and neuropsychological deficits and characteristics. This is consistent with evidence from clinical and experimental studies suggesting that children of normal intelligence who experience difficulties with arithmetic can be divided into subtypes.

According to the Multiple *Deficit* Model, comorbidity results when the *genetic* and *environmental* risk factors that increase the liability for a *disorder* are domain-general. To explore the role of domain-general etiological risk factors in the co-occurrence of learning-related difficulties, *meta-analysis* studies of third- through ninth-grade children have been conducted. These studies try to estimate the average *genetic*, shared *environmental*, and non-shared environmental correlations between reading and *attention-deficit/hyperactivity disorder (ADHD) symptoms*, and *reading* and *math*, as well as their potential moderators. Results have revealed average *genetic*, shared and non-shared environmental correlations between *reading* and ADHD symptoms of .42, .64, and .20, but high correlations of (.71, .90, and .56) and *reading* and *math* of .71, .90, and .56. This suggests that reading and math may have more domain-general risk factors than *reading* and ADHD symptoms. A number of significant sources of heterogeneity also have been found and discussed. These results McGrath et

al. (2010) have important implications for both intervention and classification of learning disabilities.

This shows that a multiple deficit model of neurodevelopmental disorders holds that any confluence of disorders is an outcome of a combination of specific, common, and combined deficits that become responsible for comorbidity of disorders, such as Reading, ADD, and Numeracy disorders. For example, phonological awareness (decomposition/recomposition in numeracy), naming speed, working memory, and inhibition are some of the shared conditions amongst reading, numeracy, and ADD.

The existence of a group of arithmetic-and-reading difficulty individuals shows that some arithmetic difficulties result from difficulties with reading. On the other hand, several children with normal reading scores obtain low scores on arithmetic tests, which shows that not all arithmetic difficulties can be attributed to a general deficit in language-related processing. In addition, there exists a substantial group of poor readers of normal intelligence whose performance on arithmetic tests is higher. Therefore, mathematics learning problems in general and the types of problems exhibited by dyscalculics, acalculics, and or dyslexics are heterogeneous in nature.

Because of the heterogeneous nature of mathematics learning problems, language-related problems in mathematics fall into three major domains:

- a. Mathematics problems related to and originating from language processing difficulties,
- b. Mathematics problems that have the same basis as reading problems because of underlying learning mechanism such as sequencing, visual-perceptual integration, working memory, organization, spatial orientation, etc.
- c. Mathematics problems that originate from the combination of language and reasoning deficits.

K. Mathematics Problems Related to Language

Some students' mathematics disability or difficulty is driven by problems with language. The same difficulties that they experience in reading and other language concepts interfere in learning mathematical concepts. In mathematics, however, language problems are confounded by difficult terminology, some of which children hear nowhere outside the mathematics classroom, and by the language-based logic of mathematics.

The way that numbers are represented linguistically is also significant and can cause mathematics difficulty. We have observed that many elementary and even early middle school students do not see how ten hundred (1000) and one thousand (1,000) are same. Often, it is only through giving examples and generating these numbers with the help of Base Ten and Dienes blocks that they begin to see them properly. When they are taught purely by the use of language and counting, they have difficulty in seeing the sameness; otherwise, they think they are different numbers. The reason is clear. For example, twelve hundred and one thousand two hundred appear to be handled differently in the brain. Twelve hundred is understood as the product of twelve and a hundred (repeated count by 100) whether consciously or not, whereas one thousand two hundred is processed as the sum of one thousand and two hundred. Asking for the solution to compute fifteen hundred plus one hundred most frequently brings the response sixteen hundred, but one thousand five hundred plus one hundred is one thousand six hundred. Although the underlying numerical values are the same, the brain appears to process the numbers differently according to their linguistic representation. The format of the question in the first problem is asking for simple addition whereas the statement in the second problem invokes place value also.

Unlike language, where one engages only with words and their combinations, in the case of mathematics one uses symbols, words, reasoning, and their combinations. In mathematics, *words appear as operators and as operand*. This differentiation has a major impact on a child's ability to learn mathematics. In this context, understanding the role of three words and their meaning is very important: *operator*, *operand*, and *operation*. *What is even more important is their integration*.

Operator (mathematical symbols, digits, numbers, +, —, =, ', ÷, (), <, £, etc.): Each symbol is packed with concept and meaning. In mathematics, both symbols and words act as *operators* (mathematical operations) that are to be performed on quantity, words, and space. Examples: Solve for x : $3x + 7 = 11$. Find the product of .34 and 67. If $f(x) = x^2$, find $f(2)$. Divide the circle into halves. The area of the circle is given by: $A = \pi r^2$, where r is the radius of the circle.

Operand: Number (quantity) and shapes are the arguments and objects of the operators. Example: Find the square root of 144. Find the product of 23 and 15. Circle the figure that represents half. Circumscribe the square in the diagram. Integrate the function $\sin(3x + 5)$ with respect to x .

To complicate matters, sometimes there is a combination of the two. For example, find the square root of the sum of 19 and 6.

Difficulty in understanding the operator and operand in mathematics is different from the simple but related problems of the dyslexic. Many students find it difficult to separate the operator and the operand. For example, in an integer problem such as: $-3 + -7$, they cannot determine the roles of signs '−' and '+' before -7 as operator or operand. Students face the same issues when there are several operations in an expression or an equation. For example, in simplifying the expression: $-4y(-2y + 3y)$, using the distributive property of multiplication over addition, many students have difficulty deciding which signs are being used as operands or operators. Examples of this type abound in mathematics and increase once the multiplicity of operations have been introduced.

In elementary school, most teachers emphasize the role of operations only and that too as isolated, separate, narrow meanings of operations of addition, subtraction, multiplication, and division (addition is counting up, subtraction as counting down, multiplication as skip counting up and division is skip counting down), or addition and multiplication make the outcome larger and subtraction and division make the outcome smaller rather than general relationship of these three terms: operator, operand, and operation. We should constantly revisit the definitions and schemas of operations periodically as they are constantly becoming broader in scope.

For example, multiplication as repeated addition, groups of, or an array is limiting; these definitions and schemas do not work for multiplications of all fractions, all decimal numbers, integers, and algebraic expressions. We need to include the area model definition of multiplication early on, so that teachers do not make multiplication of fractions and decimals just a collection of ‘tricks’—in multiplying fractions just multiply ‘numerator times numerator over denominator times denominator’ and ‘multiplying decimals treat the multiplication as if you are multiplying whole numbers and then count the number of digits and place the decimal in the right place.’ That is not teaching mathematics.

1. Lexical Entries (Naming, Labeling, and Language Containers)

One of the major characteristics of dyslexics that they share with dyscalculics is the inability to recall arithmetic facts at an automatized level because of fewer lexical entries for numbers. Reading skills are acquired by creating images of letters, words, combinations of words, objects, and ideas in our minds. These images are like names and labels or language containers for words, numbers, combinations of words and numbers, combinations of numbers, symbols, and conceptual schemas. This naming process facilitates the recall of immediate knowledge. Just as it is possible to build lexical entries for single letters and word parts as well as for words, so it seems reasonable to suppose that there are lexical entries which represent numbers, combinations of numbers (facts), and symbols. It is becoming evident that for the conceptualization of numbers (digits) up to 9 or 10, we form visual clusters (lexical entries for numbers—numberness) in our brain. It is reasonable to assume that there are lexical entries for arithmetic facts also. When the number of lexical entries is small, dyslexics face a major problem of labeling (i.e., tasks of visual and auditory discrimination) phonemes, words, objects, or facts. They do not retain them. Further, in the absence of lexical entries, students construct the facts each time they encounter them. For example,

$$8 + 7 = 8 + 2 + 5 = 10 + 5 = 10 + 5 = 15;$$

$8 \times 7 = 8 (5 + 2) = 8 \times 5 + 8 \times 2 = 40 + 16 = 40 + 10 + 6 = 56$. In both problems, the key conceptual operation is a '*script*' based on decomposition/re-composition, properties of numbers and operations, and chunking of the information into manageable (and familiar) bits to reduce the cognitive load on working memory. A *script* is a *storage-rich* and *retrieval-rich*⁹ process to perform an operation in a systematic way utilizing the fundamental concepts and operations involved. Therefore, it aids in the transfer of information from short-term and working memory to long-term memory and vice-versa.

Most dyslexics are less proficient than mental age-matched controls on tests of object naming and slower on rapid automatized naming of visual and verbal stimuli. Reading-disabled children are less accurate in labeling objects and have difficulty with low frequency and polysyllabic words. For example, young dyslexic children aged between 7 and 9 years are no different from the matched controls in tasks that do not require labeling. They are appreciably weaker than the controls in word analysis tasks and labeling, and in some of these tasks, they are weaker than the poor readers believed not to be dyslexics. This disability affects mathematics learning. They may understand the logic of arithmetic operations, but they show difficulty and inability to perform simple calculations because they cannot recall the needed facts automatically; they resort to counting every time. This also applies to naming geometric shapes and describing terms in arithmetic and mathematics.

It seems dyslexics cannot rapidly access verbal labels and arithmetic facts as they have problems in retrieval from long-term memory or even from working memory. The received sensory information in the short-term memory needs to relate to some information from the long-term memory or rehearsed to be sent to long-term memory. Since dyslexics have fewer lexical items in the long-term memory, they need to rehearse the incoming information or keep it alive in the long-term memory by constructing facts. They end up rehearsing the information at a construction level rather than at a consolidation level. Because they have problems with short-term memory function and with the retrieval of stored memory traces, they have difficulty

holding intermediate steps in calculations in their minds. However, by identifying and relating *patterns*, mastery of decomposition/re-composition, and appropriate scripts for the operation, one can improve connections between new information and learning new information and thereby improving lexical recalls. A strong association seems to exist between mathematics performance and response time on rapid automatized naming or lexical recall.

Because dyslexics have fewer lexical entries, they have problems in most common areas of arithmetic: memorizing and recalling simple addition and subtraction facts, learning multiplication tables by rote, and sequencing and direction both in space and time. Over time, the inability to retain complex information in the memory system gets in the way of learning mathematics at an expected rate. Accordingly, one would expect a lesser range of immediate knowledge of facts and information by memory on the part of dyslexics. Unlike non-dyslexics, they have fewer number facts available: thus, if the question is, ‘What is 6×7 ?’ a non-dyslexic of suitable age and ability can respond ‘42’ in one response, whereas many dyslexics can reach the answer only by working it out by using scripts and knowing essential tables of 2, 5, and adding simple numbers. Most non-dyslexics are aware intuitively of the difference between situations when they can instantly give the answer ($72 + 9 = 81$) to a calculation and when they need to work this out ($127 \times 23 = 2,921$).

The non-dyslexic will learn, for instance, that $8 \times 7 = 56$ after a relatively small number of exposures to stimuli whereas the dyslexic, because of slowness at naming and labeling, will be unable to make use of the presented learning opportunities. Consequently, he needs more exposures to stimuli before these stimuli take on symbolic significance. The dyslexic’s slowness results in longer naming times, which has consequences for the recall of digits, facts, terms, formulas and therefore creates difficulties in learning and using calculation systems and hence the importance of decomposing and re-composing.

The development of lexical entries is dependent on two factors: the underlying prerequisite skills and the amount, type, and quantity and quality

of early training in number conceptualization. This varies considerably among individuals with dyslexia since the amount and type of compensatory early training and the level of mastery of prerequisite skills are likely to be different.

This does not mean dyslexics or dyscalculics cannot learn arithmetic and other mathematics facts and recall them fast; they just need special methods and efficient strategies. Ordinarily, for automaticity and faster recall we focus on rehearsal—increasing the number and frequency of exposures. The non-dyslexic will learn, for instance, that $6 \times 7 = 42$ and automatize this fact, whereas the dyslexics, because of their slowness at naming will construct the fact and will have to pay attention to other intermediate information. This construction may take the form of laborious counting or recall of the sequence: 6, 12, 18, 24, 30, 36, and 42. The construction of facts distracts and dilutes the cumulative effect of the exposure to stimuli. Moreover, the habit of constructing facts carries into higher mathematics. When they are older, these children have difficulty memorizing even the simplest formulas in algebra and geometry.

If not treated at the appropriate age, the naming time problem has consequences later for the recall of digits and facts. It causes difficulty with calculations and with generating quantitative and spatial knowledge for use in problem solving. The extent to which numbersense and mathematical conceptualization are impaired and facts (lexical entries) and appropriate procedures are missing can vary considerably.

Many of the problems that dyslexics face in arithmetic can be put right with appropriate teaching; in particular, the use of concrete instructional materials may help to generate the appropriate lexical entries. Multi-sensory and special approaches such as the Orton-Gillingham reading program build lexical entries for letters, combination of letters, and words and teach efficient strategies. Similarly, the visible and tangible presence of, say, seven objects in a visual cluster, the mark ‘7’ on paper, the written letters ‘seven’ and the sound ‘seven’ or the use of Cuisenaire rods jointly contribute to the formation of the same lexical entry for the number 7. We consider CPVA/S^{[10](#)} as a parallel pedagogy for mathematics.

In some cases, extra exposures and more meaningful stimuli that match students' mathematics learning personalities are needed. For example, many dyslexics with qualitative mathematics learning personality may also have sequencing difficulty—a characteristic of this type of learning personality; finding addition or subtraction facts by sequential 'counting up' and 'counting down' is difficult for them. Memorizing the tables in the usual sequential order and teaching them in the usual way through exposure to the counting process will not result in a successful experience. See the chapter on CPVA/S for efficient strategies for arithmetic fact mastery.

2. Mathematics Disability Subtypes

Acquiring the skill of reading is demanding. Reading in a specialized content area such as mathematics is even more complex. Many children have difficulty mastering the core set of skills research has identified for reading. For example, the reading aloud of novel words is achieved by the phonological decoding processes, in which grapheme-to-phoneme conversion rules are applied to 'sound out' a word's spoken representation. Phonological decoding deficits have been identified as symptoms of reading disabilities (RD). These core deficits are evident across various subtypes of RD. Understanding these core deficits leads diagnosticians to identify RD and special educators to provide effective remediation. Understanding RD helps curriculum planners and classroom teachers to design appropriate programs and provide preventive instruction.

Because of extensive research studies in understanding the reading process, reading skills are well defined; therefore, it is possible to identify a set of core skills and related symptoms when explaining RD. We know what constitutes fluent reading with comprehension and when to expect mastery and what mastery looks like. Unfortunately, no such core symptoms and skills have yet been identified to explain mathematics disabilities (MD) as research on mathematics disability is less extensive than RD research. Although a definition of core skills for dyscalculia is beginning to emerge that helps to diagnose MD, there is wide discrepancy

in the symptoms, definitions, and diagnostic process, for even dyscalculia is absent for several reasons. Although we are beginning to recognize, socially and intellectually, the prevalence of learning problems in mathematics and understand the nature of the impact it has on learners, mathematics education and society, yet, as [Butterworth \(2025\)](#) very eloquently brings to our attention, there is a plethora of definitions, symptoms, diagnostic and recognition criteria for dyscalculia and individuals suffering from dyscalculia. We agree with him. The reasons are many.

First, we do not have any agreement about what constitutes the core skills in mathematics. There is no one skill that is at the core of every mathematical operation. Despite a general agreement on the wider contours of mathematics concepts to be mastered by children in elementary school, there is no agreement on the specific nature and type of skills, the level of mastery and fluency, and the timetable for achieving the skills for mathematics achievement. We think it is critical for the construction of definitions, but is overlooked.

Second, because of the cumulative nature of mathematics, we cannot identify the core skills. Unlike the basic processes that underlie reading achievement, beyond the early elementary school years (kindergarten and grades 1 and 2), mathematical achievement is cumulative and comprehensive throughout, with quantitative and qualitative changes occurring within and across grade levels. Almost 30% of curriculum material at each grade is new or expanded substantially. Yet politicians seem to think that rote learning of 'key' facts and procedures is the root cause.

Third, after acquisition of the key skill (number conceptualization, number relationships, and outcome of numerical operations—the components of skills that underlie dyscalculia), learning and applying mathematics depend on a diverse set of skills. These skills are among several different domains of functions (e.g., perceptual, quantitative, visuo-spatial, cognitive—processing, memory, logical, and linguistic). Therefore, the mastery of arithmetic and mathematics skills, concepts, and procedures and poor math achievement are linked to several skills: language (native

and mathematics), memory, visuo-spatial, affective, and prerequisite skills such as sequencing, pattern recognition, and/or executive skills. However, the definition of ‘mastery’ is also contentious among researchers, educators, parents, and policy decision makers and the students themselves.

Mathematics disability (MD) is emerging as a collection of subtypes that cluster around several problem areas: (a) related to number, (b) related to language, (c) conceptual/procedural, (d) visuo-spatial, (d) related to executive function, and (e) affective/behavioral. Because of the complexity of mathematics skills and their varied nature, it is unlikely that MD subtypes will share a unifying core deficit in skills and its manifestations. As mentioned earlier, phonological processing/lexical labeling have been associated with computational math skills in children in earlier grades and with poor math achievement. However, factors other than reading also influence MD. For example, language-specific difficulties in children with MD and RD have been reported relative to children with MD only. Many children with only MD outperform their peers with both MD and RD on exact arithmetic tasks, whereas both groups demonstrate comparable difficulty on estimation tasks.

Math performance levels are also linked to executive function skills, such as organizing, selecting and focusing on relevant information, transferring learning from short-term and working memory to long-term memory, analyzing tasks, setting hierarchies of tasks, connecting ideas, flexibility of thought and action, etc. Different aspects of executive functions appear to account for some of the variability in children’s math performance levels, with poor inhibition of irrelevant information and poor working memory, particularly as it relates to visualization. Because visualization takes place in the working memory, the working memory plays a critical role in mathematics learning and performance ([Chinn and Ashcroft, 2017](#)).

For example, when working out a problem, a simple calculation such as 12×8 may be a subsidiary problem that needs to be solved mentally before a student can solve the main problem. If the student knows the fact, there is no digression. The focus remains on the main problem. When they do not have the fact automatized, they must construct it and thus digresses. That

construction takes place in the working memory space or on paper. When they do it on paper, although the problem is not difficult, it is a distraction nonetheless, but it is not done in the working memory and, therefore, it does not help in automatizing this fact and there is no strengthening the working memory and no reinforcement of the facts. On the other hand, when the student does it in their mind, the student must keep several pieces of information in their mind: 10×8 is 80 and 2×8 is 16, so 12×8 is 80 plus 16, which is equal to 96. Therefore, $12 \times 8 = 96$. It helps to strengthen connections, improves working memory, and accesses the facts with little expense of cognitive energy.

Although this strategy is better than rote memorization, reciting the facts in order, or constructing a sequence of additions, it is still a digression from the main problem, and one pays the price in efficiency and effectiveness. The student must manipulate this information mentally, which requires visualization. This repeated visualization adds strength to the working memory. Therefore, working memory deficits in children with learning disabilities, including children with reading or math difficulties, keep positively affecting their ability to solve problems efficiently.

3. Mathematics Problems Related to Reading

One of the obvious connections between dyslexia and mathematics difficulties is the reading of mathematics material: reading instructions, definition, reading procedures, reading descriptions of settings for a problem and word problems. Although many children with reading problems may not have problems in acquiring mathematical concepts. This is particularly so with straight computational and procedural aspects of mathematics and when the instructions are straightforward. However, mathematics problems involving language, particularly reading, pose problems for them, for they may not have the facility to comprehend the words and expressions properly.

For this reason, some children with poor reading skills are also below average in arithmetic problem-solving skills. Below-average performance

can exist in different areas, from simple language-based symbolic conceptualization to complex problem solving such as word problems, making conjectures, writing definitions and proofs in algebra and geometry, as well as communicating their mathematics competence. These children frequently perform below average on the arithmetic subtests of the Wechsler Intelligence Scale for Children (WISC).

When developmental dyscalculics who are good readers are compared with those who are poor readers, the good readers may misread signs, align rows and columns inappropriately, and miss entire calculation steps. The poor readers avoid unfamiliar words, word problems, and operations; they have problems with tables and in recalling appropriate calculation procedures. Specific areas of difficulties exist in which failure in mathematics is coupled with reading disability. In these areas, students have

- Difficulty with the vocabulary and terminology of mathematics, understanding directions and explanations or translating word problems,
- Difficulty with irrelevant information included in the word problem or out-of-sequence information,
- Trouble learning or recalling concepts, definitions, and meanings of abstract terms,
- Difficulty reading texts to direct their own learning and communicating mathematics, including asking and answering questions,
- Lack of information concerning mathematical facts due to the failure to make normal school progress (since the child with reading problem may be taken out of the mainstream class or placed in special classes where the emphasis is on reading progress, the child may not get enough instruction in mathematics and therefore has limited exposure to mathematics and mathematics instruction), and
- Emotional blocking originally due to reading disability but eventually extended to mathematics.

The demands on reading in mathematics extend far beyond story problems. They include reasoning in solving and using equations, mathematics conceptualization, arriving at definitions and the concept inherent in them, etc. Research in problem-solving demonstrates that a relationship exists between the reading of algebraic symbols, instructions, and concepts and performance accuracy. In our remedial work in mathematics ([Sharma, 1980](#), [1988](#), [2004](#)) with children and adults, we have found that many students do not read and comprehend the vocabulary of algebra—properties of operations, expressions, systems—nor do they read and comprehend an expression such as $3x + 7 > 2(x + 5)$ fluently and accurately. Although the symbols are not phonetic, each symbol represents a vocabulary word whose meaning must be understood.

L. Difficulties Due to Common Issues in Mathematics and Language

The brain's perceptual and cognitive mechanisms in the discrimination of quantities, language processing in counting and problem solving, decoding of mathematical notational systems, and procedural knowledge in calculations have commonalities with skills that facilitate reading and language development. In some cases, the same difficulties that children experience in reading and other language concepts begin to interfere with learning mathematics, and as a result, some children have difficulties in learning even the simplest arithmetic concepts such as number relationships, basic number facts, and algorithms.

The poor performance in arithmetic among dyslexics could be attributable to some underlying dysfunction such as defective symbolization and visual perceptual integration, a deficit in lexical encoding, cognitive and spatial difficulties, or a minimal brain dysfunction. Because of these and many other underlying similarities in dysfunction in both arithmetic and language, dyslexic students may begin to manifest problems in number conceptualization and its usage. Problems can exist as early as number recognition and its manipulations.

1. Perceptual Issues in Language and Mathematics: Visual and Spatial

Visual processing weakness appears to be the most common cause of language and mathematics difficulties. One of the hypotheses being forwarded for the existence of dyscalculia is a visual perceptual deficit. Both reading and mathematics conceptualization are highly correlated with visual perceptual integration. Visual perception deficits affect reading as well as number work (confusions between 3 and 8 or 6 and 9; $>$ or $<$; $\frac{3}{4}$ or $\frac{4}{3}$; x^2 or x_2), etc.). For example, many students may confuse '+' with '×' sign as they confuse b and d. Locating points (3, 4) or (4, 3) on the coordinate plane. One can find examples of rotations, inversions, and distortions in the mathematics work samples of children with language difficulties.

2. Role of Visualization in Language and Mathematics

Visualization is the ability to hold and manipulate data (numbers, symbols, shapes, mathematics situations, intermediate steps in problems such as carrying or 'borrowing', estimating, etc.) in the mind's eye. Difficulty in visualization has a marked effect on calculations (written and oral) because the child cannot remember the order of appearance of specific numerals in a multi-digit number and the specific number sequence in a problem. The ability to visualize is needed to be successful in aspects of mathematics, including calculations. The child can copy and may be able to verbalize numbers but has difficulty in reproducing the numbers and operational symbols in the written form from memory. Moreover, the process of calculation is almost always visually based, and the result of this process needs to be ultimately visualized in the mind's eye, consequently visualization plays an important role.

For example, one must effectively perceive visual clusters of objects to represent a number in one's mind. Students who have a visualization problem may be unable to judge the relative sizes among dissimilar objects or collections. In the absence of visualization, a student relies almost

entirely on sequential counting of objects in the collection, rote memorization of verbal or written descriptions of mathematics concepts. The student must digress from the main problem to write subsidiary problems, even in the simplest of problems, as he is not able to visualize the intermediate steps in the problem.

Some mathematical problems also require students to combine higher-order cognition with visual perceptual and visualization skills, for instance, to determine what shape will result when a complex 2D or 3D figure is rotated or several simple figures are combined to form a complex figure. Visual perceptual integration affects the ability to combine parts into wholes (e.g., place value, decimals) and vice versa (e.g., fractions).

Proofs and calculations in geometry become difficult due to complex figures and diagrams that need to be visualized into components figures, reasoned with intra- and inter-relationships of component parts or calculations needs to be done only after their decomposition into smaller, well-recognized geometrical objects are done. The sequence in proofs is both parts to whole and whole to parts: One must visualize the main steps of the proof or procedure (perceiving the whole) and then the parts (reasoning and their representations) are organized and placed in proper sequence in the mind's eye to achieve the whole (parts-to-whole). The integration of deductive and inductive reasoning is easier with strong visualization abilities.

3. Spatial and Temporal Order and Mathematics Difficulties (Spatial Dyscalculia)

Skills such as spatial and temporal order are important for reading and mathematics, including arithmetic. Involvement of spatial concepts—spatial orientation, space organization and spatial-constructive abilities, perception, and mental representation of space are most obvious in arithmetic skills and written calculations. Calculation errors and deficits, which result because of misplacements, distortions, size, rotations, omissions, left-right confusion, and perseverations, are examples of *spatial dyscalculia* in children and

acalculia in the case of people with brain injuries in specific areas of the brain, particularly in the right hemisphere, and with patients with Alzheimer and dementia.

Numerical information, particularly place value, fractions, decimals, and exponents, contain a considerable spatial component. Concepts such as place value, fractions, etc. and procedures and algorithms (aligning numbers, organization of the numbers for operation, steps of the process, etc.) are dependent on spatial orientation and space organization. And when a spatial component is mixed with conceptual demands of understanding of number concept and place value, it becomes quite difficult for many children. For example, numbers and digits organized differently, such as 23, 2^3 , 2_3 , $\frac{2}{3}$, $\frac{3}{2}$, .23, 2.3, (2, 3), and (3, 2), are different in value and meaning because of the spatial order and organization of the numerals 2 and 3.

Children with spatial dyscalculia may also have difficulty over the ‘greater than’ and ‘less than’ symbols and in more advanced work of geometry and algebraic symbol manipulation that involve organization and arrangements. Reversals (of numbers and numerals in a multi-digit number or of diagrams) and misalignment of numbers and operations are clear examples of spatial difficulties.

Misalignment refers to numbers that are not placed in the normal order according to their values or even when they are placed incorrectly, either because of their misunderstanding of the number values, by poor understanding of the concept, or the physical/spatial arrangement looks odd spatially, during written calculations; as a result, the operation is performed inaccurately. For example: (1) Find the sum of 6.24, .345, 23, 3.985, and .87 and (2) Find the product of 14 and 12). The child writes as:

<i>First</i>	<i>Second</i>	<i>Third</i>
6.24	6.24	6.240
.345	.345	.345
23	23	23.000

<i>First</i>	<i>Second</i>	<i>Third</i>
3.985	3.985	3.985
<u>.87</u>	<u>.87</u>	<u>.870</u>
5.064.		34.440

In the first case, the operations (writing numbers and adding) are performed incorrectly because of misplacement of operations and numbers although the student knows the correct arithmetic facts. In the Second Case, the student did write the problem correctly but did not perform the operation as the arrangement looked wrong to him. In the Third Case, the student did everything correctly, except when asked why he annexed the ‘zeros’ in numbers, and why in some and not in the others. He did not have any answer, except that we do that. ‘I was told to do that.’

Many students perform all the operations in the problem correctly, only to realize that they have read an operation sign or symbol incorrectly. For example, misreading the direction sign ($<$) in an inequality, or reading a decimal point in the wrong place, will result in an incorrect answer.

Every written or mental numerical and arithmetical task comprises performances that necessarily imply the knowledge of spatial relations, sequencing, pattern recognition, visualization, estimation, and reasoning. It would therefore appear that because calculation is a function naturally linked to spatially organized thinking and manipulation, visuo-spatial and spatial-constructional deficits are at the core of spatial dyscalculia.

Disorder in mental calculation is due either to an impairment of visual memory or to a deficit in the visual analysis and/or synthesis of numerical relations and arithmetical operations.

4. Sequencing and Mathematics Difficulties

Because many dyslexics may have sequencing difficulty, memorizing the order in which operations and organization of the flow of events that are to be performed may not be easy. These children struggle learning facts by sequential counting. Success with the procedural component of

mathematics is dependent on following sequential directions involved in the procedure. Procedures such as finding the product of 58 and 79, dividing 359 by .58, and solving three simultaneous equations $x + y + z = 11$, $2x + 3y + 4z = 15$, and $x - y - z = 3$ are examples of multi-step operations. Processes such as long division and operations with fractions are particularly difficult for these children. To keep track of the steps involved requires following sequential directions. For example, just to multiply 58 and 79 by using the standard procedure involves more than 25 steps if the student has not mastered the facts, place value, and the steps of the procedure. There are several places and examples where the possibility of error looms large.

5. Memory Difficulties and Mathematics Difficulties

The memory shortcomings associated with dyslexia obviously cause problems with mental arithmetic. Even if children learn their times tables, they cannot recall the facts in sequence backwards and forwards—so if asked what six times four is, they will say: 4, 8, 12, 16, 20, 24, and then count how many times they counted four to reach 24. Then if asked what five times four is; they will have to start again counting from 1×4 . They have difficulty in skip counting starting from an arbitrary number. Many researchers have suggested a relationship between reading and short-term and working memory. In the case of mathematics, this kind of relationship is quite apparent as in mathematics procedures and solving problems, several intermediate steps and outcomes of subsidiary operations must be held in the working memory. Children with working memory problems find holding intermediate steps in calculations in the working memory very difficult.

Poor digit span, an indication of the size of the working memory, is one of the clinical characteristics listed by [Denckla \(1979\)](#) for ‘pure’ developmental dyslexia and is also implicated in number and calculation difficulties. The size of the working memory space has a great deal of impact on learning arithmetic and mathematics. For example, when students

have not automatized arithmetic facts, they must work out the fact by counting in their heads, thus fully occupying the working memory space and losing the conceptual component of the problem they are working on, thus creating digressions from the main problem. Because their working memory space is occupied by these subsidiary reconstructions of facts, they cannot devote their complete attention to the conceptualization of new ideas and making connections between concepts, facts, and ideas.

6. Mathematics Learning Problems Due to Visual Dyslexia

Visual dyslexia is an aspect of dyslexia that commonly affects arithmetic and interferes a great deal with mathematics learning. Visual dyslexia is the inability to translate printed language symbols, including mathematics symbols, into meaning and, therefore, their proper usage and applications.

The underlying cause in visual dyslexia is the child's inability to comprehend order or sequence. Generally, the impact of this disability is acute when the dyslexic deals with time, classification, arrangement, or relationships of entities along a continuum or sequence ([Jordan, 1972](#); [Sharma, 2008](#)). This may disturb work in areas such as place value, reading large numbers, locating numbers on the number line, or on coordinate plane, fractions, and decimals. Children with visual dyslexia have difficulty carrying out sequential instructions—an essential element in learning elementary mathematics and in the execution of procedural mathematics.

Numerals and numbers, of course, no less than letters, must be interpreted as symbols, and therefore many dyslexic students tend to have difficulty with mathematics, particularly as mathematics symbols are packed with information. This information is not just the appearance value—the represented value but the concept, and the action value—it is calling for an action to be performed as represented by the expression. For example, 12 divided by 4 is written as:

$$\frac{12}{4}$$

$$12 \div 4$$

$$4 \overline{)12}$$

Figure 3.2 Three ways of writing division

The first on the left is read as how many groups of 4 are there in 12

The middle one is read as 12 divided by 4

The last is read as 4 divides into 12 how many times

Then, there is the further complication of the actual calculation, in this case, long division or short division. Therefore, the area where dyslexics may be challenged is with calculations/computational mathematics. Computational problems have been noticed due to difficulty in recognizing, comprehending, interpreting, and acting as suggested by symbols. For example: Just like the division symbol: '÷,' the absolute value symbol $|x|$ has several meanings—a geometrical meaning: the distance of number x from the origin; it is a grouping symbol $2|3(x - 7) + 3|$, it represents a unary function $x \rightarrow |x|$, it is an algebraic expression. It invokes many meanings, purposes, and representations. That is what makes algebra difficult for many students. The symbol has to be interpreted in its context.

The inability to read and understand the meaning of symbols is both due to conceptual reasons and visuo-spatial difficulties. It has also been suggested that different parts of the brain are activated using different verbal and visual association networks. Changing a word may change a whole network of brain connections. As mentioned earlier, when students do not see mathematics expressions in multiple forms, they can have difficulty interpreting the information.

There is further evidence of an anomaly in the fact that when symbolic material must be processed at certain speed, dyslexics have been found to be slower than age-matched controls.

Most visual dyslexics see certain letters, numbers, and symbols backward and upside down. To read whole words in the context of a sentence is a jumbled process for such a child. Not only does he perceive individual letters incorrectly, but he also sees parts of words in reverse. Because of this scrambled perception, visual dyslexics are forced to work very slowly. As a

result, their arithmetic work is slow and deliberate. They may not finish assignments and tests in the allowed time.

Visual dyslexics are generally hampered in any situation that requires them to comprehend order or sequence; the major handicap is the inability to visualize printed symbols in the correct sequence or position, particularly in algebraic expressions. Visual dyslexics have difficulty carrying out sequential instructions: an essential element in learning and executing procedural components of mathematics, dealing with time, arrangement, or relationships of entities along a continuum. The specific concepts and skills affected are place value, fractions, integers, exponents, and algebraic manipulations with several terms and equations. This is a short-term memory problem.

M. Dysgraphia and Mathematics Problems

Dysgraphia, a related problem of some dyslexic and dyscalculic children, may interfere with learning mathematics. However, due to very little writing being demanded in classrooms, many students are presenting themselves as dysgraphics. In most cases, we have found that it is a lack of discipline, poor handwriting, teaching or no teaching, not having access to squared paper, or a lack of practice in writing. It is a rare child we see in practice who does not improve upon his writing with proper care and practice.¹¹ Dysgraphia affects a person's writing ability and fine motor skills. Writing is a huge challenge and extremely laborious for a person with dysgraphia. Students with dysgraphia have problems learning and producing the motor patterns for writing and making drawings and illustrations for mathematical concepts.

Dysgraphia involves faulty control of the muscle systems used to encode letters, numbers, and figures accurately. Even when there are no motor coordination difficulties, faulty visual perception can cause writing difficulties. When an undistorted image is picked up by the eye and sent to the seeing portion of the brain and is distorted there through improper visual perception, this distorted image will then be reproduced in this

improper fashion by the hand doing the writing. Or, if a distorted visual image is stored in a visual memory area, the hand will reproduce it later in its distorted form even though the brain's motor mechanism for fine coordination is functioning properly. Students who are so afflicted have difficulty placing their knowledge in concrete form. They also have trouble when taking notes and must rely on memory to a greater degree than normal to recall what was covered in class. Then, because they have few notes to refer to, they have little of substance to aid them when they attempt to study or do homework.

The dysgraphic student may have a clear mental image of what he intends to encode, but he finds himself 'forgetting' how to write specific symbols and draw figures, shapes, and diagrams. Children with dysgraphia may have varying levels of difficulty in written aspects of arithmetic and mathematics. The 'shape and space' areas of the curriculum are challenging for them.

1. Geometry

A frequently observed flaw when dysgraphics copy shapes is the tendency to draw 'ears' at the corners of simple figures and difficulty in identifying shapes in superimposed and overlapping situations. They distort simple geometric shapes and fail to close shapes. They have difficulty reproducing simple designs from memory. Proofs are not easy because seeing the relationships of parts-to-whole and whole-to-parts in complex figures is difficult for students with dysgraphia.

2. Spatial Orientation and Classification

Dysgraphics confuse directionality and have difficulty with less than or greater than ($>$) signs in expressions and inequalities. Because of a lack of practice in writing, they cannot differentiate between when to use upper-case or lower-case letters. In geometry and algebra, lower-case letters are used for representing variables, and upper-case letters for points, angles, parameters, place holders, unknowns, etc.; dysgraphic and students without

any training in proper writing mix these. They have difficulty writing fractions, formulas, exponents, equations, and expressions. They have difficulty grouping or classifying objects by shape, size, color, texture, or other physical characteristics. Both grouping and classifying are important skills for concepts and procedures in mathematics. Grouping is essential for learning place value and addition procedures whereas classification is for learning subtraction, division, fractions, and geometry.

3. Early Number Concepts¹²

Early number conceptualization is difficult for dysgraphics as they are awkward in tasks involving the manipulation of small objects—they over or under count objects in early number learning and in finding arithmetic facts. Forming, sequencing, aligning numbers are a challenge for them.

4. Number Work and Operations

A dysgraphic writes numerals poorly, has difficulty staying on lines and aligning numbers and procedures, writes one thing and reads it as something else, and their work deteriorates toward the end of a mathematics exercise. Children with dysgraphia, therefore, may have varying levels of difficulty in written aspects of arithmetic and mathematics. A mathematics examination or test could be an ordeal for these children.

Technology can mitigate some of the problems of dysgraphics with the representation of information.

N. Aphasia and Anarithmetia

Another term used in the literature is *anarithmetia*. However, its use is quite infrequent. As the name suggests, it relates to a child's problems in arithmetic. Even though *anarithmetia* is a disorder that is not explicable in terms of aphasic disorder, its close relationship to impairment in language function is undeniable. Almost all *anarithmetia* cases have been noted as showing mild aphasic defects, such as word-finding difficulties and related

reading and language difficulties. Thus, it is not unreasonable to view anarithmetia as a particular form of linguistic impairment related to disease of the language-dominant hemisphere.

O. Strengths of Dyslexics

Not every child who has a reading and language disability exhibits a mathematical learning disability. Many students with reading and language disabilities may possess characteristics that make them above rather than below average in mathematical skills. Since mathematics conceptualization requires a high degree of visualization and working memory skills, visualization helps a dyslexic student in mathematics.

Sometimes the visual-spatial nature of mathematics itself is helpful; some dyslexic students may possess good visualization abilities and therefore can learn mathematics more easily. For many visual and kinesthetic learners, it is easier to think about many mathematics ideas in models other than language. This usually, but not always, means thinking in terms of pictures. Many dyslexics think in terms of pictures. It is reasonably easy for them to create images in their minds and to manipulate them. This process develops into effective visual memories. For example, in building and using concrete models, they try out many possible arrangements in their minds before assembling the pieces. It is nonetheless true that the difficulty in reading frequently complicates the situation in such a way as to make the individual appear deficient in mathematics.

1. Mathematically Gifted Dyslexic Children

Mathematically gifted dyslexic children score only as well as average non-dyslexic children on tests on computation. The memory skills of dyslexic children (both gifted and average) are inferior to those of non-dyslexic children. A significant positive relationship exists between abstract reasoning and mathematical computation. There is a significant positive

relationship between memory and mathematical computation and between memory and abstract reasoning.

Despite some memory deficits, these gifted dyslexic children can compensate for their dyslexia in mathematics by reasoning logically in order to arrive at correct solutions to problems rather than relying on rote memory for retrieval of facts.

2. Arithmetic vs. Mathematics Difficulties Due to Dyslexia

We have already mentioned numerical anxiety and math anxiety in the previous chapter. In evaluating learning problems in mathematics and then planning remedial instruction, it is important to differentiate between the nature of problems in arithmetic and the nature of problems in mathematics. In some cases, the child may understand concepts in mathematics quite well, but the manipulation of numbers is a problem. Many dyslexics may be able to grasp abstract and complicated mathematical ideas and yet have difficulty over what to the rest of us seem simple calculations.

Not all mathematics problems are due to students' number difficulties. Some of them are due to the interaction between concepts, language, and number. Therefore, some of the structures, strategies, and approaches that have shown promise in addressing reading and language disabilities have implications for understanding and designing solutions to mathematics learning disabilities and difficulties, particularly the problems of dyslexic students.

P. Instructional Suggestions: Teaching Principles

As we observed earlier, because of the heterogeneous nature of the group of children with dyslexia, not all of them have problems in learning mathematics, but many do. Because of their unique learning and disability profiles, each one of these subgroups may learn mathematics differently. Mathematics conceptualization, as we have seen, involves the recognition of patterns, which become bits of knowledge that are then organized into

larger and more meaningful units. Learning for some children becomes more difficult because of deficits that interfere with the ready recognition of patterns. Other children, on the other hand, learn mathematics information differently because of their difficulty with spatial and temporal relationships. They may have difficulty with the organization of parts into wholes, or because of unique problems with integration, they may have difficulty with sequencing or memory.

Consequently, different groups of children process mathematics information differently and present different mathematics learning personalities ([Sharma, 1980](#); [Chinn, 2021c](#)). They fall on a continuum of mathematics processing styles from the quantitative mathematics learning personality on the one hand to the qualitative mathematics learning personality on the other.

Quantitative Mathematics Learning Personality (Inchworm)

These students have a good memory for sequencing and execute procedures as a ‘recipe,’ i.e., following step-by-step methods. They find the procedural components of mathematics easier to learn. They can remember formulas but may not understand why the formula makes sense. They prefer to do paper and pencil tasks and are attentive to details, but they do not see the big picture.

Qualitative Mathematics Learning Personality (Grasshopper)

These students see the big picture and have insight into the patterns of mathematics, but they are poor at computation and have problems with remembering step-by-step procedures. They understand mathematical concepts better and like to solve problems mentally and quickly; as a result, their answers may be inaccurate and not fully documented. They are stronger in the conceptual aspects of mathematics. These individuals may have difficulty in verbalizing and explaining their answers. They may not show their work. Many of these children have more difficulty with mathematics than their peers because they persist longer in their tendency to

rely on holistic cognitive strategies to deal with arithmetic problems that are best solved analytically. These children may not realize that the analytical operations they favor may not be relevant to every task in mathematics.

Teachers, academic therapists, and tutors who provide remedial instruction in mathematics must understand the nature of dyslexia, the learning approach and how it affects mathematics learning, not only mathematics language but also for specific mathematics content.

Further, the teacher needs to have a deeper understanding of the mathematics content; the ability to use a variety of instructional techniques that are multi-sensory and to provide explicit instruction for that content. The instruction should be diagnostic, systematic, cumulative, as well as inductive (to address the needs of qualitative learners) and analytical (to address the needs of quantitative learners). The instructor should also be familiar with current research in mathematical instruction and mathematics and reading disabilities.

For dyslexics, facility in learning mathematics, more than any other content area, is tied closely to the teacher's or academic therapist's knowledge of mathematics and to the way these concepts are learned and taught. Therefore, some individuals with dyslexia may continue to have problems in learning mathematics not just because of their dyslexia or dyscalculia but because their instructors are inadequately prepared in mathematics content, remedial principles, and/or in approaches to teaching mathematics to dyslexics and dyscalculics.

Competence in mathematics is not enough. The teacher and academic therapist need to understand that mathematics is problem solving which involves reasoning and the ability to read, write, discuss, and convey ideas using mathematical signs, symbols, and terms. This requires an understanding of mathematical knowledge: linguistic (the language of mathematics—its vocabulary, syntax, and skills of translation from mathematics to language and from language to mathematics), conceptual (relationships constructed internally and connected to already existing ideas) and procedural (knowledge of symbols used to represent

mathematics, and the rules and procedures that are used to carry out mathematical tasks).

Emphasis on procedural knowledge is not enough. To acquire procedural knowledge, the most important connection is to the conceptual knowledge that supports it. If procedural knowledge is learned rigidly and used narrowly, regression becomes a real possibility. When there is a connection to concepts and language, the learner not only understands the procedure but also gains access to other ideas associated with the concept ([Van de Walle, 1994](#)). For individuals with dyslexia, this linkage is critical and language plays an important role.

To assist individuals with dyslexia in making this linkage, it is essential that teachers and academic therapists provide instruction that allows the learner to work through the appropriate levels of knowing when teaching mathematical concepts at all grade levels: intuitive, concrete, pictorial, symbolic, abstract, applications, and communications.

Many individuals with dyslexia learn best when provided with concrete manipulatives with which they can work or experiment to conceptualize. These help build memory as well as allow revisualization in the absence of automatization. The next stage, pictorial, is one that may be brief, but is essential for beginning the transition away from the concrete. This is where individuals recognize or draw pictures to represent concrete materials without the materials themselves. Symbols, i.e., numerals, plus signs, etc., are introduced when individuals understand the basic concept, thereby making the connection to procedural knowledge. Finally, the abstract stage is where individuals can think about concepts and solve problems without the presence of manipulatives, pictures, and symbols.

It is important that the levels of knowing are linked through language. Language facilitates the transition from one stage of mastery to another. Three kinds of language allow one to fully integrate mathematical learning. First is the individual's own language. No matter how imperfect this language is, it is important that the individual discusses, questions, and states what she/he has learned. Second is the language of the instructor, or standard English, which clarifies the learner's own language and links to the

third language, the language of mathematics. Language allows the instructor to determine if the learner understands the concept and, even at the concrete stage, is not just blindly following steps.

For the aforementioned reasons, teachers and academic therapists working with dyslexics must be well-trained in multi-sensory structured remedial techniques both in language and mathematics instruction. They must not only demonstrate competencies in knowledge and skills in teaching language but also demonstrate their competencies in mathematics. Effective teachers and academic therapists should understand:

- The development of conceptual schemas for specific mathematics content.
- The role of developmental mathematics milestones.
- The use of appropriate methodology, models, technology, and manipulatives within the content.
- The sequential nature of, and the structures inherent in, the mathematics content.
- The connections among mathematical concepts and procedures and their practical applications.
- The use of the four processes (becoming mathematical problem-solvers, reasoning mathematically, communicating mathematically, and making mathematical connections at different levels of complexity).
- Can select, adapt, evaluate, and use instructional materials and resources, including technology.
- Can diagnose student errors and use strategies to manage, assess, and monitor student learning, and
- Have the ability, skills, and strategies to teach diverse learners.

For students to take advantage of remedial instruction, effective teachers make every effort to help them to develop the following skills:

- Visualize problems—drawing a picture to help understand the problem and do mathematics problems on graph paper to help with aligning numbers.
- Examine visual information (picture, chart, graph, etc.) and work neatly to avoid clutter so that the extra visual information does not overwhelm.
- Read the problem out loud and listen carefully to take advantage of auditory skills (which may be a strength). Make the attempt at access multi-sensory.
- Ask to see an example like the given/current problem and to think of a real-life situation that would involve this type of problem.
- Spend extra time automatizing math facts. Use rhythm or music to help memorize. And use decomposition/re-composition strategies.

Notes

1. The term ‘*script*’ is very important to us, in our teaching and interventions, and the reasons will be given in later chapters. ↵
2. A visual image, an object, action, or memory could also instigate thinking. All of these can be considered a symbol or collection of symbols. ↵
3. For effective questioning process see *The Questioning Process: Its Importance in Learning and Teaching Mathematics* ([Sharma, 2010, 2019](#)). ↵
4. More on this in the chapter on Number Concepts. ↵
5. In the chapter on Tools of Mathematics, we discuss the quality of instructional tools. ↵
6. <https://www.doe.mass.edu/mcas/testitems.html> ↵
7. *The Mathematics Vocabulary List: Kindergarten through Sixth Grade in Mathematics as a Second Language: Reading, Writing, and Doing Mathematics* by [Sharma, 2004](#). The vocabulary list is similar to word lists such as the Dolch List of commonly used sight words. ↵
8. See the chapter on CPVA/S. ↵

9. More on scripts in the later chapters. The development of the concept of scripts is based on principles and research on memory, learning, and the behaviors of experts. [!\[\]\(83eb2aa26b610eb6a9dca7cf4702d681_img.jpg\)](#)
10. CPVA/S, to be seen in later chapters, is to numeracy as the Orton Gillingham method is to the acquisition of reading skills. However, CPVA/S is much more versatile and general as it applies to instruction in mathematics. [!\[\]\(94dfacbf937cdd7da4837a6fcd8fc785_img.jpg\)](#)
11. More on this in the chapter on Number Concepts. [!\[\]\(dae8c3c5fa7c80febd6526a5e8a853bf_img.jpg\)](#)
12. More on this in chapter on Number Concepts. [!\[\]\(8f38ab9775d1331a4e1fd6648d0a83f1_img.jpg\)](#)

4 The CPVA/S Model of Mathematics Instruction

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When Einstein was asked how he came to invent the theory of relativity he replied, ‘by challenging an axiom.’ So, Galileo challenged the axiom that the heavier body falls faster. Copernicus challenged the axiom that the earth is the center of the solar system, Lobachevsky and Bolyai challenged Euclid’s axiom of parallels, Einstein challenged the axiom that of two distinct instants one must precede the other. Challenging an axiom means imagining the functioning of the world, far away from everyday experience.

The role of the teacher is not just to hand out easy answers to her students. What they should give, instead, is a sharper lens—a way of perceiving, observing, and seeing the world that allows them to ask better, braver, and more difficult questions.

Every evolving organism must adapt to its environment to survive and thrive. This need and the effort to survive is continuous; therefore, it must learn constantly. Developing human intelligence also undergoes a process of adaptation to fit and deal with circumstances and then assimilate it to become and remain viable. It must learn more for this to continue. Above everything else, as teachers, our objective in teaching is to help make our students better learners. They should also feel and experience that we are learners with them.

Most mathematics teachers assume that mathematics is a fully formed and perfectly finished body of knowledge and that it only requires effective transmission of this knowledge from the teacher/book to the students. In truth, for learners, this knowledge needs to be constructed anew for themselves. Of course, this happens with the help of knowledgeable and empathic adults and assisted by the content they already know, the skills they already possess. The innate and cultural tools and methods they have already gathered from their acculturation process are the starting point. In that sense, the eighteenth-century mathematician Vico was right when he said: ‘We can know nothing that we have not made’ (Vico, 1710, p. 76). Most of us come to understand the intricacies, depth, and reach of school mathematics only when we start teaching it. Then, we begin to own the content as we are teaching it. We need to help every child, every student of mathematics, to make it their own. They need to understand its language, its conceptual schemas, and its strategies, skills, and procedures.

All knowledge to be acquired by a learner, including mathematics knowledge, is initially constructed by the learner—in one form or another, and is fallible and malleable till it reaches a level of stability. With use, reflection, and practice, the learner achieves a level of competence and proficiency. This constructive activity ranges from individual and/or group-initiated decisions and actions to socially mediated informal and formal interactions with mathematics knowledge. Knowledge is thus constructed and reconstructed, and one converts it into contextual competence.

A. Need for a New Instruction Model for Math Instruction

Piaget and other constructivists strongly suggest that a child should interact with his world to acquire knowledge, and therefore, it calls for actions on the part of the child to make meaning of acquired knowledge and the world (*Discovery Method*). However, in this implied idiosyncratic construction of making meaning, one may end up acquiring some systematic errors, misinformation and misconceptions, and create and develop alternative conceptual schemas, and non-standard language—names, syntax, semantics, and expressions that may conflict with cultural, historical, and standard meanings and concepts and procedures ([Piaget, 1952w](#), [1953](#), [1962](#), [1972](#)). Later, as a result, in some cases, this may call for interventions and diagnostic teaching, including remediation if necessary. This creates opportunities for the development of novel strategies and innovative approaches.

On the other hand, we also need to consider the knowledge of the individual as well as of the group as historical and evolutionary; it is always growing and changing to meet emerging challenges and contradictions. The role of the culture—the group—is to provide opportunities and challenges on one hand, and ‘measured and appropriate’ guidance and support on the other (*Guided Discovery*).

Through meeting challenges and mediating contradictions, one inevitably arrives at situations of cognitive dissonance. Through resourceful support and the guidance of experiences of the culture, the resolution of the dissonance is mediated and resolved. The resolutions of dissonances result in new learnings at multiple levels and forms. These learnings are indications and markers of a child’s growth and learning. For example, enactive learning manifests in content, behaviors, and active representations of ideas in concrete to abstract and symbolic forms and uses. That content finally converts into language, concepts, and procedures related to the concept that is being focused upon.

These processes modify an individual’s psychomotoric, affective, and cognitive functions and capacities. This implies idiosyncratic as well as socially acceptable levels of constructions of meaning that lead to cognitive activities and outcomes of various kinds and at multiple levels. Everyone’s learning is the integration of individual and social resources—individual effort and socio-cultural opportunities—encouragements, modifications, support, validations, and challenges. Thus, teaching should be a well-choreographed dance ensemble.

Over the last several decades, many educators and policy makers, particularly those who are in technology and related fields (most of them non-educators) have written about the ‘promise’ of *individualized learning*. The promise of putting students on laptops or in connected mode, where ongoing digital assessments and the individually catered corresponding lesson plans will ensure that learners receive commensurate learning resources that are appropriate for their development—as with an *a la carte* menu, a prescription, or a full multi-course dinner served on request and taste.

Yet some of the most popular individualized learning programs and software, so far, have failed to clearly demonstrate significant growth results with a majority of students in their efficacy studies. Of course, the efficient and self-motivated learners will learn in any situation, and they will learn in this situation also. For example, a nationwide study of 62 schools engaged in this kind of individualization showed a very small effect size in mathematics and none in reading, at the cost of diminished feelings of social isolation and lack of socialization—a rich source of human development—intellectual, social, and physical. We do not yet know whether their ‘learnability’ increased or not.

The promise of individualization receives new energy with every new technology—from the advent of writing to books to radios to movies to television to the computer (iPad, iPhone, etc.) to the internet and, of course, to generative AI in the present and in its future iterations or reincarnations. To most onlookers into the education scene—commentators and consumers—it all seems to be so exciting and

inevitable. The destination—full accessibility, participation, involvement, competence, and contribution—is always just around the next bend in the road.

It is a small fraction of children in K-12 who can self-start, self-manage, and self-learn in the way these commentators and reformers imagine and expect. The other, much larger fraction of students needs more people embedded more tightly into their learning process, not fewer. Online programs and computer-aided individualized learning may benefit most of the kids who need it least—the 5 to 10% of the children who are already savvy with mathematics. Students with special needs, from the whole spectrum from learning disabled and excluded to gifted and talented, need better interventions—from initial teaching, to interventions, to special educators and admittedly, well-trained individuals at all levels. Moreover, during the ages of 3 to 10–11, they need more concrete experiences compared to looking at a screen for hours, day after day.

Optimal learning may be the outcome of intelligent and well-thought-out delivery of learning—interactions of human flexibility and the rich information resources available through the technology of the time, in a variety of settings and diverse modes.

B. How Children Learn

To learn and to become proficient in mathematics and to meet personal and socio-cultural expectations, children need to achieve *mastery* in key developmental milestones in their mathematics development and learning at appropriate age/grade levels (as mentioned in introduction). These outcomes should be the results of a solid curriculum—with breadth, depth, and rigor, a responsive, effective, appropriate, and efficient classroom instructional practice, and a comprehensive and timely assessment system with a timely, constructive feedback loop built in amongst teachers, parents, and from teacher to students. Once students achieve mastery in a developmental milestone, with practice and usage, they can convert it into fluent, efficient, usable skills—applicable to other parts of mathematics, other disciplines, and real-life problem solving contexts.

Different philosophers and practitioners of mathematics education, at least for the last 50 to 75 years have suggested diverse pedagogies, and these have resulted in significantly different educational implications and outcomes ([Ball & Bass, 2000](#)). The two senior authors have been there as full participants in this process for three-quarters of a century. The number of people having difficulty in learning mathematics and those who have a dislike for mathematics have increased rather than decreased. When we see schools as institutions, we don't think as much attention has been paid to 'implementing' the 'curriculum' as it has been to the advents of pedagogies to teach it. It is a bit like going into a restaurant where all the effort has gone into making it look amazing: tables beautifully set, glasses sparkling, gorgeous lighting, everything looking inviting, and staff expertly trained in creating a great experience for diners.

Then the food comes out. No one has given the same thought, care, or attention to what is on the plate. We think that there are parallels with implementing the curriculum.

Plenty of thought has gone into the processes of teaching a great lesson. We do not see similar discussions about the quality of the lesson as delivered and its content being taught. This is important because if we do not shift the lens and consider the quality of the lesson—and every lesson at that—then we are likely to end up with content which might or might not be accurately received, properly mastered, and adequately used. And for the most part, missed opportunities to convey the energy and vibrancy of the subject—the exciting content we plan to teach, and for children to learn.

In our pedagogical approach, we are guided by certain learning and teaching principles with clear objectives and aims. These are to help children develop the mastery demonstrating:

- understanding of the concepts and procedures with appropriate mathematics language.
- strong and generalizable conceptual schemas.
- developing and using effective, efficient, and elegant strategic processes.
- fluency and applicability of this understanding into problem-solving situations in diverse settings of key developmental milestones in mathematics.

In such a pedagogical system, each curricular element in an instructional setting is moderately challenging yet accessible to each student with or without scaffolding, and with possibilities of vertical and horizontal accelerations for each student—needing help to navigate the curriculum to those who are self-starters and gifted learners.

We believe that for a pedagogical approach to be responsive to the curriculum and the children, it should aspire to a responsive delivery to the individual student and the collective class and respond to an effective and meaningful *diagnostic/assessment system*. The assessment methods, instruments, and standards must be reflective of children's potential with high standards of achievement. Every instructional and assessment activity should be intentional, deliberate, and ultimately a learning process and experience. The pedagogy that emphasizes meaningful summative assessments at appropriate junctures (e.g., chapter, unit, and topic levels), and an ongoing formative assessment system (to inform the student, the teacher for future lessons and interventions, and planning). The assessment/diagnostic systems should include structures and instruments of monitoring children's progress so they can achieve goals—at the school level, at the grade level, and at the individual level. Then they welcome any opportunity to show their competence—in any setting.

These concrete experiences (enactive learning), in the form of diverse models and modes along with proper language, symbols, and appropriate questioning, facilitate the formation of comprehensive conceptual models. *Enactive* learning is learning by doing and experiencing the consequences of one's actions. It contrasts vicarious learning, where one learns by observing others which provides information for learning.

In *enactive* learning, along with content learning, one learns self-regulating, goal-directed, and self-monitoring behaviors. The teacher then uses these concrete experiences and takes the child to a little higher level of thinking called the pictorial/representational level of thinking. These enactive learning activities—actions and movement—are translated into representations: *Enactive*—> *iconic* —> *symbolic/abstract*.

Research and extensive experience in teaching at various levels has confirmed our belief that teachers have a substantial impact on their students' academic and lifelong success ([Chetty et al., 2014](#); [Levinson, 2013](#)). Effective teachers transform lives. Effective teachers are more than simply knowledgeable in their subject. Knowledge of the subject matter is only a necessary condition, not a sufficient condition for effective teaching. Putting knowledge into practice is an endeavor that a teacher must effectively translate to delivering a quality curriculum with rigor¹ and with fidelity to mathematics and to the children. It is about effective and motivating communication and, thus, teaching.

Fidelity to mathematics content means that the content is at the grade level, and it is accessible and appropriately, moderately challenging to each child. *Fidelity to students* means it is being taught meaningfully so it meets the needs of all of them, with and without learning disabilities in

mathematics. Such teaching respects mathematics and children; therefore, it has higher expectations from the teacher and students and creates a chance for both to succeed. Effective teachers meet these goals. To be an effective teacher—to deliver a quality curriculum with higher level of student learning—requires a high level of content and pedagogical knowledge on the part of teachers ([Bartos et al., 2014](#)) and keen awareness of the students and maximum communication skills.

Teachers need to know their mathematics well to teach mathematics well.

Although it is difficult to decide what knowledge teachers must possess and what must be its nature, it is important that the teacher knows the conceptual connections between the different stages of the development of the trajectory of a concept being taught. That means what comes before the concept being taught and where it is going. During their planning, effective teachers seek answers to questions, such as: Why this concept now? What is the developmental trajectory of this concept, and how does it relate with other concepts in the mathematics curriculum? How to handle the transitions of the different phases of the development of a particular concept—from its introduction to its mastery by children and transitions from one concept to the other. How to interweave this concept with other related concepts at this grade level and the knowledge they already have. They also consider accelerating the knowledge of those who need it and intervene with others who need intervention. There is a healthy tension between the two diametrically opposed demands in teaching. And they manage it.

For example, teachers from kindergarten to third grade—the foundation years for children’s learning and foundations of the edifice of mathematics—should constantly contemplate to answer: How to manage the transitions within additive reasoning—from number concept to addition to subtraction, and equally importantly, from additive reasoning to multiplicative reasoning—from addition to multiplication and from subtraction to division? To accomplish this important milestone, researchers and practitioners ([Shulman, 1986](#); [Ball et al., 2007](#); [Chinn, 2021a](#), [Sharma, 2008](#); [Ma, 1999](#); [Kaput and Blanton, 2001](#); [Warren, 2006](#)) have suggested a whole host of content and content-specific pedagogical knowledge on the part of teachers to be executed at the lesson level—in each and every lesson, during each learning encounter by each child.

C. Teacher’s Knowledge of Mathematics Content

There is definite mathematics content (specific to the level one teaches) and general and content-specific pedagogical knowledge needed for teaching each concept in school mathematics effectively, efficiently, and elegantly. It is not only embedded in the teaching methods, but also in lesson plans and in their execution, and in descriptions of concepts, procedures, and the sequence of tasks planned. It is done with care and deliberation so that the fundamentals of school mathematics can be effectively acquired by and transferred to children. It includes helping those learners who may have missed them the first time around, acquire them in the classroom,

To teach mathematics well, a teacher needs to know that mathematics is more than just its procedures.

The nature and extent of mathematics teachers’ knowledge specific to teaching mathematics is of ongoing concern in mathematics education research ([Chapman, 2015](#)). Studies have explored elementary teachers’ pre-service training, classroom teaching experience, teacher self-efficacy, and the type and nature of teaching certification in relation to student achievement. We are concerned with them, but our emphasis is on improving the quality of classroom practices.

Preservice teachers learn mathematics in a variety of situations. The most common form is exposing teachers to learn mathematics in formal methods, courses dealing in a disciplinary-structured way with

the limited content of mathematics. There is a very brief clinical internship/supervised teaching period before graduation. This may be one of the weaker elements of teacher training. Moreover, this may or may not have any significant mathematics content. The setting may not provide much in the way of training effectively. The receiving teacher may not have enough skills or content to guide teacher trainees to learn effective teaching methods. It is the luck of the draw. After pre-service teachers finish their education, they seek certification to teach.

Most elementary teachers receive a generalist certification, which enables them to teach any subject from early childhood through sixth grade.

With this kind of training, at times teachers find themselves in positions in which they may not feel comfortable teaching the mathematics content, although they are certified to teach the subject. This is where teachers' self-efficacy may play a role in student achievement.

Once teachers are in the field of education, they are encouraged to continue professional development. Professional development continues to be the primary vehicle to trigger the increase of standards-based pedagogies in mathematics classrooms ([Polly, 2014](#)). However, just like pre-service training, the quality of professional development varies greatly from very poor to highly useful and from very general to very specific.

Other than organized professional development, there are few organized opportunities to learn on the job. Few schools provide mentors or master teachers to emulate for an appreciable time for new teachers to learn from. We suggest:²

Pre-service:

- College and university faculties demonstrate examples of effective teaching by presenting demonstration lessons for their teacher trainees in school classrooms.
- Schools of education either have a formal teacher training school attached to them or have a formal collaboration with an associated school where prospective teachers benefit from different levels of exposure to teaching, from preventative to initial teaching to interventions and remedial teaching. This exposure lasts for the whole period of their teacher training similar to the experiences the medical personnel are exposed.

1. Professional Development Opportunities

On entering the profession, a new teacher should have:

- At least two to three hours a week to observe a master teacher in action in class teaching and receive consultations with the master teacher.
- Require that the new teacher sit in a class, as a student, of an experienced and recognized master teacher who is teaching a grade higher than the new teacher plans to teach. The new teacher focuses on all the components of teaching—content, pedagogy, and student reactions and behaviors.
- The receiving classroom teacher should be well-versed in the curriculum, instructional practices, and in the use of instructional tools at $N \pm 3$ grades levels—where N is the grade the teacher currently teaches. Schools of education train classroom teachers in the aforementioned areas.
- No teacher should teach the same grade for more than three years. They should rotate their teaching through the range of their teaching certification. For example, the teacher with a certification from kindergarten through sixth grade should move every two to three years to a

different grade in that range. This will give the teacher a good understanding of the trajectory of the development of the concepts and procedures.

They will know what essential and non-negotiable concepts and procedures are, in order to be ready for the next grade and the next concept and thus increase the chances of student success.

Researchers and practitioners ([Shulman, 1986](#); [Ball et al., 2007](#); Sharma,—; [Ma, 1999](#); [Kaput and Blanton, 2001](#); [Warren, 2006](#)) have suggested a whole host of content and content-specific pedagogical knowledge on the part of teachers. Most emphasize that a teacher's pedagogical knowledge has these components:

- a. Mathematics content knowledge,
- b. Mathematics content-specific pedagogical knowledge,
- c. General pedagogical knowledge,
- d. An understanding of children and their learning, and
- e. High expectations from children and from self.

It is the judicious integration of these elements that answers the question of what an appropriate pedagogical knowledge is for becoming an effective teacher and being prepared for the teaching situations one faces in a typical classroom with children of varying abilities, learner differences, starting points, and aspirations.

Another element that a teacher should realize is that in a classroom, students are not the only people who can learn; the teacher is the senior learner there. We are convinced by decades of classroom experience that teachers can definitely learn from listening to themselves while teaching, not just to the other teachers and administrators, but particularly to the children in the classroom. This comes only through a well-planned routine of reflection.

One of the authors was observing a teacher who was working with a 10-year-old student on subtraction. The intervention seemed to focus on the student trying to solve the subtraction problem from a worksheet. It was not going well. The author suggested that the teacher could explain how 'adding on' could be a useful alternative to the traditional written method. He tried, and the student began to experience success and was learning about the links between subtraction and addition. Once he gained confidence, he also learned the standard procedure. This was an example of developing the flexibility of thought as one of the components of the executive function skills to increase the learnability, not just for the child but for himself as a teacher.

The author learned very early in his career not to overdo the 'quick speed of response' *modus operandi*, which he found to be a demotivating experience for many learners. This technique, or *modus operandi*, seems to be very prevalent when testing students' recall of multiplication facts. This event is an excellent example of the 'because we always do it that way' rationale. It might be useful to discuss with colleagues why a particular strategy or way of doing something became dominant, even if there was little experience of success in schools to confirm this.

D. Pedagogical Content Knowledge

This is the compact of a collection of skills that make the particular content to penetrate and inform the lesson. As translated to the perspective of additive reasoning, for example, it is:

- Knowing the trajectory of the development of addition and subtraction—from number concept to the current concept, vocabulary, and language of addition and subtraction to strategies for mastering addition and subtraction facts to become competent in addition and subtraction procedures, and finally, preparing children to be able to access multiplicative reasoning and its components.
- Knowing and being cognizant of the fact that the child in these grades is preparing and commencing for a long journey in mathematics and this is definitely not the end of the journey, not to treat the child as a ‘terminal student’;
- The prerequisite skills needed for learning number concept and addition and subtraction.
- Knowing the various linguistic, conceptual, and symbolic representations and formulations of addition and subtraction concepts.
- Knowledge of instructional tools,³ approaches, and strategies for the subject matter in ways that students can understand, learn, and master them.
- Knowledge of the levels of knowing additive reasoning.
- Deep knowledge of cognitive behaviors and their development in children from pre-kindergarten through second grade and beyond (at least up to fourth grade)—what they can do and not do during this age range, knowing the difference between novice and expert behaviors in this content area; and finally,
- Deep understanding of the symptoms and characteristics of dyscalculia and other number and language related difficulties that children experience related to number concept and additive reasoning.

For example, the success of and progression from number concept to additive reasoning varies according to various modes of representation and the nature of mastery of number and addition and subtraction that are used for different tasks and at different times over the development of the conceptual framework of additive reasoning. The transition to addition and subtraction also depends upon the type of conceptual framework the students have about number concept. Similarly, the type of strategies to be developed and the chance of their success also depends on the nature of students’ strategies and mastery of number concept.

We have observed and research studies also show ([Chinn, 2019](#); [Sharma, 2008](#)) that it is important for teachers to attend to both structure and action within a particular mode of representation when interpreting learners’ understanding of additive relations. For example, conceptual schemas of addition and subtraction and the strategies developed and used are dependent on the language—vocabulary, syntax, and semantics used in introducing addition and subtraction concepts and the knowledge of the number concept children bring to the addition tasks. Appropriate vocabulary and initial representations of a concept by concrete, iconic, and pictorial instructional models have profound effects on children’s abstract and symbolic conceptualization of the concept.

When a teacher knows the particular mathematical content—topic or procedure—in a simplistic manner, she may not be able to reach all children and all aspects of the cognitive behaviors that need to be developed in children relating to that concept—language, conceptual schemas, strategies, and procedures, particularly its applications. For example, for a teacher, if addition is only ‘counting forward’ and the only instructional tools are counting objects and the number line, then children’s fluency, flexibility of strategies, and numbersense will be either delayed or will take an enormous amount of time and effort. As a result, some children may never reach an acceptable level of

understanding and mastery. They may begin to show symptoms of dyscalculia or other arithmetic difficulties and show anxiety about mathematics.

In the absence of the integration of content and proper pedagogical knowledge, the school mathematics instruction will be reduced to a focus on teaching facts and procedures in compartments (i.e., if addition is counting forward, then subtraction is counting backward) without any deeper understanding and rigor on the part of children, and they will quickly reach a plateau in their mathematics achievement. Making connections amongst concepts and procedures and transitions to higher mathematical ideas become difficult or impossible. Their attitudes about mathematics become negative and, in most cases, it turns to debilitating mathematics anxiety.

Most international curricular experts in school mathematics and teacher trainers ([NCTM, 2000](#); [CCSS, 2010](#); Numeracy Project, 200–, UK; Singapore Mathematics Curriculum, 2–; South Africa's revised curriculum, India's National Standards, etc.) agree that school mathematics must not be reduced to only routine facts and procedures. At the same time, most research literature shows us that the shifting to innovative and effective teaching practices stressing student comprehension and involvement will only be successful if teachers' mathematical and pedagogical content knowledge is sufficient to construct connections (*vertical* and *horizontal*)⁴ among concepts through a rich approach of the integration of linguistic, conceptual, and procedural components of a mathematics idea.

E. Content-Specific Pedagogy

Just defining what mathematics teachers must know is not enough to solve this problem. *The map is not the territory*; a description of curriculum standards and frameworks is not the lesson plan, and in turn, the lesson plan is not necessarily the actual classroom action and its outcomes of student learning. Curriculum standards and frameworks are essential, but what is even more important is how to translate them into meaningful lessons that implement the intent, the essence, and meet the needs of children. Many researchers in the field ([Ball et al., 2001](#)) claim that what many researchers lack is

- a. the knowledge of the practices of why, how, when, and what form mathematics knowledge teachers use in their classroom practices,
- b. how children react to these practices, and
- c. how these two are integrated in the daily classroom practice—the lessons. This lack of experience and knowledge in teaching practices results in many research projects and studies that focus on finding answers with only narrow understandings of the concepts or procedures and classroom practices. For example, they might focus on researching whether the answer to the addition problems of the type $M + N$ (M and N being two whole numbers less than 10) is by counting the smaller whole number forward after the bigger whole number, and whether it makes any difference in learning, retaining, and applying effectively and efficiently. The underlying assumption by many researchers is still 'addition is counting forward,' and one has not given serious thought to the idea of whether it is effective or just an adequate conceptual schema for addition. With such research we find that counting on the smaller whole number after the larger quantity does produce better accuracy and results. Does that mean we should now make that approach an exclusive teaching method for addition (and should any method be exclusive)? A teacher may agree with it as it is intuitively appealing and has been proven by research to be a good educational practice to count the smaller number onto the bigger number. It is a better strategy than counting the second number onto the first number. In very simple cases, it is a better

strategy, but not as a general strategy. This type of research does not add much to the effective teaching process of addition or subtraction.

On the other hand, the research and practice show that addition and multiplication strategies based on decomposition/re-composition of numbers as needed by the situation are more effective and have more impact on the child's learning. A student with such strategies can develop flexibility of thought and ease with number manipulations and master numeracy easily and effectively. It shows that decomposition/re-composition is to numeracy as phonemic awareness is to the reading process.⁵

Thus, the ways in which mathematical ideas are conceived, represented, and defined are fundamental to how people can understand and use these ideas ([NCTM, 2000](#)). A particular mathematical idea can often be represented in various forms both by children informally, formally by the teacher, and the textbooks in use ([Hiebert et al., 1996](#)). Effective teachers take advantage of all these elements.

For example, digits (the alphabet of mathematics) and numbers are generally perceived by children through counting and are mostly taught in that form, but they can also learn these concepts as *perceptual activities*, like many other skills they learn in other areas of their lives. This is particularly so when it comes to *subitizing*. Subitizing is a powerful concept, but it is converted by many to a counting activity rather than a perceptual activity.

The other impediments to teaching effectively and efficiently are dichotomies of different kinds. We believe *the generation of language, development and construction of a concept, forming conceptual schemas, and arriving at strategies and procedures* of additive reasoning should be integrated so that justice can be done to additive reasoning and all other concepts.

We believe, in any instructional approach, the notions of 'conceptual knowledge vs procedural knowledge,' 'episodic memory vs semantic memory,' 'practice vs application,' and 'children generated vs teacher initiated,' and other such dichotomies are not practiced separately but are emphasized, integrated, and are given equal and appropriate importance during instruction. For example, in subtracting a multi-digit number from a multi-digit number (with regrouping/renaming/borrowing), several procedures are derived that reflect aspects of additive reasoning and showcase the importance of understanding of certain key features of numbers, operations, and additive reasoning. For example: Find: $291 + 339 = ?$ can be found by procedure, but also by an understanding of number concept: $291 + 339 = 290 + 1 + 39 + 300 = 290 + 40 + 300 = 640$; $291 + 339 = 330 + 300 = 630$; $291 + 339 = 290 + 1 + 39 + 300 = 290 + 40 + 300 = 330 + 300 = 630$ (sight facts, making ten, and decomposition/re-composition) makes problems like this and many others simple, and pupils can see the power of mental mathematics. Such methods and understandings reduce the load on working memory.

Furthermore, there is a need for constructing structures and schemas associated with understanding—that are more general than specific to a particular aspect of additive reasoning or for any other concept, for that matter. Accordingly, these early cognitive schemas and strategies reflect fundamental epistemic elements of inquiry and problem structures that are developed in the context of additive reasoning. They are equally important for the learning of mathematics in any context and level. For example, we cannot add or subtract any two quantities (later to include fractions, decimals, number terms, and other mathematics elements such as determinants, matrices, functions, etc.) unless they share some common property, common attribute, membership to the same group/class, or have a common characteristic. Giving problem-specific methods is useful sometimes, but when they are a common occurrence, then we treat a student as a terminal student in the numeracy class; we should

expect and prepare every student for a long journey in mathematics or a world where a learner is reluctant to travel because of their fear of mathematics.

To achieve effective teaching in the classroom, we believe that a teacher should have a deep understanding of the trajectory of the concepts being taught in the curriculum (a deeper and wider *knowledge of content*), should be aware of the interrelationships of concepts and procedures (*content-specific pedagogy*), know one's tools and their usage well—at the expert level (*general pedagogy*), and understand children and how they learn mathematics (*principles of pedagogy*). As a result, the teacher has an attitude and approach that emphasizes underlying connectedness in concepts and procedures that prevent a fragmented teaching of isolated topics.

As mentioned earlier, historically, most school mathematics, particularly numeracy, has been taught emphasizing facts and procedures whereas our brain learns more effectively via making connections, discerning patterns, forming and articulating conjectures, identifying and growing relationships, watching others' strategies and approaches and learning from them—particularly the expert behaviors of classmates, the teacher, and others, incorporating them into one's approach and experiencing positive emotions from learning and success. We make guesses, we make false starts, commit errors, and make mistakes during learning. And we learn from our errors and mistakes, if we reflect on them. We seek and receive feedback and when it is immediately available, we make use of it to improve our previous performance. The more formative (meaningful, constructive, pointed) the feedback, the more valuable it is for us toward becoming a real learner—a better learner. Children's and even adults' addiction to video games is a perfect example of this: They make mistakes, but know where the mistake is and what the impact was of that mistake. They get immediate feedback, they play again to improve it and get feedback again, and they succeed. They become experts. Their mistakes are part of their learning.

Technically this is known as *statistical learning*. This is how evolutionary change has taken place. Human statistical learning refers to the ability to identify and extract patterns or regularities from given information, often using prior knowledge, allowing predictions—guesses, estimates, and or classifications based on observations and prior learning and skills. Learning is an iterative process and with each feedback, each approximation, each attempt, and each success, it gets better. Success builds trust; success and trust result in a development of interest and motivation and then learners will engage in learning.

Most mathematics, whether at the elementary or high school level, even at community colleges and in some undergraduate classes is still being taught emphasizing facts and procedures. Procedural mathematics is important, but with today's technology, it is not as important a skill. We can easily make calculations of all kinds. However, when the student knows the language, conceptual schemas, strategies, and has very good numbersense, then he can make better use of the results obtained from procedures through technology. For example, students who have good estimation skills can use technology tools for better applicability in mathematics. And students are better estimators when they have mastery of (a) number concept, (b) a mastery of arithmetic facts, (c) properties of number and operations, and (d) know how to derive strategies and procedures.

In today's social and economic systems, businesses are hiring workers at all levels for skills such as looking for patterns and trends, concept forming, understanding, and collaboration and critical thinking skills for creativity and innovation. *All of which students can learn in mathematics classes, where there is an emphasis on deeper conceptual learning, not just on procedures or getting answers from apps and calculators.*

In the absence of the integration of content and pedagogical knowledge, school mathematics instruction is reduced to a focus on teaching facts and procedures in a compartmental form. That cannot hold student attention too long.

School mathematics must not be reduced to routine facts and procedures, according to most international standards and experts ([NCTM, 2000](#); [CCSS, 2010](#)). But research literature also shows us that shifting to innovative teaching practices stressing student comprehension and involvement will only be successful if teachers' mathematical and pedagogical content knowledge is sufficient to construct connections among concepts in this new approach. Most new programs and innovations do not reach their potential without affecting classroom practices ([Baroody, 2004](#)).

Brains always seek to connect, integrate, and consolidate learning and experiences to make them their own. The integration of new information with existing knowledge and skills and developing insights help form new conceptual schemas—new cognitive frameworks that help us to better observe, perceive, receive, organize, interpret, connect, and convert the information into knowledge and be ready to apply for more.

Schemas are mental models and reside in our long-term memory. The brain utilizes such models to organize information about the world. Schemas are essentially built from our memories of our unique experiences.

The conceptual schemas provide us with tools for new learning. Applying new skills to learning new mathematics content, solving problems in other disciplines, and practical problems from realistic situations converts it into usable knowledge. In the process, children's ability to perceive and to cognate improves; therefore, their learnability also improves. In the process it modifies the capabilities of the brain and its functions. That is the most important information from neurosciences about brain plasticity—that with experience, particularly rich, effective, and efficient strategies, a child's cognition improves; therefore, the learnability skills improve. We believe a child should have acquired, after each lesson (individual, group, or class), not only some more content but also become a better learner than before they came to that experience.

Our *brain constantly forms and uses conceptual schemas* and reorganizes previous schemas to become more efficient. Through these conceptual schemas we take shortcuts in interpreting the vast amount of information that is available in our environment. These new schemas help to further improve our patterns of thinking and the behaviors that we use to make sense of the quantitative and spatial information in the classroom and the world around us. Students' brains will build new schemas when they make rich connections between ideas in their heads and integrate them with new information, strategies, and models. That can only happen if effective, efficient, and elegant strategies are presented to them.

One of the major objectives of writing this book is to identify, specify, and describe the content and the related pedagogical knowledge needed in teaching school mathematics effectively, particularly the foundations of numeracy, and how to bring it to the earliest grades and to every lesson where we engage them. In our approach, these principles are not only embedded in the teaching methods, lesson descriptions, and descriptions of concepts, procedures, and sequence of tasks planned, but it is done so that the fundamentals of school mathematics can easily be transferred to children and help older students who missed it the first time around learn effectively.

The pedagogy we advocate and describe here, is informed by current cognitive and neuroscience research, the role of language in mathematics learning, the importance of effective, efficient, and elegant instructional tools—concrete, pictorial, and linguistic—in conceptualizing mathematics ideas.

Moreover, it is profoundly influenced by more than a century of combined experience in teaching children and adults and extensive teacher training by the authors.

We know that deeper learning is an intentional decision in selecting strategies for building different components of the selected content. It is an exercise in ‘guided and inspired’ discovery rather than a free-flowing self-discovery process or a fully teacher-determined orchestrated activity with pre-determined goals. CPVA/S pedagogy, on the other hand, is marked by an intention of animating the teacher as she they interposes themselves between the children (or a child in an intervention situation) and the source of stimuli created by them—the materials, the tasks, the questions, and the representations. The teacher makes sure that the children have perceived the stimulus. They are given enough time, action, and experience to grasp the meaning of the stimulus activities. The teacher must make sure that the child or children have noticed and acknowledged the most relevant attributes, the primary concepts, and the pre-requisite skills involved in the content. For example, it is achieved through integration of proper concrete and visual modeling, diverse and extensive language usage, and effective questioning for *schema-building*, efficient strategies and their meaningful practice for facts and conceptual mastery, appropriate scaffolding to make transitions from one developmental milestone to the other and, on the way, making connections—both horizontal and vertical. *Horizontal connections* mean establishing relationships with other content elements at the age and grade level, and *vertical connections* mean making connections between different elements and representations of the concept along its trajectory.

In such a pedagogical environment, the schemas the students build help them solve problems (in mathematics, in other disciplines, and in the external world outside of class) they have not seen before, which is the whole point of learning and mastering numeracy and mathematics. Such teaching begins with a clear understanding of how the brain receives, makes sense of, and converts and stores new information and makes it available when needed.

Instructional activities that aim for deeper learning are not only a powerful way to teach math; such pedagogy is a collection of practices and effective methods of inspiring lifelong learners capable of making meaningful contributions to society.

However, there is a fear that these mental frameworks sometimes also cause us to exclude pertinent information and to focus instead only on things that conform to our pre-existing beliefs and ideas. Schemas thus can also contribute to stereotypes and make it difficult to retain new information that does not conform to our established ideas about the world. In such situations, the *Socratic role*⁶ of the teacher is to create cognitive dissonance in children’s minds and then help them resolve these dissonances to broaden their schematic network to prepare them to be receptive to new strategies and new procedures. We have found that many students become rigid in their methods, approaches, and thoughts. They need to work on expanding their repertoire of strategies. With such a pedagogy, teachers need to provide opportunities to help children develop *flexibility of thought* by exposing them to more than one efficient method of solving a problem ([Gattegno, 1960](#), 1974; [Gattegno & Cuisenaire, 1954](#)).

Most of our learning is the results of the residues of our experiences in our memory systems and networks. So, an understanding of how to provide experiences so that they leave rich residue in our memory systems that are ‘*storage-rich*’ and ‘*retrieval-rich*.’ To be able to do that, it is important that teachers know how our memory system works. Different kinds of memory traces from different classroom practices are left in different components/locations of the memory system. Many teachers and children do not know how to place information in memory networks properly and many find it

difficult to be able to recall the placed information. Every component and center of our memory system is malleable, if not immensely.

Many teachers assume that our brains function like card catalogs: We learn a series of disconnected facts, our brains file them away, and we can easily access them when needed. However, the brain is not a card catalog. This is because by emphasizing memorizing facts and procedures without the language, concepts, and necessary questioning processes, it finds it difficult to store and retrieve the information. And children get the wrong impression that they cannot learn mathematics and teachers assume, many times inaccurately, that the child has learning disabilities or memory issues, or no interest in learning mathematics.

Neuroscientists have come to understand that this is not how the brain stores information most efficiently. We learn better and use that language, fact, or schema better when our brain makes connections between experiences, concepts, facts, procedures, and builds a web of ideas. There are effective ways of placing different parts of information and learning in particular places, in the brain, where they naturally happen.

Math educators, for example, can facilitate this process by engaging students with big ideas and then slightly smaller ones. For example, the teacher puts 5-rod and 4-rod together end-to-end, and parallel to this, she places a 4-rod and then a 5-rod. This arrangement forms an equation. She asks children to share their observations.

Many children report that the two arrangements are the same. Some will say: purple rod and yellow rod and yellow rod and purple are the same. Someone will say: ' $4 + 5 = 5 + 4$ and I know that is true.' Many other children agree with the child and the teacher adds: 'If white is one.' Another child might say: 'Purple + yellow = yellow + purple.' The teacher agrees and says: 'Let me see if I can write it a little differently: $p + y = y + p$, where p = purple and y = yellow. Do you agree?' And they have a lively discussion. The teacher extends it further: 'What if, I call white equals 10, then what will be the new names of purple and yellow rods?' Most children answer, ' $40 + 50 = 50 + 40$.' Another child raises her hand and says: 'If I call white as 100, then the equation will be: $400 + 500 = 500 + 400$.'

Teacher then asks: 'Can each "learning pair"⁷ of you make a similar arrangement using two different-colored rods that look like mine? ([Gattegno & Cuisenaire 1954](#)) One condition: every pair's arrangement should be different than their neighbors' and mine. And you will write 5 equations for your arrangements, and be creative in your equations.'

Children make their arrangements. The teacher walks around observing children's work and approves and informally asks a guiding, leading, suggesting, or probing question that instigates some thinking in children's minds relating to the problem. After every pair has made their equation. The teacher asks more questions, and whenever someone, either independently or by the help of questions, arrives at the fact that: 'When you turn around the order, the answer of the addition remains the same.' The teacher talks about how the big idea of additive reasoning is that the addition operation is *commutative* in its nature. They discuss why this is important. The teacher answers, 'You have to do just half the work to learn all the addition facts.' The 100 facts of additions up to 20 become only 55 facts. Mastering 55 facts is much easier and quicker than mastering 100 addition facts. She asks many mediating questions. *The activity is simple, but the idea is big. They have formed a schema that will be used many times.*

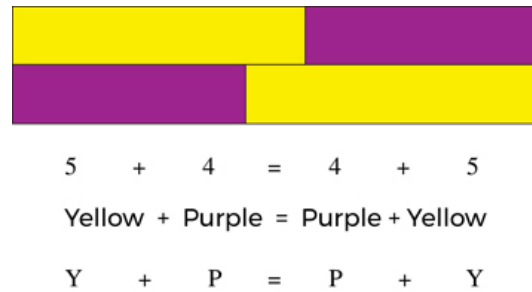


Figure 4.1 Commutative property of Addition

When students use these ideas to build a schema, and apply it to an unfamiliar problem and find a solution, they experience an ‘Aha!’ moment and their brain gets a dopamine surge. In that moment, *math is fun*.

Like other neuromodulators, dopamine promotes *neural plasticity*,⁸ and its release tends to be concentrated in several areas of the brain that are important for helping us to learn how to get rewards—success in learning arithmetic facts is a real reward for a child. Through the strategies derived by this approach, every child we work with experiences the feeling: ‘Is this all there is to it to learning facts?’

During learning, the *amygdala* and *hippocampus* work in tandem. Dopamine in the amygdala—the same brain structure that mediates math anxiety—fear of learning mathematics—helps us learn about ‘cues that signal’ an upcoming reward—*success*. In the hippocampus, dopamine helps us learn about contexts, conditions, and situations, where we are likely to get a reward—‘when I apply the schema—the *script* of learning addition facts, I get the fact right all the time.’ And finally, in the nucleus accumbens, dopamine helps us learn what we need to do to get that reward—‘I applied the script: $8 + 7 = ?$ What is the bigger number? 8. How do I get to 10? I add 2. Where is the 2 coming from? From 7. What is left in from 7? 5. I add $(8 + 2) + 5 = 10 + 5 = 15$.’ Because dopamine helps us form memories that associate cues, contexts, schemas, and actions that lead to rewards, it sets expectations that shape our experiences when we get them. ‘I can always add any two numbers. With practice, I do it quickly and I can apply it in new situations, even difficult situations: To find $68 + 7 = ?$ What is the next tens to 68? 70. How do I get to 70? I add 2. Where is 2 coming from? From 7. What is left in 7 after taking away 2? 5. So, $68 + 2 = 70$ and $70 + 5 = 75$. So, $68 + 7 = 75$. I also know $7 + 68$ is also 75. I feel successful. I taste success. I love it; I get it every time. I want to repeat this activity. I get status in the class. I would like to do it again.’

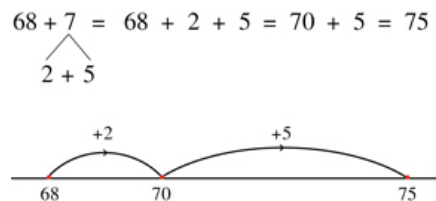


Figure 4.2 Making next tens and adding

The child looks forward to the next success. *Success is the greatest motivator at any age*. Success creates expectations and the cycle is repeated. Dopamine is produced only when we meet expectations. Therefore, the higher the expectations, and then, if success is reached, have a rush of dopamine. That

rush is a motivating factor for learning. Dopamine also helps in the binding of the neurons related to that information with other related information.

When students with learning difficulties, dyscalculia, and acquired dyscalculia learn and execute strategies, initially they hesitate and resist new strategies and methods because they have to try to learn strategies, whereas counting is effortless. Simple counting activities do not require any effort. It is human nature to take the 'path of least resistance.' But when they begin to experience success, their engagement in the tasks increases and they make more of an effort. They begin to understand the value associated with effort. They begin to realize that effort, success, and value go together.

The author asked a middle school student to simplify a multi-step, multi-operations expression using Order of Operations (GEMDAS)⁹ by hand rather than the calculator (he was allowed to use a calculator in the classroom). We had worked earlier on all the individual operations. After two or three examples, he was enjoying the results of his efforts, and he made a comment: 'Doing real mathematics is fun after all. It takes effort, but I realize that I am doing it myself, not the calculator, where I did not know what I was doing and why and even whether I was right or not. It is like scoring points myself on the basketball court rather than watching it on television. I think calculator use is like watching the game, not playing the game.'

Many students say: 'I can do real mathematics. I can do difficult things. I was always afraid of it; now I am not.' They would say: 'Can you give me a real mathematics problem, the kind other kids do? My special education teacher never gives me a real problem. She gives me a star when I can do even a tiny thing like counting the rods or I get the answer with a calculator. I have been doing it for eight years. I hate it.' Many parents report many other changes in their children's behaviors when they begin to do better in mathematics with effort. 'They are less irritable, less disagreeable, they finish their work more often, they take the initiative now.' Another parent said: 'Homework is not a big hassle. He still finds it difficult to finish the homework. But he tries much more now and gets some of the problems right and that makes him very happy.'

Emotions impact on how strongly the brain makes connections. If students have math anxiety that creates a negative emotional response to being in a math class, their amygdala will release hormones that make their brain less effective. Rather than focusing on the math problem, their brain will focus on the issue of 'How soon can I get out of this class?'

In [Chapter Two](#), talking about mathematics anxiety, we discussed several strategies. One way to counteract math anxiety is to create a classroom environment that encourages productive failure and the meaning of effort. Students need immediate feedback on their thinking that is delivered without any sense of judgment. A game-based or puzzle-based learning environment can be extremely helpful in this regard. In this case, the feedback on the student's ideas is not coming from the teacher or any person but from the game.¹⁰ This can reduce the level of emotional anxiety students feel when trying out solutions about which they may be very uncertain. During the game, the moves of other students are acting as (a) feedback (b) a lesson, and (c) expectations of winning. This feedback loop is the motivating factor.

The reward is both intrinsic and extrinsic because teaming up with classmates to solve problems is fun. It is not doing math and then playing games as a reward. It is having the math discourse become so exciting and the game so engaging that there is no delay between the flash of understanding students get when they figure something out and the emotional reaction created by the release of chemicals in the brain. For this to happen, the brain requires immediate feedback. Applying oral strategies, using scripts with the help of the teacher provides a good start in being independent.

All of this teaches them to be industrious. Research shows that being industrious in one domain that generally translates into other settings. Feeling the positive effects of effort is a generalized feeling. Research shows that some specific skills may not be transferable, but the habit of trying when doing mathematics is transferable to other topics, other content, and other domains. The experience of effort is common to all topics and domains: effort that is effective, efficient, and, if the success is immediate, it is transferable and is measurable.

Another element that is important for a teacher to practice is higher expectations accompanied by supporting help and quality questioning. Many teachers, particularly special educators, approach children with low expectations. They think their students will not be able to cope with demanding assignments. Yes, they can. Sometimes we are inclined to make things too easy for too many students in the mistaken belief that they cannot cope with hard work. Yes, they can! All students, regardless of their starting points, deserve an ambitious curriculum and effective teaching. Let us hear from a special needs student, David, who shared why he enjoys demanding work. It happened when he arrived earlier for his appointment and was waiting for his session. The session was for another student, his friend from school, but in a higher grade. He expressed an interest in sitting in on the session with his friend. His friend agreed, and I included him in the session. After that, he would request I give him hard work rather than the simple work he was getting in his special education class. After several months, he said: 'I like hard work because it is real learning, because if it is really easy you can just do it; you are not learning; but when it is hard you are thinking, you are learning, so it stays in your head.' We have heard that from almost every one of our students.

One question should always be on our minds: how can we be truly inclusive in our classrooms? Truly inclusive means all students, including those with special needs like David, have the chance to get into the big ideas of a rich curriculum and enjoy the joys of learning. Our pedagogy should be inclusive and responsive to all children in the classroom. Here the research is helpful (all effective teachers know it intuitively) that:

- Human beings are curious by nature,
- curiosity is a great driver for learning,
- background knowledge is important for learning new ideas,
- our brains privilege story. explain??

One of the key mechanisms that all mammals have is the perception-action cycle: (*Observation* → *Perception* → *Prediction* → *Expectation* → *Action* → *Outcome* → *Immediate and meaningful Feedback* → *Changed Perception* → *Reformed Prediction* → *New Expectations* → *Reformed Actions* → *Better Outcome* > ...).

It has been a cycle of actions for survival as well as a learning technique—a learning cycle that has helped us build language, schemas, structures, strategies, and ultimately useful skills and procedures. When we practice this cycle for a concept, strategy, or a procedure, we form new and more refined and comprehensive schemas for doing things—solving problems and achieving desired goals. When we practice them in problem-solving, we reform our schemas and convert them into better schemas; we become better learners. Our brain uses these very natural learning mechanisms to build a picture of how to interact with the world around us—whether it is learning mathematics, a sport, or a new habit.

Formative assessment and feedback are an immediate response that contains information that can help the brain to change perception, acquire a new perspective and refine its schemas, and therefore take a new action. In basketball, for example, there is a difference between missing a shot by a lot and

missing by a little. You can find your range by continuing to shoot, but a coach sitting there says only, 'No, that is a miss,' every time you miss he is giving useless feedback. You already know that. On the other hand, if the coach immediately after the missing shot says, 'More arc!' or 'Bend your knees,' those inputs become part of the process right away. With such input, one wants to reshape their action. In this process, our expectations and actions change. This type of immediate feedback is what makes classroom teaching, training for a competition, and working with real as well as virtual manipulatives powerful learning tools. 'Immediate' is important in that it makes it less likely that the, all important first learning experience becomes a dominant entry in the brain.

The resolution of the *perception-action cycle* is that the schema is held in the hippocampus, which compares the prediction and the reality. If the prediction and reality match, the brain releases chemicals that physically strengthen the neural connections that made that prediction, so that schema becomes physically stronger. If the schema is incorrect, it becomes less connected and weaker. The role of the teacher is to help students change the nature of the perception-action cycle and thereby the nature of the action and the result. That happens with useful feedback that reflects the need at that instant.

This cycle explains why empowering students to discover and make connections—both vertical and horizontal—leads to deeper learning than just telling them how to perform a procedure. Suppose a student has a misconception about something and finds that it does not solve the problem they are trying to solve. In that case, that schema—their misconception in this case—will become weaker if the action is abandoned; however, by feedback and reformed action it will eventually be replaced by a stronger schema. This reforming, strengthening, replacement, and creating schemas is true learning.

Learning is the growth and strengthening of 'wiring (neuroplasticity)' in our brains. Educational programs designed to work this way are effective and fun because they complement how our brains learn. They are enjoyable because they ignite students' intrinsic love of learning: They are mastering the content, but they are also learning that solving problems is exciting, a mindset that will serve them well throughout their lives and careers.

In this pedagogy, the teacher's pedagogical knowledge is built from the following considerations:

- a. Clarity in and a *deep understanding of the nature of mathematics* one teaches. Knows the *trajectory of the concepts* being taught in the curriculum and is always aware of the interrelationships of concepts, procedures, facts, the primary concepts and skills involved in the development of each secondary concept.
- b. *Understands children*—what do I know about my students—their *cognitive, affective, and psychomotoric skills*: how do they learn mathematics, what are they capable of, what are their limitations, what motivates them, what cognitive skills, such as executive functions, memory, sequencing of tasks, and tools facilitate their learning?
- c. Knows one's *instructional tools* and their *usage well* at the expert level: what tools, language, sequence of activities, practices, assessments, and feedback make this mastery visible and usable.
- d. Knows how to gauge the success of her students: constant *assessment—oral, formative, summative*, and how to give *motivational feedback* to children—how to discern errors and mistakes, how to help them learn from their mistakes, how to improve their knowledge base, not just in quantity but also in quality.
- e. An overarching principle that should guide any teaching is the *teacher's own belief system about role of teaching*: Is it to accept the student as he/she is and deliver instruction that suits her/his present cognitive state and help develop the content that fits this cognitive state of the students, or the continual mobilization of classroom and teaching resources in order to *optimally enhance not*

only the individual's current capacity but his capacity to become modified in the potential? In other words, to provide instruction to each child so that their present content needs are satisfied, but also help them to be modified in their content mastery and become transformed as learners in skills, potential, adaptational capacities for future learning.

It is the judicious integration of these elements that answers the question of how to achieve optimal learning for all children. In other words, the teacher has an attitude that emphasizes underlying connectedness in concepts and procedures and knows that a fragmented teaching of isolated topics prevents higher achievement and does not make students better learners. The teacher is keenly aware of what language, what vocabulary, what type of questions, what tools, what sequence of activities will connect the learners with the content and improve their learnability.

In such a classroom, instruction for all children is based on the

approach that considers novelty and innovation as a necessity to enhance the individual's level of functioning... to experience success under new and more demanding conditions requires more effort and more intensive work from both the educator and the student, but this goal is obtainable if appropriate intervention is introduced.

([Feuerstein et al., 2006a](#), p. 22)

These principles can be implemented, if there is an ongoing critical dialogue between the teacher and the learner, teacher and learners, among children, between children, and between teachers, both at the same and at different grade levels of teaching. Invite parents to share your understanding of their child. As tutors and professional development providers we have always invited parents to sit in on the sessions to observe the learning process. Parents are an integral part of this process.

Because mathematical thinking can be observed mostly in the process of students' problem solving and listening to dialogues among other students, the teachers' ability in learning from children's efforts, their mistakes, their initiatives, and successes is an important skill that will improve their mathematical thinking, but more importantly, will shape and sharpen their pedagogy. *One's pedagogy is always being shaped; it should never be a finished process.*

There are two conditions necessary for a fruitful, critical dialogue: a *mutual quest for improvement* and an *admission of human fallibility*. However, that depends on the physical, social, intellectual, and learning environment that a teacher aims for and then strives to achieve with the help of children, parents, and administrators.

An important mark of an effective teacher is to learn from children's ideas and perceptions—their informal strategies, their errors, their discussions, and their initial representations of the concepts being taught and observing, particularly the *behaviors of the outliers* related to the concept ([Chinn and Ashcroft, 2017](#)). A specific mathematical idea can often be represented in various forms both by children informally and formally by the teacher and the textbook (Hiebert and Carpenter, 1992), but it is what we learn from children's reaction to that is important to note. The teacher should share children's work and their thinking with the class and if the idea is incompletely represented, then effort should be made on how to improve it. If a child's idea is fully represented and its aspects are exemplary in that representation, then the child should be invited to present it to the class.

However, to derive benefit from this, the teacher must be very good at error-analysis of children's work in the process of learning, their oral responses, and on test papers and their daily work and efforts. The *error analysis* should focus on *concept* (definition, primary concepts involved, conceptual

schema, the tools used to understand it), *strategies used* (with a focus on effectiveness, efficiency, and its generalizability), or *procedure* (conceptual schema behind it, the primary concepts involved, how it was derived, the sequence/order of its execution). *Error analysis* can enhance both feedback and discussion of children's work and their understanding of errors and their corrective measures. *Error analysis*, therefore, is dependent on five key factors:

- a. Are the errors related to the misunderstanding of the trajectory of the development of the concept or procedure?
- b. Were the tools used in developing the concepts inadequate or inappropriate to conceptualize the idea?
- c. Does the lack of development of the native language or the mathematics language (vocabulary, syntax, semantics, and translation from native language to English language or vice versa) related to learning that concept play any role in the presence of the error?
- d. Does the absence or misunderstanding of the primary concepts, prerequisite skills, arithmetic facts, and earlier procedures involved in the current errors present in the concept and/or procedure?
- e. Are there cognitive, executive functions, or prerequisite skills related reasons for the errors?
- f. Is it a common error present in most children's work in the class, grade, or school? If this is the case, then the cause may be the teacher's skills, textbook-related, or common pedagogy at the school or grade level.

For the improvement of children's mathematics achievements and the refining of the related pedagogy, at a school level, much importance must be placed on the mathematical knowledge improvement of the teachers. Focus should also be on the analysis of the *teachers' classroom practices* as the starting point of teacher education and professional development. There should be a critical examination of the nature of *student learning outcomes*—the nature of individual and common errors, type of strategies often used or misused, the language usage in answers and discussions, and spread of the performance levels in the classroom. Student performance data is not just for administrators; it can be and is an important means of improving pedagogy and therefore daily lessons. All these elements will point to the consideration of the concrete needs of the teachers related to their teaching practice and collaborative work among peers and instructors as means to improvement of the pedagogical principles and practices.

Throughout this book, we bring observations and instructional tools that help teachers to focus on students' thinking and present questions and investigations with children to show how the children's thinking impacts the learning outcome. We claim that the emphasis on student learning from emphasizing their thinking will continually remind teachers how important it is for them to understand students' ideas and help bring the visions of reform in mathematics education—curriculum and instruction—into their classrooms.

This book intends to develop not only the pedagogical knowledge directly connected with teaching practices of additive reasoning, but also the questioning process needed for developing the proper *scripts* for *strategies* and *procedures*.

Teachers, curriculum planners, and researchers will find that this information is helpful for children in learning future concepts. Therefore, it is not intended for just teachers of K through 2, but all teachers of school mathematics, including special education teachers and researchers. It integrates

mathematical knowledge and curriculum management associated with tasks and materials selection and their application in the classroom.

However, as mentioned in an earlier chapter, the quality of instruction and intervention is dependent on the competence of individual teachers. Teacher certification for pre-K through 3rd grade should therefore emphasize both mastery of the content knowledge of the subject (specifically, a deeper knowledge of the arithmetic—number concept, numbersense, and numeracy) and strength in mathematics content-specific pedagogy, and familiarity with effective, efficient, and elegant strategies and tools.

From many years of direct experience in working with teachers, parents, and children, individually and in classrooms, we know that mathematics instruction—both initial and interventions—is far more effective when delivered by a teacher who understands both the subject matter and the most effective ways in which young children learn math.

Because the conceptual complexity of elementary mathematics is underrated, successful programs should ensure that early math instructors specialize in these areas. One solution may be for a school to designate a teacher in each grade who is responsible for teaching only math to all students of that grade and should provide quality interventions to children who need it at that grade level. In a small school, one teacher could specialize in teaching numeracy to more than one grade, say from kindergarten to second grade and another one from third grade to fourth grade. These two teachers should be responsible for initial instruction and then also for intervention. Their teaching load should be $\frac{3}{4}$ for initial/regular instruction and $\frac{1}{4}$ for interventions and remediations. This kind of arrangement brings better levels of achievement in children. This arrangement prevents failures from developing. However, to give it a chance to succeed, the teachers should be trained in general teaching, but should also be trained as mathematics specialists as reading specialists in schools are.

Early instruction with quality activities that develop a comprehensive number concept and numbersense can minimize and prevent failure in numeracy and in later mathematics. For example, teaching the integration of numbersense activities with an increased focus on ‘sight number facts’ automaticity and decomposition/re-composition will better prepare children for numeracy activities. Teaching these skills in isolation and without effective strategies has minimal effect on the reduction of difficulties in mathematics both for the general and LD population. Quality instruction benefits students with and without learning disabilities.

Early mathematics learning experiences can help teachers in implementing the positions of the [National Council of Teachers of Mathematics \(2016\)](#), National Research Council recommendations, and Common Core State Standards ([CCSS, 2010](#)) and other international studies.

F. A Model of Instruction in School Mathematics: CPVA/S

The authors have devoted time and effort to bringing out an effective method of teaching and learning practices in mathematics, culled from the related research from the fields of neurosciences, linguistics, science of reading, psychology and neuropsychology, artificial intelligence, math education, anthropology, education, technology, observations of practicing teachers in their classrooms all over the world, and our own extensive work with researchers, teachers, children, and parents.

We have been involved directly in all the phases of children’s demonstration of learning difficulties in numeracy and mathematics, from a diagnosis of dyscalculia to the preparation of recommendations to preparation of Independent Learning Plans (ILPs) for intervention and remediation of students’ learning challenges to reassessments and alternative educational plans and placements. After reading

many ILPs, prepared in a variety of settings and cases, we have come to the realization that the interventions and remedial methods do not produce the desired results in a very large number of cases.

Students once identified as learning disabled, special needs, dyscalculics, dyslexics, and other labels rarely come out of those situations.

We collect a large amount of data during evaluations. For example, the accurate diagnosis of dyscalculia involves a comprehensive assessment of personal, familial, scholastic, and developmental history, along with a general examination, psychometric evaluation, working memory tasks, and standardized tests to evaluate numeracy/arithmetic abilities. Various tests in the English language, such as the Wechsler Individual Achievement test, the Key Math 3 Diagnostic Assessment, and the Dyscalculia Math 3 Diagnostic Assessment, can aid in diagnosing developmental dyscalculia. All of these produce a substantial amount of information, but when it comes to prescriptions for interventions and remediations, they end up having small groups or individual settings, but the recommended pedagogy and intervention methods are all the same. The same classroom strategies are provided, only in individual settings. This has informed our work greatly in how to address those findings.

Along with our colleagues, we use this teaching methodology—techniques, strategies, skills, and instructional materials, in our courses, workshops, webinars, classroom demonstrations, and in private tutoring and consultations. We have helped thousands of educators from pre-service to veteran teachers in improving their instruction so that children can develop a mastery of mathematics ideas—the language, concepts, procedures, and skills.

We believe that children should develop the mastery of mathematics ideas with clear and specific emphasis on the language, concepts, strategies, and procedures. Ultimately, by applying them in various settings and forms—intra-mathematical, interdisciplinary, and extracurricular—these concepts and procedures should become automatized skills as tools and ideas for further learning and problem solving. We believe children’s mastery of curricular elements is possible only if lessons are delivered in such a way that students experience the concepts through the natural progression of six levels of knowing: beginning with *intuitive* level of knowing to *concrete*, *pictorial* and *abstract/symbolic* levels, then these ideas are *applied* in problem solving to reach the *communications* level of knowing. This pedagogical approach is called *CPVA/S*.

CPVA/S stands for *Concrete-Pictorial-Visualization-Abstract* and are interconnected and interwoven together by a well-designed *Script* that is based on sound principles of mathematics concept development and the mathematics developmental milestones. The script integrates, relates, and advances the development of concepts and procedures from intuitive to communications. The *script* connects and binds all the other components and makes them into a working tool. The script is an abstraction and generalization of the concrete and pictorial models of the concept and procedures. The architecture of a Script is common for most arithmetic strategies based on the number concept (45 sight facts, making ten, teen’s numbers, place value (both canonical and non-canonical expansion), next ten and properties of numbers and operations. Specific scripts are based on the definition of the concepts and strategy and operation. Properly developed “*scripts*” help integrate the language, concept, and the steps involved in the procedures.

The elements of the *CPVA/S* process are developed in a very systematic sequence based on a task analysis of each concept, strategy, or procedure and steps involved in the corresponding procedure, the *trajectory of a concept* (beginning, growth, expansion, maturation and fruition/mastery) and *composition of the concept* (prior knowledge, primary and secondary concepts), and the *development of language* (vocabulary, syntax, semantics, and translation from mathematical idea into language) to

conceptual models and representation—*concrete* and *pictorial*, *symbolic*, and *abstract*) and finally to *applicable procedures* and *tools* that can *communicate* the *power*, the *beauty* and *reach* of numeracy and mathematics.

This approach differs from the conventional CPA or CRA (suggested by [Bruner, 1979, 1966, 1960 and 1977](#); [Bruner et al., 1956](#); [Anghileri, 2001a, 2001b](#), 165; [Gravemeijer, 1997](#), and many others), where it was assumed that if a student knows a concept or a procedure at *concrete* and *pictorial* levels, he will easily transition into the *abstract* level. We have keenly observed that only a few students can do it by themselves and a few more with the help of effective teachers. However, many students do not transition from concrete and pictorial and counting processes to abstract level easily and many give up, making little or no effort or remain at the concrete and pictorial levels forever. Even those who do, they do so with a great deal of effort and an unusual amount of time spent on it. Many of them carry on with their misconceptions, inefficient strategies, and limited understanding. Misconceptions and inefficient skills develop and remain and a proper transfer to abstract and symbolic levels does not take place easily for many children.

There are three reasons for this difficulty:

- a. Many teachers use inefficient concrete models that make it difficult for a child to move from concrete to pictorial and then to abstract. For example, counting objects (*concrete*) and then counting hashmarks or tally marks (*pictorial*). Tally marks or counting on fingers is slightly better than counting hash (tick) marks as there is a sense of grouping of 5. Counting on a number line is considered by many as an abstract activity. It is a pictorial activity. These activities do not promote the transfer to abstract concepts easily for all children, unless they devote an inordinate amount of time to practice and memorizing.
- b. Many teachers, in their teaching methodology, remain at the concrete level too long and students get accustomed to, and become dependent on, the concrete models for solving all problems. In fact, special education students are sometimes perceived as being totally concrete learning-oriented; that is, they are assumed to not be capable of using higher-order mental processes to solve problems. They have hardly any chance to grow cognitively. Purely concretized educational practices are sometimes justified by claims that students are being offered a world in which they can act adequately without having to experience challenges and perhaps failure and frustration.

The problem with this is that by binding the child only to a concrete world and by training him to act—even successfully—within the limits of concrete learning opportunities, not only are we not enhancing the individual's coping and learning capacities, but we are also limiting or reducing them substantially in the development of their potential. It is also contrary to the neurological realities of the brain's plasticity. All of life's realities, even the simplest of them, are not just concrete. Change is not defined by concreteness. On the contrary, adaptation requires a continuous activation and extension to abstract thinking. Consequently, the continuous reference to concrete learning may cause the student to be incapacitated in real life situations, incapable of coping with a constantly changing environment ([Feuerstein et al., 2009](#), p. 22).

- c. On the other hand, many teachers move from concrete and/or pictorial to abstract and symbolic too quickly and inadequately. That transfer is neither automatic nor easy. As a result, many students either remain at the concrete and pictorial level, or they memorize the procedural steps independent of the concrete and pictorial representation. They do not appreciate the importance of concrete and pictorial models in learning mathematics, and they do not see the connections

between them. Mathematics remains artificial, symbolic, and of no use to them. They begin to believe that a dynamic and robust memory is essential for success in simplest of mathematics.

1. How and Why the CPVA/S Model Works

The *CPVA/S* approach makes use of concretization in its lessons and practice, but this is of temporary use only. The goal of this teaching process is to embrace the concrete first and then transcend the world of the concrete to richer and more effective levels of pictorial and abstract thinking rather than to create an ‘artificial world’ to which the student may easily adapt. The transfer and development from concrete to pictorial and then to abstract is achieved by the help of several important processes. *First*, in the *CPVA/S*, there is a *congruence* of language, questioning, conceptual models, and activities in representations at the concrete, pictorial, and abstract models. *Second*, the concrete and pictorial models and tools selected and used are such that they end up being *visualized* in the mind’s eye easily and thus they are an aid in expanding the working memory functioning rather than a limitation. *Third*, the language used at the concrete, pictorial, and abstract levels is the same and results in a concise and actionable *script* that can be used for the execution of steps at all levels in the same way. As a result, the student only needs to use the internalized script to solve problems at the abstract level. *Fourth*, once a student can visualize and apply the script, then all practice is on the abstract level rather than solving many problems at the concrete and pictorial levels. Thus, in the *CPVA/S* approach, the emphasis is on the proper use of *visualization* and *language (the script)*, and this integration plays a big role in bridging the gap between the concrete, pictorial/representation, and the abstract and symbolic.

The role of *visualization* in mathematics learning¹¹ is very important. Visualization means a person’s ability to hold information in the mind’s eye and manipulate it flexibly. Visualization takes place in the working memory. In the working memory, the incoming information from the short-term memory is mixed with the related ideas, concepts, language, and skills brought, when and if needed, from the long-term memory and are converted into new knowledge—language, concepts, schemas, strategies, and procedures.

Effective and appropriate concrete and pictorial models leave a strong learning residue (images and content) in the working memory, but for them to be used, they must be rehearsed. *Visualization and a clear, well-organized script help in this process.*

These related ideas are easily brought to the working memory when in the process of using concrete and pictorial models, efficient strategies with appropriate language—vocabulary, syntax, semantics, and translation processes—are used. Without visualization, many students cannot hold and manipulate the essence of concrete and pictorial steps in their working memory. It neither converts into the abstract/symbolic form, nor does it reach the long-term memory.

The visualization process acts as a bridge between pictorial representation and abstraction process (conversion into finished knowledge—mentally transportable, in symbolic form with schemas and abstract representations). In the brain’s learning process, a concept is first accepted, accommodated, and assimilated by extending prior and intuitive knowledge and its conceptual schemas are turned into new or expanded conceptual schemas to make sense of the new information. This information thus becomes usable knowledge. Sometimes, several related and even unrelated prior schemas are amalgamated into new schemas and the new composite schemas facilitate the acceptance of the new concept. When these two situations do not exist, the teacher must help create new schemas for the concept using effective, efficient, concrete instructional models that facilitate the creation of new

schemas and take the concept to a new level by using concrete models so that the child forms a concrete conceptual schema.

After being introduced to concrete models and being exposed to the language, concept, and strategies with the guidance of the teacher or working with a partner. The teacher introduces them to representations in pictorial form. At each level, the process of conceptualization is being facilitated by an emerging ‘*script*.’ The script is the annotation of the action in linguistic form with an emphasis on connecting and using the concrete or pictorial models. As a result, first, the child represents the ideas in the form of iconic and non-iconic diagrams or figures. Concrete and pictorial representations with the help of visualization and language translate into abstract and symbolic representation.

Along with the development of symbolic levels of knowing, the teacher also helps students to develop and apply the ‘*script*’ to be able to visualize it. This script helps the child to visualize not only the actions from the concrete manifestations, but their purpose, relationships, and results of those actions (see the script in the $8 + 7$ episode). This is a fundamental shift in the child’s thinking. This is possible only if there is a *congruence* between concrete, pictorial, visualization, and abstract representations and the teacher is using the appropriate language and questions connecting the steps. The purpose of developing the script is to enhance the decomposition/re-composition of the quantity/number process, but also connects the three brain centers: *calculation*, *language*, and *tool making and using* ([Henschen, 1926](#); [Dehaene, 2020](#)). *Script is the integration of four very important skills: language, quantity, decomposition/re-composition, and tool making and using.*

The elements of and the actions described in the script making and using are common at all levels—concrete, pictorial, visualization, and abstract/symbolic. Once the child has an image of the concept in the form of pictures, then the teacher guides the child to manipulate this picture in the mind’s eye with the help of the script, with the help of *visualization*. Once the child can hold the image of the concept, it is easier for the child to represent it in abstract form and manipulate and work with symbols. It ensures the transfer of this learned information from working memory to long-term memory. Script making and using is a ‘storage- and retrieval-rich’ activity. This is an important step emphasized in the CPVA/S methodology.

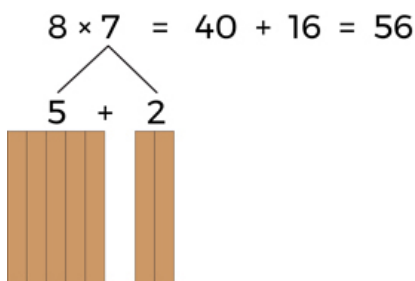


Figure 4.3 Deriving 8×7

For example, the multiplication sign $\times$ in the expression 3×4 means: (a) 3 repeatedly added 4 times, (b) 3 groups of 4, (c) a 3 by 4 array, and (d) the area of a 3 by 4 rectangle. Using the Number rods to make the area model, the teacher helps children to master the multiplication tables of 1, 2, 10, and 5, then using commutative and distributive properties of multiplication, pattern recognition, and decomposition/recognition, children derive all the other multiplication facts easily, effectively, and with competence, and master them in a very short time.

This multi-model approach not only makes the concept comprehensive but also prepares children for applying the concept and the related procedures easily. The teacher's questions and the script developed during the process help students to communicate their understanding and mastery.

In the CPVA/S model, we strongly emphasize the teaching and learning of mathematics with *rigor* and with *meaningful practice* to help students master the concept so that they achieve at higher levels in mathematics and begin to like mathematics from the very beginning. The definition of '*knowing or mastery*' means:

The student has acquired the language, conceptual schemas, automatization with understanding, and confidence in their application of the procedures and concepts. We have developed or modified existing instructional materials into special tools (skill mastery instruments—concrete, pictorial, and linguistic) and resources to help students master their concepts and achieve automaticity.

We believe in *focused work* and *vertical and horizontal acceleration* which help students in *achieving numeracy* and *mathematics developmental milestones* at the appropriate time or earlier. As mentioned in introductory chapter, we have identified key non-negotiable skills which are fundamental to being proficient and successful, first in numeracy and then helping them learn mathematics with understanding so that they feel confident and appreciate the *beauty, power, and reach* of mathematics in our world. As mentioned earlier, the developmental milestones in the quantitative and spatial domains are as follows:

1. *Quantitative Thinking*: The study of patterns and relationships in number and its manifestations.

- *Number concept,*
- *Additive reasoning,*
- *Multiplicative reasoning,*
- *Proportional reasoning, and*
- *Algebraic thinking.*

2. *Spatial Thinking*: The study of patterns and relationships between and within shapes and their manifestations. *Spatial sense*: Spatial orientation/Space Organization (Whole-to-Parts and Parts-to-Whole):

- *Identifying objects in the environment using one's body as a reference*
- *Identifying objects in the environment using opposites as a reference*
- *Identifying objects in the environment using any object as a reference*
- *Geometrical thinking*

3. *Measurement, applications, and problem solving integrates the two.*

These milestones are based on the latest research in cognitive sciences and the logical development of mathematics ideas. Our work is dedicated to guiding teachers and parents with a firm belief that all children have the potential of becoming numerate and competent users of numeracy and mathematics and many can become mathematicians and/or consumers of mathematics. CPVA/S methodology helps them to achieve this.

The research on learning is clear. From mathematics to chess, even every sport, and video games—a distinguishing factor between beginners and experts in any field is that experts organize information in their brains and connect the new information with the old, constantly refine their conceptual schemas,

strategies, and practice efficient and elegant strategies. Unfortunately, we rarely plan our instruction to help reveal that organizational structure of learning, not just in mathematics but for any discipline, for our students. This organization structure should be reflected in the instructional tasks we plan and execute. Although building mastery and expertise is quite complex, we can nevertheless use a relatively simple way to think about it to yield enormous results.

First, and importantly, understand the trajectory of the development of a concept—where and how it begins, how it develops, and what behaviors one should look for when one demonstrates mastery of the concept.

Second, identify its key features (language, the component primary concepts involved, and the nature of the conceptual schemas) and the pre-requisite cognitive skills involved in learning it.

Third, select the instructional models and tools—linguistic, concrete, pictorial—that can help develop the prerequisite skills and the primary concepts effectively.

Finally, help to develop the concept using the three roles of teaching—*didactic* (direct instruction), *Socratic* (Student participation and initiatives), and *coaching* (support and guidance) that help a teacher to achieve two goals: (a) improving the content level and mastery of the student, and (b) improving a student's learning potential and learnability.

Effective teaching combines *direct instruction* (teacher-directed tasks, creating settings for learning, clearly defined outcomes with discernible, measurable behaviors), deep and meaningful *student involvement* (discussions and exploring and using concrete models, interaction—discussion, healthy give and take, heavy doses of a range of questions and proper mathematics language usage, and exploring and using effective, efficient, and elegant concrete/pictorial models), *constructing and developing* language of concepts, strategies and their scripts, and procedures with student help, *organized and effective practice* (appropriate classroom practice—group and strategic individual or dyad (in pairs), scaffolded, supervised and then individual), *effective, diverse assessment strategies* (instant feedback to exploit error-driven learning, prevent erroneous information from entering the brain, timely and intermittent self- and peer, formative, and summative assessment), and *meaningful, appropriate, and timely homework* (homework that contributes to learning and develops responsibility and is useful to new learning, not just 'busy work' (and we must remember that homework is not supervised by us).

Each component in this process is important. It means helping them to learn ways to derive and learn efficient strategies that help them in deriving facts, strategies, language, and procedures. It also means helping them become better learners—more productive, more effective, more efficient, and with an attitude that is conducive to learning with interest. This means learning and memorization techniques for arithmetic facts, making connections, seeing patterns, acquiring study skills and metacognition—so that learners *identify strategies that help them to learn*. These are important principles in instruction for all children, but particularly for students with learning-difficulties in mathematics. It is adhering to the *six levels of learning mathematics*.

We assert that the clinical experiences where future and new teachers have the experience of observing and trying practices with experts must be the starting point of teacher education. Professors of education should demonstrate to the future teachers who will be teaching children in real schools, not just in wonderful clinics, at the universities, rather than real-life classrooms and visiting schools occasionally. The medical training model is a good example. Also, special education trainees get to see successful tutors working with children. In these settings they observe and understand how student learning becomes the focus for the outcome. The lessons described in this book, using the CPVA/S,

stress certain processes and key dimensions of pedagogical knowledge that are used in the development of numeracy, which are to:

- a. understand and know the clear and concise parameters of the content—its language, concepts and their schemas, and procedures—informal and formal, and their trajectories of development from pre-K to second grade and beyond.
- b. know the sequence of tasks, activities, and experiences to be provided, practiced, and emphasized in the classroom that demonstrate the understanding—the nature of children’s errors, behaviors that indicate a lack of or a presence of mastery of learning.
- c. know the processes needed for deepening and connecting the learning (both horizontally—concepts at the grade level, and vertically—concepts before and after the grade level) by showing commonalities and differences.
- d. have a deep understanding of the role of instructional tools—their role, their strengths, their limitations, and their congruence with mathematical representations in different forms—concrete, pictorial, mental, and abstract levels, and their interactions. Do they work for you? Do they work for this child? Do they work for all children?
- e. have a clear understanding of the role of language and questioning in learning and teaching.
- f. place due emphasis on how to reach all children *with and without learning disabilities*, and
- g. provide and practice strategies that *motivate* (providing success and the conditions for success are the greatest motivators), *inclusion* (beginning with an effective concrete model engages all children, including children with learning difficulties, including dyscalculia), and *differentiation* (asking differentiated and diverse forms of questions to each student related to the concept being taught is the best way of differentiating and motivating children).

Improvement in problem solving and investigative tasks with students which, besides promoting concept understanding, reasoning, and communication development, promote making connections among ideas and concepts.

We believe that the language related to a concept, the associated conceptual schemas, and the ensuing strategies, facts, and procedures emanating from the language and concept are integrated and mastered with each other’s help, rather than in isolation, that help produce results in learning and build confidence.

All arithmetic/numerical operations—formal and informal—are dependent on three key fundamental concepts: *number concept*, *arithmetic facts*, and *place value*. The integration of these three is true numbersense. They are mutually inter-dependent. These are the three areas where individuals with dyscalculia, acquired dyscalculia, and other numerical difficulties show deficits. When children only have a narrow and rigid conception of any of these topics and lack mastery of them, the condition can hinder their later mathematical progress in becoming fluent when executing arithmetic procedures and generalizing, and applying them to algebraic and geometrical applications.

We strive to improve problem-solving and investigative tasks with students which, besides promoting concept understanding, reasoning, and communication development, promote making connections among ideas and concepts. On the other hand, it prescribes the planning of teaching sequences and encourages the sharing of teaching experiences among teachers.

This book is not only for teacher training but also for planning lessons, executing effective lessons, designing assessments, and monitoring progress within this structure. We feel very strongly that the book is also for researchers to review the numerical and mathematics concepts in the light of this

information. That should motivate educators to explore the research questions that link theoretical questions about numeracy to practice situations, thereby helping to develop a science of numeracy similar to the progress we have made in developing the science of reading. The following describes our pedagogical and numeracy principles that we think can effectively guide instruction to achieve the goals, described in previous chapters as part of a rigorous and viable curriculum, during kindergarten to second grade and beyond.

Each one of these components is important for higher student achievement in numeracy and mathematics. It means, the instruction should be such that it helps learners to find ways to derive and learn efficient strategies that facilitate in generating the *language of the concept* under discussion, creating *conceptual schemas* for that concept, and deriving *facts, strategies, and procedures*.

It also means helping children become better learners by helping them to improve their cognitive and executive skills. Instructional processes should focus on learning, rehearsing, and on techniques for mastering arithmetic facts, making connections, seeing patterns, acquiring study skills based on executive skills and metacognition—so that learners identify strategies that help them to learn.

The fundamental principle that guides our pedagogy is based on the idea of *neuroplasticity* and *human modifiability*. The brain is constantly prioritizing what it thinks is important and letting us forget what is not. We tend to remember the events associated with intense emotions. Our emotions, as well as the actions and choices they influence, are shaped by events that intensely activate neural circuits that are related to important goals, such as *gaining success* (thriving) or *failure* (missing the goal, or loss of self-esteem), because they provide valuable information that we can use in the future to repeat that success or avoid that failure or humiliation.

When a circuit faces failure—failing a test or not being able to answer questions—such as facts in the classroom, the brain re-experiences the trauma repeatedly, at any instance where quantitative information is encountered, and it may in many cases evolve into *mathematics anxiety*. When we recall a traumatic experience, we do not just remember what happened, we re-experience the sensations vividly and with immediacy. Mathematics anxious students have described these humiliating experiences with vivid details and deep emotions. Their memories are stored up in their bodies as well as in their minds.

Similarly, when such a circuit in the brain is revved up due to success, our brain gets flooded with *neuromodulators*. These neuromodulators want us to remember and experience that success again. Once again, the remembrance is not just that of the content and related success, but also of the circumstances and the emotions that were related to that experience.

The power of emotion's influence on memory can be negative or positive.

Emotional arousal works both ways. In both cases, the visceral experience of remembering emotional events hinges on a key part of the brain: the *amygdala*. The amygdala has a close line of communication with the different neuromodulatory systems in the brain, such as peripheral glands and the hippocampus.

Amygdala (the seat of emotional experience—how to feel and the outcome —feelings) and *hippocampus* (the cognitive content of the experience—what and how much) work as a team. They work in close collaboration. The hippocampus forms the memories that capture the context at that moment—the concrete and pictorial activity and the related experiences—the amygdala is connecting those memories with survival circuits that generate the raw sensations and order neuromodulators to be released, and they do their job: turning us away from learning mathematics or asking for more of it.

The division of labor between the hippocampus and the amygdala suggests that there is an important connection between *what* we remember and how we *feel* when we remember it.

To form new memories about mathematics or to recover from past failures (i.e., mathematics anxiety), we need to pay attention to both the *content*, how it is *taught* and how it is *learned*, and under *what conditions*. The recovery from mathematics anxiety is not about just forgetting the past but rather about managing the intense emotions brought about by those past failures and how those emotions about learning are converted into new learning and new strategies that are more productive, and that result in positive emotions.

Neuromodulators are chemicals that influence the functioning of neurons, but they do not simply increase or decrease neural activity. Neuromodulators fundamentally change how neurons process information. The success of related neuromodulators causes us to sit up and pay attention. Neuromodulators also promote *plasticity*, meaning they enable significant long-lasting changes in the connections between neurons in the cell assemblies that are activated when we learn something new. The same cognitive task may bring about changes in the neural network again, or even in a different focal area of the brain, nearby, or in a related network in response to a meaningful experience, particularly, when the learner tastes success through a challenging task.

We have seen many children become very good mathematics students when they are taught with strategies that help them succeed in learning concepts that they could not learn earlier. [Butterworth \(2025\)](#) calls it '*the learning brain*.' He says: 'The structure of brain regions and their connections change in response to experiences, including learning.' The tasks that are simple and may qualify as a habit and do not involve frontal lobes, as there may not be any decision making involved and this may not activate many neural connections. That is why number operation outcomes based on pure counting without strategies may not improve children's thinking, because they have become pure habits.

When we read a new, long, and difficult word then we pause and call upon phonemic awareness and knowledge of already learned other words; that is when the frontal cortex is involved and new learning is taking place. Similarly, when we learn a new concept, solve a problem, write a proof to a theorem, simplify a complex expression, we break the new concept into familiar elements and try to see the patterns and try to combine the elements in different combinations (decomposition/re-composition), and we are learning and making new connections. The simplest examples are completing puzzles of different kinds.

For example, *noradrenaline* (also known as *norepinephrine*) is a neuromodulator that influences several of the components of the brain that help us in how we learn and remember. *The emotional arousal* (because of success, and also associated with novel experiences) heightens our attention, gives us better focus, and that makes us perceive better those salient things that are important or stand out in some way when we are engaged in new learning. So, emotional arousal influences the outcome of those 'neural elections' that determine what we will perceive, learn, and feel about during that learning, thus funneling resources to the strongest candidates (possible standby neural assemblies), resulting in new, qualitatively better, deeper, and quantitatively more learning. More connections and associations are formed, therefore, resulting in possibly better retrieval.

For example, take the process of mastering the table of multiplication facts of 9 right after the mastery of multiplication tables of 1, 2, 10, 5, and knowing the sight facts of 9 (what two numbers make 9 and one less than a number), and using the pattern recognition ability of the brain. To see a pattern, most times, to get into action, the brain needs at least four examples to observe and try to arrive at a pattern: because, the *first example* of a phenomenon may be just an accident, the *second*

example may be co-incident, the *third example* alerts the brain for a possible pattern, the *fourth example* finds the brain possibly confirming or forming the contours of the pattern, and after that it is expanding that pattern in most cases. Children by now already know: $9 \times 1 = \mathbf{09}$; $9 \times 2 = \mathbf{18}$; $9 \times 5 = \mathbf{45}$; and $9 \times 10 = \mathbf{90}$. By asking questions, the teacher helps the child to see the emerging patterns:

- a. the sum of digits in the products—the answer is always 9 ($0 + 9$, $1 + 8$, $4 + 5$, $9 + 0$); and
- b. the tens' place in the product is always 1 less than the number being multiplied with 9 as in $9 \times 1 = 09$; $9 \times 2 = 18$, $9 \times 5 = 45$, $9 \times 10 = 90$. Using this (there are four examples). Now the teacher asks:

Q: Use the same pattern you just observed, tell me: What might be $9 \times 3 = ?$ A: 9×3 ? I need some help.

Q: Remember the pattern in the facts you saw: 9×5 ? What is 9×5 ? A: 45. Yes! Now, I remember.

Q: Yes! Let us do 9×3 ? What are we multiplying 9 by? A: 3.

Q: What is 1 less than 3? A: 2.

Q: Good! What number makes 2 into 9? A: 7.

Q: So, what is 9×3 ? A: I think, it is 27.

Q: What is, then, 3×9 ?

A: 27!

Q: What two numbers multiplied give you 27? A: $9 \times 3 = 27$ and $3 \times 9 = 27$.

Q: What about 9×7 ?

A: 9×7 ? One less than 7 is 6 and 3 makes it 9, so the answer is 63. Is that right?

Q: Yes! Let us find the remaining facts of 9. Let us practice this on a Multiplication Ladder.

All the other facts are then derived in the same way and then practiced. The script and the pattern recognition become an important part of it.

$9 \times 1 = \overset{\text{1 Less}}{\underline{01}}$	$0 + 9 = 9$
$9 \times 2 = \overset{\text{1 Less}}{\underline{18}}$	$1 + 8 = 9$
$9 \times 3 = \overset{\text{1 Less}}{\underline{27}}$	$2 + 7 = 9$
$9 \times 4 = \underline{\quad}$	$\_ + \_ = 9$
$9 \times 5 = \underline{\quad}$	$\_ + \_ = 9$
$9 \times 6 = \underline{\quad}$	$\_ + \_ = 9$
$9 \times 7 = \underline{\quad}$	$\_ + \_ = 9$
$9 \times 8 = \underline{\quad}$	$\_ + \_ = 9$
$9 \times 9 = \underline{\quad}$	$\_ + \_ = 9$
$9 \times 10 = \underline{\quad}$	$\_ + \_ = 9$

Figure 4.4 Deriving table of 9

Intuitive → Concrete → Pictorial → Abstract → Applications → Communication

Figure 4.5 Six levels of knowing

Our brains are designed to hold onto the events that have revved us up; we remember them better and we talk about them, ostensibly because remembering those events has survival/thriving value. In modern society, as always, they provide status, importance, and become sources of pleasure. That is why adults and children become addicted to social media and video games. The more of these experiences we have, the more we want those experiences, and the more learning takes place in that area. We even seek out such learning experiences. That is why gifted and talented individuals are drawn to learning more and doing more about the areas they are interested in.

These are important principles in instruction for all children, with and without learning disabilities, but particularly for students with specific learning difficulties in mathematics. Pedagogically, it translates into our adherence to the *six levels of learning mathematics* while initial teaching (classroom level), tutoring, and during intervention and remediation. This pedagogy for developing number concepts, number relationships, and numeracy consists of teaching and learning at *Intuitive, Concrete, Pictorial, Visualization, and Abstract* levels supported by *Scripts*.

CPVA/S

The development of the CPVA/S method is based on the principles of teaching requirements identified and sound neuropsychological research on concept formation related to quantity and space, the role of cognitive and executive functions in learning, the science of language and reading development, and strategies for enhancing memory (storage- and retrieval-rich) and the timing, nature, and type of practice. However, the most important steps for initial teaching and learning in this developmental sequence of a concept or a procedure formation are:

- *Concrete* modeling of the concept/strategy/procedure

If this is done with efficient and effective instructional materials and with effective use of language and questioning, the teacher with the help of the learner develops the relevant schemas and scripts from the manipulation of this equipment—generates the language of the concepts and related conceptual models and provides a structure to ideas, applies *scripts* and learns from the experience.

It should be remembered that the purpose of concrete models in a classroom is more than just tactile involvement and engagement. The purposes and usage are threefold: To help:

- a. generate the language related to the concept (vocabulary, terminology, syntax, and semantics)—through these materials children can then talk about the mathematics,
- b. develop the conceptual schemas of that mathematical idea, and
- c. derive the strategies and procedures emanating from that concept.

If these processes are not happening during the use of concrete modeling, then that concrete model is not effective, or its use is inefficient and poor. The use of concrete models is not just for counting.

When enough understanding of the language, conceptual models, strategies, skills, and procedures has been achieved, children practice the lesson concretely in pairs. After that, the teacher asks questions about the representation of the concrete models and leads them into visual/pictorial models, and the same processes are repeated at this level.

- *Pictorial representation* of the concept/strategy/procedure

Visual and representational models that are efficient and effective are congruent to concrete models and processes. Since the pictorial/visual models are congruent to concrete models, the same processes: language, conceptual schema, strategies, procedures and the script are repeated in lessons and practiced by children in pairs. Judiciously selected *concrete* models, with proper usage, help develop the *concept* and *language* and the *sequence of activities* and begin constructing and refining the *script* for emerging strategies and the language.

The *process of transferring* from the *concrete* to *pictorial* helps the students to effectively perform the *task analysis*, *scale up* the *numbers* and *scope of problems*, and helps them complete the creation and polishing of the *script* for implementing and understanding the tasks involved in the concept and related procedures.

Concrete to pictorial representation (iconic to representational) is a very important step. If this does not happen, children remain dependent on concrete models and have difficulty in transitioning to the abstract. This is particularly so for dyscalculic children and many adults who have experienced poor initial instruction. Visualization helps reinforce understanding and the transfer from concrete/pictorial to abstract and solidifies the script. In the Bruner suggested model of instruction Concrete to Pictorial to Abstract (CPA or CRA), the transition from Pictorial to Abstract was not happening for many children. The transfer from pictorial to abstract visualization and the script are essential elements. For these transfers to be effective and smooth, the representations (concrete, pictorial, and abstract) should be congruent and the script from one representation to the other should be consistent, but more refined and succinct at each succeeding level.

Effective, efficient, and elegant concrete and visual representation tools, properly used, can help produce rich multi-sensory, multi-model, and multi-modal information that develops *metacognition* skills—‘how do I learn? what material helps me learn? how should I use them to learn? in what order should I use them to be more productive?’ In turn, those supported by effective *questioning processes* and *appropriate language usage* convert the experience into rich and meaningful learning. All these activities improve the *executive functions*: working memory, attention and focus, organization, and flexibility of thought. That is the beginning of a child’s transformation into a better learner.

- **Visualization**

Picturing the models—concrete and pictorial—and rehearsing the related *script* in the mind’s eye is practiced orally. A team of two students is asked to describe the process. One member of the team describes the process and the activity that took place during the concrete and pictorial activities, and their partner enacts the concrete or pictorial representation. The teacher, with the help of the learners then converts the actions into the abstract symbolic form, and introduces the symbols, expressions, and steps of the skills, strategy, or procedures.

Visualization takes place in the working memory—the brain's *sketchpad*. The visualization process uses and strengthens the working memory at the same time. Visualizations help reinforce the strategies and the scripts developed during the learning process. It works on the incoming information and transforms it by mixing and connecting it with the relevant related information called for from the long-term memory. This 'mixing' and 'connecting' is both vertical and horizontal. Vertical, in the sense that it suggests new and higher ideas and connects them with earlier incarnations of the concept at previous levels; horizontal as it connects with existing information and creates networks of neurons amongst related ideas at that level.

Explorations and related research in the relationship between working memory and dyscalculia in children has shown that working memory deficits are one of the key factors in their mathematics learning difficulties. Therefore, it is essential to identify effective interventions for expanding and enhancing the functioning of working memory to improve children's mathematical abilities and the performance of dyscalculic children.

- *Abstract* recording is done in the symbolic form—introducing and relating to symbols, formulas, procedures, equations, etc.
- *Script*: A well-defined set of steps to implement the procedure or strategy.

The script is the linguistic representation of the activities, concepts, language, and conceptual schema. Initially, it is a dialogue between the teacher and the student. Later, it is a dialogue between the student and themselves. Ultimately, it is a monologue by the student with himself through the models—concrete, pictorial/visual, imagined in the mind's eye by the help established script. It is received by the hippocampus and develops into an *episodic memory* and with constant usage. It is freed from the context and specific spatial settings of the learning situation or the tools. It becomes 'a knowledge unit.'

The execution of the script is directed by the *pre-frontal cortex* as it is guided by the integration of executive functions using the principal software of learning mathematics content, the questioning process. It ultimately resides in the *basal ganglia*; because of constant use it becomes a habit, and with the introduction of newer concepts and procedures it is constantly reformed and reshaped and it appears automatically in Default Mode Network (DMN) in the memory system, where all schemas are constantly reshaping, reframing, and are easily recalled. A script accompanying a calculation, procedure, or strategy is the integration of language and questioning and is 'storage- and retrieval-rich.' It is developed and associated with the language and conceptual schemas emerging from the questioning process embedded within these different level models. The language being developed with the use of concrete and visual models is very rich and helps in building understanding, efficient strategies, and conceptual schemas, resulting in effective strategies and procedures. These prepare children well for applications such as word problems. The script and the strategies provide effective support to the executive functions in learning mathematics procedures.

The script is the conceptual schema of the component primary and secondary concepts articulated as a sequence of steps to execute the strategy or procedure. Effective *scripts* are the integration of the language, primary and secondary concepts, and the steps involved in a procedure tied into a series of prompts in question form that can be remembered and executed easily.

Scripts are developed by using the fundamental principles of numeracy: decomposition/recomposition, number concept (sight facts, making tens, teens numbers), next ten, place value, and the

properties of the concept and operations, under discussion. For example, here is the sequence of questions that form *the script* that will be used in the process of finding the sum of 8 and 7.

Let us find the sum: $8 + 7$? Is the answer (sum) bigger than 10 or less than 10? More than 10.

Which is the bigger number, 8 or 7? 8 How do you make 8 into 10? Add 2. Where does 2 come from? From 7.

When you take 2 from 7 what is left of 7? 5 What is $8 + 2$? 10.

What is $10 + 5$? 15.

What is the sum of $8 + 7$? 15.

What is the commutative property of addition? Yes! Changing of the order of numbers in addition does not change the answer. $8 + 7 = 7 + 8$.

What is the sum of $7 + 8$? 15.

What plus 7 is 15? 8.

8 plus what is 15? 7.

15 is equal to what plus what? $8 + 7$ or $7 + 8$. Great!

It is then expanded to $28 + 7$ or $88 + 7$ and for a child who is ahead, $128 + 7 = ?$ In the sequence, if the child or children answer any of the questions incorrectly, the digression will add extra questions to clarify the error or re-teach some earlier concept(s) that were not mastered properly and fluently.

Once the script for a strategy or a particular procedure has been developed, it can be practiced on an abstract level, first under the teacher's guidance and then in pairs. Many people believe that after that, children should practice independently. It is true that students need independent practice on simple arithmetic facts and other basic formulas and rules to learn and automatize these facts. And some even believe that practice should be rote practice. We strongly believe that the initial practice should be done orally so that the children can get immediate feedback to their responses and recognition of their efforts. Corrections, misconceptions, and errors are immediately corrected. This is particularly the case before they can access demanding arithmetic facts. This belief comes from our work with children and from the evolutionary perspective. According to the evolutionary perspective, our hearing comprehension is more developed than our visual decoding and encoding abilities, particularly for novice learners. Chinn researched using 'self-voice echo' in the 1980s and found it efficacious for some children, but not all. We have found that when we ask about arithmetic facts aloud, our students' brains are bathed in rich scripts and strategies for deriving the facts that they might not otherwise have access to. Not only that, children themselves also come out with very good strategies during these discussions. However, for this to succeed, the choice of strategy for fact derivation is effective and efficient. Moreover, if the accompanying questions are asked with proper expertise and with clear and comprehensive task analysis. After that children should practice in pairs with the full script with problems posed at the abstract level by the teacher and each other. With understanding, effective strategies, practice, and applications, information is placed in the memory networks located in the appropriate places in the brain.

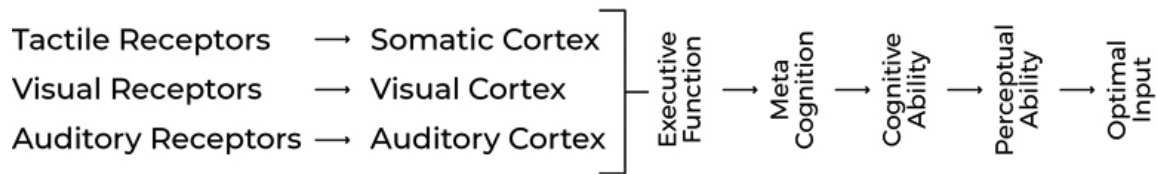
For example, different components of mathematics learning are stored in different parts of the cortical areas of the human brain. A learner attends to learning tasks readily if they are presented through concrete models and their manipulations are easy to learn, and are colorful, spatial, multi-sensory, and transparent in their outcomes. To address the various concepts (from counting to measurement), the concrete models¹² should be a combination of: (a) *discrete*, (b) *semi-discrete* (pattern-oriented), and (c) *continuous*.¹³ Interesting tools attract the attention of students more easily,

and when they notice them, the information enters short-term memory more easily and in a richer form. Students become engaged, attentive, and focused on the richly generated information. In such situations, concrete modeling of mathematical concepts becomes *a multi-sensory experience* (especially when closely linked to the symbols involved).

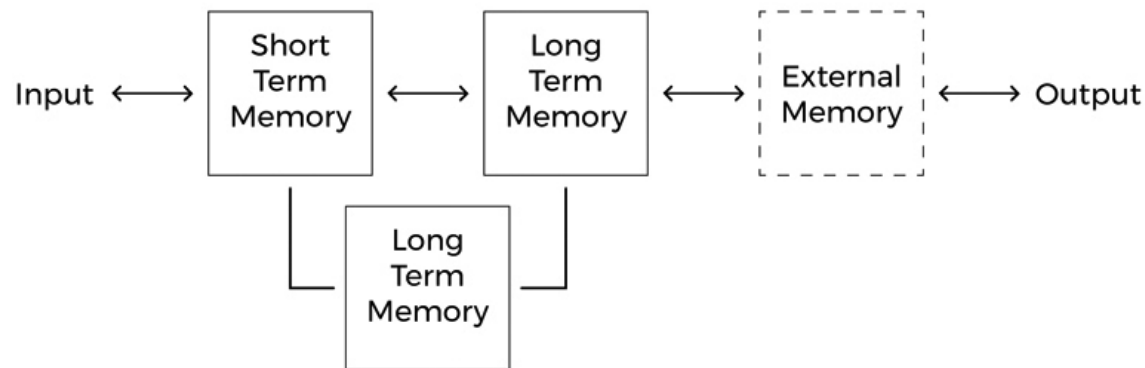
Multi-sensory information enters the short-term memory easily and the resulting inputs to the brain reach multiple areas of the cortex. For example, tactile and kinesthetic receptors send information to the *somatic cortex*, *visual receptors* to the *visual cortex*, and *auditory receptors* to the auditory cortex. When the multi-sensory information is combined with *multi-modal* and *multi-model* forms, it becomes more interesting and attractive—richer in content and appeal. When such rich information reaches the brain, executive functions become active and decision making is taken over by the frontal cortex. Multi-sensory, multi-model, and multi-modal information reaches the short-term memory, and a triage begins to take place: The information trifurcates in three specific directions—to *long-term memory*, *working memory*, and some *dissipates*. This trifurcation is facilitated by paying self-attention to the information, the teacher's questions, or by personal and social need. The teacher plays the biggest role here. Moreover, the optimal information in the *short-term memory* (the multi-sensory, multi-modal, and multi-model information) is held in the short-term and working memory easily, more in quantity, quality, and can stay active longer, the trifurcation becomes easier.

When more than one model of a concept is involved, for example, a collection of 5 random objects shown, called upon, or pointed to on a 5-domino, 5-dice, a cluster of 5 objects, a 5-card, fingers of one hand, or combination of fingers from two hands (2 fingers and 3 fingers), 5 unit number-rods, the 5-rod with 5 markers on it, Visual Cluster card with a cluster of 5 hearts, 5 hash-marks, 5 beats on a drum, 5 drops of water, 5 birds flying in a formation, etc., the information becomes multi-model and multi-modal. These models range from discrete collection of objects, in unorganized fashion to a pattern of, for example, 5 objects on a cake with the candles organized in a pattern, dominos, dice, and visual cluster cards, are semi-discrete because of patterns and grouping, a 5-number rod is a continuous model. *Because of the different models, the information is a multi-model for perceiving and understanding the concept of subitizing. It becomes a perceptual activity rather than a counting activity.* And since we (the teacher and children) are using multiple modes of communication, it is *multi-modal*. A *mode* of communication is something that can be socially or culturally shaped to give something meaning. Images, pieces of writing, and speech patterns are all examples of modes. Multi-modal means using more than one mode or method to convey information or complete a task.

1. Maximize Input to the Short-term memory



2. Maximize Transfer from Short-term memory to working and long-term memories



3. Maximize Output/Retrieval from Working and Long-term memories

Figure 4.6 Optimizing learning cycle

Once the information reaches the short-term memory, by the nature of the concrete materials (because of the shape, size, tools and models used to depict concepts, strategies and the script developed and used), it is easier to visualize them in the memory system.

- The Long-Term Memory route:* If some related information is already present in the long-term memory, the relevant part of the new information is recognized by the cues in the information, by the learner and those information pieces/components of the new and the old get connected. The teacher may provide some cues about prior knowledge that might be related to this information or ask questions that might direct the learner in that direction. When paid attention to, the new and old information automatically or intentionally interact. As a result, the old information—knowledge base, conceptual schemas, structures—are strengthened, extended, reformed, or replaced.
- The Working Memory route:* Some of the information is sent to the working memory based on interest, inquisitiveness, or need to make use of it. By the help of executive functions and the teacher's questions and language usage with the children, the mixing of new and old information begins to take place and new patterns emerge, new concepts, new schemas, and new relationships are observed and formed. Relevant and needed information is called from the long-term memory centers and mixed with the new information and new concepts, then facts, and procedures begin to be formed. The concrete and pictorial models help the student to visualize the beginning of the formation of new knowledge.

This new knowledge begins to emerge clearly and is even abstracted with proper guidance. The life and form of this newly formed information is dependent on practice and usage. With discussion and usage, the new knowledge is converted into usable language, concepts, schemas, skills, strategies, and procedures with proper scripts. When the script and the whole schema can be visualized, understood, and the script can be rehearsed, the information in the script form is easily transferred to appropriate places in the brain's storage areas of long-term memory. With the help of different functions of the *pre-frontal cortex*, it begins to be transferred to appropriate long-term memory places.

Ordinarily, when the information enters the short-term memory and is brought to the working memory, the presence of new information, then executive function activities of focusing, holding, organizing, manipulating, and comparing and contrasting begin to take place. It is a mutually beneficial relationship between visualization and executive functions. Better executive functions improve essential skills for learning, and the process of learning itself and visualization improves executive functions. As a result of this symbiosis, a cascade of other activities also begins to take place, such as *metacognition, improvement in memory functions of different kinds improvement of perceptual and other cognitive activities happen and one feels empowered*. This provides the opportunity for the pre-frontal cortex to bring out related information from the long-term memory sources to the working memory. The teacher's questions and input from other students keep these processes active and new relationships and connections are being made.

- c. *Self-destruction*: Other components of the incoming information to short-term memory may not have any related information in the long-term memory that either recognizes it or has any connections with it and no attention is paid to, or no actions are taken by the receiver, then it is defused and lost.

Some new information is so powerful, interesting, arresting of attention, emotion-laden, or essential for survival or growth that some component of the brain will focus on it.

However, most new knowledge—a fact, strategy, language element, concept, or procedure—is formed when one is intentionally prepared and pays attention to the incoming information. Paying attention means the person brings this information intentionally to the working memory space, holds it there to examine it, try it, recognize it, and manipulate it, then shapes it or mixes it with the existing information and new learning takes place in some form.

On the other hand, the person may recognize it, but the previous negative experiences related to this information impel the learner to discard it or avoid it. Then no learning takes place.

The concrete models help develop both *episodic memories* (because of the active processes, and guided and voluntary actions are episodes) and *semantic memories* (the conceptual schemas, strategies, knowledge, and scripts that come out of these actions). The *conceptual schemas* are developed by actions guided by the questioning process and student interaction with each other and observing actions and strategies of others, particularly 'experts'. The facts and formulas are derived in the working memory, but must be practiced for understanding and automatization, and when practiced and applied they become like habits. The pre-frontal cortex is the executive which manages all these functions and the interplay of language, concepts, and actions. All these activities are the components of learning and teaching in the methodology of *CPVA/S*.

Executing the sequence of *CPVA/S* for a strategy or procedure is the key to reaching all children, including those with dyscalculia and/or dyslexia, in a mathematics class, in individual settings, and during intervention sessions.

These processes and instructional practices are coordinated and connected by proper questioning and language usage that help develop the *script* for connecting these concepts and applying them in the future.

The following example: Finding 17—9? using CPVA/S illustrates the sequence of steps and suggested instructional materials and their role in the process of deriving the strategies or procedures for number concept and additive reasoning:

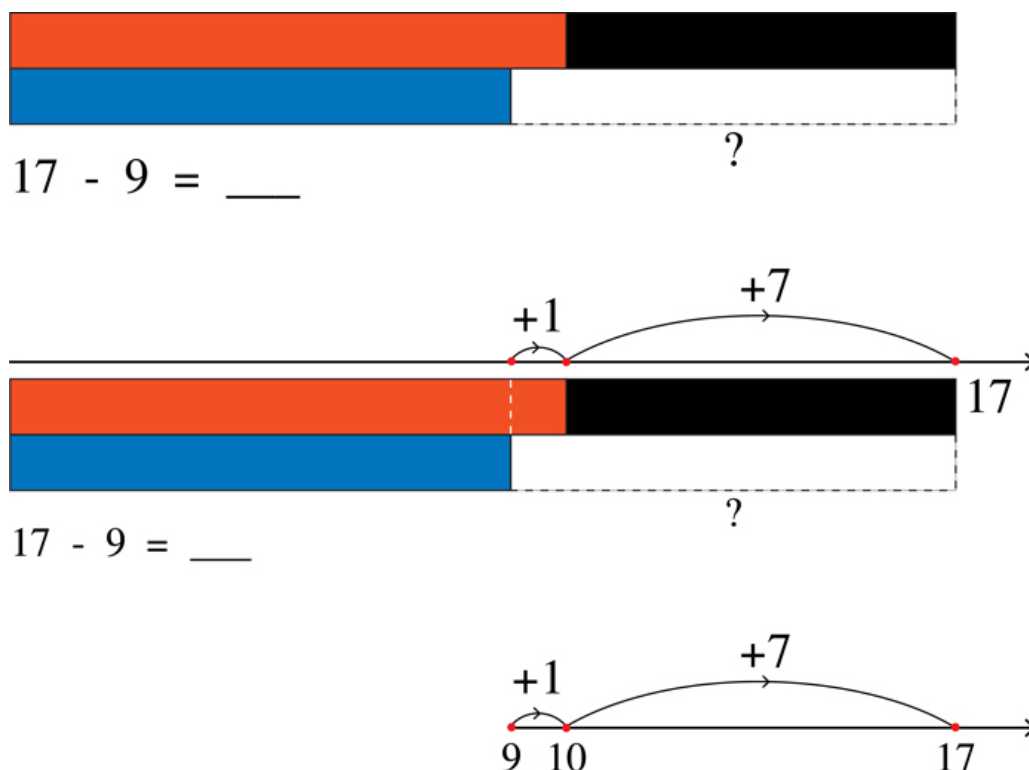


Figure 4.7 17-9 using cuisenaire rods and ENL

Cuisenaire rods or *Montessori colored rods (Concrete)*: setting the problem and the modeling of the problem (by translating the symbols into a concrete model for the subtraction $17 - 9$ by Cuisenaire rods)

- *Visual Cluster Cards* and *Empty Number Line (Pictorial)*: setting the same problem by using the Visual Cluster Cards and Empty Number Line,
- *Visualizing/Imagining Visual Cluster Cards patterns* and *Cuisenaire rods arrangements* or *constructing Empty Number Line in the mind's eye (Visualization)*,
- *Recording on paper the outcomes of previous steps using mathematical symbols (Abstract)*: recording the decomposition and re-composition of symbolically presented numbers and digits to see the result of the subtraction,
- *Practice with visualization* and at an *abstract level* using the *script* developed for a particular concept during these activities from Concrete to Abstract.

2. Role of Memory in CPVA/S Model

The role of memory in learning is a key factor. We believe the quantity and quality of learning can be defined as the amount and type of residue of an experience left in the memory system. Mathematical abilities have been shown to rely significantly on working memory, particularly executive processing, fluid reasoning, visual processing, processing speed, and planning. Neuropsychological and other brain research shows that different components of learning and different types of resultant memories of the experiences reside in different locations in the brain. [Swanson \(2008\)](#) found that children with all types of learning disabilities exhibit poor working memory performance, particularly in verbal and executive working memory—an important component for mathematics learning, as the literature suggests that working memory significantly contributes to math skill performance in areas such as counting, magnitude understanding, simple arithmetic, word problems, and procedural knowledge. Students, teachers, and parents all complain about not being able to remember learned material. In our teaching we have paid special attention to the memory problems that students from elementary to university levels face in learning mathematics.

Consequently, the processes involved in CPVA/S, particularly in the working memory space, are extremely important. The actions in the working memory, by the help of visualization, questioning process, language usage, and decomposition/re-composition, facilitate the transfer of the newly transformed information by interacting with the information received from long-term memory. The activity at the concrete/pictorial level during the process of mixing in the working memory and the script makes the activity in the form of both *episodic* and *semantic memories* and *schemas* thus easily transfer to *long-term memory*. The *episodic memory* resides in the hippocampus, whereas the *semantic memory* resides in the frontal cortex. The abstract nature of the information (in *symbolic/procedural* form) as a *script* goes to default mode network (DMN) and the pre-frontal cortex acts on it when needed and called upon. The frontal cortex is the main actor who is making these decisions in the working memory. The pre-frontal cortex and its multiple actions, in the form of executive functions is responsible for coordinating all these activities. For example, within Baddeley's multicomponent framework, the central executive and two domain-specific short-term memory subsidiary systems are proposed to have distinct roles in mathematical calculations and applied problem-solving skills. The phonological and visuospatial subsystems are responsible for temporarily storing verbal and visual information related to math problems, while the central executive acts as a general processor that updates information in short-term memory, retrieves task-relevant information from long-term memory, and assists in transitioning between mathematical operations.¹⁴ During the various activities in the *working memory* and then in the *long-term memory*, the *executive functions help make connections*. As mentioned earlier, these connections can be both vertical and horizontal in different content areas and different components of mathematical concepts, such as linguistic, conceptual, and procedural.

For the transfer of new learning from the working memory to long-term memory to be effective, the *schematic representations* in different forms (concrete, pictorial, and abstract) should be congruent with each other and the *scripts* formed from one representation to the other should also be consistent, precise, and concise. The strength of these functions is the mark of the learnability of the individual.

We have observed that a teacher's choice of instructional materials, the nature of language used, type and sequence of questions (convergent vs divergent, probing vs encouraging, evaluative vs feedback, etc.) asked and the scripts developed during the conceptualization of a mathematics idea—concept, strategy, or a procedure to determine the effectiveness of instruction—from initial to interventions to remediation. That also determines the strength and versatility of the conceptual schemas, level of understanding, and the transfer of the information to the different systems of the long-term memory.

Thus, the strength and effectiveness of instruction is dependent on the choices of activities for appropriate purposes and in appropriate sequence and occurrence.

In summary, for example, the *script* derived, formed, and generated through teaching/learning activities help integrate the relevant language, concepts, strategies, conceptual schemas, facts, and procedures, and help transfer the outcome to long-term memory. It happens in three forms: (a) through the script by its repetition, due to its constant usage, becomes like a habit and transfers the result to the *basal ganglia* and resides there, (b) the script, because of its connection with activities at the concrete level, acts like a story and as such is seen and experienced by the brain as an *episodic memory*; as a result, information also goes to the *hippocampus*, (c) the script is also like a schema and is easily adapted from situation to situation; as a result, it resides in the *Default Mode Network* (DMN) and constantly reforms the schemas even when the mind is at rest.

Since the script is reached by using reasoning and decisions are made constantly, at several levels: concrete, pictorial, and visualization and appropriate models and language usage, the involvement of the *pre-frontal cortex* is a key factor in its success.

All these factors produce success, and therefore, strong memories and connections are formed. Involvement of so many components of the brain assures that the related executive and cognitive functions are being developed and are strengthened. Thus, the student becomes a better learner and user of the learned material.

In the development of CPVA/S, we have tried to incorporate these elements in instruction, intervention, and remediation activities. For all these reasons, CPVA/S is an efficient, effective, and elegant way of making all of this happen. The choreographer of all this is a well-trained, effective, and responsive teacher who, by the help of CPVA/S, mediates with the child, creates and helps transfer all these functions to the child. In the process, children not only learn rigorous mathematics at all levels, but they also become better learners, thereby improving their mathematical thinking and developing their enthusiasm for future mathematical tasks.

Effective questioning and proper language usage by the teacher and children help develop the *script* efficiently. However, the teacher's weaving of the script into her teaching with other strategies and concepts is the key. Such teaching should include:

- *Sequencing and task-analysis* (breaking down the task into its parts—fundamental components: primary concepts and prerequisite skills) and then by synthesizing the parts into a whole, at the same time providing step-by-step prompts as scaffolding.
- *Repetition and practice* (automatizing arithmetic facts, daily testing on informal inventories without penalties, providing immediate feedback, recalling an appropriate efficient strategy for correcting the error, rehearsing the sequenced script, the interweaving of facts and establishing relationships between concepts and procedure, and making connections between prior knowledge and new learning).
- *Socratic questioning* and responses (structured convergent and divergent questioning where the teacher asks process or content questions to scaffold learning and develops scripts for tasks and steps under guidance),
- *Control of task difficulty* (the teacher provides scaffolding and necessary assistance for tasks that are sequenced from easy to difficult, interspersed with learned material).
- *Proper use of technology* (only after mastering facts, concepts, and estimation can technology be used for discovering patterns and relationships between facts, concepts, and procedures; however,

extensive use of technology in the earlier years is counterproductive as the child needs multi-sensory, manipulative experiences for the development of many of the cortical areas),

- *Teacher-modeled problem solving* (before collaboration activities and individual practice: teacher modeling as an integral part of instruction, then teacher and class modeling together, after that student pairs¹⁵ modeling, individuals modeling for class, and individual practice for personal competence),
- *Strategy cues* (reminders and scripts to be used in strategies and their usage),
- *Making connections* (each concept and procedure after it has been learnt should be connected to other related concepts, procedures, and skills), and
- *Frequent practice and assessment* of tasks to improve accuracy, fluency, recall, performance, and self-esteem.

In the case of students, where individual interventions and remediation are warranted, extra elements are added. For example, every remedial/instructional intervention session should have the following components, which are constantly choreographed by a trained practitioner in CPVA/S:

- *Developing the prerequisite and support skills*
- *Learning arithmetic facts orally with the appropriate scripts*
- *All activities are introduced with effective language, schemas, and models*
- *Visualizing problems and information*
- *Verbalizing and recording procedures and estimation*
- *Helping the child form story problems relating to the given concept*
- *Counting forward*¹⁶ beginning with a given number e.g. begin with 53 and count forward by one's, by twos, by fives, or tens, hundreds, thousands (K to grade 2),
- *Counting backward* beginning with a given number, e.g., begin with 97 and count backward, by 1s, 2s, 5s, and 10s. By the end of second grade, a child should be able to count forward and backward by 1, 2, 5, 10, and 100 from any number to any number, etc.
- *Showing patterns* of number facts, e.g., $4 + 4 = 8$, then $4 + 5 = \underline{\quad}$; $8 + 10 = 18$, then $18 + 10 = \underline{\quad}$, $78 + 10 = \underline{\quad}$, etc.
- *Given an addition equation (or a fact), write all the other related addition, subtraction, and missing addend equations.*

The three components of a mathematical idea: *linguistic*, *conceptual*, and *procedural* should be carefully included in all mathematical instruction. Before the student performs any arithmetic operation, he should be asked about the language, procedure, and the conceptual model of the problem that he plans to apply. When he has completed the problem, he should be asked whether he knows any other problems that are like the one he just completed.

When the initial teaching in a classroom (Tier I) is problematic—inefficient and ineffective—there the mastery of non-negotiable concepts and skills will not be achieved by a group of children, in that grade. This condition has definite impact on future learning by that group of children. In the future, with effective teachers, it may take at least three years to make up that gap. For example, if the concept of addition is not mastered by children in first grade, the deficit will continue to affect these children's mathematics future achievement. These children begin to be treated as Tier II students. That means their difficulties are not due to any learning disabilities, but gaps in previous learning.

The difficulties experienced by this group of students (Tier II)¹⁷ are because of environmental reasons (poor teaching, inefficient instructional materials—poor or inappropriate books—ineffective and inefficient instructional manipulatives, lack of appropriate practice, low expectations, poor classroom management, poor monitoring, lack of timely constructive feedback to the child, poor sequencing of tasks, lack of differentiation, limited time of instruction received by individual children because of grouping, and frequent absences, apathetic parents, etc.). These factors are under our (teacher and school) control (except for the last two). We can reduce or eliminate these factors, thereby minimizing the impact of these factors on children's achievement in mathematics. Thus, initial instruction is very important for children and for teachers. And that initial teaching must be quality teaching. The criteria of quality instruction in a grade is that no new Tier II students are created during the year. The criteria of interventions are that no Tier II students are going to the next grade.

A great deal of emphasis is placed on the curriculum and its quality, but not enough on the quality of initial teaching and the quality of interventions. A regular mathematics education program is called effective when it is implemented in every classroom of the school in such a way that it does not produce Tier II students and the problems of Tier III students also have the focus and attention of the teacher in the classroom instruction—in the teaching and learning processes. We should remember that 'a map is not the territory.' A great curriculum is great only when it is implemented with fidelity and rigor and reflects the learning needs of each student daily. Otherwise, it is just a collection of documents.

The progress and success of an adopted mathematics program/curriculum or pedagogy in a school should be monitored to see if the number of Tier II students is being reduced or not. This should be a criterion for the retention and continuation of any instructional program. The training of special education teachers should be in the same strategies as the classroom teachers engage in. It is also very important that they have strategies that are more refined than the regular teacher. At present, both groups need in-depth and on-going professional development. A chain is as strong as its weakest link. When my personal physician sends me to a specialist, I always ask: 'Does he/she know more than you? Does s/he have special training and expertise in this area that will augment your knowledge about my condition?' If the answer is yes, I will go or he convinces me to go. If the special education teacher or interventionists do not know any more than the classroom teacher in mathematics content, strategies, and know-how, then that special education teacher cannot make up the gaps.

Both Tier II and III students need and should be provided with extra intensive remedial mathematics work outside of the mathematics class from the beginning of the academic year, in addition to the classroom instruction and involvement. However, Tier II students suffer from gaps in their academic areas due to environmental factors; therefore, the extra support is needed only till they catch up. This intervention work should focus primarily on making up the gaps and preventive teaching to minimize the impact of the current gaps on the emerging arithmetic skills. It is also to ensure that the teaching is such that future gaps do not occur and the current gaps do not increase in nature and impact. This extra support should be provided by special educators who are trained in working with students with learning disabilities of different kinds in mathematics, including dyscalculia, not just as special educators. They should know the etiology of and remedial strategies for meeting the needs of students with mathematics learning problems, such as dyscalculia, acquired dyscalculia, and dyslexia.

During the extra sessions, only effective strategies (based on learning sight facts, making ten, forming teens numbers, and decomposition/re-composition) must be used, as emphasized in the regular classroom (assuming, of course, that the regular education teacher is using effective, efficient, and

elegant strategies).¹⁸ Repeating counting strategies to derive number facts or procedures is not going to achieve any of the intervention goals.

Tier III students, on the other hand, have gaps in their preparation, cognition, or achievement due to both environmental factors and due to their own learning disabilities and/or learning differences. A student's particular and specific disability may affect his/her learnability. It may affect (a) the learning skills, and (b) the prerequisite skills needed for effective learning of mathematics, and (c) the rate and quality of their progress.¹⁹ These students need extra specialized instruction that addresses their needs. This extra instruction should be provided in addition to and outside of the classroom. It should be provided by a specialist, in addition to the characteristics described in the previous paragraph. The interventionist should also understand the developmental trajectory of concepts, the nature of learner differences in mathematics, and is constantly in touch with what is happening in the classroom. The work they do is preparing the child for classroom instruction by helping the child attain the prerequisite and early numeracy skills needed for learning the current classroom topics helps fill the gaps, reviews the mathematics concepts needed for the classroom topics, and provides practice in select exercises with immediate feedback for the correction of errors in real time. They also weave in executive function skill preparation in the practice by the selection of exercises, asking appropriate questions and feedback.

G. Role of Learning Skills in Mathematics Achievement

The aim of any classroom instruction and intervention is twofold: Helping students to

- a. acquire the appropriate content with mastery and flexibility, and
- b. become better learners by focusing on the development of cognitive and executive functions, remembering that not everyone becomes an effective learner easily and automatically.

In the context of learning mathematics, the learning skills could be divided into three categories: (i) General cognition related learning skills; (ii) pre-requisite skills for mathematics learning; and (iii) content-specific learning skills.

Therefore, along with the planning of teaching the content, a teacher must also plan to include strategies to help students in acquiring the appropriate learning skills. The important general cognition-related learning skills are:

- *Executive functions*: Organization skills, working memory, attention and focus/inhibition control, and flexibility of thought,
- *Memory functions*:²⁰ Maximizing the reception of information to short-term memory (STM); transfer of information from short-term memory to long-term (LTM) and transfer and consolidation of information from working memory (WM) to long-term memory. Thus, the aim of instruction is to maximize the transfer from STM and WM to LTM and then help the student mix and interact the new information with previously learned materials from the long-term memory, and maximize the retrieval or relevant prior knowledge from the long-term memory to working memory for use in learning (maximize the output),²¹
- *Perseverance*: Stick-to-it-ness, perseveration
- *Resilience*: Emotional and coping skills with failure and difficulty, and
- *Task analysis*: Ability to break a task into its primary components.²²

Effective use of these skills makes a student a better learner. Other skills that are necessary for effective mathematics learning are prerequisite skills. The *prerequisite learning skills*²³ for effective learning of mathematics are:

- Ability to follow sequential directions,
- Spatial orientation/space organization,
- Pattern analysis: ability to (i) recognize, (ii) extend, (iii) apply, and (iv) create patterns (mathematics is the study of patterns),
- Visualization and visual perceptual integration,
- Estimation,
- Deductive, and
- Inductive reasoning.

Since numeracy, as mentioned earlier, is cumulative, the number concept and numbersense pre-skills for learning addition and subtraction facts and procedures are number concept (difference between digit, numberness, numeral, and number),²⁴ decomposition/re-composition of number, 45 sight facts, making ten, teens numbers, next ten, and decomposition/re-composition (both canonical and non-canonical decomposition) of two-digit numbers or more.

Strategies, particularly efficient strategies, help a child to acquire mental arithmetic skills quickly.²⁵ They help a child make connections between concepts and skills and between different skills. For example, children learn the ‘double of a number’ very early on as their symmetry in visual representation of objects; in daily usage, things occurring in doubles abound. Children observe that even numbers as outcomes are ‘good numbers’ and the doubles (two hands, two ears, a pair of shoes, etc.) abound in their environment.

When the child knows the doubles strategy, she knows 10 important addition facts: $1 + 1$, $2 + 2$, $3 + 3$, $4 + 4$, $5 + 5$, $6 + 6$, $7 + 7$, $8 + 8$, $9 + 9$, and $10 + 10$. The first five facts are learned sight facts of numbers up to 10; therefore, they should already have been mastered in kindergarten. If they are not mastered, then they are best learned by the help of Visual Cluster Cards and Cuisenaire rods, and they should be mastered first before the mastery of the last 4, and interventionists must make sure to do that. The double number mastery can easily be extended to ‘near doubles’ strategy. That means: Since I know $7 + 7$ is 14, then, $8 + 7$ will be one more than the double of 7, or I know $8 + 8 = 16$, and therefore $8 + 7$ will be one less than the double of 8. Knowing near doubles and the commutative property of addition enables the student to know 18 more important facts: $2 + 1$, $1 + 2$; $2 + 3$, $3 + 2$; $3 + 4$, $4 + 3$; $4 + 5$, $5 + 4$; $5 + 6$, $6 + 5$; $6 + 7$, $7 + 6$; $7 + 8$, $8 + 7$; $8 + 9$, $9 + 8$; and $9 + 10$, $10 + 9$. With that many facts known, the child is in a good position to master the others by relating them to these facts.

The following chapters will illustrate the application of CPVA/S for *Teaching Additive Reasoning: Number Concept, Addition and Subtraction Facts, Concepts of addition and Subtraction, Strategies, and Procedures*.

H. Classroom Management in a CPVA/S Classroom

A classroom is a community with a purpose and a goal—the whole development of its members. Every community is a dynamic whole with evolving culture. For smooth functioning, it should be noted that

that there is a developmental sequence, form, and structure for organizing activities and ‘thinking about’ and thinking in these lessons.

Within the first few weeks of the school, the teacher should clearly convey, through action, the structure and organizing principles of their teaching to children so that the classroom management is meaningful, productive, and conducive to learning. It is done by consistent (inconsistency can be confusing and de-motivating) actions, language, routines, and display of expectations. As a result, within the life of the academic year, children should have acquired the ‘form,’ ‘structure,’ ‘discipline,’ and the ‘flow’ of the lessons with certain milestones of academic life.

The presence of these elements in every lesson and other activities in the classroom instigates purposeful thinking on the part of every student.²⁶ The success emerging from well-defined and executed processes builds motivation and children begin to think of learning and problems in structured ways. The integration of these activities represents the *triune brain functions*.

The triune brain theory is an early evolutionary theory of brain development that suggests that the brain is made up of *three distinct layers* that function independently and interdependently. They interact and learning is facilitated and accelerated with the interactions of these functions. The theory was introduced by Paul D. MacLean in the 1960s, but modern neuroscience research has superseded and shown that it doesn’t accurately explain how the brain works. However, the model of triune brain theory provides a nice framework for lesson planning and by the teacher’s questions, activities, language usage, and organization of tasks in an effective lesson plan and delivery.

The three layers of the *triune brain* and their functions have a profound effect on student learning and organizing lessons with these elements in mind. The three layers are:

- *Reptilian or Primal Brain*: The basal ganglia, which governs basic survival, habits, and routine functions. It craves order, structure, safety, and predictability. Any learning situation requires order, structure, and a feeling of safety. ‘Maximal learning takes place in organized chaos.’ When a classroom does not have predictable order, form, structure, routines, organization, safety, and a level of certainty, the needs of the reptilian brain are not satisfied, then disorder and chaos prevail. There must be rules, but not too many and not oppressive rules. A teacher should begin with more rules (albeit manageable) in the beginning and relax them as children begin to own and display structure. The imposition of rules later is almost bound to create problems, if not a rebellion.
- *Paleomammalian or Emotional Brain*: The limbic system, which includes the hippocampus, amygdala, and cingulate gyrus and other centers that relate to and involve the emotional/affective development and satisfaction of emotional needs. In this, the individual craves for belonging to the group with respect and valid involvement. It craves responsibility with respect. It wants to contribute to the welfare of the group. In such a setting, there is constant need, even craving, for success and a reward system. When a child feels genuine membership in the group and experiences success with respect, on a regular basis, that student will be involved in the learning process willingly and will achieve.
- *Neomammalian or Rational Brain*: The neocortex, a cognitively sophisticated system where the pre-frontal cortex, frontal cortex, and other related components with its executive function roles take charge of learning activities and decision-making processes and make the integration of all activities possible that result in learning—the content and the learnability skills.

In a lesson, the teacher helps children to use the different components to interact with each other in a competitive and collaborative way, with the neocortex suppressing the reptilian brain’s primitive

thoughts when necessary. As modern neuroscience research shows that emotion and cognition are interdependent, and that the limbic system and neocortex are not purely emotional or cognitive centers, we need to provide experiences that make the integration possible.

In *CPVA/S* pedagogy, as these activities demonstrate, the aim is to help children to take charge of their own learning. Therefore, as and when opportunities arise, they take initiatives in their own learning and problem solving. They learn not only the new content but the ways of learning better—effective, efficient, and elegant strategies to make learning easier for them.

Throughout the book, we bring ideas and tools for teachers to focus on student thinking and questions and investigate how their thinking impacts their learning outcomes. The emphasis on student learning in the pages that follow will continually remind teachers how important it is for them to understand students' ideas and help to bring the visions of reform in mathematics education—curriculum and instruction, and ultimately into the lessons and their delivery in the classrooms.

We have developed not only the pedagogical knowledge directly connected with the teaching practice of a particular concept, such as additive reasoning, but also the questioning process needed for developing the proper *scripts* in developing any concept, strategy, and procedure. The process of developing and the importance of such scripts is illustrated and discussed with the goal of improving classroom instruction.

Teachers, curriculum planners, and researchers will find that this information is helpful for children to master not only the current concepts, facts, and procedures, but provides a strong foundation as preparation for future mathematics concepts. At the same time, it strengthens the mastery of the previously learned concepts. Therefore, it is not intended only for numeracy teachers of early grades, but all teachers of school mathematics, including special education teachers and researchers. It integrates mathematical knowledge and curriculum management associated with tasks and materials selection and their application in the classroom.

Notes

1. Components of rigor: (a) fidelity to the mathematics content—understanding of the language of the concept, its conceptual schemas, its developmental trajectory, with vertical and horizontal connections, arriving at procedures with conceptual understanding, and applications in diverse settings; (b) meeting the needs of children by engaging them in multiple levels of mastery—intuitive, concrete, pictorial, abstract, application, and communication. [↩](#)
2. The authors have international experience in teaching (pre-K through higher education), training teachers, and have interacted with children and teachers in more than 75 countries. They have experience in analyzing curricula in many countries, observed classroom teaching, and have demonstrated lessons in teaching in school classrooms in many countries. They have observed both excellent teaching and poor teaching in many countries. [↩](#)
3. The teacher should know the range of tools needed to develop the language, conceptual schemas, and the strategies of addition and subtraction from discrete models to continuous models. They should know what tools are optimal for developing which aspects of the additive reasoning. (See the continuum of Instructional Materials for developing number concept, addition/subtraction and beyond, in the chapter on Tools.) [↩](#)
4. **Vertical Connection** means connecting the current content form and manifestations of a particular concept to its development by grade and age and along its onward trajectory, and

Horizontal Connection means connecting a concept with other concepts at the grade and age level. Both add to understanding, rigor, mastery, and facility in its application. [↗](#)

5. More on this is in the chapters on Number Concept and Additive Reasoning. [↗](#)
6. In a classroom situation, a teacher has three roles: *didactic*, *Socratic*, and as a *coach*. In the *didactic* role they bring new and relevant information to their students' attention that they might have forgotten, and that is a direct teaching role. However, the information should be minimal and should start a conversation about the concept or procedure. In the *Socratic* role they ask questions and through questions of different kinds help to create, explore, form, and shape ideas in their minds. The questioning process takes many forms, and it is done at several levels.
Competence in the questioning process to generate ideas is the mark of a 'good teacher.' In the *Coaching* role, the teacher sets standards, guides them, goads them, provides feedback to their contributions and performance and helps them reach the standards. For details, see [Sharma \(2019\): The Questioning Process: The Basis of an Effective Lesson](#). [↗](#)
7. Children learn better when they work together—they talk, they quiz each other, they help each other, and they use more language. [↗](#)
8. It is the ability of the nervous system to change its activity in response to intrinsic or extrinsic stimuli by reorganizing its structure, functions, or connections after an experience, learning, or injury. This means, levels of cognitive functioning are alterable. It offers real hope to everyone from stroke victims to dyslexics and dyscalculics. [↗](#)
9. GEMDAS: Grouping, Exponents, Division and Multiplication from left to right, and Addition and Subtraction from left to right. G represents grouping and is indicated by symbols such as: (), {}, [], |, √; some language expressions give grouping symbols, etc. [↗](#)
10. See Appendix—for Number Games. [↗](#)
11. In any learning. [↗](#)
12. See the Continuum of Instructional Materials in the future chapters. [↗](#)
13. See the chapter on Tools. [↗](#)
14. [11]. [↗](#)
15. We believe two children working together is a powerful process. For learning to happen, students must participate in a joint productive activity, establish a shared focus of attention and participation, and ultimately develop shared meanings for concepts and tools. Within this 'give and take' process, students take over each other's ideas 'what someone else produces during joint activity for one's own use in subsequent productive activity and learning takes place for both. Each pair shows the uniqueness of this learning process.' [↗](#)
16. The choice of numbers to count should be according to grade level number work. See Sharma, M. (2017). Numeracy II: Additive Reasoning. CT/LM: Framingham, MA. [↗](#)
17. Tiers I, II, and III are classification schemes for intervention. Tier I is the regular classroom instruction accessible to all children. According to this classification system, the initial classroom instruction should be accessible and responsive to every student. Each student should be a participating member of the classroom, and every effort must be made to ensure responsive participation in all aspects of instruction. A Tier II student has missed some concepts, skills, and prerequisite skills and therefore has developed some gaps in their preparation and achievement. They need extra intervention to catch up. However, these students do not have any learning disabilities and/or specific learning disabilities in arithmetic. The Tier III students have identified learning disabilities, and they need extra special intervention by specialists in that field to reach

these children and meet their learning needs. These students need extra remedial sessions with a specialist in addition to the regular classes. [↗](#)

18. Instructional materials and strategies must satisfy three key characteristics: they should be *effective* (make the concept transparent and help give accurate answer); they should be *efficient* (give correct answers easily and quickly); and they should be *elegant* (can be generalized and extrapolated to larger numbers, other number systems, and can be abstracted to the written form easily and correctly with clear understanding). [↗](#)
19. Some Tier II students may also lack some executive functions or cognitive or prerequisite skills. Without quality intervention early on, they can easily turn into Tier III students. They need early success. [↗](#)
20. See *Memorization is not a dirty word: Role of memory in mathematics*, Sharma (1994, 2022). [↗](#)
21. The strategies for learning facts and procedures should be both storage- and retrieval-efficient. [↗](#)
22. Task Analysis (TA) is effective, efficient, and elegant if one is aware of (a) Developmental milestones of mathematics concept, (b) the trajectory of the concept, (c) primary concept involved in the concept or procedure, and (d) understanding one's students. Effective instruction is the execution of proper task analysis. [↗](#)
23. Prerequisite skills are non-mathematical in nature, but their presence in a child's repertoire is essential. Different prerequisite skills are needed for different components of learning a mathematics concept or procedure. For description, see *Games And Their Uses in Mathematics Learning* (Sharma 1990); *Prerequisite Skills for Mathematics Learning* (Sharma, 1994). [↗](#)
24. At the elementary school level, there is no difference between numeral and number; both represent quantity or outcome of counting and/or measurement. However, later, at the integer level, numeral is a representation of quantity, and number is a directed numeral—it has direction/sign and quantity. [↗](#)
25. See the Sequence of Strategies for Teaching Addition Facts in the later sections. [↗](#)
26. 'We do not live to think, but, on the contrary, we think in order that we may succeed in living and thriving.'
— Jose Ortega y Gasset [↗](#)

5 Nature of Tools in School Mathematics

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“I witnessed a miracle in education where there are few.” Said Dr. Caleb Gattegno on his first observation of Cuisenaire Rods being used by Georges Cuisenaire in the classroom. I felt the same when I saw Dr. Gattegno using them as a demonstration with children. At that time, I was not involved with school mathematics, but this tool made a great impression on me. Later when the opportunity to use them happened, I adopted them for teaching school mathematics. I have used them from number concept to algebra with severely dyscalculic and many other children, including gifted and talented. They love them equally.

Mahesh C. Sharma

Every profession and vocation is defined by its tools and its products. The performance and the creativity—artifacts, skills, and the expressed ideas are manifested through their tools. The use of one’s tools differentiates a novice from an expert. Tool building and their uses have been one of the major achievements and, at the same time, means for human survival, evolution, and success. The nature of our tools and their uses distinguish us from other species.

One version of human history can be written through the story of the need, invention, development, and usage of tools. The tools have defined and shaped cultures. The tools of culture have sustained cultures and their absence has dissolved cultures. The tools in human evolution have played three major roles: to

- a. maximize the survival and expansion of human race,
- b. develop and use unique cultural tools that bring out and give shape to the human evolutionary skills related to quantity, space, perception, memory, and expression,
- c. provide outlet, meaning, and expression to human skills and abilities, and
- d. satisfy inquisitiveness, the urge to innovate, seek pleasure, and enrich life.

All artists know their tools well and know how to use them to express their skills, knowledge, and mastery and meet the needs of others effectively, efficiently, and elegantly.

Mathematicians' tools are symbols and language. A mathematics teacher, therefore, is distinguished by his skills and tools—physical, intellectual, social, and psychological. However, for transferring their knowledge, competence, and skills to children, the teacher must use such tools that the transfer and development of skills and knowledge is easy, effective, efficient, and elegant. It is to make learning a pleasure and accessible and facilitated for all—the teacher and the students. Instructional tools properly used can transform a child's learning experience. We have spent most of our careers in this endeavor with some success.

In this chapter, we explore the need, purpose, type, and usage of instructional tools for teaching school mathematics to all children effectively, easily, and elegantly.

There is a well-documented need, and almost everyone agrees that we need to do better in teaching school mathematics. Higher mathematics achievements in schools are on every nation's list of priorities, from local school boards' agendas to legislative actions for nation building. A variety of suggestions and ideas are forwarded. At the same time, every teacher, parent, and researcher believes and advocates that children will learn better if we use better methods and tools, such as concrete manipulative materials in teaching mathematics at school level. At the same time, the education market has a teaching game, instructional tool, and computer/iPad/iPhone app out almost every day. The inventors/proponents claim that new inventions in technology will take care of all the problems of learning numeracy, if everyone was trained in it and if they used it. Of course, all these approaches will work for some children and even for many children. But tools and books alone do not teach. Some can learn from all or any of these tools, books, and technology. However, we have spent a lifetime in the field of mathematics education at all levels, and we have seen curricula, gadgets, and tricks come and go, but children's 'math problems' remain unsolved.

They say an artist and craftsman is only as good as his tools. The opposite is also true: effective teachers and great artists can create wonders out of anything. This is true of effective teachers all over the world whom we have observed with and without tools, creating wonders. The teachers in the second category are rare. We have observed effective teachers demonstrating their expertise through tools. Effective teachers using effective tools and with effective strategies can teach all children—irrespective of their limitations and difficulties.

Over the years, many approaches have been tried. As mentioned in earlier chapters, there are vast differences in the approaches to this problem; therefore, the advocacy of

different prescriptions. Many of these approaches have not worked. Teachers have been using manipulatives in their classrooms for more than 50 years—a long time to observe trends and patterns in any arena. Primary classrooms abound with materials. More manipulatives are not the answer; mindless practice on apps is not the answer, either. Then what will work? That is the discussion in the next few sections. This discussion is based on current research on children's learning and mastering arithmetic facts and our many decades of experience teaching and helping thousands of children, teachers, and parents in hundreds of schools and in several countries.

Low achievement in mathematics has its roots in the way we introduce number to young children at home and in school. Most parents and teachers make decisions about children's learning number and number relationships based on their own numbersense. These decisions are reflected in the choice of curricula—content, strategies, models, and the use of materials in teaching number. It is reflected in their sense of what the mastery of number means and how to assess it. These choices and criteria play important roles in the nature of instruction, experiences we provide, and the level of children's acquisition of number concept, numbersense, and numeracy.

The prevalent method of teaching arithmetic, as mentioned in earlier chapters, has been sequential counting—the counting of objects from pebbles to fingers to Unifix cubes, to Numicon boards, to dots on cards to jumping characters in apps (Andres et al., 2007, 193). It has not worked for a significant number of children for decades. We need to understand when to use counting and how and what is missing that is important for the development of true numeracy in children with and without learning disabilities, such as dyscalculia. There is a high reliance on counting in early school mathematics, and it continues even beyond elementary school into algebra or is sadly replaced only by calculators.

To successfully move from just one-to-one counting to deriving and mastering arithmetic facts, children need to be introduced to manipulatives which allow them to develop effective numbersense using their innate perceptual, quantitative, and spatial skills and the proper use of their effective instructional and cultural tools such as language, toys, and games. We need to use tools that help conceptualize numerical ideas easily, learn number relationships better, observe number patterns, appreciate properties and characteristics of numbers (e.g., decomposition/re-composition), and patterns and properties of operations (e.g., commutative, associative, and distributive properties of addition and multiplication), and arrive at efficient strategies to manipulate numbers easily. Then, they will master major basic mathematics concepts more easily, efficiently, and effectively. Such mastery will help them move to higher mathematics concepts more easily with engagement in higher-order thinking.

It is important to think about the efficiency of strategies when we are evaluating the resources and tools we use for teaching. This is not about taste; it is about value in maximizing what is being taught and what is being learned. It is return on investment. Time spent and cognitive skills involved and the content developed in the child's mind must have positive and significant relationships. Consider this one example out of thousands that any one of us can relate:

One of the authors was invited to a school to present a clinical professional development experience for teachers that includes teaching a few classes at different grade levels in different numeracy content. Before the session, the head of the school very proudly shared a mathematics project that children were engaged in during the current term. The project was: 'To learn how big is a million?' To find the answer to this question, their method was to collect bottlecaps in a big container till it was full, and their assumption was that there would be a million bottle caps in it. After that, the plan was to count them by engaging all children in the counting process. The author asked: 'How long would it take to count them?' He said: 'I do not know. I suppose it will take time.' The author said: 'It will take total of about 80 hours. How will that marathon counting teach a child how big is a million? Is it a good investment of time? Of course, there is a great deal of social value, but very little mathematical value. And then they would know how big a pile of million bottle caps is, and that will hardly help in any actual place value understanding.' Many of our methods and tools are like that.

Our students, at all levels, deserve the best. Too often poor-quality resources with sketchy images and low-level text are given to students to learn from, or they spend countless minutes and hours pushing buttons on apps. Mindless counting. Many of these tasks offer too little challenge for learners or understanding. Dull, irrelevant visuals, close exercises, filling in the blanks, and coloring in are unlikely to make students think deeply about their learning or mathematics. Why do this when there are gorgeous tools available? We just need to know where and what they are. However, this is not about aesthetics; it is about learning. High-quality models and visuals, great texts, and interesting discussions draw students in. This is because they provoke curiosity, and the return is immediate. A great text can provide pupils with high-quality information and interesting vocabulary. Beautiful resources and tools break down barriers to learning. This is particularly important for students with additional and specific needs.

A. Why Concrete Instructional Tools?

1. Stimulate Cognitive Function

The practice of effective instructional tools engages multiple areas of the brain at once, requiring significant cognitive effort and outcome. Children's motor cortex and cerebellum, associated with fine motor coordination, are activated when they move the pieces of the manipulatives/tools. Their prefrontal cortex kicks in as they concentrate on a pattern or script, and their somatic, visual, and auditory cortex get involved as they interact with them and discuss the construction and activities and observe and repeat the patterns. Even their hippocampus—which is tied to memory—is active when they are learning new strategies and concepts and sculpting their creations—patterns and steps of procedures that provide them with instant success and a good discussion with their classmates.

Effective choice and use of instructional materials demand planning, memory, sequencing, and problem-solving—all skills that are essential for keeping a sharp mind that is needed for mathematics learning. An engaging use of instructional materials regulates mood and helps children reduce their stress and ultimately boosts their mood. It may even help their body switch from a state of 'fight or flight' into a more restful state. It helps children lower cortisol (the stress hormone) and boosts attention and focus. Using effective instructional materials is a form of creative self-expression that can elicit a sense of accomplishment and pride—increasing serotonin and dopamine.

In addition to boosting children's mood, attention, and engagement, the proper and organized use of interesting instructional materials is recognized as calming and less stressful, more focused thought, which can help transition children into a 'flow state.' The process promotes social and academic engagement. Building a community through shared materials fosters a sense of belonging and identity in individuals—both of which are essential for overall well-being. It provides not only social connection but also emotional support. Forming a learning community is a huge component of our learning, restorative, and healing experience, offering a range of mental health benefits.

Questions that have engaged us, our colleagues, teachers, and parents are: which manipulatives or instructional tools work effectively, efficiently, and with definite purpose and outcomes? How can we use them effectively and efficiently with all children? When? In what sequence? And how long should children stay on a particular concrete activity? How do we facilitate transitions from concrete to pictorial and to abstract? ([Nickson, 2000](#)).

For the last 50 years, we have been experimenting with many instructional manipulatives that we have found to be useful for our students and teachers. Parents also have found them to be immensely beneficial.

We have found that use of appropriate manipulatives enables children to develop mathematics language, explore and learn concepts, take a flexible approach to learning, create robust conceptual schemas, arrive at effective strategies, and connect concrete activities with visual, semi-abstract, and abstract representations, and standard algorithms and procedures seamlessly.

The materials we have found useful are: Visual Cluster Number Cards (VCNC), Sight Facts Cards (SFC), Cuisenaire Rods, BaseTen blocks, Number Balance (particularly an Invicta Balance), Empty Number Line (ENL), paper fraction strips, graph paper, attribute blocks, and algebra tiles. They allow children to move beyond the one-to-one correspondence (e.g., counting) to see number relationships effectively, efficiently, and elegantly.

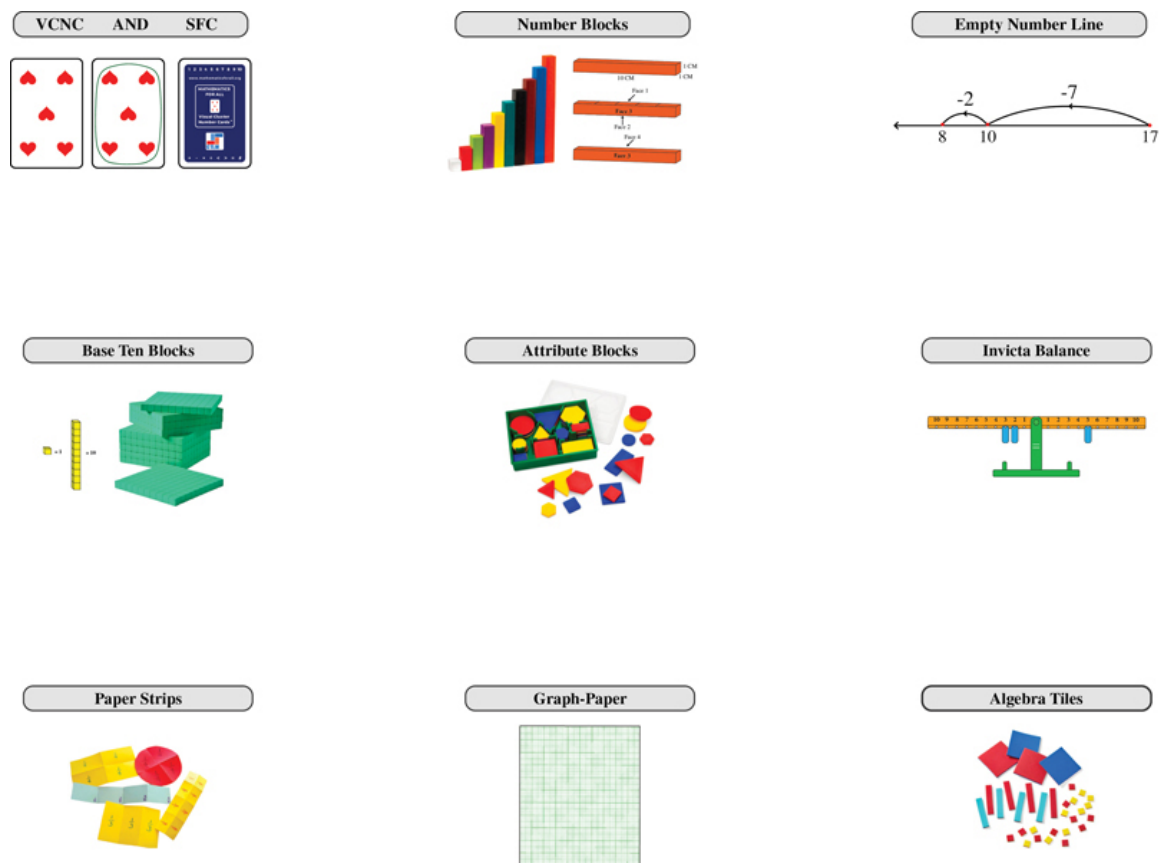


Figure 5.1 Effective, efficient and elegant tools for teaching numeracy

Developmental activities using Visual Cluster Cards and Cuisenaire Rods help develop and promote decomposition/re-composition for better mastery of numbersense skills than generally achieved through activities that involve counting. With the use of

these materials with proper questioning and appropriate language usage, students are better able to gain fluency with basic arithmetic operations and with less effort.

This book explains how to effectively use the Cuisenaire rods, Visual Cluster Number Cards, BaseTen blocks, Number Balance, and Empty Number line (ENL), applying the principles derived from recent research from cognitive psychology, neurology, neuropsychology, linguistics, anthropology, educational psychology, and on learning problems in numeracy and mathematics to assist students in fluently mastering basic facts. The purpose is that they do not become dependent on concrete models and assist them to move from the concrete to the abstract with understanding of number and numbersense for arithmetic operations and problem solving. In the process, this helps them become better learners. This sets up a strong foundation for deriving multiplication and division facts and procedures easily and effectively. The methods not only develop mastery of the content, but also help them to develop learning skills, executive functions, and improve applications of memory capabilities both for retention and recall of learned material efficiently.

To reach all children in any class requires a teacher's will, mastery of content, and effective, efficient, and elegant teaching approaches and tools of instruction. The use of instructional tools is the life and strength of a lesson. And a master teacher makes the tools come live and effective for their students. 'Beautiful' tools make work a joyful and fulfilling experience for both the teacher and the student.

B. Mathematics Tools

Ensuring that every classroom has the tools needed for student success is a high priority for teachers and school leaders alike. That said, how does one know which tools are the 'right' tools and where to spend precious resources? And how do you ensure the investment in these tools pays off? Is there congruence between the tools and the content of the grade one is teaching? To access, receive, and utilize mathematical concepts, students must master specific cognitive skills related to the topic at hand. Research and classroom experience indicate that students need to develop these foundational skills to be successful in mathematics at various levels. These prerequisite skills include:

- General cognition-related learning skills
- Pre-requisite skills for mathematics learning
- Content-specific learning skills

To achieve these skills, successful instruction tools fall into four overlapping and supporting categories:

- a. Content-specific tools,
- b. cognitive and executive function tools,
- c. language tools,
- d. physical and visual tools: concrete and visual models

It is the judicious integration and use of these tools that is the key to an effective lesson and, therefore, optimal learning. For a lesson to be effective, the teacher must display his mastery of the content, understanding of children and how they learn, and mastery of his tools and then something to show as the outcome—student learning.

1. Content Tools

Every mathematics content item/idea—language, concept, and procedure—begins as a new concept, new language, and new procedures emerge. For example,

- a. Mastery of addition is a first-grade focus concept. By the end of first grade, children should know the addition facts of sums up to 20. For its successful conclusion at the end of first grade or before, children should have mastered the following content tools. For example, they should be able to answer the problem: $68 + 7 = ?$ mentally. For this to happen, the content tools the child should have are:

- Number concept
- Decomposition/re-composition and 45 sight facts

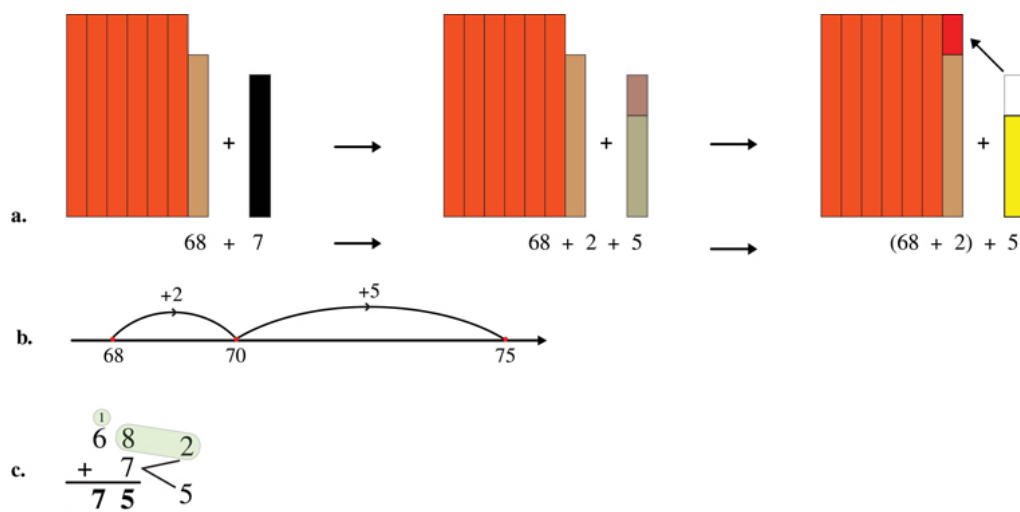


Figure 5.2 Script for addition of 68 and 7

- Making ten
- Teens numbers
- Tens' numbers and making tens numbers
- Commutative property
- The script for addition

T: Read the problem. S: $68 + 7$. (Number Concept and K-level place value)

T: Which number is bigger? S: 68. (Same as above)

T: What is the next tens' number to 68? S: 70. (Tens' Numbers)

T: How do you get to 70? S: Add 2. (Making 10 and its extension)

T: Where will 2 come from? S: From 7. (Sight Facts)

T: Then, what is left in 7? S: 5. (Sight Facts)

T: What is $68 + 2 + 5 =$? S: 75. (Making Tens)

T: What is $68 + 7 =$? S: 75.

T: Good! What is $7 + 68 =$? S: 75. (Commutative property of addition)

The result of this exchange is learning using several component skills and the development of a strategy and is learned in the form of a *script* that helps retrieve the procedure and concepts. All the questions by the teacher form the structure of the script and help practice mental arithmetic and development of numbersense. The teacher, after this kind of work, can easily move on to derive the standard procedure of addition (algorithm), which children will find to be quite easy.

- b. The concept of multiplication is a focus concept for the third grade, but for successful learning and teaching, it is important that children have acquired certain non-negotiable skills from the earlier grades to be able to access, understand, and master this concept in the third grade. For example, the following numerical concept, facts, and skills are the necessary content tools, but not sufficient for success in teaching and learning all the multiplication facts (10×10). Those content tools are:

- Number Concept: Subitizing and decomposition/re-composition
- Mastery of Addition and Subtraction Facts
- Making Ten and Teens' Numbers
- Tens' Numbers
- What is the next Tens' number to a given number.

- Concept schemas of multiplication (repeated addition, groups of, array, and area of a rectangle) and the mastery of tables 1, 2, 5, and 10; commutative and distributive properties of multiplication. The rest of them can be derived from the application of these content tools.

These content tools can be considered part of *prior knowledge*, but this is selected, focused, and directly related to the strategies to be developed for multiplication and division concepts, facts, and procedures. In the tool building part of the lesson, the teacher must devote time to the review, development, and mastery of these particular concepts, language, strategies, and procedures so that children can access and learn the main concept of the lesson.

Example, what is $8 \times 7 = ?$

T: Read the problem. S: $8 \times 7 = ?$.

T: Do you know the answer? S: No! Never done it.

T: Do you know the tables of 2 and 5?

The teacher checks the mastery of tables of 2 and 5 by the help of Multiplication Ladders. On being reassured she moves back to finding the product: 8×7 , and makes the learner aware of the decomposition/re-composition process.

T: Which number to break and why?

S: $7 = 5 + 2$, because I know the tables of 2 and 5.

T: Decompose this multiplication fact into two multiplication facts. S:
 $8 \times 7 = 8 (5 + 2)$

T: What are the two multiplication problems?

S: $8 \times 7 = 8 (5 + 2) = 8 \times 5 + 8 \times 2 = 40 + 16 = 40 + (10 + 6) = 50 + 6 = 56$

T: What is $7 \times 8 = ?$

S: $7 \times 8 = 56$.

T: Good!

All the other multiplication facts can be derived in a similar way.

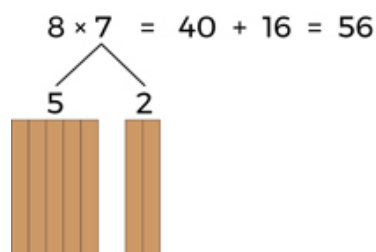


Figure 5.3 Derivation of 8×7 using Cuisenaire rods

Identifying and properly practicing the cognitive tools needed for a particular concept, fact, procedure, or solving a problem is mark of an effective teacher.

2. Cognitive Tools

To be able to access, receive, and make use of a mathematics idea, a student should have mastered certain cognitive skills that are related to the current concept. For example, research and our experience suggests that children should develop the following prerequisite skills to be successful in mathematics at the appropriate levels. As discussed in [Chapter Three](#) (p.), the learning skills could be divided into three categories: (i) General cognition-related learning skills; (ii) Prerequisite skills for mathematics learning; and (iii) Content-specific learning skills. These skills act as cognitive tools.

For example, any mathematical idea—language, concept, or procedure—can be brought to children’s attention and focus by the help of four approaches:

- a. Modeling the concept or procedure through concrete or visual examples,
- b. deriving by the help of developing a pattern,
- c. considering analogues examples and situations, or
- d. applying logical reasoning (deductive or inductive).

Most elementary school mathematics teaching falls into the first three categories.

Therefore, along with the planning of teaching the content, a teacher must also plan to include the development of these strategies to help students in acquiring the appropriate cognitive tools. The key general cognitive tools are:

- *Executive functions*: Organization skills, working memory, attention and focus/inhibition control, and flexibility of thought.

Use of executive functions is reflected in a the selection of physical tools, the type and nature of questions being asked, the sequence and organization of activities, and

exploring multiple ways of arriving at the answers.

- *Memory functions*: The aim of instruction is to maximize the input to short-term memory and to maximize the transfer from short-term memory to working memory and then to long-term memory, and thus help the student to mix and interact the new information with already learned materials from the long-term memory in the working memory. The strategies used to maximize the transfer must be such that they also maximize the storage and retrieval and related relevant prior knowledge from the long-term memory to working memory for use in learning (to maximize the output).
- *Perseverance* (Stick-to-it-ness and resilience): The task, the model, the language, the questioning, and the strategies must be such that the problem is made accessible, moderately challenging, and provides early and frequent success. These conditions make it possible to try and stay with the task. During the activity, the teacher's questions will support and instigate engagement. We have found that behind perseverance is a set of processes and the instigator to all of these processes are the teacher's questions. We have found that

Questions instigate language
Language instigates models
Models instigate thinking
Thinking instigates understanding
Understanding produces competent performance
Competent performance builds success
Success builds high self-esteem, and
Self-esteem is the motivating factor for all learning.

- *Resilience* (Emotional and intellectual skills to cope with failure): Failure is inevitable, but the suggestions and feedback during the task performance and after the completion of the task must be such that it is a learning experience for the student and results in some success and willingness/motivation to try again.
- *Task analysis*: An ability to break a task into its primary components and sequential tasks.

The effective use of cognitive skills in instruction leads to more organized teaching, engagement, and enhances students' ability to learn. In addition to these general cognitive tools, there exists another group of skills that are necessary, but not

sufficient, for effective mathematics teaching. Some *prerequisite learning skills* for learning mathematics are:

- An ability to follow sequential directions,
- Spatial orientation/space organization (parts-to-whole and whole-to-parts),
- Pattern analysis: ability for (i) discerning/recognizing, (ii) extending, (iii) applying, and (iv) creating patterns,
- Visualization and visual perceptual integration,
- Estimation,
- Deductive, and
- Inductive reasoning.

These skills serve as cognitive tools, enabling students to engage with mathematical ideas —whether language, concepts, or procedures—through approaches such as concrete or visual modeling, pattern development, analogues, and logical reasoning.

Therefore, when planning lessons, teachers should incorporate strategies that help students acquire these prerequisite skills alongside the mathematics content. By doing so, instruction becomes more effective, and students are better equipped to learn and apply mathematical concepts.

The description of the vignettes in the content tools and in the lessons in the later chapters shows the development of language, conceptual schemas, strategies, procedures, and the scripts for them. These tools are essential for the development of the scripts. For example, the two examples cited earlier emanate from the concrete manipulation, questions being asked, task analysis, and the key focus on appropriate executive functions.

3. Language Tools

As we saw in the chapter on the role of language in learning mathematics, every mathematics idea has three components: linguistic, conceptual, and procedural. Without linguistic and physical tools, no mathematics conceptual schemas can be developed. For example, the multiplication fact 3×4 can be read as: 3 added repeatedly 4 times, then the concrete representation is 3 objects + 3 objects + 3 objects + 3 objects and the conceptual schema becomes ‘repeated addition.’ 3×4 can also be seen as ‘3 groups of 4’ and expressed concretely as: 4 objects + 4 objects + 4 objects, and the conceptual schema becomes ‘groups of.’ 3×4 can also be seen as ‘3 rows of 4 objects each’ and the conceptual schema is expressed concretely as a ‘3 by 4 array.’ The first two schemas of multiplication are linear models and therefore one-

dimensional. They are also mostly operating only on whole numbers or simple fractions and decimals. They are limited models.

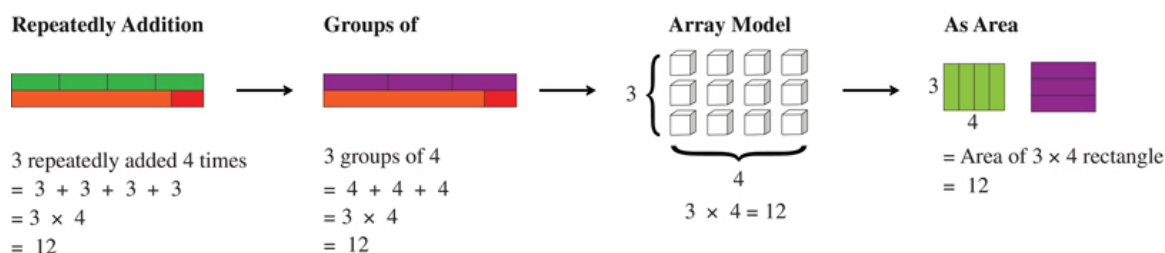


Figure 5.4 Four models of multiplication

The schema of array models of multiplication not only absorbs the previous two schemas of multiplication, but it has expanded the concept of multiplication to a two-dimensional representation. This further expands the definition of multiplication to the area of a 3 by 4 rectangle. The conceptual movement from ‘repeated addition’ and ‘groups of’ to the ‘an array’ and ‘area of a rectangle’ shows the developmental nature of multiplication. The four models of multiplication: repeated addition, groups of, array model, and the area model demonstrate the developmental nature of multiplication from one-dimensional concept (as in repeated addition and groups of) to a two-dimensional concept (as in the array and area of a rectangle models). The concept of exponents further extends it to multi-dimensional concept.

Whereas in the first three linguistic models of multiplication, the conceptual schemas were applicable mostly to whole numbers, the language, concrete, and pictorial representations of the area model expand the schema of multiplication to fractions, decimals, algebraic expressions, and easily to the concepts of exponents and exponential functions. The area model also shows the concepts of multiplication and division as inverse operations directly and the language and conceptual schemas of multiplication are clearly related to division. The sides of the rectangle are factors of the product, and the area is the product of the sides. These language tools for multiplication provide easy developmental entry into algebra.

4. Physical Tools

As we saw in the previous section, the tools (Cuisenaire rods to represent the concept of multiplication in four forms) that can make a huge difference in student learning are proper instructional manipulatives (concrete and visual models). Broadly defined, instructional *manipulatives* are physical and visual representational objects teachers and students can use to help discover, develop, derive, illustrate, and model

mathematical concepts to arrive at the proper language, conceptual schemas, facts, strategies, and procedures. Manipulatives and visual methods are critical to this sense-making process because they make mathematics tangible to students. They allow students to observe, to touch, to see, to manipulate as they generate the language, build their conceptual understanding, and develop strategies, arrive at procedures, and the skills of mathematics. These activities not only improve their perceptual skills, but they make a rich imagery to enhance the functioning of memory systems: short-term, working, and long-term memory.

We know this is important for young learners, and it is equally important for middle grades and even high school students as concepts get increasingly complex. The authors have successfully used them even with college and university students who have arrived at university with poor preparations exacerbated by a great deal of mathematics anxiety.

Most elementary classrooms have diverse collections of manipulative materials for helping students learn mathematics concepts. Teachers realize the importance of the role of these materials in learning foundation concepts and skills in mathematics. Their use, however, is dependent on the willingness and competence of the teacher. To convert these materials into effective tools as instruments—the vehicle of learning—takes, motivation, training, practice, skills, and usage. Then, a deeper understanding of mathematics takes place.

One can easily notice the nature of teaching and learning in a classroom by observing what is on display in the classroom—intellectually, linguistically, physically, and socially—from concrete objects to visual representations, to definitions, diagrams, examples, and books. They tell the nature and type of classroom activities that might happen there. The idea that physical objects might play an important role in the learning process is a relatively new idea. Until the 19th century, formal education focused almost exclusively on lectures and recitations. One of the first advocates for ‘hands-on learning’ was the Swiss educator Johann Heinrich Pestalozzi (1746–1827). Pestalozzi asserted that students need to learn through their senses and through physical activity, arguing for ‘things before words, concrete before abstract.’

Friedrich Froebel, who created the world’s first kindergarten in Germany in 1837, was strongly influenced by Pestalozzi’s ideas. Froebel’s kindergarten was filled with objects for children to play with. Froebel developed a specific set of 20 ‘gifts’—physical objects such as balls, blocks, and sticks—for children to use in the kindergarten. He carefully designed these gifts/activities to help children recognize and appreciate the common patterns and forms found in nature. Froebel’s gifts were

eventually distributed throughout the world, deeply influencing the development of generations of young children. Indeed, Frank Lloyd Wright credited his boyhood experiences with Froebel's gifts as the foundation of his architecture.

Maria Montessori, another visionary, extended Froebel's ideas, developing materials even for older children and inspired a network of schools in which specific manipulative materials play a central role even today. To create an 'education of the senses,' Montessori developed new materials and activities (now known as the Montessori apparatus and approach) to help children develop their sensory capabilities and learning-specific content. Montessori hoped that her materials would put children in control of their learning process, enabling them to learn through personal investigation and exploration.

Jean Piaget provided an epistemological foundation for these educational ideas. Piaget theorized that children must first construct knowledge through 'concrete operations' before moving on to 'formal operations.' During the past few decades, a new wave of research has suggested that Piaget, if anything, understated the importance of concrete operations. Sherry Turkle and Seymour Papert, for example, have argued for a 'revaluation of the concrete,' suggesting that 'abstract reasoning' should not be viewed as more advanced than (or superior to) concrete manipulations and is not reserved for only kinesthetic learners, but for all children—all learners.

The sixties and seventies were the renaissance of instructional materials in primary schools throughout Europe and the United States. It was the time of experimentation in education. In the seventies and eighties, both the early and the second-round adapters of concrete instructional materials in schools had the good fortune of learning about manipulatives by working with the people who had invented or created the manipulatives—people like William Hull, creator of Attribute Blocks; Zoltan Dienes, creator of Multi-base Blocks; Pat Davidson and Al Bennett, creators of Chip Trading Materials; Dr. William Fitzgerald, coauthor of *Laboratory Manual for Elementary Mathematics*; Dr. Carol Greenes and Dr. Bob Wilcutt, authors of *Problem Solving in the Mathematics Laboratory*; and Dr. Robert Reys and Thomas Post, coauthors of *A Mathematics Laboratory in the Classroom*. These men and women and others supported the manipulative movement and served as mentors to teachers. The authors have redesigned several of these materials and their usage and some new ones that have revolutionized the learning of school mathematics. For example, Visual Cluster Number Cards and Sight Fact Cards (Sharma) were first designed for dyscalculics to teach sight facts and are now used to teach all children not only arithmetic, but also integer operations. Cuisenaire rods have been redesigned as Number rods (Sharma and Raghav) to expand their use to almost all arithmetic concepts.

In the past 50 years, the demand for the use of manipulatives in elementary mathematics teaching has grown significantly. This is particularly true for the commercially manufactured products that have been developed specifically as aids in mathematics instruction. Once regarded as supplementary resource materials in the classroom, commercially produced mathematics manipulatives are today viewed as central, core instructional aids in quality school mathematics programs.

Unfortunately, the dramatic increase in the supply and demand for commercially manufactured mathematics manipulatives has not yet resulted in significant changes or improvements in many schools' mathematics instructional programs. There are several groups of children for whom achievement in mathematics has not improved significantly. Anecdotal evidence indicates that, although math manipulatives are now more widely available, their effective use in teaching may have declined. Also, the incidence of inappropriate use or underutilization of commercially manufactured mathematics manipulatives seems to be on the increase. Another trend that is emerging is the inordinate amount of time being spent on computer programs or apps for a great deal of drill and repetition of problems (old strategies in technological formats).

However, the existence of concrete and visual materials in a classroom is not enough to make any changes. When their role is understood well and their proper applications are made, and they are linked to the appropriate symbols, then, and only then, is any impact on teaching and children's achievement possible. For a long-term change, their use must be consistent with the focus on effective and elegant use of appropriate materials for concepts. Appropriate means which material is most effective and efficient in teaching a particular concept, procedure, or skill to a particular child, group of children, a class, or a grade. Efficient use of materials can save an enormous amount of time and effort on the part of teachers as well as children.

With the effective use of instructional materials, the transition from one developmental milestone to another is easily achieved. For example, we have found that modified Number rods and Base Ten blocks and Visual Cluster Number cards can effectively teach Sight Facts to Additive Reasoning to Multiplicative Reasoning, to Proportional Reasoning to operations on Algebraic manipulations and operations.

To us, concrete instructional materials are not adjuncts, aids, or engagement/entertainment items to teaching; they are fundamental tools of instruction and learning. They are the backbone of teaching. After bringing the prior knowledge of the students out regarding a new concept, the concrete level of knowing a mathematical idea is the first level of knowing a new mathematical idea. The strong foundation at this level ensures that a child will climb the other levels easily and effectively and understand the development of an idea, its trajectory, its

interconnection from concrete to pictorial, to visualization to abstract, to applications, and to its communications in multiple forms and with ease and confidence.

We have found that use of concrete instructional materials in initial teaching is a key part of developing children's understanding of mathematics concepts and procedures. In teaching most mathematics concepts and procedures, manipulative materials (such as Visual Cluster Number cards, Number Rods, Base Ten blocks, Pattern Blocks, Unifix cubes, Geoboards, an Invicta balance, fraction strips, algebra tiles, etc.) play an important role in making mathematics concepts transparent and maximizing children's learning and achievement.

The modeling of mathematics concepts by concrete materials is the most effective way of acquiring conceptual understanding—the language, concepts, and strategies—thereby increasing the engagement of children in learning. These models and manipulations make abstract concepts and ideas about numbers and shapes concrete and easy to understand.

Exploring and observing tools used by elementary school teachers, we have found that effective teachers, but rare ones, can make any tool an effective tool. There are always a few like this. In my village, a musician with a one-stringed instrument (*Ektara*—*Ek*-one, *tara*-string) was able to enthrall crowds to tears. He used to come to the school only once a week. We used to look forward to his visits. They were a rare treat indeed. At one time, almost every child in the school wanted to learn the *Ektara*. He even taught them how to make the instrument. One of the authors' interest in sitar and music was due to this artist, and later majoring in mathematics took place because of an extraordinary teacher.

In our observations, we have found that many teachers have all these instruments in their classrooms and children are using them; however, since they do not have enough proper training and experience in using them, children are using them to find the answers by counting, not developing concepts and efficient strategies. They are heavily dependent on these materials and as a result have reached their plateau in mathematics learning. Since teachers lack training in teaching and applying effectively, their students are lacking in the same things.

The absence of the right training in use of tools and mathematics creates a situation where their ability to help children to learn numeracy is limited, and we find that many children become 'tool-dependent' and use them ineffectively, and their thinking about numbers, numbersense, and numeracy has become 'tool-specific.' Many of them cannot find answers without the use of these tools. In such situations, many teachers help children use the tools to find the answers, but children do not learn the language or concepts of mathematics and do not derive efficient strategies. For example, when

they use only counting materials to learn number concept—names of numbers and sizes of collections, deriving arithmetic facts, and executing procedures, then their strategies, their facts mastery, and their problem solving strategies are counting-based: Addition is ‘counting forward,’ subtraction is ‘counting backward,’ multiplication is ‘skip counting forward,’ and division is ‘skip counting backward.’ Their conceptual schemas for number, numbersense, and numeracy and later, even for higher-order concepts. For example, we have observed that many middle school students, when they encounter concepts related to integers, also use counting-based strategies.

The strategies, methods, and lessons in this book offer a way for teachers, interventionists, special teachers, and prospective teachers to obtain significant insight into the information they need to teach effectively and to see what role instructional materials play.

Each strategy, method, and lesson we present in this book has been classroom-tested with teachers in many countries, prospective teachers, and school children at each grade and age level, with and without learning disabilities. Each unit and each vignette includes a thorough exploration and discussion of a mathematical idea using the most appropriate models and instructional tools with the appropriate script for application. The script is based on intensive usage with teachers and children and on neuropsychological principles of learning and mastering a skill, concept, or a procedure. Although a very tight script is provided for teaching and learning a skill/strategy/concept/procedure, it can easily be amended, based on observations, diagnosis, and rate of learning of a student or a group. Also included in the lessons are observations, comments, and suggestions that may elaborate or extend the ideas presented. The comments in the footnotes elaborate the context and other relations and connections ([Fruedenthal Institute, 2001](#)).

Our belief is that effective teacher training and professional development should clearly focus on proper and effective use of instructional materials, not just curriculum content. One should know that the purpose of using concrete models is not just answer-finding; it is multi-fold:

- To effectively, efficiently, and elegantly make mathematics accessible to all children, with or without learning disabilities, including dyscalculia/dyslexia.
- To promote active versus passive learning: everyone learns best when they are active learners rather than passive learners.
- To utilize guided discovery methods: Guided discovery provides an excellent method for teaching with manipulatives. Through the guided discovery method, using manipulatives, the teacher has the opportunity to apply all her skills and

roles—didactic, Socratic, and coaching—and reach all children and emphasize all the components of a mathematical idea—language, concept, and procedure.

The purpose of all concrete materials in mathematics is to make quantitative meaning of the environment and events around us, and they serve that purpose to varying extents. However, as mentioned earlier, the use of concrete models is primarily to (a) generate the language of mathematics, (b) construct conceptual schemas, and (c) derive efficient and effective strategies and procedures:

- To help generate the language (vocabulary, syntax, semantics, and translation of English language into mathematics language and vice versa) related to the given mathematics idea: concept, strategy, or procedure,
- To create, develop, and shape the conceptual schema related to the given mathematics idea, and
- To derive the fact, concept, strategy, procedure, or methodology related to the mathematical idea.

Effective concrete instructional materials share five basic features that are useful in the teaching and learning process. They help the teacher in:

- a. Making the concept, strategy, and procedure transparent,
- b. generating the language of the concept, strategy, or procedure clearly,
- c. constructing the conceptual schema(s) effectively,
- d. arriving at efficient strategies easily, and
- e. deriving the procedures related to the concept(s) elegantly.

Not all instructional materials being used in many classrooms satisfy these conditions. For example, not all addition strategies and procedures, derived by using different materials, develop these characteristics. Later, we describe the continuum of instructional materials and their place and usefulness in the teaching and learning process of additive reasoning, and the lessons in the later chapters are based on these principles. See [Figure 5.5](#) Continuum of Instructional Materials for Number Concept.

Effective concrete and pictorial materials help a child to be able to visualize the concept in their mind's eye (activating and improving the working memory) and transfer it to the abstract level and to the long-term memory (with *strength of storage* and *retrieval*).

In the light of the criteria just described that we should use concrete materials that can make ideas transparent and can help children learn these concepts, we should

examine the prevalent instructional materials.

Teachers in primary/elementary schools use a variety of instructional materials. Some of their choices are successful with one or more aspects of the concept (for example, the teaching of number concept) being taught, but very few of them work for all components of mathematics ideas. This is particularly true if more than one primary concept is involved in the idea or the procedure. In addition and subtraction and in other arithmetic operations, we will use several primary concepts.

The use of appropriate manipulatives enables children to develop mathematics language, explore and learn concepts, arrive at efficient strategies, develop numbersense, and connect concrete with standard algorithms and procedures easily.

To successfully move beyond one-to-one counting, children need to be introduced to manipulatives which allow them to conceptualize, learn, see relationships and patterns in numbers, arrive at efficient strategies, and better master all major basic mathematics concepts. These materials are: Visual Cluster Number Cards, Cuisenaire rods, and Base Ten blocks that allow moving beyond one-to-one correspondence to seeing number relationships effectively. Developmental activities using Cuisenaire rods promote better mastery of numbersense skills than generally achieved through counters. With the use of these materials, students are better able to gain fluency with the basic arithmetic operations with less effort.

Conceptually, *number* (representation of quantity) is conceptualized in two forms: (a) representation of a collection of objects, (b) representation of continuous quantity and measurement. To manipulate numbers from these two perspectives, we have instruction tools for numeracy that fall into three categories: discrete/discontinuous, continuous/visual spatial, and mixed.

- *Discrete materials* (counting-based): counting objects of various kinds—counters, cubes, beans, buttons, counting blocks, fingers, etc. The underlying feature in their usage is counting to perform the intended computation,
- *Semi-discrete*: in this category, the discrete objects are organized into patterns of some kind. The quality and extent of patterns vary. Examples of such materials are dice, dominoes, dot patterns, playing cards, Visual Cluster Number cards, etc. They are substantial improvements on counting materials and are known as mixed modality materials.
- *Continuous materials*: length-, color-, size-, and pattern-based. As mentioned earlier, they fall on a continuum. The underlying features are color, size, length, pattern, etc. The number rods, the Invicta Balance, paper strips, etc. are examples of continuous materials. The length of a particular rod represents a number. These are effective materials and can be used to teach all major arithmetic concepts

from number concept to algebra. Children who have sequencing problems and have not yet mastered number concepts properly can do so in a very short time using these materials.

For example, the instructional materials used in teaching additive reasoning vary from:

Level 1: *Discrete concrete models* (single object counting): fingers, hands, blocks, pebbles, beans, coffee-stirrers, straws, Number Line, etc.,

Level 2: *Counting groups models* (counting in groups): dice, dominos, Five-Frames, Ten-Frames, Rek-n-Rek, Numicon, Abacus, Visual Cluster Cards, Cuisenaire rods, Base Ten blocks, Dienes Apparatus, coins, Number Line, etc.,

Level 3: *Strategic counting* (using decomposition/re-composition: breaking the collection into recognizable clusters),

Level 4: *Semi-discrete models* (discerning/recognizing patterns and regularity): Visual Cluster cards, Five-Frames, Ten-Frames, Cuisenaire rods, abacus, etc.),

Level 5: *Continuous and Pictorial Models* (no counting and focusing on the length or measurement aspect of collection): Cuisenaire rods, Tens and Hundreds from Base Ten blocks (no ones), abacus, Empty Number Line, Bar Model, etc.,

Level 6: *Standard and Alternative Procedures* used with strategies and place value.

In most classrooms throughout the world, additive reasoning and even multiplicative reasoning are taught using Level 1 or 2 models and strategies.

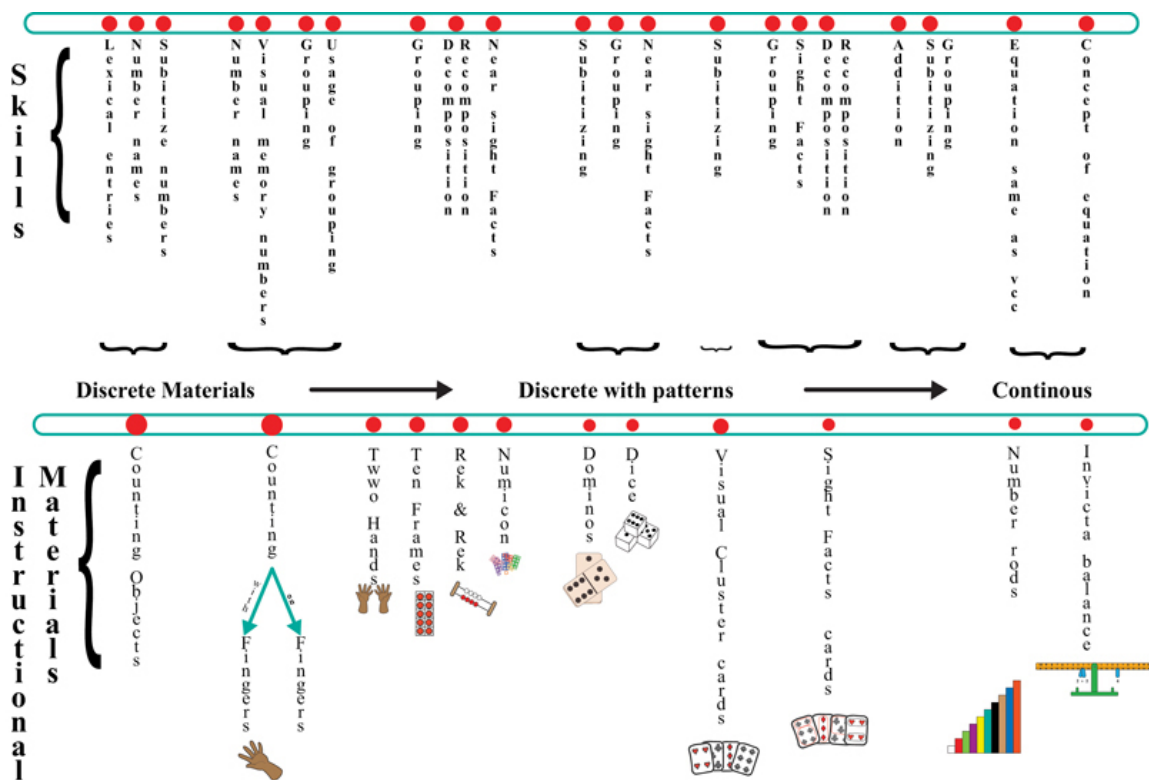


Figure 5.5 Instructional Materials Continuum for teaching of Number concept

5. Multi-modal, Multi-sensory, and Multi-model Teaching/Learning

Various studies in mathematics education have focused on the design and implementation of instructional activities significantly based on bodily experiences and on the manipulation of concrete objects with the aim of fostering the development of conceptual schemas and particular mathematical meanings. They highlight how sensorimotor, perceptive, and kinesthetic-tactile experiences are fundamental for the formation of mathematical concepts—even highly abstract ones.

Learning involves all aspects of our being—physical, intellectual, emotional, and social. Optimal learning is a total engagement—integration of all of them. For learning to be optimal, we should bring ‘the whole of us’ to this process. Teaching and learning are the two sides of the coin. In teaching, as teachers, we should bring ‘the whole of us’ to this endeavor and we need to engage the whole child. Our instructional activities should aim at engaging the whole complex of the brain by making the activities multi-sensory, multi-modal, and multi-model ([Frith, 2007](#)).

Teaching and learning strategies using physical and pictorial models are:

- a. *multi-sensory* (e.g., engagement of tactile input (touch and feel)—*somatic cortex*, visual input (color, shape, size, layout, organization, gestalt)—*visual cortex*, and auditory (questions, discussion, descriptions)—*auditory cortex*),
- b. *multi-modal* (employment of language, movement, organization/reorganization, gestures, writing, representation—concrete, pictorial, symbolic, application, communication, action, etc.), and
- c. *multi-model* (e.g., addition is joining, relating part-part-whole, combining; it is represented as the joining of discrete, continuous, or a combination of the two, etc.) to display addition in multiple ways (*counting two numbers*: representing addition as collection of two piles of objects, fingers, on a number line, Numicon, dots, etc.; *combining by using the principle of decomposition/re-composition*: joining two Visual Cluster Number Cards, joining two Number rods). The key conceptual schema, in addition and subtraction is relevant decomposition/re-composition of numbers using strategies. It is parallel to the reading process, where the key elements are decoding and encoding with the awareness of phonemic awareness rather than teaching reading by sounding out each letter.

Attention to these elements reflects recent views on human cognition, especially regarding the body, language, and the classroom culture in shaping how mathematics is thought about, taught, and learned. Distinct from traditional approaches, these conceptions highlight the cognitive role of semiotics and embodiment in mathematics thinking, teaching, and learning. Teaching, with concrete tools and with the proper role of language and questions, takes a new meaning. These multi-sensory material resources are not considered mere epiphenomena (This is an example of embodied cognition in mathematics—a paradigm arguing that mathematical thinking is rooted in physical actions, bodily tools (like fingers), and sensory engagement, rather than solely occurring in isolation inside the brain. Not all mathematics, particularly, school mathematics thinking is an example of mental cognition. In school room it begins with action on concrete objects of teaching and learning: they are conceptualized as central elements of thinking and engagement. Fingers and other objects become extensions of mental thinking tools between the students and teachers, and students and students, and their mathematical thinking.

In this view, teaching mathematics takes on a broader meaning: a sign is defined as any movement (counting steps, beats on a drum, gesture, image, pattern, symbol, or an event that communicates meaning). Teaching is not limited to verbal explanation or symbolic representation. Rather, it incorporates a wide range of communicative resources, including gestures, body positions, kinesthetic actions, physical artifacts, spoken and written language, questioning processes and interaction with others. All

these elements serve as meaningful resources in the classroom, supporting both teachers and students as they construct mathematical understanding.

Obviously, fingers and hands naturally embody part-whole relationships of simple addition with respect to 5 and 10 and therefore can and should be used to foster such awareness. That is because it has been the first choice when a human decides to carry out any calculations. For example, the key role attributed to the use of fingers in the development of number sense seems to be highly resonant with the frame of embodied cognition. In Gerstmann syndrome—a rare neurological disorder that can occur as the result of a brain injury and is characterized by the loss or absence of four cognitive abilities—the loss of the ability to express thoughts in writing (agraphia, dysgraphia), to perform simple arithmetic problems (acalculia), to recognize or indicate one's own or another's fingers (finger agnosia), and to distinguish between the right and left sides of one's body. Gerstmann syndrome phenomenon became the beginning of the study of other number-related problems such as dyscalculia.

The didactical potential of hands and fingers in their natural positions has been exploited over the centuries in many ways (e.g. see [Baccaglini-Frank, 2015](#)), including through multi-touch technology, particularly before formal schooling starts (e.g. [Baccaglini-Frank and Maracci, 2015](#)).

Many teachers, therefore, use fingers and hands or other instructional materials based on them, in one form or the other. Such materials are: *Five-Frame*, *Ten-Frame*, *Rek-n-Rek*, *dominoes*, *dice*, *Numicon*, *Montessori Bead necklaces*, *Touch-points cards*, *playing cards*, and apps of various kinds and forms for the purpose of helping students learn addition and subtraction, and later multiplication and division. They are natural extensions of the use of fingers. However, they should be a means for exploration of number relationships and as a starting point, not the endpoint, for additive reasoning. Prolonged use of fingers for deriving addition, subtraction, multiplication, and division facts, delays mastery—fluency, understanding, and flexibility of thought and discovering strategies.

We argue that it is important to keep in mind that in fostering the development of mathematical meanings, an essential component is the students' sharing, comparing, and evolving of strategies (which can be accomplished in different ways). These mathematical meanings, of course, can include the development of mathematics language, structure sense (e.g., constructing conceptual schemas), and relationships with other concepts, and this can be promoted through a variety of different mathematical content. The prolonged use of concrete models, however, limits many students in acquiring efficient strategies and insights. As a result, many students do not transcend the use of concrete materials and remain overly dependent on strategies such

as counting and do not reach the level of automatization and abstraction needed for mastery of arithmetic and higher mathematics.

Fingers and other counting methods do not achieve that goal effectively. Proper choice of concrete materials, and linking them to the appropriate symbols can do that, and once those aims have been achieved, the emphasis should be on abstract strategies and their practice. One should revert to concrete materials when a student is learning new concepts or when there are deficits in the language, conceptual schemas, and efficient procedures.

Each instructional material is a tool, and it should be selected, used, reused, refashioned, and rethought according to the objectives of one's instructional and learning needs and its usefulness, effectiveness, and efficiency for your learner(s). No one tool can be used for all purposes; similarly, no one instructional tool can work for all mathematics concepts and even all components of a concept, but a combination of instructional tools, or the same tool with modifications and form with proper combination and planning, can work in several situations. It is particularly so in early elementary school. It is unfortunate that the tool usage, except calculators and computers, in mathematics classes decreases drastically, denying many students the means of understanding many difficult concepts.

The pedagogy for teaching school mathematics, *CPVA/S*, as described in the previous chapter, is the integration of several tools at concrete, pictorial, visualization, and abstract levels in multiple roles of the teacher and her actions. The key component of *CPVA/S*, the language usage in the form of a natural and appropriate *script* for the concept and procedure, with its versatility of usage, helps children in transitioning from one level to the other, and at the same time binds and relates all the components together.

In the later chapters, we implement some of these principles for teaching and acquiring competence in numeracy skills. Here we illustrate the use of some of the materials that are used in our classrooms for children to learn mathematics. For example, Visual Cluster cards representing the ten counting numbers in well-organized patterns in two colors and four suits (spades, hearts, clubs, and diamonds) are used for learning number concepts (subitizing, decomposition/recomposition, facts including operations on integers) efficiently. The patterns on the cards help children to visualize the numbers that they represent and allow them to compose and decompose numbers easily to acquire mastery of arithmetic facts easily.

Another very effective instructional material, Number rods (a redesign of Cuisenaire rods and fraction strips), brightly colored wooden or plastic rods of varying lengths, help children learn number relationships much more effectively than a number line,

the random counting of objects, including fingers, Ten-Frames, and hash marks on paper. The colors and lengths of the Cuisenaire rods are carefully chosen to engage children in their explorations of arithmetic concepts and relationships. Children quickly discover that each brown rod is the same length as two purples or four reds.

All kinds of blocks or small objects for counting and sorting, particularly dice and dominoes, are very good for children to learn the concept of one-to-one correspondence and for students to categorize by colors, size, or shape.

Pattern blocks, polygon-shaped tiles, can be used to create mosaic-like patterns—and in the process, learn important geometric relationships and fraction concepts.

An Invicta scale is used for number concept (decomposition/re-composition), numbersense, numeracy, learning equations and arithmetic and algebraic procedures.

Base Ten materials teach the concept of place value, and all arithmetic and several algebraic procedures.

As we will see, the Base Ten blocks and the Cuisenaire rods can also be used in combination.

6. Other Tools

There is another category of tools: This class of manipulatives includes two types of real-world objects. In one group are those manipulatives that either occur in nature or are extremely inexpensive manufactured products. Examples of such manipulatives include string, beans, coins, pebbles, tongue depressors, and paper. In the other group of manipulatives are commercially manufactured products that may have possible educational applications but exist primarily for some other purposes. Examples of such manipulatives include Tinker Toys, Legos, jigsaw puzzles, children's building blocks, and many other children's games and toys. Both types of objects, the naturally occurring and the non-educational manufactured products, can be used in creative ways to teach selected mathematical topics. They exist in a culture for other reasons for children to grow and develop the prerequisite skills mentioned earlier: following sequential directions, pattern recognition and their extensions, spatial orientation and space organization, visualization, estimation, deductive/cause and effect, and inductive reasoning.

This book explains how to effectively use some of the specific mathematics instructional materials: the Cuisenaire rods, Visual Cluster Number Cards, Sight Facts cards, Base Ten blocks, paper strips, algebra tiles, Invicta Balance, attribute blocks, etc. to assist students in fluently mastering basic facts and to assist them in moving from the concrete to the abstract with understanding for operations and problem solving.

- *Visual Cluster Cards*
- *Empty Number Line*
- *Cuisenaire Rods*
- *Invicta Balance*
- *Base Ten Blocks*

C. Visual Cluster Cards

In a typical kindergarten or first grade class, children are sitting on the floor in a semi-circle (ready for the morning calendar activity) and the teacher is sitting on a low chair or on the floor as part of that circle. The teacher brings out a deck of Visual Cluster Cards and introduces children to the cards by identifying different suits (hearts, diamonds, clubs, and spade). There is a whiteboard or flip-chart next to her to write on. From time to time, she and the children write the numbers when children have visualized the visual cluster for a number.

She introduces the relevant words, terms, and phrases from the vocabulary list (mathematical terms used throughout the year) about the content and tools. Vocabulary is introduced in the context of the activities, rather than isolated. The vocabulary introduced relates the content, actions, and the tools, such as about, add, adding, addition, after, all together, arrange, arrangement, around, at the back, away; back, back of, before, behind the card, bottom, break; card, check, circle, close, closer, closest, club, cluster, cluster card, column, combine, commutative, correct, correction, count, counting, cover; decide, deck, decompose, diamond, dot; face, face card, facing, fact, farther, find, fold, first, form, front; great, greater, group, grouping; heart, hide, higher, hold, how many, how much; icon, idea, identify, inside; join, joint, just like; large, larger, largest, last, left, like, little, locate, location, long, longer, longest; makes, many, middle, meaning, more, most; next, number, number fact, number line; objects, open, on, other, other side, over; place, placement, pick, pip, prove, put, putting together; recall, recompose, relationship, remember, represent, ring, ring around, rotate, rotation, round, row; show, side, similar, similar to, small, smaller, strategy, subtract, subtraction, suit (club, diamond, heart, spade); take, take away, think, token, top, trace, track, turn, turn around, turn around fact, type of card; uncover, uncovering, under, undo; visual cluster, visualize, visualizing; write, written, etc.

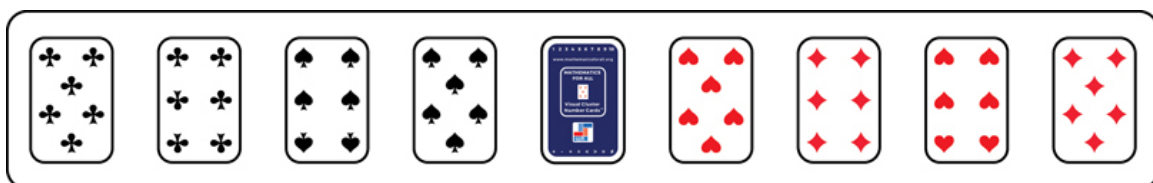


Figure 5.6 Visual Cluster Number Cards for 6

The relevant vocabulary to the concept, skill, or procedure is introduced every day in the same manner, with concrete materials, proper language, scaffolding questions, and in context—when the term is needed to communicate the mathematics ideas.

The teacher keeps two vocabulary wall charts, one for the running cumulative vocabulary and the other for the current topic vocabulary. Words are added by the teacher, and contributions are made by the teacher and children.

1. Introducing Visual Cluster Cards

Children with or without mathematics difficulties or risk of number difficulties and even those who experience difficulty with written and/or oral language processing can learn to use mental images of visual clusters to get to what they already know about certain numbers and apply that information to what they need to understand in number relations. The human brain's nature and innate ability to:

- a. See, discern, recognize, and visualize patterns,
- b. Perceive and recognize small collection of objects,
- c. Perceive the location, space relationships—spatial orientation and space organization, and
- d. Deduce the cardinality of a larger collection of objects by decomposing/recomposing the larger collection into smaller ones and then recomposing smaller collections into the larger one

are the key principles behind the design of each card: number, placement of 'pips'; organization of clusters, decomposition/recomposition of the clusters, and use of these cards—the language and the questions asked. We call them Visual Cluster Cards, as all of these derivations are done by visually perceiving the quantity displayed on the cards in the form of different collections of pips in size, colors, and organizations.

As learners gain skills in the process of visualizing and then verbalizing what they see, they begin to independently match conscious images of clusters (gestalt) to the concepts (numbers) expressed by a group of pips in clusters.

A deck of Visual Cluster Cards (VCC™) consists of 4 cards each with 1 pip to 10 pips in 4 suites (heart, diamond, club, and spade); a card without any pips, which represents the number zero; 2 jokers that can be given any value during card games such as 'Number War Game.' Cards showing 3, 6, 8, 9, and 10 have two to four types of representations of pips on them (in the standard form of clusters with slight

variations and alternative arrangements). For example, see the 6-cards (displayed earlier), where the clusters of six pips represent the number 6.

The following 7-card shows the 7 pips in a particular arrangement; after a few exposures and creating the configuration of the card a few times, the child can recognize the card easily in various orientations without counting the pips. It is just like how a child recognizes a letter after a few exposures and calls its name. Both actions are perceptual actions. The pips on number cards (e.g., 2 through 10) are organized in such a manner that the sub-clusters of smaller numbers than the given number (2 through 10) can be instantaneously recognized on the card. For example, on the 7-card in the VCNC collection or on the Sight Fact Cards collection, one can see sub-clusters of 3 and 4 or 2 and 5 and 1 and 6. In other words, the child is able to decompose it into smaller clusters and recompose into bigger clusters.

Thus, the previous diagrams demonstrate the relationship between the visual cluster of number 7 and the sub-clusters representing smaller numbers that make 7 (i.e., $3 + 4$; $5 + 2$; $6 + 1$). Recognizing visual clusters on these cards develops in children the decomposition/re-composition skill and the sight facts of that number. The organization of pips on each card has been tested with children for its ease to *perceptually* recognize the cluster and sub-clusters with ease and accuracy without counting.

The skilled questioning of a professional can cause the individual to notice where his or her images of clusters do not match the displayed clusters so that needed adjustments in images (and understanding) can be made. When images can be clearly organized in the ‘mind’s eye’ and verbally described without looking at the cards, then decomposition/re-composition can take place and higher-order tasks such as generalizing, making inferences, or drawing conclusions about number relationships become possible. Then, children are able to automatize addition and subtraction facts with ease.

The power of imagery enables a person to ‘see what you mean’ and respond by communicating about ‘the way I see things.’ What is taking place in this process is an example of eidetic (photographic) memory and the process of visualization. The patterns on the cards create a very rich, vivid, complete imprint of visual clusters on the person’s ‘mind’s eye’ (*the working memory*). With a few exposures, the images are transferred to long-term memory.

The practice of recognizing the clusters and sub-clusters not only facilitates the subitizing, decomposition/re-composition process, and the mastery of sight facts, but it also improves the functioning of the working memory. With the help of these clusters, we can recall the sight facts quickly and sometimes with only a little prompting. It is

like learning sight words with visual exposures. This means that the student who couldn't master simple addition and subtraction sight facts without spending hours counting on fingers, number line, hash marks, and objects is able to master them easily with the help of Visual Cluster Number Cards. Later, (a) using decomposition/recomposition, (b) making Ten, and (c) mastery of teens numbers, they can expand sight facts to other arithmetic facts with ease.

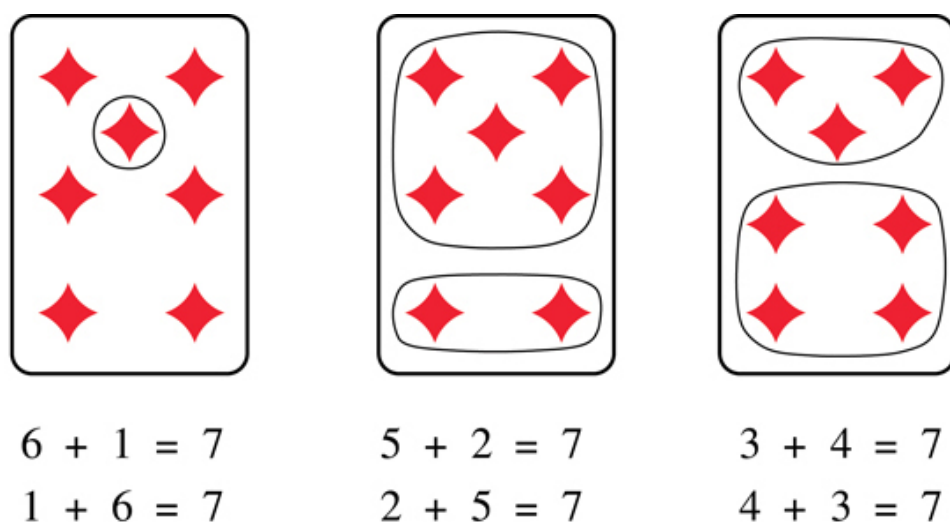


Figure 5.7 Sight facts of Number 7

Children who are taught number concept only through counting—counting discrete objects, number line, fingers, hash marks, etc. struggle with number fact recall, as they have not formed these rich connections, complete eidetic imprints. Most of the interventions focus on counting, speeding up recall, and some general cognitive processing. The efforts made and time spent on the use of multi-sensory aspects of VCNC and Cuisenaire rods help create deeper, richer eidetic imprints and the learn much more efficiently.

Research ([Rayner, 2014](#); [Raynor and Pollatsek, 1989](#)) has shown that skilled readers of English language typically look at words for about 250 milliseconds, make regressions (e.g., look back at words) 10% to 15% of the time, and skip about 25% to 33% of words. Poor readers, in contrast, spend more time on individual letters in the words, regress more, and skip fewer words. Similarly, children with little fluency in arithmetic facts focus on every number (as they count), whereas children with fluent arithmetic facts do not focus on each number but focus on clusters of numbers and use strategies derived from decomposition and re-composition of numbers. This suggests that it is particularly important for elementary and middle school teachers to select

appropriate materials so children can progress steadily rather than risk slowing their fluency and understanding/comprehension development.

All readers must learn the sound/symbol correlations that make up the written code in order to read and spell. Intuitive readers mostly pick up these patterns through exposure and some direct instruction while those with dyslexia need overt, intensive, multi-sensory (see it, say it, write it) instruction in order to internalize the code. Of course, meaning and context matter. Similarly, all children must learn the quantity/symbol correlations that make up the numeracy code to become fluent in mathematics. Most above-average and many average children pick up the patterns of number relationships through exposure and some direct instruction, while a large number of average and below-average children and those with mathematics difficulties such as dyscalculia need overt, intensive, efficient, multi-sensory (construct it, touch it, manipulate it, see it, visualize it, write it) instruction in order to internalize the quantitative relationships—number concept, arithmetic facts, place value, and arithmetic operations. These are better facilitated by effective and efficient instructional tools and a well-thought-out sequence of activities.

Multi-sensory teaching applied in the context of mathematics teaching and intervention means making mathematics concepts transparent and accessible on one hand, and seeing patterns easily and making smooth connections with other concepts and strategies on the other hand. The role of concrete materials is misunderstood by most teachers; to many it only means concrete materials are ‘to touch and manipulate.’ Thus, most of the time, they end up counting objects.

True multi-sensory usage of materials implies that one is working on improving information processing based on the input to sensory pathways rather than just improving the ability to associate quantity and symbols. It is to intentionally manipulate them with a purpose. The purpose is to generate the language, create conceptual schemas, and arrive at procedures and strategies through them.

These aims are achieved through interactions between students and materials under the keen guidance of the teacher—questioning process, manipulation of objects, discussions, generating information, seeking patterns, observing relationships, framing conjectures, making inferences, and arriving at meaningful language, concepts, strategies, and procedures. The setting, the choreography, in which every class member has a role under the directorship of the teacher. The outcome is optimal learning for all.

Instructional approaches that are effective and efficient use direct, explicit teaching of quantity-symbol relationships, number patterns, decomposition/re-composition

strategies, and meaningful number relationships, and provide a great deal of successful practice of skills, with and without problem-solving.

As mentioned earlier, the Visual Cluster Cards are organized in such a manner that the sub-clusters of all the smaller numbers are easy to recognize in the cluster of a larger number. For example, when we begin with the 8-card (the left-most card shown) and when we add one pip to it (see the second 9-card in the sequence), now we have decomposed 9 into two numbers: 8 and 1, and one can easily see the *sight facts*¹ of 9: $8 + 1 = 9$, $1 + 8 = 9$, $9 = 8 + 1$ or $1 + 8$.

Similarly, if we see the sub-cluster of two pips on the bottom of the same 9-card and the cluster of 7 above the sub-cluster of 2, one can see that the 9 is decomposed into two numbers 2 and 7, and it gives us the two other sight facts: $7 + 2 = 9$, $2 + 7 = 9$, $9 = 2 + 7$ or $7 + 2$.

If we look at the 9-card, where the one pip is added to the 8-card in the middle column at the top (see the second card in the sequence), we can see the cluster of the 5-card on the top and the 4-card below it on the 9-card, and thus 9 is decomposed into 5 and 4 or 4 and 5, giving us the sight facts: $4 + 5 = 9$, $5 + 4 = 9$, $9 = 5 + 4 = 4 + 5$.

On the other hand, if we put the 5-card and the 4-card together, we re-compose them into the 9-card.

These two 9-cards do not give us the decomposition of 9 into 6 and 3 easily. To get the decomposition of 9 as 6 and 3, we form a new 9-card starting with the 6 card and adding a group of 3 pips to the 6-card. This gives a 9-card with a different arrangement of clusters. This 9-card (the last card in the sequence) has the 9 pips that are organized into 3 groups of 3s or can be seen as one group of 3 and a cluster of 6. Now 9 is decomposed into 6 and 3 and we get the sight facts: $9 = 6 + 3 = 3 + 6$.

The different arrangements of the 9 into these three 9-cards facilitate all the possible decompositions of 9 and therefore, all the possible sight facts related to 9 ($9 = 8 + 1 = 1 + 8$; $9 = 7 + 2 = 2 + 7$; $9 = 6 + 3 = 3 + 6$; $9 = 5 + 4 = 4 + 5$). These sight facts relate 9 to all the smaller numbers. The same process can be applied for all the other numbers up to 10. In order to derive all the possible decomposition and re-composition, the numbers 8 and 10 also have alternative cluster arrangements.

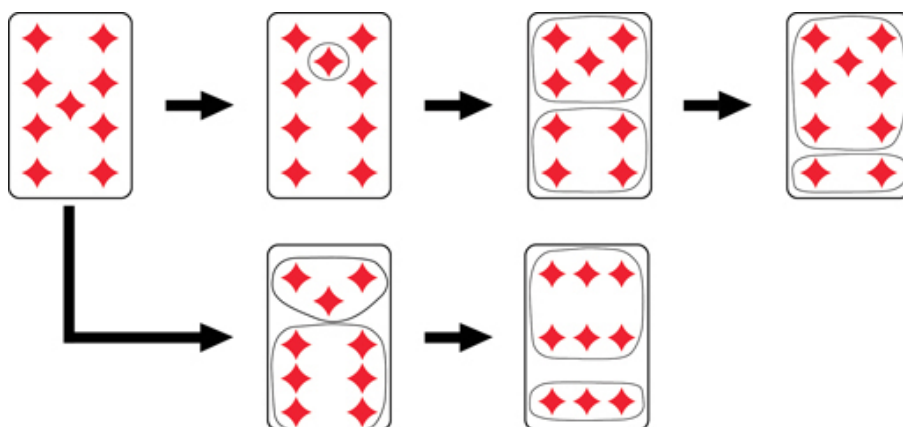


Figure 5.8 Sight Facts of 9 using decomposition and recomposition

However, if we see the next display of 9 pips on the card, there one can see all the sub-clusters on one card.

Thus, subitizing and the decomposition/re-composition of clusters and numbers is at the heart of number conceptualization and learning arithmetic facts (particularly, mastery of addition and subtraction facts). The distributive, associative, and commutative properties of operations and their application are also dependent on understanding the process of decomposition and re-composition.

The decomposition/re-composition of numbers is achieved by a careful introduction of cards, using Cuisenaire rods, and children practicing the process under supervision. Many above-average and average children achieve the decomposition and re-composition of number skills using many other materials, even counting materials, but many average and most below-average and special needs children have great difficulty in arriving at the decomposition/re-composition of number using counting strategies and materials. That becomes the stumbling block for learning number concept and arithmetic facts for them. That is the real reason for thier difficulty in masterig arithmetic facts.

In our extensive work with children with and without learning disabilities, the Visual Cluster Number Cards (with the help of Cuisenaire rods, Invicta Balance, and Empty Number Line) are the most effective materials to achieve decomposition/re-composition skills and mastering arithmetic facts. The following section describes the development of the visual clustering and decomposition/re-composition process. The chapter on Number Concept describes the process of achieving this goal and should form the basis of teaching in primary school.

2. Introducing Numerals/Numbers 1, 2, 3, and 4

The teacher introduces the first three Visual Cluster Number Cards, and the identification of cards is done by counting. Although most children already know the first few numbers, each card is introduced formally. The card with one pip (an icon of heart, diamond, club, or spade) is called out as 1. And from then onwards this card is a 1-card or number 1. And the card with two pips is first introduced by counting the two pips. The card is shown in different orientations (pips are horizontal, vertical, or at an angle) with the following questions:

How many objects are there? ‘2’

How do you know? Raise your hand, if you know the answer. (Points to a child.)

‘I just know it. I can also count.’

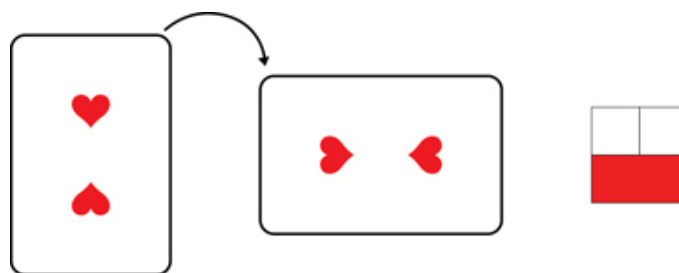


Figure 5.9 Conservation of Number 2

Please count!

‘One, two! See, there are two.’

Good! You counted from top to bottom. What if you counted from bottom to top?

‘It is still 2.’

What if I turn it this way? (Turns the card)

‘It is still 2.’

Good! What about now? ‘2’

Now, children will probably say: ‘It is always 2.’

Yes! It is 2. Now on, we will call this card 2-card or number 2-card.

Using the white rods from your Cuisenaire rods tray, make the 2-card on your desk.

This is the beginning of the Piagetian concept of *conservation of number*. We believe conservation of number is not a prerequisite for acquiring number concept, but is an outcome of proper experiences with quantity as children explore their environment quantitatively.

3. Sight Facts of Numbers from 2 to 4

Now the teacher discusses: 2 is 1 and 1 or two ones (corresponding sight fact: $1 + 1 = 2$ and $2 = 1 + 1$). They hide the 2-card and asks children: What makes 2? The children answer in a chorus: $1 + 1 = 2$, and then the teacher asks individually. They do this with different 2-cards (different suits).

The teacher repeats the same process for the 3-card, using the same types of questions as before. They take the 3-card and each child identifies the card by counting ($3 = 1 + 1 + 1$ defines the number 3). Then they draw a circle around the group of two pips and one pip to show that the 3-card can be seen as: 2 and 1 or 1 and 2 (thus deriving the sight facts for 3: $1 + 2 = 3$, $2 + 1 = 3$, $3 = 1 + 2$, and $3 = 2 + 1$).

A sight fact is decomposing or re-composing a number into two numbers. It is important to emphasize that each larger number is made up of smaller numbers, but we want to break it into two smaller numbers only, at present (not into 3 numbers as in the fourth frame following). For example:

Teacher: (The teacher holds the 3 of diamonds). How many diamonds are there on the card? (The teacher shows one of the 3-cards, one by one.)

Students: 3.

Teacher: You are right. Now look at the card (the Sight Fact card for number 3) with a circle around the diamonds. How many diamonds are circled?

Children: 2.

Teacher: How many are not circled?

Children: 1.

Teacher: What two numbers make 3?

Child One: 2 and 1.

Child Two: 1 and 2.

Child Three: $1 + 1 + 1$.

Teacher: You are right. $1 + 1 + 1$ makes 3 (shows and traces the three individual circled diamonds by hand in the fourth card); $2 + 1$ makes 3 (shows and traces by hand the two circled diamonds and the one diamond in the second card); and $1 + 2$ makes 3 (shows and traces by hand the one individual circled diamond and then the diamond by hand in the third card). We will call $2 + 1 = 3$ and $1 + 2 = 3$ the 'sight facts' of 3. We now have four *equations* as shown here and $2 + 1$.

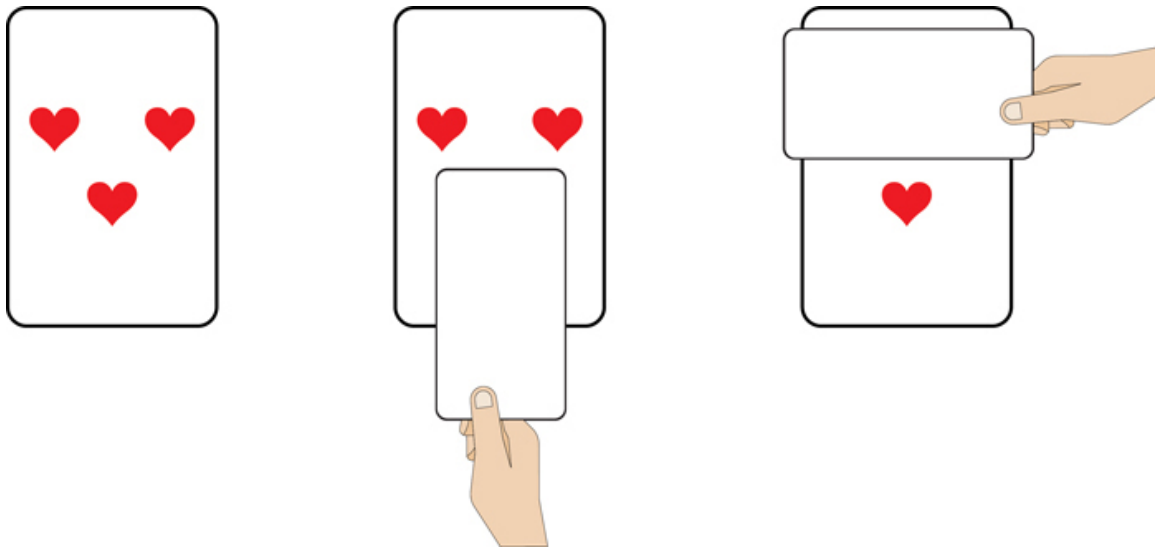


Figure 5.10 Sight fact of Number 3

Once children have created the idealized image of the number 3 in the standard form (visualized in the mind's eye), they should do the same with the three objects in other non-standard forms and any color and shape. For example:

It is important that children create the images of the clusters for number 3 in both forms (standard—as we saw in the red cards) and the alternative arrangement (black cards here).

The process of deriving the sight facts for 3 is repeated using the alternative card for 3. As we will see, the alternative arrangement of the 3-card helps children see the decomposition of higher numbers into smaller numbers (particularly 5, 7, 8, 9, and 10).

This process is repeated with the visual cluster card for 4. The teacher selects one card each day and repeats this process until the children have mastered the decomposition/re-composition of each number into its components (*sight facts*). Thus, knowing number 4 ($1 + 1 + 1 + 1 = 4$ ones $= 2 + 1 + 1 = 2$ twos) $2 + 2$ means that the child sees $4 = 2 + 2 = 3 + 1 = 1 + 3$ and can hold these mentally and can recall, identify, and write these relationships without looking at the icons on the cards. During the recall, they first say them systematically: $1 + 3 = 4$, $2 + 2 = 4$, $3 + 1 = 4$. Then randomly.

A child knows the sight facts of a number when he can recite these relationships without any concrete or pictorial objects.

The Visual Cluster Number cards for the first five numbers and their sight facts can be derived even with 3- and 4-year-olds. We have done this with hundreds of 3- and 4-year-olds.

4. Visual Cluster Card: Deriving its Sight Facts

(a) Tool Building

The teacher has introduced the children to the visual cluster cards. The children can recognize the cards representing 1, 2, 3, and 4. Every day, about 25% of the lesson time should be devoted to the review of earlier work to reinforce it, to make it automatic, to develop confidence, and in preparation for the new topic. This is called the *Tool Building* process: mastering previous skills, language, concepts.

(b) Main Concept of the Day

About 50% of the lesson time should be devoted to the main concept of the day. And 25% should be devoted to practicing, reviewing, and mastering the new concept—either in small groups, individually, or supervised by adults in the classroom. At this time, the teacher and other adults can engage in interventions and remediations with needy children, including gifted and talented children.

This day, the teacher introduces the card representing number 5. The teacher picks up the 5-card from the deck of visual cluster cards and displays the card by holding it in their hand, placing it under the document camera, or on the overhead projector. (A document camera is the best technology for this purpose.)

Teacher: I am going to show a card to all of you. Look at the card. How many diamonds are there? If you know the number, raise your right hand. When everyone has done so, I will ask one of you to tell me the number represented by the card.

Some children easily identify the card and immediately raise their hand. Others count the diamonds on the card. She asks one of these children.

Child: There are five diamonds on the card.

Teacher: Does everyone agree? (She ascertains that by asking several children.) Good! Yes, there are five diamonds on the card. We are going to call this card the 5-card. It represents the number 5. It represents five ‘things.’ ‘Five objects.’

She takes a 5 of spades, 5 of clubs, and 5 of hearts and asks them to tell her what she should call them. Most children, without counting, are able to identify the cards. She places them in different configurations and asks children to identify them. They are able to recognize them. She asks children to create the configuration of the cluster of 5, using the 5 white rods or any five objects on their desks.

It is important that the teacher encourages children to use complete sentences in their answers. She should ask several children to do the same till all children are able to identify the card in less than 2 seconds. She should also ask children to identify the 5-card from the 4 suits (diamond, club, spade, heart) to make sure that the children's answers are not suit- or representation-specific. Children should now practice clusters of 5 with dice, dominos, and cards with other objects on them.

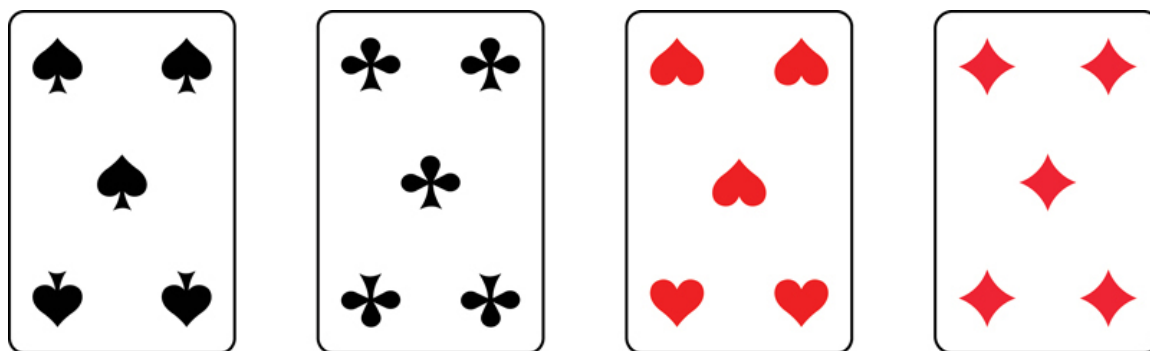


Figure 5.11 VCNC for Number 5

The objective is to help children create the idealized image of this number in their mind (working memory and ultimately transferring it to the long-term memory) and learn the decomposition and re-composition of 5. Without the idealized image of these numbers in their mind, children have difficulty in developing fluency in number relationships and arithmetic facts. This can be done by doing the following activities.

The teacher takes the 5-card and points to the first column on the 5-card (see the following picture) and moves their hand around the two diamonds in the first column on the card. The teacher first asks the whole class and then individual children. On a laminated visual card, the teacher could even circle the first column, asking:

Teacher: How many diamonds are there in the first column of the card (touching and tracing the diamonds in the first column.)

Children: 2.

Teacher: Very good! Yes, there are two diamonds in the first column. It represents the number 2.

Teacher: Now, how many diamonds are there in the last column? (touching and tracing the diamonds in the last column.)

Children: 2.

Teacher: Very good! Yes, there are two diamonds in the last column. It also represents the number 2.

Teacher: Are there the same number of diamonds in the first and the last column?

Children: Yes!

Teacher: How many are there in each column?

Children: 2.

Teacher: Very good! Yes, there are two diamonds in the first and the last column.

Both represent the number 2. We have 2 in the first column and 2 in the last column.

Teacher: What number do these two columns make?

Children: 4.

Teacher: Does this look like the 4-card?

Students: Yes!

Teacher: Now look at the middle column. How many diamonds are in the middle column?

Children: One.

Teacher: What if the middle diamond was not there? What number would the card represent?

Children: 4.

Teacher: Very good! (If a child had difficulty answering this, the teacher can display the 5-card again and point to the child by covering the middle pip with her finger. Or she can pick the visual cluster card representing number 4.) What two numbers make 5?

Children: 1 and 4 or 4 and 1.

Teacher: Yes, $1 + 4 = 5$ and $4 + 1 = 5$.

The teacher now helps children derive the other sight facts of 5.

Teacher: What if the first column was not there? What number would the rest of the card represent? (She hides the cluster of 2 by a white strip.)

Children: 3.

Teacher: Very good! What if the last column was not there? What number would the rest of the card represent?

Children: 3.

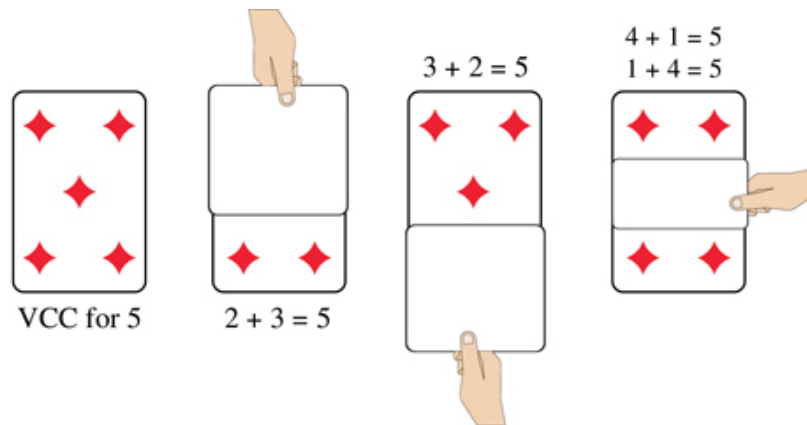


Figure 5.12 Decomposition/recomposition of 5

The teacher should ask many such questions till this concept is really understood. Now children have created an idealized image of the visual cluster of the number 5 in their minds. The teacher asks these questions first as a choral activity and then every child one by one. The teacher should make sure that they have created the image of the number 5 as a visual cluster and its component sub-clusters in their mind. They should be able to tell that:

- There are five pips (objects, things, diamonds, hearts, clubs, or spades);
- The card has a cluster of 1 and 4;
- The card has a cluster of 2 and 3;
- The card has a cluster of 3 and 2;
- The card has a cluster of 4 and 1;
- The card has a cluster of 2, another 2, and then 1.

First, children should be able to give these answers by looking at the cards and then without seeing the cards. Ultimately, they should be able to tell the number represented by the card instantaneously. That means the child knows the card. She shows other cards till the child no longer has to count the pips to recognize the number related to the card. To achieve this level of automaticity in recognizing the cards, the teacher engages in the following activity.

The teacher turns the card around so that the pips are not visible to the child, and asks the same questions once more. Children answer these questions from memory.

Teacher: How many are in the first column of the card? (The teacher touches and traces the first column on the back of the card as if the diamonds in the first column are visible.)

Children: 2.

Teacher: Very good! Yes, there are two diamonds in the first column. It represents the number 2. (And can even show those diamonds on the card by turning the card facing them, however, keeping the other pips hidden by their hand.)

If a child has difficulty recalling the number of diamonds (clubs, spades, or hearts), the teacher should ask another child. If no child is able to answer, the teacher can show the card again to that child and then to other children.

Teacher: Now, how many are in the last column? (The teacher touches and traces the diamonds in the last column.)

Children: 2.

Teacher: Very good! Yes, there are two diamonds in the last column. It also represents the number 2.

Teacher: Are there the same number of diamonds in the first and last column?

Children: Yes!

Teacher: How many are in each column?

Children: 2.

Teacher: Very good! Yes, there are two diamonds in the first and last column. Both represent the number 2. We have 2 in the first column and 2 in the last column.

Teacher: Now look at the middle column.

Children: 1!

Teacher: What if the diamond in the middle was not there? What number would the card represent?

Children: 4.

Teacher: Very good! So, 4 and 1 make 5, right?

(If a child had difficulty answering this, the teacher can display the card again and point to the child by covering the middle pip with a finger.) What if the first column was not there? What number would the rest of the card represent?

Children: 3.

Teacher: Very good! What if the last column was not there? What number would the rest of the card represent?

Children: 3.

When children are able to answer these questions correctly, the teacher engages in the following set of questions. They turn the card around with a portion of the card hidden.

Teacher: Remember the card we have been seeing.

(First, the teacher ascertains the card by asking: What was the card we have been looking at?). So now that you remember the card, I am going to show the card to you, but a portion of the card will be hidden by a strip. You need to tell

me the number of diamonds missing. (Hiding the first column.) Look at the card: How many are visible to you?

Children: 3.

Teacher: How many are hidden?

Children: 2.

Teacher: What two numbers make 5?

Children: 2 and 3.

Teacher: Great! (The teacher uncovers the hidden part of the card and shows the 5-card.) Yes, 2 and 3 make 5. (or 2 and 3 make 5; $2 + 3 = 5$.)

When children have created the image of the cluster of 5 in their heads, the teacher asks them to practice writing the number 5. Ultimately, each child has the idealized image of the number 5 as a figure (5), as a visual cluster, and its relationship with other numbers, and they should understand number 5 as $4 + 1$; $3 + 2$; $2 + 2 + 1$; and $1 + 1 + 1 + 1 + 1$. When the child recognizes the card instantaneously and says ‘five’ and can tell that 5 is five ones, is $4 + 1$, $1 + 4$, $2 + 3$, and $3 + 2$, then the child knows the number 5. Then the child also knows the sight facts of 5. This should happen for all ten numbers.

Just like reading a letter (say, M, which one recognizes by its configuration—by its strokes), similarly, a child should recognize a number (up to ten) represented by the configuration of its cluster, instantaneously. Of course, the child is introduced to the language and writing through cultural tools and rituals; similarly, the child is introduced and helped to recognize the clusters in multiple settings and contexts.

Knowing a number means that children should be able to recognize the visual cluster and its representation as a number (written, symbol form 5) and its component clusters. The whole purpose of Visual Cluster Number cards is to be able to create mental images of clusters of objects with vividness and detail.

A pupil who is able to form or recall eidetic images (as shown in the following visual cluster of 5) of visual clusters can learn the sight facts easily. Eidetic refers to the process of memories becoming unusually vivid, accurate, and reliable. Many dyscalculics lack this skill. The eidetic memory for a cluster is developed by multiple exposures to patterns of objects—dominos, cards, etc., and the accompanying questions and discussion. The more pattern-oriented the instructional materials for teaching number are, the more efficient the learning of the number concept. Some children need much more guidance and focus during these experiences. Others can create that experience for themselves. The VCNCs are a good means of achieving this. Playing cards and games using dice and dominos reinforce the development of these eidetic images also, with some limitations.

Having first identified a cluster by counting or breaking it into components, a child needs to make the identification of that cluster automatic (eidetic) for mastering arithmetic facts. Without the eidetic memory for the clusters, the child will forever labor/struggle as they try to master arithmetic facts.

Holding a Visual Cluster Number Card in ones hand and asking a child to identify the sub-clusters on the card, either by asking the child to circle the sub-cluster on the card by using finger(s) or covering it with a little strip is more helpful than using the cards where the clusters are already marked. Just like the diacritic marks on words are either ignored or make reading more difficult, already circled sub-clusters on the cards do not have an optimal effect. Identification of sub-cluster by finger is like finger tracking in the reading process.

Children with or without mathematics difficulties or at risk for number difficulties and even those who experience difficulty with written and/or oral language processing can learn to use mental images of visual clusters to get to what they already know about certain numbers and apply that information to what they need to understand in number relations. The human brain's natural ability (innate numbersense and spatialsense) to see and visualize well-organized patterns is the principle behind these cards. As students gain more skill in the process of visualizing and then verbalizing what they see, they begin to independently match conscious images of clusters (gestalt) to the concepts (numbers) expressed by a group of pips in clusters.

D. Empty Number Line and Additive Reasoning

[Gray and Tall \(1994\)](#) in one of their research studies explored children's addition and subtraction of two-digit numbers and found three levels of performance in 7- to 8-year-olds, where they moved from limited number facts and limited procedural competence (Group 1) to methods using excessive use of counting (Group 2) to the top level of achievement, where there was a spectrum of performance characterized by the flexibility in the methods used (Group 3).

They found that two kinds of information processing were evident: one based on rearrangement of numbers (decomposition and re-composition of numbers to facilitate efficient strategies) and the other based on the use of instructions and counting. Where the tasks (in both addition and subtraction) are presented horizontally (e.g., $24 + 43$, $54 - 28$), children visualized rearrangements of individual numbers, but when tasks were presented vertically, they fell back on the use of algorithms. In the case of visual arrangements, children made use of their knowledge of strategies and the patterns of numbers and used a variety of methods. When presented vertically, however, they

exhibited ‘rule-like’ behavior, which they clearly did not always understand. (Algorithms do not always have beneficial outcomes).

The reversal of success when moving from horizontal to vertical presentation of the problems is one of the more interesting outcomes of this study and is evidence of the result of constraints placed on flexible thinkers when they feel they must proceed according to given data and a given procedure.

The flexibility children can exhibit when given the freedom to do so is illustrated in an example from a study by [Hiebert et al. \(1996\)](#):

Find the difference in the heights of two children. Jorge and Paulo, who are 62 inches and 37 inches tall, respectively (as cited in de Lange, 1996, p. 96).

The question was expressed horizontally for children to think of it not as procedural, but to find the difference by other methods. One child counted up from 37 to 62 by ones (recorded by dots) and tens (recorded by ‘sticks’), so that it looked like this:



Figure 5.13a ENL to show 62–37

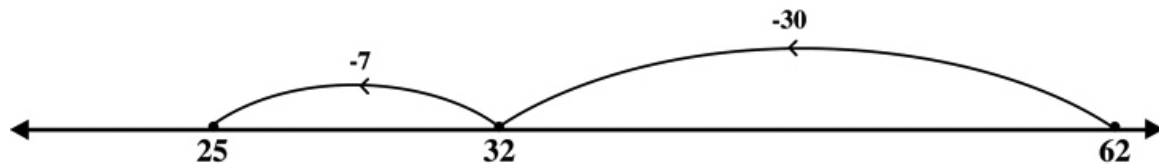


Figure 5.13b ENL to show 62–37

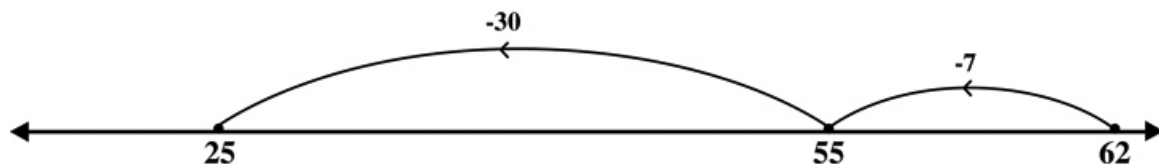


Figure 5.13c ENL to show 62–37

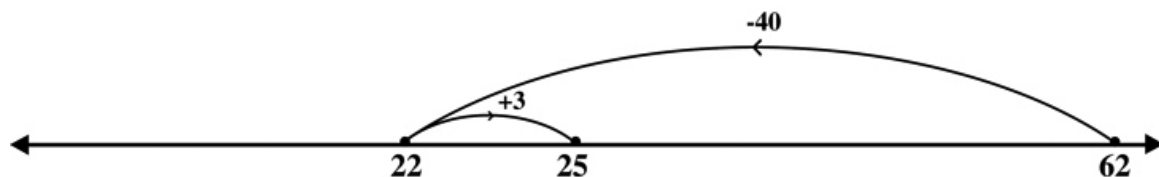


Figure 5.13d ENL to show 62–37

Another child used the ‘Empty Number Line’ (ENL) ([Gravemeijer, 1994](#)) and produced the following: Another child chose to subtract: started at 62, took away 30 to arrive at 32, and then taking away 7 more left them at 25.

Here are a few more examples for the same problem. The diversity of the ENLs shows the flexibility of thought and facility with decomposition/re-composition of numbers, place value, and number relationships.

When children are introduced to standard arithmetic procedures too early and only in the standard vertical format, the development of numbersense, mental math, the concepts of estimation and approximation, and applications of these concepts are negatively affected.

We have observed that when children are introduced to the Empty Number Line early on (first grade), the problems are presented horizontally, decomposition/re-composition of number, and, when both canonical and non-canonical decomposition of place value are emphasized, children have better understanding of numbersense and numeracy. Notably, using a standard Number Line does not produce the same results.

1. Empty Number Line as a Tool/Model

The empty number line (ENL) is a relatively new model to support children in developing mental calculation strategies for addition and subtraction, and it was developed in Holland in the early 1990s. The three components of numbersense in additive reasoning: (a) number concept, (b) addition and subtraction, and (c) place value are reinforced by using Empty Number Lines for addition and subtraction operations. It is also a superior model than the number line and 100 square for numbers up to 100, particularly for children having problems in mathematics ([Beishuizen, 1997](#)). We have also successfully used it for fractions, decimals, and integer operations.

In the use of empty number line (ENL), the lesson should begin with a simple problem to recall addition strategies. Here is the beginning of the use of Empty Number Line in the beginning of the first grade. For example, the teacher may ask:

Teacher: I am going to give you an addition problem. You will find the answer in your mind. When you have the answer, you will give me a signal of some kind, like raising your hand, to let me know that you have the answer. I will ask you to give me the answer. When I have collected answers from most of you and recorded them on the whiteboard, then I will ask you to explain the answer you

gave and the strategy you used to find your answer. Here is the question: What is the sum: $9 + 7$?

The teacher waits till all the children get the answer. Then they elicit their answers and writes them on the board. Then they invites children, one by one, to give their strategies. This is what happened when I gave this problem in a second-grade classroom recently. About a minute passed by the time all the children raised their hands. I invited them to give me their answers. All but one child gave 16 as the answer. I asked a child to give their strategy.

Laura: I thought of 9 as 10 and 7 as 6. I know 10 and 7 is 17. So my answer for $9 + 7$ is 16.

Me: That is a very good strategy. What is the name of this strategy? How many of you thought of the strategy that Laura used?

Several hands went up and students said: 10 plus a number. As the example shows, the children were familiar with the key strategies for addition and have been using these strategies regularly both orally and in writing.

David: I did the same thing but a little differently. I gave one from 7 to 9 to make it 10 and then I had $10 + 6$ and I got 16.

Me: Very good. Yes, it is a different strategy. Did anyone use some other strategy?

Randy: I know my doubles. $7 + 7$ is 14, so $9 + 7$ is 2 more so the answer for $9 + 7 = 16$.

Several pupils said they used that strategy. One of them said: 'I thought of adding 9 and 9. I know $9 + 9$ is 18 and then I know $9 + 7$ is 16.' Several children said that they used the same strategy.

Me: I am glad that you used the doubles strategy. Anyone have another strategy?

Missy: I have a different strategy. I gave 1 from 9 to 7 and I have $8 + 8$ and I know $8 + 8$ is 16.

Several children said that was a 'cool' strategy. One child said: 'I want to use it. Can you give us one more problem so I can use this strategy?'

Unlike the other children, one child said: 'I counted 7 after 9 on my fingers and got the answer. My method is not very good.'

Me: Why do you think your 'method is not very good?'

Child: It takes too long and is not very interesting. It is just counting.

Me: Did you like any of the other methods?

Child: I like several of them. From now on I am going to use one of them.

We continued this process for a few more problems till all the children were using these strategies with ease. We need to devote a few minutes a day to the use of these strategies up to fourth grade, and even later with students who have not automatized

addition and subtraction facts. Facility with addition strategies provides them with the most important skill in acquiring a strong numbersense—decomposition and re-composition of numbers. In the previous example, except for the student with the ‘counting on strategy,’ they were using decomposition and re-composition of numbers.

It is important to extend the decomposition/re-composition process to larger numbers before we introduce the standard procedures. When children are introduced to standard algorithms before they have a strong numbersense, they learn a rote procedure without knowing and understanding the number properties and without knowing what to expect as an answer. They may not even know whether their answer is right or wrong. For example, several students were asked to find the difference between 302 and 298, applied the procedure without thinking, and got the wrong answer:

$$\begin{array}{r}
 9 \\
 2 \text{ } \cancel{10} \text{ } 12 \\
 302 \quad \cancel{302} \\
 -298 \quad -298 \\
 \hline
 196 \quad 4
 \end{array}$$

In the first case, the student did not apply the procedure correctly and cannot even detect that the answer is not even reasonable. This shows no numbersense at the additive reasoning level.

In the second case, the student applied the procedure correctly but does not see the inappropriateness of the procedure and demonstrates that they have no numbersense.

These two solution approaches show that introduction of the procedure before an appropriate numbersense is developed results in such numerical and calculation behaviors that are symptomatic of the lack of numbersense. To avoid this situation, it is important to develop a robust numbersense.

The empty number line is a very good tool to facilitate the development of numbersense. For example, after several simple addition and subtraction problems involving numbers less than 20, the author gave a new problem in the class mentioned earlier. He said:

‘In the game, I scored 37 points in the first week and 91 points in the next week.
How many more points did I score the second week than in the first week?’

Symbolically, the problem is easily translated into $91 - 37$ and can be seen as ‘take away,’ ‘the difference type of subtraction.’ Almost all children in the class were able

to translate the English-language expressions into the mathematics language (symbols) and read it as: ‘91 take away 37’ or ‘take 37 away from 91.’ Some children even read it as: ‘91 is how much more than 37?’ Two children thought this was an addition problem. When I changed the problem into a simpler problem with simple numbers that were more contextual to them:

In the game, I scored 3 points the first week and 7 points in the next week. How many more points did I score the second week than the first week?

they were able to give the author the answer ‘4 more points’ correctly almost instantly. More importantly, most of them saw this new problem as subtraction, but they found the answer by adding 4 to 3. Then the author asked them to read the original problem, and they were able to see the problem as a subtraction problem.

The given problem can be written in the vertical form, and the procedure can be applied. However, rather than introducing them to the standard algorithm, the author wanted them to extend the strategies they have learned to combine smaller numbers with larger numbers. The author asked them to think of the problem in their heads and then describe their process of arriving at the answer.

A number of children put up their hands, and one of them came to the board with her solution. Using one of the several pre-drawn empty number lines (number lines without numbers pre-located) on the board, her calculation began with 91 and ended with the answer 54. The author located the numbers as she narrated them. It looked like the following.

The author talked through Nancy’s solution approach with the class, saying, ‘Nancy starts with 91, then she subtracts 10, and another 10, and another 10 and she ended with 61, then she subtracted 1 from 61. So, how much altogether?’ The class responded that she had taken 31 away from 91 at this point.

‘Very good! What has she left to subtract?’

Several children answered: ‘6 more.’

‘Yes, 6 to make the subtraction by 37. Good! Nancy has used the ‘subtraction by 10’ method. Let us call it “S10 method.” We can show this as follows:

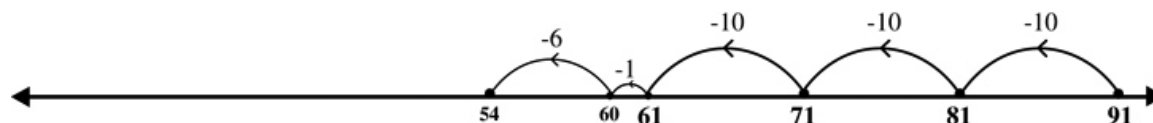


Figure 5.14 Nancy’s solution to **91–37**

‘Could anyone solve this problem any other way?’

Robert responded: ‘Oh yes, you could have started by adding 10 to 37 instead of subtracting 10 from 91. Look at it.’

He showed the class the solution by using the empty number line.

The author talked through Robert’s steps with the whole class. Then he said: ‘Can we make Robert’s 5 jumps of 10 and Nancy’s 5 jumps of 10 each into one jump of 50?’ Almost all children agreed. Some said: ‘It is easier to add 10 and then 10 and so on.’ ‘You are right,’ the author said. ‘But which one is more efficient?’ With some discussion they agreed. The author showed them the revised number line.

Another child said: ‘My answer is also 54.’ The author asked: ‘Why did you decide to add 3 to 37?’ He said: ‘I know 7 and 3 are pairs to make 10.’ He showed the work on the empty number line as follows:

Another child said: ‘I did the same thing. I added 50 to 37 to get to 87, but then I just added 4 directly to 87 to 91 as I know that 7 plus 4 is 11, so 87 plus 4 is 91.’

The author said: ‘It is a very good strategy. Does anyone else have another way of arriving at the answer?’

Another child said: ‘I did it a little differently. I added 3 to 37 to make it 40. And then I added 1 to 40 to make it 41. And then I added 50 to 41 to get 91. I also got 54.’



Figure 5.15 Robert’s solution



Figure 5.16 A more efficient way for Robert’s earlier method



Figure 5.17 ENL to Demonstrate 91-37



Figure 5.18 ENL to demonstrate 91-37: Alternate Strategy

Another said: 'I added 4 to make 41 and then added 50 to get 91.'

Another said: 'I added 3 to get 40 and then 50 to get 90 and then added 1 to get 91. And the answer is the same, 54.'

Another one said: 'I used Nancy's method but a little differently. I subtracted 1 from 91 to make it 90 and then subtracted 30 from 90 to get 60 and then subtracted the remaining 6 from 60 to get 54. I know 6 and 4 are pairs to make 10. I took 37 away in all. My answer is 54.'

David was all excited. He said: 'I have a better method. I subtracted 40 from 91 to get 51 and then added 3 to 51 to make sure that I actually subtracted 37. I added and also subtracted.'

We discussed all these strategies and tried to decide which ones were 'accurate' and 'effective.' Which ones were also 'efficient?' And which ones were 'elegant?'



Figure 5.19 ENL to Demonstrate 91-37: Alternate Strategies

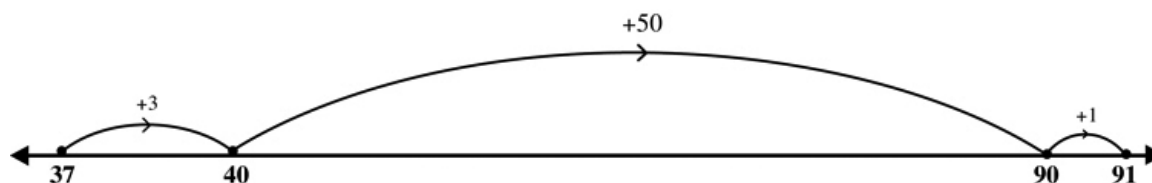


Figure 5.20 ENL to Demonstrate 91-37: Alternate Strategies



Figure 5.21 ENL to Demonstrate 91-37: Alternate Strategies

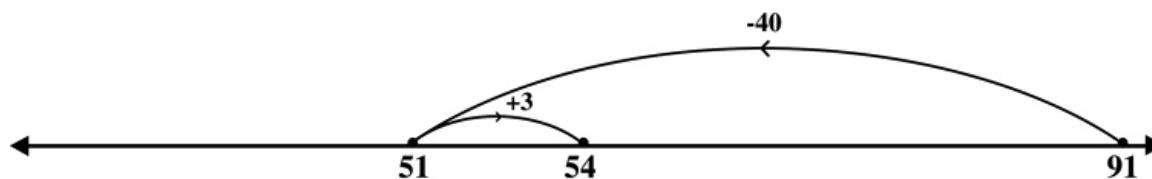


Figure 5.22 ENL to Demonstrate 91-37: Alternate Strategies

Hardly anyone counted on fingers, and after one or two examples, most of them dropped the strategy of successively adding or subtracting by 10s.

At no point was anyone vitally interested in the ‘answer’ of 54 apart from the circle drawn around it. Most children recognized why this was the answer and the indication that it was the solution. Instead, children kept their attention and focus on the methods and strategies that they were using or had used. The point was that they were using a variety of strategies and were interested in generating more strategies. At the end, we summarized and named the strategies (see following names).

When we observe the mental mathematics behavior of experts, one finds that they use a great deal of decomposition/re-composition, both at the single-digit level and the multi-digit level. We find that children who excel in numerical calculations were also exposed to decomposition/re-composition either by effective teachers, parents, and/or able siblings and friends. And some discovered this principle themselves. We started the application of decomposition/re-composition for children to use, first in written form and then in oral form. The senior author remembers using a similar kind of approach in his village school for mental addition and subtraction, as paper was in short supply in Indian village schools. Students were encouraged and helped to learn these mental mathematics strategies. Mental math methods begin with 45 sight facts. Those who do not have effective mental methods for addition and subtraction also struggle with understanding written procedures. Children who have secure mental methods cope easily with written methods (even standard ones like decomposition at a later stage).

Dutch mathematics educators at the Fruedenthal Institute at the University of Utrecht encourage the use of Empty Number Line in an informal and somewhat disorganized way. There, as well, it grew out of a need to serve their overall aim of improving children’s proficiency in mental addition and subtraction to 100 in order to develop a robust numbersense.

There the goal was always to seek flexible mental methods for addition and subtraction up to 100 that could be successively applied by children, with understanding, in a range of simple contexts. ‘Flexible’ means that the children should

know a range of methods and, from them, choose the one most suited to the numbers in each problem.

It has been demonstrated and we have extensively observed that the transition from mental calculations to written procedures is not an easy one for many children, but ENL can bridge it and discussions of ENL strategies are key in this transition.

We have extended their use to adding and subtracting multi-digit numbers, making change in money transactions (what we called, ‘shopkeeper maths’ in England), calculating time intervals, adding and subtracting fractions, and decimals quite effectively.

The names of strategies and moves made on ENL arose in Holland as an administrative device aimed at classifying methods for teacher training and for teachers to convey children’s thinking, but some schools and teachers quickly began to share them with children ([Rousham, 2009](#)), and we agree with that decision. Discussions using the names of strategies are always richer.

When we share strategies and their names with children, they use them more effectively, and discussions of problems and their solutions among children and between children and the teacher are much richer. The use of labels as a tool by both teachers and children encourages children to function at a higher, meta-cognitive level. When children’s cognitive level is improved, they have better idea of numbers and their relationships. Ordinarily, when children are faced with a problem like $91 - 37$, they tend to dive straight into the written calculation procedure. Children who have experienced ENL and discussion that is part of the ENL (therefore higher metacognition) say, ‘What would be the best strategy here? Perhaps N10C, as 37 is only 3 away from 40 ... but maybe it would be easier to apply N10 (to take a jump of 50 to get nearer to 91)’ This is a much more mature and quantitative discussion than just going to direct procedure. These children are not cleverer than the children who count on fingers or use the standard procedure as a first choice, but they have been exposed to better strategies.

2. Empty Number Line Moves: N10

When children begin with the ‘counting forward or back by ten or multiples of 10’ method, this is jumping forward or backwards on the Empty Number Line. This is seen as the basic method for addition and subtraction in Holland and is codenamed N10. The ‘10’ refers to jumping in whole tens. This is one component of the Empty Number Line (ENL) representation.

3. Empty Number Line Moves: A10

This method can be translated as ‘Add to the nearest 10.’ In this method the first step is not to start with tens-jumping but instead to add or subtract units to reach a multiple of 10. When a multiple of ten is reached, it is much easier to jump by tens. The underlying strategy that needs to be mastered by children is ‘making ten.’ In other words, children should master the pairs of numbers that make ten—from now on we will call them ‘complements of each other’ in ten. In our classes we practice it by saying: ‘Give me the complement of 3.’ Children say: ‘7,’ and so on. This practice should be a daily routine in the class, until children automatize it. All other facts are much easier to automatize after this strategy. Automatization means that they can give the response in under 3 seconds without counting them. This is also one component of the Empty Number Line (ENL) representation. Once the children know how to count by ten. The ‘making ten’ is then extended to ‘making the next tens’ number. After that we ask: ‘what is the next hundreds’ number?’ ‘How do we get to the next 100 number?’ The Cuisenaire rods are the best instructional materials to teach this concept. For example,

‘37 is more than 30 or less than 30?’

‘More than 30.’

‘Please make 37 using Cuisenaire rods.’

Child makes 37 using 3 10-rods and a 7-rod.

‘37 is less than 40 or more than 40?’

‘Less than 40.’

‘How do you make 37 into 40?’

‘Add 3.’

A great deal of time should be devoted to making the next tens’ number. After this, they are easily able to make the next tens’ number using Cuisenaire rods by representing it on the Empty Number Line.

4. Empty Number Line Moves: N10C

In this method some combination of A10 and N10 is used.

ENL is just a ‘blank canvas’ upon which any sequential (as against the 100-chart) strategy can be recorded and displayed. Addition and subtraction strategies are based on decomposition/re-composition of number, place value, and addition and subtraction facts. In essence, the key prerequisite skills (already mentioned in deriving addition and subtraction strategies) for success in applying Empty Number Line (ENL) are:

- a. Decomposition/re-composition of number,
- b. 45 Sight Facts,
- c. Making Ten,
- d. Teens Numbers,
- e. What is the next Tens' number,
- f. Canonical and non-canonical decomposition/re-composition of multi-digit numbers,
- g. Adding multiple tens or hundreds to a number.

In our method, as described in the previous chapter: CPVA/S, we have acknowledged five representations of quantity: (a) Concrete (C), (b) Pictorial (P), (c) Imagining and manipulating/visualization (V), (d) Abstract/symbolic (A), and (e) Script (S) Words/Phrases.

The first four (C, P, V, and A) are hierarchical, and the script exists at all of these levels to (a) consolidate thinking, (b) facilitate communication, (c) codify the sequence of activities, (d) facilitate memory (storage and retrieval), and (e) supports each one of the steps and integrates all of them.

Empty Number Line is a very strong pictorial form of representation, and it supports and facilitates the script development and is congruous to the concrete model and then facilitates the transition through visualization to the abstract level. Moreover, it is equally strong for addition and subtraction problems and gives rise to several alternative procedures. Therefore, it presents a unifying method for both operations and for developing a high degree of numbersense. After gaining facility in this method, the formal procedure can be introduced. We have found this to be the most suitable method for the development of facility and fluency with mental math.

It also helps avoid many of the misconceptions and pitfalls of standard procedures that children routinely apply without understanding. For example, suppose that instead of $35 + 26$ it was $35 - 26$. Pathological splitters would say '30 take away 20 is 10, and 5 take away 6 is ...' Many will now go on to perform that 'false reversal,' which says, '5 take away 6 you can't, so 6 take away 5 is 1 ... so the answer is 11.'

Children with facility with ENL just do not make such a mistake as routinely because they are not 'splitters' but 'decomposers and re-composers': using N10 they keep the first number whole (just as they do for addition): '35 take away 20 is 15, and now 15 take away 6 is 9, so the answer is 9.' Or '26 and 4 is 30 and 30 and 5 is 35, therefore, 35 is 9 more than 26.'

Before children are introduced to the place value-based standard methods of addition and subtraction, we should spend a great deal of time on ENL to develop the numbersense needed for understanding the standards procedures properly. We found

that children become proficient in mental arithmetic easier and faster when they practice Empty Number Line early and more often.

The mastery of numbersense helps children to believe that mathematics makes sense. When it stops making sense, learners tend to give up on it and react with dislike, which is a natural and intelligent survival response.

Subtraction is the first big hurdle in making sense of mathematics. Of course, there are many other hurdles that they will inevitably face later: operation of division, operations on fractions, integers, and algebra.

There is no need for us to lose so many children at the first hurdle or even at later hurdles. Actually, teaching additive reasoning and its procedures using better strategies may prepare them to face the later hurdles better.

The relationship between numerical representations and operational calculations is subtle and crucial to the way both teachers and children operate in classrooms. It is the job of teachers to make sure mathematics makes sense to children. In this respect, we need to make sure that we reorganize the teaching of addition and subtraction as interweaving, interconnecting, and inverse operations and connected concepts.

The ENL activity above illustrates that we should emphasize that children should not only have a variety of models available to them to carry out number operations, but they should also be allowed to use their own strategies within these models. When they use their own strategies, we should also discuss with them the efficiency of these strategies.

Real conceptualization of number, the fluent knowledge of number facts, and understanding of place value are necessary to develop fluency and flexibility in operations such as subtraction. For example, in the earlier exercise ($91 - 37$), children operated on the problem confidently. They got the right answers in more than one way, and when asked, they were able to explain their thinking. Finally, none of them were counting on their fingers or on a number line.

Ultimately the progression of instruction on subtraction should lead towards the standard compact decomposition algorithm. This is introduced in most programs via the ‘expanded method,’ initially using an example that requires no ‘exchanging’ or ‘borrowing.’ ‘Borrowing’ is a word that makes many educators cringe when it is used in this context. The word ‘exchanging’ is more accurate when talking about decomposition. This language has developed from a popular approach to teaching the subtraction algorithm using Base Ten materials, where we ‘exchange a ten rod for ten unit cubes.’ We prefer to use a combination of BaseTen and Cuisenaire rods for making numbers and not to exchange a ten-rod with ten ones, but to place the ten-rod next to the number-rod in the ones digit. Therefore, the language becomes: ‘open a

bundle of ten into ten ones.’ It makes the teens number easier to see, and it also provides the situation for non-canonical decomposition. For example, $54 - 37 =$

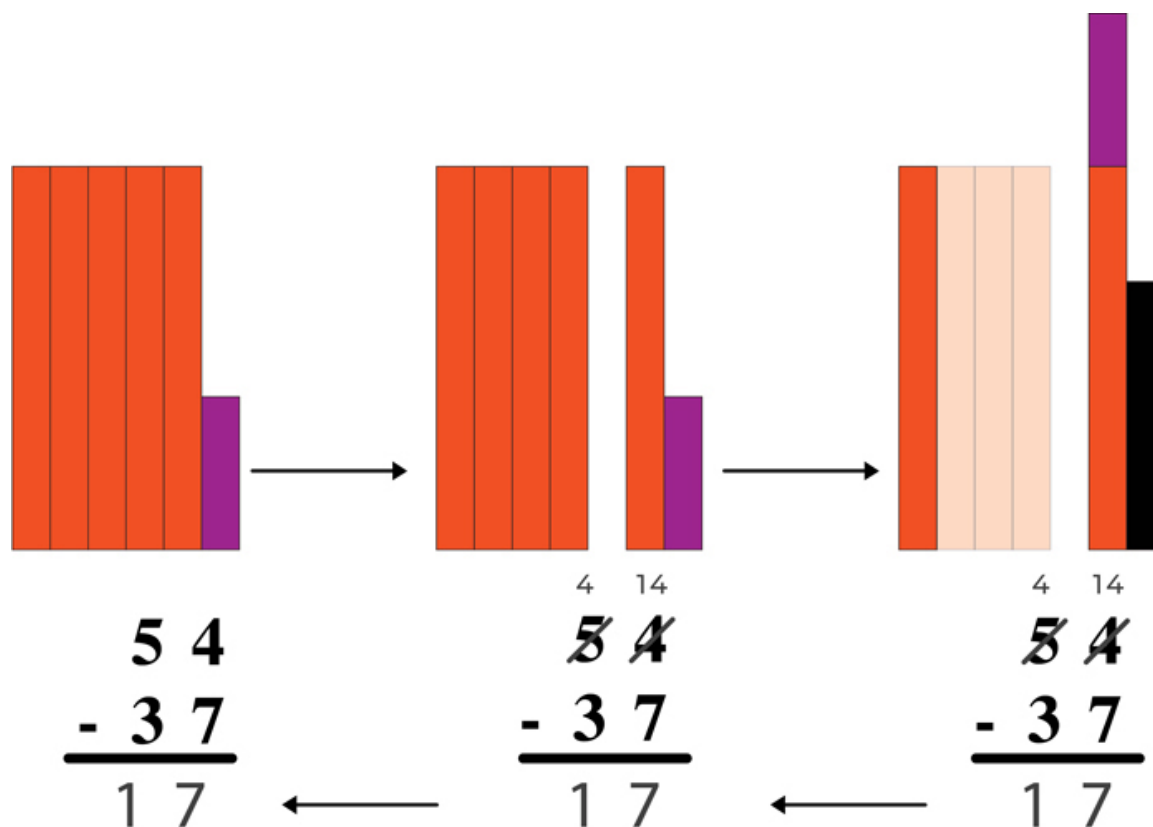


Figure 5.23 Subtraction using regrouping

Here, the 1 ten-rod + 4-rod is easily seen as 14. Non-canonical decomposition is easily achieved.

In most textbooks and programs, the approach to subtraction algorithm is, for example: $563 - 278$

$$\begin{array}{r} 500 + 60 + 3 \quad 563 \\ -(200 + 70 + 8) \end{array}$$

moving on to

$$\begin{array}{r} 150 \\ 400 - 50 - 13 \text{ or } 400 + 150 + 13 \text{ or } 563 \\ -(200 + 70 + 8) - 200 - 70 - 8 - 278 \\ 200 + 80 + 5 \quad 285 \end{array}$$

Leading to 14

$$\begin{array}{r} 4 \cancel{5} 3 \\ - \cancel{5} \cancel{6} \cancel{3} \\ \hline -2 \ 7 \ 8 \\ 2 \ 8 \ 5 \end{array}$$

Now this seems a perfectly logical sequence for arriving at the compact algorithm. However, we know from research and a wealth of teaching that children have difficulty with decomposition unless they have flexibility with canonical and non-canonical decomposition of place value, mental mathematics, and if teachers have used manipulative materials that are congruent with the compact standard algorithm.

We have found that with the help of Cuisenaire rods and Base Ten apparatus, decomposition/re-composition can be achieved much more easily. The key element to achieve here is children's practice with non-canonical decomposition ($73 = 60 + 13$) compared to canonical composition ($73 = 70 + 3$). Unfortunately, in most cases, this is the reason for their difficulty in subtraction involving 'borrowing'. Exercises such as Empty Number Line provide practice and the opportunity to master non-canonical decomposition/re-composition. A great deal of work must be done with activities involving non-canonical decomposition.

The previous problem can be represented by ENL as follows:

$$278 + 285 = 563$$

Vertically it can be arranged as:

$$\begin{array}{r} 563 \\ -278 \\ \hline \end{array}$$

$$278 + 285 = 563$$



Figure 5.24 Demonstration of $278+285$ using ENL

$$278 + 285 = 563$$

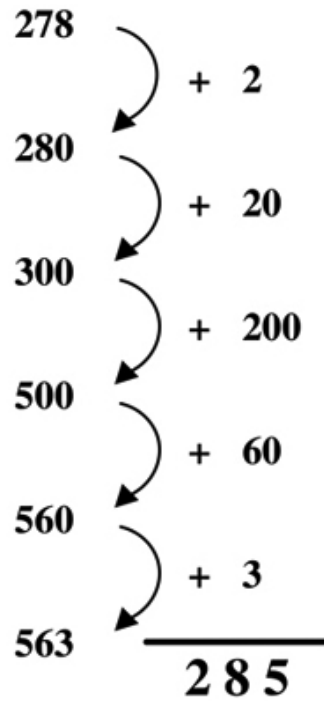


Figure 5.25 $278 + 285 = 563$

Begin with 278

$$278 + 285 = 563.$$

This method of complimentary addition is parallel to ENL. The steps are decreased as children's mental calculations improve. For example,

563

–278

Begin with 278

+22 (300)

+200 (500)

+63 (563)

285

Some books introduce an alternative strategy using negative numbers as follows:

842
-276
600
-30
-4
566

However, it is difficult to believe that most children will be able to understand the operations with negative numbers as they have not been introduced to them yet.

Number Rods

As we discussed in the previous sections, numbers and many other concepts in mathematics manifest in discrete, semi-discrete, continuous, or sometimes in all forms. For example, when we want to know the number of students in a class or find the number of pencils or people in a room, we say a number which shows the number of discrete entities, whereas when we talk about area, time, length, and so on, these are continuous in nature. The child needs to make switches in their linguistic, conceptual, and perceptual schemas to accommodate these variations. However, in most math classrooms and learning situations the language, conceptual schema, and even the material become centered around one or few limited aspects, like dominos or Ten-Frames are all counting materials with few patterns and thereby become ineffective in the instruction of concepts like area, length, etc.

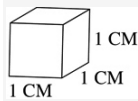
This limits the child's faculty and facility of extending and generalizing certain ideas. When we see the existing tools to teach numeracy, we find that children work with only one modality and usually the countable materials. These tools are either mostly discrete (Dienes blocks, Ten-Frames, dominos, etc.) or continuous (measurable). These tools leave very little residue in the experience of teaching and learning. We realized and saw the importance for the tool to inform and equip a learner about the relationships, related conceptual structures, and different modes and models of a mathematical idea.

Number rods are a tool that we developed while keeping in mind these fundamental transitions and relationships between discrete and continuous aspects of the concepts. Number rods are a linguistic, conceptual, and a pedagogical tool that aids and augments a child's numeracy and later its extensions to proportional, algebraic, and geometric reasoning.

These are a set of 10 different colored blocks with graded lengths from 1 cm to 10 cm. Each rod has also been divided based on the factor of its length. For example, the

10-rod has 1 flat face, one face with 2 divisions, another with 5, and the last with 10 divisions. These divisions are the factors of 10. Hence, it becomes extremely transparent and easy to see that the 10 is made of 10 ones, 5 twos, 2 fives and so on. This fundamental change gives freedom and flexibility to expand our understanding and attention from inter-conceptual relationships to intra-mathematical relationships to multi-models. A few of them are shown here, and their specifications and applicability are briefly described.

These are cuboids with the following dimensions and number representations.

Number rods	Specifications	Conceptual extensions
 Figure 5.26	It is a 1cm ×1 cm×1 cm block. It's white in color and represents a unit/one, and while teaching of numbers, it represents 1.	Since these blocks have no numbers written on them, we can easily extend and impose new meanings on this. For example, it can denote a one, a cube, and hence a unit for measuring volumes, and also one of its edges can be used to measure length of the objects as it is 1 cm in length and acts as a unit of measure.
 Figure 5.27	It is a 1cm ×1 cm×2 cm block. It's red in color. It is as big as two white blocks. In the new variation it has 2 parts shown by a line dividing one of the faces in two equal parts as shown in the figure. Also, other faces are continuous and have no divisions.	The longer face shows a 1 × 2 rectangle. It shows that two ones make one whole and also shows that the continuous part is just two as an entity in itself. So this integrates both the discrete and the continuous nature of the concepts.

Number rods

Specifications

Conceptual extensions



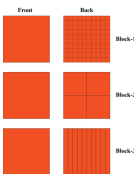
Figure

5.28

Number
10 rod

It is the tenth and longest rod in the sequence. This is an extension of the Cuisenaire blocks, where there are further divisions which represent the factors of 10 like 1, 2, 5, and 10.

This is one of the most effective tools that can help understand number concept, number relationships, and the four operations effectively, efficiently, and elegantly. Moreover, the divisions on the block make it versatile to develop concepts like factors, fractions, prime numbers, and so on.



Figure

5.29

The
Modified
100
Block

These are set of 3 blocks which represent quarters, tenths, hundredths, and so on. One of the blocks has one flat surface and the opposite surface has 100 squares on it. The second block in the set has the 10×10 square divided into 4 equal parts; the opposite side has the same 100 squares and the third block has 10 equal parts. and the opposite face shows 100 squares.

The 100s block as generally seen is also extended to include a flat surface which can model area, fractions, decimals, percentages, and so on very well. It is an extension of the existing Dienes block, which has 100 small squares on them.

Teaching of Sight-Facts Using Number Rods

We represent the teaching through the dialogue demonstrating the following questioning process. (Adam is just a representative student.)

Teacher: Hi Adam, what do you see in front of you?

Student: I see blocks of different sizes.

Teacher: Good! Can you tell me which one is the smallest block here?

Student: Here. (Picks up the white block.)

Teacher: Good. What color is that?

Student: That is white.

Teacher: Can you tell me what colors you see?

Student: (Tells the colors, and the teacher can scaffold if the child misses any color.)

Teacher: Can you give me one of each color?

Student: (Takes out different colored blocks and checks if all colors have been taken. The teacher gives reasonable time for the activity.)

Teacher: How many blocks do you see?

(The child counts the blocks.)

Teacher: Can you give me the smallest from this?

(Gives the smallest block, and the teacher asks to place it on the table.)

Teacher: Can you give me the smallest now from the remaining blocks?

(The teacher repeats this, and then after 3–4 times, the child automatizes the script.)

This is a tool building exercise to help a child remember the sequence for successful experience of the tasks and activities further. Once the child has put all the blocks in sequence, the teacher can then assign names to these in order from 1 to 10 and keep this arrangement on the side for reference in case the child is not familiar with the blocks. For children with prior experience with the blocks, this activity can be done independently.

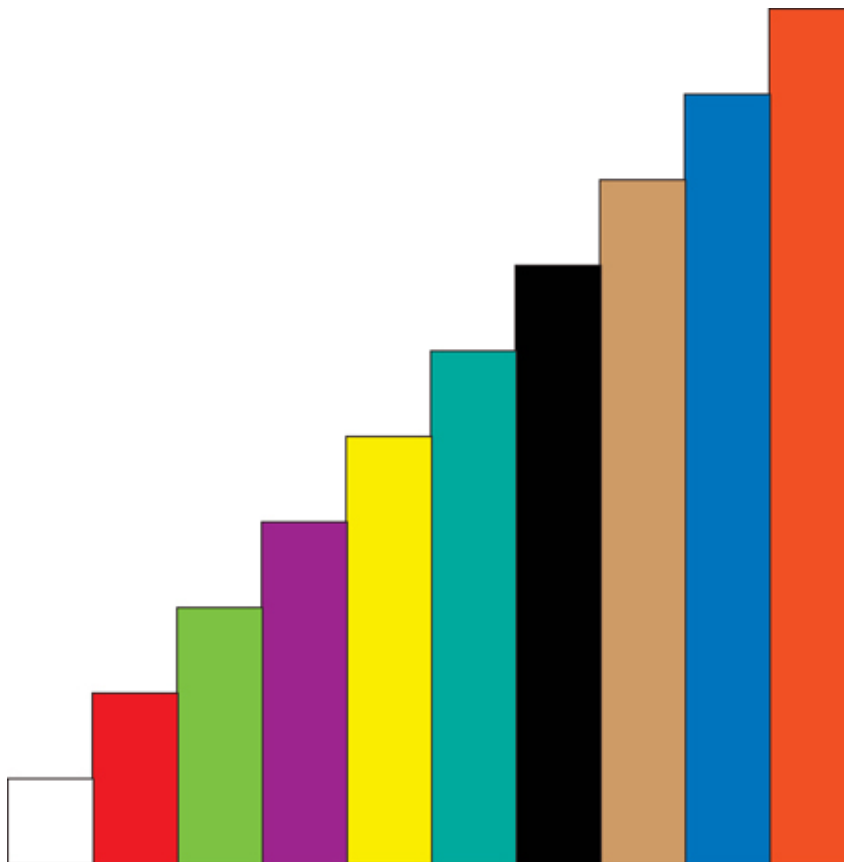


Figure 5.30 Number rods from 1–10

Teacher: Can you take the 10-block and put it horizontally?

Adam: What does horizontally mean?

Teacher: (Demonstrates—didactic teaching)

Adam puts the block horizontally.

Teacher: Can you take a 1 and put it below the 10?

Child: What does ‘below’ mean? (Language is a cultural tool and develops with interaction with people in a meaningful context. Also in teaching, we must have minimum assumptions. As a child may not know the meaning of these words, this calls for appropriate explanation and at times explicit instruction to help understand the language of the instruction.)

Teacher: ‘Below’ means down here (and the teacher points out the position and the child then puts the block below). This on the top is called above (these linguistics schemas become crucial tools and mediums for further instruction).

Now tell what does a ‘1’ need here to become as big as/as long as/as much as a 10?

The child may put the 9-block immediately or may do some trial and error. We must see the difference between strategic behavior or random directionless behavior and build awareness of the task by asking good questions. Suppose a child puts a 7 block; then rather than saying that it is wrong, it is more helpful to point out to the child whether the child needs a bigger/longer block than that (Socratic questioning).

When the child is able to answer correctly then the teacher can ask supportive questions so that the child can develop a much more comprehensive understanding of the idea. (Prof. Sharma states that when we ask different and thinking questions, then the information reaches different centers of the brain and more neural pathways are created for the learner.) Such experience and corresponding conceptual schema developed then becomes both storage- and retrieval-rich. Questions are the gateways and supporters of thinking.



Figure 5.31 Deriving sight fact of 10



Figure 5.32 Deriving sight fact of 10



Figure 5.33 Deriving sight fact of 10

Some suggestive questions can be:
what did the 1 need to become 10 or
what does the 9 need to become 10 or
1 plus what gives 10 and so on.

This language becomes the container for holding the concept. If the experience of the child is not relayed to brain centers using proper language, then it becomes dissipated and the understanding and fluency is impeded.

Teacher: Can you read this equation?

Adam: What is an equation?

Teacher: A mathematical statement that shows equality is called an equation. Like this arrangement of 1 and 9 together equal to 10?

Adam: Yes!

Teacher: So this is called an equation. We read it as ‘one plus nine equals ten.’

The reading and the gesture of fingers while reading are congruent.

Teacher: Can you read that another way?

Adam: No.

Teacher: (Now reads from the right-hand side and reads $9 + 1 = 10$ and asks if the child can repeat, with emphasis on corresponding gestures.)

The teacher reads the equation in two more ways:

$$10 = 1 + 9$$

$$10 = 9 + 1$$

In the $10 = 1 + 9$, the teacher touches 10 first, makes a gesture of ‘equals’ with fingers, and then touches 1 and 9.

The child (Adam) is expected to imitate and repeat the gesture. If the child is unable to repeat, then the teacher holds the child’s finger and does the gesture.

The goal is to develop linguistic, conceptual, and procedural schema.

Also, when the child touches the blocks in a particular order, they also develop psycho-motoric intelligence and self-regulation, which improves executive function and is a strong predictor and prerequisite of mathematical achievement and competence.

The teacher then asks what is 1 more than 1, to which the child must reply with '2'.

If the child does not reply with 2, then the teacher can get a 1 rod and a 1 rod and prove that it is a 2.

The teacher puts the 2-block on the left as shown and asks again a similar sequence of questions as in the previous part. This sequence of instructions becomes the script, which is a succinct and sequential list of steps to be taken to perform a task. These scripts become linguistic containers to hold the process of doing the operation or concept.

Proper questioning helps a child to think and anticipate things rather than sudden surprises. When the child starts seeing patterns in the questioning process and becomes aware, then cognition and metacognition improve as a result.

Teacher: Good. Can you tell me what I am going to ask you?

Adam: Do you want me to make a 2 as big as a 10?

Teacher: Well done.



Figure 5.34 The sequence of putting the number blocks to derive sight facts of 10

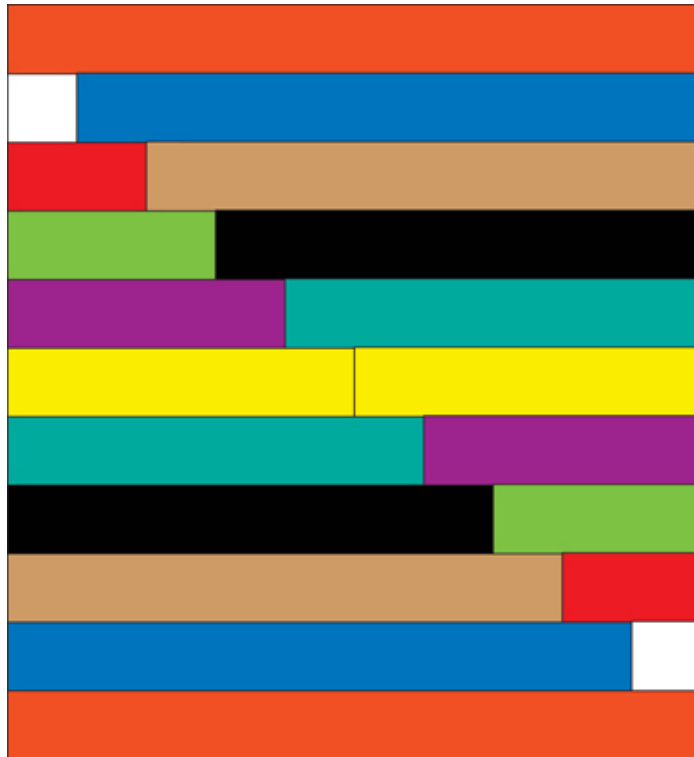


Figure 5.35 Sight facts of 10 using Cuisenaire rods

Adam puts the 8-block in the missing space and continues to form the pattern shown here.

This process is repeated in similar fashion and is repeated for automaticity.

The child generates the following arrangement through scaffolded questions.

The child then repeats this exercise sequentially and forms the arrangement shown here.

Rich dialogue and questioning processes are the key to helping a child internalize these relationships.

These are questions like what is 2 and 8 together, what does a 3 need to become 10, if 3 needs a 7 to become 10, then what does a 7 need to become a 10. If we are going in decreasing order on the left, then what number will you put next after 3, and so on. These questions improve metacognition and when cognition improves, the child becomes a better learner. This concrete experience is then carefully scaffolded to let the child internalize these relationships which become eidetic memory and the child is then able to recall these relationships effectively and efficiently. Also, these relationships and the associated sight facts become automatized and the child can commit them to memory. It is also worth remembering that oral practice of these facts with an adult is much more meaningful initially as the constructive feedback received

from the adult during the process creates neural pathways for sound understanding, hence errors in this process are stepping stones to mastery rather than something to be detested and shamed. It is often supportive to learning if the appropriate symbols (numbers) are shown next to the materials.

Note

1. A sight fact is an arithmetic fact that is learned by just sight, visual exposure, or observation of a cluster. It is just like sight words. More on this in the chapter on Number Concept. [!\[\]\(95b42f0077faf7439a26242a54e021ec_img.jpg\)](#)

6 Mastering the Concept of Number

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I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely in your thoughts advanced to the state of Science, whatever the matter may be.

Lord Kelvin, 1883

In most societies, many parents individually, and the larger society, compared to reading, undervalue and underestimate the development of mathematics skills in children, particularly during infancy. Parents, teachers, and schools place an early and heavy emphasis on children acquiring linguistic and reading skills. In advanced societies, (pre-electronics and the internet) almost every parent reads, and most of them read for and with their children. They watch movies with them and discuss them with their children. They share ideas with them. Very few parents make their children aware of quantity, number, space, and their uses.

When a child is having trouble acquiring language and reading skills, required screening, assessment, evaluation, and diagnosis follow and the search for solutions to rectify the problem ensues.

Letter names are needed for various purposes—spelling students' names, finding objects with letters on them, singing/saying various songs and rhymes, completing puzzles, etc. These activities are typical for Pre-K before formal reading instruction begins. Teachers and administrators therefore recommend making sure that children entering kindergarten are familiar with letter shapes and names for the first few weeks—and then, formal reading instruction proceeds. Numbers also abound in children's environments, and children have the capacity and ability to learn about numbers and their use. Very few parents have discussions with their infant children about numbers and making them aware of number skills.

There is enough research evidence suggesting that children can and should start to read and know about quantity and space as soon they become aware of parts-to-whole, and spell words as soon as they know the associations for a vowel and a few consonants.

And they can begin to learn forming letters and numbers as soon as they can put food in their mouth with the help of a spoon or a fork.

The National Research Council's Committee on Early Childhood Mathematics released a summary of research and recommendations to address the crucial national need to prepare all children to do well in kindergarten. This report, *Mathematics learning in early childhood: Paths toward excellence and equity* ([National Research Council, 2009](#)), provided the major research basis for the early introduction of number study and math expressions and the development of resources for early number learning. Some of the NRC's major recommendations are summarized and exemplified in the publication known as *Focus in Prekindergarten* (2010), published by the National Council of Teachers of Mathematics and the National Association for the Education of Young Children ([Thompson, 2001](#) and [2009](#); [National Research Council, 2001](#); [National Council of Teachers of Mathematics, 2001](#)).

A. Letters Vs Numbers/Digits

Reading is the priority of parents and schools when a child enters school. If a child encounters problems in acquiring reading skills, in most schools, there is a reading specialist to deal with the issue. That is not the case with numeracy. Concerns about mathematics difficulties are raised much later. Further, the methods of teaching mathematics in the early grades have hardly changed in most schools. They are still inefficient, and teachers have significantly low expectations for children in mathematics. Too many children, as a result, not only start behind, but they also begin with a negative attitude toward mathematics, with long-term effects. For example, most teachers of the early years of school mathematics believe that numerical competence begins with counting and just recognizing the orthographic representation of whole numbers. They do so because that is how language acquisition, reading, and writing begin. They should not forget that they can do so in reading as the child already has experience with language for the past 4 to 5 years. By this time, the child speaks fluently and understands spoken language. The child already knows sounds and how to break them down and even combining them to some extent. That is not the case with number. The child does not have such extensive experience in recognizing and breaking down and putting quantities together. Teachers and parents need to realize that experiences in recognition of quantity are to number as recognition of sound is to the formal learning of letter or letter combinations.

In recent years, federal and state policymakers have emphasized the need to improve children's reading skills early on, because a lack in this essential skill is a strong predictor of low student performance and increased high school dropout rates.

While the emphasis on reading proficiency and attention is critical, research also shows that early knowledge of and mastery of numeracy skills not only predicts later success in mathematics but also later reading achievement, even better than do early reading skills. More importantly, math skills at kindergarten are the strongest predictor of later achievement and development of executive function—an essential skill for academic success ([Duncan et al., 2007](#)). Some of the research shows that early math skills have the greatest predictive power, followed by reading and then attention skills. More importantly, mathematics knowledge about quantity and spatial reasoning, acquired in preschool and kindergarten, predicts achievement even into high school. In fact, research shows that doing more mathematics increases oral language abilities, even when measured during the following school year. These include vocabulary, inference, independence, and grammatical complexity.

Given the importance of early mathematics skills to academic success in all subjects, all children need a robust knowledge of numeracy skills early on. Therefore, in kindergarten, equal importance should be given to numeracy, literacy, and spatial sense. At present that is not the case. There is a need to expand the mindset of parents, teachers, district leaders, and policymakers from a ‘reading only’ early intervention strategy to one that incorporates and even emphasizes early mathematical thinking and reasoning, because children who enter kindergarten with higher levels of knowledge of quantity and space do better both in math and in reading in later grades.

The capacity to represent and reason about numbers is part of the inherent structure of the human mind. Number, like color or movement, is a basic property of the environment. Recently, scientists have found that single neurons tuned to number have been observed in animals. They have used both psychophysics and neuroimaging to examine whether a similar neural coding scheme is present in humans. The answer is yes. When infants viewed sets of items with variable numbers, their bilateral intra-parietal sulci responded selectively to number change. Functionally, the shape of this response indicates that humans, like other animal species, encode approximate number on a compressed internal scale.

Anatomically, the intra-parietal site coding for number in humans is compatible with that observed in macaque monkeys. These results therefore suggest an evolutionary basis for human elementary arithmetic. In the first few months of life, human infants can enumerate sets of entities and perform numerical computations. One school of thought is that these abilities arise from general cognitive capacities not specific to number. On the other hand, researchers propose that humans possess a specialized mental mechanism for number, one which we share with other species, and which has evolved through natural selection. This mechanism is inherently restricted in the kinds of quantitative knowledge it can support, leading to some striking limitations in early competence.

The earliest years of a child's education—from age three through third grade—set the foundations upon which future mathematics learning is built. If one cannot read by the third grade, then the negative impact on one's future in most societies is profound. The ability to read beyond a functional level is critical to an emotionally and physically healthy, safe, and fulfilling life. It is also clear now that if a child does not have strong numeracy skills by fourth grade, their chance for future success is also limited.

B. Quantity and Number

It is now widely acknowledged that the typical human brain is *endowed by evolution* with a mechanism for mentally representing and discriminating quantity. Initially, the development of such perceptual skills is limited to recognizing collections of objects consisting of up to about 4 or 5. However, exposure to these kinds of activities allows children to understand and manipulate small quantities with precision, and much larger quantities as an approximation. These abilities are, of course, a far cry from the abilities—society, teachers, and parents—expect and hope to develop in children. To achieve societal goals these natural skills are the base upon which teachers must and can build the edifice of arithmetic and mathematics.

Representing them in our familiar symbols—counting words and 'Hindu-Arabic' numerals, or any other representation of the number of items in a collection takes place by *cultural input*. In other words, since quantifying skill is innate and it is made real by cultural input, the more quantitative experiences we provide to children, the more we formally represent the cardinality of the set, as we know. In the process, unnamed mental representations are taking place in the child's mind. The endowed ability to quantify is made real by the help of cultural tools: language, symbols, rituals, games, toys, and other formal and informal interactions with the culture and their environment. Evidence for such advancement comes from a variety of sources.

Acquiring number concepts and their usage is an important cultural concept and a versatile tool. Learning and mastering it is an intentional, deliberate cultural process. An enduring question in philosophy and psychology, in the school room, and in every home is that of how we come to possess knowledge of number and use of this tool to such a level of sophistication and ease as we see all around us ([Milinkovic, 2015](#)).

Number concept is one of the first of a series of tools necessary, along with reading skills, for becoming a productive citizen, the ability to solve problems for survival, and one of the ways to demonstrate one's cognitive potential. Just like learning to use any tool, the outcome can be enhanced if we know how to learn these tools efficiently and effectively. For this reason, an introduction to, and a learning of, and the proper use of numeracy tools by children should begin as early and as efficiently as possible.

In the first few months of life, human infants can enumerate sets of entities and perform numerical computations. One school of thought is that these abilities arise from general cognitive capacities not specific to number. On the other hand, researchers propose that humans possess a specialized mental mechanism for number, one which we share with other species, and which has evolved through natural selection. This mechanism is inherently restricted in the kinds of numerical language it can support, leading to some striking facilities and limitations to early competence.

A line of evidence which has emerged indicates that children may have an innate proclivity for quantity which comes from the studies of other species. Many of those in whom numerical abilities have been tested show performance comparable with or significantly better than human infants. For example, chimpanzees are able to match the correct digit to a random display of dots up to at least ten ([Matsuzawa, 1985](#); [Tomonaga and Matsuzawa, 2002](#)). Monkeys are able to select the larger numerosity of two displays even when the elements in the display are novel. Moreover, they show a very similar ‘distance effect’ to humans—that is, the more different the numbers, the more likely they are to select the larger correctly ([Brannon and Terrace, 1998](#)). Birds have been known to be good at number tasks. Numerical abilities have been demonstrated in elephants, cats, rats, dogs, pigs, salamanders, and even fish ([Agrillo et al. 2012](#)).

There is also evidence for individual differences in various measures of innate number ability, at least in older children ([Geary et al., 2009](#); [Piazza et al., 2010](#); [Reeve et al., 2012](#)). Twin studies suggest that these differences appear to be at least partly genetic ([Geary et al., 2009](#); [Piazza et al., 2010](#); [Reeve et al., 2012](#)). The genetic factor is reinforced by the finding that certain kinds of genetic anomalies, such as Turner’s Syndrome, affect numerical abilities, including very basic abilities such as selecting the larger of two numbers or giving the number of dots in an array, even when general cognitive ability is normal or even superior ([Bruandet et al., 2004](#); [Butterworth et al., 1999](#); [Temple and Marriott, 1998](#)).

Numeracy, reading, and writing skills are essential adaptive skills for daily life, enabling individuals to function effectively in their society. Research results suggest that inhibition, working memory, and cognitive flexibility have direct and indirect effects not only on reading comprehension of narrative and expository texts, but the direct effect of working memory is highlighted, both for reading comprehension of expository and narrative texts and numeracy skills and their usage. Additionally, academic vocabulary is an important mediating variable between executive functions, especially inhibition, and reading comprehension and numeracy skills in students of this age.

Today, numeracy is the way to participate fully in a technology-dependent world. For example, success in any profession and career, in our knowledge-based, technological

society, one needs to be both literate and numerate. In fact, numerates have a distinct advantage in the modern world—*numeracy is the new literacy*.

Being numerate means having a keen sense of quantity and space, flexibility with the use of numbers and number relationships, and having the ability to use number skills to cope with the quantitative demands of everyday life. This means, at least, the mastery¹ of numeracy—number concept, numbersense, and computational fluency (the ability to execute the four whole number operations, in the standard and alternative forms, correctly, consistently, efficiently, and with understanding). It involves, for instance, observing numerical trends in information—in graphs, charts, tables, or in reference to percentage increase or decrease. This facility in applying quantitative reasoning and the resultant confidence in numbersense strongly predicts later success in STEM fields.

In most cultures, most children, as part of the acculturation/socialization process, develop a basic sense of number and arithmetic (rudimentary number relations and procedures) without explicit instruction and few even go beyond. Modern societies, however, emphasize the mastery of arithmetic as the focus of early school experience for all. The goal of teaching mathematics to young children (even very young children) is in developing a sound foundation in numbersense, inculcating the mathematical way of thinking, and acquiring the ability to integrate numbersense and spatial sense to prepare them not only for arithmetic but also for higher-order mathematics and other related disciplines.

C. Children's Concept of Number

Quantification, paying attention to quantity, and the use of numbers beyond a handful is a uniquely human evolutionary trait. Acquiring the number concept or numberness—understanding number and its manifestations as quantity, counting, and its simple applications is a fundamental skill. Mastering early numbers (digits and numerals) is like acquiring the alphabet of the mathematics language and the usage of this alphabet to make larger numbers (and later numbers smaller than one) is essential for robust numeracy skills. Expressing and communicating that knowledge as language, following certain rules and conventions of the language, is an important challenge and achievement for a child during school years. Fluent use of number and its language, in its multiple forms, results in the development of numbersense and numeracy. Whereas number concept is like the alphabet, numbersense is like forming words with meaning by combining letters. The use of language and reading with fluency and comprehension can be compared with the process of deriving, manipulating, and using strategies, procedures, and operations with understanding. Productive living in the modern world is dependent on the effective use of reading and numbersense.

From a pedagogical perspective, numbers and operations have two aspects: *conceptual* (e.g., what numbers are) and *nominal* (how we name and denote numbers). At least two possible cultural tools and pathways exist for making innate skills of subitizing, approximate number sense (ANS and Spelke, 2000), and decomposition/recomposition into the development of whole numbers: *counting* (discrete) and *measurement* (continuous).

Conceptually, numbers arise from making sense of quantity when one deals with some collection of objects, in two ways: (a) *approximation* (estimate or count a set or collection), or (b) *measurement* (measure or estimate the distance between objects, area, volume, time, rate, etc.). Opportunities for doing both abound in daily living and by being aware of one's environment.

To develop a robust conceptual understanding of number, we approach by integrating notions of *discrete quantity* (e.g., count) and *measurement*. In discrete quantity, an *object is the key*. In the measurement approach the 'unit of measurement' is the key to knowing how much (or many) of the unit is needed to constitute the given quantity while measuring one quantity by another.

Introduction and usage of both approaches concurrently to number has many advantages and is essential for comprehensive understanding of number and its usage. In Western languages, the *cardinal* and *measure* numbers appear very different from each other, as measuring numbers requires the choice of a unit. For example, this is not the case in the Chinese language, where both cardinality and measure numbers are considered in the same way. And Chinese children, on average, are three times faster than their Western counterparts in number operations, particularly additive reasoning.

To see how naturally children can learn mathematics, two important findings from the last 20 years are relevant: humans:

- (1) are born with abilities to appreciate the quantity (concept of number), and (2) seem to be born with a sense that quantity (numbers) and space are interrelated ([Willingham, 2009](#), 15).

Children, once again, intuitively know that a longer piece of wood has a larger number associated to it—whether it is length, area, volume, or weight, even when they have no experience in these concepts. They seem to be integrating the quantity and measurement. This phenomenon of integrating *quantity*, *naming*, and *shape* into one concept is called *numerosity* or *numberness*.

Mathematics learning uses mental representations that are natural, but the original evolutionary representations have limitations and may not naturally and adequately support mathematics in the way and at levels that advanced societies now need them. The human brain, originally developed for different purposes—survival and procreation

—has been reshaping and organizing for many other functions, different than from the original ones. The evolution of using it for purposes other than the two functions mentioned requires ‘hacking’ its functions for purposes such as reading and mathematics. For example, with the parts that were developed for face recognition, some parts of them are recycled for reading and mathematics. For optimal use this natural system requires formal—effective, efficient, and flexible—strategies.

We need to make this process as natural as possible. However, for some educators, that means to not expect all students to learn all aspects of mathematics with ease and with the same rate or level. Rather, we should expect that mathematical proficiency requires careful cultivation and, in some cases, may develop slowly and may need specific language, activities, and instructional models. At the same time, we should keep in mind that students are born with the ability to learn mathematics, and we should not let students give up by concluding that they are just ‘no good’ at mathematics.

D. Trajectory of the Development of Number and its Usage

Children have a natural tendency to classify and quantify small collections of objects and events around them and represent them concretely, and eventually linguistically and symbolically, to meet their personal and societal needs. Number concept (quantification) and numerical procedures arise from children’s need to understand and express quantitative and spatial aspects of their world. Over time, with usage and practice, quantification processes and the ensuing numeracy skills become part of their social intercourse ([Wright, 1991](#); [Wright et al., 2006](#)).

These processes, on the one hand, satisfy their need to relate to objects and events, and on the other hand, the quantitative knowledge helps them solve problems they encounter to understand their environment. These are also means of communicating quantitative explorations and the resultant understandings. Initially, children themselves invent words and symbols to express their quantitative knowledge (e.g., 4 straight fingers and a bent finger to show he is four and half years old); later through the acculturation process, they learn and use standard methods of communication—symbols, signs, gestures, words, models, operations, and processes to receive, manipulate, comprehend, understand, and express quantitative and spatial ideas to understand and solve problems.

Although informal experiences with quantity can provide a sense of number, appropriate formal instruction introduces children to the standard, acceptable, and universal forms of representations and communication of quantification processes. These are more efficient, elegant, and effective ways of using and expressing quantitative reasoning.

In the normal growth process, through interactions with other children, family members, socialization, teachers, and problem-solving situations involving quantitative reasoning, most children acquire efficient number concept strategies and economy of effort. Continued success in these processes creates the desire to learn more and better strategies, extend, and apply these strategies more and in better ways.

Given the fundamental role of quantitative reasoning in applications of mathematics as well as the innate human attraction to numbers, it is not surprising that number concepts and number-related skills and procedures form the basis of early school mathematics. The settings and invitations for fundamental number usage abound:

- Counts of discrete collections (numberness, cardinality of collections, populations, etc.)
- Measures of continuous quantities (time, length, area, volume, mass, etc.)
- Ratio and comparisons (more/less, discounts, probabilities, miles per hour, map scales, density, rate of change, etc.)
- Records and locations (temperature, timeline, test scores, mapping, etc.)
- Codes (highway, telephone, product model numbers, jersey numbers, etc.), and a little later
- Derived formulas and constants (π in the formula $A = \pi r^2$).

Robust (fluent and flexible) numbersense is based on acquiring efficient (flexible and fluent use of) number concept, number relationships (arithmetic facts), and strategies. It comes from understanding the number systems and their structures (e.g., place value, even and odd number, prime number, composite, square numbers, etc.). The result is making sense of quantity and numerical information in various contexts.

Application of number and numbersense through well-defined procedures/algorithms and using them efficiently converts these skills into numeracy. Mastering numeracy is an important achievement as a culmination and integration of several key developmental milestone: Number concept, Place value, Additive and Multiplicative Reasonings) in a child's mathematics journey. Numeracy is the child's manifestation of the ability and facility in executing the four whole number operations (addition, subtraction, multiplication, and division), in various forms and contexts (including the standard forms), correctly, consistently, fluently, and applying them to problem solving with understanding.

In this chapter, we describe the first and most important milestone in this journey of mastering strategies to make it easier for children to acquire numbersense. This developmental marker and the launching pad for this journey is *number concept*. The inadequate preparation and learning and or deficits in this process result in problems

such as learning difficulties in mathematics with or without learning disabilities, such as *dyscalculia*.

Dyscalculia is a child's difficulty in conceptualizing, mastering, and using number, number relationships, and outcomes of numerical operations.

Without exposure to efficient and effective strategies and methods children may not acquire proper number concept, number relationships/facts, and standard procedures. They may experience limited or no success in mathematics and problem-solving. As a result, they may not develop enough interest and meaningful progress in science, technology, engineering, and mathematics (STEM) and related disciplines. A lack of success in numeracy is detrimental to their preparation for vocational opportunities and competence in the professional world such as science and engineering, even areas of humanities. Children's failure in number-related situations and mathematics procedures can also result in *mathematics anxiety*.

Low achievement in mathematics has its roots in the way we introduce number to young children at home and in school. Most parents and teachers make decisions about children's learning number and number relationships based on their own numbersense. These decisions are reflected in the choice of curricula—content, strategies, models, and use of materials and tools in teaching number. It is reflected in their sense of what the mastery of number means and how to assess it. These choices and criteria play important role in the nature of instruction, learning experiences we provide, type of tools we use, and the level and nature of children's acquisition of number concept, numbersense, and numeracy.

Recent research on the innate mechanisms available to humans propose two foundational 'core systems' that do not initially involve whole numbers, per se. These core systems are part of our evolutionary skill set and are manifested by children as *perceptual skills*. The first two deal with quantitative sense and the third with spatial sense. Deficiencies in these core systems—it has been argued—could contribute to low numeracy and difficulties with arithmetic, such as dyscalculia and other related deficiencies. The core systems are:

1. A mechanism for keeping track of the objects of attention. This is sometimes referred to as the 'object-tracking system' (OTS) and initially has a limit of three or four objects, if the objects are displayed sequentially. The number range 3 to 4 is an artifact of the underdeveloped working memory system in early childhood. However, if the displayed objects are in some kind of pattern or involve symmetry

and well-defined organization structures, OTS could move up to paying attention to and recognition of about 5 or 6 objects. It is thought to underlie behind the phenomenon of ‘subitizing’—making an accurate estimate of one to about 5–6 objects, *perceptually* without serial enumeration ([Feigenson et al., 2004](#); [Sharma, 2019](#)). It is proposed that the objects to be enumerated are held in working memory and that they constitute a representation with ‘numerical content’ ([Carey, 2009](#); [Le Corre and Carey, 2007](#)).

2. A mechanism for the analogue representation of the approximate number objects in a display. This is referred to as the ‘analogue number system’ (ANS). The internal representations of different numerical magnitudes can be thought of as Gaussian distributions of activation on a ‘mental number line.’ It is typically tested by tasks involving clouds of dots (or other objects), typically too numerous to enumerate exactly in the time available. One common task is to compare two clouds of dots. (Addition and subtraction tasks for which the solution is compared with a third cloud of dots are also used.) Individual differences are described in terms of a psychometric function, such as the Weber fraction, the smallest proportional difference between two clouds that can be reliably distinguished by the individual ([Feigenson et al., 2004](#)).
3. Both subitizing and estimations are influenced by the *spatial arrangements* of in the display of objects: their spread, organization—sequential vs patterns, density of spread, symmetry, area occupied, and even the size of the cluster of these objects.

The first two deal with the quantitative strand of evolutionary skills that will ultimately be responsible for *number*, *numbersense*, and *numeracy*—understanding one’s world in terms of quantity. The third is related to the arrangement and organization of the quantitative elements, such as whether the child focuses on whole-to-parts or parts-to-whole—the *spatialsense*, which will ultimately grow into understanding the environment focusing on its spatial/geometrical aspects.

It is to be noted that both numbersense and spatialsense begin as perceptual skills. Therefore, early quantitative experiences should be perceptual rather than strict counting, as a starter. However, perceptual knowledge is incomplete. To make it complete, make it a whole, we have to add reason to it. Reason is cultural artifact—a cultural tool. Reason is conveyed by language and/or symbols. Subitizing is a perceptual activity—perceiving the size of a collection of objects. Subitizing is made whole by adding counting (language—the names to the collections) to that perceptual action. When these two meet, we have the knowledge of what that cluster of objects represents. We recognize that cluster instantly and call its name—the number. *Numberness*—associating a name to a collection or cluster and communicating it orally and/or writing. Thus, for learning numberness, we need perceptual skills (that are innate) and cultural skills (language—

words and symbols—the shape of numbers). *This integration (numberness) is crucial for learning number concept.*

Many researchers (Butterworth, Gelman, and Gallistel) have concluded that deficits in these fundamental concepts put natural limitations on many children in learning arithmetic effectively. However, they have argued that the genetic research is supported by neurobehavioral research that numbersense begins with identifying the representation of numerosities—the number of objects in a set—as a foundational capacity in the development of arithmetic. Where this capacity is weak, education should seek to strengthen this capacity using sets of real or virtual objects and linking the sets to the spoken and written numbers until the learner can use numbers fluently and confidently. This will provide a sound basis for developing arithmetic. And the answer to this has been using counting as a means to develop arithmetic.

We agree with the thesis and the argument is sound and that is where the effective cultural tools come into play. The number of people having difficulty with arithmetic facts, shows that prevalent method of identifying the sets by counting has not worked for a very large number of children. Appropriate modifications in the instructional methods and effective use of cultural tools are what is needed to provide the solution to the problem. In this book, we show effective methods and activities to do that ([Chinn and Ashcroft, 2017](#); [Sharma, 2019](#)).

Every culture develops language, experiences, systems, and tools to deal with the quantitative and spatial aspects of their environment and understanding the role of quantity on its survival, development, and its continuation. These systems range from very simple to elaborate and complex. With use, feedback, and innovations, these tools are reformed and new tools emerge to prepare the child to have the competence to continue and improve the aims of the culture and to be able to expand and realize their individual potential.

The enumeration and quantification systems in every culture demand that its members learn and use them with fluency and confidence, for their own well-being and to forward the interest of the larger society. The individual growth and acquisition of these numeration systems is the outcome of active interactions between a child's innate quantitative structures (neuro-physio-psychological systems—the hardware, the equipment, and the evolutionary concepts related to number) and physical and cultural experiences (operating systems—the software, algorithms, the cultural tools, rituals, expectations, and usage).

On the one hand, we have the child's innate evolutionary quantity awareness encompasses the ideas of subtilizing skill, estimation, and comparison of two collections, and rough and approximate representation of number as a rudimentary number line.

On the other, we believe, and we project it from what anthropological studies strongly suggest, that there exists a parallel strand, in the human brain, innate evolutionary spatial awareness that helped shape the quantitative strand. This rudimentary spatial sense aids in estimating the quantity by focusing on the organization and arrangements of the objects in the collection in relation to each other and in relation to the whole. In other words, in estimating the size of a collection and in the process of perceiving the sizes of clusters of objects, children take into consideration the spread of objects—larger the spread higher the estimate, the organization, such as patterns, orientation, symmetry—vertical versus horizontal arrangement. Thus, early number concept development is dependent on few perceptual skills of estimating the size of the collection and its organization.

The integration of these concepts—approximate sense of number and spatial sense, as manifested in the form of subitizing—in children form the starting point of the number concept. The question is to what extent can this current neuropsychological and cognitive perspective, act as an appropriate theoretical scope to think about early mathematical development (and the processes of stimulation of that development) or whether this perspective needs to be nuanced and enriched by other perspectives?

The cultural inputs/tools/traditions/and expectations play a big role in making the innate skills—numberness and spatial sense real and usable. We believe *it is perceiving quantity (subitizing) and understanding it by decomposing/recomposing easily is the starting point of number*. The nature of these interactions (the interactive interweaving between nature and nurture) and their quality are central to the progress of number development. How do we take advantage of this knowledge is the key to helping children learn number and number-related concepts.

One thing is sure: we need to start coordinating the learning activities—informal and formal—that make maximum advantage of the *innate evolutionary skills* (subitizing and decomposition/recomposition) and the *cultural tools* (language and counting). And these activities must start as early as possible, as [Dehaene \(2020\)](#) points out, ‘to profoundly recycle our visual cortex and become excellent readers, we must take advantage of the period of maximum plasticity that early childhood offers’ (p. 138). However, the idea of plasticity is not just related to literacy, it is equally possible and desirable for music and numeracy.

Early reading uses the same areas of the brain that used to recognize faces, in other words, in the process of recycling of the brain ‘while literacy merely drives faces away from left hemisphere and forces them to the right hemisphere, intense practice [like reading] with numbers and equations interferes with the representations of faces on both sides’ (p. 139).

Luckily, our schools are designed to make the most of the plasticity of the brain. Education relies heavily on the spectacular flexibility of the child's brain to recycle some of the circuits and reorient them toward new activities such as reading and mathematics. When schooling begins early, it can transform lives. The fact that our schools typically teach children to read and do early mathematics between the ages of five and eight nicely dovetails with the evidence for intense brain plasticity during this period. In our work with very young children ages three and seven, we find that they can acquire subitizing, visual clustering, and recognition of shapes and objects quite easily. As Dehaene points out,

cortical competition is a genuine phenomenon, and the representation of faces in our brains turns out to be remarkably sensitive to education and schooling to the point where it can provide a marker of whether a child has received training in mathematics, music, or reading. Neural recycling is a reality.

(p. 140)

E. Innate Nature of Quantitative Understanding

1. Knowing Numerosity by Observation: Role of Visual Perception

Before entering school and without formal education (by exposure to random and unorganized quantitative experiences), children do develop techniques and skills for quantification. For example, by the age of four, most children can easily tell without counting which of two sets, randomly arranged, has 'more' objects than the other. They perform this task by visually comparing their relative areas (if the objects are scattered on a flat surface in relative proximity) or lengths (if the objects are arranged in a line in relative proximity). This strategy is reasonably effective because area and length are correlated (imperfectly) with numerosity/numberness, particularly when the collections have up to five/six objects and the objects are of relatively the same size and are not unusually scattered far apart. They also have rudimentary sequential counting (mostly rote and finger reckoning) ability when a relatively small number of objects are in the collection.

Neuropsychologists have sought to understand how activities and brain functioning influence cognition, including mathematics and whole number learning ([Dehaene, 2011](#)). Simple models associated with number processing have been proposed and tested, and the main areas of the brain identified as being activated in number processing have been refined. [Butterworth \(2005\)](#), for example, calls this as a 'starter kit' for early number learning that are primarily focused on cardinal aspects of number, initially as subitizing and estimation and its importance for later mathematical development.

2. Subitizing and Visual Clustering

This innate rudimentary ability of recognizing a cluster of objects by a visual scan is called *subitizing*. It is a perceptual activity, and it functions just like recognizing a face, a color, or an object. In essence, subitizing is the ability to instantly recognize the number of objects in a small group without the need to count them. It is important to note that the process is to *perceive* the quantity of a group of items at a glance and without counting: the maximum number of items that can be subitized is about five to six. The word ‘subitizing’ comes from a Latin word that means ‘to appear suddenly.’ It makes sense, then, that subitizing is about having the number ‘appear suddenly’ in a child’s brain; for example, when a child rolls a die. Subitizing in preschool and kindergarten is all about helping students to relate numbers to actual items or groups of items.

Subitizing, as an innate activity—an evolutionary skill—was originally needed for survival from predators; however, it is not limited to human children. They needed to perceive their quantity instantly whether the group in the distance is larger than ours or smaller than us. That informed them to take the appropriate action.

Most children come to pre-kindergarten and kindergarten with this ability unless they have been taught to count objects one-to-one and the child has repeatedly practiced this process of counting sequentially as their first number experience. Most children by the end of kindergarten lose it as most of our instruction in our schools for number is done by sequential counting. They get habituated into counting and do not rely on *subitizing*. Just like the child knows the sound before he knows the letter representing it, subitizing is the recognition of the collection before there is a symbol or word to represent it. Through the native language or the adopted language of the child, it is manifested as a usable skill.

On the other hand, with very little ‘proper instruction,’ not only can it easily be enhanced, but for humans subitizing can be expanded to *visual clustering*. Visual clustering is the ability to recognize by visual scan and decomposition/recomposition a cluster up to ten objects. Visual clustering is a learned behavior—a generalization of subitizing that only humans can do, and it is dependent on learning another important skill: *decomposition/recomposition*. Subitizing and decomposition/recomposition in learning number and recognizing a letter, groups of letters and then ‘chunking and blending’ the sound of the letters and words are parallel skills. Visual clustering is an important activity as our number system is based on ten.

Subitizing—a representation of *quantity* (just like the phonological representation of sound) —can be accessed by the native language of the child as verbally spoken as *the number name* (just like a letter and combination of letters spoken as a word as a sound), and as an *orthographic form of the number* visually (as a written word). This integration of a collection of objects (quantity), the number name (orally), and a written form (a

symbol) can be thought by an individual (just like a word in language) in their mental patterns (just like lexicon) in their working memory and with some practice is placed in the long-term memory.

Once accessed, the quantity representation (just like phonological representation of that word) becomes active as a visual representation of quantity in the working memory (just like phonological working memory) and is available for manipulation during number awareness (just like phonological awareness) tasks including identifying, segmenting, blending, manipulating, etc. the subitizing units and visual clusters (just like the phonological units of the word (phonemes, syllables, rimes ...), then *it becomes the foundations of numberness*.

Identifying, segmenting, sequencing, blending, and manipulating of clusters and sub-clusters takes place within the working memory as an individual consciously reflects on and manipulates the clusters and sub-clusters (ultimately all of this work representing numbers) for the completion of a computation task. The subitizing task of identifying a cluster/number (the number identified in the working memory) is then sent to an output buffer and automatically converted into an identified word associated with that numberness.

These innate skills of *subitizing* and *decomposition/recomposition* are realized and converted into useful skills, in the form of *number concept* by the help of cultural tools—language in the form of *counting* and *assigning names* to clusters and is practiced through toys, games, rituals, and other applications in daily uses. These skills form the basis of learning the number concept and number relationships with competence. In the literature, these skills are named as: perceptual subitizing and conceptual subitizing.

- *Perceptual Subitizing*—ability to instantly recognize the number of items in a small set, usually up to about 4 or 5, even 6 with very little exposure to organized patterns.
- *Conceptual Subitizing*—being able to break down larger clusters into smaller clusters and combine smaller clusters into larger clusters up to 5 and then to 10; this process of decomposition/recomposition is called conceptual subitizing.²

In the first few weeks of life, infants are sensitive to changes in the number of things they are looking at. They will look longer at a display when we change the number of things. If the baby sees a doll placed behind a screen and then another doll is placed behind it, the baby expects there to be two dolls when the screen is removed. Because the child expects two dolls, he will look longer compared to if he sees one doll or three dolls—change attracts children’s attention. They become excited or curious when presented with ‘impossible’ situations, such as having three toys appear when only two were initially present.

A child of age two or three, for example, learns to distinguish between two and three toy cars or fingers merely by looking at the group of cars in just the same way as he learns to use his visual perception to distinguish between a car and a bus, between a red car and a green truck, or between two dissimilar-looking symbols, even letters. This process is based on visual perceptual discrimination and has important pedagogical implications for learning numbers. Children who play with toys, games, and puzzles are better prepared for later visual perceptual discrimination skills that are needed when they encounter letters, numbers, geometrical shapes, and other symbols in formal instructional situations.

Recognizing the clusters of objects—the ability to discern the clusters of objects representing numbers in patterns such as: dominos, dice, and Visual Cluster Number Cards (VCNC) is fundamental to learning numerosity/numberness. We should capitalize on this visual/spatial perceptual capability and develop it to recognize patterns and visual clusters representing number of objects.

Most teachers and parents use sequential counting as a default strategy for developing number concept and later numbersense and numeracy. Almost every textbook on numeracy for teaching number recognition emphasizes mostly sequential counting for number recognition—‘count and color four tulips on the page,’ ‘count five trucks on the page and circle them.’ The belief in designing these activities is: that number concept means one-to-one correspondence and sequential counting (and they hope it will be transformed into conservation of number and numberness easily). This is further extended to adding as ‘counting forward,’ ‘count all,’ etc. as the strategy for learning arithmetic facts. This is particularly so in special education situations where counting is a default strategy for number conceptualization and deriving arithmetic facts.

In contrast, a few weeks of playing with Visual Cluster Number Cards, dice, and dominos can help children to recognize the visual clusters and learn the numbers associated with them. When children play with dominos, playing cards, and games such as Mankalah (a game found in many countries and variously known as Kalah, Mankalah, Chhonka, etc.), they are able to recognize the collection of objects even up to ten by visual observation. This is possible due to the strong appeal of patterns of clusters, thus developing a strong numberness early and easily.

3. Introducing Number by Counting and its Limitations

Many parents, along with reading to and with their children, also provide quantitative experiences to their children. These are important parental activities that help provide their children with the two most important cultural tools—introduction to the structure and nature of native language and quantitative awareness through counting objects, thereby introducing children to the alphabet and the words of the language of

mathematics. The other important tools of acculturation are introducing children to toys, games, rituals, and expectations—acceptable codes and conducts of their culture.

Counting various kinds of objects is a good starting point as it introduces the cultural names of numbers that bring out and shape the innate perceptual skills of subitizing and estimating and comparing sizes of small collections. However, it should not be limited to just counting. Unfortunately, with the only method of knowing the size of a collection through counting and continuing to emphasize counting on fingers or objects for number acquisition and number relationships is an inefficient strategy for learning number and other related number concepts. For some children, this counting does serve as a stepping-stone for other number strategies. However, the counting should very quickly be connected with subitizing and visual clustering. Preferably, begin with identifying the numerosities of clusters.

Most average and above-average children transcend finger counting and develop strong numberness with self-discovered strategies to transform it into numbersense. However, many children remain dependent on rote counting, fingers, and objects reckoning beyond an acceptable age. Many of them never achieve fluency in number relationships even when they reach high school.

Some children do have quantitative experiences before they enter school, but many come to kindergarten with little or no experience in number work. However, their formal introduction to number is, again, when they enter school or engage in formal number experiences is sequential counting only as many pre-kindergarten and kindergarten teachers think that number concept means sequential one-to-one counting.

This instruction takes the form of: the teacher arranges a collection of objects in a straight line or pointing to a number line and says: ‘touch this and say ‘one,’ she touches the second and says: ‘touch the next one and say ‘two,’ and so on. The child does this and ultimately is told that ‘the last number counted is the name of that collection.’ This counting often is done in a rote manner, and this becomes the major number activity for them.

The conscious act of counting, associating a number to a collection by sight, usage of number words to represent, and number relationships (to count, add, and subtract up to about 4 or 5) are outcomes of acculturation. These are learned skills facilitated through quantitative experiences and models provided by one’s culture.

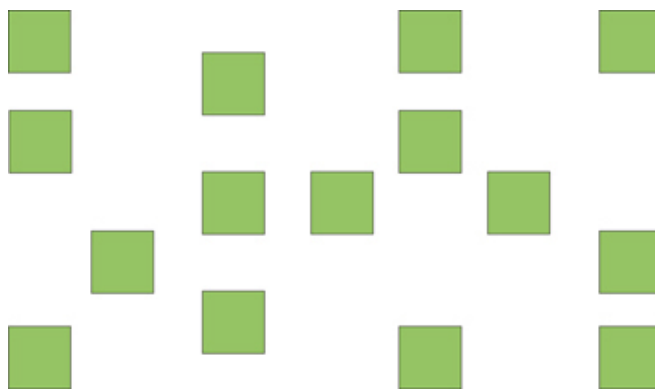


Figure 6.1 Random objects for Counting

The proper development and manifestation of children’s innate numerical ability and its transformation into formal quantitative thinking are dependent on ‘appropriate’ experiences, effective instructional models, visual images, rich language usage in instruction, and the clarity of demands made on them by the culture regarding number and numbersense. Thus, in developing instructional activities for young children related to number development, we need to recognize that there are key processes and behavioral markers involved in the development of number ability. These markers should be evident as the outcome of these experiences. Mathematical concepts, at least initially, should be grounded in ordinary human activities—interactions with the physical world.

4. Conservation of Number

Children exposed to concepts such as: organizing and classifying objects, recognizing patterns in collections of objects up to ten and assigning numbers to them, representing, relating, comparing, and operating on small numbers are better prepared for kindergarten mathematics and for later learning.

Number concept is not just that children are able to count *sequentially*. It is not even *one-to-one correspondence*. These are necessary condition but are not sufficient conditions. Meaningful sequential counting attaches a *number word* uttered to the last object counted and vice versa. This helps a child to associate a *vocabulary for numbers* (lexical entries for number), *one-to-one correspondence*, and *sequencing*.

When counting is associated to collections of objects with *spatial patterns* (clusters on cards, dominos, dice, and naturally organized and found clusters and collection of objects), it integrates pattern recognition and sequential counting—a fundamental process to number concept. It integrates the innate skills of subitizing and spatial sense—the organization of patterns. For example, the idea of ‘twoness’ is around us in abundance on a child’s body parts—two hands, two eyes, etc., and children learn it

almost instinctively. Similarly, all children sooner or later recognize ‘fiveness’ and ‘tenness’ through their hands.

Meaningful integration of these *four* skills: (a) exposure to collections of objects in patterns, (b) estimation, (c) counting, and (d) one-to-one counting results in children acquiring the concept of *subitizing*, *conservation of number*, and the fact that the number associated with a collection—its ordinality is not the property of the collection rather than the artifact of counting. Conservation of number refers to a child knowing that the *numerosity/numberness* of a collection is not changed when the collection is spatially reorganized or the counting is done in a different order. It means that the child realizes that *numerosity/numberness* is the property of the collection rather than the outcome of his/her counting.

Number concept transcends the subjective act of counting to an objective action of numberness. Numberness achieved through subitizing is a consolidation of counting. It is an act of abstraction and a higher level of concept than counting. This transcending means that the child realizes that a fixed number represents the collection. ‘I do not have to count fingers of my hand each time to know there are five of them.’ When counting results in the realization of this fact (assigning meaning to counting and fluency in counting), then the child will transcend counting. Subitizing and visual clustering—recognizing a collection by its *number name is an abstraction of counting*. It is higher skill than counting.

Conservation of number is both a cognitive and a content skill. Teachers in pre-kindergarten, kindergarten, and first grade should devote enough time for this concept to be well-established.

This transcendence from counting is a bidirectional activity: (a) number-words can be produced in sequence in either direction—to count forward and backward, and (b) can produce/generate a collection of objects for a given number and assign a number to that collection. A child’s acquisition of and fluency with the number-word sequence (e.g., one, two, three ...) is a necessary prerequisite for the ability to assigning numerosity to the collection.

Most children, with little practice, transcend the counting process and acquire strong number concept and can use it for mastering arithmetic facts. *However, many remain at the counting level and do not master arithmetic facts; that is true for many learning-disabled children, particularly dyscalculics.*

There is some research to the effect that children should have conservation of number before number conceptualization. Conservation of number is necessary, but it is not a prerequisite skill for a number. It is part of it. We have found that by using the patterns and clusters on the Visual Cluster Number Cards, dominos and dice, and the size and

color of the Cuisenaire rods, children can acquire conservation of number easily and effectively.

5. Counting and Self-Regulation

In reading, sometimes there is a mismatch between how fast a student is pronouncing the phoneme sequences in a word and how quickly his eyes scan the word. Eyes scan very quickly, and the child leaves out medial phonemes. In addition, many learning-disabled children struggle in writing what they pronounce. They speak faster than they can write, resulting in them leaving out graphemes (phonemes in the written form). Some just look at the first few phonemes and think they just ‘know the rest’ and take a guess at what they are reading.

In number recognition, the same phenomenon occurs—a child associates more or less number words (they may say, for example, “1-2-3-4” four number words and they touched only three objects or maybe five objects). In early counting, there is a need to intentionally regulate their counting action. This problem almost disappears in a very short time if we use visual cluster cards for counting and subitizing.

Self-regulation is achieved earlier and more easily when it is initially aided by an adult. For example, many teachers assist their students in reading and writing by using the method of placing dashes on a line or tapping and have the students say each phoneme as they write. This helps regulate their eyes and auditory/verbal coordination, assisting in recognition of orthographic images. Thus, tapping out the phonemes or putting the dashes for phonemes on spelling words helps better regulate written language and sends a clear message that concentration and organization are important. This is particularly so in the case of children with executive function difficulties.

Similarly, in counting or identifying objects or the digits in multi-digit numbers, the same kind of regulation is needed. When one asks children to count, one should modulate the speed of one’s speech by putting the finger on the object or the written number as we are counting/saying. The child needs to pay attention to the rate of movement of the adult’s finger—in the middle of counting one should accelerate or decelerate the rate of movement of the finger pointing at the objects. This teaches children to modulate their speed when they count. This also reduces children’s miscounting of objects or numbers.

F. Transition from Instinctive to Intellectual Numeracy

Acquiring alphabets/orthographic characters of a language is a necessary but not sufficient prerequisite skill for reading and writing. One can develop alphabetical knowledge by informal frequent exposures, but the development of appropriate

alphabetical skills and their usage for formal reading is facilitated by organized instruction, frequent exposures, and supervised practice. Similarly, effective strategies connect innate recognition of quantity and perceptual numerosity (subitizing and counting) to a well-developed sense of number. Most children can achieve this as a matter of course through exposure and a little direction from a knowledgeable and sympathetic adult. However, to achieve this for all children and in optimal ways, one requires efficient strategies through organized instruction and practice. A definite set of experiences, strategies, and skills is necessary for the development of numbers and a robust numbersense in children ([Hattie, 2009](#)).

1. Informal Experiences for Developing Numberness

Quantitative experiences at home and in childcare situations affect children's acquisition of number sense. For example, children from homes where they have exposures to numbers and number concepts (discussion of time as in scheduling, planning for quantitative activities, games and toys involving number of moves using dice, dominos, playing cards, objects, etc.) than children from homes with limited or no exposure to quantitative discussion succeed in early elementary school. For instance, kindergartners and even pre-kindergartners from these families are more able to compare numbers and answer correctly, for example, whether 5 is bigger than 4 than children with limited experiences. In low-quantitative experience families, children count only up to 15 on average before they enter kindergarten, whereas in others, they count beyond 50 and are encouraged to count more often. Such counting develops number word recognition (and *language containers* for quantity). They can also distinguish between words and phrases that stand for quantity and the words and phrases that are not related to quantity (e.g., seventy-three seems like a number, but 'sevening' does not seem to be a number).

Families with an emphasis on quantitative reasoning also expose children to concrete and experiential number work—referring to dates on the calendar (Let me put this on the calendar. Six months have already passed in this year. I do not believe half a year is already gone), planning the day with reference to time (The game is at 4:00 PM on Saturday. You have two appointments this week. We need to make sure we have enough time for practice for the game in the next three days.), scheduling of events by mentioning the hour of the appointment (It is getting late; it is already 3 o'clock; the appointment is in an hour at 4:00 pm.), discussion of weather reports with a mention of temperature (Temperature went up this morning by almost 15 degrees.), predictions and estimations (There is good chance of rain. The weather report says there is 60% chance of rain today, you better take your raincoat to work.), awareness of weight, height, days in the month, and events (The doctor said, 'the baby is gaining weight nicely. She gained

2 and ½ pounds since the last visit a year ago.’). These quantitative exchanges are rich experiences for children, even if they are not addressed to them.

Children hear more conversations with quantitative content and use of spatial and quantitative words in these families (We have to take a right turn at the next bend. The book is upstairs on the table, to the left of the bed. We need to cut it into four pieces for each one of us to have a piece. Can you put two more plates here on the left so there is one for each person? I have finished half the book. It has almost 200 pages. There were enough spaces in the parking lot for all the guests’ cars. We have enough change for the toll booth. Here, take this change and keep it ready for the tolls. How many coins did I give you? Do you have enough coins? I need only three coins.) This kind of exposure plays an important part in seeing the role of number and spatial relationships. It is good preparation for later formal work with quantity and space.

Children growing up in literacy- and numeracy-rich environments, where reading, writing, quantifying, and recording daily events are encouraged, begin to link meaning, number, and value to objects and their collections. Children, there, know such ideas as ‘one,’ ‘two,’ and ‘five’ are quantities of daily use and hundreds and thousands are large quantities. They begin to contextualize number usage and become familiar with the lexical entries for number and quantitative qualifiers when parents, siblings, and other adults use expressions such as:

‘I will have only one more piece of pizza.’

‘Two more cups are needed for this recipe, but one less teaspoon of sugar.’

‘In a year she will be ten.’

‘Only a few days are left for the book to be returned to the library.’

‘There were hundreds of people at the fair.’

‘It is more than three hours until the next train’s arrival.’

‘We had only a few children at the fair today compared to last time, when we had a lot of children at the fair.’

‘I wish they scored four more points; they would have won.’

They also hear that ‘pound’ has something to do with substance, an hour deals with time, a dollar with money, years with age, and an inch with length. They engage in informal and formal counting and measurements and their representations. They begin to represent the result of these measurements and quantification through action and pictures. They attach spoken numbers and later provide graphical (written) forms to these representations. They amass lexical entries (vocabulary—language containers for concepts) for number-related activities. This is mathematics acculturation process on display. Proper and earlier introduction of language, appropriate materials, and efficient

strategies and tools of learning make it easier for the child to begin formal quantification and its representation.

This early quantification is a preparation for learning higher mathematics concepts where the child will apply numbersense and develop mathematical tools for situations that call for:

- a. quantifying (studying and applying number),
- b. modeling problems using mathematics resulting in measurements and relationships,
- c. spotting patterns (and extrapolation), and
- d. observing ideas, concepts, and their incidence of distribution and the nature of errors.

When young children ‘play,’ they are often doing much more than just playing. Preschoolers can learn to invent solutions to simple arithmetic problems. They engage in substantial and significant amounts of pre-mathematical activity in their games and play. In fact, early childhood programs that include more number-related and strategy games have more opportunities for higher-level thinking activities that promote self-regulation and executive function. Through higher-level play, children explore patterns, shapes, and spatial relations; compare magnitudes; and count objects. Research shows that this is true regardless of the children’s income background or gender. It is an example of high-quality education that can help all children utilize their inherent skills for developing mathematical thinking. However, if high-quality mathematics education does not start in preschool and continue through the early years, many children are on a trajectory of gaps in mathematics knowledge that is difficult but not impossible to make up later.

Here is an example of early informal quantification one of the authors observed in two four-year-olds who were playing with toys. Their toy collection included plastic animals, birds, and insects. During the play, one child said to the other:

‘Let’s draw them on paper just like the book.’

As they were drawing, one said: ‘Bird two legs.’ ‘Spider more legs.’ The other said: ‘Yeah! I have two legs like a bird.’

As one can see, they already are involved in quantification. They have realized that to draw a bird, they need to draw two legs, and a spider should have more legs. It is a rich experience for them (for learning and teaching with each other) and an important cognitive inference on their part.

Each time they drew the legs for a toy, they counted the legs on the toy. One child said: ‘Look, my cat has more legs than the bird, more than me.’

Another child said: ‘Yeah! It has four legs, two in front and two in the back.’ They realized that they could ‘draw’ a toy spider, and they came to know that it had many legs, but birds had only two, just like people. There is even a sense of space: ‘two in front and two in back.’ Even rudimentary addition: ‘Four legs, two in front and two in the back.’

‘We can just draw a line, like this, for each leg of the spider.’ This sentence shows beginning of representation: from concrete to pictorial.

The conversation was rich in quantitative and spatial words, and several key mathematics concepts were involved in the activity. They were developing number vocabulary (lexical entries—language containers), representation of quantity and spatial information, one-to-one correspondence, sequencing, serial counting, comparison of quantities, estimation, and communication. At the same time, they were using reasoning for their statements and communicating their observations and learning. These explorations through play are pre-mathematical and such informal activities are the beginning of the development of numberness and numbersense, creating language containers for number concepts, and a preparation for formal instruction in quantitative concepts in school.

As we see, when children engage in play and games, they often quantify and relate to numbers and spatial characteristics of objects. They attach quantitative meaning to drawings and sounds. In talking to each other and to themselves, they use language relating to measurement and quantity, for example,

- I am older than you.
- It is heavy.
- That is bigger.
- I am four, and I will be five next Wednesday.
- I have two sisters.
- That is longer than that.
- It is getting smaller.
- I want two big pieces.

They even develop some rudimentary concept of fractions as early as age four or five when they say, ‘I want half of the cookie,’ ‘my half is smaller,’ and write part of a numeral or letter for the fraction or demand a certain portion of something.

With support, children choose appropriate methods to deal with numbers that emerge from quantification in their world. Early strong numbersense will help them to tackle mathematics problem solving without fear. To achieve such goals, the instructional approach should be a concept-based skill development. That means we want children to view mathematics not as a series of tricks, formulas, calculations and computational

procedures, or even problem-solving techniques, but as a way of knowing and learning about the world, as a collection of patterns involving quantitative and spatial sense. In other words, we want them to master mathematics concepts with a mathematical way of thinking—the study of patterns in quantity and space and their integration. *Arithmetic is the study of patterns in numbers and number relationships* (DeSmelt et al., 2011; [Gattegno, 1960, 1964](#)).

2. The Role of Exploration and Concrete Experiences in Learning Number

Historical and anthropological research about the development of number concept shows that children develop counting-based techniques such as counting on fingers and other parts of their body as a result of their interaction with their environment. Such counting activities are successful for simple quantification and quantitative operations. Indeed, counting on fingers and counting as an activity is such a basic skill that formal education is not a prerequisite for acquiring it.

Individuals in some societies (where formal education has not reached many) have a good quantitative sense and highly effective informal calculation techniques without experiences with formal and written arithmetic. In many other societies, counting and arithmetic skills are deeply embedded in their social, linguistic, and cultural activities, and children in these societies develop strong numbersense earlier. For example, several East Asian cultures have developed effective methods of finger reckoning such as Chisonbop (FingerMath) and abacus usage for problem solving. In Chinese, the teens number names are suggestive of 10+ numbers. In India, children count on fingers effectively. Each finger (outstretched palm) has four joints (including the tip of the finger—you begin at the base of the finger and stop at each joint and say 1, 2, 3, and 4. 4 is at the tip of the finger. You repeat it for each finger.). Therefore, they count up to 16 on four fingers of the right hand and then each joint on fingers in the outstretched left palm represents five. Each joint on the right-hand thumb (there are only three of them, including the tip of the thumb) could represent a ten. The same applies to the left thumb. In my village, illiterate villagers were able to calculate accurately and keep track of large numbers on their fingers. In Western countries, counting on fingers is common, but is done inefficiently—each finger just representing one number. It is often not regarded, but as an indication of being functionally innumerate.

On entering school, most children possess informally and spontaneously developed powerful quantitative processes and skills on which formal arithmetic learning can successfully be built. This informal mathematics, with and without counting, represents cognitive strengths in children. Teachers need to build on children's naturally developed counting and ethno-mathematics skills to develop formal number concepts. For example, problem solving should begin using informal number techniques and should be followed

by formalizing them into systematic symbolic representations by recording these experiences and quantitative outcomes. From these experiences, children should arrive at and teachers should help them to see patterns related to numbers. These number patterns then should be converted into efficient strategies (arithmetic facts: to add, subtract, multiply, and divide), concepts, and methods to acquire a strong numbersense.

Without efficient strategies, children become efficient in inefficient strategies such as counting or develop major gaps in their numbersense. For example, methods that rely on ‘touching points on number shapes—the child who imagines or places five touch points on 5’ and then counts them to find additions and subtractions does not achieve automatization—fluency and meaningful and generalizable strategies. Children can complete scores of ‘worksheets’ using these methods without any retention of facts and understanding. Such methods are inefficient. They work in a limited way but do not produce children with proficient skills.

When we compare children with mathematics learning disabilities (MD), with and without comorbid reading disabilities (MD/RD), and normally (N) achieving peers in tasks assessing basic numerical skills they show differences in processing numerical quantity. For example, children with mathematics disabilities are only impaired when comparing Hindu-Arabic digits (i.e., the commonly used symbolic number magnitude) but not when comparing collections (i.e., non-symbolic number magnitude). Moreover, they automatically process number magnitude when comparing the physical size of Hindu-Arabic digits in multiple settings. However, no evidence is found for differential patterns of performance between MD and MD/RD children in these tasks. It is important to note that children with mathematics learning disabilities have difficulty in accessing number magnitude from symbols rather than in processing numerosity per se. It has implications to other numerical operations, later on.

Without efficient and generalizable strategies, by the third grade, many children have evident gaps in achievement due to misconceptions about different areas of their number-related knowledge and skills. The most common type of gap is between the child’s written arithmetic and his or her informal counting-based procedures or invented strategies. Many children, for example, can add or subtract on their fingers (or using number line, hash marks, counters, touch points, etc.) to arrive at the answer on paper, but not without this counting. This results in lack of fluency in number relations and arithmetic facts. Even when children have fluency, they do not have efficient strategies behind this fluency, so they are not able to extend this knowledge to problem solving and conceptualization of higher-order concepts (multiplication and division) and procedures (algorithms). It is harder for them to interlink facts.

Development of numberness (quantity-number symbol relationships) is an essential component to successful numbersense and numeracy, and as the student progresses to

complex topics in mathematics. The basic math skills children bring to first grade from kindergarten affect their mathematics learning throughout elementary school years. For example, children who understand number can go back and forth easily and quickly, breaking (decomposition/recomposition) a larger number into smaller parts and putting back together, giving them a head start in numbersense. Children at the end of kindergarten who understand the number line and can place and locate numbers (at least up to 30) on the empty number line (when dictated a series of numbers randomly) and who know or can access quickly the 45 sight facts show faster growth in math skills than their counterparts. These are the factors that make a difference in the first grade beyond intelligence and other abilities, particularly when intelligence, working memory, and other abilities are factored out to determine the most critical beginning of school mathematics skills.

It is therefore important for parents, curriculum planners, and teachers to know the nature of the beginning of school knowledge needed to learn mathematics. In order to improve basic instruction, we have to know what to instruct and how to instruct. Mathematics knowledge is incremental, and without a good foundation, a student will not do well because mathematics gets more complex. In this chapter, we focus on: What does it mean to know number concept?

G. Reading vs Numeracy: Phonemic Awareness and Numberness

To become fluent in reading one needs, among others, two important processes: Phoneme awareness and completion tasks. *Phoneme awareness (PA)* is the ability to segment a spoken word into its separate fundamental speech sounds—the phonemes. It is the chunking and blending required for decoding or uttering a string of memorized letter-sound links. *Completion tasks* involve trying to recognize the word from the partial information uttered. In this, the brain jumps to a pronunciation when it completes the partial information into a recognized word. Both are the outcome of well-orchestrated, supervised, and individual practice. Thus, phonemic awareness, a good repository of sight words, ‘chunking and blending’ with the help of PA and sight words, and consistent supervised practice with a knowledgeable and sympathetic adult set the child on the way to becoming a reader—with fluency and comprehension.

Our increasing knowledge of phonemic awareness (PA) and its importance to the development of strong reading skills informs our understanding of *numberness* and forms a basis for developing strategies to help children acquire a robust *numbersense*. Just as phonemic awareness is critical to teaching and learning to read, the building of numberness/number concept is critical to number relationships and arithmetic

instruction. Such an emphasis on numeracy instruction benefits all students, with or without learning disabilities. The question is:

1. How Can We Best Help Children Overcome and Master This Difficult Hurdle?

Just as sounding out each letter cannot make the child a reader, the common practice of recognizing a collection of objects by one-to-one counting, arriving at arithmetic facts by sequential counting on number line, hash marks, or counters, doing number drills, and flashcards (and their modern versions the use of computer or iPad apps) cannot help *all* children arrive at numberness, number concepts, and arithmetic facts. It has not worked for most children. Research has shown it; teachers have seen it. We need to look at it from a fresh perspective. Lack of mastery of arithmetic facts is a *national* and *international problem*.

We need to learn from what research has shown us in teaching reading during the last 30 to 40 years. The research on understanding of and interventions in learning problems in mathematics is in its infancy. To answer the earlier question, we need to focus on these following key questions:

- a. What can we learn from research in the science of teaching reading?
- b. What are the implications of this information to teaching number concept?
- c. What is the nature of number conceptualization?
- d. What are the component concepts?
- e. What is the trajectory of the development of number concept?
- f. How do we teach it?
- g. What is the sequence of tasks?
- h. What are the most effective and transparent instruction materials?

2. Parallels: Letter Recognition and Number Concept

Before identifying the behavioral and cognitive markers for the mastery of number concept, we need to examine the behavioral and cognitive markers for recognizing a letter.

Knowing Letters and Reading Skills

We say that a child has the mastery of a letter when he shows the following behavioral/cognitive markers:

- a. Can recognize a letter instantly (e.g., says: ‘EM’ when he can recognize the standard form of the letter *M* instantly),

- b. Can recognize the letter in its varied forms—different and variant, or even slightly deformed (e.g., M, *M*, M, M, M, M, M, M, M, etc.),
- c. Can recognize the letter in the midst of other letters and symbols (e.g., M in the midst of other letters as in CALM, MILK, WARMER, \$M#569A, etc.),
- d. Associates a sound to the letter (e.g., the sound of the letter is like M in *monkey*), and
- e. Can write the letter without any prompts (e.g., can write M on an imaginary board with eyes closed and can describe the writing process—delineates the various ‘strokes’ of the letter in the proper sequence).

When the child demonstrates these behavioral markers for all the letters of the alphabet of the English language, with the same level of mastery (it does not matter whether the letter is A or Z, lower or upper case), then we say that he has mastered the alphabet information. Once the child demonstrates this letter mastery, then reading skill is acquired by the integration of the following:

- a. Letter Concept,
- b. A large sight vocabulary,
- c. Phonemic Awareness (phoneme/grapheme connection),
- d. Chunking and blending using (a)-(c)

(*Word attack skills*: recognize sight word; recognize the segments of a sight word; ‘chunking’ of a new and big word into smaller sound components using acceptable rules, and blending’ of smaller word/sound segments into bigger words using effective strategies),

- e. Integration of these elements requires deliberate supervised practice,
- f. Intentional, supervised, and individual practice.

These objectives are achieved by multiple and repeated exposures to quality instruction based on the science of reading, organized by knowledgeable and sympathetic adults and practiced by the child, both supervised and individually.

There is a similar process in developing the idea of *numberness* (one component of number concept) and then *numbersonsense*. A child has to show similar mastery markers (behaviors) for the first ten counting (natural) numbers and zero (the digits—the alphabet of mathematics language).

[Dehaene et al. \(2003\)](#) proposed a triple-code model of working with number, consisting of three components: verbal, visual (numerals), and magnitude. The model postulates three main representations of numbers:

- A *verbal code* in which numbers are represented as a parsed sequence of words.
- A *visual code* (Hindu-Arabic) in which numbers are represented as identified strings of digits.
- An *analogical quantity or magnitude* code.

The integration of these codes, termed by [Sharma \(2019\)](#) as ‘*numberness*.’ Each part of the model of the formation of this numberness has been associated with increased activity in a particular part of the brain. For example, the horizontal segment of the intra-parietal sulcus has been suggested as the region involved in encoding the analogical representation of numerical magnitude ([Dehaene et al., 2003](#)).

Just as the understanding of phonemic awareness has revolutionized the teaching of beginning reading to children, we believe that the influence of proper development of number concept (numberness) carries implications for instruction not only for children who struggle with mathematics but for all children (learning from the ‘outliers’).

The problem, however, is that mathematics educators and psychologists have not clearly defined and understood the meaning of number concept or numberness, in particular, and numbersense, in general. Because of this limited understanding, they have failed to clearly define and assess its absence and the nature of the deficits presented by children in their number work. Without agreement on these matters, no agreement can be reached on the methods and strategies to effectively teach it.

Therefore, the current methods of teaching number are thus limited to the person’s (the teacher’s) definition of ‘*knowing*’ number. For most people, it means one-to-one counting and recognizing and writing numerals.

Rather than focusing primarily on what students need to know, we should also focus on how they effectively learn number concept. Understanding exactly how numberness and automaticity of number facts develop—thereby opening the door for higher-order fluency and comprehension. That will provide an effective numeracy pedagogy, not for those having difficulty, but for all children. Fortunately, research in cognitive science has already identified powerful principles of learning to help us get to some answers for these and other important questions in mathematics instruction.

The objective, therefore, is to develop and focus on the clear understanding of numberness, number concept, and numbersense. Let us begin with the end in mind. Children with good numbersense can

- move seamlessly between the real world of quantities and the mathematical world of numbers and numerical expressions.
- represent and use the same number in multiple ways/forms depending on the context and purpose and using decomposition/recomposition.

- combine numbers with ease using effective, efficient, and elegant strategies based on decomposition/recomposition, properties of numbers and operations.
- recognize benchmark numbers, compare numbers, and number patterns.
- recognize gross numerical errors and have a good sense of numerical magnitude; and
- think or talk in a sensible way about the general properties of a numerical problem, operations, or expression, without doing any precise computation.

In order to operationalize the definition of numbersense for instruction, we should keep in mind the parallels and differences in the nature of reading skills (phonemic awareness, sight words, and chunking/blending) and number concept (numberness, sight facts, decomposition/recomposition).

3. Knowing the First Ten Counting Numbers (Digits—The Alphabet of Mathematics Language)

Acquiring numberness and then numbersense and fluency in arithmetic (numeracy) means understanding and integrating key processes:

- lexical entries* for number (having a good store of number names—can recite number names in order, forward and backward, knows the difference between number words and non-number words, e.g., ‘seventy-three’ is a number word, but ‘seventy-cuke’ is not a number word),
- can *meaningfully count* (one-to-one correspondence + sequencing = conservation of number)—can assign a number to a collection of objects.
- can instantly *recognize visual clusters* (extension of *subitizing*) of each number up to ten,
- can represent quantity of the collection in its *orthographic form* (giving a number name to the visual cluster as ‘5 or f-i-v-e’) and write the number—a *visual/graphical* representation, e.g., 5), and can assign a name to the graphical representation (sees the number 5 and says: f-i-v-e) and the *sound* (hears f-i-v-e and says the number 5, e.g., f-i-v-e), and,
- decompose and recompose a collection* (number)—knowing that 5 can be expressed: first by circling the two sub-clusters that make the cluster of 5 in all possible combinations, e.g., 5 is 4 plus 1, 1 plus 4, 3 and 2, and 2 and 3; then can say the combination of 2 numbers that make 5; and finally, can write: $5 = 1 + 4$, $5 = 2 + 3$, $5 = 3 + 2$, and $5 = 4 + 1$;
- can *form number shapes properly*. Just like writing letters, where we emphasize proper formation, we should emphasize the proper formation of numbers.



Figure 6.2 The five card from VCNC

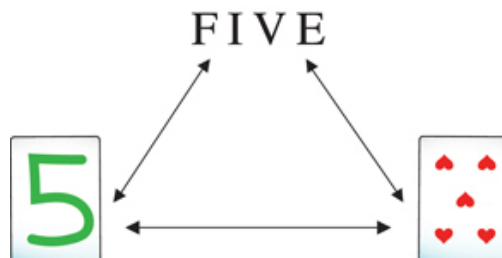


Figure 6.3 Fiveness: The property of a collection

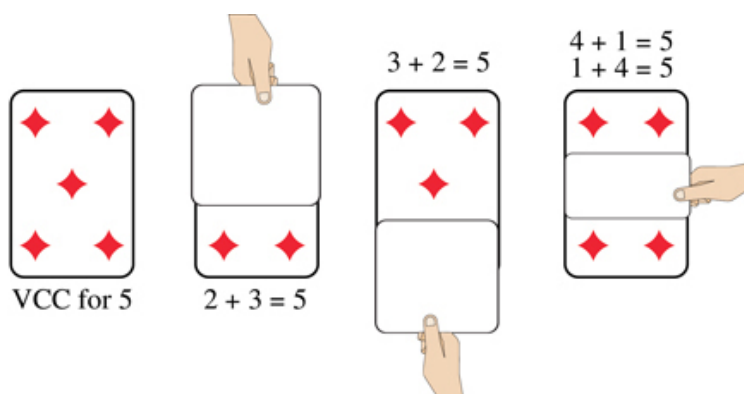


Figure 6.4 Decomposition/Recomposition of 5 using VCNC

Integration of these elements is called number concept. It requires deliberate practice in the integration of these elements, particularly the decomposition/recomposition process.

For example, a collection of seven objects should visually invoke in a child's mind: (a) the collection representing that quantity, (b) the number name, (c) its visual and auditory representation, and (d) the different combinations of numbers, e.g., $6 + 1$, $5 + 2$, $4 + 3$, $3 + 4$, $2 + 5$, and $1 + 6$. On the other hand, the number '7' should invoke the idea that it represents seven objects and a word that represents it.

Similarly, on hearing the number word 'seven,' one should instantly associate a collection of seven 'non-specific' objects and the visual representation '7'.

This integration of *visual clustering* (cluster of 7 objects), *graphical representation* of number (7), *decomposition/recomposition* of number 7 (breaking 7 into its sight facts:

1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, and 6 + 1), and the associated phoneme to that number (saying the number s-e-v-e-n) is called *sevenness*.

Presence and integration of the skills from (a) to (e) for number 7 is knowing the number 7. These are the ‘behavioral and skill markers for knowing number 7.’

Fluent integration of these markers is true numberness, mastery of number 7. This is essential to understanding what the number 7 means. This should be true for all the numerals from 1 to 10. *When that happens, the child has mastered the number concept.*

This establishes that the process of reading with fluency with understanding and acquiring number concept are parallel.

4. Implications of ‘This Parallel’ Phenomenon to Numeracy Instruction

The development of numberness (particularly the decomposition/recomposition of number), numbersense and proficiency in arithmetic (numeracy), on one hand, and phonemic awareness and the ability to read with understanding and purpose, on the other, are comparable activities.

Numberness (particularly, the decomposition/recomposition of number) is to arithmetic as phonemic awareness is to reading. Nevertheless, it is also important to recognize the differences between the two processes. In reading, phonological decoding associates graphemes and phonemes, but numberness is the integration of three elements: *grapheme* (visual representation of number), *phoneme* (saying the number), and the *numerosity* (visual cluster representing the quantity). Associating quantity to grapheme and phoneme is the difficult part.

The experiences, skills, and knowledge that children bring to school place them at different starting points, suggesting that they might need differing amounts and/or pace of instruction. But that instruction should be quality instruction with the same goal for all children, with or without learning disabilities.

In our special education programs, generally, the strategies used to teach number are inefficient. They are mostly counting-based. For example, even the sight facts are derived by counting. Many of the interventionists, including special education teachers, are not aware that the knowledge of visual clustering and decomposition/recomposition shifts the cognitive load from focusing on each component number, in addition or subtraction problem, to number relationships quickly. Children who develop automatic integration of these skills can move on to focus on fluency, automatization, and comprehension while our special education children and many other children continue to struggle, even in later grades, laboring on deriving the facts by counting.

H. Number Conceptualization: Numberness/Numerosity

To most people, knowledge and use of the first ten natural numbers {i.e., the digits: 0, 1, 2, 3, ..., and 9—the group of ten objects, and then number 10—the first number word, made from two digits} appear to be a simple and straightforward process. To them, learning to count is merely a matter of reciting a string of words like a nursery rhyme, a feat that most young children can master surprisingly early and easily. Most children are able to progress naturally from working with objects and then to represent these experiences in pictures and icons, and finally they represent and manipulate numerical symbols for objects and collections of objects.

Early humans observed that three apples and two apples make five apples, and also that three and two sheep make five sheep; and then they were able to make the important leap—an abstraction: that the ideas of threeness and twoness, whenever combined, result in fiveness. The numbers 3, 2, and 5 themselves do not physically exist in the real world, but they describe how many objects we may have in some set. The important thing is that we can also reason about them, independently of apples or sheep. This generalization—going from concrete representation of objects to an abstraction—is the basis of number concept. *Cognitively, it is a higher-level skill and a concept than just counting objects and adding objects to get another number.*

As letters are represented orthographically (graphic), verbally (the name of the letter), and the associated contextual sound, similarly, numbers (orthographic images) were invented to represent the numberness/numerosity and to communicate that information about quantity.

Numbers can be represented in three main formats: *Hindu-Arabic orthographic form* (grapheme-numerical format), *verbal form* (phonological word), and *magnitude-related* (quantity) ([Dehaene, 1992](#); [Riccomini et al., 2015](#)). Learning number means functional relationships between these different representations and their processing characteristics. Of particular interest is understanding magnitude information because magnitude information is the semantic aspect of numerical processing. This is so because each number, whatever its format, is a symbolic representation of a magnitude, quantity, or measurement. This understanding sets apart literacy and numeracy skills. The quantities represented get more and more complex and the numbers that represent them also become more and more complex as we move along the journey of learning mathematics ([McCloskey et al., 1985](#)).

The magnitude-related representation is the most difficult one and the absence of this creates number-related difficulties, such as *dyscalculia*. To facilitate this, we want to introduce the initial representation of quantity in the form of *visual clusters*. Visual clusters represent the magnitude aspect of number. Thus, understanding the representation of number is the integration of (a) Hindu-Arabic representation

(orthographic/graphemic), (b) verbal (phonemic), and (c) visual cluster representation of the number (knowing the magnitude of the number).

An average child in a normal course of the learning process takes around five years, from about age two to six, to learn to handle numbers and to apply them to everyday situations to solve simple quantitative problems accurately and consistently. Yet many children have difficulty in mastering and applying this skill according to a socially acceptable timetable or an acceptable level of mastery. For a variety of reasons, this process may be longer and difficult for some of them.

1. Lexical Entries: Number Names and Counting

Most children come to school reciting the string of the alphabet as a song, a rhyme. Similarly, long before school, many children begin counting by rote—the recitation of an ordered sequence of responses using concrete objects. In many cases, this counting does not mean real understanding of number or its effective use. It is acquired by observing others using a series of number words. Exposure to other people's/children's counting adds to the development of a larger vocabulary for number names. Just like a child should know the letters of the alphabet by rote first and then the meaning, similarly, children should acquire the string of number names, particularly to about 30 to 40, preferably 50, before they come to kindergarten.

2. Counting

Counting by rote is an indication of the presence of lexical entries relating to number (number words). These entries are essential, but they are not sufficient for number conceptualization and number usage. Counting as a rote activity can be taught easily, but the child may or may not know what he is doing except saying the words in a sequence—a string of words. Let us consider the following example of a four-and-half year child working with number.

Teacher: Can you tell me how many cubes there are on the table? (Points to a collection of cubes on the table.)

□ □ □ □ □ □ □

Child: (Child sequentially counts by touching each cube once) Seven.

Teacher: When you were counting, what number came just before seven?

Child: I do not know. Is it one? Three? Five? I don't know.

The child thinks and says: 'Lemme see.'

Teacher: Please count the cubes again. (Points to the same collection.)

Child: (The child counts the cubes again using the same strategy of touching each one as he counts them.) Seven.

Teacher: Can you give me six cubes?

Child: I don't know. I don't think I have enough here. Do I have enough to give you?
Maybe I do. (He counts six cubes one by one and gives them to the teacher.)

Teacher: That is right. (She gives them back to the child.) Please put them back the way they were.

The child rearranges them.

Teacher: How many cubes are there on the table now?

The child counts them by touching each one once and answers.

Child: Seven.

Pauses and says: 'Oh! Seven again. The same seven, I guess.'

Teacher: Yes!

This example shows that the child knows the number words but does not know what those number words mean. He has learned these number words by rote. Thus, rote counting may not help a student in problem-solving such as the comparison of two sets or operations on numbers. Though this rote counting is not true numbersense, it is a starting point for number conceptualization. This rote counting process is essential as it develops the lexical entries for numbers. However, with rote counting as the only skill, children may not use the simple one-to-one counting as a strategy to compare two sets even when they are given perceptual reminders of this correspondence. To extend the lexical entries for number 13:

Teacher: That is right.

Child: (As the teacher acknowledges the child's answer, the child is thinking about something and then touches the cubes. The child counts the cubes again using the same strategy of touching each one as he counts them, first from left to right and then right to left.)

'Seven! It is the same thing. It's seven. It doesn't matter how you count; it is the same number. It's seven. See!'

Counts again once from left to right and then right to left.

'I guess it is always seven. There are seven cubes.'

In the first example, the child is only reciting a string of words. He is not connecting the different number words with each other and with the property of the collection. The number (cardinality of the set) arrived at by counting only shows that the child thinks the number is the property of the counting process not of the collection. This is called *egocentric counting*. The child thinks that the number produced is the outcome of that counting. Many young children when asked to count the same collection again may produce a different answer and may not even question themselves. Most children with

exposure to counting, problem solving, and discussion transcend this type of egocentric outcome of the counting process.

In the second case, the child acquired an important concept: the number (*cardinality of the set*) is not the property/function of the counting process but the property of the collection.

Associating a number to a collection and considering that number as the property of the collection is the first step in the direction of conceptualizing number properly. The type of questions asked in these two activities are the means to converting a child's concrete experiences and egocentric counting into abstract concept—*numberness* (number represents the cardinality of the set—it is the property of the set not of the counting process). The prior examples illustrate how the structure of a problem and nature of questions asked influence the strategies and concepts children develop to solve problems.

I. Concept of Visual Clusters

Each of the three representational codes (verbal, Hindu-Arabic, and magnitude information) is a starting point for different arithmetic activities ([Dehaene, 1992](#)). However, the verbal and Hindu-Arabic representations are meaningless without the magnitude associated with those representations. Many children end up in these situations. Associating a magnitude using counting objects or number line representation is a good starting point, but does not work for all children, particularly for learning disabled and dyscalculic children and adults. We have found the use of visual cluster cards to be more effective for all children, with and without disabilities.

A visual cluster associates magnitude to the number easily. And it is much easier to recognize a cluster than counting a collection. Recognizing a visual cluster is just like recognizing letters and words instantly. It helps in forming *mental orthographic/graphemic representation (MGR) of a number in the mind*. The MGR is conceptually supported by the magnitude of the visual cluster. The associated MGR thus formed is not in a particular font or size or particular icon—it is a fluid and malleable representation. Even the pattern of the visual cluster is not fixed. It is also malleable. When we see it in the mind's eye, we can recognize it, recall it, say it, and can write it in Hindu-Arabic numeral form. We activate this representation independent of size, color, and font. When the image is clear and the three forms are integrated, we can call it easily. However, to be able to do it fluently needs practice.

Just like the orthographic patterns of letters are abstractly stored, the numberness concept and image formed by the help of visual cluster of a collection of objects in our brain and then associating mental orthographic abstract image for the number is easily

formed and stored in the mind's eye. If one can form mental orthographic image of letters and words, then one can also understand and embrace the concept of visual clusters. With practice, we should be able to see the image of the cluster of objects in the mind's eye and should be able to recognize it as the associated number.

Teachers need to help children form the images of clusters as mental graphemic representation using transparent effective instructional materials and practice. In other words, there should be congruence between the instructional models and the learning task of forming images of clusters and numbers. It is the familiarity and practice that helps the child in forming the mental orthographic images associated with the clusters. In fact, we should begin visual cluster activities using the experience of visualizing a familiar scene (their kitchen, classroom ...) to help them understand the concept of seeing something in their mind's eye. Children taught with an emphasis on counting do not form the images of clusters and therefore have difficulty learning and automatizing concept of number, mastering *sight facts* and later other arithmetic facts.

The brain, as it learns to speak, begins to regard words as single sounds (processing the whole rather than the parts. To them, it is not made up of parts. However, when a child learns to read, the brain must reprogram these pathways in order to be able to identify the individual bits in spoken words that are represented by letters/phonemes. Brain can perceive and recognize speech sounds does not mean that four- or five-year-old children can naturally identify the individual sounds in words. Therefore, we have to teach them to consciously identify and manipulate these sounds. This process is new to them. It is different from infants learning to speak. The brain needs to learn how to identify these bits in order to link them to a visual shape—a letter. This is essential new learning that is part of the new skills we call writing and reading.

The process is not very different in learning number ([Starkey and Cooper, 1980](#)). Initially, the child sees the five fingers of one hand as separate entities and adults help him to count them to find out how many. They provide the names as he counts. Eventually, the child sees the five fingers of the hand as a collection, not made up of components. We teach children it is made up of five individual fingers. But with experience, he knows it instantly that each hand has five fingers. That is 'fiveness.' Up to this point, there are similarities between reading and arithmetic. The extra complicating item in arithmetic is that it is cumulative in the sense that every number (as a representation of quantity) also subsumes previous numbers (as a combination of collection of quantities). A collection of five also has a sub-collection of 4, 3, 2, and 1. The development of these skills calls for effective methods of teaching. In that sense, it is more like a word than a letter: not only the collection of five individual objects is 5, but it is also made of smaller numbers. This is another important abstraction that is new to the child.

So, the question is how do we teach this collection of new skills? If it does not come naturally to struggling children, what are the effective ways to call their attention to the fact that the words and numbers they say are actually not a single bit, but strings of different bits? The brain has to be trained to deal with this new set of skills.

The alphabet was invented to make speech visible. Similarly, number names and their representations were developed to make the quantity (magnitude) visible when the collection is not there. In teaching number, we duplicate one of the most fundamental flaws found in almost all phonics instruction today, including traditional ones, is that they teach the code backwards. That is, they go from letter to sound rather than from sound to letter. The print-to-sound (conventional phonics) approach leaves gaps, invites confusion, and creates inefficiencies. We create the same problem in teaching number—most teachers begin teaching number from forming orthographic images (number shapes) to quantity. It should ideally be from quantity to orthographic representation.

As in reading, there are two elements involved—the speech-bits and the sound-bits. In learning number there are three elements involved: recognizing a cluster of objects as a whole (*the magnitude*), the orthographic image (*Hindu-Arabic numeral*), and the spoken name for the number (*verbal name*).

1. Visual Clustering

There is abundant research which tells us that struggling readers and writers (spellers) may not have developed fully specified MGRs as easily as typically developing students can. Therefore, we cannot a priori assume that most children can visualize what the (printed) word looks like.

The same process applies to teaching how to visualize visual clusters. When we think about intervention for students who have ‘fuzzy’ images of visual clusters, it is two-pronged: activities that help students establish robust images of collections and activities that teach students how to apply these images when creating a visual cluster in the mind.

Counting objects is a natural activity for children. They are exposed to counting in abundance—at home, in school, in their games, and playing with toys. However, for proper number concept, the child must master it and then transcend counting. Just as many teachers and the curricula used to focus on letters only rather than phonemes and graphemes, similarly, many teachers and curricula keep on focusing on counting. Letters are not the basic units for decoding, sounds (phonemes), and their spellings (graphemes); similarly, instead of focusing on ‘counting up’ for adding and ‘counting down’ for subtracting, instruction must focus on strategies derived from decomposition/recomposition of visual clusters for ‘sight facts’ for numbers up to 10. However, understanding the processing details of visual clusters (magnitude or quantity) in relation to Hindu-Arabic and verbal representations is an imperative. Forming an

image of the magnitude plays an important role in developing numberness—estimation, counting, number comparison, and mental arithmetic.

2. *Counting*—the ability and a resource to naming the collections of objects—a unique cultural gift to a child, is an important activity and a versatile tool; however, it is a useful tool in the initial stages of learning digits (the basis of number of our number system)—making subitizing real, understanding the structure of our number system (e.g., understanding number as the property of a collection—cardinality, location, and comparison of number—making the innate number line into a real and usable³ and ordinal number, recognizing the repeating cycles of numbers in the decades—beginning of the concept of place value, increasing number vocabulary—extending the concept of sight facts, beginning of understanding and discovering properties of number, etc.), but when it is used as the basis of deriving facts—addition, subtraction, multiplication, and division, there it is an inefficient activity, strategy, and a tool.

Access to magnitude information represented by a number is necessary before any other numerical processing can take place ([McCloskey et al., 1985](#); [McCloskey, 1992](#)). But it should transcend 1–1 counting. It should be in terms of patterns like visual clusters (dominos, dice, die, playing cards, VCCs, etc.). Introducing children to visual clusters facilitates the other representations and children take off using any of the three representations. And during processing, information can be exchanged between the systems. In other words, they can go from visual clusters to cardinality of the set to location of a number on the number line.

The ability to connect language to images in the mind (visual/verbal integration) seems so basic that it is often assumed that everyone does it. However, significant improvement in language comprehension occurs when mental imagery is stimulated, suggesting that forming and using imagery is not automatic for many people. Similarly, carefully sequenced activities about number visualizing and verbalizing using Visual Cluster cards (VCCs) can help children hold clusters in the mind's eye, therefore, numerosity of the collection in the mind's eye. This provides a helpful framework for stimulating the ability to 'see with the mind's eye' resulting in forming number relationships and transcending counting.

After initial exposure to counting materials of different types (e.g., counting blocks, hash marks, number line, fingers, pennies, buttons, marbles, etc.), materials such as dominos, dice, and playing cards are better models for the development of the number concept. Visual Cluster cards are the most effective instructional material that helps develop the number concept—sequencing, one-to-one correspondence, visual clustering, and decomposition/recomposition.⁴

Children with or without mathematics difficulties or risk for number difficulties and even those who have trouble with written and/or oral language processing can learn to use mental images of visual clusters to get to what they already know about certain numbers and apply that information to what they need to understand in number relations. The human brain's nature and ability to see and visualize patterns is the principle behind these cards. As children gain skill in the process of visualizing and then verbalizing about what they see, they begin to independently match conscious images of clusters (gestalt of the cluster) to the concepts (numbers) expressed by a group of pips forming the clusters.

A deck of Visual Cluster Number Cards (VCNC) consists of 4 cards with each with 1 to 10 pips in 4 suites (heart, diamond, club, and spade); a card without any pips, which represents the number zero; 2 jokers represent variables—can be given any value, contextually, during card games such as Number War Game.⁵ Cards showing 3, 8, 9, and 10 have two to four different types of patterns as representations of pips on them (in the standard form of clusters with slight variations and alternative arrangements).

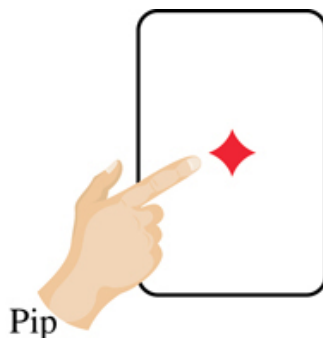


Figure 6.5 An example of a ‘pip’ or ‘icon’ on a card. The card representing number 1 has 1 (one) pip in the middle of the deck.

3. Decomposition/Recomposition

The pips on number cards (e.g., 2 through 10) are organized in such a manner that the sub-clusters of smaller numbers than the given number can be instantaneously recognized on the same card. For example, on the 7 of diamonds in the VCC collection (shown in the following), one can see a cluster of ‘5 and 2’ or ‘6 and 1’ and ‘4 and 3’ easily.

The following diagram demonstrates the relationship between the visual cluster of number 7 and the sub-clusters representing smaller numbers that make 7 (as $3 + 4$, $4 + 3$; $5 + 2$, $2 + 5$; $6 + 1$, $1 + 6$). The ability to see two sub-clusters on the visual cluster of 7 and bringing them together to make it 7 is the key concept in understanding number relationships. It is called *decomposition/recomposition*.

Recognizing visual clusters on the cards develops in children the key concept of numbersense. With the help of decomposition/recomposition skill one develops the 45 sight facts—the beginning of number relationships. For example, from the decomposition/recomposition of visual cluster of 7, one develops six sight facts: $1 + 6$, $2 + 5$, $3 + 4$, $4 + 3$, $5 + 2$, and $6 + 1$. Later on, decomposition/recomposition is responsible for developing the flexibility of use of numbers in arithmetic operations.

The skilled questioning of a professional can cause the individual to notice where his or her images of clusters do not match the displayed clusters so that needed adjustments in images (and understanding) can be made. When images can be clearly organized in the mind's eye and described verbally without looking at the cards, then decomposition/recomposition can take place and higher-order tasks such as generalizing, making inferences, or drawing conclusions about number relationships become possible. Then, children are able to automatize addition and subtraction facts with ease.

The patterns on the cards create a very rich, vivid, complete imprint of visual clusters on the brain. The power of imagery enables a person to see these visual clusters. What is taking place in this process is an example of *eidetic memory*. By the help of these clusters, a child can recall the sub-clusters and form sight facts quickly and sometimes with only a little prompting. It is like learning sight words with visual exposures. This means that the student who could not master simple addition and subtraction sight facts without spending hours counting on fingers, number line, hash marks, and objects and re-counting can learn first the sight facts. Later, using decomposition/recomposition, making ten, and mastery of teens numbers, they can expand sight facts to other facts. We call these *sight facts* as they are learned by sight—visual exposure rather than counting.

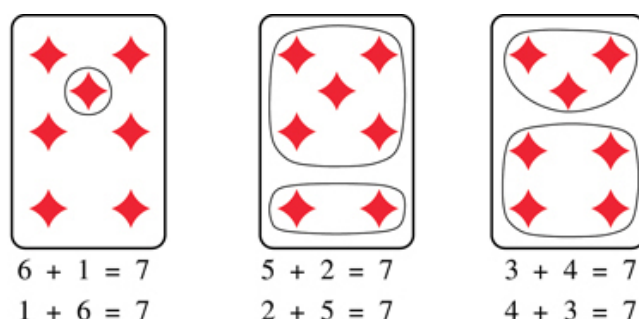


Figure 6.6 Visualising Sight Facts of Number 7 Using VCNC

Children who are taught number concept only through counting—counting discrete objects, number line, fingers, hash marks, etc., struggle with number fact recall, as they have not formed these rich, complete, eidetic imprints and the most important concept about number, the concept of decomposition/recomposition. Most of the interventions, generally, focus on counting, ‘seeing’ numbers as made up from ones, speeding up recall

(flash cards, MadMinutes, computer, iPad, iPhone apps) cannot achieve what children can achieve with decomposition/recomposition using Visual Cluster cards. The efforts made and time spent on the use of multi-sensory aspects of VCC and Cuisenaire rods help create deeper, richer eidetic imprints and flexibility in conceptualizing number.

Research (Rayner, 2014) has shown that skilled readers of English language typically look at words for about 250 milliseconds, make regressions (e.g., look back at words) 10% to 15% of the time, and skip about 25% to 33% of words. Poor readers, in contrast, spend more time on individual letters in the words, regress more, and skip fewer words. Similarly, children with little fluency in arithmetic facts focus on every number (as they count), whereas children with fluent arithmetic facts do not focus on each number but focus on clusters of numbers and use strategies derived from decomposition and recomposition of numbers. This suggests that it is particularly important for elementary- and middle-school teachers to select appropriate materials so children can progress steadily rather than risk slowing their fluency and understanding/comprehension development.

All readers must learn the sound/symbol correlations that make up the written code in order to read and spell. Intuitive readers mostly pick up these patterns through exposure and some direct instruction while those with dyslexia need overt, intensive, multi-sensory (see it, say it, write it) instruction to internalize the code. Of course, meaning and context also matter.

Similarly, all children must learn the quantity/symbol correlations that make up the numeracy code to become fluent in mathematics. Most above-average and many average children pick up the patterns of number relationships through exposure and some direct instruction, while a large number of average and below-average children and those with mathematics difficulties such as dyscalculia need overt, intensive, focused, efficient, multi-sensory (construct it, touch it, manipulate it, see it, visualize it, say it, write it) instruction in order to internalize the quantitative relationships—number concept, arithmetic facts, place value, and arithmetic operations. These are better facilitated by effective and efficient materials and a well-thought-out sequence of activities that are easily converted into ‘working scripts’ ([Gelman & Gallistel, 1978](#)).

Multi-sensory teaching applied in the context of mathematics teaching and intervention means making mathematics concepts transparent, easy to understand, and helpful in arriving at the conceptual schema and procedure efficiently. It is a misunderstood concept by most math educators. To many teachers concrete materials are just objects that we touch, see, and manipulate, and learning happens. And most of the time they end up counting objects and finding the answer with them (add numbers, count them, subtract a number from another, count the smaller number from larger number). The purpose of concrete materials is to:

- a. generate the language of the idea,
- b. create the conceptual schema, and
- c. arrive at efficient and elegant strategies and procedures.

True multi-sensory apparatus implies that one is working on improving information processing based on sensory pathways rather than just improving the ability to associate quantity and symbols.

Instructional approaches that are effective and efficient use direct, explicit teaching of quantity-symbol relationships, number patterns, decomposition/recomposition strategies, develop conceptual schemas, and meaningful number relationships, and provide a great deal of successful practice of skills and procedures, with and without problem solving.

No one learns to read English or similar languages without phonics instruction. The only difference is whether we teach it to them, or they teach it to themselves. We all do some of the latter, and most of us need some or more of the former, but no one learns to read without phonics instruction. If they do, it requires a great deal of effort. The two authors have the firsthand experience of learning the English language. Similarly, no one learns arithmetic facts and number relationships well without decomposition/recomposition. The only difference is whether we teach it to them, or they teach it to themselves. We all do some of the latter, and most of us need a lot of the former, but no one learns numeracy meaningfully or fluently without decomposition/recomposition process. But some do not acquire decomposition/recomposition without intentional, direct, efficient instruction. Formal, intentional instruction in decomposition/recomposition facilitates and accelerates learning number concept, arithmetic facts, numbersense, and ultimately, numeracy—a major goal of mathematics instruction in elementary school. When a student in higher grades has not mastered numeracy, we need to make sure that he has mastered decomposition/recomposition first.

4. Sight Facts

Decomposition/Recomposition Process and Sight Facts for Addition

Sight facts are the addition facts that are related to number concept, but they are not the outcome of counting nor even of addition. They are learned by exposure, sight, and without counting—it is a perceptual activity. They are perceived by a child as a whole-and-parts simultaneously. It is truly a perceptual activity, like perceiving color and letters. If arrangements of objects are in a pattern, it is much easier to recognize that collection. For example, when a child visualizes the arrangement of five pips on the

Visual Cluster Number card,⁶ s/he automatically (by observation through perceptual skills of decomposition/recomposition) recognizes that:

$$\begin{aligned} 5 &= 1 + 4 \\ &= 2 + 3 \\ &= 3 + 2 \\ &= 4 + 1 \end{aligned}$$

Initially, the child does not derive these sight facts in this order, but once she has generated the four sight facts of 5, the teacher should encourage her to organize them in order. When any task is organized and displayed as a pattern, it is easier to visualize it, and then transferring it to long-term memory is also easier. This process is an early example of combinatorial logic. Not only does the child know what strategy to apply but can recite all the sight facts of any number by following the sequence of numbers in order. In such a situation, a child also knows when the task is done. Otherwise, they randomly generate them by counting and sometimes it becomes a loop; they begin to repeat them and some sight facts might be left out. And they constantly ask: ‘Am I done?’

The sight facts can be learned effectively and efficiently by the help of Visual Cluster Number and Sight Fact Cards (an instructional model for discrete objects with the added advantage of patterns in the arrangement of objects on the cards) and Cuisenaire rods (an instructional model for continuous quantities). Both instructional materials are the basis of deriving strategies for learning arithmetic facts involving addition, subtraction, multiplication, and division.

5. Deriving the Sight Facts of Number 5

(a) Using Visual Cluster Number and Sight Fact Cards

The following diagram illustrates the 5-VCN card (from now on, this card will be known by the number: *5-card*) with its sight facts identified: $3 + 2 = 5$ and $2 + 3 = 5$; $1 + 4 = 5$ and $4 + 1 = 5$. The number 5 thus has 4 sight facts.

The diagram also illustrates the key concept of numeracy: *decomposition/recomposition* (breaking the cluster of 5 into two sub-clusters of 2 and 3, 3 and 2, 4 and 1, and 1 and 4 and then recognizing that $1 + 4 = 5$, $2 + 3 = 5$, $3 + 2 = 5$, and $4 + 1 = 5$). The concept of decomposition/recomposition in mathematics is parallel to the concept of phonemic awareness.⁷

(b) Deriving Sight Facts Using Cuisenaire Rods

The teacher demonstrates⁸ (and children engage in the same activity on their desks) that the yellow 5-rod is the same length as the purple (4-rod) plus the white (1-rod); the light green (3-rod) plus the red (2-rod), the red (2-rod) and the light green (3-rod), and the white (1-rod) plus the purple (4-rod) are equal in length, demonstrating the 4 sight facts of number 5. Then, the teacher points to the rods showing:



Figure 6.7 Sight facts of 5 using number rod and VCNC

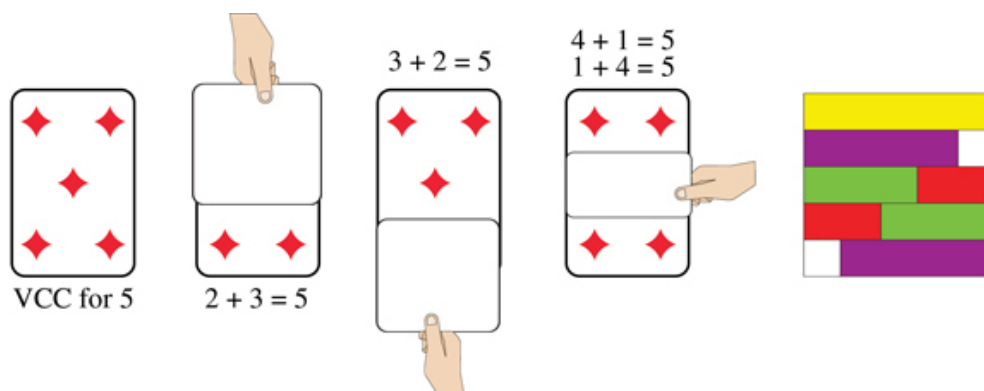


Figure 6.8 Demonstration of sight facts of Number 5 using VCNC and Cuisenaire rods/Number rods

$$4 + 1 = 5, 3 + 2 = 5, 2 + 3 = 5 \text{ and } 1 + 4 = 5$$

These equations show the additive relationships between 1, 2, 3, 4, and 5 and identify and define the 4 sight facts for number (numeral) 5 as a continuous model and the concept of equality is clearly evident. A similar process can be used for finding the sight facts for the first ten natural numbers.⁹ Every number from 2 to 10 has 1 less than the number as the number of sight facts associated with it. These sight facts and their roles are similar to sight words in the reading process.

(c) Using Invicta Balance

The teacher demonstrates¹⁰ (and children engage in the same activity doing it on their desks) that the unit weight is placed on 5 on one side of the balance, representing

number 5, and then invites children to take 2-unit weights and place unit weights at 1 and 4 to help achieve the equilibrium on the balance. This shows that $1 + 4 = 5$ and asks children to read all the equations involving the sight facts of 5. The same is repeated with other equations of sight facts of 5. Then, the teacher demonstrates or asks for volunteers to demonstrate the equations:

$$4 + 1 = 5, 3 + 2 = 5, 2 + 3 = 5 \text{ and } 1 + 4 = 5$$

These equations show the additive relationships between 1, 2, 3, 4, and 5 and identify and define the 4 sight facts for number (numeral) 5 as a continuous, but abstract model. The Invicta balance can be used to draw attention to the idea/concept of ‘equals.’ Most children miss the concept of equality. They treat it as a one-way implication. Invicta balance and Cuisenaire rods are the most efficient models to show the two-way implication of equality.

By the help of Cuisenaire rods and the Invicta Balance, the concept of equality is clearly evident. Concept of equality is very important in mathematics and is a difficult one for children to understand ([Shoecraft, 1989](#)). A similar process can be used for finding the sight facts for the first ten natural numbers.¹¹ Every number from 2 to 10 has 1 less than the number as the number of sight facts associated with it.

As noted earlier, that number should be conceptualized as a (a) count and/or (b) measurement. As a result, in our instructional approach (VAVA/S) to number, we use both models: Visual Cluster cards take advantage of the number concept as a perceptual activity, and therefore, the child realizes it as the cardinality of sets/collections, and through the exploration of Cuisenaire rods, she realizes the measurement aspect of number.

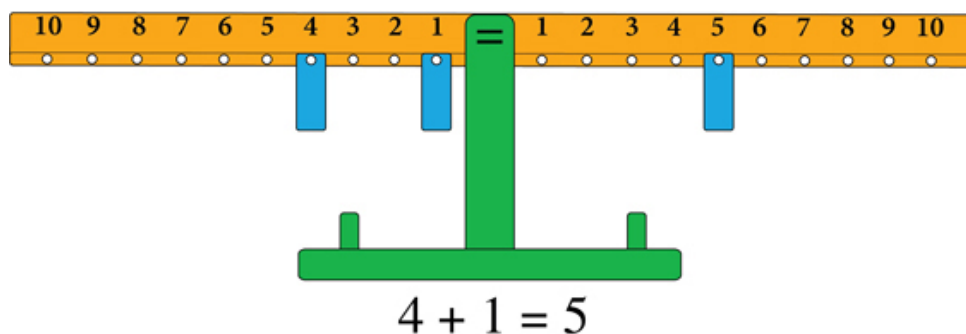


Figure 6.9 Sight fact using Invicta Balance

As we derive the sight facts, it is important to emphasize the commutative property of addition at this point so children can realize that once they have one sight fact, they also have another one. Cuisenaire rods and Invicta Balance, along with the questioning and

practice, are the best means of teaching and learning the commutative property of addition.

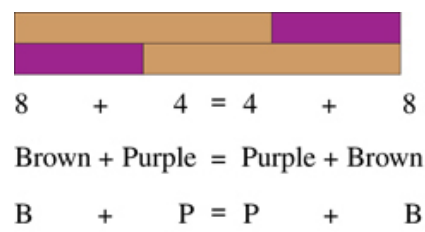


Figure 6.10 Commutative property of addition

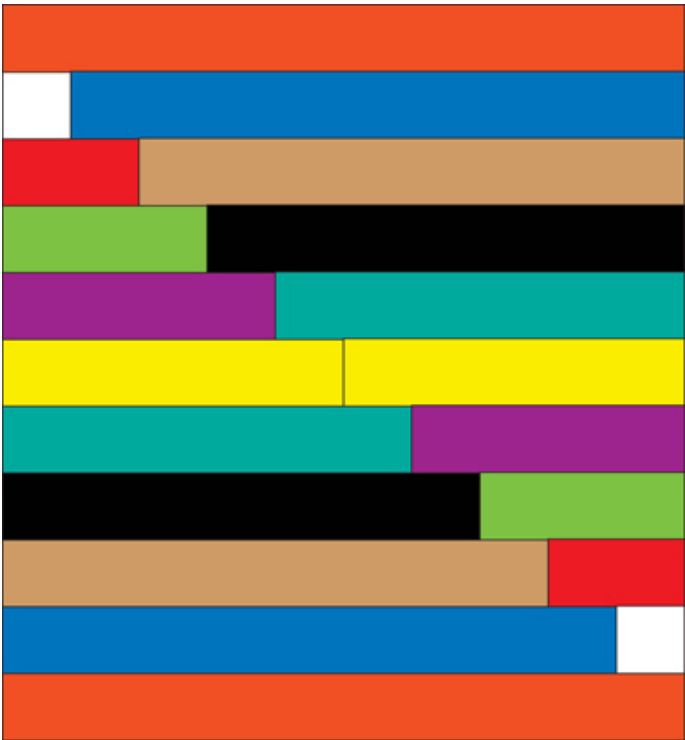


Figure 6.11 Sight facts of 10

2 = 1 + 1	(1 sight fact)
3 = 2 + 1 = 1 + 2	(2 sight facts)
4 = 3 + 1 = 1 + 3 = 2 + 2	(3 sight facts)

$5 = 4 + 1 = 1 + 4 = 3 + 2 = 2 + 3$	(4 sight facts)
$6 = 5 + 1 = 1 + 5 = 2 + 4 = 4 + 2 = 3 + 3$	(5 sight facts)
$7 = 6 + 1 = 1 + 6 = 2 + 5 = 5 + 2 = 3 + 4 = 4 + 3$	(6 sight facts)
$8 = 7 + 1 = 1 + 7 = 2 + 6 = 6 + 2 = 3 + 5 = 5 + 3 = 4 + 4$	(7 sight facts)
$9 = 8 + 1 = 1 + 8 = 2 + 7 = 7 + 2 = 3 + 6 = 6 + 3 = 5 + 4 = 4 + 5$	(8 sight facts)
$10 = 9 + 1 = 1 + 9 = 8 + 2 = 2 + 8 = 3 + 7 = 7 + 3 = 6 + 4 = 4 + 6 = 5 + 5$	(9 sight facts)
TotalNumberOfSightFacts = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45	

Figure 6.12 List of sight facts of numbers up to 10

The above graphic demonstrates the sight facts of 10 or the process of making ten using the Cuisenaire rods. The key mathematics strategy used in learning and mastering sight facts is decomposition/recomposition.

J. Teens Numbers

The role of teens numbers is very important in all arithmetic procedures. Most children think of a teens number, for example, 14, as fourteen 1s, till they understand teens numbers as the beginning of place-value. That means that the digit ‘1’ in number 14 is a group of 10, and being in that place makes it show that it represents a ten and the 4 is a group of 4 ones and it can be ‘broken’ into ‘1 and 3,’ ‘2 and 2,’ and ‘1 and 3.’ Therefore, we may teach them first beginning with a group of 10 and 4, but very soon it should turn into an effective and efficient model using Number rods.

The teens number should be derived and mastered by using Visual Cluster cards and Cuisenaire rods. However, the Cuisenaire rods are the most effective means of making

the teens number. To make 15, for example, one uses the 10-rod and the 5-rod as shown in the following: We begin with a very informal activity with the whole class. The teacher asks children in the early days of kindergarten, when they have been introduced to number-rods and they have learned their values, to pick up the number-rod representing the number she calls out. She begins with 1, 5, 7, 4, 8, and then she says: 14. Most children would say: 'There is no number rod that is 14.' However, one or two children will show the teacher 14. As soon as they do it, almost everyone shows the teacher one 10-rod and a 4-rod in their hands. To confirm it, the teacher asks them to make the train of the 10-rod and the 4-rod and place 1-rods parallel to the train and count. They all confirm the fact, and the teacher checks the train of a few children; for those got the correct answer, she deputizes them to be her assistants, and they check the work of rest of the children. The game continues for about 15 or 20 minutes. Occasionally she would ask, 'I will say a number in the twenties or thirties,' and then she would ask the same question individually. The numbers she calls out will depend on what she knows about the children during this game. By the end of the activity, they all know teens numbers. The following is the approach to teach teens numbers and how to write them and the idea of two-digit numbers.

After children have made the number 15 by rods, a child is asked to take her rods, and she finds the number 15 on the calendar or the number line. She is asked to take the number by her dominant finger and places the orange rod (10-rod) above the digit 1 in 15 and the yellow rod (5-rod) above the digit 5. The important idea here is that the children know and understand the difference between 'digit' and 'numeral/number.' They should be able to answer the following questions:

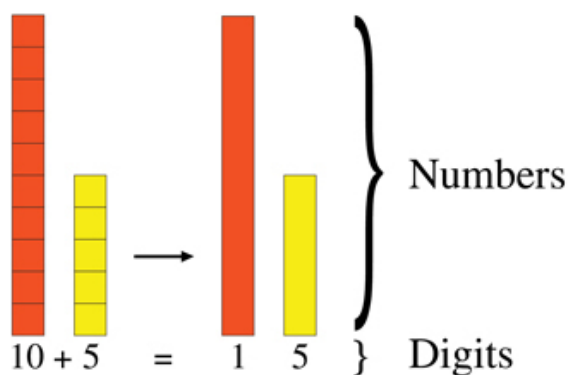


Figure 6.13 Representing number 15 using number rods

- What two digits make 15? 1 and 5.
- Digit 1 is in what place? Tens place.
- How do you know? The 10-rod is placed above it.
- Digit 5 is in what place? Ones place.

- How do you know? The 5-rod is above it. And 5-rod is 5 ones.
- What two numbers make 15? 10 and 5.
- What is the value of the digit 1? 1 Ten or 10.
- What is the value of the digit 5? 5 ones or 5.

If these questions are answered correctly and consistently, then *the child knows 19 more addition facts as part of the 100 addition facts (the sums up to 20)*. As mentioned earlier, the number of sight facts and number of facts based on teens numbers account for $45 + 19 = 64$ out of 100 addition facts.¹² As a result, there are only 36 addition facts left from the list of addition facts to be mastered in the first grade. And because of the commutative property of addition, there are only about 20 facts remaining to be mastered—not a large number for the whole year of instruction. Some kindergartners can go further than these 64 addition facts, just like some kindergartners can read fluently beyond kindergarten level. However, the important part is to acquire fluency and understanding with an ability to apply them, rather than arriving or learning them just by rote counting or using only concrete or pictorial methods.

All children, with or without disability, can learn, master, and apply the 45 sight facts plus 19 teens' number facts by the end of kindergarten.

Remaining facts: $100 - 45 = 55$ facts

By commutative property of addition, they are reduced to 23. (excluding $6 + 6, 7 + 7, 8 + 8, 9 + 9, 2 + 9, 3 + 6, 3 + 7, 3 + 8, 3 + 9, 4 + 7, 4 + 8, 4 + 9, 5 + 6, 5 + 7, 5 + 8, 5 + 9, 6 + 7, 6 + 8, 6 + 9, 7 + 8, 7 + 9, 8 + 9, 9 + 9$).

We can definitely manage 23 facts in first grade in a year.

Strategies based on decomposition/recomposition and sight facts take students to the level where they can easily apply their number concept skills to derive other number relationships, arithmetic facts, and understand the concept behind place value better and more deeply ([Sinclair and Coles, 2015](#)). Moreover, in teaching any mathematics concept, one should remember that learning mathematics is to integrate the linguistic, conceptual, and procedural components of a mathematics idea.

Mathematics learning, including numeracy, is not just learning a collection of facts and procedures. This is achieved by effective questioning by the teacher and proper use of efficient and effective instructional, cognitive, and language tools. The objective is to develop understanding of all of these, which helps formalize the '*language script*'¹³ for the number attack skills, conceptual schemas, and procedural methods. This learned

material can be sent to long-term memory easily; it resides there to be used for forming other facts and then retrieved easily for applications.

The important question, then, is:

- Are the prevalent methods of teaching numeracy effective?
- Do they achieve these objectives?
- If not, why not?
- What are their strengths and weaknesses?
- What are the implications, if we do not use efficient strategies?

Let us discuss each one of these approaches in detail: Their strengths and their limitations.

The Visual Cluster cards are organized in such a manner that the sub-clusters of a given number are easy to recognize on the standard cluster of that number. For example, as demonstrated in the Tools chapter, when we begin with the 8-card and add one pip to it we have the 9-card (see three 9-cards; first 9-card is made by adding the new pip in the middle of the 8-card; if we add that pip on the top on the 8-card, we get the second 9-card; and if we add the pip in the bottom, we have the third 9-card in the sequence), now we have decomposed 9 into two numbers: 8 and 1, and one can easily see the sight facts: $8 + 1 = 9$, $1 + 8 = 9$, $9 = 8 + 1$ or $1 + 8$. Similarly, if we see the sub-cluster of two pips on the bottom of the first two 9-cards in the sequence, and the cluster of 7 above the sub-cluster of 2, one can see that the 9 is decomposed into two numbers, 2 and 7, and it gives us the sight facts: $7 + 2 = 9$, $2 + 7 = 9$, $9 = 2 + 7$ or $7 + 2$.

If we look at the 9-card, where one pip is added to the 8-card in the middle column at the top (see the second card in the sequence), we can see the cluster of the 5-card on the top and the 4-card below it on the 9-card and thus 9 is decomposed into 5 and 4 or 4 and 5 giving the sight facts: $4 + 5 = 9$, $5 + 4 = 9$, and, $9 = 5 + 4 = 4 + 5$. On the other hand, if we put the 5-card and the 4-card together, we recompose it into the 9-card.

The 9-cards in the last row can be seen as cluster of 6 on the top and then a cluster of 3 below the 6-cluster. To get the decomposition of 9 as 6 and 3 better, we form a new 9-card starting with the 6 card and adding a group of 3 pips to the 6-card. This gives a 9-card with a different arrangement of clusters. This 9-card (the last card in the sequence) has the 9 pips that are organized into 3 groups of 3s or can be seen as one group of 3 and a cluster of 6. Now 9 is decomposed into 6 and 3, and we get the sight

facts: $9 = 6 + 3 = 3 + 6$.

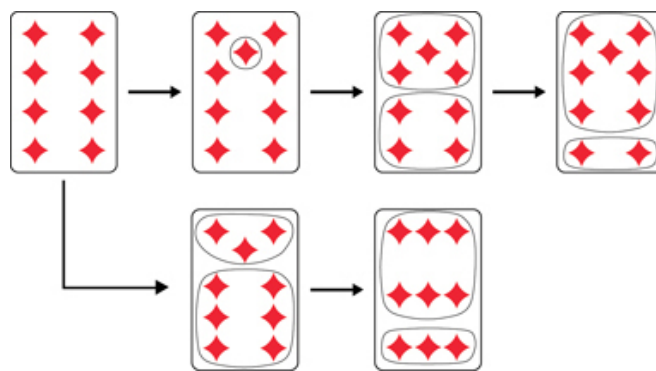


Figure 6.14 Display of Sight Facts of 9 on VCNC 

In [Figure 6.14](#) Sight facts of 9, one can identify all the sub-clusters clearly. Children are able to see the sub-clusters with very little training. However, the different arrangements of 9 into these three 9-cards facilitate all the possible decompositions of 9 (as a sum of two numbers) and therefore all the possible sight facts (8 sight facts) related to 9 ($9 = 8 + 1 = 1 + 8$; $9 = 7 + 2 = 2 + 7$; $9 = 6 + 3 = 3 + 6$; $9 = 5 + 4 = 4 + 5$). These sight facts relate 9 to all the smaller counting numbers.

The same process can be applied for all the other numbers up to ten. In order to derive all the possible combinations of decomposition and recomposition, the numbers 3, 6, 8, and 10 also have alternative cluster arrangements. We find that the number 9 is the most difficult number to derive the sight facts from; therefore, 9 has four variations.

Decomposition/recomposition and sight facts are at the heart of number conceptualization and learning arithmetic facts. The distributive, associative, and commutative properties of operations and their application are also dependent on understanding the process of decomposition and recomposition. They can easily be shown by the help of Visual Cluster cards and number rods.

The decomposition/recomposition of numbers is achieved by a careful introduction to cards, using Cuisenaire rods and children practicing the process under supervision. Many above-average and average children achieve the decomposition and recomposition of number skills independently, using many even by counting, but many average and most below-average and special needs children have great difficulty in arriving at the decomposition/recomposition of number using counting strategies and materials.

In our extensive work with children with and without learning disabilities, the Visual Cluster Cards and Cuisenaire rods are the most effective materials to achieve decomposition/recomposition skills and mastering arithmetic facts. One begins with VCC and then moves to Cuisenaire rods. The next section describes the development of the visual clustering and decomposition/recomposition process using Visual Cluster Cards and Cuisenaire rods.

With true numberness with decomposition and recomposition as the basis, one can easily learn the arithmetic facts, particularly addition/subtraction facts. The concept of visual cluster and the decomposition/recomposition is the phonological equivalent of numbersense. The weaker a child's decomposition/recomposition skill and automatic fact recognition skills, the more the child has to rely on inefficient strategies such as counting.

Interventions that provide compensatory use of inefficient strategies will never entirely help children make up for their weak numbersense. Similarly, rote practice of arithmetic facts and procedures is not productive in the long run. Practicing a skill over and over in the same way may teach students to acquire the skill, but it won't necessarily lead them to apply that skill flexibly in other contexts. It does not encourage comprehension. Students need to practice the skill in a variety of settings and in different ways using efficient strategies to be able to retain, generalize, and apply that information.

K. Social and Academic Expectations About Numeracy

1. What is Numbersense?

To be fluent in numeracy, one of the components is having good numbersense. One's open, positive, and flexible attitude toward number and the ability to display that flexibility and proficiency in handling quantity is called *numbersense*. As a skill set, numbersense refers to a person's ability to look at the world quantitatively and make quantitative and spatial judgments and decisions with mental calculations using the properties of numbers and operations. It is the integration of (a) *number concept*, (b) *number relationships* (arithmetic facts), and (c) *place value*.

Numbersense consists of a cluster of ideas such as the meaning and ways of representing numbers, developing and seeing relationships among numbers, observing the relative magnitude of numbers, and proficiency in working with them (ultimately leading to mastering arithmetic facts and their usage). Numbersense is not a set of discrete skills but a set of integrative skills. Students with good numbersense can move effortlessly between the real world of numbers and formal numerical expressions. They can represent the same number in multiple ways depending on the context and purpose. In operations with numbers, children with a good sense of number can decompose and recompose numbers flexibly, with ease and fluency. When asked to give the sum $7 + 8$, a person produces 15. But she would also be able to work out the sum using efficient strategies as: $7 + 7$ is 14 and then add 1 to make 15, or two 8s is 16 and subtract 1 and get 15, or, I take the 3 from 8 and add to 7 and $7 + 3 = 10$ and 5 more is 15, $8 + 2$ is 10

and 5 more is 15, or, $7 + 8 = 5 + 2 + 3 + 5 = 5 + 5 + 5 = 15$. When she knows $7 + 8 = 15$, she can extend it easily to: $70 + 80 = 150$, $700 + 800 = 1500$, $7x + 8x = 15x$, etc.

Multiple and purposeful use of these strategies gives them fluency ($7 + 8 = 15$ in 2 seconds or less orally and 3 seconds or less in writing) with understanding. Exploring number relationships in this manner, they become so familiar with this fact that it is committed to memory, automatically. There is a need to automatize facts, but there is no need to memorize via rote learning. Automatized math facts, in themselves, are a small part of mathematics, and they are best learned through the strategic use of numbers in different and flexible ways and situations. Unfortunately, many teachers and parents focus only on the rote memorization of math facts and use of unproductive strategies. This gives students the wrong impression that math facts are the essence of mathematics and that the fast recall means being a strong mathematics student. This is a misunderstanding and misrepresentation of mathematics. We should minimize the exclusive role of rote memory in mathematics learning, as it may produce math-anxious and disaffected students. We should emphasize accuracy and fluency with understanding, using efficient strategies for deriving them and then automatization with practice.

Numbersense is age- and grade-specific. It becomes increasingly complex and sophisticated as a person encounters new number systems and operations. For example, at the end of kindergarten the expected mastery of numbersense is:

- a. Number concept,
- b. 45 sight facts, and
- c. Place value of 2-digits.

Similarly, mastery of numbersense expected by the end of first grade is:

- a. Number concept,
- b. 100 Addition facts (sums up to 20), and
- c. Place value of 3-digits.

And numbersense at the end of fourth grade is:

- a. Number concept,
- b. All arithmetic facts (addition, subtraction, multiplication, and division), and
- c. Place value of any number of digits in whole numbers and up to the hundredth place.

At the end of sixth grade, it is:

- a. Mastery of numeracy,
- b. Flexible and effective use of whole numbers, fractions, decimals, and percents, and
- c. Extended place value of whole numbers and decimals.

Through organized practice and experiences in various forms, such as algorithms/procedures, this proficiency and fluency in numbersense is translated into numeracy.

Numeracy is a child's ability and facility in arithmetic operations (addition, subtraction, multiplication, and division) on whole numbers correctly, consistently, fluently, in multiple forms of procedures, including the standard algorithm with understanding.

By the end of fourth grade, a child should have mastered numeracy. If a student has deficits in it at the end of fourth grade, he needs intervention from specialists. That intervention should focus on developing number concept, arithmetic facts, place value, and then numeracy using efficient, effective, and elegant strategies.

L. Dyscalculia and Acquired Dyscalculia

As mentioned earlier, there are several reasons for the incidence of specific mathematics learning difficulties, from language-related to neuropsychological and cognitive reasons to environmental factors. However, *understanding the development of numbersense provides a window into children's arithmetic difficulties, particularly dyscalculia.*

Most of the difficulties in mathematics, particularly arithmetic, emanate from the difficulties in the learning of number-related difficulties.

Essentially, whatever the reasons,

the problem of dyscalculia is manifested as a child's difficulty in conceptualizing and using number, mastering number relationships, and producing outcomes of number operations in one form or other. It is lack of the formation of a 'meaningful' numbersense.

This description of dyscalculia is our working definition. It is based on our observations of a very large number of students over the whole range of mathematics education—from the early years of schooling to university students over a span of more than 50 years in the UK and the United States. Our methods have evolved over the years to respond to their needs.

The difficulty with numbersense may be the result of a child's assets (or lack thereof)—neurological, neuropsychological, and cognitive reasons, and/or environmental factors—poor teaching, poor curriculum, or lower expectations. When these difficulties may

exist in spite of a child having intact neurological, neuropsychological, or other cognitive factors, then it is purely because of environmental factors and this will be termed *acquired dyscalculia*.

Dyscalculia or acquired dyscalculia thus are the manifestation of difficulties in the integration of number concept, numbersense, and numeracy. However, just like effective teaching methods for reading can mitigate the impact of dyslexia, similarly, one can have dyscalculia or acquired dyscalculia, but effective and efficient teaching methods can give students skills so that the effects of dyscalculia and acquired dyscalculia (and other specific mathematics difficulties) are minimized or mitigated. Dyscalculia¹⁴ is a quantity/number (with some overlap of spatial orientation/space organization) based disorder, so the intervention and remedial programs should focus on the development of number concept, numbersense, and numeracy. That means the interventions and remedial instruction for children with learning disabilities (including dyscalculics) initially should focus on the mastery of number concept—visual clustering, decomposition and recomposition, and acquiring the sight facts.

To design effective methods for intervention and remediation, a keen understanding of the definitions and characteristics of dyscalculia is critical. For example, children, especially gifted children, may be able to compensate for even massive deficits using one or more of their other equally massive strengths. A child with tremendous memory and fantastic oral comprehension might be able to get around abysmal arithmetic fact fluency for a while to produce an adequate arithmetic result when mathematics is still fairly simple. But if they have deficits in the understanding, fluency and applicability of number concept, numbersense, and procedures, they will have difficulty in future mathematics. Similarly, a child who derives arithmetic facts using only counting methods may be able to do well on tests and exams in the early grades, but will have difficulty later. Many children do well on elementary assessments, but many of them begin to perform poorly in later grades. Many primary school teachers, administrators, and parents sometimes misunderstand the ‘good scores’ on tests as by the help of inefficient methods; however, those methods are not effective in later grades.

They are not developmental ([Chinn, 2021a](#)). As a result, many students get disenchanted with middle school mathematics.

Unfortunately, many school personnel often get this wrong as their own understanding of dyscalculia and arithmetic disability may be limited, thus affecting resource allocations, availability of professional development in general mathematics teaching, and training of special educators in effective interventions and remediations. This is particularly so in the inadequate preparation of special education teachers in mathematics and mathematics learning disabilities, including dyscalculia, compounding the environmental factors related to dyscalculia. This training is often in its infancy.

Many parents and school personnel mistakenly assume that dyscalculia equates to a permanent condition of deficit or disability. Since true number concept is at the basis of the development of fluent numbersense, the condition of dyscalculia and acquired dyscalculia can be compounded without proper teaching of number concept. It is very difficult to master (understanding, fluency, and the ability to apply) arithmetic facts and procedures without proper number concept teaching.

1. Catch Them Before They Fall

Research in reading processes indicates that certain domain-specific deficits such as measures of a child's phonemic awareness and phonological sensitivity are the best predictors of early reading performance (better than IQ tests, readiness scores, or socioeconomic levels). It is widely accepted that deficits in phonological processing are the proximal cause of Reading Disability (RD) ([Vellutino et al., 2004](#))).

Similarly, a domain-specific deficit in processing numerosities (numberness) has been implicated in mathematics disabilities (MD), such as dyscalculia ([Butterworth, 2010](#); [Wilson and Dehaene, 2007](#)). And, of the several sub-skills of numberness, the most important skills involved as deficits are subitizing and decomposition/recomposition.

Therefore, early screening, identification, and assessment of number concept and numbersense are essential to prevent numeracy failure in young children and avoiding future mathematics difficulties. Students without adequate and early mastery of sight facts and decomposition/recomposition continue to demonstrate poor numbersense and numeracy skills, even into the middle grades and high school ([Sharma, 2009](#)).

There is a strong predictive validity to early number concept (particularly subitizing and decomposition/recomposition) to later mathematics achievement. The contribution, importance, and the role of sight facts and decomposition/recomposition does not diminish with grades. Children continue to use the number concept and decomposition/recomposition whenever they encounter numeracy problems. For example, when students encounter work on fractions, integers and rational numbers, they continue to need and use them for new number relationships.

In addition, domain-general cognitive risk factors such as slow processing speed and working memory might be shared between the two disorders and could possibly explain why they may co-occur. Research shows that almost 40% of dyslexics also exhibit symptoms of dyscalculia. This comorbidity between RD and MD indicates that the fundamental components in both of them that may be responsible are: phonemic awareness in reading and decomposition/recomposition in numeracy, respectively.

We have observed with thousands of children that fluent numberness (integration of one-to-one correspondence with sequencing, subitizing, visual clustering, and decomposition/recomposition) is a predictor of future proficiency and fluency in

arithmetic and even in higher mathematics. Remediation of numberness-related key skills results in better understanding and mastery of numeracy.

On the other hand, high comorbidity rates between reading disorder (RD) and mathematics disorder (MD) also indicate that, although the cognitive core deficits underlying these disorders are distinct, additional domain-general risk factors might be shared between these disorders. Three domain-general cognitive abilities, processing speed, temporal processing, and working memory, are studied in RD and MD. Since attention problems frequently co-occur with learning disorders, the three factors, which are known to be associated with attention problems, account for the comorbidity between these disorders. However, of the attention problems observed in the case of MD, some of them are secondary in the sense that they might be the by-product of consistent failure in mathematics and/or mathematics anxiety. Consistent failures in numeracy may create aversion to quantitative activities; therefore, lower attention and inefficient methods such as counting may overwhelm the working memory.

Research on measures of processing speed, temporal processing, and memory with primary school children with RD, children with MD, children with both disorders (RD + MD), and typically developing children (TD controls) show that all three risk factors are associated with poor attention. After controlling for attention, associations with RD and MD differ: Although deficits in verbal memory are associated with both RD and MD, reduced processing speed is related to RD, but not to MD; and the association with RD is restricted to processing speed for familiar nameable symbols. In contrast, impairments in temporal processing and visuospatial memory are associated with MD, but not RD. Visuospatial memory is essential for subitizing, visual clustering, decomposition/recomposition, and therefore with development of sight facts.

2. Interventions and Remedial Strategies for Dyscalculia

Reading specialists and diagnosticians know that the more solid and deep the underlying phonological, orthographic, semantic/morphological representations of a word (the more robust the connections between the word's linguistic components of sounds, letters, and meanings), the faster the word is recalled. This leads to automatic sight word recognition in reading. This is the power of multi-linguistic word study instruction and intervention.

Thus, to help LD students, particularly poor readers, become fluent readers, in intervention work a few key components are typically involved: (a) constantly increasing sight vocabulary, (b) sustained, systematic work on phonics and phonological sensitivity as a means to 'breaking the code' and build proficiency by focusing on phonemic awareness, and (c) repeated readings using efficient strategies for blending sounds to build fluency under supervision and then individually.

As one can see, the key element is the constant and intense practice in phonemic awareness that helps students to connect graphemes and phonemes. This process helps children to move from the decoding of individual letters to chunking and blending sounds. This insight helps to interrupt the cycle of failure for poor readers. This insight from reading intervention has been greatly motivating to our work with children with and without learning disabilities in mathematics, including dyscalculia. This means that both in initial instruction and interventions, there should be a real and consistent emphasis on the fundamental skills of: (a) subitizing, visual clustering, and numberness, (b) decomposition/recomposition and the 45 sight facts, (c) making ten, teens' numbers, next tens, making next tens, (d) place value with a focus on both canonical and non-canonical decomposition of a multi-digit number, and (e) the development of effective use of executive functions, particularly the development of working memory and flexibility of thought.

Many planners of mathematics instruction for young children often do not fully take into account that to increase proficiency, competence, and fluency with basic addition and subtraction facts, children need to develop solid number concept (numberness) and flexible numbersense. They stop short: as soon as children are able to count one-to-one, they assume the child has the concept of number. Sequential counting and one-to-one correspondence, even when it is converted into conservation of number is not enough for competence in numbersense.

For example, explicit teaching of phonemic awareness skills and sound blending skills is important, but instruction that integrates 'how to blend phonemes together' and 'how to "pull apart"' or 'segment words into phonemes' makes more robust the connections between the word's linguistic components of sounds, letters, and meanings, and the more useful it is to a student to acquire reading skills. The organized and intense supervised practice in building these components of reading and language converts novices into fluent readers.

Similarly, learning subitizing, sequential counting, one-to-one correspondence, visual clustering, building sight facts, and decomposition/recomposition in isolation are useful to some extent, but what is even more important and productive is instruction that focuses on their integration. An organized and early intensive supervised practice (where there is emphasis on the language—vocabulary, syntax, and semantics, the questioning process using proper language, invoking conceptual schemas that have been developed with efficient tools, *the script* and *the graphic organizers* that have been guided by decomposition/recomposition for developing strategies and procedures, and the development of executive functions) in the integration of these component skills develops true *number concept* (numberness).

With proper number concept, future concepts, such as place value, operations, and their integrations that result in the development of numbersense and numeracy, are easier to achieve. As mentioned earlier, the integration of number concept, arithmetic facts, and place value results in optimal development of robust of *numbersense*. It is with numbersense that we navigate the daily encounters with quantity. The same process is repeated when all four operations have been introduced and when we want children to be proficient in *numeracy*—where number concept, numbersense, and all four operations are integrated.

When one hears or sees a number, one does not see discrete objects; instead, one sees or hears a collection represented in its abstract symbolic form. Children can count objects one-by-one, but they have difficulty recognizing visual clusters of objects as represented by specific numbers. Counting is like recognizing and decoding individual letters and their sounds, and recognizing visual clusters for individual numbers is like recognizing phonemes and their integration into words ([Krajewski & Schneider, 2009](#)).

Decoding individual letters does not make a child a fluent reader. For proficiency and fluency in reading with comprehension, one needs grapheme-phoneme connection and practice in blending sounds. Similarly, counting does not make a child fluent in numberness or numbersense. One has to blend the numberness of two numbers (sight and addition facts) to produce new numbers. Arithmetic facts (the blending of two numbers) and place value (the blending of two or more numbers) are like identifying the phonemes in a big word and then the blending of those sounds in reading that word.

Weaknesses in phoneme awareness, rapid automatized naming, and working memory are strong and persistent correlates of literacy problems, particularly spelling, even in adults.

Similarly, decomposition/recomposition, working memory, and rapid automatized naming are related to addition and subtraction facts and then with multiplication and division facts.

Strategies and instruction in *arithmetic facts* and procedures are much more productive when a child has acquired the number concept properly. Further, true number concept and decomposition/recomposition is the basis of deriving addition and subtraction strategies.

[Making ten, the commutative property of addition, the associative property of addition, and special strategies such as: $N + 9$ (add $10 - 1$), $N + N$ (doubles), $N + (N + 1)$ (doubles plus 1), $N + (N - 1)$ (doubles minus 1), $N + 2$ (2 more), $(N + 1) + (N - 1)$ (2 apart)].

Mastery of arithmetic facts thus is dependent on number concept, decomposition/recomposition, and strategies based on these. For example, $8 + 6 = 8 + 2 + 4$ (applying the knowledge that we need 2 to make 10 and 6 is

decomposed into $2 + 4 = 10 + 4$ (8 and 2 are recomposed into 10 and with the knowledge of teens numbers—place value to get 14), or $8 + 6 = (7 + 1) + 6 = 7 + (1 + 6) = 7 + 7 = 14$, or $8 + 6 = 2 + 6 + 6 = 2 + (6 + 6) = 2 + 12 = 14$, or $8 + 6 = 8 + 8 - 2 = (8 + 8) - 2 = 16 - 2 = 14$. All of these strategies are built on *sight facts*, *making ten*, and *decomposition/recomposition*.

Examining an arithmetic fact using multiple strategies not only develops a strong numbersense, but also flexibility of thought and strengthens the ability to apply arithmetic knowledge to problem solving and critical thinking.

As phonemic awareness is to reading, decomposition/recomposition is to numbersense and numeracy. The reading research demonstrates phonemic awareness (PA) is one of the biggest building blocks of the reading success. We have found that the decomposition/recomposition process is the basic building block not just for numeracy, but also for mathematics. PA should not be abandoned until the child has demonstrated advanced levels of PA skills. However, decomposition/recomposition, as a process, is present in mathematics at all levels; it is never abandoned and should not be abandoned. It will be used, later on, in operations on fractions, decimals, integers, and algebraic manipulations.

A skilled reader is able to read almost every word without activating the phonological processor. A fluent reader reaches that point because she did use the phonological processor which allowed her to make words automatic. There is a real difference between a beginning reader and a fluent reader.

Similarly, looking at a cluster of objects (up to ten) and recognizing it and giving it a numerical name instantly is the goal, but to get there we need to focus on recognizing smaller clusters and even some counting. As in reading a word, we do not focus on each letter, nor do we begin with the whole word memorization, similarly, in numberness, we neither focus on one object at a time nor on the whole cluster to start with. How every child can do it in a reasonable time frame and with fluency is the goal of the next several sections and chapters.

Thus, the acquisition of number concept and then acquiring arithmetic facts, at one level, is similar to learning to read: one acquires the meaning of symbols, decodes the symbols, and then combines the symbols to relate to other symbols. In reading, it is the process of associating phonemes to grapheme and vice-versa and the blending of sounds.

When a child encounters a new word, she breaks the word into ‘chunks’ and then ‘blends’ these chunks into the word. This chunking and blending is the manifestation of the integration of grapheme-phoneme and meaning relationship. A child cannot become a fluent reader without being fluent in grapheme-phoneme association and the process of ‘chunking’ and ‘blending.’

The parallel process in gaining numberness is also three-fold: (a) instantly associate a visual cluster with its number representation, (b) associating a sound with number—the number name, and (c) decomposition and recomposition according to context.

The recognition of a visual cluster and decomposing a cluster into its sub-clusters and combining sub-clusters into a larger cluster and instantly recognizing the larger cluster as the combination of the sub-clusters are the key processes to acquiring the fluency in number concept and numbersense and their applications.

In most situations, in teaching number, teachers help children acquire the concept of number by connecting the phoneme (sound s-e-v-e-n) with grapheme (symbol 7). But that is not enough, because acquiring numberness is more than learning to read a number or even count a number of objects. The lack of the integrations mentioned here become the causes and the manifestations of the symptoms of *dyscalculia* and *acquired dyscalculia* in most children.

In teaching the letters of the English Alphabet, there are two schools/traditions: in England and commonwealth countries, the sound of letters are taught first and then the letters; in most American schools, the letters first and then their sounds. In reading research, there is higher correlation between sound/letter to reading than letter/sound relationship. There is a great deal of evidence that both are important. Similarly, in the United States, the number names are taught first and then the numerosity associated with them. In contrast, in Asia and England, it is the numerosity first and then the number shapes. Both are essential for learning number concept. However, when numerosity and oral representation are learned before the writing of numbers, children develop number concept faster. The judicious integration of the three expedites the process.

3. Sight Words vs Sight Facts

In reading, it is important to acquire a large number of sight and high-frequency words.

Similarly, we have found that it is important to acquire *sight facts* or *high-frequency arithmetic facts*. The teaching of sight facts is important for learning addition and subtraction facts efficiently. High-frequency words are best taught and acquired in context and with frequent exposure. As sight words are not learned by focusing on each letter in the word, sight facts are not learned by counting discrete objects. That method is counterproductive. Many average and above children can learn sight facts in spite of using fingers, TenFrames, Numicon, Number Line, and other counting materials.

The most effective method for all children—with or without learning difficulties in mathematics—is decomposition/recomposition using Visual Cluster Cards and Cuisenaire rods.

Knowing the first ten numbers (e.g., digits) means that children have idealized images of visual clusters for the ten numbers and can apply decomposition/recomposition to

these numbers with fluency. In other words, with this knowledge/skill, they are ready and should be able to derive relationships between these ten numbers and by the help of the cultural tools, such as language, games, toys, rituals, and other social exchanges, they will learn new relationships between these numbers and come to know the number system of their culture. It is a robust example of acculturation.

Teachers of emerging and struggling readers know that teaching decoding and encoding are complementary skills. Teaching them as complementary skills means that good spelling instruction is good reading instruction and vice versa. The same is true about decomposition/recomposition of numbers. *To become fluent in addition and subtraction facts, one must be fluent in decomposition/recomposition.* The encoding and decoding processes in reading are parallel to decomposition and recomposition process in numeracy. That means a child should be able to give not only the sum of two numbers (e.g. $4 + 6 = 10$, recomposing), but also the fact given the number 10, it can be decomposed into two numbers (e.g., $10 = 9 + 1 = 1 + 9 = 2 + 8 = 8 + 2 = 3 + 7 = 7 + 3 = 4 + 6 = 6 + 4 = 5 + 5$). When they can do this reversible process with all ten digits {1, 2, 3, 4, 5, 6, 7, 8, 9, and 0}, they have mastered the alphabet of Mathematics Language and when they can do this with 10, they have come to know the ‘facts’ (the words of mathematics). Then with the knowledge of number concept (numberness—digit, decomposition/recomposition, and sight facts), making ten, and making teens numbers, they are ready to learn arithmetic facts.

Many teachers use a variety of activities for children to acquire sight words. They use a multi-sensory approach in which students say the word, say the names of the letters, trace the letters on a screen, sky-write them, and then repeat the name of the word. The key is multiple exposures. The same approach works for sight facts: Using Visual Cluster cards or Cuisenaire rods, they say the number represented by the Visual Cluster card (say 7), picture (visualize) the cluster on the card (3 pips in the first column, 1 in the middle, and 3 in the third column), visualize two sub-clusters on that card (6 and 1; 5 and 2; 4 and 3; 3 and 4; 2 and 5; and 1 and 6) by drawing a ring around the two sub-clusters by finger in the air or on imaginary whiteboard (sky writing), say the corresponding sight facts as equations ($6 + 1 = 7$; $5 + 2 = 7$; $4 + 3 = 7$; $3 + 4 = 7$; $2 + 5 = 7$; $1 + 6 = 7$; $7 = 1 + 6 = 6 + 1 = 2 + 5 = 5 + 2 = 3 + 4 = 4 + 3$), then write these equations on paper.

4. List of Sight Facts

There are more than 200 sight words (the number depends upon the program being used) that children must acquire. Most schools and programs expect children to master between 35 to 75 sight words at the end of kindergarten. Then with the help of *sight*

words and *phonemic awareness*, they begin to ‘*chunk*’ and ‘*blend*’ learn the rules of using phonemes and with practice (supervised and individual) they learn to read.

Ultimately, fluent reading is converting all words into sight words with meaning.

Inspired by the role of sight words in the reading process, we developed ([Sharma, 2008](#)) the concept of sight facts in helping children understand the relationships between numbers. A sight fact is seeing the interrelationship of numbers up to 10. This relationship is acquired by exposure, observing their configuration, and the quantity they represent. As sight words are learned by exposure, the frequency of observations, and their spatial arrangements, in arithmetic, we call them sight facts. To expedite this process, we developed Visual Cluster Number Cards. Sight facts play the same role in acquiring and mastering arithmetic facts as sight words play in the reading process. The following is the list of sight facts.

The key element in the acquisition, mastery, and application is the *decomposition/recomposition process*. Without this process, children do not acquire understanding and fluency in addition and subtraction facts, easily and smoothly. Once children have the concept of number, the decomposition/recomposition process can also be accomplished and reinforced with Cuisenaire rods. For example, the number 10 can be shown as the combination of two numbers as follows (the same process can be used for all other numbers 2, 3, 4, 5, 6, 7, 8, and 9): The rods show the links between the different facts.

Once children have formed these combinations (all the possible sight facts of 10), the teacher helps them to make these combinations fluent and provides opportunities for applying these sight facts in addition, understanding other numbers, and applications in games and toys. For example, the most appropriate game/toy to practice sight facts of 10 is the game *Shut the Box*.

When children have learned the sight facts of a number—recognized the combinations of sub-clusters on Visual Cluster Cards, formed them using Cuisenaire rods or Invicta Balance, can recite them—then they should be asked to record them with the help of these materials. When children can supply the decomposition/recomposition of a number in several forms, they are ready to write the sight fact equations as pointed out earlier in the sight fact equations for number.

Repeated exposures to making combinations of numbers (sight facts) by using Visual Cluster Number Cards and Cuisenaire rods are important as a starting point for learning other arithmetic facts. Children should move from oral to written form with specific, positive, and corrective feedback both for making combinations and acquiring fluency. The practice should involve only strategies using decomposition/recomposition. The best activities to practice these sight facts are dice, dominos, and Number War Games.^{[15](#)}

The role of Cuisenaire rods in teaching number concept is very important. The use of Cuisenaire rods provides the opportunity and invitation to see number as a prelude to measurement. This book describes two approaches to developing *concepts of number* using general notions of *quantity as a collection of discrete objects* (as described earlier through Visual Cluster cards and Sight Fact cards and other counting objects) and their *measurement* notion of number—*number as represented as a continuous model* (through the introduction of Cuisenaire rods).

Number as a discrete collection is the traditional approach to number and discrete models have been the models to teach children the concept of number. Instructional methods based on discrete models are the most popular methods and are found in almost every school and when parents introduce their children to number concept. The measurement notion of number or seeing the number concept through the use of a continuous model is a fairly new one. This approach was most prominently articulated by Russian mathematician Davydov and his colleagues. The discrete model approach leads most children to learn number concept, but many children are not able to reach the fluency of facts through this conceptual approach. They develop difficulties in learning number concepts and carry that experience of having difficulty with them to learning other arithmetic facts. Many of them begin to demonstrate the symptoms of dyscalculia and acquired dyscalculia.

We have found that the use of the combination of the two approaches—using both discrete models and continuous models is a better approach to teaching not only arithmetic, but also other concepts such as: arithmetic operations, fractions, decimals, integers, and algebraic manipulations.

This dual approach prepares a child to for future number systems better. Number as measurement is a direct way of providing coherent connections in the development of whole numbers and fractions. Second, and even more important, is that it makes the geometric number line continuum present from the start of the school curriculum as a useful mathematical object, instructional tool, and concept into which real numbers can be eventually explicitly developed and located and understood. Third, in the dual approach, there are real opportunities for some early algebraic thinking. And finally, this context and approach provide for the development of place value as a numeration system modeled on the invention of a place value system of number representation.

we assumed that the students' creation of a detailed and thorough conception of a real number, underlying which is the concept of quantity, is the purpose of this entire subject, from grade 1 to 10 ... the teacher, relying on the knowledge previously acquired by the children, introduces number as a ... representation of a general relationship of quantities, where one of the quantities is taken as a measure and is computing the other.

(Vasily [Davydov, 1990](#))

This idea of ‘one of the quantities is taken as a measure and is computing the other’ is the real understanding of number. Till then the discreteness of number shows that it is just a collection of objects and its count identifies the number. For understanding number as a continuous quantity, one needs to convert one of them to be the unit of measurement and then measuring other collections by that unit is the first step in the direction of understanding the number as being a measurement also and then it becomes a continuous quantity. That is a real growth in numbersense. The quantity and its measure are at the basis of understanding the number concept. The movement from perceptually recognizing a visual cluster and recognizing a particular Cuisenaire rod as representing a number is the foundation of number concept. Many children, particularly dyscalculic, miss this idea. As a result, to make this concept clear to children, one of the exercises we engage children in is the construction of their own number lines.

The process of constructing the number line (measuring and locating number lines) is a very powerful concept for children to experience as it later forms the basis of locating fractions and decimals on the number line. Research from the analysis of NAEP testing shows that locating of numbers, fractions, decimals, rational numbers, and real numbers is a very difficult concept at all levels of mathematics development.

When students learn arithmetic facts and procedures with the help of strategies based on decomposition/recomposition, they develop mathematics and conceptual understanding with robust knowledge structures behind them rather than learning isolated facts and routine procedures by counting. *Knowledge structures* here refer to conceptual schemas that students use to organize and relate language, concepts, and facts related to number. The difference between ‘*novice*’ and ‘*expert*’ learners of quantity is that experts have developed complex knowledge structures with multiple and flexible interconnections based on fundamental concepts while novices and poor students have inefficient, simpler, disjointed knowledge structures with fewer connections that make it difficult for them to assimilate new concepts or an ability to recall these facts. For example, they may think addition is just ‘counting up’ and subtraction is ‘counting down.’ These children end up working harder in order to master arithmetic facts with little or no payoff or encouragement. We find that without the conceptual schemas, dyscalculic and other students having difficulty in numeracy are given ‘poor strategies’ that keep them struggling. For example, when they reach higher grades and one asks them: ‘Count by $\frac{1}{4}$ to locate the number $3\frac{3}{4}$ on the number line.’ Many children have great difficulty in such tasks. Because they think a fraction is a part of something, how can you locate that part on a number line; just like a number is a collection, how can it be a point on a number line?

A child without efficient strategies needs help so that he does not develop gaps in his preparation for learning higher skills. This requires timely and effective interventions.

As soon as a teacher observes that a child is struggling with numberness, she must arrange for interventions. It is important that during interventions, we help them develop efficient strategies, not just repeating the same counting strategies. They have not worked. All children benefit from effective math interventions, including gifted and talented children, when they are not reaching their potential. We have found that many of them develop idiosyncratic and inefficient strategies that they have difficulty changing later on. We have found many underachieving gifted and talented children in our careers.

Notes

1. We define mastery in detail later. Simply, mastery means (a) the language, conceptual schemas, strategies, fluency, and ability to apply them. ↵
2. It is to be noted that this is not dependent on counting. The only time counting may come up is when we introduce a cluster for the first time. ↵
3. Neuroscience research shows that the brain has an approximate line to represent the approximate size of a collection. A number line is the integration of innate and cultural skills—two innate skills: (a) subitizing and (b) spatial sense and patterns, and two cultural tools: Approximate Number Sense (ANS), and (b) measurement aspect of number (there must be same distance between any two consecutive whole numbers)—the location of numbers on the line. The increasing size of numbers corresponding to the location of the number is the key element of the number line. Locating a number on the Number Line and comparing numbers are key skills that will be important throughout the explorations of numbers from whole numbers to the real number system. Without understanding these aspects, children have problems understanding rational numbers (fractions and decimals) and irrational numbers. ↵
4. Visual Cluster Number Cards are available from Center for Teaching/Learning of Mathematics www.mathematicsforall.org. ↵
5. For number games see: *Games and Their Uses for Mathematics Learning* by [Sharma \(2008\)](#) (CT/LM). ↵
6. Visual Cluster Number Cards are a specially designed deck of 60 cards for teaching number concept for all children, with and without learning disabilities. However, they are particularly effective for dyscalculic individuals. To learn more about how to acquire concept, the idea of sight facts, and how to use the Visual Cluster Number Cards, see the eBook on *How to Teach Number Concept* and videos on YouTube: <https://youtu.be/fuSPkrEwsxo> <https://youtu.be/1zd9BUDWfNs>

<https://youtu.be/KTUdQgTF7IQ> and the video tutorials at www.mathsexplainedco.uk. ↵

7. See the videos (attached link above) for the comparison between Numeracy and Literacy. ↵
8. The term ‘demonstration’ is used here for a teacher-guided activity using CPVA/S. Teacher and children engage in the same activity. The teacher asks questions and children respond to the process and use the concrete materials. When a child has formed the equation, the teacher assures that it is correct by asking questions, and when all children agree, she forms that equation. ↵
9. For details of this process, read: *How to Teach Number Effectively and Easily: Preventing Learning Difficulties in Mathematics* ([Sharma 2019](#)). ↵
10. Same as endnote 8. ↵
11. For details of this process read: *How to Teach Number Effectively and Easily: Preventing Learning Difficulties in Mathematics* ([Sharma 2019](#)). ↵
12. This is part of the goal for kindergarten, as mentioned in Introduction. It is an immensely doable task with the strategies mentioned in the eBook, *How to Teach Number Concept Easily and Effectively* ([Sharma 2019](#)). ↵
13. Arithmetic Facts, such as: addition/subtraction and multiplication/division are memorized as language in the long-term memory. And the methods used to put them in the long-term memory must be ‘storage-rich’ and ‘retrieval-rich.’ The ‘scripts’ developed based on conceptual schemas and rich language and supported by concrete models satisfy these two conditions. Therefore, the practice must be done with the script, not just the facts. For example, $8 + 7 = ?$ Which is the bigger number? 8. How do I make it into 10? Add 2. Where is the 2 coming from? From 7. What is left from 7 when 2 is given to 8 to make 10? 5. What is $10 + 5$? 15. What is $8 + 7$? 15. What is $7 + 8 = ?$ 15. What two numbers make 15? 8 and 7. Great! Practice of this script again and again provides the integration of strategy, language, and procedure and remains in the long-term memory and can be retrieved. ↵
14. See *Dyscalculia* by [Sharma and Loveless \(1988\)](#). ↵
15. See *Number War Games* by [Sharma \(2008\)](#). ↵

7 (Part One) Understanding Additive Reasoning

Its Language, Concept, Procedures: Addition

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Starting from Piaget's logical operations framework (e.g. [Piaget and Szeminska, 1952](#); [Dehaene et al., 1994](#)), there has been a great deal of research attention devoted to the development of children's quantitative and spatial reasoning skills. It includes attention to such issues as children's acquisition and understanding of the number concept, the additive composition of number (additive reasoning), and the multiplicative reasoning skills. As well as to the importance and preparation of children for the later mathematical learning in school mathematics (e.g. [Clements and Sarama, 2011b](#); [Coles et al., 2014](#); [Nunes et al., 2008, 2012](#); [Chinn, 2021](#); [Sharma, 2019](#)).

In the next few chapters, we focus on additive reasoning—how children develop the concepts of addition and subtraction—their language, conceptual schemas, strategies to master facts, and the addition and subtraction procedures. We believe, if the first developmental milestones of numeracy—*number concept*, *additive reasoning*, and *place-value*—are properly developed and mastered by children, then not only will the other numeracy skills and preparation for higher mathematics concepts be sound, but the effects of any possible dyscalculia issues are mitigated or nullified and possibilities of other learning problems in numeracy are also greatly minimized.

A. Current Status of Additive Reasoning

The major method of introducing additive reasoning (addition and subtraction) in most primary/elementary classrooms all over the world is the use of sequential counting. In this approach, not only do addition methods, strategies, and their procedures emerge, but all other arithmetic procedures are also derived from these strategies, and almost all of them are some variant of ‘sequential counting’ as a conceptual schema. The concept and procedure of addition begin with a situation that presents two collections of objects in the child’s environment. The situation or a person poses the question: *How many in all? Or altogether?*

This question: ‘How many altogether?’ either emerges in a natural setting in a child’s experiences, or he is a witness to such a situation and observes someone answering the question by counting all the objects. Or, for the child, a situation of this type comes up when he is either playing by himself and faces this question, or a group of children may be playing a game or with toys. In either case, for the child, resolving this situation is removing a dilemma; it is solving an immediate problem of importance. All mathematics procedures have emerged as solutions to a problem situation.

In the classroom, the context is more formal. The child is informed, ‘in a certain situation, you are given two collections of objects, you are asked to determine ‘How many in all?’ The child knows how to convert the collection of objects into their abstract forms—number—by counting the two collections. To resolve the situation, the instructions given to the child in different situations are: (i) count all, (ii) count second number to the first number, or, (iii) count the smaller number onto the bigger number, etc.

Sometime in the future, the strategy of counting is used to introduce the concept of subtraction—counting away a smaller collection from the given collection as subtraction. For example, if the problem involves finding the sum $8 + 7$ or the difference $12 - 5$, children are taught to count discrete objects such as beans, blocks, chips, fingers, counting numbers on a number line, etc. Some children move on to ***Five-Frames, Ten-frames, Rek-n-Rek, or Numicon, making hash marks*** ($++++ // + +++ //$ = 15, etc.) to

display the process as well as the thinking behind it. However, the underlying feature is still counting. Most children remain at the counting level, or they become dependent on counting of objects, on one's fingers, on counting tools, on a number line, or with hash marks. Once derived, they are asked to memorize them by practice—and such counting strategies needs relatively much more practice is needed. Occasionally certain strategies may emerge as they memorise the facts but the main conceptual schema still remains 'counting-up' for addition and inversely 'counting-down' for subtraction which is highly inefficient. Certain strategies might emerge as they memorize the facts. Ultimately the *conceptual schema* for addition becomes: '*counting up*' and for subtraction it becomes '*counting down*' (Baroody, 2004; [Gattegno, 1960, 1964](#); [Mulligan & Woolcott, 2015](#)).

These approaches get them the correct answer most of the time. But each one of these is an inefficient or semi-efficient strategy and an uninteresting activity for learning arithmetic facts. These activities teach children only counting or limited 'grouping the quantities' activity and are only useful initially. And most children do learn by these methods, but it takes a longer time to master these facts with a heavy cost in boredom and loss of interest and a considerable load on a child's cognitive resources—cognition, executive functions, memory, time, physical, and emotional. Many of them eventually transcend to better strategies or become fluent by *sheer practice* and *repetition*. However, many children do not transcend these initial counting-based strategies. Over time, they become only proficient counters. Some never master arithmetic facts. And some of them develop fear and dread of arithmetic facts. This fear and dread converts into mathematics anxiety—from mild annoyance to a debilitating deficit of not being able to become proficient in numeracy. These strategies are examples of the prevalent pedagogy—CRA/CPA, as mentioned in the previous chapters.

The CRA or CPA are evident in textbooks, lesson plans, and teaching methodologies worldwide. They are entrenched in lesson development—schools of education emphasize them and classroom lesson plans demonstrate their presence. However, even after more than three-quarters of a century of applications of CRA, many children still have problems with

the transfer from CR or CP to Abstract and applications with fluency, competence, and confidence.

The question arises: Why do so many children have difficulty in transferring the understanding achieved at the concrete and representational/pictorial levels to the abstract/symbolic level? Why do they remain dependent on concrete and pictorial levels? Children begin to believe the task is too difficult or unmanageable. They give up. We find four elements that contribute to this situation:

- Counting places a heavy cognitive and memory load on the working memory;
- Because of counting, it is difficult to *discern, extend, create, and use patterns*, therefore, there is a difficulty seeing relationships and making connections between numbers in the working memory as it is overloaded;
- Counting and their iconic versions and the strategies derived from them do not emphasize relationships and patterns; and
- There is no script or conceptual schema, tools, or mental algorithm accompanying the counting process, therefore, there is nothing to help visualize the number relationships in the mind's eye (working memory).

As mentioned in earlier chapters, the working memory plays a big role in learning numeracy and even higher mathematics. There is hardly any language usage in counting that connotes the addition and subtraction operation. And no script or schema accompanies each stage of representation that facilitates visualization of the process. The child must reinvent the outcome of even the smallest of facts again and again. Therefore, the transfer of newly learned information from working memory to long-term memory is not achieved. This is particularly so with children with poor executive functions, who are lacking in the key prerequisite skills, such as pattern analysis, visualization, task analysis, and children with dyscalculia and/or dyslexia ([Frith, 2007](#); [Flavel, 1979, 1987](#)).

The reason for the lack of this transfer is the nature and use of concrete models used for learning addition and subtraction. We have already identified the optimal learning tools for these activities in the previous chapters. Here, we need to focus on two aspects of the nature of models:

- a. There should be congruence between the models at the three levels: *concrete*, *pictorial*, and *abstract*. When there is congruence in models and rich language usage to arrive at a strategy with a script, children can move along the continuum: concrete—> pictorial—> visualization—> abstract/symbolic levels easier. The experience leaves a rich residue in the working memory. Quality of learning is dependent on the amount of residue left from the experience.
- b. The concrete model should be such that their memory and script aid in visualizing the information in the absence of the model. Neurology has proven that neurons cannot distinguish the actual experience and the vividly imagined experience. The difference is only in intensity.
- c. In counting-only strategies, quantity (number) is being treated as a collection of discrete elements rather than its presence as discrete, continuous, and mixed. The role of this phenomenon was fully described in the previous chapter.

The first condition makes the transition easier, and the second helps in visualizing the process more easily and then the information acquired is *storage-* and *retrieval-rich*. The third condition extends the understanding of the quantity to a richer and diverse level and prepares the child to answer questions such as: David is 17 years old and his sister is 9 years old. How much older is David than his sister? Etc.

When these three conditions are not satisfied, children have difficulty in remembering the information and the strategies are not effective. Then children's conceptual schema or strategy based on counting strategies work at the concrete and iconic levels only and do not get translated to the abstract symbolic level. These children either remain at the concrete and pictorial levels for a very long time and keep on deriving arithmetic facts

only by comfortably counting or they resort to rote memorizing. Neither are tremendously helpful.

Based on these observations and examining the neuropsychological profiles of many of these children, we find that when children were taught number concept using the strategies derived from and based on the principles identified in the previous chapters on number concept on addition strategies, emphasis on patterns (Visual Cluster Number and Sight Fact cards), use of both discrete and continuous models, and developed with an accompanying script for the strategies, they were able to visualize the strategies, remember them, and also apply them at all levels, including abstract levels. The CPVA/S ([Chapters Four](#) and [Five](#)) includes these elements for arithmetic calculations. We have observed and strongly believe ([Sharma, 2015](#)) that the missing link in most of the current pedagogies is the lack of visualization of the concept or strategies and the absence of *scripts* that can be used without the concrete and pictorial models.

Through the lessons and teaching strategies developed in this book, there is a real effort to develop activities that demonstrate innovative aspects of structural development into broad levels of progression from concrete to pictorial, to visualization to abstract thinking, as well as consideration of embodied action. In the following pages, there is clearly an emphasis on the use of congruence at all levels, partitioning (decomposition/recomposition), language, questioning, and scripts on learning arithmetic facts and procedures. The CPVA/S approach, with its simplicity of application, raises relevant research questions about how the internalization of the structural features occur within and across the instructional dimensions over time and how this promotes abstraction and generalization in developing arithmetic relationships into robust numbersense.

B. Effect of Number Competence on Additive Reasoning

Based on our experience working with children on number concept, we have found that once children have conceptualized the first ten numbers really well (with the help of effective instructional materials, such as: Visual

Cluster Number and Sight Fact cards, dominos, dice, and Cuisenaire rods), addition and its language, concept, and facts can be taught efficiently and effectively, and children can learn and master them well, even those children with learning disabilities.

In order to do so, children should know the specific conditions that must be satisfied in the case of each operation. For example, the child should know when to add, subtract, multiply, and divide and under what conditions a particular conceptual schema and model, strategy, operation, or a combination of operations is called for in solving the problem. In other words, what does a particular concept or operation mean? They should learn the language—*vocabulary*: terms, expressions, definitions, *semantics*: the context, the meaning of terms, symbols, and operations, and *syntax* and *structure of native language and mathematics language* and order of operations. They should learn and apply the conceptual schema(s) for learning and applying operations. The language, the conceptual schema, and efficient and effective strategies will help them arrive at the *procedures* with understanding and the ability to apply them proficiently.

C. How Should We Teach Additive Reasoning?

The analysis of early numeracy-related competencies show us that children's early numerical competencies that are prerequisites for success in future developmental milestones and, in particular, learning strategies and numerical operations, are:

- a. *Subitizing and Visual Clustering skills*: An innate set of skills—the ability to perceptually recognize a small collection of objects up to about 5 or 6; ([Dehaene, 2020](#); [Schleifer and Landerl, 2011](#); [Sharma, 2019](#); [Butterworth, 2025](#)) and because of our BaseTen system, the ability to extend subitizing to the ability of recognizing clusters of objects up to 10 (*visual clustering*);
- b. *Understanding the idea of digits and number*—a culturally endowed and learned knowledge of the ten digits {1, 2, 3, 4, 5, 6, 7, 8, 9, 0} as

the alphabets of numeracy akin to letters as the alphabet of a language; knowing the difference between digits and numbers (e.g., the number 15 is made up of digits 1 and 5 and is the sum of numbers 10 and 5; the digits 1 and 5 in 15 represent numbers 10 and 5, respectively) is crucial for learning arithmetic operations. It is like a child being expected to know the difference between letters and phonemes and words;

- c. *Counting skills*: One-to-one sequential counting and arriving at the ability to recognize that a number—*cardinality of the set is the property of the collection of objects*. It can be achieved by sequential counting and is identified by the last number of that count—an important cultural tool, but it is more than the outcome of one's action of counting. The counting experiences and instruments are artifacts of the native language endowed by the culture of the child ([Geary et al., 1992](#); [Sharma, 2019](#); [Butterworth, 2025](#));
- d. *Approximate Number System (ANS)*: Ability to compare numerical magnitudes (innate for small numbers, and learned with the acquisition of language—thus being aware of approximate comparison of the cardinality of two sets of small number of objects up to 5 or 6 in them ([Griffin, 2004](#); [Dehaene, 2020](#); [Sharma, 2019](#); [Butterworth, 2025](#))) and it is a perceptual activity ([Dehaene, 2020](#); [Sharma, 2019](#))
- e. *Ability to position numerical magnitudes on a number line, including on an empty number line*: an innate spatial and sequential skill and mostly a cultural skill; [Siegler and Booth, 2004](#); [Dehaene, 2020](#); Freudenthal, 1967; [Sharma, 2019](#)); and
- f. *Decomposition/recomposition of a number*: The innate ability to realize that a collection is made of smaller sub-collections and smaller collections put together make a bigger collection (i.e., part-to-whole and whole-to-parts awareness). It is as important to numeracy as phonemic awareness is to reading ([Sharma, 2019](#)).

These skills are important and are manifested as mastery of: (i) *knowing 45 sight facts*, (ii) *making ten*, (iii) *the teens numbers*, (iv) *place value*, (v)

decomposition/recomposition. These skills should be considered prerequisite skills and content-related tools for mastering additive reasoning.

Empirical evidence has demonstrated that the presence of these abilities and skills in children facilitates the process and ability to acquire the multi-componential nature of number concept and related numbersense, and numeracy competencies of additive and multiplicative reasoning ([Dowker, 2008](#); [Sharma, 2019](#); [Dehaene, 2020](#)).

Additive reasoning (integration of addition and subtraction concepts) is the first organized attempt children make to understand the adult system of operations with numbers. Up to this point the informal and formal work children have done with number has been to try to understand, explore their environment from the quantitative perspective, and work with and relate the objects they come in contact with. Additive reasoning is a natural extension of their naturally learned number relationships. Success in learning and using additive reasoning is dependent on children's mastery of number concept and development of proper language and conceptual schema of addition and subtraction. Conceptual schema for a concept is dependent on the language of the concept. Concepts reside in the language containers.

As mentioned earlier, the number competence includes the mastery of: *subitizing*, *number recognition*, and *representation*, *systematic counting* and understanding the *relationship* between *quantity*, *digits*, and *number* (i.e., small quantity collections give rise to digits—smallest number components and then digits make numbers—a digit is like a letter and numbers are like words—made up of digits satisfying the rules, such as place value, of making number); '*numberness*'—cardinality of a collection and its different representations (verbal code, visual code—orthographic representation, and an analogical quantity or magnitude code), understanding *order of numbers* and *locating them on a number line*, and *estimating a collection* and *approximate comparison of quantity*. More precisely, for successful learning and mastery of addition and subtraction facts and procedures, the content-related tools are: (a) Subitizing, (b) 45 sight facts, (c) Making Ten, (d) the teens numbers, and (e) decomposition/recomposition.

With additive reasoning, students truly begin to see some of the interesting possibilities they discover while playing with number: about joining, separating, and comparing numbers using their earlier skills related to number. This later skill naturally extends to multiplicative reasoning.

Learning the concepts of addition and subtraction is natural for children in any culture and social strata. These actions have been found in all cultures for a very long time. Even some other species practice the actions of addition and subtraction. To a great extent, most children develop them informally and easily—in their day-to-day living experiences—playing games, comparing scores, number of days left to the next birthday, sizes of things, knowing their age, knowing their hobby collections, etc. However, the formal, abstract, and symbolic representations of addition and subtraction pose some difficulties for many children, particularly the concept, procedures, and applications of subtraction. Without the help and formal introduction to these concepts, most children do not know how to adequately represent these actions/operations and apply them effectively and efficiently. Initially, on their own, they only represent the final outcome quantity.

Many mathematics educators overly rely on children using a ‘pure discovery’ method for learning arithmetic, or anything else for that matter. First, pure discovery is not possible. It does not happen. Even the setting of learning conditions for children, choices of materials presented, types of games and activities made available are in one form or the another, are somebody else’s contribution to the child’s learning. Those choices may color the process of discovery. Moreover, on their own, children may or may not discover several important principles and properties of addition and subtraction operations and strategies for achieving fluency and applicability—for example, commutative and associative properties of addition, additive inverse, and the additive identity. These properties are important in developing fluency, understanding, and competence in both mental and written arithmetic and developing more concepts in arithmetic. However, giving a procedure without exploration or experimentation is also not good. We believe in using ‘*guided discovery*’ to derive arithmetic facts

and procedures. Posing questions, presenting problems, creating situations to be resolved, supporting their decisions, and scaffolding (giving hints) where they will be productive is a better way of teaching.

Several authors have explored the emergence and early development of the properties of operations and numbers in the two forms of quantitative reasoning (additive and multiplicative). As far as additive reasoning is concerned, various principles including the additive composition of number, the commutativity of addition ($a + b = b + a$), the addition-subtraction inverse ($a + b - b = a$), and the addition-subtraction complement principle ($a - b + b = a$) have been intensively studied ([Baroody et al., 2009](#); [Bryant et al., 1999](#); [Gilmore and Bryant, 2006](#); [Robinson et al., 2006](#)), sometimes also in relation to children's actual use of these principles in their mental arithmetic ([Baroody, 1999](#); [Peters et al., 2010](#)). However, only a few studies have explicitly addressed the question of how children's understanding of these principles is acquired and how it affects their (later) achievement in whole-number arithmetic and its effect on number fluency and numbersense. The limited available evidence from these few studies suggests that quantitative reasoning of this sort makes a specific contribution to achievement in whole-number arithmetic. These studies highlight the importance of focusing on the *relational aspects* (properties of operations) over and above the operational aspect of quantitative reasoning as critical principles that contribute to strong mental computational flexibility.

Additive reasoning is addition and subtraction, but conceptually it includes far more. Researchers studying the mathematical content in addition and subtraction, and the way young learners interact with it, have said that there are five important elements in additive relationships:

- [Joining¹](#)
- Separating
- Part-part-whole, and whole-part-part
- Comparing (directly)
- Nullifying an operation

D. Additive Reasoning: Joining of Two Sets/Groups/Parts

As the name suggests, ‘joining is the act of putting quantities together to achieve a sum.’ Most teachers and even researchers use the expression: ‘joining is the act of putting quantities together to achieve a *larger* sum’ instead of ‘joining is the act of putting quantities together to achieve a sum.’ With the addition of whole numbers, fractions, and decimals, it is true that the act of joining does produce a larger sum. However, the conceptual schema created by that expression stays in many children’s minds and when they deal with integers, rational numbers, irrational numbers, they carry that image and the conceptual schema. Once again, it emphasizes the importance of a first experience about a concept or a procedure, and the role language plays in creating the language containers and the conceptual schemas.

The basic tenet of this model of problems involving ‘joining’ is that in the problem situation, three quantities are involved: two quantities to be joined and the third quantity, which is obtained by joining. In any situation of joining, the two of the three quantities are given, and one piece of information is missing.

In additive reasoning (addition and subtraction), three quantities are involved, either one of the quantities to be joined is missing, or the result of the joining is missing. *To realize which quantity is missing is the kernel of the concepts of addition and subtraction* in this model of additive reasoning. Here are two examples:

Sam had 6 baseball cards. He got some more. Now he has 13. How many more did he get?

Maria walked 4 blocks, stopped for rest for some time, and then walked 6 blocks more. How far/how many blocks did Maria walk?

The first problem deals with one of the ‘missing quantity’ situations. It can also be considered as, by extension, a subtraction problem. This is sometimes referred to mathematically as a ‘missing addend’ problem. It can be solved, of course, by using addition as well as subtraction operations.

Young learners, however, seldom take the subtraction approach. The underlying conceptual element in this additive reasoning is joining, because this is the initial conceptual strategy that children use to solve when they ask, ‘What can I join (or add to) to 6 to get 13?’ However, a student who understands the inverse relationship of addition and subtraction has other options for strategies.

The second problem is a ‘missing result problem,’ probably the most common type of joining problem students encounter. Missing result problems are the foundation for many important skills (like number facts), but they are often overemphasized. The result of this overemphasis is that students develop a misconception about the equal symbol. These kinds of problems are two-way implication problems. All additive reasoning problems are ‘two-way implication problems.’ In fact, ‘*equal*’ implies two-way implication:

$$8 + 2 \Rightarrow 10 \text{ and } 8 + 2 \Leftarrow 10.$$

This two-way implication gives us $8 + 2 = 10$ or $10 = 8 + 2$. Most students do not see this two-way implication, as most teachers do not emphasize it. Lack of this understanding creates misconceptions and problems later. For example, rather than understanding “=” symbol as a symbol that means that both sides are ‘balanced,’ students take the symbol to mean ‘and the answer is’ This leads to a very common error in the test question:

$$6 + 4 = \_\_\_ + 5$$

where almost 40% of students, even in the fourth grade in U.S. schools, frequently put ‘10’ in the blank space. For this reason, it is important that students working on additive reasoning get the opportunity to work on joining problems and equations that are not always of the ‘result missing’ variety.

Both instances of this misconception or difficulty are easily rectified if the concept of equality is properly introduced using instructional tools, such

as: *Number rods* and the *Math Balance* (Invicta Balance).

Mathematically, ‘separating problems’ conceptually are quite similar to joining problems. Both can be described by a *part/part/whole* (partitioning) relationship. The thinking in separation, however, is closer to the idea of ‘pulling apart,’ the opposite of joining. This kind of thinking is conceptually related to the operation of subtraction. It is important to emphasize this conceptual (inverse) relationship between addition and subtraction.

Here, as with joining, there are some common misconceptions that develop because of the limited opportunities students have to explore the concept fully. *Separating* (and subtraction) is often conceptualized (by students and teachers) as ‘take away.’ This is a very useful analogy for subtraction, but it has its limitations—both as a conceptual model and in terms of action steps. However, this method and the associated language, as practiced by most teachers, as ‘*take away*’ *only*, does produce a group of children who have difficulty in mastering additive reasoning flexibly, and they show lower achievement when it comes to fractions, decimals, and integers. Separating is a much better way to think of subtraction because it covers a much broader range of possibilities. Putting into two (or more) groups is a way that quantities can be separated without removing anything.

1. Additive reasoning: Part-Part-Whole Relationship

Consider the problems:

The soccer team had 18 players. The coach separated the team into two squads. One squad had 12 players. How many players did the other squad have?

There are 20 candies in a box. 14 of them have nuts. How many don’t have nuts?

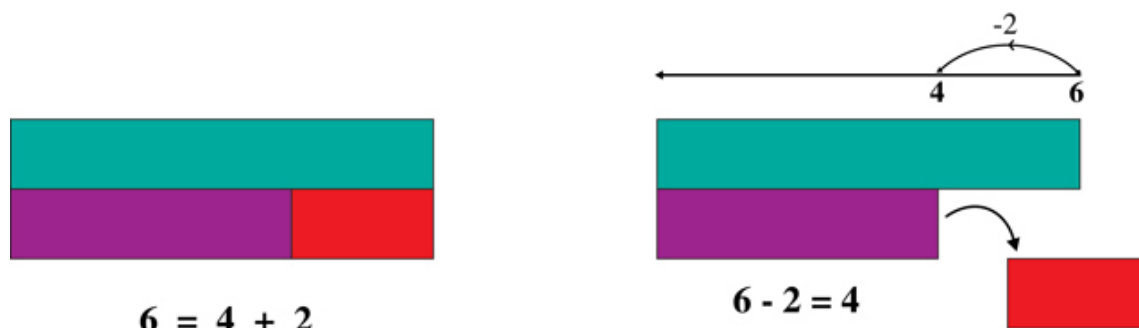


Figure 7.1a Inverse relationship between addition and subtraction

Both problems call for realizing that they involve a *part-part-whole relationships*. We strongly believe that teaching subtraction and addition together is a better strategy than teaching just addition and then subtraction. We have found that practice with the joining and separation activities together leads easily to the realization of part-part-whole relationships and to learning additive reasoning more easily. Part-part-whole also leads to other common conceptual models that teachers adopt for addition and subtraction. The idea of separating leads naturally into the ‘part-part-whole’ relationship, which brings together joining and separating as well as addition and subtraction. Part/part/whole thinking in additive reasoning represents a conception that is larger than and inclusive of both joining and separating approaches to computation and easily generalizable to other number systems (fractions, integers, decimals, and algebraic expressions). It is the idea that a *sum* is made up of two quantities, and difference also involves two quantities. The part-part-whole understanding is the recognition that putting quantities together and partitioning a quantity into groups creates mathematical relationships. For example, the addition and subtraction equations can easily be read from the following arrangement in concrete or visual representational models that lead quickly to the abstract and symbolic models for addition, subtraction, and missing addend:

- $4 + 2 = 6$, $2 + 4 = 6$, $6 = 2 + 4$, $6 = 4 + 2$;
- $6 - 2 = 4$, $6 - 4 = 2$;
- $4 + \square = 6$, $\square + 2 = 6$, $4 + 2 = \square$.

The relationships can be seen as joining in some instances and as separating in others. Part/part/whole expresses the connectedness of ‘joining and separating.’ It is a more complete way of viewing the additive relationship. It also demonstrates the inverse relationship of these operations easily. We believe children should solve many addition and subtraction problems using these concrete and visual models before they solve problems purely at abstract/symbolic levels.

Unfortunately, students rarely get enough experience viewing addition and subtraction from this perspective. When the numbers get large enough, students have difficulty seeing the relationship at all. A first-grade student, for example, who understands that 16 can be broken into parts containing 9 and 7, will often flounder in third or fourth grade when confronted with breaking 245 into 190 and 55. As a result, many struggling learners see addition and subtraction as mechanical operations, rather than the *expression of numerical relationships* having definite and fascinating conceptual schemas.

However, the concept of part-part-whole should be presented carefully and with proper and diverse use of language with appropriate concrete and visual models that are transparent in presenting the ideas.² The concept of part-whole relations arises from what Resnick and colleagues ([Resnick et al., 1991](#)) have described as proto-quantitative part-whole schemas ‘that organize children’s knowledge about the ways in which material around them comes apart and goes together’ (p. 32). For example, part-whole thinking helps students recognize the idea that although numbers³ are abstract units,^{4,5} they can be partitioned and then recombined in different ways, just like concrete objects to facilitate numerical calculation ([Britt and Irwin, 2011](#)). Such students develop better mental mathematics strategies and flexibility of thought.

As mentioned earlier, addition is a part-to-whole concept and subtraction is a whole-part concept; when we understand and are able to use the part-part-whole conceptual schema, the inverse relationship between these two becomes transparent. Moreover, part-whole thinking is fundamental for higher mathematical reasoning also. For example, the pre-algebraic

literature highlights how if attention is drawn to the development of part-whole relations earlier, no longer do ‘addition and subtraction appear as separate operations, but rather as dialectically interrelated actions that arise from the part-whole relation between quantities’ ([Schmittau, 2011](#), p. 77; [Sharma, 2010](#)), and the two operations together can be termed as: ‘Combining integers/numbers’ and later as ‘combining like terms’ ([Sharma, 2016](#)). As a matter of fact, in algebra $-7 + 8$ to some children looks like an addition problem, and they act on it that way by giving the answer -15 and the same problem: $-7 - -8$ is a subtraction problem.

However, when subtraction is taught alone and after addition, because of the whole-to-part conceptual framework more children have difficulty in learning and mastering subtraction. Interestingly, the same thing happens with division and fractions. Both are whole-to-part relationships. We have much more success helping children learn subtraction when they are taught addition and subtraction together,⁶ and similarly with division when multiplication and division are taught together using continuous models such as Cuisenaire rods and Empty Number Line.

[Baccaglini-Frank \(2015\)](#) refers to the part-whole relationship as a construct, highly resonant with the research on *awareness of mathematical pattern and structure* (AMPS) ([Mulligan and Mitchelmore, 2009, 2013](#)). They refer to the critical feature of AMPS that is characterized by the child’s structuring of groups to represent quantities; this involves part-whole relationships. Similarly, the ability to structure quantities is discussed in other research, where the focus is on instruction that needs to provide opportunities for structural relationships to be associated with fundamental properties of objects, collections, and numbers.

2. Additive Reasoning: Absolute Comparison of Quantities

‘Comparison’ of two quantities is another aspect of additive reasoning that differs slightly from the other aspects in that it asks the student to judge, in absolute terms, between two quantities. There will be other comparisons in the future, such as the ‘relative comparison,’ that are made in proportional

terms (e.g., proportional reasoning). *The Empty Number Line* is the most effective tool to make this concept of absolute comparison transparent and easy to learn for children.

Young children don't often have difficulty telling which quantity has/is more (or less), as long as the quantities are not representing large collections. The ability to compare collections of small numbers of objects (if the objects are about the same size) is an evolutionary ability. It was needed for survival. However, where children often get stumped is on the questions: 'How many more?' 'How many less?' 'How much more?' Or, 'How much less?' The difference between these two terms, 'how many more' and 'how much more,' is the difference between seeing number as a representation of discrete objects or as a continuous quantity. As we saw in the previous chapter, these are two forms in which quantity is manifested/occurs or number is represented. This is why we advocate for the use of three kinds of models: discrete (counting based), continuous (measurement based—Number rods), and mixed (pattern-based—Visual Cluster Number Cards).

The answers to the first two questions (examples of discrete model) require either counting or decomposition/recomposition, a culturally learned skill. The last two questions (examples of continuous model) require a continuous model or we must mentally or physically convert it into a continuous model (for example, *Number Line* or *Empty Number line*). Both types of problems require a formal introduction of strategies. To answer the second set of questions here, children need to learn that number is also a representation of a continuous quantity. For example, converting discrete unit blocks into a larger block as a line-segment using Cuisenaire rods converts discrete count into a continuous (measurement) count. This transition is a big advancement in the understanding of number concept as it makes absolute comparison very easy. The next step is to develop the concept of the number line and the empty number line.

As adults, we can see that these questions can be answered from either an 'adding on' or a 'subtracting off' approach. For a young learner, though, the

idea that the red stack includes the quantity in the green stack and some more is a novel and difficult concept.

To develop this idea conceptually requires significant experience with language, spatial models, and problem-solving skills. Often learners struggle because they are rushed to use a subtraction algorithm when comparing, without understanding why that works. Actually, these kinds of problems are best solved first through the use of Visual Cluster Number cards, Number rods, the Empty Number Line, and the bar model. Converting discrete models into continuous models helps children acquire the language, conceptual schema, strategies, and efficient procedures—for both standard algorithms and alternative methods of solving addition and subtraction problems.

E. Definition and Meaning of Addition

In deriving the definition of addition and subtraction, the first important question is:

- When can we add or subtract quantities (numbers)?
- Under what conditions can we add or subtract?

Obviously, children need to know at the outset that we can only add those quantities which belong to the same class/group/type of objects. In other words, ‘they have some characteristic, property, or attribute in common.’⁷ 8 apples + 6 bananas cannot be answered until we define the membership of these objects into a class: Either as fruits or as an objects. We can say: 8 fruits + 6 fruits = 14 fruits; 8 objects + 6 objects = 14 objects.

This knowledge has profound implications later when we are adding fractions (e.g., fractions with the same denominators can be added or should be converted into same denominators), integers (e.g., integers with same signs can be easily added), algebraic expressions (e.g., like terms can be collected together),⁸ etc.

Initially, two quantities representing two collections of the ‘same type,’ with some common characteristic, property, attribute, or measurement unit can be ‘combined, joined, put together, or added’ with respect ‘to that property, characteristic, attribute, or unit of measurement.’ This act of collecting these objects or putting those sets together with that common property calls for an action: either ‘counting’ (if the objects are discrete) or making some kind of ‘measurement,’ (if the objects are continuous) therefore, must result in a ‘new quantity.’

The resulting collection is represented by a number that denotes the cardinality of this new collection with the same characteristic and property. The new number represents either the *count*, if we are combining objects, or the *measure* of the combined objects, if we are adding/combining two measurements. Children should, therefore, understand that the concept or process of addition is ‘putting things of the same type together.’

Ultimately, it is another abstraction process—addition is combining numbers, but it is not just combining objects. Numbers 5 and 3 represent any collection of five and three objects, respectively. We saw *it is an abstraction*. Now when we combine 5 objects of any kind and 3 objects of the same kind, we get 8 objects of the same type. After some experience, children transcend the type of objects and they only focus on quantity.

This abstract process is called *addition* (the action), and the result is called the *sum* or *total*. And the symbol for this operation/process is ‘+’. Thus, the *sum is the result of addition*. We should refrain from using the term ‘the answer is’ or the ‘solution is.’ The word ‘solve’ should be used for solving a problem, an equation, inequality, or a system of equations/inequalities. When the recognition of the symbol invokes that action at the orthographic level without counting, then that understanding has been reached.

Since we are equipping children for a long journey in mathematics, it is important that we think of the trajectory of a concept. In other words, how this operation will unfold, be used and expanded in future, so we should not treat this operation as a terminal operation and the student as a ‘terminal student.’ Therefore, technically, in numeracy and mathematics, it is

important that we emphasize from the very beginning the importance of three operational terms important for translating from English language to mathematics language: operator, operand, and operation. To operationalize ‘what is 9 more than 7?’ we need to identify ‘who’ is *operating* on ‘whom’ and ‘how.’ Here 9 is the *operator*, 7 is the *operand* and addition is the *operation*. Similarly, to operationalize ‘What is 3 less than 9?’, we need to know that 9 is the *operand*, 3 is the *operator*, and *subtraction is the operation*. This information will be vitally important, when we go to higher mathematics. Let us take a preview: If a student translates ‘9 more than a number’ as $9 + n$, then he would translate ‘9 less than a number’ as $9 - n$ which is incorrect and many students do this as $9 + n$ was accepted by the teacher earlier, although $9 + n$ and $n + 9$ are equal. We need to respect the syntax of the expressions. More importantly, the application of the understanding that the number is subject and object and operations are being done by one number on the other and a new number is emerging. Understanding of this concept takes the child to becoming a mathematician and an astute consumer of mathematics.

This phrase ‘*collections of the same things or same kind of measurements together*’ is important in understanding the concept of addition. This phrase will be crucial later, when we try to ‘combine or add,’ any kind of numbers, for example, the same question would be raised: ‘Can I add these fractions, decimals, integers, rational numbers, radical numbers, complex numbers, algebraic expressions, etc.?’ In the addition/subtraction process, one needs to focus on several aspects:

- Can I add them?
- When can I add them?
- How do I add them?
- Do I have a strategy(ies) to add these numbers/quantities?
- What is the outcome/answer?
- What does the outcome/answer represent?
- What is the outcome called?
- Can I call it by another name?

- How do I express this outcome/answer?

When a student can answer these questions correctly and fluently, not just at the concrete and pictorial levels but at the abstract level, and can give reasons/strategies for her answers, she understands the meaning and process of addition. She will be able to apply this knowledge for solving problems related to addition. To summarize the process, the addition is:

Recognizing any collection perceptually (subitizing)—> Numberness
—> Number—> Adding <—> Subtraction.

Each transition here leads to another level of abstraction.

This combination of numbers (in addition and in other numerical operations) is facilitated and made easier if the children are shown effective strategies using decomposition and recomposition, place value, and properties of numbers and the corresponding operation. Therefore, **mastery of number concept⁹** is a prerequisite for learning all arithmetic facts: addition, subtraction, multiplication, and division. With a strong number concept, children can master arithmetic facts easily. As we mentioned earlier, the primary teachers' goal is:

- *By the end of the first grade, children should have mastered the 10×10 addition facts (100 addition facts within 20) with rigor: understanding (language and effective strategies), fluency, and with an ability to apply them.*
- *By the end of the second grade, they should have mastered corresponding subtraction facts and understood the inverse relationship between addition and subtraction concepts, facts, and procedures with rigor: understanding (language and effective strategies), fluency, and with an ability to apply.*

In other words, they should be able to fill the entries in the 10×10 Addition Grid and respond correctly to any addition and subtraction facts randomly selected from the sums of the form $m + n$, where m and n are any numbers up to 10 and where $m > n$ and m and n are numbers up to 20, fluently and with understanding.

These are non-negotiable skills for first and second grades. With effective, efficient, and elegant strategies, these goals can be met for all children, with or without learning disabilities in numeracy.

To teach and learn these 100 addition and corresponding subtraction facts, one needs to learn effective, efficient, and elegant strategies. Effective strategies for addition and subtraction are arrived at when we use effective language and conceptual schemas derived from using transparent models (from concrete to pictorial to abstract levels). With effective strategies and proper practice, the task is doable. Children can learn, master, and use these 100 addition facts (sums up to 20). With proper task analysis, these facts are arrived at easily, some during kindergarten (45 sight facts) and then the remaining facts in the first grade.

The following pages describe the process of doing just that—to help every child learn, master, and acquire the ability to apply them to learn more facts such as subtraction, solve problems in other areas of learning and problems they encounter in navigating and understanding their environment in school and outside of school—games, transactions, and practical activities.

For learning and attaining the mastery of addition facts, effective strategies¹⁰ are based on the important principle of *decomposition/recomposition* of number as a part of the proper development of number concept.

In this book, we have identified, first a universal effective strategy to derive all the addition facts—using the concept of making ten, and then, eleven specific strategies to derive particular groups of addition facts (based on certain patterns observed in number groups). And then the same strategies are used for subtraction with minor modifications.

When using these strategies, we have hardly come across any child who has not learned these facts with mastery and understanding within a reasonable time and with confidence (about five to ten weeks of work in a regular classroom). However, the rate of mastery may vary with some children because of other factors, such as cognitive delay, language deficit, neuropsychological developmental reasons, specific mathematics learning disability such as dyscalculia, acquired dyscalculia or their effect, and mathematics anxiety, but they still are able to master the addition and subtraction facts or derive them with understanding in efficient ways when they need them.

Before one can design instructional, intervention, or remedial activities for a concept or procedure for any child, with or without disabilities, one should know:

- What does a learned behavior or mastery of a skill 'look like'?
- What are the components of the behavior and mastery of that skill?
- What actions does the master-learner need to perform to demonstrate mastery and what will be the markers of the quality of the expected behavior?
- What actions does the master-learner display in that state?
- What pre-requisite skills and primary concepts are necessary to achieve that behavior or mastery of that skill?
- Does the individual have these prerequisite skills or concepts?
- What instructional methods, approaches, sequence of activities, experiences, and type and amount of practice and reinforcement will make it possible effectively, efficiently, and elegantly?
- Are there any concepts or procedures on which the individual currently demonstrates mastery and competence?

For example, when we observe a child reading with fluency and with comprehension, we know exactly what behavior the child is displaying. We know the same reading behavior is expected from every and any child. The expectations are clear. We would want to see these same behaviors being

demonstrated before we declare any child a reader. In such a situation, the method used by the child to acquire the skill of reading may not be detectable. In teaching, we should focus on the input elements—*the method* (the assumptions, strategies, tools, frequency of treatment, etc.), the *information about the child*—so the method can be modified to meet the child's specific needs, *the sequence of activities* and *the amount and frequency of practice*, and *the nature of the behavior that these actions or input produce*. Then, in the assessment of mastery, we need to focus on the outcome behaviors—*demonstration of skill level, attitudes to the task, and dispositions*.

Similarly, when a student knows his addition facts, the method of learning those facts need not be detectable in the final behavior. It should be the *display of mastery—fluency with understanding and the ability to apply it in different contexts*. If a child is giving us the answer by counting on fingers, that is not mastery. Then we should plan instruction that achieves the goal of mastery of content, not just the mastery of a method, unless we want the child to know that method. Similarly, the focus of assessment of mastery of content should be on *accuracy, fluency, understanding, and the ability to apply*,¹¹ not the emphasis on the assessment of the method or instrument of learning. For example, asking a child to find $8 + 7$ by using a number line or unifix cubes.

As mentioned earlier, the number related pre-requisite skills and concepts for learning and mastering addition facts with fluency and understanding are: (a) *Subitizing/visual clustering*; (b) *One-to-one sequential counting*; (c) *Decomposition/recomposition of numbers*; and (d) *45 sight facts* (e) *Making ten*; (f) *Teens' numbers*; and *next tens number*.

These strategies for deriving addition facts should be mastered by children during kindergarten.

In the case of older students struggling with arithmetic facts, particularly the addition facts, we always begin with the mastery of these prerequisite

skills. When students come to first grade without these prerequisite skills, the teacher should begin with these skills, and even during teaching addition facts, she should refer to and even find time to show these skills and help students to master them.

Similarly, the interventionists and remedial teachers should focus on the mastery of these skills first and during intervention and remediation devote enough time to develop these skills. In the introductory part of the lesson on arithmetic facts, before they begin formal work on mastery of other arithmetic facts, including addition and subtraction, teachers should make sure that children have mastered these prerequisite skills. All the effective strategies for deriving arithmetic facts, addition/subtraction and multiplication/division, are dependent on the properties of number and particular operation, language, the sequence of the complexity of the fact and the operation, and the questioning process. In particular, the derivation of addition/subtraction facts involve properties of number and operations, specifically the following strategies and principles:

- The commutative property of addition ($\mathbf{m} + \mathbf{n} = \mathbf{n} + \mathbf{m}$),¹²
- The associative property of addition $(\mathbf{m} + \mathbf{n}) + \mathbf{p} = \mathbf{m} + (\mathbf{n} + \mathbf{p})$,
- Number plus one ($\mathbf{n} + \mathbf{1}$ and $\mathbf{1} + \mathbf{n}$)¹³ and number minus 1 ($\mathbf{n} - \mathbf{1}$)
- Making ten ($\mathbf{m} + \mathbf{n} = \mathbf{10}$),¹⁴ $\mathbf{m}$ and $\mathbf{n}$ are counting numbers up to 10, and
- Adding ten to a number ($\mathbf{n} + \mathbf{10}$ and $\mathbf{10} + \mathbf{n}$)¹⁵ where $\mathbf{n}$ is a number up to 10.

The concept of number and these five strategies help children to acquire the key skills needed to learn and master arithmetic facts easily and effectively.

Just like sight vocabulary (most commonly found sight words—the words learned by sight/observation/exposure) helps children to enter and participate in the language acquisition and reading process, particularly initial reading, similarly, children need to acquire the *45 sight facts* to start the process of addition of numbers comprehensively with meaningful strategies.

With the help of sight vocabulary and phonemic awareness, children begin to acquire ‘*word attack*’ skills and with supervised practice and with meaningful immediate feedback and corrections with good strategies, they are well on their way to acquiring reading skills; similarly, with the help of decomposition/recomposition, 45 sight facts, making ten, and teens numbers, children learn ‘*number attack*’ skills and they make a solid entry into the process of acquiring the numeracy skills.

In both cases, under the tutelage and care of a knowledgeable and empathic adult—a teacher, tutor, or a parent—children make effective and efficient progress toward reading, writing, and arithmetic. Due to these observations, we believe if we can train parents, particularly mothers, the problems of mastering arithmetic facts can be solved effectively. At the Center, we always invite parents to sit in on the remedial sessions and take notes on the strategies, and then to help their child in practicing these skills at home using the same instructional materials and scripts for applying strategies and procedures to show the importance of consistency. Parents and children are trained in the use of instructional materials, how to apply these tools, the strategies, and when and how to practice using the specially designed workbooks and paper and pencil instruments for practice. The result is that our students reach fluency with understanding quickly.

From this perspective, the knowledge and skills of the teacher in (a) how children acquire number concepts, (b) what strategies are effective for all children, (c) what is the developmental trajectory of a particular concept, (d) what instructional materials, tools, and strategies are effective, efficient, and elegant, and (e) how, how much, and when to practice for mastery—achieving accuracy, fluency, and ability to apply in a problem solving are crucial and fundamental.

F. How Do We Introduce the Addition of Whole Numbers to Three-, Four-, Five- and Six-Year-Olds?

Traditional practices of teaching arithmetic and even research related to teaching arithmetic tell us that most children at the ages of three and four

are not ready for rote learning addition facts. Even [Piaget \(1959\)](#) believed that children that young do not have the language and the understanding of the fundamental structures of number. Our observation is that the problem is not with children, it is with the teaching methods. New information about the brain, research in neurology and neuropsychology, advances in the understanding of language development, processes of learning, and new technology help us to think otherwise. With the effective methods even some three- and many four-year-olds can read; they can also learn number concept and then to add and subtract.

As described earlier, the first organized addition related concepts that should be taught to children are the basic number combinations—the 45 Sight Facts. We begin with introducing such a lesson. For this, one assumes that the children are familiar with the Cuisenaire/Number rods.^{[16](#)} It is also important that they are quite familiar with the Visual Cluster Number Cards (VCNC) and Sight Fact Number cards^{[17](#)} representing the first ten digits/numbers.

Children should be familiar with the BaseTen blocks (Dienes' Blocks) during kindergarten, and by the end of kindergarten they should be able to recognize two-digit numbers easily. In kindergarten, they have already constructed a Number Line and should know the location of numbers and the idea should be extended to Empty Number Line (or Open Number Line).

As we saw in the chapter on tools, the Empty Number Line (ENL) is a very efficient and versatile tool, not only for addition and subtraction of whole numbers (even multi-digit addition and subtraction), but also for addition and subtraction of fractions, decimals, and integers. It is also a very effective tool for developing mental mathematics and robust numbersense. The ENL should be introduced in the beginning of the first grade with the use of addition strategies.

Since the beginning of the kindergarten academic year, children would have been playing with the Visual Cluster Number cards (learning decomposition/recomposition of numbers), making figures with Cuisenaire rods,^{[18](#)} doing different activities to discern their sizes and colors and

building staircases, sight fact equations, etc. They should be able to identify colors and associate numbers with colors and vice versa. If children have never used Visual Cluster Number cards or Cuisenaire rods, they can be introduced within this lesson.¹⁹

In the first grade and later, the Number rods²⁰ and empty number line (ENL) would be used to understand and perform the operation of addition by taking two rods and finding a bigger one that would fit the two smaller ones together in length and derive and learn the related arithmetic facts in multiple forms. Visual Cluster Number Cards, Addition Ladders, and number games would be used to practice, master, and apply the addition facts.

1. Introductory Activities for Teaching Addition

The first set of activities/lessons should involve the development of the language of addition; however, it should be in the context of the addition activity, rather than just memorizing the vocabulary words of addition or subtraction, in isolation, or as a vocabulary learning activity. Language and terminology related to a concept should always be introduced by the help of concrete models or reference to concrete/pictorial models. Therefore, the purpose of the use of concrete models in mathematics instruction is threefold:

- a. generating *the language* of the concept (here it is the development of language related to addition/subtraction),
- b. developing the *conceptual schemas* of the related conceptual idea (here the development of the schemas for addition), and
- c. deriving the *strategies and procedures* related to the concept of addition and subtraction.

These three components can be developed if the teacher assumes the three roles of the teacher: didactic, Socratic, and coaching. The following activity is in preparation for learning addition and subtraction concepts, strategies, facts, and procedures. Timelines for when individual children are ready for

this engagement, able to learn, challenged by the task, or can master it will vary with each child, but most children are ready for it in kindergarten.

As a starting point, the teacher should explain the meaning of addition (combining groups/sets/collections/quantities/numbers); we can call this: ‘plus,’ ‘add,’ ‘combine,’ ‘join,’ ‘putting together,’ etc.), and demonstrate each word and term related to the concept with the help of concrete objects. These concrete materials may vary from discrete objects, such as counting blocks, pencils, the pips on the Visual Cluster Number cards, to continuous objects like Cuisenaire rods.

Each collection is then related to numbers involved in the addition problem. The addition problem is modeled by these materials. For example, beginning with a posed problem: $3 + 5 = ?$ The numbers of the problem are related to the appropriate collections (3 1-rods or three objects and 5 1-rods or five objects or 2 Cuisenaire rods (3 = light green rod, 5 = yellow rod) and then the rods are put together end-to-end to show the joining or adding of two numbers. Now the rods and the collections can be related to each other, then joining them and comparing the collections with Number rods, in that order. As mentioned earlier, Addition is combining a collection of objects that have some characteristic or attribute in common.

This can be done either by (a) putting groups of objects together, (b) placing two Visual Cluster Number Cards, corresponding to the collections, next to each other and reading the two together as the sum of two numbers represented by the two piles of objects, or (c) connecting two Cuisenaire rods together (putting two Cuisenaire rods end to end), then finding the sum by the help of another Cuisenaire rod(s) that is equal to the length of the two rods placed end-to-end.

Activity One:

Teacher: Let us consider an addition problem: We are going to add two numbers. Let us take four white rods (or put four Unifix cubes or

buttons, etc. in a horizontal line). You do the same on your desk with your partner. You can choose objects of your choice.

The teacher, then asks several children:

‘How many objects are there in front of you?’

‘If you need to, you can count them.’

‘Do you have the same number of objects as the next table team has?’

How do you know?’

Answers to these questions will indicate their preparedness for addition activities. Some will assert the fact by just counting each collection. Some might be shown it by one-to-one correspondence.

She examines the work of each pair. She introduces several terms relating to addition as she introduces the idea of addition. She constantly emphasizes what each word means: adding, combining, horizontal, line, in a row, straight, together, join, end-to-end, next to, put to the right of, put to the left of, make a train, etc.

For example, the teacher looks around and points to an arrangement of two cards (the 4-card and 3-card), 4 gummy bears and 3 gummy bears, or a pair of Cuisenaire rods (4-rod and 3-rod), then she might say: ‘Good, that is a good representation for the addition problem $4 + 3$. Does anyone else have any other combination?’

Children share their representations with the whole class. Each team of children is acknowledged and engaged in relevant conversations.

It is important that there is congruence in her non-verbal actions and her word usage. The teacher should realize that the mathematics language should be transparent, precise and concise. Development of proper language—vocabulary, syntax, semantics, and translation—²¹ will help children conceptualize the ideas better. All these words are new to many children. She solicits the answers to these words. First, she asks them to count the collections from left to right and then she asks them to count from right to left.²²

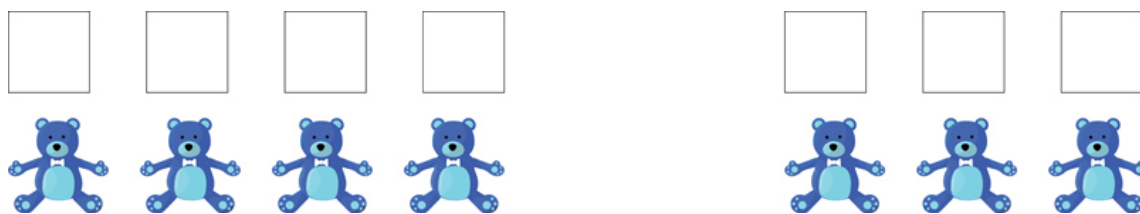


Figure 7.2 One to one correspondence



Figure 7.3 One to one correspondence

‘Now pretend that each white-rod (1-rod) represents a teddy bear. Place one teddy bear for each rod in the next row. Make sure you put one teddy bear for each rod. Do not forget to align them.’

She asks each pair of the students the question: ‘How many teddy bears are there?’ She points to the rods and says: ‘These represent four teddy bears’ (pointing to the collection as she places these objects).²³

Teacher: ‘Let us take another set of three white rods. These also represent a group of three teddy bears. Place one teddy bear for each rod.’

‘Let us put the two groups of rods and the two groups of teddy bears together. Let us find out how many teddy bears are there altogether when we put them together.’

‘Can someone tell me what I should do now?’

Some children may say: Put them together and count all of them. Most children will count them and give the answer: 7.

Teacher: That is right. How many are there? Seven! You found the *sum* of the two groups of teddy bears. *Sum means the same thing as putting the two groups together.*

Teacher: How many teddy bears are in the first group?

Children: 4.

Teacher: How many teddy bears are in the second group?

Children: 3.

Teacher: What is the sum of the two groups of teddy bears?

Children: 7.

Teacher: What is the sum of 4 and 3?

Children: 7.

Teacher: Good! Let us try another way of finding the sum.

At this point, she should introduce the technical terms: sum, total, answer to an addition problem, equation, equal, etc. The words **sum** or *total* represent answers to addition problems. The arrangement ' $3 + 4 = 7$ ' is called an equation. Let us read the equation in four ways: '3 plus 4 equals 7; 4 plus 3 equals 7; 7 equals 3 plus 4; and 7 equals 4 plus 3.' Notice the 'two-way implication' as the meaning of equal sign.

The activity just concluded is introducing addition using discrete objects. In this form, it is dependent on counting, and it promotes counting. Be aware that if we remain too long on counting, few children will transcend the counting process and automatize arithmetic facts, and many will remain at the counting level and struggle with the mastery of arithmetic facts.

The teacher should repeat the same problem or a similar problem using the Visual Cluster Number cards. Visual Cluster Number cards also use discrete representation; however, because of the patterns on the cards, children are able to see the sum much easier and faster, even without counting. They are a very effective means by which we begin to learn decomposition and recomposition of numbers, and they mediate between discrete (quantity) and continuous (measurement) models of number.²⁴ She can also show the two numbers: 3 and 4 on the 7-card and develop the equation $3 + 4 = 7$, etc.

The following activity is to introduce addition using a continuous model, in this case the Cuisenaire rods, emphasizing the measurement model of number for addition. We have found Cuisenaire rods to be very effective,

not only for learning number concept, but for number relationships, additive reasoning, multiplicative reasoning, and their generalizations into algebra.

Activity Two:

The teacher asks children to pick four 1-rods (white rods from the Cuisenaire rods set) and organize them horizontally, end-to-end, in a train on their desks.

Teacher: Look at the 4 white rods (1-rods) you have organized in a straight line on your desk. Now, let us examine these four 1-rods placed together (end-to-end), touching each other. It looks like you are adding four 1s. You also know that it could be 2 and 2, 3 and 1, or 1 and 3. Let us quickly check these arrangements.

Notice that by this action, the teacher has reviewed the sight facts of 4. This is called ‘on the spot remediation’ or ‘filling the gaps.’ The teacher should never miss these opportunities. She should also have done this in the beginning of the lesson, in the ‘*tool building*’ section of the lesson. She should show the symbols, 2, 3, and 4, alongside the rods.

‘Now somebody tell me: What one rod (which color rod from your staircase) will fit the train of 4 white rods?’

If necessary, the teacher can remind them of the relationship. ‘For example, you already know which rod is made up of four white rods. If you do not remember, you can consult your staircase of the rods. Then put the 4-rod parallel to the train of 4 white rods (1-rods).’ The teacher should explain and demonstrate or establish the meaning of parallel by the help of children.



Figure 7.4 Naming a 4



Figure 7.5 Naming a 3 rod



Figure 7.6 Deriving the $4+3$ equals 7 instead of the caption mentioned

Some children will say: *purple*. Some will say: *4-rod*. She should acknowledge both answers as correct, but she should lead them to say 4-rod or just 4.

Teacher: That is right. The 4-rod will fit the four 1-rods. Both have the same length. That means they are equal in value. Now, let us put the three 1-rods together.

What one rod (which color rod from your staircase) will fit the train of three white rods? Some children will say: *light green*. Some will say: *3-rod*.

Teacher: That is right. The 3-rod will fit the three 1-rods. Both have the same length. Now, let us put the four 1-rods train and the three 1-rods train together. Are the new train of whites and the train of purple and light green the same length?

Students: Yes!

Teacher: Great! So, 4 white rods train and the 3 white rods train is same length as the train of purple and light green rods. To find the answer for the white rods train, we will find the answer for purple and light green rods train. Now let us put the purple and light green rods together.

Which rod from your staircase is as long as the two of them together ($4 + 3$)? Some children will say: ‘the black rod or the 7-rod.’

Teacher: Let us check it. We have two trains of rods ($3 + 4$) and the 7-rod ($= 7$), and they are equal in length and so they are equal. We have: $3 + 4 = 7$. We call this arrangement making an equation, as in the word ‘equal.’

She should show words in written form as symbols to compare the ‘equal’ in both.

‘When two things’ “expressions” are equal, they define an equation and we write or read that the first expression is equal to the second expression. And the second expression is equal to the first expression.’

When we read these two trains, we get an expression $\backslash 3 + 4 /$, and we see that 3 and 4 together equals the 7-rod. The two rows of trains are equal in length.

Can someone tell me how to read this equation?’

Some children will say: ‘purple plus light green equals black.’

Some will say: ‘4 plus 3 equals 7’ or $\backslash 3 + 4 = 7 /$. Some might say: $\backslash 7 = 3 + 4 /$

Teacher: Great! Let us read and write these relationships as equations.

4 white rods and 3 white rods = 7 white rods.

Purple rod + Light Green rod = 1 black rod.

$$4 + 3 = 7$$

Read these three equations to your partner out loud.

Teacher: When we read it from left to right it is read as:

Students: $4 + 3 = 7$.

Teacher: Let us read this equation form right to left.

Many students will read it as: $3 + 4 = 7$. Some will read it as: $7 = 4 + 3$.

Teacher: The correct way of reading it from right to left is: $7 = 4 + 3$.

Teacher: Great! Can we rearrange the purple and Light Green rods? Will they together still be equal to the black rod? Please take another pair

of these rods and show me the new equation.

The teacher should make the new equation of the three rods and go through the same process of questioning.

Activity Three:

Make a new arrangement of equations of the three rods.

Purple + Light green = Black Rod

Light green + Purple = Black Rod Back

Black Rod = Purple + Light green

Black Rod = Light green + Purple.

Teacher: If we read the equation on the top, we have: $4 + 3$, below that is $3 + 4$ and then we have those $= 7$. We have $4 + 3 = 3 + 4$; $4 + 3 = 7$; $3 + 4 = 7$; $7 = 4 + 3$; and $7 = 3 + 4$. Now we have five equations:

$3 + 4 = 4 + 3$; $4 + 3 = 7$, $3 + 4 = 7$, $7 = 3 + 4$, and $7 = 4 + 3$.

Now everyone should generate another set of equations by taking another pair of rods. And, in then write the five equations. After choosing the rods, you may write the equations on paper and then check the equations by making the equations with the help of rods.

When you find several children who you think are up to the challenge, suggest it to them and let their work ‘spread across the classroom’ as other children see what they are doing. They should find a partner and share their work with each other by making new trains. All equations should involve two rods for the problem and 1 rod for the answer or rods where one of them is a ten-rod, and then take one rod and break/show as two rods.

If children are ready for this level, they will copy and start making their own equivalent trains. Other children will be content to continue making their own structures or alternate between this and other activities. Some

children will be ready to record their equivalent train discoveries and their recording can also vary according to their developmental levels.

After this, using one-centimeter grid paper, children can color (matching the rods) to show equivalent trains and write the five equations.

Give a new example of addition, ask the children to put the 2-rod next to a 3-rod.

Teacher: If we read it from left to right it will be read as 2 plus 3, and from right to left it will be read as 3 and 2. If we read it or write it from right to left. We say 2 and 3 are being added, or we are finding the sum of 2 and 3. Or, 3 is being added to 2. Everyone should read it out loud with me from left to right in three ways: 2 plus 3; we are adding 2 and 3; we are finding the sum of 2 and 3.

Teacher: I want you to make an arrangement just like mine. Now, find one rod that is as long as these two together, e.g., 2-rod and 3-rod. Find one rod that will fit the two together when you put that one parallel to them.

Students: The yellow rod. The 5-rod.

If a child has found the rod (the yellow rod), show the equation to all children by placing the yellow rod parallel to this arrangement.

Teacher: Now let's read it. Red plus green equals yellow. Green plus red equals yellow. Yellow equals red plus green. Yellow equals green plus red.

Children read their equation with the teacher.

'Now read the equation in numbers as 3 plus 2 equals 5, 2 plus 3 equals 5, 5 equals 3 plus 2, 5 equals 2 plus 3. Who can read it using the number names?'

Children read their equation with the teacher.

The expectation of this exercise is that first, the students will read the equations in color and then they will read all four combinations in numbers.

As the student is reading his/her equation to the class, the teacher will be pointing to the appropriate rods.

$$2 + 3 = 3 + 2; 2 + 3 = 5; 3 + 2 = 5; 5 = 2 + 3; \text{ and } 5 = 3 + 2.$$

The teacher writes these equations on the board as children read them.

Repeat this process with several other combinations. The last four equations are essential and need lots of repetition through questioning for the students to remember the fact. Students must hear this at least 15 to 20 times for it to enter their long-term memory. Here are some of the ways that the questions can be asked. It is important to vary the way that the teacher presents the question to students.



Figure 7.7 Deriving the Name of the 5 rod



Figure 7.8 Deriving the Name of the 5 rod

- $2 + 3$ equals what?
- $3 + 2$ equals what?
- What equals $2 + 3$?
- What equals $3 + 2$?
- $2 +$ what equals 5?
- $3 +$ what equals 5?
- If I have 2, how many more must I add to it to get 5?
- If I have 3, how many more do I need to get 5?
- What is the sum of 2 and 3?
- If I have 2, how many more do I need to get to 5?

Children need to *hear, see, say, and write* these relationships several times. After this, the teacher should ask children to make their individual equations, and then they can share their equations by reading them to the class. Every student should get a chance to share and read their equation to the class. Some students may read their equations with someone's help.

The major objective of these kinds of exercises is the integration of language, concept, and procedure. It is not enough to give the answer. It is important to know the language associated with the equations and results. This kind of integration is the preparation for applications, particularly word problems.

Activity Four: Addition with Zero

Just like when we add numbers ($5 + 7$) together, we place two VCNCs or Cuisenaire rods together and then, either by counting or some other strategy, we find the sum of 5 and 7 as 12.

For addition with zero, for example $5 + 0$, we take the 5-card plus the blank-card (the card which has no pips on it) and see that the white card did not change the outcome. The number of pips with both cards together is still the same. Thus $5 + 0 = 5$. In other words, all the child needs to remember now is that when we add zero to any number, the result is the number itself. Thus, we have:

- $5 + 0 = 5$
- $70 + 0 = 70$
- $570 + 0 = 570$

2. Role of Flash Cards in Learning Arithmetic Facts

Mastering arithmetic facts using strategies develops numbersense, numeracy and applications of numeracy. Memorizing facts by using methods such as: using *Counting*,²⁵ *Flash Cards*, *Mad Minutes*, *iPad apps*, and other similar devices alone may not develop true numbersense. There is nothing wrong with the use of these strategies and devices. In the hands of a professional, they are wonderful. However, there are several assumptions

inherent in the expectations from the use of these methods (e.g., flash cards, etc.).

First, one assumes that every child can derive arithmetic facts with the help of just flash cards; and that flash cards are enough for learning facts with understanding, accuracy, automatization, and ability to apply;

Second, learning facts is an easy task and can be accomplished by anyone, and then they can apply them easily in learning new concepts and solving problems. If it is so easy to teach any child arithmetic facts and this is the only strategy a teacher can provide a child, then why do we need to have so many years of training and degrees.

All of these assumptions may be erroneous. Even if these methods work for some, the cost of the effort and emotions is enormous. These assumptions also indicate that, to them, learning facts is not an important mathematical activity; it is an adjunct to other mathematics concepts and procedures. The problems we see with the sole use of flash cards are:

First, because of the rote nature of using the flash cards, the child may not know arithmetic facts with understanding and may not acquire flexibility of thought;

Second, the flash card method forces and requires one to overlearn the material;

Third, flash cards practice places an individual fact in one particular place in the memory bank (memory is not organized like a *filing system*²⁶ at the same time, and no network connections are made in such learning) where the information remains isolated. Therefore, often it is less useful.

Finally, flash cards do not help children see connections and relationships between numbers and concepts. *Memorizing facts using flash cards is done at a high cost of time, effort, and motivation.* This practice unduly taxes an individual's working memory—it is then filled with this mindless practice and no space is left for higher-order thinking and learning.

The teacher's belief that learning is just memorization and expectations alone can make it happen are erroneous. Moreover, the belief that children will be able to learn them by just sheer repetition is also erroneous ([Lane](#)

[and Chinn, 1986](#)). In this process, they repeat and repeat and memorize. It is a poor use of children's time. However, leaving this automatization to children or their parents to teach them is also not enough. In most cases, these are unrealistic expectations from the perspective that not all parents can or are able to do this.

The teaching of how to arrive at facts needs to be done well, efficiently, and by professionals—the teachers who are trained to do this. If teachers use effective strategies, the task can be achieved in the classroom itself, in a relatively short time. We have found that during kindergarten through upper elementary school, very few children are taught strategies such as:

How to study? How to consolidate learning? How to make even proper flash-cards? How to memorize? How to retrieve information? How to prepare for tests? In other words, how to take advantage of one's enormous resources? Teaching learning skills is a bonafide teaching activity.

We believe that learning arithmetic facts is one of the most important and fundamental skills in mathematics that every child needs to acquire—effectively, efficiently, and elegantly—even if later one has the availability of modern computing and thinking technology. Learning facts and automatizing them is a critical activity in the numeracy and mathematics learning process, and should be taken seriously. It is parallel to a child learning to read. Most children need efficient strategies and methods from knowledgeable and sympathetic adults to become fluent readers. Once they are fluent readers, they can then read to learn anything. Similarly, a child needs the help, support, and supervision of a knowledgeable and sympathetic adult to learn arithmetic facts effectively, efficiently, and using elegant methods so that she can use them to learn other important mathematics concepts.

Through the sole use of flash cards, mathematics unfortunately becomes uninteresting to children very early on. Rote memorization activities should be used as adjuncts to other methods of teaching, not as the only method. They have limited effectiveness, and only after language, concept, and strategies have been acquired. In most cases, there will not be any need for flash cards, however, limited and occasional use for review might be useful.

Mastery of arithmetic facts and then the ability to learn other arithmetic concepts opens doors for not just mathematics, but all STEM fields. Arithmetic facts are important because their use is all-pervasive in mathematics and in other related disciplines. Arithmetic facts are hardly ever used in isolation. Children should be shown connections and their embedded usage in other concepts. And not solely by rote-learning.

It is true that ultimately, one has to commit arithmetic facts to memory, but the key is that this needs to be done with understanding. In order to access a large number of concepts and procedures and in order to develop fluency in mathematical thinking, students need to master arithmetic facts with the use of strategies that help in understanding and applicability. Whatever we put in our memory bank should be meaningful and useful and easily retrievable. The key element is the use of appropriate and effective models and methods (approaches) to help learn with understanding and then to memorize arithmetic facts through use and by making connections. Real understanding of future arithmetic and mathematics concepts, such as fractions, decimals, laws of exponents, etc., will be difficult, if not impossible, for a child if he has not first understood and committed the basic arithmetic facts to memory *with understanding*. All children should be exposed to the role of memory and strategies in learning anything, and then particular and appropriate strategies for mathematics learning.

3. Language, Strategies, and Questioning Process: Their Role in Mastering Arithmetic Facts

Entry into the domain of addition is not just ‘counting forward’ and subtraction is just ‘counting backward.’ Additive reasoning is more than just counting forward or backward. It involves specific language, conceptual schemas, strategies, skills, and applications of these elements. Language and conceptual schemas developed and formed in earlier grades become the basis of understanding the operations on future number systems. Mathematics is very developmental and cumulative in nature.

Addition and subtraction as operations emerge from certain situations, actions, contexts, and in the process of answering certain questions. Therefore, the language and setting of that language act as instigators or ‘prompts’ for certain conceptual schemas and resulting operations. Without the language, context, and purpose, arithmetical operations become operations on ‘naked numbers.’ However, one should inculcate the language properly and fully; just underlining the prompt words to instigate operations is not enough. Let us consider two examples:

- a. $8 + 7 = ?$, and
- b. $17 - 8 = ?$

These two equations are abstractions of many concrete, linguistic, and contextual situations. It is a major achievement on the part of children to reach this point.

- a. $8 + 7 = ?$

What is $8 + 7$? This question and the related equations that may emerge in multiple real-life and academic situations define the related operations. Finding the answer to this question and solving the problem related to this question means understanding that one is answering one of the following questions in some context:

- What is the sum of 8 and 7?
- What is the total of 8 and 7?
- What is 7 more than 8?
- What is 8 and 7?
- What do we get when we add 8 and 7?

In arithmetic/mathematics, as mentioned earlier, three entities/numbers are related by three terms: operand, operator, and operation. In the problem: $8 + 7 = ?$, the number 8 is the operand, 7 is the operator, and the operation is addition.

What do we get when we combine 8 objects and 7 objects of the same type?

How old is David if he is 7 years older than his sister?

His sister is 8 years old.

When children are only given prompts rather than emphasis on the language of addition and subtraction, they have difficulty in solving word problems that may not have any of the prompts as in the two previous word problems. For example, many parents and teachers will tell the children: ‘When you see the ‘+’ just count up.’ ‘When you see the ‘–’ just count down.’ They can add or subtract at the time, but they may not go too far with that kind of limited language and the related conceptual schemas.

b. $17 - 8 = ?$

What is $17 - 8$? This question may emerge in multiple situations. Finding the answer to this question and solving the problem related to this question means understanding that one is answering the question:

- What is the difference between 17 and 8?
- What is 17 take away 8?
- What is left when we subtract 8 from 17?
- What is 17 minus 8?
- What is 8 less than 17?
- How much more is 17 than 8?
- How much less is 8 than 17?
- What is 17 subtract 8? Etc.

What do we get when we ‘take away’ 8 objects from 17 objects of the same type?

How old is David if he is 8 years younger than his sister? His sister is 17 years old.

In the process of finding the answer to these problems, one encounters several words and phrases and specific contexts that are associated with

these equations. They are learning that several words and phrases give rise to addition and subtraction, not just that addition is ‘counting up’ or just ‘add’ and subtraction is ‘counting down’ or just ‘take away.’

Anyone who works with children from kindergarten through second grade should realize that with age, children will encounter even more words and phrases in the context of addition: increase, combine, join, union, etc. Children should encounter language, contexts, and situations (appropriate to age and grade) when they are learning addition and subtraction.

Accessing facts from memory is essential. However, this memorizing process is automatically taking place when children learn efficient, effective, and elegant strategies and when they use them for solving problems. Such a natural process of memorization has many advantages. When we have facts in our memory bank, we are able to see new and related facts easily. Many other related facts become associated with that fact. If we know a fact, other facts related to that fact get ‘anchored’ to that fact. They become easier to recall. More facts become part of the memory system. Then we are able to discern patterns in the incoming quantitative information. For example, if children know $8 + 7 = 15$, they are likely to see that ‘eight hundreds plus seven hundreds is fifteen hundreds,’ or ‘eight thousands plus seven thousands is fifteen thousands,’ or ‘eight-nineteenths plus seven-nineteenths is fifteen-nineteenths,’ or $8x + 7x = 15x$ or $8nyz + 7nyz = 15nyz$. Facts can, and should, be mutually supportive.

As one can see, if children find the answer to the problem $8 + 7 = ?$ by always counting up, then they are not likely to generalize that fact or that knowledge. When we learn a fact by repetition it gets lodged in the memory bank at one particular location (a place called the basal ganglia)—it is stored in one file. On the other hand, if we find the fact $8 + 7$ in several ways, using multiple strategies:

- $8 + 7 = 8 + 8 - 1 = 16 - 1 = 15$;
- $8 + 7 = 1 + 7 + 7 = 1 + 14 = 15$;
- $8 + 7 = 8 + 2 + 5 = 10 + 5 = 15$;
- $8 + 7 = 5 + 3 + 7 = 5 + 10 = 15$;

- $8 + 7 = 5 + 3 + 5 + 2 = 5 + 5 + 5 = 15.$

then the fact is stored in several files, in this case, ‘doubles file,’ ‘near doubles,’ ‘making ten file,’ ‘decomposition/recomposition file,’ etc.

We always ask students to share their strategies with the class. As they are sharing their strategies, we record them and ask students’ opinion to vote which strategy was most used and which one was the most efficient strategy. This sharing improves other children’s approaches to problem solving. They find that they can adopt efficient strategies and even invent new strategies.

When one of the authors asked one student this question, he said: The answer is 15. He asked him, ‘how did you get that?’ ‘Oh! That is easy. I know $10 + 10$ is 20. I took out 5 from 20 as the two tens I used are 2 and 3 more than 8 and 7 respectively.’ It is on the extreme side of the answers, but it shows that the child is still using decomposition/recomposition and shows the inverse relationship between addition and subtraction to get the answer. Another student answered: ‘I used $10 + 8 = 18$ and then take away 3. That equals 15.’ After that, a few more children wanted to share their unique solutions.

Asking children to give their reasoning helps develop the key executive function of ‘flexibility of thought,’ and many connections are made. With these strategies and connections, one is able to see more patterns, and children are able to learn more, in the process improving their numbersense.

For children who have not mastered facts, each fact, whether old or new, is a new fact and they have to spend the same energy and effort each time they encounter that fact. When children do not know their facts, they do not see patterns in the incoming information and then they have a much heavier load on their memory bank, particularly on the working memory, in learning new concepts, procedures, and skills. They have to take digressions more often in deriving the simplest of facts to receive and understand the new information or to derive/construct a new fact. On the other hand, if they know more facts with efficient strategies, then more information enters

their short-term and working memories and more is transferred to long-term memory, facilitating better learning.

We need to help children discover and acquire efficient strategies. If they do not discover these strategies, we should lead them to these discoveries by asking ‘scaffolding’ questions. In some cases, when this does not work, we should teach these strategies using appropriate concrete materials.

As we have already mentioned, at present the major way of teaching facts in elementary schools is to use sequential counting and ‘add on,’ ‘count on,’ or ‘count all’ methods. These methods have an advantage only in very simple situations, but they are mostly disadvantageous. These methods are particularly useful with children who process information very sequentially and have very limited cognitive ability. Only in extreme cases should we limit teaching counting methods. When a teacher uses only sequential methods, students become sequentially oriented and even as adults continue to use their fingers for counting, even for adding small numbers. Children who learn arithmetic facts by counting have a heavier load on their memory network. This heavier load robs them of appreciating mathematics learning. For example, for such a child, $8 + 7$ and $7 + 8$ are two different addition facts; 8×7 and 7×8 are two different multiplication facts. The counting method does not promote the smooth development of commutative property of addition and multiplication.

To be effective, the approach which we suggest in this book should be incorporated in the mathematics teaching methodology from the very beginning of teaching numbers and number facts. The objective is to help children discern, extend, create, and apply patterns in exploring and learning number relationships rather than just learn isolated facts by sequential counting and memorizing by rote.

Our emphasis should be on conceptualization (language and conceptual schema), fact development strategies using decomposition/recomposition, and facts mastery at a level of automatization. That means mastering arithmetic facts but at the same time knowing the language and concepts behind them and being able to apply them meaningfully. For example, when a child knows a fact: $8 + 6 = 14$, then he can automatically write

$6 + 8 = 14$; $14 = 8 + 6$; $14 = 6 + 8$; and a little later, $14 - 6 = 8$, $14 - 8 = 6$; $8 + \square = 14$; $\square + 8 = 14$; $14 = 6 + \square$; $14 = 8 + \square$. He can also solve the following problem without counting, which means without drawing xs or circles on the paper.

There were 8 apples and 6 oranges in the basket.

How many fruits were in the basket?

When students are urged and helped to master arithmetic facts in this manner—by using strategies, they can retrieve them forward and backward—that means randomly. If each time they see a problem like this, they draw xs in circles, collect objects in two piles of 8 and 6 objects, count on a number line, or count on fingers, they will remain at the counting level.

One can even teach effective strategies using counting blocks, but it is easier and more efficient to teach them using appropriate concrete materials. Examples of effective methods for conceptual development, deriving strategies, and mastering facts are to teach arithmetic facts using Visual Cluster Number cards, Cuisenaire rods, Empty Number line, and the Invicta Balance, with the underlying principles of decomposition/recomposition and the concept of equality.

The choice of appropriate models for teaching a particular concept is an important decision and is the mark of an effective teacher. Conceptual Models—concrete, visual, or abstract—presuppose some kind of isomorphism between the models and what is being modeled (the concept)—the concrete, the picture, and abstract must be congruent to each other (and used alongside symbols). The models fail when the isomorphism fails. Many models teachers use do not transition to abstract level as there is no isomorphism between their concrete and representational models and between representational and the abstract and symbolic representations.

Any concept's representation by the concrete and pictorial models should be transparent and effective. Transparent means that the concept is accessible and understandable to the child and is effective when they are able to form a conceptual schema easily. An instructional material or model

should have three major characteristics: it should be *effective*; it should help a student to get the *exact* answer; it should be *efficient* in finding the answer; and it should be *elegant*—can help in arriving at a strategy or a procedure easily. The method can be generalized, can be extrapolated, and helps us in abstracting the concept. In the case of arithmetic facts, elegant means a method that helps the child to see the patterns to generalize and take it to the abstract and symbolic level of knowing.

For example, using Cuisenaire rods, one can see the commutative property of addition easily, efficiently, and elegantly. With Visual Number Cluster cards, one sees the sight facts effortlessly. The idea of equality and the properties of equality are seen concretely and intuitively using the Number (Invicta) Balance. The ‘counting on’ or other counting strategies and materials do not satisfy these characteristics. They are not elegant, they are not efficient, and sometimes they are not even exact and effective.

When we are planning to work with any child, a small group, or a whole class to help them learn a new concept or improve upon their mastery of arithmetic facts, here is what we would do:

- Evaluate their strengths and weaknesses and put a plan in place
- Improve essential prerequisite skills for becoming a good learner
- Follow a personalized plan for an individualized child and a well-organized lesson plan that is based on the child’s or the class needs
- Deliver the plan or the lesson using CPVA/S
- Test the child or the class orally first and then in writing, frequently and provide immediate feedback both in strategies and how to practice.

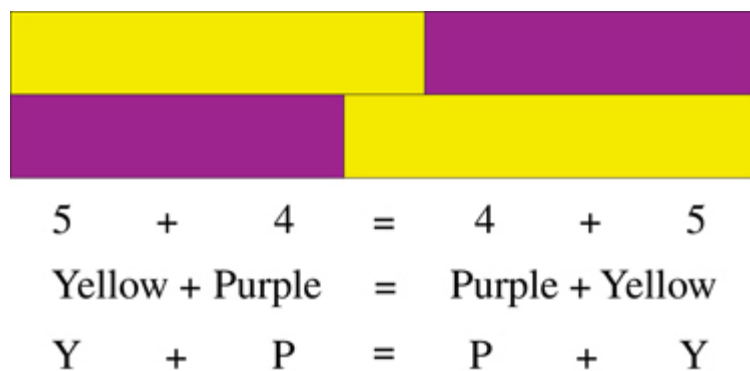


Figure 7.9 Commutative property of addition

We have used this process to improve thousands and thousands of children's numeracy and mathematics skills at different levels.

The quality of the implementation of an instructional approach greatly influences its impact on student learning. For example, the value of using the manipulative materials mentioned here to investigate number concept and then to learn arithmetic facts depends on how they are used and how the facts are derived with the students. Hence, as teachers implement any of the strategies described here, it is essential that the teachers themselves first learn the instructional materials and strategies, understand and internalize the questioning process necessary for teaching these strategies, master them, and then constantly monitor and adjust the way the strategy is being implemented to optimize improvements in student achievement.

Notes

1. Using a variety of words that express the idea of addition or subtraction. [!\[\]\(cdf2842d82858164c68c92720a337fb9_img.jpg\)](#)
2. See the section on Subtraction for the diverse language expressions to be used. [!\[\]\(3973dad7f2f3eafb2c144deb79d5c822_img.jpg\)](#)
3. Counting is an abstraction of subitizing. A number is an abstraction of counting and is a representation (oral, orthographic, or quantitative) of a collection. It is the property of a collection. Therefore, each number as a representation is a second-order abstraction. Addition, therefore, is the next level of abstraction. [!\[\]\(78688513da7a924039ac16e546d7bf8b_img.jpg\)](#)
4. It is important to note that here the transition of number as a qualifier to subject/noun is important as only when number is perceived by children as such can they operate on them in multiple ways. And arithmetic operations, of various kinds, may be performed. [!\[\]\(508a1136d49b79584e89f03b586dfa9e_img.jpg\)](#)
5. It also shows that the numbers are now beginning to act as subjects and objects—operators and operands and operation is producing a third

- object—an outcome. ↵
6. In interventions and remedial sessions with older students, we always teach these pairs of concepts together. We ask for every problem to be expressed as both ways. ↵
 7. Some of the repetition of information in different sections is intentional from the point of view of emphasis. ↵
 8. Only those quantities that have some common characteristic or attribute can be added or subtracted. Similarly, two terms are called Like Terms, if they involve the same variables and the variables have the same powers (exponents/index). ↵
 9. See *The Concept of Number: Preventing Mathematics Learning Problems* ([Sharma, 2019](#)), an eBook. The number concept prerequisites for success in additive reasoning are: (a) difference between subitizing, digit, counting, and number, (b) 45 sight facts, (c) making ten, (d) teens' numbers, (e) next tens number, and (f) the process of decomposition/recomposition. ↵
 10. *How to Teach Arithmetic Facts Effectively and Easily* by [Sharma \(2008\)](#) and *How to Teach Number Effectively* by [Sharma \(2019\)](#), an eBook. ↵
 11. See *Understanding and Comprehension: Numeracy and Literacy* by [Sharma \(2012\)](#). ↵
 12. It reduces the number of 100 facts of addition to 55, a manageable task for any child. ↵
 13. These addition facts are included in the list of 45 sight facts covered, derived, and mastered in kindergarten. ↵
 14. These 9 addition facts are part of the sight facts. ↵
 15. These 19 addition facts covered by this strategy make the total number of addition facts covered to 64. ↵
 16. With a few months' practice, even three- to four-year-olds are able to recognize Cuisenaire rods by color, number, size, and their relative sizes. Similarly, they are able to recognize the Visual Cluster cards. ↵
 17. The Sight Fact cards are modified Visual Cluster cards, where the decomposition of a sub-collection is already marked so that children

- who may have difficulty perceiving the whole to parts and parts to whole can see the sight facts of numbers easily. [↵](#)
18. See *Math Note Book: Cuisenaire Rods and Mathematics Learning* by Sharma (1999) [↵](#)
 19. For the introductory lessons in Visual Cluster Number cards and Cuisenaire rods, see the eBook *How to Teach Number Concept Effectively* ([Sharma, 2019](#)) or watch the videos mentioned earlier. [↵](#)
 20. From now on, the Cuisenaire rods will be referred to as Number Rods; White rod as 1-rod, Red = 2-rod, and so on. [↵](#)
 21. For more on mathematics language, particularly mathematics vocabulary, see *Dyscalculia, Dyslexia, and Other Language Related Mathematics Difficulties* by [Sharma \(2008, 2014\)](#). [↵](#)
 22. For classroom management, I (Sharma) always establish two rules from the very beginning of the class and keep following them.

Rule # 1: If you want to answer my question, ask me a question, want my attention, or anything else, you must raise your right hand always. Knowing left and right hands is a very important skill for learning mathematics. It is the first overt sign of the development of spatial orientation/space organization. Therefore, I practice recognizing left and right hand multiple times in the day till every child recognizes his or her right/left hand instantly.

Rule # 2: You should answer my question only when I call upon you by your name.

I practice these two rules strictly and within three weeks of the school year starting, all children, even in kindergarten, know their right from their left and respect the decorum of the class. All children should sit in a semi-circle at the same level. Exceptions are allowed only in the case of physical disability, not based on choice. The major concept should be taught to the whole class with due emphasis on individualization, involvement, and responsiveness. Then reinforcement and

further individualization are practiced in small groups or in helping an individual child. ↵

23. Each activity the teacher demonstrates should be simultaneously performed by children in groups of two. However, the grouping should be ad-hoc and properly formed to have some compatibility. From the beginning, the rules of the activity and behavior are explained and demonstrated. ↵
24. Number is conceptualized as recognition of quantity (innate subitizing—evolutionary, anthropological, neuropsychological view—the number starter perspective—Butterworth, Dehaene, Sharma, etc.) and measurement (cultural contribution—counting and measurement tools, Davydev, Sharma, etc.) ↵
25. The term ‘counting’ here applies to any counting—fingers, objects, number line, mental line, etc. ↵
26. For effective learning and retrieval of information, the information should be placed in several files in the long-term memory to make connections, intentional and unintentional, and to make learning and retrieval easier. ↵

7 (Part Two) Understanding Additive Reasoning

Its Language, Concept, Procedures: Addition

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A. Difficulties and Misconceptions in Additive Reasoning

Figuring out a word problem involving perimeter, making change, or finding the total number of points scored by a team in three games with scores of 23, 15, and 12 all involve additive reasoning. All these tasks and more draw on an essential foundation of mathematics knowledge—*additive reasoning*. Everything else a child does in mathematics in primary school and beyond relies on his/her ability to use the knowledge of addition and subtraction.

Difficulties children face in additive reasoning are focused on three important pieces of early mathematics instruction:

- a. Learning addition and subtraction facts with fluency,
- b. Place value of multi-digit numbers,
- c. Applying fact fluency in problem-solving, particularly word problem-solving.

From our perspective, these are not separate problems; as we will see in this and coming chapters, they are intricately related.

Learning addition and subtraction facts is a ‘multi-year progression and program of action’ from kindergarten to second grade. The end goal of math fact practice is for students to be able to recall them ‘automatically,’ or calculate them ‘within a reasonable time—a second or two and apply them in learning more mathematics, in other disciplines and outside of school, in real-life situations.’ All students, with and without learning problems in mathematics, should reach this goal by the end of second grade.

We have outlined in this book a three-year progression of learning and mastering addition and subtraction facts and procedures with effective, efficient, and elegant strategies.

In *Chapter Zero*, we identified the non-negotiable skills kindergarten children should learn as: how to do basic work with numbers—subitizing, decomposition/recomposition, 45 sight facts, making ten, teens numbers, and two-digit place value. Then, teachers introduce the idea that it is possible to join and separate numbers. From there, students learn strategies for joining and separating numbers in a more efficient way. For instance, students can use their knowledge of reaching 10 by adding to 9 as easier by pulling 1 from the other number to transform the problem into one where students are adding ten and the teens number. For example, students are adding $9 + 7 = ?$

Students ask: “How do I turn 9 into 10?”

Answer: “Take 1 from 7 to give to 9. Break the 7 into $1 + 6$ ”

The problem becomes: $9 + (1 + 6)$

That is transformed into: $9 + 7 = (9 + 1) + 6$.

That is further transformed into: $9 + 7 = 9 + 1 + 6$.

Student: “I know $9 + 1 = 10$.”

The problem is transformed into $10 + 6 = 16$. So, $9 + 7 = 16$.

Then, fluently recalling these facts becomes the goal. The reason we are doing so is that the practice of remembering these strategies is actually what is going to strengthen a child’s memory over time. Math facts

should be taught as part of an ‘interconnected network’ of number relationships. Automaticity with math facts is important, but students should have one or more strategies to help with understanding the ‘why’ behind them.

It is not surprising to see students who find the fact: $8 + 5$, (of course, by counting) and then when we ask: ‘what is $8 + 6$?’ They will say: ‘I do not know what $8 + 6$ is?’ Another student might say: ‘I know.’ He proceeds to count 5 after 8 and gives you the answer. And they count again when you ask them: ‘how to find what $8 + 6$ is.’ Our instruction should be such that they see the relationships easily and without prompts. We want children to notice and feel secure that if they know $8 + 5$, they automatically also know $8 + 6$, $8 + 7$, or $18 + 5$, $18 + 6$, or, $18 + 7$.

This is one reason why we believe calculators can’t replace practice with mastery and understanding of math facts; if children are relying on calculators and addition charts, they are not developing this relational knowledge. When students are fluent in math facts, it becomes part of an interconnected network. However, an interface between humans and technology in this process has promise. But we are not there yet.

However, ‘bombarding’ children with failure when they constantly practice ‘mad minutes,’ ‘1-minute fact checkup,’ ‘3-minute speed test,’ endless practice on apps, and other such rituals are not learning situations. They are testing situations; they are regularly occurring stressful events for a child. They are anxiety-producing situations. The chronic effect of these is ‘math anxiety,’ ‘math trauma,’ and ‘mathophobia.’ They create avoidance of mathematics from an early age.

Arithmetic facts practice should be done first orally with immediate constructive and strategic feedback on errors and immediate self-corrections. When a child has mastered them orally, then one should practice them in written form. One should only formally assess his/her students on math fact recall once a week. The teacher should also intersperse facts that students already can recall easily with ones that they are trying to memorize. You don’t want to bombard them with ‘I don’t know, I don’t know, I don’t know.’ You want to build a lot of success in there. Children are only developing automaticity with a few facts at a time with an emphasis on strategies. Do not practice all the facts at one time. It is important that students have mastery of strategies, like making 10, to compute the facts that they can use if they can’t recall the answer automatically. In the following pages, we provide systematic strategies for fact mastery.

1. What Struggles Do Students Have with Additive Reasoning?

Children can have a wide variety of struggles with addition and subtraction. They present certain specific behaviors that can be considered symptoms and behavioral markers that students with developmental dyscalculia, innate dyscalculia, acquired dyscalculia, and other numeracy difficulties show on most assessments and evaluations. Some of the most common include:¹

- *An unclear sense of magnitude*—the size of the collection it represents; the idea that cardinal number is the property of the collection; an inability to locate the number on a number line or create a number on an Empty Number Line in relation to other numbers—in size, in distance, and relative position.
- Thinking and *using numbers as a collection of ones* without thinking of numbers as a group or a collection of groups—for example, thinking of 14 as one number (with the awareness that it has a place on the number line rather than locating it by counting from 1s or being aware of 14 as $10 + 4$, one 10 and 4 rather than as 14 ones or a group of ten ones and four ones.
- *Lack of understanding of place-value*: Difficulty in regrouping and place value: They think 45 is made up of two ‘numbers 4 and 5’ rather than seeing $45 = 40 + 5$ (*Canonical decomposition*), and also as *non-canonical decomposition*: $45 = 40 + 5 = 30 + 15 = 20 + 25 = 10 + 35$.
- *Lack of Facts Mastery*: Addition and subtraction facts up to 20 are not mastered with fluency, understanding, and strategies.
- *Poor transitions* (backward and forward) to and from decade, century, etc. finding $N \pm 1$, where N is a multiple of 10, 100, 1,000, etc.

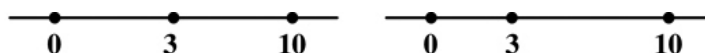
An unclear sense of magnitude means that students are not really sure how much greater one number is from another, in what sense, and why. They are not sure of the order between numbers and unsure about their location on the number line. For example, when asked to put a '3' on an Empty Number Line between 0 and 10, many students will put 3 in the middle. That is why helping children to construct a number line is an important activity in kindergarten as well as the construction of an empty number line in the first grade.

Or when we ask: what is 2 less than 10 and the child counts up to 10 and then counts two less, not knowing the importance of key numbers: 0, 1, 5, 15, 20, etc. They should know the tens' numbers or the next tens to a given number, say 63 (it is 70 or the nearest ten to 63 (60)). Knowing the location of numbers on the number line in relation to these numbers is very important.

When these issues about misunderstanding about place value are not dealt with properly at the primary level, then later on, they will experience the same types of difficulties as adolescents in the context of fractions and decimals. Many older students experience difficulty with magnitude and the real meaning of place value—that each place value to the left increases from one place to the other by a multiple of 10 and to the right it becomes a multiple of ($\div$ 10). When a group of students in the seventh grade was told that a million seconds is $11\frac{1}{2}$ days, and asked how long a billion seconds would be, many students gave answers that were small multiples of 11, like three or four months. They do not have a sense of how much bigger 'a thousand times' is.

Another common difficulty that young students have is not recognizing the groups of ten that make up place value. Students may conceptualize 18 as a group of ten ones and 8 ones or they may think of '18' as a symbol only, and that means to them 18 is 18 individual units/ones. When reasoning additively, students must learn to create groups of 10 whenever they can.

They also need to make the connection between the number in the tens' place, and its meaning as the 'number of groups of ten.' They should be clear about: (a) 18 is made up of two digits: 1 and 8; (b) 18 is made of two numbers 10 and 8 or 1 group of 10 and a group of 8 ones; the digit 1 is in the tens' place and it represents a 10 or 1 ten and 8 is in the ones place and its value is 8 or 8 ones. They should also know that 300 is 3 hundreds; or 30 tens; and 300 ones. One more hundred will affect the digit in the hundreds place only, unless there is a 9 in the hundreds place. They can give correct answer to these questions without writing these numbers.



ADDITIVE REASONING

Figure 7.1 Misconception about relative position

Most of these ideas are best achieved during the initial instruction if they are emphasized with both the discrete and continuous nature of number and if the teacher uses both discrete and continuous models that are effective instructional tools: Visual Cluster Number cards, and a hybrid of Cuisenaire rods and the BaseTen blocks, an Invicta Balance, and the Empty Number Line. The teacher should use the ten-rod from the Cuisenaire rods and not use the ten-rod from the BaseTen blocks, as the BaseTen block has ten markers on the ten-rod and students keep thinking of it as 10 ones. Similarly, the digit 8 in the ones place, using unit/one's blocks is represented by 8 unit/one's blocks, and students keep thinking of them as 8 distinct units/ones. In the Cuisenaire rods, the digit 8 in the ones place is represented by the 8-rod and therefore, the digit 8 in the ones place is conceptualized as a group of 8 ones. The 8-number rod is seen as one number: 8. This advancement of conceptualizing 18 as the combination of two groups—1 group of ten and a group of 8 is a big jump in thinking of numbers as groups.

The following *script* helps achieve this level of understanding. (Kindergarten activity.)

(Children have been making teens' numbers for some time and the teacher has introduced the two places—tens place and ones place in a teens' number.)

At present, each child has a representation of number 18 by the help of Cuisenaire rods displayed on their desks. The teacher asks one of the children to bring her a representation of number 18 by Cuisenaire rods and asks her to place the rods under the document camera. She points to her representation of 18 under the document camera and asks her to write 18 under the rods (the digit 1 under the 10-rod and digit 8 under the 8-rod). She asks the child:

Teacher: What number is represented by the rods?

Child: 18.

Teacher: What digits represent the number 18?

Child: 1 and 8.

Teacher: The orange rod is in what place?

Child: In the tens place.

Teacher: What is the digit in the tens place?

Child: 1.

Teacher: What is the value of this digit?

Child: 10. It is a group of 10.

Teacher: What is the digit in the ones place?

Child: 8.

Teacher: What is the value of this digit?

Child: 8. It is a group of 8 ones.

The teacher writes the number 18 on the board and asks the child to read the number.

Teacher: What is this number?

Child: 18.

Teacher: What two digits make the number 18?

Child: 1 and 8.



Figure 7.2 Teens number using number rods

Teacher: What two numbers make 18?

Child: 10 and 8.

Teacher: What is the value of the digit 1?

Child: 10. One group of 10.

Teacher: What is the value of the digit 8?

Child: 8. Eight ones

Teacher: Great! What will happen if you reverse the places of the digits?

Child: It will be the same number.

Teacher: Everybody reverse the digits of this number and look at the number line on the wall and see if it is at the same place on the number line.

Children examine the number line and several hands are raised. The teacher points to one child and asks the child.

Child 1: It is a much bigger number. It is far away on the number line. He points to the number. Every child is allowed to touch the numbers 18 and 81 on the wall.

Child 2: It is 81. Eighty and one. I can make it.

The teacher invites the child to the front of the room. She asks all the children's pairs to make both numbers (18 and 81) by using Number rods. She examines all samples of the numbers and asks one of the children to make both numbers on the floor as two trains (two number line segments) with the starting points of the line segments as the same starting point. She asks the children to make another pair of numbers.

For the purpose of differentiation and for formative assessment, she asks for different two-digit numbers from each child. For children who are at a higher level of understanding of the number concept, she asks for any two-digit or a three-digit number, not limiting to just teens' numbers. For children who make a mistake, she helps them correct them and she provides immediate feedback. She now forms ad-hoc pairs of children to practice this exercise by asking these questions in turn. She moves around to check and help. This supervised practice is then augmented by practice in pairs and then individual practice.

Differentiation is not just teaching children in smaller groups and instructing them in these small groups or individuals all the time. Differentiation is meeting the needs of all children in the classroom. Special cases need augmentation of instruction by the specialist or the classroom teacher in addition to this instruction. Asking oral questions at different levels gives the chance for children who are just learning the concept to have the opportunity of observing examples of expert behavior—the language, concepts, and their actions in using the tools. When children see expert behavior, they try to mimic that behavior and want to reach that level. That is one reason why teaching should take place at a whole-class level first, but the practice (individual and small groups) should be in small groups and individual levels to shape their individual behaviors.

Ongoing formative assessment is the backbone of teaching. It is to assess the child's ongoing learning and to gather information about individual children (making mental notes) and giving immediate feedback. It is also to assess one's own teaching and a cause to reflect on one's planning for the next step, immediately as well as in the future. It helps the child and the teacher to move from one level to another, and it should also inform the teacher to change her instruction on the go to help children and improve both the child's learning and her instruction.

B. The Role of 'Counting Based' Methods in Learning Facts

The history of teaching number relations is long and varied. Every culture, society, and country develops their unique approaches and tools for children to master arithmetic facts, particularly addition and multiplication facts. However, the problem of children not being able to master the arithmetic facts is universal, particularly reaching fluency and learning them in meaningful ways. This situation has reached acute levels in the case of learning multiplication tables. For the longest time in history, a mastery of numeracy was limited to a few people in most societies. In today's global technological society, having a numbersense has distinct advantages, more now than ever. Numbersense, in spite of all the technology available to us, is helpful in every arena of a person's life—from personal, to professional, to civic, to social. And 'meaningful' numbersense makes a person even more powerful with technology.

Reaching a higher level of individual and collective numbersense in a society is a function of *the realization of its importance* and *the methods used for teaching it in homes and in schools*. Let us examine some of the approaches and tools to teach arithmetic facts.

1. Is Finger Counting to Acquiring Facts Efficient?

Most teachers have experienced the following situation: you ask a child a simple question involving an addition or subtraction fact. You look at the child. There does not seem to be any reaction on the child's face; s/he simply stares at you or off into space. You repeat the question; you change the language of the question; you explain the relevant information to the question again. You are in the midst of this process and all of a sudden the child blurts out the answer, most of the time a correct answer. Naturally, you wonder what was happening and what had happened. The answer is evident. During all this time that you were explaining the question and reassigning the question, the child was busy calculating the answer on his fingers behind his back or under the table or mentally counting on fingers or an imaginary number line. The thinking is visualized, but the wrong thing is being visualized. Because the emphasis on 'counting forward' to find the sum of two numbers is one of the major strategies many teachers and textbooks employ, children as a result begin to use fingers as one of the concrete and easy models for finding addition facts. Using fingers is a natural and an evolutionary process. Our BaseTen system is based on human beings having ten fingers. They are quite 'handy.' As a matter of fact, one of the components of Gerstmann syndrome,² acalculia, and/or dyscalculia is awareness of one's fingers—the subject's ability to recognize the place on a finger touched by the diagnostician when the subject's eyes are closed.

We do not have to tell children to use their fingers to count. It is natural for them to do so. In the absence of efficient strategies, that strategy is their 'default' strategy. Children should use fingers for many things in their learning; they are essential for myriad activities—counting, playing, building, constructing, etc. But in the case of numeracy learning, children should know finger and number line counting, but need to transcend the counting process to derive and master arithmetic facts. In addition to finger counting, beyond counting, they must learn more efficient strategies. The type of strategy used by a child determines which part and/or parts of the human cortex will be used for its processing and storage and how easy it will be to use learn, store, and recall learning.

Some parents and teachers even encourage them to use their fingers: 'You will always have your fingers with you.' Unfortunately, they continue to use finger counting well beyond the time that the skill has served its purpose and even when they know that these models are inappropriate and inefficient. Most children who use counting on fingers as the only method for learning arithmetic facts never really reach fluency levels.

In many classrooms, the only means of helping learners to automatize arithmetic facts might be either flash cards or some variations (app, repetitive tasks, etc.) of them. These materials and teaching strategies encourage children, particularly those who still need concrete examples to conceptualize number, to use their fingers to find the results of simple addition and subtraction problems. They find counting on their fingers quite convenient and dependable. It is a strategy of default. It is not wrong, but it is quite inefficient and not developmental to use fingers to count and arrive at arithmetic facts.³ The issue here is not whether the use of fingers is appropriate or not; instead, we need to ask:

What is the level of mastery achieved in arithmetic facts by the student with the use of fingers?

When using fingers to count, the child is just constructing the fact by counting, again and again, whenever he needs that result. It shows that the child does not have any strategy other than counting to store the information or recall. The result of this process is that he has to always count. It always takes the same 'longer time' to get the answer. There is very little retention of the outcome, and very little progress toward numbersense is made, as relations are not formed, nor are patterns amongst numbers and their outcomes observed. The child has not automatized the corresponding addition or subtraction facts.

Children need the use of appropriate⁴ strategies with concrete, manipulative materials to help conceptualize number concept, see relations amongst numbers, observe patterns, and master mathematical ideas. To achieve these aims, they need to be exposed to and helped to master efficient strategies,⁵ and have enough and meaningful practice in the use of a particular strategy in problem situations.

Mastering addition facts means a student can give the fact in two seconds or less (orally) and three seconds or less (in writing), using at least one strategy other than counting and can demonstrate an understanding of the concept and process. Any kind of counting, whether that counting is on a number line, on fingers, hash-marks, or random objects such as blocks, pennies, etc. is not an efficient and generalizable strategy.

The use of fingers is not necessarily a cause for the child's numeracy problems; it may only be one of the symptoms of such a problem. Moreover, using a finger to represent a number is a poor use of fingers. As soon as one has counted up to ten, one has run out of fingers and therefore has difficulty in keeping the tally if one has to continue the count. Counting on fingers places much heavier demands on the working memory.

For example, notice the difficulties encountered in the process of adding by the counting process:

- a. In the process of deriving the sum $9 + 8$ by counting, a child needs to *keep track of 16 numbers* in his working memory (8 to count as a tally to keep track that one has counted the required number 8, and 8 to count the numbers beyond 9). This puts a heavy cognitive load on the child's working memory. Working memory capacity is limited. It can usually hold at any time 7 ± 2 'chunks' of information.
- b. The process of counting encourage children to focus on *each intermediate number*. The child begins with 9 and proceeds to count 8 more numbers after 9 and says (vocally or sub-vocally): '10, 11, 12, 13, 14, 15, 16, and 17.' And then counts the counted fingers to make sure, he did count 8 fingers. Then he declares, '9 plus 8 is 17.' Notice, along with the final number as the answer, the child also uttered all the intermediate numbers between 9 and 17 during counting. All these intermediate numbers get connected with the problem and the answer. In this way, there is nothing inherently important or special about the number 17; moreover, in this process, the outcome 17 or the fact: $9 + 8 = 17$ do not get transferred to long-term memory with any specific characteristic. There is no distinguishing feature in the final outcome of the number and the intermediate numbers counted.

The working memory in such a situation is overloaded and none of this information is being transferred to long-term memory. As a result, every time you ask a child to find the sum of 9 and 8, he counts, once again. There is no residue of this counting in the long-term memory as these strategies/methods are neither '*storage-rich*' nor '*retrieval-rich*.'⁶ It taxes the working memory to the extent that because of the limited capacity of the working memory, it is completely occupied by the counting process. As a result, they do not have space left in the working memory to *observe, discern and extend patterns, seek relationships amongst numbers, develop strategies, or even apply strategies*. This also precludes space and inclination for higher-order thinking.

There is nothing inherently wrong with counting or counting strategies. They work for some children. Most average and above-average children discover and derive efficient and effective strategies from this counting themselves. Many of them eventually transcend counting. However, many other children are not able to discover efficient strategies by themselves because they become bogged down by the counting and do not develop the ability to see patterns and groupings in numbers, particularly the concept of decomposition/recomposition—the foundation of mathematics. Thus the counting and accompanied verbalizing of the counting prevents them from focusing on the answer and discovering strategies. The verbalizing of the count of numbers is not using language effectively and richly. This delays understanding, accuracy, applications, and automatization. For most children adding as 'counting up' and subtraction is 'counting down' are ineffective and inefficient strategies. Counting is a limited strategy and is not likely to develop children's understanding of numbers, nor their knowledge and recall of basic facts.

At the same time, many children with sequential problems find counting and keeping track of this count to be much more difficult. They either *miscount, overcount, or count all numbers*, including the *first number*. That results in many more errors. Analysis of errors of results shows that many incorrect answers on tests and examinations are either one less or one more than the desired answer.

2. What About the Use of the Number Line?

The sequential counting up and down on a number line, just like counting objects also eventually promotes finger reckoning because this counting is transferable to fingers automatically. Thus, an over-reliance on finger counting or counting on a number line precludes learning efficient strategies in most cases. After a while, children resist learning new and efficient strategies. They do not learn the major pre-skills: 'decomposition/recomposition,' 'making ten,' 'teens' numbers, to learn addition, subtraction, or multiplication and division. Even on the number line, during counting, they do not pay attention to important 'milestone' numbers: 0, 5, 10, 15, 20, etc. Just as in reading, we learn strategies to blend sounds using *phonetic*, *morphological*, and *language principles*; there are several effective strategies for finding the sums of two numbers, such as making tens, doubles, and near doubles, missing double, near ten, etc. But they do not become part of their repertoire easily. Those children, who are not dependent on counting, learn these strategies early on.

The problem is not just that the efficient strategies are not taught. In most of our classrooms, instructional materials such as number line, abacus, fingers, Unifix cubes (used as counting blocks), etc. are used only for sequential counting. And sometimes teachers may not even use any concrete instructional materials in introducing arithmetic concepts, let alone use appropriate and efficient materials for teaching them effectively ([Peters et al., 2015](#)).

3. How Can Fingers Be Used Efficiently?

Using fingers could be made an efficient activity if one decides that it is an effective way of counting. It is also true that in some cases, particularly in the case of those children who are kinesthetically oriented, the use of fingers may be one of the ways of learning arithmetic. There are excellent methods that use fingers effectively for counting. However, these methods are *culture-specific*. In several Asian countries, fingers are used in very creative ways for calculations and counting. The most popular method for finger use in calculation is *Chisonbop* or *Fingermath*.⁷ In this method, specific values are assigned to fingers on both hands and they are used according to predetermined rules for calculations.

Several variations of these techniques are present in almost all Asian countries. They are successful methods because they are cultural tools and children see their usage outside of school. For example, I (Sharma) still use my fingers when I want to keep count in certain situations. In Hindu temples, we do *parikrama*, going around the inner temple 108 times or more and we keep tallies on our fingers without any error. In Asian countries, they say their prayers and keep their count on their fingers. And in some countries, there are cultural-specific instruments that are fashioned around fingers and place value. For example, the *abacus* is a very effective and efficient tool in China, Japan, and South Korea; it is present everywhere; children use it in such ways that their numbersense gets stronger. But in Western countries, it does not make sense as it is not found outside of schools.

Even in these settings, the use of fingers is not a very efficient method. We should provide effective, efficient, and elegant strategies, independent of counting, from the very beginning so that all children can achieve the automatization of arithmetic facts with understanding and fluency.

When children are provided with appropriate concrete manipulative experiences and effective methods for their use, they learn numeracy much more quickly and effectively and they reach a level of context, materials, and culture-free methods that are effective, efficient, and elegant. If a child or a parent brings a cultural tool or method to the classroom that is efficient, effective, and is an elegant tool for mathematics, it should be welcomed, studied, adopted and practiced, in addition to other efficient methods, as this will improve the flexibility of the thinking processes.

As a result of effective concrete experiences, children should realize that there are better strategies and they usually drop the use of counting on fingers automatically. Force and nagging do not eliminate the habit of calculating using one's fingers. The longer children are kept from more effective methods and strategies, the more difficult it becomes for them to move away from using their fingers.

With older children who are still using finger counting for deriving facts in middle and high school mathematics classes, with the help of manipulative materials such as Cuisenaire rods, Visual Cluster Number

cards, and Empty Number Line, we have been able to help these students to drop the use of fingers for calculations and acquire efficient strategies.

For example, when the concept of integers⁸ and operations on integers are introduced in the sixth and seventh grades, it is a wonderful opportunity to introduce efficient strategies for operations on integers that will automatically fill the arithmetic gaps. In such situations, we have been very successful in helping them acquire numbersense and develop efficient strategies and models for calculations, once again with the same methods we used with younger students. Methods are so simple and focus on a current topic that they are willing to learn them in the interest of doing well on integer operations at that time. The success with operations on integers and related numbersense gives them a renewed motivation to improve their numbersense further. Success is the greatest motivator for all students, but more so for older students. In the case of older students, one should be persistent in helping them change their inefficient habits and inculcate thinking strategies that pay generous dividends by relating them to operation on integers and algebraic expressions. This also strengthens their understanding of the order of operations. We have found that they carry their success to algebra and other higher mathematics concepts.

4. What About *Rek-N-Rek*, *Five-Frame*, *Ten-Frame*, and *Numicon*?

This family of instructional materials is visual organization and representations of hands as concrete or semi-concrete number organization. A *Five-Frame* is literally a hand represented. Similarly, a *Ten-Frame* is a representation of two hands. *Rek-N-Rek* and *Numicon* are slight variations of the two hands. Because of the emphasis on grouping of 5 and 10, they are much better than one-to-one counting of objects, on a number line, or on fingers. As a matter of fact, they help a child to realize the grouping of '5.' And because of visual representation of objects in two rows, there is a good possibility that with proper use of the language of subitizing, a child will be able to see some elements of the decomposition/recomposition process. In a sense, they are good beginning instruments combining several of the fundamental elements of number concept: Counting, grouping by five, and some decomposition/recomposition. However, the sight facts, decomposition/recomposition, making ten, and teens numbers are not learned as efficiently by the help of these materials. These instructional materials are not as versatile in the development of addition and subtraction strategies as: *Visual Cluster Number cards*, Cuisenaire rods, *Number Balance*, and *Empty Number line*. For their role and location in the development of instructional materials, see the *Continuum of Instructional Materials for Number Concept and Additive Reasoning* in the chapter on tools.

Earlier, we provided children with effective strategies for addition and efficient tools and models earlier and more easily, so that they will learn and master arithmetic facts and procedures better. They can then develop higher-order thinking and can learn higher mathematics concepts, such as fractions, decimals, integers, and other pre-algebraic concepts and procedures, more easily. They experience success and therefore positive attitudes toward mathematics are formed earlier. We have almost three years (kindergarten, first grade, and second grade—a total of 540 days in American schools and much more in European and Asian schools) to reach the goal of mastering additive reasoning. We believe, with proper planning and effective strategies, meaningful practice, and applications, this amount of time for instruction is more than adequate. If we miss this period, it becomes much more difficult for a child to achieve the same level of mastery.

Prolonged work and practice with problems involving discrete objects or word problems related to or describing discrete objects will keep children on counting processes for a long time, beyond the time it is natural for children to master addition and subtraction facts. This is an opportunity loss—we did not use this important time in the child's life effectively. It is also poor ROI (return on investment of time). When this time is lost, it is difficult to regain this time. After this period it becomes, if not impossible, much more difficult to learn the facts for reasons such as:

Apart from that there is an optimal time for learning fundamental skills of reading and specific skills in numeracy. When this time is missed, extra difficulties are posed in learning the same skills such as:

- a. Children learn much more from watching and observing age-mates and less from us, including special educators.
- b. Learning and practicing something that you have failed earlier creates mental blocks, stigma, and feelings of inadequacy in a child, and therefore improving the quality of initial teaching is very important.
- c. The amount of dedicated time to learn arithmetic facts the first time around is much more productive as the focus of learning, at that time, are these concepts only and therefore focused learning is easier. Children's first-learning experiences have an enduring and shaping influence on their attitudes toward future learning.
- d. The age-level expert behavior has much more motivational impact on a child's learning than when they know that they have some kind of inadequacy.
- e. Repetition of inefficient methods has turned students off to learning arithmetic facts; later, because of failures, the child's attitude toward mathematics also works against the child; motivation to learn them may not be present.
- f. Exclusive time to learn them may not be available due to the pressures of more, newer, and higher concepts; the child has many more extracurricular demands on his time and intellect that prevents them from achieving success at the same rate.
- g. Teachers' and interventionists' eagerness to introduce grade-level content and to support that, they end up recommending the use of calculators, addition, or multiplication tables/charts as a reference to make up the deficits of arithmetic facts. Or they may provide 'tricks' and 'short-cuts' for specific situations that are not generalizable.

As they say, there is always a right time for things to be done. That time is now! The age from $3\frac{1}{2}$ – 4 to 9-10 years is the right age for mastering arithmetic facts—addition, subtraction, multiplication, and division. After that, one uses these skills to learn more mathematics, applying the knowledge to learn other subjects, and solve problems, at a variety of levels.

There is one more very important reason for examining this pedagogy of learning arithmetic facts, based on recent neurological research related to addition and subtraction on one hand, and multiplication and division on the other. [Prado et al. \(2011\)](#) proposed a *neural recycling hypothesis* that states that an important part of the brain mechanisms in the *intraparietal sulcus (IPS)*, originally involved in nonverbal magnitude comparison (the 'approximate number system'), are recycled for symbolic arithmetic. Thus, recent cultural inventions such symbolic arithmetic recycle evolutionary older neural mechanisms ([Prado et al., 2011](#)) that were used for other purposes before, particularly for survival and assuring futures.

[Dehaene \(2020\)](#) also mentions that original functions of certain parts of the brain are 'hacked' to perform newer functions, such as reading and doing arithmetic. For example, the initial function of this evolutionary older mechanism was likely related to the perception and nonverbally quantifying of objects (numerosity), a capacity observed in both humans and many non-human animals, essential for survival functions and looking for the future, like foraging and social interactions.

The *neural recycling hypothesis* posits that the degree of this recycling depends on the similarity between this original function and the novel task being performed now. In this hypothesis⁹ a central assumption is that the degree to which a pre-existing brain mechanism is recycled depends upon the degree of similarity between its initial function and the new functions and skills, like reading a face or the mark of a predators' foot (old skill) and reading a word, a number, and a figure (new skills).

[Prado et al. \(2011\)](#) tested this by comparing brain activity during two different arithmetic operations:

- *Subtraction problems*, which inherently involve numbersense (numerical comparison) showed greater activity in the IPS, demonstrating a higher degree of recycling of the magnitude comparison system. And by extension, addition problems as addition and subtraction are inverse activities and they can be performed by either process.
- *Multiplication problems*, which rely more on retrieving verbal facts from memory, elicited greater activity in language-related regions (such as the middle temporal gyrus and inferior frontal gyrus), and less in the

IPS, illustrating how different types of arithmetic recruit different recycled systems based on functional similarity.

The researchers ([Prado et al., 2011](#)) found that the brain region involved and localized for magnitude (numerosity comparison, perceptually identifying the size of a collection—subitizing, ANS, comparison of two collections, etc.) is in the intra-parietal sulcus (IPS)—place for a set of evolutionary skills is recruited to a greater degree by arithmetic problems that involve number comparison (e.g., single-digit subtractions) than by problems that involve retrieving facts from memory (single-digit multiplications). Results confirmed that subtractions are associated with greater activity in the IPS than multiplications, whereas multiplications elicit greater activity than subtractions in regions involved in verbal processing including the middle temporal gyrus and inferior frontal gyrus that were localized by a phonological processing task.

Pattern analyses further indicated that the neural mechanisms more active for subtraction than multiplication in the IPS overlap with those involved in numerosity comparison, and that the strength of this overlap predicts inter-individual performance in the subtraction task. These findings provide novel evidence that elementary arithmetic relies on the co-option of evolutionary older neural circuits. *Therefore, one can theorize that teaching strategies that are based on subitizing, comparisons of discrete and continuous quantities, decomposition/recomposition and groupings are better methods of learning addition and subtraction than the counting processes as they are more focused on language processing and heavily dependent on working memory functions.* As we discussed in the chapters on CPVA/S and the tools of teaching numeracy that children learn easier and faster when they tap the evolutionary skills and the cultural tools and integrate them. In this chapter we capitalize on these components and develop strategies for addition and subtraction—in the development of language, constructing conceptual schemas, and deriving strategies and procedures.

5. What About Apps and Other Approaches to Mastering Addition?

We want children to start to notice that ... $8 + 5$ is almost $8 + 6$, which is almost $8 + 7$ and they can extend their knowledge $8 + 5$ to deriving $8 + 6$ and others based on what they already know. Similarly, $8 + 5$ has something to do with $88 + 5$. That's the kind of space we want them to be in. They should begin to see interrelationships. This is one reason why calculators and apps cannot replace prowess with math facts learned by seeing relationships. If children are relying on calculators and apps, they are not developing this relational knowledge. When students are fluent in math facts, it becomes part of an interconnected network. This interconnected network idea is what develops a mathematical way of thinking in children.

C. What does 'Mastery' of Arithmetic Facts Mean?

The word 'mastery' is fraught with historical baggage. Everyone has a definition and meaning of this word. In spite of this, we want to venture into defining and describing it.

- What does the State of Mastery Look Like?
- What Are the Behavioral Markers of Mastery?

Let us begin with a non-controversial situation. We call a child a reader when he can read the reading material fluently with comprehension and confidence. It is the goal of all reading instruction programs. It should be clearly evident when the child reads and can answer a variety of questions about the reading material. All component skills that need to be mastered are visible; they are evident. Even an average person can pass judgment on whether a child is a reader or not. With practice and introduction to the variety of reading materials, the child uses this new skill to learn other new materials, and ultimately reads for learning and problem solving and for his own pleasure. Even the illiterate villagers of my village would say when they saw their child reading: 'Please read such and such page of such and such scripture.' The child would read and they would pass judgment because they knew that passage by heart, and the child did read those words and then they

would ask some questions and when satisfied, they would say: 'I am so glad my grandson is such a good reader.' Mastery is always visible!

When reading skill is present: there is an action being performed to the standard defined, which is a priority. When a child reads at that standard—the action of reading, the reading behavior, and the outcome—the comprehension can also be made clearly visible by asking questions to ascertain that. And even the reader knows that she is a reader. It is evident to everyone, *both to self and others*. I (Sharma) always ask a child when they have completed a multiplication table in certain time: 'Are you satisfied with your performance?' The child invariably says: 'Yes! I need to work on my speed.' This is when the child has given the answer quite fluently.

Should we not expect the same kind of action, behavior, and outcomes as we do with reading skills about fact mastery in numeracy? The answer should be obvious. Similar to reading behavior, a child has mastery of arithmetic facts when he can give the fact fluently, and when asked can provide a strategy based on sound principles of number and number operations, for example, decomposition/recomposition, other than counting, and can apply the strategy to extend these facts when needed.

Mastery also includes that the student feels confident and competent in her methods of acquiring the knowledge and the skills. This requires sound instruction and development of skills with rigor both in instruction and in the child's responses. The components of rigor that produce fluency and understanding are: the language of the concept under discussion, understanding of the appropriate conceptual schemas related to the concept, strategies, accuracy, fluency, and applicability.

- *Language of the Concept and Understanding*: Can read the problem (on addition and subtraction) even when given in multiple ways and forms (with varying vocabulary, syntax, context, and can formulate number story problems for that fact or equation); knows the language of additive reasoning, definitions, terms, vocabulary, syntax (e.g., knows the difference between '7 subtracted from 9' and '9 subtract 7'; knows that 2 more than 8 is $8 + 2$ and 8 more than 2 is $2 + 8$ and also knows that they are same because of the commutative property of addition; also knows that addition and subtraction are inverse operations.
- *Effective, Efficient and Elegant Strategies*: Can provide strategies based on properties of number (e.g., decomposition/recomposition of number, sight facts, even/odd numbers, making tens, teens' numbers, the role of key numbers), the properties of operations (e.g., commutative and associative properties of addition), and knows that subtraction is not commutative; the sum of two even numbers or two odd numbers is even and adding an odd and even number results in an odd number.
- *Accuracy*: Can derive the fact correctly and consistently using efficient strategy(ies). Accuracy is not just giving a correct answer as a reflex action. It is that the fact can be asked in different forms and the answer is forthcoming and if the child makes an error, she knows that this is not the correct answer and can correct it immediately.
- *Fluency*: After achieving accuracy orally and on paper, with appropriate, and meaningful practice, the child is able to answer a question about a fact (addition or subtraction) in a reasonable amount of time (e.g., on the average, a fact in 2 seconds orally and 3 seconds in writing; can derive a fact in more than one way, write a fact, identify a fact, locate a fact.
- *Applicability*: Can apply the knowledge of addition and subtraction in problem solving situations (e.g., money, time, measurement, and in real-life situations) and learning other mathematics concepts, and facts by applying the given fact (for example, $8 + 8$ is double number fact and if problem is $8 + 9$, then this is 1 more than $8 + 8$, so the answer is 17 as $8 + 8$ is 16; it is 1 less than $9 + 9$, so it is 1 less than 18; and it is close to $10 + 10$, 8 is 2 away from 10 and 9 is 1 away from 10, so $8 + 9$ is 3 away from 20).

1. What is Fluency?

The criteria for 'what is fluency' is determined by the culture—academic and social. For example, we believe an addition fact should be given in two seconds or less orally and three seconds or less in writing. However, at the same time, we should know first that fluency is not speed. Fluency can be compared with the flow of water

—smooth, continuous, effortless. In arithmetic, we seek physical, linguistic, conceptual, and procedural fluency at the conscious level and then at the unconscious level.

The conscious and unconscious fluency may not happen at the same time. First, it should be at the conscious level. That means I should know how to derive a fact with efficient means and with understanding of the process and the reasons for those steps. The more efficient the process of deriving a fact and the better the understanding, the shorter the time to achieve the acceptable level of fluency. Eventually, with practice and usage, the fluency should be achieved at the unconscious level. This is achieved progressively and with effective practice, with constant, creative, and positive feedback.

To achieve fluency, a student should have access to observing the behaviors, models, and examples of the demonstration of true fluency just like in cooking, music and sports. The pitch should be from ‘how can I do what they are doing to how do they do it.’ Then a knowledgeable adult helps the child plan the process of practicing. These observations make the criteria for fluency clear to the child.

Fluency is built and maintained by practice, but the practice should also include applications of those skills and facts. Building and achieving fluency is an ongoing process. Applications are one of the most important means of improving fluency and flexibility of thought.

Systematic and intermittent practice of efficient and effective strategies, with conceptual understanding of the connections between different components of the concept, with proper script integrating and binding the steps and concepts of the task and constant self and supported testing enhances the process of achieving fluency that we envision.

D. How to Master Basic Arithmetic Facts with Understanding and Fluency?

This is a major issue facing every primary school and every primary teacher as they work to help children become proficient learners in numeracy and mathematics. As a matter of fact, it is a national and even an international issue. Many children are having difficulty in learning mathematics because they lack mastery of arithmetic facts. There are several reasons for this problem:

One reason, which we have already identified in detail, is the choice of instructional materials and pedagogical approach—most teachers and textbook programs use counting materials to arrive at the addition facts by counting.

1. Choices for Introduction to Quantity, Number, and Addition

The other reason may be specific mathematics disabilities of children that may hinder a child’s acquiring mastery of arithmetic facts, but that number is small compared to the number due to poor teaching and poor expectations.

As mentioned earlier, most primary teachers introduce addition, subtraction, multiplication, and division concepts and facts using discrete counters, and they emphasize that children use them in counting and then memorize them by various means: rote memorization, flash cards, rhymes, songs, practice with apps, mad minutes, etc. This is a natural way to start, but when used exclusively and for prolonged times, it leaves many children having difficulty in moving beyond deriving number relationships by one-to-one counting (Gobel et al., 2014).

Unfortunately, in most cases, the use of only simple counting as a means of arriving at arithmetic facts does not work. The answer is not rote memorization with ‘flash cards’ or apps or computer programs that act as flash cards. The answer is *efficient strategies* and *proper practice*.

The efficiency of the methods can be categorized in terms of the models as follows:

- a. Rote memorization (the traditional flash cards, Mad Minutes, song and dance routines, etc.); repeated writing of facts; modern incarnation of rote memorization: apps, computer prompts, rote-inspired games and toys, etc.

- b. Rote memorization (flash cards, Mad Minutes, song and dance, etc.) written under the guidance of an adult: selective flash card practice, etc.
- c. Rote memorization with immediate feedback (orally)
- d. Practicing with strategies orally
- e. Practicing with strategies orally with immediate oral feedback¹⁰
- f. Practicing with CPVA/s strategies with fact ladders and with immediate feedback, with the recall of the script.

We perform addition and/or subtraction operations on collections of objects (on the cardinality of the set of discrete elements) and on continuous objects (some kind of measurement on two or more continuous objects). In the first case, combining the two collections produces a new collection, therefore a new number—the cardinality of the new set.

Example 1: There are pictures of two collections of cubes here. How many cubes are in each collection? How many cubes are there altogether?

Example 2: There are 6 green cubes in one circle and 7 red cubes in the second circle. How many cubes will be altogether if we combine the two groups of cubes?

In the first example, the child has to count the cubes individually in two collections to answer the first question. But they do not have to count the whole collection to find the answer to the second question in the problem. However, if the child does not have any other strategy other than counting to find the sum, she will necessarily count both collections together. In the second example, again, there is no need to find the answer by counting. However, if the child does not have any other strategy, the child will find the answer by counting. Now, let us consider the following example.

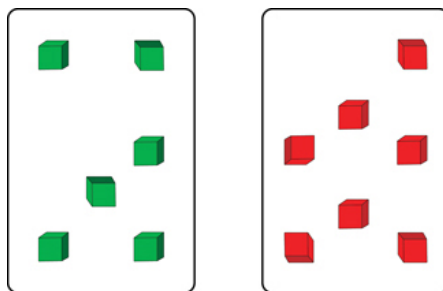


Figure 7.3 Combining two collections arranged randomly

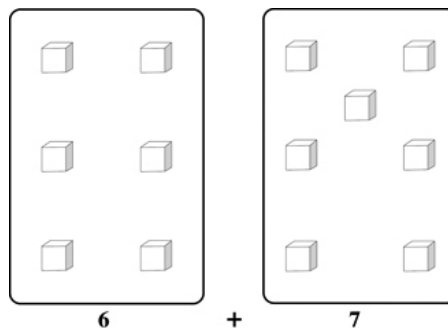


Figure 7.4 Combining organized collections



Figure 7.5 Demonstrating $6+7$ using Number blocks

Example 3: There are two colored rods in the picture. The dark green rod is 6 cm long and the black rod is 7 cm long. If we put the two rods together, end-to-end to make a train, how many centimeters long will the train be?

In this example, there is no need to count; one has to directly apply some kind of addition strategy. Children who remain on using counting make even the continuous examples (measurement) into a ‘counting all’ or a ‘counting on’ activity. That delays their mastery of facts. When we use inefficient methods of teaching addition and subtraction, they create achievement gaps. All of these factors become responsible for the gap increasing with passing time rather than decreasing. Thus, to achieve the goals of children mastering arithmetic facts, we need to:

- to bridge the gap between discrete (number represented by counting objects) and continuous (number represented by length, color, or a pattern) concrete models into strategies and to the use of mathematics symbols, and
- to develop strategies that are based on decomposition/recomposition of number and their properties and the relationships between numbers.

We suggest that this progression can easily be achieved by moving emphasis from discrete collections to organized collections (patterns—dice, die, Visual Cluster Cards, Sight Fact cards) to continuous/measurement models (Cuisenaire rods and Number Balance) to pictorial models (iconic representation of Cuisenaire rods to an Empty Number Line) to graphic representation (decomposition/recomposition diagram) to abstract representations (equations), and finally to representing it on a formal Number line.

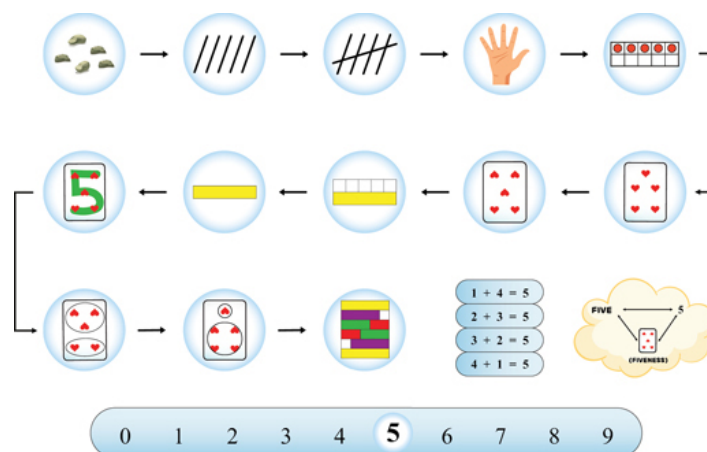


Figure 7.6 Trajectory of the number concept

(a) What Type of Differentiation or Individualization Is Most Effective?

There is a universal consensus for not segregating children with learning differences from the regular classroom instruction. The assumption in such a decision is that all children, with and without learning disabilities, can benefit from the current instructional methods and curricular offerings. That is far from true. It is a rare teacher who is able to make this happen. There are several pedagogical methodologies that have emerged from this decision (Clements et al., 2004).

Based on this information, there is an emergence of instructional strategies whose goal is to make the main lesson and its contents available and accessible to every student in the classroom and the techniques used for that purpose may be called *differentiation in instruction*.

In a class with students with homogenous background, ability, and preparation, many teachers believe that the need for differentiation of instruction is not warranted. That may not be true. We believe on the one hand that

every group is diverse from some cognitive perspective. And on the other, even if we start with quite a uniform and homogeneously functioning group, in a very short time, the group may become diverse if proper instruction and reinforcement are not provided and if every child is not reached and helped and held accountable for the work and participation in the instruction.

The process of differentiation and individualization is not just making subgroups in the class based on their academic levels or even randomly in a classroom and instructing them separately in different concepts and at different levels or the same concept to small groups at different times. It is almost like providing instruction at different grades or instructing them in the same topic at different times.

This type of differentiation does not have many advantages: teaching in small groups is always better in most learning situations, but because of the smallness of the group, the range of questions asked and answered is narrower and the depth and breadth of discussion is limited. But more importantly, the total amount of instruction time received by each student in the class during the academic year is a fraction of the instruction time needed to do justice to the topics, and all their components: introduction, development, and reinforcement of the concept. For example, the weekly time of 'x' hours allotted to a subject to deliver the curriculum makes the assumption that each child will get 'x' number of hours of instruction/learning time. When we divide the class into small groups (say 3 or 4) for instruction, then each student in the class will get only one-third or one-fourth of the total instructional time. That is not adequate by any standards to deliver the curriculum effectively and adequately. It is adequate only for a few children.

For a lesson to be effective and to do justice to the prescribed curriculum, each lesson should be divided into three components: Part I: Introduction/tool building (25% of the total lesson), Part II: Main Concept/procedure (50% of the total lesson), and Part III: Review/practice/small groups (based on needs and observations) (25% of the total time).

From the pedagogical perspective, the first two components: introduction/tool building and the development of concept/procedure must be presented to the whole class, but reinforcement and extra practice should be individualized or should be in small groups based on needs.

During the whole class time, the choice of tasks (for the introduction of the same concept), the nature and number of questions being asked, the type and range of discussions, and the level of difficulty of task or practice exercises varies. There should be different questions in difficulty and complexity for each child. The worksheets should have levels of tasks and should be offered to children according to their ability and understanding. However, they should be on the same topic, using the same language, same instructions, same diagrams, but at different levels—number, complexity. This variety is a positive contribution to the child's development. Children learn from other children—in their answers, they are exposed to a range of thinking and they see novice and expert behaviors.

Differentiation is not just working individually; it is meeting the individual needs. Differentiation is of three kinds: (a) deepening the understanding—solving a problem with multiple strategies, (b) broadening the understanding—making connections with other concepts horizontally, and (c) vertical acceleration—when a child has mastered the current concept with understanding, fluency, and applicability, then we should offer him/her problems according to his/her needs.

This arrangement does not preclude small groups. It means making smaller groups when there is need for extra practice and for deepening their understanding and making connections.

Therefore, the focus on selecting and applying strategies that promote differentiation and individualization should be a key component of any instruction. However, as mentioned earlier, any differentiation or individualization does not mean instruction based on the decision to working with small groups of children exclusively. That reduces the per capita time spent on instruction for each child to a small fraction of the time they deserve, need, and are allotted by the schedule. For example, if a class is to get 5 hours of mathematics per week, then each child should get 5 hours of quality learning time.

E. Preparation for Mastering Addition Facts

The goal of this section is to help teachers see the contours and trajectory of the lessons in teaching and helping students learn arithmetic facts, place value, and procedures with mastery (acquiring its language, an understanding of the concept, developing and acquiring conceptual schemas for the concepts, applying efficient strategies to derive them, reaching accuracy, responding with fluency and confidence, and engaging in applications of these concepts and skills in diverse forms and type—intra-mathematical (learning more mathematics), interdisciplinary (applying in other courses), and extracurricular (outside of classroom—in their games, toys, and day-to-day living)).

Combining/joining two numbers—the first ten natural numbers with the first ten natural numbers (placing in addition each symbol, forming this list with every other symbol once, e.g., $1 + 1$, $1 + 2$, $1 + 3$, etc.) results in one hundred basic addition/combinations facts. The goal is that each child learns and masters these 100 addition facts (the sums of whole numbers up to 20) by the end of first grade, if not earlier. Of course, some children will learn them earlier. Many of them will learn easily, and some will not acquire specific and effective strategies.

The following describes the sequence of 11 strategies that are based on the properties of number, operations, and decomposition/recomposition for the addition of two single-digit numbers (sums up to 20; 100 addition facts—the cells to be filled by addition facts on this 10×10 grid). These addition strategies are organized, developed, practiced, and mastered in effective ways to learn arithmetic facts easily, effectively, and efficiently, without counting. The results of each strategy are filled in the *previous table* by an indicative color for easy reference.

At our Center, working with children and adults with and without learning problems in mathematics including dyscalculia, professional development courses with teachers at universities and colleges, and in model class lessons in schools, we introduce these strategies using Visual Cluster cards, Cuisenaire rods, the Invicta Balance, and the Empty Number Line (ENL) before we move on to abstract/symbolic form.

With these instruments, these strategies are almost magical. Although these strategies are useful even when using counting materials, they are most effective, efficient, and generalizable if we use appropriate instructional tools, described throughout this book. Therefore, we suggest that a teacher model these strategies using Visual Cluster cards, Cuisenaire rods, and the Invicta Balance (in this order). Using these materials, anyone, with little training, can achieve the goal of arithmetic facts easily, effectively, efficiently, and with elegance.

There are some specific cognitive constructs that people forget while helping children develop arithmetic facts, particularly addition and subtraction facts. These are: Conservation of number and length, use of decomposition/recomposition of number/quantity, understanding the concept of equality (what does equal sign “=” mean?), the commutative property of addition, and most importantly, effective, efficient, and elegant tools for instruction. These concepts are fundamental to the development of effective processes and strategies for addition and subtraction.

For example, Visual Cluster cards and Cuisenaire rods extend, bypass, subsume, and abstract and symbolize the counting process and the conservation of number—a prerequisite skill for numbersense ([Piaget, 1952](#)). And the Number (Invicta) Balance makes the concept of equality so clear and therefore the concept of ‘*equation*’ is brought to the child’s attention much earlier than it is usually done. These instruments satisfy the requirements and characteristics of successful instructional and interventional materials. Children can find the answer and understand it without counting. These tools and strategies make the addition operation transparent: provide exact answers (effective) to addition problems quickly (efficiency) and can be used to extrapolate (elegance) the strategies to large numbers and other number systems.^{[11](#)}

These instruments make deriving and understanding the role of strategies in arithmetic operations more natural, and easier for all children, with or without disabilities. These strategies are, in the numeracy context, like the rules for blending sounds in the reading process. With the help of these strategies, children are able to acquire addition facts easily and can apply them to new situations and become proficient in deriving, mastering, and applying arithmetic facts. It is, however, important that children have a good command of number concept.^{[12](#)}

1. Extensions: Decomposing Teens Numbers

As mentioned earlier, along with number concept—subitizing, numberness, decomposition/recomposition, and 45 sight facts—the other prerequisite skills for learning addition and subtraction easily is mastery of teens' numbers. Mastery of the teens' numbers is achieved very easily using Visual Cluster cards and Cuisenaire rods. Children find all the combinations of 11, 12, 13, ... 20 by making two-rod trains for the teens' numbers by organizing 'orange plus white,' 'orange and red,' 'orange and green,' ... orange and orange trains. Children should record and practice using Cuisenaire rods (concrete models) and then by coloring on grid paper and using Visual Cluster cards (visual models) and then writing the number equations for the teens numbers (symbols). Again, these should be automatized; for example, for the orange-purple train (combinations of 14), the trains would look like the following.

After the teens' numbers are automatized orally, they should practice them in writing. Teens' numbers are automatized when a child can answer the following questions for any teens' number, without counting of any kind:

- What two numbers make 14? 10 and 4 or 4 and 10.
- 10 plus what number is 14? 4.
- 4 plus what number is 14? 10
- What two digits make 14? 1 and 4 in that order?
- What is the value of digit 1, in number 14? 10. Etc.

Then children are asked to make all the combinations of two numbers that make a given teens number and write the corresponding addition equations, emphasizing the idea that '=' has two-way implications. For example, the equations for the concrete model are:

$$10 + 4 = 14, 4 + 10 = 14, 14 = 10 + 4, \text{ and } 14 = 4 + 10.$$

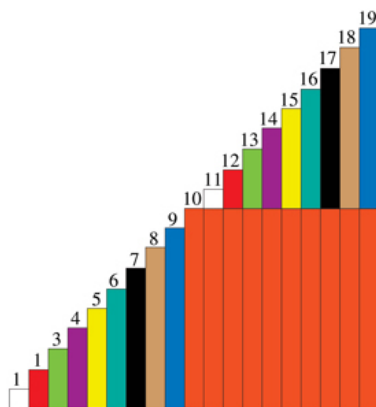


Figure 7.7 Representing teens number using number rods



Figure 7.8 Demonstrating $9 + 5 = 10 + 4 = 14$

After this, children are asked to take the blue rod (9-rod) and find out what rod added to the blue rod will make 14? $9 + 5 = 14$

Have the child (each child gets a different number) choose a number from 11 through 20, and have the students show the representation with Cuisenaire rods, using an orange and another color rod.

Example: $14 = \text{Orange} + \text{purple}$.

Now have students find as many combinations for 14 as they can, encouraging them to reverse the order and emphasizing to make them systematically, so that they have:

$14 = \text{orange} + \text{purple}$ and $14 = \text{purple} + \text{orange}$

$14 = 10 + 4$; $14 = 4 + 10$; $4 + 10 = 14$; $10 + 4 = 14$.

$14 = \text{blue} + \text{yellow}$ and $14 = \text{yellow} + \text{blue}$

$14 = 9 + 5$; $14 = 5 + 9$; $9 + 5 = 14$; $5 + 9 = 14$.

$14 = \text{brown} + \text{dark green}$ and $14 = \text{dark green} + \text{brown}$

$14 = 8 + 6$; $14 = 6 + 8$; $8 + 6 = 14$; $6 + 8 = 14$.

Etc.

The teacher asks each student to read their equation. Each student gives his/her examples both in the color equivalent and the number/symbol form.

After random facts for a number (14 as aforementioned) have been identified and established, children should be asked to organize them in order, first with the help of Cuisenaire rods and then in writing. Ultimately, examples are written and displayed sequentially to demonstrate the decomposition/recomposition. What will make 14 can be seen as the sum of two rods where the larger number is decreasing by one and the smaller of the numbers is increasing by one, as can be seen in the following. Such a pattern helps them to memorize the facts quite easily (an example of combinatorial logic).

$14 = 10 + 4$ and $4 + 10$

$14 = 9 + 5$ and $5 + 9$

$14 = 8 + 6$ and $6 + 8$

$14 = 7 + 7$. Etc.

Ask them to look at the arrangement and read all the combinations, then close their eyes and recite the combinations. When a child can give the facts automatically orally, then you ask them to write them without looking at the rods. This should be true for all numbers up to 20.

The same exercise is repeated with the Visual Cluster Cards.

F. Organizing Instruction Sequence for Basic Addition Combinations

Based on the important property of decomposition/recomposition of number and a collection, the following sequence of strategies for teaching and learning families of addition facts have been found to be easy to learn, effective to reach fluency, and elegant in generalizations. The sequence of these strategies is very important as it recognizes that each succeeding strategy is built on the combination of preceding strategies as an extension of or deepening of previous strategies. Since mathematics is cumulative and numeracy is a collection of concepts and strategies related to numbers, we need to remember that knowing one strategy provides energy, motivation, and an opening into learning new concepts and strategies. Knowing some mathematics helps learners to know more mathematics. Knowing mathematics with efficient strategies gets success and motivation and a will to learn more mathematics.

Each succeeding strategy can be derived from combinations of the previous strategies. As a result, the student has several ways of arriving at the family of facts for a number or numbers. Along with the content (addition facts), this process develops two key components of executive function: ‘flexibility of thought’ and ‘deeper understanding’ and even vertical acceleration. This process also helps in providing differentiated instruction for children in a diverse group of children by cognition, special needs, etc.

G. Additive Reasoning: Concept, Facts, and Procedures

The research on learning and expertise is clear. From mathematics to chess or even sports—a distinguishing factor between beginners and experts in any field is that experts organize information in *conceptual schemas* in their brains, connect the new information with the old, and constantly refine their schemas, structures, and strategies in the light of new information and experiences. In the case of experts, the structures and organization in their learning, and in the process of improving skills, their performance makes their learning productive. But we rarely plan our instruction to help reveal these kinds of organizational structures for our students. This organization should be reflected in the instructional tasks we plan, organize, set behavioral standards of performance, execute, assess them constantly, and connect these structures with each other ([Geary & Brown, 1991](#); [Geary et al., 2004](#)).

Although building mastery and expertise is quite complex, nevertheless, we can use a relatively simple way to think about it. It is to know what do we want in the form of results—language, concepts, strategies, procedures, and applications. To do this we need a plan.

First, we should understand the trajectory of the development of a concept—where and how it begins, how it develops, and what behaviors one should look for when one demonstrates the mastery of this concept.

Second, identify its key features (language, component primary concepts involved, and the nature of the conceptual schemas) and the pre-requisite cognitive skills involved in learning it.

Third, select the instructional models and tools that can best help develop the prerequisite skills and the concept effectively.

Finally, help develop the concept using the three roles of teaching—didactic, Socratic, and coaching.

Additive reasoning is the integration of concepts of addition and subtraction, mastery of addition and subtraction facts, fluency in executing addition and subtraction procedures, understanding that addition and subtraction are inverse operations, and an ability to apply these components in solving problems and learning new concepts in mathematics (Frudenthal Institute, 2001).

1. Prerequisite Number Concept Skills for Mastering Addition and Subtraction Facts

When children have mastered the following numbersense skills and concepts about number concept, they are ready for driving the addition facts up to 20 and formal and informal procedures on addition and subtraction. Numeracy, in fact, all mathematics learning is cumulative in nature. It builds on the mastery of previous language, concepts, procedures, and skills. To master addition facts, one needs mastery of:

Number Concept:

- *Subitizing and visual clustering*
- Difference between *digits* and *number* and that digits can make different numbers according to their location in the number
- *Decomposition/recomposition*
- *45 sight facts*
- *Making ten*
- *Commutative property of addition*
- *Teens' numbers* ($N + 1$ and $10 + N$)

When students have these pre-requisite number skills, the addition facts can be mastered very quickly and effectively. Here is the sequence for strategies. The goal is to:

- Derive the strategies,
- Achieve accuracy,
- Achieve fluency, and
- Practice by applying the facts in problem solving.

And in the process automatize these facts with understanding and confidence. Let us focus on the structure and organization for achieving this goal for each child. An organization and structure of the task makes the work easier and manageable.

The strategies developed here are not only preventive in nature, but they are also effective strategies for the initial teaching for all children. With initial teaching being organized effectively, learning problems in mathematics are minimized. In our schools, the Tier II children are increasing as the initial teaching is not effective and the problems of learning disabled children are magnified. These strategies are also effective, efficient, and elegant for regular instruction and for remediation, not only for children who are learning for the first time or for those who are in higher grades, but never mastered the arithmetic facts earlier.

Kindergarten

Here is the sequence of activities to help children learn these arithmetic facts when it is optimal to teach them. These activities are most successful in achieving their goals when children work in pairs. Children should have comparable skills and cognitive levels and the pairs should change every day and for every exercise. From time to time, the teacher can also give a choice for them to select their partners as long as children with learning differences are not segregated. The teacher can also use the expertise of children to help the teacher by becoming teacher's helpers to a group. When a child has finished his work and volunteers to help, the teacher decides the person to work with. However, a child can request help from one of her classmates. The teacher demonstrates to the whole class how to help their classmates. The role of help is to ask questions and answer questions not to solve the problem for them. The volunteers can also help by demonstrating the method of solving as long as they give a reason for every step they derive. The volunteers can also help by demonstrating the method of solving as long as they give reasons for every step they derive. In a sense, these children are the teacher's assistants, and they take turns. New assistants are created every day. It is an honor and the role is formally rewarded by recognition by the teacher and finally by the principal.

A. COMMUTATIVE PROPERTY OF ADDITION (TURN AROUND FACTS) ($M + N = N + M$)

The first step in the development of both addition and multiplication facts is the concept of the commutative property. The commutative property of addition should be introduced and mastered by the end of kindergarten. This can easily be shown with the help of Cuisenaire rods and Visual Cluster Number cards. As we mentioned earlier, the commutative property of addition reduces the 100 addition facts to 55 individual facts to be learned, and therefore makes the task much more manageable to master. Later on, the same principle applies to the 100 multiplication facts. In some school textbooks, this property is known as 'turn around' facts.¹³ It is important that children not only know the strategy but also refer to it whenever they use it. For example, 'I know that $8 + 7 = 15$, and by the commutative property of addition, I also know $7 + 8 = 15$.' The following table shows the relationship and arrangement of the facts organized by the commutative property. If a student knows the facts in the top triangle where the cells are marked by 'xx' have a corresponding partner in the lower triangle where the cells are marked by 'x.' The diagonal on the other hand represents the facts derived by recognizing the doubles, here marked by '0.'

2. The 45 Sight Facts

As mentioned earlier, because of learning the 45 sight facts as part of the number concept, the task is reduced to only 30 addition facts (55 addition facts—the 25 sight facts). This helps the teacher to achieve much more in much less time with better conceptual understanding on the part of children and in the process, they gain a great deal of confidence and numeracy stamina. This also gives more time to review so that children can really automatize the addition facts with required fluency and understanding. They should leave first grade knowing addition facts with mastery—understanding, fluency, and ability and confidence to apply to learn new arithmetic concepts (subtraction, place value, etc.) and solving application problems—money, time, measurement, perimeter, etc.

45 Sight Facts

+	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	
2	3	4	5	6	7	8	9	10		
3	4	5	6	7	8	9	10			
4	5	6	7	8	9	10				
5	6	7	8	9	10					
6	7	8	9	10						
7	8	9	10							
8	9	10								
9	10									
10										

If we introduce the concept of the commutative property of addition and help students master sight facts, even before we actually develop the further addition strategies and the addition grid formally, it becomes easier for children to understand, derive, and then memorize the addition facts to achieve automaticity.

The understanding and mastery of the commutative property of addition can be achieved by several very effective and efficient instructional materials: Cuisenaire rods, Invicta Balance, Visual Cluster Number cards, etc.

With Cuisenaire rods, the teacher can help children see the commutative relationship easily. To demonstrate the commutative property, the teacher organizes the following equation using Cuisenaire rods and with the help of children derives the commutative equation.¹⁴ It is better to start with small numbers so that the children can easily perceive the relationship and visualize it easily. The teacher gives the following instruction:

Teacher: Look at my equation with number rods (previous). She reads the equation in three forms: (a) in color, (b) in numbers, and (c) in letters (using the first letter of the color: r = red, lg = light green). We call this the commutative property of addition or commutative property of addition operation. Let us read it together, in unison.

They read it.

$$\begin{aligned}
 \text{red} + \text{light green} &= \text{light green} + \text{red} \\
 2 + 3 &= 3 + 2 \\
 r + lg &= lg + r
 \end{aligned}$$

Teacher: Now, I want one person to read the equation in three ways.

Several children in turn are invited to read the same equation in three forms.

Teacher: It demonstrates the commutative property of addition. It is, sometimes, also called ‘turned around fact.’ It means that in adding order does not matter. Now, I want you to make another commutative equation for another addition problem and equation like mine, but different than mine and different than your neighbor. Of course, you are welcome to look at your neighbor’s equation or visit anyone, in the classroom to observe their equation, and you can even ask them how they made it. However, your equation should be different from your neighbor and the person you visited.

The children make their equations. The teacher asks them to read their equations in turn, in different forms: using colors, using numbers, and using the first letters of the colors.

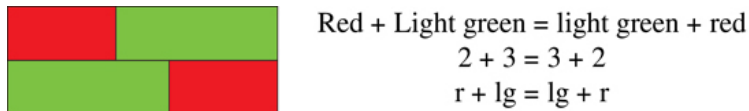


Figure 7.9 Commutative property of addition using number rods

The whole class is working on the equations. Some children, if they want to, can also work in pairs. As children are making the equation, the teacher observes them and provides advice, support, and directions to construct the commutative equation. Then the teacher gives the following instruction:

Teacher: I want you to make another equation like the previous one but different than that one, different than mine, and different than your neighbors.

The teacher moves around supporting and providing necessary hints for children to model their own commutative equations for addition problem. The teacher should help children explore these relations with other rods/numbers. Once they can construct addition facts concretely in terms of color the children should then translate these equations into number equations, like the following.

Students begin with

$$\text{red} + \text{light green} = \text{light green} + \text{red}$$

at the end of the lesson they will be able to see:

$$3 + 2 = 2 + 3$$

They will also know that:

$$3 + 2 = 5 \text{ and } 2 + 3 = 5.$$

At the end of this series of lessons they will recognize and know the concept in its more general/abstract form, such as:

$$lg + r = r + lg,$$

or

$$a + b = b + a.$$

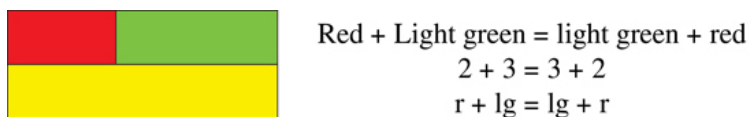


Figure 7.10 Demonstration of $2 + 3 = 5$ using number blocks



Figure 7.11 Commutative property of addition

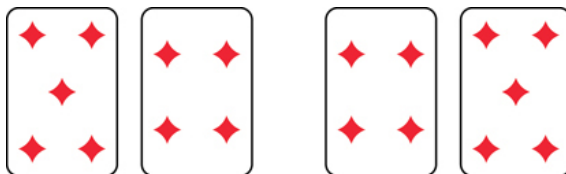


Figure 7.12 Commutative property using VCNC [🔗](#)



Figure 7.13 Commutative property using one card of VCNC

Commutative property of addition can also be learned and practiced by using Visual Cluster Number cards. It is particularly beneficial as those children who have not mastered the number concept properly have one more chance of learning the Visual Cluster cards, and therefore the numbers and sight facts well. The teacher shows two cards, such as:

See [Figure 7.12](#) above and concludes $5 + 4 = 4 + 5$, without counting. Using the Visual Cluster cards, children can practice it in pairs. This activity also helps children, if they have weakness in identifying visual clusters and sub-clusters of numbers. This reinforces the decomposition/recomposition of numbers.

The commutative property can also be demonstrated by using only one Visual Cluster Number Card or Sight Fact Card, as shown in the following. The teacher demonstrates it and uses the language of creating this in the following example.

Teacher: I put the yellow rod next to the purple rod (two rods are placed in the same row, end to end), and then I put the purple rod and then the yellow rod parallel to the first row. Now, you can see that these two rows have the same length. The two rows are equal. Therefore, we have an equation. Once again, I will call this ‘commutative equation or turn around facts.’ Similarly, this property of addition is very simply shown by placing the

$$4\text{-card} + 5\text{-card} = 5\text{-card} + 4\text{-card}$$

Now you do the same on your table with your partner. Children construct the two equations using cards and the Number rods.

Teacher: These equations shown by our arrangements (Cuisenaire rods and cards) show the equality of $4 + 5$ and $5 + 4$ and can be written as (points to the equations written on the board): ‘We can read your

arrangements as equations.'

yellow + purple = purple + yellow, or

$$5 + 4 = 4 + 5, \text{ or}$$

$$y + p = p + y.$$

Teacher: Now, I want you to make your own 'commutative addition fact equation' or 'turn around fact equation' different from mine and different from your neighbors using Visual Cluster Cards. When you have made it, raise your hand, and I will come and check it.

Children get busy and the teacher observes children's arrangements and makes her observations known to children. She helps those pairs who are struggling. She identifies experts and asks them to help other children who need help. When every pair has their equation, she gets the children's attention.

Teacher: Now, I want everyone to read your equation/fact in three ways: using the colors of the rods, using numbers, and using the first letter of the number rods. Then, we will write each equation using numbers. For example, my equation is:

$$4 + 5 = 5 + 4.$$

She also points out that the equation is also demonstrating the two sight facts of 9, that is $9 = 4 + 5$ and $9 = 5 + 4$.

The teacher goes around and checks the work. Then, children take turns to read their equations/facts. As children read their equations, the teacher writes these equations on the board. After this, the teacher asks children to write their equation/facts on their papers.

This process is repeated until every child can make, read, and write a turnaround fact/commutative equation correctly. Then the teacher helps them practice it orally.

She asks one child if $4 + 5$ is the same thing as: ____? The child should say: $5 + 4$. She asks another child in turn to complete the equation involving a turnaround fact. Once they have mastered this, the teacher extends it to examples such as:

'If you know that 4 plus 6 is 10, what will 6 and 4 be?'

'If you add 4 to 16 to make 20, what will be $16 + 4$ or what two numbers will make 20?'

To master each strategy, we have found that it should be practiced through a series of activities: (a) choral activity—the whole class gives the answer orally as a group, (b) individual oral response, and (c) individual written response. For example, for commutative property here are some suggested questions.

Choral Activity:

Teacher: Look at this equation (the teacher forms a commutative equation using jumbo or magnetic Cuisenaire rods, or using the document camera). I want all of you to read this equation together.

Children read the equation.

Teacher: Now I am going to read part of the equation, and you are going to complete the equation. $4 + 5$ is equal to what?

Children: $4 + 5 = 5 + 4$.

The teacher should ask dozens of questions like this. Initially, some children may be hesitant to participate, but the teacher may focus on them or encourage them to participate. Choral activity is a powerful activity. In this

activity, there is no pressure on individual children. They are responding to a concrete or oral input and with other children also producing oral output. However, they are hearing the same problem and language for the concept, and they are getting ample reinforcement to take a chance in answering. The teacher monitors each child's response and provides immediate feedback. The teacher could ask for one volunteer from the class who could lead the choral activity. When the teacher feels that every child is participating in the choral activity, she can start the individual oral activity.

Individual Oral Activity:

Teacher: You did a very good job as a group. Now we are going to do it individually. I will ask each pair in turn to read their commutative equation for addition. The children will still have their equations displayed in front of them. (The teacher asks each child the same kind of question, but she gives a different equation to each child by pointing to an equation being demonstrated by another pair. She varies the level and nature of questions (direct vs indirect; convergent vs divergent; conceptually based vs a straight fact). The choice of question reflects the individual student's competence, level of language, preparation in the concept, and emotional/social maturity and confidence.) For example:

For Child A, the question might be:

$5 + 6$ is the same as what plus what? Or $1 + 7 = \text{what} + \text{what}$?

For Child B, it might be:

$17 + 13 = ?$ or, glob + slob = ? Or, $b + c = c + ?$

For Child C: it might be:

if $8 + 9 = 17$, then, what is $9 + 8 = ?$

If a child has difficulty answering the question posed, the teacher quickly forms the equation using the Cuisenaire rods and helps the child to get the correct answer or supports her partial efforts by scaffolding the questions.

In the case of any children who are struggling, the teacher can ask them to repeat the equation given by other children. She keeps on asking questions of individual children until each child is able to give the equation, correctly, fluently, and independently.

The teacher does this for several rounds. Each round is focused on a particular aspect of the concept, strategy, or procedure. On successive rounds, the questions become more and more difficult and complex, but they are still accessible to the individual child. For example:

Teacher: If I told you $8 + 7 = 15$, then what is $7 + 8 = ?$ If I told you $3 + 4 = 7$, then what connected fact can you give me? Etc.

In the last problem, the child is now applying the commutative property of addition to arrive at the answer. A strategy is not mastered until it is applied and communicated contextually.

Written Supervised Practice

The goal of all school mathematics teaching is that individual children, without assistance, are able to demonstrate their learned skills in concrete, visual, abstract, and symbolic forms, depending on the context and nature of the mathematics problem. For that to happen, after the concept, skill, and/or procedure has been

introduced and understood by the student, then the appropriate and necessary group, supervised and individual practice is organized and provided.

The practice should be productive and adequate so it should take the child to the automatization level. For normally growing children in mathematics, giant steps in the skill continuum are possible and desirable, but for mathematically challenged children carefully designed steps and skills are advised that result in small successes and useful diversions to enhance their motivation.

Therefore, the supervised skill practice has to be carefully choreographed (based on efficient task-analysis of the concept and its developmental trajectory), supported (by proper use of language, questioning process, and sequence of activities), and monitored (checking the mastery of each sub-skill—concretely, pictorially, through visualization, and abstractly).

Cycles of frequent oral and short written testing and practice is the answer to reach a level of confidence and mastery.

Answering these problems using counting of objects, reading equations formed from Cuisenaire rods are examples of a *concrete level of knowing*. However, as frequently mentioned earlier, answering and practicing using counting skills and models is a poor way of seeking and reaching mastery; it is not a demonstration of addition, but of counting.

The ability and facility to answer questions using Number Line, Empty Number line, or equations formed from Visual Cluster Number cards are examples of the *pictorial level of knowing*;

Visualizing the equations in the mind's eye (in working memory space) and reading them is the *visualization level of knowing*. When an equation, model, or idea can be visualized, transfer of the visualized information to the long-term memory is much higher. Effective concrete and visual models facilitate this transfer and retrieval of information when needed.

The ability to write the equations symbolically, without the help of concrete and pictorial models is an example of the *abstract level of knowing*. True mastery is reached when a student can apply new learning to problem-solving and communicate the process and outcome meaningfully.

In order to accommodate the mathematically handicapped child, one must carefully examine the presence of prerequisite skills in the child's repertoire. The prerequisite skills and appropriate concrete manipulatives facilitate the easier development of a strategy and procedure.

In planning every level and form of practice, the teacher should determine the existing hierarchy of skills and perform a task analysis for planning effective activities and forming good questions. It could be that further subdivision and reordering of the skills in the sequence is warranted to address the needs of students to provide small, measurable, and usable successes for children with special needs due to learning disabilities and or dyscalculia. The process of translating the hierarchy into the smallest possible steps and skills, practicing it properly and adequately, can generate greater potential for a successful learning experience, retention of skills for all children, and the development of a more positive attitude.

Individual Written Practice

When the teacher is sure that every child is orally able to communicate the commutative equation for addition in different forms and fluently, then the children should be invited to write the equations.

The teacher now gives a sheet of paper with commutative property equation problems. The written sheet (with about 15 problems with 5 problems at each level of difficulty—as earlier, in the three cases of children: Child A, Child B, and Child C, organized in columns) should have three levels of questions: (a) simple, (b) medium, and (c) moderately challenging but accessible. However, the heading of the columns in the worksheet should only show categories as I, II, and III, not simple, medium, and difficult.

Examples for practicing commutative property of addition at each level:

Level I: $5 + 6 = 6 + ?$; $1 + 7 = ? + 1$; $6 + \square = 9 + 6$;

Level II: $9 + 8 = \square + \triangle$; $17 + 13 = ? + ?$; $\text{glob} + \text{slob} = \triangle + \text{glob}?$; $b + c = ?$

Level III: if $10 + 20 = 30$, then $20 + 10 = ?$; $8 + 9 = 17$, then, what is $9 + 8$?

Since the commutative property of addition reduces the number of addition facts to be learned to almost half of the total addition facts (55 out of 100, to be exact), children should have extensive practice with this concept when saying all four equations for two-rod equivalence trains for addition. When children have ‘really’ mastered the commutative property of addition, then and only then the should the teacher move to the next strategies of addition.

10 x 10 grid

An example of addition grid using specific addition strategies

+	1	2	3	4	5	6	7	8	9	10
1	2	2	2	2	2	2	2	2	2,3,5	2,3,4,4
2	2							3	5	4
3	2						3		5	4
4	2					3			5	4
5	2				3				5	4
6	2			3					5	4
7	2		3						5	4
8	2	3							5	4
9	2,3,5	5	5	5	5	5	5	5	5	4
10	2,3,4	4	4	4	4	4	4	4	4	4

Figure 7.13a Display of addition facts by strategies

The same kind of lessons, strategies, language, questioning process, and practice is needed when teachers (interventionist or special educator) work with children in the remedial or intervention situations. Most of these children have been introduced to addition facts through sequential counting (addition as counting up and subtraction is counting down). As mentioned, this is an inefficient strategy and is not helpful in understanding, deriving, mastering, memorizing the facts, and applying them efficiently. Children with learning disabilities should be introduced to the most efficient strategies, rather than the most inefficient strategies. Because of these counting strategies, for them, the addition facts $9 + 4$ and $4 + 9$ are two different facts. They derive them separately. As a result they are overwhelmed in mastering the 100 facts of addition (sums up to 20). And if they do not have the addition facts, then mastering subtraction and all other later facts become difficult, or even impossible.

As might have been noticed throughout the activity, there were enough examples of differentiation of instruction to meet the needs of children. We believe that differentiation is not when we form groups of children by their ability and focus and teach them in small groups at different levels. By doing so we are providing only a portion of allotted time of instruction to all children. In such a situation, we can not expect to achieve one year’s progress in a short time of instruction. Differentiation should be practiced during teaching the whole class. Yes,

from time to time, they might be working with a small group of children on a particular concept that impinges on the main concept, and these children will need an extra effort to be able to participate in the class activity.

Differentiation consists of paying attention to the needs of each child—knowing how they learn, their potential (which changes with every bit of learning), their limitations, their modifiability, asking questions appropriate to the child's current ability—modifying questions by the level of content and complexity, scaffolding questions, giving direction for their movement, restating a question, etc. But the questions and activities must deal with the common concepts being taught in the classroom at that time.

Differentiation means modifying the sequence of questions and tasks for practice, homework, and assessment dependent on the need of the moment and the students. Differentiation also requires that the child's instruction is augmented with extra instruction and practice and extended time for testing and assessment. Frequent assessment/diagnosis to inform you to be able to give the right kind of feedback. However, the expectation should always be that each child leaves with mastery of the material at their grade level.

The next lesson/activity would review the language of addition and have more demonstrations. Children should solve problems with multiple materials for acquiring the flexibility of thought (a key component of executive function and higher-order thinking) but should master the concepts and facts using efficient strategies and appropriate materials with the appropriate script accompanying the work. Again, the teacher should orally ask all children simple problems involving the commutative property of addition. Each child will have the opportunity to answer several questions of this type. From here on children would spend individual time with efficient and effective materials and answer questions relating to addition concepts.

The teacher will give a problem, and they will demonstrate the concrete model for that problem using any of the materials utilized in the introduction of the concept. Children should also work as partners, forming and answering their own problems by the help of concrete materials used earlier and on paper.

After some practice with adding and manipulating the materials, the teacher should ask them to write these facts as addition equations either on their whiteboards or on their papers¹⁵ individually. Through these activities, a teacher has ample opportunities to do ongoing *formative assessments* and give appropriate and constructive feedback.

Based on the formative assessment, the teacher should plan a summative assessment on the commutative property of addition similar to the written practice. This time, the problems to be included on the assessment are questions/items that are mixed by difficulty. The comments section on the worksheet for entering the formative assessment should include the succinct information about formative assessment observations for future ideas for differentiation of learning.

To automatize these facts, tell children that you are going to ask them to make a commutative equation using Cuisenaire rods or a written equation, so they should be prepared to write both kinds of equations.

Note and Caution: It should be noted that Commutative Property of Addition is an algebraic concept, and it should be derived and experienced by children as an algebraic concept—as a general concept rather than a property arrived at by counting. For example, answering '4 + 5 and 5 + 4 are equal' by counting (enumerating the collection of objects in the two collections) transforms the 'algebraic reasoning' task into a numerical-reasoning task, and that too at a low level. Success in such special cases may represent a localized rather than a general understanding of the commutative property. This caution and principle applies to all strategies, concepts, and procedures.

3. $N + 1$, $1 + N$ (Adding One More to a Number Means the Next Number)

The idea that one more than a given number is the next number and one less than the number is the previous number is fundamental for operations on numbers. It is the key to addition and generating other numbers. The $N + 1$ concept is fundamental to generating the natural number system. Many children have difficulty with this concept. Cuisenaire rods arranged in a staircase, by putting the number one rod next to a given rod, can very easily show that one plus a given number is the next number. By this strategy, the following facts will be derived:

[$1 + 1$; $1 + 2$, $2 + 1$; $3 + 1$, $1 + 3$; $1 + 4$, $4 + 1$; $5 + 1$, $1 + 5$; $1 + 6$, $6 + 1$; $1 + 7$, $7 + 1$; $8 + 1$, $1 + 8$; $9 + 1$, $1 + 9$; $10 + 1$, $1 + 10$]

Incidentally, all these facts are sight facts, which are already mastered in kindergarten. So, this is a review strategy. This represents 19 facts learned on the Addition Grid: 10 on the first row and 9 on the first column in the grid.

This collection of addition facts is dependent on the child knowing his/her numbers. Daily counting, experience with Visual Cluster cards, dominos, and dice provides a very strong background for the mastery of this strategy. Furthermore, when children make the staircase, they see that the next rod is one bigger than the previous one and they begin to realize the property that when you add one to a number you get the next number. To introduce this strategy, the teacher asks children to make a staircase using one rod of each color.

The addition by 1 to a number can be demonstrated by adding the 1-rod to any step of the staircase and helping them to perceive and derive the idea that adding one means the next number. Here you see 1 added to each step of the staircase. It is important to emphasize that the child answers the question: ‘*What is $6 + 1$?*’ by saying $6 + 1 = 7$, and other similar questions. What is even more important is that the students see that: ‘by adding one to any number, we get the next number.’ This is a profound discovery, for the child, which forms the basis of generating all whole numbers. Or, more clearly, ‘adding 1 to a whole number means the next number.’ Knowing this means that the child can generalize it to any number. One more than N means $N + 1$, and that means the next number.

This strategy should be addressed in the earlier part of the kindergarten class and reinforced with the visual cluster cards. By this practice, children should form the mental image that adding one to any number results in the next higher number, visually the next stair (rod or number) on the staircase. The subtraction of one can be demonstrated by comparing any ‘stair’ with the next smaller ‘stair.’

Eventually children should be provided with practice for ‘any number plus one becomes the next number’ (any rod plus a white rod will equal the next number in the sequence). For this, children should automatize the $N + 1$ equations and commutative property as follows:

$1 + 1 = 2$
 $2 + 1 = 3$ and $1 + 2 = 3$
 $3 + 1 = 4$ and $1 + 3 = 4$
 $4 + 1 = 5$ and $1 + 4 = 5$
 $5 + 1 = 6$ and $1 + 5 = 6$
 $6 + 1 = 7$ and $1 + 6 = 7$
 $7 + 1 = 8$, and $1 + 7 = 8$
 $8 + 1 = 9$ and $1 + 8 = 9$
 $1 + 10 = 11$ and $10 + 1 = 11$

Without the commutative property, many children, particularly those who have learned addition by counting, can give the answer to $6 + 1$ rather easily, because they are just adding one to 6. However, when it comes to $1 + 7$, they begin to count 6 more from 1. It should be noted that if we move backward on the Cuisenaire rods staircase, we can introduce the ‘take away 1,’ ‘subtract 1,’ ‘number minus 1’ right after adding one to a number. We can also bring the phrases of ‘one more than a number’ and ‘one less than a number’ to children’s repertoire. These phrases pose difficulty to most children.

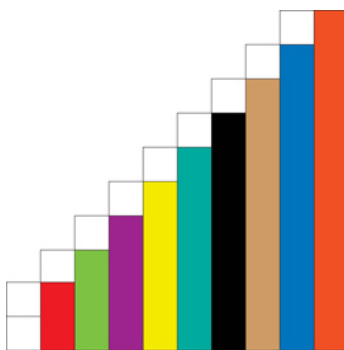


Figure 7.14 $N + 1$ or $1 + N$ strategy

Deducting/removing one cube from any of the numbers (from the step) will give the subtraction ‘minus one equations’:

$$(2 - 1 = 1; 3 - 1 = 2; 4 - 1 = 3; 5 - 1 = 4; 6 - 1 = 5; 7 - 1 = 6; 8 - 1 = 7; 9 - 1 = 8).$$

Once again, we need to use the three activities: Choral, individual oral, and written individual. During these activities, the teacher asks questions such as the following to move these basic facts into their long-term memory. However, at this level we add a fourth activity: Cumulative practice activity. In this activity, the problems for practice also include facts from the current strategy and also from previous strategy of commutative property of addition.

What is one more than 3? Show me on your staircase.

How would you say that?

$$3 + 1 = 4, 1 + 3 = 4, 4 = 1 + 3, \text{ and } 4 = 3 + 1.$$

Repeat this for other combinations and involve all children.

Choral Activity for $N + 1$ and $N - 1$

Teacher: I am going to say a number and the class will say the number that is one more. Three!

Students: 4.

Teacher: One less than 3!

Students: 2.

Practicing: One more than 3 is 4. One less than 4 is 3.

This is repeated dozens of times. The teacher says the number and children give one more than the teacher’s number. Then the teacher says a number and children say one less than the number. The teacher says a number and asks to add one or subtract one. Children give the number that is one more than the number given. When the teacher is sure that children can give the answer correctly for number plus 1, she asks the same questions individually.

Individual Oral Activity

Now, the teacher asks one question or more from each child in turn. However, the questions asked are appropriate to the child. For example, from one child she wants the answer for $4 + 1$, but from another one she wants $17 + 1$ or $43 + 1$. From another one she wants the answer for $49 + 1$ or $69 + 1$ ¹⁶ or $1 + 49$. From yet another, she wants the answer for $10 - 1$, $50 - 1$ or $30 - 1$. This is an example of differentiation.

The reinforcement of this strategy can also be achieved by practicing this using the Visual Cluster Number cards. Pick a card and ask them to pick the card that is one more than your card. This same procedure can be used with subtraction.

Practice: Make it into a game. Give a deck of Visual Cluster Cards and children play ‘One More’ and ‘One Less’ games. Children take turns and answer ‘what is one more than this?’ Or ‘What is one less than this number?’ Answering a question 1 more than a number where the number is up to 5 gets one point; if the one more than the number, and the number is more than 5 this gets two points.

4. Making Ten (A Pair of Numbers That Make 10)

[2 + 8, 8 + 2; 3 + 7, 7 + 3; 4 + 6, 6 + 4; 5 + 5]

(These are sight facts of 10.)

(7 new facts added on the Addition Grid)

Because of our number system being ‘BaseTen,’ the number 10 plays a key role in all operations. Without the mastery of number ‘ten,’ nothing is possible in mathematics. Place value in this system is based on multiples of ten (and, more importantly, powers of 10), so we make groups of 10 when we are adding or subtracting. Therefore, making a group of ten by adding two numbers is essential for extending the concept of addition or subtraction of numbers beyond ten. Recognizing the ‘groups of ten or tens’ is the key to all arithmetic operations.

Most children who continue to rely on counting have not yet learned this relationship even when they reach higher grades. These children have great difficulty in mastering the addition facts. In order to add two numbers whose sum is more than ten requires that the student has a good grasp of the ‘making ten’ strategy. We have observed that children as well as adults who have not mastered these seven arithmetic facts (two numbers whose sum is ten) continue to count on fingers. When we help these individuals to master these facts, they stop counting on fingers. Fortunately, learning this strategy does not typically take more than one session.

The ‘making ten’ strategy is fundamental to the understanding of our BaseTen system. For example, once children know the fact $6 + 4 = 10$, they can extend this mastery to adding numbers such as: $60 + 40 = 100$ and $600 + 400 = 1,000$, and this forms the basis for ‘near 10’ calculations, e.g., $6 + 4 = 10$ leads to learning $6 + 5 = 11$ or $6 + 3 = 9$.

By working with Visual Cluster cards, children very quickly realize that the facts (which two numbers make ten) can be seen as sight facts of 10. For example, $5 + 5 = 10$ can readily be seen on the 10-card or by the help of Cuisenaire rods. Applying the important number concept skill of decomposition/recomposition for the sum of higher numbers is dependent on mastering making ten and its extensions. If the child remembers the 10-visual Cluster card, he can construct and then remember these 7 very important sight facts. (It is to be remembered that the other 2 sight facts of 10 are already learned in $N + 1$ and $1 + 9 = 10$ and $9 + 1 = 10$.) The arrangement on the card on the left easily shows: $5 + 5 = 10$. The same thing happens with other combinations of two numbers making 10.

Based on our observations of children’s work, particularly those who struggle with arithmetic facts and are habitual counters, we find that what was in the way of achieving fluency in arithmetic facts is lack of fluency in making ten. Once we are able to help them achieve this skill, they take off. Therefore, we consider this the most important strategy in learning arithmetic facts and procedures. If a child does not know what two numbers make 10, they will have difficulty moving forward.

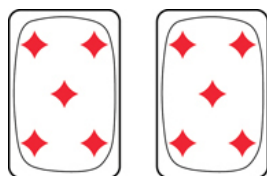


Figure 7.15 Decomposition/recomposition of 10

(a) *Decomposition and Recomposition of Ten*

Here is another example of constructing this strategy: Have children place the orange rod (10-rod) down on the mat/table. Ask them to find as many different combinations using any two rods and place them parallel and under the orange rod to make sure that they add to exactly ten. Let us say a child made the combination of 6-rod and 4-rod to give ten; encourage them to discover the relationships between these three numbers: $10 = 6 + 4$; $10 = 4 + 6$; $6 + 4 = 10$; $4 + 6 = 10$.

Do the same thing for other combination: $10 = 7 + 3$, etc. Then ask them to make all the combinations of two numbers that make ten.

Children may have different orders and combinations of two numbers that make ten. Several children may make an Arrangement A that may look like the one on the left. The

Arrangement A on the left shows a child's display that is randomly organized, whereas the Arrangement B on the right is organized in a systematic pattern.

Arrangement B, an organized pattern on the right, helps children learn and memorize the systematic combinations of two numbers that make ten much more easily as they can be visualized in the working memory. Arrangement B has several advantages and teaches children much more than the sight facts of ten. It teaches the child: (a) *decomposition/recomposition* property of a number (in this case ten), (b) the *pattern* helps to visualize the order and the ordered combinations helps them to memorize the sight facts (*combinations*), and (c) helps them to learn the important cognitive skill of combinatorial logic. Combinatorial logic means the child knows that (a) there is a system of organizing the information (possible combinations), (b) the system helps him to generate the combinations easily, and (c) he knows when the organizing process is completed. In other words, all the possible organizations are accounted for.

The teacher should discuss the advantages of the pattern on the right and help them realize why these elements are so important for learning. After this, the teacher should ask children to change their arrangements to look like the one on the right.

Once every child has successfully made the arrangement in the systematic form (arrangement B) to have understanding and mastery, the teacher should ask children to dismantle their arrangements and make it again and again till they can make it fluently. This helps them to know combinations of ten at the concrete level.

After this, the teacher introduces the idea of the concept of the 'complement of a number with respect to 10.' The pair of numbers with the sum of 10 are called complements of each other with respect to 10.

The teacher should make sure that the children respond once again in three forms: Choral, individual oral, and individual written activities relating to this strategy. The following briefly describes the process.

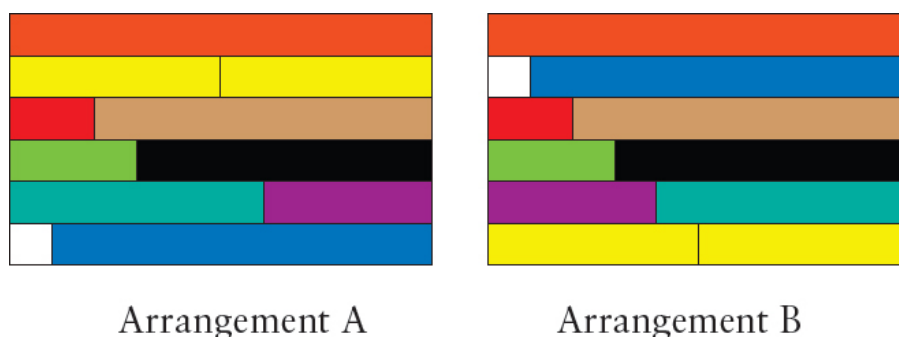


Figure 7.16 Random versus organized arrangements for sight facts of 10

Teacher: I will pick a rod and I will ask you to give me the 'complement' of this rod with respect to the 10-rod. That means the other rod—the other member of the pair of rods that make 10. The number that

makes ten when you add your number to mine will be called the complement of number 9. For example, I show you 6-rod, you pick up the 4-rod. Let us do this. Do they together make 10? If yes, then 6 is the complement of 4 and 4 is the complement of 6. You can also call them ‘partners to make ten.’

The teacher asks the students to construct the equation showing these combinations, first with number rods, with Visual Cluster Number cards, with numbers, and then write their results on paper for all combinations of ten. Then children reverse the order so that they state that $6 + 4 = 10$, $4 + 6 = 10$, $10 = 4 + 6$, and $10 = 6 + 4$. Children perform this for all the combinations of ten.

After children have made all the combinations of ten using the rods, the teacher introduces the idea of complements of ten orally without rods. The teacher asks children to give her the definition of ‘complement.’ She repeats the definition of complement: Two numbers that make 10 are called complements of each other. For example, the complement of 6 is 4 and the complement of 4 is 6.

Students may give the answer first in color (showing the Cuisenaire rod) or the Visual Cluster Number card, and then as a number. The teacher should encourage them to answer the questions without rods. If a student has difficulty, let him/her use the rods, derive the equation, and then encourage them to work without the rods. This is done first in chorus with all answering as one voice and then individually. Continue this process until the children have mastered making ten.

Teacher: This time I will say a number, and you say the complement of my number (the other part of the number ten). For example, if I say 1, you say 9. Right?

To automatize these facts, tell children that you are going to say a number, and they have to give the answer as the ‘complement of the number’ (for example, if you say ‘6’, then they say ‘4’; if you say ‘7’, then they say ‘3’).

This process is repeated till children learn the combinations of ten at an automatized level. Now the teacher asks them to hide the rods and write all the equations for ten.

After students have mastered the combinations of ten at the oral level, now they should practice this on the written level. Again, the practice sheets should have three levels, just as we had for $N + 1$.

Learning 10 as a combination of two numbers is crucial and merits several class periods till all children have memorized them (orally and in writing) with fluency. Throughout kindergarten and first grade, a teacher should open the class with oral practice of making ten. Children who count on fingers to get the sum of two numbers as 10 do not know the combinations of ten at an automatization level. Later on they will have difficulty in mastering other facts resulting in more than 10, and then knowing subtraction facts.

When the combinations of ten have been mastered alone, children should have practice sheets with problems involving combinations of *ten*, $N + 1$, and the *commutative property of addition*.

This practice should continue until the students have mastered these three strategies with rigor—fluency, understanding, and applicability. Fluency means responding with a fact orally in 2 seconds and in writing within 3 seconds. The table shows examples of what the children should be practicing. The comment section is for formative assessment comments—time taken, attitude, progress, etc. Similar sheets can be used for progress monitoring after each strategy and then for cumulative summative assessment.

This process is repeated until every child can give the complement of a given number. Once they have mastered this, you extend it to examples such as, ‘If you add 4 to 6 to make it 10, what will you add to 16 to make it 20, or what will you add to 66 to make it 70,’ etc. If some children have hesitance in this, then the teacher should use the Cuisenaire rods to help children understand it better. One should reach the level where almost all children recognize the relationship right away. After this, the teacher should say 10 and each child individually gives two numbers that make ten. For example, the teacher says: ten. One child says: 6 and 4. Another says: 7 and 3, etc. Then, the teacher may extend it to other tens numbers. However, before this exercise, the teacher should practice the tens numbers using Cuisenaire rods: 10, 20, 30, ..., 100, 110, ...

After these four strategies are introduced and children have mastered them, the teacher should begin the mathematics class every day with practicing these strategies orally for about 5 minutes each day.

5. $N + 10$, $10 + N$ (The Teens Numbers)

[$10 + 2$, $2 + 10$; $3 + 10$, $10 + 3$; $10 + 4$; $10 + 5$, $5 + 10$; $6 + 10$, $10 + 6$; $10 + 7$, $7 + 10$; $8 + 10$, $10 + 8$; $9 + 10$, $10 + 9$; $10 + 10$]

(17 new facts on the Addition Grid)

This strategy is the basis of learning place value. These facts—Teens numbers—are the first step in learning the concept of place value. Our number system is based on ten digits (symbols), making groups of ten objects. Adding and subtracting up to ten, making ten, and adding ten to a number is essential for mastery in the development of numbersense and numeracy. Thus, this collection of facts is really the beginning of learning the concept of place value. This concept should be learned in kindergarten and at the latest, in early first grade.

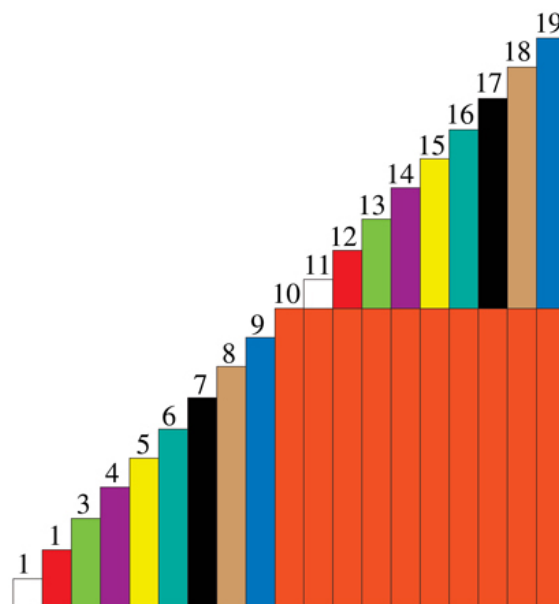


Figure 7.17 Teens numbers arrangement

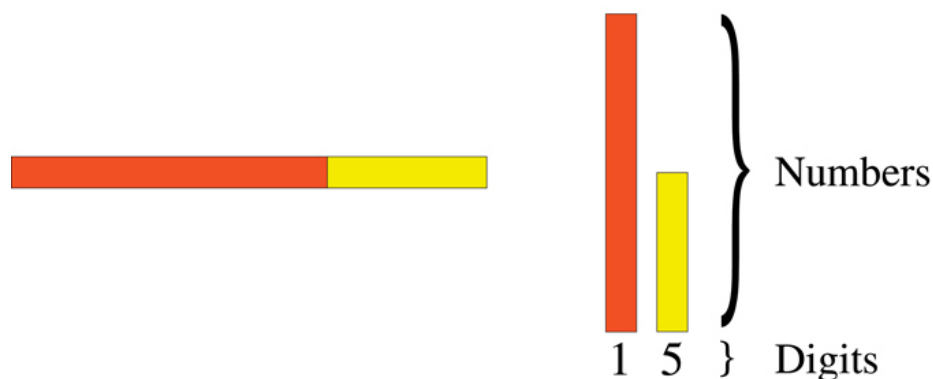


Figure 7.18 Showing number 15 using number blocks

In essence, it is the addition of the ten rod to all of the other rods to form the ‘teens’ numbers. The Place Value concept is covered in detail in the section on place value.¹⁷

The best way of learning this set of facts is to make a staircase using Cuisenaire rods and then to ask children to make numbers more than ten by using the ten rod and another rod.

For example, ask children to show you 15, and one of the rods should be the ten rod. Children should show you the ten rod (orange rod) and the five rod (yellow rod).

Point to the rods that are used for making 15, both in horizontal and vertical forms, and then ask: what two numbers make 15? Children should tell you 10 and 5 or 5 and 10. What two digits make 15? The answer should be 1 and 5. $10 + 5 = 15$. Repeat this for all of the 17 facts listed earlier.

The teacher writes the number 15 on the white board, displays it under the document camera, or points to each digit of the number 15 on the number line or on a calendar. This 15 is also made by the help of the orange rod and the yellow rod. Then the teacher asks the following questions:

Teacher: What two digits make 15?

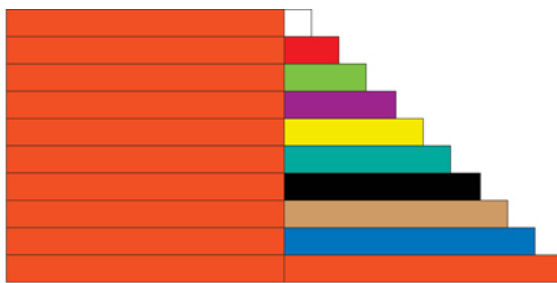
Students should answer: 1 and 5.

(If the students do not, the teacher should devote some time in making clear the difference between digits, numerals, and numbers.)¹⁸

Teacher: Digits 1 and 5 used together represent the number or numeral 15. Now, look at the number 15 as made by the rods. What two **numbers** make 15?

Students: 10 and 5.

Teacher: Very good! Look at the number '15' written on the board. What is the value of the digit 1? What is the value of the digit 5?



$$10 + 1 = 11$$

$$10 + 2 = 12$$

$$10 + 3 = 13$$

*and continue through
all the numbers to 20*

Teens number arranged horizontally

Figure 7.19 Teens numbers from 11 to 20

Students: 10 and 5. And 10 and 5 equals 15.

Teacher: Very good! As you can see, just like the place of the letter in the word determines the sound of the letter or letters, similarly, the place of the digit determines the value of the digit.

At this point, the teacher could give a mini lesson to the children on the tens numbers: 10, 20, 30, ..., 60 (for example) to introduce the idea that there is 0 in a certain place as against 12, 17, or 19. What does this mean?

Children's knowledge of teens number mean that a teens number is both the sum of 10 and a 1-digit number and in its place-value form: the tens place and the ones place have certain values. It is more than the sum of two numbers. The place of the digit is important, just as sound of a letter is determined by the place. This is a great advancement.

Another important aspect of teens numbers is their location. It is not just by counting from 0 to a given teens number. It is in relationship to the ten. ‘How far away from 10 is 15, how far from 20?’

To achieve this, the teacher should introduce the activity of constructing an individual number line up to number 30 by each child using a $1\text{cm} \times 1\text{cm}$ graph paper and the Cuisenaire rods. In making the number line, the teacher introduces the idea of units and repetition of the unit (ones). They should realize that each number is 1 unit away from the previous number. Adding one more unit makes the next number. The previous number is one unit to the left on the number line. Each number up to 30 is located at the appropriate place. They should realize that the ‘.’ on the number line is the place where the number resides, not where it is written. After the construction of the number line, the teacher conducts the following lesson.

Now ask them to locate the number 15 on their number line.

Category I	Category II	Category III	Comments
$3 + 7 = \underline{\quad}$	$29 + 1 = \underline{\quad}$	$39 + 1 = \underline{\quad}$	
$5 + \underline{\quad} = 10$	$7 + 33 = \underline{\quad}$	$51 + \underline{\quad} = 60$	
$4 + 6 = \underline{\quad}$	$15 + \underline{\quad} = 20$	$14 + \square = 20$	
$\underline{\quad} + \underline{\quad} = 10$	$24 + \underline{\quad} = 20$	$30 - 5 = \underline{\quad}$	
$8 + 2 = \underline{\quad}$	$10 = \underline{\quad} + 3$	$100 - 50 = \underline{\quad}$	
$8 + 10 = \underline{\quad}$	$67 + 10 = \underline{\quad}$	$25 + \underline{\quad} = 35$	
$14 + \underline{\quad} = 15$	$16 - 1 = \underline{\quad}$	$10 - 6 = \underline{\quad}$	
$7 + \underline{\quad} = 10$	$20 - \underline{\quad} = 10$	$17 - 10 = \underline{\quad}$	

Figure 7.20 Differentiated teaching and accelerated remediation

At the same time, the teacher locates the same number on the line on the board or screen. Then she asks them to point to the number 10 and asks them:

Teacher: Is number 15 to the right of 10 or to the left of 10 on your number line?

Children 1: To the right.

Teacher: How far away is 15 from 10?

Students: 5.

Ask children to make other teens’ numbers using the Cuisenaire rods (as earlier). This is just like making the teens numbers where we wrote the digits of the number above the rods for them to see that the digit 1 in the number 14 represents one ten. Ask questions such as:

10 and what number makes 14?

4 and what number makes 14?

14 is made of 4 and what number?

Take the 9 rod and make it 19.

Take the 2 rod and make it 12.

Take the 7 rod and make it 17.

Take the 10 rod and make it 15.

Take the 10 rod and make it 20.

If I add 5 to 10 what number do I get?

What is the sum of 10 and 6?

If I have 10 and add 7 more, what is the answer?

The teacher should instruct the students to look at the arrangement on the screen.

Teacher: Look at the screen: there is an extended staircase (with ten rods). As you can see, I have arranged a 10-rod and another number rod to make a new number. It is called a teens number (except for eleven and twelve). I am going to touch a combination; you will read the corresponding teens number. See the examples in the picture. I will touch the first row, that is made up of the 10-rod and 1-rod and makes the number 11. We will read this as: $10 + 1 = 11$.

The teacher continues moving down the rows, saying the name of each number. Children respond by saying: 11, 12, 13, etc. After they can say that correctly in sequence, the teacher moves randomly: for example, the tenth row, then fifth row, etc. When children are fluent in identifying the teens numbers, then the teacher asks the same thing as addition facts.

Teacher: I want you now to make the teens combinations as I made them, in the same order. You will make them with your partner, and you will read them in turn with your partner. When you have made it and read it, raise your hand and let me know. Remember, both partners should raise their hands.

The teacher moves around helping children arrange the rods and asking different combinations as family of facts: $10 + 6 = 16$, $6 + 10 = 16$, $16 = 10 + 6$ and $16 = 6 + 10$.

Teacher: Now I will say a teens number, and you will touch the corresponding combinations. 19

Children touch the combination 10 and 9.

Teacher: Good! What two numbers make 19? Raise your hand.

Student 1: 10 and 9.

Student 2: 9 and 10.

Teacher: That is good.

The teacher continues until students' responses are fluent in this exercise.

Teacher: Now remove the rods, and now I am going to ask you: What two numbers make a particular teens number?

Teacher: What two numbers make 18?

Student 1: 10 and 8.

Student 2: 8 and 10.

Teacher: That is good!

Teacher: Here is a list of some numbers; make them using the ten-rod and another rod: (She calls these numbers and asks the response from particular students to do formative assessments) 12, 18, 17, 11, 19, 10, 14, 16, 18, 15, etc. No student is left out from this activity.

Then the teacher makes numbers using the Cuisenaire rods or BaseTen blocks and children write the corresponding numbers. And the teacher follows the reinforcement through the usual four activities: *Choral*, *individual oral*, *individual written*, and *cumulative written activity*.

At each strategy level, there should be oral/formative assessments to make sure that the strategies have been mastered. However, after these four strategies, the teacher should assess each student individually by asking each student to come to her in a corner of the class, and in about 2 to 3 minutes, the teacher does the assessment. In a team-teaching situation, the two teachers together can manage this activity. This is one of the most important activities the teacher performs to boost her teaching. This is even more important before the teacher introduces the other strategies, as the mastery of future strategies is dependent on these four strategies. There should be a summative assessment after these four strategies. An assessment sheet may look like the following:

Name: _____

Strategies: 1 - 4

Strategy 1: Commutative Property

Strategy 3: Making 10a

Name: _____			
$A + B = C$	$\_\_\_ + \_\_\_ = C$	N	$N + \_\_\_ = 10$
Strategy 2: N + 1			
N	+1	N	+10
5		4	
6		10	
9		3	
2		5	

Figure 7.21 Worksheet Template for Differentiated learning and accelerated remediation

After the initial four strategies are learned, where most important idea was the sight facts and the beginning of the place value. Formal addition of numbers begins with the following addition strategies. These next sets of strategies will form the basis of numbersense and the foundations of formal multi-digit operations and therefore numeracy.

H. Sequence of Strategies for Addition Facts in the Beginning of First Grade

The following strategies are built on the key properties of number concept, definition, and properties of the concepts and procedure of addition, the learnings from the science of reading and language development, how our brain perceives, processes, makes connections and relationships, stores different kinds of information places them in our memory network, and then how it retrieves it when needed to be used for applying in various situations and contexts. The key skills and tools needed to make it happen are:

- i. The 45 Sight facts,
- ii. Principle of decomposition/recomposition,
- iii. Making Ten,
- iv. Teens Numbers, and
- v. Empty Number Line.

These are the ‘tools’ each child should have before proceeding further. Just like a carpenter sharpens his tools before he begins the day’s job, the teacher must put these tools in children’s minds before she begins the introduction and use of next strategies. In the tool building section of the lesson, she helps children review these orally. Through these strategies, we have not met a single child who could not master addition facts fluently with understanding and confidence.

The teacher should make sure that these skills/tools are mastered before she proceeds to the introduction of the following strategies or any other strategies. Once again, the order and name of these strategies is important. In developing these addition facts, the first strategy should always be ‘the making ten strategy.’ After the fact has been derived by this strategy, then one should explore other interesting shortcuts and extensions of strategies. Ultimately, children should be exposed to several strategies to derive a given fact to develop (i) flexibility of thought (an important executive function), (ii) robust numbersense, and (iii) familiarity in observing and recognizing patterns in number relationships in order to learn new properties of numbers and operations.

Grade One

5. $N + 9$, $9 + N$ (Adding 9 to a Number or adding a Number to 9)

[$9 + 2$, $2 + 9$; $3 + 9$, $9 + 3$; $4 + 9$, $9 + 4$; $5 + 9$, $9 + 5$; $9 + 6$, $6 + 9$; $7 + 9$, $9 + 7$; $9 + 8$, $8 + 9$; $9 + 9$]

(15 new facts on the Addition Grid)

This collection of facts is extending the list of previous facts. There are several ways/strategies of generating these facts: (a) Making Ten, and (b) $10 + N$ and then calculating $9 + N$, by calculating $10 + N - 1$.

Teacher: We want to find what is $9 + 6$ (the sum of 9 and 6).¹⁹ Does anyone know the answer? If you know the answer, please raise your hand. But do not announce the answer. You should consult with your partner to make sure both of you are right. (There is visible activity in the classroom of talking with each other and some discussions.)

Several hands go up and many do not volunteer, although the teacher knows that they have the correct answer. In any group of children of that age, some children always know the answer. They learn the facts by themselves, or their parents are heavily involved in their education. Some learn from their siblings. Do not call on these students first. Calling on these students means you are letting your class and teaching be ‘hijacked’ by a few students. You want to ask them, but after a few other students have attempted to answer. The teacher identifies a child and asks her:

Teacher: Renata, give me your answer to $9 + 6$, please!

Renata gives the correct answer. The teacher writes the answer on the whiteboard but refrains from asking her strategy for deriving the fact, nor does she declare whether it is correct or not.

She solicits the answers from many other children and writes the answers on the board without any commentary. When she has collected all the possible answers, she asks the children (owners of the answers) to share the strategies used in arriving at their answers.

Most of the answers will fall in the categories of: (a) ‘I found the answer by counting on fingers, number line, or in my head, etc.’, (b) ‘I just know it,’ or (c) ‘I memorized it, and my grandmother helped me.’

Some would give the answer as: ‘I know $10 + 6$ is 16, so $9 + 6$ is one less than 16.’

Teacher: Let us check the answers now. (She acknowledges all the correct answers. Children with incorrect answers are asked to reexamine the answer and give the correct answer by applying one of the strategies articulated by the other children.) Then she asks: Which one of these strategies is efficient? (During the discussion, children arrive at a criteria of efficient strategies. This discussion will take place during and after each one of the strategies brought to discussion in the class.)

And most children acknowledge that the last strategy is efficient.

She demonstrates the strategy by the help of Cuisenaire rods and the Visual Cluster Cards.

a. Making Ten and the Teens Number: Let us consider $9 + 6$ or $6 + N$.

Teacher: Let us take another strategy.

Let us consider the same example: $6 + 9$. We know by commutative property that $6 + 9$ is same as $9 + 6$. So if we find $9 + 6$, that will give us $6 + 9$ also. I would like to make a train for $9 + 6$ by putting the 9-rod and 6-rod end to end.



Figure 7.22 Recognizing that $9 + 6$ is 1 less than $10 + 6$



Figure 7.23a Demonstration of $9 + 6$ part 1 concrete



Figure 7.23b Demonstration of $9 + 6$ part 2 concrete

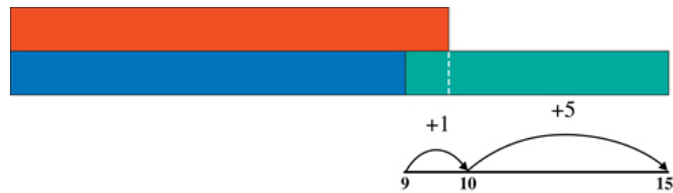


Figure 7.23c Making a ten and adding



Figure 7.23d Demonstration of $9 + 6$ part 3 concrete

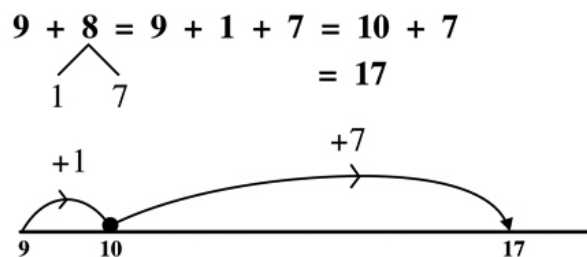


Figure 7.24 Making ten and adding abstract form of the script

Teacher: Look at this train of two rods. With your partner, please make this train on your desk. Could someone read this train in different ways?

Several children volunteer and give their answers (9 plus 6, 6 plus 9; 6 added to 9, sum of 9 and 6; total of 9 and 6, etc.). The teacher records all of these. More phrases would be added to these.

Teacher: Is the train (the sum or total, etc.) shorter/smaller than, equal to, or longer/bigger than 10?

Students: Bigger than 10.

Some children actually put the 10-rod next to and parallel to the train even before the question is asked. The teacher should acknowledge those children and point it to the whole class.

Teacher: That is right. Place the 10-rod parallel to the $9 + 6$ train and the ten-rod next to the 9-rod. Look at my arrangement.

‘Now you do the same. When you are done, let me know by raising your right hand.’

The teacher waits for all children to complete the task. She helps those who either did not understand the instructions or did not understand the problem, the task, or the language of the task.

Teacher: How did you know that it is more than ten?

Children give their explanations. She listens to all of them in turn and asks each child to defend their answers concretely using proper mathematical language and expressions.

The teacher either points to someone’s explanation or explains it to the class or helps children to derive the facts that to make 10, 1 was taken from 6 and added to 9 so that $9 + 1$ became 10. And she helps children realize that 5 is left from the 6. It is to be noted the key skills being used here are: decomposition/recomposition, the sight facts, integrating the discrete and continuous aspects of quantity, the idea of equality, and comparison. It is a great deal of new and key learning for children.

The purpose of this discussion is to reinforce ‘how to make 10’ and ‘the sight facts of 10 and 6.’

Teacher: Now, will $9 + 6$ be more than 10 or less than 10. How will you find the sum of $9 + 6$? What number will fit in the empty space? (Points to the model being displayed.)

Student 1: It is 5.

Teacher: How do you know?

Student 1: I first tried 6. That did not fit. So I knew it was going to be 5. Then I tried and that worked.

Teacher: That is good! Did others try the same strategy?

Many hands go up.

Teacher: Did anyone else try another strategy?

Student 2: I looked at it and I thought it was 5. I tried 5 directly and it worked.

Teacher: How did you know to try 5?

Student 2: I just knew.

Teacher: Does anyone know the reason for trying 5?

Several hands go up.

Student 3: Well, I knew that I took 1 away from 6 to make 9 into a 10, so 5 was left from 6. I tried 5.

Several other children also agreed with her. Several children raise a hand.

Teacher: Let us hear another reason. Karen, what is your reason?

Several children are eager to share their reasons.

Teacher: I will hear everyone’s reason. Let Karen express her reasons.

Karen: I knew that 1 is taken from 6 and given to 9 to make it 10. Then I knew the sight facts of 6 and I knew that it will be 5.

Several children shout out in agreement.

Teacher: Yes, the 5-rod is right. And the reason is also correct.

It is the knowledge of the sight facts of 6 is that gives the best answer. It is the most efficient answer. That is why it fits there. Let us put it there. Everyone does the same thing. Now, let us read the two trains as an equation. Does someone want to read the addition equation?

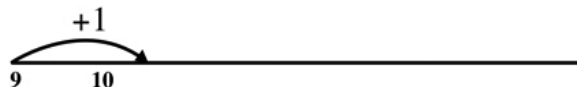


Figure 7.25 Making a ten and adding: Abstract Form of the CPVA Script

Student 1: $9 + 6 = 10 + 5$

Student 2: $9 + 6 = 15$. (Several children shout out the answer.)

Teacher: That is right. Now read the equation made by the two trains. Talk to your partner and when you have the answer, write the equation on your whiteboard or paper and raise your right hand. Remember the right hands of both partners must be up to be recognized for the answer.

Several hands go up. The teacher asks several children to show their white boards. She identifies several correct answers, reads their answers from the displayed answers, and invites one of the teams to come to the board. She asks one of the team members to write the equation and the other partner to read it in four ways and explain how they arrived at the answer.

The team produces the equations:

$$9 + 6 = 10 + 5, 6 + 9 = 10 + 5, 10 + 5 = 9 + 6, \text{ and } 10 + 5 = 6 + 9.$$

Teacher: That is correct. There is another way of writing these equations.

The teacher identifies another team and invites the team to write the equations on the whiteboard in another form. The team writes and reads the equations: $9 + 6 = 15$, $6 + 9 = 15$, $15 = 9 + 6$, and $15 = 6 + 9$.

Teacher: How did you arrive at the answer? Who is going to explain? (She identifies a team.)

Student: I saw that 10 is one more than 9, so I thought of taking the one from 6 and that leaves 5 and 5 and 10 is 15.

Teacher: That is right. Let us try your approach to another problem. What is $9 + 4$?

Several students raise their hands. The teacher identifies one of them.

Student: 13.

Teacher: That is right. Explain your strategy.

Student: I took 1 from 4 and gave it to 9. That became 10 and then I added 3 to 10. The answer is 13. $9 + 4$ is 13.

Teacher: Very good! Give this team a very big hand. Everyone write the four equations for this problem.

After several practice exercises, the teacher articulates the ‘*script*’ of the strategy and applies it to another similar problem.

Script for the strategy:

Problem: Find $9 + 8 = ?$

Q: Which is the bigger number?

A: 9.

Q: How do I make it into 10?

A: Add 1.

Q: Where is the 1 coming from?

A: From number 8.

Q: What is left from 8?

A: 7.

Q: What is $9 + 1$?

A: 10.

Q: What is $10 + 7$?

A: 17.

Q: What is $9 + 8$?

A: $9 + 1$ is 10 and $10 + 7$ is 17. So $9 + 8 = 17$.

Q: What is $8 + 9$?

A: 17.

Q: Recite the four equations for the family of facts of $9 + 8$.

A: $9 + 8 = 17$; $8 + 9 = 17$; $17 = 9 + 8$; $17 = 8 + 9$.

The script is derived from efficient concrete modeling. The concrete molding and the resultant script is the integration of spatial, temporal, tactile, kinesthetic, language, pre-requisite skills, executive functions, and the primary concepts involved in the main concept of deriving the strategy. It is a multi-sensory, multi-modal, and multi-model experience.

This forms a rich *episodic memory* in the brain, and converting this process into a ‘*script*’ has abstracted and generalized the process and therefore, it has been converted into *semantic memory*. Therefore, the *process of the script* is *conceptually strong* and *storage- and retrieval-rich*. This information will be *retained* and can be *retrieved easily*.

The teacher practices the script of the strategy by the usual three activities using the Cuisenaire rods: Choral, individual oral, and individual written, till the strategy is mastered by all children. For the first few examples, the concrete display is there on the board/screen and in front of the children. After that they describe the strategy from *visualizing* the *concrete* and *pictorial* models. That helps in transferring the script from concrete/pictorial to abstract. The emphasis during the practice is not just getting the correct answer, it is to know the strategy being used through visualizing it in the mind’s eye and producing the answer with fluency. The practice continues till all children can give the correct answer first, and then the strategy they used.

(a) Alternative Scenario

Another approach that many teachers take is the following:

Teacher: Now, what other number can fit in the empty place in the train with the 10-rod? (She points to the empty place.)

Children suggest several numbers. The teacher places the suggested number in the empty place. If the answer is incorrect, she asks for another try by asking whether the particular child is going to choose the next one to be smaller or larger than the current choice, and asks for the next choice till the correct answer is found. When a correct answer is found, she acknowledges it. Several children suggest the 5-rod.

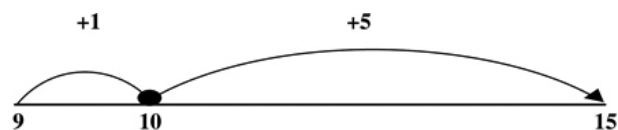


Figure 7.26 Demonstration of $9+6$ Using ENL: Pictorial level of Knowing

This approach produces the correct result, but the approach does not emphasize the use of pre-requisite skills: sight facts, making ten, teens numbers and as a result, they may not transit from concrete to abstract level easily. They may stay at concrete level for a long time. And the approach does not develop a strategy; it only derives the answer. Therefore, this is not an optimal approach.

(b) Differentiation

To differentiate, some children are asked the questions such as:

$$59 + 6 = ?, 19 + 9 = ?, \text{ or } 16 - 9 = ?, 17 - 9 = ?$$

When they give the correct answer, they are asked to explain their strategies. It is important that it is done in public rather than in a small group. When it is brought into public, other children are exposed to novel strategies

and higher order of thinking. Children observe the examples of *expert behavior* being displayed right in front of them.

Practicing a strategy or method without seeing the outcome behaviors and without the display of efficient strategies takes children longer to realize, know, or display such a behavior. The display of expert behavior by teacher alone is not sufficient for many children to acquire efficient and expert strategies. This exposure produces many more higher-order thinkers in any classroom.

(c) Representational Level of Knowing

Teacher: Let us represent this strategy as a picture on an Empty Number Line (ENL). In adding $9 + 6$. I am drawing a line on the paper (rather an array, with the end-point on the left). I want everyone to do the same. We place a dot on the left end and write 9 below it.

The dot on the number line indicates the location of the number 9. That is where number 9 lives. Every number you know has a place on the number line. No number is homeless. As you go from grade to grade, you will see that every point on the number line represents a number and every real number has a place on the number line. At present you are locating only the *counting numbers*. More numbers will join the counting numbers. You should visit me when you are in higher grades and tell me what new numbers you have added to the *number line*. Because numbers live on this line, the name of this line is the ‘*number line*.’

We start with 9, so we locate 9, by indicating it by the dot, and writing 9 below the dot on the empty number line. You do the same. Then we use the script we derived earlier.

Teacher: How do we make 9 into 10?

A: We take 1 from 6 and add to 9.

Teacher: So we take a jump of 1 unit from 9 on the number line and land at another point. Where do we land? What is $9 + 1$? Remember it is a very short jump.

Students: 10.

Teacher: So we write 10 below at the end of the arrow. See my picture and you do the same. We have used 1 from 6, so what is left from 6?

Students: 5.

Teacher: Good! So what is our next jump?

Students: A jump of 5 from 10.

Teacher: Good! Let us show it on the Empty Number Line. You do the same. Remember, 5 is bigger than 1, so it is a longer jump than the previous jump. However, it is not a high jump. What number do we end at? Place [Figure 7.27](#) here.

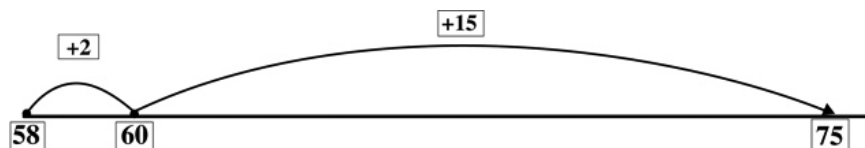


Figure 7.27 Using ENL to add 58 and 17 [📄](#)

Students: 15.

The teacher makes sure that all children have the appropriate ENL. This is the time that the teacher shapes children's behavior about drawing ENL:

- there is a dot on the number line indicating the place (location) for the number on the number line,
- there is an arrow indicating the direction of the jump ($\longrightarrow$ for addition) and ($\longleftarrow$ for subtraction), and

- the approximate proportional size of the jump (for example, look at the ENL displayed the +5 jump is bigger than the +1 jump; the +5 jump is almost 5 times the size of the 1-jump).

Later, we will add few more characteristics of the ENL.

Teacher: So, we have: $9 + 6 = 9 + 1 + 5 = 10 + 5 = 15$.



Teacher: Let us practice few more problems.

The teacher helps children practice a few more problems of the type: $9 + 6$ with the help of the Empty Number Line (ENL). When children have acquired a level of fluency, the teacher introduces the second strategy for adding 9 and a number.

Teacher: Let us take another example with two 2-digit numbers. $58 + 17 = ?$

	9	+	6	=	1	5	

Figure 7.28 $9 + 6$ in the abstract form



Figure 7.29 $N + 9$ alternate strategy part 1 concrete

Before any formal procedure is introduced, the problem should be solved at the concrete and pictorial levels. This sum can be found very easily by the help of Empty Number Line, as earlier.

(d) Abstract Level of Knowing

Teacher: Let us represent this strategy for adding $9 + 6$ on paper. Let us write the problem on your paper, and I will write it on the board. Make sure you are writing each digit and operation symbols in a square on the graph paper.

One of the important components of executive function and parts of higher-order thinking is flexibility of thought. We want children to make connections between concepts, procedures, and skills. We also do not want children to be dependent on a particular strategy, even if it is efficient. When children have more than one strategy, they will experiment and take risks. They become innovative. They invent their own strategies and want to share their strategies. Therefore, we should expose them to alternative strategies. By this practice, they will begin to look for efficient strategies. When they invent/discover a new or change a strategy to make it more efficient, the teacher should name that strategy after the student's name and announce it on the school's public address system. We have found that this boosts the morale of the students.

(e) *Alternative Strategy for $(N + 9$ and $9 + N$)*

Teacher: All of you now know how to find the answer for $9 +$ a number. We want to take the same problem ($9 + 6$) and find the answer another way. You also know that $9 + 6 = 6 + 9$ by commutative property of addition.

Do you remember that one of you told me another way of finding the answer for $9 + 6$? We want to see why it works.

The teacher asks children to make a train for $6 + 9$ by putting them end to end. Therefore, if we know $9 + 6$, we already know $6 + 9$.

Teacher: We have a train for $6 + 9$. Look at your train. Is it smaller than, equal to, or bigger than 10?



Figure 7.30 Demonstrating $N+9$ strategy using Number Rods: Concrete level of Knowing



Figure 7.31 Demonstrating $N+9$ strategy using Number Rods: Concrete level of Knowing



Figure 7.32 Demonstrating $N+9$ strategy using Number Rods: Concrete level of Knowing

Children: Bigger than 10.

Some children actually put the 10-rod next and parallel to the train even before the question is asked. The teacher should acknowledge it and point it out to the whole class. Sharing children's work with the whole class is one of the best ways of developing not only flexibility of thought, but also inviting many children to innovate for themselves.

Teacher: That is right. Place the 10-rod parallel to the train and the 10-rod next to the 9-rod. Look at my arrangement. The starting edge of the orange rod and blue rod are matching.

'Now you do the same. When you are done, let me know by raising your right hand.'

The teacher waits for the children to complete the task. During this time, she helps those who either did not understand the instruction or did not understand the problem or task. She can ask those children who have correctly made the arrangement to help some of these children.

This is the time when other adults in the room (if there are any) can help children, not during the time when the teacher is teaching. At that time, children listen to only one adult, the classroom teacher. During the teacher's instruction time, there is no role for any other adult (paraprofessionals, special educators, volunteers, other children, etc.). During the lesson, it is a whole-class activity, not a small-group activity.

Teacher: Now, take the 6-rod and place it in the empty place in the train. (She points to the empty place.)

Children follow the directions. The teacher makes sure that every child has followed the instruction and they have the same display in front of them as the teacher.

Teacher: Now we have two trains. What do you observe?

Pay attention to the response of children. This may be time to do some formative assessment. Most children will answer it correctly. The next set of questions are introducing a great deal of mathematics language: vocabulary, syntax, semantics, and comparative words. This vocabulary is the basis of preparing for word problems and other applications.

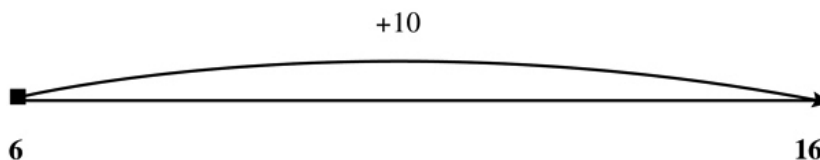


Figure 7.33 Alternate strategies

Teacher: Is the $6 + 10$ train is longer than $6 + 9$ train? What is $6 + 10$?

Students: 16.

Teacher: How much longer is the $6 + 10$ train than the $6 + 9$ train?

Student A: One more.

Teacher: How much shorter (less) is $6 + 9$ train than $6 + 10$?

Students: 1 less.

Teacher: What is 1 less than 16?

Students: 15.

Teacher: Great! Now someone can give me the answer for $9 + 6$.

Student B: 15.

Teacher: Does someone wants to read the addition equation?

Students: $6 + 9 = 15$.

Teacher: How did you arrive at the answer? Who is going to explain?

She identifies a team.

Student C: I saw that 10 is 1 more than 9, so I thought of adding 10 and 6 and I know that $6 + 10$ is 16. That means $9 + 6$ is 1 less than 16. So $9 + 6 = 15$.

Another student raises his hand.

Teacher: What do you want to say?

Student D: To add 9 and another number, first add 10 and the number and then just take away 1 from that.

$10 + 6$ is 16 so $9 + 6$ is 1 less. So $9 + 6 = 15$.

Teacher: Great! Does everyone understand the new strategy?

She practices the strategy and the script for this strategy. After she is satisfied with the display of understanding, she continues the practice.

Teacher: Who is going to give me the family of facts (the four equations) for this problem?

Several children raise their right hands; she identifies one team. She invites them to the board; one writes the equations and the other reads the equations: $6 + 9 = 15$, $9 + 6 = 15$, $15 = 9 + 6$, and $15 = 6 + 9$.

Teacher: That's right. Let us try some problems this way. I will give you a problem; you find the answer and then raise your hand. After that I will ask you, 'How did you arrive at the answer?'

(After they have practiced the strategy by using the rods, she practices it with the Empty Number Line.)

Teacher: Let us draw the Empty Number Line for these problems using the new strategy. Let us add 6 and 9 again. Can someone take the lead? Who wants to be a leader for this problem? What do you start with?

What did we do first?

Students: We added 6 and 10 first.

Teacher: Good! Let us draw that.

She draws it on the board and children draw on their whiteboards or paper.

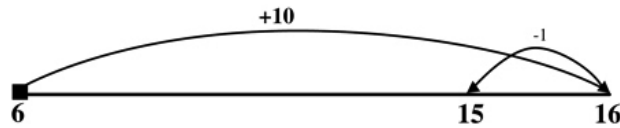


Figure 7.34 Demonstrating N+9 strategy using Number Rods: pictorial level of knowing



Figure 7.35 Demonstrating alternate strategies using Number Rods: Concrete level of knowing

Teacher: So we have $6 + 10 = 16$ on our ENL. We wanted $6 + 9$. Now what do we do? Does anyone have an idea?

Several children raise their right hand. She corrects some children who had their left hand up. She identifies one child to complete the task on the board. The child comes and draws it.

The teacher emphasizes the direction and the result. Children explain their reasoning and again write the family of facts. The teacher practices several problems of this type and strategy using the ENL.

The teacher practices the two strategies by the usual three activities: Choral, individual oral, and individual written, till both of these strategies are mastered—*concretely*, on ENL, and *abstractly*. Children should know both strategies and at all levels (concrete, pictorial—ENL, and abstractly) to find the answer for $9 + N$. This develops flexibility of thought. Later they should be able to find the answer for $9 + 9$ by other strategies also. Until the children have several ways of arriving at the fact correctly, fluently, with an efficient strategy, and are able to apply it, a fact is not mastered completely. When children have this kind of fluency, there is no or minimal regression during the weekend, long vacations, and over summer or during transitions.

The major complaint from teachers, parents, and administration is that many children forget what they have learned fairly quickly after learning the information. They do not remember much after a long weekend, winter breaks, or the summer vacation. Mathematics education literature even has a term for it: ‘summer slide.’ Some summer slide is inevitable due to the nature of memory itself. From the evolutionary perspective, we were not supposed to remember everything we were exposed to. We had to prioritize what to remember; that is a learned behavior. Our memories follow certain rules. Prioritizing is one of them. The *forgetting curve* is well-known.

However, in the case of many children, forgetting is more pronounced. This problem is real, but the cause of forgetting is not the breaks. The cause is the nature of teaching, methods of teaching arithmetic facts, and lack of mastery. The mastery shown by these children before the breaks might have been ‘superficial,’ ‘ill-formed,’ ‘improper or inadequate practice,’ ‘not placed properly in the long-term memory,’ or ‘it was not storage-rich’ or ‘retrieval-rich.’

The mastery might have not reached all component areas of the learning of a concept—understanding, fluency, and applicability. When children have understanding (the concept and related strategies), they have applied the problem in only in a few and in very simple cases and did not have enough practice to reach a level of fluency. They might have had few or no opportunities to apply what they have learned in diverse situations to develop flexibility of thought (more than one strategy to arrive at the answer). In other words, when students’

learning is deeper and more connected, they have deeper roots. Then there is very little chance of regression. The amount of regression is very small when the ‘learning script’ for the method has been internalized.

The other alternative for minimizing learning regression is *over-learning* and *rote memorization*. Over-learning and rote memorization without understanding does not take children far in mathematics. They quickly reach a plateau or lose interest in mathematics. Moreover, the time and effort needed for over-learning is too high to pay in time and human resources. Except for a few self-motivated children, someone has to devote a great deal of time and effort to this. This kind of learning does not engender interest in and love for mathematics.

(f) *Subtracting Nine from a Number*

When children have mastered adding 9 to a number, they are ready for subtracting 9 from a teens number. The teacher briefly reviews the first four strategies and then begins this strategy.

The teacher starts with any teens number and ask the children to subtract 10 from the number and then asks them:

Teacher: Let us consider number 15. What two numbers make 15?

Students: 10 and 5.

Teacher: Great! What is 15 take away 10?

Student: 5.

Teacher: Now if I take 9 from 15, will that be more than 5 or less than 5 or equal to 5?

Student: More.

If students answer this as less, then the teacher should go to the concrete model.

Teacher: How much more?

Student: 1 more.

Teacher: So, if 15 take away 10 is 5, then 15 take away 9 is what?



Figure 7.36 Demonstrating alternate strategies using Number Rods: Concrete level of knowing

Student: 6.

Teacher: Great! Can some one summarize this strategy?

Student: Yes. *Take 9 from a number, first take 10 from it and then add back 1 more.*

The teacher practices using the three activities and brings their attention to the starting number and the ending number. Have children describe what they do to subtract 9 from another number, such as $14 - 9 = 5$.

A student raises her hand. The teacher acknowledges her and asks her to speak. She describes the process of subtracting 9 from 13 by the help of concrete model using Cuisenaire rods, as shown in the following.

Student 1: My mother told me to subtract 9 from a number means just make one more than the ones place number you started with and tens place is 1 less. Let me show you: $18 - 9 = 9$. Let me show you even with bigger numbers: $26 - 9 = 17$, one more than 6 is 7 and one less than 2 is 1, so the answer is 17.

Teacher: That is right. Your mother's strategy works. Can you tell me why does this strategy work?

Student: No! I only know that it works. She did not tell me and I did not ask her.

Another student raises her hand. Teacher responds to the student.

Teacher: Yes! Do you know why this strategy works? Go ahead!

Student 2: It is really the same thing. *You take 10 from the number that makes the tens digit one less and then since we took away 1 too much than 9, therefore, we need to add 1 to the ones digit.*

Teacher: Yes, that is wonderful! Give both of them a big hand.

When a child comes up with an 'approach' of solving a problem that looks like a 'trick,' 'unorthodox,' 'does not seem to be supported by apparent reason,' 'it is different than the strategy in the book' or 'different than the teacher is trying to teach,' it is important for it to be acknowledged.

After the acknowledgment, the teacher should give the child a chance to explain his/her reasoning. If the child does not know the reasoning, the teacher should ask other children to either provide reasoning or provide a counter example where the 'approach' does not work and supply reasoning why it does not work. If children are not able to do so, the teacher should provide a counter example, supply the reasoning, or prove that the 'approach' is equivalent to the book's or her approach. If the teacher is not able to do any of these, she should ask them to hold the thought and advise the class to return to the idea later with appropriate reasoning and discussion.

When children's approaches are not recognized or discussed, they may not develop an interest in mathematics and mathematical reasoning and stop taking risks—an important ingredient for developing a mathematical way of thinking is taking risks. If a child is arriving at a strategy, it is imperative on the part of the teacher to listen to the child.

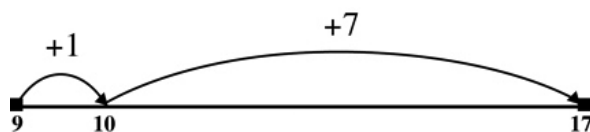


Figure 7.37 Demonstrating alternate strategies using ENL: pictorial level of knowing

The strategy for adding any two numbers, which will end up with a sum more than 10 and less than 20, is to start with the bigger number and take a part of the smaller number to make the first number into 10 and then the remainder of the second number is added to 10 to give the sum of the two numbers as a teens number.

Teacher: Now, we have three ways of adding to or subtracting 9 from a number. Let us practice the three ways. Let us also complete ENL for subtraction for a problem like the ones we have been doing for addition. Let us consider the subtraction problem: $17 - 9$. See if you can complete one for me. As soon as you have one ENL completed, raise your right hand, and I will come and check your ENL. Remember to display and demonstrate all the necessary elements for the ENL process.

The teacher walks around the classroom watching children complete an ENL for the problem, helping some children start an ENL, making comments to others so that they look at the problem or approach it more deeply. For non-starters, the teacher provides them with minimal hints but a lot of encouragement. This is the time for the teacher to do some formative assessments and differentiation by observing children's work and listening to their discussions.

One student raises her hand to show her ENL. The teacher checks it. The child's ENL and the discussion between them is as follows:

Teacher: Evelyn, your ENL is perfect for this problem. Great job! Now write the equations for this problem: Can you share with me your reasoning? (The child provides the reasoning and writes the equations: $17 - 9 = 8$, $8 = 17 - 9$.)

Teacher: Can you write your subtraction equation as an addition equation?

Student: Yes! $8 + 9 = 17$ and $9 + 8 = 17$.

Teacher: Great! Later I would like you to share your ENL with the class. Now, can you create another ENL for the same problem with another decomposition/recomposition strategy?

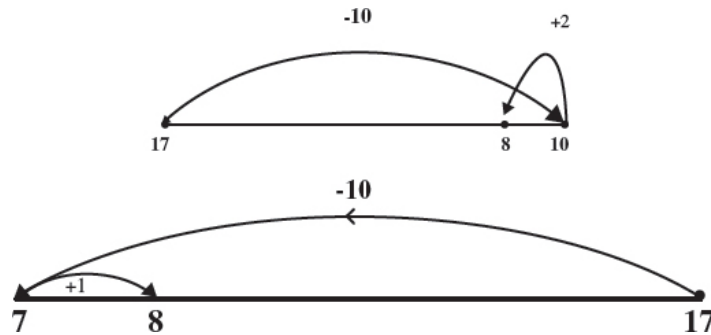


Figure 7.38 Demonstrating alternate strategies using ENL: pictorial level of knowing

The teacher continues her work with the children. Several other children raise their hands. The teacher checks their work. Most of them have the same kind of ENL. She asks all of them to meet with Evelyn and see if they share the same reasoning. One of the children had a different ENL. She asks for her reasoning. Her ENL is as shown above:

Teacher: Your answer is correct and the process is very good. However, you have shown 17 to the left of 10 and 8 is between 17 and 10. Is that possible? Could you correct the orientation of the ENL and show me the corrected one?

Another child raised his hand. She recognizes him.

Child: Can I do it as an addition problem? I found the answer to the subtraction problem, but I did it by using addition. Is this right?

Teacher: Yes! Actually, David has already done one ENL that way. Share yours with David and see if this is different or the same. If it is different, let me know. If they are the same, then together, you can come up with another strategy.

She connects children with each other and ask them to share their ideas. When several ENLs of different types have emerged, she asks children to share their ENLs, their approach, and their reasons for that approach. All the ENLs are displayed on the whiteboard or under the document camera so that all children are exposed to all possible ENLs in the class.

The teacher practices adding and subtracting 9 to or from a number, using the three activities: Choral, individual oral, and individual written. She does a summative assessment of this strategy when children have become fluent in the strategies at Cuisenaire rods, Visual Cluster Number cards, ENL, oral, and written form as a whole class. The children who are having trouble are provided with extra practice and they will continue to receive this from her and the interventionists. The teacher asks the pupils to discuss what is similar in the different strategies and what is the same.



Figure 7.39 Doubles facts of 2, 3, 4, and 5

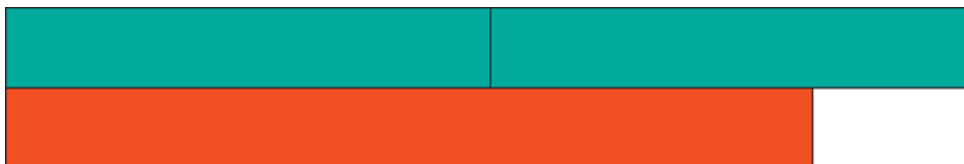


Figure 7.40 Doubles using number rods

6. $N + N$ (Double Numbers)

[$2 + 2$, $3 + 3$, $4 + 4$, $6 + 6$, $7 + 7$, $8 + 8$]

(6 new facts on the Addition Grid; 3 are sight facts and 3 are new.)

Most children learn ‘double numbers’ quite easily since the use of doubles is quite common in our culture. There are many words in the English language for two {pair, double, couple, duo, twice, etc.}. These are sometimes the first set of addition facts that they memorize quite easily. These facts are also a good way to introduce the concept of even numbers.

The addition facts $2 + 2$, $3 + 3$, and $4 + 4$ are sight facts. The teacher can review the sight facts by going back to the Visual Cluster Number cards.

The doubles of 1, 2, 3, 4, and 5 can easily be seen by the Visual Cluster cards and Sight Fact cards because of the symmetry of displays on the 2-, 4-, 6-, 8-, and 10-cards. We have already discussed the doubles of 1 and 5, during the sight facts study and later the double of 5 in making ten also.

Because of the symmetry of the arrangements of numbers on many objects in real life, children find adding double numbers to be easier than other addition facts. This is particularly so with the earlier double numbers. Therefore, it is easier to teach the double strategy.

Once these facts are mastered the teacher helps children to derive the other three double facts using Cuisenaire rods. The teacher asks children to show $6 + 6$ by making the train.

The teacher introduces the idea of double numbers by giving and asking for examples of familiar situations. She derives the first few double number facts using Visual Cluster Number cards and then introduces the process for finding the doubles of 6, 7, and 8. She asks children to model $6 + 6$ on their desks in pairs. She examines that everyone has made the train of 2 ‘6-rods.’ Then she asks:

Teacher: Is the $6 + 6$ train longer (more) than, equal, or less than 10?

Students: More than 10.

Teacher: Place the 10-rod parallel to the $6 + 6$ train just like mine.

We can now say that $6 + 6$ is greater/more than 10. The $6 + 6$ train is how much more than 10? Remember the strategy of making ten and also the sight facts of 10. How do you make the 6-rod into a 10-rod?

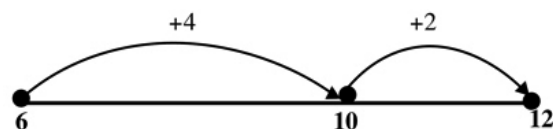


Figure 7.41 Doubles using making ten and adding



Figure 7.42 Doubles using making ten and adding

Students: $6 + 4$.

Teacher: Good! Where does 4 come from?

Students: From 6.

Teacher: Good! What is left in 6?

Students: 2.

Teacher: Very good! So 10 is how much smaller than $6 + 6$?

Students: 2!

Teacher: What rod can fit in the empty place in the $6 + 6$ train? (Points to the empty place).

Children: The 2-rod.

Teacher: Place the 2-rod in the empty place. The $6 + 6$ train is now how long?

Children: $10 + 2$ or 12.

Teacher: Good! So $6 + 6 = 10 + 2 = 12$.

Now, the teacher asks the students to say it in two ways. Students write the two equations:

$$6 + 6 = 12, 12 = 6 + 6.$$

Teacher: As we can see, it seems that to make 6 from 10, we are cutting off (taking away) 4 from the second 6. Since we already know that 4 and 2 make 6, we have $6 + 6 = 6 + 4 + 2 = 10 + 2 = 12$.

So, we have a strategy for adding double numbers: Let us read the two trains together as equations:

- 6 plus 6 equals 12
- Double of 6 is 12
- 6 and 6 is 12
- Two groups of 6 is 12

Let us write the 'script' for the strategy:

Problem: Find the sum: $8 + 8 = ?$

Q: Which is the bigger number?

A: Both are same. We can take either one. 8!

Q: How do I make 8 into 10?

A: Add 2.

Q: Where is the 2 coming from?

A: From the second 8.

Q: What is left from 8?

A: 6.

Q: What is $10 + 6$?

A: 16

Q: What is $8 + 8$?

A: $8 + 2$ is 10 and $10 + 6$ is 16. So $8 + 8 = 16$.

Teacher: I forgot something very important about double numbers. Do you want to know?

Children: Yes! Yes! What is it? Please! Please!

Teacher: OK! Does anyone know the word half or one-half? (Almost every hand goes up.) Can someone give me an example of half or one-half? Several children raise their hands.

Teacher: Yes, Lydia! Give me your example.

Lydia: I shared half of my cookie with Karen at lunchtime.

Several examples are solicited by the teacher, and examples are shared with the class.

Teacher: Can someone tell me, if the train of 12 ($6 + 6$) was a big, long cookie, what is half of 12?

Several children laugh and someone makes comment about the two green cookies visible on the screen. The teacher asks one of the children to come to the front and point to the half-cookie. She quickly identifies one of the 6-rods as the half-cookie.

Teacher: What is half of 12?

Student: 6!

Teacher: Can you say this as sentence?

Student: A half of 12 is 6.

Teacher: Great! Half of 12 is 6. Let us say all the equation aloud.

Students:

- 6 plus 6 equals 12
- Double of 6 is 12
- 6 and 6 is 12
- Sum of 6 and 6 is 12
- Two groups of 6 is 12
- Half of 12 is 6.

Once a strategy has been derived with understanding, then one should practice the strategy with the concrete material, Empty Number Line, and then abstractly. And finally, only abstractly.

To understand and to memorize these facts, the following is helpful. Let us use the strategy to generate the Empty Number Line.

Teacher: Let us show this by ENL. Everyone should draw the Empty Number line for $6 + 6$. What do we start with?

Students: Start at number 6 and take a jump of 4 from 6, because we take 4 from the second 6.

Teacher: What do we do next?

Students: We take a jump of 2 from 10 for the 2 left from the 6. We land at 12.

Teacher: So we get $6 + 6 = 12$. Now draw the ENL for $7 + 7$.

For any concept and procedure to be mastered by children, we need to provide supervised and then independent practice. Deriving $6 + 6$ was an example of demonstration. Whereas during the demonstration process, the teacher is practicing *didactic* and *Socratic* roles of a teacher, in the practice activity, the emphasis is on the *coaching* and *Socratic* roles. The role of language and the questioning process are the key elements in both situations.²⁰ The next example is to make sure that the process has been understood by children, and that the script is being memorized and all of the behavior markers of competent performance are being shaped. To find, for example, $7 + 7$, along with the questions already introduced in the demonstration process, we can ask the following questions.

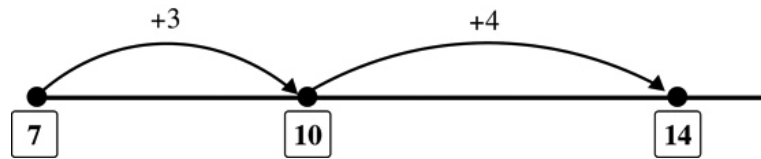


Figure 7.43 7+7 using decomposition/recomposition

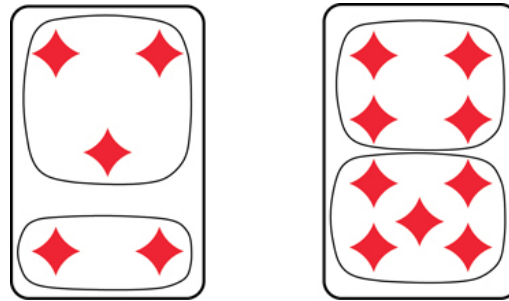


Figure 7.44 Doubles + 1 Pictorial

Teacher: What do we need to make 7 as 10?

Students: Add 3.

Teacher: If we take the 3 from the second 7 and give it to 7 to make it 10, what is left of the second 7?

Students: 4.

Teacher: So now, what is $7 + 7 = ?$

Students: 14. Make the first 7 as 10. We take 3 from the second 7 and 4 is left in the second 7. I know $3 + 4 = 7$. Then add 10 and 4.

Teacher: That is right. $7 + 7 = 7 + (3 + 4) = (7 + 3) + 4 = 10 + 4 = 14$. Let us see the ENL for $7 + 7$.

Teacher: Now, every pair will practice $9 + 9$ by showing to their partner at four levels: (a) with Visual Cluster Number cards, (b) Cuisenaire rods, (c) Empty Number Line, and (d) Abstract level on paper. Remember, we have already done it in strategy #5.

The teacher practices adding and subtracting double numbers using the activities: Choral, individual oral, individual written, and cumulative written exercises.

7. $N + N + 1$ and $N + N - 1$ (Near Doubles Strategy)

[$2 + 3$, $3 + 2$; $3 + 4$, $4 + 3$; $4 + 5$, $5 + 4$; $5 + 6$, $6 + 5$; $6 + 7$, $7 + 6$; $7 + 8$, $8 + 7$]

(12 new facts on the Addition Grid)

These are extensions of the facts from the double's strategy. Once again $2 + 3$, $3 + 2$, $3 + 4$, $4 + 3$, $4 + 5$, and $5 + 4$ are sight facts and are best seen by the help of Visual Cluster Number cards. Therefore, the development of this strategy should begin with the Visual Cluster cards and Sight Fact Cards. Show the Sight Fact cards that are doubles: The 2-card ($1 + 1$), 4-card ($2 + 2$), 6-card ($3 + 3$), 8-card ($4 + 4$), and 10-card ($5 + 5$). Then show the Sight Fact cards that are near doubles ($2 + 3$; $3 + 4$; $4 + 5$) and have a discussion about the relationship of doubles with near doubles and relate them with the doubles card: By asking them to show doubles on the near doubles cards, by hiding a pip or adding one pip.



Figure 7.45 Demonstrating near doubles strategy using number rods : concrete level of knowing

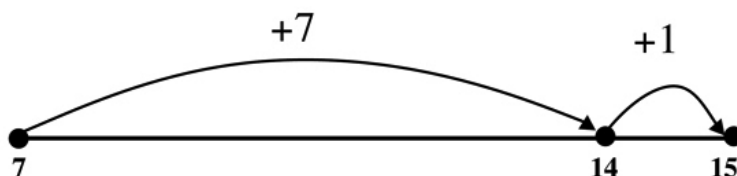


Figure 7.46 Demonstrating near doubles strategy using ENL: pictorial level of knowing

In this case, there are two strategies for deriving the remaining near double facts: $5 + 6$, $6 + 5$; $6 + 7$, $7 + 6$; $7 + 8$, $8 + 7$. And because of the commutative property of addition, we need to derive only three facts: $6 + 5$, $7 + 6$, and $8 + 7$.

a. Using Making Ten Strategy

Let us take $7 + 6$.

Let us apply the Script for adding using the ‘making ten’ strategy.

Script for the strategy:

Problem: Find $7 + 6 = ?$

Q: Which is the bigger number?

A: 7.

Q: How do I make 7 into 10?

A: Add 3.

Q: Where is the 3 coming from?

A: From the 6.

Q: What is left from 6?

A: 3.

Q: What is $10 + 3$?

A: 13.

Q: What is $7 + 6$?

A: $7 + 3$ is 10 and $10 + 3$ is 13. So $7 + 6 = 13$.

The Making of Ten strategy is a ‘generic strategy,’ and every child should know this strategy for any two numbers whose sum is under 20.

(a) Near Doubles Strategy (Doubles + 1 Strategy)

Since children know the doubles, those facts are used to learn the near doubles facts. The first goal in adding the two numbers that are near doubles is to introduce the ‘concept of near doubles’ by using as examples the sight facts that a child already knows. The following questions can help children to realize the nature of near doubles.

Teacher: Give me an example of a double fact.

Students respond with several pairs: $4 + 4$, $9 + 9$, $1 + 1$, etc.

Teacher: Yes, you are right. If $4 + 4$ is a double, then what should we call $3 + 4$ or $4 + 5$?

Student A: They are like doubles. They look like close to doubles or near doubles.

Student B: They will become doubles if we change one of the numbers by adding 1 to it.

Student C: One is before the double, the other is after the double.

Student D: I know! I know! They are called near doubles.

Teacher: You are right! *They are called near doubles.* They can also be called next to doubles. Give me another example of near doubles and the doubles fact near it.

Student E: $7 + 8$ is a near double; it is near the doubles $7 + 7$ and $8 + 8$.

Student F: $1 + 2$ is a near double it is near the doubles $1 + 1$ and $2 + 2$.

Student G: They can also be big numbers, like: $20 + 21$ or $100 + 101$.

Teacher: You are right. These are examples of near doubles. Let us take the near double pair: $7 + 8$. Please take the 7-rod and 8-rod and make a train by putting them end to end.

The teacher and children form the $7 + 8$ train.

Teacher: What double pair is near the pair 7 and 8?

Students: $7 + 7$ and $8 + 8$.

Teacher: Now place the $7 + 7$ train parallel to the $7 + 8$ train and make sure the *joints line up*. Look at my arrangement. What do you see?

Student A: This shows that $7 + 8$ is one more than $7 + 7$.

Teacher: What is $7 + 7$?

Student B: 14.

Teacher: Then what is $8 + 7$?

Student C: One more than 14. So $8 + 7$ is 1 more than 14. $8 + 7 = 15$.

Teacher: What if we used the other double: $8 + 8$. Let form the trains of $8 + 7$ and also $8 + 8$ in the same way as last time—with the joint being the same.

Teacher: Look at my arrangement. What do you see?

Student C: This shows that $7 + 8$ is one less than $8 + 8$.

Teacher: What is $8 + 8$?

Student D: 16.

Teacher: Then what is $8 + 7$?

Student E: One less than 16. So $8 + 7$ is 1 less than 16. $8 + 7 = 15$.

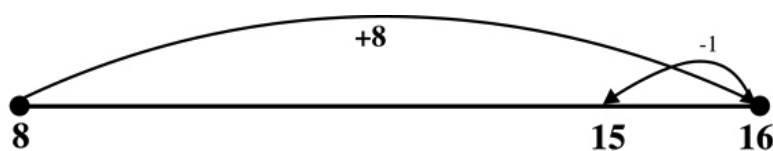


Figure 7.47 Demonstrating doubles+1 strategy using ENL: pictorial level of knowing

The teacher uses the same type of questioning process as used in the previous strategies and arrives at the four addition equations for near doubles: $7 + 8 = 15$, $8 + 7 = 15$, $15 = 7 + 8$, $15 = 8 + 7$.

Students predict and then prove the sum for near doubles in several cases. The teacher then relates the sum of the near doubles to the sum of the nearby doubles: $8 + 7 = (7 + 1) + 7 = (7 + 7) + 1 = 14 + 1 = 15$; $8 + 7 = 8 + (8 - 1) = (8 + 8) - 1 = 16 - 1 = 15$.

To Find the Sum of Near Doubles

First determine if the two numbers are near doubles.

Next determine the pairs of doubles that are near it.

Recall the sum of doubles and then add one more or one less.
Read the four equations for that pair of near doubles.

The script for near doubles is:

Let us show by the help of ENL.

Teacher: Let us construct ENLs for the near doubles pairs. Let us take the same problem: $7 + 8$. What do you start with?

Students: 7.

Teacher: What do we do next? What are the doubles near it?

Students: $7 + 7$ and $8 + 8$.

Teacher: What do you do next?

Student: You take a jump of 7 from 7 and then a jump of 1.

Teacher: That is very good. Any other way we can construct another ENL?

Student: We could start with 8 and then a jump of 8 and then a jump back of 1.

Teacher: That is right. $7 + 8$ can be found in two ENLs. We are going to practice a few more problems using more ENLs.

(b) Doubles in Subtraction

The teacher should use questions like the following,

Teacher: What two same rods make eight?

Students: $4 + 4$.

Teacher: Can we then say that: $8 - 4 = ?$

Students: 4.

Teacher: What two same rods make ten? So $10 - 5 = ?$ Etc.

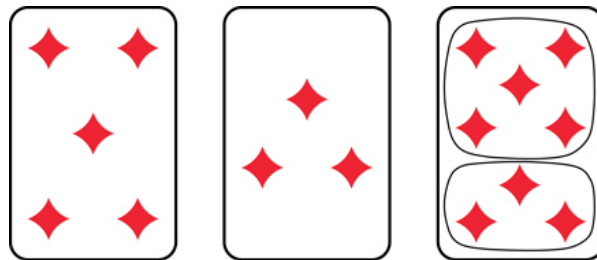


Figure 7.48 Demonstrating double of the middle strategy using VCNC : Pictorial level of knowing

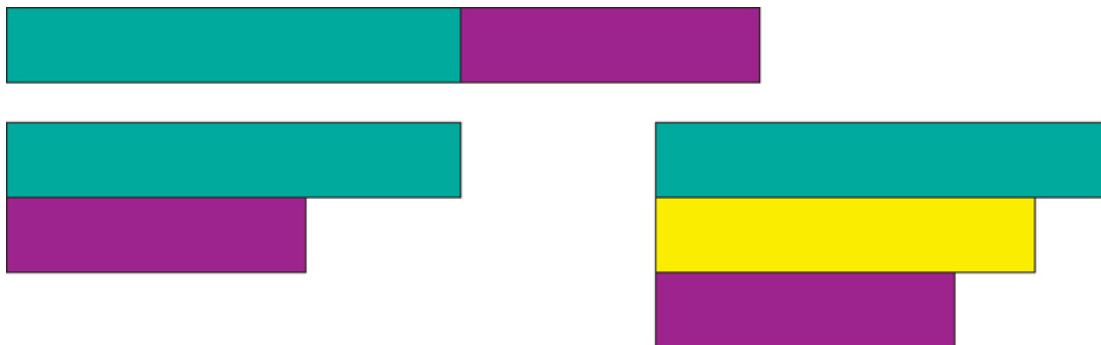


Figure 7.49 Demonstrating double of the middle strategy using number rods : concrete level of knowing

Repeat with many other double number pairs. Repeat it with visual cluster cards. Ask many questions about double and half. What is double of 4? What is $\frac{1}{2}$ of 12?

Then the teacher practices adding and subtracting near double numbers, using the three activities: choral, individual oral, and individual written.

7. $N + (N - 2)$ (Adding Numbers That Are Two Apart, Then Their Sum = Double of the Middle)

[2 + 4, 4 + 2; 3 + 5, 5 + 3; 5 + 7, 7 + 5; 6 + 8, 8 + 6]

(8 new facts on the Addition Grid)

These are extensions of the facts from the doubles strategy. It is again important to notice that 2 + 4, 4 + 2, 3 + 5, and 5 + 3 are sight facts and are best learned by the help of Visual Cluster cards. Therefore, there are only four new facts to be learned: 5 + 7, 7 + 5; 6 + 8, 8 + 6. By using the commutative property of addition, these four facts are reduced to learning only two facts: 5 + 7 and 6 + 8.

(a) Using Making Ten Strategy

The Making Ten strategy is the first step of this strategy. It is the default strategy, and other alternative strategies are developed after this strategy has been mastered.

We need to derive only two of them: 7 + 5 and 8 + 6. This information makes the task manageable, and children feel that it is not difficult to learn them. The script for this strategy is the same.

Script for the strategy:

Problem: Find $8 + 6 = ?$

Q: Which is the bigger number?

A: 8!

Q: How do I make 8 into 10?

A: Add 2.

Q: Where is the 2 coming from?

A: From 6.

Q: What is left from 6?

A: 4.

Q: What is $10 + 4$?

A: 14

Q: What is $8 + 6$?

A: $8 + 2$ is 10 and $10 + 4$ is 14. So $8 + 6 = 14$.

(b) Double of the Middle Strategy

The strategy is dependent on knowing doubles. We saw an example of this already when we added 4 and 6; the sum was 10. $4 + 6 = 10 = 5 + 5$. Here 4 and 6 are two apart and their sum is the double of the number in the middle. Similarly, $2 + 4 = 3 + 3 = 6$.

These pairs of numbers are special. Here are the pairs that are two apart. There is a number missing between them. In this list, the pair 3 + 5 and 5 + 3 are examples of sight facts. This fact can be seen by observing the Visual Cluster cards for 5, 3 and then 8. We can clearly see the 5-cluster and the 3-cluster on the 8-card, showing that $3 + 5 = 8$ and $5 + 3 = 8$. But another form of the 8-card is also $4 + 4$. Thus $5 + 3 = 4 + 4 = 8$.

Before adding the remaining facts in the prior list, first we want to take a pair that satisfies the same characteristic, but the sum is already known to us as we have already added these numbers by a previous strategy.

Let us take the 4 and 6 pair and take the corresponding rods. We already know that $6 + 4$ is 10 from the previous strategy: Making 10. Now we want to find the answer by another strategy. Just like the other pairs in this section, the two rods are two apart. There is a number between them. The number missing between 4 and 6 is 5.

Teacher: For $6 + 4$, which rod is between the 6 and 4 rods?

Students: 5 is between 4 and 6.

Teacher: Good! That is right. This is called the missing middle number. Each one of the pairs in this section has a missing double. What is the missing double between 5 and 7 or 7 and 5?

Students: 6.

Teacher: Good! What is the missing double between 6 and 8 or 8 and 6?

Students: 7.

Teacher: Good! What is the missing double between 7 and 9 or 9 and 7?

Students: 8.

Teacher: Good! Remember we already know how to find the sum $9 + 7$ by the adding 9 strategy. We will find this again by this new strategy.

Teacher: To add 6 and 4, we put the rods end to end. Look at my arrangement and make a train just like mine.

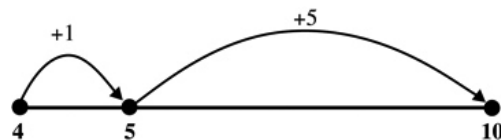


Figure 7.50 Demonstrating near doubles addition strategy using ENL: Pictorial/Visualisation level of knowing

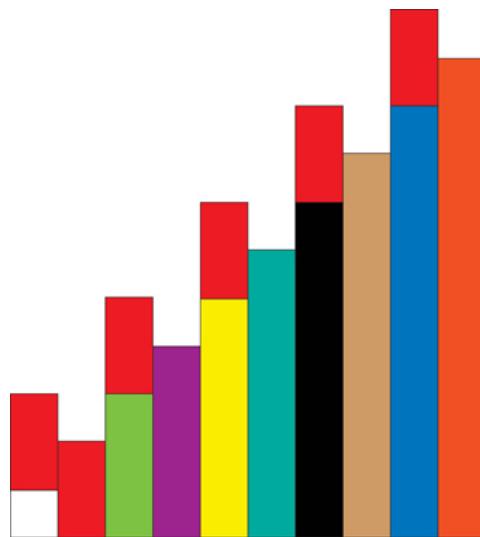


Figure 7.51a Demonstrating $N+2$ addition strategy using number rods: concrete level of knowing



Figure 7.51b Demonstrating sight facts of number 5 using number rods: concrete level of knowing



Figure 7.52 Demonstrating sight facts of number 5 using number rods: concrete level of knowing

‘Now, place the missing middle number 5 parallel to the two rods and next to the 6-rod.

Now you can guess what rod can fit there in the empty place.’ (Points to the empty place.) ‘What rod will fit there?’

Students: 5.

Teacher: That is right. Let us put the 5-rod in the empty place. What is the answer?

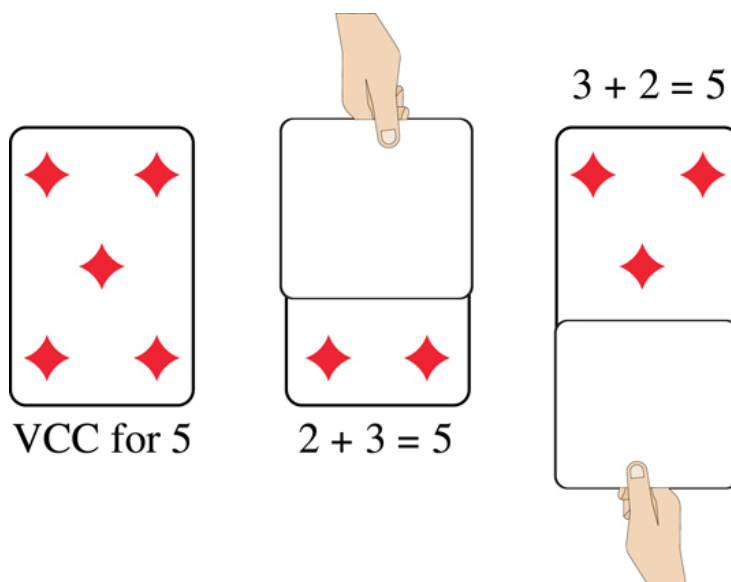


Figure 7.53 Demonstrating N-2 strategy of addition using VCNC : Pictorial level of knowing

Students: 10. We already know that $5 + 5$ is 10, so $6 + 4$ is also 10.

Teacher: We were able to change the problem into a doubles problem. Whenever there are two numbers that are 2 apart, their sum is the double of the middle number. You can also see that if you take 1 from 6 and give it to 4, then 4 becomes 5 and 6 becomes 5. $6 + 4 = (5 + 1) + 4 = 5 + (1 + 4) = 5 + 5$.

The teacher should help children to derive this relationship rather than just give it to them. The discussion and the questioning process is very important. They should finally arrive at the strategy:

For the numbers that are two apart, the sum is double the middle number. Then the teacher should practice this strategy: When the numbers are 2 apart, take 1 from the bigger number and give it to the smaller number. Now both numbers are equal and the sum is equal to the double of the missing number.

Teacher: Using this strategy, let us see this on ENL. Look at these Empty Number lines and explain to me how we arrived at our answer.

When children are able to provide reasoning for these ENLs, she takes larger numbers with the missing double strategy. To differentiate, she can take even two- and three-digit numbers.

Children themselves discover some of their own strategies. Sometimes, those are quite efficient, effective, and elegant. After testing the strategy suggested by a child or their mother/father/previous teacher/tutor (the source of the strategy), I write it on a poster paper and name the strategy after the source. For example, the child said $8 + 6 = 8 + 8 - 2 = 16 - 2 = 14$. I named it as:

*Megan's strategy for adding two numbers that are two apart:
Double the bigger number and 'take two' away from the sum.*



Figure 7.54 Demonstrating near tens strategy of addition using VCNC : Pictorial level of knowing

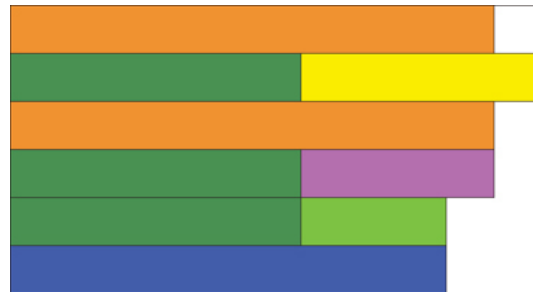


Figure 7.55 Demonstrating near tens strategy of addition using number rods : concrete level of knowing

The poster remained there for several months—till the parent/teacher conference. Children compete to form conjectures, rules, methods, and formulas. Sometimes they would go to other textbooks and find alternate formulas, methods, and rules. Children whose posters are displayed brought their parents to see their contribution to the parent/teacher conferences. Whenever possible, I will ask teachers to make sure that they make announcements on the public address system of these contributions to mathematics by her children.

8. $N + 2$, $2 + N$ (Adding 2 to a Number Means Skip a Number)

[$5 + 2$, $2 + 5$; $6 + 2$, $2 + 6$; $7 + 2$, $2 + 7$]

(6 new facts on the Addition Grid)

All of these facts are sight facts; if a child knows his/her number concept, s/he automatically knows these facts. Remember, number concept means:

- number names in sequence,
- one-to-one correspondence in sequence,
- having the images of the visual clusters for numbers 1 through 10 in the mind's eye, and
- able to decompose and recompose numbers from 1 to 10 at will

(e.g. , $9 = 1 + 8 = 2 + 7 = 3 + 6 = 4 + 5 = 5 + 4 = 6 + 3 = 7 + 2 = 8 + 1$).

Again, reviewing the Visual Cluster Number cards is the best way to achieve these sight facts.

We will still look at these facts using Cuisenaire rods.

Teacher: Make your staircase and place the 2-rod on the 1-rod, 5-rod, 6-rod, and 7-rod. Observe the following arrangement, when we add 2 to 1-rod, it is equal to the 3-rod and the 10-rod. What do you observe?

Student A: The placing of the 2-rod on a rod, it makes the number 2 more.

Student B: They have become 2 more than the rod where we place them.

Student C: You are adding 2 each time.

Teacher: That is right! We have $1 + 2 = 3$. We skipped number 2. Add 2 to 4, we skipped 5 and we get 6. Add 2 to 7, we skipped 8 and we get 9. This means adding 2 to a number, we skip a number and the next number is the sum.

Student: That means adding to a number means skip the next number, and the answer is the number after that. So, add 2 to 7, skip 8 and the number 9 is the answer to $7 + 2$. $7 + 2 = 9$, $2 + 7 = 9$.

Teacher: That is great. To find a number plus two or two plus a number, we just skip the next number (after the bigger number). What is two more than five ($5 + 2$)?

Students: 7.

Teacher: Yes! You know one more is the next number, then 2 more is the answer. What number did we skip to get to our answer for $5 + 2$?

Students: We skipped 6.

Teacher: Show me on your staircase by placing the 2-rod on top of the 5-rod. How tall are the two rods together? What is the sum of the two rods together?

To practice, the teacher picks a rod and asks: ‘What number is two more than this?’ She repeats it with many more numbers. She repeats the activity with Visual Cluster number cards, using questions like the following:

- What is two more than 3?
- Show me the card that is two more than 5.
- Show me the fact by the help of your staircase.

S/he should repeat the activities for many more examples. S/he should ask one question or more from each child in turn. The practice should follow the same format: choral, individual oral, and individual written and differentiation is done by changing the questions.

(a) Mastering Number Minus Two

The first objective in teaching the concept of subtraction is the language itself. The language of subtraction is not always direct. Many times it is difficult and complex.

In the English language, there are many more words and phrases for subtraction than addition and, moreover, it is a ‘whole-to-part’ activity rather than ‘part and part to whole’ like addition. The teacher should relate subtracting 2 from a number to adding 2 to a number. S/he can use the questions like the following:

- What is 2 less than 4?
- How do you write 2 less than 4?

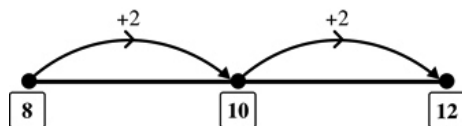


Figure 7.56 $8 + 4$ on ENL

- Show me on your staircase 2 less than 4.
- Locate the 4-rod on your staircase, and then touch the rod that is 2 less than the 4-rod.
- What is 2 less than 5 ($5 - 2$)?

Teacher: Take the 5-rod from your collection and place two rods that make the 5-rod; make sure one of the rods is the 2-rod. Your arrangement should look like mine. You have two trains: 5-rod and $2 + 3$ train.

What is the other rod?

Students: 3-rod.

Teacher: Now read it in four ways. $2 + 3 = 5$, $3 + 2 = 5$, $5 = 2 + 3$, and $5 = 3 + 2$.

Now, you can take either one of the 2 or 3 from the $2 + 3$ train. If you take the 2-rod away from the $2 + 3$ train, what rod is left in that train?

Students: 3-rod.

Teacher: If you take 2 away, what number is left?

Students: 3.

Teacher: Very good! Now let us write this as a ‘take away’ or ‘subtraction equation.’

Students: $5 - 2 = 3$.

Teacher: Now, if you take the 3-rod away from the $2 + 3$ train, what rod is left in that train?

Students: The 2-rod.

Teacher: Now say it in terms of numbers.

Students: If we take 3 away from 5, what number is left?

Teacher: Very good! Now let us write this as a take away or subtraction equation.

Students: $5 - 3 = 2$.

Teacher: Using the numbers 5, 3, and 2, we have two subtraction equations. Write the two subtraction or take away equations.

Students: $5 - 2 = 3$ and $5 - 3 = 2$.

Teacher: Let’s write the addition and subtraction family of facts: Remember, there were four equations with addition and two with subtraction. There are six equations. I will write them on the whiteboard and then I will erase them and then you will write them in pairs and then individually.

The teacher writes the equations as students suggest them to her. She helps them to generate them.

$2 + 3 = 5$, $3 + 2 = 5$, $5 = 2 + 3$, $5 = 3 + 2$, $5 - 2 = 3$, and $5 - 3 = 2$.



Figure 7.57 A Concrete example of Sight Facts of 5

Then children write them on their white boards. Some children still need the rods to create the equations. The teacher, other children, or other adults²¹ can help them. This is repeated for several problems.

Teacher: Take the 5-rod from your collection. Place the 3-rod parallel to the 5-rod. There is an empty place.

We can write this situation as: $5 = 3 + \underline{\quad}$ or $3 + \underline{\quad} = 5$. What rod will fit there?

Students: The 2-rod.

The teacher demonstrates it by bringing the 2 rod into the empty place. S/he repeats this exercise using many other examples. Ask one or more questions from each child in turn. Practice this with Visual Cluster cards as well.

The teacher picks a card and asks the students to pick the card that is two less than her card. She hides a portion of the card and asks: What number is hiding? What number with the number showing will make it 5?

Teacher: I am showing a number (holds a card in her hand or places it under the document camera); pretend that you are covering the bottom two of the five stars showing on the card. What is visible to you then? What is left if you cover the bottom 2 objects on the card? What is that number? What if you cover the top two of the five stars of the card, what is left? What is that number? What if you cover the first column of the card; what is left? What if you covered the last column; how many are left? What is that number? Please pick up the card that is two less than my card. Please pick a rod that is two less than my card.

These questions are converted as a lesson plan—an oral script for action to be transferred to children.

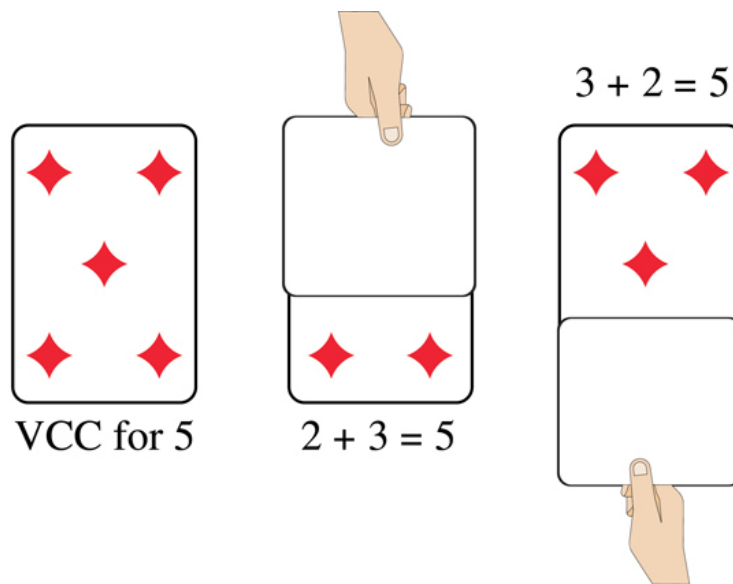


Figure 7.58 An example of N - 2 strategy

Count if the numbers are very close: Using two rods, which differ by one or two, e.g., $6 + 2$, $6 - 2$ or $7 + 2$, $7 - 2$, children can count up or down. If children have experience with Visual Cluster cards and Cuisenaire rods, they will not need to count.

However, it should be remembered that the counting may be encouraged only in these initial situations. One should move to look for patterns, and the result should be derived by observing a pattern. One should use the same strategies as we used for learning sight facts originally. After all, the facts in this section are sight facts. Whenever sight facts are involved in an addition or subtraction problem (and later in multiplication and division), we should always go to the Visual Cluster Number cards and review the sight facts of the number involved). For example, $6 + 1$ and $5 + 2$ are sight facts of 7; they should be automatic with no counting needed. If a child has regressed or counting is his default system, proper intervention is needed right away. So we should get the 7-card and review the sight facts of 7 ($1 + 6$, $2 + 5$, $3 + 4$, $4 + 3$, $5 + 2$, $6 + 1$), not just those two facts in question.

9. Near Tens (Sum of Two Numbers, That is 1 More or 1 Less than 10; the Sum is 9 or 11)

[$3 + 6$, $6 + 3$; $4 + 7$, $7 + 4$; $3 + 8$, $8 + 3$]

(6 new facts on the Addition Grid)

It should be noted that $3 + 6$ and $6 + 3$ are sight facts. Therefore, there are only four more new facts. This strategy could be introduced immediately after children have learned the strategy of making ten. As they give the facts with the sum being 9 or 11, the teacher should ask the children to refer to the ‘making ten’ strategy and the sight facts of 9.

The facts: $6 + 3$ and $3 + 6$ are sight facts of 9 and should therefore be reviewed with the help of the 9-Visual Cluster card. For the other facts, we use the Cuisenaire rods and arrive at the strategy.

The following sample shows the comparison of 10-rod to the $6 + 4$ rods that are also equal to 10. By taking one away from the 4 to make 3, the problem is $6 + 3$ and since we took one away the sum is 1 less than the previous sum, it becomes clear that $6 + 3 = 9$. Similar displays can be used to form the other equations. Similarly, if we make 4 as 5, then the sum should be 1 more than 10. That means $6 + 5 = 11$.

These facts are the ‘near ten’ facts. Their sum is either 9 or 11. These facts are learned by relating them to the ten-rod and the combinations of ten.



Figure 7.59 An example of Near Tens Strategy

10. Remaining Four Facts

[$8 + 4$, $4 + 8$; $8 + 5$, $5 + 8$]

Examining the results of school system wide and statewide data from state examinations, our research and observations in many classrooms, and comments from veteran teachers show that many children have extra difficulty learning these four facts. They resort to counting in deriving their answers. These ‘remaining four facts’ need to be derived individually, but whenever possible, relate the new fact to a known fact.

The key strategy for deriving these facts, once again, is the *Making Ten strategy* that calls for: the decomposition of the smaller number (4) to make the larger number as ten (8 into 10) in deriving ($8 + 4$). And then recomposition of ten and the remainder from the smaller number.

However, because of the commutative property of addition, we have to derive only two facts: $8 + 4$ and $8 + 5$. We can derive each one of them individually and master them independently. But $8 + 4$ is only one less than $8 + 5$, therefore deriving only $8 + 5$ is enough.

For example, let us consider $8 + 5$. We use the familiar script.

$$8 + 5$$

Which one is the bigger number? 8.

How do we make it 10? Add 2.

Where is the 2 coming from? 5.

What is left after taking 2 away? 3.

What is $8 + 2$? 10.

What is $10 + 3$? 13.

What is $8 + 5$? 13.

What is $5 + 8$? 13.

Now, let us add these two numbers by using Empty Number Line.

We need to break 5 into 2 and 3, because 2 and 8 make 10 and then 10 and 3 make 13. $8 + 5 = 8 + (2 + 3) = (8 + 2) + 3 = 10 + 3 = 13$. Therefore, $8 + 5 = 13$.

Using ENL and applying decomposition/recomposition, we have:

Same strategy applies to the other combination: $8 + 4 = 8 + (2 + 2) = (8 + 2) + 2 = 10 + 2 = 12$.

For example:

Teacher: $8 + 2$ equals what number?

Student: 10.

Teacher: If $8 + 2$ equals 10, then $8 + 3$ equals what?

Student: 11.

Teacher: If $8 + 3 = 11$, then $3 + 8$ equals what?

Student: 11.

Teacher: If $8 + 2$ equals 10 and $8 + 3$ equals 11, then what two numbers may make 12?

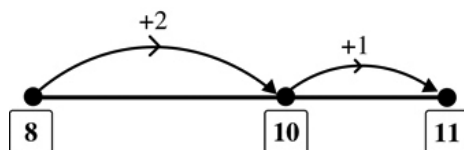


Figure 7.60 $8 + 4$ on ENL

Student: $8 + 4$.

Hopefully, they answer that $8 + 4$ equals 12. If not, you should direct them to the combination of the facts discussed earlier.

Script of the Strategy

- Read the problem: $8 + 4$.
- Which one is the bigger number? 8.
- How do you make it into 10? Add 2.
- Where does 2 come from? From 4.
- What is $8 + 2$? 10.
- What is left in from 4 after giving 2 to 8? 2.
- What is $10 + 2$? 12.
- What is $8 + 4$? 12.
- What is $4 + 8$? 12.
- What property of addition did you use?
- Commutative Property of Addition.
- Read the four equations related to $8 + 4$.

The following represents the $10 + 10$ Grid of the sums up to 20. The goal is for all the facts (100) to be filled in on this Grid.

Addition Facts Table Organized by Strategies

To represent all the addition strategies in a comprehensive manner the following addition grids are developed. In Figure 7.13a, each strategy for developing addition facts (strategies from 1 - 11) is represented by a different color. The process is very simple, As the teacher is introducing and developing these 11 strategies, in order, s/he

gives each child a blank Addition Grid. They fill in the entries in the Addition table Grid in the appropriate colors as each strategy is completed.

+	1	2	3	4	5	6	7	8	9	10
1	2	2	2	2	2	2	2	2	2,3,5	2,3,4,4
2	2							3	5	4
3	2						3		5	4
4	2					3			5	4
5	2				3				5	4
6	2			3					5	4
7	2		3						5	4
8	2	3							5	4
9	2,3,5	5	5	5	5	5	5	5	5	4
10	2,3,4	4	4	4	4	4	4	4	4	4

The teacher should also fill the entries in the Addition Grid in the prescribed color as the strategies are being developed. The teacher begins with a strategy, derives it, children practice it, and then enter the facts in the color indicated to see their progress in developing addition facts. The process and progress also provide the scope of the task done and remaining. We have found that children find it easy to grasp the scope of the work, and it motivates them to complete the task. For example, when the second strategy ($N + 1$ and $1 + N$) is being developed, as the 19 facts are completed, the first column and the first row are colored. That shows that almost 1/5 of the Grid is filled (see the brown color in the Grid). Different colors represent strategies. The commutative property of addition is reflected in the symmetry that is roughly diagonal from top left corner to bottom right hand corner.

Completed Grid

After each strategy is learned, mastered, and filled in on the Grid, children take an individual oral test to show their mastery—fluency and understanding (not a written one) of that strategy. When all the facts are entered, the teacher gives a cumulative summative assessment for all of the addition facts.

Notes

1. These can be considered as symptoms and behavioral markers of dyscalculics (students with developmental dyscalculia, innate dyscalculia and acquired dyscalculia) and occur on most assessments and evaluations. [🔗](#)
2. *Gerstmann syndrome* is a neurological disorder that is characterized by a constellation of symptoms: Gerstmann syndrome is characterized by four primary symptoms, collectively referred to as a *tetrad*:
 - Dysgraphia/agraphia: deficiency in the ability to write

- Dyscalculia/acalculia: difficulty in learning or comprehending numeracy/mathematics
- Finger agnosia: inability to distinguish the fingers on the hand (when the subjects are covered)
- Left-right disorientation that suggests the presence of a lesion usually near the junction of the temporal and parietal lobes at or near the angular gyrus.

Gerstmann syndrome is typically associated with damage to the inferior parietal lobule of the dominant hemisphere. It is classically considered a left-hemisphere disorder, although right-hemisphere damage has also been associated with components of the syndrome. [↗](#)

3. In many cultures, fingers are used for doing many intricate calculations, very effectively and efficiently. I (Sharma) learned various strategies of the use of fingers for calculations in my village and sometimes, I still use them very effectively. [↗](#)
4. In previous chapters, we defined what appropriate, efficient, effective, and elegant materials are and how to use them. [↗](#)
5. A strategy should be effective, efficient, and elegant. *Effective* means it does what it supposed to do. *Efficient* means it helps an individual to achieve the outcome in less time, effort, energy, or digression from the task. It is *elegant* if it is generalizable, extrapolated, and can be abstracted. [↗](#)
6. A strategy/method is *storage-rich* if it makes connections with other information (or other areas in the cortex) in the long-term memory. It is *retrieval-rich* if it invokes the information relevant to the concept. If the information is not stored in many places of the brain—such as the language area, or the tools area, or spatial area, it is not storage and retrieval rich. It has not left with a conceptual schema. [↗](#)
7. FingerMath issue of Focus. On Learning Problems in Mathematics, Vol 4 (3&4): Fingermath: Pedagogical Implications for Classroom Use. [↗](#)
8. See *Operations on Integers* by [Sharma 2013](#). [↗](#)
9. *Human Brain Map*. 2011 Nov;32(11):1932–47. doi: [10.1002/hbm.21159](#). Epub 2011 Jan 18.
Distinct representations of subtraction and multiplication in the neural systems for numerosity and language by Jérôme Prado, Rachna Mutreja, Hongchuan Zhang, Rucha Mehta, Amy S Desroches, Jennifer E Minas, James R Booth. [↗](#)
10. See the blog post *How to Give Feedback* by Sharma on [www.mathematicsforall.org](#) [↗](#)
11. For a demonstration of Cuisenaire rods, you can see the videos on the YouTube channel: Lessons with Mahesh Sharma and Numeracy DVDs that I prepared while working with children in actual classrooms. They are available from the Center for Teaching/Learning of Mathematics ([info@mathematicsforall.org](#)). Also see Lessons on the Use of Visual Cluster Cards and Cuisenaire Rods by Professor Sharma on YouTube. [↗](#)
12. *How to Teach Number Concept Effectively and Easily* (see [Sharma, 2019](#), an eBook). For teaching children the concept of number, you can refer to my Math Notebooks on the **Concept of Number** Part I and II: **Numeracy: Developing Number Sense**, and several posts on my blog: [www.mathlanguage.wordpress.com](#). [↗](#)
13. Which is not a true mathematical term; it is a colloquial term. We should, as much as possible, use exact, precise, and concise mathematics language. [↗](#)
14. In later grades, the rods may represent larger numbers and even variables. For example, if the white rod is 10, then the red becomes 20 and light green becomes 30, the equation becomes $20 + 30 = 30 + 20$, or if white is $1/10$, then red is $2/10$ and light green is $3/10$ and our equation becomes $2/10 + 3/10 = 3/10 + 2/10$, etc. [↗](#)
15. Children should do the writing work only on graph paper. In kindergarten, the graph paper should have $1\text{cm} \times 1\text{cm}$ grids; from the middle of the first grade, the graph paper should have smaller-size grids. The child should write one digit or operational and other symbols in each square. Most children with little practice develop into well-organized with beautiful handwriting skills. For learning to happen, we should

- aim at helping children to provide meaningful *form* to their learning. *Form* translates into *structure*; *structure* into *discipline*. With *form*, *structure*, and *discipline*, *productive thinking* is possible. [↗](#)
16. This is a more difficult question, as the question asks for the transition to the next decade. [↗](#)
 17. For the details on how to teach place value effectively, see the video on *How to Teach Place Value* and the Mathematics Notebook on *How To Teach Place Value* ([Sharma, 2008](#)). [↗](#)
 18. The ten *digits* are the alphabets of mathematics. In alphanumeric languages, such as English, *letters* are used to form words. Letters, combinations of letters, and their placements inside the word represents their sounds. Knowing the letters and their associated sounds based on their placements are necessary for reading, spelling, and knowing the meaning of words (comprehension). Similarly, for writing a numeral and number, we use digits. Digits are like letters. *Numeral* and *number* are like words. They are representations of quantity in context. At the elementary level, in arithmetic, there is no difference between numeral and number. However, later on in mathematics, *numeral* is a *representation of quantity* and *number* is a *directed numeral*—a *number represents quantity and a direction*. For example, +125 and −125, both represent numbers. In both, the signs ‘+’ and ‘−’ represents the direction of the quantity and the symbol ‘125’ represent the amount of the quantity. The digits used to represent these numbers, +125 and −125 and the numeral 125 are 1, 2, and 5. [↗](#)
 19. During this activity and in all future concepts, the teacher keeps on bringing the different words and expressions in context and keeps a running tally of the vocabulary words, phrases, and definitions on a separate board or a flipchart. The whole classroom is covered by these flipcharts. The new words related to the new concept are written in a particular color—the color of the day. The already introduced terms, words, and expressions are grouped according to concepts on separate flip charts. The words and terms coined by children (with the child’s name attached to the word or term) are written on a separate chart and consulted from time to time to see if we have arrived at a more appropriate term or word. Attribution should be like: Ron’s term for sight facts of ten: ‘Ten’s family.’ [↗](#)
 20. For improving the questioning process, see *The Role of Questioning Process in Mathematics Learning* ([Sharma, 2019](#)), an eBook. [↗](#)
 21. I urge teachers that from time to time they invite parents, grandparents, other teachers—colleagues from other local schools, principal, or generator to visit her class for as much as time they can spare. [↗](#)

8 Understanding Additive Reasoning

Subtraction

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Throughout history, in different countries, different ways to perform subtraction have evolved, and many are still evolving;¹ these include: the traditional method (borrowing/regrouping/exchange, etc.), Empty/Open Number Line model, Take-away model, Comparison model, ‘Make a ten’ strategy, Complement method, Counting up, and Using manipulatives like counters or fingers, where you physically remove objects to represent subtraction. Children should experience these, and many others that they discover or are exposed to by teachers and other children.

Additive reasoning involves number concept, number relationship, addition and subtraction concepts and procedures, and understanding the inverse relationship between addition and subtraction. When students can successfully convert an addition relationship into a subtraction relationship and vice-versa, they have acquired additive reasoning.

The two related processes: addition and subtraction are parallel to ‘decoding’ and ‘encoding’ skills in reading. It is the understanding that every number is made up of other numbers and it can be broken into components, and the components, when combined properly, can give you back the number. Practice with this inverse relationship leads children to the most important mathematical skill—decomposition and re-composition of large and small numbers (the basis of future mathematics). This is truly a preparation for higher arithmetic concepts such as multiplication, division, fractions, and then other number systems.

A. Difficulties with the Operation of Subtraction

Most primary school teachers and parents report that they find that children experience more difficulties in learning and mastering the language, concept, and

procedures of subtraction than the related addition concepts. Subtraction is the inverse of addition, thus knowing and having the mastery of addition should be enough to learn and master subtraction more easily. For most students, that is what happens. Their knowledge of addition gives them enough skills and strategies to learn and master subtraction—its language, concept, and procedures, but many students find the concept, automatizing subtraction facts, and the operation of subtraction much more difficult than addition.

Our observations from many years of experience working with students at different levels of schooling and higher education show that many more individuals experience and express they have difficulty with the concepts and procedures of subtraction and division of whole numbers, fractions, and decimals. For a variety of reasons, this is true all over the world. Examination and test results and research in these areas show the same kinds of results. Most children experience difficulty even in initial learning, mastering, and applying the concepts and procedures of subtraction, division, and fractions. There are several reason for the phenomenon. They range from teaching strategies to cognition to the concept itself. The nature and extent of their difficulties with subtraction fall into six major categories:

- a. the nature of the concept of subtraction itself,
- b. cognitive (student preparation for the concept),
- c. neuropsychological (role of executive functions in learning),
- d. pedagogical (teaching models, approach, and sequencing of tasks),
- e. linguistic (adequate language related to the concept), and
- f. emotional (experiences with earlier concepts and skills) and pedagogical factors.

First, pedagogically, we devote more time to addition and multiplication than subtraction and division (time on task: 7 problems vs. 3 problems) in the curricula and actual instruction time on subtraction and division. One of the reasons this happens is that when children arrive in second grade, many of them have not mastered the addition facts and procedures by the end of the first grade, although we had almost two years (kindergarten and first grade) to do the task. The focus of mathematics instruction in second grade should be on subtraction; however, about one-third of the allotted instruction time is still being devoted to addition facts and procedures and place value; therefore, once again many children leave second grade without mastering subtraction. And this affects all other topics in the future.

Moreover, in second grade, subtraction is not taught as related to addition—an extension and inverse of addition. It is taught as a new concept—‘counting back’ is subtraction as counting forward was addition. Most teachers and parents think that ‘counting back’ is a parallel skill to ‘counting forward’ and cognitively at the same level. That is not the case; most children find counting backwards very difficult compared to counting forward. Children go through not only the same problems as they faced in addition, but new problems arise because of the new difficulty of counting backward. For example, transiting a decade, from 9 to 10; 19 to 20; 49 to 50; 79 to 80, etc. are difficult, but 11 to 9; 31 to 29; 71 to 69; etc. are much more difficult to achieve.

Many teachers and parents teach children addition as ‘*counting up*’ and subtraction as ‘*counting down*,’ because it is much easier to teach both addition and subtraction this way. They have to give only one instruction. And because of extra practice and longer exposure to the counting up activity, they think this process is an easier process for children (they have been counting up for a long time) to learn, but it is a difficult strategy to help them to automatize subtraction facts. Counting down (backwards, back) is not a natural process of learning and therefore, it is a difficult process for many children to generalize to automatization. They have limited experiences in counting down. During the evaluation of children’s learning difficulties, we have found that absence of counting backwards (at all levels—from whole numbers to fractions and decimals) is a sure sign of difficulty in learning mathematics—particularly the operations of subtraction and division.

Counting forward is practiced by children all the time, whereas counting backward is practiced very little. We encourage teachers to do both types of counting up to eighth grade. When children learn subtraction as ‘counting back,’ they are not taking advantage of the already mastered skills of addition to learn subtraction. The amount of practice then needed is the same or more as for addition. By children, it is not seen as the derivation of addition—a pay-off on addition. Rather, they think of this as a new concept, unrelated to addition.

The *second* reason for difficulty arises from the nature of the subtraction concept itself—its language is varied and complex (e.g., the list of vocabulary words for subtraction in English language is much larger—there are more subtraction words and phrases than addition words and phrases). Thus, the conceptual schema is difficult to grasp (e.g., it is the whole-to-parts concept compared to addition, which is a parts-to-whole concept), and the strategies, phrases, and processes are direct as well as indirect (e.g., passive voice vs active voice; direct vs indirect statements).

There are far too many words and expressions describing subtraction, not just in the English language but in almost all major languages. For example, subtraction appears when any of these words appear in the phrase: How much more (bigger, greater, or larger) is 17 than 9? What is the difference between 17 and 9? Etc. Each one of the many word/phrases such as less than, fewer, smaller, decrease, etc. call for a different conceptual schema. Children see subtraction in many more places and in many more forms.

The *third* reason for children's difficulty with subtraction relates to executive functions: addition is a 'parts-to-whole' concept; subtraction, on the other hand, is 'whole-to-parts' concept. The whole-to-parts concepts² make more demands on the working memory and are difficult to focus on as the 'whole' and one of the 'parts' disappears in the process. This is particularly so when the only definition of subtraction given to children is the 'take away' and the model of teaching is to literally removing the group of objects from the whole. Moreover, the language we use for subtraction is not intuitive. Secondly, in the case of the addition process, the two groups/collections/quantities to be added are present in the sum (e.g., if you do not mix them). They are still visible to some extent, or one can still visualize them in one's mind as two parts being put together. In the case of subtraction, the group or part to be subtracted and the group/whole from which the number/group is being subtracted are transformed, and a great deal of this work is performed in the working memory. As a result, cognitively/perceptually, it is a difficult process.

The *fourth* reason is that subtraction shows up in many disguises, and sometimes those forms do not directly relate to subtraction. For example, 'how much more is 34 than 19?' does not sound like subtraction to many children as it involves the word 'more.' To really understand subtraction, children need to know as many manifestations of addition and related subtraction as possible occurring in different settings. Therefore, one problem in learning subtraction, just as in addition, is to acquire the language of subtraction first and more comprehensively, and connecting to the conceptual schemas of subtraction before applying it. However, the acquisition of language of subtraction is only a necessary condition; it is not a sufficient condition.

The *fifth* reason is that many people in most societies have a psychological bias toward 'more' rather than 'fewer,' 'taller' rather than 'shorter,' and so on. People express, in front of children, elation when they score, quantity, points, cookies, etc. has become bigger than it has become smaller (it happens only in golf that lower score is better, but that is much later). Many children also associate subtraction with

negative actions, states, and relationships. Losing something is traumatic to some children.

The *sixth* reason for difficulties that children face in learning subtraction is that in some situations subtraction can be performed, whereas addition can be performed only on those objects which have some attributes or characteristics in common. That means, subtraction can be performed on dissimilar things or objects. In other words, subtraction is more general an operation than addition. For example,

There are 9 boys and 7 girls in the school band. How many more boys than girls are in the school band?

As we see in this case, the two groups do not belong to the same category; still, we can do the subtraction. However, to add, we will ask: ‘How many people or members are in the band?’ We need to convert the boys and girls to a more general category of ‘people or members’ before we can add them. We could answer another question of the similar type about subtraction.

What is the difference between the number of boys and girls in the school band?

Finally, conceptually when we add two quantities, the two components (the *addends*—the two or more numbers being added) are indirectly and sometimes directly present in the sum and can still be discerned. However, in subtraction, particularly in the ‘take away’ model, the amount being subtracted is gone. This needs visualization by the child and that challenges many children who have difficulty in holding the *action* and *the objects* or *the group* in their minds. That places more demands on the working memory.

Based on these reasons, the pedagogies, models, tools, and language used in teaching and mastering subtraction facts and strategies should take these factors into account.

We suggest two interdependent and non-exclusive approaches to teaching subtraction.

To teach young children subtraction there are some basic thoughts to keep in mind. *First*, the various forms are not particularly difficult if taken in isolation, and, indeed, many young children can tackle at least two forms from an early age without formal schooling whatsoever. For example, they can do ‘take away’ tasks if

presented in an appropriate manner ([Hughes, 1986](#)). These types of problems compare the original number of items with the final number.

They can usually also do ‘difference between’ tasks because they generally know that if you have five cards and I have four cards then you have more than I do, and they can often tell you by ‘how many.’ Or, the problems may involve *comparing before and after situations*, such as ‘we had 9 apples in the morning and now there are only 6 apples.’

The *second* point to remember is that the process of subtraction can sometimes be instigated by symbols other than ‘−.’ For example, $4 + \square = 9$ does not involve the ‘−’ symbol, yet it instigates that in this situation subtraction needs to be performed. Because the problem will be solved by applying the concept of subtraction. These kinds of problems occur where we compare two sets of objects. For example, I have written 9 letters but I have only 4 envelopes; how many more envelopes do I need? Or when we compare two items: for example, David’s score is 8 points and Matt’s score is 4 points; what is the difference between the two scores?

These kinds of problems (examples of direct comparisons) are common in young children’s everyday life and generally are informally presented; therefore, they are unlikely to cause conceptual difficulties. It is important that young children experience these types of problems and that they are properly and correctly represented. However, presenting the same problems involving ‘more than’ or ‘lower than’ is far more difficult for children as they involve indirect comparisons. Without making connections with direct actions such as with the ‘take away’ schema, these formal language presentations create difficulty for many children.

To address the difficulty with subtraction, children should first know the concept of addition and then understand the concept of subtraction in relation to addition.

For students to acquire the concept and procedures of subtraction, teachers need to help students develop multiple linguistic expressions for the same arithmetical operation. For example, it is important that the students see problems such as: $17 - 9 = ?$ in its linguistic expression form in multiple forms:

- 17 minus 9 equals what?
- 17 take away 9 equals what?
- What is 17 reduced by 9?
- What is 17 subtract 9?
- What is 9 take away from 17?
- What is 9 fewer than 17?

- Subtract 9 from 17?
- Deduct 9 from 17? Etc.
- What is the difference between 17 and 9?
- 17 is how much more than 9?
- 9 is how much less than 17?
- How much more is 17 than 9?
- How much less than 17 is 9?
- What do you get when you subtract 9 from 17?

As one can see, these sentences/phrases use several vocabulary terms in different places to signify the complexity and the reach of the operation of subtraction. The sentences are in different *syntax, form, and structure* (e.g., *direct vs indirect; active voice vs passive voice; context of the origin of the problem, discrete vs continuous*, etc.). The range of meaning derived (e.g., *semantics*) from the expressions is also quite broad. The language of the sentences also suggest a range of conceptual schemas for developing the idea of subtraction, solution approaches for solving subtraction problems, and use of instructional approaches and materials/tools. If children are only taught that subtraction involves only the ‘taking away’ form, they may experience difficulties (perseveration), later, when they are confronted with ‘the difference between’ or ‘subtract -4.5 from -3.4 ’ or subtraction of algebraic expressions. Our instructional approach should reflect all of these elements. The instruction must be effective, efficient, and generalizable to future subtraction situations.

The conceptual and linguistic schema formed by the limited language of ‘take away’ is adequate for whole numbers only when we are subtracting a smaller number from a larger number. It is, however, inadequate for word problems that are not based on that schema and later instances of subtraction problems dealing with fractions, decimals, integers, real numbers, and algebraic expressions. It is also not helpful where subtraction is not directly evident as in ‘ $\square + 2 = 8$.’ Similarly, problems such as: $-32 - (-7)$ or $-1.04 - 3.9$ do not lend themselves to the ‘take away’ model.

As has been mentioned, a ‘comprehensive’ vocabulary is a necessary condition for learning any meaningful concept, including mathematical ideas. *Language helps give rise to conceptual schemas*. However, simple vocabulary is not sufficient for forming robust conceptual schemas and arriving at efficient and generalizable procedures. Similarly, even an extensive vocabulary of subtraction is not sufficient

for developing adequate and proper cognitive schemas for subtraction; one has to understand and develop the syntactical structure of the language of subtraction as well. It is with the help of vocabulary, syntax, form, and context that one develops the semantics of a concept. Then the conceptual schemas formed for the concept are stronger, elastic, transparent, effective, and generalizable.

Many students never understand and are not able to master the subtraction of integers or operations on algebraic expressions where ‘combining like terms (CLT)’ are concerned because they have experienced subtraction only in the take-away model. The take-away model is inadequate to conceptualize such problems. This lack of ability for an extension of subtraction facts from whole numbers to fractions and integers creates a major difficulty for students in algebra. Our experience indicates, many teachers report, and even research shows that more than 70% of problems students face in algebra deal with arithmetic facts, an understanding of concepts and procedures on integers and fractions. A poor understanding of language, concepts, and procedures of subtraction, division, and fraction is a major cause of those errors in algebra.

We have emphasized that the mathematics idea is truly understood and we can apply it when we have the language, conceptual schema, and procedure and we understand the interrelationship of these elements about a concept/idea. The native language plays a very important role in learning mathematics ideas till we are fully fluent in the mathematics language itself.

Language helps us form cognitive schemas—conceptual models for mathematical ideas; limited language models limit the conceptualization process, producing limited and limiting schemas. The predominant role of language is demonstrated by a very remarkable experiment by Russian psychologist Lev Vygotsky. He set some very young children to try to sketch butterfly wings that they have not done before. Those children who had words in their vocabulary for ‘dot,’ ‘triangle,’ ‘slash,’ and the like were able to draw the butterfly wings even from their memory;³ generally those children who did not have these *words* were unable to *draw* those wings even while looking at them.

Continuing his experiment, Vygotsky took half of those children who had been unable to draw the wings, and in a wholly different context he taught those words and the related meaning and representation.

In the final phase of the experiment, with the children again attempting to draw butterfly wings, those children who now had those words were able to draw those wings quite effectively even from memory. Those children who still did not have

those words showed very little improvement from the first round and generally were quite unable to draw them (Vygotzky, 1962, 1978, and 1986).

To see the role of language in our context, let us consider the problem $17 - 9 = ?$, we read this in several ways. When it is read as ‘17 *minus* 9 equals what?’, ‘17 *subtract* 9 equals what?’, or ‘17 *take away* 9 equals what?’ all of these linguistic expressions create a particular conceptual model and schema and suggest a certain kind of action to take in the mind suggesting a specific procedure. The language model (*minus*, *subtract*, and *take away*) and the related conceptual schema invoke a particular type of activity/action to complete that task. In this model, you begin with ‘17’ objects and then you *take away/subtract/minus* ‘9’ objects from the collection of 17 objects. We begin with ‘17’ objects and perform a direct action (taking away, subtracting, ‘minusing,’⁴ etc.) on ‘9’ of those 17 objects.

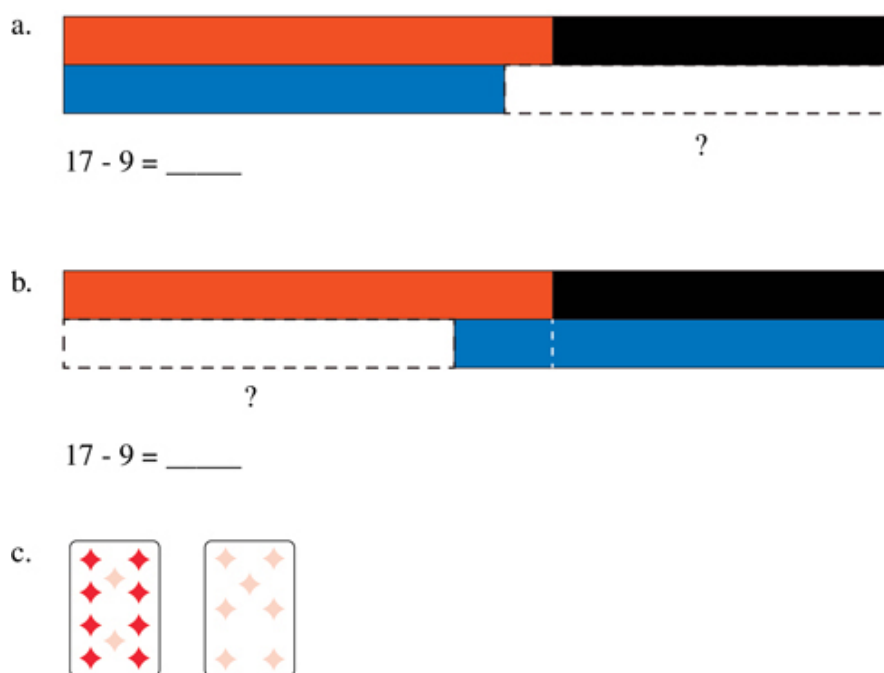


Figure 8.1 Modeling 17–9 using number rods and VCNC

As seen earlier, the basic problem with subtraction is the number of meanings something as seemingly simple as $17 - 9 = ?$ can have. As a result, the word problems associated with subtraction become difficult for children. For example, subtraction represents any one of the following ‘linguistic structures’ for word problems or subtraction stories:

- I made 17 brownies. The children ate 9 of them. How many brownies were left for others?
- I made 17 brownies. Sarah made only 9 brownies. How many more brownies did I make than Sarah?
- I made 17 brownies. Sarah made only 9 brownies. How many fewer brownies did Sarah make than me?
- I made 17 brownies. Sarah made only 9 brownies. What is the difference in the number of brownies we each made?
- I made 17 brownies. 9 were taken away from the pan. How many were not taken from the pan?
- There were 17 ounces of orange juice in the jug. I drank 9 ounces. How many ounces of juice were left in the jug?
- I made 17 ounces of juice and Sarah made 9 ounces of juice. How many more ounces of juice did I make than Sarah?
- David is 17 years old. His sister is 9 years old. How much older is David than his sister?
- David is 17 years old. His sister is 9 years old. How much younger than David is his sister.
- David is 17 years old. His sister is 9 years old. In how many years will she be as old as David is now?
- David is 17 years old. His sister is 9 years old. How much older was David when his sister was born?
- David is 17 years old. His sister is 9 years old. How many years ago was David same age as his sister is now? ETC.

In all of these problems, the operation of subtraction appears in a variety of forms. As discussed earlier, an overemphasis on the ‘take away’ form of subtraction accompanied by little or no mention of the other forms can result in limited understanding of subtraction, which in the later years of schooling can inhibit—and possibly even prevent—greater insight into the wider concept and trajectory of subtraction involving the extended number systems from whole numbers to real number system and algebraic expressions.

Associated with this potential problem are the different structural representations, which use either the discrete model—*counting objects* (17 brownies) or the *continuous model* (17 ounces or 17 years). For example, the phrases: ‘how many’ and ‘how much’ connote certain choices of materials—‘how many’ suggests

counting discrete objects for a child and ‘how much’ suggests a continuous model. Children can count the discrete objects but have difficulty with the continuous models as they do not lend themselves to counting, particularly if the child is still working on the concrete level of knowing using on discrete objects. As mentioned in earlier chapters, they are forming *episodic memories* that only goes to the hippocampus. It has not been converted into ‘knowledge’ which becomes semantic memory and more generalizable and is directed by the pre-frontal cortex and has wider ‘availability’ and therefore wider ‘usability’ ([Behr et al., 1984](#)).

Another important potential problem could be that subtraction problems often include specialized words which frequently have other meanings in everyday usage. A common example are words such as: ‘difference,’ ‘more,’ ‘fewer,’ etc. As mentioned, language plays a huge role in the formation of conceptual models and approaches to solving the problem. Children bring to the task their common meaning of the word ‘different.’ ‘Different’ to them means that an attribute is being emphasized: ‘17 is a bigger number than 7 and 7 is a smaller number.’ ‘17 is to the right of 9 on the number line.’ ‘17 is a 2-digit number and 9 is a 1-digit number.’ As mentioned in the chapter on language, it is important to note that mathematics vocabulary consists of a variety of words, and their proper introduction in proper form and context is important.

Mathematics vocabulary consists of five categories of words: (a) words that are used with the same meaning as in ordinary life and mathematics classroom, i.e., ‘take away,’; (b) words have different meanings in the mathematics classroom, i.e., ‘difference’; (c) words used in mathematics that may not be commonly used, i.e., ‘deduct,’ ‘subtrahend,’ ‘minuend,’ (d) compound words—mathematics words formed by combining two or more words, i.e., ‘missing addend’; (e) words formed by combining different words, terms, operations, and actions or the meaning changes because of the combination, i.e., ‘how much more?’ ‘subtraction table.’

As demonstrated earlier, from the very beginning, children should see addition and subtraction as inverse relationships. Teachers from kindergarten through second grade should work together with the aim that children by the end of second grade must master additive reasoning. Teachers of kindergarten through second grade children should keep the developmental trajectory of additive reasoning in mind during the planning of their lessons. During planning and teaching number concept in kindergarten, addition in first grade, and teaching subtraction in the second grade, they should anticipate and reinforce this developmental trajectory of additive reasoning.

If the student knows the inverse relationship between addition and subtraction facts, then the given subtraction problem can be solved by converting the subtraction problem into an addition problem and performing the corresponding addition operation. The ability to transform a subtraction problem into an addition problem and vice versa shows that the child understands the inverse relationship of addition and subtraction. This opens up possibilities for diverse models. This understanding should therefore be one of the three key goals of the K-2 mathematics education relating to number and number relationships. The other two are recognition of common shapes and figures in their environment and performing simple measurements.

The inverse relationship between addition and subtraction is possible if care is taken when representing subtraction problems using concrete materials and models. To reduce the likelihood of later difficulties, it is important that children are given plenty of opportunities to represent subtraction problems using a variety of models and materials. They may begin with objects such as pencils, straws, buttons, or pennies so that they can relate concepts to informal situations. Later, as children gain in confidence and experience, it is important that they are presented with materials and models such as Cuisenaire rods, BaseTen blocks, Invicta Balance, Dienes blocks, etc. as well as a variety of linguistic expressions for the concept of subtraction. Visual Cluster Number Cards, due to their patterns and structure, serve as intermediaries between discrete and continuous models ([Booth and Siegler, 2008](#); [Sharma, 2019c](#)).

B. How Should We Teach Subtraction and Subtraction Facts?

Subtraction and addition as concepts should be related very early. For example, in kindergarten, as children gain confidence in addition sight facts of a number, the teacher can introduce sight facts of subtraction using the same approach, strategies and methods with the help of Visual Cluster Number cards and Number-rods. Subtraction should be introduced as the inverse of addition by creating a context for it. It is advisable to teach related subtraction facts right after teaching an addition strategy. When most children in the class or in the group have automatized the addition facts related to a strategy (mastery with a high level of proficiency and understanding), begin teaching subtraction facts relating to that addition strategy. For example, as soon as they know the strategy $N + 1$, they should learn $N - 1$ facts ($N =$ a number less than 20). It is important to realize here that we refer to

mastery as automatization of facts, not constructing facts by counting blocks, fingers, or on the number line. They should know this strategy but not remain on it exclusively. For mastery, use only effective strategies. For example, when students can give each of the addition facts family for a fact relating to the strategy, and begin to read them as subtraction facts also ($8 + 1 = 9$; $1 + 8 = 9$; $9 - 1 = 8$; $9 - 8 = 1$; $8 + \square = 9$; $\square + 1 = 9$; $9 - \square = 8$; etc.).

In other words, arrive at an addition strategy, master it, and then introduce subtraction facts related to that addition strategy. Practice the addition strategy separately first. For example, after learning the family of facts for addition, introduce the subtraction facts into the addition fact family, and then augment this list with missing addend facts:

- $6 + 4 = 10$; $4 + 6 = 10$; $10 = 6 + 4$; $10 = 4 + 6$
(Addition Fact Family)
- $10 - 4 = 6$; $10 - 6 = 4$.
(Subtraction Fact Family)
- $6 + 4 = 10$; $4 + 6 = 10$; $10 = 6 + 4$; $10 = 4 + 6$; $10 - 4 = 6$; $10 - 6 = 4$
(Addition and Subtraction Facts Family)

After this, this family of facts is extended to include the missing addends:

- $\square + 4 = 10$; $4 + \square = 10$; $10 = 4 + \square$; $10 = \square + 4$; $\square + 6 = 10$;
 $6 + \square = 10$; $10 = 6 + \square$; $10 = \square + 6$.

By the end of second grade, students should master all the addition and subtraction facts in a comprehensive manner. The list articulated here is the first objective of introducing and then mastering subtraction facts and their relationship with addition facts on one hand, and the missing addend on the other.

The teacher must help children to develop the 45 sight facts of subtraction using Visual Cluster Number cards and Cuisenaire rods. This should be achieved in the same way as we developed the addition sight facts.⁵ In the process, the teacher should emphasize the inverse nature of the two operations: Just as in addition, the most effective instructional materials for teaching subtraction concept, facts, and procedures are: (a) *Visual Cluster Cards* and *Sight Fact Cards*, (b) *Number rods*, (c) *BaseTen Blocks*, and (c) *Empty Number Line*.

As mentioned in the previous section, many children have difficulty in conceptualizing subtraction and then mastering subtraction facts and the related

procedures. Specifically, there are two levels of difficulty:

- a. learning subtraction facts, and
- b. learning and mastering the procedure for subtraction of multi-digit numbers.

Both are dependent on understanding and having facility with additive reasoning—a child’s lack of understanding of the idea of inverse relationship between addition and subtraction concepts and procedures. It is easier to expand the addition facts and procedures from known addition facts. For example, it is easier to learn the concept of doubles of numbers such as $6 + 6 = 12$, because $5 + 5 = 10$; and then to sums such as $26 + 6$, $76 + 6$ but the idea of subtraction $72 - 6$, $42 - 6$, $22 - 6$, and $12 - 6$ are not that simple as it does not directly relate to $5 + 5 = 10$.

Similarly, if we ask children to add a pair of two-digit numbers like 35 and 26, they tend immediately to ‘*decompose*’ both numbers (i.e., 30 and 20, that is 50, and 5 and 6 is 11, and so that makes 61). This ‘decomposition of numbers’ by place value can take them to the standard addition procedures rather quickly; however, this habitual splitting of numbers does not easily take them to the standard procedure of subtraction, and in some cases this habit creates misconceptions and errors, particularly when the standard procedures are introduced too early and without conceptual understanding. In such situations, they apply these standard procedures even when they could find the answer easily without splitting. The following examples are responses given by fourth-grade students on the state examination in one of the states in the USA. These examples demonstrate that the students are performing/executing the subtraction procedure incorrectly and inefficiently without any understanding of the components for the mastery of addition, subtraction, and place value. These kinds of errors are prevalent when a child does not have good numbersense.

	9 12	
302	~~302~~	
–298	– 298	
196	104	

We have found that the use of the *Empty Number Line (ENL)* for subtraction of bigger numbers will be a much better strategy before and during the standard procedures for subtraction are introduced. Only after modeling several problems on subtractions by an ENL should the teacher go to the standard procedures by using number rods to model the standard procedure.

Problems in learning subtraction concepts and procedures emanate from several sources. For example, from an early age, children have been encouraged to partition any three-digit and two-digit number into their hundreds, tens, and units components, and this tradition is emphasized within all Standards-based curricula and textbooks: arrow cards, place value charts, hundreds charts, etc. This is called ‘canonical’ *decomposition/recomposition of a multi-digit number*, which is the basis of place value system (e.g., $234 = 200 + 30 + 4$). It is important to know this ‘canonical’ decomposition/recomposition, but children should also know *non-canonical decomposition/recompositions* (e.g., $234 = 200 + 34 = 230 + 4 = 210 + 24$, $100 + 134 = 234$, etc.), which is the basis of a great deal of our mental mathematics activities and developing fluency in numbersense.

The use of canonical decomposition in procedural methods work well for addition. Addition is ‘easier’ than subtraction (‘easier’ here means that you will get a higher success rate if you give ten two-digit additions to 7-, 8- or 9-year-old children than if you give them ten subtraction problems). In this sense, subtraction is harder for them. The point is that the difficulty with subtraction is not wholly with subtraction itself, and certainly not always with the child, but in most cases, it is with the way they are taught to do it.

Misconception in standard algorithm

$\begin{array}{r} 302 \\ - 298 \\ \hline 196 \end{array}$	$\begin{array}{r} 912 \\ 302 \\ - 298 \\ \hline 104 \end{array}$

Figure 8.2 Inverse relationship between addition and subtraction

Our experience teaching foundation concepts shows us that the first step in teaching subtraction is to teach the 45 subtraction sight facts. Deriving the

subtraction facts should be done by the help of Cuisenaire rods and Visual Cluster cards. Here is an example.

Teacher: Look at the equation.

First, please construct the equation with your partner with the help of Cuisenaire rods. Give me an addition fact or a subtraction fact using the equation. The teacher demonstrates a few examples.

Each child makes the aforementioned equation with Cuisenaire rods, and in turn reads them with the help of the Cuisenaire rods equation and then they give the facts without this concrete model. When students have shown oral fluency, the teacher asks them to write all the facts on paper.

$$9 + 5 = 14, 5 + 9 = 14, 14 = 9 + 5, 14 = 5 + 9,$$

$$14 - 9 = 5, 14 - 5 = 9,$$

$$9 + ? = 14, ? + 9 = 14.$$

This should be done for all of the strategies and all of the facts one by one. After the subtraction facts for an individual strategy have been mastered, then the mastery should take place in a cumulative fashion.

Although addition strategies were extended to subtraction operation, it is important to discuss the nuances and subtleties of these extensions. Here is an example:

1. Subtracting Ten

The corresponding subtraction activity is for subtracting ten from a teens number and then extending it to subtracting 10 from any two-digit number. Children should be engaged in this activity only after they have really mastered the addition strategy for making teens numbers.

Teaching subtraction begins with children developing the concept of: Just like we can add ‘things’ only if they are of the same type, similarly, most of the time, we can subtract or take away only if the two numbers are of same kind. Let us begin:

Teacher: I am holding a number in my hand (say 14, using the jumbo Cuisenaire rods, magnetic rods on the white board, or a display of rods on the screen

through projector of the rods under the document camera), what is the number?⁶



Figure 8.3 Subtracting from teens numbers

Students: 14.

Teacher: What if I want to take away 4 from 14, what rod should I remove from my number arrangement?

Students: The 4-rod.

Teacher: If I remove 4 from 14, what rod will be left in my hand?

Students: A 10-rod.

Teacher: If I take away 4 from 14, what number will be left?

Students: 10.

Teacher: What if I take away 10 from 14, what number will be left?

Students: 4.

Teacher: Children, this is called taking away or subtraction. When you take some objects from a collection of objects, it is called *subtraction*. That is the same as: When you take away a number from another number, it is also called subtraction. If you have 5 cookies and you eat 3 of them, how many cookies are left?

Students: 2.

Teacher: This is also an example of subtraction. Let us look at another number (she holds the 10-rod and the 8-rod). What number (made by these rods) do I have in my hand?

Students: 18.

Teacher: Now, what if I take 8 away, what number is left?

Students should say: 10. (If they do not answer correctly, the teacher demonstrates by action and establishes the answer.)

Teacher: Watch me: if I take 10 away from 18, what is left now? What if I take 18 away from 18, what is left? (Most children do not say anything.)

Teacher: What number name do we give to nothing?

Students: Zero.

Teacher: How do we write zero?

Student: It is like an '0.'

Teacher: Let us write the number 0 on the whiteboard. Can someone write the symbol for zero or nothing on the board for me for what I am demonstrating in my hands?

The teacher shows her two hands. She holds the 10-rod in one hand and the other hand empty.

Teacher: What two numbers do I have in my hands?

Students: 10 in one hand and nothing in the other hand.

She asks for a volunteer to come forward. A child comes forward.

Teacher: Here, take this number (hands the 10-rod), and now also take this number (empties her empty hand in the volunteer's hand). What two numbers did I give you?

Student: 10 and 0.

Teacher: How do we write this action?

Student: $10 + 0$.

Teacher: What two numbers do you have?

Student: 10. $10 + 0 = 10$.

Teacher: Good! If we add 0 to 19, what do we get?

Students: 19.

Teacher: Good! Write the equation showing this.

Students write $19 + 0 = 19$.

Teacher: What if we subtract 0 from 19, what do we get?

Students: 19.

Teacher: Can someone write the equation for this?

A student writes: $19 - 0 = 19$. The teacher should establish the addition and subtraction of 0 with a number well.

After this the teacher practices (a) subtraction of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 from 10, and subtraction of 10 from the teens numbers ($10 + N$) and subtraction of N from $10 + N$.

The first part of the lesson was an extension of addition sight facts to subtraction sight facts. Subtraction sight facts should be practiced along with addition. Then, we study subtraction facts by extending the family of addition facts (

$4 + 6 = 10$, $6 + 4 = 10$, $10 = 4 + 6$, and $10 = 6 + 4$) to additive reasoning (a family of facts with addition and subtraction: $4 + 6 = 10$; $6 + 4 = 10$; $10 = 4 + 6$; and $10 = 6 + 4$; $10 - 4 = 6$; $10 - 6 = 4$; $4 + \square = 10$; $6 + \square = 10$; $\square + 4 = 10$; $\square + 6 = 10$; etc.)

However, in subtracting 10 from $10 + N$, where N is 10 or less than 10 and subtracting N from $10 + N$ do not fall in the category of sight facts. They should be practiced till they are mastered as they are essential for all four arithmetic operations. Therefore, the teacher should repeat this with several other teens numbers.

The teacher should start with any teens number (say, 19) and ask a child to subtract (use several other words, such as ‘take away’) 10 (compare with the ten-rod) from 19. Call their attention to the starting number (19) and the ending number (9). Have children describe what happens when they subtract ten, and then write it, e.g., $19 - 10 = 9$ (some child will point out that the one’s place number stays the same). Now, the 2 subtraction facts: $19 - 10 = 9$ and $19 - 9 = 10$.

After the teacher’s introduction of a concept, strategy, or procedure, as we did in the case of addition, she should practice this with the whole class by engaging in the three activities: *choral*, *individual oral*, and *individual written*. When addition and subtraction facts are mastered, then ask children to read the addition and subtraction facts together. For example:
 $10 + 9 = 19$, $9 + 10 = 19$, $19 = 10 + 9$, $19 = 9 + 10$, $19 - 10 = 9$, $19 - 9 = 10$. The teacher should practice it with several examples by asking children addition and subtraction problems at random. Once again, facts practice should be done in three forms: *choral*, *individual oral*, *individual written*. After the whole-class activities, they should repeat the practice in pairs and then individually.

All the remaining strategies for subtraction are also based on four key concepts: (a) number concept (visualization, decomposition/recomposition, sight facts), (b) commutative property of addition,⁷ (c) making 10, and (d) teens numbers. From this perspective, these are the key skills for the mastery of addition and subtraction facts. These skills must be developed during the kindergarten year.

Support for children who are lagging in these concepts should be provided as soon as one finds that a child is having trouble, rather than waiting for the assessment and formal evaluations at the end of the kindergarten. Title One and special educators and other interventionists should use the same strategies and materials, rather than counting strategies and discrete instructional materials, and should aim for mastery rather than coverage and generating answers.

Any addition equation can be written as a subtraction equation and any subtraction equation can be written as an addition equation. The same strategies of decomposition in addition can be used for subtraction.

We suggest four interdependent and non-exclusive strategic approaches to be used for teaching the subtraction facts and procedures.

First, starting from kindergarten to eighth grade, at least five minutes a day should be devoted to *counting forward* and *backward* starting from any number by key numbers, such as: 1, 2, 10, 5, 50, 100, unit fraction, proper fraction, mixed fraction, integer, etc.

- Kindergarten: children should be able to count by: 1, 2, 10 by the end of the year;
- First grade: counting by: 1, 2, 5, 10, 100 by the end of the year;
- Second grade: counting by: 1, 2, 5, 10, 100, 1000 by the end of the year.
- Third grade: all of the aforementioned and counting by unit fractions (e.g., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, etc.) from any number.
- Fourth grade: all of the aforementioned and counting by proper fractions (e.g., $\frac{2}{3}$, $\frac{3}{4}$, etc.) from any number.
- Fifth grade: all of the aforementioned and counting by mixed fractions (e.g., $1\frac{1}{2}$, $1\frac{1}{3}$, $2\frac{1}{5}$, etc.) and decimal numbers starting from any number.
- Sixth grade: all of the aforementioned and counting by mixed fractions, decimals, and integers (e.g., -2, -5, etc.) from any number. The same applies in seventh and eighth grades.

An inability and a difficulty in counting backward is one of the key factors in many children having problems in mathematics not just in subtraction, but in many other areas of mathematics.

Second, use Visual Cluster cards for introducing sight facts for addition and subtraction. When children are introduced to addition and subtraction using Visual Cluster Number cards, they should be able to see the relationship between addition and subtraction sight facts quite easily. They also understand the concept of decomposition/recomposition with clarity. Decomposition/recomposition connects addition and subtraction as inverse operations and is the basis of learning efficient strategies for addition and subtraction.

Third, teachers should introduce children to the developmental trajectory of and different ways of conceptualizing subtraction: Let us consider the problem: ' $14 - 6 = ?$ ' To demonstrate this idea.

- 14–6 (the ‘take away’ model): Counting off or down 6 one by one, starting either with a group of 14 or a group of 4 and 10. It is important to point out from which number the 6 is counted off. If the counting is taking place from a collection of 14 objects, then the method emphasizes only counting and no decomposition/recomposition takes place. That does not develop efficient strategies. However, if the counting down is taking place either from 10 first, then last step involves combining 4 and 4 and therefore decomposition and recomposition. Similarly, if the learner counts 6 first 4 from 4 and then rest 2 from the remaining 10, there is also decomposition/recomposition taking place. In either of the later cases, we should make sure that the corresponding addition fact is emphasized, and they see the concepts of addition and subtraction together using Visual Cluster Number cards and Cuisenaire rods.
- 14–6: count up from 6 while keeping track of numbers counted. This connects addition and subtraction.
- 14–6: Think of this as an addition problem. 6 plus what number is 14? To achieve this, first one asks: 6 plus what equals 10, 6 and 4 is 10, and then 4 more is 14, so $14 - 6 = 4 + 4 = 8$. These two actions develop an efficient strategy and use decomposition/recomposition, making ten and then the teens number.
- 14–6 : $14 - 6 = (10 + 4) - 6 = (10 - 6) + 4 = 4 + 4 = 8$. Notice the ‘making ten’ strategy from addition and the teens number strategies are being used here very effectively.
- 14–6 : $14 - 4 = 10$ and then $10 - 2 = 8$, therefore, $14 - 6 = 8$. (Once again, the knowledge of teens numbers, making ten, and the decomposition/recomposition is being used.)

These strategies are fundamental to subtraction and are based on the strategies discussed in the section on addition and on a good understanding of making ten—what two numbers make 10?

Fourth, as soon as a child has mastered a family of addition facts ($8 + 6 = 14$; $6 + 8 = 14$; $14 = 8 + 6$; $14 = 6 + 8$), introduce the child to its extension, the family of subtraction facts ($14 - 8 = 6$; $14 - 6 = 8$) and then to the family of facts with missing addend: $6 + \square = 14$; $8 + \square = 14$; $\square + 6 = 14$; $\square + 8 = 14$; $14 = 6 + \square$; $14 = + 6$; $14 = \square + 8$; $14 = \square + 8$).

Once the subtraction family facts and missing addend family facts are automatized, then mix all of them: addition family facts, subtraction family facts,

and the missing addend family facts. Practice their mastery with these group facts.

The next pages demonstrate a multiplicity of strategies developed based on decomposition/recomposition, sight facts, making ten, teens numbers, the language of addition and subtraction, and using Number rods, Visual Cluster Number Cards, and the Empty Number Line.

Cuisenaire rods, Visual Cluster Number Cards, and the Empty Number Line are the most effective and efficient models for integrating addition, subtraction, the family of addition facts, the family of subtraction facts, and then their interrelationship.

C. The Trajectory of the Development of Subtraction in Children

Over the history of teaching arithmetic, several approaches and methods have been developed to teach subtraction. We need to see the development of the concept of subtraction in children. Our teaching methods should reflect that information.

First, we should introduce children to the developmental stages of conceptualizing subtraction. Subtraction emerges in multiple forms, and there are several processes involved in arriving at the outcome.

- a. *Counting—subtraction method* ($14 - 6$): counting off or down 6 one by one, starting either with a group of 14 or a group of 4 and of 10. It is important to point out ‘from which number the 6 are counted off’. In either case, we want to make sure the corresponding addition fact is emphasized.
- b. *Supplement—addition method* ($14 - 6$): count up from 6 while keeping track of numbers counted.
- c. *Subtraction—addition method* ($14 - 6$): $14 - 6 = (10 + 4) - 6 = (10 - 6) + 4 = 4 + 4 = 8$. (Notice the ‘making ten’ strategy from addition strategies is being used here very effectively.)
- d. *Subtraction—subtraction method* ($14 - 6$): $14 - 6 = (10 + 4) - (5 + 1) = (10 - 5) + (4 - 1) = 5 + 3 = 8$. $13 - 9 = (10 + 3) - 9 = (10 - 9) + 3 = 1 + 3 = 4$.

These four strategies are common in Asian countries. They are based on the strategies discussed in the addition facts section and on a good understanding of making ten—what two numbers make 10? Japanese educators have specific names for these concepts, and they want children to know the names of specific processes and strategies. It indicates that once you have the language container for a concept, you can master it easily.

It is also important, in the initial stage of the concept development, to point out that if we are subtracting a number that is smaller than 5 from another number, then counting down is a good strategy. For example, $9 - 4$. In this case, counting down is a good start-up strategy.

If the number being subtracted is closer to the number from which it is being subtracted, then counting up is a good strategy. For example, $9 - 7$. Here, counting up is a better strategy. On the other hand, if a number is being subtracted from a teens number, for example, $14 - 3$, then subtract 3 from 4 and make it $10 + 1 = 11$.

On the other hand, if the problem is: $14 - 7$, then we should subtract 7 from 10 ($10 - 7 = 3$) and the answer is composed by adding 3 and 4. The following pages demonstrate these strategies. Making these decisions strengthens executive functions compared to the default strategy for subtraction as always counting down.

1. Missing Addend

The concept of missing addend has already been introduced in several places; however, to introduce it formally, we suggest the following exercise.

Have each student compete with his/her partner and ask them to take turns finding what rod is missing when one of them takes a rod away from an addition equation and hides it. The student finds the answer and proves it by placing the correct number rod in the empty place. Then each student reverses position with his partner so they take turns hiding the number rod and discovering which number rod is missing. This game should be played almost every day for a few minutes till they achieve automaticity. Here is a sample, using the Number rods:

Have the student translate orally what the rods would correspond to in numbers and write the equation again making sure that the students give four different ways of writing a given problem:

$$10 = 7 + \underline{\quad\quad}.$$

$$\underline{\quad\quad} + 7 = 10;$$

$$10 = \underline{\quad\quad} + 7;$$

$$7 + \underline{\quad\quad} = 10.$$



Figure 8.4a Missing addends

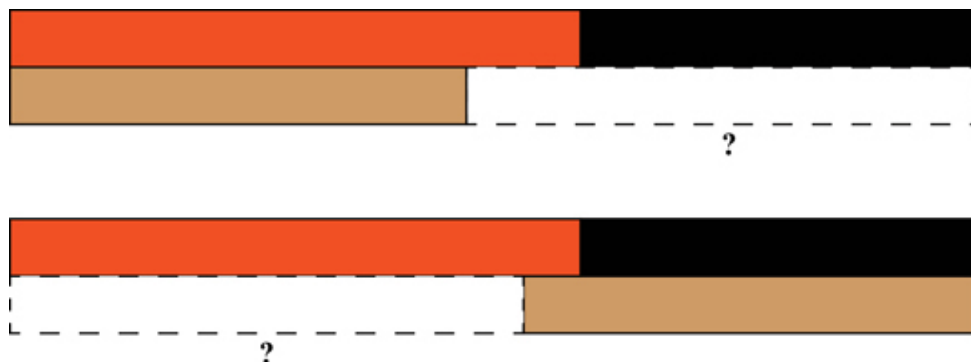


Figure 8.4b Modeling 17-9 using number rods and VCNC

After, children have practiced the missing addend problem with concrete models for numbers less than 10, they should practice with numbers between 11 and 20. A more complicated step might be for the students to find the missing addend for a number between 11 and 20, such as:

$$17 = \text{Brown} + \text{_____} (\text{Blue});$$

$$17 = 8 + \text{_____} (9).$$

The same exercise whether before rods or after is equally useful for conceptualizing this concept.

2. How Do You Teach Adding or Subtracting by Zero?

Although we introduced addition and subtraction by zero, in context. We believe addition and subtraction by zero needs to be carefully introduced only after children have good understanding of addition/subtraction facts (non-zero facts) and the addition/subtraction process. Here is an activity to formally introduce the concept.

Teacher: You have a cookie on your plate in front of you. I also have a cookie in my plate. I would like you to eat your cookie.

The child eats the cookie.

Teacher: What is left on your plate?

Student: Nothing!

Teacher: We also call nothing zero. How many cookies were there in your plate before?

Student: One!

Teacher: How many are there now?

Student: None!

Teacher: Is that also called zero?

Student: Yes!

Hold the 5-rod in your palm of your right hand and nothing in the other hand and ask:

Teacher: What number do I have in my right hand (point to it)?

Children: 5.

Teacher: What number do I have in my left hand (point to it)?

Children: Nothing!

Teacher: Another way of saying nothing is zero.

Teacher: What number do I get when I put them together (combine or add them)?

Children: Just 5.

Teacher: So, 5 plus 0 equals what? (Show the action by pointing to the hands).

Children: 5.

Teacher: Good! Let us write the addition equation for the sum of 5 and 0. Who will write it for me?

One child is asked to write it on the board. He writes:

$$5 + 0 = 5.$$

Ask another child to write all the other forms.

$$0 + 5 = 5, 5 = 5 + 0, 5 = 0 + 5.$$

Ask all students to write the four equations:

$$5 + 0 = 5, 0 + 5 = 5, 5 = 5 + 0, 5 = 0 + 5.$$

Teacher: What numbers can I add to zero?

Children: Any number.

Teacher: So any number plus zero equals what?

Children: That number.

Teacher: Good! I have this rod (picks up the 8-rod). What number will it become if you add (nothing) zero to it? Please add zero to it (points to the 8-rod). What is the answer?

Student: 8.

Repeat the activity with other numbers. Do the same activities with Visual Cluster Cards. Rehearse it using the three activities: *chorally*, *individual oral*, and *individual written*. Differentiate the questions for children by using different levels of numbers (from one digit to three digits).

Now, the teacher holds the 5-rod in the palm of her right hand.

Teacher: What number do I have in my right hand (point to it)?

Children: 5.

Teacher: Take the number from my hand (one child takes the five rod away). What number is left in my hand?

Children: Nothing or zero.

Teacher: Let us write the equation for this action. 5 minus 5 equals what?

Children: $5 - 5 = 0$.

Teacher: Great! What number do I get when I take nothing from it?

(Holds the 5-rod in the right hand and points to it).

Children: Just 5.

Teacher: So, five take away zero equals what?

(Show the action by pointing to the hand).

Children: 5.

Teacher: Good! Let us write the subtraction equation. Who will write it for me?

Child: $5 - 0 = 5$.

Teacher: Let us write all the equations we just created:

$5 + 0 = 5$, $0 + 5 = 5$, $5 - 0 = 5$, $5 - 5 = 0$.

The activity is repeated in the same way with the visual cluster cards. The only difference is that there is an empty Visual Cluster card that has no icon or pips on it, so this card represents zero.

The teacher holds the five-card in one hand and the blank card in the other hand and follows the same sequence of questions.

Children can make a zero with their thumb and forefinger and combine this with any rod.

3. The Importance of the Inverse Relationship between Addition and Subtraction

Children should understand that just as the addition operation is performed on the same kind of ‘things’; similarly, subtraction operations can be performed only on objects of the same type, except when we are comparing two different types of objects (e.g., there are 7 boys and 2 girls in the room; how many more boys than girls are there in the room?). This additional component in the subtraction context creates a great deal of difficulty for children in solving word problems involving subtraction. Therefore, the role of language is even more important than for addition.

Just as in teaching addition, we have three kinds of models: ***discrete*** (counting objects, e.g., buttons, cubes of different kinds, multiple types of blocks, pennies, chips, straws, number line, etc.), ***continuous*** (visual spatial materials: Number rods, strips, number line, rulers, Empty Number Line, etc.), ***mixed*** (discrete objects organized into patterns, e.g., Visual Cluster Number cards, dominos, dice, BaseTen blocks, Ten Frames, fingers, etc.).

We should teach subtraction facts with addition facts, but introduce an addition strategy first and practice it. A few days after an addition strategy has been introduced and children feel secure with the concept of addition and mastery with rigor has been achieved for a particular strategy, then we should introduce the reverse part of that strategy in the case of subtraction. For example, when $N + 1$ strategy has been mastered, then begin the subtraction strategy of $N - 1$. Or $N + 1$ minus 1 equals what?

Practice these strategies till children have mastered it well with rigor. Rigor means: understanding the strategy (an accurate answer is arrived at by meaningful reasoning), fluency (the answer is arrived at in acceptable speed—two seconds orally and three seconds in writing), and applicability (can use it in solving problems). Then, integrate corresponding addition and subtraction strategies. The addition strategies in transformed form by reversal are the subtraction strategies. Meaningful strategies in subtraction are not arrived at by counting back; they should be based on efficient addition strategies. Therefore, one should begin by reviewing addition strategies and equations before subtraction facts are introduced. For example, using Visual Cluster Cards and/or Cuisenaire rods, one should ask:

- What plus 3 is 8?
- 3 plus what is 8?

- 8 take away 3 is what?
- 8 minus 3 is what?
- How much more (or greater) is 8 than 3?
- How much less is 3 than 8?

Now there are six equations for the parallel trains with Cuisenaire rods:

$$3 + 5 = 8$$

$$5 + 3 = 8$$

$$8 = 3 + 5$$

$$8 = 5 + 3$$

$$8 - 5 = 3$$

$$8 - 3 = 5.$$

Using these six equations and asking questions like those asked here are the means of exposing them to see the patterns in addition and subtraction. Then, since the children have heard the collection of similar facts 18—25 times. They are ready to visualize the facts, so have the children hide the rods and write all six equations.

The lesson should always move from the *intuitive level* (can they give the answer based on their addition fact knowledge?) to the *concrete level* (Cuisenaire rods) to the *pictorial level* (Visual Cluster Cards, Sight Fact Cards, with cm grid paper which children color to show the sum of the rods, and then show the strategy on an Empty Number Line) to the *symbolic level* (Graphic organizer—the decomposition/recomposition equation and then with the written equations) to the *application level* (solving problems: applying to other mathematics problems, real life situations, etc.) to the *communication level* (able to explain the strategy and reasoning in applying the subtraction facts and strategies). When a child has reached the communications level, then they will have little or no regression.

For the application level, the teacher should be sure that several types of word problems are given to be solved during their instruction, from simplest to complex. Before children are introduced to word problems, they should be encouraged to construct number stories from subtraction equations, first orally, and then by asking them to write them, assisted by the teacher and the other adults present in the classroom. This is an exercise in translation from mathematics language to English language, whereas solving word problems is an exercise in translation from English language to mathematics language. When they are able to perform this dual

translation, they have the facility in mathematics language—its vocabulary, syntax, semantics, and translation.

After they are comfortable in constructing number stories, formal word problems should be introduced as an extension of these number stories. Word problems can be constructed in several ways based on *language*—vocabulary and syntax (direct statements to indirect statements), type of numbers used, placement of the questions, and the presence or absence of extraneous information. The first time around, the word problems can be illustrated with the rods. There are five problem types:

- a. *Number stories where data is placed first and the question is last.*

Example:

John had 8 apples and gave away 3.

How many apples are left?

- b. *Number stories where the question is first and data is last.*

Example:

How much taller is Jan than Joel?

Joel is 120 cm tall and Jan is 135 cm tall.

- c. *Number stories where the question is embedded in the story.*

Example:

Linda has 35 cookies. How many more cookies does she have than Sally? Sally has 24 cookies.

- d. *Number stories with extraneous information included in the story.*

Example:

Today is May 2, 2013, and I have 10 red stickers; each sticker is worth 5 points. I have 6 blue stickers, and each one of them is worth 10 points. How many stickers do I have?

- e. *Number stories with extraneous information and units that are not same in the story or more than one operation is involved.*

Example:

Linda has 35 cookies. How many more cookies than Sally does she have? Sally has two trays of cookies with 10 cookies on each tray.

D. Writing Number Stories and Word Problem-Solving Strategies

To conceptualize mathematics ideas and solve word problems, students need competence in two-way translation: English to Mathematics as well as Mathematics

to English. Similarly, in learning and teaching mathematics, we are also constantly moving from the native language to the mathematics language and vice versa.

As mentioned in the chapter on language, in the English language, number is a qualifier (an adjective) of nouns: ‘3 plates,’ ‘5 children,’ or an object: ‘it is 5 o’clock.’ In mathematics, the number is an *operator* and *operand* and there is always an *operation* between them. In the problem: $17 - 9 = ?$ 17 is an operand and 9 is an operator, and subtraction is the operation. The word problems are stories involving numbers that describe operations and relationships between numbers in a certain context, where one or more numbers are operators and one or more numbers are operands and one operation or more are taking place. It is a story about numbers and their actions upon each other, and there is always an outcome.

Children should know how addition and subtraction equations emerge—the context, problem situations, the role of numbers, and the type of problem. This can be done by focusing on the language associated with a particular addition or subtraction equation. For example: Let us consider the familiar subtraction equation we have encountered already:

$$17 - 9 = \square$$

First, ask students to read this equation with different English words and expressions for the operation involved in the equation. The teacher can suggest a word or phrase to be used to generate the language for the equation. The teacher can also do this exercise in pairs and together the partners come up with expressions. Then the teachers collect them on the board. When they run out of expressions, the teacher can interject the words and expressions for them to use. After they can give an oral expression for the equation, the teacher can write or display the expression/story on the board for children to read and possibly write, with the teacher’s help.

Before they write stories or solve problems, it is important to translate the expression $17 - 9$ in many different ways using a multiplicity of language expressions:

$$17 - 9 = ?$$

How can we write this as a sentence in English language? The teacher should write one or two statements and then ask students to generate and write more sentences with their partners.

- a. 17 subtract 9 equals what?

Let students work together and then share out. Record all the sentences generated on the board. Discuss them and then reorganize them according to the patterns. For example:

- b. 17 minus 9 equals what?
- c. 17 take away 9 equals what?
- d. 17 decreased by 9 equals what?

Here the syntax is direct: *number, operation, number*. It is the form in which subtraction may occur or be called for in most of the word problems children are given. In many classrooms these are the only forms of questions children experience and, as a result, when the word problems use different question form, expressions for operations, or order of words (syntax), meaning, and number forms, they find the word problems difficult. They have difficulty even initiating the solution process, let alone solving it.

- e. If 9 is taken away from 17, what do you have?
- f. 9 less than 17 is what?
- g. Subtract 9 from 17 is what? Or,
- h. What is the answer when we subtract 9 from 17?

Here the syntax is indirect: the order of placement of numbers is changed without changing the meaning of or the operation involved in the problem. The placement of the question in this case does not change the syntax and of mathematics operation.

- i. What is the difference between 17 and 9?
- j. 17 is how much more than 9?
- k. 9 is how much less than 17?

Here the vocabulary, syntax, and the structure of the sentence is more complex than the other two patterns of language. There is no direct word or expression that refers to 'subtraction,' 'minus,' or 'take away.' These expressions are most difficult for children to grasp, so only the context of the story helps them to grasp it. For example:

David is 17 years old and Ron is 9 years old. What is the differences in their ages?

The concept of subtraction is more difficult than addition and division is more difficult than multiplication for several reasons, but the complexity and diversity of language used in these concepts is one of the key reasons. The analysis of this structure and its understanding is the beginning of algebraic reasoning.

Now let's do the reverse translation; let us go from English language to mathematical expressions.

Example: 8 decreased by 5 translates into $8 - 5$ as an arithmetical expression.

Example: 9 increased by 7 translates to $9 + 7$.

Example: 12 increased by 6 translates to $12 + 6$

Example: 8 less than 15 translates to $15 - 8$.

When they are able to translate these mathematical expressions fluently, they will be able to construct, understand, solve, and apply this knowledge to solving problems emerging from many situations in their environment. After this, the teacher should give numbers in a story and then ask students to generate questions to be answered. For example,

Teacher: Listen to this story. David is 17 years old. His sister is 9 years old. What question should we add to this story so that it will be a subtraction problem?

Student 1: How much older is David than his sister?

Student 2: How much younger is David's sister than David?

Student 3: What is the difference in their ages?

Children begin to repeat themselves. At this point the teacher should introduce few more questions.

Teacher: Any more? Let me give you my question:

How many years ago was David as old as his sister is now?

Student: In how many years will his sister be as old as David is now?

Teacher: Very good! Any more? Let me give you another way of asking a question so that it will be a subtraction problem.

How old was David when his sister was born?

Teacher: Now, I want you to complete the story (she distributes a sheet of a paper with the story without the question written on it) by adding one of these questions. Each one of you must complete the story with one question. When you are done, show me your story.

The teacher roams around in the classroom watching their work, helping them with spelling, writing, and checking their stories. When a child has completed a story, she asks him/her to complete the story with another question. At the end of this process, some children have written four to six stories, but everyone has written at least one story. This is an example of differentiation. This kind of differentiation meets the needs of all children, including the gifted and talented children. After this, s/he asks children to write their own stories with the same data.

Teacher: You did a good job. Now on the same paper, I want you to write a new story, a new setting, and new characters for a subtraction problem using the same numbers in the equation ($17 - 9 = ?$). You cannot use David and his sister as characters or their ages in your story.

Once again, s/he roams around in the classroom watching them work, helping them with spelling, writing, and checking their stories. When a child has completed a story, s/he asks him/her to complete the story with another question. In this process, some children have written four to six stories, but everyone has written at least one story. Examples are then shared in the class.

Example 1: There were 9 pencils on the desk. How many more pencils are needed if there are 17 children in the room and every child will be given a pencil

Example 2: I have pledges to walk 17 miles for the hunger-walk. I have already walked 8 miles. How many more miles do I have to walk?

Example 3: My team made 17 points on Monday and 9 points on Tuesday. What is the difference of points between Monday and Tuesday?

Then children are asked to write their own stories with their numbers and questions in the story.

Example 1: There were 9 boys in the class. How many more boys should join so that there are 17 children in the classroom?

Equation: $9 + \square = 17$ Or $17 - 9 = ?$

Example 2: I want to have 19 stickers in my collection. I already have 8 stickers in my collection. How many more stickers do I need to have?

Equation: $8 + \square = 19$ Or $19 - 8 = ?$

Example 3: The book has 54 pages in it. I already have read 29 pages. How many more pages do I have to read?

Equation: $54 - 29 = ?$ or $29 + ? = 54$.

This process of writing equations from ‘number stories’ is the simplest and surest way of providing skills in understanding and solving word problems and preparing students for applications in multiple settings:

- a. *intra-mathematical*,
- b. *interdisciplinary*, and
- c. *extra-curricular*.

These activities provide opportunities for developing the *vocabulary*, *syntax*, *semantics*, and *translation skills* from English to mathematics and from mathematics to English.

In each of these examples, it is important that the students see parts to whole and whole to parts relationships. Through these examples and with Cuisenaire rods and other materials (e.g., Unifix cubes, Visual Cluster cards, Empty Number Line, BaseTen Blocks, etc.), students will be able to progress through the six levels of knowing.

Understanding and integration of several primary concepts involved in the missing addend problems related to additive reasoning (e.g., addition/subtraction, missing element, equality, part-to-whole relationship, decomposition/recomposition) are the key for mastery of additive reasoning and for a solid foundation for the next concepts, such as multiplicative reasoning.

In general, in primary numeracy classrooms most first and second grade children may be encouraged to focus on not just on single operations (addition or subtraction, missing element, or equality) at a time, but on the integration of all of these much earlier. Students need to acquire the ability to reverse and integrate operations. That provides them with flexibility of thought and their learnability increases.

4. Setting and Extending Patterns of Addition

It is not just the act of counting and repetition of numbers that is important; it is seeing the pattern in the emerging numbers and their relationships that helps children to see connections and therefore helps them in sending information to and retrieving information from their long-term memory. This can be achieved very easily as follows: Ask a student a fact (addition or subtraction) orally. For example, what is $7 + 5$? One student answers it. Ask the student to give you the strategy used. When several strategies have been explored, then you ask another student what is $17 + 5$? Write several problems on the board of the following type:

37 47 67 7 87 77 127 27 427 57 137

+5 +5 +5 +5 +5 +5 +5 +5 +5 +5

Which of these questions you ask a particular child depends upon what you know about that child. A child who is struggling should be asked a simple question ($7 + 5$, $17 + 5$, etc.), and the student who has understood the strategy or pattern should be asked a more difficult question ($87 + 5$, $127 + 5$, $237 + 15$, $347 + 145$ etc.).

Ask each child one of these questions. You want to make sure that the child is not arriving at the answer by counting. If the child begins to count, bring the strategy to the child's attention by asking scaffolding questions: $7 + 5 = ?$

T: Let us use the Empty Number Line. Is the answer more than 10 or less than 10?

C: More!

T: Good! Which is the bigger number?

C: 7.

T: How do get to 10 from 7?

C: Add 3.

T: Where do you get the 3?

C: From 5.

T: Very good! When 3 from 5 is used, what is left in the 5?

C: 2. Now I know. It will be 12.

T: Great. Wonderful! Now, can you give me any other strategy?

C: Yes! I already know $5 + 5$ is 10 so, 2 more is 12, because $7 + 5 = 2$.

With the student who is asked $127 + 5$, the questions may go as follows. However, this discussion should be for the whole class to observe.

T: How would you find the sum of $127 + 5$? I see David's hand is up. How would you do this, David?

D: My brother showed me how to solve this kind of problem. I write them like this:

$$\begin{array}{r} 1 \\ 127 \\ +5 \\ \hline 132 \end{array}$$

T: That is correct! He taught you well and you learned it well. Very good! This is a procedure and everyone will learn this in this class. However, can you apply the same strategy as others applied with their problems?

D: I do not know! I can try!

T: Does anyone know what are tens numbers?

Several hands go up. She asks David: 'Can you count by tens?'

D: Yes! It is so easy. 10, 20, 30, 40, ..., ..., 80, 90, 100, 110, 120, ...

T: Great! What is the next tens to 127?

D: 130.

T: Great! How do you get there from 127?

D: Add 3 more!

T: Where is 3 coming from?

D: From 5! And the answer is 132 as $127 + 3$ is 130 and $130 + 2$ is 132. So, $127 + 5 = 132$.

T: Great! So when adding a one-digit number to a multi-digit number, the only thing we will change is, instead of making ten, we would say: What is the next ten to the bigger number?

At this point, she invites other children who want to try adding a two-digit and a one-digit number or a three-digit number and a one-digit number. Invariably, several children volunteer. To make it collaborative, we always invite a team to try the problems, then even more children volunteer to solve difficult problems. In our experience in such an environment, we have found that many more children are willing to try problems that are harder than they thought they could do earlier. Contrary to belief that by nature children are born cooperative and are not competitive, our experience shows that children are highly competitive and equally cooperative. It depends on how they are approached. Many children believe that if

David can do the problem, I will try, too! Because of this the performance level of the whole class goes up.

E. How to Address Students with Learning Differences

Our approach to teaching-learning processes (pedagogy) in the form of *CPVA/S*, as described and used earlier, in developing concepts, strategies, and procedures is multi-sensory, multi-modal, and multi-model; therefore, every child has the opportunity and possibility to learn and optimize their potential.

However, it is more important to pay attention to (a) how to maximize this input and (b) what the student does once he receives this input information. So the goal of instruction must be to help children maximize the processing of this incoming information and facilitate to use of it in a multitude of applications.

The nature and quality of processing of information defines one's level of mathematics learning as it also defines the *mathematics learning personality* of a student. Each one of us falls on a continuum of mathematics learning personalities. Therefore, in every classroom and in every group, there are children with a range of different mathematics learning personalities. To reach that range of mathematics learning personalities, not only different modalities but also different types of instructional materials, models, and modes of presentations should be used and we need to maximize their usage.

This *continuum* ranges from *quantitative* to *qualitative mathematics learning personalities*⁸ ([Bath et al., 1986](#); [Sharma, 1988](#)). *Quantitative mathematics learning personality* children process information sequentially—parts-to-whole, step-by-step, procedurally oriented. They learn more when emphasis in instruction is more on *deductive reasoning*, through *direct instruction*, *more language-oriented*, and *properly categorized* by the nature of the problems. Their approach is more *parts-to-whole* and *procedurally/sequentially-oriented*, whereas, with *qualitative mathematics learning personality* students process information more *visually/spatially*, look for *patterns*, and *whole-to-parts*. They learn more and better when emphasis on instruction is more on *inductive reasoning*, through pattern recognition, through experimentation, visually oriented, and with an *emphasis on concepts before any procedure is derived*.

Quantitatively oriented mathematics learning personality students make errors of *commission*—lack of or poor strategies, incorrect or inappropriate use of a procedure or strategy, doing what is not needed. They do not see the big picture.

They do 10 problems correctly of the same type, but when the problem is slightly changed, they may say: ‘We have not covered these problems yet.’ Qualitatively oriented students make errors of *omission*—leaving out a sign, not paying attention to a operational symbol, mis-interpreting the problem, lack of memorization of facts, over-generalization, not paying attention to the unit of measurement, etc. They look for patterns, try to do too much in ‘their head’—not showing their work. They are less organized. We have found that quantitatively oriented children do better on procedural aspects of mathematics (e.g., arithmetic, algebra, coordinate geometry, etc.) and qualitatively oriented students do better on spatially oriented concepts and geometry.

In every classroom, there is a range of mathematics learning personalities represented from extreme quantitative to extreme qualitative. The needs of all children can be met if the teacher uses a multiplicity of instructional models that satisfy the conditions of: (a) from discrete to continuous, (b) multi-sensory, (c) multi-modal, and (d) multi-model. For example, during the primary years, to meet the needs of the quantitative learners, the teacher may begin the lesson with discrete materials and tools (e.g., counting blocks, Unifix cubes, number line and counting chips, etc.). But she quickly moves to semi-continuous and continuous materials (e.g., Dominos, Dice, Visual Cluster cards, Sight Fact cards, Cuisenaire rods, BaseTen blocks, Invicta Balance, etc.) to meet the needs of the qualitative learners and all those who lie in between the continuum of mathematics learning personalities. By using a range of materials and models in the activities, the needs of both groups of learners are addressed. Similarly, she should mix the deductive and inductive reasoning in her approach to explaining ideas and arriving at procedures and strategies. The teacher’s questions must range from convergent to divergent. She should use all of her three roles as a teacher: didactic, Socratic questioning, and coaching in an appropriate manner to integrate their learning and achieving mastery.

In any given situation, children function at different levels of mathematics understanding from intuitive to concrete manipulations to visual and pictorial representations to abstract and symbolic to applications and finally to communications. Therefore, the amount of time devoted to each level, for example, on work with concrete materials and use of computer instruction (it can play the role of visual representation of materials) and the time at the abstract level (paper-pencil activities dealing with addition and subtraction) work should depend on our understanding of mathematics content and the children and their needs. The child who has been identified as having difficulty or with limited preparation for these

tasks should be provided with more opportunities of manipulating materials for a longer time and with more diverse activities generated by the computer. On the other hand, the child who is ready and functioning at a higher level should be given appropriately challenging questions during the instruction time. The introductory tool building component of the lesson and the main lesson are the same for all children except the nature of numbers used and the number of problems solved is different. However, during the individual written practice time students can be given differentiated levels of problems and amount of reinforcement of instruction.

F. Reversibility of Thought (Writing Subtraction Equation from Addition Equation)

For fluent performance on mental arithmetic dealing with subtraction operations, it is important for a child to have acquired fluency in the reversibility of addition and subtraction operations. The concept of the reversibility of addition and subtraction means that if the child knows the family of addition facts, he is also able to state corresponding subtraction facts. For example, if he knows that $7 + 8 = 15$, fill the gap $8 + 7 = 15$, $15 = 7 + 8$, and $15 = 8 + 7$, then he is able to state that $15 - 7 = 8$ and $15 - 8 = 7$; $127 + 15 = 142$, then $142 - 127 = 15$ and $142 - 15 = 127$. By this facility in reversibility between addition and subtraction, the student can use all of the addition strategies to generate subtraction strategies and transform subtraction problems into addition problems and is able to mentally solve them. The reversibility of thought also provides opportunities for strategies that may make use of a combination of addition and subtraction strategies. For example: $80 - 39$ can be found as: $80 - 40 = 40$ and then $80 - 39 = 80 - 40 + 1 = 40 + 1 = 41$. At face value, this is a difficult concept for many children, but by the help of Cuisenaire rods, this concept is easily learned.

Notes

1. You just have to explore the Internet and you will have a plethora of methods, lessons, approaches, and pedagogies, tools, etc. for teaching and learning subtraction. [!\[\]\(065aacad479feea1b3f501fa02b79a7a_img.jpg\)](#)
2. Children will encounter similar difficulties with division vs multiplication, subtraction and division of fractions compared to addition and subtraction of

fractions, and factoring of polynomials compared to multiplying two binomials or polynomials. [!\[\]\(08a82c22d89d6b027ff69762ad096586_img.jpg\)](#)

3. It is important to note why these words and the related sketches, in their minds, represent the basic structures of the task at hand. This also shows the role of task analysis and primary concepts in learning a new secondary concept. [!\[\]\(6841ca9b0e023296428e7c9e683b9367_img.jpg\)](#)
4. Children keep saying ‘minusing’ for all of these operations/actions. This substitution of words for all the words used in the form avoids understanding the original words and delays the applications and becoming proficient in word problems. [!\[\]\(e258e347e7683f87061f627f84598eb5_img.jpg\)](#)
5. For deeper understanding of the process, see the chapter on Number Concept and the eBook *How to Teach Number Concept Effectively and Easily* ([Sharma, 2019](#)), an eBook. [!\[\]\(1233990ad3f0b7475c568d7bf16af31f_img.jpg\)](#)
6. It is important that we do not begin with apps and visual representations or even Virtual Cuisenaire rods for deriving facts, strategies, and procedures as our aim is to make the experience multi-sensory. [!\[\]\(18570b67a4686b081406cd3de636c1c3_img.jpg\)](#)
7. However, commutative property for subtraction does not exist. It is important to demonstrate this by the help of Cuisenaire rods. [!\[\]\(411af059a517db12f1097bc63c4fbe36_img.jpg\)](#)
8. Chinn () identifies these, in much more general ways and applicable to other academic learning, as cognitive process styles: ‘*inchworms*’ and ‘*grasshoppers*.’ Others have tried to associate these with hemispheric specialization. There is some correlation, but the phenomenon of preference is more than just innate abilities. It is also a function of environment. The instruction by teachers of one type or the other in a row of three to four years influences students’ approach to learning mathematics in certain ways. [!\[\]\(ed2b7fb1e3bd6514676d2ab3c70d5776_img.jpg\)](#)

9 Understanding Procedures of Additive Reasoning

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The number concept and addition and subtraction facts and relationships have their uses in the larger world, particularly in their applications. Therefore, the long journey of what we are learning today is due to the applications of few key evolutionary skills: language, spatial sense, numbersense, tool building, and inventing and manipulating symbols. In numeracy, the integration of these skills is exemplary. The best examples are the tools of mathematics. They begin with the procedures—algorithms. In the case of most tools in mathematics, ontogeny recapitulates phylogeny loosely kept. This marvelously succinct and elegant process almost becomes our school mathematics curricula—what the human race experimented, discovered, invented, developed, honed, applied, and improved; we want our children to learn in a span of few years. Most children go through this process quite successfully, some struggle. To understand, learn, apply, and appreciate these procedures/algorithms, we need to provide our students opportunities to experiment and taste this creative process. In other words, create a semblance of creativity, shaping ideas, practice, and mastery.

Once children have mastered the facts (e.g., addition and subtraction and their inverse relationship), they are ready to extend their knowledge to add numbers beyond 20 and learn different algorithms including the standard algorithms for addition and subtraction. It should be remembered that understanding and flexibility in numbersense—fluency, mastery, speed, and the elegance of number usage continues to develop during the procedural level also ([Fuson, 1988, 1992](#)).

Many teachers introduce, teach, and practice algorithms at the abstract symbolic level only and without properly deriving them. The polished, effective, efficient, and elegant algorithms are the outcome of a long process. Students should be given the opportunity and taste of this process.

Some teachers do use bundles of straws and BaseTen blocks (either concrete or pictorial forms) for showing the standard procedures of addition and subtraction. However, their instructions and these materials continue to emphasize counting for grouping of numbers into tens, groups of tens, and more. We have found that the combination of BaseTen blocks and Cuisenaire rods as one instructional tool is much more appropriate and efficient for

teaching multi-digit addition and subtraction procedures rather than bundles of straws, connecting cubes, or BaseTen blocks alone, Numicon, Rek-n-Rek, etc. Using Cuisenaire rods and BaseTen pieces (minus the unit cubes) together help children understand the procedure and the move to abstract, symbolic level easily and effectively.

Initially, students should have the opportunity and support to explore and develop their own algorithms for addition and subtraction. The use of proper manipulatives with the *guided explorations* in collaborative settings and answering thoughtful instructional questions from the teacher are key to helping students develop and derive their own procedures. It is important to know that quality of questions from teachers and classmates, diverse but precise/language usage, exposure, and the presence of ‘multiple purpose’ tools instigate better explorations and results in understanding and applicability.

Similarly, mental numbersense strategies and their integration are needed to bridge one-digit addition and subtraction to two-digit (and then multi-digit) addition and subtraction procedures. We have found the following strategies and prerequisite skills to be very helpful for adding and subtracting two-digit numbers mentally and then practicing them using Empty Number Line and then relating them to addition and subtraction procedures, including standard algorithms:

- Identifying and locating a number between the two tens numbers: 27 is between 20 and 30.
- Find the distance from a number to the next tens or the next multiple of 10, 100, or 1000: 27 is 3 away from 30; 270 is 30 away from 300; 2700 is 300 away from 3000.
- What is: $10 - 1$; $20 - 1$; $70 - 1$; $90 - 1$; $100 - 1$; $190 - 1$; $200 - 1$; $1000 - 1$; etc.
- Count on or back in repeated steps of 1, 10, 100, or 1000.
- Partition into hundreds, tens, and ones: canonical and non-canonical expansion of a number.
- Flexibility with addition strategies into decomposing and re-composing numbers,
- Add or subtract the nearest multiple of 10, 100, or 1000 and then adjust the result.
- Use the inverse relationship between addition and subtraction.

Contrary to common belief that the objectives of mental math (learn a repertoire of strategies from which to select) and written arithmetic (gaining proficiency in a compact method of procedures) are different, we believe they are complementary and build on each other. We have found that when connections are made between mental calculations and standard procedures, children do not commit as many procedural errors as we see in the case of a problem such as: $302 - 298 = 196$ or 104 . The mental strategies help them to see the estimate of the number: ‘300 is 2 away from 298 and 302 is 2 away from 300; therefore, the answer is 4.’ The following is a series of activities that can help children see these connections:

- Establish mental methods, based on decomposing and recomposing of numbers to form sums and differences, properties of numbers and operations, and a good understanding

of place value,

- Practice ‘canonical’ and ‘non-canonical’ expansion of a multi-digit number from *standard form* to *expanded form* and *place-value form*,
- Present problems in horizontal format as well as vertical format and ask them to convert from one form to the other,
- Show children how to set out calculations vertically, initially using expanded layouts that record their mental methods (such as Empty Number Line, 100-chart, arrows, graph/squares paper, etc.),
- Practice addition (in first and second grades) and subtraction (in second and third grades) of multi-digit numbers by using Empty Number Line for several months,
- As children become more confident, refine the written record into a more compact methods and develop compact and efficient scripts.

Research shows that children from several Asian countries and some European countries do much better than US and UK children on international tests and competitions.¹ In recent years, the outstanding performance of Chinese students in Shanghai in the OECD PISA mathematics assessment and Macau SAR’s rise from 15th position in 2009 to 6th position in 2013 have attracted much interest from educators around the world. One of the reasons seems to be that educators in these countries emphasize the automatization of arithmetic facts and mastery of mental operations before formal procedures are introduced and practiced. They consider teaching strategies important and they insist that children know and articulate those strategies and their applications during their usage. In the earlier chapters, we have introduced several activities the teacher can use for developing facility with mental arithmetic involving addition and subtraction concepts and operations ([Berg and Hutichonson, 2010](#)).

A. The Role of the Properties of Numbers and Number Operations on Mastery of Arithmetic Facts and Arithmetic Procedures

Mental arithmetic and written procedures are highly dependent on and are facilitated when students understand and are proficient in the properties of numbers, operations, and place value.

- Commutative Property of Addition* (discussed earlier)
 $a + b = b + a$, for any number a and b
- Associative Property of Addition*
 $(a + b) + c = a + (b + c)$ for any number a , b , and c
- Combination of Commutative and Associative Properties of Addition*
 $(a + b) + c = a + b + c = a + c + b = (a + c) + b$

$$\text{Ex: } 57 + 80 + 23 = 57 + 23 + 80 = 57 + 20 + 3 + 80 = 57 + 3 + 20 + 80 = 60 + 100 = 100 + 60 = 160$$

Cuisenaire rods and Visual Cluster cards are the most effective instructional materials to make these properties clear, understandable, and usable for and by children. These properties greatly facilitate mastering the addition and subtraction facts. This involves decomposition and rearrangement of numbers into smaller numbers that are manageable and compatible in some form and then recomposition of these numbers using known number relationship strategies.

- d. *Understanding Even and Odd Numbers*: The sum of even numbers is always even; the sum of an even and odd is odd, the sum of two odd numbers is even, etc.
- e. *Observing patterns of number relationships*, e.g., $7 + 3 = 10$; $17 + 3 = 20$; $97 + 3 = 100$; $10 - 8 = 2$; $20 - 8 = 12$; $90 - 8 = 82$; $760 - 8 = 752$; etc.
- f. *Making ten and teens numbers* is very useful strategy in adding and subtracting procedures.

For example, when one of us, in a demonstration class, asked a group of first graders: What is the sum of: $9 + 6 + 3 + 1 + 5 + 4 + 5 + 2 + 7 + 8$? First, they said: ‘it is too big a problem.’ ‘Too many numbers.’ Then, one of the students said: ‘I can do it. Look, there is a 9 and a 1 that is 10, there is 6 and 4 ...’ As soon as this student said that, several other students said: ‘That is right. We can do it.’ They transformed the problem to:
 $9 + 6 + 3 + 1 + 5 + 4 + 5 + 2 + 7 + 8$ as:
 $6 + 4 + 9 + 1 + 7 + 3 + 5 + 5 + 2 + 8 = 10 + 10 + 10 + 10 + 10 = 50$ by applying the commutative property of addition, making ten—the sight facts of ten.

The commutative and associative properties helped them to see the connection and then the addition strategy of ‘making ten’ helped them to get the answer. Now, they wanted to find the sum of even bigger and more numbers and several problems. Some of them even found the sum $12 + 7 + 25 + 8 + 23 + 5$ easily by regrouping and using commutative and associative properties as $12 + 8 + 7 + 23 + 25 + 5 = 20 + 30 + 30 = 80$. Similarly, some of them found: $137 + 253 = 390$ by mentally adding: $100 + 37 + 200 + 53 = 100 + 200 + 30 + 50 + 7 + 3 = 300 + 80 + 10 = 390$. Children in classes where addition strategies and properties of operations are not emphasized or mastered do not even try these kinds of problems, and if they try they use only sequential counting. Many teachers underestimate the abilities of children.

B. What Should the Level of Mastery of Procedures Be by the End of Second Grade?

By the end of second grade, students should have mastered the additive reasoning. Additive reasoning means students have mastered addition and subtraction—their language, the

conceptual schemas, facts, and procedures including the standard algorithms. They are able to see the inverse relationship between addition and subtraction language operations—given a subtraction fact or a procedure, they are able to write the corresponding addition equation and vice versa.

In this grade, children's attention should be primarily focused on consolidating and mastering conceptual and procedural aspects of the addition and subtraction concepts. To master algorithms, students need to master addition and subtraction facts, place value, and then the procedures themselves.

Before formal algorithms (both alternatives and the standard procedures) are introduced, it is important that addition and subtraction facts are extended (see Addition and Subtraction Ladders). Children should practice Empty Number line exercises on adding and subtracting one-, two-, and three-digit numbers for a month or two, both procedurally and in applications. The Empty Number Line (ENL) is one of the most effective and efficient ways to achieve a strong sense of number and number relationships and therefore numbersense. It is an antidote to acquired dyscalculia and provides freedom from counting. Extending the use of facts is the basis of exercises using ENL. For example:

Teacher: What is $8 + 7$?

Students: 15

Now the teacher should ask: what is $18 + 7$? And then, $48 + 7$? $28 + 7$?, $68 + 7$, $288 + 27$?

This kind of extension of patterns by selection of numbers should be repeated for every answer. Keep increasing the sizes of and selection of numbers so children can try different strategies. As children give their answers, the teacher should write on the board without any comments. After all the answers have been collected, the teacher begins asking for their strategies. She should start with the answers that are not correct. As they proceed to describe their strategies in response to the questions, the moment, the reason for the incorrect answer emerges, the teacher should ask questions that bring their attention to the errors, without pointing out the error, so that they themselves realize the presence of errors and identify their errors. Through pertinent questioning by the teacher's and other children's help, they arrive at the correct answer. This effort should be rewarded by the class applauding for the student (for motivation). After all the correct answers are identified and gathered, by all children and different strategies acknowledged, the teacher should ask for alternate answers and discuss the effectiveness, efficiency, and elegance of their strategies and procedures. The second set of problems given to children should be differentiated in terms of the difficulty of the problems as well as the selection of strategies. Students who are functioning at a higher level can be given even bigger and interesting numbers.

As a pedagogy, before we come to the standard (most effective, efficient, and elegant) algorithms, we must explore the concept in multiple ways so that the children's intuitive

sense is given exposure (He, 2015). We need to explore all the possible ways of performing that task. Children will make decisions, of course, with the teacher's help and direction and when they come to the standard algorithm, they appreciate all that represents and the efficiency and power it offers.

Children should be asked to identify the most efficient strategies to get their answers. Through these types of discussions, guided by the teacher, they should arrive at criteria for efficient strategies—some might be generalizable, and some might be efficient only in particular situations. These discussions make children 'future' mathematicians and scientists—whether exploring natural, physical or social world. They become thinking people rather than applying procedures through rote, without knowing what they are really doing. After this, the teacher should extend the problem to subtraction.

Extend it to subtraction facts such as this.

Teacher: What is $15 - 8 = ?$

Students: 7.

Teacher: What is $15 - 7 = ?$

Students: 8.

If a student gives an incorrect answer, the teacher should ask for the use of the Empty Number line. The teacher produces the Empty Number Line representation with the help of the script provided by the child. Then the child produces the ENL for a similar problem.

Now the teacher should extend the subtraction exercises and ask:

What is $25 - 8 = ?$ $75 - 8 = ?$, $35 - 8 = ?$, $95 - 8 = ?$, $85 - 8 = ?$

What is $25 - 7 = ?$ $55 - 7 = ?$ $35 - 7 = ?$ $95 - 7 = ?$

Each child individually produces the ENL for the problem and raises her hand, the teacher looks at it and, if it is properly done, she asks her to do it another way. This process continues till every child has produced at least one ENL for the problem in question. All of the ENLs are represented/displayed on the board/screen.

This work, using the Empty Number line, should be regular feature of a mathematics lessons in first and second grades. The teacher should encourage students to solve the addition or subtraction problem in several ways and then ask the student which one of her approaches is most efficient. The ENL uses all the strategies and skills learned so far in their math curriculum. All the components of number concept, number relationships, place value and the outcomes of addition and subtraction that are culprits in the symptoms of dyscalculia and acquired dyscalculia are addressed by the ENL and the accompanying conceptual models, language, and scripts are practiced in these ENL exercises.

Other strategies for subtraction should be explored. Some examples are:

1. **Translation Method:** $63 - 39$ is equivalent to $64 - 40$ (translation of the problem to the right by 1 unit). By this transformation into a very simple problem, similarly, a translation of one unit to the left makes another difficult problem quite easy.

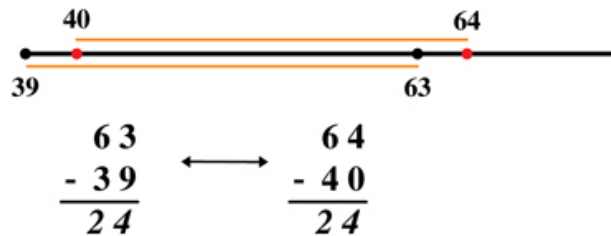


Figure 9.1 Modeling translation on ENL

$70 - 59$ is equivalent to $69 - 58$ (translation of the whole problem by one unit to the left.)

2. Successive Subtraction Down Method

Let's do $81 - 47$.

It involves two steps: First, $81 - 40 = 41$, and then $41 - 7 = 34$. To support this method, it can be easily and very effectively represented by the Empty Number Line.

3. Subtract by Place Value Method

It is similar to the earlier method, with emphasis on place values of the digits. Do subtraction in the tens separately and then the subtraction of the remaining number.

$$81 - 47 = ?$$

$$80 - 40 = 40$$

$$40 - 7 = 33$$

$$33 + 1 = 34$$

4. Subtract in Each Place

$$81 - 47 = ?$$

$$80 - 40 = 40$$

$$1 - 7 = -6$$

$$40 - 6 = 34$$

5. Strategic Compensation Method

It is an extension of the translation method.

$$81 - 47 = ? \text{ Add 3 to both numbers}$$

$$84 - 50 = 34$$

Or, subtract 2 from both numbers

$$79 - 45 = 34.$$

6. Break Apart Method

The *break-apart method of subtraction* is a strategy where one or both numbers in a subtraction problem are broken down (partitioned) by their place values to make the calculation easier, often for mental math or to build a strong understanding of place value.

This method is also known as the *partitioning method* or the *decomposition method*, which often involves ‘borrowing’ or ‘regrouping’ when necessary.

How it Works (Breaking Apart the Subtrahend)

The most common approach for basic subtraction is to break the *subtrahend* (the number being subtracted) into its place values (tens and ones).

Example: Subtract: $65 - 27$

Begin with 65 and break 27 into 20 and 7.
 $60 - 20 = 40$, $40 - 7 = 33$ + $10 - 7 = 3$ + $3 = 33$. $33 + 5 = 38$; $65 - 27 = 38$.

Alternative Approach (Breaking Apart to a ‘Friendly’ Number)

Another way to use the break-apart method is to adjust the numbers to reach a ‘friendly’ number (a multiple of ten).

Example: Subtract $65 - 27$

Start with the first number: 65.

Break the second number (7) into parts so that one part helps you reach the nearest multiple of ten from the first number. The number in the ones place of 27 is 7, so you can break 27 into 25 and 2 (because $25 + 2 = 27$).

Subtract the first part to reach a multiple of ten:

$$65 - 25 = 40$$

Subtract the remaining part from the new result:

$$40 - 2 = 38$$

The final answer is 38. This strategy is excellent for mental math.

This method is also known as the *partitioning method* or the *decomposition method*, which often involves ‘borrowing’ or ‘regrouping’ when necessary.

7. Reverse Method (Indian Algorithm)

One of the earliest and more popular subtraction algorithms came from India. This is called the reverse method, and it has some similarities to our traditional algorithm. It involves borrowing. Here is an example:

$$783 - 298$$

Begin at the left-hand side.

Left column furthest to the left (*hundreds*): $7 - 2 = 5$ Write down the 5.

Middle column (*tens*). $8 - 9$. Since we cannot subtract and obtain a positive number, we cross out the 5 and change it to a **4**. Place a 1 in front of the 8 to make 18. $18 - 9$. Write down 9.

Last column (*ones*). $3 - 8$. Since we cannot subtract and obtain a positive number, we cross out the 9 and change it to an **8**. Change the 3 to 13 in the last column. $13 - 8 = 5$.

Your final answer is 485.

8. Treviso Method (European Algorithm)

$$783 - 298$$

Starting from the right side (ones column), because 8 is larger than 3, what number do you need to add to 8 to obtain a 10? 2.

Also add this number 2 to 3 to get 5. Write down 5 on the bottom. Decrease the 8 in the second column by 1 to get 7.

For the middle column, you have $7 - 9$. Since 9 is the larger than 7, what number do you need to add to 9 to obtain a 10? 1. Add this number to 7, to get 8. Write the 8 on the bottom. Decrease the 7 in the first column to a 6.

For the left column, we now have $6 - 2 = 4$. Write down the 4 at the bottom.

Your final answer is 485.

In this situation, we use the word ‘complement.’ The complement of 1 is 9. The complement of 2 is 8, and so on. Whenever the bottom number is larger than the top number, you add the complement of the bottom number to the top number. (Think of ten, 10.)

9. European Method

It is a variation of the translation method with a slight twist.

63 6¹3 one ten is added to the subtrahend in the one/s place
 – 47 – 5 7 one ten is added to the minuend in the ten's place,
 1 6 therefore, it is a translation by 10 to the whole
 problem. It is like balancing an equation.

C. How to Introduce the Standard Algorithm

When children are quite proficient in understanding the concept of addition or subtraction and fact mastery, the teacher must introduce the standard algorithm, at the concrete level. The teacher should model multi-digit addition problems by the help of Cuisenaire rods and BaseTen blocks and develop the language, the conceptual schema, and the strategies and steps involved in deriving the standard addition algorithm.

First the teacher should take problems where no regrouping is involved and then problems where grouping at the ones level is involved: Let us take an example:

1. Addison Procedure, Without Regrouping

Find the Sum

$$\begin{array}{r} 23 \\ +35 \end{array}$$

The two numbers to be added are displayed on the Place Value mat and aligned in place value with each other.

$$\begin{array}{rcl} 23 & = & 20 + 3 \quad 23 \\ +35 & = & 30 + 5 \quad +35 \\ 50 + 8 & = & 58 \quad 58 \end{array}$$

The first student trades the 3-rod and the 5-rod for the 8-rod and then all the 10-rods are combined and the answer is recorded. Notice: no counting, and the teacher should remind the pupil that they are using the sight facts of 8 and 5.

Many problems like this are computed by children until they have achieved understanding, fluency, and applications of these kinds of problems. After that is the family of addition problems, with regrouping is introduced.

For example, for a problem like $27 + 35$, have children place two tens at the top of the tens (middle section) of the BaseTen place mat and a seven rod at the top of the ones section (right hand section). Below, in the second row of the mat put 3 tens under the tens section and a five rod under the ones section (see the following arrangement).

Example 2: Addition Procedure with regrouping from ones place

$$\begin{array}{r} 27 \\ +35 \end{array}$$

Allow children to decide how to put the two together and how to record. Let them discover their own algorithm approaches.

After sufficient examples like the aforementioned where students have found the answer themselves by using their own strategies and procedure, a discussion of these procedures can take place. The teacher asks them if the ones could be added. If the children ask why (and some children are always interested in whys)? Explain that it is possible that the sum of the ones for the two numbers might be more than 10, as is the case here, so rather than just adding the tens before we can add in the ten from $7 + 5$, so we can add all the tens in one time.

Through this, the teacher correlates and directs the discussion toward the standard algorithm and its advantages. The recording would be in two steps: Recording the partial sums and then the standard procedure.

Partial Sum Recording

Understanding the partial sum recording is the first step of making sense not only of the evolution and development of the procedure, but the role that place value, arithmetic facts, and the addition/strategies play in the standard procedure. There are many alternative procedures for addition and subtraction, as we discussed earlier, but the standard procedures, if understood properly, are effective, efficient, and universal, therefore they are powerful tools. The following describes the essential steps in the *Partial Product* recording procedures.

The sum $7 + 5$ (as shown here) can be derived either by the Cuisenaire rods or by the abstract strategy (also shown here).

$$\begin{array}{rcl} 27 & 27 = 20 + 7 \\ +35 & + 35 = 30 + 5 \\ 12 & = 50 + 12 = 50 + (10 + 2) = (50 + 10) + 2 \\ 50 & = 60 + 2 \\ 62 & = 62 \end{array}$$



Figure 9.2 Modelling $7+5$ using number rods

The Partial Sum Method leads to the standard algorithm which is illustrated in the following manner with the rods:

When adding the digits in the ones places in $27 + 35$, the teacher should help the children to make a train of the two addends (7-rod = Black and 5-rod = yellow) and then ask: Is the train longer than or shorter than ten?

Students: Bigger/longer/more.

Teacher: What color is the 10-rod?

Students: Orange!

Teacher: Please place the 10-rod above the train of $7 + 5$.

They then put a 10-rod and the extra Cuisenaire rod (2-rod) as a parallel train to show that $7 + 5$ is equal to $10 + 2$.

The 10-rod can then be placed in the second section, slightly raised as it would be in the recording of the standard algorithm. The red rod indicates 2 in the ones place of the answer. The original rods (5- and 7-rods) should be removed. Abstractly, using decomposition/recomposition, the same addition can be represented as:

$$7 + 5 = 5 + 7 = 5 + 5 + 2 = 10 + 2 = 12.$$

The sum of digits in the tens' places in the problem can be completed by considering the 10-rods in both numbers. 2-tens in 27 and 3-tens in 35, and 1-ten from 12 gives us $2\text{-tens} + 3\text{-tens} + 1\text{-ten} = 6\text{-tens} = 60$. Therefore, the sum ($27 + 35 = 60 + 2 = 62$).

Recording of the concrete activity in terms of abstract symbols should be done simultaneously as you manipulate the concrete objects so that they can realize the congruence of the concrete model and the symbolic representation. *This is a key learning factor.*

The hardest part of this connecting or recording-as-you-go phase is getting the children to remember to write down each step as they do it after the teacher has demonstrated this recording. The tendency is to put down the pencil and simply finish the task with the blocks. One suggestion is to have the children work in pairs, with one child in charge of the blocks and the other in charge of the recording of the results of the manipulations on concrete models. Children reverse roles with each problem or from one step to another. Two children working together also gives the opportunity for discussion, the use of mathematics language, and the 'script' of the procedures developed during the concrete manipulations are very important. Every time the script is used, it revisits and reviews the language, the concepts, and their relationships.

To determine if children have adequately made the connections with the models, the first step is to see if they can use the manipulatives and record the actions in proper sequence, and if they have written the tens column which represents the renaming of the ten from the ones column. The second step is to ask children to visualize the steps and describe the

actions properly, without hesitation and without looking at the concrete model. For some, the teacher may allow the recitation of steps by the help of the rods and then without them.

For each problem, the teacher should articulate a number story. For example, in the previous problem, she could create a situation, with children, such as:

‘I have two books on my desk. One book has 27 pages, and the other book has 35 pages. Can someone ask me a question about this information, so that I will add these two numbers?’

Children begin to generate questions about the situation with their partners. Someone will come up with the right response; you just have to ask the right questions to scaffold the task for the students. The teacher should walk around and watch students’ progress in forming and discussing questions. When she observes that there are enough responses, not necessarily all correct, she should ask them to share their responses with the class. She should write the problem and the questions on the board for all to see. This will instigate more responses and more discussions. After collecting all the responses and writing them on the board, the teacher discusses the number story with each question added to the story.

Writing number stories and generating appropriate questions is a clear mark of understanding. This is the beginning of applications of the procedure.

When the addition algorithm is mastered with rigor and fluency, they understand ‘what’ and ‘why’ they are doing; they are accurate and fluent in what they are doing consistently (they can do several problems correctly and efficiently). That shows that they have mastered the addition procedure. This also culminates the sequence of movement on the Levels of Knowing continuum: from intuitive to concrete to pictorial to abstract to applications and finally to communication.

Mastery here means: They have the *language, strategies, accuracy, fluency*, and *they can apply the strategies and procedures to problem solving* (Verschaffel et al., 2015), including *word problems*. Then they are ready for the subtraction algorithms and ultimately the standard subtraction procedure, including regrouping from both ones’ and tens’ places and then extending it to multi-digit situations.

However, if there is a gap between the time they completed addition procedures and when they were introduced to subtraction procedures, particularly before introducing them to the standard procedure, the teacher should ascertain that the students have the numerical prerequisite skills for subtraction. These are, once again: (a) 45 subtraction facts, (b) making ten, (c) teens’ numbers, (d) decomposition/recomposition, (e) what is next ten, and (f) the inverse relationship between addition and subtraction.

D. Subtraction Procedures: Standard Algorithm

For the algorithms, use of BaseTen blocks in addition to the Cuisenaire rods is most effective. Use place value mats with three sections, one of which can be folded under to be out of the way. The first section on the right would have the outline of a 10-rod at the bottom of the section. This outline is used to determine whether the sum of the two rods (representing the numbers in the ones place) is greater than ten and therefore requires a trade. The second section (or middle section) would have the outline of a flat at the bottom of the section. This would be used to determine if the sum of the tens is greater than ten tens and therefore requires a trade.

Before, the standard algorithm for subtraction is introduced children should practice: (a) Empty Number Line (ENL) exercises on addition and subtraction for a month or two, (b) Practice extended addition and subtraction facts on Addition Practice Ladders and Subtraction Practice Ladders. For example:

Teacher: What is $17 - 9 = ?$

Students: 8.

Teacher: How did you find the answer?

Student 1: I just knew it: $17 - 9 = 8$.

Student 2: I know $8 + 9$ is 17, so $17 - 9 = 8$.

Student 3: $9 + 1 = 10$ and 7 more is 17, so $9 + 8 = 17$.

Student 4: $17 - 7 = 10$ and $10 - 2 = 8$, so $17 - 9 = 8$.

Student 5: I asked myself: What is $9 + 9$? It is 18. So, $9 + 8 = 17$. The answer, therefore, is 8.

Teacher: All of you are right. Let us see if we can make the problem a slightly challenging one.

Now the teacher should extend their thinking by using these types of problems: What is $27 - 9 = ?$ $57 - 9 = ?$ $37 - 9 = ?$ $97 - 9 = ?$

Then the teacher should solve a subtraction problem by the help of Cuisenaire rods and BaseTen blocks. First the teacher should take problems where no regrouping is involved and then problems where regrouping at the ones level is involved: Let us take an example of subtraction without regrouping: Find the difference:

$$\begin{array}{r} 37 \\ -25 \\ \hline \end{array}$$

Use BaseTen blocks and Cuisenaire rods to represent the problem. Show both the minuend and the subtrahend with the blocks/rods (seen on the side). The teacher should read this problem in multiple ways. However, first she should ask the children if anyone knows how to read this problem. When a child reads the problem, she should acknowledge whether it was read correctly or not. If it was partially correct, she should help the child to shape it. If it is incorrect, ask the other children if they can help the child to shape it. When all the ways of

reading from children are exhausted, then she should *systematize* all the ways of reading the problem. The process of systematizing is the most important part of teaching. All children are system seekers, and one of the processes of system seeking is looking for patterns. A pattern emerges if there are several examples of that phenomenon. Therefore, the teacher should ask a lot of questions.

Teacher: Let us read the problem.

Student A: 37 take away 25.

Student B: 37 minus 25.

Student C: 37 subtract 25.

Teacher: Another way? We can also say what you just said as: 25 subtract from 37. It seems I should also show you to read another way: 37 is how much more than 25?

Student D: I can read it another way: 25 is how much less than 37?

Teacher: What should we add to 25 so that we have 37?

Teacher: Am I correct if I say: Take 25 away from 37? Or, subtract 25 from 37?

Some children will agree and others may not. At this point the teacher must emphasize the role of the words here, particularly ‘*from.*’

After these discussions, the teacher comes back to the procedure.

Teacher: So we can use any of these expressions. With your rods you can see that it is really two subtraction problems, in one. So we can subtract two 10-rods of 25 from the three 10-rods of 37 and we can subtract the 5-rod from the 7-rod. So we get a 10-rod and a 2-rod. That means the answer is $10 + 2 = 12$. Therefore, $37 - 25 = 12$.

$$\begin{array}{r} 37 \\ -25 \\ \hline 12 \end{array}$$

The work with the rods and blocks should be recorded on the board and on individual student papers concurrently. The teacher should solve several problems of this type and children practice more problems till they are proficient in the procedure.

Teacher: Let us find the answer to this problem by the help of the Empty Number Line.

First we treat this problem as an addition problem. Can someone express this subtraction problem as an addition problem?

Student 1: 25 plus what number is 37?

Student 2: What do I add to 25 to get 37?

Teacher: Let us show these statements by the Empty Number Line. What do we start with?

The teacher draws an ENL and demonstrates the action. Other students' suggestions about representing this by the ENL are shown on the board. See the following ENL diagrams.

After several problems of this type have been solved using ENL, the teacher begins subtraction problems with 'regrouping,' using ENL (see following diagram).

After students have achieved a fair level of proficiency in these kinds of problems, using the ENL, the teacher introduces the standard procedure, first concretely and then abstractly.

Now let us take a problem where regrouping is involved: $42 - 26 = ?$

The same type of questions are asked again to provide the linguistic component for the procedure.

For this subtraction problem, we use the same place value mat, but just put the first number on the mat and use numeral cards to indicate the number to be subtracted. For example, for $42 - 26$, show four 10-rods in the tens section and the 2-rod for the ones section.

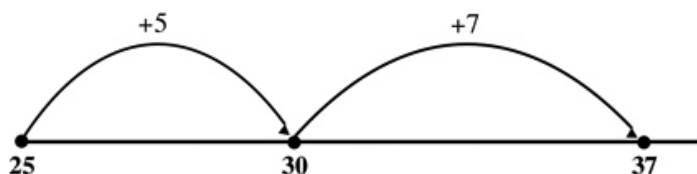


Figure 9.3 ENL to show $25 + \underline{\quad} = 37$

Below the mat show respective numeral rods for two tens (two 10-rods) and six ones (6-rod). This leaves one ten and six for the answer. The recording would be the same as that done with the BaseTen blocks.

Teacher: Is this problem different in any way than the other subtraction problems that you have already computed?

Children: Yes!

Teacher: Ok! How is it different?

Child One: The one's place in the second number is bigger than the one's digit in the first number. You cannot subtract 6 from 2.

Teacher: Yes! That is true! This problem is different from the previous subtraction problems. We cannot treat this problem as two separate subtraction problems as we did earlier: Ones place problem and tens place problem. We cannot take away or subtract 6 from 2. What do we do, then?

Child 2: We can do the same thing we did with the Empty Number Line. 26 plus 10 is 36 and then 4 more is 40 and then 2 more is 42. The answer is: $10 + 4 + 2 = 16$.

Teacher: Yes, that was great and most times that is what I want you to do, but when the numbers are very large and not the kind of numbers we are working now, we need to have another great method. At present, we cannot subtract 6 from 2, but of course, we can make 2 bigger. Can someone tell me how do we make 2 bigger?

Children suggest a variety of ways. One child raises his hand. The teacher recognizes her.

Teacher: Now, can we take 6 away from 12? Look at my arrangement. To subtract 6 from 12, a trade will need to be made from the tens section to the ones section. Since six will be subtracted, ten will need to be shown as 6 and 4 (dark green and purple rods). Then six can be removed. Place the dark green rod on the numeral six below the mat. That leaves a purple rod and a red rod for a result of six. Then two tens can be removed from the tens section and placed on the numeral two.

A child may say: let us make the 2 bigger. However, no child comes forward. The teacher introduces this idea that we can subtract 6 if we make 2 bigger.

The Child: We can make 2 bigger by taking one of the tens from the other tens of 42.

Teacher: Yes! You are right, Melanie. We can make the 2-rod bigger by taking a 10-rod from four 10-rods of 40 from the tens section and joining it with the 2-rod to make it 12. Look at the new arrangement of rods. Do we still have 42?

Most children agree with the statement. The teacher should make sure that all children understand this arrangement clearly.

Teacher: How do we know that we have the same quantity?

Student: Because $30 + 12$ is 42. The first row is still 42.

Teacher: Great! We should give Ken a big hand. Now we have two subtraction problems. What are the two subtraction problems?

Student: We can subtract 6 away from 12 and we can subtract 20 from 30.

Teacher! Yes! Now we can take 6 away from 12. Look at my arrangement of your own arrangement. Does anyone know what is $12 - 6$?

Almost all of them shout out $12 - 6 = 6$.

Now we have two subtraction problems: We can subtract 6 from 12, and we can subtract 20 from 30.

Teacher: Can someone remind us of the script for subtracting 6 from 12?

Student: I ask myself: How do I get to 10? Add 4. How do I get from 10 to 12? Add 2. How do I get from 6 to 12? $4 + 2$. So $12 - 6 = 6$.

Teacher: Fantastic! Everyone should give Ken a big hand. Can someone else give me another way of finding $12 - 6$?

Student: $12 - 10$ and then $10 - 4$ is 6.

Teacher: Great! Give Michael a big hand. What is $30 - 20$?

Student: 10.

Teacher: Great! Let us complete the problem. What is $42 - 26$?

Student: 16.

Teacher: Great! Give Jack and all the others who helped get the answer a big hand.

Then the teacher records this concrete activity into the symbolic procedural form. The teacher provides enough practice in the standard procedure till the students achieve mastery with rigor. Mastery means each student can execute the standard procedure for subtraction correctly, consistently, and fluently with understanding. This practice should include application of the procedure in solving word problems.

Teacher: Now think of a story problem relating to this problem. Work with your partner and come up with a story problem that uses these two numbers and somehow asks to subtract 26 from 42.

Children discuss and work on writing their stories, the teacher helps some children and provides support to others. As soon as one team writes their story and the teacher has checked their story, she asks the two students to go and help two teams to help arrive at the story problem. The same is repeated with other children till everyone has written a story. The teacher provides enough practice in the standard procedure till the students achieve mastery with rigor. Mastery means each student can execute the standard procedure for subtraction correctly, consistently, and fluently with understanding. This practice should include the application of the procedure in solving word problems.

By the end of second grade, children should have acquired additive reasoning, the addition and subtraction facts, and competence in executing standard procedures for addition and subtraction.

E. Numbersense and Mental Mathematics

Mental math plays an important role in acquiring fluency and facility in mathematics learning, particularly the mastery of arithmetic facts. First, let us understand what we mean by mental math. We can define mental math in various ways. For some, mental math means rote memorization of addition flash cards, times tables, and remembering the solutions to problems. For others, mental mathematics is the ability to perform simple calculations in one's head. We define it as: When a child can compute the answer to a simple arithmetic operation without pencil and paper, elaborate procedures, or other physical and electronic aids.

Mental mathematics is the ability to see relationships between numbers and their properties and operations, their interrelationships and their properties. The purpose of mental mathematics is to improve numbersense. Thus, it is a group of skills that allow a student to do math 'in their head' without using pencil and paper or a calculator. Some of the skills:

- a. Mastery of arithmetic facts: for example, $8 + 5 = 13$ or $8 \times 5 = 40$

- b. Properties of numbers: Product of an even number and any other number will be an even number.
- c. Properties of operations: Commutative property of addition and multiplication makes mastering addition and multiplication easier.
- d. Decomposition and recomposition: Every operation is an exercise in decomposition and recomposition.
- e. Estimation and approximation.

The following examples illustrate the integration of these skills.

Examples 1: $85 + 17 = ?$

- $85 + 17$ is thought of as : $85 + 10 = 95$, $95 + 5 = 100$, and $100 + 2 = 102$;
Or, $85 + 15 = 100$, $100 + 2 = 102$;
Or $85 + 20$ and $105 - 3 = 102$; Or, $85 + 5 = 90$ and $90 + 12 = 102$.

Example 2: $1004 - 647$

- $1004 - 647$ is thought of as: $647 + 3 = 650$, $650 + 50 = 700$, $700 + 300 = 1000$, and $1000 + 4 = 1004$.

Therefore, the answer is: $3 + 50 + 300 + 4 = 357$;

- $1004 - 647 = 357$;
- Or, $647 + 300 = 947$, $947 + 50 = 997$, and $997 + 3 = 1000$, and $1000 + 4 = 1004$.

Practice in mental mathematics builds confidence in dealing with numbers, fluency in seeing relationships between numbers, ability to make sense of quantities in personal and professional lives with confidence, and strong numbersense. Other skills that are also enhanced include rounding numbers and estimating calculations.

Mental math is useful in everyday life to answer questions like:

- What is the sale price for this item?
- Do I have enough cash to buy everything in my cart?
- Am I getting the correct amount of change from the cashier?
- When should I leave in order to arrive on time?

The two concepts of mental mathematics and numbersense are intricately related. Progress and fluency in one improve the other. Mental math and numbersense help students understand mathematics concepts better. Using and practicing mental math regularly helps students improve their numbersense. Understanding of numbersense—number concept,

arithmetic facts, and place value—improves mental mathematics. However, in the current context, we want to limit numbersense and mental mathematics only related to additive reasoning.

Mental math is useful in other ways, too. It keeps math facts from getting rusty. It can also help you get to the answers faster.

For example, what is $47 + 65$? It is easier to do this in your head if you break the numbers down into their digits' values: $40 + 60 = 100$ and $7 + 5 = 12$. Add those two totals and you get the answer of 112. Better still: Let me pretend 47 is 45. Then $45 + 65$ is 110 and 2 more is 112. Or, 65 and 35 is 100 and 12 more is 112. This facility is dependent on effective numbersense.

Some children develop mental mathematics skills at their own pace by themselves. When children struggle with mental math, it does not mean they are not smart. Difficulty with mental math could be a matter of needing more time and practice to “catch up” and learning strategies and skills to improve short-term and working memories.

Trouble with mental math could also be a sign that the child needs more support to get better at math. Several methods are available to learn and practice mental mathematics. The traditional method for developing mental mathematics is called learning ‘by heart,’ or it is better known as the rote memorizing method, where teachers ask students to memorize addition and multiplication tables through repetition of information with the help of flash cards or some such device. Rote memorizing has some utility and works for many children to some extent. This system gives some degree of success initially as one can answer easy problems, but then when the bigger application problems come, one is not conceptually prepared for them. In many cases, this method, if used exclusively, not only kills the interest of children in mathematics but may also develop a dislike for the subject for the rest of the years they have to study it. Rote memorizing does not work effectively for all students.

The second method is quite popular in Asia, particularly in China and Japan. In these cultural settings, it provides a good degree of success, and we would highly recommend it for younger children because of its kinesthetic value. It hails from China and is popularly known as ‘the abacus’ (also known as the ‘Soroban’ in Japan). An abacus is a calculating tool, often constructed as a wooden frame with beads sliding on wires. With the use of this tool, one can perform calculations relating to addition, subtraction, multiplication, and division with ease. Gradually one practices with the tool in one’s hand and later on when experienced, one learns to do it without the tool. A visual image of the tool helps the child to add, subtract, multiply, and divide in seconds.

The main drawback of this system is that one needs it for a longer time to be able to fully grasp the system; hence you see courses in the abacus stretching to over 2 years, which leads the child to boredom and then quitting the course.

The next mental math system was developed during the Second World War in a Nazi concentration camp by Ukrainian mathematician Jakow Trachtenberg to keep his mind occupied. It is known as the Trachtenberg Speed System of Mathematics and consists of

rapid mental methods of doing mathematics. The system consists of a number of readily memorized patterns that allow one to perform arithmetic computations very quickly. The method focuses on patterns for multiplication by particular numbers, say 5, 6, 7, and even 11 and 12. It then gives a general method for rapid multiplication and a special two-finger method.

Another mental math method is known as Kumon Math, founded by Japanese educator Toru Kumon in the 1950s. Students do not work together as a class but progress through the curriculum at their own pace, moving to the next level when they have achieved mastery of the previous level. This sometimes involves repeating the same set of worksheets until the student achieves a satisfactory score within a specified time limit. In North American Kumon Centers, the mathematics program starts with basic skills such as pattern recognition and counting.

However, the Kumon Math program is not mental math. It does not offer any special methods to conceptualizing mathematics and one does not improve one's speed of doing mathematics. There is a set curriculum of worksheets on each topic, which one does until one achieves mastery in the topic. These sheets are designed on the principle that if we break the problem into its simple steps and each step is practiced, the child will acquire proficiency in mathematics. So, for example, when completing a sheet on division, one would continue to do division by the conventional method until one gets a satisfactory score, and then one moves on to a higher level. This certainly does not make division any faster, and the process is certainly not mental.

Another method is Vedic Mathematics. It has its roots in ancient Indian scriptures called the Vedas, meaning 'the fountain head of knowledge.' With it, not only can you add, subtract, multiply, or divide which is the limiting factor of the abacus but you can also solve complex mathematics such as algebra, geometry, calculus, and trigonometry. Some of the most advanced, complex, and arduous problems can be solved with extreme ease using the Vedic Mathematics method. All this is done by memorizing 16-word formulas called Sutras.

All of the methods mentioned have some limitations and drawbacks because of the methodology or the cultural roots. To use these mental math methods, teachers have to transcend cultural and idiosyncratic methods. We have integrated the principles from all of these methods in our approach to teaching arithmetic facts. The approach integrates mental math principles with research on the psychology of learning mathematics. It is based on the fundamental idea that mathematics is the study of patterns in quantity and space and the strategies based on 'decomposition/recomposition' are easier to learn and more effective to apply in more cases/examples/tasks. From this perspective, arithmetic is the study of patterns in numbers and their relationships. Addition and multiplication facts are the recognition of patterns, and the understanding and mastery of these patterns helps with understanding the patterns in higher mathematics.

This approach has resulted in the development of a sequence of strategies and methods of teaching arithmetic facts—addition, subtraction, multiplication, and division that prepare

children and give them confidence to apply this mastery of arithmetic facts to higher-order mathematics such as fractions, integers, and algebra, and later apply their mathematics skills to solving problems in and out of mathematics.

This system does not focus on learning by just repetition as in the Kumon Method or sheer memorization of some formulas as in Trachtenberg or Vedic Mathematics. Our objective and focus are to help children improve their mathematical way of thinking. The purpose is not limited to improving performance in school or on tests, but instead we provide a broader outlook, resulting in improved mathematical intelligence, problem solving competence, and mental sharpness.

How to Improve Mental Math

Achieving competence in mental math requires strong *memory skills*, particularly the working memory. Working memory is the sketchpad of the brain. Working memory is where the problem-related information coming from short-term memory is held and mixed with the information (related to the problem) that is brought from the long-term memory. The pre-frontal cortex executes the scanning of the new information from short-term memory and searches for relevant information from the long-term memory that is related to the given problem, such as math facts, formulas, definition, strategies, scripts, and other concepts and retrieves them and brings them to the working memory space. There the relationships, connections, and integration between these two categories of information are formed, some involuntarily and some by intentions. Here, some new connections are made, some are strengthened and the new information is used or sent back to the long-term memory as new learning. The results of the work in the working memory are always transferred to the long-term memory. This includes partial solutions and experiences from the solution process and efforts, whether the problem is solved or not. All this information, over time, just being there in the long-term memory makes further connections and transforms some of the information into new forms and new skills. Every piece of new information entering the long-term memory transform some old memories ([Peter et al., 2010](#); [Ranganath, 2024](#)).

In our working memory, we keep track of the steps and processes needed to organize, rearrange, and solve a particular math problem. Working memory is one of the brain's executive functions. It is a skill and a place that allows us to work with information without losing track of what we are doing. Think of working memory as a temporary sticky note in the brain. It holds new information in place so the thinking brain can work with it briefly and connect it with other information. For example, in mental mathematics, working memory lets a student 'see' in their head the numbers that one is dealing with. One may not remember any of the numbers by the next problem even 10 minutes later. But the essence (the schemas formed, reformed, newly learned) of number relationships formed in solving problems are constantly transferred from the working memory to long-term memory. Working memory has done its short-term job by helping them tackle the task at hand.

Working memory is not just for short-term use. It also helps the brain to organize (i.e., seeing patterns, seeing connections, making conjectures, and converting information into knowledge and skills) the new information so that it is ‘storage- and retrieval-rich’ before it is sent for long-term storage from the working memory to long-term memory. When students have trouble with working memory, they may store information in the long-term memory in a jumbled way that may not be storage- and retrieval-rich. Or they may not be stored in the long-term memory at all. That is why regression in our learning takes place—that means we forget what we learned.

Sometimes, what may look like trouble with or deficit in working memory is an intention, attention, and/or focus issue: For example, if the information was never received properly in the working memory, the intention was not there to process it effectively. Or no attention was paid, and it was not put into the long-term storage system in the first place. Intention, attention, focus, working memory, and practice are crucial for everyday living and learning. They are very important aspects of executive functioning that help us take the information in and make sense of the new information and convert it into useful knowledge that needs to be stored for future retrieval. But while they are closely linked, these functions are not the same thing.

The twin processes of *attention* and *focus* make it possible for the information to be noticed and paid attention to and alerts the pre-frontal cortex to take notice of the new information. Pre-frontal cortex then allows us to take information in and send it to the working memory. However, there is a step before information enters the short-term memory. If the information is multi-modal, multi-sensory, and multi-model, it enters the memory system optimally. Even after it enters the memory system, we still select useful components from the information received. We can think of it as a funnel. We gather what we need, find interesting and useful to know, and channel it into other components of the memory and executive systems.

After gathering information, the attention funnel feeds it into the brain’s short-term storage bucket. This is where new information is first held. Experts call this process encoding. This is also where the brain manipulates new information, so it is made useful. This process takes place in the ‘working memory.’ The quality of work in the working memory is a function of attention.

There are five important parts of paying attention well. A learner may have trouble with any or all of these components.

1. *Intention and alertness.* Learner must have an intention to pay attention to and needs to be ready to pay attention to the incoming information—*be present*.
2. *Focus.* Not all the information we are paying attention to needs to be attended to at any given time. A learner’s goals and priorities help the pre-frontal cortex to be able to identify what deserves attention and what to focus on in the incoming information. We

need to pay attention to the information that is useful for us in the present or for our future needs.

3. *Selection.* With focused attention one selects what one needs for a current task and for future usage. For example, one must focus on the sequence of tasks to select the information when a particular strategy is to be used for solving a problem. At the same time, if some decisions are to be made, for example, whether a particular strategy works or does not work. Under what conditions does it work and where it does not work? These evaluative tasks cannot be performed without focus and attention.
4. *Sustaining attention.* The learner must stay reasonably attentive over time. This could be for a three-minute presentation or for a 40-minute lecture. The hippocampus notes the beginning and ending of a task, but the pre-frontal cortex brings the attention on specific tasks, specific strategies, and specific sequence of tasks to be executed. The Default Mode Network (DMN) is always behind the scenes forming schemas of activities. These are the strategies, procedures, scripts, and routines of practice. Practicing a task brings it to the habit level—automatized forms and transfers it to the Basal Ganglia, where habits reside. Procedures and methods are schemas and reside in DMN. Skills and arithmetic facts should be converted into habits.
5. *Shifting.* The learner should be able to shift attention briefly when important new information is introduced. For example, problems involving multiple steps, and multiple strategies require shifting of attention and a focus on specific part of the strategies or procedure.

Learners with attention issues may not recall what they have been taught. That's because it never 'got into their head' in the first place. Learning is a participatory sport, not a spectator sport. Every part of the brain is actively involved; one is engaged.

Working memory is an active and instant process. It allows us to use new and learned information while we are in the middle of an activity.

Imagine an introductory lesson on subtraction. The teacher is talking about the language of subtraction or a situation where the operation of subtraction is called for. As students listen, their working memory does things with the information they hear so it can have meaning and relevance. To make meaning of this information, they have to bring information from the long-term memory such as: what do these words and symbols mean, what do they know about the number—the nature and properties of the number, the definition of the addition or subtraction, addition and subtraction facts that they already know, what script/strategy/method or skill to use, etc.?

This often involves putting pieces of information into some kind of order. So that it could be acted upon in chronological order. For example, they might have to recall a strategy, a fact, or what they know about decomposition/recomposition or teens' numbers.

Once this new information is processed, it leaves the brain's short-term storage bucket. Next, it moves into a larger long-term 'tank.' If the teacher quizzes students at the end of the

lesson, the information is retrieved from the long-term memory tank.

When children have working memory issues, the information they put into long-term storage may be disorganized in their minds. It might not carry much meaning because it wasn't packaged correctly, and so it may not be very useful. Working memory issues may also cause information to be lost and not enter the long-term tank.

While attention and working memory are different, both are important for learning. Learners with ADHD and executive functioning issues struggle with attention and working memory. And many learners with language-based learning differences may struggle in these areas, too.

Children with multiple challenges like these may find it tougher to learn. But understanding exactly where they have trouble makes it easier to help them succeed.

Note

1. Since 1995, TIMSS has monitored trends in mathematics and science achievement every four years, at the fourth and eighth grades. TIMSS 2023 is the eighth such assessment, providing nearly three decades of trend data. TIMSS 2023 completes the transition to a fully digital administration, concluding a process that began in TIMSS 2019, and features new and interactive item formats. TIMSS Advanced studies achievement in advanced mathematics and physics for students in their final year of secondary school. TIMSS Advanced was conducted in 1995, 2008, and 2015. Trends in International Mathematics and Science Study (2000, 2004, 2020). Boston College, TIMSS & PIRLS International Study Center. [🔗](#)

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Extended Descriptions for Chapter 1

Extended Description for Figure 1.1

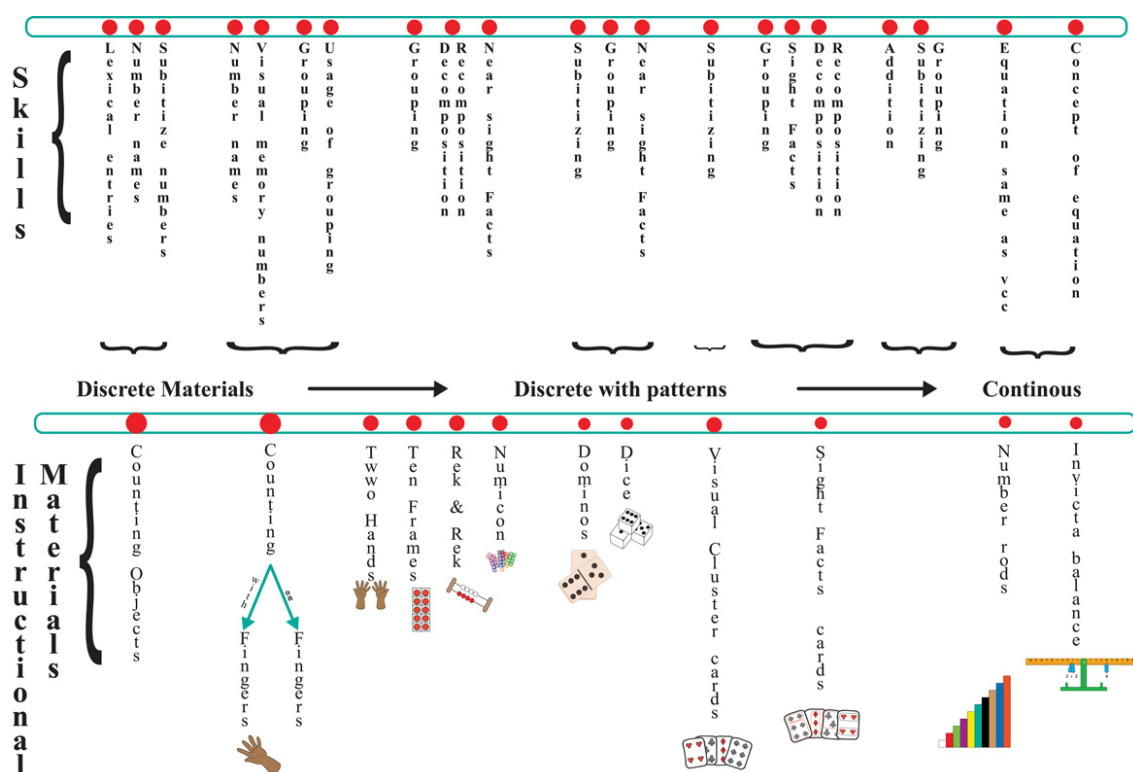


Figure 1.1 Instructional materials continuum for number concept

The number concept continuum has two horizontal sections labeled Skills and Instructional Materials. The Skills section progresses from discrete skills toward continuous concepts. From left to right, it lists Lexical entries, Number names, Subitizing numbers, Visual memory numbers, Grouping, Use of grouping, Decomposition, Recombination, Number sight facts, Subitizing, Grouping, Near sight facts, Subitizing, Grouping, Sight facts, Decomposition, Addition, Subitizing, Grouping, Equation same as VCC, and Concept of equation, with curly brackets grouping related skills. The

Instructional Materials section progresses from Discrete Materials through Discrete with patterns to Continuous, with right-pointing arrows connecting the three stages. Discrete Materials include Counting Objects, Counting Fingers, Two Hands, Ten Frames, Reck and Reck, and Numicon. Discrete with patterns includes Dominos, Dice, Visual Cluster cards, and Sight Facts cards. Continuous includes Number rods and Inequality balance. Counting Fingers branches into two instances labeled Fingers. Each material is paired with a corresponding pictorial representation, and curly brackets group the instructional materials under the three stages.

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